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Volume 4

Part II

D.H.Fremlin

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MEASURE THEORY

Volume 4

Topological Measure Spaces

Part II

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Contents

Chapter 46

Pointwise compact sets of measurable functions

This chapter collects results inspired by problems in functional analysis. §§461 and 466 look directly at measures on linear topological spaces. The primary applications are of course to Banach spaces, but as usual we quickly find ourselves considering weak topologies. In §461 I look at 'barycenters', or centres of mass, of probability measures, with the basic theorems on existence and location of barycenters of given measures and the construction of measures with given barycenters. In §466 I examine topological measures on linear spaces in terms of the classification developed in Chapter 41. A special class of normed spaces, those with 'Kadec norms', is particularly important, and in §467 I sketch the theory of the most interesting Kadec norms, the 'locally uniformly rotund' norms.

In the middle sections of the chapter, I give an account of the theory of pointwise compact sets of measurable functions, as developed by A.Bellow, M.Talagrand and myself. The first step is to examine pointwise compact sets of continuous functions (§462); these have been extensively studied because they represent an effective tool for investigating weakly compact sets in Banach spaces, but here I give only results which are important in measure theory, with a little background material. In §463 I present results on the relationship between the two most important topologies on spaces of measurable functions, *not* identifying functions which are equal almost everywhere: the pointwise topology and the topology of convergence in measure. These topologies have very different natures but nevertheless interact in striking ways. In particular, we have important theorems giving conditions under which a pointwise compact set of measurable functions will be compact for the topology of convergence in measure (463G, 463L).

The remaining two sections are devoted to some remarkable ideas due to Talagrand. The first, 'Talagrand's measure' (§464), is a special measure on $\mathcal{P}I$ (or $\ell^\infty(I)$), extending the usual measure of $\mathcal{P}I$ in a canonical way. In §465 I turn to the theory of 'stable' sets of measurable functions, showing how a concept arising naturally in the theory of pointwise compact sets led to a characterization of Glivenko-Cantelli classes in the theory of empirical measures.

461 Barycenters and Choquet's theorem

One of the themes of this chapter will be the theory of measures on linear spaces, and the first fundamental concept is that of 'barycenter' of a measure, its centre of mass (461Aa). The elementary theory (461B-461E) uses non-trivial results from the theory of locally convex spaces (§4A4), but is otherwise natural and straightforward. It is not always easy to be sure whether a measure has a barycenter in a given space, and I give a representative pair of results in this direction (461F, 461H). Deeper questions concern the existence and nature of measures on a given compact set with a given barycenter. The Riesz representation theorem is enough to tell us just which points can be barycenters of measures on compact sets (461I). A new idea (461K-461L) shows that the measures can be moved out towards the boundary of the compact set. We need a precise definition of 'boundary'; the set of extreme points seems to be the appropriate concept (461M). In some important cases, such representing measures on boundaries are unique (461P). I append a result identifying the extreme points of a particular class of compact convex sets of measures (461Q-461R).

461A Definitions (a) Let X be a Hausdorff locally convex linear topological space, and μ a probability measure on a subset A of X. Then $x^* \in X$ is a **barycenter** or **resultant** of μ if $\int_A g\,d\mu$ is defined and equal to $g(x^*)$ for every $g \in X^*$. Because X^* separates the points of X (4A4Ec), μ can have at most one barycenter, so we may speak of 'the' barycenter of μ.

(b) Let X be any linear space over $\mathbb{R}$, and $C \subseteq X$ a convex set (definition: 2A5E). Then a function $f : C \to \mathbb{R}$ is **convex** if $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$ for all x, $y \in C$ and $t \in [0,1]$. (Compare 233G, 233Xd.)

(c) The following elementary remark is useful. Let X be a linear space over $\mathbb{R}$, $C \subseteq X$ a convex set, and $f : C \to \mathbb{R}$ a function. Then f is convex iff the set $\{(x,\alpha) : x \in C,\ \alpha \ge f(x)\}$ is convex in $X \times \mathbb{R}$ (cf. 233Xd).

461B Proposition Let X and Y be Hausdorff locally convex linear topological spaces, and $T : X \to Y$ a continuous linear operator. Suppose that $A \subseteq X$, $B \subseteq Y$ are such that $T[A] \subseteq B$, and let μ be a probability measure on A which has a barycenter x^* in X. Then Tx^* is the barycenter of the image measure μT^{-1} on B.

proof All we have to observe is that if $g \in Y^*$ then $gT \in X^*$ (4A4Bd), so that

$$g(Tx^*) = \int_A g(Tx)\mu(dx) = \int_B g(y)\nu(dy)$$

by 235G[1].

461C Lemma Let X be a Hausdorff locally convex linear topological space, C a convex subset of X, and $f : C \to \mathbb{R}$ a lower semi-continuous convex function. If $x \in C$ and $\gamma < f(x)$, there is a $g \in X^*$ such that $g(y) + \gamma - g(x) \le f(y)$ for every $y \in C$.

proof Let D be the convex set $\{(x, \alpha) : x \in C, \alpha \ge f(x)\}$ in $X \times \mathbb{R}$ (461Ac). Then the closure $\overline{D}$ of D in $X \times \mathbb{R}$ is also convex (2A5Eb). Now D is closed in $C \times \mathbb{R}$ (4A2B(d-i)), and $(x, \gamma) \notin D$, so $(x, \gamma) \notin \overline{D}$.

Consequently there is a continuous linear functional $h : X \times \mathbb{R} \to \mathbb{R}$ such that $h(x, \gamma) < \inf_{w \in \overline{D}} h(w)$ (4A4Eb). Now there are $g_0 \in X^*$, $\beta \in \mathbb{R}$ such that $h(y, \alpha) = g_0(y) + \beta\alpha$ for every $y \in X$ and $\alpha \in \mathbb{R}$ (4A4Be). So we have

$$g_0(x) + \beta\gamma = h(x, \gamma) < h(y, f(y)) = g_0(y) + \beta f(y)$$

for every $y \in C$. In particular, $g_0(x) + \beta\gamma < g_0(x) + \beta f(x)$, so $\beta > 0$. Setting $g = -\frac{1}{\beta} g_0$,

$$f(y) \ge \tfrac{1}{\beta} g_0(x) + \gamma - \tfrac{1}{\beta} g(y) = g(y) + \gamma - g(x)$$

for every $y \in C$, as required.

461D Theorem Let X be a Hausdorff locally convex linear topological space, $C \subseteq X$ a convex set and μ a probability measure on a subset A of C. Suppose that μ has a barycenter x^* in X which belongs to C. Then $f(x^*) \le \underline{\int}_A f\,d\mu$ for every lower semi-continuous convex function $f : C \to \mathbb{R}$.

proof Take any $\gamma < f(x^*)$. By 461C there is a $g \in X^*$ such that $g(y) + \gamma - g(x^*) \le f(y)$ for every $y \in C$. Integrating with respect to μ,

$$\underline{\int}_A f d\mu \ge \gamma - g(x^*) + \int_A g\,d\mu = \gamma.$$

As γ is arbitrary, $f(x^*) \le \underline{\int}_A f\,d\mu$.

Remark Of course $\underline{\int}_A f\,d\mu$ might be infinite.

461E Theorem Let X be a Hausdorff locally convex linear topological space, and μ a probability measure on X such that (i) the domain of μ includes the cylindrical σ-algebra of X (ii) there is a compact convex set $K \subseteq X$ such that $\mu^* K = 1$. Then μ has a barycenter in X, which belongs to K.

proof If $g \in X^*$, then $\gamma_g = \sup_{x \in K} |g(x)|$ is finite, and $\{x : |g(x)| \le \gamma_g\}$ is a measurable set including K, so must be conegligible, and $\phi(g) = \int g\,d\mu$ is defined and finite. Now $\phi : X^* \to \mathbb{R}$ is a linear functional and $\phi(g) \le \sup_{x \in K} g(x)$ for every $g \in X^*$; because K is compact and convex, there is an $x_0 \in K$ such that $\phi(g) = g(x_0)$ for every $g \in X^*$ (4A4Ef), so that x_0 is the barycenter of μ in X.

461F Theorem Let X be a complete locally convex linear topological space, and $A \subseteq X$ a bounded set. Let μ be a τ-additive topological probability measure on A. Then μ has a barycenter in X.

proof (a) If $g \in X^*$, then $g{\restriction}A$ is continuous and bounded (3A5N(b-v)), therefore μ-integrable. For each neighbourhood G of 0 in X, set

$$F_G = \{y : y \in X, |g(y) - \int_A g\,d\mu| \le 2\tau_G(g) \text{ for every } g \in X^*\}$$

where $\tau_G(g) = \sup_{x \in G} g(x)$ for $g \in X^*$. Then F_G is non-empty. **P** Set $H = \operatorname{int}(G \cap (-G))$, so that H is an open neighbourhood of 0. Because A is bounded, there is an $m \ge 1$ such that $A \subseteq mH$. The set $\{x + H : x \in A\}$ is an open cover of A, and μ is a τ-additive topological measure, so there are $x_0, \dots, x_n \in A$ such that $\mu(A \setminus \bigcup_{i \le n}(x_i + H)) \le \frac{1}{m}$. Set

[1] Formerly 235I.

$$E_i = A \cap (x_i + H) \setminus \bigcup_{j<i}(x_j + H)$$

for $i \le n$, and $y = \sum_{i=0}^{n}(\mu E_i)x_i$; set $E = A \setminus \bigcup_{i\le n}(x_i + H)$. Then, for any $g \in X^*$,

$$|g(y) - \int_A g\,d\mu| \le \sum_{i=0}^{n} |\mu E_i g(x_i) - \int_{E_i} g\,d\mu| + \int_E |g|d\mu$$
$$\le \sum_{i=0}^{n} \int_{E_i} |g(x) - g(x_i)|\mu(dx) + m\tau_G(g)\mu E$$

(because $E \subseteq mH$, so $g(x) \le m\tau_G(g)$ and $g(-x) \le m\tau_G(g)$ for every $x \in E$)

$$\le \sum_{i=0}^{n} \tau_G(g)\mu E_i + m\tau_G(g)\mu E$$

(because if $i \le n$ and $x \in E_i$, then $x - x_i$ and $x_i - x$ belong to G, so $|g(x) - g(x_i)| = |g(x - x_i)| \le \tau_G(g)$)

$$\le 2\tau_G(g).$$

As g is arbitrary, $y \in F_G$ and $F_G \ne \emptyset$. **Q**

(c) Since $F_{G\cap H} \subseteq F_G \cap F_H$ for all neighbourhoods G and H of 0, $\{F_G : G$ is a neighbourhood of $0\}$ is a filter base and generates a filter $\mathcal{F}$ on X. Now $\mathcal{F}$ is Cauchy. **P** If G is any neighbourhood of 0, let $G_1 \subseteq G$ be a closed convex neighbourhood of 0, and set $H = \frac{1}{4}G_1$. **?** If y, $y' \in F_H$ and $y - y' \notin G$, then there is a $g \in X^*$ such that $g(y - y') > \tau_{G_1}(g)$ (4A4Eb again). But now

$$\tau_H(g) = \frac{1}{4}\tau_{G_1}(g) < \frac{1}{4}(|g(y) - \int_A g\,d\mu| + |g(y') - \int_A g\,d\mu|)$$
$$\le \frac{1}{4}(2\tau_H(g) + 2\tau_H(g)). \text{ **X**}$$

This means that $F_H - F_H \subseteq G$; as G is arbitrary, $\mathcal{F}$ is Cauchy. **Q**

(d) Because X is complete, $\mathcal{F}$ has a limit x^* say. Take any $g \in X^*$. **?** If $g(x^*) \ne \int_A g\,d\mu$, set $G = \{x : |g(x)| \le \frac{1}{3}|g(x^*) - \int_A g\,d\mu|\}$. Then G is a neighbourhood of 0 in X, and

$$0 < |g(x^*) - \int_A g\,d\mu| = \lim_{x\to\mathcal{F}} |g(x) - \int_A g\,d\mu|$$
$$\le \sup_{x\in F_G} |g(x) - \int_A g\,d\mu| \le 2\tau_G(g) \le \frac{2}{3}|g(x^*) - \int_A g\,d\mu|. \text{ **X**}$$

So $g(x^*) = \int_A g\,d\mu$; as g is arbitrary, x^* is the barycenter of μ.

461G Lemma Let X be a normed space, and μ a probability measure on X such that every member of the dual X^* of X is integrable. Then $g \mapsto \int g\,d\mu : X^* \to \mathbb{R}$ is a bounded linear functional on X^*.

proof Replacing μ by its completion if necessary, we may suppose that μ is complete, so that every member of X^* is Σ-measurable, where Σ is the domain of μ. (The point is that μ and its completion give rise to the same integrals, by 212Fb.) Set $\phi(g) = \int g\,d\mu$ for $g \in X^*$. For each $n \in \mathbb{N}$ let E_n be a measurable envelope of $B_n = \{x : \|x\| \le n\}$; replacing E_n by $\bigcap_{i\ge n} E_i$ if necessary, we may suppose that $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-decreasing. If $n \in \mathbb{N}$ and $g \in X^*$ then $\{x : x \in E_n, |g(x)| > n\|g\|\}$ is a measurable subset of E_n disjoint from B_n, so must be negligible, and $|\int_{E_n} g| \le n\|g\|$. We therefore have an element ϕ_n of X^{**} defined by setting $\phi_n(g) = \int_{E_n} g$ for every $g \in X^*$. But also $\phi(g) = \lim_{n\to\infty} \phi_n(g)$ for every g, because $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of measurable sets with union X. By the Uniform Boundedness Theorem (3A5Ha), $\{\phi_n : n \in \mathbb{N}\}$ is bounded in X^{**}, and $\phi \in X^{**}$.

461H Proposition Let X be a reflexive Banach space, and μ a probability measure on X such that every member of X^* is μ-integrable. Then μ has a barycenter in X.

proof By 461G, $g \mapsto \int g\,d\mu$ is a bounded linear functional on X^*; but this means that it is represented by a member of X, which is the barycenter of μ.

461I Theorem Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact set. Then the closed convex hull of K in X is just the set of barycenters of Radon probability measures on K.

proof (a) If μ is a Radon probability measure on K with barycenter x^*, then

$$g(x^*) = \int_K g(x)\mu(dx) \le \sup_{x\in K} g(x) \le \sup_{x\in\overline{\Gamma(K)}} g(x)$$

for every $g \in X^*$; because $\overline{\Gamma(K)}$ is closed and convex, it must contain x^* (4A4Eb once more).

(b) Now suppose that $x^* \in \overline{\Gamma(K)}$. Let $W \subseteq C(X)$ be the set of functionals of the form $g + \alpha\chi X$, where $g \in X^*$ and $\alpha \in \mathbb{R}$. Set $U = \{g{\restriction}K : g \in W\}$, so that U is a linear subspace of $C(K)$ containing χK.

If g_1, $g_2 \in W$ and $g_1{\restriction}K = g_2{\restriction}K$, then $\{x : g_1(x) = g_2(x)\}$ is a closed convex set including K, so contains x^*, and $g_1(x^*) = g_2(x^*)$; accordingly we have a functional $\phi : U \to \mathbb{R}$ defined by setting $\phi(g{\restriction}K) = g(x^*)$ for every $g \in W$. Of course ϕ is linear; moreover, $\phi(f) \le \sup_{x\in K} f(x)$ for every $f \in U$, by 4A4Eb yet again. Applying this to $\pm f$, we see that $|\phi(f)| \le \|f\|_\infty$ for every $f \in U$. We therefore have an extension of ϕ to a continuous linear functional ψ on $C(K)$ such that $\|\psi\| \le 1$, by the Hahn-Banach theorem (3A5Ab). Now

$$\psi(\chi K) = \phi(\chi K) = \chi X(x^*) = 1;$$

so if $0 \le f \le \chi K$ then

$$|1 - \psi(f)| = |\psi(\chi K - f)| \le \|\chi K - f\|_\infty \le 1,$$

and $\psi(f) \ge 0$. It follows that $\psi(f) \ge 0$ for every $f \in C(K)^+$. But this means that there is a Radon probability measure μ on K such that $\psi(f) = \int f\,d\mu$ for every $f \in C(K)$ (436J/436K). As $\mu K = \psi(\chi K) = 1$, μ is a probability measure; and for any $g \in X^*$

$$\int_K g\,d\mu = \psi(g{\restriction}K) = \phi(g{\restriction}K) = g(x^*),$$

so x^* is the barycenter of μ, as required.

461J Corollary: Kreĭn's theorem Let X be a complete Hausdorff locally convex linear topological space, and $K \subseteq X$ a weakly compact set. Then the closed convex hull $\overline{\Gamma(K)}$ of K is weakly compact.

proof Give K the weak topology induced by $\mathfrak{T}_s(X, X^*)$. Let P be the set of Radon probability measures on K, so that P is compact in its narrow topology (437R(f-ii)). By 461F, every $\mu \in P$ has a barycenter $b(\mu)$ in K. If $g \in X^*$, $g(b(\mu)) = \int_K g\,d\mu$, while $g{\restriction}K$ is continuous, so $\mu \mapsto \int_K g\,d\mu$ is continuous, by 437Kc. Accordingly $b : P \to X$ is continuous for the narrow topology on P and the weak topology on X, and $b[P]$ is weakly compact. But $b[P]$ is the weakly closed convex hull of K, by 461I applied to the weak topology on X. By 4A4Ed, $\Gamma(K)$ has the same closure for the original topology of X as it has for the weak topology, and $\overline{\Gamma(K)} = b[P]$ is weakly compact.

461K Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X, and P the set of Radon probability measures on K. Define a relation $\preccurlyeq$ on P by saying that $\mu \preccurlyeq \nu$ if $\int f d\mu \le \int f d\nu$ for every continuous convex function $f : K \to \mathbb{R}$.

(a) $\preccurlyeq$ is a partial order on P.

(b) If $\mu \preccurlyeq \nu$ then $\int f d\mu \le \int f d\nu$ for every lower semi-continuous convex function $f : K \to \mathbb{R}$.

(c) If $\mu \preccurlyeq \nu$ then μ and ν have the same barycenter.

(d) If we give P its narrow topology, then $\preccurlyeq$ is closed in $P \times P$.

(e) For every $\mu \in P$ there is a $\preccurlyeq$-maximal $\nu \in P$ such that $\mu \preccurlyeq \nu$.

proof (a) Write Ψ for the set of continuous convex functions from K to $\mathbb{R}$. Note that if f, $g \in \Psi$ and $\alpha \ge 0$ then αf, $f + g$ and $f \vee g$ all belong to Ψ. Consequently $\Psi - \Psi$ is a Riesz subspace of $C(K)$. **P** $\Psi - \Psi$ is a linear subspace because Ψ is closed under addition and multiplication by positive scalars. If f, $g \in \Psi$ then

$$|f - g| = (f - g) \vee (g - f) = 2(f \vee g) - (f + g) \in \Psi;$$

by 352Ic, $\Psi - \Psi$ is a Riesz subspace. **Q**

It follows that $\Psi - \Psi$ is $\| \|_\infty$-dense in $C(K)$. **P** Constant functions belong to Ψ, and if x, $y \in K$ are distinct there is an $f \in X^*$ such that $f(x) \ne f(y)$, in which case $f{\restriction}K$ belongs to Ψ and separates y from x. By the Stone-Weierstrass theorem (281A), $\Psi - \Psi$ is dense. **Q**

The definition of $\preccurlyeq$ makes it plain that it is reflexive and transitive. But it is also antisymmetric. **P** If $\mu \preccurlyeq \nu$ and $\nu \preccurlyeq \mu$, then $\int f d\mu = \int f d\nu$ for every $f \in \Psi$, therefore for every $f \in \Psi - \Psi$, therefore for every $f \in C(K)$, and $\mu = \nu$ by 416E(b-v). **Q**

So $\preccurlyeq$ is a partial order.

(b)(i) Now suppose that $f : K \to \mathbb{R}$ is a lower semi-continuous convex function, and $x \in K$. Then $f(x) = \sup\{g(x) : g \in \Psi,\ g \le f\}$. **P** If $\gamma < f(x)$ there is a $g \in X^*$ such that $g(y) + \gamma - g(x) \le f(y)$ for every $y \in K$, by 461C. Now $g\restriction K$ belongs to Ψ, $g\restriction K \le f$ and $(g\restriction K)(x) = \gamma$. **Q**

(ii) It follows that if $f : K \to \mathbb{R}$ is lower semi-continuous and convex, $\int f d\mu = \sup\{\int g\,d\mu : g \in \Psi,\ g \le f\}$ for every $\mu \in P$. **P** Because Ψ is closed under $\vee$, $A = \{g : g \in \Psi,\ g \le f\}$ is upwards-directed. Because μ is τ-additive and $f = \sup A$, $\int f d\mu = \sup_{g\in A} \int g\,d\mu$ by 414Ab. **Q**

So if μ, $\nu \in P$ and $\mu \preccurlyeq \nu$, then

$$\int f d\mu = \sup_{g\in\Psi,g\le f} \int g\,d\mu \le \sup_{g\in\Psi,g\le f} \int g\,d\nu = \int f d\nu;$$

as f is arbitrary, (b) is true.

(c) By 461E, applied to the Radon probability measure on X extending μ, μ has a barycenter $x \in K$. If $g \in X^*$ then $g\restriction K$ belongs to Ψ, so $\int_K g\,d\mu \le \int_K g\,d\nu$; but the same applies to $-g$, so

$$g(x) = \int_K g\,d\mu = \int_K g\,d\nu.$$

As g is arbitrary, x is also the barycenter of ν.

(d) As noted in 437Kc, the narrow topology on P corresponds to the weak* topology on $C(K)^*$. So all the functionals $\mu \mapsto \int f d\mu$, for $f \in \Psi$, are continuous; it follows at once that $\preccurlyeq$ is a closed subset of $P \times P$.

(e) Any non-empty upwards-directed $Q \subseteq P$ has an upper bound in P. **P** For $\nu \in Q$ set $V_\nu = \{\lambda : \nu \preccurlyeq \lambda\}$. Then every V_ν is closed, and because Q is upwards-directed the family $\{V_\nu : \nu \in Q\}$ has the finite intersection property. Because P is compact (437R(f-ii) again), $\bigcap_{\nu\in Q} V_\nu$ is non-empty; now any member of the intersection is an upper bound of Q. **Q**

By Zorn's lemma, every member of P is dominated by a maximal element of P.

461L Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X, and P the set of Radon probability measures on K. Suppose that $\mu \in P$ is maximal for the partial order $\preccurlyeq$ of 461K.

(a) $\mu(\frac{1}{2}(M_1 + M_2)) = 0$ whenever M_1, M_2 are disjoint closed convex subsets of K.

(b) $\mu F = 0$ whenever $F \subseteq K$ is a Baire set (for the subspace topology of K) not containing any extreme point of K.

proof (a) Set $M = \{\frac{1}{2}(x+y) : x \in M_1, y \in M_2\}$. Set $q(x, y) = \frac{1}{2}(x+y)$ for $x \in M_1$, $y \in M_2$, so that $q : M_1\times M_2 \to M$ is continuous. Let μ_M be the subspace measure on M induced by μ, so that μ_M is a Radon measure on M (416Rb). Let λ be a Radon measure on $M_1 \times M_2$ such that $\mu_M = \lambda q^{-1}$ (418L), and define $\psi : C(K) \to \mathbb{R}$ by writing

$$\psi(f) = \int_{K\setminus M} f\,d\mu + \int_{M_1\times M_2} \tfrac{1}{2}(f(x) + f(y))\lambda(d(x, y)).$$

Then ψ is linear, $\psi(f) \ge 0$ whenever $f \in C(K)^+$, and

$$\psi(\chi K) = \mu(K \setminus M) + \lambda(M_1 \times M_2) = \mu(K \setminus M) + \mu_M(M) = 1.$$

Let $\nu \in P$ be such that $\int f d\nu = \psi(f)$ for every $f \in C(K)$ (436J/436K again). If $f : K \to \mathbb{R}$ is continuous and convex, then

$$\begin{aligned}\int f d\nu = \psi(f) &= \int_{K\setminus M} f\,d\mu + \int_{M_1\times M_2} \frac{1}{2}(f(x) + f(y))\lambda(d(x, y))\\ &\ge \int_{K\setminus M} f\,d\mu + \int_{M_1\times M_2} f(\frac{1}{2}(x + y))\lambda(d(x, y))\\ &= \int_{K\setminus M} f\,d\mu + \int_{M_1\times M_2} fq\,d\lambda = \int_{K\setminus M} f\,d\mu + \int_M f\,d\mu_M\end{aligned}$$

(235G)

$$= \int_K f \times \chi(K \setminus M)d\mu + \int_K f \times \chi M\,d\mu$$

(131Fa)

$$= \int_K f\,d\mu.$$

So $\mu \preccurlyeq \nu$; as we are assuming that μ is maximal, $\mu = \nu$.

Because M_1 and M_2 are disjoint compact convex sets in the Hausdorff locally convex space X, there are a $g \in X^*$ and an $\alpha \in \mathbb{R}$ such that $g(x) < \alpha < g(y)$ whenever $x \in M_1$ and $y \in M_2$ (4A4Ee). Set $f(x) = |g(x) - \alpha|$ for $x \in K$; then f is a continuous convex function. If $x \in M_1$ and $y \in M_2$, then

$$\begin{aligned} f(\tfrac{1}{2}(x+y)) &= |g(\tfrac{1}{2}(x+y)) - \alpha| = \tfrac{1}{2}|(g(x)-\alpha) + (g(y)-\alpha)| \\ &< \tfrac{1}{2}(|g(x)-\alpha| + |g(y)-\alpha|) = \tfrac{1}{2}(f(x)+f(y)). \end{aligned}$$

Looking at the formulae above for $\psi(f)$, we see that we have

$$\int f\,d\nu = \int_{K\setminus M} f\,d\mu + \int_{M_1\times M_2} \frac{1}{2}(f(x)+f(y))\lambda(d(x,y)),$$

$$\int f\,d\mu = \int_{K\setminus M} f\,d\mu + \int_{M_1\times M_2} f(\frac{1}{2}(x+y))\lambda(d(x,y)).$$

Since these are equal, and $f(\frac{1}{2}(x+y)) < \frac{1}{2}(f(x)+f(y))$ for all $x \in M_1$ and $y \in M_2$, we must have $\mu M = \lambda(M_1 \times M_2) = 0$, as required.

(b)(i) Consider first the case in which F is a zero set for the subspace topology. Since $F \subseteq K$ is a closed G_δ set in K, $K \setminus F$ is expressible as a union $\bigcup_{n\in\mathbb{N}} F_n$ of compact sets. For any $n \in \mathbb{N}$, $z \in F$ and $y \in F_n$, there is a $g \in X^*$ such that $g(z) \ne g(y)$; since F and F_n are compact, there is a finite set $\Phi_n \subseteq X^*$ such that whenever $z \in F$ and $y \in F_n$ there is a $g \in \Phi_n$ such that $g(z) \ne g(y)$. Set $\Phi = \bigcup_{n\in\mathbb{N}} \Phi_n \cup \{0\}$; then Φ is countable; let $\langle g_n\rangle_{n\in\mathbb{N}}$ be a sequence running over Φ, and define $T : X \to \mathbb{R}^{\mathbb{N}}$ by setting $(Tx)(n) = g_n(x)$ for $n \in \mathbb{N}$, $x \in X$. Then $T[F] \cap T[F_n] = \emptyset$ for every n, so $F = K \cap T^{-1}[T[F]]$.

Now $T[F]$ is a compact subset of the metrizable compact convex set $T[K]$, and does not contain any extreme point of $T[K]$, by 4A4Gc.

Let $\mathcal{U}$ be the set of convex open subsets of $\mathbb{R}^{\mathbb{N}}$. Because the topology of $\mathbb{R}^{\mathbb{N}}$ is locally convex, $\mathcal{U}$ is a base for the topology of $\mathbb{R}^{\mathbb{N}}$; because it is separable and metrizable, $\mathcal{U}$ includes a countable base $\mathcal{U}_0$ (4A2P(a-iii)), and $\mathcal{V} = \{T[K] \cap U : U \in \mathcal{U}_0\}$ is a countable base for the topology of $T[K]$ (4A2B(a-vi)). Set $\mathcal{M} = \{K \cap T^{-1}[\overline{V}] : V \in \mathcal{V}\}$, so that $\mathcal{M}$ is a countable family of closed convex subsets of K.

If $z \in F$, then there are distinct u, $v \in T[K]$ such that $Tz = \frac{1}{2}(u+v)$. Now there must be V, $V' \in \mathcal{V}$, with disjoint closures, such that $u \in V$ and $v \in V'$, so that $z \in \frac{1}{2}(M+M')$, where $M = K \cap T^{-1}[\overline{V}]$ and $M' = K \cap T^{-1}[\overline{V'}]$ are disjoint members of $\mathcal{M}$. Thus

$$F \subseteq \bigcup\{\tfrac{1}{2}(M+M') : M,\ M' \in \mathcal{M},\ M \cap M' = \emptyset\}.$$

But (a) tells us that $\frac{1}{2}(M+M')$ is μ-negligible whenever M, $M' \in \mathcal{M}$ are disjoint. As $\mathcal{M}$ is countable, $\mu F = 0$, as required.

(ii) Now consider the Baire measure $\mu \restriction \mathcal{B}\mathfrak{a}(K)$, where $\mathcal{B}\mathfrak{a}(K)$ is the Baire σ-algebra of K. This is inner regular with respect to the zero sets (412D). If $F \in \mathcal{B}\mathfrak{a}(K)$ contains no extremal point of K, then (i) tells us that $\mu Z = 0$ for every zero set $Z \subseteq F$, so μF must also be 0.

461M Theorem Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X and E the set of extreme points of K. Let $x \in X$. Then there is a probability measure μ on E with barycenter x. If K is metrizable we can take μ to be a Radon measure.

proof Let P be the set of Radon probability measures on K and $\preccurlyeq$ the partial order on P described in 461K. By 461Ke, there is a maximal element ν of P such that $\delta_x \preccurlyeq \nu_0$, where $\delta_x \in P$ is the Dirac measure on X concentrated at x. By 461Kc, x is the barycenter of ν.

Let $\lambda = \nu \restriction \mathcal{B}\mathfrak{a}(K)$ be the Baire measure associated with ν. By 461Lb, $\lambda^* E = 1$. So the subspace measure λ_E on E is a probability measure on E. Let μ be the completion of λ_E.

If $g \in X^*$ then $g \restriction K$ is continuous, therefore $\mathcal{B}\mathfrak{a}(K)$-measurable, so

$$g(x) = \int_K g\,d\nu = \int_K g\,d\lambda = \int_E g\,d\lambda_E$$

(214F)

$$= \int_E g\,d\mu$$

(212Fb). So x is the barycenter of μ.

Now suppose that K is metrizable. In this case E is a G_δ set in K. **P** Let ρ be a metric on K inducing its topology. Then

$$K \setminus E = \bigcup_{n\ge 1}\{tx + (1-t)y : x,\, y \in K,\, t \in [2^{-n}, 1-2^{-n}],\, \rho(x,y) \ge 2^{-n}\}$$

is K_σ, so its complement in K is a G_δ set in K. **Q** So E is analytic (423Eb) and μ is a Radon measure (433Cb).

461N Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X, and P the set of Radon probability measures on K. Let E be the set of extreme points of K and suppose that $\mu \in P$ and $\mu^*E = 1$. Then μ is maximal in P for the partial order $\preccurlyeq$ of 461K.

proof (a) For $\mu \in P$ write $b(\mu)$ for the barycentre of μ. Then $b : P \to K$ is continuous for the narrow topology of P and the weak topology of X, as in 461J. For $f \in C(K)$ define $\bar f : K \to \mathbb{R}$ by setting $\bar f(x) = \sup\{\int f d\mu : \mu \in P,\, b(\mu) = x\}$ for $x \in K$.

(i) Taking δ_x to be the Dirac measure on X concentrated at x, we see that

$$f(x) = \int f d\delta_x \le \bar f(x)$$

for any $x \in K$.

(ii) For any $x \in K$ there is a $\mu \in P$ such that $b(\mu) = x$ and $\int f d\mu = \bar f(x)$. **P** The set $\{\mu : \mu \in P,\, b(\mu) = x\}$ is compact, so its continuous image $\{\int f d\mu : b(\mu) = x\}$ is compact and contains its supremum. **Q**

$\bar f$ is upper semi-continuous. **P** For any $\alpha \in \mathbb{R}$, the set

$$\{(\mu, x) : \mu \in P,\, x \in K,\, b(\mu) = x,\, \int f d\mu \ge \alpha\}$$

is compact, so its projection onto the second coordinate is closed; but this projection is just $\{x : \bar f(x) \ge \alpha\}$. **Q**

(iii) $\bar f : K \to \mathbb{R}$ is concave. **P** Suppose that x, $y \in K$ and $t \in [0,1]$. Take μ, $\nu \in P$ such that $b(\mu) = x$, $\bar f(x) = \int f d\mu$, $b(\nu) = y$ and $\bar f(y) = \int f d\nu$. Set $\lambda = t\mu + (1-t)\nu$. Then

$$\int_K g\,d\lambda = t\int_K g\,d\mu + (1-t)\int_K g\,d\nu = tg(x) + (1-t)g(y) = g(tx + (1-t)y)$$

for any $g \in X^*$, so $b(\lambda) = tx + (1-t)y$ and

$$\bar f(tx + (1-t)y) \ge \int f d\lambda = t\int f d\mu + (1-t)\int f d\nu = t\bar f(x) + (1-t)\bar f(y).$$

As x, y and t are arbitrary, $\bar f$ is concave. **Q**

(iv) If $x \in K$ and $\bar f(x) > f(x)$ then $x \notin E$. **P** There is a $\mu \in P$ such that $b(\mu) = x$ and $\int f d\mu > f(x)$. We cannot have $\mu\{x\} = 1$ because $\int f d\mu \ne f(x)$, so there is a point y of the support of μ such that $y \ne x$. Let $g \in X^*$ be such that $g(y) > g(x)$ and set $G = \{z : z \in K,\, g(z) > g(x)\}$, $t = \mu G$. Then $t > 0$; also $\int_G g\,d\mu > tg(x) = t\int g\,d\mu$, so $t \ne 1$. Define ν_1, $\nu_2 \in P$ by setting $\nu_1 H = \frac{1}{t}\mu(G\cap H)$ whenever $H \subseteq K$ and μ measures $G\cap H$, $\nu_2 H = \frac{1}{1-t}\mu(H\setminus G)$ whenever $H \subseteq K$ and μ measures $H \setminus G$. Let x_1, x_2 be the barycenters of ν_1, ν_2 respectively. For any $h \in X^*$,

$$h(tx_1 + (1-t)x_2) = th(x_1) + (1-t)h(x_2) = t\int_K h\,d\nu_1 + (1-t)\int_K h\,d\nu_2$$
$$= \int_G h\,d\mu + \int_{K\setminus G} h\,d\mu = h(x).$$

So $x = tx_1 + (1-t)x_2$. But both x_1 and x_2 belong to K and $g(x_1) > g(x)$, so $x_1 \ne x$, while $0 < t < 1$, so x is not an extreme point of K. **Q**

(b) Now take $\mu \in P$ such that $\mu^*E = 1$, and $\nu \in P$ such that $\mu \preccurlyeq \nu$. For any convex $f \in C(K)$, $f - \bar f$ is the sum of lower semi-continuous convex functions so is lower semi-continuous, and $\{x : f(x) = \bar f(x)\} = \{x : f(x) - \bar f(x) \ge 0\}$

is a G_δ set. By (a-iv), it includes E, so $\int f d\mu = \int \bar{f} d\mu$. In addition, $\int f d\mu \le \int f d\nu$ and $\int \bar{f} d\nu \le \int \bar{f} d\mu$, by 461Kb applied to $-\bar{f}$. But since $f \le \bar{f}$, we have

$$\int f d\nu \le \int \bar{f} d\nu \le \int \bar{f} d\mu = \int f d\mu;$$

as f is arbitrary, $\nu \preccurlyeq \mu$; as ν is arbitrary, μ is maximal.

461O Lemma Suppose that X is a Riesz space with a Hausdorff locally convex linear space topology, and $K \subseteq X$ a compact convex set such that every non-zero member of the positive cone X^+ is uniquely expressible as αx for some $x \in K$ and $\alpha \ge 0$. Let P be the set of Radon probability measures on K and $\preccurlyeq$ the partial order described in 461K. If μ, $\nu \in P$ have the same barycenter then they have a common upper bound in P.

proof (a) If $X = \{0\}$ then K is either $\{0\}$ or empty and the result is immediate, so henceforth suppose that X is non-trivial. Because each non-zero member of X^+ is uniquely expressible as a multiple of a member of K, no distinct members of $K \setminus \{0\}$ can be multiples of each other; as K is convex, $0 \notin K$. If $z_0, \dots , z_r \in K$, $\gamma_0, \dots , \gamma_r \ge 0$ and $z = \sum_{k=0}^r \gamma_k z_k$ belongs to K, then $\sum_{k=0}^r \gamma_k = 1$. **P** Setting $\gamma = \sum_{k=0}^r \gamma_k$, $\gamma \ne 0$ and $\frac{1}{\gamma} z = \sum_{k=0}^r \frac{\gamma_r}{\gamma} z_k$ belongs to K; accordingly $\frac{1}{\gamma} z = z$ and $\gamma = 1$. **Q**

(b) Take any $x \in K$, and let P_x be the set of elements of P with barycenter x. Write Q_x for the set of members of P_x with finite support. Then Q_x is dense in P_x. **P** Suppose that $\mu \in P_x$, $f_0, \dots , f_n \in C(K)$ and $\epsilon > 0$. Then there is a finite cover of K by relatively open convex sets on each of which every f_i has oscillation at most ϵ; so we have a partition $\mathcal{H}$ of K into finitely many non-empty Borel sets H such that every f_i has oscillation at most ϵ on the convex hull $\Gamma(H)$. If $H \in \mathcal{H}$ and $\mu H > 0$, let x_H be the barycentre of the measure $\mu_H \in P$ where $\mu_H F = \frac{1}{\mu H} \mu(F \cap H)$ whenever $F \subseteq K$ and μ measures $F \cap H$. For other $H \in \mathcal{H}$, take any point x_H of H. In all cases, $x_H \in \overline{\Gamma(H)}$ so $|f_i(y) - f_i(x_H)| \le \epsilon$ whenever $i \le n$ and $y \in H$.

Consider $\nu = \sum_{H \in \mathcal{H}} \mu H \cdot \delta_{x_H}$, where $\delta_{x_H} \in P$ is the Dirac measure on K concentrated at x_H. If $g \in X^*$ then

$$\int_K g\, d\nu = \sum_{H \in \mathcal{H}} \mu H \cdot g(x_H) = \sum_{H \in \mathcal{H}} \int_H g\, d\mu = \int_K g\, d\mu = g(x);$$

as g is arbitrary, $\nu \in P_x$ and $\nu \in Q_x$. Next, for $i \le n$,

$$\begin{aligned} |\int f_i d\nu - \int f_i d\mu| &\le \sum_{H \in \mathcal{H}} |f_i(x_H) \mu H - \int_H f_i d\mu| \\ &\le \sum_{H \in \mathcal{H}} \int_H |f_i(x_H) - f_i(y)| \mu(dy) \le \sum_{H \in \mathcal{H}} \epsilon \mu H = \epsilon. \end{aligned}$$

As $f_0, \dots , f_n$ and ϵ are arbitrary, Q_x is dense in P_x. **Q**

(c) Suppose that $x \in K$ and μ, $\nu \in Q_x$. Then they have a common upper bound in P. **P** Express μ, ν as $\sum_{i=0}^m \alpha_i \delta_{x_i}$, $\sum_{j=0}^n \beta_j \delta_{y_j}$ respectively, where all the α_i and β_j are strictly positive, all the x_i and y_j belong to K, and $\sum_{i=0}^m \alpha_i = \sum_{j=0}^n \beta_j = 1$. If $g \in X^*$ then

$$g(x) = \int_K g\, d\mu = \sum_{i=0}^m \alpha_i g(x_i) = \sum_{j=0}^n \beta_j g(y_j),$$

so $x = \sum_{i=0}^m \alpha_i x_i = \sum_{j=0}^n \beta_j y_j$. By the decomposition theorem 352Fd there is a family $\langle w_{ij} \rangle_{i \le m, j \le n}$ in X^+ such that $\alpha_i x_i = \sum_{j=0}^n w_{ij}$ for every $i \le m$ and $\beta_j y_j = \sum_{i=0}^n w_{ij}$ for every $j \le n$. Each w_{ij} is expressible as $\gamma_{ij} z_{ij}$ where $z_{ij} \in K$ and $\gamma_{ij} \ge 0$. Now

$$\sum_{j=0}^n \frac{\gamma_{ij}}{\alpha_i} z_{ij} = x_i \in K;$$

by (a), $\sum_{j=0}^n \gamma_{ij} = \alpha_i$, for every $i \le m$. Similarly, $\sum_{i=0}^m \gamma_{ij} = \beta_j$ for $j \le n$. Of course this means that $\sum_{i=0}^m \sum_{j=0}^n \gamma_{ij} = 1$.

Set $\lambda = \sum_{i=0}^m \sum_{j=0}^n \gamma_{ij} \delta_{z_{ij}} \in P$. If $f : K \to \mathbb{R}$ is continuous and convex,

$$\int f d\mu = \sum_{i=0}^{m} \alpha_i f(x_i) = \sum_{i=0}^{m} \alpha_i f(\sum_{j=0}^{n} \frac{\gamma_{ij}}{\alpha_i} z_{ij})$$
$$\le \sum_{i=0}^{m} \alpha_i \sum_{j=0}^{n} \frac{\gamma_{ij}}{\alpha_i} f(z_{ij}) = \sum_{i=0}^{m} \sum_{j=0}^{n} \gamma_{ij} f(z_{ij}) = \int f d\lambda.$$

So $\mu \preccurlyeq \lambda$. Similarly, $\nu \preccurlyeq \lambda$ and we have the required upper bound for $\{\mu, \nu\}$. **Q**

(d) Now consider

$$\{(\mu, \nu, \lambda) : \mu,\ \nu,\ \lambda \in P,\ \mu \preccurlyeq \lambda,\ \nu \preccurlyeq \lambda\}.$$

This is a closed set in the compact set $P \times P \times P$ (461Kd), so its projection

$$R = \{(\mu, \nu) : \mu,\ \nu \text{ have a common upper bound in } P\}$$

is a closed set in $P \times P$.

If $x \in K$ then (c) tells us that R includes $Q_x \times Q_x$, so (b) tells us that R includes $P_x \times P_x$. Thus any two members of P_x have a common upper bound in P, as required.

461P Theorem Suppose that X is a Riesz space with a Hausdorff locally convex linear space topology, and $K \subseteq X$ a metrizable compact convex set such that every non-zero member of the positive cone X^+ is uniquely expressible as αx for some $x \in K$ and $\alpha \ge 0$. Let E be the set of extreme points of K, and x any point of K. Then there is a unique Radon probability measure μ on E such that x is the barycenter of μ.

proof By 461M, there is a Radon probability measure μ on E such that x is the barycenter of μ. Suppose that μ_1 is another measure with the same properties. Let ν, ν_1 be the Radon probability measures on K extending μ, μ_1 respectively. Then ν and ν_1 both have barycenter x and make E conegligible. By 461N, they are both maximal in P. By 461O, they must have a common upper bound in P, so they are equal. But this means that $\mu = \mu_1$.

461Q It is a nearly universal rule that when we encounter a compact convex set we should try to identify its extreme points. I look now at some sets which arose naturally in §437.

Proposition (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a sequentially order-continuous Boolean homomorphism. Let M_σ be the L-space of countably additive real-valued functionals on $\mathfrak{A}$, and Q the set

$$\{\nu : \nu \in M_\sigma,\ \nu \ge 0,\ \nu 1 = 1,\ \nu\pi = \nu\}.$$

If $\nu \in Q$, then the following are equiveridical: (i) ν is an extreme point of Q; (ii) $\nu a \in \{0, 1\}$ whenever $\pi a = a$; (iii) $\nu a \in \{0, 1\}$ whenever $a \in \mathfrak{A}$ is such that $\nu(a \triangle \pi a) = 0$.

(b) Let X be a set, Σ a σ-algebra of subsets of X, and $\phi : X \to X$ a (Σ, Σ)-measurable function. Let M_σ be the L-space of countably additive real-valued functionals on Σ, and $Q \subseteq M_\sigma$ the set of probability measures with domain Σ for which ϕ is inverse-measure-preserving. If $\mu \in Q$, then μ is an extreme point of Q iff ϕ is ergodic with respect to μ (definition: 372Ob[2]).

proof (a) I ought to remark at once that because $\mathfrak{A}$ is Dedekind σ-complete, every countably additive functional on $\mathfrak{A}$ is bounded (326M[3]), so that M_σ is the L-space of bounded countably additive functionals on $\mathfrak{A}$, as studied in 362A-362B.

(i)⇒(ii) Suppose that ν is an extreme point of Q and that $a \in \mathfrak{A}$ is such that $\pi a = a$. Set $\alpha = \nu a$. **?** If $0 < \alpha < 1$, define $\nu_1 : \mathfrak{A} \to \mathbb{R}$ by setting

$$\nu_1 b = \frac{1}{\alpha} \nu(b \cap a)$$

for $b \in \mathfrak{A}$. Then ν_1 is a non-negative countably additive functional, and $\nu_1 1 = 1$. Moreover, for any $b \in \mathfrak{A}$,

$$\nu_1 \pi b = \frac{1}{\alpha} \nu(\pi b \cap a) = \frac{1}{\alpha} \nu(\pi b \cap \pi a) = \frac{1}{\alpha} \nu\pi(b \cap a) = \frac{1}{\alpha} \nu(b \cap a) = \nu_1 b,$$

[2] Formerly 372Pb.
[3] Formerly 326I.

so $\nu_1\pi = \nu_1$ and $\nu_1 \in Q$. Since $\nu_1 a = 1$, $\nu_1 \ne \nu$. Similarly, $\nu_2 \in Q$, where $\nu_2 b = \frac{1}{1-\alpha}\nu(b \setminus a)$ for $b \in \mathfrak{A}$. Now $\nu = \alpha\nu_1 + (1-\alpha)\nu_2$ is a proper convex combination of members of Q and is not extreme. **X** So $\nu a \in \{0,1\}$; as a is arbitrary, (ii) is true.

(ii)⇒(iii) Suppose that (ii) is true, and that $a \in \mathfrak{A}$ is such that $\nu(a \triangle \pi a) = 0$. Then

$$\nu(\pi^n a \triangle \pi^{n+1} a) = \nu\pi^n(a \triangle \pi a) = \nu(a \triangle \pi a) = 0$$

for every $n \in \mathbb{N}$, so $\nu(a \triangle \pi^n a) = 0$ for every $n \in \mathbb{N}$. Set

$$b_n = \sup_{m \ge n} \pi^m a \text{ for } n \in \mathbb{N}, \quad b = \inf_{n\in\mathbb{N}} b_n.$$

Because π is sequentially order-continuous and ν is countably additive,

$$\nu(a \triangle b_n) = 0 \text{ for every } n \in \mathbb{N}, \quad \nu(a \triangle b) = 0.$$

Now

$$\pi b_n = \sup_{m \ge n} \pi^{m+1} a = b_{n+1} \subseteq b_n$$

for every $n \in \mathbb{N}$, so

$$\pi b = \inf_{n\in\mathbb{N}} \pi b_n = \inf_{n\in\mathbb{N}} b_{n+1} = b.$$

Consequently $\nu a = \nu b \in \{0,1\}$. As a is arbitrary, (iii) is true.

(iii)⇒(i) Suppose that (iii) is true, and that $\nu = \frac{1}{2}(\nu_1 + \nu_2)$ where ν_1, $\nu_2 \in Q$. For $\alpha \ge 0$, set $\theta_\alpha = \nu_1 - \alpha\nu \in M_\sigma$. Then we have a corresponding element $a_\alpha = [\![\theta_\alpha > 0]\!]$ in $\mathfrak{A}$ such that

$$\theta_\alpha c > 0 \text{ whenever } 0 \ne c \subseteq a_\alpha, \quad \theta_\alpha c \le 0 \text{ whenever } c \cap a_\alpha = 0$$

(326S[4]). Observe that if $b \in \mathfrak{A}$, then

$$\theta_\alpha b = \theta_\alpha a_\alpha - \theta_\alpha(a_\alpha \setminus b) + \theta_\alpha(b \setminus a_\alpha) \le \theta_\alpha a_\alpha,$$

$$\theta_\alpha \pi b = \nu_1 \pi b - \alpha\nu\pi b = \nu_1 b - \alpha\nu b = \theta_\alpha b.$$

Now $\nu(a_\alpha \setminus \pi a_\alpha) = 0$. **P?** Otherwise, setting $c = a_\alpha \setminus \pi a_\alpha$, we must have $\theta_\alpha c > 0$, so

$$\theta_\alpha(c \cup \pi a_\alpha) > \theta_\alpha \pi a_\alpha = \theta_\alpha a_\alpha. \text{ **XQ**}$$

Consequently

$$\nu(a_\alpha \triangle \pi a_\alpha) = \nu\pi a_\alpha - \nu a_\alpha + 2\nu(a_\alpha \setminus \pi a_\alpha) = 0$$

and $\nu a_\alpha \in \{0,1\}$.

If $\alpha \le \beta$ then $\theta_\beta \le \theta_\alpha$ and $a_\beta \subseteq a_\alpha$. As $\theta_0 = \nu_1$, $\nu_1(1 \setminus a_0) = 0$, $\nu_1 a_0 = 1$ and $\nu a_0 \ge \frac{1}{2}$; accordingly $\nu a_0 = 1$. As $\theta_2 \le 0$, $a_2 = 0$ and $\nu a_2 = 0$. So

$$\beta = \sup\{\alpha : \nu a_\alpha = 1\} = \sup\{\alpha : \nu a_\alpha > 0\}$$

is defined in $[0,2]$.

Now $\nu_1 = \beta\nu$. **P** Let $c \in \mathfrak{A}$. **?** If $\nu_1 c > \beta\nu c$, take $\alpha > \beta$ such that $\nu_1 c > \alpha\nu c$. Then $\nu a_\alpha = 0$, but

$$0 < \theta_\alpha c \le \theta_\alpha a_\alpha \le \nu_1 a_\alpha \le 2\nu a_\alpha. \text{ **X**}$$

? If $\nu_1 c < \beta\nu c$, take $\alpha \in [0,\beta[$ such that $\nu_1 c < \alpha\nu c$. Then $\nu a_\alpha = 1$ so $\nu(1 \setminus a_\alpha) = 0$, but

$$0 > \theta_\alpha c \ge \theta_\alpha(c \setminus a_\alpha) \ge -\alpha\nu(c \setminus a_\alpha) \ge -\alpha\nu(1 \setminus a_\alpha). \text{ **X**}$$

Thus $\nu_1 c = \beta\nu c$; as c is arbitrary, $\nu_1 = \beta\nu$. **Q**

Accordingly ν_1 is a multiple of ν and must be equal to ν. Similarly, $\nu_2 = \nu$. As ν_1 and ν_2 were arbitrary, ν is an extreme point of Q.

(b) In (a), set $\mathfrak{A} = \Sigma$ and $\pi E = \phi^{-1}[E]$ for $E \in \Sigma$; then 'ϕ is ergodic with respect to μ' corresponds to condition (ii) of (a), so we have the result.

461R Corollary Let X be a compact Hausdorff space and $\phi : X \to X$ a continuous function. Let Q be the non-empty compact convex set of Radon probability measures μ on X such that ϕ is inverse-measure-preserving for

[4]Formerly 326O.

μ, with its narrow topology and the convex structure defined by 234G and 234Xf. Then the extreme points of Q are those for which ϕ is ergodic.

proof (a) If $\mu_0 \in Q$ is not extreme, let $\mathcal{B} = \mathcal{B}(X)$ be the Borel σ-algebra of X, so that ϕ is $(\mathcal{B},\mathcal{B})$-measurable, and write Q' for the set of ϕ-invariant Borel probability measures on X. Then 461Q tells us that the extreme points of Q' are just the measures for which ϕ is ergodic. If $\mu \in Q$, then $\mu\restriction\mathcal{B} \in Q'$, and of course the function $\mu \mapsto \mu\restriction\mathcal{B}$ is injective (416Eb) and preserves convex combinations. So $\mu_0\restriction\mathcal{B}$ is not extreme in Q'. By 461Q, ϕ is not $\mu_0\restriction\mathcal{B}$-ergodic, and therefore not μ_0-ergodic.

(b) If $\mu_0 \in Q$ and ϕ is not μ_0-ergodic, let $E \in \operatorname{dom}\mu_0$ be such that $0 < \mu_0 E < 1$ and $\phi^{-1}[E] = E$. Set $\alpha = \mu_0 E$ and $\beta = 1-\alpha$, and let μ_1, μ_2 be the indefinite-integral measures over μ_0 defined by $\frac{1}{\alpha}\chi E$ and $\frac{1}{\beta}\chi(X\setminus E)$. Then μ_1 is a Radon probability measure on X (416S), so the image measure $\mu_1\phi^{-1}$ also is a Radon measure (418I). The argument of part (b) of the proof of 461Q tells us that $\mu_1\phi^{-1}$ agrees with μ_1 on Borel sets, so $\mu_1 = \mu_1\phi^{-1}$ (416Eb) and $\mu_1 \in Q$. Similarly, $\mu_2 \in Q$, and $\mu_0 = \alpha\mu_1 + \beta\mu_2$, so μ_0 is not extreme in Q.

461X Basic exercises >(a) Let X be a Hausdorff locally convex linear topological space, $C \subseteq X$ a convex set, and $g : C \to \mathbb{R}$ a function. Show that the following are equiveridical: (i) g is convex and lower semi-continuous; (ii) there are a non-empty set $D \subseteq X^*$ and a family $\langle\beta_f\rangle_{f\in D}$ in $\mathbb{R}$ such that $g(x) = \sup_{f\in D} f(x) + \beta_f$ for every $x \in C$. (Compare 233Hb.)

>(b) Let X be a Hausdorff locally convex linear topological space, $K \subseteq X$ a compact convex set, and x an extreme point of K. Let μ be a probability measure on X such that $\mu^*K = 1$ and x is the barycenter of μ. (i) Show that $\{y : y \in K,\ f(y) \ne f(x)\}$ is μ-negligible for every $f \in X^*$. (*Hint*: 461E.) (ii) Show that if μ is a Radon measure then $\mu\{x\} = 1$.

(c) For each $n \in \mathbb{N}$, define $e_n \in \boldsymbol{c}_0$ by saying that $e_n(n) = 1$, $e_n(i) = 0$ if $i \ne n$. Let μ be the point-supported Radon probability measure on $\boldsymbol{c}_0$ defined by saying that $\mu E = \sum_{n=0}^{\infty} 2^{-n-1}\chi E(2^n e_n)$ for every $E \subseteq \boldsymbol{c}_0$. (i) Show that every member of $\boldsymbol{c}_0^*$ is μ-integrable. (*Hint*: $\boldsymbol{c}_0^*$ can be identified with ℓ^1.) (ii) Show that μ has no barycenter in $\boldsymbol{c}_0$.

(d) Let I be an uncountable set, and $X = \{x : x \in \ell^\infty(I),\ \{i : x(i) \ne 0\}$ is countable$\}$. (i) Show that X is a closed linear subspace of $\ell^\infty(I)$. (ii) Show that there is a probability measure μ on X such that (α) $\mu\{x : \|x\| \le 1\}$ is defined and equal to 1 (β) $\mu\{x : x(i) = 1\} = 1$ for every $i \in I$. (iii) Show that $\int f d\mu$ is defined for every $f \in X^*$. (*Hint*: for any $f \in X^\sim$, there is a countable set $J \subseteq X$ such that $f(x) = 0$ whenever $x\restriction J = 0$, so that $f =_{\text{a.e.}} f(\chi J)$.) (iv) Show that μ has no barycenter in X.

>(e) Let X be a complete Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact set. Show that every extreme point of $\overline{\Gamma(K)}$ belongs to K. (*Hint*: show that it cannot be the barycenter of any measure on K which is not supported by a single point.)

>(f) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a metrizable compact set. Show that $\overline{\Gamma(K)}$ is metrizable. (*Hint*: we may suppose that X is complete, so that $\overline{\Gamma(K)}$ is compact. Show that $\overline{\Gamma(K)}$ is a continuous image of the space of Radon probability measures on K, and use 437Rf.)

(g) Let X be a Hausdorff locally convex linear topological space and $K \subseteq X$ a compact set. Show that the Baire σ-algebra of K is just the subspace σ-algebra induced by the cylindrical σ-algebra of X.

>(h) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact convex set; let E be the set of extreme points of K. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in X^* such that $\sup_{x\in E,n\in\mathbb{N}}|f_n(x)|$ is finite and $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in E$. Show that $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in K$.

>(i) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a metrizable compact convex set. Show that the algebra of Borel subsets of K is just the subspace algebra of the cylindrical σ-algebra of X.

(j) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact set. Let us say that a point x of K is **extreme** if the only Radon probability measure on K with barycenter x is the Dirac measure on K concentrated at x. (Cf. 461Xb.) (i) Show that if X is complete, then $x \in X$ is an extreme point of K iff it is an extreme point of $\overline{\Gamma(K)}$. (ii) Writing E for the set of extreme points of K, show that any point of K is the barycenter of some probability measure on E. (iii) Show that if K is metrizable then E is a G_δ subset of K and any point of K is the barycenter of some Radon probability measure on E.

(k) Let G be an abelian group with identity e, and K the set of positive definite functions $h : G \to \mathbb{C}$ such that $h(e) = 1$. (i) Show that K is a compact convex subset of $\mathbb{C}^G$. (ii) Show that the extreme points of K are just the group homomorphisms from G to S^1. (iii) Show that K generates the positive cone of a Riesz space. (*Hint*: 445N.)

(l) Let X be a compact metrizable space and G a subgroup of the group of autohomeomorphisms of X. Let M_σ be the space of signed Borel measures on X with its vague topology, and $Q \subseteq M_\sigma$ the set of G-invariant Borel probability measures on X. Show that every member of Q is uniquely expressible as the barycenter of a Radon probability measure on the set of extreme points of Q.

(m) Let X be a set, Σ a σ-algebra of subsets of X, and P the set of probability measures with domain Σ, regarded as a convex subset of the linear space of countably additive functionals on Σ. Show that $\mu \in P$ is an extreme point in P iff it takes only the values 0 and 1.

(n) Let $\mathfrak{A}$ be a Boolean algebra and M the L-space of bounded finitely additive functionals on $\mathfrak{A}$, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism. (i) Show that $U = \{\nu : \nu \in M,\ \nu\pi = \nu\}$ is a closed Riesz subspace of M. (ii) Set $Q = \{\nu : \nu \in U,\ \nu \ge 0,\ \nu 1 = 1\}$. Show that if μ, ν are distinct extreme points of Q then $\mu \wedge \nu = 0$. (iii) Set $Q_\sigma = \{\nu : \nu \in Q,\ \nu \text{ is countably additive}\}$. Show that any extreme point of Q_σ is an extreme point of Q. (iv) Set $Q_\tau = \{\nu : \nu \in Q,\ \nu \text{ is countably additive}\}$. Show that any extreme point of Q_τ is an extreme point of Q.

>(o) Set $S^1 = \{z : z \in \mathbb{C},\ |z| = 1\}$, and let $w \in S^1$ be such that $w^n \ne 1$ for any integer 1. Define $\phi : S^1 \to S^1$ by setting $\phi(z) = wz$ for every $z \in S^1$. Show that the only Radon probability measure on S^1 for which ϕ is inverse-measure-preserving is the Haar probability measure μ of S^1. (*Hint*: use 281N to show that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^n f(w^k z) = \int f d\mu$ for every $f \in C(S^1)$; now put 461R and 372H[5] together.)

(p) Set $\phi(x) = 2\min(x, 1-x)$ for $x \in [0,1]$ (cf. 372Xp[6]). Show that there are many point-supported Radon measures on $[0,1]$ for which ϕ is inverse-measure-preserving.

(q) Let X and Y be Hausdorff locally convex linear topological spaces, $A \subseteq X$ a convex set and $\phi : A \to Y$ a continuous function such that $\phi[A]$ is bounded and $\phi(tx+(1-t)y) = t\phi(x)+(1-t)\phi(y)$ for all x, $y \in A$ and $t \in [0,1]$. Let μ be a topological probability measure on A with a barycenter $x^* \in A$. Show that $\phi(x^*)$ is the barycenter of the image measure $\mu\phi^{-1}$ on Y. (*Hint*: show first that if $\langle E_i\rangle_{i\in I}$ is a finite partition of A into non-empty convex sets measured by μ, $\alpha_i = \mu E_i$ for each $i \in I$ and $C = \{\sum_{i\in I} \alpha_i x_i : x_i \in E_i \text{ for every } i \in I\}$, then $x^* \in \overline{C}$.)

461Y Further exercises (a) Let X be a Hausdorff locally convex linear topological space. (i) Show that if $M_0, \dots, M_n$ are non-empty compact convex subsets of X with empty intersection, then there is a continuous convex function $g : X \to \mathbb{R}$ such that $g(\sum_{i=0}^n \alpha_i x_i) < \sum_{i=0}^n \alpha_i g(x_i)$ whenever $x_i \in M_i$ and $\alpha_i > 0$ for every $i \le n$. (ii) Show that if $K \subseteq X$ is compact and $x^* \in \overline{\Gamma(K)}$ then there is a Radon probability measure μ on K, with barycenter x^*, such that $\mu(\alpha_0 M_0 + \ldots + \alpha_n M_n) = 0$ whenever $M_0, \dots, M_n$ are compact convex subsets of K with empty intersection, $\alpha_i \ge 0$ for every $i \le n$, and $\sum_{i=0}^n \alpha_i = 1$.

(b) In $\mathbb{R}^{[0,2[}$ let K be the set of those functions u such that (α) $0 \le u(s) \le u(t) \le 1$ whenever $0 \le s \le t \le 1$ (β) $|u(t+1)| \le u(s') - u(s)$ whenever $0 \le s < t < s' \le 1$. (i) Show that K is a compact convex set. (ii) Show that the set E of extreme points of K is just the set of functions of the types $\mathbf{0}$, $\chi[0,1]$, $\chi[s,1] \pm \chi\{1+s\}$ and $\chi]s,1] \pm \chi\{1+s\}$ for $0 < s < 1$. (iii) Set $w(s) = s$ for $s \in [0,1]$, 0 for $s \in]1,2[$. Show that if μ is any Radon probability measure on K with barycenter w then $\mu E = 0$.

(c) Write ν_{ω_1} for the usual measure on $Z = \{0,1\}^{\omega_1}$. Fix any $z_0 \in Z$, and let U be the linear space $\{u : u \in C(Z),\ u(z_0) = \int u\,d\nu_{\omega_1}\}$. Let X be the Riesz space of signed tight Borel measures μ on Z such that $\mu\{z_0\} = 0$, with the topology generated by the functionals $\mu \mapsto \int u\,d\mu$ as u runs over U. Let $K \subseteq X$ be the set of tight Borel probability measures μ on Z such that $\mu\{z_0\} = 0$. (i) Show that K is compact and convex and that every member of $X^+ \setminus \{0\}$ is uniquely expressible as a positive multiple of a member of K. (ii) Show that the set E of extreme points of K can be identified, as topological space, with $Z \setminus \{z_0\}$, so is a Borel subset of K but not a Baire subset. (iii) Show that the restriction of ν_{ω_1} to the Borel σ-algebra of Z is the barycenter of more than one Baire measure on E.

[5]Formerly 372I.
[6]Formerly 372Xm.

(d) Let X be a set, Σ a σ-algebra of subsets of X, and Φ a set of (Σ,Σ)-measurable functions from X to itself. Let M_σ be the L-space of countably additive real-valued functionals on Σ, and $Q \subseteq M_\sigma$ the set of probability measures with domain Σ for which every member of Φ is inverse-measure-preserving. (i) Show that if $\mu \in Q$, then μ is an extreme point of Q iff $\mu E \in \{0,1\}$ whenever $E \in \Sigma$ and $\mu(E \triangle \phi^{-1}[E]) = 0$ for every $\phi \in \Phi$. (ii) Show that if $\mu \in Q$ and Φ is countable and commutative, then μ is an extreme point of Q iff $\mu E \in \{0,1\}$ whenever $E \in \Sigma$ and $E = \phi^{-1}[E]$ for every $\phi \in \Phi$.

(e) Let X be a non-empty Hausdorff space, and define $\phi : X^{\mathbb{N}} \to X^{\mathbb{N}}$ by setting $\phi(x)(n) = x(n+1)$ for $x \in X^{\mathbb{N}}$ and $n \in \mathbb{N}$. Let Q be the set of Radon probability measures on $X^{\mathbb{N}}$ for which ϕ is inverse-measure-preserving. Show that a Radon probability measure λ on $X^{\mathbb{N}}$ is an extreme point of Q iff it is a Radon product measure $\mu^{\mathbb{N}}$ for some Radon probability measure μ on X.

(f) Let G be a topological group. Show that the following are equiveridical: (i) G is amenable; (ii) whenever X is a Hausdorff locally convex linear topological space, and $\bullet$ is a continuous action of G on X such that $x \mapsto a\bullet x$ is a linear operator for every $a \in G$, and $K \subseteq X$ is a non-empty compact convex set such that $a\bullet x \in K$ whenever $a \in G$ and $x \in K$, then there is an $x \in K$ such that $a\bullet x = x$ for every $a \in G$; (iii) whenever X is a Hausdorff locally convex linear topological space, $K \subseteq X$ is a non-empty compact convex set, and $\bullet$ is a continuous action of G on K such that $a\bullet(tx + (1-t)y) = t\,a\bullet x + (1-t)a\bullet y$ whenever $a \in G$, x, $y \in K$ and $t \in [0,1]$, then there is an $x \in K$ such that $a\bullet x = x$ for every $a \in G$. Use this to simplify parts of the proof of 449C. (*Hint*: 493B.)

(g) Let G be an amenable topological group, and $\bullet$ an action of G on a reflexive Banach space U, continuous for the given topology on G and the weak topology of U, such that $u \mapsto a\bullet u$ is a linear operator of norm at most 1 for every $a \in G$. Set $V = \{v : v \in U,\ a\bullet v = v \text{ for every } a \in G\}$. Show that $\{u + v - a\bullet u : u \in U,\ v \in V,\ a \in G\}$ is dense in U.

461 Notes and comments The results above are a little unusual in that we have studied locally convex spaces for several pages without encountering two topologies on the same space more than once (461J). In fact some of the most interesting properties of measures on locally convex spaces concern their relationships with strong and weak topologies, but I defer these ideas to later parts of the chapter. For the moment, we just have the basic results affirming (i) that barycenters exist (461E, 461F, 461H) (ii) that points can be represented as barycenters (461I, 461M). The last two can be thought of as refinements of the Kreĭn-Milman theorem. Any compact convex set K (in a locally compact Hausdorff space) is the closed convex hull of the set E of its extreme points. By 461M, given $x \in K$, we can actually find a measure on E with barycenter x; and if K is metrizable we can do this with a Radon measure. Of course the second part of 461M is a straightforward consequence of the first. But I do not know of any proof of 461M which does not pass through 461K-461L.

Kreĭn's theorem (461J) is a fundamental result in the theory of linear topological spaces. The proof here, using the Riesz representation theorem and vague topologies, is a version of the standard one (e.g., Bourbaki 87, II.4.1), written out to be a little heavier in the measure theory and a little lighter in the topological linear space theory than is usual. There are of course proofs which do not use measure theory.

In §437 I have already looked at an archetypal special case of 461I and 461M. If X is a compact Hausdorff space and P the compact convex set of Radon probability measures on X with the narrow (or vague) topology, then the set of extreme points of P can be identified with the set Δ of Dirac measures on X (437S, 437Xt). If we think of P as a subset of $C(X)^*$ with the weak* topology, so that the dual of the linear topological space $C(X)^*$ can be identified with $C(X)$, then any $\mu \in P$ is the barycenter of a Baire probability measure ν on Δ. In fact (because Δ here is compact) μ is the barycentre of a Radon measure on Δ, and this is just the image measure $\mu\delta^{-1}$.

I have put the phrase 'Choquet's theorem' into the title of this section. Actually it should perhaps be 'first steps in Choquet theory', because while the theory as a whole was dominated for many years by the work of G.Choquet the exact attribution of the results presented here is more complicated. See Phelps 66 for a much fuller account. But certainly both the existence and uniqueness theorems 461M and 461P draw heavily on Choquet's ideas.

Theorem 461P, demanding an excursion through 461N-461O, seems fairly hard work for a relatively specialized result. But it provides a unified explanation for a good many apparently disparate phenomena. Of course the simplest example is when $X = C(Z)^*$ for some compact metrizable space Z and K is the set of positive linear functionals of norm 1, so that E can be identified with Z and we find ourselves back with the Riesz representation theorem. A less familiar case already examined is in 461Xk. At the next level we have such examples as 461Xl.

Another class of examples arising in §437 is explored in 461Q-461R, 461Xm-461Xn and 461Yd-461Ye. It is when we have an explicit listing of the extreme points, as in 461Yb and 461Ye, that we can begin to feel that we understand a compact convex set.

462 Pointwise compact sets of continuous functions

In preparation for the main work of this chapter, beginning in the next section, I offer a few pages on spaces of continuous functions under their 'pointwise' topologies (462Ab). There is an extensive general theory of such spaces, described in Arkhangel'skii 92; here I present only those fragments which seem directly relevant to the theory of measures on normed spaces and spaces of functions. In particular, I star the paragraphs 462C-462D, which are topology and functional analysis rather than measure theory. They are here because although this material is well known, and may be found in many places, I think that the ideas, as well as the results, are essential for any understanding of measures on linear topological spaces.

Measure theory enters the section in the proof of 462E, in the form of an application of the Riesz representation theorem, though 462E itself remains visibly part of functional analysis. In the rest of the section, however, we come to results which are pure measure theory. For (countably) compact spaces X, the Radon measures on $C(X)$ are the same for the pointwise and norm topologies (462I). This fact has extensive implications for the theory of separately continuous functions (462K) and for the theory of convex hulls in linear topological spaces (462L).

462A Definitions (a) A regular Hausdorff space X is **angelic** if whenever A is a subset of X which is relatively countably compact in X, then (i) its closure $\overline{A}$ is compact (ii) every point of $\overline{A}$ is the limit of a sequence in A.

(A **Fréchet-Urysohn** space is a topological space in which, for any set A, every point of the closure of A is a limit of a sequence in A. So (ii) here may be written as 'every compact subspace of X is a Fréchet-Urysohn space'.)

(b) If X is any set and A a subset of $\mathbb{R}^X$, then the topology of **pointwise convergence** on A is that inherited from the usual product topology of $\mathbb{R}^X$; that is, the coarsest topology on A for which the map $f \mapsto f(x) : A \to \mathbb{R}$ is continuous for every $x \in X$. I shall commonly use the symbol $\mathfrak{T}_p$ for such a topology. In this context, I will say that a sequence or filter is **pointwise convergent** if it is convergent for the topology of pointwise convergence. Note that if A is a linear subspace of $\mathbb{R}^X$ then $\mathfrak{T}_p$ is a linear space topology on A (4A4Ba).

***462B Proposition** (Pryce 71) Let $(X, \mathfrak{T})$ be an angelic regular Hausdorff space.

(a) Any subspace of X is angelic.

(b) If $\mathfrak{S}$ is a regular topology on X finer than $\mathfrak{T}$, then $\mathfrak{S}$ is angelic.

(c) Any countably compact subset of X is compact and sequentially compact.

proof (a) Let Y be any subset of X. Then of course the subspace topology on Y is regular and Hausdorff. If $A \subseteq Y$ is relatively countably compact in Y, then A is relatively countably compact in X, so $\overline{A}$, the closure of A in X, is compact. Now if $x \in \overline{A}$, there is a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in A converging to x; but $\langle x_n \rangle_{n\in\mathbb{N}}$ must have a cluster point in Y, and (because $\mathfrak{T}$ is Hausdorff) this cluster point can only be x. Accordingly $\overline{A} \subseteq Y$ and is the closure of A in Y. Thus A is relatively compact in Y. Moreover, any point of $\overline{A}$ is the limit of a sequence in A. As A is arbitrary, Y is angelic.

(b) By hypothesis, $\mathfrak{S}$ is regular, and it is Hausdorff because it is finer than $\mathfrak{T}$. Now suppose that $A \subseteq X$ is $\mathfrak{S}$-relatively countably compact. Because the identity map from $(X, \mathfrak{S})$ to $(X, \mathfrak{T})$ is continuous, A is $\mathfrak{T}$-relatively countably compact (4A2G(f-iv)), and the $\mathfrak{T}$-closure $\overline{A}$ of A is $\mathfrak{T}$-compact.

Let $\mathcal{F}$ be any ultrafilter on X containing A. Then $\mathcal{F}$ has a $\mathfrak{T}$-limit $x \in X$. **?** If $\mathcal{F}$ is not $\mathfrak{S}$-convergent to x, there is an $H \in \mathfrak{S}$ such that $x \in H$ and $X \setminus H \in \mathcal{F}$, so that $A \setminus H \in \mathcal{F}$. Now x belongs to the $\mathfrak{T}$-closure of $A \setminus H$, because $A \setminus H \in \mathcal{F}$ and $\mathcal{F}$ is $\mathfrak{T}$-convergent to x; because $\mathfrak{T}$ is angelic, there is a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in $A \setminus H$ which $\mathfrak{T}$-converges to x. But now $\langle x_n \rangle_{n\in\mathbb{N}}$ has a $\mathfrak{S}$-cluster point x'. x' must also be a $\mathfrak{T}$-cluster point of $\langle x_n \rangle_{n\in\mathbb{N}}$, so $x' = x$; but every x_n belongs to the $\mathfrak{S}$-closed set $X \setminus H$, so $x' \notin H$, which is impossible. **X**

Thus every ultrafilter on X containing A is $\mathfrak{S}$-convergent. Because $\mathfrak{S}$ is regular, the $\mathfrak{S}$-closure $\tilde{A}$ of A is $\mathfrak{S}$-compact (3A3De).

Again because $\mathfrak{S}$ is finer than $\mathfrak{T}$, and $\mathfrak{T}$ is Hausdorff, the two topologies must agree on $\tilde{A} = \overline{A}$. But now every point of $\overline{A}$ is the $\mathfrak{T}$-limit of a sequence in A, so every point of $\tilde{A}$ is the $\mathfrak{S}$-limit of a sequence in A. As A is arbitrary, $\mathfrak{S}$ is angelic.

(c) If $K \subseteq X$ is countably compact, then of course it is relatively countably compact in its subspace topology, so (being angelic) must be compact in its subspace topology. If $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in K, let x be any cluster point of $\langle x_n \rangle_{n\in\mathbb{N}}$. If $\{n : x_n = x\}$ is infinite, then this immediately provides us with a subsequence converging to x. Otherwise, take n such that $x \neq x_i$ for $i \geq n$. Since $x \in \overline{\{x_i : i \geq n\}}$, and $\{x_i : i \geq n\}$ is relatively countably compact, there is a sequence $\langle y_i \rangle_{i\in\mathbb{N}}$ in $\{x_i : i \geq n\}$ converging to x. The topology of X being Hausdorff, $\{y_i : i \in \mathbb{N}\}$ must be infinite, and $\langle y_i \rangle_{i\in\mathbb{N}}$, $\langle x_i \rangle_{i\in\mathbb{N}}$ have a common subsequence which converges to x. As $\langle x_i \rangle_{i\in\mathbb{N}}$ is arbitrary, K is sequentially compact.

***462C Theorem** (PRYCE 71) Let X be a topological space such that there is a sequence $\langle X_n\rangle_{n\in\mathbb{N}}$ of relatively countably compact subsets of X, covering X, with the property that a function $f:X\to\mathbb{R}$ is continuous whenever $f\restriction X_n$ is continuous for every $n\in\mathbb{N}$. Then the space $C(X)$ of continuous real-valued functions on X is angelic in its topology of pointwise convergence.

proof Of course $C(X)$ is regular and Hausdorff under $\mathfrak{T}_p$, because $\mathbb{R}^X$ is, so we need attend only to the rest of the definition in 462Aa. Let $A\subseteq C(X)$ be relatively countably compact for $\mathfrak{T}_p$.

(a) Since $\{f(x):f\in A\}$, being a continuous image of A, must be relatively countably compact in $\mathbb{R}$ (4A2G(f-iv)), therefore relatively compact (4A2Le), for every $x\in X$, the closure $\overline{A}$ of A in $\mathbb{R}^X$ is compact, by Tychonoff's theorem.

? Suppose, if possible, that $\overline{A}\not\subseteq C(X)$; let $g\in\overline{A}$ be a discontinuous function. By the hypothesis of the theorem, there is an $n\in\mathbb{N}$ such that $g\restriction X_n$ is not continuous; take $x^*\in X_n$ such that $g\restriction X_n$ is discontinuous at x^*. Let $\epsilon>0$ be such that for every neighbourhood U of x^* in X_n there is a point $x\in U$ such that $|g(x)-g(x^*)|\ge\epsilon$.

Choose sequences $\langle f_i\rangle_{i\in\mathbb{N}}$ in A and $\langle x_i\rangle_{i\in\mathbb{N}}$ in X_n as follows. Given $\langle f_i\rangle_{i<m}$ and $\langle x_i\rangle_{i<m}$, choose $x_m\in X_n$ such that $|f_i(x_m)-f_i(x^*)|\le 2^{-m}$ for every $i<m$ and $|g(x^*)-g(x_m)|\ge\epsilon$. Now choose $f_m\in A$ such that $|f_m(x^*)-g(x^*)|\le 2^{-m}$ and $|f_m(x_i)-g(x_i)|\le 2^{-m}$ for every $i\le m$. Continue.

At the end of the induction, take a cluster point x of $\langle x_i\rangle_{i\in\mathbb{N}}$ in X and a cluster point f of $\langle f_i\rangle_{i\in\mathbb{N}}$ in $C(X)$. Because $|f_i(x_m)-f_i(x^*)|\le 2^{-m}$ whenever $i<m$, $f_i(x)=f_i(x^*)$ for every i, and $f(x)=f(x^*)$. Because $|f_m(x^*)-g(x^*)|\le 2^{-m}$ for every m, $f(x^*)=g(x^*)$. Because $|f_m(x_i)-g(x_i)|\le 2^{-m}$ whenever $i\le m$, $f(x_i)=g(x_i)$ for every i, $|g(x^*)-f(x_i)|\ge\epsilon$ for every i, and $|g(x^*)-f(x)|\ge\epsilon$; but this is impossible, because $f(x)=f(x^*)=g(x^*)$. **X**

Thus the compact set $\overline{A}\subseteq C(X)$ is the closure of A in $C(X)$, and A is relatively compact in $C(X)$.

(b) Now take any $g\in\overline{A}$. There are countable sets $D\subseteq X$, $B\subseteq A$ such that

whenever $I\subseteq B\cup\{g\}$ is finite, $n\in\mathbb{N}$, $\epsilon>0$ and $x\in X_n$, there is a $y\in D\cap X_n$ such that $|f(y)-f(x)|\le\epsilon$ for every $f\in I$;

whenever $J\subseteq D$ is finite and $\epsilon>0$ there is an $f\in B$ such that $|f(x)-g(x)|\le\epsilon$ for every $x\in J$.

P For any finite set $I\subseteq\mathbb{R}^X$ and $n\in\mathbb{N}$, the set $Q_{In}=\{\langle f(x)\rangle_{f\in I}:x\in X_n\}$ is a subset of the separable metrizable space $\mathbb{R}^I$, so is itself separable, and there is a countable dense set $D_{In}\subseteq X_n$ such that $Q'_{In}=\{\langle f(x)\rangle_{f\in I}:x\in D_{In}\}$ is dense in Q_{In}. Similarly, because $g\in\overline{A}$, we can choose for any finite set $J\subseteq X$ a sequence $\langle f_{Ji}\rangle_{i\in\mathbb{N}}$ in A such that $\lim_{i\to\infty}f_{Ji}(x)=g(x)$ for every $x\in J$.

Now construct $\langle D_m\rangle_{m\in\mathbb{N}}$, $\langle B_m\rangle_{m\in\mathbb{N}}$ inductively by setting

$$D_m=\bigcup\{D_{In}:n\in\mathbb{N},\ I\subseteq\{g\}\cup\bigcup_{i<m}B_i\text{ is finite}\},$$

$$B_m=\{f_{Jk}:k\in\mathbb{N},\ J\subseteq\bigcup_{i<m}D_i\text{ is finite}\}.$$

At the end of the induction, set $D=\bigcup_{m\in\mathbb{N}}D_m$ and $B=\bigcup_{m\in\mathbb{N}}B_m$. Since the construction clearly ensures that $\langle D_m\rangle_{m\in\mathbb{N}}$ and $\langle B_m\rangle_{m\in\mathbb{N}}$ are non-decreasing sequences of countable sets, D and B are countable, and we shall have $D_{In}\subseteq D$ whenever $n\in\mathbb{N}$ and $I\subseteq B\cup\{g\}$ is finite, while $f_{Ji}\in B$ whenever $i\in\mathbb{N}$ and $J\subseteq D$ is finite. Thus we have suitable sets D and B. **Q**

By the second condition on D and B, there must be a sequence $\langle f_i\rangle_{i\in\mathbb{N}}$ in B such that $g(x)=\lim_{i\to\infty}f_i(x)$ for every $x\in D$. In fact $g(y)=\lim_{i\to\infty}f_i(y)$ for every $y\in X$. **P?** Otherwise, there is an $\epsilon>0$ such that $J=\{i:|g(y)-f_i(y)|\ge\epsilon\}$ is infinite. Let n be such that $y\in X_n$. For each $m\in\mathbb{N}$, $I_m=\{f_i:i\le m\}$ is a finite subset of B, so there is an $x_m\in D_{I_mn}$ such that $|f(x_m)-f(y)|\le 2^{-m}$ for every $f\in I_m\cup\{g\}$. Let $x^*\in X$ be a cluster point of $\langle x_m\rangle_{m\in\mathbb{N}}$, and $h\in C(X)$ a cluster point of $\langle f_i\rangle_{i\in J}$. Then

because $g(x)=\lim_{i\to\infty}f_i(x)$ for every $x\in D$, $g(x_m)=h(x_m)$ for every $m\in\mathbb{N}$, and $g(x^*)=h(x^*)$;

because $|g(y)-f_i(y)|\ge\epsilon$ for every $i\in J$, $|g(y)-h(y)|\ge\epsilon$;

because $|f_i(x_m)-f_i(y)|\le 2^{-m}$ whenever $i\le m$, $f_i(x^*)=f_i(y)$ for every $i\in\mathbb{N}$, and $h(x^*)=h(y)$;

because $|g(x_m)-g(y)|\le 2^{-m}$ for every m, $g(x^*)=g(y)$;

but this means that $g(y)=g(x^*)=h(x^*)=h(y)\ne g(y)$, which is absurd. **XQ**

So $g=\lim_{i\to\infty}f_i$ for $\mathfrak{T}_p$. As g is arbitrary, A has both properties required in 462Aa; as A is arbitrary, $C(X)$ is angelic.

Remark For a slight strengthening of this result, see 462Ya.

***462D Theorem** Let U be any normed space. Then it is angelic in its weak topology.

proof Write X for the unit ball of the dual space U^*, with its weak* topology. Then X is compact (3A5F). We have a natural map $u \mapsto \hat{u} : U \to \mathbb{R}^X$ defined by setting $\hat{u}(x) = x(u)$ for $x \in X$ and $u \in U$. By the definition of the weak* topology, $\hat{u} \in C(X)$ for every $u \in U$. The weak topology of U is normally defined in terms of all functionals $u \mapsto f(u)$, for $f \in U^*$; but as every member of U^* is a scalar multiple of some $x \in X$, we can equally regard the weak topology of U as defined just by the functionals $u \mapsto x(u) = \hat{u}(x)$, for $x \in X$. But this means that the map $u \mapsto \hat{u}$ is a homeomorphism between U, with its weak topology, and its image $\widehat{U}$ in $C(X)$, with the topology of pointwise convergence.

Since $C(X)$ is $\mathfrak{T}_p$-angelic (462C), so is $\widehat{U}$ (462Ba), and U is angelic in its weak topology.

462E Theorem Let X be a locally compact Hausdorff space, and $C_0(X)$ the Banach lattice of continuous real-valued functions on X which vanish at infinity (436I). Write $\mathfrak{T}_p$ for the topology of pointwise convergence on $C_0(X)$.

(i) $C_0(X)$ is $\mathfrak{T}_p$-angelic.

(ii) A sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $C_0(X)$ is weakly convergent to $u \in C_0(X)$ iff it is $\mathfrak{T}_p$-convergent to u and norm-bounded.

(iii) A subset K of $C_0(X)$ is weakly compact iff it is norm-bounded and $\mathfrak{T}_p$-countably compact.

proof (a) Let $X^* = X \cup \{x_\infty\}$ be the one-point compactification of X (3A3O). Then $C(X^*)$ is angelic in its topology $\mathfrak{T}_p^*$ of pointwise convergence, by 462C. Set $V = \{g : g \in C(X^*),\ g(x_\infty) = 0\}$. By 462Ba, V is angelic in the subspace topology induced by $\mathfrak{T}_p^*$. Now observe that we have a natural bijection $g \mapsto g\restriction X : V \to C_0(X)$, and that this is a homeomorphism for the topologies of pointwise convergence on V and $C_0(X)$. So $C_0(X)$ is angelic under $\mathfrak{T}_p$.

(b) Since all the maps $u \mapsto u(x)$, where $x \in X$, are bounded linear functionals on $C_0(X)$, $\mathfrak{T}_p$ is coarser than the weak topology $\mathfrak{T}_s$; so a $\mathfrak{T}_s$-convergent sequence is $\mathfrak{T}_p$-convergent to the same limit, and a $\mathfrak{T}_s$-compact set is $\mathfrak{T}_p$-compact, therefore $\mathfrak{T}_p$-countably compact.

(c) If $K \subseteq C_0(X)$ is $\mathfrak{T}_s$-compact, then $f[K] \subseteq \mathbb{R}$ must be compact, therefore bounded, for every $f \in C_0(X)^*$; by the Uniform Boundedness Theorem (3A5Hb), K is norm-bounded. Applying this to $\{u\} \cup \{u_n : n \in \mathbb{N}\}$, we see that any $\mathfrak{T}_s$-convergent sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ with limit u is norm-bounded.

(d) Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-bounded and $\mathfrak{T}_p$-convergent to u. By Lebesgue's Dominated Convergence Theorem, $\lim_{n\to\infty} \int u_n d\mu = \int u\, d\mu$ for every totally finite Radon measure μ on X. But by the Riesz Representation Theorem (in the form 436K), this says just that $\lim_{n\to\infty} f(u_n) = f(u)$ for every positive linear functional f on $C_0(X)$. Since every member of $C_0(X)^*$ is expressible as the difference of two positive linear functionals (356Dc), $\lim_{n\to\infty} f(u_n) = f(u)$ for every $f \in U^*$, that is, $\langle u_n\rangle_{n\in\mathbb{N}}$ is $\mathfrak{T}_s$-convergent to u.

Putting this together with (b) and (c), we see that (ii) is true.

(e) Now suppose that $K \subseteq C_0(X)$ is norm-bounded and $\mathfrak{T}_p$-countably compact. Any sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in K has a subsequence which is $\mathfrak{T}_p$-convergent to a point of K (462Bc), and this subsequence is also $\mathfrak{T}_s$-convergent, by (c). This means that K is sequentially compact, therefore countably compact, in $C_0(X)$ for the topology $\mathfrak{T}_s$. Since $\mathfrak{T}_s$ is angelic (462D), K is $\mathfrak{T}_s$-compact, by 462Bc again.

462F Lemma Let X be a topological space, and Q a relatively countably compact subset of X. Suppose that $K \subseteq C_b(X)$ is $\|\ \|_\infty$-bounded and $\mathfrak{T}_p$-countably compact, where $\mathfrak{T}_p$ is the topology of pointwise convergence on $C_b(X)$. Then the map $u \mapsto u\restriction Q : K \to C_b(Q)$ is continuous for $\mathfrak{T}_p$ on K and the weak topology of the Banach space $C_b(Q)$.

proof We have a natural map $x \mapsto \hat{x} : X \to \mathbb{R}^K$ defined by writing $\hat{x}(u) = u(x)$ for every $u \in K$ and $x \in X$. By the definition of $\mathfrak{T}_p$, $\hat{x} \in C(K)$ for every $x \in X$, if we take $C(K)$ to be the space of $\mathfrak{T}_p$-continuous real-valued functions on K; and $x \mapsto \hat{x} : X \to C(K)$ is continuous for the given topology on X and the topology of pointwise convergence on $C(K)$ because $K \subseteq C(X)$. It follows that $\{\hat{x} : x \in Q\}$ is relatively countably compact for the topology of pointwise convergence on $C(K)$ (4A2G(f-iv)). But now $Z = \overline{\{\hat{x} : x \in Q\}}$ must be actually compact for the topology of pointwise convergence on $C(K)$, by 462C.

Next, consider the natural map $u \mapsto \hat{u} : K \to \mathbb{R}^Z$ defined by setting $\hat{u}(f) = f(u)$ for $f \in Z$ and $u \in K$. Just as in the last paragraph, this is a continuous function from K to $C(Z)$, if we give K, Z and $C(Z)$ their topologies of pointwise convergence. So $L = \{\hat{u} : u \in K\}$ is countably compact for the topology of pointwise convergence on $C(Z)$ (4A2G(f-vi)). Moreover, it is norm-bounded, because

$$\sup_{\phi\in L}\|\phi\|_\infty = \sup_{u\in K,f\in Z}|\hat u(f)| = \sup_{u\in K,f\in Z}|f(u)| = \sup_{u\in K,x\in Q}|\hat x(u)|$$
$$= \sup_{u\in K,x\in Q}|u(x)| \le \sup_{u\in K,x\in X}|u(x)| = \sup_{u\in K}\|u\|_\infty$$

is finite. So 462E(iii) tells us that L is weakly compact in $C(Z)$. (Note that $C(Z) = C_0(Z)$ because Z is compact.) Since the weak topology on $C(Z)$ is finer than the pointwise topology, while the pointwise topology is Hausdorff, the two topologies on L coincide; it follows that $u \mapsto \hat u : K \to C(Z)$ is continuous for $\mathfrak{T}_p$ and the weak topology on $C(Z)$.

Now we have an operator $T : C(Z) \to \mathbb{R}^Q$ defined by setting

$$(T\phi)(x) = \phi(\hat x)$$

for $\phi \in C(Z)$ and $x \in Q$. Because $x \mapsto \hat x : Q \to Z$ is continuous, $T\phi \in C(Q)$ for every $\phi \in C(Z)$, and of course T, regarded as a linear operator from $C(Z)$ to $C_b(Q)$, has norm at most 1. So T is continuous for the weak topologies of $C(Z)$ and $C_b(Q)$ (2A5If), and $u \mapsto T\hat u : K \to C_b(Q)$ is continuous for $\mathfrak{T}_p$ and the weak topology of $C_b(Q)$.

But if $u \in K$ and $x \in Q$,

$$(T\hat u)(x) = \hat u(\hat x) = \hat x(u) = u(x),$$

so $T\hat u = u{\restriction}Q$. Accordingly $u \mapsto u{\restriction}Q : K \to C_b(Q)$ is continuous for $\mathfrak{T}_p$ and the weak topology on $C_b(Q)$.

462G Proposition Let X be a countably compact topological space. Then a subset of $C_b(X)$ is weakly compact iff it is norm-bounded and compact for the topology $\mathfrak{T}_p$ of pointwise convergence.

proof A weakly compact subset of $C_b(X)$ is norm-bounded and $\mathfrak{T}_p$-compact by the same arguments as in (b)-(c) of the proof of 462E. In the other direction, taking $Q = X$ in 462F, we see that a norm-bounded $\mathfrak{T}_p$-compact set is weakly compact.

462H Lemma Let X be a topological space, Q a relatively countably compact subset of X, and μ a totally finite measure on $C_b(X)$ which is Radon for the topology $\mathfrak{T}_p$ of pointwise convergence on $C_b(X)$. Let $T : C_b(X) \to C_b(Q)$ be the restriction map. Then the image measure $\nu = \mu T^{-1}$ on $C_b(Q)$ is Radon for the norm topology of $C_b(Q)$.

proof (a) T is almost continuous for $\mathfrak{T}_p$ and the weak topology of $C_b(Q)$. **P** If $E \in \operatorname{dom}\mu$ and $\mu E > \gamma \ge 0$, then there is a $\mathfrak{T}_p$-compact set $K \subseteq E$ such that $\mu K > \gamma$. Since all the balls $\{f : f \in C_b(X),\ \|f\|_\infty \le k\}$ are $\mathfrak{T}_p$-closed, we may suppose that K is norm-bounded. Now $T{\restriction}K$ is continuous for $\mathfrak{T}_p$ and the weak topology, by 462F. **Q** By 418I, ν is a Radon measure for the weak topology of $C_b(Q)$.

(b) I show next that if $F \in \operatorname{dom}\nu$, $\nu F > 0$ and $\epsilon > 0$, there is some $g \in C_b(Q)$ such that $\nu(F \cap B(g,\epsilon)) > 0$, where $B(g,\epsilon) = \{h : \|h - g\|_\infty \le \epsilon\}$. **P** Since all the balls $B(g,\epsilon)$ are convex and norm-closed, they are weakly closed (3A5Ee) and measured by ν. **?** Suppose, if possible, that $F \cap B(g,\epsilon)$ is ν-negligible for every $g \in C_b(Q)$. Set $E = T^{-1}[F]$. As in (a), there is a $\|\,\|_\infty$-bounded $\mathfrak{T}_p$-compact set $K \subseteq E$ such that $\mu K > 0$. Choose $\langle K_n\rangle_{n\in\mathbb{N}}$ and $\langle f_n\rangle_{n\in\mathbb{N}}$ as follows. $K_0 = K$. Given that $K_n \subseteq E$ is non-negligible and $\mathfrak{T}_p$-compact, and that $\langle f_i\rangle_{i<n}$ is a finite sequence in $C_b(X)$, then the convex hull

$$\Gamma_n = \{\textstyle\sum_{i=0}^{n-1}\alpha_i Tf_i : \alpha_i \ge 0 \text{ for every } i < n,\ \sum_{i=0}^{n-1}\alpha_i = 1\}$$

of the finite set $\{Tf_i : i < n\}$ is norm-compact in $C_b(Q)$, so there is a finite set $D_n \subseteq \Gamma_n$ such that for every $g \in \Gamma_n$ there is a $g' \in D_n$ such that $\|g - g'\|_\infty \le \frac{1}{2}\epsilon$. Now

$$H_n = \{f : \|Tf - g\|_\infty > \epsilon \text{ for every } g \in D_n\}$$

is a $\mathfrak{T}_p$-open set and

$$E \setminus H_n = \textstyle\bigcup_{g\in D_n} T^{-1}[F \cap B(g,\epsilon)]$$

is μ-negligible, so $\mu(K_n \cap H_n) > 0$ and we can find a non-negligible $\mathfrak{T}_p$-closed set $K_{n+1} \subseteq K_n \cap H_n$; choose $f_n \in K_{n+1}$. Continue.

At the end of the induction, let $f^* \in K$ be a cluster point of $\langle f_n\rangle_{n\in\mathbb{N}}$ for $\mathfrak{T}_p$. Since $T{\restriction}K : K \to C_b(Q)$ is continuous for $\mathfrak{T}_p$ and the weak topology of $C_b(Q)$, Tf^* is a cluster point of $\langle Tf_n\rangle_{n\in\mathbb{N}}$ for the weak topology on $C_b(Q)$. The set $\Gamma = \bigcup_{n\in\mathbb{N}}\Gamma_n$ is convex, so its norm-closure $\overline{\Gamma}$ is also convex (2A5Eb), therefore closed for the weak topology (3A5Ee), and contains Tf^*. So there is a $g \in \Gamma$ such that $\|Tf^* - g\|_\infty \le \frac{1}{2}\epsilon$. Now there is some n such that $g \in \Gamma_n$. Let $g' \in D_n$ be such that $\|g - g'\|_\infty \le \frac{1}{2}\epsilon$, so that $\|Tf^* - g'\|_\infty \le \epsilon$. But $f_i \in K_{n+1}$ for every $i \ge n$, so $f^* \in K_{n+1} \subseteq H_n$ and $\|Tf^* - g'\|_\infty > \epsilon$, which is impossible. **XQ**

(c) What this means is that if we take $\mathcal{K}_n$ to be the family of subsets of $C_b(Q)$ which can be covered by finitely many balls of radius at most 2^{-n}, then ν is inner regular with respect to $\mathcal{K}_n$ (see 412Aa), and therefore with respect to $\mathcal{K} = \bigcap_{n\in\mathbb{N}} \mathcal{K}_n$ (412Ac). But $\mathcal{K}$ is just the set of subsets of $C_b(Q)$ which are totally bounded for the norm-metric ρ on $C_b(Q)$.

At the same time, ν is inner regular with respect to the ρ-closed sets in $C_b(Q)$. **P** If $\nu F > \gamma$, there is a $\| \|_\infty$-bounded $\mathfrak{T}_p$-compact set $K \subseteq T^{-1}[F]$ such that $\mu K \ge \gamma$; now $T[K]$ is weakly compact, therefore weakly closed and ρ-closed in $C_b(Q)$, while $T[K] \subseteq F$ is measured by ν and

$$\nu T[K] = \mu T^{-1}[T[K]] \ge \mu K \ge \gamma. \textbf{ Q}$$

(d) By 412Ac again, ν must be inner regular with respect to the family of ρ-closed ρ-totally bounded sets; because $C_b(Q)$ is ρ-complete, these are the ρ-compact sets. Next, every ρ-compact set is weakly compact, therefore weakly closed, and is measured by ν, by (a); and ν, being the image of a complete totally finite measure, is complete and totally finite. Consequently every ρ-closed set is measured by ν (use 412Ja) and ν is a ρ-Radon measure, as claimed.

462I Theorem Let X be a countably compact topological space. Then the totally finite Radon measures on $C(X)$ are the same for the topology of pointwise convergence and the norm topology.

proof Write $\mathfrak{T}_p$ for the topology of pointwise convergence on $C(X)$ and $\mathfrak{T}_\infty$ for the norm topology. Because $\mathfrak{T}_p \subseteq \mathfrak{T}_\infty$ and $\mathfrak{T}_p$ is Hausdorff, every totally finite $\mathfrak{T}_\infty$-Radon measure is $\mathfrak{T}_p$-Radon (418I). On the other hand, 462H, with $Q = X$, tells us that every $\mathfrak{T}_p$-Radon measure is $\mathfrak{T}_\infty$-Radon.

462J Corollary Let X be a countably compact Hausdorff space, and give $C(X)$ its topology of pointwise convergence. If μ is any Radon measure on $C(X)$, it is inner regular with respect to the family of compact metrizable subsets of $C(X)$.

proof In the language of 462I, μ is inner regular with respect to the family of $\mathfrak{T}_\infty$-compact sets; but as $\mathfrak{T}_p \subseteq \mathfrak{T}_\infty$, the two topologies agree on all such sets, and they are compact and metrizable for $\mathfrak{T}_p$.

462K Proposition Let X be a topological space, Y a Hausdorff space, $f : X \times Y \to \mathbb{R}$ a bounded separately continuous function, and ν a totally finite Radon measure on Y. Set $\phi(x) = \int f(x,y)\nu(dy)$ for every $x \in X$. Then $\phi\restriction Q$ is continuous for every relatively countably compact set $Q \subseteq X$.

proof For $y \in Y$, set $u_y(x) = f(x,y)$ for every $x \in X$. Then every u_y is continuous and bounded, because f is bounded and continuous in the first variable, and $y \mapsto u_y : Y \to C_b(X)$ is continuous, if we give $C_b(X)$ the topology $\mathfrak{T}_p$ of pointwise convergence, because f is continuous in the second variable. We therefore have a $\mathfrak{T}_p$-Radon image measure μ on $C_b(X)$, by 418I.

Let $T : C_b(X) \to C_b(Q)$ be the restriction map. By 462H, the image measure $\lambda = \mu T^{-1}$ is a Radon measure for the norm topology of $C_b(Q)$. Now recall that f is bounded. If $|f(x,y)| \le M$ for all $x \in X$, $y \in Y$, then $\|u_y\|_\infty \le M$ for every $y \in Y$, and the ball $B(0,M)$ in $C_b(Q)$ is λ-conegligible. By 461F, applied to the subspace measure on $B(0,M)$, ν has a barycenter h in $C_b(Q)$. Now we can compute h by the formulae

$$h(x) = \int_{C_b(Q)} g(x)\lambda(dg)$$

(because $g \mapsto g(x)$ belongs to $C_b(X)^*$)

$$= \int_{C_b(X)} u(x)\mu(du) = \int_Y u_y(x)\nu(dy)$$

(by 235G)

$$= \int_Y f(x,y)\nu(dy) = \phi(x),$$

for every $x \in Q$. So $\phi = h$ is continuous.

462L Corollary Let X be a topological space such that

whenever $h \in \mathbb{R}^X$ is such that $h{\restriction}Q$ is continuous for every relatively countably compact $Q \subseteq X$, then h is continuous.

Write $\mathfrak{T}_p$ for the topology of pointwise convergence on $C(X)$. Let $K \subseteq C(X)$ be a $\mathfrak{T}_p$-compact set such that $\{h(x) : h \in K, x \in Q\}$ is bounded for any relatively countably compact set $Q \subseteq X$. Then the $\mathfrak{T}_p$-closed convex hull of K, taken in $C(X)$, is $\mathfrak{T}_p$-compact.

proof If K is empty, this is trivial; suppose that $K \neq \emptyset$. Since $\sup_{h\in K} |h(x)|$ is finite for every $x \in X$, the closed convex hull $\overline{\Gamma(K)}$ of K, taken in $\mathbb{R}^X$, is closed and included in a product of closed bounded intervals, therefore compact. If $h \in \overline{\Gamma(K)}$, then there is a Radon probability measure μ on K such that h is the barycenter of μ (461I), so that $h(x) = \int f(x)\mu(df)$ for every $x \in X$.

If $Q \subseteq X$ is relatively countably compact, then $h{\restriction}Q$ is continuous. **P** Of course we may suppose that Q is non-empty. Consider its closure $Z = \overline{Q}$. We have a continuous linear operator $T : \mathbb{R}^X \to \mathbb{R}^Z$ defined by setting $Tf = f{\restriction}Z$ for every $f \in \mathbb{R}^X$. $L = T[K]$ is compact in $\mathbb{R}^Z$, and $L \subseteq C(Z)$; moreover,

$$\sup_{g\in L} \|g\|_\infty = \sup_{f\in K, x\in Z} |f(x)| = \sup_{f\in K, x\in Q} |f(x)|$$

is finite. Since $T{\restriction}K : K \to L$ is continuous, the image measure $\nu = \mu(T{\restriction}K)^{-1}$ on L is a Radon measure. If $x \in Z$, then

$$h(x) = \int_K f(x)\mu(df) = \int_K (Tf)(x)\mu(df) = \int_L g(x)\nu(dg).$$

The map $(x, g) \mapsto g(x) : Z \times L \to \mathbb{R}$ is separately continuous, because $L \subseteq C(Z)$ is being given its topology of pointwise convergence, and bounded. Also every sequence in Q has a cluster point in X which must also belong to Z, and Q is relatively countably compact in Z. By 462K, $h{\restriction}Q$ is continuous, as required. **Q**

Thus the $\mathfrak{T}_p$-compact set $\overline{\Gamma(K)}$ is included in $C(X)$, and must be the closed convex hull of K in $C(X)$.

Remark The hypothesis

whenever $h \in \mathbb{R}^X$ is such that $h{\restriction}Q$ is continuous for every relatively countably compact $Q \subseteq X$, then h is continuous

is clumsy, but seems the best way to cover the large number of potential applications of the ideas here. Besides the obvious case of countably compact spaces X, we have all first-countable spaces (for which, of course, the other hypotheses can be relaxed, as in 462Xc), and all k-spaces. (A ***k*-space** is a topological space X such that a set $G \subseteq X$ is open iff $G \cap K$ is relatively open in K for every compact set $K \subseteq X$; see Engelking 89, 3.3.18 *et seq.* In particular, all locally compact spaces are k-spaces.)

462X Basic exercises (a) (i) Show that $\mathbb{R}$, with the right-facing Sorgenfrey topology, is angelic. (ii) Show that any metrizable space is angelic. (iii) Show that the one-point compactification of an angelic locally compact Hausdorff space is angelic. (iv) Find a first-countable regular Hausdorff space which is not angelic.

>(b) Let X be any countably compact topological space. Show that a norm-bounded sequence in $C_b(X)$ which is pointwise convergent is weakly convergent.

(c) Let X be a first-countable topological space, (Y, T, ν) a totally finite measure space, and $f : X \times Y \to \mathbb{R}$ a bounded function such that $y \mapsto f(x, y)$ is measurable for every $x \in X$, and $x \mapsto f(x, y)$ is continuous for almost every $y \in Y$. Show that $x \mapsto \int f(x, y)\nu(dy)$ is continuous.

(d) Give an example of a $\mathfrak{T}_p$-compact subset K of $C([0, 1])$ such that the convex hull of K is not relatively compact in $C([0, 1])$.

462Y Further exercises (a) Let X be a topological space such that there is a sequence $\langle X_n\rangle_{n\in\mathbb{N}}$ of relatively countably compact subsets of X, covering X, with the property that a function $f : X \to \mathbb{R}$ is continuous whenever $f{\restriction}X_n$ is continuous for every $n \in \mathbb{N}$. Let $\mathfrak{T}_p$ be the topology of pointwise convergence on $C(X)$. Show that, for a set $K \subseteq C(X)$, the following are equiveridical: (i) $\phi[K]$ is bounded for every $\mathfrak{T}_p$-continuous function $\phi : C(X) \to \mathbb{R}$; (ii) whenever $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in K and $A \subseteq X$ is countable, there is a cluster point of $\langle f_n{\restriction}A\rangle_{n\in\mathbb{N}}$ in $C(A)$ for the topology of pointwise convergence on $C(A)$; (iii) K is relatively compact in $C(X)$ for $\mathfrak{T}_p$. (See Asanov & Velichko 81.)

(b) Let U be a metrizable locally convex linear topological space. Show that it is angelic in its weak topology. (*Hint*: start with the case in which U is complete, using Grothendieck's theorem and the full strength of 462C, with $X = U^*$.)

(c) In 462K, show that the conclusion remains valid for any totally finite τ-additive topological measure ν on Y which is inner regular with respect to the relatively countably compact subsets of Y.

(d) Show that if X is any compact topological space (more generally, any topological space such that X^n is Lindelöf for every $n \in \mathbb{N}$), then $C(X)$, with its topology of pointwise convergence, is countably tight.

(e)(i) Let X and Y be Polish spaces, and write $B_1(X;Y)$ for the set of functions $f : X \to Y$ such that $f^{-1}[H]$ is G_δ in X for every closed set $H \subseteq Y$ (Kuratowski 66, §31). Show that $B_1(X;Y)$, with the topology of pointwise convergence inherited from Y^X, is angelic. (*Hint*: Bourgain Fremlin & Talagrand 78.) (ii) Let X be a Polish space. Show that the space $\tilde{C}^{\Downarrow\!\Uparrow}(X)$ of 438P-438Q is angelic.

462Z Problem Let K be a compact Hausdorff space. Is $C(K)$, with the topology of pointwise convergence, necessarily a pre-Radon space? (Compare 454S.)

462 Notes and comments The theory of pointwise convergence in spaces of continuous functions is intimately connected with the theory of separately continuous functions of two variables. For if X and Y are topological spaces, and $f : X \times Y \to \mathbb{R}$ is any separately continuous function, then we have natural maps $x \mapsto f_x : X \to C(Y)$ and $y \mapsto f^y : Y \to C(X)$, writing $f_x(y) = f^y(x) = f(x,y)$, which are continuous if $C(X)$ and $C(Y)$ are given their topologies of pointwise convergence; and if X is a topological space and Y is any subset of $C(X)$ with its topology of pointwise convergence, the map $(x,y) \mapsto y(x) : X \times Y \to \mathbb{R}$ is separately continuous. I include a back-and-forth shuffle between $C(X)$ and separately continuous functions in 462H-462K-462L as a demonstration of the principle that all the theorems here can be expressed in both languages.

462Yb is a compendium of Šmulian's theorem with part of Eberlein's theorem; 462E and 462L can be thought of as the centre of Krein's theorem. There are many alternative routes to these results, which may be found in Köthe 69 or Grothendieck 92. In particular, 462E can be proved without using measure theory; see, for instance, Fremlin 74, A2F.

Topological spaces homeomorphic to compact uniformly bounded subsets of $C(X)$, where X is some compact space and $C(X)$ is given its topology of pointwise convergence, are called **Eberlein compacta**; see 467O-467P.

A positive answer to A.Bellow's problem (463Za below) would imply a positive answer to 462Z; so if the continuum hypothesis, for instance, is true, then $C(K)$ is pre-Radon in its topology of pointwise convergence for any compact space K.

463 $\mathfrak{T}_p$ and $\mathfrak{T}_m$

We are now ready to start on the central ideas of this chapter with an investigation of sets of measurable functions which are compact for the topology of pointwise convergence. Because 'measurability' is, from the point of view of this topology on $\mathbb{R}^X$, a rather arbitrary condition, we are looking at compact subsets of a topologically irregular subspace of $\mathbb{R}^X$; there are consequently relatively few of them, and (under a variety of special circumstances, to be examined later in the chapter and also in Volume 5) they have some striking special properties.

The presentation here is focused on the relationship between the two natural topologies on any space of measurable functions, the 'pointwise' topology $\mathfrak{T}_p$ and the topology $\mathfrak{T}_m$ of convergence in measure (463A). In this section I begin with results which apply to any σ-finite measure space (463B-463H) before turning to some which apply to perfect measure spaces (463I-463L) – in particular, to Lebesgue measure. These lead to some interesting properties of separately continuous functions (463M-463N).

463A Preliminaries Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ the space of all Σ-measurable functions from X to $\mathbb{R}$, so that $\mathcal{L}^0$ is a linear subspace of $\mathbb{R}^X$. On $\mathcal{L}^0$ we shall be concerned with two very different topologies. The first is the topology $\mathfrak{T}_p$ of pointwise convergence (462Ab); the second is the topology $\mathfrak{T}_m$ of (local) convergence in measure (245A). Both are linear space topologies. **P** For $\mathfrak{T}_p$ I have already noted this in 462Ab. For $\mathfrak{T}_m$, repeat the argument of 245Da; $\mathfrak{T}_m$ is defined by the functionals $f \mapsto \int_F \min(1, |f|)d\mu$, where $\mu F < \infty$, and these are F-seminorms (definition: 2A5B[7]). **Q** $\mathfrak{T}_p$ is Hausdorff (3A3Id) and locally convex (4A4Ce); only in exceptional circumstances is either true of $\mathfrak{T}_m$. However, $\mathfrak{T}_m$ can easily be pseudometrizable (if, for instance, μ is σ-finite, as in 245Eb), while $\mathfrak{T}_p$ is not, except in nearly trivial cases.

Associated with the topology of pointwise convergence on $\mathbb{R}^X$ is the usual topology of $\mathcal{P}X$ (4A2A); the map $\chi : \mathcal{P}X \to \mathbb{R}^X$ is a homeomorphism between $\mathcal{P}X$ and its image $\{0,1\}^X \subseteq \mathbb{R}^X$.

$\mathfrak{T}_m$ is intimately associated with the topology of convergence in measure on $L^0 = L^0(\mu)$ (§245). A subset of $\mathcal{L}^0$ is open for $\mathfrak{T}_m$ iff it is of the form $\{f : f^\bullet \in G\}$ for some open set $G \subseteq L^0$; consequently, a subset K of $\mathcal{L}^0$ is compact, or separable, for $\mathfrak{T}_m$ iff $\{f^\bullet : f \in K\}$ is compact or separable for the topology of convergence in measure on L^0.

It turns out that the identity map from $(\mathcal{L}^0, \mathfrak{T}_p)$ to $(\mathcal{L}^0, \mathfrak{T}_m)$ is sequentially continuous (463B). Only in nearly trivial cases is it actually continuous (463Xa(i)), and it is similarly rare for the reverse map from $(\mathcal{L}^0, \mathfrak{T}_m)$ to $(\mathcal{L}^0, \mathfrak{T}_p)$ to be continuous (463Xa(ii)). If, however, we relativise both topologies to a $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0$, the situation becomes very different, and there are many important cases in which the topologies are comparable.

463B Lemma Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Then every pointwise convergent sequence in $\mathcal{L}^0$ is convergent in measure to the same limit.

proof 245Ca.

463C Proposition (Ionescu Tulcea 73) Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on $\mathcal{L}^0$; for $A \subseteq \mathcal{L}^0$, write $\mathfrak{T}_p^{(A)}$, $\mathfrak{T}_m^{(A)}$ for the corresponding subspace topologies.

(a) If $A \subseteq \mathcal{L}^0$ and $\mathfrak{T}_p^{(A)}$ is metrizable, then the identity map from A to itself is $(\mathfrak{T}_p^{(A)}, \mathfrak{T}_m^{(A)})$-continuous.

(b) Suppose that μ is semi-finite. Then, for any $A \subseteq \mathcal{L}^0$, $\mathfrak{T}_m^{(A)}$ is Hausdorff iff whenever f, g are distinct members of A the set $\{x : f(x) \ne g(x)\}$ is non-negligible.

(c) Suppose that $K \subseteq \mathcal{L}^0$ is such that $\mathfrak{T}_p^{(K)}$ is compact and metrizable. Then $\mathfrak{T}_p^{(K)} = \mathfrak{T}_m^{(K)}$ iff $\mathfrak{T}_m^{(K)}$ is Hausdorff.

(d) Suppose that μ is σ-finite, and that $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-sequentially compact. Then $\mathfrak{T}_p^{(K)} = \mathfrak{T}_m^{(K)}$ iff $\mathfrak{T}_m^{(K)}$ is Hausdorff, and in this case $\mathfrak{T}_p^{(K)}$ is compact and metrizable.

(e) Suppose that $K \subseteq \mathcal{L}^0$ is such that $\mathfrak{T}_p^{(K)}$ is compact and metrizable. Then whenever $\epsilon > 0$ and $E \in \Sigma$ is a non-negligible measurable set, there is a non-negligible measurable set $F \subseteq E$ such that $|f(x) - f(y)| \le \epsilon$ whenever $f \in K$ and $x, y \in F$.

proof (a) All we need is to remember that sequentially continuous functions from metrizable spaces are continuous (4A2Ld), and apply 463B.

(b) $\mathfrak{T}_m^{(A)}$ is Hausdorff iff for any distinct f, $g \in A$ there is a measurable set F of finite measure such that $\int_F \min(1, |f - g|)d\mu > 0$, that is, $\mu\{x : x \in F, f(x) \ne g(x)\} > 0$; because μ is semi-finite, this happens iff $\mu\{x : f(x) \ne g(x)\} > 0$.

(c) If $\mathfrak{T}_p^{(K)} = \mathfrak{T}_m^{(K)}$ then of course $\mathfrak{T}_m^{(K)}$ is Hausdorff, because $\mathfrak{T}_p^{(K)}$ is. If $\mathfrak{T}_m^{(K)}$ is Hausdorff then the identity map $(K, \mathfrak{T}_p^{(K)}) \to (K, \mathfrak{T}_m^{(K)})$ is an injective function from a compact space to a Hausdorff space and (by (a)) is continuous, therefore a homeomorphism, so the two topologies are equal.

(d) If $\mathfrak{T}_p^{(K)} = \mathfrak{T}_m^{(K)}$ then $\mathfrak{T}_m^{(K)}$ must be Hausdorff, just as in (c). So let us suppose that $\mathfrak{T}_m^{(K)}$ is Hausdorff. Note that, by 245Eb, the topology of convergence in measure on L^0 is metrizable; in terms of $\mathcal{L}^0$, this says just that the topology of convergence in measure on $\mathcal{L}^0$ is pseudometrizable. So $\mathfrak{T}_m^{(K)}$ is Hausdorff and pseudometrizable, therefore metrizable (4A2La).

We are told that any sequence in K has a $\mathfrak{T}_p^{(K)}$-convergent subsequence. But this subsequence is now $\mathfrak{T}_m^{(K)}$-convergent (463B), so $\mathfrak{T}_m^{(K)}$ is sequentially compact; being metrizable, it is compact (4A2Lf). Moreover, the same is true of any $\mathfrak{T}_p^{(K)}$-closed subset of K, so every $\mathfrak{T}_p^{(K)}$-closed set is $\mathfrak{T}_m^{(K)}$-compact, therefore $\mathfrak{T}_m^{(K)}$-closed. Thus the identity map from $(K, \mathfrak{T}_m^{(K)})$ to $(K, \mathfrak{T}_p^{(K)})$ is continuous. Since $\mathfrak{T}_m^{(K)}$ is compact and $\mathfrak{T}_p^{(K)}$ is Hausdorff, the two topologies are equal; and, in particular, $\mathfrak{T}_p^{(K)}$ is compact and metrizable.

(e) Let ρ be a metric on K inducing the topology $\mathfrak{T}_p^{(K)}$. Let $D \subseteq K$ be a countable dense set. For each $n \in \mathbb{N}$, set

[7]Later editions only; see §4A7.

$$G_n = \{x : |f(x) - g(x)| \le \frac{1}{3}\epsilon \text{ whenever } f,\, g \in D \text{ and } \rho(f,g) \le 2^{-n}\}.$$

Because D is countable, G_n is measurable. Now $\bigcup_{n\in\mathbb{N}} G_n = X$. **P?** If $x \in X \setminus \bigcup_{n\in\mathbb{N}} G_n$, then for each $n \in \mathbb{N}$ we can find f_n, $g_n \in D$ such that $\rho(f_n, g_n) \le 2^{-n}$ and $|f_n(x) - g_n(x)| \ge \frac{1}{3}\epsilon$. Because K is compact, there is a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ such that $\langle f_{n_k}\rangle_{k\in\mathbb{N}}$ and $\langle g_{n_k}\rangle_{k\in\mathbb{N}}$ are both convergent to f, g say. Now

$$\rho(f,g) = \lim_{k\to\infty} \rho(f_{n_k}, g_{n_k}) = 0, \quad |f(x) - g(x)| = \lim_{k\to\infty} |f_{n_k}(x) - g_{n_k}(x)| \ge \frac{1}{3}\epsilon,$$

so $f = g$ while $f(x) \ne g(x)$, which is impossible. **XQ**

There is therefore some $n \in \mathbb{N}$ such that $\mu(E \cap G_n) > 0$. Since K, being compact, is totally bounded for ρ, there is a finite set $D' \subseteq D$ such that every member of D is within a distance of 2^{-n} of some member of D'. Now there is a measurable set $F \subseteq E \cap G_n$ such that $\mu F > 0$ and $|g(x) - g(y)| \le \frac{1}{3}\epsilon$ whenever $g \in D'$ and x, $y \in F$. So $|f(x) - f(y)| \le \epsilon$ whenever $f \in D$ and x, $y \in F$. But as D is dense in K, $|f(x) - f(y)| \le \epsilon$ whenever $f \in K$ and x, $y \in F$, as required.

463D Lemma Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$ for the topology of pointwise convergence on $\mathcal{L}^0$. Suppose that $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-compact and that there is no $\mathfrak{T}_p$-continuous surjection from any closed subset of K onto $\{0,1\}^{\omega_1}$. If $E \in \Sigma$ has finite measure, then every sequence in K has a subsequence which is convergent almost everywhere in E.

proof (a) Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in K. Set $q(x) = \max(1, 2\sup_{n\in\mathbb{N}} |f_n(x)|)$ for each $x \in X$. For any infinite $I \subseteq \mathbb{N}$, set

$$g_I = \liminf_{i\to I} f_i = \sup_{n\in\mathbb{N}} \inf_{i\in I, i\ge n} f_i,$$

$$h_I = \limsup_{i\to I} f_i = \inf_{n\in\mathbb{N}} \sup_{i\in I, i\ge n} f_i;$$

because $\sup_{f\in K} |f(x)|$ is surely finite for each $x \in X$, g_I and h_I are defined in $\mathcal{L}^0$, and $g_I \le h_I$. For $f \in \mathcal{L}^0$ set $\tau'(f) = \int_E \min(1, |f|/q)$, and for $I \in [\mathbb{N}]^\omega$ (the set of infinite subsets of $\mathbb{N}$) set $\Delta(I) = \tau'(h_I - g_I)$. Since $h_I - g_I \le q$, $\Delta(I) = \int_E (h_I - g_I)/q$. If I, $J \in [\mathbb{N}]^\omega$ and $J \setminus I$ is finite, then $g_I \le g_J \le h_J \le h_I$, so $\Delta(J) \le \Delta(I)$, with equality only when $g_I = g_J$ a.e. on E and $h_I = h_J$ a.e. on E.

(b) There is a $J \in [\mathbb{N}]^\omega$ such that $\Delta(I) = \Delta(J)$ for every $I \in [J]^\omega$. **P** For $J \in [\mathbb{N}]^\omega$, set $\underline{\Delta}(J) = \inf\{\Delta(I) : I \in [J]^\omega\}$. Choose $\langle I_n\rangle_{n\in\mathbb{N}}$ in $[\mathbb{N}]^\omega$ inductively in such a way that $I_{n+1} \subseteq I_n$ and $\Delta(I_{n+1}) \le \underline{\Delta}(I_n) + 2^{-n}$ for every n. If we now set

$$J = \{\min\{i : i \in I_n,\, i \ge n\} : n \in \mathbb{N}\},$$

$J \subseteq \mathbb{N}$ will be an infinite set and $J \setminus I_n$ is finite for every n. If $I \in [J]^\omega$ then, for every n,

$$\Delta(J) \le \Delta(I_{n+1}) \le \underline{\Delta}(I_n) + 2^{-n} \le \Delta(I_n \cap I) + 2^{-n} = \Delta(I) + 2^{-n};$$

as n and I are arbitrary, $\underline{\Delta}(J) = \Delta(J)$, as required. **Q**

Now for any $I \in [J]^\omega$ we have $g_I = g_J$ a.e. on E and $h_I = h_J$ a.e. on E.

(c) $\Delta(J) = 0$. **P?** Otherwise, $F = \{x : x \in E,\, g_J(x) < h_J(x)\}$ has positive measure. Write K_0 for $\bigcap_{n\in\mathbb{N}} \overline{\{f_i : i \in J,\, i \ge n\}}$, the closure being taken for $\mathfrak{T}_p$, so that K_0 is $\mathfrak{T}_p$-compact. Let $\mathcal{A}$ be the family of sets $A \subseteq F$ such that whenever L, $M \subseteq A$ are finite and disjoint there is an $f \in K_0$ such that $f(x) = g_J(x)$ for $x \in L$ and $f(x) = h_J(x)$ for $x \in M$. Then $\mathcal{A}$ has a maximal member A_0 say. If we define $\phi : \mathcal{L}^0 \to [0,1]^{A_0}$ by setting $\phi(f)(x) = \frac{\operatorname{med}(0, f(x) - g_J(x), h_J(x) - g_J(x))}{h_J(x) - g_J(x)}$ for $x \in A_0$ and $f \in \mathcal{L}^0$, $\phi[K_0]$ is a compact subset of $[0,1]^{A_0}$, and whenever L, $M \subseteq A_0$ are finite there is a $g \in \phi[K_0]$ such that $g(x) = 0$ for $x \in L$ and $g(x) = 1$ for $x \in M$. This means that $\phi[K_0] \cap \{0,1\}^{A_0}$ is dense in $\{0,1\}^{A_0}$ and must therefore be the whole of $\{0,1\}^{A_0}$. So $\{0,1\}^{A_0}$ is a continuous image of a closed subset of K.

Since $\{0,1\}^{\omega_1}$ is not a continuous image of a closed subset of K, it is not a continuous image of $\{0,1\}^{A_0}$, and cannot be homeomorphic to $\{0,1\}^A$ for any $A \subseteq A_0$. Thus no subset of A_0 can have cardinal ω_1 and A_0 is countable.

For each pair L, M of disjoint finite subsets of A_0, we have a cluster point f_{LM} of $\langle f_j\rangle_{j\in J}$ such that $f_{LM}(x) = g_J(x)$ for $x \in L$ and $f_{LM}(x) = h_J(x)$ for $x \in M$. Let $I(L, M)$ be an infinite subset of J such that $\lim_{i\to\infty, i\in I(L,M)} f_i(x) = f_{LM}(x)$ for every $x \in A_0$. Then $g_{I(L,M)} = g_J$ and $h_{I(L,M)} = h_J$ almost everywhere in E. Because $\mu F > 0$ and A_0 has only countably many finite subsets, there is a $y \in F$ such that $g_{I(L,M)}(y) = g_J(y)$ and $h_{I(L,M)}(y) = h_J(y)$ whenever L and M are disjoint finite subsets of A_0.

What this means is that if L and M are disjoint finite subsets of A_0, then there are infinite sets I', $I'' \subseteq I(L,M)$ such that $\lim_{i\to\infty,i\in I'} f_i(y) = g_J(y)$ and $\lim_{i\to\infty,i\in I''} f_i(y) = h_J(y)$; so that there are f', $f'' \in K_0$ such that

$$f'(x) = g_J(x) \text{ for } x \in L \cup \{y\}, \quad f'(x) = h_J(x) \text{ for } x \in M,$$

$$f''(x) = g_J(x) \text{ for } x \in L, \quad f''(x) = h_J(x) \text{ for } x \in M \cup \{y\}.$$

But this means that $A_0 \cup \{y\} \in \mathcal{A}$, and also that $y \notin A_0$; and A_0 was chosen to be maximal. **XQ**

(d) So $\int_E (h_J - g_J)/q = 0$ and $g_J = h_J$ almost everywhere in E. But if we enumerate J in ascending order as $\langle n_i\rangle_{i\in\mathbb{N}}$, $g_J = \liminf_{i\to\infty} f_{n_i}$ and $h_J = \limsup_{i\to\infty} f_{n_i}$, so $\langle f_{n_i}\rangle_{i\in\mathbb{N}}$ converges almost everywhere in E.

463E Proposition Let (X,Σ,μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on $\mathcal{L}^0$. Suppose that $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-compact and that there is no $\mathfrak{T}_p$-continuous surjection from any closed subset of K onto $\omega_1 + 1$ with its order topology. Then the identity map from $(K,\mathfrak{T}_p)$ to $(K,\mathfrak{T}_m)$ is continuous.

proof (a) It is worth noting straight away that $\xi \mapsto \chi\xi : \omega_1 + 1 \to \{0,1\}^{\omega_1}$ is a homeomorphism between $\omega_1 + 1$ and a subspace of $\{0,1\}^{\omega_1}$. So our hypothesis tells us that there is no continuous surjection from any closed subset of K onto $\{0,1\}^{\omega_1}$, and therefore none onto $\{0,1\}^A$ for any uncountable A.

(b) ? Suppose, if possible, that the identity map from $(K,\mathfrak{T}_p)$ to $(K,\mathfrak{T}_m)$ is not continuous at $f_0 \in K$. Then there are an $E \in \Sigma$, of finite measure, and an $\epsilon > 0$ such that $C = \{f : f \in K, \tau_E(f - f_0) \ge \epsilon\}$ meets every $\mathfrak{T}_p$-neighbourhood of f, where $\tau_E(f) = \int_E \min(1,|f|)$ for every $f \in \mathcal{L}^0$, and there is an ultrafilter $\mathcal{F}$ on $\mathcal{L}^0$ which contains C and converges to f_0 for $\mathfrak{T}_p$. Consider the map $\psi : \mathcal{L}^0 \to L^0(\mu_E)$, where μ_E is the subspace measure on E, defined by setting $\psi(f) = (f\restriction E)^\bullet$ for $f \in \mathcal{L}^0$. We know from 463D that every sequence in K has a subsequence convergent almost everywhere in E, so every sequence in $\psi[K]$ has a subsequence which is convergent for the topology of convergence in measure on $L^0(\mu_E)$. Since this is metrizable, $\psi[K]$ is relatively compact in $L^0(\mu_E)$ (4A2Le), and the image filter $\psi[[\mathcal{F}]]$ has a limit $v \in L^0(\mu_E)$. Let $f_1 \in \mathcal{L}^0$ be such that $\psi(f_1) = v$.

For any countable set $A \subseteq X$ there is a $g \in C$ such that $g\restriction A = f_0\restriction A$ and $g = f_1$ almost everywhere in E. **P** If $X = \emptyset$ this is trivial, so we may, if necessary, enlarge A by one point so that it is not empty. Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a sequence running over A. Then for each $n \in \mathbb{N}$ the set

$$\{g : g \in C, |g(x_i) - f_0(x_i)| \le 2^{-n} \text{ for every } i \le n, \tau_E(g, f_1) \le 2^{-n}\}$$

belongs to $\mathcal{F}$, so is not empty; take g_n in this set. Let $g \in K$ be any cluster point of $\langle g_n\rangle_{n\in\mathbb{N}}$. Since $\langle g_n\rangle_{n\in\mathbb{N}}$ converges to f_1 almost everywhere in E, $g = f_1$ a.e. on E and $\langle g_n\rangle_{n\in\mathbb{N}}$ converges to g almost everywhere in E. Consequently $\tau_E(g - f_0) = \lim_{n\to\infty} \tau_E(g_n - f_0)$, by the dominated convergence theorem, and $g \in C$. Since $\langle g_n(x_i)\rangle_{n\in\mathbb{N}}$ converges to $f_0(x_i)$ for every i, $g\restriction A = f_0\restriction A$. So we have the result. **Q**

In particular, there is a $g \in C$ such that $g = f_1$ a.e. on E, so $\tau_E(f_1 - f_0) = \tau(g - f_0) \ge \epsilon$ and $F = \{x : x \in E, f_0(x) \ne f_1(x)\}$ has non-zero measure. Now choose $\langle g_\xi\rangle_{\xi<\omega_1}$ in K and $\langle x_\xi\rangle_{\xi<\omega_1}$ in F inductively so that

$$g_\xi \in C, \quad g_\xi = f_1 \text{ almost everywhere in } E, \quad g_\xi(x_\eta) = f_0(x_\eta) \text{ for } \eta < \xi$$

(choosing g_ξ),

$$x_\xi \in F, \quad g_\eta(x_\xi) = f_1(x_\xi) \text{ for } \eta \le \xi$$

(choosing x_ξ). If we now set $A = \{x_\xi : \xi < \omega_1\}$,

$$K_1 = \bigcap_{\xi\le\eta<\omega_1} \{f : f \in K, \text{ either } f(x_\xi) = f_0(x_\xi) \text{ or } f(x_\eta) = f_1(x_\eta)\},$$

then K_1 is a closed subset of K containing every g_ξ and also f_0. But if we look at $\{f\restriction A : f \in K_1\}$, this is homeomorphic to $\omega_1 + 1$; which is supposed to be impossible. **X**

So we conclude that the identity map from $(K,\mathfrak{T}_p)$ to $(K,\mathfrak{T}_m)$ is continuous.

463F Corollary Let (X,Σ,μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on $\mathcal{L}^0$. Suppose that $K \subseteq \mathcal{L}^0$ is compact and countably tight for $\mathfrak{T}_p$. Then the identity map from $(K,\mathfrak{T}_p)$ to $(K,\mathfrak{T}_m)$ is continuous. If $\mathfrak{T}_m$ is Hausdorff on K, the two topologies coincide on K.

proof Since $\omega_1 + 1$ is not countably tight (the top point ω_1 is not in the closure of any countable subset of ω_1), $\omega_1 + 1$ is not a continuous image of any closed subset of K (4A2Kb), and we can apply 463E to see that the identity map is continuous. It follows at once that if $\mathfrak{T}_m$ is Hausdorff on K, then the topologies coincide.

463G Theorem (Ionescu Tulcea 74) Let (X,Σ,μ) be a σ-finite measure space, and K a convex set of measurable functions from X to $\mathbb{R}$ such that (i) K is compact for the topology $\mathfrak{T}_p$ of pointwise convergence (ii) $\{x : f(x) \ne g(x)\}$ is not negligible for any distinct f, $g \in K$. Then K is metrizable for $\mathfrak{T}_p$, which agrees with the topology of convergence in measure on K.

proof (a) Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in K. Then it has a pointwise convergent subsequence. **P** Because K is compact, we surely have $\sup_{f\in K}|f(x)| < \infty$ for every $x \in X$. Let $\langle X_k\rangle_{k\in\mathbb{N}}$ be a sequence of measurable sets of finite measure covering X, and set

$$Y_k = \{x : x \in X_k,\ |f_n(x)| \le k \text{ for every } n \in \mathbb{N}\}$$

for $k \in \mathbb{N}$,

$$q = \sum_{k=0}^{\infty} \frac{1}{2^k(1+\mu Y_k)}\chi Y_k,$$

so that q is a strictly positive measurable function and

$$\|f_n \times q\|_2 \le \sum_{k=0}^{\infty} \frac{k}{2^k} = 2$$

for every n.

Set $K' = \{f \times q : f \in K\}$, so that K' is another convex pointwise compact set of measurable functions, this time all dominated by q', so that $K' \subseteq \mathcal{L}^2(\mu)$. Setting $g_n = f_n \times q$, the sequence $\langle g_n^\bullet\rangle_{n\in\mathbb{N}}$ of equivalence classes is a norm-bounded sequence in the Hilbert space $L^2 = L^2(\mu)$. It therefore has a weakly convergent subsequence $\langle g_{n(i)}^\bullet\rangle_{i\in\mathbb{N}}$ say (4A4Kb), with limit v.

? Suppose, if possible, that $\langle f_{n(i)}\rangle_{i\in\mathbb{N}}$ is not pointwise convergent. Then there must be some $x_0 \in X$ such that $\liminf_{i\to\infty} f_{n(i)}(x_0) < \limsup_{i\to\infty} f_{n(i)}(x_0)$; let $\alpha < \beta$ in $\mathbb{R}$ be such that $I = \{i : f_{n(i)}(x_0) \le \alpha\}$, $I' = \{i : f_{n(i)}(x_0) \ge \beta\}$ are both infinite. In this case v belongs to the weak closures of both $D = \{g_{n(i)}^\bullet : i \in I\}$ and $D' = \{g_{n(i)}^\bullet : i \in I'\}$. It must therefore belong to the *norm* closures of their convex hulls $\Gamma(D)$, $\Gamma(D')$ (4A4Ed). Accordingly we can find $v_n \in \Gamma(D)$, $v'_n \in \Gamma(D')$ such that $\|v - v_n\|_2 \le 3^{-n}$, $\|v - v'_n\|_2 \le 3^{-n}$ for every $n \in \mathbb{N}$.

Setting $A = \{f_{n(i)} : i \in I\}$, $A' = \{f_{n(i)} : i \in I'\}$, we see that there must be $h_n \in \Gamma(A)$, $h'_n \in \Gamma(A')$ such that $v_n = (h_n \times q)^\bullet$, $v'_n = (h'_n \times q)^\bullet$ for every $n \in \mathbb{N}$. Now if $g : X \to \mathbb{R}$ is a measurable function such that $g^\bullet = v$, and $\tilde{h} = g/q$, we have

$$\begin{aligned}\mu\{x : |\tilde{h}(x) - h_n(x)| \ge \frac{1}{2^n q(x)}\} &= \mu\{x : |g(x) - (h_n \times q)(x)| \ge 2^{-n}\}\\ &\le 4^n\|v - v_n\|_2^2 \le 2^{-n}\end{aligned}$$

for every $n \in \mathbb{N}$, and $h_n \to \tilde{h}$ a.e. Similarly, $h'_n \to \tilde{h}$ a.e.

At this point, recall that K is supposed to be convex, so all the h_n, h'_n belong to K. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Because K is pointwise compact, $h = \lim_{n\to\mathcal{F}} h_n$ and $h' = \lim_{n\to\mathcal{F}} h'_n$ are both defined in K for the topology of pointwise convergence. For any x such that $\lim_{n\to\infty} h_n(x) = \tilde{h}(x)$, we surely have $h(x) = \tilde{h}(x)$; so $h =_{\text{a.e.}} \tilde{h}$. Similarly, $h' =_{\text{a.e.}} \tilde{h}$, and $h =_{\text{a.e.}} h'$.

Now at last we apply the hypothesis that distinct members of K are not equal almost everywhere, to see that $h = h'$. But if we look at what happens at the distinguished point x_0 above, we see that $f(x_0) \le \alpha$ for every $f \in A$, so that $f(x_0) \le \alpha$ for every $f \in \Gamma(A)$, $h_n(x_0) \le \alpha$ for every $n \in \mathbb{N}$, and $h(x_0) \le \alpha$; and similarly $h'(x_0) \ge \beta$. So $h \ne h'$, which is absurd. **X**

This contradiction shows that $\langle f_{n(i)}\rangle_{i\in\mathbb{N}}$ is pointwise convergent, and is an appropriate subsequence. **Q**

(b) Now 463Cd tells us that K is metrizable for $\mathfrak{T}_p$, and that $\mathfrak{T}_p$ agrees on K with the topology of convergence in measure.

463H Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite topological measure space in which μ is strictly positive. Suppose that

> whenever $h \in \mathbb{R}^X$ is such that $h\restriction Q$ is continuous for every relatively countably compact $Q \subseteq X$, then h is continuous.

If $K \subseteq C_b(X)$ is a norm-bounded $\mathfrak{T}_p$-compact set, then it is $\mathfrak{T}_p$-metrizable.

proof By 462L, the $\mathfrak{T}_p$-closed convex hull $\overline{\Gamma(K)}$ of K in $C(X)$ is $\mathfrak{T}_p$-compact. Because μ is strictly positive, $\mu\{x : f(x) \ne g(x)\} > 0$ whenever f and g are distinct continuous real-valued functions on X. So the result is immediate from 463G.

463I Lemma Let (X,Σ,μ) be a perfect probability space, and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ. Suppose that there is an $\epsilon>0$ such that

$$\epsilon\mu F\le\liminf_{n\to\infty}\mu(F\cap E_n)\le\limsup_{n\to\infty}\mu(F\cap E_n)\le(1-\epsilon)\mu F$$

for every $F\in\Sigma$. Then $\langle E_n\rangle_{n\in\mathbb{N}}$ has a subsequence $\langle E_{n_k}\rangle_{k\in\mathbb{N}}$ such that $\mu_*A=0$ and $\mu^*A=1$ for any cluster point A of $\langle E_{n_k}\rangle_{k\in\mathbb{N}}$ in $\mathcal{P}X$; in particular, $\langle E_{n_k}\rangle_{k\in\mathbb{N}}$ has no measurable cluster point.

proof (a) If we replace μ by its completion, we do not change μ^* and μ_* (212Ea, 413Eg), so we may suppose that μ is already complete.

(b) Suppose that $\langle E_n\rangle_{n\in\mathbb{N}}$ is actually stochastically independent, with $\mu E_n=\frac{1}{k}$ for every n, where $k\ge2$ is an integer. In this case $\mu^*A=1$ for any cluster point A of $\langle E_n\rangle_{n\in\mathbb{N}}$.

P (i) There is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $A=\lim_{n\to\mathcal{F}}E_n$ in $\mathcal{P}X$ (4A2F(a-ii)); that is, $\chi A(x)=\lim_{n\to\mathcal{F}}\chi E_n(x)$ for every $x\in X$; that is, $A=\{x:x\in X,\ \{n:x\in E_n\}\in\mathcal{F}\}$.

(ii) We have a measurable function $\phi:X\to Y=\{0,1\}^{\mathbb{N}}$ defined by setting $\phi(x)(n)=(\chi E_n)(x)$ for every $x\in X$, $n\in\mathbb{N}$. Because μ is complete and perfect and totally finite, and $\{0,1\}^{\mathbb{N}}$ is compact and metrizable, the image measure $\nu=\mu\phi^{-1}$ is a Radon measure (451O). For any basic open set of the form $H=\{y:y(i)=\epsilon_i$ for every $i\le n\}$, $\mu\phi^{-1}[H]=\tilde\nu H$, where $\tilde\nu$ is the product measure corresponding to the measure ν_0 on $\{0,1\}$ for which $\nu_0\{1\}=\frac{1}{k}$, $\nu_0\{0\}=\frac{k-1}{k}$. Since $\tilde\nu$ also is a Radon measure (416U), $\tilde\nu=\nu$ (415H(v)).

Set $B=\{y:y\in Y,\ \{n:y(n)=1\}\in\mathcal{F}\}$, so that $\phi^{-1}[B]=A$ and B is determined by coordinates in $\{n,n+1,\dots\}$ for every $n\in\mathbb{N}$. By 451Pc, $\mu^*A=\nu^*B$; by the zero-one law (254Sa), ν^*B must be either 0 or 1. So μ^*A is either 0 or 1.

(iii) To see that μ^*A cannot be 0, we repeat the arguments of (ii) from the other side, as follows. Let λ_0 be the uniform probability measure on $\{0,1,\dots,k-1\}$, giving measure $\frac{1}{k}$ to each point; let λ be the corresponding product measure on $Z=\{0,\dots,k-1\}^{\mathbb{N}}$. Let $\psi:Z\to Y$ be defined by setting $\psi(z)(n)=1$ if $z(n)=0$, 0 otherwise; then ψ is inverse-measure-preserving (254G). Since λ is a Radon measure and ψ is continuous, $\lambda\psi^{-1}$ is a Radon measure on Y and must be equal to ν. Accordingly $\nu^*B=\lambda^*\psi^{-1}[B]$, by 451Pc again or otherwise.

(iv) We have a measure space automorphism $\theta:Z\to Z$ defined by setting $\theta(z)(n)=z(n)+_k1$ for every $z\in Z$, $n\in\mathbb{N}$, where $+_k$ is addition mod k. So, writing $C=\psi^{-1}[B]$, $\lambda^*C=\lambda^*\theta^i[C]$ for every $i\in\mathbb{N}$. Now, for $z\in Z$,

$$\{n:z(n)=0\}\in\mathcal{F}\iff\{n:\psi(z)(n)=1\}\in\mathcal{F}\iff\psi(z)\in B\iff z\in C.$$

But for any $z\in Z$, there is some $i<k$ such that $\{n:z(n)=i\}\in\mathcal{F}$, so that $\theta^{k-i}(z)\in C$. Thus $\bigcup_{i<k}\theta^i[C]=Z$ and $\sum_{i=0}^{k-1}\lambda^*\theta^i[C]\ge1$ and $\lambda^*C>0$. But this means that $\mu^*A=\nu^*B=\lambda^*C$ is non-zero, and μ^*A must be 1. **Q**

(c) Now return to the general case considered in (a). Note first that μ is atomless, because if $\mu F>0$ there is some $n\in\mathbb{N}$ such that $0<\mu(F\cap E_n)<\mu F$.

Let $k\ge2$ be such that $\frac{1}{k}<\epsilon$. Then there are a strictly increasing sequence $\langle m(i)\rangle_{i\in\mathbb{N}}$ in $\mathbb{N}$ and a stochastically independent sequence $\langle F_i\rangle_{i\in\mathbb{N}}$ in Σ such that $F_i\subseteq E_{m(i)}$ and $\mu F_i=\frac{1}{k}$ for every $i\in\mathbb{N}$. **P** Choose $\langle m(i)\rangle_{i\in\mathbb{N}}$, F_i inductively, as follows. Let Σ_i be the (finite) algebra generated by $\{F_j:j<i\}$. Choose $m(i)$ such that $m(i)>m(j)$ for any $j<i$ and $\mu(F\cap E_{m(i)})\ge\frac{1}{k}\mu F$ for every $F\in\Sigma_i$. List the atoms of Σ_i as $G_{i0},\dots,G_{ip_i}$, and choose $F_{ir}\subseteq E_{m(i)}\cap G_{ir}$ such that $\mu F_{ir}=\frac{1}{k}\mu G_{ir}$, for each $r\le p_i$; 215D tells us that this is possible. Set $F_i=\bigcup_{r\le p_i}F_{ir}$; then $\mu(F_i\cap F)=\frac{1}{k}\mu F$ for every $F\in\Sigma_i$, and $F_i\subseteq E_{m(i)}$. Continue. It is easy to check that $\mu(F_{i_1}\cap\dots\cap F_{i_r})=1/k^r$ whenever $i_1<\dots<i_r$, so that $\langle F_i\rangle_{i\in\mathbb{N}}$ is stochastically independent. **Q**

If A is a cluster point of $\langle E_{m(i)}\rangle_{i\in\mathbb{N}}$, then there is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $A=\lim_{i\to\mathcal{F}}E_{m(i)}$ in $\mathcal{P}X$. In this case, $A\supseteq A'$, where $A'=\lim_{i\to\mathcal{F}}F_i$. But (b) tells us that μ^*A' must be 1, so $\mu^*A=1$.

(d) Thus we have a subsequence $\langle E_{m(i)}\rangle_{i\in\mathbb{N}}$ of $\langle E_n\rangle_{n\in\mathbb{N}}$ such that any cluster point of $\langle E_{m(i)}\rangle_{i\in\mathbb{N}}$ has outer measure 1. But the same argument applies to $\langle X\setminus E_{m(i)}\rangle_{i\in\mathbb{N}}$ to show that there is a strictly increasing sequence $\langle i_k\rangle_{k\in\mathbb{N}}$ such that every cluster point of $\langle X\setminus E_{m(i_k)}\rangle_{k\in\mathbb{N}}$ has outer measure 1. Since complementation is a homeomorphism of $\mathcal{P}X$, $\mu^*(X\setminus A)=1$, that is, $\mu_*A=0$, for every cluster point A of $\langle E_{m(i_k)}\rangle_{k\in\mathbb{N}}$. So if we set $n_k=m(i_k)$, any cluster point of $\langle E_{n_k}\rangle_{k\in\mathbb{N}}$ will have inner measure 0 and outer measure 1, as claimed.

463J Lemma Let (X,Σ,μ) be a perfect probability space, and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ. Then

either $\langle\chi E_n\rangle_{n\in\mathbb{N}}$ has a subsequence which is convergent almost everywhere

or $\langle E_n\rangle_{n\in\mathbb{N}}$ has a subsequence with no measurable cluster point in $\mathcal{P}X$.

proof Consider the sequence $\langle\chi E_n^\bullet\rangle_{n\in\mathbb{N}}$ in the Hilbert space $L^2 = L^2(\mu)$. This is a norm-bounded sequence, so has a weakly convergent subsequence $\langle\chi E_{n_i}^\bullet\rangle_{i\in\mathbb{N}}$ with limit v say (4A4Kb again). Express v as $g^\bullet$ where $g : X \to \mathbb{R}$ is Σ-measurable.

case 1 Suppose that $g(x) \in \{0,1\}$ for almost every $x \in X$; set $F = \{x : g(x) = 1\}$. Then

$$\lim_{i\to\infty}\int_F \chi E_{n_i} = \int_F g = \mu F, \quad \lim_{i\to\infty}\int_{X\setminus F}\chi E_{n_i} = \int_{X\setminus F} g = 0.$$

So, replacing $\langle\chi E_{n_i}\rangle_{i\in\mathbb{N}}$ with a sub-subsequence if necessary, we may suppose that

$$|\mu F - \int_F \chi E_{n_i}| \le 2^{-i}, \quad |\int_{X\setminus F}\chi E_{n_i}| \le 2^{-i}$$

for every i. But as $0 \le \chi E_{n_i} \le 1$ everywhere, we have $\int|\chi F - \chi E_{n_i}| \le 2^{-i+1}$ for every i, so that $\chi E_{n_i} \to \chi F$ a.e., and we have a subsequence of $\langle\chi E_n\rangle_{n\in\mathbb{N}}$ which is convergent almost everywhere.

case 2 Suppose that $\{x : g(x) \notin \{0,1\}\}$ has positive measure. Note that because $\int_F g = \lim_{i\to\infty}\mu(F\cap E_{n_i})$ lies between 0 and μF for every $F \in \Sigma$, $0 \le g \le 1$ a.e., and $\mu\{x : 0 < g(x) < 1\} > 0$. There is therefore an $\epsilon > 0$ such that $\mu G > 0$, where $G = \{x : \epsilon \le g(x) \le 1-\epsilon\}$.

Write μ_G for the subspace measure on G, and Σ_G for its domain; set $\nu = (\mu G)^{-1}\mu_G$, so that ν is a probability measure. We know that μ_G is perfect (451Dc), so ν also is (see the definition in 451Ad). Now if $F \in \Sigma_G$,

$$\lim_{i\to\infty}\nu(F\cap E_{n_i}) = (\mu G)^{-1}\int_F g$$

lies between $\epsilon\mu F/\mu G = \epsilon\nu F$ and $(1-\epsilon)\nu F$.

By 463I, there is a strictly increasing sequence $\langle i(k)\rangle_{k\in\mathbb{N}}$ such that $B \notin \Sigma_G$ whenever B is a cluster point of $\langle G\cap E_{n_{i(k)}}\rangle_{k\in\mathbb{N}}$ in $\mathcal{P}G$. If A is any cluster point of $\langle E_{n_{i(k)}}\rangle_{k\in\mathbb{N}}$ in $\mathbb{R}^X$, then $A\cap G$ is a cluster point of $\langle G\cap E_{n_{i(k)}}\rangle_{k\in\mathbb{N}}$ in $\mathcal{P}G$, so cannot belong to Σ_G. Thus $A \notin \Sigma$.

So in this case we have a subsequence $\langle E_{n_{i(k)}}\rangle_{k\in\mathbb{N}}$ of $\langle E_n\rangle_{n\in\mathbb{N}}$ which has no measurable cluster point.

463K Fremlin's Alternative (Fremlin 75a) Let (X,Σ,μ) be a perfect σ-finite measure space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of real-valued measurable functions on X. Then

either $\langle f_n\rangle_{n\in\mathbb{N}}$ has a subsequence which is convergent almost everywhere

or $\langle f_n\rangle_{n\in\mathbb{N}}$ has a subsequence with no measurable cluster point in $\mathbb{R}^X$.

proof (a) If $\mu X = 0$ then of course $\langle f_n\rangle_{n\in\mathbb{N}}$ itself is convergent a.e., so we may suppose that $\mu X > 0$. If there is any $x \in X$ such that $\sup_{n\in\mathbb{N}}|f_n(x)| = \infty$, then $\langle f_n\rangle_{n\in\mathbb{N}}$ has a subsequence with no cluster point in $\mathbb{R}^X$, measurable or otherwise; so we may suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is bounded at each point of X.

(b) Let λ be the c.l.d. product of μ with Lebesgue measure on $\mathbb{R}$, and Λ its domain. Then λ is perfect (451Ic) and also σ-finite (251K). There is therefore a probability measure ν on $X\times\mathbb{R}$ with the same domain and the same negligible sets as λ (215B(vii)), so that ν also is perfect. For any function $h \in \mathbb{R}^X$, write

$$\Omega(h) = \{(x,\alpha) : x \in X,\ \alpha \le h(x)\} \subseteq X\times\mathbb{R}$$

(compare 252N).

(c) By 463J, applied to the measure space $(X\times\mathbb{R},\Lambda,\nu)$ and the sequence $\langle\chi\Omega(f_n)\rangle_{n\in\mathbb{N}}$, we have a strictly increasing sequence $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that either $\langle\chi\Omega(f_{n(i)})\rangle_{i\in\mathbb{N}}$ is convergent ν-a.e. or $\langle\Omega(f_{n(i)})\rangle_{i\in\mathbb{N}}$ has no cluster point in Λ.

case 1 Suppose that $\langle\chi\Omega(f_{n(i)})\rangle_{i\in\mathbb{N}}$ is convergent ν-a.e. Set

$$W = \{(x,\alpha) : \lim_{i\to\infty}\chi\Omega(f_{n(i)})(x,\alpha) \text{ is defined}\}.$$

Then W is λ-conegligible, so $W^{-1}[\{\alpha\}] = \{x : (x,\alpha) \in W\}$ is μ-conegligible for almost every α (apply 252D to the complement of W). Set $D = \{\alpha : W^{-1}[\{\alpha\}]$ is μ-conegligible$\}$, and let $Q \subseteq D$ be a countable dense set; then $G = \bigcap_{\alpha\in Q}W^{-1}[\{\alpha\}]$ is μ-conegligible. But if $x \in G$, then for any $\alpha \in Q$ the set $\{i : f_{n(i)}(x) \ge \alpha\} = \{i : \chi\Omega(f_{n(i)})(x,\alpha) = 1\}$ is either finite or has finite complement in $\mathbb{N}$, so $\langle f_{n(i)}(x)\rangle_{i\in\mathbb{N}}$ must be convergent in $[-\infty,\infty]$. Since $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ is supposed to be bounded, $\langle f_{n(i)}(x)\rangle_{i\in\mathbb{N}}$ is convergent in $\mathbb{R}$. Thus in this case we have an almost-everywhere-convergent subsequence of $\langle f_n\rangle_{n\in\mathbb{N}}$.

case 2 Suppose that $\langle\Omega(f_{n(i)})\rangle_{i\in\mathbb{N}}$ has no cluster point in Λ. Let h be any cluster point of $\langle f_{n(i)}\rangle_{i\in\mathbb{N}}$ in $\mathbb{R}^X$. Then there is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $h = \lim_{i\to\mathcal{F}} f_{n(i)}$ in $\mathbb{R}^X$. Set $A = \lim_{i\to\mathcal{F}}\Omega(f_{n(i)})$, so that $A \notin \Lambda$. If $x \in X$ and $\alpha \in \mathbb{R}$, then

$$\alpha < h(x) \Longrightarrow \{i : \alpha < f_{n(i)}(x)\} \in \mathcal{F} \Longrightarrow (x,\alpha) \in A,$$

$$h(x) < \alpha \Longrightarrow \{i : \alpha < f_{n(i)}(x)\} \notin \mathcal{F} \Longrightarrow (x,\alpha) \notin A.$$

Thus $\Omega'(h) \subseteq A \subseteq \Omega(h)$, where $\Omega'(h) = \{(x,\alpha) : \alpha < h(x)\}$.

? If h is Σ-measurable, then

$$\Omega'(h) = \bigcup_{q\in\mathbb{Q}} \{x : h(x) > q\} \times]-\infty, q],$$

$$\Omega(h) = (X \times \mathbb{R}) \setminus \bigcup_{q\in\mathbb{Q}} \{x : h(x) < q\} \times [q, \infty[$$

belong to Λ, and $\lambda(\Omega(h) \setminus \Omega'(h)) = 0$ (because every vertical section of $\Omega(h) \setminus \Omega'(h)$ is negligible). But as $\Omega'(h) \subseteq A \subseteq \Omega(h)$, $A \in \Lambda$ (remember that product measures in this book are complete), which is impossible. **X**

Thus h is not Σ-measurable. As h is arbitrary, $\langle f_{n(i)}\rangle_{i\in\mathbb{N}}$ has no measurable cluster point in $\mathbb{R}^X$.

So at least one of the envisaged alternatives must be true.

463L Corollary Let (X, Σ, μ) be a perfect σ-finite measure space. Write $\mathcal{L}^0 \subseteq \mathbb{R}^X$ for the space of real-valued Σ-measurable functions on X.

(a) If $K \subseteq \mathcal{L}^0$ is relatively countably compact for the topology $\mathfrak{T}_p$ of pointwise convergence on $\mathcal{L}^0$, then every sequence in K has a subsequence which is convergent almost everywhere. Consequently K is relatively compact in $\mathcal{L}^0$ for the topology $\mathfrak{T}_m$ of convergence in measure.

(b) If $K \subseteq \mathcal{L}^0$ is countably compact for $\mathfrak{T}_p$, then it is compact for $\mathfrak{T}_m$.

(c) Suppose that $K \subseteq \mathcal{L}^0$ is countably compact for $\mathfrak{T}_p$ and that $\mu\{x : f(x) \neq g(x)\} > 0$ for any distinct f, $g \in K$. Then the topologies $\mathfrak{T}_m$ and $\mathfrak{T}_p$ agree on K, so both are compact and metrizable.

proof (a) Since every sequence in K must have a $\mathfrak{T}_p$-cluster point in $\mathcal{L}^0$, 463K tells us that every sequence in K has a subsequence which is convergent almost everywhere, therefore $\mathfrak{T}_m$-convergent. Now K is relatively sequentially compact in the pseudometrizable space $(\mathcal{L}^0, \mathfrak{T}_m)$, therefore relatively compact (4A2Le again).

(b) As in (a), every sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in K has a subsequence $\langle g_n\rangle_{n\in\mathbb{N}}$ which is convergent almost everywhere. But $\langle g_n\rangle_{n\in\mathbb{N}}$ has a $\mathfrak{T}_p$-cluster point g in K, and now $g(x) = \lim_{n\to\infty} g_n(x)$ for every x for which the limit is defined; accordingly $g_n \to g$ a.e., and g is a $\mathfrak{T}_m$-limit of $\langle g_n\rangle_{n\in\mathbb{N}}$ in K. Thus every sequence in K has a $\mathfrak{T}_m$-cluster point in K, and (because $\mathfrak{T}_m$ is pseudometrizable) K is $\mathfrak{T}_m$-compact.

(c) The point is that K is sequentially compact under $\mathfrak{T}_p$. **P** Note that as K is countably compact, $\sup_{f\in K} |f(x)|$ is finite for every $x \in K$. (I am passing over the trivial case $K = \emptyset$.) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in K, then, by (a), it has a subsequence $\langle g_n\rangle_{n\in\mathbb{N}}$ which is convergent a.e. **?** If $\langle g_n\rangle_{n\in\mathbb{N}}$ is not $\mathfrak{T}_p$-convergent, then there are a point $x_0 \in X$ and two further subsequences $\langle g'_n\rangle_{n\in\mathbb{N}}$, $\langle g''_n\rangle_{n\in\mathbb{N}}$ of $\langle g_n\rangle_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} g'_n(x_0)$, $\lim_{n\to\infty} g''_n(x_0)$ exist and are different. Now $\langle g'_n\rangle_{n\in\mathbb{N}}$, $\langle g''_n\rangle_{n\in\mathbb{N}}$ must have cluster points g', $g'' \in K$ with $g'(x_0) \neq g''(x_0)$.

However,

$$g'(x) = \lim_{n\to\infty} g_n(x) = g''(x)$$

whenever the limit is defined, which is almost everywhere; so $g' =_{\text{a.e.}} g''$. And this contradicts the hypothesis that if two elements of K are equal a.e., they are identical. **X** Thus $\langle g_n\rangle_{n\in\mathbb{N}}$ is a $\mathfrak{T}_p$-convergent subsequence of $\langle f_n\rangle_{n\in\mathbb{N}}$. As $\langle f_n\rangle_{n\in\mathbb{N}}$ is arbitrary, K is $\mathfrak{T}_p$-sequentially compact. **Q**

Now 463Cd gives the result.

463M Proposition Let $X_0, \dots, X_n$ be countably compact topological spaces, each carrying a σ-finite perfect strictly positive measure which measures every Baire set. Let X be their product and $\mathcal{B}\mathfrak{a}(X_i)$ the Baire σ-algebra of X_i for each i. Then any separately continuous function $f : X \to \mathbb{R}$ is measurable with respect to the σ-algebra $\widehat{\bigotimes}_{i\le n}\mathcal{B}\mathfrak{a}(X_i)$ generated by $\{\prod_{i\le n} E_i : E_i \in \mathcal{B}\mathfrak{a}(X_i)$ for $i \le n\}$.

proof For $i \le n$ let μ_i be a σ-finite perfect strictly positive measure on X_i such that $\mathcal{B}\mathfrak{a}(X_i) \subseteq \operatorname{dom}\mu_i$; let μ be the product measure on X.

(a) The proof relies on the fact that

(∗) if g, $g' : X \to \mathbb{R}$ are distinct separately continuous functions, then $\mu\{x : g(x) \neq g'(x)\} > 0$;

I seek to prove this, together with the stated result, by induction on n. The induction starts easily with $n = 0$, so that X can be identified with X_0, a separately continuous function on X is just a continuous function on X_0, and (∗) is true because $\mu = \mu_0$ is strictly positive.

(b) For the inductive step to $n+1$, given a separately continuous function $f : X_0 \times \ldots \times X_{n+1} \to \mathbb{R}$, set $f_t(y) = f(y,t)$ for every $y \in Y = X_0 \times \ldots \times X_n$ and $t \in X_{n+1}$, and $K = \{f_t : t \in X_{n+1}\}$ Then every f_t is separately continuous, therefore $\widehat{\bigotimes}_{i\le n}\mathcal{B}\mathfrak{a}(X_i)$-measurable, by the inductive hypothesis. So K consists of $\widehat{\bigotimes}_{i\le n}\mathcal{B}\mathfrak{a}(X_i)$-measurable functions. Moreover, again because f is separately continuous, the function $t \mapsto f_t(y)$ is continuous for every y, that is, $t \mapsto f_t : X_{n+1} \to \mathbb{R}^Y$ is continuous; it follows that K is countably compact (4A2G(f-vi)). Finally, by the inductive hypothesis $(*)$, $\nu\{y : f_t(y) \ne f_{t'}(y)\} > 0$ whenever $t, t' \in X_{n+1}$ and $f_t \ne f_{t'}$, where ν is the product measure on Y.

Since ν is perfect (451Ic) and $\widehat{\bigotimes}_{i\le n}\mathcal{B}\mathfrak{a}(X_i) \subseteq \operatorname{dom}\nu$, we can apply 463Lc to see that K is metrizable for the topology of pointwise convergence. Let ρ be a metric on K inducing its topology, and $\langle g_i\rangle_{i\in\mathbb{N}}$ a sequence running over a dense subset of K. (I am passing over the trivial case $K = \emptyset = X_{n+1}$.) For $m, i \in \mathbb{N}$ set $E_{mi} = \{t : \rho(f_t, g_i) \le 2^{-m}\}$. Because $t \mapsto f_t$ and $t \mapsto \rho(f_t, t_i)$ are continuous, $E_{mi} \in \mathcal{B}\mathfrak{a}(X_{n+1})$. Set $f^{(m)}(y,t) = g_i(y)$ for $t \in E_{mi} \setminus \bigcup_{j<i} E_{mj}$ for $m, i \in \mathbb{N}$, $y \in Y$ and $t \in X_{n+1}$. Then $f^{(m)} : X \to \mathbb{R}$ is $\widehat{\bigotimes}_{i\le n+1}\mathcal{B}\mathfrak{a}(X_i)$-measurable because every g_i is $\widehat{\bigotimes}_{i\le n}\mathcal{B}\mathfrak{a}(X_i)$-measurable and every E_{mi} belongs to $\mathcal{B}\mathfrak{a}(X_{n+1})$. Also $\langle f^{(m)}\rangle_{m\in\mathbb{N}} \to f$ at every point, because $\rho(f_t^{(m)}, f_t) \le 2^{-m}$ for every $m \in \mathbb{N}$ and $t \in X_{n+1}$. So f is $\widehat{\bigotimes}_{i\le n+1}\mathcal{B}\mathfrak{a}(X_i)$-measurable.

(c) We still have to check that $(*)$ is true at the new level. But if $h, h' : X \to \mathbb{R}$ are distinct separately continuous functions, then there are $t_0 \in X_{n+1}$, $y_0 \in Y$ such that $h(y_0, t_0) \ne h'(y_0, t_0)$. Let G be an open set containing t_0 such that $h(y_0, t) \ne h'(y_0, t)$ whenever $t \in G$. Then $\nu\{y : h(y,t) \ne h'(y,t)\} > 0$ for every $t \in G$, by the inductive hypothesis, so

$$\mu\{(y,t) : h(y,t) \ne h'(y,t)\} = \int \nu\{y : h(y,t) \ne h'(y,t)\}\mu_{n+1}(dt) > 0$$

because μ_{n+1} is strictly positive. Thus the induction continues.

463N Corollary Let $X_0, \ldots, X_n$ be Hausdorff spaces with product X. Then every separately continuous function $f : X \to \mathbb{R}$ is universally Radon-measurable in the sense of 434Ec.

proof Let μ be a Radon measure on X and Σ its domain.

(a) Suppose first that the support C of μ is compact. For each $i \le n$, let $\pi_i : X \to X_i$ be the coordinate projection, and $\mu_i = \mu\pi_i^{-1}$ the image Radon measure; let Z_i be the support of μ_i and $Z = \prod_{i\le n} Z_i$. Note that $\pi_i[C]$ is compact and μ_i-conegligible, so that $Z_i \subseteq \pi_i[C]$ is compact, for each i. At the same time, $\pi^{-1}[Z_i]$ is μ-conegligible for each i, so that Z is μ-conegligible.

By 463M, $f{\restriction}Z$ is $\widehat{\bigotimes}_{i\le n}\mathcal{B}\mathfrak{a}(Z_i)$-measurable; because Z is conegligible, f is Σ-measurable.

(b) In general, if $C \subseteq X$ is compact, then we can apply (a) to the measure $\mu \llcorner C$ (234M) to see that $f{\restriction}C$ is Σ-measurable. As μ is complete and locally determined and inner regular with respect to the compact sets, f is Σ-measurable (see 412Ja).

As μ is arbitrary, f is universally Radon-measurable.

463X Basic exercises >(a) Let (X, Σ, μ) be a measure space, $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X, $\mathfrak{T}_p$ the topology of pointwise convergence on $\mathcal{L}^0$ and $\mathfrak{T}_m$ the topology of convergence in measure on $\mathcal{L}^0$. (i) Show that $\mathfrak{T}_m \subseteq \mathfrak{T}_p$ iff for every measurable set E of finite measure there is a countable set $D \subseteq E$ such that $\mu^*D = \mu E$. (ii) Show that $\mathfrak{T}_p \subseteq \mathfrak{T}_m$ iff $0 < \mu^*\{x\} < \infty$ for every $x \in X$.

(b) Let (X, Σ, μ) be a σ-finite measure space, and $K \subseteq \mathcal{L}^0$ a $\mathfrak{T}_p$-countably compact set. Show that the following are equiveridical: (i) every sequence in K has a subsequence which converges almost everywhere; (ii) K is $\mathfrak{T}_m$-compact; (iii) K is totally bounded for the uniformity associated with the linear space topology $\mathfrak{T}_m$. Show that if moreover the topology on K induced by $\mathfrak{T}_m$ is Hausdorff, then K is $\mathfrak{T}_p$-metrizable.

(c)(i) Show that there is a set of Borel measurable functions on $[0,1]$ which is countably tight, compact and non-metrizable for the topology of pointwise convergence. (ii) Show that there is a strictly localizable measure space (X, Σ, μ) with a set K of measurable functions which is countably tight, compact, Hausdorff and non-metrizable for both the topology of pointwise convergence and the topology of convergence in measure. (*Hint*: the one-point compactification of any discrete space is countably tight.)

(d) Let X be a topological space and $K \subseteq C(X)$ a convex $\mathfrak{T}_p$-compact set. Show that if there is a strictly positive σ-finite topological measure on X, then K is $\mathfrak{T}_p$-metrizable.

(e) Use Komlós' theorem (276H) to shorten the proof of 463G.

(f) Let (X,Σ,μ) be any complete σ-finite measure space. Show that if $A\subseteq\mathcal{L}^0$ is $\mathfrak{T}_m$-relatively compact, and $\sup_{f\in A}|f(x)|$ is finite for every $x\in X$, then A is $\mathfrak{T}_p$-relatively countably compact in $\mathcal{L}^0$.

(g) Let K be the set of non-decreasing functions from $[0,1]$ to $\{0,1\}$. Show that K, with its topology of pointwise convergence, is homeomorphic to the split interval (419L). Show that (for any Radon measure μ on $[0,1]$) the identity map from $(K,\mathfrak{T}_p)$ to $(K,\mathfrak{T}_m)$ is continuous.

>(h) Let K be the set of non-decreasing functions from ω_1 to $\{0,1\}$. Show that if μ is the countable-cocountable measure on ω_1 then K is a $\mathfrak{T}_p$-compact set of measurable functions and is also $\mathfrak{T}_m$-compact, but the identity map from $(K,\mathfrak{T}_p)$ to $(K,\mathfrak{T}_m)$ is not continuous.

>(i) Let K be the set of functions $f:[0,1]\to\mathbb{R}$ such that $\max(\|f\|_\infty,\operatorname{Var}_{[0,1]}f)\le 1$, where $\operatorname{Var}_{[0,1]}f$ is the variation of f (224A). Show that K is $\mathfrak{T}_p$-compact and that (for any Radon measure on $[0,1]$) the identity map from $(K,\mathfrak{T}_p)$ to $(K,\mathfrak{T}_m)$ is continuous.

>(j) Let A be the set of functions $f:[0,1]\to[0,1]$ such that $\overline{\int}|f|d\mu\cdot\operatorname{Var}_{[0,1]}f\le 1$, where μ is Lebesgue measure. Show that every member of A is measurable and that every sequence in A has a subsequence which converges almost everywhere to a member of A, but that $\int:A\to[0,1]$ is not $\mathfrak{T}_p$-continuous, while A is $\mathfrak{T}_p$-dense in $[0,1]^{[0,1]}$.

(k) Let X be a Hausdorff space and $K\subseteq C(X)$ a $\mathfrak{T}_p$-compact set. Show that if there is a strictly positive σ-finite Radon measure on X then K is $\mathfrak{T}_p$-metrizable.

(l) Let (X,Σ,μ) be a localizable measure space and $K\subseteq\mathcal{L}^0$ a non-empty $\mathfrak{T}_p$-compact set. Show that $\sup\{f^\bullet : f\in K\}$ is defined in $L^0(\mu)$.

463Y Further exercises (a) Let (X,Σ,μ) be a probability space and V a Banach space. A function $\phi:X\to V$ is **scalarly measurable** (often called **weakly measurable**) if $h\phi:X\to\mathbb{R}$ is Σ-measurable for every $h\in V^*$. ϕ is **Pettis integrable**, with **indefinite Pettis integral** $\theta:\Sigma\to V$, if $\int_E h\phi\,d\mu$ is defined and equal to $h(\theta E)$ for every $E\in\Sigma$ and every $h\in V^*$. (i) Show that if ϕ is scalarly measurable, then $K=\{h\phi : h\in V^*,\ \|h\|\le 1\}$ is a $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0$. (ii) Show that if ϕ is scalarly measurable, then it is Pettis integrable iff every function in K is integrable and $f\mapsto\int_E f:K\to\mathbb{R}$ is $\mathfrak{T}_p$-continuous for every $E\in\Sigma$. (*Hint*: 4A4Cg.) (iii) In particular, if ϕ is bounded and scalarly measurable and the identity map from $(K,\mathfrak{T}_p)$ to $(K,\mathfrak{T}_m)$ is continuous, then ϕ is Pettis integrable. (See Talagrand 84, chap. 4.)

(b) Show that any Bochner integrable function (253Yf) is Pettis integrable.

(c) Let μ be Lebesgue measure on $[0,1]$, and define $\phi:[0,1]\to L^\infty(\mu)$ by setting $\phi(t)=\chi[0,t]^\bullet$ for every $t\in[0,1]$. (i) Show that if $h\in L^\infty(\mu)^*$ and $\|h\|\le 1$, then $h\phi$ has variation at most 1. (ii) Show that $K=\{h\phi : h\in L^\infty(\mu)^*,\ \|h\|\le 1\}$ is a $\mathfrak{T}_p$-compact set of Lebesgue measurable functions, and that the identity map from $(K,\mathfrak{T}_p)$ to $(K,\mathfrak{T}_m)$ is continuous, so that ϕ is Pettis integrable. (iii) Show that ϕ is not Bochner integrable.

(d) Let (X,Σ,μ) be a σ-finite measure space and suppose that μ is inner regular with respect to some family $\mathcal{E}\subseteq\Sigma$ of cardinal at most ω_1. (Subject to the continuum hypothesis, this is true for any subset of $\mathbb{R}$, for instance.) Show that if $K\subseteq\mathcal{L}^0$ is $\mathfrak{T}_p$-compact then it is $\mathfrak{T}_m$-compact. (See 536C[8], or Talagrand 84, 9-3-3.)

(e) Assume that the continuum hypothesis is true; let $\preccurlyeq$ be a well-ordering of $[0,1]$ with order type ω_1 (4A1Ad). Let (Z,ν) be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, and $q:Z\to[0,1]$ the canonical inverse-measure-preserving map (416V). Let $g:[0,1]\to[0,\infty[$ be any function. Show that there is a function $f:[0,1]\times Z\to[0,\infty[$ such that (α) f is continuous in the second variable (β) $f(t,z)=0$ whenever $q(z)\preccurlyeq t$ (γ) $\int f(t,z)\nu(dz)=g(t)$ for every $t\in[0,1]$. Show that f is universally measurable in the first variable, but need not be $\tilde\Lambda$-measurable, where $\tilde\Lambda$ is the domain of the product Radon measure on $[0,1]\times Z$. Setting $f_z(t)=f(t,z)$, show that $K=\{f_z : z\in Z\}$ is a $\mathfrak{T}_p$-compact set of Lebesgue measurable functions and that g belongs to the $\mathfrak{T}_p$-closed convex hull of K in $\mathbb{R}^{[0,1]}$.

[8]Later editions only.

463Z Problems (a) A.Bellow's problem Let (X,Σ,μ) be a probability space, and $K\subseteq\mathcal{L}^0$ a $\mathfrak{T}_p$-compact set such that $\{x:f(x)\ne g(x)\}$ is non-negligible for any distinct functions f, $g\in K$, as in 463G and 463Lc. Does it follow that K is metrizable for $\mathfrak{T}_p$?

A positive answer would displace several of the arguments of this section, and have other consequences (see 462Z, for instance). It is known that under any of a variety of special axioms (starting with the continuum hypothesis) there is indeed a positive answer; see §536 in Volume 5, or Talagrand 84, chap. 12.

(b) Let $X\subseteq[0,1]$ be a set of outer Lebesgue measure 1, and μ the subspace measure on X, with Σ its domain. Let K be a $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0$. Must K be $\mathfrak{T}_m$-compact?

(c) Let $X_0,\dots,X_n$ be compact Hausdorff spaces and $f:X_0\times\dots\times X_n\to\mathbb{R}$ a separately continuous function. Must f be universally measurable?

463 Notes and comments The relationship between the topologies $\mathfrak{T}_p$ and $\mathfrak{T}_m$ is complex, and I do not think that the results here are complete; in particular, we have a remarkable outstanding problem in 463Za. Much of the work presented here has been stimulated by problems concerning the integration of vector-valued functions. I am keeping this theory firmly in the 'further exercises' (463Ya-463Yc), but it is certainly the most important source of examples of pointwise compact sets of measurable functions. In particular, since the set $\{h\phi:h\in V^*,\,\|h\|\le 1\}$ is necessarily convex whenever V is a Banach space and $\phi:X\to V$ is a function, we are led to look at the special properties of convex sets, as in 463G. There are obvious connexions with the theory of measures on linear topological spaces, which I will come to in §466.

The dichotomy in 463K shows that sets of measurable functions on perfect measure spaces are either 'good' (relatively countably compact for $\mathfrak{T}_p$, relatively compact for $\mathfrak{T}_m$) or 'bad' (with neither property). It is known that the result is not true for arbitrary σ-finite measure spaces (see §464 below), but it is not clear whether there are important non-perfect spaces in which it still applies in some form; see 463Zb.

Just as in §462, many questions concerning the topology $\mathfrak{T}_p$ on $\mathbb{R}^X$ can be re-phrased as questions about real-valued functions on products $X\times K$ which are continuous in the second variable. For the topology of pointwise convergence on sets of measurable functions, we find ourselves looking at functions which are measurable in the first variable. In this way we are led to such results as 463M-463N and 463Ye. Concerning 463M and 463Zc, it is the case that if X and Y are *any* compact Hausdorff spaces, and $f:X\times Y\to\mathbb{R}$ is separately continuous, then f is Borel measurable (Burke & Pol 05, 5.2).

A substantial proportion of the questions which arise naturally in this topic are known to be undecidable without using special axioms. I am avoiding such questions in this volume, but it is worth noting that the continuum hypothesis, in particular, has many striking consequences here, of which 463Ye is a sample. It also decides 463Za and 463Zb (see 463Yd).

464 Talagrand's measure

An obvious question arising from 463I and its corollaries is, do we really need the hypothesis that the measure involved is perfect? A very remarkable construction by M.Talagrand (464D) shows that these results are certainly not true of all probability spaces (464E). Investigating the properties of this measure we are led to some surprising facts about additive functionals on algebras $\mathcal{P}I$ and the duals of ℓ^∞ spaces (464M, 464R).

464A The usual measure on $\mathcal{P}I$ Recall from 254J and 416U that for any set I we have a standard measure ν, a Radon measure for the usual topology on $\mathcal{P}I$, defined by saying that $\nu\{a:a\subseteq I,\,a\cap J=c\}=2^{-\#(J)}$ whenever $J\subseteq I$ is a finite set and $c\subseteq J$, or by copying from the usual product measure on $\{0,1\}^I$ by means of the bijection $a\mapsto\chi a:\mathcal{P}I\to\{0,1\}^I$. We shall need a couple of simple facts about these measures.

(a) If $\langle I_j\rangle_{j\in J}$ is any partition of I, then ν can be identified with the product of the family $\langle\nu_j\rangle_{j\in J}$, where ν_j is the usual measure on $\mathcal{P}I_j$, and we identify $\mathcal{P}I$ with $\prod_{j\in J}\mathcal{P}I_j$ by matching $a\subseteq I$ with $\langle a\cap I_j\rangle_{j\in J}$; this is the 'associative law' 254N. It follows that if we have any family $\langle A_j\rangle_{j\in J}$ of subsets of $\mathcal{P}I$, and if for each j the set A_j is 'determined by coordinates in I_j' in the sense that, for $a\subseteq I$, $a\in A_j$ iff $a\cap I_j\in A_j$, then $\nu^*(\bigcap_{j\in J}A_j)=\prod_{j\in J}\nu^*A_j$ (use 254Lb).

(b) Similarly, if f_1, f_2 are non-negative real-valued functions on $\mathcal{P}I$, and if there are disjoint sets I_1, $I_2\subseteq I$ such that $f_j(a)=f_j(a\cap I_j)$ for every $a\subseteq I$ and both j, then the upper integral $\overline{\int}f_1+f_2\,d\nu$ is $\overline{\int}f_1d\nu+\overline{\int}f_2d\nu$. **P** We may suppose that $I_2 = I \setminus I_1$. For each j, define $g_j : \mathcal{P}I_j \to [0,\infty[$ by setting $g_j = f_j \restriction \mathcal{P}I_j$, so that $f_j(a) = g_j(a \cap I_j)$ for every $a \subseteq I$. Let ν_j be the usual measure on $\mathcal{P}I_j$, so that we can identify ν with the product measure $\nu_1 \times \nu_2$, if we identify $\mathcal{P}I$ with $\mathcal{P}I_1 \times \mathcal{P}I_2$; that is, we think of a subset of I as a pair (a_1, a_2) where $a_j \subseteq I_j$ for both j.

Now we have

$$\overline{\int} f_1 + f_2 \, d\nu = \overline{\int} g_1 \, d\nu_1 + \overline{\int} g_2 \, d\nu_2$$

(253K)

$$= \overline{\int} g_1 \, d\nu_1 \cdot \overline{\int} \chi(\mathcal{P}I_2) d\nu_2 + \overline{\int} \chi(\mathcal{P}I_1) d\nu_1 \cdot \overline{\int} g_2 \, d\nu_2$$

$$= \overline{\int} f_1 \, d\nu + \overline{\int} f_2 \, d\nu$$

by 253J, because we can think of $f_1(a_1, a_2)$ as $g_1(a_1) \cdot (\chi\mathcal{P}I_2)(a_2)$ for all a_1, a_2. **Q**

(c) If $A \subseteq \mathcal{P}I$ is such that $b \in A$ whenever $a \in A$, $b \subseteq I$ and $a \triangle b$ is finite, then $\nu^* A$ must be either 0 or 1; this is the zero-one law 254Sa, applied to the set $\{\chi a : a \in A\} \subseteq \{0,1\}^I$ and the usual measure on $\{0,1\}^I$.

464B Lemma Let I be any set, and ν the usual measure on $\mathcal{P}I$.

(a)(i) There is a sequence $\langle m(n) \rangle_{n \in \mathbb{N}}$ in $\mathbb{N}$ such that $\prod_{n=0}^{\infty} 1 - 2^{-m(n)} = \frac{1}{2}$.

(ii) Given such a sequence, write X for $\prod_{n \in \mathbb{N}} (\mathcal{P}I)^{m(n)}$, and let λ be the product measure on X. We have a function $\phi : X \to \mathcal{P}I$ defined by setting

$$\phi(\langle\langle a_{ni} \rangle_{i < m(n)} \rangle_{n \in \mathbb{N}}) = \bigcup_{n \in \mathbb{N}} \bigcap_{i < m(n)} a_{ni}$$

whenever $\langle\langle a_{ni} \rangle_{i < m(n)} \rangle_{n \in \mathbb{N}} \in X$. Now ϕ is inverse-measure-preserving for λ and ν.

(b) The map

$$(a, b, c) \mapsto (a \cap b) \cup (a \cap c) \cup (b \cap c) : (\mathcal{P}I)^3 \to \mathcal{P}I$$

is inverse-measure-preserving for the product measure on $(\mathcal{P}I)^3$.

proof (a)(i) Choose $m(n)$ inductively so that, for each n in turn, $m(n)$ is minimal subject to the requirement $\prod_{k=0}^{n} 1 - 2^{-m(k)} > \frac{1}{2}$.

(ii) If $t \in I$, then

$$\{\mathfrak{x} : \mathfrak{x} \in X, \, t \notin \phi(\mathfrak{x})\} = \{\langle\langle a_{ni} \rangle_{i < m(n)} \rangle_{n \in \mathbb{N}} : t \notin \bigcup_{n \in \mathbb{N}} \bigcap_{i < m(n)} a_{ni}\}$$

has measure

$$\prod_{n=0}^{\infty} \lambda_n \{\langle a_i \rangle_{i < m(n)} : t \notin \bigcap_{i < m(n)} a_i\},$$

where λ_n is the product measure on $(\mathcal{P}I)^{m(n)}$ for each n. But this is just

$$\prod_{n=0}^{\infty} 1 - 2^{-m(n)} = \frac{1}{2},$$

by the choice of $\langle m(n) \rangle_{n \in \mathbb{N}}$. Accordingly $\lambda\{\mathfrak{x} : t \in \phi(\mathfrak{x})\} = \frac{1}{2}$ for every $t \in I$. Next, if we identify X with $\mathcal{P}(\{(t, n, i) : t \in I, \, n \in \mathbb{N}, \, i < m(n)\})$, each set $E_t = \{\mathfrak{x} : t \in \phi(\mathfrak{x})\}$ is determined by coordinates in $J_t = \{(t, n, i) : n \in \mathbb{N}, \, i < m(n)\}$. Since the sets J_t are disjoint, the sets E_t, for different t, are stochastically independent (464Aa), so if $J \subseteq I$ is finite,

$$\lambda\{\mathfrak{x} : J \subseteq \phi(\mathfrak{x})\} = \prod_{t \in J} \lambda E_t = 2^{-\#(J)} = \nu\{a : J \subseteq a\}.$$

This shows that $\lambda\phi^{-1}[F] = \nu F$ whenever F is of the form $\{a : J \subseteq a\}$ for some finite $J \subseteq I$. By the Monotone Class Theorem (136C), $\lambda\phi^{-1}[F] = \nu F$ for every F belonging to the σ-algebra generated by sets of this form. But this σ-algebra certainly contains all sets of the form $\{a : a \cap J = c\}$ where $J \subseteq I$ is finite and $c \subseteq J$, which are the sets corresponding to the basic cylinder sets in the product $\{0,1\}^I$. By 254G, ϕ is inverse-measure-preserving.

(b) This uses the same idea as (a-ii). Writing ν^3 for the product measure on $(\mathcal{P}I)^3$, then, for any $t \in I$,

$$\begin{aligned}\nu^3\{(a,b,c) : t \in (a\cap b)\cup(a\cap c)\cup(b\cap c)\}\\ &= \nu^3\{(a,b,c) : t \in a\cap b\} + \nu^3\{(a,b,c) : t \in a\cap c\}\\ &\qquad + \nu^3\{(a,b,c) : t \in b\cap c\} - 2\nu^3\{(a,b,c) : t \in a\cap b\cap c\}\\ &= 3\cdot\frac{1}{4} - 2\cdot\frac{1}{8} = \frac{1}{2}.\end{aligned}$$

Once again, these sets are independent for different t, and this is all we need to know in order to be sure that the map is inverse-measure-preserving.

464C Lemma Let I be any set, and let ν be the usual measure on $\mathcal{P}I$.

(a) (see SIERPIŃSKI 45) If $\mathcal{F} \subseteq \mathcal{P}I$ is any filter containing every cofinite set, then $\nu_*\mathcal{F} = 0$ and $\nu^*\mathcal{F}$ is either 0 or 1. If $\mathcal{F}$ is a non-principal ultrafilter then $\nu^*\mathcal{F} = 1$.

(b) (TALAGRAND 80[9]) If $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is a sequence of filters on I, all of outer measure 1, then $\bigcap_{n\in\mathbb{N}}\mathcal{F}_n$ also has outer measure 1.

proof (a) That $\nu^*\mathcal{F} \in \{0,1\}$ is immediate from 464Ac. If $\mathcal{F}$ is an ultrafilter, then $\mathcal{P}I = \mathcal{F} \cup \{I\setminus a : a \in \mathcal{F}\}$; but as $a \mapsto I\setminus a$ is a measure space automorphism of $(\mathcal{P}I,\nu)$, $\nu^*\{I\setminus a : a \in \mathcal{F}\} = \nu^*\mathcal{F}$, and both must be at least $\frac{1}{2}$, so $\nu^*\mathcal{F} = 1$. Equally, $\nu^*(\mathcal{P}I\setminus\mathcal{F}) = \nu^*\{I\setminus a : a\in\mathcal{F}\} = 1$, so $\nu_*\mathcal{F} = 0$. Returning to a general filter containing every cofinite set, this is included in a non-principal ultrafilter, so also has inner measure 0.

(b) Let $\langle m(n)\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that $\prod_{n=0}^\infty 1-2^{-m(n)} = \frac{1}{2}$, and let X, λ and $\phi : X \to \mathcal{P}I$ be as in 464Ba. Consider the set $D = \prod_{n\in\mathbb{N}}\mathcal{F}_n^{m(n)}$ as a subset of X. By 254Lb, $\lambda^*D = 1$. If we set $\mathcal{F} = \bigcap_{n\in\mathbb{N}}\mathcal{F}_n$, then we see that whenever $\mathfrak{x} = \langle\langle a_{ni}\rangle_{i<m(n)}\rangle_{n\in\mathbb{N}}$ belongs to D,

$$\phi(\mathfrak{x}) \supseteq \bigcap_{i<m(n)} a_{ni} \in \mathcal{F}_n$$

for every n, so that $\phi(\mathfrak{x}) \in \mathcal{F}$. Thus $D \subseteq \phi^{-1}[\mathcal{F}]$ and

$$\nu^*\mathcal{F} \ge \lambda^*\phi^{-1}[\mathcal{F}] \ge \lambda^*D = 1$$

because ϕ is inverse-measure-preserving (413Eh).

As $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, we have the result.

464D Construction (TALAGRAND 80) Let I be any set, and ν the usual Radon measure on $\mathcal{P}I$, with T its domain. Let Σ be the set

$$\begin{aligned}\{E : E \subseteq \mathcal{P}I,\text{ there are a set } F \in \mathrm{T} \text{ and a filter } \mathcal{F} \text{ on } I\\ \text{such that } \nu^*\mathcal{F} = 1 \text{ and } E\cap\mathcal{F} = F\cap\mathcal{F}\}.\end{aligned}$$

Then there is a unique extension of ν to a complete probability measure μ, with domain Σ, defined by saying that $\mu E = \nu F$ whenever $E \in \Sigma$, $F \in \mathrm{T}$ and there is a filter $\mathcal{F}$ on I such that $\nu^*\mathcal{F} = 1$ and $E\cap\mathcal{F} = F\cap\mathcal{F}$. (By 464C, we can apply 417A with

$$\mathcal{A} = \{\mathcal{P}I\setminus\mathcal{F} : \mathcal{F} \text{ is a filter on } I \text{ such that } \nu^*\mathcal{F} = 1\}.)$$

Definition This measure μ is **Talagrand's measure** on $\mathcal{P}I$.

464E Example If μ is Talagrand's measure on $X = \mathcal{P}\mathbb{N}$, and Σ its domain, then there is a set $K \subseteq \mathbb{R}^X$, consisting of Σ-measurable functions and compact for the topology $\mathfrak{T}_p$ of pointwise convergence, such that K is not compact for the topology of convergence in measure, that is, there is a sequence in K with no subsequence which is convergent almost everywhere, even though every cluster point is Σ-measurable.

proof Take K to be $\{\chi\mathcal{F} : \mathcal{F}$ is an ultrafilter on $\mathbb{N}\}$. Then K is $\mathfrak{T}_p$-compact (in fact, is precisely the set of Boolean homomorphisms from $\mathcal{P}\mathbb{N}$ to $\{0,1\}$, identified with the Stone space of $\mathcal{P}\mathbb{N}$ in 311E). By 464Ca, we see that any non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ belongs to Σ, and $\mu\mathcal{F} = 1$. On the other hand, all the principal ultrafilters $\mathcal{F}_n = \{a : n \in a \subseteq \mathbb{N}\}$ are measured by ν and therefore by μ, and form a stochastically independent sequence of

[9]The date of this paper is misleading, as there was an unusual backlog in the journal; in reality it preceded FREMLIN & TALAGRAND 79.

sets of measure $\frac{1}{2}$. So K consists of Σ-measurable functions; but the sequence $\langle \chi F_n \rangle_{n \in \mathbb{N}}$ has no subsequence which is convergent almost everywhere, and K is not relatively compact for the topology of convergence in measure.

Remark In this example, a very large number of members of K are equal almost everywhere; indeed, all non-principal ultrafilters are equal a.e., and if we look at $\{f^\bullet : f \in K\}$ in $L^0(\mu)$, it is a countable discrete set. Given that K is a set of measurable functions homeomorphic to $\beta\mathbb{N}$, something like this has to happen (see 536D[10] in Volume 5). Looking at this from a different angle, if we wish to extend the usual measure ν on $\mathcal{P}\mathbb{N}$ to measure every ultrafilter, there will have to be two distinct ultrafilters $\mathcal{F}_1$, $\mathcal{F}_2$ such that $\mathcal{F}_1 \triangle \mathcal{F}_2$ is negligible for the extended measure.

464F The L-space $\ell^\infty(I)^*$ For the next step, we shall need to recall some facts from Volume 3. Let I be any set.

(a) The space $\ell^\infty(I)$ of bounded real-valued functions on I is an M-space (354Ha), so its dual $\ell^\infty(I)^\sim = \ell^\infty(I)^*$ is an L-space (356N), that is, a Banach lattice such that $\|f+g\| = \|f\| + \|g\|$ for all non-negative f, $g \in \ell^\infty(I)^*$. Since $\ell^\infty(I)$ can be identified with the space $L^\infty(\mathcal{P}I)$ as described in §363 (see 363Ha), we can identify $\ell^\infty(I)^*$ with the L-space M of bounded finitely additive functionals on $\mathcal{P}I$ (363K), matching any $f \in \ell^\infty(I)^*$ with the functional $a \mapsto f(\chi a) : \mathcal{P}I \to \mathbb{R}$ in M.

(b) In §§361-363 I examined some of the bands in M. The most significant ones for our present purposes are the band M_τ of completely additive functionals (362Bb) and its complement $M_\tau^\perp$ (352P); because M is Dedekind complete, we have $M = M_\tau \oplus M_\tau^\perp$ (353I). In fact M_τ is just the set of those $\theta \in M$ such that $\theta a = \sum_{t \in a} \theta\{t\}$ for every $a \subseteq I$, while $M_\tau^\perp$ is the set of those $\theta \in M$ such that $\theta\{t\} = 0$ for every $t \in I$. **P** For any $\theta \in M$, we can set $\alpha_t = \theta\{t\}$ for each $t \in I$; in this case,

$$|\sum_{t \in J} \alpha_t| = |\theta J| \le \|\theta\|$$

for every finite set $J \subseteq I$, so $\sum_{t \in I} |\alpha_t|$ is finite, and we have a functional θ_1 defined by setting $\theta_1 a = \sum_{t \in a} \alpha_t$ for every $a \subseteq I$. It is easy to check that $\theta_1 \in M_\tau$. If we write $\theta_2 = \theta - \theta_1$ then $\theta_2\{t\} = 0$ for every $t \in I$. So if $\phi \in M_\tau$ and $0 \le \phi \le |\theta_2|$, we still have $\phi\{t\} = 0$ for every $t \in I$, and $\phi a = 0$ for every finite $a \subseteq I$ and $\phi = 0$; thus $\theta_2 \in M_\tau^\perp$. Now $\theta \in M_\tau$ iff $\theta_2 = 0$, that is, iff $\theta = \theta_1$ and $\theta a = \sum_{t \in a} \theta\{t\}$ for every $a \subseteq I$; while $\theta \in M_\tau^\perp$ iff $\theta_1 = 0$, that is, $\theta\{t\} = 0$ for every $t \in I$. **Q**

Observe that if $\theta \in M_\tau^\perp$ and a, $b \subseteq I$ are such that $a \triangle b$ is finite, then $\theta a = \theta b$, because $\theta(a \setminus b) = \theta(b \setminus a) = 0$.

(c) It will be useful to have an elementary fact out in the open. If $\theta \in M^+ \setminus \{0\}$, then $\{a : a \subseteq I, \theta a = \theta I\}$ is a filter; this is because $\{a : \theta a = 0\}$ is a proper ideal in $\mathcal{P}I$.

464G We also need a new result not exactly covered by those in Chapters 35 and 36.

Lemma Let $\mathfrak{A}$ be any Boolean algebra. Write M for the L-space of bounded additive functionals on $\mathfrak{A}$, and M^+ for its positive cone, the set of non-negative additive functionals. Suppose that $\Delta : M^+ \to [0, \infty[$ is a functional such that

(α) Δ is non-decreasing,

(β) $\Delta(\alpha\theta) = \alpha\Delta(\theta)$ whenever $\theta \in M^+$, $\alpha \ge 0$,

(γ) $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$ whenever θ_1, $\theta_2 \in M^+$ are such that, for some $e \subseteq I$, $\theta_1(1 \setminus e) = \theta_2 e = 0$,

(δ) $|\Delta(\theta_1) - \Delta(\theta_2)| \le \|\theta_1 - \theta_2\|$ for all θ_1, $\theta_2 \in M^+$.

Then there is a non-negative $h \in M^*$ extending Δ.

proof (a) If θ_1, $\theta_2 \in M^+$ are such that $\theta_1 \wedge \theta_2 = 0$ in M^+, then $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$. **P** Let $\epsilon > 0$. Then there is an $e \in \mathfrak{A}$ such that $\theta_1(1 \setminus e) + \theta_2 e \le \epsilon$ (362Ba). Set $\theta_1' a = \theta_1(a \cap e)$, $\theta_2'(a) = \theta_2(a \setminus e)$ for $a \in \mathfrak{A}$; then θ_1' and θ_2' belong to M^+ and $\theta_1'(1 \setminus e) = \theta_2' e = 0$, so $\Delta(\theta_1' + \theta_2') = \Delta(\theta_1') + \Delta(\theta_2')$, by hypothesis ($\gamma$). On the other hand,

$$0 \le \theta_1 a - \theta_1' a = \theta(a \setminus e) \le \theta(1 \setminus e) \le \epsilon$$

for every $a \in \mathfrak{A}$, so $\|\theta_1 - \theta_1'\| \le \epsilon$. Similarly, $\|\theta_2 - \theta_2'\| \le \epsilon$ and $\|(\theta_1 + \theta_2) - (\theta_1' - \theta_2')\| \le 2\epsilon$. But this means that we can use the hypothesis (δ) to see that

$$|\Delta(\theta_1 + \theta_2) - \Delta(\theta_1) - \Delta(\theta_2)| \le |\Delta(\theta_1' + \theta_2') - \Delta(\theta_1') - \Delta(\theta_2')| + 4\epsilon = 4\epsilon.$$

As ϵ is arbitrary, $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$. **Q**

[10]Formerly 536C.

(b) Now recall that M, being a Dedekind complete Riesz space, can be identified with an order-dense solid linear subspace of $L^0(\mathfrak{C})$ for some Boolean algebra $\mathfrak{C}$ (368H). Inside $L^0(\mathfrak{C})$ we have the order-dense Riesz subspace $S(\mathfrak{C})$ (364Ja). Write S_1 for $M \cap S(\mathfrak{C})$, so that S_1 is an order-dense Riesz subspace of M (352Nc, 353A).

If θ_1, $\theta_2 \in S_1^+ = S_1 \cap M^+$, then $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$. **P** We can express θ_1 as $\sum_{i=0}^m \alpha_i \chi c_i$, where $c_0, \dots, c_m$ are disjoint members of $\mathfrak{C}$, and $\alpha_i \ge 0$ for each i (361Ec); adding a term $0 \cdot \chi(1 \setminus \sup_{i \le m} c_i)$ if necessary, we may suppose that $\sup_{i \le m} c_i = 1$. Similarly, we can express θ_2 as $\sum_{j=0}^n \beta_j \chi d_j$ where $d_0, \dots, d_n \in \mathfrak{C}$ are disjoint, $\beta_j \ge 0$ for every j and $\sup_{j \le n} d_j = 1$. In this case, $\theta_1 = \sum_{i \le m, j \le n} \alpha_i \chi(c_i \cap d_j)$ and $\theta_2 = \sum_{i \le m, j \le n} \beta_j \chi(c_i \cap d_j)$. Re-enumerating $\{c_i \cap d_j : i \le m, j \le n\}$ as $\langle e_i : i \le k\rangle$ we have expressions of θ_1, θ_2 in the form $\sum_{i \le k} \gamma_i \chi e_i$, $\sum_{i \le k} \delta_i \chi e_i$, while $e_0, \dots, e_k$ are disjoint.

Setting $\theta_1^{(r)} = \sum_{i=0}^r \gamma_i \chi e_i$ for $r \le k$, we see that $\theta_1^{(r)} \wedge \gamma_{r+1}\chi e_{r+1} = 0$ for $r < k$, so (a) above, together with the hypothesis (β), tell us that

$$\Delta(\theta_1^{(r+1)}) = \Delta(\theta_1^{(r)}) + \Delta(\gamma_{r+1}\chi e_{r+1}) = \Delta(\theta_1^{(r)}) + \gamma_{r+1}\Delta(\chi e_{r+1})$$

for $r < k$. Accordingly $\Delta(\theta_1) = \sum_{i=0}^k \gamma_i \Delta(\chi e_i)$. Similarly, $\Delta(\theta_2) = \sum_{i=0}^k \delta_i \Delta(\chi e_i)$ and

$$\Delta(\theta_1 + \theta_2) = \sum_{i=0}^k (\gamma_i + \delta_i)\Delta(\chi e_i) = \Delta(\theta_1) + \Delta(\theta_2),$$

as claimed. **Q**

(c) Consequently, $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$ for all θ_1, $\theta_2 \in M^+$. **P** Let $\epsilon > 0$. Because S_1 is order-dense in M, and the norm of M is order-continuous (354N), S_1 is norm-dense (354Ef), and there are θ_1', $\theta_2' \in S_1^+$ such that $\|\theta_j - \theta_j'\| \le \epsilon$ for both j (354Be). But now, just as in (a),

$$|\Delta(\theta_1 + \theta_2) - \Delta(\theta_1) - \Delta(\theta_2)| \le |\Delta(\theta_1' + \theta_2') - \Delta(\theta_1') - \Delta(\theta_2')| + 4\epsilon = 4\epsilon.$$

As ϵ is arbitrary, we have the result. **Q**

(d) Now (c) and the hypothesis (β) are sufficient to ensure that Δ has an extension to a positive linear functional (355D).

464H The next lemma contains the key ideas needed for the rest of the section.

Lemma Let I be any set, and M the L-space of bounded additive functionals on $\mathcal{P}I$; let ν be the usual measure on $\mathcal{P}I$. For $\theta \in M^+$, set

$$\Delta(\theta) = \overline{\int} \theta \, d\nu.$$

(a) For every $\theta \in M^+$, $\frac{1}{2}\theta I \le \Delta(\theta) \le \theta I$.

(b) There is a non-negative $h \in M^*$ such that $h(\theta) = \Delta(\theta)$ for every $\theta \in M^+$.

(c) If $\theta \in (M_\tau^\perp)^+$, where $M_\tau \subseteq M$ is the band of completely additive functionals, then $\theta \le \Delta(\theta)$ ν-a.e., and $\nu^*\{a : \alpha \le \theta a \le \Delta(\theta)\} = 1$ for every $\alpha < \Delta(\theta)$.

(d) Suppose that $\theta \in (M_\tau^\perp)^+$ and β, $\gamma \in [0,1]$ are such that $\theta I = 1$ and $\beta\theta' I \le \Delta(\theta') \le \gamma\theta' I$ whenever $\theta' \le \theta$ in M^+. Then, for any $\alpha < \beta$,

(i) for any finite set $K \subseteq \mathcal{P}I$, the set

$$\{a : a \subseteq I,\ \alpha\theta b \le \theta(a \cap b) \le \gamma\theta b \text{ for every } b \in K\}$$

has outer measure 1 in $\mathcal{P}I$;

(ii) if $\alpha \ge \frac{1}{2}$, the set

$$R = \{(a, b, c) : a,\ b,\ c \subseteq I,\ \theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \ge 2\alpha^2 + (1 - 2\alpha)\gamma^2\}$$

has outer measure 1 in $(\mathcal{P}I)^3$;

(iii) if $\alpha \ge \frac{1}{2}$, then $2\alpha^2 + (1 - 2\alpha)\gamma^2 \le \gamma$.

(e) Any $\theta \in M^+$ can be expressed as $\theta_1 + \theta_2$ where $\Delta(\theta_1) = \frac{1}{2}\theta_1 I$ and $\Delta(\theta_2) = \theta_2 I$.

(f) Suppose that $0 \le \theta' \le \theta$ in M.

(i) If $\Delta(\theta) = \frac{1}{2}\theta I$, then $\Delta(\theta') = \frac{1}{2}\theta' I$.

(ii) If $\Delta(\theta) = \theta I$, then $\Delta(\theta') = \theta' I$.

proof (a) Of course

$$\Delta(\theta) \le \int \theta I \, d\nu = \theta I.$$

On the other hand, because $a \mapsto I \setminus a : \mathcal{P}I \to \mathcal{P}I$ is an automorphism of the measure space $(\mathcal{P}I, \nu)$, $\overline{\int} f(I \setminus a)\nu(da) = \overline{\int} f(a)\nu(da)$ for any real-valued function f (cf. 235Xn[11]). In particular,

$$\Delta(\theta) = \overline{\int}\theta(a)\nu(da) = \overline{\int}\theta(I \setminus a)\nu(da).$$

So

$$2\Delta(\theta) = \overline{\int}\theta(a)\nu(da) + \overline{\int}\theta(I \setminus a)\nu(da) \geq \overline{\int}\theta(a) + \theta(I \setminus a)\nu(da)$$

(133J(b-ii))

$$= \overline{\int}\theta I\,\nu(da) = \theta I,$$

and $\Delta(\theta) \geq \frac{1}{2}\theta I$.

(b) I use 464G with $\mathfrak{A} = \mathcal{P}I$. Examine the conditions (α)-(δ) there.

(α) Of course Δ is non-decreasing (133Jc).

(β) $\Delta(\alpha\theta) = \alpha\Delta(\theta)$ for every $\theta \in M^+$ and every $\alpha \geq 0$, by 133J(b-iii).

(γ) If θ_1, $\theta_2 \in M^+$ then $\Delta(\theta_1 + \theta_2) \leq \Delta(\theta_1) + \Delta(\theta_2)$ by 133J(b-ii), as in (a) above. If θ_1, $\theta_2 \in M^+$ and $e \subseteq I$ are such that $\theta_1(I \setminus e) = \theta_2 e = 0$, then $\theta_1 a = \theta_1(a \cap e)$, $\theta_2(a) = \theta_2(a \setminus e)$ for every $a \subseteq I$. So $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$ by 464Ab.

(δ) For every $a \subseteq I$, $\theta_2 a \leq \theta_1 a + \|\theta_2 - \theta_1\|$, so $\Delta(\theta_2) \leq \Delta(\theta_1) + \|\theta_2 - \theta_1\|$. Similarly, $\Delta(\theta_1) \leq \Delta(\theta_2) + \|\theta_1 - \theta_2\|$. So $|\Delta(\theta_1) - \Delta(\theta_2)| \leq \|\theta_1 - \theta_2\|$, and Δ satisfies condition (δ) of 464G.

(ϵ) Accordingly 464G tells us that Δ has an extension to a member of M^*.

(c) If $\theta \in M_\tau^\perp$, then $\theta a = \theta b$ whenever $a \triangle b$ is finite (464Fb), so all the sets $\{a : \theta a > \alpha\}$ have outer measure either 0 or 1, by 464Ac. But this means that if f is a measurable function and $\theta \leq_{\text{a.e.}} f$ and $\int f = \Delta(\theta)$, as in 133J(a-i), $\{a : f(a) \leq \Delta(\theta)\}$ has positive measure and meets $A = \{a : \theta a > \Delta(\theta)\}$ in a negligible set; so A cannot have full outer measure and is negligible.

On the other hand, if $\alpha < \Delta(\theta)$, then θ cannot be dominated a.e. by $\alpha\chi(\mathcal{P}I)$, so $\{a : \theta a > \alpha\}$ is not negligible and has outer measure 1. Consequently $\nu^*\{a : \alpha < \theta a \leq \Delta(\theta)\} = 1$.

(d)(i) The point is just that for any $b \subseteq I$, the functional θ_b, defined by saying that $\theta_b(a) = \theta(a \cap b)$ for every $a \subseteq I$, belongs to M^+ and is dominated by θ, so that $\beta\theta_b I \leq \Delta(\theta_b) \leq \gamma\theta_b I$, and $\{a : \alpha\theta_b I \leq \theta_b a \leq \gamma\theta_b I\}$ has outer measure 1, by (c). (If $\alpha\theta_b I = \Delta(\theta_b)$, this is because $\theta_b I = 0$, and the result is trivial; otherwise, $\alpha\theta_b I < \Delta(\theta_b) \leq \gamma\theta_b I$.) But this just says that, for any $b \subseteq I$,

$$\{a : \alpha\theta b \leq \theta(a \cap b) \leq \gamma\theta b\}$$

has outer measure 1.

Now, given any finite set $K \subseteq \mathcal{P}I$, let $\mathcal{B}$ be the subalgebra of $\mathcal{P}I$ generated by K, and $b_0, \dots, b_n$ the atoms of $\mathcal{B}$. Then all the sets

$$A_i = \{a : \alpha\theta b_i \leq \theta(a \cap b_i) \leq \gamma\theta b_i\}$$

have outer measure 1; because each A_i is determined by coordinates in b_i, and $b_0, \dots, b_n$ are disjoint, $A = \bigcap_{i \leq n} A_i$ still has outer measure 1, by 464Aa. But if $a \in A$ and $b \in K$, then $b = \bigcup_{i \in J} b_i$ for some $J \subseteq \{0, \dots, n\}$ and

$$\begin{aligned}\alpha\theta b &= \sum_{i \in J}\alpha\theta b_i \leq \sum_{i \in J}\theta(a \cap b_i)\\ &= \theta(a \cap b) \leq \sum_{i \in J}\gamma\theta b_i = \gamma\theta b.\end{aligned}$$

So $\{a : \alpha\theta b \leq \theta(a \cap b) \leq \gamma\theta b$ for every $b \in K\}$ includes A and has outer measure 1.

[11] Later editions only.

(ii) ? Suppose, if possible, otherwise; then $\nu^3_*(\mathcal{P}I\setminus R)>0$, where ν^3 is the product measure on $(\mathcal{P}I)^3$, and there is a measurable set $W\subseteq(\mathcal{P}I)^3\setminus R$ such that $\nu^3W>0$. For $a\subseteq I$, set $W_a=\{(b,c):(a,b,c)\in W\}$; then $\nu^3W=\int\nu^2W_a\nu(da)$, where ν^2 is the product measure on $(\mathcal{P}I)^2$, by Fubini's theorem (252D). Set $E=\{a:\nu^2W_a$ is defined and not zero$\}$; then $\nu E>0$. Since $\{a:\alpha\le\theta a\le\gamma\}$ has outer measure 1, there is an $a\in E$ such that $\alpha\le\theta a\le\gamma$.

For $b\subseteq I$, set $W_{ab}=\{c:(b,c)\in W_a\}=\{c:(a,b,c)\in W\}$. Then $0<\nu^2W_a=\int\nu W_{ab}\nu(db)$, so $F=\{b:\nu W_{ab}$ is defined and not $0\}$ has non-zero measure. But also, by (i),

$$\{b:\theta b\ge\alpha,\ \theta(a\cap b)\le\gamma\theta a\}$$

has outer measure 1, so we can find a $b\in F$ such that $\theta b\ge\alpha$ and $\theta(a\cap b)\le\gamma\theta a$.

By (i) again,

$$\{c:\theta(c\cap(a\triangle b))\ge\alpha\theta(a\triangle b)\}$$

has outer measure 1, so meets W_{ab}; accordingly we have a $c\subseteq I$ such that $(a,b,c)\in W$ while $\theta(c\cap(a\triangle b))\ge\alpha\theta(a\triangle b)$.

Now calculate, for this triple (a,b,c),

$$\theta((a\cap b)\cup(a\cap c)\cup(b\cap c))=\theta(a\cap b)+\theta(c\cap(a\triangle b))\ge\theta(a\cap b)+\alpha\theta(a\triangle b)$$

(by the choice of c)

$$=\alpha(\theta a+\theta b)+(1-2\alpha)\theta(a\cap b)$$
$$\ge\alpha(\theta a+\alpha)+(1-2\alpha)\gamma\theta a$$

(by the choice of b, recalling that $1-2\alpha\le0$)

$$\ge 2\alpha^2+(1-2\alpha)\gamma^2$$

by the choice of a. But this means that $(a,b,c)\in W\cap R$, which is supposed to be impossible. **X**

(iii) Now recall that the map $(a,b,c)\mapsto(a\cap b)\cup(a\cap c)\cup(b\cap c)$ is inverse-measure-preserving (464Bb). Since $\theta a\le\Delta(\theta)\le\gamma$ for ν-almost every a, we must have $\theta((a\cap b)\cup(a\cap c)\cup(b\cap c))\le\gamma$ for ν^3-almost every (a,b,c). But as R is not negligible, there must be some $(a,b,c)\in R$ such that $\theta((a\cap b)\cup(a\cap c)\cup(b\cap c))\le\gamma$, and $2\alpha^2+(1-2\alpha)\gamma^2\le\gamma$.

(e)(i) M^*, being the dual of an L-space, is an M-space (356Pb), so can be represented as $C(Z)$ for some compact Hausdorff space Z (354L). The functional h of (b) above therefore corresponds to a function $w\in C(Z)$. Any $\theta\in M^+$ acts on M^* as a positive linear functional, so corresponds to a Radon measure μ_θ on Z (436J/436K); we have $\Delta(\theta)=h(\theta)=\int w\,d\mu_\theta$. The inequalities $\frac{1}{2}\theta I\le\Delta(\theta)\le\theta I$ become $\frac{1}{2}\mu_\theta Z\le\int w\,d\mu_\theta\le\mu_\theta Z$, because the constant function χZ corresponds to the standard order unit of M^* (356Pb again), so that

$$\mu_\theta Z=\int\chi Z\,d\mu_\theta=\|\theta\|=\theta I$$

for every $\theta\ge0$. Since $0\le h(\theta)\le\|\theta\|$ for every $\theta\ge0$, $\|w\|_\infty=\|h\|\le1$ and $0\le w\le\chi Z$.

(ii) Now suppose that $\beta<\gamma$ and that $G=\{z:z\in Z,\ \beta<w(z)<\gamma\}$ is non-empty. In this case there is a non-zero $\theta_0\in M^+$ such that $\beta\theta'I\le\Delta(\theta')\le\gamma\theta'I$ whenever $0\le\theta'\le\theta_0$.

P **(α)** We have a solid linear subspace $V=\{v:v\in C(Z),\ v(z)=0$ for every $z\in G\}$ of $C(Z)$. Consider $U=\{\theta:\theta\in M,\ (\theta|v)=0$ for every $v\in V\}$, where I write $(\,|\,)$ for the duality between M and $C(Z)$ corresponding to the identification of $C(Z)$ with M^*.

(β) If $\theta\in U\cap M^+$, then $\beta\theta I\le\Delta(\theta)\le\gamma\theta I$. To see this, observe that $\int v\,d\mu_\theta=(\theta|v)=0$ for every $v\in V$, so

$$\mu_\theta(Z\setminus\overline{G})=\sup\{\mu_\theta K:K\subseteq Z\setminus\overline{G}\text{ is compact}\}$$
$$=\sup\{\int v\,d\mu_\theta:v\in C(Z),\ 0\le v\le\chi(Z\setminus\overline{G})\}$$

(because whenever $K\subseteq Z\setminus\overline{G}$ is compact there is a $v\in C(Z)$ such that $\chi K\le v\le\chi(Z\setminus\overline{G})$, by 4A2F(h-ii))

$$=0.$$

Because w is continuous, $\beta\le w(z)\le\gamma$ for every $z\in\overline{G}$; thus $\beta\le w\le\gamma$ μ_θ-a.e. and $\int w\,d\mu_\theta$ must belong to $[\beta\mu_\theta Z,\gamma\mu_\theta Z]=[\beta\theta I,\gamma\theta I]$.

(γ) The dual $M^* = M^\times$ of M is perfect (356Lb), and $C(Z)$ is perfect; moreover, M is perfect (356Pa), so the duality $(\,|\,)$ identifies M with $C(Z)^\times$. Now $V^\perp$, taken in $C(Z)$, contains any continuous function zero on $Z \setminus G$, so is not $\{0\}$; since $V^\perp$, like $C(Z)$, must be perfect (356La), $(V^\perp)^\times$ is non-trivial. Take any $\psi > 0$ in $(V^\perp)^\times$. Being perfect, $C(Z)$ is Dedekind complete (356K), so there is a band projection $P : C(Z) \to V^\perp$ (353I). Now ψP is a positive element of $C(Z)^\times$ which is zero on V, and must correspond to a non-zero element θ_0 of $U \cap M^+$.

(δ) If $0 \le \theta' \le \theta_0$ in M, then, for any $v \in V$,

$$|(\theta'|v)| \le (\theta'||v|) \le (\theta_0||v|) = 0,$$

because $|v| \in V$. So $\theta' \in U$ and $\beta\theta' I \le \Delta(\theta') \le \gamma\theta' I$, by (β). Thus θ_0 has the required property. **Q**

(iii) It follows at once that $w(z) \ge \frac{1}{2}$ for every $z \in Z$. **P?** If $w(z_0) < \frac{1}{2}$, then we can apply (ii) with $\beta = -1$, $\gamma \in \left]w(z_0), \frac{1}{2}\right[$ to see that there is a non-zero $\theta \in M^+$ such that $\Delta(\theta) \le \gamma\theta I < \frac{1}{2}\theta I$, which is impossible, by (a). **XQ**

(iv) But we find also that $w(z) \notin \left]\frac{1}{2}, 1\right[$ for any $z \in Z$. **P?** If $w(z_0) = \delta \in \left]\frac{1}{2}, 1\right[$, then $2\delta^2 + (1 - 2\delta)\delta^2 > \delta$ (because $\delta(2\delta - 1)(1 - \delta) > 0$). We can therefore find α, β and γ such that $\frac{1}{2} \le \alpha < \beta < \delta < \gamma$ and $2\alpha^2 + (1-2\alpha)\gamma^2 > \gamma$. But now $\{z : \beta < w(z) < \gamma\}$ is non-empty, so by (ii) there is a non-zero $\theta \in M^+$ such that $\beta\theta' I \le \Delta(\theta') \le \gamma\theta' I$ whenever $0 \le \theta' \le \theta$. Multiplying θ by a suitable scalar if necessary, we can arrange that θI should be 1. But this is impossible, by (d-iii). **XQ**

(v) Thus w takes only the values $\frac{1}{2}$ and 1; let H_1 and H_2 be the corresponding open-and-closed subsets of Z. Take $\theta \in M^+$. For $u \in C(Z)$, set $\phi(u) = \int u\,d\mu_\theta$ and $\phi_j(u) = \int_{H_j} u\,d\mu_\theta$ for each j. Then each ϕ_j is a positive linear functional on $C(Z)$ and $\phi_j \le \phi$. But ϕ is the image of θ under the canonical isomorphism from M to $C(Z)^\times \cong M^{\times\times}$, and $C(Z)^\times$ is solid in $C(Z)^\sim$ (356B), so both ϕ_1 and ϕ_2 belong to the image of M, and correspond to θ_1, $\theta_2 \in M$. For any $u \in C(Z) \cong M^*$,

$$(\theta_1 + \theta_2|u) = \phi_1(u) + \phi_2(u) = (\theta|u),$$

so $\theta = \theta_1 + \theta_2$. We have

$$\begin{aligned}\Delta(\theta_j) = \phi_j(w) &= \int_{H_j} w\,d\mu_\theta \\ &= \frac{1}{2}\mu_\theta(H_1) = \frac{1}{2}\theta_1 I \text{ if } j = 1, \\ &= \mu_\theta(H_2) = \theta_2 I \text{ if } j = 2.\end{aligned}$$

So we have a suitable decomposition $\theta = \theta_1 + \theta_2$.

(f) This is easy. Set $\theta'' = \theta - \theta'$; then

$$\tfrac{1}{2}\theta' I \le \Delta(\theta') \le \theta' I,$$

$$\tfrac{1}{2}\theta'' I \le \Delta(\theta'') \le \theta'' I$$

by (a), while $\Delta(\theta') + \Delta(\theta'') = \Delta(\theta)$ by (b), and of course $\theta' I + \theta'' I = \theta I$. But this means that

$$\Delta(\theta') - \tfrac{1}{2}\theta' I \le \Delta(\theta) - \tfrac{1}{2}\theta I, \quad \theta' I - \Delta(\theta') \le \theta I - \Delta(\theta),$$

and the results follow.

464I Measurable and purely non-measurable functionals As before, let I be any set, ν the usual measure on $\mathcal{P}I$, T its domain, and M the L-space of bounded additive functionals on $\mathcal{P}I$. Following Fremlin & Talagrand 79, I say that $\theta \in M$ is **measurable** if it is T-measurable when regarded as a real-valued function on $\mathcal{P}I$, and **purely non-measurable** if $\{a : a \subseteq I,\ |\theta|(a) = |\theta|(I)\}$ has outer measure 1. (Of course the zero functional is both measurable and purely non-measurable.)

464J Examples Before going farther, I had better offer some examples of measurable and purely non-measurable functionals. Let I, ν and M be as in 464I.

(a) Any $\theta \in M_\tau$ is measurable, where M_τ is the space of completely additive functionals on $\mathcal{P}I$. **P** By 464Fb, θ can be expressed as a sum of point masses; say $\theta a = \sum_{t \in a} \alpha_t$ for some family $\langle \alpha_t \rangle_{t \in I}$ in $\mathbb{R}$. Since $\sum_{t \in I} |\alpha_t|$ must be finite, $\{t : \alpha_t \neq 0\}$ is countable, and we can express θ as the limit of a sequence of finite sums $\sum_{t\in K} \alpha_t\hat{t}$, where $\hat{t}(a) = 1$ if $t \in a$, 0 otherwise. But of course every $\hat{t}$ is a measurable function, so $\sum_{t\in K} \alpha_t\hat{t}$ is measurable for every finite set K, and θ is measurable. **Q**

(b) For a less elementary measurable functional, consider the following construction. Let $\langle t_n\rangle_{n\in\mathbb{N}}$ be any sequence of distinct points in I. Then $\lim_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\, t_i \in a\}) = \frac{1}{2}$ for ν-almost every $a \subseteq I$. **P** Set $f_n(a) = 1$ if $t_n \in a$, 0 otherwise. Then $\langle f_n\rangle_{n\in\mathbb{N}}$ is an independent sequence of random variables. By any of the versions of the Strong Law of Large Numbers in §273 (273D, 273H, 273I), $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f_i = \frac{1}{2}$ a.e., which is what was claimed. **Q** So if we take any non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$, and set $\theta a = \lim_{n\to\mathcal{F}} \frac{1}{n}\#(\{i : i < n,\, t_i \in a\})$ for $a \subseteq I$, θ will be constant ν-almost everywhere, and measurable; and it is easy to check that θ is additive. Note that $\theta\{t\} = 0$ for every t, so $\theta \in M_\tau^\perp$, by 464Fb.

(c) If $\mathcal{F}$ is any non-principal ultrafilter on I, and we set $\theta a = 1$ for $a \in \mathcal{F}$, 0 otherwise, then θ is an additive functional which is purely non-measurable, by 464Ca.

For further remarks on where to look for measurable and purely non-measurable functionals, see 464P-464Q below.

464K The space M_{m}: Lemma Let I be any set, ν the usual measure on $\mathcal{P}I$, and M the L-space of bounded additive functionals on $\mathcal{P}I$. Write M_{m} for the set of measurable $\theta \in M$, M_τ for the space of completely additive functionals on $\mathcal{P}I$ and $\Delta(\theta) = \overline{\int}\theta\, d\nu$ for $\theta \in M^+$, as in 464H.

(a) If $\theta \in M_{\mathrm{m}} \cap M_\tau^\perp$ and $b \subseteq I$, then $\theta(a \cap b) = \frac{1}{2}\theta b$ for ν-almost every $a \subseteq I$.

(b) $|\theta| \in M_{\mathrm{m}}$ for every $\theta \in M_{\mathrm{m}}$.

(c) A functional $\theta \in M^+$ is measurable iff $\Delta(\theta) = \frac{1}{2}\theta I$.

(d) M_{m} is a solid linear subspace of M.

proof (a)(i) $\theta = \frac{1}{2}\theta I$ ν-a.e. **P** For any $\alpha \in \mathbb{R}$, $A_\alpha = \{a : \theta a < \alpha\}$ is measurable; but also $a' \in A$ whenever $a \in A$ and $a\triangle a'$ is finite, by 464Fb, so νA must be either 1 or 0, by 464Ac. Setting $\delta = \sup\{\alpha : \nu A_\alpha = 0\}$, we see that $\nu A_\delta = 0$, $\nu A_{\delta+2^{-n}} = 1$ for every $n \in \mathbb{N}$, so that $\theta = \delta$ a.e. Also, because $a \mapsto I \setminus a$ is a measure space automorphism, $\theta(I \setminus a) = \delta$ for almost every a, so there is some a such that $\theta a = \theta(I \setminus a) = \delta$, and $\delta = \frac{1}{2}\theta I$. **Q**

(ii) $\theta(a \cap b) = \frac{1}{2}\theta b$ for almost every a. **P** We know that $\theta a = \frac{1}{2}\theta I$ for almost every a. But $a \mapsto a\triangle b : \mathcal{P}I \to \mathcal{P}I$ is inverse-measure-preserving, so $\theta(a\triangle b) = \frac{1}{2}\theta I$ for almost every a. This means that $\theta a = \theta(a\triangle b)$ for almost every a, and

$$\theta(a \cap b) = \frac{1}{2}(\theta b + \theta a - \theta(a\triangle b)) = \frac{1}{2}\theta b$$

for almost every a. **Q**

(b)(i) If $\theta \in M_{\mathrm{m}} \cap M_\tau^\perp$, then θ^+, taken in M, is measurable. **P** For any $n \in \mathbb{N}$ we can find $b_n \subseteq I$ such that $\theta^- b_n + \theta^+(I \setminus b_n) \le 2^{-n}$, so that

$$|\theta^+ a - \theta(a \cap b_n)| = |\theta^+ a - \theta^+(a \cap b_n) + \theta^-(a \cap b_n)| \le \theta^+(I \setminus b_n) + \theta^- b_n \le 2^{-n}$$

for every $a \subseteq I$. But as $a \mapsto \theta(a \cap b_n)$ is constant a.e. for every n, by (a), so is θ^+, and θ^+ is measurable. **Q**

Consequently $|\theta| = 2\theta^+ - \theta$ is measurable.

(ii) Now take an arbitrary $\theta \in M_{\mathrm{m}}$. Because M is Dedekind complete (354N, 354Ee), $M = M_\tau + M_\tau^\perp$ (353I again), and we can express θ as $\theta_1 + \theta_2$ where $\theta_1 \in M_\tau$ and $\theta_2 \in M_\tau^\perp$; moreover, $|\theta| = |\theta_1| + |\theta_2|$ (352Fb). Now θ_1 is measurable, by 464Ja, so $\theta_2 = \theta - \theta_1$ is measurable; as $\theta_2 \in M_\tau^\perp$, (i) tells us that $|\theta_2|$ is measurable. On the other hand, $|\theta_1|$ belongs to M_τ and is measurable, so $|\theta| = |\theta_1| + |\theta_2|$ is measurable.

Thus (b) is true.

(c) Let f be a ν-integrable function such that $\theta \le_{\text{a.e.}} f$ and $\int f d\nu = \Delta(\theta)$. Then

$$\theta I - f(a) \le \theta I - \theta a = \theta(I \setminus a)$$

for almost every a, so

$$\theta I - \Delta(\theta) = \int \theta I - f(a)\nu(da) \le \underline{\int}\theta(I \setminus a)\nu(da) = \underline{\int}\theta(a)\nu(da)$$

because $a \mapsto I \setminus a$ is a measure space automorphism, as in the proof of 464Ha. So if $\Delta(\theta) = \frac{1}{2}\theta I$ then $\underline{\int}\theta\,d\nu = \overline{\int}\theta\,d\nu$ and θ is ν-integrable (133Jd), therefore (because ν is complete) $(\operatorname{dom}\nu)$-measurable. On the other hand, if θ is measurable, then

$$\Delta(\theta) = \int \theta\,d\nu = \int \theta(I \setminus a)\nu(da) = \theta I - \int \theta\,d\nu = \theta I - \Delta(\theta),$$

so surely $\Delta(\theta) = \frac{1}{2}\theta I$.

(d) Of course M_{m} is a linear subspace. If $\theta_0 \in M_{\mathrm{m}}$ and $|\theta| \le |\theta_0|$, then $|\theta_0| \in M_{\mathrm{m}}$, by (b), so $\Delta(|\theta_0|) = \frac{1}{2}|\theta_0|(I)$, by (c). Because $\theta^+ \le |\theta| \le |\theta_0|$, $\Delta(\theta^+) = \frac{1}{2}\theta^+ I$ (464H(f-i)), and θ^+ is measurable, by (c) in the reverse direction. Similarly, θ^- is measurable, and $\theta = \theta^+ - \theta^-$ is measurable. As θ and θ_0 are arbitrary, M_{m} is solid.

464L The space M_{pnm}: Lemma Let I be any set, ν the usual measure on $\mathcal{P}I$, and M the L-space of bounded additive functionals on $\mathcal{P}I$. This time, write M_{pnm} for the set of those members of M which are purely non-measurable in the sense of 464I.

(a) If $\theta \in M^+$, then θ is purely non-measurable iff $\Delta(\theta) = \theta I$.

(b) M_{pnm} is a solid linear subspace of M.

proof (a)(i) If θ is purely non-measurable, and $f \ge \theta$ is integrable, then $\{a : f(a) \ge \theta I\}$ is a measurable set including $\{a : \theta a = \theta I\}$, so has measure 1, and $\int f \ge \theta I$; as f is arbitrary, $\Delta(\theta) = \theta I$.

(ii) If $\Delta(\theta) = \theta I$, then $\Delta(\theta') = \theta' I$ whenever $0 \le \theta' \le \theta$, by 464Hf. But this means that θ' cannot be measurable whenever $0 < \theta' \le \theta$, by 464Kc above, so that $\theta' \notin M_\tau$ whenever $0 < \theta' \le \theta$, by 464Ja. Thus $\theta \in M_\tau^\perp$.

By 464Hc, $\nu^*\{a : \alpha \le \theta a \le \Delta(\theta)\} = 1$ for every $\alpha < \Delta(\theta)$. Let $\langle m(n)\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that $\prod_{n=0}^\infty 1 - 2^{-m(n)} = \frac{1}{2}$, and define X, λ and ϕ as in 464Ba. Set $\eta_n = 2^{-n}/m(n) > 0$ for each n. Consider the sets $A_n = \{a : \theta a \ge (1 - \eta_n)\theta I\}$ for each $n \in \mathbb{N}$. Then $\nu^* A_n = 1$ for each n, and $\lambda^*(\prod_{n\in\mathbb{N}} A_n^{m(n)}) = 1$. Because ϕ is inverse-measure-preserving, $\nu^*(\phi[\prod_{n\in\mathbb{N}} A_n^{m(n)}]) = 1$ (413Eh again). But if $\mathfrak{x} = \langle\langle a_{ni}\rangle_{i<m(n)}\rangle_{n\in\mathbb{N}}$ belongs to $\prod_{n\in\mathbb{N}} A_n^{m(n)}$, then

$$\theta(\bigcap_{i<m(n)} a_{ni}) \ge \theta I - m(n)\eta_n\theta I = (1 - 2^{-n})\theta I$$

for each n, and

$$\theta(\phi(\mathfrak{x})) \ge \sup_{n\in\mathbb{N}} \theta(\bigcap_{i<m(n)} a_{ni}) = \theta I.$$

Thus the filter $\{a : \theta a = \theta I\}$ includes $\phi[\prod_{n\in\mathbb{N}} A_n^{m(n)}]$ and has outer measure 1, so that θ is purely non-measurable.

(b)(i) If $\theta \in M_{\mathrm{pnm}}$ and $|\theta'| \le |\theta|$, then $\Delta(|\theta|) = |\theta|(I)$, by the definition in 464I, so $\Delta(|\theta'|) = |\theta'|(I)$, by 464H(f-ii), and $\theta' \in M_{\mathrm{pnm}}$, by (a) above. Thus M_{pnm} is solid.

(ii) If θ_1, $\theta_2 \in M_{\mathrm{pnm}}$, then

$$\Delta(|\theta_1| + |\theta_2|) = \Delta(|\theta_1|) + \Delta(|\theta_2|) = |\theta_1|(I) + |\theta_2|(I) = (|\theta_1| + |\theta_2|)(I)$$

(using 464Hb), and $|\theta_1| + |\theta_2| \in M_{\mathrm{pnm}}$; as $|\theta_1 + \theta_2| \le |\theta_1| + |\theta_2|$, $\theta_1 + \theta_2 \in M_{\mathrm{pnm}}$. Thus M_{pnm} is closed under addition.

(iii) It follows from (ii) that if $\theta \in M_{\mathrm{pnm}}$ then $n\theta \in M_{\mathrm{pnm}}$ for every integer $n \ge 1$, and then from (i) that $\alpha\theta \in M_{\mathrm{pnm}}$ for every $\alpha \in \mathbb{R}$; so that M_{pnm} is closed under scalar multiplication, and is a linear subspace.

464M Theorem (Fremlin & Talagrand 79) Let I be any set. Write M for the L-space of bounded finitely additive functionals on $\mathcal{P}I$, and M_{m}, M_{pnm} for the spaces of measurable and purely non-measurable functionals, as in 464K-464L. Then M_{m} and M_{pnm} are complementary bands in M.

proof (a) We know from 464K and 464L that these are both solid linear subspaces of M. Next, $M_{\mathrm{m}} \cap M_{\mathrm{pnm}} = \{0\}$. **P** If θ belongs to the intersection, then $\Delta(|\theta|) = \frac{1}{2}|\theta|(I) = |\theta|(I)$, by 464Kc and 464La; so $\theta = 0$. **Q**

(b) Now recall that every element of M^+ is expressible in the form $\theta_1 + \theta_2$ where $\theta_1 \in M_{\mathrm{m}}^+$ and $\theta_2 \in M_{\mathrm{pnm}}^+$; this is 464He, using 464Kc and 464La again. Because M_{m} and M_{pnm} are linear subspaces, with intersection $\{0\}$, $M = M_{\mathrm{m}} \oplus M_{\mathrm{pnm}}$. Now $M_{\mathrm{pnm}} \subseteq M_{\mathrm{m}}^\perp$, so $M_{\mathrm{m}} + M_{\mathrm{m}}^\perp = M$ and $M_{\mathrm{m}} = M_{\mathrm{m}}^{\perp\perp}$ is a complemented band (352Ra); similarly, M_{pnm} is a complemented band. Since $(M_{\mathrm{m}} + M_{\mathrm{pnm}})^\perp = \{0\}$, M_{m} and M_{pnm} are complementary bands (see 352S).

464N Corollary (Fremlin & Talagrand 79) Let I be any set, and let μ be Talagrand's measure on $\mathcal{P}I$; write Σ for its domain. Then every bounded additive functional on $\mathcal{P}I$ is Σ-measurable.

proof Defining M, M_{m} and M_{pnm} as in 464K-464M, we see that every functional in M_{m} is Σ-measurable because it is (by definition) $(\operatorname{dom}\nu)$-measurable, where ν is the usual measure on $\mathcal{P}I$. If $\theta \in M_{\mathrm{pnm}}^+$, then $\mathcal{F} = \{a : \theta a = \theta I\}$ is a non-measurable filter; but this means that $\mu\mathcal{F} = 1$, by the construction of μ, so that $\theta = \theta I$ μ-a.e. So if θ is any member of M_{pnm}, both θ^+ and θ^- are Σ-measurable, and $\theta = \theta^+ - \theta^-$ also is.

464O Remark Note that we have a very simple description of the behaviour of additive functionals as seen by the measure μ. Since $M_\tau \subseteq M_{\mathrm{m}}$, we have a three-part band decomposition $M = M_\tau \oplus (M_{\mathrm{m}} \cap M_\tau^\perp) \oplus M_{\mathrm{pnm}}$.

(i) Functionals in M_τ are T-measurable, where T is the domain of ν, therefore Σ-measurable, just because they can be built up from the functionals $a \mapsto \chi a(t)$, as in 464Fb.

(ii) A functional in M_{m} is T-measurable, by definition; but a functional θ in $M_{\mathrm{m}} \cap M_\tau^\perp$ is actually constant, with value $\frac{1}{2}\theta I$, ν-almost everywhere, by 464Ka. Thus the almost-constant nature of the functionals described in 464Jb is typical of measurable functionals in $M_\tau^\perp$.

(iii) Finally, a functional $\theta \in M_{\mathrm{pnm}}$ is equal to θI μ-almost everywhere; once again, this follows from the definition of 'purely non-measurable' and the construction of μ for $\theta \geq 0$, and from the fact that M_{pnm} is solid for other θ.

(iv) Thus we see that any $\theta \in M_\tau^\perp = (M_{\mathrm{m}} \cap M_\tau^\perp) \oplus M_{\mathrm{pnm}}$ is constant μ-a.e. We also have $\int \theta\, d\mu = \Delta(\theta)$ for every $\theta \geq 0$ (look at $\theta \in M_{\mathrm{m}}$ and $\theta \in M_{\mathrm{pnm}}$ separately, using 464La for the latter), so that if $h \in M^*$ is the linear functional of 464Hb, then $\int \theta\, d\mu = h(\theta)$ for every $\theta \in M$.

464P More on purely non-measurable functionals (a) We can discuss non-negative additive functionals on $\mathcal{P}I$ in terms of the Stone-Čech compactification βI of I, as follows. For any set $A \subseteq \beta I$ set $H_A = \{a : a \subseteq I,\ A \subseteq \widehat{a}\}$, where $\widehat{a} \subseteq \beta I$ is the open-and-closed set corresponding to $a \subseteq I$. If $A \neq \emptyset$, H_A is a filter on I. Write $\mathcal{A}$ for the family of those sets $A \subseteq \beta I$ such that $\nu^* H_A = 1$, where ν is the usual measure on $\mathcal{P}I$. Then $\mathcal{A}$ is a σ-ideal. **P** Of course $A \in \mathcal{A}$ whenever $A \subseteq B \in \mathcal{A}$, since then $H_A \supseteq H_B$. If $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$ with union A, then $\nu^*(\bigcap_{n\in\mathbb{N}} H_{A_n}) = 1$, by 464C; but $H_A \supseteq \bigcap_{n\in\mathbb{N}} H_{A_n}$. **Q** Note that if $A \in \mathcal{A}$ then $\overline{A} \in \mathcal{A}$, because $H_A = H_{\overline{A}}$. We see also that $\{z\} \in \mathcal{A}$ for every $z \in \beta I \setminus I$ (since $H_{\{z\}}$ is a non-principal ultrafilter, as in 464Jc), while $\{t\} \notin \mathcal{A}$ for any $t \in I$ (since $H_{\{t\}} = \{a : t \in a\}$ has measure $\frac{1}{2}$).

Because βI can be identified with the Stone space of $\mathcal{P}I$ (4A2I(b-i)), we have a one-to-one correspondence between non-negative additive functionals θ on $\mathcal{P}I$ and Radon measures μ_θ on βI, defined by writing $\mu_\theta(\widehat{a}) = \theta a$ whenever $a \subseteq I$ and $\theta \in M^+$ (416Qb). (This is not the same as the measure μ_θ of part (e) of the proof of 464H, which is on a much larger space.) Now suppose that θ is a non-negative additive functional on $\mathcal{P}I$. Then $\mathcal{F}_\theta = \{a : \theta a = \theta I\}$ is either $\mathcal{P}I$ or a filter on I. If we set $F_\theta = \bigcap\{\widehat{a} : a \in \mathcal{F}_\theta\}$, then $\mathcal{F}_\theta = H_{F_\theta}$ (4A2I(b-iii)). Since $a \in \mathcal{F}_\theta$ iff $\theta a = \theta I$,

$$\begin{aligned} F_\theta &= \bigcap\{\widehat{a} : a \in \mathcal{F}_\theta\} = \bigcap\{\widehat{a} : \theta a = \theta I\} \\ &= \bigcap\{\widehat{a} : \mu_\theta \widehat{a} = \mu_\theta(\beta I)\} = \beta I \setminus \bigcup\{\widehat{a} : \mu_\theta \widehat{a} = 0\}. \end{aligned}$$

But this is just the support of μ_θ (411N), because $\{\widehat{a} : a \subseteq I\}$ is a base for the topology of βI.

Thus we see that $\theta \in M^+$ is purely non-measurable iff the support F_θ of the measure μ_θ belongs to $\mathcal{A}$. If you like, θ is purely non-measurable iff the support of μ_θ is 'small'.

(b) Yet another corollary of 464C is the following. Since M is a set of real-valued functions on $\mathcal{P}I$, it has the corresponding topology $\mathfrak{T}_p$ of pointwise convergence as a subspace of $\mathbb{R}^{\mathcal{P}I}$. Now if $C \subseteq M_{\mathrm{pnm}}$ is countable, its $\mathfrak{T}_p$-closure $\overline{C}$ is included in M_{pnm}. **P** It is enough to consider the case in which C is non-empty and $0 \notin C$. For each $\theta \in C$, $\mathcal{F}_{|\theta|} = \{a : |\theta|(a) = |\theta|(I)\}$ is a filter with outer measure 1, so $\mathcal{F} = \{a : |\theta|(a) = |\theta|(I) \text{ for every } \theta \in C\}$ also has outer measure 1, by 464Cb. Now suppose that $\theta_0 \in \overline{C}$. If $a \in \mathcal{F}$, then $|\theta|(I \setminus a) = 0$ for every $\theta \in C$, that is, $\theta b = 0$ whenever $\theta \in C$ and $b \subseteq I \setminus a$ (362Ba). But this means that $\theta_0 b = 0$ whenever $b \subseteq I \setminus a$, so $|\theta_0|(I \setminus a) = 0$ and $|\theta_0|(a) = |\theta_0|(I)$. Thus $\mathcal{F}_{|\theta_0|}$ includes $\mathcal{F}$, so has outer measure 1, and θ_0 also is purely non-measurable. **Q**

(c) If $\theta \in M$ is such that $\theta a = 0$ for every countable set $a \subseteq I$, then $\theta \in M_{\mathrm{pnm}}$. **P** ν is inner regular with respect to the family $\mathcal{W}$ of sets which are determined by coordinates in countable subsets of I, by 254Ob. But if $W \in \mathcal{W}$ and $\nu W > 0$, let $J \subseteq I$ be a countable set such that W is determined by coordinates in J; then $|\theta|(J) = 0$, so if a is any member of W we shall have $a \cup (I \setminus J) \in W \cap \mathcal{F}_{|\theta|}$. As W is arbitrary, $\nu^* \mathcal{F}_{|\theta|} = 1$ and θ is purely non-measurable. **Q**

In particular, if $\theta \in M_\sigma \cap M_\tau^\perp$, where M_σ is the space of countably additive functionals on $\mathcal{P}I$ (362B), then $\theta \in M_{\mathrm{pnm}}$. (For 'ordinary' sets I, $M_\sigma = M_\tau$; see 438Xa. But this observation is peripheral to the concerns of the present section.)

In the language of (a) above, we have a closed set in βI, being $F = \beta I \setminus \bigcup\{\widehat{a} : a \in [I]^{\le\omega}\}$; and if θ is such that the support of μ_θ is included in F, then θ is purely non-measurable.

464Q More on measurable functionals (a) We know that M_{m} is a band in M, and that it includes the band M_τ. So it is natural to look at the band $M_{\mathrm{m}} \cap M_\tau^\perp$.

(b) If θ is any non-zero non-negative functional in $M_{\mathrm{m}} \cap M_\tau^\perp$, we can find a family $\langle a_\xi\rangle_{\xi<\omega_1}$ in $\mathcal{P}I$ which is independent in the sense that $\theta(\bigcap_{\xi\in K} a_\xi) = 2^{-\#(K)}\theta I$ for every non-empty finite $K \subseteq I$. **P** Choose the a_ξ inductively, observing that at the inductive step we have to satisfy only countably many conditions of the form $\theta(a_\xi \cap b) = \frac{1}{2}\theta b$, where b runs over the subalgebra generated by $\{a_\eta : \eta < \xi\}$, and that each such condition is satisfied ν-a.e., by 464Ka; so that ν-almost any a will serve for a_ξ. **Q**

In terms of the associated measure μ_θ on βI, this means that μ_θ has Maharam type at least ω_1 (use 331Ja). If $\theta I = 1$, so that μ_θ is a probability measure, then $\langle(\widehat{a}_\xi)^\bullet\rangle_{\xi<\omega_1}$ is an uncountable stochastically independent family in the measure algebra of μ_θ (325Xf),

Turning this round, we see that if λ is a Radon measure on βI, of countable Maharam type, and $\lambda I = 0$, then the corresponding functional on $\mathcal{P}I$ is purely non-measurable.

[For a stronger result in this direction, see 521S in Volume 5.]

(c) Another striking property of measurable additive functionals is the following. If $\theta \in M_{\mathrm{m}} \cap M_\tau^\perp$, and $n \in \mathbb{N}$, then $\theta(a_0 \cap a_1 \cap \ldots \cap a_n) = 2^{-n-1}\theta I$ for ν^{n+1}-almost every $a_0, \ldots, a_n \subseteq I$, where ν^{n+1} is the product measure on $(\mathcal{P}I)^{n+1}$. **P** For $K \subseteq \{0, \ldots, n\}$ write $\psi_K(a_0, \ldots, a_n) = I \cap \bigcap_{i\in K} a_i \setminus \bigcup_{i\le n, i\notin K} a_i$ for $a_0, \ldots, a_n \subseteq I$; for $\mathcal{S} \subseteq \mathcal{P}\{0, \ldots, n\}$ write $\phi_{\mathcal{S}}(\mathfrak{x}) = \bigcup_{K\in\mathcal{S}} \psi_K(\mathfrak{x})$ for $\mathfrak{x} \in (\mathcal{P}I)^{n+1}$. Then, for any $t \in I$,

$$\nu^{n+1}\{\mathfrak{x} : t \in \phi_{\mathcal{S}}(\mathfrak{x})\} = \textstyle\sum_{K\in\mathcal{S}} \nu^{n+1}\{\mathfrak{x} : t \in \phi_K(\mathfrak{x})\} = 2^{-n-1}\#(\mathcal{S}).$$

Moreover, for different t, these sets are independent. So if $\#(\mathcal{S}) = 2^n$, that is, if $\mathcal{S}$ is just half of $\mathcal{P}\{0, \ldots, n\}$, then $\phi_{\mathcal{S}}$ will be inverse-measure-preserving, by the arguments in 464B. (In fact 464Bb is the special case $n = 2$, $\mathcal{S} = \{K : \#(K) \ge 2\}$.)

Accordingly we shall have $\theta(\phi_{\mathcal{S}}(\mathfrak{x})) = \frac{1}{2}\theta I$ for ν^{n+1}-almost every $\mathfrak{x}$ whenever $\mathcal{S} \subseteq \mathcal{P}\{0, \ldots, n\}$ has 2^n members, as in 464Ka. Since there are only finitely many sets $\mathcal{S}$, the set E is ν^{n+1}-conegligible, where

$$E = \{\mathfrak{x} : \mathfrak{x} \in (\mathcal{P}I)^{n+1}, \, \theta(\phi_{\mathcal{S}}(\mathfrak{x})) = \frac{1}{2}\theta I \text{ whenever } \mathcal{S} \subseteq \mathcal{P}\{0, \ldots, n\} \text{ and } \#(\mathcal{S}) = 2^n\}.$$

But given $\mathfrak{x} = (a_0, \ldots, a_n) \in E$, let $K_1, \ldots, K_{2^{n+1}}$ be a listing of $\mathcal{P}\{0, \ldots, n\}$ in such an order that $\theta(\psi_{K_i}(\mathfrak{x})) \le \theta(\psi_{K_j}(\mathfrak{x}))$ whenever $i \le j$, and consider $\mathcal{S} = \{K_1, \ldots, K_{2^n}\}$; since

$$\textstyle\sum_{i=1}^{2^n} \theta(\psi_{K_i}(\mathfrak{x})) = \theta(\phi_{\mathcal{S}}(\mathfrak{x})) = \frac{1}{2}\theta I = \frac{1}{2}\sum_{i=0}^{2^{n+1}} \theta(\psi_{K_i}(\mathfrak{x})),$$

we must have $\theta(\psi_{K_i}(\mathfrak{x})) = 2^{-n-1}\theta I$ for every i. In particular, $\theta(\psi_{\{0,\ldots,n\}}(\mathfrak{x})) = 2^{-n-1}\theta I$, that is, $\theta(\bigcap_{i\le n} a_i) = 2^{-n-1}\theta I$. As this is true whenever $a_0, \ldots, a_n \in E$, we have the result. **Q**

464R A note on $\ell^\infty(I)$ As already noted in 464F, we have, for any set I, a natural additive map $\chi : \mathcal{P}I \to \ell^\infty(I) \cong L^\infty(\mathcal{P}I)$, giving rise to an isomorphism between the L-space M of bounded additive functionals on $\mathcal{P}I$ and $\ell^\infty(I)^*$. If we write $\tilde\mu$ for the image measure $\mu\chi^{-1}$ on $\ell^\infty(I)$, where μ is Talagrand's measure on $\mathcal{P}I$, and $\tilde\Sigma$ for the domain of $\tilde\mu$, then every member of $\ell^\infty(I)^*$ is $\tilde\Sigma$-measurable, by 464N. Thus $\tilde\Sigma$ includes the cylindrical σ-algebra of $\ell^\infty(I)$ (4A3T). We also have a band decomposition $\ell^\infty(I)^* = \ell^\infty(I)^*_{\mathrm{m}} \oplus \ell^\infty(I)^*_{\mathrm{pnm}}$ corresponding to the decomposition $M = M_{\mathrm{m}} \oplus M_{\mathrm{pnm}}$ (464M).

In this context, M_τ corresponds to $\ell^\infty(I)^\times$ (363K, as before). Since we can identify $\ell^\infty(I)$ with $\ell^1(I)^* = \ell^1(I)^\times$ (243Xl), and $\ell^1(I)$, like any L-space, is perfect, $\ell^\infty(I)^\times$ is the canonical image of $\ell^1(I)$ in $\ell^\infty(I)^*$. Because any functional in $M_\tau^\perp$ is μ-almost constant (464O), any functional in $(\ell^\infty(I)^\times)^\perp$ will be $\tilde\mu$-almost constant.

464X Basic exercises (a) Let I be any set and λ a Radon measure on βI. Show that if the support of λ is a separable subset of $\beta I \setminus I$, then the corresponding additive functional on $\mathcal{P}I$ is purely non-measurable.

(b) Let I be any set, and $\tilde{\mu}$ the image of Talagrand's measure on $\ell^\infty(I)$, as in 464R. Show that $\tilde{\mu}$ has a barycenter in $\ell^\infty(I)$ iff I is finite.

464Y Further exercises (a) Show that there is a sequence $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ of distinct non-principal ultrafilters on $\mathbb{N}$ with the following property: if we define $h(a) = \{n : a \in \mathcal{F}_n\}$ for $a \subseteq \mathbb{N}$, then $\{h(a) : a \subseteq \mathbb{N}\}$ is negligible for the usual measure on $\mathcal{P}\mathbb{N}$.

(b) Let μ be Talagrand's measure on $\mathcal{P}\mathbb{N}$, and λ the corresponding product measure on $X = \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$. Define $\Phi : X \to \ell^\infty$ by setting $\Phi(a, b) = \chi a - \chi b$ for all a, $b \subseteq \mathbb{N}$. Show that Φ is Pettis integrable (463Ya) with indefinite Pettis integral Θ defined by setting $(\Theta E)(n) = \int_E f_n d\lambda$, where $f_n(a, b) = \chi a(n) - \chi b(n)$. Show that $K = \{h\Phi : h \in \ell^{\infty*}, \|h\| \le 1\}$ contains every f_n, and in particular is not $\mathfrak{T}_m$-compact, so the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is not continuous.

(c) Let I be any set. Write $\boldsymbol{c}_0(I)$ for the closed linear subspace of $\ell^\infty(I)$ consisting of those $x \in \mathbb{R}^I$ such that $\{t : t \in I, |x(t)| \ge \epsilon\}$ is finite for every $\epsilon > 0$; that is, $C_0(I)$ if I is given its discrete topology (436I). Show that, in 464R, $M_\tau^\perp$ can be identified as Banach lattice with $(\ell^\infty(I)/\boldsymbol{c}_0(I))^*$.

(d)(i) Let $\theta : \mathcal{P}\mathbb{N} \to \mathbb{R}$ be an additive functional which is T-measurable in the sense of 464I. Show that $\{\theta\{n\} : n \in \mathbb{N}\}$ is bounded. (ii) Let $\theta : \mathcal{P}\mathbb{N} \to \mathbb{R}$ be an additive functional which is universally measurable for the usual topology of $\mathcal{P}\mathbb{N}$. Show that θ is bounded. (iii) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\theta : \mathfrak{A} \to \mathbb{R}$ an additive functional which is universally measurable for the order-sequential topology on $\mathfrak{A}$ (definition: 393L). Show that θ is bounded and θ^+ is universally measurable.

(e) Show that there is a T-measurable finitely additive functional $\theta : \mathcal{P}\mathbb{N} \to \mathbb{R}$ which is not bounded.

464Z Problem Let I be an infinite set, and $\tilde{\mu}$ the image on $\ell^\infty(I)$ of Talagrand's measure (464R). Is $\tilde{\mu}$ a topological measure for the weak topology of $\ell^\infty(I)$?

464 Notes and comments The central idea of this section appears in 464B: the algebraic structure of $\mathcal{P}I$ leads to a variety of inverse-measure-preserving functions ϕ from powers $(\mathcal{P}I)^K$ to $\mathcal{P}I$. The simplest of these is the measure space automorphism $a \mapsto I \setminus a$, as used in the proofs of 464Ca, 464Ha, 464Ka and 464Kc. Then we have the map $(a, b, c) \mapsto (a \cap b) \cup (a \cap c) \cup (b \cap c)$, as in 464Bb, and the generalization of this in the argument of 464Qc; and, most important of all, the map $\langle\langle a_{ni}\rangle_{i<m(n)}\rangle_{n\in\mathbb{N}} \mapsto \bigcup_{n\in\mathbb{N}} \bigcap_{i<m(n)} a_{ni}$ of 464Ba. In each case we can use probabilistic intuitions to guide us to appropriate formulae, since the events $t \in \phi(\mathfrak{x})$ are always independent, so if they have probability $\frac{1}{2}$ for every $t \in I$, the function ϕ will be inverse-measure-preserving. Of course this depends on the analysis of product measures in §254. It means also that we must use the 'ordinary' product measure defined there; but happily this coincides with the 'Radon' product measure of §417 (416U).

Talagrand devised his measure when seeking an example of a pointwise compact set of measurable functions which is not compact for the topology of convergence in measure, as in 464E. The remarkable fact that it is already, in effect, a measure on the cylindrical σ-algebra of ℓ^∞ (464R) became apparent later, and requires a much more detailed analysis. An alternative argument not explicitly involving the Riesz space structure of the space M of bounded additive functionals may be found in Fremlin & Talagrand 79. The proof I give here depends on the surprising fact that, for non-negative additive functionals, the upper integral $\overline{\int} d\nu$ is additive (464Hb), even though the functionals may be very far from being measurable. Once we know this, we can apply the theory of Banach lattices to investigate the corresponding linear functional on the L-space M. There is a further key step in 464Hd. We there have a non-negative $\theta \in M_\tau^\perp$ such that $\theta I = 1$ and $\beta\theta' I \le \overline{\int}\theta' d\nu \le \gamma\theta' I$ whenever $0 \le \theta' \le \theta$. It is easy to deduce that $\{a : \alpha\theta b \le \theta(a \cap b) \le \gamma\theta b$ for every $b \in K\}$ has outer measure 1 for every finite $K \subseteq \mathcal{P}I$ and $\alpha < \beta$. If α and γ are very close, this means that there will be many families (a, b, c) in $\mathcal{P}I$ which, as measured by θ, look like independent sets of measure γ, so that $\theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \doteq 3\gamma^2 - 2\gamma^3$. But since we must also have $\theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \le \gamma$ almost everywhere, we can get information about the possible values of γ.

Once having noted this remarkable dichotomy between 'measurable' and 'purely non-measurable' functionals, it is natural to look for other ways in which they differ. The position seems to be that 'simple' Radon measures on $\beta I \setminus I$ (e.g., all measures with separable support (464Xa) or countable Maharam type (464Qb)) have to correspond to purely non-measurable functionals. Of course the simplest possible measures on βI are those concentrated on I, which give rise to functionals which belong to M_τ and are therefore measurable; it is functionals in $M_{\mathrm{m}} \cap M_\tau^\perp$ which have to give rise to 'complicated' Radon measures.

The fact that every element of $M_\tau^\perp$ is almost constant for Talagrand's measure leads to an interesting Pettis integrable function (464Yb). The suggestion that Talagrand's measure on ℓ^∞ might be a topological measure for the weak topology (464Z) is a bold one, but no more outrageous than the suggestion that it might measure every continuous linear functional once seemed. Talagrand's measure does of course measure every Baire set for the weak topology (4A3U). I note here that the usual measure on $\{0,1\}^I$, when transferred to $\ell^\infty(I)$, is actually a Radon measure for the weak* topology $\mathfrak{T}_s(\ell^\infty,\ell^1)$, because on $\{0,1\}^I$ this is just the ordinary topology.

465 Stable sets

The structure of general pointwise compact sets of measurable functions is complex and elusive. One particular class of such sets, however, is relatively easy to describe, and has a variety of remarkable properties, some of them relevant to important questions arising in the theory of empirical measures. In this section I outline the theory of 'stable' sets of measurable functions from Talagrand 84 and Talagrand 87.

The first steps are straightforward enough. The definition of stable set (465B) is not obvious, but given this the basic properties of stable sets listed in 465C are natural and easy to check, and we come quickly to the fact that (for complete locally determined spaces) pointwise bounded stable sets are relatively pointwise compact sets of measurable functions (465D). A less transparent, but still fairly elementary, argument leads to the next reason for looking at stable sets: the topology of pointwise convervence on a stable set is finer than the topology of convergence in measure (465G).

At this point we come to a remarkable fact: a uniformly bounded set A of functions on a complete probability space is stable if and only if certain laws of large numbers apply 'nearly uniformly' on A. These laws are expressed in conditions (ii), (iv) and (v) of 465M. For singleton sets A, they can be thought of as versions of the strong law of large numbers described in §273. To get the full strength of 465M a further idea in this direction needs to be added, described in 465H here.

The theory of stable sets applies in the first place to sets of true functions. There is however a corresponding notion applicable in function spaces, which I explore briefly in 465O-465R. Finally, I mention the idea of 'R-stable' set (465S-465U), obtained by using τ-additive product measures instead of c.l.d. product measures in the definition.

465A Notation Throughout this section, I will use the following notation.

(a) If X is a set and Σ a σ-algebra of subsets of X, I will write $\mathcal{L}^0(\Sigma)$ for the space of Σ-measurable functions from X to $\mathbb{R}$, as in §463.

(b) I will identify $\mathbb{N}$ with the set of finite ordinals, so that each $n\in\mathbb{N}$ is the set of its predecessors, and a power X^n becomes identified with the set of functions from $\{0,\dots,n-1\}$ to X.

(c) If (X,Σ,μ) is any measure space, then for finite sets I (in particular, if $I=k\in\mathbb{N}$) I write μ^I for the c.l.d. product measure on X^I, as defined in 251W. (For definiteness, let us take μ^0 to be the unique probability measure on $X^0=\{\emptyset\}$.) If (X,Σ,μ) is a probability space, then for any set I μ^I is to be the product probability measure on X^I, as defined in §254.

(d) We shall have occasion to look at free powers of algebras of sets. If X is a set and Σ is an algebra of subsets of X, then for any set I write $\bigotimes_I\Sigma$ for the algebra of subsets of X^I generated by sets of the form $\{w:w(i)\in E\}$ where $i\in I$ and $E\in\Sigma$, and $\widehat{\bigotimes}_I\Sigma$ for the σ-algebra generated by $\bigotimes_I\Sigma$.

(e) Now for a new idea, which will be used in almost every paragraph of the section. If X is a set, $A\subseteq\mathbb{R}^X$ a set of real-valued functions defined on X, $E\subseteq X$, $\alpha<\beta$ in $\mathbb{R}$ and $k\ge1$, write

$$D_k(A,E,\alpha,\beta)=\bigcup_{f\in A}\{w:w\in E^{2k},f(w(2i))\le\alpha,$$

$$f(w(2i+1))\ge\beta\text{ for }i<k\}.$$

(f) In this context it will be useful to have a special notation. If X is a set, $k\ge1$, $u\in X^k$ and $v\in X^k$, then I will write $u\#v=(u(0),v(0),u(1),v(1),\dots,u(k-1),v(k-1))\in X^{2k}$. Note that if (X,Σ,μ) is a measure space then $(u,v)\mapsto u\#v$ is an isomorphism between the c.l.d. product $(X^k,\mu^k)\times(X^k,\mu^k)$ and (X^{2k},μ^{2k}) (see 251Wh).

We are now ready for the main definition.

465B Definition Let (X,Σ,μ) be a semi-finite measure space. Following TALAGRAND 84, I say that a set $A\subseteq\mathbb{R}^X$ is **stable** if whenever $E\in\Sigma$, $0<\mu E<\infty$ and $\alpha<\beta$ in $\mathbb{R}$, there is some $k\ge 1$ such that $(\mu^{2k})^*D_k(A,E,\alpha,\beta)<(\mu E)^{2k}$.

Remark I hope that the next few results will show why this concept is important. It is worth noting at once that these sets D_k need not be measurable, and that some of the power of the definition derives precisely from the fact that quite naturally arising sets A can give rise to non-measurable sets $D_k(A,E,\alpha,\beta)$. If, however, the set A is countable, then all the corresponding D_k will be measurable; this will be important in the results following 465R.

465C I start with a list of the 'easy' properties of stable sets, derivable more or less directly from the definition.

Proposition Let (X,Σ,μ) be a semi-finite measure space.

(a) If $A\subseteq\mathbb{R}^X$ is stable, then any subset of A is stable.

(b) If $A\subseteq\mathbb{R}^X$ is stable, then $\overline{A}$, the closure of A in $\mathbb{R}^X$ for the topology of pointwise convergence, is stable.

(c) Suppose that $A\subseteq\mathbb{R}^X$, $E\in\Sigma$, $n\ge 1$ and $\alpha<\beta$ are such that $0<\mu E<\infty$ and $(\mu^{2n})^*D_n(A,E,\alpha,\beta)<(\mu E)^{2n}$. Then

$$\lim_{k\to\infty}\frac{1}{(\mu E)^{2k}}(\mu^{2k})^*D_k(A,E,\alpha,\beta)=0.$$

(d) If A, $B\subseteq\mathbb{R}^X$ are stable, then $A\cup B$ is stable.

(e) If $A\subseteq\mathbb{R}^X$ is stable, then $\gamma A=\{\gamma f:f\in A\}$ is stable, for any $\gamma\in\mathbb{R}$.

(f) If $A\subseteq\mathbb{R}^X$ is stable and $g\in\mathcal{L}^0=\mathcal{L}^0(\Sigma)$, then $A+g=\{f+g:f\in A\}$ is stable.

(g) If $A\subseteq\mathcal{L}^0$ is finite it is stable.

(h) If $A\subseteq\mathbb{R}^X$ is stable and $g\in\mathcal{L}^0$, then $A\times g=\{f\times g:f\in A\}$ is stable.

(i) If $\hat\mu$, $\tilde\mu$ are the completion and c.l.d. version of μ, then $A\subseteq\mathbb{R}^X$ is stable with respect to one of the measures μ, $\hat\mu$, $\tilde\mu$ iff it is stable with respect to the others.

(j) Let ν be an indefinite-integral measure over μ (234J[12]). If $A\subseteq\mathbb{R}^X$ is stable with respect to μ, it is stable with respect to ν and with respect to $\nu\restriction\Sigma$.

(k) Let $h:\mathbb{R}\to\mathbb{R}$ be a continuous non-decreasing function. If $A\subseteq\mathbb{R}^X$ is stable, so is $\{hf:f\in A\}$.

(l) If $A\subseteq\mathbb{R}^X$ is stable, so is $\{f^+:f\in A\}\cup\{f^-:f\in A\}$.

(m) If $A\subseteq\mathbb{R}^X$ is stable, and $Y\subseteq X$ is such that the subspace measure μ_Y is semi-finite, then $A_Y=\{f\restriction Y:f\in A\}$ is stable in $\mathbb{R}^Y$ with respect to the measure μ_Y.

(n) A set $A\subseteq\mathbb{R}^X$ is stable iff $A_E=\{f\restriction E:f\in A\}$ is stable in $\mathbb{R}^E$ with respect to the subspace measure μ_E whenever $E\in\Sigma$ has finite measure.

proof (a) This is immediate from the definition in 465B, since $D_k(B,E,\alpha,\beta)\subseteq D_k(A,E,\alpha,\beta)$ for all k, E, α, β and $B\subseteq A$.

(b) Given E such that $0<\mu E<\infty$, and $\alpha<\beta$, take α', β' such that $\alpha<\alpha'<\beta'<\beta$. Then it is easy to see that $D_k(\overline{A},E,\alpha,\beta)\subseteq D_k(A,E,\alpha',\beta')$, so

$$(\mu^{2k})^*D_k(\overline{A},E,\alpha,\beta)\le(\mu^{2k})^*D_k(A,E,\alpha',\beta')<(\mu E)^{2k}$$

for some $k\ge 1$. As E, α and β are arbitrary, $\overline{A}$ is stable.

(c) For any $m\ge 1$ and $l<n$, if we identify $X^{2(mn+l)}$ with $(X^{2n})^m\times X^{2l}$, we see that $D_{mn+l}(A,E,\alpha,\beta)$ becomes identified with a subset of $D_n(A,E,\alpha,\beta)^m\times E^{2l}$. (If $w\in D_{mn+l}(A,E,\alpha,\beta)$, there is an $f\in A$ such that $f(w(2i))\le\alpha$, $f(w(2i+1))\ge\beta$ for $i<mn+l$. Now $(w(2rn),w(2rn+1),\dots,w(2rn+2n-1))\in D_n(A,E,\alpha,\beta)$ for $r<m$.) So

$$\begin{aligned}\frac{1}{(\mu E)^{2(mn+l)}}&(\mu^{2(mn+l)})^*D_{mn+l}(A,E,\alpha,\beta)\\&\le\frac{1}{(\mu E)^{2(mn+l)}}\bigl((\mu^{2n})^*D_n(A,E,\alpha,\beta)\bigr)^m(\mu E)^{2l}\end{aligned}$$

(251Wm)

$$=\Bigl(\frac{1}{(\mu E)^{2n}}(\mu^{2n})^*D_n(A,E,\alpha,\beta)\Bigr)^m\to 0$$

[12] Formerly 234B.

as $m \to \infty$.

(d) Note that $D_k(A \cup B, E, \alpha, \beta) = D_k(A, E, \alpha, \beta) \cup D_k(B, E, \alpha, \beta)$ for all E, k, α and β. Now, given that $0 < \mu E < \infty$ and $\alpha < \beta$, there are m, $n \geq 1$ such that $(\mu^{2m})^* D_m(A, E, \alpha, \beta) < (\mu E)^{2m}$ and $(\mu^{2n})^* D_n(A, E, \alpha, \beta) < (\mu E)^{2n}$. So, by (c) above,

$$\frac{1}{(\mu E)^{2k}} (\mu^{2k})^* D_k(A \cup B, E, \alpha, \beta)$$
$$\leq \frac{1}{(\mu E)^{2k}} \big((\mu^{2k})^* D_k(A, E, \alpha, \beta) + (\mu^{2k})^* D_k(B, E, \alpha, \beta) \big) \to 0$$

as $k \to \infty$, and there is some k such that $(\mu^{2k})^* D_k(A \cup B, E, \alpha, \beta) < (\mu E)^{2k}$. As E, α and β are arbitrary, $A \cup B$ is stable.

(e)(i) If $\gamma > 0$, $D_k(\gamma A, E, \alpha, \beta) = D_k(A, E, \alpha/\gamma, \beta/\gamma)$ for all k, E, α and β, so the result is elementary. Similarly, if $\gamma = 0$, then $D_k(\gamma A, E, \alpha, \beta) = \emptyset$ whenever $k \geq 1$, $E \in \Sigma$ and $\alpha < \beta$, so again we see that γA is stable.

(ii) If $\gamma = -1$ then, for any k, E, α and β,

$$\begin{aligned} D_k(-A, E, \alpha, \beta) &= \bigcup_{f \in A} \{w : w \in E^{2k},\ f(w(2i)) \geq -\alpha, \\ &\qquad\qquad f(w(2i+1)) \leq -\beta \ \forall\, i < k\} \\ &= \phi[D_k(A, E, -\beta, -\alpha)], \end{aligned}$$

where $\phi : X^{2k} \to X^{2k}$ is the measure space automorphism defined by setting

$$\phi(w) = (w(1), w(0), w(3), w(2), \dots, w(2k-1), w(2k-2))$$

for $w \in X^{2k}$. So, given $E \in \Sigma$ and $\alpha < \beta$, there is a $k \geq 1$ such that

$$(\mu^{2k})^* D_k(-A, E, \alpha, \beta) = (\mu^{2k})^* D_k(A, E, -\beta, -\alpha) < (\mu E)^{2k}.$$

So $-A$ is stable.

(iii) Together with (i) this shows that γA is stable for every $\gamma \in \mathbb{R}$.

(f) Take E such that $0 < \mu E < \infty$, and $\alpha < \beta$. Set $\eta = \frac{1}{2}(\beta - \alpha) > 0$. Then there is a $\gamma \in \mathbb{R}$ such that $F = \{x : x \in E, \gamma \leq g(x) \leq \gamma + \eta\}$ has non-zero measure. Set $\alpha' = \alpha - \gamma$, $\beta' = \beta - \gamma - \eta$. Then $D_k(A + g, F, \alpha, \beta) \subseteq D_k(A, F, \alpha', \beta')$, while $\alpha' < \beta'$. So if we take $k \geq 1$ such that $(\mu^{2k})^* D_k(A, F, \alpha', \beta') < (\mu F)^{2k}$, then

$$(\mu^{2k})^* D_k(A + g, E, \alpha, \beta) \leq (\mu^{2k})^* D_k(A + g, F, \alpha, \beta) + \mu^{2k}(E^{2k} \setminus F^{2k}) < (\mu E)^{2k}.$$

(g) If A is empty, or contains only the constant function $\mathbf{0}$ with value 0, this is trivial. Now (f) tells us that $\{g\} = \{\mathbf{0}\} + g$ is stable for every $g \in \mathcal{L}^0$, and from (d) it follows that any finite subset of $\mathcal{L}^0$ is stable.

(h) Let $E \in \Sigma$, $\alpha < \beta$ be such that $0 < \mu E < \infty$. Set

$$E_0 = \{x : x \in E,\ g(x) = 0\}, \quad E_1 = \{x : x \in E,\ g(x) > 0\},$$

$$E_2 = \{x : x \in E,\ g(x) < 0\}.$$

(i) Suppose that $\mu E_0 > 0$. Then $D_1(A \times g, E, \alpha, \beta)$ does not meet E_0^2, so $\mu^* D_1(A \times g, E, \alpha, \beta) < \mu E$.

(ii) Suppose that $\mu E_1 > 0$. Let $\eta > 0$ be such that

$$\max(\alpha, \tfrac{\alpha}{1+\eta}) = \alpha' < \beta' = \min(\beta, \tfrac{\beta}{1+\eta}).$$

Let $\gamma > 0$ be such that $\mu F > 0$, where $F = \{x : x \in E, \gamma \leq g(x) \leq \gamma(1+\eta)\}$. If $x \in F$ then

$$f(x)g(x) \leq \alpha \Longrightarrow f(x) \leq \frac{\alpha}{g(x)} \leq \frac{\alpha'}{\gamma},$$

$$f(x)g(x) \geq \beta \Longrightarrow f(x) \geq \frac{\beta}{g(x)} \geq \frac{\beta'}{\gamma}.$$

So

$$D_k(A\times g,F,\alpha,\beta)\subseteq D_k(A,F,\frac{\alpha'}{\gamma},\frac{\beta'}{\gamma})$$

for every k. Now, because A is stable, there is some $k\ge 1$ such that

$$(\mu^{2k})^*D_k(A,F,\frac{\alpha'}{\gamma},\frac{\beta'}{\gamma})<(\mu F)^{2k},$$

and in this case $(\mu^{2k})^*D_k(A\times g,E,\alpha,\beta)<(\mu E)^{2k}$, just as in the argument for (f) above.

(iii) If $\mu E_2>0$, then we know from (e) that $-A$ is stable, so (ii) tells us that there is a $k\ge 1$ such that $(\mu^{2k})^*D_k((-A)\times(-g),E,\alpha,\beta)<(\mu E)^{2k}$.

Since one of these three cases must occur, and since E, α and β are arbitrary, $A\times g$ is stable.

(i) The product measures μ^{2k}, $\hat\mu^{2k}$ and $\tilde\mu^{2k}$ are all the same (251Wn), so this follows immediately from the definition in 465B.

(j) Let h be a Radon-Nikodým derivative of ν with respect to μ (234J). Suppose that $0<\nu E<\infty$ and $\alpha<\beta$. Then there is an $F\in\Sigma$ such that $F\subseteq E\cap\operatorname{dom}h$, $h(x)>0$ for every $x\in F$, and $0<\mu F<\infty$. There is a $k\ge 1$ such that $(\mu^{2k})^*D_k(A,F,\alpha,\beta)<(\mu F)^{2k}$, that is, there is a $W\subseteq F^{2k}\setminus D_k(A,F,\alpha,\beta)$ such that $\mu^{2k}W>0$. In this case, $\nu^{2k}W>0$. **P** Set $\tilde h(w)=\prod_{i=0}^{2k-1}h(w(i))$ for $w\in(\operatorname{dom}h)^{2k}$. Then ν^{2k} is the indefinite integral of $\tilde h$ with respect to μ^{2k} (253I, extended by induction to the product of more than two factors), and $\tilde h(w)>0$ for every $w\in F^{2k}$. **Q** Since $W\subseteq E^{2k}\setminus D_k(A,E,\alpha,\beta)$, $(\nu^{2k})^*D_k(A,E,\alpha,\beta)<(\nu E)^{2k}$; as E, α and β are arbitrary, A is stable with respect to ν.

Since ν is the completion of its restriction $\nu\restriction\Sigma$ (234Lb[13]), A is also stable with respect to $\nu\restriction\Sigma$, by (i).

(k) Write $B=\{hf:f\in A\}$. Suppose that $0<\mu E<\infty$ and $\alpha<\beta$. If either $\alpha<h(\gamma)$ for every $\gamma\in\mathbb{R}$ or $h(\gamma)<\beta$ for every $\gamma\in\mathbb{R}$,

$$\mu^*D_1(B,E,\alpha,\beta)=\mu\emptyset=0<\mu E.$$

Otherwise, because h is continuous, the Intermediate Value Theorem tells us that there are $\alpha'<\beta'$ such that $\alpha<h(\alpha')<h(\beta')<\beta$. In this case $D_k(B,E,\alpha,\beta)\subseteq D_k(A,E,\alpha',\beta')$ for every k. Because A is stable, there is some $k\ge 1$ such that $(\mu^{2k})^*D_k(A,E,\alpha',\beta')<(\mu E)^{2k}$, so that $(\mu^{2k})^*D_k(B,E,\alpha,\beta)<(\mu E)^{2k}$. As E, α and β are arbitrary, B is stable.

(l) From (k) we see that $\{f^+:f\in A\}$ is stable; now from (e) and (d) we see that $\{f^-:f\in A\}=\{f^+:f\in -A\}$ and $\{f^+:f\in A\}\cup\{f^-:f\in A\}$ are stable.

(m) Writing Σ_Y for the subspace σ-algebra, take $F\in\Sigma_Y$ such that $\mu_YF=\mu^*F$ is finite and non-zero, and $\alpha<\beta$ in $\mathbb{R}$. Let $E\in\Sigma$ be a measurable envelope of F. Then there is a $k\ge 1$ such that $(\mu^{2k})^*D_k(A,E,\alpha,\beta)<(\mu E)^{2k}$. Consider $D_k(A_Y,F,\alpha,\beta)=F^k\cap D_k(A,E,\alpha,\beta)$. The identity map from F to E is inverse-measure-preserving for the subspace measures μ_F and μ_E (214Ce), so the identity map from F^{2k} to E^{2k} is inverse-measure-preserving for the product measures μ_F^{2k} and μ_E^{2k} (apply 254H to appropriate normalizations of μ_E, μ_F). Also μ_E^{2k} is the subspace measure on E^{2k} induced by μ^{2k} (251Wl), and similarly μ_F^{2k} is the subspace measure on F^{2k} induced by μ_Y^{2k}, so

$$(\mu_Y^{2k})^*D_k(A_Y,F,\alpha,\beta)=(\mu_F^{2k})^*D_k(A_Y,F,\alpha,\beta)\le(\mu_E^{2k})^*D_k(A,E,\alpha,\beta)$$

(413Eh)

$$=(\mu^{2k})^*D_k(A,E,\alpha,\beta)<(\mu E)^{2k}=(\mu_YF)^{2k}.$$

As F, α and β are arbitrary, A_Y is stable, as claimed.

(n) If A is stable, then (m) tells us that A_E will be stable for every $E\in\Sigma$. Conversely, if A_E is stable for every E of finite measure, take $E\in\Sigma$ such that $0<\mu E<\infty$ and $\alpha<\beta$ in $\mathbb{R}$. Then there is a $k\ge 1$ such that

$$(\mu E)^{2k}=(\mu_EE)^{2k}>(\mu_E^{2k})^*D_k(A_E,E,\alpha,\beta)=(\mu^{2k})^*D_k(A,E,\alpha,\beta).$$

As E, α and β are arbitrary, A is stable.

[13]Formerly 234D.

465D Now for the first result connecting the notion of 'stable' set with the concerns of this chapter.

Proposition Let (X,Σ,μ) be a complete locally determined measure space, and $A\subseteq\mathbb{R}^X$ a stable set.

(a) $A\subseteq\mathcal{L}^0=\mathcal{L}^0(\Sigma)$ (that is, every member of A is Σ-measurable).

(b) If $\{f(x):f\in A\}$ is bounded for each $x\in X$, then A is relatively compact in $\mathcal{L}^0$ for the topology of pointwise convergence.

proof (a) ? Suppose, if possible, that there is a non-measurable $f\in A$. Then there is an $\alpha\in\mathbb{R}$ such that $D_0=\{x:f(x)>\alpha\}\notin\Sigma$. Because μ is locally determined, there is an $F_0\in\Sigma$ such that $\mu F_0<\infty$ and $D_0\cap F_0\notin\Sigma$. Let $F_1\subseteq F_0$ be a measurable envelope of $D_0\cap F_0$ (132Ee). Then $D_0\cap F_1=D_0\cap F_0$ is not measurable; because μ is complete, $F_1\setminus D_0$ cannot be negligible. Now $D_0=\bigcup_{n\in\mathbb{N}}\{x:f(x)\ge\alpha+2^{-n}\}$, so there is some $\beta>\alpha$ such that $D_1=F_1\cap\{x:f(x)\ge\beta\}$ is not negligible. Let E be a measurable envelope of D_1. Then, setting $P=\{x:x\in E,\,f(x)\le\alpha\}$, $Q=\{x:x\in E,\,f(x)\ge\beta\}$ we have $\mu^*P=\mu^*Q=\mu E>0$.

Now suppose that $k\ge1$. Then $D_k(\{f\},E,\alpha,\beta)\supseteq(P\times Q)^k$, so

$$(\mu^{2k})^*D_k(\{f\},E,\alpha,\beta)=(\mu^*P\cdot\mu^*Q)^k=(\mu E)^{2k}$$

(251Wm again). Since this is true for every k, $\{f\}$ is not stable, and (by 465Ca) A cannot be stable; which contradicts our hypothesis. **X**

(b) Because $\{f(x):x\in A\}$ is bounded for each x, $\overline{A}$, the closure of A in $\mathbb{R}^X$, is compact for the topology of pointwise convergence. But $\overline{A}$ is stable, by 465Cb, so is included in $\mathcal{L}^0$, by (a).

465E The topology $\mathfrak{T}_s(\mathcal{L}^2,\mathcal{L}^2)$ Some of the arguments below will rely on ideas of compactness in function spaces. There are of course many ways of expressing the method, but a reasonably accessible one uses the Hilbert space L^2, as follows. Let (X,Σ,μ) be any measure space. Then $L^2=L^2(\mu)$ is a Hilbert space with a corresponding weak topology $\mathfrak{T}_s(L^2,L^2)$ defined by the functionals $u\mapsto(u|v)$ for $v\in L^2$. In the present section it will be more convenient to regard this as a topology $\mathfrak{T}_s(\mathcal{L}^2,\mathcal{L}^2)$ on the space $\mathcal{L}^2=\mathcal{L}^2(\mu)$ of square-integrable real-valued functions, defined by the functionals $f\mapsto\int f\times g$ for $g\in\mathcal{L}^2$. The essential fact we need is that norm-bounded sets are relatively weakly compact. In L^2, this is because Hilbert spaces are reflexive (4A4Ka). In $\mathcal{L}^2$, given an ultrafilter $\mathcal{F}$ containing a $\|\,\|_2$-bounded set $B\subseteq\mathcal{L}^2$, $v=\lim_{f\to\mathcal{F}}f^\bullet$ must be defined in L^2 for $\mathfrak{T}_s(L^2,L^2)$, and now there is a $g\in\mathcal{L}^2$ such that $g^\bullet=v$; in which case

$$\lim_{f\to\mathcal{F}}\int f\times h=\lim_{f\to\mathcal{F}}(f^\bullet|h^\bullet)=(g^\bullet|h^\bullet)=\int g\times h$$

for every $h\in\mathcal{L}^2$. Note that we are free to take g to be a Σ-measurable function with domain X (241Bk).

465F Lemma Let (X,Σ,μ) be a measure space, and $B\subseteq\mathcal{L}^2=\mathcal{L}^2(\mu)$ a $\|\,\|_2$-bounded set. Suppose that $h\in\mathcal{L}^2$ belongs to the closure of B for $\mathfrak{T}_s(\mathcal{L}^2,\mathcal{L}^2)$. Then for any $\delta>0$ and $k\ge1$ the set

$$W=\bigcup_{f\in B}\{w:w\in X^k,w(i)\in\operatorname{dom}f\cap\operatorname{dom}h$$
$$\text{and }f(w(i))\ge h(w(i))-\delta\text{ for every }i<k\}$$

is μ^k-conegligible in X^k.

proof (a) Since completing the measure μ does not change the space $\mathcal{L}^2$ (244Xa) nor the product measure μ^k (251Wn), we may suppose that μ is complete.

(b) The first substantive fact to note is that there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in B converging to h for $\mathfrak{T}_s=\mathfrak{T}_s(\mathcal{L}^2,\mathcal{L}^2)$. **P** Setting $C=\{f^\bullet:f\in B\}$, C is a bounded set in $L^2=L^2(\mu)$ and $h^\bullet$ belongs to the $\mathfrak{T}_s(L^2,L^2)$-closure of C. But L^2, being a normed space, is angelic in its weak topology (462D), and C is relatively compact in L^2, so there is a sequence in C converging to $h^\bullet$. We can represent this sequence as $\langle f_n^\bullet\rangle_{n\in\mathbb{N}}$ where $f_n\in B$ for every n, and now $\langle f_n\rangle_{n\in\mathbb{N}}\to h$ for $\mathfrak{T}_s$. **Q**

(c) The second component of the proof is the following simple idea. Suppose that $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ such that $\bigcap_{n\in I}E_n$ is negligible for every infinite set $I\subseteq\mathbb{N}$. For $m\ge1$ and $I\subseteq\mathbb{N}$ set

$$V_m(I)=\bigcap_{n\in I}\{w:w\in X^m,\,\exists\,i<m,\,w(i)\in E_n\}.$$

Then $\mu^mV_m(I)=0$ for every infinite $I\subseteq\mathbb{N}$. **P** Induce on m. For $m=1$ this is just the original hypothesis on $\langle E_n\rangle_{n\in\mathbb{N}}$. For the inductive step to $m+1$, identify μ^{m+1} with the product of μ^m and μ, and observe that

$$V_{m+1}(I)^{-1}[\{x\}] = \{w : (w,x) \in V_{m+1}(I)\} = V_m(\{n : n \in I,\, x \notin E_n\})$$

for every $x \in X$. Now, setting $I_x = \{n : n \in I,\, x \notin E_n\}$, $F = \{x : I_x \text{ is finite}\}$, we have

$$F = \bigcup_{r\in\mathbb{N}} \bigcap_{n\in I\setminus r} E_n,$$

so F is negligible, while if $x \notin F$ then $V_m(I_x)$ is negligible, by the inductive hypothesis. But this means that almost every horizontal section of $V_{m+1}(I)$ is negligible, and $V_{m+1}(I)$, being measurable, must be negligible, by Fubini's theorem (252F). Thus the induction proceeds. **Q**

(d) Now let us return to the main line of the argument from (b). For each $n \in \mathbb{N}$, set $E_n = \{x : x \in \operatorname{dom} f_n \cap \operatorname{dom} h,\, f_n(x) < h(x) - \delta\}$. If $I \subseteq \mathbb{N}$ is infinite, then $\bigcap_{n\in I} E_n$ is negligible. **P** Setting $G = \bigcap_{n\in I} E_n$, μG is finite (because $\delta\chi E_n \le_{\text{a.e.}} |h - f_n|$, so $\mu E_n < \infty$ for every n) and $\int_G f_n \le \int_G h - \delta\mu G$ for every $n \in I$. But $\lim_{n\to\infty} \int_G f_n = \int_G h$ and I is infinite, so

$$\delta\mu G \le \inf_{n\in I} |\int_G h - \int_G f_n| = 0. \text{ **Q**}$$

By (c), it follows that

$$V = \bigcap_{n\in\mathbb{N}} \{w : w \in X^k,\, \exists\, i < k,\, w(i) \in E_n\}$$

is negligible. But if we set $Y = \bigcap_{n\in\mathbb{N}} \operatorname{dom} f_n \cap \operatorname{dom} h$, Y is a conegligible subset of X, Y^k is a conegligible subset of X^k, and $Y^k \setminus V \subseteq W$, so W is conegligible, as required.

465G Theorem Let (X, Σ, μ) be a semi-finite measure space, and $A \subseteq \mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ a stable set of measurable functions. Let $\mathfrak{T}_p$ and $\mathfrak{T}_m$ be the topologies of pointwise convergence and convergence in measure, as in §463. Then the identity map from A to itself is $(\mathfrak{T}_p, \mathfrak{T}_m)$-continuous.

proof ? Suppose, if possible, otherwise.

(a) We must have an $f_0 \in A$, a set $F \in \Sigma$ of finite measure, and an $\epsilon > 0$ such that for every $\mathfrak{T}_p$-neighbourhood U of f_0 there is an $f \in A \cap U$ such that $\int \chi F \wedge |f - f_0| d\mu \ge 2\epsilon$. Set

$$B = \{\chi F \wedge (f - f_0)^+ : f \in A\} \cup \{\chi F \wedge (f - f_0)^- : f \in A\}.$$

Then

$$B = \{\chi F - (\chi F - (f - f_0)^+)^+ : f \in A\} \cup \{\chi F - (\chi F - (f - f_0)^-)^+ : f \in A\}$$

is stable, by 465Cf, 465Cl and 465Ce, used repeatedly. Setting $B' = \{f : f \in B,\, \int f \ge \epsilon\}$, B' is again stable (465Ca). Our hypothesis is that f_0 is in the $\mathfrak{T}_p$-closure of

$$\{f : f \in A,\, \int \chi F \wedge |f - f_0| \ge 2\epsilon\} \subseteq \{f : f \in A,\, \int \chi F \wedge (f - f_0)^+ \ge \epsilon\}$$
$$\cup \{f : f \in A,\, \int \chi F \wedge (f - f_0)^- \ge \epsilon\};$$

since $f \mapsto \chi F \wedge (f - f_0)^+$ and $f \mapsto \chi F \wedge (f - f_0)^-$ are $\mathfrak{T}_p$-continuous, 0 belongs to the $\mathfrak{T}_p$-closure of B'.

(b) Let $\mathcal{F}$ be an ultrafilter on B' which $\mathfrak{T}_p$-converges to 0. Because B' is $\| \|_2$-bounded (since $0 \le f \le \chi F$ for every $f \in B'$), $\mathcal{F}$ also has a $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$-limit h say, as noted in 465E; and we can suppose that h is measurable and defined everywhere. We must have

$$\int_F h = \lim_{f\to\mathcal{F}} \int_F f \ge \epsilon > 0.$$

So there is a $\delta > 0$ such that $E = \{x : x \in F,\, h(x) \ge 3\delta\}$ has measure greater than 0.

(c) Because B' is stable, there must be some $k \ge 1$ such that $(\mu^{2k})^* D_k(B', E, \delta, 2\delta) < (\mu E)^{2k}$. Let $W \subseteq E^{2k} \setminus D_k(B', E, \delta, 2\delta)$ be a measurable set of positive measure. By Fubini's theorem, there must be a $u \in X^k$ such that $\mu^k V$ is defined and greater than 0, where

$$V = \{v : v \in X^k,\, u\#v \in W\}.$$

Set $C = \{f : f \in B',\, f(u(i)) \le \delta \text{ for every } i < k\}$; then $C \in \mathcal{F}$, because $\mathcal{F} \to 0$ for $\mathfrak{T}_p$. Accordingly h belongs to the $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$-closure of C. But now 465F tells us that there must be a $v \in V$ and an $f \in C$ such that $f(v(i)) \ge h(v(i)) - \delta$ for every $i < k$.

Consider $w = u\#v$. We know that $w \in W$ (because $v \in V$), so, in particular, $w \in E^{2k}$ and $h(w(i)) \geq 3\delta$ for every $i < 2k$; accordingly

$$f(w(2i+1)) = f(v(i)) \geq h(v(i)) - \delta \geq 2\delta$$

for every $i < k$. On the other hand, $f(w(2i)) = f(u(i)) \leq \delta$ for every $i < k$, because $f \in C$. But this means that f witnesses that $w \in D_k(B', E, \delta, 2\delta)$, which is supposed to be disjoint from W. **X**

This contradiction shows that the theorem is true.

465H We shall need some interesting and important general facts concerning powers of measures. I start with an important elaboration of the strong law of large numbers.

Theorem Let (X, Σ, μ) be any probability space. For $n \in \mathbb{N}$, write Λ_n for the domain of the product measure μ^n. For $w \in X^{\mathbb{N}}$, $k \geq 1$, $n \geq 1$ write ν_{wk} for the probability measure with domain $\mathcal{P}X$ defined by writing

$$\nu_{wk}(E) = \frac{1}{k}\#(\{i : i < k,\, w(i) \in E\})$$

for $E \subseteq X$, and ν_{wk}^n for the corresponding product measure on X^n.

Then whenever $n \geq 1$ and $f : X^n \to \mathbb{R}$ is bounded and Λ_n-measurable, $\lim_{k\to\infty} \int f d\nu_{wk}^n$ exists, and is equal to $\int f d\mu^n$, for $\mu^{\mathbb{N}}$-almost every $w \in X^{\mathbb{N}}$.

proof (a) For $k \geq n$ and $w \in X^{\mathbb{N}}$, set

$$h_k(w) = \tfrac{(k-n)!}{k!}\sum_{\pi:n\to k \text{ is injective}} f(w\pi).$$

Setting $M = \sup_{v\in X^n} |f(v)|$, we have

$$\begin{aligned}\bigl|h_k(w) - \int f d\nu_{wk}^n\bigr| &= \bigl|\tfrac{(k-n)!}{k!} \sum_{\pi:n\to k \text{ is injective}} f(w\pi) - \tfrac{1}{k^n} \sum_{\pi:n\to k} f(w\pi)\bigr| \\ &\leq M\bigl(\tfrac{(k-n)!}{k!} - \tfrac{1}{k^n}\bigr) + \tfrac{M}{k^n}\bigl(k^n - \tfrac{k!}{(k-n)!}\bigr) \to 0\end{aligned}$$

as $k \to \infty$, for every $w \in X^{\mathbb{N}}$.

(b) Write Λ for the domain of $\mu^{\mathbb{N}}$, and for $k \geq n$ set

$$\mathrm{T}_k = \{W : W \in \Lambda,\, w\pi \in W \text{ whenever } w \in W \text{ and } \pi \in S_k\},$$

where S_k is the set of permutations $\psi : \mathbb{N} \to \mathbb{N}$ such that $\psi(i) = i$ for $i \geq k$. Then $\langle \mathrm{T}_k\rangle_{k\geq n}$ is a non-increasing sequence of σ-subalgebras of Λ. For any injective function $\pi : n \to k$ there are just $(k-n)!$ extensions of π to a member of S_k. So

$$h_k(w) = \tfrac{1}{k!}\sum_{\pi\in S_k} f(w\pi\restriction n)$$

for every w. Observe that $h_k(w\psi) = h_k(w)$ for every $\psi \in S_k$, $w \in X^{\mathbb{N}}$, so h_k is T_k-measurable. (Of course we need to look back at the definition of h_k to confirm that it is Λ-measurable.)

(c) For any $k \geq n$, h_k is a conditional expectation of h_n on T_k. **P** If $W \in \mathrm{T}_k$, then

$$\int_W h_k(w)dw = \tfrac{1}{k!}\sum_{\pi\in S_k} \int_W f(w\pi\restriction n)dw = \tfrac{1}{k!}\sum_{\pi\in S_k} \int_W g(w\pi)dw$$

where $g(w) = f(w\restriction n)$ for $w \in X^{\mathbb{N}}$. Now observe that for every $\pi \in S_k$ the map $w \mapsto w\pi$ is a measure space automorphism of $X^{\mathbb{N}}$ which leaves W unchanged, because $W \in \mathrm{T}_k$; so that $\int_W g(w\pi)dw = \int_W g(w)dw$, by 235Gc. So $\int_W h_k = \int_W g$. But (since $W \in \mathrm{T}_k \subseteq \mathrm{T}_n$) $\int_W h_n$ is also equal to $\int_W g$, and $\int_W h_k = \int_W h_n$. As W is arbitrary, h_k is a conditional expectation of h_n on T_k. **Q**

(d) By the reverse martingale theorem (275K), $h_\infty(w) = \lim_{k\to\infty} h_k(w)$ is defined for almost every $w \in X^{\mathbb{N}}$. Accordingly $\lim_{k\to\infty} \int f d\nu_{wk}^n$ is defined for almost every w.

(e) To see that the limit is $\int f d\mu^n$, observe that if $W \in \mathrm{T}_\infty = \bigcap_{k\in\mathbb{N}} \mathrm{T}_k$ then $\mu^{\mathbb{N}}W$ must be either 0 or 1. **P** Set $\gamma = \mu^{\mathbb{N}}W$. Let $\epsilon > 0$. Then there is a $V \in \bigotimes_{\mathbb{N}} \Sigma$ (notation: 465Ad) such that $\mu^{\mathbb{N}}(W \triangle V) \leq \epsilon$ (254Fe). There is some k such that V is determined by coordinates in k. If we set $\pi(i) = 2k - i$ for $i < 2k$, i for $i \geq 2k$, then

$V' = \{w\pi : w \in V\}$ is determined by coordinates in $2k \setminus k$, so $\mu^{\mathbb{N}}(V \cap V') = (\mu^{\mathbb{N}} V)^2$. On the other hand, because $W \in \mathrm{T}_{2k}$, the measure space automorphism $w \mapsto w\pi$ does not move W, and $\mu^{\mathbb{N}}(W \setminus V') = \mu^{\mathbb{N}}(W \setminus V)$. Accordingly

$$\gamma = \mu^{\mathbb{N}}(W \cap W) \le \mu^{\mathbb{N}}(V \cap V') + 2\mu^{\mathbb{N}}(W \setminus V) \le (\mu^{\mathbb{N}} V)^2 + 2\epsilon \le (\gamma + \epsilon)^2 + 2\epsilon.$$

As ϵ is arbitrary, $\gamma \le \gamma^2$ and $\gamma \in \{0, 1\}$. **Q**

(e) Now 275K tells us that h_∞ is T_∞-measurable, therefore essentially constant, and must be equal to its expectation almost everywhere. But, setting $W = X^{\mathbb{N}}$ in the proof of (c), we see that $\int h_k = \int f(w \restriction n) dw = \int f d\mu^n$ for every k, so

$$\lim_{k \to \infty} \int f d\nu^n_{wk} = h_\infty(w) = \int f d\mu^n$$

for almost every w, as claimed.

465I Now for a string of lemmas, working towards the portmanteau Theorem 465M. The first is elementary.

Lemma Let X be a set, and Σ a σ-algebra of subsets of X. For $w \in X^{\mathbb{N}}$ and $k \ge 1$, write ν_{wk} for the probability measure with domain $\mathcal{P}X$ defined by writing

$$\nu_{wk}(E) = \frac{1}{k}\#(\{i : i < k,\, w(i) \in E\})$$

for $E \subseteq X$. Then for any $k \in \mathbb{N}$ and any set I, $w \mapsto \nu^I_{wk}(W)$ is $\widehat{\bigotimes}_{\mathbb{N}}\Sigma$-measurable (notation: 465Ad) for every $W \in \widehat{\bigotimes}_I \Sigma$.

proof Write $\mathcal{W}$ for the set of subsets W of X^I such that $w \mapsto \nu^I_{wk}(W)$ is $\widehat{\bigotimes}_{\mathbb{N}}\Sigma$-measurable. Then $X^I \in \mathcal{W}$, $W' \setminus W \in \mathcal{W}$ whenever W, $W' \in \mathcal{W}$ and $W \subseteq W'$, and $\bigcup_{n \in \mathbb{N}} W_n \in \mathcal{W}$ whenever $\langle W_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}$. Write $\mathcal{V}$ for the set of Σ-cylinders in X^I, that is, sets expressible in the form $\{v : v(i) \in E_i$ for every $i \in J\}$, where $J \subseteq I$ is finite and $E_i \in \Sigma$ for $i \in J$. Then $\mathcal{V} \subseteq \mathcal{W}$. **P** If $J \subseteq I$ is finite and $E_i \in \Sigma$ for $i \in J$, then

$$w \mapsto \nu_{wk} E_i = \frac{1}{k}\sum_{j=0}^{k-1} \chi E_i(w(j))$$

is $\widehat{\bigotimes}_{\mathbb{N}}$-measurable for every $i \in J$. So

$$w \mapsto \nu^I_{wk}\{v : v(i) \in E_i \ \forall\, i \in J\} = \prod_{i \in J} \nu_{wk} E_i$$

is also measurable. **Q**

Because $V \cap V' \in \mathcal{V}$ for all V, $V' \in \mathcal{V}$, the Monotone Class Theorem (136B) tells us that $\mathcal{W}$ must include the σ-algebra generated by $\mathcal{V}$, which is $\widehat{\bigotimes}_I \Sigma$.

465J The next three lemmas are specifically adapted to the study of stable sets of functions.

Lemma Let (X, Σ, μ) be a probability space. For any $n \in \mathbb{N}$ and $W \subseteq X^n$ I say that W is **symmetric** if $w\pi \in W$ whenever $w \in W$ and $\pi : n \to n$ is a permutation. Give each power X^n its product measure μ^n.

(a) Suppose that for each $n \ge 1$ we have a measurable set $W_n \subseteq X^n$, and that $W_{m+n} \subseteq W_m \times W_n$ for all m, $n \ge 1$, identifying X^{m+n} with $X^m \times X^n$. Then $\lim_{n \to \infty}(\mu^n W_n)^{1/n}$ is defined and equal to $\delta = \inf_{n \ge 1}(\mu^n W_n)^{1/n}$.

(b) Now suppose that each W_n is symmetric. Then there is an $E \in \Sigma$ such that $\mu E = \delta$ and $E^n \setminus W_n$ is negligible for every $n \in \mathbb{N}$.

(c) Next, let $\langle D_n \rangle_{n \ge 1}$ be a sequence of sets such that

$D_n \subseteq X^n$ is symmetric for every $n \ge 1$,

whenever $1 \le m \le n$, $v \in D_n$ then $v \restriction m \in D_m$.

Then $\delta = \lim_{n \to \infty}((\mu^n)^* D_n)^{1/n}$ is defined and there is an $E \in \Sigma$ such that $\mu E = \delta$ and $(\mu^n)^*(D_n \cap E^n) = (\mu E)^n$ for every $n \in \mathbb{N}$.

proof (a) For any $\eta > 0$, there is an $m \ge 1$ such that $\mu^m W_m \le (\delta + \eta)^m$. If $n = mk + i$, where $k \ge 1$ and $i < m$, then (identifying X^n with $(X^m)^k \times X^i$) $W_n \subseteq (W_m)^k \times X^i$, so

$$\mu^n W_n \le (\delta + \eta)^{mk} \le \gamma(\delta + \eta)^{mk+i} = \gamma(\delta + \eta)^n,$$

where

$$\gamma=\max_{i<m}(\tfrac{1}{\delta+\eta})^i.$$

So

$$\limsup_{n\to\infty}(\mu^nW_n)^{1/n}\le(\delta+\eta)\limsup_{n\to\infty}\gamma^{1/n}=\delta+\eta.$$

As η is arbitrary,

$$\delta\le\liminf_{n\to\infty}(\mu^nW_n)^{1/n}\le\limsup_{n\to\infty}(\mu^nW_n)^{1/n}\le\delta,$$

and $\lim_{n\to\infty}(\mu^nW_n)^{1/n}=\delta$.

(b) It is enough to consider the case $\delta>0$.

(i) Consider the family $\mathbb{V}$ of sequences $\langle V_n\rangle_{n\ge1}$ such that

for each $n\ge1$, V_n is a symmetric measurable subset of X^n and $\mu^nV_n\ge\delta^n$,

if $1\le m\le n$ then $v\restriction m\in V_m$ for every $v\in V_n$.

Observe that $\boldsymbol{W}=\langle W_n\rangle_{n\ge1}\in\mathbb{V}$. Order $\mathbb{V}$ by saying that $\langle V_n\rangle_{n\ge1}\le\langle V'_n\rangle_{n\ge1}$ if $V_n\subseteq V'_n$ for every n. For $\boldsymbol{V}=\langle V_n\rangle_{n\ge1}$ in $\mathbb{V}$, set $\theta(\boldsymbol{V})=\sum_{n=1}^\infty2^{-n}\mu^nV_n$. Any non-increasing sequence $\langle\langle V_{kn}\rangle_{n\ge1}\rangle_{k\in\mathbb{N}}$ in $\mathbb{V}$ has a lower bound $\langle\bigcap_{k\in\mathbb{N}}V_{kn}\rangle_{n\ge1}$ in $\mathbb{V}$, so there must be a $\boldsymbol{W'}=\langle W'_n\rangle_{n\ge1}\in\mathbb{V}$ such that $\boldsymbol{W'}\le\boldsymbol{W}$ and $\theta(\boldsymbol{V})=\theta(\boldsymbol{W'})$ whenever $\boldsymbol{V}\in\mathbb{V}$ and $\boldsymbol{V}\le\boldsymbol{W'}$; that is, whenever $\langle V_n\rangle_{n\ge1}\in\mathbb{V}$ and $V_n\subseteq W'_n$ for every $n\ge1$, then $\mu^nV_n=\mu^nW'_n$ for every n.

(ii) For $x\in X$, $n\ge1$ set $V^{(x)}_n=\{w:(x,w)\in W'_{n+1}\}$. Then $V^{(x)}_n$ is measurable for almost every x; let $X_1\subseteq X$ be a conegligible set such that $V^{(x)}_n$ is measurable for every $x\in X_1$ and every $n\ge1$. Every $V^{(x)}_n$ is symmetric, and if $1\le m\le n$ and $v\in V^{(x)}_n$ then $v\restriction m\in V^{(x)}_m$. It follows that if m, $n\ge1$ then $V^{(x)}_{m+n}$ becomes identified with a subset of $V^{(x)}_m\times V^{(x)}_n$.

From (a) we see that $\delta_x=\lim_{n\to\infty}(\mu^nV^{(x)}_n)^{1/n}$ is defined for every $x\in X_1$. The map $x\mapsto\mu^nV^{(x)}_n:X_1\to[0,1]$ is measurable for each n, by Fubini's theorem (252D), so $x\mapsto\delta_x$ is also measurable. Since $V^{(x)}_n\subseteq W'_n\subseteq W_n$ for every x and n, $\delta_x\le\delta$ for every $x\in X_1$.

(iii) Set $E=\{x:x\in X_1,\,\delta_x=\delta\}$. Then $\mu E\ge\delta$. **P?** Otherwise, there is some $\beta<\delta$ such that $\mu F<\beta$, where $F=\{x:x\in X_1,\,\delta_x\ge\beta\}$. Now

$$\begin{aligned}X_1\setminus F&=\{x:x\in X_1,\,\lim_{n\to\infty}(\mu^nV^{(x)}_n)^{1/n}<\beta\}\\&\subseteq\bigcup_{m\in\mathbb{N}}\bigcap_{n\ge m}\{x:x\in X_1,\,\mu^nV^{(x)}_n\le\beta^n\},\end{aligned}$$

so there is some $m\in\mathbb{N}$ such that $\mu H\ge1-\beta$, where

$$H=\bigcap_{n\ge m}\{x:x\in X_1,\,\mu^nV^{(x)}_n\le\beta^n\}.$$

Set $\gamma_n=\mu^nW'_n\ge\delta^n$ for each n. Then, for any $n\ge m$,

$$\begin{aligned}\gamma_{n+1}=\mu^{n+1}W'_{n+1}&=\int_{X_1}\mu^nV^{(x)}_n\mu(dx)\\&=\int_H\mu^nV^{(x)}_n\mu(dx)+\int_{X_1\setminus H}\mu^nV^{(x)}_n\mu(dx)\le\beta^n+\beta\gamma_n\end{aligned}$$

because $V^{(x)}_n\subseteq W'_n$ for every x, and $\mu(X_1\setminus H)\le\beta$. An easy induction shows that $\gamma_{m+k}\le k\beta^{m+k-1}+\beta^k\gamma_m$ for every $k\in\mathbb{N}$. But this means that

$$\delta^k=\delta^{-m}\delta^{m+k}\le\delta^{-m}\gamma_{m+k}\le\beta^k\delta^{-m}(k\beta^{m-1}+\gamma_m)$$

for every k; setting $\eta=(\delta-\beta)/\beta>0$,

$$\tfrac{k(k-1)}{2}\eta\le(1+\eta)^k=\bigl(\tfrac{\delta}{\beta}\bigr)^k\le\delta^{-m}(k\beta^{m-1}+\gamma_m)$$

for every k, which is impossible. **XQ**

(iv) Next, for any $x\in E$, $\boldsymbol{V}^{(x)}=\langle V^{(x)}_n\rangle_{n\ge1}\in\mathbb{V}$ and $\boldsymbol{V}^{(x)}\le\boldsymbol{W'}$, so $\mu^n(W'_n\setminus V^{(x)}_n)=0$ for every $n\ge1$. This means that, given $n\ge1$, every vertical section of $(E\times W'_n)\setminus W'_{n+1}$ (regarded as a subset of $X\times X^n$) is negligible; so $(E\times W'_n)\setminus W'_{n+1}$ is negligible. We are assuming that $\delta>0$, so

$$E \subseteq \{x : x \in X_1,\, V_1^{(x)} \neq \emptyset\} \subseteq \{x : \{w : (x, w) \in W_2'\} \neq \emptyset\} \subseteq W_1'.$$

Now a simple induction shows that $E^n \setminus W_n'$ is negligible for every $n \geq 1$, so that $E^n \setminus W_n$ is negligible for every n, and we have an appropriate E. (Of course $\mu E = \delta$ exactly, because $(\mu E)^n \leq \mu W_n$ for every n.)

(c) For each $n \in \mathbb{N}$ let V_n be a measurable envelope of D_n in X^n. Define $\langle W_n \rangle_{n \geq 1}$ inductively by saying

$$W_1 = V_1,$$

$$W_{n+1} = \{w : w \in X^{n+1}, w\pi{\restriction}n \in W_n,$$
$$w\pi \in V_{n+1} \text{ for every permutation } \pi : n+1 \to n+1\}$$

for each $n \geq 1$. Then an easy induction on n shows that W_n is measurable and symmetric and that $D_n \subseteq W_n \subseteq V_n$, so that W_n is a measurable envelope of D_n and $\mu^n W_n = (\mu^n)^* D_n$.

Now $\langle W_n \rangle_{n \geq 1}$ satisfies the hypotheses of (b), so

$$\delta = \lim_{n\to\infty} (\mu^n W_n)^{1/n} = \lim_{n\to\infty} ((\mu^n)^* D_n)^{1/n}$$

is defined and there is a set $E \in \Sigma$, of measure δ, such that $E^n \setminus W_n$ is negligible for every n; but this means that

$$(\mu^n)^*(E^n \cap D_n) = \mu^n(E^n \cap W_n) = \delta^n$$

for every n, as required.

465K Lemma Let (X, Σ, μ) be a complete probability space, and $A \subseteq [0,1]^X$ a stable set. Suppose that $\epsilon > 0$ is such that $\int f d\mu \leq \epsilon^2$ for every $f \in A$. Then there are an $n \geq 1$ and a $W \in \widehat{\bigotimes}_n \Sigma$ (notation: 465Ad) and a $\gamma > \mu^n W$ such that $\int f d\nu \leq 3\epsilon$ whenever $f \in A$ and ν is a probability measure on X with domain including Σ such that $\nu^n W \leq \gamma$.

proof (a) For $n \in \mathbb{N}$ write $\tilde{C}_n = \bigcup_{f \in A} \{x : f(x) \geq \epsilon\}^n$. Then $\langle \tilde{C}_n \rangle_{n \in \mathbb{N}}$ satisfies the conditions of 465Jc, so $\delta = \lim_{n\to\infty} ((\mu^n)^* \tilde{C}_n)^{1/n}$ is defined, and there is an $E \in \Sigma$ such that $\mu E = \delta$ and $(\mu^n)^*(E^n \cap \tilde{C}_n) = \delta^n$ for every $n \in \mathbb{N}$. Now, for $B \subseteq [0,1]^X$ and $n \in \mathbb{N}$, write $C_n(B) = \bigcup_{f \in B} \{x : x \in E,\, f(x) \geq \epsilon\}^n$, so that $C_n(A) = E^n \cap \tilde{C}_n$, and $(\mu^n)^* C_n(A) = \delta^n$ for every n. For any $B \subseteq [0,1]^X$, $\langle C_n(B) \rangle_{n \in \mathbb{N}}$ also satisfies the conditions of 465Jc, and $\delta_B = \lim_{n\to\infty} ((\mu^n)^* C_n(B))^{1/n}$ is defined; we have $\delta_A = \delta$.

(b) If B, $B' \subseteq [0,1]^X$, then

$$(\mu^n)^* C_n(B) \leq (\mu^n)^* C_n(B \cup B') \leq (\mu^n)^* C_n(B) + (\mu^n)^* C_n(B')$$

for every n, so $\delta_B \leq \delta_{B \cup B'} \leq \max(\delta_B, \delta_{B'})$. It follows that if

$$\mathcal{G} = \{G : G \subseteq [0,1]^X \text{ is } \mathfrak{T}_p\text{-open},\, \delta_{G \cap A} < \delta\},$$

where $\mathfrak{T}_p$ is the usual topology of $[0,1]^X$, no finite subfamily of $\mathcal{G}$ can cover A. Accordingly, since the $\mathfrak{T}_p$-closure $\overline{A}$ of A is $\mathfrak{T}_p$-compact, there is an $h \in \overline{A}$ such that $\delta_{G \cap A} = \delta$ for every $\mathfrak{T}_p$-open set G containing h.

(c) At this point recall that every function in $\overline{A}$ is measurable (465Cb, 465Da) and that $\int : \overline{A} \to [0,1]$ is $\mathfrak{T}_p$-continuous (465G). So $\int h \leq \epsilon^2$ and $\mu\{x : h(x) \geq \epsilon\} \leq \epsilon$.

? Suppose, if possible, that $\delta > \epsilon$. Then there is some $\eta > 0$ such that $\mu F > 0$, where $F = \{x : x \in E,\, h(x) < \epsilon - \eta\}$. For $k \in \mathbb{N}$, $u \in F^k$ set $G_u = \{f : f \in [0,1]^X,\, f(u(i)) < \epsilon - \eta \text{ for every } i < k\}$. Then G_u is an open neighbourhood of h, so $\delta_{G_u \cap A} = \delta$ and $(\mu^k)^*(C_k(G_u \cap A)) \geq \delta^k$. But because $F \subseteq E$ and $C_k(G_u \cap A) \subseteq E^k$, this means that $(\mu^k)^*(F^k \cap C_k(G_u \cap A)) = (\mu F)^k$.

In the notation of 465Af,

$$C_k(G_u \cap A) \cap F^k \subseteq \{v : u\#v \in D_k(A, F, \epsilon - \eta, \epsilon)\}$$

for any $u \in F^k$. So $(\mu^k)^*\{v : u\#v \in D_k(A, F, \epsilon - \eta, \epsilon)\} = (\mu F)^k$ for every $u \in F^k$. But this means that $(\mu^{2k})^* D_k(A, F, \epsilon - \eta, \epsilon) = (\mu F)^{2k}$. Since this is so for every $k \geq 1$, A is not stable. **X**

(d) Thus $\delta \leq \epsilon$. There is therefore some $n \geq 1$ such that $(\mu^n)^* \tilde{C}_n < (2\epsilon)^n$. Let $W \in \widehat{\bigotimes}_n \Sigma$ be a measurable envelope of $\tilde{C}_n$, and try $\gamma = (2\epsilon)^n$. If ν is any probability measure on X with domain including Σ such that $\nu^n W \leq \gamma$, then for any $f \in A$ we have

$$\{x : f(x) \geq \epsilon\}^n \subseteq \tilde{C}_n \subseteq W, \quad (\nu\{x : f(x) \geq \epsilon\})^n \leq \nu^n W \leq (2\epsilon)^n,$$

so that $\nu\{x : f(x) \geq \epsilon\} \leq 2\epsilon$. As $0 \leq f(x) \leq 1$ for every $x \in X$, $\int f d\nu \leq 3\epsilon$, as required.

465L Lemma (Talagrand 87) Let (X,Σ,μ) be a complete probability space, and $A\subseteq[0,1]^X$ a set which is not stable. Then there are measurable functions h_0, $h_1:X\to[0,1]$ such that $\int h_0\,d\mu<\int h_1\,d\mu$ and $(\mu^{2k})^*\tilde D_k=1$ for every $k\ge1$, where

$$\tilde D_k=\bigcup_{f\in A}\{w:w\in X^{2k},\,f(w(2i))\le h_0(w(2i)),$$
$$f(w(2i+1))\ge h_1(w(2i+1))\text{ for every }i<k\}. \qquad (*)$$

proof The proof divides into two cases.

case 1 Suppose that there is an ultrafilter $\mathcal{F}$ on A such that the $\mathfrak{T}_p$-limit g_0 of $\mathcal{F}$ is not measurable. Let h_0', $h_1':X\to[0,1]$ be measurable functions such that $h_0'\le g_0\le h_1'$ and $\int h_0'=\underline{\int}g_0$, $\int h_1'=\overline{\int}g_0$ (133Ja). Then $\delta=\frac15\int h_1'-h_0'>0$ (133Jd). Set $h_0=h_0'+2\delta\chi X$, $h_1=h_1'-2\delta\chi X$, so that $\int h_0<\int h_1$.

Set $Q_0=\{x:x\in X,\,g_0(x)\le h_0(x)-\delta\}$. Then $\mu^*Q_0=1$, by 133J(a-i). Similarly, $\mu^*Q_1=1$, where $Q_1=\{x:g_0(x)\ge h_1(x)+\delta\}$.

If $k\ge1$, $u\in Q_0^k$ and $v\in Q_1^k$, then there is an $f\in A$ such that $|f(u(i))-g_0(u(i))|\le\delta$, $|f(v(i))-g_0(v(i))|\le\delta$ for every $i<k$. But this means that $f(u(i))\le h_0(u(i))$ and $f(v(i))\ge h_1(v(i))$ for every $i<k$, and $u\#v\in\tilde D_k$. Thus

$$\tilde D_k\supseteq Q_0\times Q_1\times Q_0\times Q_1\times\ldots\times Q_0\times Q_1,$$

which has full outer measure, by 254L. As k is arbitrary, we have found appropriate h_0, h_1 in this case.

case 2 Now suppose that for every ultrafilter $\mathcal{F}$ on A, the $\mathfrak{T}_p$-limit of $\mathcal{F}$ is measurable.

(i) We are supposing that A is not stable, so there are $E\in\Sigma$ and $\alpha<\beta$ such that $\mu E>0$ and $(\mu^{2k})^*D_k(A,E,\alpha,\beta)=(\mu E)^{2k}$ for every $k\ge1$. The first thing to note is that if $\mathcal{I}$ is the set of those $B\subseteq A$ for which there is some $k\in\mathbb{N}$ such that $(\mu^{2k})^*D_k(B,E,\alpha,\beta)<(\mu E)^{2k}$, then $\mathcal{I}$ is an ideal of subsets of A. **P** Of course $\emptyset\in\mathcal{I}$ and $B\in\mathcal{I}$ whenever $B\subseteq B'\in\mathcal{I}$. Also 465Cc tells us that, if $B\in\mathcal{I}$, then $\lim_{k\to\infty}\frac{1}{(\mu E)^{2k}}(\mu^{2k})^*D_k(B,E,\alpha,\beta)=0$. It follows easily (as in the proof of 465Cd) that $B\cup B'\in\mathcal{I}$ for all B, $B'\in\mathcal{I}$. **Q**

(ii) $\mathcal{I}$ is a proper ideal of subsets of A (by the choice of E, k, α and β), so there is an ultrafilter $\mathcal{F}$ on A such that $\mathcal{F}\cap\mathcal{I}=\emptyset$. Let g_0 be the $\mathfrak{T}_p$-limit of $\mathcal{F}$. Then g_0 is measurable.

Set $\delta=\frac13(\beta-\alpha)\mu E>0$, and define h_0, h_1 by setting

$$\begin{aligned}h_0(x)&=\alpha\text{ if }x\in E,\\&=g_0(x)+\delta\text{ if }x\in X\setminus E,\\h_1(x)&=\beta\text{ if }x\in E,\\&=g_0(x)-\delta\text{ if }x\in X\setminus E.\end{aligned}$$

Then

$$\int h_1-\int h_0\ge(\beta-\alpha)\mu E-2\delta\mu(X\setminus E)>0.$$

(iii) We shall need to know a little more about sets of the form $D_k(B,E,\alpha,\beta)$ for $B\in\mathcal{F}$. In fact, if $B\subseteq A$ and $B\notin\mathcal{I}$, then for any finite sets I and J

$$(\mu^I\times\mu^J)^*D_{I,J}(B,E,\alpha,\beta)=(\mu E)^{\#(I)+\#(J)},$$

where $D_{I,J}(B,E,\alpha,\beta)$ is

$$\textstyle\bigcup_{f\in B}\{(u,v):u\in E^I,\,v\in E^J,\,f(u(i))\le\alpha\text{ for }i\in I,\,f(v(j))\ge\beta\text{ for }j\in J\}.$$

P We may suppose that $I=k$ and $J=l$ where k, $l\in\mathbb{N}$. Take $m=\max(1,k,l)$. Then we have an inverse-measure-preserving map $\phi:X^{2m}\to X^I\times X^J$ defined by saying that $\phi(w)=(u,v)$ where $u(i)=w(2i)$ for $i<k$ and $v(i)=w(2i+1)$ for $i<l$. **?** If $(\mu^I\times\mu^J)^*D_{I,J}(B,E,\alpha,\beta)<(\mu E)^{k+l}$, there is a non-negligible measurable set $V\subseteq(E^I\times E^J)\setminus D_{I,J}(B,E,\alpha,\beta)$. Now $\phi^{-1}[V]$ is non-negligible and depends only on coordinates in $\{2i:i<k\}\cup\{2i+1:i<l\}$, so

$$\mu^{2m}(E^{2m}\cap\phi^{-1}[V])=\mu^{2m}(\phi^{-1}[V])\cdot(\mu E)^{2m-k-l}>0.$$

But $\phi^{-1}[V]\cap D_m(B,E,\alpha,\beta)=\emptyset$, so $(\mu^{2m})^*D_m(B,E,\alpha,\beta)<(\mu E)^{2m}$, and $B\in\mathcal{I}$. **X** So $(\mu^I\times\mu^J)^*D_{I,J}(B,E,\alpha,\beta)=(\mu E)^{k+l}$, as required. **Q**

(iv) ? Suppose, if possible, that $k\ge 1$ is such that $(\mu^{2k})^*\tilde D_k<1$, where $\tilde D_k$ is defined from h_0 and h_1 by the formula (*) in the statement of the lemma. Let $W\subseteq X^{2k}$ be a measurable set of positive measure disjoint from $\tilde D_k$. For I, $J\subseteq k$ write W_{IJ} for

$$\{w:w\in W, w(2i)\in E \text{ for } i\in I,\ w(2i)\notin E \text{ for } i\in k\setminus I,\\ w(2i+1)\in E \text{ for } i\in J,\ w(2i+1)\notin E \text{ for } i\in k\setminus J\}.$$

Then there are I, $J\subseteq K$ such that $\mu^{2k}W_{IJ}>0$.

We can identify X^{2k} with $X^I\times X^J\times X^{k\setminus I}\times X^{k\setminus J}$, matching any $w\in X^{2k}$ with (w_0,w_1,w_2,w_3) where

$$w_0(i)=w(2i) \text{ for } i\in I,$$
$$w_1(i)=w(2i+1) \text{ for } i\in J,$$
$$w_2(i)=w(2i) \text{ for } i\in k\setminus I,$$
$$w_3(i)=w(2i+1) \text{ for } i\in k\setminus J.$$

Write $\tilde W$ for the image of W_{IJ} under this matching. The condition $W_{IJ}\cap\tilde D_k=\emptyset$ translates into

(†) whenever $(w_0,w_1,w_2,w_3)\in\tilde W$, $f\in A$,
either there is an $i\in I$ such that $f(w_0(i))>\alpha$
or there is an $i\in J$ such that $f(w_1(i))<\beta$
or there is an $i\in k\setminus I$ such that $f(w_2(i))>g_0(w_2(i))+\delta$
or there is an $i\in k\setminus J$ such that $f(w_3(i))<g_0(w_3(i))-\delta$.

(v) By Fubini's theorem, applied to $(X^I\times X^J)\times(X^{k\setminus I}\times X^{k\setminus J})$, we can find $w_2\in X^{k\setminus I}$, $w_3\in X^{k\setminus J}$ such that $(\mu^I\times\mu^J)(V)$ is defined and greater than 0, where $V=\{(w_0,w_1):(w_0,w_1,w_2,w_3)\in\tilde W\}$. Set

$$B=\{f:f\in A, |f(w_2(i))-g_0(w_2(i))|\le\delta \text{ for } i\in k\setminus I,\\ |f(w_3(i))-g_0(w_3(i))|\le\delta \text{ for } i\in k\setminus J\}.$$

Then $B\in\mathcal{F}$, because $\mathcal{F}\to g_0$ for $\mathfrak{T}_p$. So $(\mu^I\times\mu^J)^*D_{I,J}(B,E,\alpha,\beta)=(\mu E)^{\#(I)+\#(J)}$, by (iii) above. Since $W_{IJ}\subseteq W$, V is included in $E^I\times E^J$ and meets $D_{I,J}(B,E,\alpha,\beta)$; that is, there are $f\in B$ and $(w_0,w_1)\in V$ such that $f(w_0(i))\le\alpha$ for $i\in I$ and $f(w_1(i))\ge\beta$ for $i\in J$. But because $f\in B$ we also have $f(w_2(i))\le g_0(w_2(i))+\delta$ for $i\in k\setminus I$ and $f(w_3(i))\ge g_0(w_3(i))-\delta$ for $i\in k\setminus J$; which contradicts the list in (†) above. **X**

(vi) Thus $(\mu^{2k})^*\tilde D_k=1$ for every k, and in this case also we have an appropriate pair h_0, h_1.

465M Theorem (Talagrand 82, Talagrand 87) Let (X,Σ,μ) be a complete probability space, and A a non-empty uniformly bounded set of real-valued functions defined on X. Then the following are equiveridical:

(i) A is stable.

(ii) Every function in A is measurable, and $\lim_{k\to\infty}\sup_{f\in A}|\frac{1}{k}\sum_{i=0}^{k-1}f(w(i))-\int f|=0$ for almost every $w\in X^{\mathbb{N}}$.

(iii) Every function in A is measurable, and for every $\epsilon>0$ there are a finite subalgebra T of Σ in which every atom is non-negligible and a sequence $\langle h_k\rangle_{k\ge1}$ of measurable functions on $X^{\mathbb{N}}$ such that

$$h_k(w)\ge\sup_{f\in A}\frac{1}{k}\sum_{i=0}^{k-1}\bigl|f(w(i))-\mathbb{E}(f|\mathrm{T})(w(i))\bigr|$$

for every $w\in X^{\mathbb{N}}$ and $k\ge1$, and

$$\limsup_{k\to\infty}h_k(w)\le\epsilon$$

for almost every $w\in X^{\mathbb{N}}$. (Here $\mathbb{E}(f|\mathrm{T})$ is the (unique) conditional expectation of f on T.)

(iv) $\lim_{k,l\to\infty}\sup_{f\in A}|\frac{1}{k}\sum_{i=0}^{k-1}f(w(i))-\frac{1}{l}\sum_{i=0}^{l-1}f(w(i))|=0$ for almost every $w\in X^{\mathbb{N}}$.

(v) $\lim_{k,l\to\infty}\overline{\int}\sup_{f\in A}|\frac{1}{k}\sum_{i=0}^{k-1}f(w(i))-\frac{1}{l}\sum_{i=0}^{l-1}f(w(i))|\mu^{\mathbb{N}}(dw)=0$.

proof All the statements (i)-(v) are unaffected by translations (by constant functions) and scalar multiplications of the set A, so it will be enough to consider the case in which $A \subseteq [0,1]^X$.

As in 465I-465H, I write $\nu_{wk}(E) = \frac{1}{k}\#(\{i : i < k,\, w(i) \in E\})$ for $w \in X^{\mathbb{N}}$, $k \ge 1$ and $E \subseteq X$; so for any function $f : X \to \mathbb{R}$, $\int f d\nu_{wk} = \frac{1}{k}\sum_{i=0}^{r-1} f(w(i))$.

(a)(i)⇒(iii)(α) If A is stable, then every function in A is measurable, by 465Da. Let $\epsilon > 0$. Set $\eta = \frac{1}{108}\epsilon^2 > 0$. By 465Db, the $\mathfrak{T}_p$-closure $\overline{A}$ of A in $\mathbb{R}^X$ is a $\mathfrak{T}_p$-compact set of measurable functions, and by 465G it is $\mathfrak{T}_m$-compact; because A is uniformly bounded, it must be totally bounded for the pseudometric induced by $\| \|_1$. So there are $f_0, \dots, f_m \in A$ such that for every $f \in A$ there is an $i \le m$ such that $\|f - f_i\|_1 \le \eta$. Let T_0 be the finite subalgebra of Σ generated by the sets $\{x : j\eta \le f_i(x) < (j+1)\eta\}$ for $i \le m$ and $j \le \dfrac{1}{\eta}$. Then T_0 may have negligible atoms, but if we absorb these into non-negligible atoms we get a finite subalgebra T of Σ such that $|f_i(x) - \mathbb{E}(f_i|\mathrm{T})(x)| \le \eta$ for almost every $x \in X$, every $i \le m$. (Because T is a finite algebra with non-negligible atoms, two T-measurable functions which are equal almost everywhere must be identical, and we have unique conditional expectations with respect to T.) Since $\|\mathbb{E}(f|\mathrm{T}) - \mathbb{E}(g|\mathrm{T})\|_1 \le \|f - g\|_1$ for all integrable functions f and g (242Je), $\|f - \mathbb{E}(f|\mathrm{T})\|_1 \le 3\eta$ for every $f \in A$.

(β) Set $A' = \{f - \mathbb{E}(f|\mathrm{T}) : f \in A\}$. Then A' is stable. **P** Suppose that $\mu E > 0$ and $\alpha < \beta$. The set $B = \{\mathbb{E}(f|\mathrm{T}) : f \in A\}$ is a uniformly bounded subset of a finite-dimensional space of functions, so is $\| \|_\infty$-compact. So there are $g_0, \dots, g_r \in B$ such that $B \subseteq \bigcup_{i \le r}\{g : \|g - g_i\|_\infty \le \frac{1}{3}(\beta - \alpha)\}$. By 465Cf and 465Cd, $C = \bigcup_{i \le r} A - g_i$ is stable. So there is a $k \ge 1$ such that

$$(\mu^{2k})^* D_k(C, E, \tfrac{2}{3}\alpha + \tfrac{1}{3}\beta, \tfrac{1}{3}\alpha + \tfrac{2}{3}\beta) < (\mu E)^{2k}.$$

But for every $g \in A'$ there is an $h \in C$ such that $\|g - h\|_\infty \le \frac{1}{3}(\beta - \alpha)$, so

$$D_k(A', E, \alpha, \beta) \subseteq D_k(C, E, \tfrac{2}{3}\alpha + \tfrac{1}{3}\beta, \tfrac{1}{3}\alpha + \tfrac{2}{3}\beta), \quad (\mu^{2k})^* D_k(A', E, \alpha, \beta) < (\mu E)^{2k}.$$

As E, α and β are arbitrary, A' is stable. **Q**

(γ) By 465Cl,

$$A'' = \{g^+ : g \in A'\} \cup \{g^- : g \in A'\}$$

is stable. By 465K there are an $n \ge 1$, a $W \in \widehat{\bigotimes}_n \Sigma$ and a $\gamma > \mu^n W$ such that $\int h\, d\nu \le 3\sqrt{3\eta} = \frac{1}{2}\epsilon$ whenever $h \in A''$ and ν is a probability measure on X, with domain including Σ, such that $\nu^n W \le \gamma$. So $\int |g| d\nu \le \epsilon$ whenever $g \in A'$ and ν is such a measure. If $w \in X^{\mathbb{N}}$ and $k \in \mathbb{N}$, ν_{wk} is a probability measure on X; set $q_k(w) = \nu_{wk}^n(W)$. Applying 465H to the indicator function of W, we see that $\lim_{k\to\infty} q_k(w) = \mu^n W$ for almost every $w \in X^{\mathbb{N}}$. Also, because $W \in \widehat{\bigotimes}_n \Sigma$, every q_k is measurable, by 465I.

Set $h_k(w) = 1$ if $q_k(w) > \gamma$, ϵ if $q_k(w) \le \gamma$. Then every h_k is measurable and $\lim_{k\to\infty} h_k(w) = \epsilon$ for almost every w.

For any $w \in X^{\mathbb{N}}$ and any $f \in A$, $g = f - \mathbb{E}(f|\mathrm{T}) \in A'$ and $\|g\|_\infty \le 1$. So either $h_k(w) = 1$ and certainly $\int |g| d\nu_{wk} \le h_k(w)$, or $h_k(w) = \epsilon$, $\nu_{wk}^n(W) \le \gamma$ and $\int |g| d\nu_{wk} \le \epsilon$. Thus we have

$$\frac{1}{k}\sum_{i=0}^{k-1}\big|f(w(i)) - \mathbb{E}(f|\mathrm{T})(w(i))\big| = \int \Big|f - \mathbb{E}(f|\mathrm{T})\Big| d\nu_{wk} \le h_k(w)$$

for every $w \in X^{\mathbb{N}}$ and every $f \in A$, as required by (iii).

(b)(iii)⇒(ii) & (v) Set

$$g_k(w) = \sup_{f\in A} |\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \int f d\mu|,$$

$$g'_{kl}(w) = \sup_{f\in A} |\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))|,$$

for $w \in X^{\mathbb{N}}$ and k, $l \ge 1$. Let $\epsilon > 0$. Let T and $\langle h_k \rangle_{k\ge1}$ be as in (iii), and let $E_0, \dots, E_r$ be the atoms of T. For $w \in X^{\mathbb{N}}$, $k \ge 1$, $j \le r$ set $q_{kj}(w) = |\mu E_j - \nu_{wk} E_j|$. Then for any $f \in A$, $\mathbb{E}(f|\mathrm{T})$ is expressible as $\sum_{j=0}^{r} \alpha_j \chi E_j$ where $\alpha_j \in [0,1]$ for every j (remember that $A \subseteq [0,1]^X$), so

$$\begin{aligned}
\big|\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \int f d\mu\big| &= \big|\int f d\nu_{wk} - \int f d\mu\big| \\
&\le \big|\int f d\nu_{wk} - \int \mathbb{E}(f|\mathrm{T}) d\nu_{wk}\big| + \big|\int \mathbb{E}(f|\mathrm{T}) d\nu_{wk} - \int \mathbb{E}(f|\mathrm{T}) d\mu\big| \\
&\le \frac{1}{k}\sum_{i=0}^{k-1}\big|f(w(i)) - \mathbb{E}(f|\mathrm{T})(w(i))\big| + \sum_{j=0}^{r}\alpha_j\big|\mu E_j - \nu_{wk}E_j\big| \\
&\le h_k(w) + \sum_{j=0}^{r} q_{kj}(w), \\
\big|\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))\big| & \\
&\le h_k(w) + \sum_{j=0}^{r} q_{kj}(w) + h_l(w) + \sum_{j=0}^{r} q_{lj}(w).
\end{aligned}$$

Taking the supremum over f, we have

$$g_k(w) \le h_k(w) + \sum_{j=0}^{r} q_{kj}(w),$$

$$g'_{kl}(w) \le h_k(w) + \sum_{j=0}^{r} q_{kj}(w) + \sum_{j=0}^{r} q_{lj}(w) + h_l(w).$$

But, for each $j \le r$, $\lim_{k\to\infty} q_{kj}(w) = 0$ for almost every w, by 465H (or 273J) applied to the indicator function of E_j. So

$$\limsup_{k\to\infty} g_k(w) \le \limsup_{k\to\infty} h_k(w) \le \epsilon$$

for almost every w. At the same time,

$$\limsup_{k,l\to\infty} \overline{\int} g'_{kl} \le \limsup_{k,l\to\infty} \int h_k + \sum_{j=0}^{r}\int q_{kj} + \sum_{j=0}^{r}\int q_{lj} + \int h_l \le 2\epsilon.$$

As ϵ is arbitrary, $\{w : \limsup_{k\to\infty} g_k(w) \ge 2^{-i}\}$ is negligible for every $i \in \mathbb{N}$, and $\lim_{k\to\infty} g_k = 0$ almost everywhere, as required, while equally $\lim_{k,l\to\infty} \overline{\int} g'_{kl} = 0$. Thus (ii) and (v) are true.

(c)(ii)⇒(iv) is trivial, since

$$\big|\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))\big| \le \Big|\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \int f\Big| + \Big|\frac{1}{l}\sum_{i=0}^{l-1} f(w(i)) - \int f\Big|.$$

(d) not-(i)⇒not-(iv) & not-(v) Now suppose that A is not stable.

(α) In this case, by 465L, there are measurable functions h_0, $h_1 : X \to [0,1]$ such that $\int h_0 d\mu < \int h_1 d\mu$ and $(\mu^{2k})^*\tilde{D}_k = 1$ for every $k \in \mathbb{N}$, where

$$\begin{aligned}
\tilde{D}_k = \bigcup_{f\in A}\{w : w \in X^{2k}, f(w(2i)) \le h_0(w(2i)),& \\
f(w(2i+1)) \ge h_1(w(2i+1)) \text{ for every } i < k\}.&
\end{aligned}$$

(β) Set $\delta = \frac{1}{4}\int h_1 - h_0 > 0$. Let $k_0 \ge 1$ be so large that

$$\mu^k\{w : w \in X^k, |\int h_j - \frac{1}{k}\sum_{i=0}^{k-1} h_j(w(i))| \le \delta\} \ge \frac{1}{2}$$

for both j and for every $k \ge k_0$ (273J or 465H. The point is that

$$\begin{aligned}
\mu^k\{w : w \in X^k, |\int h_j - \frac{1}{k}\sum_{i=0}^{k-1} h_j(w(i))| \le \delta\}& \\
= \mu^{\mathbb{N}}\{w : w \in X^{\mathbb{N}}, |\int h_j - \frac{1}{k}\sum_{i=0}^{k-1} h_j(w(i))| \le \delta\} &\to 1 \text{ as } k \to \infty.)
\end{aligned}$$

Let $\langle k_n\rangle_{n\ge 1}$ be such that $k_n \ge \frac{2}{\delta}\sum_{i=0}^{n-1} k_i$ for every $n\ge 1$. Since $\delta\le\frac{1}{4}$, every k_n is at least as large as k_0.

(γ) Set $m_n = 2\sum_{i<n} k_i$ for each $n\in\mathbb{N}$. Then we have a measure space isomorphism $\phi:\prod_{n\in\mathbb{N}} X^{2k_n}\to X^{\mathbb{N}}$ defined by setting

$$\phi(w)(m_n+i)=w(n)(2i),\quad \phi(w)(m_n+k_n+i)=w(n)(2i+1)$$

for $n\in\mathbb{N}$ and $i<k_n$. For each $n\in\mathbb{N}$, $\tilde D_{k_n}$ has outer measure 1 in X^{2k_n}, so $\tilde D=\prod_{n\in\mathbb{N}}\tilde D_{k_n}$ has outer measure 1 in $\prod_{n\in\mathbb{N}} X^{2k_n}$, and $\phi[\tilde D]$ has outer measure 1 in $X^{\mathbb{N}}$. Note that $\phi[\tilde D]$ is just the set of $w\in X^{\mathbb{N}}$ such that, for every $n\in\mathbb{N}$, there is an $f\in A$ such that

$$f(w(i))\le h_0(w(i)) \text{ for } m_n\le i<m_n+k_n,$$
$$f(w(i))\ge h_1(w(i)) \text{ for } m_n+k_n\le i<m_n+2k_n=m_{n+1}.$$

If we set

$$V_{n0}=\{w: w\in X^{\mathbb{N}},\ |\tfrac{1}{k_n}\textstyle\sum_{i=m_n}^{m_n+k_n-1} h_0(w(i))-\int h_0|\le\delta\},$$

$$V_{n1}=\{w: w\in X^{\mathbb{N}},\ |\tfrac{1}{k_n}\textstyle\sum_{i=m_n+k_n}^{m_n+2k_n-1} h_1(w(i))-\int h_1|\le\delta\},$$

every V_{n0} and V_{n1} has measure at least $\frac{1}{2}$, because $k_n\ge k_0$.

(δ) Now suppose that $n\in\mathbb{N}$ and $w\in V_{n0}\cap V_{n1}\cap\phi[\tilde D]$. Then

$$\sup_{f\in A}\Bigl|\frac{1}{m_n+2k_n}\sum_{i=0}^{m_n+2k_n-1} f(w(i))-\frac{1}{m_n+k_n}\sum_{i=0}^{m_n+k_n-1} f(w(i))\Bigr|\ge\frac{\delta}{(2+\delta)(1+\delta)}.$$

P Since $w\in\phi[\tilde D]$, there must be an $f\in A$ such that

$$f(w(i))\le h_0(w(i)) \text{ for } m_n\le i<m_n+k_n,$$
$$f(w(i))\ge h_1(w(i)) \text{ for } m_n+k_n\le i<m_n+2k_n=m_{n+1}.$$

Set $s=\sum_{i=0}^{m_n-1} f(w(i))$, $t=\sum_{i=m_n}^{m_n+k_n} f(w(i))$ and $t'=\sum_{i=m_n+k_n}^{m_n+2k_n-1} f(w(i))$. Then

$$\begin{aligned}\frac{1}{m_n+2k_n}&\sum_{i=0}^{m_n+2k_n-1} f(w(i))-\frac{1}{m_n+k_n}\sum_{i=0}^{m_n+k_n-1} f(w(i))\\ &=\frac{s+t+t'}{m_n+2k_n}-\frac{s+t}{m_n+k_n}=\frac{(t'-t-s)k_n+t'm_n}{(m_n+2k_n)(m_n+k_n)}\\ &\ge\frac{(t'-t-m_n)k_n}{(m_n+2k_n)(m_n+k_n)}\ge\frac{t'-t-\delta k_n}{(2+\delta)(1+\delta)k_n}\end{aligned}$$

because $m_n\le\delta k_n$, by the choice of k_n.

To estimate t and t', we have

$$t=\textstyle\sum_{i=m_n}^{m_n+k_n-1} f(w(i))\le\sum_{i=m_n}^{m_n+k_n-1} h_0(w(i))\le k_n(\int h_0+\delta)$$

because $w\in V_{n0}$. On the other hand,

$$t'=\textstyle\sum_{i=m_n+k_n}^{m_n+2k_n-1} f(w(i))\ge\sum_{i=m_n+k_n}^{m_n+2k_n-1} h_1(w(i))\ge k_n(\int h_1-\delta)$$

because $w\in V_{n1}$. So

$$t'-t\ge k_n(\textstyle\int h_1-h_0-2\delta)=2k_n\delta,$$

and

$$\begin{aligned}\frac{1}{m_n+2k_n}&\sum_{i=0}^{m_n+2k_n-1} f(w(i))-\frac{1}{m_n+k_n}\sum_{i=0}^{m_n+k_n-1} f(w(i))\\ &\ge\frac{2\delta k_n-\delta k_n}{(2+\delta)(1+\delta)k_n}=\frac{\delta}{(2+\delta)(1+\delta)}.\ \mathbf{Q}\end{aligned}$$

(ϵ) The V_{n0} and V_{n1} are all independent. So $\mu^{\mathbb{N}}(V_{n0}\cap V_{n1})\ge\frac{1}{4}$ for every n, and

$$W=\{w:w\in X^{\mathbb{N}},\ w\in V_{n0}\cap V_{n1}\text{ for infinitely many }n\}$$

has measure 1 (by the Borel-Cantelli lemma, 273K, or otherwise). Accordingly $W\cap\phi[\tilde{D}]$ has outer measure 1 in $X^{\mathbb{N}}$. But if $w\in W\cap\phi[\tilde{D}]$, then ($\delta$) tells us that

$$\limsup_{k,l\to\infty}\sup_{f\in A}\Bigl|\frac{1}{k}\sum_{i=0}^{k-1}f(w(i))-\frac{1}{l}\sum_{i=0}^{l-1}f(w(i))\Bigr|\ge\frac{\delta}{(2+\delta)(1+\delta)},$$

because there are infinitely many n such that $w\in V_{n0}\cap V_{n1}\cap\phi[\tilde{D}]$. So (iv) must be false.

(ζ) We see also that, for any $n\in\mathbb{N}$,

$$\overline{\int}g'_{m_n+k_n,m_n+2k_n}\ge\frac{\delta}{(2+\delta)(1+\delta)}\mu^{\mathbb{N}}(V_{n0}\cap V_{n1})\ge\frac{\delta}{4(2+\delta)(1+\delta)},$$

writing

$$g'_{kl}(w)=\sup_{f\in A}|\tfrac{1}{k}\sum_{i=0}^{k-1}f(w(i))-\tfrac{1}{l}\sum_{i=0}^{l-1}f(w(i))|$$

for k, $l\in\mathbb{N}$. So $\limsup_{k,l\to\infty}\overline{\int}g'_{kl}>0$, and (v) is false.

Remark If (X,Σ,μ) is a probability space, a set $A\subseteq\mathcal{L}^1(\mu)$ is a **Glivenko-Cantelli class** if $\sup_{f\in A}|\frac{1}{k}\sum_{i=0}^{k-1}f(w(i))-\int f|\to 0$ as $k\to\infty$ for $\mu^{\mathbb{N}}$-almost every $w\in X^{\mathbb{N}}$. Compare 273Xi.

465N Theorem Let (X,Σ,μ) be a semi-finite measure space.

(a) (Talagrand 84) Let $A\subseteq\mathbb{R}^X$ be a stable set. Suppose that there is a measurable function $g:X\to[0,\infty[$ such that $|f(x)|\le g(x)$ whenever $x\in X$ and $f\in A$. Then the convex hull $\Gamma(A)$ of A in $\mathbb{R}^X$ is stable.

(b) If $A\subseteq\mathbb{R}^X$ is stable, then $|A|=\{|f|:f\in A\}$ is stable.

(c) Let A, $B\subseteq\mathbb{R}^X$ be two stable sets such that $\{f(x):f\in A\cup B\}$ is bounded for every $x\in X$. Then $A+B=\{f_1+f_2:f_1\in A,\ f_2\in B\}$ is stable.

(d) Suppose that μ is complete and locally determined. Let $A\subseteq\mathbb{R}^X$ be a stable set such that $\{f(x):f\in A\}$ is bounded for every $x\in X$. Then $\Gamma(A)$ is relatively compact in $\mathcal{L}^0(\Sigma)$ for the topology of pointwise convergence.

proof (a)(i) Consider first the case in which $\mu X=1$ and $A\subseteq[-1,1]^X$. In this case,

$$\sup_{f\in\Gamma(A)}|\tfrac{1}{k}\sum_{i=0}^{k-1}f(w(i))-\int f|=\sup_{f\in A}|\tfrac{1}{k}\sum_{i=0}^{k-1}f(w(i))-\int f|$$

for every $k\ge 1$ and $w\in X^{\mathbb{N}}$. So $\Gamma(A)$ satisfies condition (ii) of 465M whenever A does, and we have the result.

(ii) Now suppose just that $\mu X=1$. Set $A'=\{\frac{f}{g+\chi X}:f\in A\}$. Then A' is stable (465Ch), so $\Gamma(A')$ is stable, by (i), and $\Gamma(A)=\{f\times(g+\chi X):f\in A'\}$ is stable.

(iii) If $\mu X=0$, the result is trivial. If $\mu X<\infty$, apply (ii) to a multiple of the measure μ. For the general case, write $A_E=\{f\restriction E:f\in A\}$ for $E\subseteq X$. Then A_E is stable for the subspace measure on E, by 465Cm. It follows that $\Gamma(A_E)$ is stable whenever $\mu E<\infty$. But $\Gamma(A_E)=(\Gamma(A))_E$, so 465Cn tells us that $\Gamma(A)$ is stable.

(b)(i) I begin with a basic special case of (c). If A, $B\subseteq\mathbb{R}^X$ are stable and uniformly bounded, then $A+B$ is stable. **P** Putting 465Cd, (a) of this theorem, 465Ce and 465Ca together, we see that $A\cup B$, $\Gamma(A\cup B)$ and $A+B\subseteq 2\Gamma(A\cup B)$ are stable. **Q**

(ii) Adding this to 465Cl, $|A|\subseteq\{f^+:f\in A\}+\{f^-:f\in A\}$ is stable whenever $A\subseteq\mathbb{R}^X$ is stable and uniformly bounded.

(iii) For the general case, set $h(\alpha)=\tanh\alpha$ for $\alpha\in\mathbb{R}$. Then 465Ck and (ii) here tell us that if $A\subseteq\mathbb{R}^X$ is stable then

$$\{hf:f\in A\},\quad\{|hf|:f\in A\},\quad\{h^{-1}|hf|:f\in A\}=|A|$$

are stable.

(c)(i) By 465Cn, it is enough to consider the case of totally finite μ, so let us suppose from now on that $\mu X < \infty$. We may also suppose that neither A nor B is empty; finally, by 465Ci, we can suppose that μ is complete, so that $A \cup B \subseteq \mathcal{L}^0(\Sigma)$ (465Da).

I introduce some temporary notation: if $E \subseteq X$, $k \ge 1$, $\epsilon > 0$ and $A \subseteq \mathbb{R}^X$, set

$$\tilde{D}_k(A, E, \epsilon) = \bigcup_{f \in A}\{u : u \in E^k,\ |f(u(i))| \ge \epsilon \text{ for } i < k\}.$$

(ii) We need to know that if $A \subseteq \mathbb{R}^X$ and every $f \in A$ is zero a.e., then A is stable iff whenever $E \in \Sigma$, $0 < \mu E < \infty$ and $\epsilon > 0$ there is a $k \ge 1$ such that $(\mu^k)^*\tilde{D}_k(A, E, \epsilon) < (\mu E)^k$.

P$(\boldsymbol{\alpha})$ If A is stable, then $|A|$ is stable, by (b), so if $0 < \mu E < \infty$ and $\epsilon > 0$ there is a $k \ge 1$ such that $(\mu^{2k})^*D_{2k}(|A|, E, 0, \epsilon) < (\mu E)^{2k}$. Let $W \in \widehat{\bigotimes}_{2k}\Sigma$ be such that $D_{2k}(|A|, E, 0, \epsilon) \subseteq W \subseteq E^{2k}$ and $\mu^{2k}W < (\mu E)^{2k}$. Because $(u, v) \mapsto u\#v$ is a measure space isomorphism,

$$\mu^{2k}W = \int \mu^k\{u : u\#v \in W\}\mu^k(dv),$$

so if we set $V = \{v : v \in E^k,\ \mu^k\{u : u\#v \in W\} = (\mu E)^k\}$ we must have $\mu^k V < (\mu E)^k$. If $v \in \tilde{D}_k(A, E, \epsilon)$, there is an $f \in A$ such that $|f(v(i))| \ge \epsilon$ for every $i < k$; now

$$\{u : u\#v \in W\} \supseteq \{u : u \in E^k,\ f(u(i)) = 0 \text{ for every } i < k\}$$

has measure $(\mu E)^k$, because $f = 0$ a.e. So $\tilde{D}_k(A, E, \epsilon) \subseteq V$ and $(\mu^k)^*\tilde{D}_k(A, E, \epsilon) < (\mu E)^k$. As E and ϵ are arbitrary, A satisfies the condition.

$(\boldsymbol{\beta})$ Now suppose that A satisfies the condition. Take $E \in \Sigma$ such that $\mu E > 0$, and $\alpha < \beta$ in $\mathbb{R}$. If $\beta > 0$, set $\epsilon = \beta$; otherwise, set $\epsilon = -\alpha$. Then there is a $k \in \mathbb{N}$ such that $(\mu^k)^*\tilde{D}_k(A, E, \epsilon) < (\mu E)^k$. As $D_k(A, E, \alpha, \beta)$ is included in $\{(u, v) : u \in E^k,\ v \in \tilde{D}_k(A, E, \epsilon)\}$ (if $\beta > 0$) or $\{(u, v) : u \in \tilde{D}_k(A, E, \epsilon),\ v \in E^k\}$ (if $\beta \le 0$), $(\mu^{2k})^*D_k(A, E, \alpha, \beta) < (\mu E)^{2k}$. As E, α and β are arbitrary, A is stable. **Q**

(iii) Suppose that A and B are stable sets such that $f = 0$ a.e. for every $f \in A \cup B$. Then $A + B$ is stable. **P** Set $A' = \{|f| \wedge \chi X : f \in A\}$, $B' = \{|f| \wedge \chi X : f \in B\}$. Then A' and B' are stable, so $A' + B'$ is stable, by (b)(i) above. But now observe that if $u \in \tilde{D}_k(A + B, E, \epsilon)$, where $E \subseteq X$, $\epsilon > 0$ and $k \ge 1$, then there are $f_1 \in A$, $f_2 \in B$ such that $|f_1(u(i)) + f_2(u(i))| \ge \epsilon$ for every $i < k$. In this case, setting $f_j' = |f_j| \wedge \chi X$ for both j, $g = f_1' + f_2'$ belongs to $A' + B'$ and $g(u(i)) \ge \min(1, \epsilon)$ for every $i < k$. This shows that $\tilde{D}_k(A + B, E, \epsilon) \subseteq \tilde{D}_k(A' + B', E, \min(1, \epsilon))$. Also every function in either $A + B$ or $A' + B'$ is zero a.e. So (ii) tells us that $A + B$ also is stable. **Q**

(iv) Suppose that A, $B \subseteq \mathbb{R}^X$ are stable, that $|f| \le \chi X$ for every $f \in A$, and that $g = 0$ a.e. for every $g \in B$. Then $A + B$ is stable. **P** For $g \in B$ set $g'(x) = \operatorname{med}(-2, g(x), 2)$ for $x \in X$; set $B' = \{g' : g \in B\}$. Then B' is stable, by 465Ck, and both A and B' are uniformly bounded, so $A + B'$ is stable. Take $E \in \Sigma$ such that $\mu E > 0$, and $\alpha < \beta$ in $\mathbb{R}$.

If $\beta > 1$, then, by (ii), there is a $k \ge 1$ such that $(\mu^k)^*\tilde{D}_k(B, E, \beta - 1) < (\mu E)^k$. Now if $w \in D_k(A + B, E, \alpha, \beta)$ there are $f \in A$, $g \in B$ such that $f(w(2i + 1)) + g(w(2i + 1)) \ge \beta$ for every $i < k$; accordingly $g(w(2i + 1)) \ge \beta - 1$ for $i < k$ and $w = u\#v$ for some $u \in E^k$, $v \in \tilde{D}_k(B, E, \beta - 1)$. So

$$(\mu^{2k})^*D_k(A + B, E, \alpha, \beta) \le (\mu E)^k \cdot (\mu^k)^*\tilde{D}(B, E, \beta - 1) < (\mu E)^{2k}.$$

Similarly, if $\alpha < -1$, then

$$(\mu^{2k})^*D_k(A + B, E, \alpha, \beta) \le (\mu E)^k \cdot (\mu^k)^*\tilde{D}(B, E, -1 - \alpha) < (\mu E)^{2k}$$

for some k.

On the other hand, if $-1 \le \alpha < \beta \le 1$, there is a $k \ge 1$ such that $(\mu^{2k})^*D_k(A + B', E, \alpha, \beta) < (\mu E)^{2k}$. If now $w \in D_k(A + B, E, \alpha, \beta)$, take $f \in A$ and $g \in B$ such that $f(w(2i)) + g(w(2i)) \le \alpha$ and $f(w(2i+1)) + g(w(2i+1)) \ge \beta$ for $i < k$. In this case, for each $i < k$,

— either $g'(w(2i)) \le g(w(2i))$ and $f(w(2i)) + g'(w(2i)) \le \alpha$, or $g'(w(2i)) = -2$ and $f(w(2i)) + g'(w(2i)) \le -1 \le \alpha$,

— either $g'(w(2i + 1)) \ge g(w(2i + 1))$ and $f(w(2i + 1)) + g'(w(2i + 1)) \ge \beta$, or $g'(w(2i + 1)) = 2$ and $f(w(2i + 1)) + g'(w(2i + 1)) \ge 1 \ge \beta$.

So $w \in D_k(A + B', E, \alpha, \beta)$. Accordingly $(\mu^{2k})^*D_k(A + B, E, \alpha, \beta) < (\mu E)^{2k}$.

As E, α and β are arbitrary, $A + B$ is stable. **Q**

(v) Now suppose that $|f| \le_{\text{a.e.}} \chi X$ for every $f \in A \cup B$. For $f \in A \cup B$ and $x \in X$, set $f_0(x) = \operatorname{med}(-1, f(x), 1)$, $f_1(x) = \max(0, f(x) - 1)$ and $f_2(x) = \max(0, -1 - f(x))$; then $f = f_0 + f_1 - f_2$, $|f_0| \le \chi X$ and f_1, f_2 are zero a.e. Also $A_0=\{f_0:f\in A\}$, $A_1=\{f_1:f\in A\}$, $A_2=\{f_2:f\in A\}$, $B_0=\{f_0:f\in B\}$, $B_1=\{f_1:f\in B\}$ and $B_2=\{f_2:f\in B\}$ are all stable, by 465Ck. Accordingly A_0+B_0 is stable, by (ii); by (iii), $A_1-A_2+B_1-B_2$ is stable; by (iv),

$$A+B\subseteq A_0+B_0+A_1-A_2+B_1-B_2$$

is stable.

(vi) Finally, turn to the hypothesis stated in the proposition: that A and B are stable and pointwise bounded. Let $h:X\to[0,\infty[$ be such that $|f(x)|\le h(x)$ for every $f\in A\cup B$ and $x\in X$; note that I do *not* assume here that h is measurable. However, we are supposing that μ is totally finite, so there must be a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $A\cup B$ such that $|f|\le_{\text{a.e.}}\sup_{n\in\mathbb{N}}|f_n|$ for every $f\in A\cup B$. **P** For each $q\in\mathbb{Q}$, choose a countable set $C_q\subseteq A\cup B$ such that $\{x:|f(x)|\ge q\}\setminus\bigcup_{g\in C_q}\{x:|g(x)|\ge q\}$ is negligible for any $f\in A\cup B$ (215B(iv)); let $\langle f_n\rangle_{n\in\mathbb{N}}$ run over $\bigcup_{q\in\mathbb{Q}}C_q$. **Q** Set $h_1=\chi X+\sup_{n\in\mathbb{N}}|f_n|$; then h_1 is finite-valued, strictly positive and measurable, and $|f|\le_{\text{a.e.}}h_1$ for every $f\in A\cup B$. By 465Ch, $A_1=\{f/h_1:f\in A\}$ and $B_1=\{f/h_1:f\in B\}$ are stable; by (v) here, A_1+A_2 is stable; by 465Ch again, $A+B=\{g\times h_1:g\in A_1+A_2\}$ is stable. So we're done.

(d) Since A is pointwise bounded, the closure $\overline{\Gamma(A)}$ of $\Gamma(A)$ in $\mathbb{R}^X$ for the topology of pointwise convergence is compact. **?** Suppose, if possible, that there is a $g\in\overline{\Gamma(A)}\setminus\mathcal{L}^0(\Sigma)$. Then there must be a measurable set E of finite measure and $\alpha<\beta$ in $\mathbb{R}$ such that $\mu^*P=\mu^*Q=\mu E>0$, where

$$P=\{x:x\in E,\,g(x)\le\alpha\},\quad Q=\{x:x\in E,\,g(x)\ge\beta\}$$

(see part (a) of the proof of 465D). Set $Y_n=\{x:x\in E,\,|f(x)|\le n\text{ for every }f\in A\}$; then $\langle Y_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with union E, so there is an $n\in\mathbb{N}$ such that $\mu^*(P\cap Y_n)$ and $\mu^*(Q\cap Y_n)$ are both at least $\frac{2}{3}\mu E$. Let F', F'' be measurable envelopes of $P\cap Y_n$ and $Q\cap Y_n$ respectively, and $Y=F'\cap F''\cap Y_n$; then

$$\begin{aligned}0&<\mu(F'\cap F'')=\mu^*(F'\cap F''\cap P\cap Y_n)\\&=\mu^*(F'\cap F''\cap Q\cap Y_n)=\mu^*Y=\mu^*(P\cap Y)=\mu^*(Q\cap Y).\end{aligned}$$

Let μ_Y be the subspace measure on Y and Σ_Y its domain, and consider the set $A_Y=\{f\restriction Y:f\in A\}$. With respect to the measure μ_Y, this is stable (465Cm). Also it is uniformly bounded. So $\Gamma(A_Y)$ is μ_Y-stable, by (a) of this theorem. As μ_Y is complete and totally finite, the closure $\overline{\Gamma(A_Y)}$ for the topology of pointwise convergence in $\mathbb{R}^Y$ is included in $\mathcal{L}^0(\Sigma_Y)$ (465Da). Since $f\mapsto f\restriction Y:\mathbb{R}^X\to\mathbb{R}^Y$ is linear and continuous for the topologies of pointwise convergence, $g\restriction Y\in\overline{\Gamma(A_Y)}$ and $g\restriction Y$ is Σ_Y-measurable. But

$$\mu_Y^*(P\cap Y)=\mu^*(P\cap Y)=\mu^*(Q\cap Y)=\mu_Y^*(Q\cap Y)=\mu_YY\in\,]0,\infty[$$

(214Cd), so this is impossible. **X**

Thus $\overline{\Gamma(A)}\subseteq\mathcal{L}^0(\Sigma)$ and $\Gamma(A)$ is relatively compact.

465O Stable sets in L^0 The notion of 'stability' as defined in 465B is applicable only to true functions; in such examples as 465Xl, the irregularity of the set A is erased entirely if we look at its image in the space L^0 of equivalence classes of measurable functions. We do, however, have a corresponding concept for subsets of function spaces, which can be expressed in the language of §325. If $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra, and $k\ge1$, I write $(\widehat{\bigotimes}_k\mathfrak{A},\bar\mu^k)$ for the localizable measure algebra free product of k copies of $(\mathfrak{A},\bar\mu)$, as described in 325H. If $Q\subseteq L^0(\mathfrak{A})$, $k\ge1$, $a\in\mathfrak{A}$ has finite measure and $\alpha<\beta$ in $\mathbb{R}$, set

$$\begin{aligned}d_k(Q,a,\alpha,\beta)=\sup_{v\in Q}\bigl((a\cap[\![v\le\alpha]\!])\otimes(a\cap[\![v\ge\beta]\!])\otimes\ldots\\ \otimes(a\cap[\![v\le\alpha]\!])\otimes(a\cap[\![v\ge\beta]\!])\bigr)\end{aligned}$$

in $\widehat{\bigotimes}_{2k}\mathfrak{A}$, taking k repetitions of the formula $(a\cap[\![v\le\alpha]\!])\otimes(a\cap[\![v\ge\beta]\!])$ to match the corresponding formula

$$D_k(A,E,\alpha,\beta)=\textstyle\bigcup_{f\in A}((E\cap\{x:f(x)\le\alpha\})\times(E\cap\{x:f(x)\ge\beta))^k.$$

(Note that the supremum $\sup_{v\in Q}\ldots$ is defined because $a^{\otimes2k}=a\otimes\ldots\otimes a$ has finite measure in the measure algebra $(\widehat{\bigotimes}_{2k}\mathfrak{A},\bar\mu^{2k})$. Of course I mean to take $d_k(Q,a,\alpha,\beta)=0$ if $Q=\emptyset$.) Now we can say that Q is **stable** if whenever $0<\bar\mu a<\infty$ and $\alpha<\beta$ there is a $k\ge1$ such that $\bar\mu^{2k}d_k(Q,a,\alpha,\beta)<(\bar\mu a)^{2k}$; that is, $d_k(Q,a,\alpha,\beta)\ne a\otimes\ldots\otimes a$.

We have the following relationships between the two concepts of stability.

465P Theorem Let (X,Σ,μ) be a semi-finite measure space, with measure algebra $(\mathfrak{A},\bar\mu)$.

(a) Suppose that $A\subseteq\mathcal{L}^0(\Sigma)$ and that $Q=\{f^\bullet:f\in A\}\subseteq L^0(\mu)$, identified with $L^0=L^0(\mathfrak{A})$ (364Ic[14]). Then Q is stable in the sense of 465O iff every countable subset of A is stable in the sense of 465B.

(b) If μ is strictly localizable and $Q\subseteq L^0(\mu)$ is stable, then there is a stable set $B\subseteq\mathcal{L}^0$ such that $Q=\{f^\bullet:f\in B\}$.

proof (a)(i) Suppose that all countable subsets of A are stable, and take $a\in\mathfrak{A}$ such that $0<\bar\mu a<\infty$ and $\alpha<\beta$ in $\mathbb{R}$. For each $k\in\mathbb{N}$ there is a countable set $Q_k\subseteq Q$ such that $d_k(Q_k,a,\alpha,\beta)=d_k(Q,a,\alpha,\beta)$, because $a^{\otimes 2k}$ has finite measure in $\widehat{\bigotimes}_{2k}\mathfrak{A}$. Now there is a countable set $A'\subseteq A$ such that $\{f^\bullet:f\in A'\}=\bigcup_{k\in\mathbb{N}}Q_k$. Let $E\in\Sigma$ be such that $E^\bullet=a$. As $\mu E=\bar\mu a\in\,]0,\infty[$ and A' is stable, there is a $k\ge 1$ such that $(\mu^{2k})^*D_k(A',E,\alpha,\beta)<(\mu E)^{2k}$. Because A' is countable, $D_k(A',E,\alpha,\beta)$ is measurable. But 325He tells us that we have an order-continuous measure-preserving Boolean homomorphism π from the measure algebra of μ^{2k} to $\widehat{\bigotimes}_{2k}\mathfrak{A}$, such that $\pi(\prod_{i<2k}F_i)^\bullet=F_0^\bullet\otimes\ldots\otimes F_{2k-1}^\bullet$ for all $F_0,\ldots,F_{2k-1}\in\Sigma$; accordingly

$$\begin{aligned}\bar\mu^{2k}d_k(Q,a,\alpha,\beta)&=\bar\mu^{2k}d_k(Q_k,a,\alpha,\beta)\le\bar\mu^{2k}d_k(\bigcup_{i\in\mathbb{N}}Q_i,a,\alpha,\beta)\\&=\bar\mu^{2k}(\pi D_k(A',E,\alpha,\beta)^\bullet)=\mu^{2k}D_k(A',E,\alpha,\beta)\\&<(\mu E)^{2k}=\bar\mu^{2k}a^{\otimes 2k}.\end{aligned}$$

As a, α and β are arbitrary, Q is stable.

(ii) Now suppose that Q is stable and that A' is any countable subset of A. Take $E\in\Sigma$ such that $0<\mu E<\infty$, and $\alpha<\beta$ in $\mathbb{R}$. Set $a=E^\bullet\in\mathfrak{A}$. This time, writing Q' for $\{f^\bullet:f\in A'\}$, we have

$$\pi D_k(A',E,\alpha,\beta)^\bullet=d_k(Q',a,\alpha,\beta)\subseteq d_k(Q,a,\alpha,\beta)$$

for every $k\ge 1$. There is some k such that $d_k(Q,a,\alpha,\beta)\ne a^{\otimes 2k}$, and in this case $\mu^{2k}D_k(A',E,\alpha,\beta)<(\mu E)^{2k}$; as E, α and β are arbitrary, A' is stable.

(b)(i) If $\mu X=0$ this is trivial; suppose that $\mu X>0$. Replacing μ by its completion does not change either $L^0(\mu)$ or the stable subsets of $\mathbb{R}^X$ (241Xb, 465Ci), and leaves μ strictly localizable (212Gb), so we may suppose that μ is complete. Let $\langle E_i\rangle_{i\in I}$ be a decomposition of X into sets of finite measure. Amalgamating any negligible E_i into other non-negligible ones, we may suppose that $\mu E_i>0$ for each i. Writing μ_i for the subspace measure on E_i, we have a consistent lifting ϕ_i for μ_i (346J). Set $\phi E=\bigcup_{i\in I}\phi_i(E\cap X_i)$ for $E\in\Sigma$; then ϕ is a lifting for μ. Let θ be the corresponding lifting from $\mathfrak{A}$ to Σ (341B) and $T:L^\infty(\mathfrak{A})\to\mathcal{L}^\infty(\Sigma)$ the associated linear operator, defined by saying that $T(\chi a)=\chi(\theta a)$ for every $a\in\mathfrak{A}$ (363F). Since $\theta(a)^\bullet=a$ for every $a\in\mathfrak{A}$, $(Tv)^\bullet=v$ for every $v\in L^\infty$.

(ii) We need to know that if $v\in L^\infty$ and $\alpha<\alpha'$, then $\{x:(Tv)(x)\le\alpha\}\subseteq\phi(\{x:(Tv)(x)\le\alpha'\})$. **P** Let $v'\in S(\mathfrak{A})$ be such that $\|v-v'\|_\infty\le\frac{1}{2}(\alpha'-\alpha)$ (363C), and set $\gamma=\frac{1}{2}(\alpha+\alpha')$. Express v' as $\sum_{i=0}^n\alpha_i\chi a_i$ where $a_0,\ldots,a_n\in\mathfrak{A}$ are disjoint. Then $Tv'=\sum_{i=0}^n\alpha_i\chi(\theta a_i)$. Now

$$\|Tv-Tv'\|_\infty\le\|v-v'\|_\infty\le\gamma-\alpha=\alpha'-\gamma,$$

so

$$\begin{aligned}\{x:(Tv)(x)\le\alpha\}&\subseteq\{x:(Tv')(x)\le\gamma\}\\&=\bigcup\{\theta a_i:i\le n,\,\alpha_i\le\gamma\}=\phi(\bigcup\{\theta a_i:i\le n,\,\alpha_i\le\gamma\})\end{aligned}$$

(because $\phi(\theta a)=\theta a$ for every $a\in\mathfrak{A}$)

$$=\phi(\{x:(Tv')(x)\le\gamma\})\subseteq\phi(\{x:(Tv)(x)\le\alpha'\}),$$

as claimed. **Q**

Similarly, if $\beta'<\beta$ then $\{x:(Tv)(x)\ge\beta\}\subseteq\phi(\{x:(Tv)(x)\ge\beta'\})$.

(iii) For the moment, suppose that $Q\subseteq L^\infty(\mathfrak{A})$, which we may identify with $L^\infty(\mu)$ (363I). Set $B=T[Q]$, so that $Q=\{f^\bullet:f\in B\}$. Then B is stable. **P** Let $E\in\Sigma$ be such that $0<\mu E<\infty$, and $\alpha<\beta$. Let $i\in I$ be such that $\mu(E\cap E_i)>0$, and α', $\beta'\in\mathbb{R}$ such that $\alpha<\alpha'<\beta'<\beta$. Setting $a=(E\cap E_i)^\bullet$, we have $0<\bar\mu a<\infty$, so there is some $k\in\mathbb{N}$ such that $d_k(Q,a,\alpha',\beta')\ne a^{\otimes 2k}$. Let π be the measure-preserving Boolean homomorphism from the

[14] Formerly 364Jc.

measure algebra of μ^{2k} to $\widehat{\bigotimes}_{2k}\mathfrak{A}$ described in part (a) of this proof; as noted in 325He, the present context is enough to ensure that π is an isomorphism. So there is a $W\in\operatorname{dom}\mu^{2k}$ such that $\pi W^\bullet=d_k(Q,a,\alpha',\beta')$; since

$$d_k(Q,a,\alpha',\beta')\subseteq a^{\otimes 2k}\subseteq(E_i^{2k})^\bullet,$$

we may suppose that $W\subseteq E_i^{2k}$. If $f\in B$, then

$$\pi D_k(\{f\},E,\alpha',\beta')^\bullet=d_k(\{f^\bullet\},a,\alpha',\beta')\subseteq d_k(Q,a,\alpha',\beta'),$$

so $D_k(\{f\},E,\alpha',\beta')\setminus W$ is negligible.

At this point, recall that ϕ_i was supposed to be a consistent lifting for μ_i. So we have a lifting ϕ' of μ_i^{2k} such that $\phi'(\prod_{j<2k}F_j)=\prod_{j<2k}\phi_iF_j$ for all $F_0,\dots,F_{2k-1}\in\Sigma_i$. In particular, if $f\in B$ and we set

$$F_{2j}=\{x:x\in E\cap E_i,\ f(x)\le\alpha\},\quad F_{2j+1}=\{x:x\in E\cap E_i,\ f(x)\ge\beta\},$$

$$F'_{2j}=\{x:x\in E\cap E_i,\ f(x)\le\alpha'\},\quad F'_{2j+1}=\{x:x\in E\cap E_i,\ f(x)\ge\beta'\},$$

for $j<k$, we shall have

$$\prod_{j<2k}F_j\subseteq\prod_{j<2k}\phi F'_j$$

(by (ii) above, because $f=Tv$ for some v)

$$=\prod_{j<2k}\phi_iF'_j=\phi'(\prod_{j<2k}F'_j)\subseteq\phi'W$$

because $\prod_{j<2k}F'_j=D_k(\{f\},E,\alpha',\beta')$. As f is arbitrary, $D_k(B,E\cap E_i,\alpha,\beta)\subseteq\phi'W$. But now

$$(\mu^{2k})^*D_k(B,E\cap E_i,\alpha,\beta)\le\mu^{2k}(\phi'W)=\mu_i^{2k}(\phi'W)$$

(251Wl)

$$=\mu_i^{2k}W=\mu^{2k}W=\bar\mu^{2k}d_k(Q,a,\alpha',\beta')$$
$$<\bar\mu^{2k}(a^{\otimes 2k})=\mu(E\cap E_i)^{2k}.$$

As usual, it follows that $(\mu^{2k})^*D_k(B,E,\alpha,\beta)<(\mu E)^{2k}$; as E, α and β are arbitrary, B is stable, as claimed. **Q**

(iv) Thus the result is true if Q is included in the unit ball of L^∞. In general, set $h(\alpha)=\tanh\alpha$ for $\alpha\in\mathbb{R}$, and consider

$$A=\{f:f\in\mathcal{L}^0(\Sigma),\ f^\bullet\in Q\},\quad A'=\{hf:f\in A\},\quad Q'=\{(hf)^\bullet:f\in A\}.$$

By (a), every countable subset of A is stable, so every countable subset of A' is stable (465Ck) and Q' is stable. But Q' is included in the unit ball of L^∞, so there is a stable set $B'\subseteq\mathcal{L}^0$ such that $Q'=\{g^\bullet:g\in B'\}$. Setting $B=\{h^{-1}g:g\in B'\}$, B is stable and $\{f^\bullet:f\in B\}=Q$. So we have the general theorem.

465Q Remarks Using 465Pa, we can work through the first part of this section to get a list of properties of stable subsets of L^0. For instance, the convex hull of an order-bounded stable set in L^0 is stable, as in 465Na. It is harder to relate such results as 465M to the idea of stability in L^0, but the argument of 465Pb gives a line to follow: if (X,Σ,μ) is complete and strictly localizable, there is a linear operator $T:L^\infty(\mu)\to\mathcal{L}^\infty(\Sigma)$, defined from a lifting, such that, for $Q\subseteq L^\infty$, $T[Q]$ is stable iff Q is stable. So when μ is a complete probability measure, we can look at the averages $\psi_{wk}(v)=\frac{1}{k}\sum_{i=0}^{k-1}(Tv)(w(i))$ for $v\in Q$, $w\in X^{\mathbb{N}}$ to devise criteria for stability of Q in terms of the linear functionals ψ_{wk}.

Working in L^1, however, we can look for results of a different type, as follows.

465R Theorem (Talagrand 84) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $T:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{B},\bar\nu)$ a bounded linear operator. If Q is stable and order-bounded in $L^1(\mathfrak{A},\bar\mu)$, then $T[Q]\subseteq L^1(\mathfrak{B},\bar\nu)$ is stable.

proof (a) To begin with (down to (d) below) let us suppose that

$(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are the measure algebras of measure spaces (X,Σ,μ) and (Y,T,ν),

so that we can identify $L^1_{\bar\mu}$, $L^1_{\bar\nu}$ with $L^1(\mu)$ and $L^1(\nu)$ (365B),

Q is countable,

so that Q can be expressed as $\{f^\bullet : f \in A\}$, where $A \subseteq \mathcal{L}^0(\Sigma)$ is countable and stable (465P),

T is positive,

μ and ν are totally finite,

$T(\chi X^\bullet) = \chi Y^\bullet$,

$Q \subseteq L^\infty(\mathfrak{A})$ is $\| \|_\infty$-bounded,

so that we may take $A \subseteq \mathcal{L}^\infty$ to be $\| \|_\infty$-bounded,

$\nu Y = 1$,

$\mu X = 1$,

and that $\|T\| \le 1$.

(b) The idea of the argument is that for any $n \ge 1$ we have a positive linear operator $U_n : L^1(\mu^n) \to L^1(\nu^n)$ defined as follows.

If $f_0, \dots , f_{n-1} \in \mathcal{L}^1(\mu)$, set $(f_0 \otimes \dots \otimes f_{n-1})(w) = \prod_{i=0}^n f_i(w(i))$ whenever $w \in \prod_{i<n} \operatorname{dom} f_i$. Now we can define $u_0 \otimes u_1 \otimes \dots \otimes u_{n-1} \in L^1(\mu^n)$, for $u_0, \dots , u_{n-1} \in L^1(\mu)$, by saying that $f_0^\bullet \otimes \dots \otimes f_{n-1}^\bullet = (f_0 \otimes \dots \otimes f_{n-1})^\bullet$ for all $f_0, \dots , f_{n-1} \in \mathcal{L}^1(\mu)$, as in 253E.

Define the operators U_n inductively. $U_1 = T$. Given that $U_n : L^1(\mu^n) \to L^1(\nu^n)$ is a positive linear operator, then we have a bilinear operator $\psi : L^1(\mu^n) \times L^1(\mu) \to L^1(\nu^{n+1})$ defined by saying that $\psi(q, u) = U_n q \otimes Tu$ for $q \in L^1(\mu^n)$, $u \in L^1(\mu)$, where $\otimes : L^1(\nu^n) \times L^1(\nu) \to L^1(\nu^n \times \nu) \cong L^1(\nu^{n+1})$ is the operator of 253E. By 253F, there is a (unique) bounded linear operator $U_{n+1} : L^1(\mu^{n+1}) \to L^1(\nu^{n+1})$ such that $U_{n+1}(q \otimes u) = \psi(q, u)$ for all $q \in L^1(\mu^n)$, $u \in L^1(\mu)$. To see that U_{n+1} is positive, use 253Gc. (Remember that we are supposing that T is positive.) Continue.

Now it is easy to check that

$$U_n(u_0 \otimes \dots \otimes u_{n-1}) = Tu_0 \otimes \dots \otimes Tu_{n-1}$$

for all $u_0, \dots , u_{n-1} \in L^1(\mu)$. Moreover, $\|U_{n+1}\| \le \|U_n\|\|T\|$ for every n (see 253F), so $\|U_n\| \le 1$ for every n.

For $i < n \in \mathbb{N}$, we have a natural operator $R_{ni} : L^1(\mu) \to L^1(\mu^n)$, defined by saying that $R_{ni} f^\bullet = (f\pi_{ni})^\bullet$ for every $f \in \mathcal{L}^1(\mu)$, where $\pi_{ni}(w) = w(i)$ for $w \in X^n$. Similarly, we have an operator $S_{ni} : L^1(\nu) \to L^1(\nu^n)$. Observe that

$$R_{ni}u = e \otimes \dots \otimes e \otimes u \otimes e \otimes \dots \otimes e$$

where $e = \chi X^\bullet$ and the u is put in the position corresponding to the coordinate i. Since $Te = (\chi Y)^\bullet = e'$ say,

$$U_n R_{ni} u = e' \otimes \dots \otimes Tu \otimes \dots \otimes e' = S_{ni} Tu$$

for every $u \in L^1(\mu)$.

(c) Let $B \subseteq \mathcal{L}^\infty(\mathrm{T})$ be a countable $\| \|_\infty$-bounded set such that $T[Q] = \{g^\bullet : g \in B\}$. ($T[Q]$ is $\| \|_\infty$-bounded because T is positive and $T(\chi X)^\bullet = (\chi Y)^\bullet$.) I seek to show that B is stable by using the criterion 465M(v). Let $\epsilon > 0$. Then there is an $m \ge 1$ such that $\int f_{kl}(w)\mu^{\mathbb{N}}(dw) \le \epsilon$ for any k, $l \ge m$, writing

$$f_{kl}(w) = \sup_{f\in A} |\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))|$$

for $w \in X^{\mathbb{N}}$; note that f_{kl} is measurable because A is countable.

Take any k, $l \ge m$ and consider

$$g_{kl}(z) = \sup_{g\in B} |\frac{1}{k}\sum_{i=0}^{k-1} g(z(i)) - \frac{1}{l}\sum_{i=0}^{l-1} g(z(i))|$$

for $z \in Y^{\mathbb{N}}$. I claim that $\int g_{kl} d\nu^{\mathbb{N}} \le \epsilon$. **P** Set $n = \max(k, l)$. Then $\int g_{kl} d\nu^{\mathbb{N}} = \int \tilde{g}\, d\nu^n$, where

$$\tilde{g}(z) = \sup_{g\in B} |\frac{1}{k}\sum_{i=0}^{k-1} g(z(i)) - \frac{1}{l}\sum_{i=0}^{l-1} g(z(i))|$$

for $z \in Y^n$. If we look at $\tilde{g}^\bullet$ in $L^1(\nu^n)$, we see that it is

$$\sup_{g\in B} |\frac{1}{k}\sum_{i=0}^{k-1} S_{ni} g^\bullet - \frac{1}{l}\sum_{i=0}^{l-1} S_{ni} g^\bullet|$$

where $S_{ni} : L^1(\nu) \to L^1(\nu^n)$ is defined in (b) above. Thus

$$\tilde{g}^\bullet = \sup_{v\in T[Q]} |\frac{1}{k}\sum_{i=0}^{k-1} S_{ni} v - \frac{1}{l}\sum_{i=0}^{l-1} S_{ni} v|.$$

Similarly, setting

$$\tilde{f}(w) = \sup_{f\in A} |\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))|$$

for $w \in X^n$,

$$\tilde{f}^\bullet = \sup_{u\in Q} |\frac{1}{k}\sum_{i=0}^{k-1} R_{ni}u - \frac{1}{l}\sum_{i=0}^{l-1} R_{ni}u|.$$

Now consider $U_n\tilde{f}^\bullet$. For any $v \in T[Q]$, we can express v as Tu where $u \in Q$, so

$$\begin{aligned} |\frac{1}{k}\sum_{i=0}^{k-1} S_{ni}v - \frac{1}{l}\sum_{i=0}^{l-1} S_{ni}v| &= |\frac{1}{k}\sum_{i=0}^{k-1} S_{ni}Tu - \frac{1}{l}\sum_{i=0}^{l-1} S_{ni}Tu| \\ &= |\frac{1}{k}\sum_{i=0}^{k-1} U_nR_{ni}u - \frac{1}{l}\sum_{i=0}^{l-1} U_nR_{ni}u| \end{aligned}$$

(because $U_nR_{ni} = S_{ni}T$, as noted in (b) above)

$$\begin{aligned} &= |U_n(\frac{1}{k}\sum_{i=0}^{k-1} R_{ni}u - \frac{1}{l}\sum_{i=0}^{l-1} R_{ni}u)| \\ &\le U_n(|\frac{1}{k}\sum_{i=0}^{k-1} R_{ni}u - \frac{1}{l}\sum_{i=0}^{l-1} R_{ni}u|) \end{aligned}$$

(because U_n is positive)

$$\le U_n\tilde{f}^\bullet.$$

As v is arbitrary, $\tilde{g}^\bullet \le U_n\tilde{f}^\bullet$, and

$$\int g_{kl}d\nu^{\mathbb{N}} = \int \tilde{g}\,d\nu^n = \|\tilde{g}^\bullet\|_1 \le \|U_n\tilde{f}^\bullet\|_1 \le \|\tilde{f}^\bullet\|_1$$

(because $\|U_n\| \le 1$)

$$\le \epsilon$$

because k, $l \ge m$. **Q**

(d) As ϵ is arbitrary, B satisfies the criterion 465M(v), and is stable. So $T[Q]$ is stable, by 465Pa in the other direction.

(e) Now let us seek to unwind the list of special assumptions used in the argument above. Suppose we drop the last two, and assume only that

$(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are the measure algebras of measure spaces (X, Σ, μ) and (Y, T, ν),
Q is countable,
T is positive,
μ and ν are totally finite,
$T(\chi X^\bullet) = \chi Y^\bullet$,
$Q \subseteq L^\infty(\mathfrak{A})$ is $\| \|_\infty$-bounded,
$\nu Y = 1$.

Then $T[Q]$ is stable. **P** Define $\mu_1 : \Sigma \to [0, \infty[$ by setting $\mu_1 E = \int T(\chi E^\bullet)$ for every $E \in \Sigma$. Then μ_1 is countably additive because T is (sequentially) order-continuous (355Ka). If $\mu E = 0$ then $\chi E^\bullet = 0$ in $L^1(\mu)$ and $\mu_1 E = 0$, so μ_1 is truly continuous with respect to μ (232Bb) and has a Radon-Nikodým derivative (232E). By 465Cj, A is stable with respect to μ_1, while $\mu_1 X = \nu Y = 1$, because $T(\chi X)^\bullet = \chi Y^\bullet$.

Let $(\mathfrak{A}_1, \bar{\mu}_1)$ be the measure algebra of μ_1. If $E \in \Sigma$ and $\mu_1 E = 0$, then $T(\chi E^\bullet) = 0$. Accordingly we can define an additive function $\theta : \mathfrak{A}_1 \to L^1(\nu)$ by setting $\theta E^\bullet = T(\chi E^\bullet)$ for every $E \in \Sigma$. (Note that the two $^\bullet$s here must be interpreted differently. In the formula $\theta E^\bullet$, the equivalence class $E^\bullet$ is to be taken in $\mathfrak{A}_1$. In the formula $\chi E^\bullet = (\chi E)^\bullet$, the equivalence class is to be taken in $L^0(\mu)$. In the rest of this proof I will pass over such points without comment; I hope the context will always make it clear how each $^\bullet$ is to be read.) Because T is positive, θ is non-negative, and by the definition of μ_1 we have $\|\theta a\|_1 = \bar{\mu}_1 a$ for every $a \in \mathfrak{A}_1$. So we have a positive linear operator $T_1 : L^1(\mathfrak{A}_1, \bar{\mu}_1) \to L^1(\nu)$ defined by setting

$$T_1(\chi E^\bullet) = \theta E^\bullet = T(\chi E^\bullet)$$

for every $E \in \Sigma$ (365K).

If $f : X \to \mathbb{R}$ is simple (that is, a linear combination of indicator functions of sets in Σ), then $T_1 f^\bullet = T f^\bullet$. So this is also true for every $f \in \mathcal{L}^\infty(\Sigma)$; in particular, it is true for every $f \in A$, so that $T[Q] = T_1[Q_1]$, where $Q_1 = \{f^\bullet : f \in A\} \subseteq L^1(\mu_1)$. But μ_1, Q_1 and T_1 satisfy all the conditions of (a), so (b)-(d) tell us that $T_1[Q_1]$ is stable, and $T[Q]$ is stable, as required. **Q**

(f) The next step is to drop the condition '$\nu Y = 1$'. But this is elementary, since we are still assuming that ν is totally finite, and multiplying ν by a non-zero scalar doesn't change $L^0(\nu)$ or the stability of any of its subsets, while the case $\nu Y = 0$ is trivial. So we conclude that if

$(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are the measure algebras of measure spaces (X, Σ, μ) and (Y, T, ν),
Q is countable,
T is positive,
μ and ν are totally finite,
$T(\chi X^\bullet) = \chi Y^\bullet$,
$Q \subseteq L^\infty(\mathfrak{A})$ is $\| \|_\infty$-bounded,

then $T[Q]$ is stable.

(g) We can now attack what remains. We find that if

$(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are the measure algebras of measure spaces (X, Σ, μ) and (Y, T, ν),
Q is countable,
T is positive,

then $T[Q]$ is stable. **P** At this point recall that we are supposing that Q is order-bounded. Let $u_0 \in L^1(\mu)$ be such that $|u| \le u_0$ for every $u \in Q$; let $f_0 \in \mathcal{L}^0(\Sigma)^+$ be such that $f_0^\bullet = u_0$. Setting $A' = \{(f \wedge f_0) \vee (-f_0) : f \in A\}$, A' is still stable, because its image in $L^1(\mu)$ is still Q, or otherwise. Set $v_0 = Tu_0$, and let $g_0 \in \mathcal{L}^0(\mathrm{T})^+$ be such that $g_0^\bullet = v_0$. Because T is positive, $|Tu| \le T|u| \le v_0$ for every $u \in Q$. So we can represent $T[Q]$ as $\{g^\bullet : g \in B\}$, where $B \subseteq \mathcal{L}^0(\mathrm{T})$ is a countable set and $|g| \le g_0$ for every $g \in B$. Set $F_0 = \{y : g_0(y) \ne 0\}$.

Define measures μ_1, ν_1 by setting $\mu_1 E = \int_E f_0 d\mu$ for $E \in \Sigma$, $\nu_1 F = \int_F g_0 d\nu$ for $F \in \mathrm{T}$. Then both μ_1 and ν_1 are totally finite. By 465Cj, A' is stable with respect to μ_1. Set $A_1 = \{\frac{f}{f_0} : f \in A'\}$, interpreting $\frac{f}{f_0}(x)$ as 0 if $f_0(x) = 0$; then A_1 is stable with respect to μ_1, by 465Ch, and $\|f\|_\infty \le 1$ for every $f \in A_1$. Take $Q_1 = \{f^\bullet : f \in A_1\} \subseteq L^1(\mu_1)$, so that Q_1 is stable.

We have a norm-preserving positive linear operator $R : L^1(\mu_1) \to L^1(\mu)$ defined by setting $Rf^\bullet = (f \times f_0)^\bullet$ for every $f \in \mathcal{L}^1(\mu_1)$ (use 235A). Observe that $R[Q_1] = Q$ and $R(\chi X)^\bullet = u_0$. Similarly, we have a norm-preserving positive linear operator $S : L^1(\nu_1) \to L^1(\nu)$ defined by setting $Sg^\bullet = (g \times g_0)^\bullet$ for $g \in \mathcal{L}^1(\nu_1)$. The set of values of S is just

$$\{g^\bullet : g \in \mathcal{L}^1(\nu),\, g(y) = 0 \text{ whenever } g_0(y) = 0\},$$

which is the band in $L^1(\nu)$ generated by v_0. So

$$\{u : u \in L^1(\mu_1),\, TR|u| \in S[L^1(\nu_1)]\}$$

is a band in $L^1(\nu_1)$ containing $\chi X^\bullet$, and must be the whole of $L^1(\mu_1)$. Thus we have a positive linear operator $T_1 = S^{-1}TR : L^1(\mu_1) \to L^1(\nu_1)$, and $T_1(\chi X)^\bullet = \chi Y^\bullet$ in $L^1(\nu_1)$.

By (f), $T_1[Q_1]$ is stable in $L^1(\nu_1)$. Observe that $T_1[Q_1] = \{S^{-1}g^\bullet : g \in B\} = \{g^\bullet : g \in B_1\}$, where $B_1 = \{\frac{g}{g_0} : g \in B\}$, interpreting $\frac{g}{g_0}(y)$ as 0 if $y \in Y \setminus F_0$. Consequently B_1 and $B = \{g \times g_0 : g \in B_1\}$ are stable with respect to ν_1. By 465Cj, once more, B is stable with respect to ν_0, where

$$\nu_0 F = \int_F \frac{1}{g_0} d\nu_1 = \nu(F \cap F_0)$$

for any $F \in \mathrm{T}$. But because $g(y) = 0$ whenever $g \in B$ and $y \in Y \setminus F_0$,

$$(\nu^{2k})^* D_k(B, F, \alpha, \beta) = (\nu^{2k})^* D_k(B, F \cap F_0, \alpha, \beta) = (\nu_0^{2k})^* D_k(B, F, \alpha, \beta)$$

whenever $F \in \mathrm{T}$, $\alpha < \beta$ and $k \geq 1$; so B is also stable with respect to ν, and $Q = \{g^\bullet : g \in B\}$ is stable in $L^1(\nu)$. **Q**

(h) The worst is over. If we are not told that T is positive, we know that it is expressible as the difference of positive linear operators T_1 and T_2 (371D); now $T_1[Q]$ and $T_2[Q]$ will be stable, by the work above, so $T[Q] \subseteq T_1[Q] - T_2[Q]$ is stable, by 465Nc. If we are not told that Q is countable, we refer to 465P to see that we need only check that countable subsets of $T[Q]$ are stable, and these are images of countable subsets of Q. Finally, the identification of the abstract measure algebras $(\mathfrak{A}, \bar{\mu}_1)$ and $(\mathfrak{B}, \bar{\nu}_1)$ with the measure algebras of measure spaces is Theorem 321J.

***465S R-stable sets** The theory above has been developed in the context of general measure (or probability) spaces and the 'ordinary' product measure of measure spaces. For τ-additive measures – in particular, for Radon measures – we have an alternative product measure, as described in §417. If $(X, \mathfrak{T}, \Sigma, \mu)$ is a semi-finite τ-additive topological measure space such that μ is inner regular with respect to the Borel sets, write $\tilde{\mu}^I$ for the τ-additive product measure on X^I, as described in 417C (for the product of two spaces) and 417E (for the product of any family of probability spaces); we can extend the construction of 417C to arbitrary finite products (417D). Now say that $A \subseteq \mathbb{R}^X$ is **R-stable** if whenever $0 < \mu E < \infty$ and $\alpha < \beta$ there is a $k \geq 1$ such that $(\tilde{\mu}^{2k})^* D_k(A, E, \alpha, \beta) < (\mu E)^{2k}$. Because we have a version of Fubini's theorem for the products of τ-additive topological measures (417H), all the arguments of this section can be applied to R-stable sets, yielding criteria for R-stability exactly like those in 465M.

Because the τ-additive product measure extends the c.l.d. product measure, stable sets are always R-stable. (We must have

$$(\tilde{\mu}^{2k})^* D_k(A, E, \alpha, \beta) \leq (\mu^{2k})^* D_k(A, E, \alpha, \beta)$$

for all k, A, E, α and β.) For an example of an R-stable set which is not stable, see 465U.

The concept of 'R-stability' is used in Talagrand 84 in applications to the integration of vector-valued functions. I give one result, however, to show how it is relevant to a question with a natural expression in the language of this chapter.

***465T Proposition** (Talagrand 84) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a semi-finite τ-additive topological measure space such that μ is inner regular with respect to the Borel sets. If $A \subseteq C(X)$ is such that every countable subset of A is R-stable, then A is R-stable.

proof For any $\alpha < \beta$ and $k \geq 1$,

$$D_k(A, X, \alpha, \beta) = \bigcup_{f \in A} \{w : w \in X^{2k}, f(w(2i)) < \alpha, \\ f(w(2i+1)) > \beta \text{ for } i < k\}$$

is open. Suppose that $0 < \mu E < \infty$. Because all the product measures $\tilde{\mu}^{2k}$ are τ-additive, we can find a countable set $A' \subseteq A$ such that

$$\tilde{\mu}^{2k} D_k(A, E, \alpha, \beta) = \tilde{\mu}^{2k} D_k(A', E, \alpha, \beta)$$

for every $k \geq 1$ and all rational α, β. Now, if $\alpha < \beta$, there are rational α', β' such that $\alpha < \alpha' < \beta' < \beta$, and a $k \geq 1$ such that $\tilde{\mu}^{2k} D_k(A', E, \alpha', \beta') < (\mu E)^{2k}$; in which case

$$\begin{aligned}(\tilde{\mu}^{2k})^* D_k(A, E, \alpha, \beta) &\leq \tilde{\mu}^{2k} D_k(A, E, \alpha', \beta') = \tilde{\mu}^{2k} D_k(A', E, \alpha', \beta') \\ &\leq \tilde{\mu}^{2k} D_k(A', E, \alpha', \beta') < (\mu E)^{2k}.\end{aligned}$$

As E, α and β are arbitrary, A is R-stable.

***465U** I come now to the promised example of an R-stable set which is not stable. I follow the construction in Talagrand 88, which displays an interesting characteristic related to 465O-465P above.

Example There is a Radon probability space with an R-stable set of continuous functions which is not stable.

proof (a) Let (X, Σ, μ) be an atomless probability space (e.g., the unit interval with Lebesgue measure). Define $\langle r_n \rangle_{n \in \mathbb{N}}$ by setting $r_0 = 1$, $r_1 = 2$, $r_{n+1} = 2^{nr_n}$ for $n \geq 1$; then $2^n \leq r_n < r_{n+1}$ for every n. Let $\langle \Sigma_n \rangle_{n \in \mathbb{N}}$ be an increasing sequence of finite subalgebras of Σ, each Σ_n having r_n atoms of the same size; this is possible because r_{n+1} is always a multiple of r_n. Write $\mathcal{H}_n$ for the set of atoms of Σ_n. Next, for each $n \in \mathbb{N}$, let $\langle G_H \rangle_{H \in \mathcal{H}_n}$ be an independent family in Σ_{n+1} of sets of measure 2^{-n}; such a family exists because r_{n+1} is a multiple of 2^{nr_n}.

Let $\mathcal{E}$ be the family of all sets expressible in the form $E = \bigcup_{i\in\mathbb{N}} H_i$ where, for some strictly increasing sequence $\langle n_i\rangle_{i\in\mathbb{N}}$ in $\mathbb{N}$, $H_i \in \mathcal{H}_{n_i}$ and $H_j \subseteq G_{H_i}$ whenever $i < j$ in $\mathbb{N}$. Set $A = \{\chi E : E \in \mathcal{E}\} \subseteq \mathcal{L}^0(\Sigma)$.

(b) A is stable. **P** Suppose that $F \in \Sigma$ and that $\mu F > 0$. Take $n \in \mathbb{N}$ so large that $3 \cdot 2^{-n} < (\mu F)^2$. Set $\mathcal{H} = \{H : H \in \mathcal{H}_n,\ \mu(H \cap F) > 0\}$; enumerate $\mathcal{H}$ as $\langle H_i\rangle_{i<m}$; set $F_1 = F \cap \bigcup_{i<m} H_i$, $V = \prod_{j<m}(H_j \cap F_1) \subseteq F_1^m$. Because $\mu F_1 = \mu F$, $m \ge r_n \mu F \ge 3$.

Consider

$$U_k = \bigcup_{H\in\mathcal{H}_k}(H \times H) \cup (H \times G_H) \cup (G_H \times H) \subseteq X^2$$

for $k \in \mathbb{N}$. Then

$$\mu^2 U_k \le r_k\Bigl(\frac{1}{r_k^2} + \frac{1}{2^k r_k} + \frac{1}{2^k r_k}\Bigr) \le 3 \cdot 2^{-k},$$

so

$$\mu^2(F_1^2 \setminus \bigcup_{k>n} U_k) \ge (\mu F)^2 - 3\sum_{k=n+1}^{\infty} 2^{-k} > 0.$$

Set

$$V = \{w : w \in F_1^{2m},\ w(2i) \in H_i \text{ for every } i < m,\ (w(1), w(3)) \notin \bigcup_{k>n} U_k\}.$$

Then

$$\mu^{2m}(V) \ge \prod_{i=0}^{m-1} \mu(F_1 \cap H_i) \cdot \mu^2(F_1^2 \setminus \bigcup_{k>n} U_k) \cdot (\mu F)^{m-2} > 0.$$

? Suppose, if possible, that there is a point $w \in V \cap D_m(A, F, 0, 1)$. Then there is an $E \in \mathcal{E}$ such that $w(2i) \notin E$, $w(2i+1) \in E$ for $i < m$. Express E as $\bigcup_{j\in\mathbb{N}} E_j$ where $E_j \in \mathcal{H}_{n_j}$ for every $j \in \mathbb{N}$, where $\langle n_j\rangle_{j\in\mathbb{N}}$ is strictly increasing, and $E_k \subseteq G_{E_j}$ whenever $j < k$. Because $w(2i) \in F_1 \cap H_i \setminus E$, E_j cannot include H_i for any $i < m$, $j \in \mathbb{N}$; so $E_j \cap H_i = \emptyset$ whenever $i < m$ and $n_j \le n$. Because $w(1)$, $w(3) \in E$, there are j_0, $j_1 \in \mathbb{N}$ such that $w(1) \in E_{j_0}$ and $w(3) \in E_{j_1}$. Since both $w(1)$ and $w(3)$ belong to $F_1 \subseteq \bigcup_{i<m} H_i$, n_{j_0} and n_{j_1} are both greater than n. But now

— if $j_0 = j_1$, then $(w(1), w(3)) \in E_{j_0}^2$,

— if $j_0 < j_1$, then $(w(1), w(3)) \in E_{j_0} \times E_{j_1} \subseteq E_{j_0} \times G_{E_{j_0}}$,

— if $j_1 < j_0$, then $(w(1), w(3)) \in E_{j_0} \times E_{j_1} \subseteq G_{E_{j_1}} \times E_{j_1}$,

so in any case $(w(1), w(3)) \in U_k$ where $k = \min(n_{j_0}, n_{j_1}) > n$, which is impossible. **X**

Thus $D_m(A, F, 0, 1)$ does not meet V, and $(\mu^{2m})^* D_m(A, F, 0, 1) < (\mu F)^{2m}$.

Now if $\alpha < \beta$, then $D_m(A, F, \alpha, \beta) = D_m(A, F, 0, 1)$ if $0 \le \alpha < \beta \le 1$, $\emptyset$ otherwise; so in all cases $(\mu^{2m})^* D_m(A, F, \alpha, \beta) < (\mu F)^{2m}$. As F also is arbitrary, A is stable, as claimed. **Q**

(c) Now let $(Z, \mathfrak{G}, \mathrm{T}, \nu)$ be the Stone space of the measure algebra of (X, Σ, μ) (321K), so that ν is a Radon measure (411Pe). For $E \in \Sigma$, write E^* for the corresponding open-and-closed set in Z, so that $E \mapsto E^* : \Sigma \to \mathrm{T}$ is a measure-preserving Boolean homomorphism. Set $A^* = \{\chi E^* : E \in \mathcal{E}\} \subseteq C(Z)$. Write ν^m for the c.l.d. product measure on Z^m for $m \ge 1$. We already know that ν^2 is not a topological measure (419E, 419Xc).

(d) The point is that $(\nu^{2m})^* D_m(A^*, Z, 0, 1) = 1$ for every $m \ge 1$. **P?** Suppose, if possible, otherwise. Then there is a set $\tilde{W} \subseteq Z^{2m}$ such that $\nu^{2m}(\tilde{W}) > 0$ and $\tilde{W} \cap D_m(A^*, Z, 0, 1) \ne \emptyset$. There is an $\epsilon > 0$ such that

$$\tilde{V} = \{v : v \in Z^m,\ \nu^m\{u : u \in Z^m,\ u\#v \in \tilde{W}\} \text{ is defined and greater than } m\epsilon\}$$

has non-zero inner measure for ν^m. Now there are sets $\tilde{F}_{ij} \in \mathrm{T}$, for $i \in \mathbb{N}$ and $j < m$, such that

$$Z^m \setminus \tilde{V} \subseteq \bigcup_{i\in\mathbb{N}} \prod_{j<m} \tilde{F}_{ij}, \quad \sum_{i=0}^{\infty} \prod_{j=0}^{m} \nu \tilde{F}_{ij} < 1.$$

Enlarging the $\tilde{F}_{ij}$ slightly if need be, we may suppose that they are all open-and-closed (322Rc), therefore expressible as F_{ij}^* where $F_{ij} \in \Sigma$ for $i \in \mathbb{N}$, $j < m$. Set $V = X^m \setminus \bigcup_{i\in\mathbb{N}} \prod_{j<m} F_{ij}$, so that

$$\mu^m V = \nu^m(Z^m \setminus \bigcup_{i\in\mathbb{N}} \prod_{j<m} F_{ij}^*) > 0.$$

I seek to choose $\langle n_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$, $\langle H_k\rangle_{k\in\mathbb{N}}$ and $\langle E_{kj}\rangle_{k\in\mathbb{N},j<m}$ in Σ inductively, in such a way that

$$2^{-n_0} \le \tfrac{1}{2}\epsilon,$$

$$E_{kj} \subseteq E_{ij} \cap G_{H_i} \text{ for } i < k \text{ and } j < m,$$

$$\prod_{j<m} E_{kj} \cap \prod_{j<m} F_{kj} = \emptyset,$$

$$\mu^m(V \cap G^m_{H_k} \cap \prod_{j<m} E_{kj}) > 0,$$

$$n_i < n_k \text{ for every } i < k,$$

$$H_k \in \mathcal{H}_{n_k},$$

$$\text{if } k = i_k m + j_k, \text{ where } i_k \in \mathbb{N} \text{ and } j_k < m, \text{ then } \mu(H_k \cap E_{kj_k}) > 0,$$

for every $k \in \mathbb{N}$. The induction proceeds as follows. Set $E'_{kj} = X$ if $k = 0$, $G_{H_{k-1}} \cap E_{k-1,j}$ otherwise, so that $\mu(V \cap \prod_{j<m} E'_{kj}) > 0$. Because $V \cap \prod_{j<m} F_{kj}$ is empty, we can find $E_{kj} \subseteq E'_{kj}$, for $j < m$, such that $\mu^m(V \cap \prod_{j<m} E_{kj}) > 0$ and $\prod_{j<m} E_{kj} \cap \prod_{j<m} F_{kj} = \emptyset$. Set $\eta = \mu E_{kj_k}$, $\delta = \mu^m(V \cap \prod_{j<m} E_{kj})$, so that η and δ are both strictly positive.

Now take n_k so large that

$$(\text{if } k = 0)\ 2^{-n_k} \le \tfrac{1}{2}\epsilon, \quad n_k > n_i \text{ for } i < k, \quad (1 - 2^{-mn_k})^{\eta r_{n_k}} < \delta.$$

(This is possible because $\lim_{n\to\infty} 2^{-mn} r_n = \infty$.) Set $\mathcal{H} = \{H : H \in \mathcal{H}_{n_k}, \mu(H \cap E_{kj_k}) > 0\}$; then $\#(\mathcal{H}) \ge \eta r_{n_k}$. Consider the family $\langle G^m_H \rangle_{H\in\mathcal{H}}$. These are stochastically independent sets of measure 2^{-mn_k}, so their union has measure $1 - (1 - 2^{-mn_k})^{\#(\mathcal{H})} > 1 - \delta$, and there is an $H_k \in \mathcal{H}$ such that $\mu^m(V \cap G^m_{H_k} \cap \prod_{j<m} E_{kj}) > 0$. Thus the induction continues.

Look at the sequence $\langle H_k \rangle_{k\in\mathbb{N}}$ and its union E. We have $H_k \in \mathcal{H}_{n_k}$ for every k; moreover, if $i < k$, then $\mu(H_k \cap E_{kj_k}) > 0$, while $E_{kj_k} \subseteq G_{H_i}$; since H_k is an atom of Σ_{n_k}, while $G_{H_i} \in \Sigma_{n_k}$, $H_k \subseteq G_{H_i}$. Thus $E \in \mathcal{E}$. Next, whenever $i \le k \in \mathbb{N}$,

$$\prod_{j<m} E_{kj} \cap \prod_{j<m} F_{ij} \subseteq \prod_{j<m} E_{ij} \cap \prod_{j<m} F_{ij} = \emptyset,$$

so $\prod_{j<m} E_{kj} \cap \bigcup_{i\le k} \prod_{j<m} F_{ij} = \emptyset$. At the same time, we know that

$$E^m \cap \prod_{j<m} E_{kj} \supseteq \prod_{j<m} H_{km+j} \cap E_{kj} \supseteq \prod_{j<m} H_{km+j} \cap E_{km+j,j}$$

has non-zero measure. So $\mu^m(E^m \setminus \bigcup_{i\le k} \prod_{j<m} F_{ij}) > 0$.

Moving back to Z, this translates into

$$\nu^m((E^*)^m \setminus \bigcup_{i\le k} \prod_{j<m} F^*_{ij}) > 0.$$

But this means that $(E^*)^m$ is not included in $\bigcup_{i\le k} \prod_{j<m} F^*_{ij}$, for any $k \in \mathbb{N}$. Because E^* is compact and every F^*_{ij} is open, $(E^*)^m$ is not included in $\bigcup_{i\in\mathbb{N}} \prod_{j<m} F^*_{ij}$, and there is some $v \in (E^*)^m \cap \tilde{V}$.

By the definition of $\tilde{V}$,

$$\nu^m\{u : u\#v \in \tilde{W}\} > m\epsilon \ge m \sum_{k=0}^{\infty} 2^{-n_0-k} \ge m \sum_{k=0}^{\infty} 2^{-n_k}$$

$$= m \sum_{k=0}^{\infty} \mu H_k \ge m\mu E = m\nu E^*.$$

So there must be some u such that $u\#v \in \tilde{W}$ and $u(j) \notin E^*$ for every $j < m$. But now, setting $w = u\#v$, we have $w(2j) \notin E^*$, $w(2j+1) \in E^*$ for $j < m$, and $w \in D_m(A^*, Z, 0, 1) \cap \tilde{W}$; which is supposed to be impossible. **XQ**

(e) This shows that A^* is not stable. It is, however, R-stable. **P** We have a measure algebra isomorphism between the measure algebras of μ and ν defined by the map $E \mapsto E^* : \Sigma \to \mathrm{T}$. The corresponding isomorphism between $L^0(\mu)$ and $L^0(\nu)$ takes $\{f^\bullet : f \in A\}$ to $\{h^\bullet : h \in A^*\}$. By 465Pa and (b) above, $\{f^\bullet : f \in A\}$ is stable in $L^0(\mu)$, so $\{h^\bullet : h \in A^*\}$ is stable in $L^0(\nu)$, and every countable subset of A^* is stable. Since $A^* \subseteq C(X)$, it follows that A^* is stable (465T). **Q**

***465V Remark** This example is clearly related to 419E. The argument here is significantly deeper, but it does have an idea in common with that in 419E, besides the obvious point that both involve the Stone spaces of atomless probability spaces. Suppose that, in the context of 465U, we take $\mathcal{E}_0$ to be the family of sets E expressible as the union of a *finite* chain $H_0, \dots, H_k$ where $H_i \in \mathcal{H}_{n_i}$ for $i \le k$ and $H_j \subseteq G_{H_i}$ for $i < j \le k$. Then we find, on repeating the argument of (b) in the proof of 465U, that the countable set $A^*_0 = \{\chi E^* : E \in \mathcal{E}_0\}$ is stable, so that, setting $W_m = D_m(A^*_0, Z, 0, 1)$, $\nu^{2m} W_m$ is small for large m. On the other hand, setting

$$\tilde{W}_m = \bigcup\{V : V \subseteq Z^m \text{ is open}, V \setminus W_m \text{ is negligible}\},$$

we see that $D_m(A^*,Z,0,1) \subseteq \tilde{W}_m$, so that $(\nu^{2m})^*\tilde{W}_m = 1$ for all $m \ge 1$. Of course, writing $\tilde{\nu}^{2m}$ for the Radon product measure on Z^{2m}, we have $\tilde{\nu}^{2m}(\tilde{W}_m) = \nu^{2m}W_m < 1$ for large m, just as in 419E.

Both $A \subseteq \mathcal{L}^0(\Sigma)$ and $A^* \subseteq \mathcal{L}^0(\mathrm{T})$ are relatively pointwise compact. Note that while I took (X,Σ,μ) and (Z,T,ν) to be quite separate, it is entirely possible for them to be actually the same space. In this case it is natural to take every Σ_n to consist of open-and-closed sets, so that every member of $\mathcal{E}$ is open, and E^* becomes identified with the closure of E for $E \in \mathcal{E}$.

465X Basic exercises (a) Let (X,Σ,μ) be a semi-finite measure space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of measurable real-valued functions on X which converges a.e. Show that $\{f_n : n \in \mathbb{N}\}$ is stable.

(b) Let $\mathcal{C}$ be the family of convex sets in $\mathbb{R}^r$. Show that $\{\chi C : C \in \mathcal{C}\}$ is stable with respect to Lebesgue measure on $\mathbb{R}^r$, but that if $r \ge 2$ there is a Radon probability measure ν on $\mathbb{R}^r$ such that $\{\chi C : C \in \mathcal{C},\ C \text{ is closed}\}$ is not stable with respect to ν.

(c) Show that, for any $M \ge 0$, the set of functions $f : \mathbb{R} \to \mathbb{R}$ of variation at most M is stable with respect to any Radon measure on $\mathbb{R}$. (*Hint*: show that if μ is a Radon measure on $\mathbb{R}$ and $E \in \operatorname{dom}\mu$ has non-zero finite measure, and $(2k-1)(\beta-\alpha) > M$, then $(\mu^{2k})^*D_k(A,E,\alpha,\beta) < (\mu E)^{2k}$.)

(d) Let (X,Σ,μ) be a semi-finite measure space, and $A \subseteq \mathbb{R}^X$. Show that A is stable iff $\{f^+ : f \in A\}$ and $\{f^- : f \in A\}$ are both stable.

(e) Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function. Show that if $B \subseteq \mathbb{R}^Y$ is stable with respect to ν, then $\{g\phi : g \in B\}$ is stable with respect to μ.

(f) Let (X,Σ,μ) be a semi-finite measure space and $A \subseteq \mathbb{R}^X$. Suppose that μ is inner regular with respect to the family $\{F : F \in \Sigma,\ \{f \times \chi F : f \in A\} \text{ is stable}\}$. Show that A is stable.

(g) Let (X,Σ,μ) be a totally finite measure space and T a σ-subalgebra of Σ. Let $A \subseteq \mathcal{L}^0(\mathrm{T})$ be any set. (i) Show that if A is $\mu\restriction\mathrm{T}$-stable then it is μ-stable. (ii) Give an example to show that A can be μ-stable and pointwise compact without being $\mu\restriction\mathrm{T}$-stable. (*Hint*: take μ to be Lebesgue measure on $[0,1]$ and T the countable-cocountable algebra.)

(h) Let (X,Σ,μ) be a semi-finite measure space, and $A \subseteq \mathbb{R}^X$. For $g : X \to [0,\infty[$ set $A_g = \{(f \wedge g) \vee (-g) : f \in A\}$. Show that A is stable iff A_g is stable for every integrable $g : X \to [0,\infty[$.

(i) Let (X,Σ,μ) be a semi-finite measure space. A set $A \subseteq \mathbb{R}^X$ is said to have the **Bourgain property** if whenever $E \in \Sigma$, $\mu E > 0$ and $\epsilon > 0$, there are non-negligible measurable sets $F_0,\dots,F_n \subseteq E$ such that for every $f \in A$ there is an $i \le n$ such that the oscillation $\sup_{x,y\in F_i}|f(x)-f(y)|$ of f on F_i is at most ϵ. Show that in this case A is stable.

(j) Let X be a topological space, and μ a τ-additive effectively locally finite topological measure on X. Show that any equicontinuous subset of $C(X)$ has the Bourgain property, so is stable.

(k) Let (X,Σ,μ) be a locally determined measure space and $A \subseteq \mathbb{R}^X$ a stable set. Let $\overline{A}$ be the closure of A for the topology of pointwise convergence. Show that $\{f^\bullet : f \in \overline{A}\}$ is just the closure of $\{f^\bullet : f \in A\} \subseteq L^0(\mu)$ for the topology of convergence in measure.

(l) Show that there is a disjoint family $\mathcal{I}$ of finite subsets of $[0,1]$ such that $A = \{\chi I : I \in \mathcal{I}\}$ is not stable, though A is pointwise compact and the identity map on A is continuous for the topology of pointwise convergence and the topology of convergence in measure.

(m) Let (X,Σ,μ) be a probability space. Show that $A \subseteq \mathbb{R}^X$ is stable iff

$$\inf_{m\in\mathbb{N}}\bigl((\mu^{2m})^*D_m(A,X,\alpha,\beta)\bigr)^{1/m} = 0$$

whenever $\alpha < \beta$ in $\mathbb{R}$.

(n) Let (X,Σ,μ) be a semi-finite measure space. Show that a countable set $A \subseteq \mathcal{L}^0(\Sigma)$ is not stable iff there are $E \in \Sigma$ and $\alpha < \beta$ such that $0 < \mu E < \infty$ and $\mu^m\tilde{D}_m(A,E,\alpha,\beta) = (\mu E)^m$ for every $m \ge 1$, where $\tilde{D}_m(A,E,\alpha,\beta)$ is the set of those $w \in E^m$ such that for every $I \subseteq m$ there is an $f \in A$ such that $f(w(i)) \le \alpha$ for $i \in I$, $f(w(i)) \ge \beta$ for $i \in m \setminus I$. (*Hint*: see part (iii) of case 2 of the proof of 465L.)

(o) Let (X,Σ,μ) be a semi-finite measure space, and $A\subseteq\mathcal{L}^0(\Sigma)$ a set which is compact and metrizable for the topology of pointwise convergence. Show that A is stable. (*Hint*: otherwise, apply the ideas of case 2 in the proof of 465L to a countable dense subset of A to obtain a sequence which contradicts the conclusion of 465Xa.)

(p) Let (X,Σ,μ) be a probability space and $A\subseteq\mathcal{L}^0(\Sigma)$ a uniformly bounded set of functions. Show that A is stable iff

$$\lim_{n\to\infty}\overline{\int}\sup_{f\in A,k,l\ge n}\Bigl|\frac{1}{k}\sum_{i=0}^{k-1}f(w(i))-\frac{1}{l}\sum_{i=0}^{l-1}f(w(i))\Bigr|\mu^{\mathbb{N}}(dw)=0.$$

(q) Show that there is a set $A\subseteq\mathbb{R}^{[0,1]}$ such that $g(x)=\sup_{f\in A}|f(x)|$ is finite for every x, and A is stable with respect to Lebesgue measure, but its convex hull $\Gamma(A)$ is not. (*Hint*: take $A=\{g(x)\chi\{x\}:x\in[0,1]\}$, where g is such that for every $m\ge1$, every non-negligible compact set $K\subseteq[0,1]^m$ there is a $w\in K$ such that $g(w(i))=2^{i+1}$ for every $i<m$.)

(r) Show that there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of functions from $[0,1]$ to $\mathbb{N}$ such that $\{f_n:n\in\mathbb{N}\}$ is stable for Lebesgue measure on $[0,1]$, but $\{f_m-f_n:m,\,n\in\mathbb{N}\}$ is not.

(s) Let (X,Σ,μ) be a strictly localizable measure space and $Q\subseteq L^0(\mu)$ a set which is stable in the sense of 465R. Show that the closure of Q (for the topology of convergence in measure) is stable. (*Hint*: 465Xk.)

(t) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $A\subseteq\mathbb{R}^X$ a countable stable set of Σ-measurable functions such that $\int\sup_{f\in A}|f(x)|\mu(dx)<\infty$. Show that if for each $f\in A$ we choose a conditional expectation g_f of f on T (requiring each g_f to be T-measurable and defined everywhere on X), then $\{g_f:f\in A\}$ is stable.

(u) Give an example of a probability algebra $(\mathfrak{A},\bar\mu)$, a conditional expectation operator $P:L^1_{\bar\mu}\to L^1_{\bar\mu}$ (365R), and a uniformly integrable stable set $A\subseteq L^1_{\bar\mu}$ such that $P[A]$ is not stable. (*Hint*: start by picking a sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ in $P[L^1_{\bar\mu}]$ which is norm-convergent to 0 but not stable, and express this as $\langle Pu_n\rangle_{n\in\mathbb{N}}$ where $\bar\mu[\![u_n>0]\!]\le2^{-n}$ for every n.)

465Y Further exercises (a) Find an integrable continuous function $f:[0,1[^2\to[0,\infty[$ such that, in the notation of 465H, $\limsup_{k\to\infty}\int f d\nu^2_{wk}=\infty$ for almost every $w\in[0,1[^{\mathbb{N}}$. (*Hint*: arrange for $f(x,x)$ to grow very fast as $x\uparrow1$.)

(b) Show that in 465M we may replace the condition 'A is uniformly bounded' by the condition '$|f|\le f_0$ for every $f\in A$, where f_0 is integrable'. (*Hint*: Talagrand 87.)

(c) Let (X,Σ,μ) be a probability space, and $A\subseteq\mathbb{R}^X$ a uniformly bounded set. Show that A is stable iff for every $\epsilon>0$ there are a stable set $B\subseteq\mathbb{R}^X$, a sequence $\langle h_k\rangle_{k\in\mathbb{N}}$ of measurable functions on $X^{\mathbb{N}}$, and a family $\langle g_f\rangle_{f\in A}$ in B such that

$$h_k(w)\ge\frac{1}{k}\sum_{i=0}^{k-1}|f(w(i))-g_f(w(i))|\text{ for every }w\in X^{\mathbb{N}},\ k\ge1\text{ and }f\in A,$$

$$\limsup_{k\to\infty}h_k(w)\le\epsilon\text{ for almost every }w\in X^{\mathbb{N}}.$$

(*Hint*: 465M(iii)$\Rightarrow$465M(v), with a little help from 465Nc.)

(d) Let (X,Σ,μ) be a semi-finite measure space, and $A,\,B\subseteq\mathbb{R}^X$ pointwise bounded stable sets. Show that $\{f\times g:f\in A,\,g\in B\}$ is stable. (*Hint*: for the uniformly bounded case use 465M(iii) and 465Yc; then extend as in 465Nc.)

(e) Let (X,Σ,μ) and (Y,T,ν) be semi-finite measure spaces, with c.l.d. product $(X\times Y,\Lambda,\lambda)$. Suppose that $A\subseteq\mathbb{R}^X$ and $B\subseteq\mathbb{R}^Y$ are pointwise bounded stable sets. Show that $\{f\otimes g:f\in A,\,g\in B\}$ is stable with respect to λ, where $(f\otimes g)(x,y)=f(x)g(y)$.

(f) Let (X,Σ,μ) be a semi-finite measure space, $A\subseteq\mathbb{R}^X$ a uniformly bounded stable set and $h:\mathbb{R}\to\mathbb{R}$ a continuous function. Show that $\{hf:f\in A\}$ is stable. (*Hint*: use 465Yd, 465Nc and the Stone-Weierstrass theorem.)

(g) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a τ-additive topological measure space such that μ is inner regular with respect to the Borel sets. Show that a countable R-stable subset of $\mathbb{R}^X$ is stable.

(h) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete τ-additive topological probability space such that μ is inner regular with respect to the Borel sets, and $A\subseteq[0,1]^X$ an R-stable set. Suppose that $\epsilon>0$ is such that $\int f d\mu\le\epsilon^2$ for every $f\in A$. Show that there are an $n\ge 1$ and a Borel set $W\subseteq X^n$ and a $\gamma>\tilde\mu^n W$ (writing $\tilde\mu^n$ for the τ-additive product measure on X^n) such that $\int f d\nu\le 3\epsilon$ whenever $f\in A$, $\nu:\mathcal{P}X\to[0,1]$ is a point-supported probability measure and $\nu^n W\le\gamma$.

(i) Set $X=\prod_{n=2}^\infty\mathbb{Z}_n$, where each $\mathbb{Z}_n$ is the cyclic group of order n with its discrete topology; let μ be the Haar probability measure on X. Let $\bullet_l$ be the left shift action of X on $\mathbb{R}^X$, so that $(a\bullet_l f)(x)=f(x-a)$ for a, $x\in X$ and $f\in\mathbb{R}^X$ (i) Show that for any $n\in\mathbb{N}$ there is a compact negligible set $K_n\subseteq X$ such that for every $I\in[X]^n$ there are uncountably many $a\in X$ such that $I\subseteq K_n+a$. (ii) Show that there is a negligible set $E\subseteq X$ such that $\{a\bullet_l\chi E:a\in X\}$ is dense in $\{0,1\}^X$ for the topology of pointwise convergence. (iii) Show that if $f\in C(X)$ then $\{a\bullet_l f:a\in X\}$ is stable. (iv) Show that there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $C(X)$ such that $\{f_n:n\in\mathbb{N}\}$ is stable but $\{a\bullet_l f_n):a\in X,\,n\in\mathbb{N}\}$ is not. (v) Find expressions of these results when X is replaced by the circle group or by $\mathbb{R}$.

465 Notes and comments The definition in 465B arose naturally when M.Talagrand and I were studying pointwise compact sets of measurable functions; we found that in many cases a set of functions was relatively pointwise compact because it was stable (465Db). Only later did it appear that the concept was connected with Glivenko-Cantelli classes in the theory of empirical measures, as explained in Talagrand 87.

It is not the case that all pointwise compact sets of measurable functions are stable. In fact I have already offered examples in 463Xh and 464E above. In both cases it is easy to check from the definition in 465B that they are not stable, as can also be deduced from 465G. Another example is in 465Xl. You will observe however that all these examples are 'pathological' in the sense that either the measure space is irregular (from some points of view, indeed, any measure space not isomorphic to Lebesgue measure on the unit interval can be dismissed as peripheral), or the set of functions is uninteresting. It is clear from 465R and 465T, for instance, that we should start with countable sets. So it is natural to ask: if we have a *separable* pointwise compact set of real-valued measurable functions on the unit interval, must it be stable? It turns out that this is undecidable in Zermelo-Fraenkel set theory (Shelah & Fremlin 93); I hope to return to the question in Volume 5. (If we ask for 'metrizable', instead of 'separable', we get a positive answer; see 465Xo.)

The curious phrasing of the statement of 465M(iii), with the auxiliary functions h_k, turns on the fact that all the expressions '$\sup_{f\in A}\ldots$' here give rise to functions which need not be measurable. Thus the simple pointwise convergence described in (ii) and (iv) is not at all the same thing as the convergence in (v), which may be thought of as a kind of $\|\,\|_1$-convergence if we write $\|g\|_1=\overline{\int}|g|$ for arbitrary real-valued functions g. (Since the sets A here are uniformly bounded, it may equally be thought of as convergence in measure.) Similarly, 465M(iii) is saying much more than just

$$\lim_{k\to\infty}\sup_{f\in A}\frac{1}{k}\sum_{i=0}^{k-1}\bigl|f(w(i))-\mathbb{E}(f|\mathrm{T})(w(i))\bigr|=0\text{ a.e.,}$$

though of course for countable sets A these distinctions disappear. On the other hand, since the convergence is certainly not monotonic in k, $\|\,\|_1$-convergence does not imply pointwise convergence. So we can look for something stronger than either (iv) or (v) of 465M, as in 465Xp. But this is still nowhere near the strength of 465M(iii), in which $|\sum\ldots|$ is replaced by $\sum|\ldots|$. For further variations, see Talagrand 87 and Talagrand 96.

If we wish to adapt the ideas here to spaces of equivalence classes of functions rather than spaces of true functions, we find that problems of measurability evaporate, and that (because the definition of stability looks only at sets of finite measure) all the relevant suprema can be interpreted as suprema of countable sets. Consequently a subset of $L^0(\mu)$ or $L^0(\mathfrak{A})$ is stable if all its countable subsets are stable (465Pa). It is remarkable that, for strictly localizable measures μ, we can lift any stable set in L^0 to a stable set in $\mathcal{L}^0$ (465Pb, 465Q). By moving to function spaces we get a language in which to express a new kind of permanence property of stable sets (465R, 465Xt). See also Talagrand 89.

The definition of 'stable set' of functions seems to be utterly dependent on the underlying measure space. But 465R tells us that in fact the property of being an order-bounded stable subset of $L^1(\bar\mu)$ is invariant under normed space automorphisms. ('Order-boundedness' is a normed space invariant in L^1 spaces by the Chacon-Krengel theorem, 371D.) Since stability can be defined in terms of order-bounded sets (465Xh), we could, for instance, develop a theory of stable sets in abstract L-spaces.

The theory of stable sets is of course bound intimately to the theory of product measures, and such results as 465J have independent interest as theorems about sets in product spaces. So any new theory of product measures will give rise to a new theory of stable sets. In particular, the τ-additive product measures in §417 lead to R-stable sets (465S). It is instructive to work through the details, observing how the properties of the product are employed. Primarily, of course, we need 'associative' and 'commutative' laws, and Fubini's theorem; but some questions of measurability need to be re-examined, as in 465Yh.

I have starred 465U because it involves the notion of R-stability. In fact this appears only in the final stage, and the construction, as set out in part (a) of the proof, is an instructive challenge to any intuitive concept of what stable sets are like.

466 Measures on linear topological spaces

In this section I collect a number of results on the special properties of topological measures on linear topological spaces. The most important is surely Phillips' theorem (466A-466B): on any Banach space, the weak and norm topologies give rise to the same totally finite Radon measures. This is not because the weak and norm topologies have the same Borel σ-algebras, though this does happen in interesting cases (466C-466E, §467). When the Borel σ-algebras are different, we can still ask whether the Borel measures are 'essentially' the same, that is, whether every (totally finite) Borel measure for the weak topology extends to a Borel measure for the norm topology. A construction due to M.Talagrand (466H, 466Ia) gives a negative answer to the general question.

Just as in $\mathbb{R}^r$, a totally finite quasi-Radon measure on a locally convex linear topological space is determined by its characteristic function (466K). I end the section with a note on measurability conditions sufficient to ensure that a linear operator between Banach spaces is continuous (466L-466M), and with brief remarks on Gaussian measures (466N-466O).

466A Theorem Let $(X,\mathfrak{T})$ be a metrizable locally convex linear topological space and μ a σ-finite measure on X which is quasi-Radon for the weak topology $\mathfrak{T}_s(X,X^*)$. Then the support of μ is separable, so μ is quasi-Radon for the original topology $\mathfrak{T}$. If X is complete and μ is locally finite with respect to $\mathfrak{T}$, then μ is Radon for $\mathfrak{T}$.

proof (a) Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a base of neighbourhoods of 0 in X, and for $n\in\mathbb{N}$ set $V_n^\circ=\{f:f\in X^*,\ f(x)\le 1$ for every $x\in V_n\}$. Then each V_n° is a convex $\mathfrak{T}_s$-compact subset of X^* (4A4Bf), and $X^*=\bigcup_{n\in\mathbb{N}}V_n^\circ$. Let $Z\subseteq X$ be the support of μ, and for $n\in\mathbb{N}$ set $K_n=\{f\restriction Z:f\in V_n^\circ\}$. Since the map $f\mapsto f\restriction Z:X^*\to C(Z)$ is linear and continuous for $\mathfrak{T}_s$ and the topology $\mathfrak{T}_p$ of pointwise convergence on Z, each K_n is convex and $\mathfrak{T}_p$-compact. Next, the subspace measure μ_Z on Z is σ-finite and strictly positive, so by 463G each K_n is $\mathfrak{T}_p$-metrizable.

Let $\mathcal{U}$ be the base for $\mathfrak{T}_p$ consisting of sets of the form

$$U(I,h,\epsilon)=\{f:f\in C(Z),\ |f(x)-h(x)|<\epsilon\text{ for every }x\in I\},$$

where $I\subseteq Z$ is finite, $h\in\mathbb{R}^I$ and $\epsilon>0$. For each $n\in\mathbb{N}$, write $\mathcal{V}_n=\{K_n\cap U:U\in\mathcal{U}\}$, so that $\mathcal{V}_n$ is a base for the subspace topology of K_n (4A2B(a-vi)). Since this is compact and metrizable, therefore second-countable, there is a countable base $\mathcal{V}_n'\subseteq\mathcal{V}_n$ (4A2Ob). Now there is a countable set $\mathcal{U}'\subseteq\mathcal{U}$ such that $\mathcal{V}_n'\subseteq\{K_n\cap U:U\in\mathcal{U}'\}$ for every $n\in\mathbb{N}$, and a countable set $D\subseteq Z$ such that

$$\mathcal{U}'\subseteq\{U(I,h,\epsilon):I\subseteq D\text{ is finite, }h\in\mathbb{R}^I,\ \epsilon>0\}.$$

Let $D'\subseteq X$ be the linear span of D over the rationals. Then D' is again countable, and its $\mathfrak{T}$-closure Y is a linear subspace of X (4A4Bg). **?** If $Z\not\subseteq Y$, then $\mu(X\setminus Y)>0$. But, writing

$$\begin{aligned}Y^\circ&=\{f:f\in X^*,\ f(x)\le 1\text{ for every }x\in Y\}\\&=\{f:f\in X^*,\ f(x)=0\text{ for every }x\in Y\},\end{aligned}$$

$X\setminus Y$ must be $\bigcup_{f\in Y^\circ}\{x:|f(x)|>0\}$, by 4A4Eb. Because μ is τ-additive, there must be an $f\in Y^\circ$ such that $\mu\{x:|f(x)|>0\}>0$. Let $n\in\mathbb{N}$ be such that $f\in V_n^\circ$. Since $f(x)=0$ for every $x\in D$, $f\restriction Z$ belongs to the same members of $\mathcal{V}_n'$ as the zero function; since $\mathcal{V}_n'$ is a base for the Hausdorff subspace topology of K_n, $f\restriction Z$ actually is the zero function, and $\{x:|f(x)|>0\}\subseteq X\setminus Z$, so $\mu Z<\mu X$, contrary to the choice of Z. **X**

Thus $Z\subseteq Y$. Because $Y=\overline{D'}$ is $\mathfrak{T}$-separable, and $\mathfrak{T}$ is metrizable, Z also is $\mathfrak{T}$-separable (4A2P(a-iv)).

(b) The subspace topology $\mathfrak{T}_Y$ induced on Y by $\mathfrak{T}$ is a separable metrizable locally convex linear space topology, so the Borel σ-algebras of $\mathfrak{T}_Y$ and the associated weak topology $\mathfrak{T}_s(Y,Y^*)$ are equal (4A3V). But $\mathfrak{T}_s(Y,Y^*)$ is just the subspace topology induced on Y by $\mathfrak{T}_s$ (4A4Ea), Accordingly the subspace measure μ_Y on Y is $\mathfrak{T}_s(Y,Y^*)$-quasi-Radon (415B). Since every $\mathfrak{T}_Y$-open set is $\mathfrak{T}_s(Y,Y^*)$-Borel, μ_Y is a topological measure for $\mathfrak{T}_Y$. Since $\mathfrak{T}_Y$ is finer than $\mathfrak{T}_s(Y,Y^*)$, μ_Y is effectively locally finite for $\mathfrak{T}_Y$ and inner regular with respect to the $\mathfrak{T}_Y$-closed sets, and therefore is a quasi-Radon measure for $\mathfrak{T}_Y$ (415D(i)).

By 415J, there is a measure $\tilde{\mu}$ on X, quasi-Radon for $\mathfrak{T}$, such that $\tilde{\mu}E = \mu_Y(E \cap Y)$ whenever $\tilde{\mu}$ measures E. But as Y is $\mathfrak{T}$-closed and μ-conegligible, and μ is complete, we have $\mu = \tilde{\mu}$, and μ is quasi-Radon for $\mathfrak{T}$.

(c) If X is complete and μ is locally finite with respect to $\mathfrak{T}$, then $(X,\mathfrak{T})$ is a pre-Radon space (434Jg), so μ is a Radon measure for $\mathfrak{T}$ (434Jb).

466B Corollary (Compare 462I.) If X is a Banach space and μ is a totally finite measure on X which is quasi-Radon for the weak topology of X, it is a Radon measure for both the norm topology and the weak topology.

proof By 466A, μ is a Radon measure for the norm topology; by 418I, or otherwise, it is a Radon measure for the weak topology.

Remark Thus Banach spaces, with their weak topologies, are pre-Radon.

466C Definition A normed space X has a **Kadec norm** (also called **Kadec-Klee norm**) if the norm and weak topologies coincide on the sphere $\{x : \|x\| = 1\}$. Of course they will then also coincide on any sphere $\{x : \|x-y\| = \alpha\}$.

Example For any set I and any $p \in [1,\infty[$, the Banach space $\ell^p(I)$ has a Kadec norm. **P** Set $S = \{x : \|x\|_p = 1\}$. If $x \in S$ and $\epsilon > 0$, take $\eta \in \,]0,1]$ such that $2\eta + (2p\eta)^{1/p} \le \epsilon$. Let $J \subseteq I$ be a finite set such that $\sum_{i\in I\setminus J} |x(i)|^p \le \eta^p$. Set $H = \{y : y \in \ell^p(I),\ \sum_{i\in J} |y(i) - x(i)|^p < \eta^p\}$; then H is open for the weak topology of $\ell^p(I)$. If $y \in H \cap S$, then, writing x_J for $x \times \chi J$, etc.,

$$\|x_J\|_p \ge 1 - \|x_{I\setminus J}\|_p \ge 1 - \eta,$$

$$\|y_J\|_p \ge \|x_J\|_p - \eta \ge 1 - 2\eta, \quad \|y_J\|_p^p \ge 1 - 2p\eta,$$

$$\begin{aligned}\|y - x\|_p &\le \|y_J - x_J\|_p + \|x_{I\setminus J}\|_p + \|y_{I\setminus J}\|_p \\ &\le \eta + \eta + (1 - \|y_J\|_p^p)^{1/p} \le 2\eta + (2p\eta)^{1/p} \le \epsilon.\end{aligned}$$

Thus $\{y : y \in S, \|y-x\|_p \le \epsilon\}$ is a neighbourhood of x for the subspace weak topology on S; as x and ϵ are arbitrary, the weak and norm topologies agree on S. **Q**

For further examples, see 467B *et seq.*

466D Proposition (Hansell 01) Let X be a normed space with a Kadec norm. Then there is a network for the norm topology on X expressible in the form $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$, where for each $n \in \mathbb{N}$ $\mathcal{V}_n$ is an isolated family for the weak topology and $\bigcup \mathcal{V}_n$ is the difference of two closed sets for the weak topology.

proof Let $\mathcal{U}$ be a σ-disjoint base for the norm topology of X (4A2L(g-ii)); express it as $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ where every $\mathcal{U}_n$ is disjoint. For rational numbers q, q' with $0 < q < q'$, set $S_{qq'} = \{x : q < \|x\| \le q'\}$, and for $A \subseteq X$ write $W(A,q,q')$ for the interior of $A \cap S_{qq'}$ taken in the subspace weak topology of $S_{qq'}$. Set $\mathcal{V}_{nqq'} = \{W(U,q,q') : U \in \mathcal{U}_n\}$, so that $\mathcal{V}_{nqq'}$ is a disjoint family of relatively-weakly-open subsets of $S_{qq'}$, and is an isolated family for the weak topology. Now $\bigcup_{n\in\mathbb{N},q,q'\in\mathbb{Q},0<q<q'} \mathcal{V}_{nqq'}$ is a network for the norm topology on $X \setminus \{0\}$. **P** If $x \in X \setminus \{0\}$ and $\epsilon > 0$, then take $n \in \mathbb{N}$, $U \in \mathcal{U}_n$ such that $x \in U \subseteq \{y : \|y-x\| \le \epsilon\}$. Let $\delta > 0$ be such that $\{y : \|y-x\| \le \delta\} \subseteq U$. Next, because $\|\ \|$ is a Kadec norm, there is a weak neighbourhood V of 0 such that $\|y-x\| \le \frac{1}{2}\delta$ whenever $y \in x - V$ and $\|y\| = \|x\|$. Let V' be an open weak neighbourhood of 0 such that $V' + V' \subseteq V$. Let $\eta \in \,]0,1[$ be such that $\eta\|x\| \le \frac{1}{2}\delta$ and $y \in V'$ whenever $\|y\| \le \eta\|x\|$. If $y \in x - V'$ and $(1-\eta)\|x\| \le \|y\| \le (1+\eta)\|x\|$, then

$$\|y - \frac{\|x\|}{\|y\|}y\| = |1 - \frac{\|x\|}{\|y\|}|\,\|y\| = |\frac{\|y\|}{\|x\|} - 1|\,\|x\| \le \eta\|x\| \le \frac{1}{2}\delta,$$

$$x - \frac{\|x\|}{\|y\|}y = (x-y) + (y - \frac{\|x\|}{\|y\|}y) \in V' + V' \subseteq V,$$

so $\|x - \frac{\|x\|}{\|y\|}y\| \le \frac{1}{2}\delta$ and $\|x-y\| \le \delta$ and $y \in U$. This means that if we take q, $q' \in \mathbb{Q}$ such that $(1-\eta)\|x\| \le q \le \|x\| \le q' \le (1+\eta)\|x\|$, then $(x - V') \cap S_{qq'} \subseteq U$ and $x \in W(U,q,q') \in \mathcal{V}_{nqq'}$. Since of course $W(U,q,q') \subseteq U \subseteq \{y : \|y-x\| \le \epsilon\}$, and x and ϵ are arbitrary, we have the result. **Q**

To get a σ-isolated family for the weak topology which is a network for the norm topology on the whole of X, we just have to add the singleton set $\{0\}$. To see that the union of each of our isolated families is the difference of two weakly open sets, observe that $\bigcup \mathcal{V}_{nqq'}$ is a relatively weakly open subset of $S_{qq'}$, which is the difference of the weakly open sets $\{x : \|x\| > q\}$ and $\{x : \|x\| > q'\}$.

466E Corollary Let X be a normed space with a Kadec norm.

(a) The norm and weak topologies give rise to the same Borel σ-algebras.

(b) The weak topology has a σ-isolated network, so is hereditarily weakly θ-refinable.

proof (a) Write $\mathcal{B}_{\| \|}$, $\mathcal{B}_{\mathfrak{T}_s}$ for the Borel σ-algebras for the weak and norm topologies. Of course $\mathcal{B}_{\mathfrak{T}_s} \subseteq \mathcal{B}_{\| \|}$. Let $\langle \mathcal{V}_n \rangle_{n \in \mathbb{N}}$ be a sequence covering a network for the norm topology as in 466D. Because $\mathcal{V}_n$ is (for the weak topology) isolated and its union belongs to $\mathcal{B}_{\mathfrak{T}_s}$, $\bigcup \mathcal{W} \in \mathcal{B}_{\mathfrak{T}_s}$ for every $n \in \mathbb{N}$ and $\mathcal{W} \subseteq \mathcal{V}_n$. But this means that $\bigcup \mathcal{W} \in \mathcal{B}_{\mathfrak{T}_s}$ for every $\mathcal{W} \subseteq \mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$; and as $\mathcal{V}$ is a network for the norm topology, every norm-open set belongs to $\mathcal{B}_{\mathfrak{T}_s}$, and $\mathcal{B}_{\| \|} \subseteq \mathcal{B}_{\mathfrak{T}_s}$. Thus the two Borel σ-algebras are equal.

(b) Of course $\mathcal{V}$ is also a network for the weak topology, so the weak topology has a σ-isolated network; by 438Ld, it is hereditarily weakly θ-refinable.

466F Proposition Let X be a Banach space with a Kadec norm. Then the following are equiveridical:

(i) X is a Radon space in its norm topology;

(ii) X is a Radon space in its weak topology;

(iii) the weight of X (for the norm topology) is measure-free in the sense of §438.

proof (a)(i)⇔(ii) By 466Ea, the norm and weak topologies give rise to the same algebra $\mathcal{B}$ of Borel sets. If X is a Radon space in its norm topology, then any totally finite measure with domain $\mathcal{B}$ is inner regular with respect to the norm-compact sets, therefore inner regular with respect to the weakly compact sets, and X is Radon in its weak topology. If X is a Radon space in its weak topology, then any totally finite measure μ with domain $\mathcal{B}$ has a completion $\hat{\mu}$ which is a Radon measure for the weak topology, therefore also for the norm topology, by 466B; as μ is arbitrary, X is a Radon space for the norm topology.

(b)(i)⇔(iii) is a special case of 438H.

466G Definition A partially ordered set X has the **σ-interpolation property** or **countable separation property** if whenever A, B are non-empty countable subsets of X and $x \le y$ for every $x \in A$, $y \in B$, then there is a $z \in X$ such that $x \le z \le y$ for every $x \in A$ and $y \in B$. A Dedekind σ-complete partially ordered set (314Ab) always has the σ-interpolation property.

466H Proposition (Jayne & Rogers 95) Let X be a Riesz space with a Riesz norm, given its weak topology $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$. Suppose that ($\alpha$) X has the σ-interpolation property (β) there is a strictly increasing family $\langle p_\xi \rangle_{\xi < \omega_1}$ in X. Then there is a $\mathfrak{T}_s$-Borel probability measure μ on X such that

(i) μ is not inner regular with respect to the $\mathfrak{T}_s$-closed sets;

(ii) μ is not τ-additive for the topology $\mathfrak{T}_s$;

(iii) μ has no extension to a norm-Borel measure on X.

Accordingly $(X, \mathfrak{T}_s)$ is not a Radon space (indeed, is not Borel-measure-complete).

proof (a) Let K be the set

$$\{f : f \in X^*,\, f \ge 0,\, \|f\| \le 1\} = \{f : \|f\| \le 1\} \cap \bigcap_{x \in X^+} \{f : f(x) \ge 0\},$$

so that K is a weak*-closed subset of the unit ball of X^* and is weak*-compact. Because X^* is a solid linear subspace of the order-bounded dual $X^\sim$ of X (356Da), every member of X^* is the difference of two non-negative members of X^*, and K spans X^*. We shall need to know that if $x < y$ in X, there is an $f \in K$ such that $f(x) < f(y)$; set $f = \frac{1}{\|g\|}|g|$ where $g \in X^*$ is such that $g(x) \ne g(y)$. (Recall that the norm of X^* is a Riesz norm, as also noted in 356Da.)

Set

$$A = \bigcup_{\xi < \omega_1} \{x : x \in X,\, x \le p_\xi\};$$

then every sequence in A has an upper bound in A, but A has no greatest member. It follows that if $B \subseteq K$ is countable there is an $x \in A$ such that $f(x) = \sup_{y \in A} f(y)$ for every $f \in B$, so that $f(x) = f(y)$ whenever $x \le y \in A$ and $f \in B$.

Let $\mathcal{I}$ be the family of those sets $E \subseteq X$ such that $A \cap E$ is bounded above in A. Then $\mathcal{I}$ is a σ-ideal of subsets of X. **P** Of course $\emptyset \in \mathcal{I}$. If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{I}$ and $E \subseteq \bigcup_{n\in\mathbb{N}} E_n$, there is for each $n \in \mathbb{N}$ an $x_n \in A$ which is an upper bound for $E_n \cap A$. Let $x \in A$ be an upper bound for $\{x_n : n \in \mathbb{N}\}$; then x is an upper bound for $E \cap A$ in A. So $E \in \mathcal{I}$. **Q**

(b) For $G \in \mathfrak{T}_s$ and $k \in \mathbb{N}$, let $W(G,k)$ be the set of those $x \in X$ for which there are $f_0, \dots , f_k \in K$ such that $\{y : y \in X, \, |f_i(y) - f_i(x)| \le 2^{-k}$ for every $i \le k\}$ is included in G. Because K spans X^*, $G = \bigcup_{k\in\mathbb{N}} W(G,k)$. So if $G \in \mathfrak{T}_s \setminus \mathcal{I}$ there is a $k \in \mathbb{N}$ such that $W(G,k) \notin \mathcal{I}$.

(c) (The key.) If $\langle G_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{T}_s \setminus \mathcal{I}$, then $\bigcap_{n\in\mathbb{N}} G_n \notin \mathcal{I}$. **P** Start from any $z^* \in A$. For each $n \in \mathbb{N}$, take $k_n \in \mathbb{N}$ such that $W(G_n,k_n) \notin \mathcal{I}$. For each $z \in A$ and $n \in \mathbb{N}$, choose $w_{zn} \in A \cap W(G_n,k_n)$ such that $w_{zn} \ge z$; now choose a family $\langle f_{nzi} \rangle_{z\in A, i\le k_n}$ in K such that

$$\{y : |f_{nzi}(y - w_{zn})| \le 2^{-k_n} \text{ for every } i \le k_n\} \subseteq G_n.$$

Let $\mathcal{F}$ be any ultrafilter on A containing $\{x : x \in A, \, x \ge z\}$ for every $z \in A$, and write $f_{ni} = \lim_{z\to\mathcal{F}} f_{nzi}$ for $n \in \mathbb{N}$ and $i \le k_n$, the limit being taken for the weak* topology on K. Let $z_1^* > z^*$ be such that $z_1^* \in A$ and $f_{ni}(x - z_1^*) = 0$ whenever $x \in A$, $x \ge z_1^*$, $n \in \mathbb{N}$ and $i \le k_n$.

Choose sequences $\langle x_n \rangle_{n\in\mathbb{N}}$, $\langle y_n \rangle_{n\in\mathbb{N}}$ and $\langle z_n \rangle_{n\in\mathbb{N}}$ in A inductively, as follows. Set $y_0 = z_1^*$. Given that $y_n \ge z_1^*$, the set

$$C_n = \{z : z \in A, \, z \ge y_n, \, |(f_{nzi} - f_{ni})(y_n - z_1^*)| \le 2^{-k_n} \text{ for every } i \le k_n\}$$

belongs to $\mathcal{F}$, so is not empty; take $z_n \in C_n$. Because $y_n \ge z_1^*$, we have $f_{ni}(y_n - z_1^*) = 0$ and $|f_{nz_ni}(y_n - z_1^*)| \le 2^{-k_n}$, for every $i \le k_n$. Set $x_n = w_{z_nn}$, so that $x_n \in A$, $x_n \ge z_n$ and $\{y : |f_{nz_ni}(y - x_n)| \le 2^{-k_n}$ for every $i \le k_n\}$ is included in G_n. Now let $y_{n+1} \in A$ be such that $y_{n+1} \ge x_n$ and $f_{nz_ni}(y - y_{n+1}) = 0$ whenever $y \in A$ and $y \ge y_{n+1}$. Of course

$$y_{n+1} \ge x_n \ge z_n \ge y_n \ge z_1^*.$$

Continue.

At the end of the induction, let z_2^* be an upper bound for $\{y_n : n \in \mathbb{N}\}$ in A. For $n \in \mathbb{N}$, set

$$u_n = z_1^* + \textstyle\sum_{j=0}^{n} x_j - y_j, \qquad v_n = u_n + z_2^* - y_{n+1}.$$

Then $\langle u_n \rangle_{n\in\mathbb{N}}$ is non-decreasing and $u_n \le v_n \le z_2^*$ for every $n \in \mathbb{N}$; moreover,

$$v_n - v_{n+1} = y_{n+1} - x_{n+1} - y_{n+1} + y_{n+2} \ge 0$$

for every n, so $\langle v_n \rangle_{n\in\mathbb{N}}$ is non-increasing, and $u_m \le v_n$ for all m, $n \in \mathbb{N}$. Because X has the σ-interpolation property, there is an $x \in X$ such that $u_n \le x \le v_n$ for every $n \in \mathbb{N}$. Since $z_1^* \le x \le z_2^*$, $x \in A$ and $x > z^*$.

Take any $n \in \mathbb{N}$. Then

$$0 \le f_{nz_ni}(x - u_n) \le f_{nz_ni}(v_n - u_n) = f_{nz_ni}(z_2^* - y_{n+1}) = 0$$

for every $i \le k_n$. On the other hand,

$$x_n - u_n = (y_n - z_1^*) - \textstyle\sum_{j=0}^{n-1} (x_j - y_j)$$

lies between 0 and $y_n - z_1^*$, so

$$0 \le f_{nz_ni}(x_n - u_n) \le 2^{-k_n}$$

for every $i \le k_n$, and $|f_{nz_ni}(x - x_n)| \le 2^{-k_n}$ for every i. Thus $x \in G_n$. As n is arbitrary, $x \in \bigcap_{n\in\mathbb{N}} G_n$.

As $x > z^*$, this shows that z^* is not an upper bound of $\bigcap_{n\in\mathbb{N}} G_n \cap A$. As z^* is arbitrary, $\bigcap_{n\in\mathbb{N}} G_n \notin \mathcal{I}$. **Q**

(d) Set

$$\mathcal{K}_0 = \{G \setminus H : G, \, H \in \mathfrak{T}_s, \, G \notin \mathcal{I}, \, H \in \mathcal{I}\}.$$

Then $\bigcap_{n\in\mathbb{N}} E_n \ne \emptyset$ for any sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}_0$. **P** Express each E_n as $G_n \setminus H_n$ where G_n, H_n are $\mathfrak{T}_s$-open, $G_n \notin \mathcal{I}$ and $H_n \in \mathcal{I}$. Then $\bigcup_{n\in\mathbb{N}} H_n \in \mathcal{I}$, as noted in (a), while $\bigcap_{n\in\mathbb{N}} G_n \notin \mathcal{I}$, by (c); so $\bigcap_{n\in\mathbb{N}} E_n = \bigcap_{n\in\mathbb{N}} G_n \setminus \bigcup_{n\in\mathbb{N}} H_n$ is non-empty. **Q**

It follows that $\mathcal{K} = \mathcal{K}_0 \cup \{\emptyset\}$ is a countably compact class in the sense of 413L. Moreover, $E \cap E' \in \mathcal{K}$ for all E, $E' \in \mathcal{K}$ (using (a) and (c) again), so if we define $\phi_0 : \mathcal{K} \to \{0,1\}$ by writing $\phi_0(E) = 1$ for $E \in \mathcal{K}_0$ and $\phi_0(\emptyset) = 0$, then $\mathcal{K}$ and ϕ_0 will satisfy all the conditions of 413M. There is therefore a measure $\hat{\mu}$ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}_\delta$, the family of sets expressible as intersections of sequences in $\mathcal{K}$. The domain of $\hat{\mu}$ must include every member of $\mathcal{K}$; but if $G \in \mathfrak{T}_s$ then either G or $X \setminus G$ belongs to $\mathcal{K}_0$, so is measured by $\hat{\mu}$, and $\hat{\mu}$ is a topological measure.

We need to observe that, because ϕ_0 takes only the values 0 and 1, $\hat{\mu}E \le 1$ for every $E \in \mathcal{K}_\delta$, and $\hat{\mu}X \le 1$; since $\phi_0 X = 1$, $\hat{\mu}X = 1$ and $\hat{\mu}$ is a probability measure.

(e) We may therefore take μ to be the restriction of $\hat{\mu}$ to the algebra $\mathcal{B}$ of $\mathfrak{T}_s$-Borel sets, and μ is a $\mathfrak{T}_s$-Borel probability measure. Now μ is not inner regular with respect to the $\mathfrak{T}_s$-closed sets. **P** For each $\xi < \omega_1$, $p_\xi < p_{\xi+1} < p_{\xi+2}$, so there are g_ξ, $h_\xi \in K$ such that $g_\xi(p_\xi) < g_\xi(p_{\xi+1})$ and $h_\xi(p_{\xi+1}) < h_\xi(p_{\xi+2})$. Let $D \subseteq \omega_1$ be any set such that D and $\omega_1 \setminus D$ are both uncountable, and set

$$G = \bigcup_{\xi \in D} \{x : g_\xi(p_\xi) < g_\xi(x),\, h_\xi(x) < h_\xi(p_{\xi+2})\}.$$

Then $G \in \mathfrak{T}_s$, and $p_{\xi+1} \in G$ for every $\xi \in D$, so $G \notin \mathcal{I}$ and $\mu G = \phi_0 G = 1$. On the other hand, if $\eta \in \omega_1 \setminus D$, then for every $\xi \in D$ either $\xi < \eta$ and $h_\xi(p_{\xi+2}) \le h_\xi(p_{\eta+1})$, or $\xi > \eta$ and $g_\xi(p_{\eta+1}) \le g_\xi(p_\xi)$; thus $p_{\eta+1} \notin G$ for any $\eta \in \omega_1 \setminus D$, and $X \setminus G \notin \mathcal{I}$. But this means that if $F \subseteq G$ is closed then $X \setminus F \in \mathcal{K}_0$ and $\mu F = 0$. Thus $\mu G > \sup_{F \subseteq G \text{ is closed}} \mu F$. **Q**

(f) Because $\mathfrak{T}_s$ is regular, μ cannot be τ-additive, by 414Mb. It follows at once that $(X, \mathfrak{T}_s)$ is not Borel-measure-complete, and in particular is not a Radon space. To see that μ has no extension to a norm-Borel measure, we need to look again at the set A. For each $\xi < \omega_1$, set $F_\xi = \{x : x \le p_\xi\}$. Then every F_ξ is norm-closed (354Bc) and $\langle F_\xi \rangle_{\xi < \omega_1}$ is an increasing family with union A. Consequently, A is norm-closed (4A2Ld, 4A2Ka). At the same time, every F_ξ is convex (cf. 351Ce), so A is also convex. It follows that A, like every F_ξ, is $\mathfrak{T}_s$-closed (3A5Ee). So μ measures A and every F_ξ. Because $X \setminus F_\xi$ is a $\mathfrak{T}_s$-open set not belonging to $\mathcal{I}$, $\mu F_\xi = 0$, for every $\xi < \omega_1$; because $X \setminus A$ is a $\mathfrak{T}_s$-open set belonging to $\mathcal{I}$, $\mu A = 1$.

But ω_1 is a measure-free cardinal (438Cd), so 438I tells us that $\lambda A = \sup_{\xi < \omega_1} \lambda F_\xi$ for any semi-finite norm-Borel measure λ on X. Thus μ has no extension to a norm-Borel measure, and the proof is complete.

466I Examples The following spaces satisfy the hypotheses of 466H.

(a) (Talagrand 78a, or Talagrand 84, 16-1-2) $X = \ell^\infty(I)$ or $\{x : x \in \ell^\infty(I),\, \{i : x(i) \ne 0\} \text{ is countable}\}$, where I is uncountable. **P** X has the σ-interpolation property because it is Dedekind complete, and if $\langle i_\xi \rangle_{\xi < \omega_1}$ is any family of distinct elements of I, we can set $p_\xi(i_\eta) = 1$ for $\eta \le \xi$, $p_\xi(i) = 0$ for all other $i \in I$ to obtain a strictly increasing family $\langle p_\xi \rangle_{\xi < \omega_1}$ in X. **Q**

(b) (de Maria & Rodriguez-Salinas 91) $X = \ell^\infty / \boldsymbol{c}_0$, where $\boldsymbol{c}_0$ is the space of real sequences converging to 0.

P (i) To see that X has the σ-interpolation property, let A, $B \subseteq X$ be non-empty countable sets such that $u \le v$ for all $u \in A$, $v \in B$. Let $\langle x_n \rangle_{n \in \mathbb{N}}$, $\langle y_n \rangle_{n \in \mathbb{N}}$ be sequences in ℓ^∞ such that $A = \{x_n^\bullet : n \in \mathbb{N}\}$ and $B = \{y_n^\bullet : n \in \mathbb{N}\}$. Set $\tilde{x}_n = \sup_{i \le n} x_i$, $\tilde{y}_n = \inf_{i \le n} y_i$ for $n \in \mathbb{N}$; then $\tilde{x}_n^\bullet \le \tilde{y}_n^\bullet$, so $(\tilde{x}_n - \tilde{y}_n)^+ \in \boldsymbol{c}_0$. Set

$$k_n = \max(\{n\} \cup \{i : \tilde{x}_n(i) \ge \tilde{y}_n(i) + 2^{-n}\})$$

for $n \in \mathbb{N}$, and define $x \in \ell^\infty$ by writing

$$\begin{aligned} x(i) &= 0 \text{ if } i \le k_0, \\ &= \tilde{x}_n(i) \text{ if } k_n < i \le k_{n+1}. \end{aligned}$$

Then it is easy to check that $u \le x^\bullet \le v$ for every $u \in A$, $v \in B$; as A and B are arbitrary, X has the σ-interpolation property.

(ii) To see that X has the other property, recall that there is a family $\langle I_\xi \rangle_{\xi < \omega_1}$ of infinite subsets of $\mathbb{N}$ such that $I_\xi \setminus I_\eta$ is finite if $\eta \le \xi$, infinite if $\xi < \eta$ (4A1Fa). Setting $p_\xi = \chi(\mathbb{N} \setminus I_\xi)^\bullet$, we have a strictly increasing family $\langle p_\xi \rangle_{\xi < \omega_1}$ in X. **Q**

466J Theorem Let X be a linear topological space and Σ its cylindrical σ-algebra. If μ and ν are probability measures with domain Σ such that $\int e^{if(x)} \mu(dx) = \int e^{if(x)} \nu(dx)$ for every $f \in X^*$, then $\mu = \nu$.

proof Define $T : X \to \mathbb{R}^{X^*}$ by setting $(Tx)(f) = f(x)$ for $f \in X^*$, $x \in X$. Then T is linear and continuous for the weak topology of X. So if $F \subseteq \mathbb{R}^{X^*}$ is a Baire set for the product topology of $\mathbb{R}^{X^*}$, $T^{-1}[F]$ is a Baire set for the weak topology of X (4A3Kc) and belongs to Σ (4A3U). We therefore have Baire measures μ', ν' on $\mathbb{R}^{X^*}$ defined by saying that $\mu' F = \mu T^{-1}[F]$ and $\nu' F = \nu T^{-1}[F]$ for every Baire set $F \subseteq \mathbb{R}^{X^*}$.

If $h : \mathbb{R}^{X^*} \to \mathbb{R}$ is a continuous linear functional, it can be expressed in the form $h(z) = \sum_{i=0}^n \alpha_i z(f_i)$ where $f_0, \dots , f_n \in X^*$ and $\alpha_0, \dots , \alpha_n \in \mathbb{R}$. So

$$h(Tx) = \sum_{i=0}^n \alpha_i (Tx)(f_i) = \sum_{i=0}^n \alpha_i f_i(x) = f(x)$$

for every $x \in X$, where $f = \sum_{i=0}^n \alpha_i f_i$. This means that

$$\int e^{ih(z)} \mu'(dz) = \int e^{ih(Tx)} \mu(dx) = \int e^{if(x)} \mu(dx) = \int e^{if(x)} \nu(dx) = \int e^{ih(z)} \nu'(dz).$$

As h is arbitrary, $\mu' = \nu'$ (454Pa).

Now let Σ' be the family of subsets of X of the form $T^{-1}[F]$ where $F \subseteq \mathbb{R}^{X^*}$ is a Baire set. This is a σ-algebra and contains all sets of the form $\{x : f(x) \ge \alpha\}$ where $f \in X^*$ and $\alpha \in \mathbb{R}$. So every member of X^* is Σ'-measurable and Σ' must include the cylindrical σ-algebra of X. Since μ and ν agree on Σ' they must be identical.

466K Proposition If X is a locally convex linear topological space and μ, ν are quasi-Radon probability measures on X such that $\int e^{if(x)} \mu(dx) = \int e^{if(x)} \nu(dx)$ for every $f \in X^*$, then $\mu = \nu$.

proof Write $\mathfrak{T}$ for the given topology on X and $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$ for the weak topology. By 466J, μ and ν must agree on the cylindrical σ-algebra Σ of X. Since Σ includes a base for $\mathfrak{T}_s$, every weakly open set G is the union of an upwards-directed family of open sets belonging to Σ; as μ and ν are τ-additive, $\mu G = \nu G$. Consequently μ and ν agree on $\mathfrak{T}_s$-closed sets, and therefore on $\mathfrak{T}$-closed convex sets (4A4Ed). Write $\mathcal{H}$ for the family of $\mathfrak{T}$-open sets which are expressible as the union of a non-decreasing sequence of $\mathfrak{T}$-closed convex sets; then μ and ν agree on $\mathcal{H}$. If τ is a $\mathfrak{T}$-continuous seminorm on X, $x_0 \in X$ and $\alpha > 0$, then $\{x : \tau(x - x_0) < \alpha\} \in \mathcal{H}$; and sets of this kind constitute a base for $\mathfrak{T}$ (4A4Cb). Also the intersection of two members of $\mathcal{H}$ belongs to $\mathcal{H}$. By 415H(v), $\mu = \nu$.

Remark This generalizes 285M and 454Xk, which are the special cases $X = \mathbb{R}^r$ (for finite r) and $X = \mathbb{R}^I$; see also 445Xq.

466L Proposition Suppose that X and Y are Banach spaces and that $T : X \to Y$ is a linear operator such that $gT : X \to \mathbb{R}$ is universally Radon-measurable, in the sense of 434Ec, for every $g \in Y^*$. Then T is continuous.

proof ? Suppose, if possible, otherwise. Then there is a $g \in Y^*$ such that gT is not continuous (4A4Ib). For each $n \in \mathbb{N}$, take $x_n \in X$ such that $\|x_n\| = 2^{-n}$ and $g(Tx_n) > 2n$. Define $h : \{0,1\}^{\mathbb{N}} \to X$ by setting $h(t) = \sum_{n=0}^\infty t(n)x_n$ (4A4Ie). Then h is continuous, because $\|h(t) - h(t')\| \le \sum_{n=0}^\infty 2^{-n}|t(n) - t'(n)|$ for all t, $t' \in \{0,1\}^{\mathbb{N}}$. Let ν be the usual measure on $\{0,1\}^{\mathbb{N}}$; then the image measure $\mu = \nu h^{-1}$ is a Radon measure on X (418I), so gT must be $\operatorname{dom}\mu$-measurable, and $\phi = gTh$ is $\operatorname{dom}\nu$-measurable. In this case there is an $m \in \mathbb{N}$ such that $E = \{t : |\phi(t)| \le m\}$ has measure greater than $\frac{1}{2}$. But as $g(Tx_m) > 2m$, we see that if $t \in E$ then $t' \notin E$, where t' differs from t at the mth coordinate only, so that $|\phi(t) - \phi(t')| = g(Tx_m)$. Since the map $t \mapsto t'$ is an automorphism of the measure space $(\{0,1\}^{\mathbb{N}}, \nu)$, $\nu E \le \frac{1}{2}$, which is impossible. **X**

466M Corollary If X is a Banach space, Y is a separable Banach space, and $T : X \to Y$ is a linear operator such that the graph of T is a Souslin-F set in $X \times Y$, then T is continuous.

proof It will be enough to show that $T\restriction Z$ is continuous for every separable closed linear subspace Z of X (because then it must be sequentially continuous, and we can use 4A2Ld). Write $\Gamma \subseteq X \times Y$ for the graph of T. If $H \subseteq Y$ is open, then $\Gamma \cap (Z \times H)$ is a Souslin-F set in the Polish space $Z \times Y$, so is analytic (423Eb), and its projection $(T\restriction Z)^{-1}[H]$ also is analytic (423Ba), therefore universally measurable (434Dc). Thus $T\restriction Z : Z \to Y$ is a universally measurable function, and $gT\restriction Z$ must be universally measurable for any $g \in Y^*$ (434Df). By 466L, $T\restriction Z$ is continuous; as Z is arbitrary, T is continuous.

466N Gaussian measures Some of the ideas of §456 can be adapted to the present context, as follows.

Definition If X is a linear topological space, I will say that a probability measure μ on X is a **centered Gaussian measure** if its domain includes the cylindrical σ-algebra of X and every continuous linear functional on X is either zero almost everywhere or a normal random variable with zero expectation. (Thus a 'centered Gaussian distribution' on $\mathbb{R}^I$, as defined in 456A, is a distribution in the sense of 454K which is a centered Gaussian measure when $\mathbb{R}^I$ is thought of as a linear topological space.)

Warning! many authors reserve the phrase 'Gaussian measure' for strictly positive measures.

466O Proposition Let X be a separable Banach space, and μ a probability measure on X. Suppose that there is a linear subspace W of X^*, separating the points of X, such that every element of W is $\operatorname{dom}\mu$-measurable and either zero a.e. or a normal random variable with zero expectation. Then μ is a centered Gaussian measure with respect to the norm topology of X.

proof (a) As W separates the points of X, $X \setminus \{0\} = \bigcup_{f \in W}\{x : f(x) \neq 0\}$. Because X is Polish, therefore hereditarily Lindelöf, there is a countable set $I \subseteq W$ still separating the points of X. Let W_0 be the linear subspace of X^* generated by I.

Define $T : X \to \mathbb{R}^I$ by setting $(Tx)(f) = f(x)$ for $f \in I$. Then T is an injective linear operator, and is continuous for $\mathfrak{T}_s(X, W_0)$ and the usual topology of $\mathbb{R}^I$. Let λ be the distribution of the family $\langle f \rangle_{f \in I}$; T is inverse-measure-preserving for $\hat\mu$ and λ, where $\hat\mu$ is the completion of μ (454J(iv)). If $g : \mathbb{R}^I \to \mathbb{R}$ is a continuous linear functional, then $gT \in W_0$ (use 4A4Be); now the distribution of g, with respect to the probability measure λ, is just the distribution of gT with respect to $\hat\mu$ and μ, and is therefore either normal or the Dirac measure concentrated at 0. So λ is a centered Gaussian distribution in the sense of 456Ab. Because I is countable, λ is a Radon measure (454J(iii)).

(b) If $\epsilon > 0$, there is a norm-compact $K \subseteq X$ such that $\hat\mu K$ is defined and is at least $1 - \epsilon$. **P** As X and $\mathbb{R}^I$ are analytic (423B), there is a Radon measure μ' on X such that $\lambda = \mu' T^{-1}$ (432G). Of course $\mu' X = \lambda \mathbb{R}^I = 1$. There is a compact set $K \subseteq X$ such that $\mu' K \geq 1 - \epsilon$; now $T[K]$ is compact and $K = T^{-1}[T[K]]$, so

$$\hat\mu K = \mu T^{-1}[T[K]] = \lambda T[K] = \mu' K \geq 1 - \epsilon. \quad \mathbf{Q}$$

(c) Now suppose that $g \in X^*$. Then there is a sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ in W_0 such that $\langle g_n \rangle_{n \in \mathbb{N}} \to g$ μ-a.e. **P** For each $n \in \mathbb{N}$, there is a compact set $K_n \subseteq X$ such that $\hat\mu K_n \geq 1 - 2^{-n-1}$; we can suppose that $K_{n+1} \supseteq K_n$ for each n. W_0 is dense in X^* for the weak*-topology $\mathfrak{T}_s(X^*, X)$ (4A4Eh); being convex, it is dense for the Mackey topology $\mathfrak{T}_k(X^*, X)$ (4A4F), and there is a $g_n \in W_0$ such that $\sup_{x \in K_n} |g_n(x) - g(x)| \leq 2^{-n}$. Now $g(x) = \lim_{n \to \infty} g_n(x)$ for every x in the μ-conegligible set $\bigcup_{n \in \mathbb{N}} K_n$. **Q**

Set $\sigma_n = \sqrt{\operatorname{Var}(g_n)}$ for each n. Then $\{\sigma_n : n \in \mathbb{N}\}$ is bounded. **P** Set $M = \sup_{x \in K_0} |g(x)|$. If $n \in \mathbb{N}$ and $\sigma_n \neq 0$, $|g_n(x)| \leq M + 1$ for every $x \in K_0$, and

$$\tfrac{1}{2} \leq \mu K_0 \leq \Pr(|g_n| \leq M + 1) = \tfrac{1}{\sigma_n \sqrt{2\pi}} \int_{-M-1}^{M+1} e^{-t^2/2\sigma_n^2} dt \leq \tfrac{2(M+1)}{\sigma_n \sqrt{2\pi}},$$

so $\sigma_n \leq \frac{4(M+1)}{\sqrt{2\pi}}$. **Q**

We therefore have a strictly increasing sequence $\langle n_k \rangle_{k \in \mathbb{N}}$ such that $\sigma = \lim_{k \to \infty} \sigma_{n_k}$ is defined in $[0, \infty[$. For each k, let ν_k be the distribution of g_{n_k} and ϕ_k its characteristic function; let ν, ϕ be the distribution and characteristic function of g. Since $\langle g_{n_k} \rangle_{k \in \mathbb{N}} \to g$ a.e.,

$$\int h\, d\nu = \int hg\, d\mu = \lim_{k \to \infty} \int hg_{n_k} d\mu = \lim_{k \to \infty} \int h\, d\nu_k$$

for every bounded continuous function $h : \mathbb{R} \to \mathbb{R}$, and $\phi(t) = \lim_{k \to \infty} \phi_k(t)$ for every $t \in \mathbb{R}$, by 285L. But, for each k,

$$\phi_k(t) = \exp(-\frac{1}{2}\sigma_{n_k}^2 t^2)$$

by 285E if $\sigma_{n_k} > 0$ and by direct calculation if $\sigma_{n_k} = 0$, as then $g_{n_k} = 0$ almost everywhere.

Accordingly $\phi(t) = \exp(-\frac{1}{2}\sigma^2 t^2)$ for every t. But this means that ν is either the Dirac measure concentrated at 0 (if $\sigma = 0$) or a normal distribution with zero expectation (if $\sigma > 0$).

(d) As g is arbitrary, μ is a centered Gaussian measure.

466X Basic exercises (a) Let $(X, \mathfrak{T})$ be a metrizable locally convex linear topological space and μ a totally finite measure on X which is quasi-Radon for the topology $\mathfrak{T}$. Show that μ is quasi-Radon for the weak topology $\mathfrak{T}_s(X, X^*)$.

>(b) Let $\langle e_n \rangle_{n \in \mathbb{N}}$ be the usual orthonormal basis of ℓ^2. Give ℓ^2 the Radon probability measure ν such that $\nu\{e_n\} = 2^{-n-1}$ for every n. Let I be an uncountable set, and set $X = (\ell^2)^I$ with the product linear structure and the product topology $\mathfrak{T}$, each copy of ℓ^2 being given its norm topology. (i) Let λ be the τ-additive product of copies of ν (417G). Show that λ is quasi-Radon for $\mathfrak{T}$ but is not inner regular with respect to the $\mathfrak{T}_s$-closed sets. (ii) Write

$\mathfrak{T}_s$ for the weak topology of X. Let λ_s be the τ-additive product measure of copies of ν when each copy of ℓ^2 is given its weak topology instead of its norm topology. Show that λ_s is quasi-Radon for $\mathfrak{T}_s$ but does not measure every $\mathfrak{T}$-Borel set. (*Hint*: setting $E = \{e_n : n \in \mathbb{N}\}$, $\lambda(E^I) = 1$ and E^I is relatively $\mathfrak{T}_s$-compact.)

>(c) Let X be a metrizable locally convex linear topological space and μ a τ-additive totally finite measure on X with domain the cylindrical σ-algebra of X. Show that μ has an extension to a quasi-Radon measure on X. (*Hint*: 4A3U, 415N.)

(d) Let X be a metrizable locally convex linear topological space which is Lindelöf in its weak topology, and Σ the cylindrical σ-algebra of X. Show that any totally finite measure with domain Σ has an extension to a quasi-Radon measure on X.

>(e) Let X be a separable Banach space. Show that it is a Radon space when given its weak topology.

(f) Let K be a compact metrizable space, and $C(K)$ the Banach space of continuous real-valued functions on K. Show that the σ-algebra of subsets of $C(K)$ generated by the functionals $x \mapsto x(t) : C(K) \to \mathbb{R}$, for $t \in K$, is just the cylindrical σ-algebra of $C(K)$. (*Hint*: 4A2Pe.) Examine the connexions between this and 454Sa, 462Z and 466Xd.

(g) Let K be a scattered compact Hausdorff space. Show that the weak topology and the topology of pointwise convergence on $C(K)$ have the same Borel σ-algebras.

(h) Re-write part (d) of the proof of 466H to avoid any appeal to results from §413.

(i) Let X be a locally convex linear topological space and μ, ν two totally finite quasi-Radon measures on X. Show that if μ and ν give the same measure to every half-space $\{x : f(x) \geq \alpha\}$, where $f \in X^*$ and $\alpha \in \mathbb{R}$, then $\mu = \nu$.

(j) Let X be a Hilbert space and μ, ν two totally finite Radon measures on X. Show that if μ and ν give the same measure to every ball $B(x, \delta)$, where $x \in X$ and $\delta \geq 0$, then $\mu = \nu$. (*Hint*: every open half-space is the union of a non-decreasing sequence of balls.)

(k) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = 1$ and $f(x + y) = f(x) + f(y)$ for all x, $y \in \mathbb{R}$. Show that the following are equiveridical: (i) $f(x) = x$ for every $x \in \mathbb{R}$; (ii) f is continuous at some point; (iii) f is bounded on some non-empty open set; (iv) f is bounded on some Lebesgue measurable set of non-zero measure; (v) f is Lebesgue measurable; (vi) f is Borel measurable; *(vii) f is bounded on some non-meager G_δ set; *(viii) f is $\widehat{\mathcal{B}}$-measurable, where $\widehat{\mathcal{B}}$ is the Baire-property algebra of $\mathbb{R}$. (*Hint*: 443Db.)

(l) Set $X = \{x : x \in \mathbb{R}^{\mathbb{N}}, \{n : x(n) \neq 0\}$ is finite$\}$, and give X any norm. Show that any linear operator from X to any normed space is universally measurable.

(m) Let X be a linear topological space and μ a centered Gaussian measure on X. (i) Let Y be another linear topological space and $T : X \to Y$ a continuous linear operator. Show that the image measure μT^{-1} is a centered Gaussian measure on Y. (ii) Show that $X^* \subseteq \mathcal{L}^2(\mu)$. (iii) Let us say that the **covariance matrix** of μ is the family $\langle \sigma_{fg} \rangle_{f,g \in X^*}$, where $\sigma_{fg} = \int f \times g \, d\mu$ for f, $g \in X^*$. Suppose that ν is another centered Gaussian measure on X with the same covariance matrix. Show that μ and ν agree on the cylindrical σ-algebra of X.

(n) Let $\langle X_i \rangle_{i \in I}$ be a family of linear topological spaces with product X. Suppose that for each i we have a centered Gaussian measure μ_i on X_i. Show that the product probability measure $\prod_{i \in I} \mu_i$ is a centered Gaussian measure on X.

(o) Let X be a linear topological space. Show that the convolution of two quasi-Radon centered Gaussian measures on X is a centered Gaussian measure.

(p) Let X be a separable Banach space, and μ a complete measure on X. Show that the following are equiveridical: (i) μ is a centered Gaussian measure on X; (ii) μ extends a centered Gaussian Radon measure on X; (iii) there are a set I, an injective continuous linear operator $T : X \to \mathbb{R}^I$ and a centered Gaussian distribution λ on $\mathbb{R}^I$ such that T is inverse-measure-preserving for μ and λ; (iv) whenever I is a set and $T : X \to \mathbb{R}^I$ is a continuous linear operator there is a centered Gaussian distribution λ on $\mathbb{R}^I$ such that T is inverse-measure-preserving for μ and λ.

(q) Let X be a Banach space, and μ a Radon measure on X. Show that, with respect to μ, the unit ball of X^* is a stable set of functions in the sense of §465. (*Hint*: 465Xj.)

>(r) Let I be an infinite set. Show that Talagrand's measure, interpreted as a measure on $\ell^\infty(I)$ (464R), is not τ-additive for the weak topology.

466Y Further exercises (a) Give an example of a Hausdorff locally convex linear topological space $(X, \mathfrak{T})$ with a probability measure μ on X which is a Radon measure for the weak topology $\mathfrak{T}_s(X, X^*)$ but not for the topology $\mathfrak{T}$. (*Hint*: take $C = C([0,1])$ and $X = C^*$ with the Mackey topology for the dual pair (X, C), that is, the topology of uniform convergence on weakly compact subsets of C.)

(b) Let X be a normed space and $\mathfrak{T}$ a linear space topology on X such that the unit ball of X is $\mathfrak{T}$-closed and the topology on the unit sphere S induced by $\mathfrak{T}$ is finer than the norm topology on S. (i) Show that every norm-Borel subset of X is $\mathfrak{T}$-Borel. (ii) Show that if $\mathfrak{T}$ is coarser than the norm topology, then it has a σ-isolated network.

(c) (i) Let X be a Banach space. Set $S = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ and suppose that $\langle K_\sigma\rangle_{\sigma\in S}$ is a family of non-empty weakly compact convex subsets of X such that $K_\sigma \subseteq K_\tau$ whenever σ, $\tau \in S$ and σ extends τ. (α) Show that there is a weakly Radon probability measure on X giving measure at least 2^{-n} to K_σ whenever $n \in \mathbb{N}$ and $\sigma \in \{0,1\}^n$. (β) Show that there are a $\sigma \in S$ and $x \in K_{\sigma^\frown<0>}$, $y \in K_{\sigma^\frown<1>}$ such that $\|x - y\| \le 1$. (ii) Let X be a locally convex Hausdorff linear topological space. and $\langle A_\sigma\rangle_{\sigma\in S}$ a family of non-empty relatively weakly compact subsets of X such that $A_\sigma \subseteq A_\tau$ whenever σ, $\tau \in S$ and σ extends τ. For $\sigma \in S$, set $C_\sigma = A_{\sigma^\frown<1>} - A_{\sigma^\frown<0>}$. Show that $0 \in \overline{\bigcup_{\sigma\in S} C_\sigma}$.

(d) Find Banach spaces X and Y and a linear operator from X to Y which is not continuous but whose graph is an F_σ set in $X \times Y$.

(e) Let X be a complete Hausdorff locally convex linear topological space and μ a Radon probability measure on X. Suppose that there is a linear subspace W of X^*, separating the points of X, such that every member of W is either zero a.e. or a normal random variable with zero expectation. Show that μ is a centered Gaussian measure.

466Z Problems (a) Does every probability measure defined on the $\mathfrak{T}_s(\ell^\infty, (\ell^\infty)^*)$-Borel sets of $\ell^\infty = \ell^\infty(\mathbb{N})$ extend to a measure defined on the $\| \|_\infty$-Borel sets?

It is by no means obvious that the Borel sets of ℓ^∞ are different for the weak and norm topologies; for a proof see Talagrand 78b.

(b) Assume that $\mathfrak{c}$ is measure-free. Does it follow that ℓ^∞, with its weak topology, is a Radon space?

Note that a positive answer to 464Z (with $I = \mathbb{N}$) would settle this, since Talagrand's measure, when interpreted as a measure on ℓ^∞, cannot agree on the weakly Borel sets with any Radon measure on ℓ^∞ (466Xq, 466Xr).

466 Notes and comments I have given a proof of 466A using the machinery of §463; when the measure μ is known to be a Radon measure for the weak topology, rather than just a quasi-Radon measure or a τ-additive measure on the cylindrical algebra (466Xc), the theorem is older than this, and for an instructive alternative approach see Talagrand 84, 12-1-4. Another proof is in Jayne & Rogers 95.

On any Banach space we have at least three important σ-algebras: the norm-Borel σ-algebra (generated by the norm-open sets), the weak-Borel algebra (generated by the weakly open sets) and the cylindrical algebra (generated by the continuous linear functionals). (Note that the Baire σ-algebras corresponding to the norm and weak topologies are the norm-Borel algebra (4A3Kb) and the cylindrical algebra (4A3U).) If our Banach space is naturally represented as a subspace of some $\mathbb{R}^I$ (e.g., because it is a space of continuous functions), then we have in addition the σ-algebra generated by the functionals $x \mapsto x(i)$ for $i \in I$, and the Borel algebra for the topology of pointwise convergence. We correspondingly have natural questions concerning when these algebras coincide, as in 466E and 4A3V and 466Xf, and when a measure on one of the algebras leads to a measure on another, as in 466A-466B and 466Xc.

The question of which Banach spaces are Radon spaces in their norm topologies is, if not exactly 'solved', at least reducible to a classical problem in set theory by the results in §438. It seems much harder to decide which non-separable Banach spaces are Radon spaces in their weak topologies. We have a simple positive result for spaces with Kadec norms (466F), and after some labour a negative result for a couple of standard examples (466I), but no effective general criterion is known. Even the case of ℓ^∞ seems still to be open in 'ordinary' set theories (466Zb). ℓ^∞ is of particular importance in this context because the dual of any separable Banach space is isometrically isomorphic to a linear subspace of ℓ^∞ (4A4Id). So a positive answer to either question in 466Z would have very interesting consequences – and would be correspondingly surprising.

466L and 466M belong to a large family of results of the general form: if, between spaces with both topological and algebraic structures, we have a homomorphism (for the algebraic structures) which is not continuous, then it is wildly irregular. I hope to return to some of these ideas in Volume 5. For the moment I just give a version of the classical result that an additive function $f : \mathbb{R} \to \mathbb{R}$ which is Lebesgue measurable must be continuous (466Xk). The definition of 'universally measurable' function which I gave in §434 has a number of paradoxical aspects. I have already remarked that in some contexts we might prefer to use the notion of 'universally Radon-measurable' function; this is also appropriate for 466L. But when our space X, for any reason, has few Borel measures, as in 466Xl, there are correspondingly many universally measurable functions defined on X. Of course 466M can also be thought of as a generalization of the closed graph theorem; but note that, unlike the closed graph theorem, it needs a separable codomain (466Yd).

The point of 466O is that the most familiar separable Banach spaces are presented with continuous linear embeddings into $\mathbb{R}^{\mathbb{N}}$, and of course any separable Banach space X has such a presentation. We can now describe the centered Gaussian Radon measures on X in terms of centered Gaussian distributions on $\mathbb{R}^{\mathbb{N}}$, as in 466Xp. But perhaps the most important centered Gaussian measure is Wiener measure (477Yj), which is not in fact on a Banach space.

A curious geometric question concerning measures on metric spaces is the following. If two totally finite Radon measures on a metric space agree on balls, must they be identical? It is known that (even for compact spaces) the answer, in general, is 'no' (Davies 71); in Hilbert spaces the answer is 'yes' (466Xj); and in fact the same is true in any normed space (Preiss & Tišer 91).

*467 Locally uniformly rotund norms

In the last section I mentioned Kadec norms. These are interesting in themselves, but the reason for including them in this book is that in a normed space with a Kadec norm the weak topology has the same Borel sets as the norm topology (466Ea). The same will evidently be true of any space which has an equivalent Kadec norm. Now Kadec norms themselves are not uncommon, but equivalent Kadec norms appear in a striking variety of cases. Here I describe the principal class of spaces (the 'weakly K-countably determined' Banach spaces, 467H) which have equivalent Kadec norms. In fact they have 'locally uniformly rotund' norms, which are much easier to do calculations with.

Almost everything here is pure functional analysis, mostly taken from Deville Godefroy & Zizler 93, which is why I have starred the section. The word 'measure' does not appear until 467P. At that point, however, we find ourselves with a striking result (Schachermayer's theorem) which appears to need the structure theory of weakly compactly generated Banach spaces developed in 467C-467M.

467A Definition Let X be a linear space with a norm $\| \|$. $\| \|$ is **locally uniformly rotund** or **locally uniformly convex** if whenever $\|x\| = 1$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\|x - y\| \le \epsilon$ whenever $\|y\| = 1$ and $\|x + y\| \ge 2 - \delta$.

If X has a locally uniformly rotund norm, then every subspace of X has a locally uniformly rotund norm. Of course any uniformly convex norm (definition: 2A4K) is locally uniformly rotund.

467B Proposition A locally uniformly rotund norm is a Kadec norm.

proof Let X be a linear space with a locally uniformly rotund norm $\| \|$. Set $S_X = \{x : \|x\| = 1\}$. Suppose that G is open for the norm topology and that $x \in G \cap S_X$. Then there is an $\epsilon > 0$ such that $G \supseteq B(x, \epsilon) = \{y : \|y - x\| \le \epsilon\}$. Let $\delta > 0$ be such that $\|x - y\| \le \epsilon$ whenever $\|y\| = 1$ and $\|x + y\| \ge 2 - \delta$. Now there is an $f \in X^*$ such that $f(x) = \|f\| = 1$ (3A5Ac). So $V = \{y : f(y) > 1 - \delta\}$ is open for the weak topology. But if $y \in V \cap S_X$, then $\|x + y\| \ge f(x + y) \ge 2 - \delta$, so $\|x - y\| \le \epsilon$ and $y \in G$. As x is arbitrary, $G \cap S_X$ is open for the weak topology on S_X; as G is arbitrary, the norm and weak topologies agree on S_X.

467C A technical device (a) I will use the following notation for the rest of the section. Let X be a linear space and $p : X \to [0, \infty[$ a seminorm. Define $q_p : X \times X \to [0, \infty[$ by setting

$$q_p(x, y) = 2p(x)^2 + 2p(y)^2 - p(x + y)^2 = (p(x) - p(y))^2 + (p(x) + p(y))^2 - p(x + y)^2$$

for $x \in X$.

(b) A norm $\| \|$ on X is locally uniformly rotund iff whenever $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that $\|x-y\| \le \epsilon$ whenever $q_{\| \|}(x,y) \le \delta$.

P(i) Suppose that $\| \|$ is locally uniformly rotund, $x \in X$ and $\epsilon > 0$. (α) If $x = 0$ then $q_{\| \|}(x,y) = \|y\|^2 = \|x-y\|^2$ for every y so we can take $\delta = \epsilon^2$. (β) If $x \ne 0$ set $x' = \frac{1}{\|x\|}x$. Let $\eta > 0$ be such that $\|x'-y'\| \le \frac{1}{2}\epsilon\|x\|$ whenever $\|y'\| = 1$ and $\|x'+y'\| \ge 2-\eta$. Let $\delta > 0$ be such that

$$\delta + 2\sqrt{\delta}\|x\| \le \eta\|x\|^2, \quad \sqrt{\delta} < \|x\|, \quad \sqrt{\delta} \le \frac{1}{2}\epsilon\|x\|^2.$$

Now if $q_{\| \|}(x,y) \le \delta$, we must have

$$(\|x\| - \|y\|)^2 \le \delta < \|x\|^2, \quad (\|x\| + \|y\|)^2 - \|x+y\|^2 \le \delta,$$

so that $y \ne 0$ and

$$\Bigl|\frac{1}{\|x\|} - \frac{1}{\|y\|}\Bigr|\|y\| = \frac{|\|y\|-\|x\||}{\|x\|} \le \frac{\sqrt{\delta}}{\|x\|},$$

$$\|x\| + \|y\| - \|x+y\| \le \frac{\delta}{\|x\|+\|y\|+\|x+y\|} \le \frac{\delta}{\|x\|}.$$

Set $y' = \frac{1}{\|y\|}y$, $y'' = \frac{1}{\|x\|}y$. Then $\|y'\| = 1$, and

$$\|y' - y''\| = \Bigl|\frac{1}{\|y\|} - \frac{1}{\|x\|}\Bigr|\|y\| \le \frac{\sqrt{\delta}}{\|x\|} \le \frac{1}{2}\epsilon\|x\|.$$

Accordingly

$$\begin{aligned}\|x'+y'\| &\ge \|x'+y''\| - \|y'-y''\| \ge \frac{1}{\|x\|}\|x+y\| - \frac{\sqrt{\delta}}{\|x\|}\\ &\ge \frac{1}{\|x\|}(\|x\|+\|y\| - \frac{\delta}{\|x\|}) - \frac{\sqrt{\delta}}{\|x\|} = 1 + \frac{\|y\|}{\|x\|} - \frac{\delta}{\|x\|^2} - \frac{\sqrt{\delta}}{\|x\|}\\ &\ge 1 + \frac{\|x\|-\sqrt{\delta}}{\|x\|} - \frac{\delta}{\|x\|^2} - \frac{\sqrt{\delta}}{\|x\|} = 2 - \frac{\delta}{\|x\|^2} - \frac{2\sqrt{\delta}}{\|x\|} \ge 2 - \eta.\end{aligned}$$

But this means that $\|x'-y'\| \le \frac{1}{2}\epsilon\|x\|$, so that $\|x'-y''\| \le \epsilon\|x\|$ and $\|x-y\| \le \epsilon$. As x and ϵ are arbitrary, the condition is satisfied.

(ii) Suppose the condition is satisfied. If $\|x\| = 1$ and $\epsilon > 0$, take $\delta \in]0,2]$ such that $\|x-y\| \le \epsilon$ whenever $q(x,y) \le 4\delta$; then if $\|y\| = 1$ and $\|x+y\| \ge 2-\delta$, $q(x,y) = 4 - \|x+y\|^2 \le 4\delta$ and $\|x-y\| \le \epsilon$. As x and ϵ are arbitrary, $\| \|$ is locally uniformly rotund. **Q**

(c) We have the following elementary facts. Let X be a linear space.

(i) For any seminorm p on X, $q_p(x,y) \ge (p(x) - p(y))^2 \ge 0$ for all x, $y \in X$. **P** $(p(x)+p(y))^2 - p(x+y)^2 \ge 0$ because $p(x+y) \le p(x) + p(y)$. **Q**

(ii) Suppose that $\langle p_i \rangle_{i \in I}$ is a family of seminorms on X such that $\sum_{i\in I} p_i(x)^2$ is finite for every $x \in X$. Set $p(x) = \sqrt{\sum_{i\in I} p_i(x)^2}$ for $x \in X$; then p is a seminorm on X and $q_p = \sum_{i\in I} q_{p_i}$. **P** Of course $p(\alpha x) = |\alpha|p(x)$ for $\alpha \in \mathbb{R}$ and $x \in X$. If $x \in X$, then $p(x) = \|\phi(x)\|_2$, where $\phi(x) = \langle p_i(x) \rangle_{i\in I} \in \ell^2(I)$. Now for x, $y \in X$,

$$0 \le \phi(x+y) \le \phi(x) + \phi(y)$$

in $\ell^2(I)$, so

$$p(x+y) = \|\phi(x+y)\|_2 \le \|\phi(x)+\phi(y)\|_2 \le \|\phi(x)\|_2 + \|\phi(y)\|_2 = p(x) + p(y).$$

Thus p is a seminorm. Now the calculation of $q_p = \sum_{i\in I} q_{p_i}$ is elementary. **Q** In particular, $q_p \ge q_{p_i}$ for every $i \in I$.

(iii) If $\| \|$ is an inner product norm on X, then $q_{\| \|}(x,y) = \|x-y\|^2$ for all x, $y \in X$. **P**

$$\begin{aligned}2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 &= 2(x|x) + 2(y|y) - (x+y|x+y)\\ &= (x|x) + (y|y) - (x|y) - (y|x) = (x-y|x-y).\ \mathbf{Q}\end{aligned}$$

467D Lemma Let $(X, \| \|)$ be a normed space. Suppose that there are a space Y with a locally uniformly rotund norm $\| \|_Y$ and a bounded linear operator $T : Y \to X$ such that $T[Y]$ is dense in X and, for every $x \in X$ and $\gamma > 0$, there is a $z \in Y$ such that $\|x - Tz\|^2 + \gamma\|z\|_Y^2 = \inf_{y\in Y} \|x - Ty\|^2 + \gamma\|y\|_Y^2$. Then X has an equivalent locally uniformly rotund norm.

proof (a) For each $n \in \mathbb{N}$, $x \in X$ set

$$p_n(x) = \sqrt{\inf_{y\in Y} \|x - Ty\|^2 + 2^{-n}\|y\|_Y^2}.$$

Then $p_n : X \to [0, \infty[$ is a norm on X, equivalent to $\| \|$. **P** (i) The functionals $(x, y) \mapsto \|x-Ty\|$, $(x, y) \mapsto 2^{-n/2}\|y\|_Y$ from $X \times Y$ to $[0, \infty[$ are both seminorms, so the functional $(x, y) \mapsto \phi(x, y) = \sqrt{\|x - Ty\|^2 + 2^{-n}\|y\|_Y^2}$ also is, by 467C(c-ii). (ii) If $x \in X$ and $\alpha \in \mathbb{R}$, take $z \in Y$ such that $p_n(x) = \phi(x, z)$; then

$$p_n(\alpha x) \le \phi(\alpha x, \alpha z) = |\alpha|\phi(x, z) = |\alpha|p_n(x).$$

If $\alpha \ne 0$, apply the same argument to see that $p_n(x) \le |\alpha|^{-1}p_n(\alpha x)$, so that $p_n(\alpha x) = |\alpha|p_n(x)$. (iii) Now take any x_1, $x_2 \in X$. Let z_1, $z_2 \in Y$ be such that $p_n(x_i) = \phi(x_i, z_i)$ for both i. Then

$$\begin{aligned} p_n(x_1 + x_2) &= 2p_n(\frac{1}{2}x_1 + \frac{1}{2}x_2) \le 2\phi(\frac{1}{2}x_1 + \frac{1}{2}x_2, \frac{1}{2}z_1 + \frac{1}{2}z_2) \\ &\le 2(\frac{1}{2}\phi(x_1, z_1) + \frac{1}{2}\phi(x_2, z_2)) = p_n(x_1) + p_n(x_2). \end{aligned}$$

Thus p_n is a seminorm. (iv) $p_n(x) \le \phi(x, 0) = \|x\|$ for every $x \in X$. (vi) For any $x \in X$ and $y \in Y$, either $\|Ty\| \ge \frac{1}{2}\|x\|$ and

$$\phi(x, y) \ge 2^{-n/2}\|y\|_Y \ge \frac{1}{2\cdot 2^{n/2}\|T\|}\|x\|,$$

or $\|Ty\| \le \frac{1}{2}\|x\|$ and

$$\phi(x, y) \ge \|x - Ty\| \ge \frac{1}{2}\|x\|;$$

this shows that $p_n(x) \ge \min(\frac{1}{2}, \frac{1}{2}2^{-n/2}\|T\|^{-1})\|x\|$. (I am passing over the trivial case $X = \{0\}$, $\|T\| = 0$.) In particular, $p_n(x) = 0$ only when $x = 0$. Thus p_n is a norm equivalent to p. **Q**

(b) For any $x \in X$, $\lim_{n\to\infty} p_n(x) = 0$. **P** Let $\epsilon > 0$. Let $y \in Y$ be such that $\|x - Ty\| \le \epsilon$; then

$$\limsup_{n\to\infty} p_n(x)^2 \le \limsup_{n\to\infty} \|x - Ty\|^2 + 2^{-n}\|y\|_Y^2 \le \epsilon^2.$$

As ϵ is arbitrary, $\lim_{n\to\infty} p_n(x) = 0$. **Q**

(c) Set $\|x\|' = \sqrt{\sum_{n=0}^{\infty} 2^{-n}p_n(x)^2}$ for $x \in X$. The sum is always finite because $p_n(x) \le \|x\|$ for every n, so $\| \|' \le \sqrt{2}\| \|$ is a seminorm; and it is a norm equivalent to $\| \|$ because p_0 is. Now $\| \|'$ is locally uniformly rotund. **P** Take $x \in X$ and $\epsilon > 0$. Let $n \in \mathbb{N}$ be such that $p_n(x) \le \frac{1}{4}\epsilon$. Choose $y \in Y$ such that $p_n(x)^2 = \|x - Ty\|^2 + 2^{-n}\|y\|_Y^2$. (This is where we really use the hypothesis that the infimum in the definition of p_n is attained.) Let $\delta > 0$ be such that $2^n\delta \le (\frac{1}{4}\epsilon)^2$ and $\|T\|\|y' - y\| \le \frac{1}{4}\epsilon$ whenever $q_{\| \|_Y}(y', y) \le 2^{2n}\delta$.

If $q_{\| \|'}(x, x') \le \delta$, then $q_{2^{-n/2}p_n}(x, x') \le \delta$, by 467C(c-ii), that is, $q_{p_n}(x, x') \le 2^n\delta$. Let $y' \in Y$ be such that $p_n(x')^2 = \|x' - Ty'\|^2 + 2^{-n}\|y'\|_Y^2$. Then

$$\begin{aligned} p_n(x + x')^2 &\le \|x + x' - Ty - Ty'\|^2 + 2^{-n}\|y + y'\|_Y^2 \\ &\le (\|x - Ty\| + \|x' - Ty'\|)^2 + 2^{-n}\|y + y'\|_Y^2, \end{aligned}$$

so

$$\begin{aligned} q_{p_n}(x, x') &= 2p_n(x)^2 + 2p_n(x')^2 - p_n(x + x')^2 \\ &\ge 2(\|x - Ty\|^2 + 2^{-n}\|y\|_Y^2) + 2(\|x' - Ty'\|^2 + 2^{-n}\|y'\|_Y^2) \\ &\qquad - (\|x - Ty\| + \|x' - Ty'\|)^2 - 2^{-n}\|y + y'\|_Y^2 \\ &= (\|x - Ty\| - \|x' - Ty'\|)^2 + 2^{-n}(2\|y\|_Y^2 + 2\|y'\|_Y^2 - \|y + y'\|_Y^2). \end{aligned}$$

This means that

$$q_{\| \|_Y}(y, y') \le 2^n q_{p_n}(x, x') \le 2^{2n}\delta,$$

so $\|T\|\|y-y'\|_Y \le \frac{1}{4}\epsilon$, while also

$$\|x-Ty\|+\|x'-Ty'\| \le 2\|x-Ty\|+\sqrt{q_{p_n}(x,x')} \le 2p_n(x)+2^{n/2}\sqrt{\delta} \le \frac{3}{4}\epsilon.$$

Finally

$$\frac{1}{\sqrt{2}}\|x-x'\|' \le \|x-x'\| \le \|T\|\|y-y'\|_Y+\|x-Ty\|+\|x'-Ty'\| \le \frac{1}{4}\epsilon+\frac{3}{4}\epsilon=\epsilon.$$

As x and ϵ are arbitrary, this shows that $\| \|'$ is locally uniformly rotund. **Q**

This completes the proof.

467E Theorem Let X be a separable normed space. Then it has an equivalent locally uniformly rotund norm.

proof (a) It is enough to show that the completion of X has an equivalent locally uniformly rotund norm; since the completion of X is separable, we may suppose that X itself is complete. Let $\langle x_i\rangle_{i\in\mathbb{N}}$ be a sequence in X running over a dense subset of X. Define $T:\ell^2\to X$ by setting

$$Ty=\sum_{i=0}^{\infty}\frac{y(i)}{2^i(1+\|x_i\|)}x_i$$

for $y\in\ell^2=\ell^2(\mathbb{N})$; then Ty is always defined (4A4Ie); T is a linear operator and

$$\|Ty\| \le \sum_{i=0}^{\infty}2^{-i}|y(i)| \le \sqrt{\sum_{i=0}^{\infty}2^{-2i}}\,\|y\|_2$$

for every $y\in\ell^2$, by Cauchy's inequality (244Eb). So T is a bounded linear operator.

(b) T satisfies the conditions of 467D. **P** $T[\ell^2]$ is dense because it contains every x_i. Given $x\in X$, $\gamma>0$ and $\alpha\ge 0$, the function $y\mapsto\sqrt{\|x-Ty\|^2+\gamma\|y\|_2^2}$ is convex and norm-continuous, so the set

$$C_\alpha(x)=\{y:y\in\ell^2,\,\|x-Ty\|^2+\gamma\|y\|_2^2\le\alpha^2\}$$

is convex and norm-closed. Consequently, $C_\alpha(x)$ is weakly closed (4A4Ed); since $\|y\|_2\le\gamma^{-1/2}\alpha$ for every $y\in C_\alpha(x)$, $C_\alpha(x)$ is weakly compact (4A4Ka). Set $\beta=\inf_{y\in\ell^2}\|x-Ty\|^2+\gamma\|y\|_2^2$. Then $\{C_\alpha(x):\alpha>\beta\}$ is a downwards-directed set of non-empty weakly compact sets, so has non-empty intersection; taking any $z\in\bigcap_{\alpha>\beta}C_\alpha(x)$, $\beta=\|x-Tz\|^2+\gamma\|z\|_2^2$. **Q**

(c) So 467D gives the result.

467F Lemma Let $(X,\| \|)$ be a Banach space, and $\langle T_i\rangle_{i\in I}$ a family of bounded linear operators from X to itself such that

(i) for each $i\in I$, the subspace $T_i[X]$ has an equivalent locally uniformly rotund norm,
(ii) for each $x\in X$, $\epsilon>0$ there is a finite set $J\subseteq I$ such that $\|x-\sum_{i\in J}T_ix\|\le\epsilon$,
(iii) for each $x\in X$, $\epsilon>0$ the set $\{i:i\in I,\,\|T_ix\|\ge\epsilon\}$ is finite.

Then X has an equivalent locally uniformly rotund norm.

proof (a) Let $\| \|_i$ be a locally uniformly rotund norm on $X_i=T_i[X]$ equivalent to $\| \|$ on X_i. Reducing $\| \|_i$ by a scalar multiple if necessary, we may suppose that $\|T_ix\|_i\le\|x\|$ for every $x\in X$ and $i\in I$. By (iii), $\sup_{i\in I}\|T_ix\|$ is finite for every $x\in X$; by the Uniform Boundedness Theorem (3A5Ha), $M=\sup_{i\in I}\sup_{\|x\|\le 1}\|T_ix\|$ is finite. (This is where we need to suppose that X is complete.) For finite sets $J\subseteq I$ and $k\ge 1$, set

$$p_{Jk}(x)=\sqrt{\sum_{i\in J}\|T_ix\|_i^2+\tfrac{1}{k}\sum_{K\subseteq J}\|x-\sum_{i\in K}T_ix\|^2};$$

for $n\in\mathbb{N}$ and $k\ge 1$ set

$$p_k^{(n)}(x)=\sup\{p_{Jk}(x):J\subseteq I,\,\#(J)\le n\}.$$

By 467C(c-ii), as usual, all the p_{Jk} are seminorms, and it follows at once that the $p_k^{(n)}$ are seminorms. Observe that if $K\subseteq I$ is finite, then $\|x-\sum_{i\in K}T_ix\|\le(1+M\#(K))\|x\|$ for every x, so if $J\subseteq I$ is finite then

$$p_{Jk}(x)\le\sqrt{\#(J)+2^{\#(J)}(1+M\#(J))}\|x\|,$$

and $p_k^{(n)}(x)\le\sqrt{n+2^n(1+Mn)}\|x\|$ whenever $n\in\mathbb{N}$ and $k\ge 1$. Setting $\beta_{nk}=2^{2n+k}$ for n, $k\in\mathbb{N}$,

$$\|x\|'=\sqrt{\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\beta_{nk}^{-1}p_k^{(n)}(x)^2}$$

is finite for every $x \in X$, so that $\| \|'$ is a seminorm on X; moreover, $\|x\|' \le \beta\|x\|$ for every $x \in X$, where

$$\beta = \sqrt{\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \beta_{nk}^{-1}(n + 2^n(1 + Mn))}$$

is finite. Since we also have

$$\|x\|' \ge \frac{1}{\sqrt{2}} p_1^{(0)}(x) = \frac{1}{\sqrt{2}} p_{\emptyset 1}(x) = \frac{1}{\sqrt{2}} \|x\|$$

for every x, $\| \|'$ is a norm on X equivalent to $\| \|$.

(b) Now $\| \|'$ is locally uniformly rotund. **P** Take $x \in X$ and $\epsilon > 0$. Let $K \subseteq I$ be a finite set such that $\|x - \sum_{i \in K} T_i x\| \le \frac{1}{4}\epsilon$; we may suppose that $T_i x \ne 0$ for every $i \in K$. Set $\alpha_1 = \min_{i \in K} \|T_i x\|_i$, $J = \{i : i \in I, \|T_i x\|_i \ge \alpha_1\}$ and $\alpha_0 = \sup_{i \in I \setminus J} \|T_i x\|_i$. (For completeness, if $K = \emptyset$, take $J = \emptyset$, $\alpha_0 = \sup_{i \in I} \|T_i x\|_i$ and $\alpha_1 = \alpha_0 + 1$.) Then J is finite and $\alpha_0 < \alpha_1$, by hypothesis (iii) of the lemma. Set $n = \#(J)$. Let k be so large that $\frac{2^n(Mn+1)}{k}\|x\|^2 < \frac{1}{2}(\alpha_1^2 - \alpha_0^2)$. Let $\eta > 0$ be such that

$$(\beta_{nk} + 1)\eta \le \min(\frac{\epsilon^2}{16k}, \alpha_1^2 - \alpha_0^2),$$

$$\|T_i x - z\| \le \frac{\epsilon}{1+4n} \text{ whenever } i \in K,\ z \in X_i \text{ and } q_{\| \|_i}(T_i x, z) \le (\beta_{nk} + 1)\eta;$$

this is where we use the hypothesis that every $\| \|_i$ is locally uniformly rotund (and equivalent to $\| \|$ on X_i).

Now suppose that $y \in X$ and $q_{\| \|'}(x, y) \le \eta$. Then $q_{p_k^{(n)}}(x, y) \le \beta_{nk}\eta$, by 467C(c-ii). Let $L \in [I]^n$ be such that $p_k^{(n)}(x + y)^2 \le p_{Lk}(x + y)^2 + \eta$. Then

$$\begin{aligned} q_{p_{Lk}}(x, y) &= 2p_{Lk}(x)^2 + 2p_{Lk}(y)^2 - p_{Lk}(x + y)^2 \\ &\le 2p_k^{(n)}(x)^2 + 2p_k^{(n)}(y)^2 - p_k^{(n)}(x + y)^2 + \eta \le (\beta_{nk} + 1)\eta. \end{aligned}$$

We also have

$$\begin{aligned} 2p_{Lk}(x)^2 &\ge 2p_{Lk}(x)^2 + 2p_{Lk}(y)^2 - 2p_k^{(n)}(y)^2 \ge p_{Lk}(x + y)^2 - 2p_k^{(n)}(y)^2 \\ &\ge p_k^{(n)}(x + y)^2 - \eta - 2p_k^{(n)}(y)^2 \ge 2p_k^{(n)}(x)^2 - (\beta_{nk} + 1)\eta, \end{aligned}$$

so

$$\begin{aligned} \sum_{i \in J} \|T_i x\|_i^2 &\le p_{Jk}(x)^2 \le p_k^{(n)}(x)^2 \le p_{Lk}(x)^2 + \frac{1}{2}(\beta_{nk} + 1)\eta \\ &\le \sum_{i \in L} \|T_i x\|_i^2 + \frac{2^n(Mn+1)}{k}\|x\|^2 + \frac{1}{2}(\beta_{nk} + 1)\eta \\ &< \sum_{i \in L} \|T_i x\|_i^2 + \alpha_1^2 - \alpha_0^2. \end{aligned}$$

Since $\#(L) = \#(J)$ and

$$\|T_i x\|_i^2 \le \alpha_0^2 < \alpha_1^2 \le \|T_j x\|_j^2$$

whenever $j \in J$ and $i \in I \setminus J$, we must actually have $L = J$. In particular, $K \subseteq L$. But this means that (by 467C(c-ii) again)

$$q_{\| \|_i}(T_i x, T_i y) \le q_{p_{Lk}}(x, y) \le (\beta_{nk} + 1)\eta$$

and (by the choice of η) $\|T_i x - T_i y\| \le \frac{\epsilon}{1+4n}$ for every $i \in K$, so that $\|\sum_{i \in K} T_i x - \sum_{i \in K} T_i y\| \le \frac{1}{4}\epsilon$.

The last element we need is that, setting $\tilde{p}(z) = \frac{1}{\sqrt{k}}\|z - \sum_{i \in K} T_i z\|$, $\tilde{p}$ is a seminorm on X and is one of the constituents of p_{Lk}; so that

$$\begin{aligned} \frac{1}{k}(\|x - \sum_{i \in K} T_i x\| - \|y - \sum_{i \in K} T_i y\|)^2 &\le q_{\tilde{p}}(x, y) \le q_{p_{Lk}}(x, y) \\ &\le (\beta_{nk} + 1)\eta \le \frac{\epsilon^2}{16k}, \end{aligned}$$

and $\left|\|x-\sum_{i\in K}T_ix\|-\|y-\sum_{i\in K}T_iy\|\right|\le\frac{1}{4}\epsilon$. It follows that

$$\|y-\sum_{i\in K}T_iy\|\le\tfrac{1}{4}\epsilon+\|x-\sum_{i\in K}T_ix\|\le\tfrac{1}{2}\epsilon.$$

Putting these together,

$$\|x-y\|\le\|x-\sum_{i\in K}T_ix\|+\sum_{i\in K}\|T_ix-T_iy\|+\|y-\sum_{i\in K}T_iy\|\le\epsilon.$$

And this is true whenever $q_{\|\,\|'}(x,y)\le\eta$. As x and ϵ are arbitrary, $\|\,\|'$ is locally uniformly rotund. **Q**

467G Theorem Let X be a Banach space. Suppose that there are an ordinal ζ and a family $\langle P_\xi\rangle_{\xi\le\zeta}$ of bounded linear operators from X to itself such that

(i) if $\xi\le\eta\le\zeta$ then $P_\xi P_\eta=P_\eta P_\xi=P_\xi$;

(ii) $P_0(x)=0$ and $P_\zeta(x)=x$ for every $x\in X$;

(iii) if $\xi\le\zeta$ is a non-zero limit ordinal, then $\lim_{\eta\uparrow\xi}P_\eta(x)=P_\xi(x)$ for every $x\in X$;

(iv) if $\xi<\zeta$ then $X_\xi=\{(P_{\xi+1}-P_\xi)(x):x\in X\}$ has an equivalent locally uniformly rotund norm.

Then X has an equivalent locally uniformly rotund norm.

Remark A family $\langle P_\xi\rangle_{\xi\le\zeta}$ satisfying (i), (ii) and (iii) here is called a **projectional resolution of the identity**.

proof For $\xi<\zeta$ set $T_\xi=P_{\xi+1}-P_\xi$. From condition (i) we see easily that $T_\xi T_\eta=T_\xi$ if $\xi=\eta$, 0 otherwise; and that $T_\xi P_\eta=T_\xi$ if $\xi<\eta$, 0 otherwise.

I seek to show that the conditions of 467F are satisfied by $\langle T_\xi\rangle_{\xi<\zeta}$. Condition (i) of 467F is just condition (iv) here. Let Z be the set of those $x\in X$ for which conditions (ii) and (iii) of 467F are satisfied; that is,

for each $\epsilon>0$ there is a finite set $J\subseteq\zeta$ such that $\|x-\sum_{\xi\in J}T_\xi x\|\le\epsilon$, and $\{\xi:\|T_\xi x\|\ge\epsilon\}$ is finite.

Then Z is a linear subspace of X. For $\xi\le\zeta$, set $Y_\xi=P_\xi[X]$. Then $Y_\xi\subseteq Z$. **P** Induce on ξ. Since $Y_0=\{0\}$, the induction starts. For the inductive step to a successor ordinal $\xi+1\le\zeta$, $Y_{\xi+1}=Y_\xi+X_\xi\subseteq Z$. For the inductive step to a non-zero limit ordinal $\xi\le\zeta$, given $x\in Y_\xi$ and $\epsilon>0$, we know that there is a $\xi'<\xi$ such that $\|P_\eta x-P_\xi x\|\le\frac{1}{3}\epsilon$ whenever $\xi'\le\eta\le\xi$. So $\|T_\eta x\|\le\frac{2}{3}\epsilon$ whenever $\xi'\le\eta<\xi$, and

$$\{\eta:\|T_\eta x\|\ge\epsilon\}=\{\eta:\eta<\xi',\,\|T_\eta x\|\ge\epsilon\}=\{\eta:\|T_\eta P_{\xi'}x\|\ge\epsilon\}$$

is finite, by the inductive hypothesis. Moreover, there is a finite set $J\subseteq\xi'$ such that $\|P_{\xi'}x-\sum_{\eta\in J}T_\eta P_{\xi'}x\|\le\frac{2}{3}\epsilon$, and now $\|x-\sum_{\eta\in J}T_\eta x\|\le\epsilon$. As x and ϵ are arbitrary, $Y_\xi\subseteq Z$. **Q**

In particular, $X=Y_\zeta\subseteq Z$ and conditions (ii) and (iii) of 467F are satisfied. So 467F gives the result.

467H Definitions (a) A topological space X is **K-countably determined** or a **Lindelöf-Σ** space if there are a subset A of $\mathbb{N}^{\mathbb{N}}$ and an usco-compact relation $R\subseteq A\times X$ such that $R[A]=X$. Observe that all K-analytic Hausdorff spaces (§422) are K-countably determined.

(b) A normed space X is **weakly K-countably determined** if it is K-countably determined in its weak topology.

(c) Let X be a normed space and Y, W closed linear subspaces of X, X^* respectively. I will say that (Y,W) is a **projection pair** if $X=Y\oplus W^\circ$ and $\|y+z\|\ge\|y\|$ for every $y\in Y$, $z\in W^\circ$, where

$$\begin{aligned}W^\circ&=\{z:z\in X,\,f(z)\le1\text{ for every }f\in W\}\\&=\{z:z\in X,\,f(z)=0\text{ for every }f\in W\}.\end{aligned}$$

467I Lemma (a) If X is a weakly K-countably determined normed space, then any closed linear subspace of X is weakly K-countably determined.

(b) If X is a weakly K-countably determined normed space, Y is a normed space, and $T:X\to Y$ is a continuous linear surjection, then Y is weakly K-countably determined.

(c) If X is a Banach space and $Y\subseteq X$ is a dense linear subspace which is weakly K-countably determined, then X is weakly K-countably determined.

proof (a) Let $A\subseteq\mathbb{N}^{\mathbb{N}}$, $R\subseteq A\times X$ be such that R is usco-compact (for the weak topology on X) and $R[A]=X$. Let Y be a (norm-)closed linear subspace of X; then Y is closed for the weak topology (3A5Ee). Also the weak topology on Y is just the subspace topology induced by the weak topology of X (4A4Ea). Set $R'=R\cap(A\times Y)$.

Then R' is usco-compact whether regarded as a subset of $A \times X$ or as a subset of $A \times Y$ (422Da, 422Db, 422Dg). Since $Y = R'[A]$, Y is weakly K-countably determined.

(b) Let $A \subseteq \mathbb{N}^{\mathbb{N}}$, $R \subseteq A \times X$ be such that R is usco-compact for the weak topology on X and $R[A] = X$. Because T is continuous for the weak topologies on X and Y (3A5Ec),

$$R_1 = \{(\phi, y)\text{: there is some } x \in X \text{ such that } (\phi, x) \in R \text{ and } Tx = y\}$$

is usco-compact in $A \times Y$ (422Db, 422Df). Also $R_1[A] = T[R[A]] = Y$. So Y is weakly K-countably determined.

(c) Let $A \subseteq \mathbb{N}^{\mathbb{N}}$, $R \subseteq A \times Y$ be such that R is usco-compact (for the weak topology on Y) and $R[A] = Y$. Then, as in (a), R is usco-compact when regarded as a subset of $A \times X$. By 422Dd, the set

$$R_1 = \{(\langle \phi_n \rangle_{n\in\mathbb{N}}, \langle y_n \rangle_{n\in\mathbb{N}}) : (\phi_n, y_n) \in R \text{ for every } n \in \mathbb{N}\}$$

is usco-compact in $A^{\mathbb{N}} \times Y^{\mathbb{N}}$. Now examine

$$S = \{(\langle y_n \rangle_{n\in\mathbb{N}}, x) : x \in X,\ y_n \in Y \text{ and } \|y_n - x\| \le 2^{-n} \text{ for every } n \in \mathbb{N}\}.$$

Then S is usco-compact in $Y^{\mathbb{N}} \times X$. **P** (Remember that we are using weak topologies on X and Y throughout.) If $\boldsymbol{y} = \langle y_n \rangle_{n\in\mathbb{N}}$ is a sequence in Y and $(\boldsymbol{y}, x) \in S$, then $x = \lim_{n\to\infty} y_n$; so $S[\{\boldsymbol{y}\}]$ has at most one member and is certainly compact. Let $F \subseteq X$ be a weakly closed set and $\boldsymbol{y} \in Y^{\mathbb{N}} \setminus S^{-1}[F]$.

case 1 If there are m, $n \in \mathbb{N}$ such that $\|y_m - y_n\| > 2^{-m} + 2^{-n}$, let $f \in Y^*$ be such that $\|f\| \le 1$ and $f(y_m - y_n) > 2^{-m} + 2^{-n}$. Then $G = \{\boldsymbol{z} : f(z_m - z_n) > 2^{-m} + 2^{-n}\}$ is an open set in $Y^{\mathbb{N}}$ containing $\boldsymbol{y}$ and disjoint from $S^{-1}[F]$, because $S[G]$ is empty.

case 2 Otherwise, $\boldsymbol{y}$ is a Cauchy sequence and (because X is a Banach space) has a limit $x \in X$, which does not belong to F. Let $\delta > 0$ and $f_0, \dots , f_r \in X^*$ be such that $\{w : |f_i(w) - f_i(x)| \le \delta \text{ for every } i \le r\}$ does not meet F. Let $n \in \mathbb{N}$ be such that $2^{-n}\|f_i\| \le \frac{1}{3}\delta$ for every $i \le r$. Then $G = \{\boldsymbol{z} : |f_i(z_n) - f_i(y_n)| < \frac{1}{3}\delta \text{ for every } i \le r\}$ is an open set in $Y^{\mathbb{N}}$ containing Y. If $\boldsymbol{z} \in G$ and $(\boldsymbol{z}, w) \in S$, then $\|z_n - w\| \le 2^{-n}$ so

$$\begin{aligned}|f_i(w) - f_i(x)| &\le |f_i(w) - f_i(z_n)| + |f_i(z_n) - f_i(y_n)| + |f_i(y_n) - f_i(x)| \\ &\le 2^{-n}\|f_i\| + \frac{1}{3}\delta + 2^{-n}\|f_i\| \le \delta\end{aligned}$$

for every $i \le r$, and $w \notin F$. Thus again $G \cap S^{-1}[F]$ is empty.

This shows that there is always an open set containing y and disjoint from $S^{-1}[F]$. As y is arbitrary, $S^{-1}[F]$ is closed. As F is arbitrary, S is usco-compact. **Q**

It follows that $SR_1 \subseteq A^{\mathbb{N}} \times X$ is usco-compact (422Df), while

$$(SR_1)[A^{\mathbb{N}}] = S[R_1[A^{\mathbb{N}}]] = S[Y^{\mathbb{N}}] = X$$

because Y is dense in X. Finally, $A^{\mathbb{N}}$ is homeomorphic to a subset of $\mathbb{N}^{\mathbb{N}}$ because it is a subspace of $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \cong \mathbb{N}^{\mathbb{N}}$. So X is weakly K-countably determined.

467J Lemma Let X be a weakly K-countably determined Banach space. Then there is a family $\mathcal{M}$ of subsets of $X \cup X^*$ such that

(i) whenever $B \subseteq X \cup X^*$ there is an $M \in \mathcal{M}$ such that $B \subseteq M$ and $\#(M) \le \max(\omega, \#(B))$;

(ii) whenever $\mathcal{M}' \subseteq \mathcal{M}$ is upwards-directed, then $\bigcup \mathcal{M}' \in \mathcal{M}$;

(iii) whenever $M \in \mathcal{M}$ then $(\overline{M \cap X}, \overline{M \cap X^*})$ (where the closures are taken for the norm topologies) is a projection pair of subspaces of X and X^*.

proof (a) Let $A \subseteq \mathbb{N}^{\mathbb{N}}$, $R \subseteq A \times X$ be such that R is usco-compact in $A \times X$ and $R[A] = X$. Set $S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ and for $\sigma \in S$ set $F_\sigma = R[I_\sigma]$, where $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$; set $S_0 = \{\sigma : \sigma \in S,\ F_\sigma \ne \emptyset\}$.

Let $\mathcal{M}$ be the family of those sets $M \subseteq X \cup X^*$ such that (α) whenever x, $y \in M \cap X$ and $q \in \mathbb{Q}$ then $x + y$ and qx belong to M (β) whenever f, $g \in M \cap X^*$ and $q \in \mathbb{Q}$ then $f + g$ and qf belong to M (γ) $\|x\| = \max\{f(x) : f \in M \cap X^*, \|f\| \le 1\}$ for every $x \in M \cap X$ (δ) $\sup_{x\in F_\sigma} f(x) = \sup_{x\in F_\sigma\cap M} f(x)$ for every $f \in M \cap X^*$, $\sigma \in S_0$.

(b) For each $x \in X$, choose $h_x \in X^*$ such that $\|h_x\| \le 1$ and $h_x(x) = \|x\|$; for each $f \in X^*$ and $\sigma \in S_0$ choose a countable set $C_{f\sigma} \subseteq F_\sigma$ such that $\sup\{f(x) : x \in C_{f\sigma}\} = \sup\{f(x) : x \in F_\sigma\}$. Given $B \subseteq X$, define $\langle B_n \rangle_{n\in\mathbb{N}}$ by setting

$$B_{n+1}=B_n\cup\{x+y:x,\,y\in B_n\cap X\}\cup\{qx:q\in\mathbb{Q},\,x\in B_n\cap X\}$$
$$\cup\{f+g:f,\,g\in B_n\cap X\}\cup\{qf:q\in\mathbb{Q},\,f\in B_n\cap X\}$$
$$\cup\{h_x:x\in B_n\cap X\}\cup\bigcup\{C_{f\sigma}:f\in B_n\cap X^*,\,\sigma\in S_0\},$$

for each $n\in\mathbb{N}$. Then $M=\bigcup_{n\in\mathbb{N}}B_n$ belongs to $\mathcal{M}$ and has cardinal at most $\max(\omega,\#(B))$.

(c) The definition of $\mathcal{M}$ makes it plain that if $\mathcal{M}'\subseteq\mathcal{M}$ is upwards-directed then $\bigcup\mathcal{M}'$ belongs to $\mathcal{M}$.

(d) Now take $M\in\mathcal{M}$ and set $Y=\overline{M\cap X}$, $W=\overline{M\cap X^*}$. These are linear subspaces (2A5Ec). If $y\in M\cap X$ and $z\in W^\circ$, then there is an $f\in M\cap X^*$ such that $\|f\|\le 1$ and $f(y)=\|y\|$, so that

$$\|y+z\|\ge f(y+z)=f(y)=\|y\|.$$

Because the function $y\mapsto\|y+z\|-\|y\|$ is continuous, $\|y+z\|\ge\|y\|$ for every $y\in Y$ and $z\in W^\circ$. In particular, if $y\in Y\cap W^\circ$, $\|y\|\le\|y-y\|=0$ and $y=0$, so $Y+W^\circ=Y\oplus W^\circ$. If $x\in\overline{Y+W^\circ}$, then there are sequences $\langle y_n\rangle_{n\in\mathbb{N}}$ in Y and $\langle z_n\rangle_{n\in\mathbb{N}}$ in W° such that $x=\lim_{n\to\infty}y_n+z_n$; now $\|y_m-y_n\|\le\|(y_m+z_m)-(y_n+z_n)\|\to 0$ as $n\to\infty$, so (because X is a Banach space) $\langle y_n\rangle_{n\in\mathbb{N}}$ is convergent to y say; in this case, $y\in Y$ and $x-y=\lim_{n\to\infty}z_n$ belongs to W°, so $x\in Y+W^\circ$. This shows that $Y\oplus W^\circ$ is a closed linear subspace of X.

(e) ? Suppose, if possible, that $Y\oplus W^\circ\ne X$. Then there is an $x_0\in X\setminus(Y\oplus W^\circ)$. By 4A4Eb, there is an $f\in X^*$ such that $f(x_0)>0$ and $f(y)=f(z)=0$ whenever $y\in Y$ and $z\in W^\circ$; multiplying f by a scalar if necessary, we can arrange that $f(x_0)=1$. By 4A4Eg, f belongs to the weak* closure of W in X^*.

Let $\phi\in A$ be such that $(\phi,x_0)\in R$. Then $K=R[\{\phi\}]$ is weakly compact. Now the weak* closure of W is also its closure for the Mackey topology of uniform convergence on weakly compact subsets of X (4A4F). So there is a $g\in W$ such that $|g(x)-f(x)|\le\frac{1}{7}$ for every $x\in K$. Next, because K is bounded, and g belongs to the norm closure of $M\cap X^*$, there is an $h\in M\cap X^*$ such that $|h(x)-g(x)|\le\frac{1}{7}$ for every $x\in K$. This means that $|h(x)-f(x)|\le\frac{2}{7}$ for every $x\in K$, and K is included in the weakly open set $G=\{x:|h(x)-f(x)|<\frac{3}{7}\}$, that is, ϕ does not belong to $R^{-1}[X\setminus G]$, which is relatively closed in A, because R is usco-compact regarded as a relation between A and X with the weak topology. There is therefore a $\sigma\in S$ such that $\phi\in I_\sigma$ and $I_\sigma\cap R^{-1}[X\setminus G]=\emptyset$, that is, $F_\sigma\subseteq G$. In this case, $x_0\in F_\sigma$, so $\sigma\in S_0$, while $h(x)-f(x)<\frac{3}{7}$ for every $x\in F_\sigma$. But, because $M\in\mathcal{M}$, there is a $y\in M\cap X\cap F_\sigma$ such that $h(y)\ge h(x_0)-\frac{1}{7}$, and as $y\in Y$ we must now have

$$0=f(y)=h(y)-(h(y)-f(y))>h(x_0)-\frac{1}{7}-\frac{3}{7}$$
$$=f(x_0)-(f(x_0)-h(x_0))-\frac{4}{7}\ge 1-\frac{3}{7}-\frac{4}{7}=0,$$

which is absurd. **X**

Thus $X=Y\oplus W^\circ$ and (Y,W) is a projection pair. This completes the proof.

467K Theorem Let X be a weakly K-countably determined Banach space. Then it has an equivalent locally uniformly rotund norm.

proof Since the completion of X is weakly K-countably determined (467Ic), we may suppose that X is complete. The proof proceeds by induction on the weight of X.

(a) The induction starts by observing that if $w(X)\le\omega$ then X is separable (4A2Li/4A2P(a-i)) so has an equivalent locally uniformly rotund norm by 467E.

(b) So let us suppose that $w(X)=\kappa>\omega$ and that any weakly K-countably determined Banach space of weight less than κ has an equivalent locally uniformly rotund norm.

Let $\mathcal{M}$ be a family of subsets of $X\cup X^*$ as in 467J. Then there is a non-decreasing family $\langle M_\xi\rangle_{\xi\le\kappa}$ in $\mathcal{M}$ such that $\#(M_\xi)\le\max(\omega,\#(\xi))$ for every $\xi\le\kappa$, M_κ is dense in X, and $M_\xi=\bigcup_{\eta<\xi}M_\eta$ for every limit ordinal $\xi\le\kappa$. **P** By 4A2Li, there is a dense subset of X of cardinal κ; enumerate it as $\langle x_\xi\rangle_{\xi<\kappa}$. Choose M_ξ inductively, as follows. $M_0=\emptyset$. GIven M_ξ with $\#(M_\xi)\le\max(\omega,\#(\xi))$, then by 467J(i) there is an $M_{\xi+1}\in\mathcal{M}$ such that $M_{\xi+1}\supseteq M_\xi\cup\{x_\xi\}$ and

$$\#(M_{\xi+1})\le\max(\omega,\#(M_\xi\cup\{x_\xi\})\le\max(\omega,\#(\xi+1)).$$

Given that $\langle M_\eta\rangle_{\eta<\xi}$ is a non-decreasing family in $\mathcal{M}$ with $\#(M_\eta) \le \max(\omega, \#(\eta))$ for every $\eta < \xi$, set $M_\xi = \bigcup_{\eta<\xi} M_\eta$; then 467J(ii) tells us that $M_\xi \in \mathcal{M}$, while $\#(M_\xi) \le \max(\omega, \#(\xi))$, as required by the inductive hypothesis. **Q**

At the end of the induction, $M_\kappa \supseteq \{x_\xi : \xi < \kappa\}$ will be dense in X.

(c) For each $\xi < \kappa$, set $Y_\xi = \overline{M_\xi \cap X}$, $W_\xi = \overline{M_\xi \cap X^*}$. Then (Y_ξ, W_ξ) is a projection pair, by 467J(iii). Since $X = Y_\xi \oplus W_\xi^\circ$, we have a projection $P_\xi : X \to Y_\xi$ defined by saying that $P_\xi(y+z) = y$ whenever $y \in Y_\xi$ and $z \in W_\xi^\circ$. Now $P_\xi P_\eta = P_\eta P_\xi = P_\xi$ whenever $\xi \le \eta$. **P** The point is that $Y_\xi \subseteq Y_\eta$ and $W_\xi \subseteq W_\eta$, so $W_\eta^\circ \subseteq W_\xi^\circ$. If $x \in X$, express it as $y + z_1$ where $y \in Y_\xi$ and $z_1 \in W_\xi^\circ$, and express z_1 as $y' + z$ where $y' \in Y_\eta$ and $z' \in W_\eta^\circ$. Then

$$P_\xi x = y \in Y_\xi \subseteq Y_\eta$$

so $P_\eta P_\xi x = P_\xi x$. On the other hand, $x = y + y' + z$, $y + y' \in Y_\eta$ and $z \in W_\eta^\circ$, so $P_\eta x = y + y'$; and as $y' = z_1 - z$ belongs to W_ξ°, $P_\xi(y + y') = y$, so $P_\xi P_\eta x = P_\xi x$. **Q**

Note that the condition

$$\|y + z\| \ge \|y\| \text{ whenever } y \in Y_\xi,\ z \in W_\xi^\circ$$

ensures that $\|P_\xi\| \le 1$ for every ξ.

Next, if $\xi \le \kappa$ is a non-zero limit ordinal, $P_\xi x = \lim_{\eta\uparrow\xi} P_\eta x$. **P** We know that

$$P_\xi x \in Y_\xi = \overline{M_\xi \cap X} = \overline{\textstyle\bigcup_{\eta<\xi} M_\eta \cap X}.$$

So, given $\epsilon > 0$, there is an $x' \in \bigcup_{\eta<\xi} M_\eta$ such that $\|P_\xi x - x'\| \le \frac{1}{2}\epsilon$. Let $\eta < \xi$ be such that $x' \in M_\eta$. If $\eta \le \eta' \le \xi$, then

$$\|P_\xi x - P_{\eta'} x\| = \|P_\xi(P_\xi x - x') - P_{\eta'}(P_\xi x - x')\|$$

(because $x' \in Y_\eta$, so $P_\xi x' = P_{\eta'} x' = x'$)

$$\le 2\|P_\xi x - x'\| \le \epsilon.$$

As ϵ is arbitrary, $P_\xi x = \lim_{\eta\uparrow\xi} P_\eta x$. **Q**

(d) Now observe that every Y_ξ is weakly K-countably determined (467Ia), while $w(Y_\xi) \le \max(\omega, \#(\xi)) < \kappa$ for every $\xi < \kappa$ (using 4A2Li, as usual). So the inductive hypothesis tells us that $Y_\xi = P_\xi[X]$ has an equivalent locally uniformly rotund norm for every $\xi < \kappa$. By 467G, $X = P_\kappa[X]$ has an equivalent locally uniformly rotund norm. Thus the induction proceeds.

467L Weakly compactly generated Banach spaces The most important class of weakly K-countably determined spaces is the following. A normed space X is **weakly compactly generated** if there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of weakly compact subsets of X such that $\bigcup_{n\in\mathbb{N}} K_n$ is dense in X.

467M Proposition (Talagrand 75) A weakly compactly generated Banach space is weakly K-countably determined.

proof Let X be a Banach space with a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of weakly compact subsets of X such that $\bigcup_{n\in\mathbb{N}} K_n$ is dense in X. Set

$$L_n = \{\textstyle\sum_{i=0}^n \alpha_i x_i : |\alpha_i| \le n,\ x_i \in \bigcup_{j\le n} K_j \text{ for every } i \le n\}$$

for $n \in \mathbb{N}$. Then every L_n is weakly compact, and $Y = \bigcup_{n\in\mathbb{N}} L_n$ is a linear subspace of X including $\bigcup_{n\in\mathbb{N}} K_n$, therefore dense. Now Y is a countable union of weakly compact sets, therefore K-analytic for its weak topology (422Gc, 422Hc); in particular, it is weakly K-countably determined. By 467Ic, X is weakly K-countably determined.

467N Theorem Let X be a Banach lattice with an order-continuous norm (§354). Then it has an equivalent locally uniformly rotund norm.

proof (a) Consider first the case in which X has a weak order unit e. Then the interval $[0, e]$ is weakly compact. **P** We have $X^* = X^\times$ and $X^{**} = X^{\times\sim}$ (356D). The canonical identification of X with its image in X^{**} is an order-continuous Riesz homomorphism from X onto a solid order-dense Riesz subspace of $X^{\times\times}$ (356I) and therefore of $X^{\times\sim} = X^{**}$. In particular, $[0, e] \subseteq X$ is matched with an interval $[0, \hat{e}] \subseteq X^{**}$. But $[0, \hat{e}] = \{\theta : \theta \in X^{**},\ 0 \le$

$\theta(f) \le f(e)$ for every $f \in (X^*)^+\}$ is weak*-closed and norm-bounded in X^{**}, therefore weak*-compact; as the weak* topology on X^{**} corresponds to the weak topology of X, $[0, e]$ is weakly compact. **Q**

Now, for each $n \in \mathbb{N}$, $K_n = [-ne, ne] = n[0, e] - n[0, e]$ is weakly compact. If $x \in X^+$, then $\langle x \wedge ne\rangle_{n\in\mathbb{N}}$ converges to X, because the norm of X is order-continuous; so for any $x \in X$, $\langle x^+ \wedge ne - x^- \wedge ne\rangle_{n\in\mathbb{N}}$ converges to x. Thus $\bigcup_{n\in\mathbb{N}} K_n$ is dense in X and X is weakly compactly generated. By 467M and 467K, X has an equivalent locally uniformly rotund norm.

(b) For the general case, let $\langle x_i\rangle_{i\in I}$ be a maximal disjoint family in X^+. For each $i \in I$ let X_i be the band in X generated by x_i, and $T_i : X \to X_i$ the band projection onto X_i (354Ee, 353Hb). Then $\langle T_i\rangle_{i\in I}$ satisfies the conditions of 467F. **P** (i) Each T_i is a continuous linear operator because the given norm $\| \|$ of X is a Riesz norm. Next, each X_i has a weak order unit x_i, and the norm on X_i is order-continuous, so (a) tells us that there is an equivalent locally uniformly rotund norm on $X_i = T_i[X]$. (ii) If $x \in X$, set $x' = \sup_{i\in I} T_i|x|$; then $(|x| - x') \wedge x_i = 0$ for every i, so, by the maximality of $\langle x_i\rangle_{i\in I}$, $x' = |x|$. If $\epsilon > 0$ then, because the norm of X is order-continuous, there is a finite $J \subseteq I$ such that

$$\|x - \sum_{j\in J} T_i x\| = \|x' - \sup_{j\in J} T_i|x|\| \le \epsilon.$$

Moreover, if $i \in I \setminus J$, then

$$\|T_i x\| \le \|x - \sum_{j\in J} T_j x\| \le \epsilon.$$

Thus conditions (ii) and (iii) of 467F are satisfied. **Q**

Accordingly 467F tells us that X has an equivalent locally uniformly rotund norm.

467O Eberlein compacta: Definition A topological space K is an **Eberlein compactum** if it is homeomorphic to a weakly compact subset of a Banach space.

467P Proposition Let K be a compact Hausdorff space.

(a) The following are equiveridical:

(i) K is an Eberlein compactum;

(ii) there is a set $L \subseteq C(K)$, separating the points of K, which is compact for the topology of pointwise convergence.

(b) Suppose that K is an Eberlein compactum.

(i) K has a σ-isolated network, so is hereditarily weakly θ-refinable.

(ii) (Schachermayer 77) If $w(K)$ is measure-free, K is a Radon space.

proof (a)(i)⇒(ii) If K is an Eberlein compactum, we may suppose that it is a weakly compact subset of a Banach space X. Set $L = \{f{\restriction}K : f \in X^*, \|f\| \le 1\}$; since the map $f \mapsto f{\restriction}K : X^* \to C(K)\}$ is continuous for the weak* topology of X^* and the topology $\mathfrak{T}_p$ of pointwise convergence on $C(K)$, L is $\mathfrak{T}_p$-compact; and L separates the points of K because X^* separates the points of X.

(ii)⇒(i) If $L \subseteq C(K)$ is $\mathfrak{T}_p$-compact and separates the points of K, set $L_n = \{f : f \in L, \|f\|_\infty \le n\}$ for each $n \in \mathbb{N}$. Then L_n is $\mathfrak{T}_p$-compact for each n. Set $L' = \{0\} \cup \bigcup_{n\in\mathbb{N}} 2^{-n} L_n$; then $L' \subseteq C(K)$ is norm-bounded and $\mathfrak{T}_p$-compact and separates the points of K. Now define $x \mapsto \hat{x} : K \to \mathbb{R}^{L'}$ by setting $\hat{x}(f) = f(x)$ for $x \in K$ and $f \in L'$. Then $\hat{x} \in C(L')$ for every x and $x \mapsto \hat{x}$ is continuous for the given topology of K and the topology of pointwise convergence on $C(L')$; so the image $\hat{K} = \{\hat{x} : x \in K\}$ is $\mathfrak{T}_p$-compact. Since it is also bounded, it is weakly compact (462E). But $x \mapsto \hat{x}$ is injective, because L' separates the points of K; so K is homeomorphic to $\hat{K}$, and is an Eberlein compactum.

(b) Again suppose K is actually a weakly compact subset of a Banach space X. As in 467M, set $L_n = \{\sum_{i=0}^n \alpha_i x_i : |\alpha_i| \le n, x_i \in K$ for every $i \le n\}$ for each $n \in \mathbb{N}$. Then $Y = \overline{\bigcup_{n\in\mathbb{N}} L_n}$ is a weakly compactly generated Banach space. (I am passing over the trivial case $K = \emptyset$.) So Y has an equivalent locally uniformly rotund norm (467M, 467K), which is a Kadec norm (467B), and Y, with the weak topology, has a σ-isolated network (466Eb). It follows at once that K has a σ-isolated network (4A2B(a-ix)), so is hereditarily weakly θ-refinable (438Ld); and if $w(K)$ is measure-free, K is Borel-measure-complete (438M), therefore Radon (434Jf, 434Ka).

467X Basic exercises (a)(i) Show that a continuous image of a K-countably determined space is K-countably determined. (ii) Show that the product of a sequence of K-countably determined spaces is K-countably determined. (iii) Show that any K-countably determined topological space is Lindelöf. (*Hint*: 422De.) (iv) Show that any Souslin-F subset of a K-countably determined topological space is K-countably determined. (*Hint*: 422Hc.)

(b) Let X be a σ-compact Hausdorff space. Show that a subspace Y of X is K-countably determined iff there is a countable family $\mathcal{K}$ of compact subsets of X such that $\bigcap\{K : y \in K \in \mathcal{K}\} \subseteq Y$ for every $y \in Y$.

(c) Show that if X and Y are weakly K-countably determined normed spaces, then $X \times Y$, with an appropriate norm, is weakly K-countably determined.

(d) Show that a normed space X is weakly compactly generated iff there is a weakly compact set $K \subseteq X$ such that the linear subspace of X generated by K is dense in X.

(e) Show that any separable normed space is weakly compactly generated.

(f) Show that any reflexive Banach space is weakly compactly generated.

>(g) Show that if X is a weakly compactly generated Banach space, then it is K-analytic in its weak topology. (*Hint*: in 467M, use the proof of 467Ic.)

(h) Show that if X is a Banach space and there is a set $A \subseteq X$ such that A is K-countably determined for the weak topology and the linear subspace generated by A is dense, then X is weakly K-countably determined.

(i) Show that the one-point compactification of any discrete space is an Eberlein compactum.

>(j) Let K be an Eberlein compactum, and μ a Radon measure on K. Show that μ is completion regular and inner regular with respect to the compact metrizable subsets of K. (*Hint*: 466B.)

467Y Further exercises (a) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of weakly K-countably determined normed spaces. Investigate normed subspaces of $\prod_{n\in\mathbb{N}} X_n$ which will be weakly K-countably determined.

(b)(i) Show that if (Ω, Σ, μ) is a probability space, then $L^1(\mu)$ has a locally uniformly rotund Riesz norm. (*Hint*: apply the construction of 467D with $Y = L^2(\mu)$ and T the identity operator; show that all the norms p_n are Riesz norms.) (ii) Show that if X is any L-space then it has a locally uniformly rotund Riesz norm. (*Hint*: apply the construction of 467F/467N, noting that if the T_i in 467F are band projections and $\| \|$ and all the $\| \|_i$ are Riesz norms, then all the norms in the proof of 467F are Riesz norms.)

(c) Let $(X, \| \|)$ be a Banach space, and $\mathfrak{T}$ a linear space topology on X such that the unit ball of X is $\mathfrak{T}$-closed. Suppose that $\langle T_i\rangle_{i\in I}$ is a family of bounded linear operators from X to itself such that

(i) for each $i \in I$, T_i is $\mathfrak{T}$-continuous as well as norm-continuous,

(ii) for each $i \in I$, the subspace $T_i[X]$ has an equivalent locally uniformly rotund norm for which the unit ball is closed for the topology on $T_i[X]$ induced by $\mathfrak{T}$,

(ii) for each $x \in X$, $\epsilon > 0$ there is a finite set $J \subseteq I$ such that $\|x - \sum_{i\in J} T_i x\| \le \epsilon$,

(iii) for each $x \in X$, $\epsilon > 0$ the set $\{i : i \in I, \|T_i x\| \ge \epsilon\}$ is finite.

Show that X has an equivalent locally uniformly rotund norm for which the unit ball is $\mathfrak{T}$-closed.

(d) Let X be a normed space with a locally uniformly rotund norm, and $\mathfrak{T}$ a linear space topology on X such that the unit ball of X is $\mathfrak{T}$-closed. Show that every norm-Borel subset of X is $\mathfrak{T}$-Borel.

(e) Let κ be any cardinal, and K a dyadic space. (i) Show that $C(K)$ has a locally uniformly rotund norm, equivalent to the usual supremum norm $\| \|_\infty$, for which the unit ball is closed for the topology $\mathfrak{T}_p$ of pointwise convergence. (See Deville Godefroy & Zizler 93, VII.1.10.) (ii) Show that the norm topology on $C(K)$, the weak topology on $C(K)$ and $\mathfrak{T}_p$ give rise to the same Borel σ-algebras. (iii) Show that $\mathfrak{T}_p$ has a σ-isolated network. (iv) Show that if $w(K)$ is measure-free, then $(C(K), \mathfrak{T}_p)$ is Radon, and every $\mathfrak{T}_p$-Radon measure on $C(K)$ is norm-Radon.

467 Notes and comments The purpose of this section has been to give an idea of the scope of Proposition 466F. 'Local uniform rotundity' has an important place in the geometrical theory of Banach spaces, but for the many associated ideas I refer you to Deville Godefroy & Zizler 93. From our point of view, Theorem 467E is therefore purely accessory, since we know by different arguments that on separable Banach spaces the weak and norm topologies have the same Borel σ-algebras (4A3V). We need it to provide the first step in the inductive proof of 467K.

Since the concept of 'K-analytic' space is one of the fundamental ideas of Chapter 43, it is natural here to look at 'K-countably determined' spaces, especially as many of the ideas of §422 are directly applicable (467Xa). But the goal of this part of the argument is Schachermayer's theorem 467P(b-ii), which uses 'weakly compactly generated' spaces (467L). 'Eberlein compacta' are of great interest in other ways; they are studied at length in Arkhangel'skii 92.

I mention order-continuous norms here (467N) because they are prominent in the theory of Banach lattices in Volume 3. Note that the methods here do *not* suffice in general to arrange that the locally uniformly rotund norm found on X is a Riesz norm; though see 467Yb. It is in fact the case that every Banach lattice with an order-continuous norm has an equivalent locally uniformly rotund Riesz norm, but this requires further ideas (see Davis Ghoussoub & Lindenstrauss 81).

The general question of identifying Banach spaces with equivalent Kadec norms remains challenging. For a recent survey see Moltó Orihuela Troyanski & Valdivia 09.

Chapter 47

Geometric measure theory

I offer a chapter on geometric measure theory, continuing from Chapter 26. The greater part of it is directed specifically at two topics: a version of the Divergence Theorem (475N) and the elementary theory of Newtonian capacity and potential (§479). I do not attempt to provide a balanced view of the subject, for which I must refer you to MATTILA 95, EVANS & GARIEPY 92 and FEDERER 69. However §472, at least, deals with something which must be central to any approach, Besicovitch's Density Theorem for Radon measures on $\mathbb{R}^r$ (472D). In §473 I examine Lipschitz functions, and give crude forms of some fundamental ineqalities relating integrals $\int \|\operatorname{grad} f\| d\mu$ with other measures of the variation of a function f (473H, 473K). In §474 I introduce perimeter measures λ_E^∂ and outward-normal functions ψ_E as those for which the Divergence Theorem, in the form $\int_E \operatorname{div}\phi\, d\mu = \int \phi\,\boldsymbol{.}\,\psi_E\, d\lambda_E^\partial$, will be valid (474E), and give the geometric description of $\psi_E(x)$ as the Federer exterior normal to E at x (474R). In §475 I show that λ_E^∂ can be identified with normalized Hausdorff $(r-1)$-dimensional measure on the essential boundary of E.

§471 is devoted to Hausdorff measures on general metric spaces, extending the ideas introduced in §264 for Euclidean space, up to basic results on densities (471P) and Howroyd's theorem (471S). In §476 I turn to a different topic, the problem of finding the subsets of $\mathbb{R}^r$ on which Lebesgue measure is most 'concentrated' in some sense. I present a number of classical results, the deepest being the Isoperimetric Theorem (476H): among sets with a given measure, those with the smallest perimeters are the balls.

The last three sections are different again. Classical electrostatics led to a vigorous theory of capacity and potential, based on the idea of 'harmonic function'. It turns out that 'Brownian motion' in $\mathbb{R}^r$ (§477) gives an alternative and very powerful approach to the subject. I have brought Brownian motion and Wiener measure to this chapter because I wish to use them to illuminate the geometry of $\mathbb{R}^r$; but much of §477 (in particular, the strong Markov property, 477G) is necessarily devoted to adapting ideas developed in the more general contexts of Lévy and Gaussian processes, as described in §§455-456. In §478 I give the most elementary parts of the theory of harmonic and superharmonic functions, building up to a definition of 'harmonic measures' based on Brownian motion (478P). In §479 I use these techniques to describe Newtonian capacity and its extension Choquet-Newton capacity (479C) on Euclidean space of three or more dimensions, and establish their basic properties (479E, 479F, 479N, 479P, 479U).

471 Hausdorff measures

I begin the chapter by returning to a class of measures which we have not examined since Chapter 26. The primary importance of these measures is in studying the geometry of Euclidean space; in §265 I looked briefly at their use in describing surface measures, which will reappear in §475. Hausdorff measures are also one of the basic tools in the study of fractals, but for such applications I must refer you to FALCONER 90 and MATTILA 95. All I shall attempt to do here is to indicate some of the principal ideas which are applicable to general metric spaces, and to look at some special properties of Hausdorff measures related to the concerns of this chapter and of §261.

471A Definition Let (X,ρ) be a metric space and $r \in \,]0,\infty[$. For $\delta > 0$ and $A \subseteq X$, set

$$\theta_{r\delta}A = \inf\{\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r : \langle D_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A,$$
$$\operatorname{diam} D_n \le \delta \text{ for every } n \in \mathbb{N}\}.$$

(As in §264, take $\operatorname{diam}\emptyset = 0$ and $\inf\emptyset = \infty$.) It will be useful to note that every $\theta_{r\delta}$ is an outer measure. Now set

$$\theta_r A = \sup_{\delta>0}\theta_{r\delta}A$$

for $A \subseteq X$; θ_r also is an outer measure on X, as in 264B; this is **r-dimensional Hausdorff outer measure** on X. Let μ_{Hr} be the measure defined by Carathéodory's method from θ_r; μ_{Hr} is **r-dimensional Hausdorff measure** on X.

Notation It may help if I list some notation already used elsewhere. Suppose that (X,ρ) is a metric space. I write

$$B(x,\delta) = \{y : \rho(y,x) \le \delta\}, \quad U(x,\delta) = \{y : \rho(y,x) < \delta\}$$

for $x \in X$, $\delta \ge 0$; recall that $U(x,\delta)$ is open (2A3G). For $x \in X$ and A, $A' \subseteq X$ I write

$$\rho(x,A)=\inf_{y\in A}\rho(x,y),\quad \rho(A,A')=\inf_{y\in A,z\in A'}\rho(y,z);$$

for definiteness, take $\inf\emptyset$ to be ∞, as before.

471B Definition Let (X,ρ) be a metric space. An outer measure θ on X is a **metric outer measure** if $\theta(A_1\cup A_2)=\theta A_1+\theta A_2$ whenever A_1, $A_2\subseteq X$ and $\rho(A_1,A_2)>0$.

471C Proposition Let (X,ρ) be a metric space and θ a metric outer measure on X. Let μ be the measure on X defined from θ by Carathéodory's method. Then μ is a topological measure.

proof (Compare 264E, part (b) of the proof.) Let $G\subseteq X$ be open, and A any subset of X such that $\theta A<\infty$. Set

$$A_n=\{x:x\in A,\,\rho(x,A\setminus G)\ge 2^{-n}\},$$

$$B_0=A_0,\quad B_n=A_n\setminus A_{n-1}\text{ for }n>1.$$

Observe that $A_n\subseteq A_{n+1}$ for every n and $\bigcup_{n\in\mathbb{N}}A_n=\bigcup_{n\in\mathbb{N}}B_n=A\cap G$. The point is that if m, $n\in\mathbb{N}$ and $n\ge m+2$, and if $x\in B_m$ and $y\in B_n$, then there is a $z\in A\setminus G$ such that $\rho(y,z)<2^{-n+1}\le 2^{-m-1}$, while $\rho(x,z)$ must be at least 2^{-m}, so $\rho(x,y)\ge\rho(x,z)-\rho(y,z)\ge 2^{-m-1}$. Thus $\rho(B_m,B_n)>0$ whenever $n\ge m+2$. It follows that for any $k\ge 0$

$$\textstyle\sum_{m=0}^{k}\theta B_{2m}=\theta(\bigcup_{m\le k}B_{2m})\le\theta(A\cap G)<\infty,$$

$$\textstyle\sum_{m=0}^{k}\theta B_{2m+1}=\theta(\bigcup_{m\le k}B_{2m+1})\le\theta(A\cap G)<\infty.$$

Consequently $\sum_{n=0}^{\infty}\theta B_n<\infty$.

But now, given $\epsilon>0$, there is an m such that $\sum_{n=m}^{\infty}\theta B_m\le\epsilon$, so that

$$\begin{aligned}\theta(A\cap G)+\theta(A\setminus G)&\le\theta A_m+\sum_{n=m}^{\infty}\theta B_n+\theta(A\setminus G)\\&\le\epsilon+\theta A_m+\theta(A\setminus G)=\epsilon+\theta(A_m\cup(A\setminus G))\end{aligned}$$

(since $\rho(A_m,A\setminus G)\ge 2^{-m}$)

$$\le\epsilon+\theta A.$$

As ϵ is arbitrary, $\theta(A\cap G)+\theta(A\setminus G)\le\theta A$. As A is arbitrary, G is measured by μ; as G is arbitrary, μ is a topological measure.

471D Theorem Let (X,ρ) be a metric space and $r>0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X, and Σ its domain; write θ_r for r-dimensional Hausdorff outer measure on X, as defined in 471A.

(a) μ_{Hr} is a topological measure.

(b) For every $A\subseteq X$ there is a G_δ set $H\supseteq A$ such that $\mu_{Hr}H=\theta_rA$.

(c) θ_r is the outer measure defined from μ_{Hr} (that is, θ_r is a regular outer measure).

(d) Σ is closed under Souslin's operation.

(e) $\mu_{Hr}E=\sup\{\mu_{Hr}F:F\subseteq E\text{ is closed}\}$ whenever $E\in\Sigma$ and $\mu_{Hr}E<\infty$.

(f) If $A\subseteq X$ and $\theta_rA<\infty$ then A is separable and the set of isolated points of A is μ_{Hr}-negligible.

(g) μ_{Hr} is atomless.

(h) If μ_{Hr} is totally finite it is a quasi-Radon measure.

proof (a) The point is that θ_r, as defined in 471A, is a metric outer measure. **P** (Compare 264E, part (a) of the proof.) Let A_1, A_2 be subsets of X such that $\rho(A_1,A_2)>0$. Of course $\theta_r(A\cup B)\le\theta_rA+\theta_rB$, because θ_r is an outer measure. For the reverse inequality, we may suppose that $\theta_r(A\cup B)<\infty$, so that θ_rA and θ_rB are both finite. Let $\epsilon>0$ and let δ_1, $\delta_2>0$ be such that

$$\theta_rA_1+\theta_rA_2\le\theta_{r\delta_1}A_1+\theta_{r\delta_2}A_2+\epsilon,$$

defining the $\theta_{r\delta_i}$ as in 471A. Set $\delta=\min(\delta_1,\delta_2,\frac{1}{2}\rho(A_1,A_2))>0$ and let $\langle D_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets of diameter at most δ, covering $A_1\cup A_2$, and such that $\sum_{n=0}^{\infty}(\operatorname{diam}D_n)^r\le\theta_{r\delta}(A_1\cup A_2)+\epsilon$. Set

$$K=\{n:D_n\cap A_1\ne\emptyset\},\quad L=\{n:D_n\cap A_2\ne\emptyset\}.$$

Because $\rho(x,y) > \operatorname{diam} D_n$ whenever $x \in A_1$, $y \in A_2$ and $n \in \mathbb{N}$, $K \cap L = \emptyset$; and of course $A_1 \subseteq \bigcup_{n\in K} D_k$, $A_2 \subseteq \bigcup_{n\in L} D_n$. Consequently

$$\theta_r A_1 + \theta_r A_2 \le \epsilon + \theta_{r\delta_1} A_1 + \theta_{r\delta_2} A_2 \le \epsilon + \sum_{n\in K} (\operatorname{diam} D_n)^r + \sum_{n\in L} (\operatorname{diam} D_n)^r$$

$$\le \epsilon + \sum_{n=0}^{\infty} (\operatorname{diam} D_n)^r \le 2\epsilon + \theta_{r\delta}(A_1 \cup A_2) \le 2\epsilon + \theta_r(A_1 \cup A_2).$$

As ϵ is arbitrary, $\theta_r(A_1 \cup A_2) \ge \theta_r A_1 + \theta_r A_2$. The reverse inequality is true just because θ_r is an outer measure, so $\theta_r(A_1 \cup A_2) = \theta_r A_1 + \theta_r A_2$. As A_1 and A_2 are arbitrary, θ_r is a metric outer measure. **Q**

Now 471C tells us that μ_{Hr} must be a topological measure.

(b) (Compare 264Fa.) If $\theta_r A = \infty$ this is trivial. Otherwise, for each $n \in \mathbb{N}$, let $\langle D_{ni} \rangle_{i\in\mathbb{N}}$ be a sequence of sets of diameter at most 2^{-n} such that $A \subseteq \bigcup_{i\in\mathbb{N}} D_i$ and $\sum_{i=0}^{\infty} (\operatorname{diam} D_{ni})^r \le \theta_{r,2^{-n}}(A) + 2^{-n}$, defining $\theta_{r,2^{-n}}$ as in 471A. Let $\eta_{ni} \in \,]0, 2^{-n}]$ be such that $(2\eta_{ni} + \operatorname{diam} D_{ni})^r \le 2^{-n-i} + (\operatorname{diam} D_{ni})^r$, and set $G_{ni} = \{x : \rho(x, D_{ni}) < \eta_{ni}\}$ for all n, $i \in \mathbb{N}$; then $G_{ni} = \bigcup_{x\in D_{ni}} U(x, \eta_{ni})$ is an open set including D_{ni} and $(\operatorname{diam} G_{ni})^r \le 2^{-n-i} + (\operatorname{diam} D_{ni})^r$. Set

$$H = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} G_{ni},$$

so that H is a G_δ set including A.

For any $\delta > 0$, there is an $n \in \mathbb{N}$ such that $3 \cdot 2^{-n} \le \delta$, so that $\operatorname{diam} G_{mi} \le \operatorname{diam} D_{mi} + 2\eta_{mi} \le \delta$ for every $i \in \mathbb{N}$ and $m \ge n$, and

$$\theta_{r\delta} H \le \sum_{i=0}^{\infty} (\operatorname{diam} G_{mi})^r \le \sum_{i=0}^{\infty} 2^{-m-i} + (\operatorname{diam} D_{mi})^r$$

$$\le 2^{-m+1} + \theta_{r,2^{-m}}(A) + 2^{-m} \le 2^{-m+2} + \theta_r A$$

for every $m \ge n$. Accordingly $\theta_{r\delta} H \le \theta_r A$ for every $\delta > 0$, so $\theta_r H \le \theta_r A$. Of course this means that $\theta_r H = \theta_r A$; and since, by (a), μ_{Hr} measures H, we have $\mu_{Hr} H = \theta_r A$, as required.

(c) (Compare 264Fb.) If $A \subseteq X$,

$$\theta_r A \ge \mu_{Hr}^* A$$

(by (b))

$$= \inf\{\theta_r E : A \subseteq E \in \Sigma\} \ge \theta_r A.$$

(d) Use 431C.

(e) By (b), there is a Borel set $H \supseteq E$ such that $\mu_{Hr} H = \mu_{Hr} E$, and now there is a Borel set $H' \supseteq H \setminus E$ such that $\mu_{Hr} H' = \mu_{Hr}(H \setminus E) = 0$, so that $G = H \setminus H'$ is a Borel set included in E and $\mu_{Hr} G = \mu_{Hr} E$. Now G is a Baire set (4A3Kb), so is Souslin-F (421L), and $\mu_{Hr} G = \sup_{F\subseteq G \text{ is closed}} \mu_{Hr} F$, by 431E.

(f) For every $n \in \mathbb{N}$, there must be a sequence $\langle D_{ni} \rangle_{i\in\mathbb{N}}$ of sets of diameter at most 2^{-n} covering A; now if $D \subseteq A$ is a countable set which meets D_{ni} whenever i, $n \in \mathbb{N}$ and $A \cap D_{ni} \ne \emptyset$, D will be dense in A. Now if A_0 is the set of isolated points in A, it is still separable (4A2P(a-iv)); but as the only dense subset of A_0 is itself, it is countable. Since $\theta_{r\delta}\{x\} = (\operatorname{diam}\{x\})^r = 0$ for every $\delta > 0$, $\mu_{Hr}\{x\} = 0$ for every $x \in X$, and A_0 is negligible.

(g) In fact, if $A \subseteq X$ and $\theta_r A > 0$, there are disjoint A_0, $A_1 \subseteq A$ such that $\theta_r A_i > 0$ for both i. **P** (i) Suppose first that A is not separable. For each $n \in \mathbb{N}$, let $D_n \subseteq A$ be a maximal set such that $\rho(x,y) \ge 2^{-n}$ for all distinct x, $y \in D_n$; then $\bigcup_{n\in\mathbb{N}} D_n$ is dense in A, so there is some $n \in \mathbb{N}$ such that D_n is uncountable; if we take A_1, A_2 to be disjoint uncountable subsets of D_n, then $\theta_r A_1 = \theta_r A_2 = \infty$. (ii) If A is separable, then set $\mathcal{G} = \{G : G \subseteq X$ is open, $\theta_r(A \cap G) = 0\}$. Because A is hereditarily Lindelöf (4A2P(a-iii)), there is a countable subset $\mathcal{G}_0$ of $\mathcal{G}$ such that $A \cap \bigcup \mathcal{G} = A \cap \bigcup \mathcal{G}_0$ (4A2H(c-i)), so $A \cap \bigcup \mathcal{G}$ is negligible and $A \setminus \bigcup \mathcal{G}$ has at least two points x_0, x_1. If we set $A_i = U(x_i, \frac{1}{2}\rho(x_i, x_{1-i}))$ for each i, these are disjoint subsets of A of non-zero outer measure. **Q**

(h) If μ_{Hr} is totally finite, then it is inner regular with respect to the closed sets, by (e). Also, because X must be separable, by (f), therefore hereditarily Lindelöf, μ_{Hr} must be τ-additive (414O). Finally, μ_{Hr} is complete just because it is defined by Carathéodory's method. So μ_{Hr} is a quasi-Radon measure.

471E Corollary If (X,ρ) is a metric space, $r>0$ and $Y\subseteq X$ then r-dimensional Hausdorff measure $\mu_{Hr}^{(Y)}$ on Y extends the subspace measure $(\mu_{Hr}^{(X)})_Y$ on Y induced by r-dimensional Hausdorff measure $\mu_{Hr}^{(X)}$ on X; and if either Y is measured by $\mu_{Hr}^{(X)}$ or Y has finite r-dimensional Hausdorff outer measure in X, then $\mu_{Hr}^{(Y)}=(\mu_{Hr}^{(X)})_Y$.

proof Write $\theta_r^{(X)}$ and $\theta_r^{(Y)}$ for the two r-dimensional Hausdorff outer measures.

If $A\subseteq Y$ and $\langle D_n\rangle_{n\in\mathbb{N}}$ is any sequence of subsets of X covering A, then $\langle D_n\cap Y\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of Y covering A, and $\sum_{n=0}^{\infty}(\operatorname{diam}(D_n\cap Y))^r\le\sum_{n=0}^{\infty}(\operatorname{diam}D_n)^r$; moreover, when calculating $\operatorname{diam}(D_n\cap Y)$, it doesn't matter whether we use the metric ρ on X or the subspace metric $\rho\restriction Y\times Y$ on Y. What this means is that, for any $\delta>0$, $\theta_{r\delta}A$ is the same whether calculated in Y or in X, so that $\theta_r^{(Y)}A=\sup_{\delta>0}\theta_{r\delta}A=\theta_r^{(X)}A$.

Thus $\theta_r^{(Y)}=\theta_r^{(X)}\restriction\mathcal{P}Y$. Also, by 471Db, $\theta_r^{(X)}$ is a regular outer measure. So 214Hb gives the results.

471F Corollary Let (X,ρ) be an analytic metric space (that is, a metric space in which the topology is analytic in the sense of §423), and write μ_{Hr} for r-dimensional Hausdorff measure on X. Suppose that ν is a locally finite indefinite-integral measure over μ_{Hr}. Then ν is a Radon measure.

proof Since $\operatorname{dom}\nu\supseteq\operatorname{dom}\mu_{Hr}$, ν is a topological measure. Because X is separable, therefore hereditarily Lindelöf, ν is σ-finite and τ-additive, therefore locally determined and effectively locally finite. Next, it is inner regular with respect to the closed sets. **P** Let f be a Radon-Nikodým derivative of ν. If $\nu E>0$, there is an $E'\subseteq E$ such that

$$0<\nu E'=\int f\times\chi E'\,d\mu_{Hr}<\infty.$$

There is a μ_{Hr}-simple function g such that $g\le f\times\chi E'$ μ_{Hr}-a.e. and $\int g\,d\mu_{Hr}>0$; setting $H=E'\cap\{x:g(x)>0\}$, $\nu_{Hr}H<\infty$. Now there is a closed set $F\subseteq H$ such that $\mu_{Hr}F>0$, by 471De, and in this case $\nu F\ge\int_F g\,d\mu_{Hr}>0$. By 412B, this is enough to show that ν is inner regular with respect to the closed sets. **Q**

Since ν is complete (234I[1]), it is a quasi-Radon measure, therefore a Radon measure (434Jf, 434Jb).

471G Increasing Sets Lemma (Davies 70) Let (X,ρ) be a metric space and $r>0$.

(a) Suppose that $\delta>0$ and that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A. Then $\theta_{r,6\delta}(A)\le(5^r+2)\sup_{n\in\mathbb{N}}\theta_{r\delta}A_n$.

(b) Suppose that $\delta>0$ and that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A. Then $\theta_{r\delta}A\le\sup_{n\in\mathbb{N}}\theta_{r\delta}A_n$.

proof (a) If $\sup_{n\in\mathbb{N}}\theta_{r\delta}A_n=\infty$ this is trivial; suppose otherwise.

(i) Take any $\gamma>\gamma'>\sup_{n\in\mathbb{N}}\theta_{r\delta}A_n$. For each $i\in\mathbb{N}$, let $\zeta_i\in\left]0,\frac{1}{4}\delta\right]$ be such that $(\alpha+2\zeta_i)^r\le\alpha^r+2^{-i-1}(\gamma-\gamma')$ whenever $0\le\alpha\le\delta$. For each $n\in\mathbb{N}$, there is a sequence $\langle C_{ni}\rangle_{i\in\mathbb{N}}$ of sets covering A_n such that $\operatorname{diam}C_{ni}\le\delta$ for every i and $\sum_{i=0}^{\infty}(\operatorname{diam}C_{ni})^r<\gamma'$; let $\langle\gamma_{ni}\rangle_{i\in\mathbb{N}}$ be such that $\operatorname{diam}C_{ni}\le\gamma_{ni}\le\delta$ and $\gamma_{ni}>0$ for every i and $\sum_{i=0}^{\infty}\gamma_{ni}^r\le\gamma'$. Since $\sum_{i=0}^{\infty}\gamma_{ni}^r$ is finite, $\lim_{i\to\infty}\gamma_{ni}=0$. Because $\gamma_{ni}>0$ for every i, we may rearrange the sequences $\langle C_{ni}\rangle_{i\in\mathbb{N}}$, $\langle\gamma_{ni}\rangle_{i\in\mathbb{N}}$ in such a way that $\gamma_{ni}\ge\gamma_{n,i+1}$ for each i.

In this case, $\lim_{i\to\infty}\sup_{n\in\mathbb{N}}\gamma_{ni}=0$. **P**

$$(i+1)\gamma_{ni}^r\le\sum_{j=0}^{i}\gamma_{nj}^r\le\gamma$$

for every n, $i\in\mathbb{N}$. **Q**

(ii) By Ramsey's theorem (4A1G, with $n=2$), there is an infinite set $I\subseteq\mathbb{N}$ such that

for all i, $j\in\mathbb{N}$ there is an $s\in\mathbb{N}$ such that either $C_{mi}\cap C_{nj}=\emptyset$ whenever m, $n\in I$ and $s\le m<n$ or $C_{mi}\cap C_{nj}\ne\emptyset$ whenever m, $n\in I$ and $s\le m<n$,

for each $i\in\mathbb{N}$, $\alpha_i=\lim_{n\in I,n\to\infty}\gamma_{ni}$ is defined in $\mathbb{R}$.

(Apply 4A1Fb with the families

$$\begin{aligned}\mathcal{J}_{ij}=\{J:J\in[\mathbb{N}]^{\omega},\ &\text{either } C_{mi}\cap C_{nj}=\emptyset\text{ whenever } m,\,n\in J\text{ and } m<n\\&\text{or } C_{mi}\cap C_{nj}\ne\emptyset\text{ whenever } m,\,n\in J\text{ and } m<n\}\\\mathcal{J}'_{iq}=\{J:J\in[\mathbb{N}]^{\omega},\ &\text{either }\gamma_{ni}\le q\text{ for every } n\in J\\&\text{or }\gamma_{ni}\ge q\text{ for every } n\in J\}\end{aligned}$$

for i, $j\in\mathbb{N}$ and $q\in\mathbb{Q}$.)

[1]Formerly 234A.

Of course $\alpha_j \le \alpha_i \le \delta$ whenever $i \le j$, because $\gamma_{nj} \le \gamma_{ni} \le \delta$ for every n. Set $D_{ni} = \{x : \rho(x, C_{ni}) \le 2\alpha_i + 2\zeta_i\}$ for all n, $i \in \mathbb{N}$, and $D_i = \bigcup_{s\in\mathbb{N}} \bigcap_{n\in I\setminus s} D_{ni}$ for $i \in \mathbb{N}$. (I am identifying each $s \in \mathbb{N}$ with the set of its predecessors.)

(iii) Set

$$L = \{(i,j) : i,\, j \in \mathbb{N},\, \forall\, s \in \mathbb{N}\ \exists\, m,\, n \in I,\, s \le m < n \text{ and } C_{mi} \cap C_{nj} \ne \emptyset\}.$$

If $(i,j) \in L$ then there is an $s \in \mathbb{N}$ such that $C_{mi} \subseteq D_{\min(i,j)}$ whenever $m \in I$ and $m \ge s$. **P** By the choice of I, we know that there is an $s_0 \in \mathbb{N}$ such that $C_{mi} \cap C_{nj} \ne \emptyset$ whenever m, $n \in I$ and $s_0 \le m < n$. Let $s_1 \ge s_0$ be such that

$$\gamma C_{mi} \le \alpha_i + \min(\zeta_i, \zeta_j), \quad \gamma C_{mj} \le \alpha_j + \min(\zeta_i, \zeta_j)$$

whenever $m \in I$ and $m \ge s_1$. Take $m_0 \in I$ such that $m_0 \ge s_1$, and set $s = m_0 + 1$. Let $m \in I$ be such that $m \ge s$.

($\boldsymbol{\alpha}$) Suppose that $i \le j$ and $x \in C_{mi}$. Take any $n \in I$ such that $m \le n$. Then there is an $n' \in I$ such that $n < n'$. We know that $C_{mi} \cap C_{n'j}$ and $C_{ni} \cap C_{n'j}$ are both non-empty. So

$$\rho(x, C_{ni}) \le \operatorname{diam} C_{mi} + \operatorname{diam} C_{n'j} \le \gamma_{mi} + \gamma_{n'j} \le \alpha_i + \zeta_i + \alpha_j + \zeta_i \le 2\alpha_i + 2\zeta_i$$

and $x \in D_{ni}$. This is true for all $n \in I$ such that $n \ge m$, so $x \in D_i$. As x is arbitrary, $C_{mi} \subseteq D_i$.

($\boldsymbol{\beta}$) Suppose that $j \le i$ and $x \in C_{mi}$. Take any $n \in I$ such that $n > m$. Then $C_{mi} \cap C_{nj}$ is not empty, so

$$\rho(x, C_{nj}) \le \operatorname{diam} C_{mi} \le \gamma_i \le \alpha_i + \zeta_j \le \alpha_j + \zeta_j$$

and $x \in D_{nj}$. As x and n are arbitrary, $C_{mi} \subseteq D_j$.

Thus $C_{mi} \subseteq D_{\min(i,j)}$ in both cases. **Q**

(iv) Set

$$D = \bigcup_{i\in\mathbb{N}} D_i, \quad J = \{i : i \in \mathbb{N},\, \exists\, s \in \mathbb{N},\, C_{ni} \subseteq D \text{ whenever } n \in I \text{ and } n \ge s\}.$$

If $i \in \mathbb{N} \setminus J$ and $j \in \mathbb{N}$, then (iii) tells us that $(i,j) \notin L$, so there is some $s \in \mathbb{N}$ such that $C_{mi} \cap C_{nj} = \emptyset$ whenever m, $n \in I$ and $s \le m < n$.

(v) For $l \in \mathbb{N}$, $\mu^*_{Hr}(A_l \setminus D) \le 2\gamma$. **P** Let $\epsilon > 0$. Then there is a $k \in \mathbb{N}$ such that $\gamma_{ni} \le \epsilon$ whenever $n \in \mathbb{N}$ and $i > k$. Next, there is an $s \in \mathbb{N}$ such that

$$C_{ni} \subseteq D \text{ whenever } i \le k,\, i \in J,\, n \in I \text{ and } s \le n,$$

$$C_{mi} \cap C_{nj} = \emptyset \text{ whenever } i,\, j \le k,\, i \notin J,\, m,\, n \in I \text{ and } s \le m < n.$$

Take m, $n \in I$ such that $\max(l, s) \le m < n$. Then

$$\begin{aligned} A_l \setminus D &= \bigcup_{i\in\mathbb{N}} A_l \cap C_{mi} \setminus D \\ &\subseteq \bigcup_{i\le k} (A_n \cap C_{mi} \setminus D) \cup \bigcup_{i>k} C_{mi} \\ &\subseteq \bigcup_{i\le k, i\notin J} (A_n \cap C_{mi}) \cup \bigcup_{i>k} C_{mi} \\ &\subseteq \bigcup_{i\le k, i\notin J, j\le k} (C_{mi} \cap C_{nj}) \cup \bigcup_{j>k} C_{nj} \cup \bigcup_{i>k} C_{mi} \\ &= \bigcup_{j>k} C_{nj} \cup \bigcup_{i>k} C_{mi}. \end{aligned}$$

Since $\operatorname{diam} C_{nj} \le \gamma_j \le \epsilon$ and $\operatorname{diam} C_{mi} \le \gamma_i \le \epsilon$ for all i, $j > k$,

$$\theta_{r\epsilon}(A_l \setminus D) \le \sum_{i=k+1}^{\infty} \gamma_{ni}^r + \sum_{i=k+1}^{\infty} \gamma_{mi}^r \le 2\gamma.$$

This is true for every $\epsilon > 0$, so $\mu^*_{Hr}(A_l \setminus D) \le 2\gamma$, as claimed. **Q**

(vi) This is true for each $l \in \mathbb{N}$. But this means that $\mu^*_{Hr}(A \setminus D) \le 2\gamma$ (132Ae). Now $\theta_{r,6\delta} D \le 5^r\gamma$. **P** For each $i \in \mathbb{N}$,

$$\begin{aligned} \operatorname{diam} D_i &\le \lim_{n\in I, n\to\infty} \operatorname{diam} D_{ni} \le \lim_{n\in I, n\to\infty} \operatorname{diam} C_{ni} + 4\alpha_i + 4\zeta_i \\ &\le \lim_{n\in I, n\to\infty} \gamma_{ni} + 4\alpha_i + 4\zeta_i = 5\alpha_i + 4\zeta_i \le 6\delta. \end{aligned}$$

Next, for any $k \in \mathbb{N}$,

$$\sum_{i=0}^{k} \alpha_i^r = \lim_{n\in I, n\to\infty} \sum_{i=0}^{k} \gamma_{ni}^r \le \gamma',$$

so

$$\sum_{i=0}^{k} (\operatorname{diam} D_i)^r \le 5^r \sum_{i=0}^{k} (\alpha_i + 2\zeta_i)^r \le 5^r (\sum_{i=0}^{k} \alpha_i^r + 2^{-i-1}(\gamma - \gamma'))$$

(by the choice of the ζ_i)

$$\le 5^r \gamma.$$

Letting $k \to \infty$,

$$\theta_{r,6\delta} D \le \sum_{i=0}^{\infty} (\operatorname{diam} D_i)^r \le 5^r \gamma. \textbf{ Q}$$

Putting these together,

$$\theta_{r,6\delta} A \le \theta_{r,6\delta} D + \theta_{r,6\delta}(A \setminus D) \le \theta_{r,6\delta} D + \mu_{Hr}^*(A \setminus D) \le (5^r + 2)\gamma.$$

As γ is arbitrary, we have the preliminary result (a).

(b) Now let us turn to the sharp form (b). Once again, we may suppose that $\sup_{n\in\mathbb{N}} \theta_{r\delta} A_n$ is finite.

(i) Take γ, γ' such that $\sup_{n\in\mathbb{N}} \theta_{r\delta} A_n < \gamma' < \gamma$. As in (a)(i) above, we can find a family $\langle C_{ni} \rangle_{n,i\in\mathbb{N}}$ such that

$$A_n \subseteq \bigcup_{i\in\mathbb{N}} C_{ni},$$

$$\operatorname{diam} C_{ni} \le \delta \text{ for every } i \in \mathbb{N},$$

$$\sum_{i=0}^{\infty} (\operatorname{diam} C_{ni})^r \le \gamma'$$

for each n, and

$$\lim_{i\to\infty} \sup_{n\in\mathbb{N}} \operatorname{diam} C_{ni} = 0.$$

Replacing each C_{ni} by its closure if necessary, we may suppose that every C_{ni} is a Borel set.

Let $Q \subseteq X$ be a countable set which meets C_{ni} whenever n, $i \in \mathbb{N}$ and C_{ni} is not empty. This time, let $I \subseteq \mathbb{N}$ be an infinite set such that

$$\alpha_i = \lim_{n\in I, n\to\infty} \operatorname{diam} C_{ni} \text{ is defined in } [0, \delta] \text{ for every } i \in \mathbb{N},$$

$$\lim_{n\in I, n\to\infty} \rho(z, C_{ni}) \text{ is defined in } [0, \infty] \text{ for every } i \in \mathbb{N} \text{ and every } z \in Q.$$

(Take $\rho(z, \emptyset) = \infty$ if any of the C_{ni} are empty.) It will be helpful to note straight away that the limit $\lim_{n\in I, n\to\infty} \rho(x, C_{ni})$ is defined in $[0, \infty]$ for every $i \in \mathbb{N}$ and $x \in \overline{Q}$. **P** If $\lim_{n\in I, n\to\infty} \rho(y, C_{ni}) = \infty$ for some $y \in Q$, then $\lim_{n\in I, n\to\infty} \rho(x, C_{ni}) = \infty$, and we can stop. Otherwise, for any $\epsilon > 0$, there are a $z \in Q$ such that $\rho(x, z) \le \epsilon$ and an $s \in \mathbb{N}$ such that C_{mi} is not empty and $|\rho(z, C_{mi}) - \rho(z, C_{ni})| \le \epsilon$ whenever m, $n \in I \setminus s$; in which case $|\rho(x, C_{mi}) - \rho(x, C_{ni})| \le 3\epsilon$ whenever m, $n \in I \setminus s$. As ϵ is arbitrary, $\lim_{n\in I, n\to\infty} \rho(x, C_{ni})$ is defined in $\mathbb{R}$. **Q**

Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ containing I, and for $i \in \mathbb{N}$ set

$$D_i = \{x : \lim_{n\to\mathcal{F}} \rho(x, C_{ni}) = 0\}.$$

Set $D = \bigcup_{i\in\mathbb{N}} \overline{D}_i$. (Actually it is easy to check that every D_i is closed.)

(ii) Set

$$A^* = \bigcup_{m\in\mathbb{N}} \bigcap_{n\in I\setminus m} \bigcup_{i\in\mathbb{N}} C_{ni} \setminus D;$$

note that A^* is a Borel set. For k, $m \in \mathbb{N}$, set

$$A_{km}^* = \bigcap_{n\in I\setminus m} \bigcup_{i\ge k} C_{ni}.$$

For fixed k, $\langle A_{km}^* \rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence of sets. Also its union includes A^*. **P** Take $x \in A^*$.

(α)? If $x \notin \overline{Q}$, let $\epsilon > 0$ be such that $Q \cap B(x, \epsilon) = \emptyset$. Let $l \in \mathbb{N}$ be such that $\operatorname{diam} C_{ni} \le \epsilon$ whenever $n \in \mathbb{N}$ and $i \ge l$; then $x \notin C_{ni}$ whenever $n \in \mathbb{N}$ and $i \ge l$. Let $m \in \mathbb{N}$ be such that $x \in \bigcup_{i\in\mathbb{N}} C_{ni}$ whenever $n \in I$ and

$n \geq m$. Then $x \in \bigcup_{i<l} C_{ni}$ whenever $n \in I$ and $n \geq m$. But this means that there must be some $i < l$ such that $\{n : x \in C_{ni}\} \in \mathcal{F}$ and $x \in D_i \subseteq D$; which is impossible. **X**

(β) Thus $x \in \overline{Q}$, so $\lim_{n\in I, n\to\infty} \rho(x, C_{ni})$ is defined for each i (see (i) above), and must be greater than 0, since $x \notin D_i$. In particular, there is an $s \in \mathbb{N}$ such that $x \notin C_{ni}$ whenever $i < k$ and $n \in I \setminus s$; there is also an $m \in \mathbb{N}$ such that $x \in \bigcap_{n\in I\setminus m} \bigcup_{i\in\mathbb{N}} C_{ni}$; so that $x \in A^*_{k,\max(s,m)}$. As x is arbitrary, $A^* \subseteq \bigcup_{m\in\mathbb{N}} A^*_{km}$. **Q**

(iii) $\mu_{Hr} A^*$ is finite. **P** Take any $\epsilon > 0$. Let $k \in \mathbb{N}$ be such that $\operatorname{diam} C_{ni} \leq \epsilon$ whenever $i \geq k$ and $n \in \mathbb{N}$. For any $m \in I$, $\theta_{r\epsilon} A^*_{km} \leq \sum_{i=k}^\infty (\operatorname{diam} C_{mi})^r \leq \gamma$. By (a),

$$\begin{aligned}\theta_{r,6\epsilon} A^* &\leq (5^r + 2) \sup_{m\in\mathbb{N}} \theta_{r\epsilon} A^*_{km} \\ &= (5^r + 2) \sup_{m\in I} \theta_{r\epsilon} A^*_{km} \leq (5^r + 2)\gamma.\end{aligned}$$

As ϵ is arbitrary, $\mu_{Hr} A^* \leq (5^r + 2)\gamma < \infty$. **Q**

(iv) Actually, $\mu_{Hr} A^* \leq \gamma' - \sum_{i=0}^\infty \alpha_i^r$. **P?** Suppose, if possible, otherwise. Take β such that $\gamma' - \sum_{i=0}^\infty \alpha_i^r < \beta < \mu_{Hr} A^*$. For $x \in A^*$ and $k \in \mathbb{N}$, set $f_k(x) = \min\{n : n \in I,\ x \in A^*_{kn}\}$; then $\langle f_k \rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence of Borel measurable functions from A^* to $\mathbb{N}$. Choose $\langle s_k \rangle_{k\in\mathbb{N}}$ inductively so that

$$\mu_{Hr}\{x : x \in A^*,\ f_j(x) \leq s_j \text{ for every } j \leq k\} > \beta$$

for every $k \in \mathbb{N}$. Set $\tilde{A} = \{x : x \in A^*,\ f_j(x) \leq s_j \text{ for every } j \in \mathbb{N}\}$; because $\mu_{Hr} A^*$ is finite, $\mu_{Hr} \tilde{A} \geq \beta$. Take $\epsilon > 0$ such that $\theta_{r\epsilon} \tilde{A} > \gamma' - \sum_{i=0}^\infty \alpha_i^r$. Let $k \in \mathbb{N}$ be such that $\theta_{r\epsilon} \tilde{A} + \sum_{i=0}^{k-1} \alpha_i^r > \gamma'$ and $\operatorname{diam} C_{ni} \leq \epsilon$ whenever $n \in \mathbb{N}$ and $i \geq k$. Take $n \in I$ such that $n \geq s_j$ for every $j \leq k$ and $\theta_{r\epsilon} \tilde{A} + \sum_{i=0}^{k-1} (\operatorname{diam} C_{ni})^r > \gamma'$. If $x \in \tilde{A}$, then

$$f_k(x) \leq s_k \leq n, \quad x \in A^*_{kn} \subseteq \bigcup_{i\geq k} C_{ni},$$

so $\theta_{r\epsilon} \tilde{A} \leq \sum_{i=k}^\infty (\operatorname{diam} C_{ni})^r$; but this means that $\sum_{i=0}^\infty (\operatorname{diam} C_{ni})^r > \gamma'$, contrary to the choice of the C_{ni}. **XQ**

(v) Now observe that

$$A \subseteq \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \bigcup_{i\in\mathbb{N}} C_{ni} \subseteq A^* \cup D.$$

Moreover, for any $i \in \mathbb{N}$, $\operatorname{diam} \overline{D}_i \leq \alpha_i \leq \delta$. **P** If x, $y \in D_i$ then for every $\epsilon > 0$

$$\rho(x, C_{ni}) \leq \epsilon, \quad \rho(y, C_{ni}) \leq \epsilon, \quad \operatorname{diam} C_{ni} \leq \alpha_i + \epsilon$$

for all but finitely many $n \in I$. So $\rho(x, y) \leq \alpha_i + 3\epsilon$. As x, y and ϵ are arbitrary, $\operatorname{diam} \overline{D}_i = \operatorname{diam} D_i \leq \alpha_i$. Of course $\alpha_i \leq \delta$ because $\operatorname{diam} C_{ni} \leq \delta$ for every n. **Q**

Now

$$\theta_{r\delta} D \leq \sum_{i=0}^\infty (\operatorname{diam} \overline{D}_i)^r \leq \sum_{i=0}^\infty \alpha_i^r.$$

Putting this together with (iv),

$$\theta_{r\delta} A \leq \theta_{r\delta} D + \theta_{r\delta} A^* \leq \theta_{r\delta} D + \mu_{Hr} A^* \leq \gamma.$$

As γ and γ' are arbitrary,

$$\theta_{r\delta} A \leq \sup_{n\in\mathbb{N}} \theta_{r\delta} A_n,$$

as required.

471H Corollary Let (X, ρ) be a metric space, and $r > 0$. For $A \subseteq X$, set

$$\theta_{r\infty} A = \inf\{\sum_{n=0}^\infty (\operatorname{diam} D_n)^r : \langle D_n \rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A\}.$$

Then $\theta_{r\infty}$ is a Choquet capacity on X.

proof (a) Of course $0 \leq \theta_{r\infty} A \leq \theta_{r\infty} B$ whenever $A \subseteq B \subseteq X$.

(b) Suppose that $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of A with union A. By (a), $\gamma = \lim_{n\to\infty} \theta_{r\infty} A_n$ is defined and less than or equal to $\theta_{r\infty} A$. If $\gamma = \infty$, of course it is equal to $\theta_{r\infty} A$. Otherwise, take $\beta = (\gamma + 1)^{1/r}$. For n, $k \in \mathbb{N}$ there is a sequence $\langle D_{nki} \rangle_{i\in\mathbb{N}}$ of sets, covering A_n, such that $\sum_{i=0}^\infty (\operatorname{diam} D_{nki})^r \leq \gamma + 2^{-k}$. But in this case $\operatorname{diam} D_{nki} \le \beta$ for all n, k and i, so the D_{nki} witness that $\theta_{r\beta}A_n \le \gamma$. By 471Gb, $\gamma \ge \theta_{r\beta}A \ge \theta_{r\infty}A$ and again we have $\gamma = \theta_{r\infty}A$.

(c) Let $K \subseteq X$ be any set, and suppose that $\gamma > \theta_{r\infty}K$. Let $\langle D_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets, covering K, such that $\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r < \gamma$. Let $\langle \epsilon_n\rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{n=0}^{\infty}(\operatorname{diam} D_n + 2\epsilon_n)^r \le \gamma$. Set $G_n = \{x : \rho(x, D_n) < \epsilon_n\}$ for each n; then G_n is open and $\operatorname{diam} G_n \le \operatorname{diam} D_n + 2\epsilon_n$. So $G = \bigcup_{n\in\mathbb{N}} G_n$ is an open set including K, and $\langle G_n\rangle_{n\in\mathbb{N}}$ witnesses that $\theta_{r\infty}G \le \gamma$. As K and γ are arbitrary, condition (iii) of 432Ja is satisfied and $\theta_{r\infty}$ is a Choquet capacity.

Remark $\theta_{r\infty}$ is **r-dimensional Hausdorff capacity** on X.

471I Theorem Let (X, ρ) be a metric space, and $r > 0$. Write μ_{Hr} for r-dimensional Hausdorff measure on X. If $A \subseteq X$ is analytic, then $\mu_{Hr}A$ is defined and equal to $\sup\{\mu_{Hr}K : K \subseteq A \text{ is compact}\}$.

proof (a) Before embarking on the main line of the proof, it will be convenient to set out a preliminary result. For $\delta > 0$, $n \in \mathbb{N}$, $B \subseteq X$ set

$$\theta_{r\delta}^{(n)}(B) = \inf\{\textstyle\sum_{i=0}^{n}(\operatorname{diam} D_i)^r : B \subseteq \bigcup_{i\le n} D_i,\ \operatorname{diam} D_i \le \delta \text{ for every } i \le n\},$$

taking $\inf\emptyset = \infty$ as usual. Then $\theta_{r\delta}B \le \theta_{r\delta}^{(n)}(B)$ for every n. Now the point is that $\theta_{r\delta}^{(n)}(B) = \sup\{\theta_{r\delta}^{(n)}(I) : I \subseteq B$ is finite$\}$. **P** Set $\gamma = \sup_{I\in[B]^{<\omega}} \theta_{r\delta}^{(n)}(I)$. Of course $\gamma \le \theta_{r\delta}^{(n)}(B)$. If $\gamma = \infty$ there is nothing more to say. Otherwise, take any $\gamma' > \gamma$. For each $I \in [B]^{<\omega}$, we have a function $f_I : I \to \{0, \dots, n\}$ such that $\sum_{i\in J}\rho(x_i, y_i)^r \le \gamma'$ whenever $J \subseteq \{0, \dots, n\}$ and x_i, $y_i \in I$ and $f_I(x_i) = f_I(y_i) = i$ for every $i \in J$, while $\rho(x, y) \le \delta$ whenever x, $y \in I$ and $f_I(x) = f_I(y)$. Let $\mathcal{F}$ be an ultrafilter on $[B]^{<\omega}$ such that $\{I : x \in I \in [B]^{<\omega}\} \in \mathcal{F}$ for every $x \in B$ (4A1Ia). Then for every $x \in B$ there is an $f(x) \in \{0, \dots, n\}$ such that $\{I : x \in I \in [B]^{<\omega},\ f_I(x) = f(x)\} \in \mathcal{F}$. Set $D_i = f^{-1}[\{i\}]$ for $i \le n$. If x, $y \in B$ and $f(x) = f(y)$, there is an $I \in [B]^{<\omega}$ containing both x and y such that $f_I(x) = f(x) = f(y) = f_I(y)$, so that $\rho(x, y) \le \delta$; thus $\operatorname{diam} D_i \le \delta$ for each i. If $J \subseteq \{0, \dots, n\}$ and for each $i \in J$ we take x_i, $y_i \in D_i$, then there is an $I \in [B]^{<\omega}$ such that $f_I(x_i) = f_I(y_i) = i$ for every $i \in J$, so $\sum_{i\in J}\rho(x_i, y_i)^r \le \gamma'$. This means that $\sum_{i\le n}(\operatorname{diam} D_i)^r \le \gamma'$, so that $\theta_{r\delta}^{(n)}(B) \le \gamma'$. As γ' is arbitrary, $\theta_{r\delta}^{(n)}(B) \le \gamma$, as claimed. **Q**

(b) Now let us turn to the set A. Because A is Souslin-F (422Ha), μ_{Hr} measures A (471Da, 471Dd). Set $\gamma = \sup\{\mu_{Hr}K : K \subseteq A \text{ is compact}\}$.

? Suppose, if possible, that $\mu_{Hr}A > \gamma$. Take $\gamma' \in\ ]\gamma, \mu_{Hr}A[$. Let $\delta > 0$ be such that $\gamma' < \theta_{r\delta}A$. Let $f : \mathbb{N}^{\mathbb{N}} \to A$ be a continuous surjection. For $\sigma \in S = \bigcup_{n\in\mathbb{N}}\mathbb{N}^n$, set

$$F_\sigma = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}},\ \phi(i) \le \sigma(i) \text{ for every } i < \#(\sigma)\},$$

so that $f[F_\emptyset] = A$. Now choose $\psi \in \mathbb{N}^{\mathbb{N}}$ and a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{N}^{\mathbb{N}}$ inductively, as follows. Given that $I_j \subseteq F_{\psi\restriction n}$ for every $j < n$ and that $\theta_{r\delta}(f[F_{\psi\restriction n}]) > \gamma'$, then $\theta_{r\delta}^{(n)}(f[F_{\psi\restriction n}]) > \gamma'$, so by (a) above there is a finite subset I_n of $F_{\psi\restriction n}$ such that $\theta_{r\delta}^{(n)}(f[I_n]) \ge \gamma'$. Next,

$$\lim_{i\to\infty}\theta_{r\delta}f[F_{(\psi\restriction n)^\frown<i>}] = \theta_{r\delta}(\bigcup_{i\in\mathbb{N}} f[F_{(\psi\restriction n)^\frown<i>}])$$

(by 471G)

$$= \theta_{r\delta}f[F_{\psi\restriction n}] > \gamma',$$

so we can take $\psi(n)$ such that $\bigcup_{j\le n} I_j \subseteq F_{\psi\restriction n+1}$ and $\theta_{r\delta}f[F_{\psi\restriction n+1}] > \gamma'$, and continue.

At the end of the induction, set $K = \{\phi : \phi \le \psi\}$. Then $f[K]$ is a compact subset of A, and $I_n \subseteq K$ for every $n \in \mathbb{N}$, so

$$\theta_{r\delta}^{(n)}(f[K]) \ge \theta_{r\delta}^{(n)}(f[I_n]) \ge \gamma'$$

for every $n \in \mathbb{N}$. On the other hand, $\mu_{Hr}(f[K]) \le \gamma$, so there is a sequence $\langle D_i\rangle_{i\in\mathbb{N}}$ of sets, covering $f[K]$, all of diameter less than δ, such that $\sum_{i=0}^{\infty}(\operatorname{diam} D_i)^r < \gamma'$. Enlarging the D_i slightly if need be, we may suppose that they are all open. But in this case there is some finite n such that $K \subseteq \bigcup_{i\le n} D_i$, and $\theta_{r\delta}^{(n)}(K) \le \sum_{i=0}^{n}(\operatorname{diam} D_i)^r < \gamma'$; which is impossible. **X**

This contradiction shows that $\mu_{Hr}A = \gamma$, as required.

471J Proposition Let (X,ρ) and (Y,σ) be metric spaces, and $f : X \to Y$ a γ-Lipschitz function, where $\gamma \ge 0$. If $r > 0$ and $\theta_r^{(X)}$, $\theta_r^{(Y)}$ are the r-dimensional Hausdorff outer measures on X and Y respectively, then $\theta_r^{(Y)} f[A] \le \gamma^r \theta_r^{(X)} A$ for every $A \subseteq X$.

proof (Compare 264G.) Let $\delta > 0$. Set $\eta = \delta/(1+\gamma)$ and consider $\theta_{r\eta}^{(X)} : \mathcal{P}X \to [0,\infty]$, defined as in 471A. We know that $\theta_r^{(X)} A \ge \theta_{r\eta}^{(X)} A$, so there is a sequence $\langle D_n \rangle_{n\in\mathbb{N}}$ of sets, all of diameter at most η, covering A, with $\sum_{n=0}^{\infty} (\operatorname{diam} D_n)^r \le \theta_r^{(X)} A + \delta$. Now $\phi[A] \subseteq \bigcup_{n\in\mathbb{N}} \phi[D_n]$ and

$$\operatorname{diam} \phi[D_n] \le \gamma \operatorname{diam} D_n \le \gamma\eta \le \delta$$

for every n. Consequently

$$\theta_{r\delta}^{(Y)}(\phi[A]) \le \sum_{n=0}^{\infty} (\operatorname{diam} \phi[D_n])^r \le \sum_{n=0}^{\infty} \gamma^r (\operatorname{diam} D_n)^r \le \gamma^r (\theta_r^{(X)} A + \delta),$$

and

$$\theta_r^{(Y)}(\phi[A]) = \lim_{\delta\downarrow 0} \theta_{r\delta}^{(Y)}(\phi[A]) \le \gamma^r \theta_r^{(X)} A,$$

as claimed.

471K Lemma Let (X,ρ) be a metric space, and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. If $A \subseteq X$, then $\mu_{Hr} A = 0$ iff for every $\epsilon > 0$ there is a countable family $\mathcal{D}$ of sets, covering A, such that $\sum_{D\in\mathcal{D}} (\operatorname{diam} D)^r \le \epsilon$.

proof If $\mu_{Hr} A = 0$ and $\epsilon > 0$, then, in the language of 471A, $\theta_{r1} A \le \epsilon$, so there is a sequence $\langle D_n \rangle_{n\in\mathbb{N}}$ of sets covering A such that $\sum_{n=0}^{\infty} (\operatorname{diam} D_n)^r \le \epsilon$.

If the condition is satisfied, then for any ϵ, $\delta > 0$ there is a countable family $\mathcal{D}$ of sets, covering A, such that $\sum_{D\in\mathcal{D}} (\operatorname{diam} D)^r \le \min(\epsilon, \delta^r)$. If $\mathcal{D}$ is infinite, enumerate it as $\langle D_n \rangle_{n\in\mathbb{N}}$; if it is finite, enumerate it as $\langle D_n \rangle_{n<m}$ and set $D_n = \emptyset$ for $n \ge m$. Now $A \subseteq \bigcup_{n\in\mathbb{N}} D_n$ and $\operatorname{diam} D_n \le \delta$ for every $n \in \mathbb{N}$, so $\theta_{r\delta} A \le \sum_{n=0}^{\infty} (\operatorname{diam} D_n)^r \le \epsilon$. As ϵ is arbitrary, $\theta_{r\delta} A = 0$; as δ is arbitrary, $\theta_r A = 0$; it follows at once that $\mu_{Hr} A$ is defined and is zero (113Xa).

471L Proposition Let (X,ρ) be a metric space and $0 < r < s$. If $A \subseteq X$ is such that $\mu_{Hr}^* A$ is finite, then $\mu_{Hs} A = 0$.

proof Let $\epsilon > 0$. Let $\delta > 0$ be such that $\delta^{s-r}(1+\mu_{Hr}^* A) \le \epsilon$. Then there is a sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of sets of diameter at most δ such that $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and $\sum_{n=0}^{\infty} (\operatorname{diam} A_n)^r \le 1+\mu_{Hr}^* A$. But now, by the choice of δ, $\sum_{n=0}^{\infty} (\operatorname{diam} A_n)^s \le \epsilon$. As ϵ is arbitrary, $\mu_{Hs} A = 0$, by 471K.

471M There is a generalization of the density theorems of §§223 and 261 for general Hausdorff measures, which (as one expects) depends on a kind of Vitali theorem. I will use the following notation for the next few paragraphs.

Definition If (X,ρ) is a metric space and $A \subseteq X$, write $A^\sim$ for $\{x : x \in X,\ \rho(x,A) \le 2 \operatorname{diam} A\}$, where $\rho(x,A) = \inf_{y\in A} \rho(x,y)$. (Following the conventions of 471A, $\emptyset^\sim = \emptyset$.)

471N Lemma Let (X,ρ) be a metric space. Let $\mathcal{F}$ be a family of subsets of X such that $\{\operatorname{diam} F : F \in \mathcal{F}\}$ is bounded. Set

$$Y = \bigcap_{\delta>0} \bigcup \{F : F \in \mathcal{F},\ \operatorname{diam} F \le \delta\}.$$

Then there is a disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that

(i) $\bigcup \mathcal{F} \subseteq \bigcup_{F\in\mathcal{I}} F^\sim$;

(ii) $Y \subseteq \overline{\bigcup \mathcal{J}} \cup \bigcup_{F\in\mathcal{I}\setminus\mathcal{J}} F^\sim$ for every $\mathcal{J} \subseteq \mathcal{I}$.

proof (a) Let γ be an upper bound for $\{\operatorname{diam} F : F \in \mathcal{F}\}$. Choose $\langle \mathcal{I}_n \rangle_{n\in\mathbb{N}}$, $\langle \mathcal{J}_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. $\mathcal{I}_0 = \emptyset$. Given $\mathcal{I}_n$, set $\mathcal{F}'_n = \{F : F \in \mathcal{F},\ \operatorname{diam} F \ge 2^{-n}\gamma,\ F \cap \bigcup \mathcal{I}_n = \emptyset\}$, and let $\mathcal{J}_n \subseteq \mathcal{F}'_n$ be a maximal disjoint set; now set $\mathcal{I}_{n+1} = \mathcal{I}_n \cup \mathcal{J}_n$, and continue.

At the end of the induction, set

$$\mathcal{I}' = \bigcup_{n\in\mathbb{N}} \mathcal{I}_n, \quad \mathcal{I} = \mathcal{I}' \cup \{\{x\} : x \in F \setminus \bigcup \mathcal{I}',\ \{x\} \in \mathcal{F}\}.$$

The construction ensures that every $\mathcal{I}_n$ is a disjoint subset of $\mathcal{F}$, so $\mathcal{I}'$ and $\mathcal{I}$ are also disjoint subfamilies of $\mathcal{F}$.

(b) **?** Suppose, if possible, that there is a point x in $\bigcup\mathcal{F}\setminus\bigcup_{F\in\mathcal{I}}F^\sim$. Let $F\in\mathcal{F}$ be such that $x\in F$. Since $x\notin\bigcup\mathcal{I}'$ and $\{x\}\notin\mathcal{I}$, $\{x\}\notin\mathcal{F}$, and $\operatorname{diam}F>0$; let $n\in\mathbb{N}$ be such that $2^{-n}\gamma\le\operatorname{diam}F\le 2^{-n+1}\gamma$. If $F\notin\mathcal{F}'_n$, there is a $D\in\mathcal{I}_n$ such that $F\cap D\ne\emptyset$; otherwise, since $\mathcal{J}_n$ is maximal and $F\notin\mathcal{J}_n$, there is a $D\in\mathcal{J}_n$ such that $F\cap D\ne\emptyset$. In either case, we have a $D\in\mathcal{I}$ such that $F\cap D\ne\emptyset$ and $\operatorname{diam}F\le 2\operatorname{diam}D$. But in this case $\rho(x,D)\le\operatorname{diam}F\le 2\operatorname{diam}D$ and $x\in D^\sim$, which is impossible. **X**

(c) **?** Suppose, if possible, that there are a point $x\in Y$ and a set $\mathcal{J}\subseteq\mathcal{I}$ such that $x\notin\overline{\bigcup\mathcal{J}}\cup\bigcup_{F\in\mathcal{I}\setminus\mathcal{J}}F^\sim$. Then there is an $F\in\mathcal{F}$ such that $x\in F$ and $\operatorname{diam}F<\rho(x,\bigcup\mathcal{J})$, so that $F\cap\bigcup\mathcal{J}=\emptyset$. As in (b), F cannot be $\{x\}$, and there must be an $n\in\mathbb{N}$ such that $2^{-n}\gamma<\operatorname{diam}F\le 2^{-n+1}\gamma$. As in (b), there must be a $D\in\mathcal{I}_{n+1}$ such that $F\cap D\ne\emptyset$, so that $x\in D^\sim$; and as D cannot belong to $\mathcal{J}$, we again have a contradiction. **X**

471O Lemma Let (X,ρ) be a metric space, and $r>0$. Suppose that A, $\mathcal{F}$ are such that

(i) $\mathcal{F}$ is a family of closed subsets of X such that $\sum_{n=0}^\infty(\operatorname{diam}F_n)^r$ is finite for every disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$,

(ii) for every $x\in A$, $\delta>0$ there is an $F\in\mathcal{F}$ such that $x\in F$ and $0<\operatorname{diam}F\le\delta$.

Then there is a countable disjoint family $\mathcal{I}\subseteq\mathcal{F}$ such that $A\setminus\bigcup\mathcal{I}$ has zero r-dimensional Hausdorff measure.

proof Replacing $\mathcal{F}$ by $\{F:F\in\mathcal{F},\,0<\operatorname{diam}F\le 1\}$ if necessary, we may suppose that $\sup_{F\in\mathcal{F}}\operatorname{diam}F$ is finite and that $\operatorname{diam}F>0$ for every $F\in\mathcal{F}$. Take a disjoint family $\mathcal{I}\subseteq\mathcal{F}$ as in 471N. If $\mathcal{I}$ is finite, then $A\subseteq Y\subseteq\bigcup\mathcal{I}$, where Y is defined as in 471N, so we can stop. Otherwise, hypothesis (i) tells us that $\{F:F\in\mathcal{I},\operatorname{diam}F\ge\delta\}$ is finite for every $\delta>0$, so $\mathcal{I}$ is countable; enumerate it as $\langle F_n\rangle_{n\in\mathbb{N}}$; we must have $\sum_{n=0}^\infty(\operatorname{diam}F_n)^r<\infty$. Since $\operatorname{diam}F_n^\sim\le 5\operatorname{diam}F_n$ for every n, $\sum_{n=0}^\infty(\operatorname{diam}F_n^\sim)^r$ is finite, and $\inf_{n\in\mathbb{N}}\sum_{i=n}^\infty(\operatorname{diam}F_i^\sim)^r=0$. But now observe that the construction ensures that $A\setminus\bigcup\mathcal{I}\subseteq\bigcup_{i\ge n}F_i^\sim$ for every $n\in\mathbb{N}$. By 471K, $\mu_{Hr}(A\setminus\bigcup\mathcal{I})=0$, as required.

471P Theorem Let (X,ρ) be a metric space, and $r>0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. Suppose that $A\subseteq X$ and $\mu_{Hr}^*A<\infty$.

(a) $\lim_{\delta\downarrow 0}\sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam}D)^r}:x\in D,\,0<\operatorname{diam}D\le\delta\}=1$ for μ_{Hr}-almost every $x\in A$.

(b) $\limsup_{\delta\downarrow 0}\frac{\mu_{Hr}^*(A\cap B(x,\delta))}{\delta^r}\ge 1$ for μ_{Hr}-almost every $x\in A$. So

$$2^{-r}\le\limsup_{\delta\downarrow 0}\frac{\mu_{Hr}^*(A\cap B(x,\delta))}{(\operatorname{diam}B(x,\delta))^r}\le 1$$

for μ_{Hr}-almost every $x\in A$.

(c) If A is measured by μ_{Hr}, then

$$\lim_{\delta\downarrow 0}\sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam}D)^r}:x\in D,\,0<\operatorname{diam}D\le\delta\}=0$$

for μ_{Hr}-almost every $x\in X\setminus A$.

proof (a)(i) Note first that as the quantities

$$\sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam}D)^r}:x\in D,\,0<\operatorname{diam}D\le\delta\}$$

decrease with δ, the limit is defined in $[0,\infty]$ for every $x\in X$. Moreover, since $\operatorname{diam}D=\operatorname{diam}\overline{D}$ and $\mu_{Hr}^*(A\cap\overline{D})\ge\mu_{Hr}^*(A\cap D)$ for every D,

$$\begin{aligned}\sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam}D)^r}&:x\in D\subseteq X,\,0<\operatorname{diam}D\le\delta\}\\&=\sup\{\frac{\mu_{Hr}^*(A\cap F)}{(\operatorname{diam}F)^r}:F\subseteq X\text{ is closed},\,x\in F,\,0<\operatorname{diam}F\le\delta\}\end{aligned}$$

for every x and δ.

(ii) Fix ϵ for the moment, and set

$$A_\epsilon=\{x:x\in A,\,\lim_{\delta\downarrow 0}\sup\{\frac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam}D)^r}:x\in D,\,0<\operatorname{diam}D\le\delta\}>1+\epsilon\}.$$

Then $\theta_{r\eta}(A)\le\mu_{Hr}^*A-\frac{\epsilon}{1+\epsilon}\mu_{Hr}^*A_\epsilon$ for every $\eta>0$, where $\theta_{r\eta}$ is defined in 471A. **P** Let $\mathcal{F}$ be the family

$$\{F : F \subseteq X \text{ is closed, } 0 < \operatorname{diam} F \le \eta,\ (1+\epsilon)(\operatorname{diam} F)^r \le \mu_{Hr}^*(A \cap F)\}.$$

Then every member of A_ϵ belongs to sets in $\mathcal{F}$ of arbitrarily small diameter. Also, if $\langle F_n\rangle_{n\in\mathbb{N}}$ is any disjoint sequence in $\mathcal{F}$,

$$\sum_{n=0}^{\infty}(\operatorname{diam} F_n)^r \le \sum_{n=0}^{\infty}\mu_{Hr}^*(A\cap F_n) \le \mu_{Hr}^* A < \infty$$

because every F_n, being closed, is measured by μ_{Hr}. (If you like, $F_n \cap A$ is measured by the subspace measure on A for every n.) So 471O tells us that there is a countable disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that $A_\epsilon \setminus \bigcup\mathcal{I}$ is negligible, and $\mu_{Hr}^* A_\epsilon = \mu_{Hr}^*(A_\epsilon \cap \bigcup\mathcal{I})$.

Because $\theta_{r\eta}$ is an outer measure and $\theta_{r\eta} \le \mu_{Hr}^*$,

$$\theta_{r\eta}A \le \theta_{r\eta}(A\cap\bigcup\mathcal{I}) + \theta_{r\eta}(A\setminus\bigcup\mathcal{I}) \le \sum_{F\in\mathcal{I}}(\operatorname{diam} F)^r + \mu_{Hr}^*(A\setminus\bigcup\mathcal{I})$$

(because $\mathcal{I}$ is countable)

$$\le \tfrac{1}{1+\epsilon}\mu_{Hr}^*(A\cap\bigcup\mathcal{I}) + \mu_{Hr}^*(A\setminus\bigcup\mathcal{I}) = \mu_{Hr}^*A - \tfrac{\epsilon}{1+\epsilon}\mu_{Hr}^*(A\cap\bigcup\mathcal{I})$$

$$\le \mu_{Hr}^*A - \tfrac{\epsilon}{1+\epsilon}\mu_{Hr}^*(A_\epsilon\cap\bigcup\mathcal{I}) = \mu_{Hr}^*A - \tfrac{\epsilon}{1+\epsilon}\mu_{Hr}^*A_\epsilon,$$

as claimed. **Q**

(iii) Taking the supremum as $\eta \downarrow 0$, $\mu_{Hr}^*A \le \mu_{Hr}^*A - \frac{\epsilon}{1+\epsilon}\mu_{Hr}^*A_\epsilon$ and $\mu_{Hr}A_\epsilon = 0$.

This is true for any $\epsilon > 0$. But

$$\{x : x \in A,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam} D)^r} : x \in D,\ 0 < \operatorname{diam} D \le \delta\} > 1\}$$

is just $\bigcup_{n\in\mathbb{N}} A_{2^{-n}}$, so is negligible.

(iv) Next, for $0 < \epsilon \le 1$, set

$$A'_\epsilon = \{x : x \in A,\ \mu_{Hr}^*(A\cap D) \le (1-\epsilon)(\operatorname{diam} D)^r \text{ whenever } x \in D \text{ and } 0 < \operatorname{diam} D \le \epsilon\}.$$

Then A'_ϵ is negligible. **P** Let $\langle D_n\rangle_{n\in\mathbb{N}}$ be any sequence of sets of diameter at most ϵ covering A'_ϵ. Set $K = \{n : D_n \cap A'_\epsilon \ne \emptyset\}$. Then

$$\mu_{Hr}^*A'_\epsilon \le \sum_{n\in K}\mu_{Hr}^*(A\cap D_n)$$

$$\le (1-\epsilon)\sum_{n\in K}(\operatorname{diam} D_n)^r \le (1-\epsilon)\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r.$$

As $\langle D_n\rangle_{n\in\mathbb{N}}$ is arbitrary,

$$\mu_{Hr}^*A'_\epsilon \le (1-\epsilon)\theta_{r\epsilon}A'_\epsilon \le (1-\epsilon)\mu_{Hr}^*A'_\epsilon,$$

and $\mu_{Hr}^*A'_\epsilon$ (being finite) must be zero. **Q**

This means that

$$\{x : x\in A,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam} D)^r} : x\in D,\ 0<\operatorname{diam} D\le\delta\} < 1\} \subseteq \bigcup_{n\in\mathbb{N}} A'_{2^{-n}}$$

is also negligible, and we have the result.

(b) We need a slight modification of the argument in (a)(iv). This time, for $0 < \epsilon \le 1$, set

$$\tilde{A}'_\epsilon = \{x : x \in A,\ \mu_{Hr}^*(A\cap B(x,\delta)) \le (1-\epsilon)\delta^r \text{ whenever } 0 < \delta \le \epsilon\}.$$

Then $\mu_{Hr}^*\tilde{A}'_\epsilon \le \epsilon$. **P** Note first that, as $\mu_{Hr}\{x\} = 0$ for every x, $\mu_{Hr}^*(A\cap B(x,\delta)) \le (1-\epsilon)\delta^r$ whenever $x \in \tilde{A}'_\epsilon$ and $0 \le \delta \le \epsilon$. Let $\langle D_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets of diameter at most ϵ covering $\tilde{A}'_\epsilon$. Set $K = \{n : D_n \cap \tilde{A}'_\epsilon \ne \emptyset\}$, and for $n \in K$ choose $x_n \in D_n \cap \tilde{A}'_\epsilon$ and set $\delta_n = \operatorname{diam} D_n$. Then $D_n \subseteq B(x_n,\delta_n)$ and $\delta_n \le \epsilon$ for each n, so $\tilde{A}'_\epsilon \subseteq \bigcup_{n\in K} B(x_n,\delta_n)$ and

$$\mu_{Hr}^*\tilde{A}'_\epsilon \le \sum_{n\in K}\mu_{Hr}^*(\tilde{A}'_\epsilon\cap B(x_n,\delta_n))$$
$$\le \sum_{n\in K}(1-\epsilon)\delta_n^r \le (1-\epsilon)\sum_{n=0}^\infty(\operatorname{diam}D_n)^r.$$

As $\langle D_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mu_{Hr}^*\tilde{A}'_\epsilon \le (1-\epsilon)\mu_{Hr}^*\tilde{A}'_\epsilon$ and $\tilde{A}'_\epsilon$ must be negligible. **Q**

Now

$$\{x : x\in A,\ \limsup_{\delta\downarrow 0}\tfrac{\mu_{Hr}^*(A\cap B(x,\delta))}{\delta^r} < 1\} = \bigcup_{n\in\mathbb{N}}\tilde{A}'_{2^{-n}}$$

is negligible. As for the second formula, we need note only that $\operatorname{diam}B(x,\delta)\le 2\delta$ for every $x\in X$, $\delta>0$ to obtain the first inequality, and apply (a) to obtain the second.

(c) Let $\epsilon>0$. This time, write $\tilde{A}_\epsilon$ for

$$\{x : x\in X,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam}D)^r} : x\in D,\ 0<\operatorname{diam}D\le\delta\} > \epsilon\}.$$

Let $E\subseteq A$ be a closed set such that $\mu(A\setminus E)\le\epsilon^2$ (471De). For $\eta>0$, let $\mathcal{F}_\eta$ be the family

$$\{F : F\subseteq X\setminus E \text{ is closed, } 0<\operatorname{diam}F\le\eta,\ \mu_{Hr}(A\cap F)\ge\epsilon(\operatorname{diam}F)^r\}.$$

Just as in (a) above, every point in $\tilde{A}_\epsilon\setminus E$ belongs to members of $\mathcal{F}_\eta$ of arbitrarily small diameter. If $\langle F_i\rangle_{i\in I}$ is a countable disjoint family in $\mathcal{F}_\eta$,

$$\textstyle\sum_{i\in I}(\operatorname{diam}F_i)^r \le \frac{1}{\epsilon}\mu_{Hr}(A\setminus E)\le\epsilon$$

is finite. There is therefore a countable disjoint family $\mathcal{I}_\eta\subseteq\mathcal{F}_\eta$ such that $\mu_{Hr}((\tilde{A}_\epsilon\setminus E)\setminus\bigcup\mathcal{I}_\eta)=0$. If $\theta_{r\eta}$ is the outer measure defined in 471A, we have

$$\theta_{r\eta}(\tilde{A}_\epsilon\setminus A)\le\theta_{r\eta}(\bigcup\mathcal{I}_\eta)+\theta_{r\eta}(\tilde{A}_\epsilon\setminus(E\cup\bigcup\mathcal{I}_\eta))$$
$$\le\sum_{F\in\mathcal{I}_\eta}(\operatorname{diam}F)^r+\mu_{Hr}^*(\tilde{A}_\epsilon\setminus(E\cup\bigcup\mathcal{I}_\eta))\le\epsilon.$$

As η is arbitrary, $\mu_{Hr}^*(\tilde{A}_\epsilon\setminus A)\le\epsilon$. But now

$$\{x : x\in X\setminus A,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu_{Hr}^*(A\cap D)}{(\operatorname{diam}D)^r} : x\in D,\ 0<\operatorname{diam}D\le\delta\} > 0\}$$

is $\bigcup_{n\in\mathbb{N}}\tilde{A}_{2^{-n}}\setminus A$, and is negligible.

471Q I now come to a remarkable fact about Hausdorff measures on analytic spaces: their Borel versions are semi-finite (471S). We need some new machinery.

Lemma Let (X,ρ) be a metric space, and $r>0$, $\delta>0$. Suppose that $\theta_{r\delta}X$, as defined in 471A, is finite.

(a) There is a non-negative additive functional ν on $\mathcal{P}X$ such that $\nu X=5^{-r}\theta_{r\delta}X$ and $\nu A\le(\operatorname{diam}A)^r$ whenever $A\subseteq X$ and $\operatorname{diam}A\le\frac{1}{5}\delta$.

(b) If X is compact, there is a Radon measure μ on X such that $\mu X=5^{-r}\theta_{r\delta}X$ and $\mu G\le(\operatorname{diam}G)^r$ whenever $G\subseteq X$ is open and $\operatorname{diam}G\le\frac{1}{5}\delta$.

proof (a) I use 391E. If $\theta_{r\delta}X=0$ the result is trivial. Otherwise, set $\gamma=5^r/\theta_{r\delta}X$ and define $\phi:\mathcal{P}X\to[0,1]$ by setting $\phi A=\min(1,\gamma(\operatorname{diam}A)^r)$ if $\operatorname{diam}A\le\frac{1}{5}\delta$, 1 for other $A\subseteq X$. Now

whenever $\langle A_i\rangle_{i\in I}$ is a finite family of subsets of X, $m\in\mathbb{N}$ and $\sum_{i\in I}\chi A_i\ge m\chi X$, then $\sum_{i\in I}\phi A_i\ge m$.

P Discarding any A_i for which $\phi A_i=1$, if necessary, we may suppose that $\operatorname{diam}A_i\le\frac{1}{5}\delta$ and $\phi A_i=\gamma(\operatorname{diam}A_i)^r$ for every i. Choose $\langle I_j\rangle_{j\le m}$, $\langle J_j\rangle_{j<m}$ inductively, as follows. $I_0=I$. Given that $j<m$ and that $I_j\subseteq I$ is such that $\sum_{i\in I_j}\chi A_i\ge(m-j)\chi X$, apply 471N to $\{A_i : i\in I_j\}$ to find $J_j\subseteq I_j$ such that $\langle A_i\rangle_{i\in J_j}$ is disjoint and $\bigcup_{i\in I_j}A_i\subseteq\bigcup_{i\in J_j}A_j^{\sim}$. Set $I_{j+1}=I_j\setminus J_j$. Observe that $\sum_{i\in J_j}\chi A_j\le\chi X$, so $\sum_{i\in I_{j+1}}\chi A_i\ge(m-j-1)\chi X$ and the induction proceeds.

Now note that, for each $j<m$, $\langle A_i^{\sim}\rangle_{i\in J_j}$ is a cover of $\bigcup_{i\in I_j}A_i=X$ by sets of diameter at most δ. So $\sum_{i\in J_j}(\operatorname{diam}A_i^{\sim})^r\ge\theta_{r\delta}X$ for each j, and $\sum_{i\in I}(\operatorname{diam}A_i^{\sim})^r\ge m\theta_{r\delta}X$. Accordingly

$$\sum_{i\in I}\phi A_i=\gamma\sum_{i\in I}(\operatorname{diam}A_i)^r\ge 5^{-r}\gamma\sum_{i\in I}(\operatorname{diam}A_i^{\sim})^r$$
$$\ge 5^{-r}m\gamma\theta_{r\delta}X=m. \textbf{Q}$$

By 391E, there is an additive functional $\nu_0:\mathcal{P}X\to[0,1]$ such that $\nu_0X=1$ and $\nu_0A\le\phi A$ for every $A\subseteq X$. Setting $\nu=5^{-r}\theta_{r\delta}X\nu_0$, we have the result.

(b) Now suppose that X is compact. By 416K, there is a Radon measure μ on X such that $\mu K\ge\nu K$ for every compact $K\subseteq X$ and $\mu G\le\nu G$ for every open $G\subseteq X$. Because X itself is compact, $\mu X=\nu X=5^{-r}\theta_{r\delta}X$. If G is open and $\operatorname{diam}G\le\frac{1}{5}\delta$,

$$\mu G\le\nu G\le(\operatorname{diam}G)^r,$$

as required.

471R Lemma (Howroyd 95) Let (X,ρ) be a compact metric space and $r>0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. If $\mu_{Hr}X>0$, there is a Borel set $H\subseteq X$ such that $0<\mu_{Hr}H<\infty$.

proof (a) Let $\delta>0$ be such that $\theta_{r,5\delta}(X)>0$, where $\theta_{r,5\delta}$ is defined as in 471A. Then there is a family $\mathcal{V}$ of open subsets of X such that (i) $\operatorname{diam}V\le\delta$ for every $V\in\mathcal{V}$ (ii) $\{V:V\in\mathcal{V},\operatorname{diam}V\ge\epsilon\}$ is finite for every $\epsilon>0$ (iii) whenever $A\subseteq X$ and $0<\operatorname{diam}A<\frac{1}{4}\delta$ there is a $V\in\mathcal{V}$ such that $A\subseteq V$ and $\operatorname{diam}V\le 8\operatorname{diam}A$. **P** For each $k\in\mathbb{N}$, let I_k be a finite subset of X such that $X=\bigcup_{x\in I_k}B(x,2^{-k-2}\delta)$; now set $\mathcal{V}=\{U(x,2^{-k-1}\delta):k\in\mathbb{N},\,x\in I_k\}$. Then $\mathcal{V}$ is a family of open sets and (i) and (ii) are satisfied. If $A\subseteq X$ and $0<\operatorname{diam}A<\frac{1}{4}\delta$, let $k\in\mathbb{N}$ be such that $2^{-k-3}\delta\le\operatorname{diam}A<2^{-k-2}\delta$. Take $x\in I_k$ such that $B(x,2^{-k-2}\delta)\cap A\ne\emptyset$; then $A\subseteq U(x,2^{-k-1}\delta)\in\mathcal{V}$ and $\operatorname{diam}U(x,2^{-k-1}\delta)\le 2^{-k}\delta\le 8\operatorname{diam}A$. **Q**

In particular, $\{V:V\in\mathcal{V},\operatorname{diam}V\le\epsilon\}$ covers X for every $\epsilon>0$.

(b) Set

$$P=\{\mu:\mu\text{ is a Radon measure on }X,\ \mu V\le(\operatorname{diam}V)^r\text{ for every }V\in\mathcal{V}\}.$$

P is non-empty (it contains the zero measure, for instance). Now if $G\subseteq X$ is open, $\mu\mapsto\mu G$ is lower semi-continuous for the narrow topology (437Jd), so P is a closed set in the narrow topology on the set of Radon measures on X, which may be identified with a subset of $C(X)^*$ with its weak* topology (437Kc). Moreover, since there is a finite subfamily of $\mathcal{V}$ covering X, $\gamma=\sup\{\mu X:\mu\in P\}$ is finite, and P is compact (437Pb/437Rf). Because $\mu\mapsto\mu X$ is continuous, $P_0=\{\mu:\mu\in P,\,\mu X=\gamma\}$ is non-empty. Of course P and P_0 are both convex, and P_0, like P, is compact. By the Kreĭn-Mil'man theorem (4A4Gb), applied in $C(X)^*$, P has an extreme point ν say.

Note next that $\theta_{r,5\delta}(X)$ is certainly finite, again because X is compact. By 471Qb, $\gamma>0$, and ν is non-trivial. For any $\epsilon>0$, there is a finite cover of X by sets in $\mathcal{V}$ of diameter at most ϵ, which have measure at most ϵ^r (for ν); so ν is atomless. In particular, $\nu\{x\}=0$ for every $x\in X$.

(c) For $\epsilon>0$, set

$$G_\epsilon=\bigcup\{V:V\in\mathcal{V},\,0<\operatorname{diam}V\le\epsilon\text{ and }\nu V\ge\tfrac{1}{2}(\operatorname{diam}V)^r\}.$$

Then G_ϵ is ν-conegligible. **P?** Otherwise, $\nu(X\setminus G_\epsilon)>0$. Because $\mathcal{V}'_\epsilon=\{V:V\in\mathcal{V},\operatorname{diam}V>\epsilon\}$ is finite, there is a Borel set $E\subseteq X\setminus G_\epsilon$ such that $\nu E>0$ and, for every $V\in\mathcal{V}'_\epsilon$, either $E\subseteq V$ or $E\cap V=\emptyset$. Because ν is atomless, there is a measurable set $E_0\subseteq E$ such that $\nu E_0=\frac{1}{2}\nu E$ (215D); set $E_1=E\setminus E_0$.

Define Radon measures ν_0, ν_1 on X by setting

$$\nu_i(F)=2\nu(F\cap E_i)+\nu(F\setminus E)$$

whenever ν measures $F\setminus E_{1-i}$, for each i (use 416S if you feel the need to check that this defines a Radon measure on the definitions of this book). If $V\in\mathcal{V}$, then, by the choice of E,

either $E\subseteq V$ and $\nu_iV=\nu V\le(\operatorname{diam}V)^r$
or $E\cap V=\emptyset$ and $\nu_iV=\nu V\le(\operatorname{diam}V)^r$
or $0<\operatorname{diam}V\le\epsilon$ and $\nu V<\frac{1}{2}(\operatorname{diam}V)^r$, in which case $\nu_iV\le 2\nu V\le(\operatorname{diam}V)^r$
or $\operatorname{diam}V=0$ and $\nu_iV=\nu V=0=(\operatorname{diam}V)^r$.

So both ν_i belong to P and therefore to P_0, since $\nu_iX=\nu X=\gamma$. But $\nu=\frac{1}{2}(\nu_0+\nu_1)$ and $\nu_0\ne\nu_1$, so this is impossible, because ν is supposed to be an extreme point of P_0. **XQ**

(d) Accordingly, setting $H=\bigcap_{n\in\mathbb{N}}G_{2^{-n}}$, $\nu H=\nu X=\gamma$. Now examine $\mu_{Hr}H$.

(i) $\mu_{Hr}H \ge 8^{-r}\gamma$. **P** Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets covering H with $\operatorname{diam} A_n \le \frac{1}{8}\delta$ for every n. Set $K = \{n : \operatorname{diam} A_n > 0\}$, $H' = H \cap \bigcup_{n\in K} A_n$; then $H \setminus H'$ is countable, so $\nu H' = \nu H$. For each $n \in K$, let $V_n \in \mathcal{V}$ be such that $A_n \subseteq V_n$ and $\operatorname{diam} V_n \le 8 \operatorname{diam} A_n$ ((a) above). Then

$$\begin{aligned}\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^r &= \sum_{n\in K}(\operatorname{diam} A_n)^r \ge 8^{-r}\sum_{n\in K}(\operatorname{diam} V_n)^r\\ &\ge 8^{-r}\sum_{n\in K}\nu V_n \ge 8^{-r}\nu H' = 8^{-r}\gamma.\end{aligned}$$

As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary,

$$8^{-r}\gamma \le \theta_{r,\delta/8}(H) \le \mu_{Hr}^* H = \mu_{Hr}H. \ \mathbf{Q}$$

(ii) $\mu_{Hr}H \le 2\gamma$. **P** Let $\eta > 0$. Set $\mathcal{F} = \{\overline{V} : V \in \mathcal{V},\ 0 < \operatorname{diam} V \le \eta,\ \nu V \ge \frac{1}{2}(\operatorname{diam} V)^r\}$. Then $\mathcal{F}$ is a family of closed subsets of X, and (by the definition of G_ϵ) every member of H belongs to members of $\mathcal{F}$ of arbitrarily small diameter. Also $\nu F \ge \frac{1}{2}(\operatorname{diam} F)^r$ for every $F \in \mathcal{F}$, so

$$\textstyle\sum_{n=0}^{\infty}(\operatorname{diam} F_n)^r \le 2\sum_{n=0}^{\infty}\nu F_n < \infty$$

for any disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$. By 471O, there is a countable disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that $\mu_{Hr}(H\setminus\bigcup\mathcal{I}) = 0$. Accordingly

$$\textstyle\theta_{r\eta}(H) \le \sum_{F\in\mathcal{I}}(\operatorname{diam} F)^r + \theta_{r\eta}(H\setminus\bigcup\mathcal{I}) \le \sum_{F\in\mathcal{I}} 2\nu F \le 2\gamma.$$

As η is arbitrary, $\mu_{Hr}H = \mu_{Hr}^*H \le 2\gamma$. **Q**

(e) But this means that we have found a Borel set H with $0 < \mu_{Hr}H < \infty$, as required.

471S Theorem (Howroyd 95) Let (X,ρ) be an analytic metric space, and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X, and $\mathcal{B}$ the Borel σ-algebra of X. Then the Borel measure $\mu_{Hr}\restriction\mathcal{B}$ is semi-finite and tight (that is, inner regular with respect to the closed compact sets).

proof Suppose that $E \in \mathcal{B}$ and $\mu_{Hr}E > 0$. Since E is analytic (423Eb), 471I above tells us that there is a compact set $K \subseteq E$ such that $\mu_{Hr}K > 0$. Next, by 471R, there is a Borel set $H \subseteq K$ such that $0 < \mu_{Hr}H < \infty$. (Strictly speaking, $\mu_{Hr}H$ here should be calculated as the r-dimensional Hausdorff measure of H defined by the subspace metric $\rho\restriction K\times K$ on K. By 471E we do not need to distinguish between this and the r-dimensional measure calculated from ρ itself.) By 471I again (applied to the subspace metric on H), there is a compact set $L \subseteq H$ such that $\mu_{Hr}L > 0$.

Thus E includes a non-negligible compact set of finite measure. As E is arbitrary, this is enough to show both that $\mu_{Hr}\restriction\mathcal{B}$ is semi-finite and that it is tight.

471T Proposition Let (X,ρ) be a metric space, and $r > 0$.

(a) If X is analytic and $\mu_{Hr}X > 0$, then for every $s \in \left]0,r\right[$ there is a non-zero Radon measure μ on X such that $\iint \frac{1}{\rho(x,y)^s}\mu(dx)\mu(dy) < \infty$.

(b) If there is a non-zero topological measure μ on X such that $\iint \frac{1}{\rho(x,y)^r}\mu(dx)\mu(dy)$ is finite, then $\mu_{Hr}X = \infty$.

proof (a) By 471S, there is a compact set $K \subseteq X$ such that $\mu_{Hr}K > 0$. Set $\delta = 5\operatorname{diam} K$ and define $\theta_{r\delta}$ as in 471A. Then $\theta_{r\delta}K > 0$, by 471K, and $\theta_{r\delta}K \le (\operatorname{diam} K)^r < \infty$. By 471Qb, there is a Radon measure ν on K such that $\nu K > 0$ and $\nu G \le (\operatorname{diam} G)^r$ whenever $G \subseteq K$ is relatively open; consequently $\nu^*A \le (\operatorname{diam} A)^r$ for every $A \subseteq K$. Now, for any $y \in X$,

$$\begin{aligned}\int_K \frac{1}{\rho(x,y)^s}\nu(dx) &= \int_0^\infty \nu\{x : x\in K,\ \frac{1}{\rho(x,y)^s} \ge t\}dt = \int_0^\infty \nu\{x : x \in K,\ \rho(x,y) \le \frac{1}{t^{1/s}}\}dt\\ &= \int_0^\infty \nu(K\cap B(y,\frac{1}{t^{1/s}}))dt \le \int_0^\infty (\operatorname{diam}(K\cap B(y,\frac{1}{t^{1/s}})))^r dt\\ &\le \int_0^\infty (\min(\operatorname{diam} K, \frac{2}{t^{1/s}}))^r dt \le 2^r\int_0^\infty \min((\operatorname{diam} K)^r, \frac{1}{t^{r/s}})dt < \infty\end{aligned}$$

because $r > s$. It follows at once that $\int_K \int_K \frac{1}{\rho(x,y)^s}\nu(dx)\nu(dy)$ is finite. Taking μ to be the extension of ν to a Radon measure on X for which $X \setminus K$ is negligible, we have an appropriate μ.

(b)(i) We can suppose that X is separable (471Df). Since the integrand is strictly positive, μ must be σ-finite, so that there is no difficulty with the repeated integral. Replacing μ by $\mu\llcorner F$ for some set F of non-zero finite measure, we can suppose that μ is totally finite; and replacing μ by a scalar multiple of itself, we can suppose that it is a probability measure.

(ii) Let $\epsilon > 0$. Let H be the conegligible set $\{y : \int \frac{1}{\rho(x,y)^r}\mu(dx) < \infty\}$. For any $y \in X$, $\mu\{y\} = 0$, so

$$\lim_{\delta\downarrow 0} \int_{B(y,\delta)} \frac{1}{\rho(x,y)^r}\mu(dx) = 0$$

for every $y \in H$. For each $\delta > 0$,

$$(x,y) \mapsto \frac{\chi B(y,\delta)(x)}{\rho(x,y)^r} : X \times X \to [0,\infty]$$

is Borel measurable, so

$$y \mapsto \int_{B(y,\delta)} \frac{1}{\rho(x,y)^r}\mu(dx) : X \to [0,\infty]$$

is Borel measurable (252P, applied to the restriction of μ to the Borel σ-algebra of X). There is therefore a $\delta > 0$ such that $E = \{y : y \in H, \int_{B(y,\delta)} \frac{1}{\rho(x,y)^r}\mu(dy) \le \epsilon\}$ has measure $\mu E \ge \frac{1}{2}$. Note that if $C \subseteq X$ has diameter less than or equal to δ and meets E then $\mu\overline{C} \le \epsilon(\operatorname{diam} C)^r$. **P** Set $\gamma = \operatorname{diam} C$ and take $y \in C \cap E$. If $C = \{y\}$ then $\mu\overline{C} = 0$. Otherwise,

$$\mu\overline{C} \le \mu B(y,\gamma) \le \gamma^r \int_{B(y,\delta)} \frac{1}{\rho(x,y)^r}\mu(dx) \le \gamma^r\epsilon. \text{ } \mathbf{Q}$$

Now suppose that $E \subseteq \bigcup_{i\in I} C_i$ where $\operatorname{diam} C_i \le \delta$ for every i, and each C_i is either empty or meets E. Then

$$\frac{1}{2} \le \mu E \le \sum_{i=0}^{\infty} \mu\overline{C}_i \le \sum_{i=0}^{\infty} \epsilon(\operatorname{diam} C_i)^r.$$

As $\langle C_i\rangle_{i\in\mathbb{N}}$ is arbitrary, $\epsilon\mu_{Hr}E \ge \frac{1}{2}$ and $\mu_{Hr}X \ge \frac{1}{2\epsilon}$. As ϵ is arbitrary, $\mu_{Hr}X = \infty$.

471X Basic exercises (a) Define a metric ρ on $X = \{0,1\}^{\mathbb{N}}$ by setting $\rho(x,y) = 2^{-n}$ if $x\restriction n = y\restriction n$ and $x(n) \ne y(n)$. Show that the usual measure μ on X is one-dimensional Hausdorff measure. (*Hint*: $\operatorname{diam} F \ge \mu F$ for every closed set $F \subseteq X$.)

(b) Let (X,ρ) be a metric space and $r > 0$; let μ_{Hr}, $\theta_{r\infty}$ be r-dimensional Hausdorff measure and capacity on X. (i) Show that, for $A \subseteq X$, $\mu_{Hr}A = 0$ iff $\theta_{r\infty}A = 0$. (ii) Suppose that $E \subseteq X$ and $\delta > 0$ are such that $\delta\mu_{Hr}E < \theta_{r\infty}E$. Show that there is a closed set $F \subseteq E$ such that $\mu_{Hr}F > 0$ and $\delta\mu_{Hr}(F\cap G) \le (\operatorname{diam} G)^r$ whenever μ_{Hr} measures G. (*Hint*: show that $\{G^\bullet : \theta_{r\infty}G < \delta\mu_{Hr}G\}$ cannot be order-dense in the measure algebra of μ_{Hr}. This is a version of 'Frostman's Lemma'.) (iii) Let $\mathcal{C}$ be the family of closed subsets of X, with its Vietoris topology. Show that $\theta_{r\infty}\restriction\mathcal{C}$ is upper semi-continuous.

(c) Let (X,ρ) be an analytic metric space, (Y,σ) a metric space, and $f : X \to Y$ a Lipschitz function. Show that if $r > 0$ and $A \subseteq X$ is measured by Hausdorff r-dimensional measure on X, with finite measure, then $f[A]$ is measured by Hausdorff r-dimensional measure on Y.

(d) Let (X,ρ) be a metric space and $r > 0$. Show that a set $A \subseteq X$ is negligible for Hausdorff r-dimensional measure on X iff there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X such that $\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^r$ is finite and $A \subseteq \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n} A_m$.

(e) Let (X,ρ) be a metric space. (i) Show that there is a unique $\dim_H(X) \in [0,\infty]$ such that the r-dimensional Hausdorff measure of X is infinite if $0 < r < \dim_H(X)$, zero if $r > \dim_H(X)$. ($\dim_H(X)$ is the **Hausdorff dimension** of X.) (ii) Show that if $\langle A_n\rangle_{n\in\mathbb{N}}$ is any sequence of subsets of X, then $\dim_H(\bigcup_{n\in\mathbb{N}} A_n) = \sup_{n\in\mathbb{N}} \dim_H(A_n)$.

(f) Let (X,ρ) be a metric space, and μ any topological measure on X. Suppose that $E \subseteq X$ and that μE is defined and finite. (i) Show that $(x,\delta) \mapsto \mu(E \cap B(x,\delta)) : X \times [0,\infty[\to \mathbb{R}$ is upper semi-continuous. (ii) Show that

$x \mapsto \limsup_{\delta\downarrow 0} \frac{1}{\delta^r}\mu(E\cap B(x,\delta)) : X \to [0,\infty]$ is Borel measurable, for every $r \geq 0$. (iii) Show that if X is separable, then $\mu B(x,\delta) > 0$ for every $\delta > 0$, for almost every $x \in X$.

(g) Give $\mathbb{R}$ its usual metric. Let $C \subseteq \mathbb{R}$ be the Cantor set, and $r = \ln 2/\ln 3$. Show that

$$\liminf_{\delta\downarrow 0} \frac{\mu_{Hr}(C\cap B(x,\delta))}{(\operatorname{diam} B(x,\delta))^r} \leq 2^{-r}$$

for every $x \in \mathbb{R}$.

(h) Let (X,ρ) be a metric space and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X and $\tilde{\mu}_{Hr}$ its c.l.d. version (213D-213E). Show that $\tilde{\mu}_{Hr}$ is inner regular with respect to the closed sets, and that $\tilde{\mu}_{Hr}A = \mu_{Hr}A$ for every analytic set $A \subseteq X$.

(i) Suppose that $g : \mathbb{R} \to \mathbb{R}$ is continuous and non-decreasing, and that ν is the corresponding Lebesgue-Stieltjes measure (114Xa). Define $\rho(x,y) = |x-y| + \sqrt{|g(x)-g(y)|}$ for x, $y \in \mathbb{R}$. Show that ρ is a metric on $\mathbb{R}$ defining the usual topology. Show that ν is 2-dimensional Hausdorff measure for the metric ρ.

471Y Further exercises (a) The next few exercises (down to 471Yd) will be based on the following. Let (X,ρ) be a metric space and $\psi : \mathcal{P}X \to [0,\infty]$ a function such that $\psi\emptyset = 0$ and $\psi A \leq \psi A'$ whenever $A \subseteq A' \subseteq X$. Set

$$\theta_{\psi\delta}A = \inf\{\sum_{n=0}^{\infty} \psi D_n : \langle D_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A,$$
$$\operatorname{diam} D_n \leq \delta \text{ for every } n \in \mathbb{N}\}$$

for $\delta > 0$, and $\theta_\psi A = \sup_{\delta>0} \theta_{\psi\delta}A$ for $A \subseteq X$ Show that θ_ψ is a metric outer measure. Let μ_ψ be the measure defined from θ_ψ by Carathéodory's method.

(b) Suppose that $\psi A = \inf\{\psi E : E$ is a Borel set including $A\}$ for every $A \subseteq X$. Show that $\theta_\psi = \mu_\psi^*$ and that $\mu_\psi E = \sup\{\mu_\psi F : F \subseteq E$ is closed$\}$ whenever $\mu_\psi E < \infty$.

(c) Suppose that X is separable and that there is a $\beta \geq 0$ such that $\psi A^\sim \leq \beta\psi A$ for every $A \subseteq X$, where $A^\sim$ is defined in 471M. (i) Suppose that $A \subseteq X$ and $\mathcal{F}$ is a family of closed subsets of X such that $\sum_{n=0}^{\infty} \psi F_n$ is finite for every disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ and for every $x \in A$, $\delta > 0$ there is an $F \in \mathcal{F}$ such that $x \in F$ and $0 < \operatorname{diam} F \leq \delta$. Show that there is a disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that $\mu_\psi(A \setminus \bigcup\mathcal{I}) = 0$. (ii) Suppose that $\delta > 0$ and that $\theta_{\psi\delta}(X) < \infty$. Show that there is a non-negative additive functional ν on $\mathcal{P}X$ such that $\nu X = \frac{1}{\beta}\theta_{\psi\delta}(X)$ and $\nu A \leq \psi A$ whenever $A \subseteq X$ and $\operatorname{diam} A \leq \frac{1}{5}\delta$. (iii) Now suppose that for every $x \in X$, $\epsilon > 0$ there is a $\delta > 0$ such that $\psi B(x,\delta) \leq \epsilon$. Show that if X is compact and $\mu_\psi X > 0$ there is a compact set $K \subseteq X$ such that $0 < \mu_\psi K < \infty$.

(d) State and prove a version of 471P appropriate to this context.

(e) Give an example of a set $A \subseteq \mathbb{R}^2$ which is measured by Hausdorff 1-dimensional measure on $\mathbb{R}^2$ but is such that its projection onto the first coordinate is not measured by Hausdorff 1-dimensional measure on $\mathbb{R}$.

(f) Show that the space (X,ρ) of 471Xa can be isometrically embedded as a subset of a metric space (Y,σ) in such a way that (i) $\operatorname{diam} B(y,\delta) = 2\delta$ for every $y \in Y$ and $\delta \geq 0$ (ii) $Y \setminus X$ is countable. Show that if μ_{H1} is one-dimensional Hausdorff measure on Y, then $\mu_{H1}B(y,\delta) \leq \delta$ for every $y \in Y$ and $\delta \geq 0$, so that

$$\limsup_{\delta\downarrow 0} \frac{\mu_{H1}B(y,\delta)}{\operatorname{diam} B(x,\delta)} = \frac{1}{2}$$

for every $y \in Y$.

(g) Let $\langle k_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N} \setminus \{0,1,2,3\}$ such that $\sum_{n=0}^{\infty} \frac{1}{k_n} < \infty$. Set $X = \prod_{n\in\mathbb{N}} k_n$. Set $m_0 = 1$, $m_{n+1} = k_0k_1\ldots k_n$ for $n \in \mathbb{N}$. Define a metric ρ on X by saying that

$$\begin{aligned}\rho(x,y) &= 1/2m_n \text{ if } n = \min\{i : x(i) \neq y(i)\} \text{ and } \min(x(n),y(n)) = 0,\\ &= 1/m_n \text{ if } n = \min\{i : x(i) \neq y(i)\} \text{ and } \min(x(n),y(n)) > 0.\end{aligned}$$

Let ν be the product measure on X obtained by giving each factor k_n the uniform probability measure in which each singleton set has measure $1/k_n$. (i) Show that if $A \subseteq X$ then $\nu^* A \le \operatorname{diam} A$. (ii) Show that ν is one-dimensional Hausdorff measure on X. (iii) Set $E = \bigcup_{n \in \mathbb{N}} \{x : x \in X,\, x(n) = 0\}$. Show that $\nu E < 1$. (iv) Show that

$$\limsup_{\delta \downarrow 0} \frac{\nu(E \cap B(x,\delta))}{\nu B(x,\delta)} \ge \frac{1}{2}$$

for every $x \in X$. (v) Show that there is a family $\mathcal{F}$ of closed balls in X such that every point of X is the centre of arbitrarily small members of $\mathcal{F}$, but $\nu(\bigcup \mathcal{I}) < 1$ for any disjoint subfamily $\mathcal{I}$ of $\mathcal{F}$.

(h) Let ρ be the metric on $\{0,1\}^{\mathbb{N}}$ defined in 471Xa. Show that for any integer $s \ge 1$ there is a bijection $f : [0,1]^s \to \{0,1\}^{\mathbb{N}}$ such that whenever $0 < r \le s$, μ_{Hr}^* is Hausdorff r-dimensional outer measure on $[0,1]^s$ (for its usual metric) and $\tilde\mu_{H,r/s}^*$ is Hausdorff $\frac{r}{s}$-dimensional measure on $\{0,1\}^{\omega}$, then there is an $\alpha > 0$ such that $\mu_{Hr}^* f^{-1}[A] \le \mu_{H,r/s}^* A \le \alpha \mu_{Hr}^* f^{-1}[A]$ for every $A \subseteq \{0,1\}^{\mathbb{N}}$.

(i) Let (X,ρ) be a metric space, and $r > 0$; let $s \ge 1$ be an integer. Write $\mu_{Hr}^{(X)}$ and $\mu_{H,r+s}^{(X\times\mathbb{R}^s)}$ for Hausdorff r-dimensional measure on X and Hausdorff $(r+s)$-dimensional measure on $X \times \mathbb{R}^s$ respectively, and μ_{Ls} for Lebesgue measure on $\mathbb{R}^s$. (i) Show that there are non-zero constants c, c' such that $c\mu_{H,r+s}^{(X\times\mathbb{R}^s)}(E \times F) \le \mu_{Hr}^{(X)}(E) \cdot \mu_{Ls}(F) \le c'\mu_{H,r+s}^{(X\times\mathbb{R}^s)}(E \times F)$ for all Borel sets $E \subseteq X$, $F \subseteq \mathbb{R}^s$. (*Hint*: Federer 69, 2.10.45.) (ii) Write λ for the c.l.d. product of $\mu_{Hr}^{(X)}$ and μ_{Ls}, and $\tilde\mu_{H,r+s}^{(X\times\mathbb{R}^s)}$ for the c.l.d. version of $\mu_{H,r+s}^{(X\times\mathbb{R}^s)}$. Show that these have the same domain Λ and that $c\tilde\mu_{H,r+s}^{(X\times\mathbb{R}^s)}(W) \le \lambda W \le c'\tilde\mu_{H,r+s}^{(X\times\mathbb{R}^s)}(W)$ for every $W \in \Lambda$.

(j) Let (X,ρ) be a metric space and $0 \le s < t$. Suppose that there is an analytic set $A \subseteq X$ such that $\mu_{Ht} A > 0$. Show that there is a Borel surjection $f : X \to \mathbb{R}$ such that $\mu_{Hs} f^{-1}[\{\alpha\}] \ge 1$ for every $\alpha \in \mathbb{R}$.

(k) Let (X,ρ) be a separable metric space and $r > 0$. Suppose that there is an atomless Borel probability measure μ on X such that $\iint \frac{1}{\rho(x,y)^r}\mu(dx)\mu(dy)$ is finite. Show that X has infinite r-dimensional Hausdorff measure.

(l) Let (X,ρ) be a metric space, and μ, ν two non-zero quasi-Radon measures on X such that $\mu B(x,\delta) = \mu B(y,\delta)$ and $\nu B(x,\delta) = \nu B(y,\delta)$ for all $\delta > 0$ and x, $y \in X$. Show that μ is a multiple of ν. (*Hint*: 442B.)

471 Notes and comments In the exposition above, I have worked throughout with simple r-dimensional measures for $r > 0$. As noted in 264Db, there are formulae in which it is helpful to interpret μ_{H0} as counting measure. More interestingly, when we come to use Hausdorff measures to give us information about the geometric structure of an object (e.g., in the observation that the Cantor set has $\ln 2/\ln 3$-dimensional Hausdorff measure 1, in 264J), it is sometimes useful to refine the technique by using other functionals than $A \mapsto (\operatorname{diam} A)^r$ in the basic formulae of 264A or 471A. The most natural generalization is to functionals of the form $\psi A = h(\operatorname{diam} A)$ where $h : [0,\infty] \to [0,\infty]$ is a non-decreasing function (264Yo). But it is easy to see that many of the arguments are valid in much greater generality, as in 471Ya-471Yc. For more in these directions see Rogers 70 and Federer 69.

In the context of this book, the most conspicuous peculiarity of Hausdorff measures is that they are often very far from being semi-finite. (This is trivial for non-separable spaces, by 471Df. That Hausdorff one-dimensional measure on a subset of $\mathbb{R}^2$ can be purely infinite is not I think obvious; I gave an example in 439H.) The response I ordinarily recommend in such cases is to take the c.l.d. version. But then of course we need to know just what effect this will have. In geometric applications, one usually begins by checking that the sets one is interested in have σ-finite measure, and that therefore no problems arise; but it is a striking fact that Hausdorff measures behave relatively well on analytic sets, even when not σ-finite, provided we ask exactly the right questions (471I, 471S, 471Xh).

The geometric applications of Hausdorff measures, naturally, tend to rely heavily on density theorems; it is therefore useful to know that we have effective versions of Vitali's theorem available in this context (471N-471O), leading to a general density theorem (471P) similar to that in 261D; see also 472D below. I note that 471P is useful only after we have been able to concentrate our attention on a set of finite measure. And traps remain. For instance, the formulae of 261C-261D cannot be transferred to the present context without re-evaluation (471Yg).

472 Besicovitch's Density Theorem

The first step in the program of the next few sections is to set out some very remarkable properties of Euclidean space. We find that in $\mathbb{R}^r$, for geometric reasons (472A), we have versions of Vitali's theorem (472B-472C) and Lebesgue's Density Theorem (472D) for arbitrary Radon measures. I add a version of the Hardy-Littlewood Maximal Theorem (472F).

Throughout the section, $r \geq 1$ will be a fixed integer. As usual, I write $B(x, \delta)$ for the closed ball with centre x and radius δ. $\| \ \|$ will represent the Euclidean norm, and $x \centerdot y$ the scalar product of x and y, so that $x \centerdot y = \sum_{i=1}^{r} \xi_i \eta_i$ if $x = (\xi_1, \dots , \xi_r)$ and $y = (\eta_1, \dots , \eta_r)$.

472A Besicovitch's Covering Lemma Suppose that $\epsilon > 0$ is such that $(5^r + 1)(1 - \epsilon - \epsilon^2)^r > (5 + \epsilon)^r$. Let $x_0, \dots , x_n \in \mathbb{R}^r$, $\delta_0, \dots , \delta_n > 0$ be such that

$$\|x_i - x_j\| > \delta_i, \quad \delta_j \leq (1 + \epsilon)\delta_i$$

whenever $i < j \leq n$. Then

$$\#(\{i : i \leq n, \|x_i - x_n\| \leq \delta_i + \delta_n\}) \leq 5^r.$$

proof Set $I = \{i : i \leq n, \|x_i - x_n\| \leq \delta_n + \delta_i\}$.

(a) It will simplify the formulae of the main argument if we suppose for the time being that $\delta_n = 1$; in this case $1 \leq (1+\epsilon)\delta_i$, so that $\delta_i \geq \frac{1}{1+\epsilon}$ for every $i \leq n$, while we still have $\delta_i < \|x_i - x_n\|$ for every $i < n$, and $\|x_i - x_n\| \leq 1 + \delta_i$ for every $i \in I$.

For $i \in I$, define x'_i by saying that

– if $\|x_i - x_n\| \leq 2 + \epsilon$, $x'_i = x_i$;

– if $\|x_i - x_n\| > 2 + \epsilon$, x'_i is to be that point of the closed line segment from x_n to x_i which is at distance $2 + \epsilon$ from x_n.

(b) The point is that $\|x'_i - x'_j\| > 1 - \epsilon - \epsilon^2$ whenever i, j are distinct members of I. **P** We may suppose that $i < j$.

case 1 Suppose that $\|x_i - x_n\| \leq 2 + \epsilon$ and $\|x_j - x_n\| \leq 2 + \epsilon$. In this case

$$\|x'_i - x'_j\| = \|x_i - x_j\| \geq \delta_i \geq \frac{1}{1+\epsilon} \geq 1 - \epsilon.$$

case 2 Suppose that $\|x_i - x_n\| \geq 2 + \epsilon \geq \|x_j - x_n\|$. In this case

$$\begin{aligned}\|x'_i - x'_j\| &= \|x'_i - x_j\| \geq \|x_i - x_j\| - \|x_i - x'_i\| \\ &\geq \delta_i - \|x_i - x_n\| + 2 + \epsilon \geq \delta_i - \delta_i - 1 + 2 + \epsilon = 1 + \epsilon.\end{aligned}$$

case 3 Suppose that $\|x_i - x_n\| \leq 2 + \epsilon \leq \|x_j - x_n\|$. Then

$$\begin{aligned}\|x'_i - x'_j\| &= \|x_i - x'_j\| \geq \|x_i - x_j\| - \|x_j - x'_j\| > \delta_i - \|x_j - x_n\| + 2 + \epsilon \\ &\geq \delta_i - \delta_j - 1 + 2 + \epsilon \geq \delta_i - \delta_i(1 + \epsilon) + 1 + \epsilon \geq 1 + \epsilon - \epsilon(2 + \epsilon)\end{aligned}$$

(because $\delta_i < \|x_i - x_n\| \leq 2 + \epsilon$)

$$= 1 - \epsilon - \epsilon^2.$$

case 4 Suppose that $2 + \epsilon \leq \|x_j - x_n\| \leq \|x_i - x_n\|$. Let y be the point on the line segment between x_i and x_n which is the same distance from x_n as x_j. In this case

$$\|y - x_j\| \geq \|x_i - x_j\| - \|x_i - y\| \geq \delta_i - \|x_i - x_n\| + \|x_j - x_n\| \geq \|x_j - x_n\| - 1.$$

Because the triangles (x_n, y, x_j) and (x_n, x'_i, x'_j) are similar,

$$\|x'_i - x'_j\| = \frac{2+\epsilon}{\|x_j - x_n\|}\|y - x_j\| \geq (2 + \epsilon)\frac{\|x_j - x_n\| - 1}{\|x_j - x_n\|} \geq 1 + \epsilon$$

because $\|x_j - x_n\| \geq 2 + \epsilon$.

case 5 Suppose that $2+\epsilon \le \|x_i - x_n\| \le \|x_j - x_n\|$. This time, let y be the point on the line segment from x_n to x_j which is the same distance from x_n as x_i is. We now have

$$\begin{aligned}\|y - x_i\| &\ge \|x_i - x_j\| - \|x_j - y\| > \delta_i - \|x_j - x_n\| + \|x_i - x_n\| \\ &\ge \delta_j - \epsilon\delta_i - (\delta_j + 1) + \|x_i - x_n\| \\ &= \|x_i - x_n\| - 1 - \epsilon\delta_i \ge \|x_i - x_n\|(1-\epsilon) - 1,\end{aligned}$$

so that

$$\begin{aligned}\|x'_i - x'_j\| &= \tfrac{2+\epsilon}{\|x_i - x_n\|}\|y - x_i\| > (2+\epsilon)\tfrac{\|x_i - x_n\|(1-\epsilon)-1}{\|x_i - x_n\|} \\ &\ge (2+\epsilon)\tfrac{(2+\epsilon)(1-\epsilon)-1}{2+\epsilon} = 1 - \epsilon - \epsilon^2.\end{aligned}$$

So we have the required inequality in all cases. **Q**

(c) Now consider the balls $B(x'_i, \frac{1-\epsilon-\epsilon^2}{2})$ for $i \in I$. These are disjoint, all have Lebesgue measure $2^{-r}\beta_r(1-\epsilon-\epsilon^2)^r$ where β_r is the measure of the unit ball $B(\mathbf{0}, 1)$, and are all included in the ball $B(x_n, 2+\epsilon+\frac{1-\epsilon}{2})$, which has measure $2^{-r}\beta_r(5+\epsilon)^r$. So we must have

$$2^{-r}\beta_r(1-\epsilon-\epsilon^2)^r\#(I) \le 2^{-r}\beta_r(5+\epsilon)^r.$$

But ϵ was declared to be so small that this implies that $\#(I) \le 5^r$, as claimed.

(d) This proves the lemma in the case $\delta_n = 1$. For the general case, replace each x_i by $\delta_n^{-1}x_i$ and each δ_i by δ_i/δ_n; the change of scale does not affect the hypotheses or the set I.

472B Theorem Let $A \subseteq \mathbb{R}^r$ be a bounded set, and $\mathcal{I}$ a family of non-trivial closed balls in $\mathbb{R}^r$ such that every point of A is the centre of a member of $\mathcal{I}$. Then there is a family $\langle\mathcal{I}_k\rangle_{k<5^r}$ of countable subsets of $\mathcal{I}$ such that each $\mathcal{I}_k$ is disjoint and $\bigcup_{k<5^r}\mathcal{I}_k$ covers A.

proof (a) For each $x \in A$ let $\delta_x > 0$ be such that $B(x, \delta_x) \in \mathcal{I}$. If either A is empty or $\sup_{x\in A}\delta_x = \infty$, the result is trivial. (In the latter case, take $x \in A$ such that $\delta_x \ge \operatorname{diam} A$ and set $\mathcal{I}_0 = \{B(x, \delta_x)\}$, $\mathcal{I}_k = \emptyset$ for $k > 0$.) So let us suppose henceforth that $\{\delta_x : x \in A\}$ is bounded in $\mathbb{R}$. In this case, $C = \bigcup_{x\in A} B(x, \delta_x)$ is bounded in $\mathbb{R}^r$.

Fix $\epsilon > 0$ such that $(5^r + 1)(1-\epsilon-\epsilon^2)^r > (5+\epsilon)^r$.

(b) Choose inductively a sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I} \cup \{\emptyset\}$ as follows. Given $\langle B_i\rangle_{i<n}$, then if $A \subseteq \bigcup_{i<n} B_i$ set $B_n = \emptyset$. Otherwise, set $\alpha_n = \sup\{\delta_x : x \in A \setminus \bigcup_{i<n} B_i\}$, choose $x_n \in A \setminus \bigcup_{i<n} B_i$ such that $(1+\epsilon)\delta_{x_n} \ge \alpha_n$, set $B_n = B(x_n, \delta_{x_n})$ and continue.

Now whenever $n \in \mathbb{N}$, $I_n = \{i : i < n,\ B_i \cap B_n \ne \emptyset\}$ has fewer than 5^r members. **P** We may suppose that $B_n \ne \emptyset$, in which case $B_i = B(x_i, \delta_{x_i})$ for every $i \le n$, and the x_i, δ_{x_i} are such that, whenever $i < j \le n$,

$$x_j \notin B_i, \text{ i.e., } \|x_i - x_j\| > \delta_{x_i},$$

$$\delta_{x_j} \le \alpha_i \le (1+\epsilon)\delta_{x_i}.$$

But now 472A gives the result at once. **Q**

(c) We may therefore define a function $f : \mathbb{N} \to \{0, 1, \dots, 5^r - 1\}$ by setting

$$f(n) = \min\{k : 0 \le k < 5^r,\ f(i) \ne k \text{ for every } i \in I_n\}$$

for every $n \in \mathbb{N}$. Set $\mathcal{I}_k = \{B_i : i \in \mathbb{N},\ f(i) = k,\ B_i \ne \emptyset\}$ for each $k < 5^r$. By the choice of f, $i \notin I_j$, so that $B_i \cap B_j = \emptyset$, whenever $i < j$ and $f(i) = f(j)$; thus every $\mathcal{I}_k$ is disjoint. Since $B_i \subseteq C$ for every i, $\sum\{\mu B_i : f(i) = k\} \le \mu^* C$ for every $k < 5^r$, and $\sum_{i=0}^{\infty}\mu B_i \le 5^r\mu^* C$ is finite.

(d) ? Suppose, if possible, that

$$A \not\subseteq \bigcup_{k<5^r}\bigcup\mathcal{I}_k = \bigcup_{n\in\mathbb{N}} B_n.$$

Take $x \in A \setminus \bigcup_{n\in\mathbb{N}} B_n$. Then, first, $A \not\subseteq \bigcup_{i<n} B_i$ for every n, so that α_n is defined; next, $\alpha_n \ge \delta_x$, so that $(1+\epsilon)\delta_{x_n} \ge \delta_x$ for every n. But this means that $\mu B_n \ge \beta_r\bigl(\frac{\delta_x}{1+\epsilon}\bigr)^r$ for every n, and $\sum_{n=0}^{\infty}\mu B_n = \infty$; which is impossible. **X**

(e) Thus $A \subseteq \bigcup_{k<5^r}\bigcup\mathcal{I}_k$, as required.

472C Theorem Let λ be a Radon measure on $\mathbb{R}^r$, A a subset of $\mathbb{R}^r$ and $\mathcal{I}$ a family of non-trivial closed balls in $\mathbb{R}^r$ such that every point of A is the centre of arbitrarily small members of $\mathcal{I}$. Then

(a) there is a countable disjoint $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\lambda(A \setminus \bigcup \mathcal{I}_0) = 0$;

(b) for every $\epsilon > 0$ there is a countable $\mathcal{I}_1 \subseteq \mathcal{I}$ such that $A \subseteq \bigcup \mathcal{I}_1$ and $\sum_{B \in \mathcal{I}_1} \lambda B \le \lambda^* A + \epsilon$.

proof (a)(i) The first step is to show that if $A' \subseteq A$ is bounded then there is a finite disjoint set $\mathcal{J} \subseteq \mathcal{I}$ such that $\lambda^*(A' \cap \bigcup \mathcal{J}) \ge 6^{-r} \lambda^* A'$. **P** If $\lambda^* A' = 0$ take $\mathcal{J} = \emptyset$. Otherwise, by 472B, there is a family $\langle \mathcal{J}_k \rangle_{k<5^r}$ of disjoint countable subsets of $\mathcal{I}$ such that $\bigcup_{k<5^r} \mathcal{J}_k$ covers A'. Accordingly

$$\lambda^* A' \le \sum_{k=0}^{5^r-1} \lambda^*(A' \cap \bigcup \mathcal{J}_k)$$

and there is some $k < 5^r$ such that $\lambda^*(A' \cap \bigcup \mathcal{J}_k) \ge 5^{-r} \lambda^* A'$. Let $\langle B_i \rangle_{i \in \mathbb{N}}$ be a sequence running over $\mathcal{J}_k$; then

$$\lim_{n\to\infty} \lambda^*(A' \cap \bigcup_{i \le n} B_i) = \lambda^*(A' \cap \bigcup \mathcal{J}_k) \ge 5^{-r} \lambda^* A',$$

so there is some $n \in \mathbb{N}$ such that $\lambda^*(A' \cap \bigcup_{i\le n} B_i) \ge 6^{-r} \lambda^* A'$, and we can take $\mathcal{J} = \{B_i : i \le n\}$. **Q**

(ii) Now choose $\langle \mathcal{K}_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. Start by fixing on a sequence $\langle m_n \rangle_{n\in\mathbb{N}}$ running over $\mathbb{N}$ with cofinal repetitions. Take $\mathcal{K}_0 = \emptyset$. Given that $\mathcal{K}_n$ is a finite disjoint subset of $\mathcal{I}$, set $\mathcal{I}' = \{B : B \in \mathcal{I},\ B \cap \bigcup \mathcal{K}_n = \emptyset\}$, $A_n = A \cap B(\mathbf{0}, m_n) \setminus \bigcup \mathcal{K}_n$. Because every point of A is the centre of arbitrarily small members of $\mathcal{I}$, and $\bigcup \mathcal{K}_n$ is closed, every member of A_n is the centre of (arbitrarily small) members of $\mathcal{I}'$, and (i) tells us that there is a finite disjoint set $\mathcal{J}_n \subseteq \mathcal{I}'$ such that $\lambda^*(A_n \cap \bigcup \mathcal{J}_n) \ge 6^{-r} \lambda^* A_n$. Set $\mathcal{K}_{n+1} = \mathcal{K} \cup \mathcal{J}_n$, and continue. At the end of the induction, set $\mathcal{I}_0 = \bigcup_{n\in\mathbb{N}} \mathcal{K}_n$; because $\langle \mathcal{K}_n \rangle_{n\in\mathbb{N}}$ is non-decreasing and every $\mathcal{K}_n$ is disjoint, $\mathcal{I}_0$ is disjoint, and of course $\mathcal{I}_0 \subseteq \mathcal{I}$.

The effect of this construction is to ensure that

$$\lambda^*(A \cap B(\mathbf{0}, m_n) \setminus \bigcup \mathcal{K}_{n+1}) = \lambda^*(A_n \setminus \bigcup \mathcal{J}_n) = \lambda^* A_n - \lambda^*(A_n \cap \bigcup \mathcal{J}_n)$$

(because $\bigcup \mathcal{J}_n$ is a closed set, therefore measured by λ)

$$\le (1 - 6^{-r}) \lambda^* A_n$$
$$= (1 - 6^{-r}) \lambda^*(A \cap B(\mathbf{0}, m_n) \setminus \bigcup \mathcal{K}_n)$$

for every n. So, for any $m \in \mathbb{N}$,

$$\lambda^*(A \cap B(\mathbf{0}, m) \setminus \bigcup \mathcal{K}_n) \le \lambda^*(A \cap B(\mathbf{0}, m))(1 - 6^{-r})^{\#(\{j : j<n, m_j = m\})} \to 0$$

as $n \to \infty$, and $\lambda^*(A \cap B(\mathbf{0}, m) \setminus \bigcup \mathcal{I}_0) = 0$. As m is arbitrary, $\lambda^*(A \setminus \bigcup \mathcal{I}_0) = 0$, as required.

(b)(i) Let $E \supseteq A$ be such that $\lambda E = \lambda^* A$, and $H \supseteq E$ an open set such that $\lambda H \le \lambda E + \frac{1}{2}\epsilon$ (256Bb). Set $\mathcal{I}' = \{B : B \in \mathcal{I},\ B \subseteq H\}$. Then every point of A is the centre of arbitrarily small members of $\mathcal{I}'$, so by (a) there is a disjoint family $\mathcal{I}_0 \subseteq \mathcal{I}'$ such that $\lambda(A \setminus \bigcup \mathcal{I}_0) = 0$. Of course

$$\sum_{B \in \mathcal{I}_0} \lambda B = \lambda(\bigcup \mathcal{I}_0) \le \lambda H \le \lambda^* A + \tfrac{1}{2}\epsilon.$$

(ii) For $m \in \mathbb{N}$ set $A_m = A \cap B(\mathbf{0}, m) \setminus \bigcup \mathcal{I}_0$. Then there is a $\mathcal{J}_m \subseteq \mathcal{I}$, covering A_m, such that $\sum_{B \in \mathcal{J}_m} \lambda B \le 2^{-m-2}\epsilon$. **P** There is an open set $G \supseteq A_m$ such that $\lambda G \le 5^{-r} 2^{-m-2} \epsilon$. Now $\mathcal{I}'' = \{B : B \in \mathcal{I},\ B \subseteq G\}$ covers A_m, so there is a family $\langle \mathcal{J}_{mk} \rangle_{k<5^r}$ of disjoint countable subfamilies of $\mathcal{I}''$ such that $\mathcal{J}_m = \bigcup_{k<5^r} \mathcal{J}_{mk}$ covers A_m. For each k,

$$\sum_{B \in \mathcal{J}_{mk}} \lambda B = \lambda(\bigcup \mathcal{J}_{mk}) \le \lambda G,$$

so

$$\sum_{B \in \mathcal{J}_m} \lambda B \le 5^r \lambda G \le 2^{-m-2}\epsilon. \text{ } \mathbf{Q}$$

(iii) Setting $\mathcal{I}_1 = \mathcal{I}_0 \cup \bigcup_{m\in\mathbb{N}} \mathcal{J}_m$ we have a cover of A by members of $\mathcal{I}$, and

$$\sum_{B \in \mathcal{I}_1} \lambda B \le \sum_{B \in \mathcal{I}_0} \lambda B + \sum_{m=0}^{\infty} \sum_{B \in \mathcal{J}_m} \lambda B$$
$$\le \lambda^* A + \frac{1}{2}\epsilon + \sum_{m=0}^{\infty} 2^{-m-2}\epsilon = \lambda^* A + \epsilon.$$

472D Besicovitch's Density Theorem Let λ be any Radon measure on $\mathbb{R}^r$. Then, for any locally λ-integrable real-valued function f,

(a) $f(y)=\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}f d\lambda$,

(b) $\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|f(x)-f(y)|\lambda(dx)=0$

for λ-almost every $y\in\mathbb{R}^r$.

Remark The theorem asserts that, for λ-almost every y, limits of the form $\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\ldots$ are defined; in my usage, this includes the assertion that $\lambda B(y,\delta)\neq 0$ for all sufficiently small $\delta>0$.

proof (Compare 261C and 261E.)

(a) Let Z be the support of λ (411Nd); then Z is λ-conegligible and $\lambda B(y,\delta)>0$ whenever $y\in Z$ and $\delta>0$. For $q<q'$ in $\mathbb{Q}$ and $n\in\mathbb{N}$ set

$$A_{nqq'}=\{y:y\in Z\cap\operatorname{dom}f,\ \|y\|<n,\ f(y)\le q,\ \limsup_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}f d\lambda>q'\}.$$

Then $\lambda A_{nqq'}=0$. **P** Let $\epsilon>0$. Then there is an $\eta\in\,]0,\epsilon]$ such that $\int_F|f|d\lambda\le\epsilon$ whenever $F\subseteq B(\mathbf{0},n)$ and $\lambda F\le\eta$ (225A). Let E be a measurable envelope of $A_{nqq'}$ included in $\{y:y\in Z\cap\operatorname{dom}f,\ f(y)\le q,\ \|y\|<n\}$, and take an open set $G\supseteq E$ such that $G\subseteq B(\mathbf{0},n)$ and $\lambda(G\setminus E)\le\eta$ (256Bb again). Let $\mathcal{I}$ be the family of non-singleton closed balls $B\subseteq G$ such that $\int_B f\ge q'\lambda B$. Then every point of $A_{nqq'}$ is the centre of arbitrarily small members of $\mathcal{I}$, so there is a disjoint family $\mathcal{I}_0\subseteq\mathcal{I}$ such that $\lambda(A_{nqq'}\setminus\bigcup\mathcal{I}_0)=0$ (472C). Now $\lambda(E\setminus\bigcup\mathcal{I}_0)=0$ and $\lambda((\bigcup\mathcal{I}_0)\setminus E)\le\eta\le\epsilon$, so

$$\begin{aligned}q'\lambda E&\le q'\lambda(\bigcup\mathcal{I}_0)+\epsilon|q'|=\sum_{B\in\mathcal{I}_0}q'\lambda B+\epsilon|q'|\\&\le\sum_{B\in\mathcal{I}_0}\int_B f d\lambda+\epsilon|q'|=\int_{\bigcup\mathcal{I}_0}f d\lambda+\epsilon|q'|\\&\le\int_E f d\lambda+\epsilon(1+|q'|)\le q\lambda E+\epsilon(1+|q'|),\end{aligned}$$

and

$$(q'-q)\lambda^*A_{nqq'}=(q'-q)\lambda E\le(1+|q'|)\epsilon.$$

As ϵ is arbitrary, $\lambda^*A_{nqq'}=0$. **Q**

As n, q and q' are arbitrary,

$$\limsup_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}f\le f(y)$$

for λ-almost every $y\in Z$, therefore for λ-almost every $y\in\mathbb{R}^r$. Similarly, or applying the same argument to $-f$,

$$\liminf_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}f\ge f(y)$$

for λ-almost every y, and

$$\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}f\text{ exists }=f(y)$$

for λ-almost every y.

(b) Now, for each $q\in\mathbb{Q}$ set $g_q(x)=|f(x)-q|$ for $x\in\operatorname{dom}f$. By (a), we have a λ-conegligible set D such that

$$\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}g_q d\lambda=g_q(y)$$

for every $y\in D$ and $q\in\mathbb{Q}$. Now, if $y\in D$ and $\epsilon>0$, there is a $q\in\mathbb{Q}$ such that $|f(y)-q|\le\epsilon$, and a $\delta_0>0$ such that

$$\Bigl|\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}g_q d\lambda-g_q(y)\Bigr|\le\epsilon$$

whenever $0<\delta\le\delta_0$. But in this case

$$
\begin{aligned}
&\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|f(x)-f(y)|\lambda(dx)\\
&\qquad\le\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|f(x)-q|\lambda(dx)+\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|q-f(y)|\lambda(dx)\\
&\qquad\le 3\epsilon.
\end{aligned}
$$

As ϵ is arbitrary,

$$\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|f(x)-f(y)|\lambda(dx)=0;$$

as this is true for every $y\in D$, the theorem is proved.

***472E Proposition** Let λ, λ' be Radon measures on $\mathbb{R}^r$, and $G\subseteq\mathbb{R}^r$ an open set. Let Z be the support of λ, and for $x\in Z\cap G$ set

$$M(x)=\sup\{\frac{\lambda' B}{\lambda B}:B\subseteq G\text{ is a non-trivial ball with centre }x\}.$$

Then

$$\lambda\{x:x\in Z,\,M(x)\ge t\}\le\frac{5^r}{t}\lambda' G$$

for every $t>0$.

proof The function $M:Z\to[0,\infty]$ is lower semi-continuous. **P** If $M(x)>t\ge 0$, there is a $\delta>0$ such that $B(x,\delta)\subseteq G$ and $\lambda' B(x,\delta)>t\lambda B(x,\delta)$. Because λ is a Radon measure, there is an open set $V\supseteq B(x,\delta)$ such that $V\subseteq G$ and $\lambda' B(x,\delta)>t\lambda V$; because $B(x,\delta)$ is compact, there is an $\eta>0$ such that $B(x,\delta+2\eta)\subseteq V$. Now if $y\in Z$ and $\|y-x\|\le\eta$,

$$B(x,\delta)\subseteq B(y,\delta+\eta)\subseteq V,$$

so $\lambda' B(y,\delta+\eta)>t\lambda B(y,\delta+\eta)$ and $M(y)>t$. **Q**

In particular, $H_t=\{x:x\in Z\cap G,\,M(x)>t\}$ is always measured by λ. Now, given $t>0$, let $\mathcal{I}$ be the set of non-trivial closed balls $B\subseteq G$ such that $\lambda' B>t\lambda B$. By 472B, there is a family $\langle\mathcal{I}_k\rangle_{k<5^r}$ of countable disjoint subsets of $\mathcal{I}$ such that $\bigcup_{k<5^r}\mathcal{I}_k$ covers H_t. So

$$\lambda H_t\le\sum_{k=0}^{5^r-1}\sum_{B\in\mathcal{I}_k}\lambda B\le\frac{1}{t}\sum_{k=0}^{5^r-1}\sum_{B\in\mathcal{I}_k}\lambda' B\le\frac{5^r}{t}\lambda' G,$$

as claimed.

***472F Theorem** Let λ be a Radon measure on $\mathbb{R}^r$, and $f\in\mathcal{L}^p(\lambda)$ any function, where $1<p<\infty$. Let Z be the support of λ, and for $x\in Z$ set $f^*(x)=\sup_{\delta>0}\frac{1}{\lambda B(x,\delta)}\int_{B(x,\delta)}|f|d\lambda$. Then f^* is lower semi-continuous, and $\|f^*\|_p\le 2\bigl(\frac{5^r p}{p-1}\bigr)^{1/p}\|f\|_p$.

proof (a) Replacing f by $|f|$ if necessary, we may suppose that $f\ge 0$. Z is λ-conegligible, so that f^* is defined λ-almost everywhere. Next, f^* is lower semi-continuous. **P** I repeat an idea from the proof of 472E. If $f^*(x)>t\ge 0$, there is a $\delta>0$ such that $\int_{B(x,\delta)}|f|d\lambda>t\lambda B(x,\delta)$. Because λ is a Radon measure, there is an open set $V\supseteq B(x,\delta)$ such that $\int_{B(x,\delta)}|f|d\lambda>t\lambda V$; because $B(x,\delta)$ is compact, there is an $\eta>0$ such that $B(x,\delta+2\eta)\subseteq V$; and now $f^*(y)>t$ for every $y\in Z\cap B(x,\eta)$. **Q**

(b) For $t>0$, set $H_t=\{x:x\in Z,\,f^*(x)>t\}$ and $F_t=\{x:x\in\operatorname{dom}f,\,f(x)\ge t\}$. Then

$$\lambda H_t\le\frac{2\cdot 5^r}{t}\int_{F_{t/2}}fd\lambda.$$

P Set $g=f\times\chi F_{t/2}$. Because $(\frac{t}{2})^p\lambda F_{t/2}\le\|f\|_p^p$ is finite, $\lambda F_{t/2}$ is finite, $\chi F_{t/2}\in\mathcal{L}^q(\lambda)$ (where $\frac{1}{p}+\frac{1}{q}=1$) and $g\in\mathcal{L}^1(\lambda)$ (244Eb). Let λ' be the indefinite-integral measure defined by g over λ (234J); then λ' is totally finite, and is a Radon measure (416S). Set

$$M(x)=\sup\{\frac{\lambda' B}{\lambda B}:B\subseteq\mathbb{R}^r\text{ is a non-trivial ball with centre }x\}$$

for $x \in Z$. Then $f^*(x) \le M(x) + \frac{t}{2}$ for every $x \in Z$, just because

$$\int_B f d\lambda \le \frac{t}{2}\lambda B + \int_B g\, d\lambda = \frac{t}{2}\lambda B + \lambda' B$$

for every closed ball B. Accordingly

$$\lambda H_t \le \lambda\{x : M(x) > \tfrac{t}{2}\} \le \tfrac{2\cdot 5^r}{t}\lambda'\mathbb{R}^r$$

(by 472E)

$$= \tfrac{2\cdot 5^r}{t}\int_{F_{t/2}} f d\lambda. \quad \mathbf{Q}$$

(c) As in part (c) of the proof of 286A, we now have

$$\begin{aligned}\int (f^*)^p d\lambda &= \int_0^\infty \lambda\{x : f^*(x)^p > t\}dt = p\int_0^\infty t^{p-1}\lambda\{x : f^*(x) > t\}dt \\ &\le 2\cdot 5^r p\int_0^\infty t^{p-2}\int_{F_{t/2}} f d\lambda dt = 2\cdot 5^r p\int_{\mathbb{R}^r} f(x)\int_0^{2f(x)} t^{p-2}dt\lambda(dx) \\ &= 2\cdot 5^r p\int_{\mathbb{R}^r}\tfrac{2^{p-1}}{p-1}f(x)^p\lambda(dx) = \tfrac{2^p 5^r p}{p-1}\int f^p d\lambda.\end{aligned}$$

Taking pth roots, we have the result.

472X Basic exercises (a) Show that if λ, λ' are Radon measures on $\mathbb{R}^r$ which agree on closed balls, they are equal. (Cf. 466Xj.)

(b) Let λ be a Radon measure on $\mathbb{R}^r$. Let $A \subseteq \mathbb{R}^r$ be a non-empty set, and $\epsilon > 0$. Show that there is a sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ of closed balls in $\mathbb{R}^r$, all of radius at most ϵ and with centres in A, such that $A \subseteq \bigcup_{n\in\mathbb{N}} B_n$ and $\sum_{n=0}^\infty \lambda B_n \le \lambda^* A + \epsilon$.

(c) Let λ be a non-zero Radon measure on $\mathbb{R}^r$ and Z its support. Show that we have a lower density ϕ (definition: 341C) for the subspace measure λ_Z defined by setting $\phi E = \{x : x \in Z,\ \lim_{\delta\downarrow 0}\frac{\lambda(E\cap B(x,\delta))}{\lambda B(x,\delta)} = 1\}$ whenever λ_Z measures E.

(d) Let λ be a Radon measure on $\mathbb{R}^r$, and f a locally λ-integrable function. Show that $E = \{y : g(y) = \lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)} f\, d\lambda$ is defined in $\mathbb{R}\}$ is a Borel set, and that $g : E \to \mathbb{R}$ is Borel measurable.

472Y Further exercises (a)(i) Let $\mathcal{I}$ be a finite family of intervals (open, closed or half-open) in $\mathbb{R}$. Show that there are subfamilies $\mathcal{I}_0$, $\mathcal{I}_1 \subseteq \mathcal{I}$, both disjoint, such that $\mathcal{I}_0 \cup \mathcal{I}_1$ covers $\bigcup\mathcal{I}$. (*Hint*: induce on $\#(\mathcal{I})$.) Show that this remains true if any totally ordered set is put in place of $\mathbb{R}$. (ii) Show that if $\mathcal{I}$ is any family of non-empty intervals in $\mathbb{R}$ such that none contains the centre of any other, then $\mathcal{I}$ is expressible as $\mathcal{I}_0 \cup \mathcal{I}_1$ where both $\mathcal{I}_0$ and $\mathcal{I}_1$ are disjoint.

(b) Let $m = m(r)$ be the largest number such that there are $u_1, \dots , u_m \in \mathbb{R}^r$ such that $\|u_i\| = 1$ for every i and $\|u_i - u_j\| \ge 1$ for all $i \ne j$. Let $A \subseteq \mathbb{R}^r$ be a bounded set and $x \mapsto \delta_x : A \to]0,\infty[$ a bounded function; set $B_x = B(x, \delta_x)$ for $x \in A$. (i) Show that $m < 3^r$. (ii) Show that there is an $\epsilon \in]0, \frac{1}{10}]$ such that whenever $\|u_0\| = \ldots = \|u_m\| = 1$ there are distinct i, $j \le m$ such that $u_i \,\mathbf{.}\, u_j > \frac{1}{2}(1 + \epsilon)$. (iii) Suppose that u, $v \in \mathbb{R}^r$ are such that $\frac{1}{3} \le \|u\| \le 1$, $\|v\| \le 1 + \epsilon$ and $\|u - v\| > 1$. Show that the angle $u\widehat{\mathbf{0}}v$ has cosine at most $\frac{1}{2}(1 + \epsilon)$. (*Hint*: maximise $\frac{a^2+b^2-c^2}{2ab}$ subject to $\frac{1}{3} \le a \le 1$, $b \le 1 + \epsilon$ and $c \ge 1$.) (iv) Suppose that $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in A such that $x_n \notin B_{x_i}$ for $i < n$ and $(1+\epsilon)\delta_{x_n} \ge \sup\{\delta_x : x \in A\setminus\bigcup_{i<n} B_{x_i}\}$ for every n. Show that $A \subseteq \bigcup_{n\in\mathbb{N}} B_{x_n}$. (v) Take $y \in \mathbb{R}^r$. Show that there is at most one n such that $\|y - x_n\| \le \frac{1}{3}\delta_{x_n}$. (vi) Show that if $i < j$, $\frac{1}{3}\delta_{x_i} \le \|y - x_i\| \le \delta_{x_i}$ and $\|y - x_j\| \le \delta_j$ then the cosine of the angle $x_i\widehat{y}x_j$ is at most $\frac{1}{2}(1 + \epsilon)$. (vii) Show that $\#(\{i : y \in B_{x_i}\}) \le m + 1$.

Hence show that if $\mathcal{I}$ is any family of non-trivial closed balls such that every point of A is the centre of some member of $\mathcal{I}$, then there is a countable $\mathcal{I}_0 \subseteq \mathcal{I}$, covering A, such that no point of $\mathbb{R}^r$ belongs to more than 3^r members of $\mathcal{I}_0$.

(c) Use 472Yb to prove an alternative version of 472B, but with the constant 9^r+1 in place of 5^r.

(d) Let $A \subseteq \mathbb{R}^r$ be a bounded set, and $\mathcal{I}$ a family of non-trivial closed balls in $\mathbb{R}^r$ such that whenever $x \in A$ and $\epsilon > 0$ there is a ball $B(y,\delta) \in \mathcal{I}$ such that $\|x-y\| \le \epsilon\delta$. Show that there is a family $\langle \mathcal{I}_k \rangle_{k<5^r}$ of subsets of $\mathcal{I}$ such that each $\mathcal{I}_k$ is disjoint and $\bigcup_{k<5^r} \mathcal{I}_k$ covers A.

(e) Give an example of a strictly positive Radon probability measure μ on a compact metric space (X,ρ) for which there is a Borel set $E \subseteq X$ such that

$$\liminf_{\delta\downarrow 0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} = 0, \quad \limsup_{\delta\downarrow 0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} = 1$$

for every $x \in X$.

(f) Let λ be a Radon measure on $\mathbb{R}^r$, and f a λ-integrable real-valued function. Show that $\sup_{\delta>0} \frac{1}{\lambda B(x,\delta)} \int_{B(x,\delta)} |f| d\lambda$ is defined and finite for λ-almost every $x \in \mathbb{R}^r$.

(g) Let λ, λ' be Radon measures on $\mathbb{R}^r$. (i) Show that $g(x) = \lim_{\delta\downarrow 0} \frac{\lambda' B(x,\delta)}{\lambda B(x,\delta)}$ is defined in $\mathbb{R}$ for λ-almost every x. (ii) Setting $\lambda_0 = \sup_{n\in\mathbb{N}} \lambda' \wedge n\lambda$ in the cone of Radon measures on $\mathbb{R}^r$ (437Yi), show that g is a Radon-Nikodým derivative of λ_0 with respect to λ. (*Hint*: show that if λ and λ' are mutually singular then $g = 0$ λ-a.e.)

472 Notes and comments I gave primacy to the 'weak' Vitali's theorem in 261B because I think it is easier than the 'strong' form in 472C, it uses the same ideas as the original one-dimensional theorem in 221A, and it is adequate for the needs of Volume 2. Any proper study of general measures on $\mathbb{R}^r$, however, will depend on the ideas in 472A-472C. You will see that in 472B, as in other forms of Vitali's theorem, there is a key step in which a sequence is chosen greedily. This time we must look much more carefully at the geometry of $\mathbb{R}^r$ because we can no longer rely on a measure to tell us what is happening. (Though you will observe that I still use the elementary properties of Euclidean volume in the argument of 472A.) Once we have reached 472C, however, we are in a position to repeat all the arguments of 261C-261E in much greater generality (472D), and, as a bonus, can refine 261F (472Xb). For more in this direction see Mattila 95 and Federer 69, §2.8.

It is natural to ask whether the constant '5^r' in 472B is best possible. The argument of 472A is derived from Sullivan 94, where a more thorough analysis is given. It seems that even for $r=2$ the best constant is unknown. (For $r=1$, the best constant is 2; see 472Ya.) Note that even for finite families $\mathcal{I}$ we should have to find the colouring number of a graph (counting two balls as linked if they intersect), so it may well be a truly difficult problem. The method in 472B amounts to using the greedy colouring algorithm after ordering the balls by size, and one does not expect such approaches to give exact colouring numbers. Of course the questions addressed here depend only on the existence of *some* function of r to do the job.

An alternative argument runs through a kind of pointwise version of 472A (472Yb-472Yc). It gives a worse constant but is attractive in other ways. For many of the applications of 472C, the result of 472Yb is already sufficient.

The constant $2\bigl(\frac{5^r p}{p-1}\bigr)^{1/p}$ in 472F makes no pretence to be 'best', or even 'good'. The only reason for giving a formula at all is to emphasize the remarkable fact that it does not depend on the measure λ. The theorems of this section are based on the metric geometry of Euclidean space, not on any special properties of Lebesgue measure. The constants *do* depend on the dimension, so that even in Hilbert space (for instance) we cannot expect any corresponding results.

473 Poincaré's inequality

In this section I embark on the main work of the first half of the chapter, leading up to the Divergence Theorem in §475. I follow the method in Evans & Gariepy 92. The first step is to add some minor results on differentiable and Lipschitz functions to those already set out in §262 (473B-473C). Then we need to know something about convolution products (473D), extending ideas in §§256 and 444; in particular, it will be convenient to have a fixed sequence $\langle \tilde{h}_n \rangle_{n\in\mathbb{N}}$ of smoothing functions with some useful special properties (473E).

The new ideas of the section begin with the Gagliardo-Nirenberg-Sobolev inequality, relating $\|f\|_{r/(r-1)}$ to $\int \|\operatorname{grad} f\|$. In its simplest form (473H) it applies only to functions with compact support; we need to work much harder to get results which we can use to estimate $\int_B |f|^{r/(r-1)}$ in terms of $\int_B \|\operatorname{grad} f\|$ and $\int_B |f|$ for balls B (473I, 473K).

473A Notation For the next three sections, $r \ge 2$ will be a fixed integer. For $x \in \mathbb{R}^r$ and $\delta \ge 0$, $B(x,\delta) = \{y : \|y-x\| \le \delta\}$ will be the closed ball with centre x and radius δ. I will write $\partial B(x,\delta)$ for the boundary of $B(x,\delta)$, the sphere $\{y : \|y-x\| = \delta\}$. $S_{r-1} = \partial B(\mathbf{0},1)$ will be the unit sphere. As in Chapter 26, I will use Greek letters to represent coordinates of vectors, so that $x = (\xi_1,\dots,\xi_r)$, etc.

μ will always be Lebesgue measure on $\mathbb{R}^r$. $\beta_r = \mu B(\mathbf{0},1)$ will be the r-dimensional volume of the unit ball, that is,

$$\begin{aligned}\beta_r &= \frac{\pi^k}{k!} \text{ if } r = 2k \text{ is even,}\\ &= \frac{2^{2k+1}k!\pi^k}{(2k+1)!} \text{ if } r = 2k+1 \text{ is odd}\end{aligned}$$

(252Q). ν will be normalized Hausdorff $(r-1)$-dimensional measure on $\mathbb{R}^r$, that is, $\nu = 2^{-r+1}\beta_{r-1}\mu_{H,r-1}$, where $\mu_{H,r-1}$ is $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$ as described in §264. Recall from 265F and 265H that $\nu S_{r-1} = 2\pi\beta_{r-2} = r\beta_r$ (counting β_0 as 1).

473B Differentiable functions (a) Recall from §262 that a function ϕ from a subset of $\mathbb{R}^r$ to $\mathbb{R}^s$ (where $s \ge 1$) is differentiable at $x \in \mathbb{R}^r$, with derivative an $s \times r$ matrix T, if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|\phi(y) - \phi(x) - T(y-x)\| \le \epsilon\|y-x\|$ whenever $\|y-x\| \le \delta$; this includes the assertion that $B(x,\delta) \subseteq \operatorname{dom}\phi$. In this case, the coefficients of T are the partial derivatives $\frac{\partial\phi_j}{\partial\xi_i}(x)$ at x, where $\phi_1,\dots,\phi_s$ are the coordinate functions of ϕ, and $\frac{\partial}{\partial\xi_i}$ represents partial differentiation with respect to the ith coordinate (262Ic).

(b) When $s = 1$, so that we have a real-valued function f defined on a subset of $\mathbb{R}^r$, I will write $(\operatorname{grad} f)(x)$ for the derivative of f at x, the **gradient** of f. If we strictly adhere to the language of (a), $\operatorname{grad} f$ is a $1 \times r$ matrix $\begin{pmatrix}\frac{\partial f}{\partial\xi_1} & \dots & \frac{\partial f}{\partial\xi_r}\end{pmatrix}$; but it is convenient to treat it as a vector, so that $\operatorname{grad} f(x)$ (when defined) belongs to $\mathbb{R}^r$, and we can speak of $y\,.\operatorname{grad} f(x)$ rather than $(\operatorname{grad} f(x))(y)$, etc.

(c) Chain rule for functions of many variables I find that I have not written out the following basic fact. Let $\phi : A \to \mathbb{R}^s$ and $\psi : B \to \mathbb{R}^p$ be functions, where $A \subseteq \mathbb{R}^r$ and $B \subseteq \mathbb{R}^s$. If $x \in A$ is such that ϕ is differentiable at x, with derivative S, and ψ is differentiable at $\phi(x)$, with derivative T, then the composition $\psi\phi$ is differentiable at x, with derivative TS.

P Recall that if we regard S and T as linear operators, they have finite norms (262H). Given $\epsilon > 0$, let $\eta > 0$ be such that $\eta\|T\| + \eta(\|S\|+\eta) \le \epsilon$. Let δ_1, $\delta_2 > 0$ be such that $\phi(y)$ is defined and $\|\phi(y)-\phi(x)-S(y-x)\| \le \eta\|y-x\|$ whenever $\|y-x\| \le \delta_1$, and $\psi(z)$ is defined and $\|\psi(z)-\psi\phi(x)-T(z-\phi(x))\| \le \eta\|z-\phi(x)\|$ whenever $\|z-\phi(x)\| \le \delta_2$. Set $\delta = \min(\delta_1, \frac{\delta_2}{\eta+\|S\|}) > 0$. If $\|y-x\| \le \delta$, then $\phi(y)$ is defined and

$$\|\phi(y)-\phi(x)\| \le \|S(y-x)\| + \|\phi(y)-\phi(x)-S(y-x)\| \le (\|S\|+\eta)\|y-x\| \le \delta_2,$$

so $\psi\phi(y)$ is defined and

$$\begin{aligned}&\|\psi\phi(y) - \psi\phi(x) - TS(y-x)\|\\ &\qquad\le \|\psi\phi(y) - \psi\phi(x) - T(\phi(y)-\phi(x))\| + \|T\|\|\phi(y)-\phi(x)-S(y-x)\|\\ &\qquad\le \eta\|\phi(y)-\phi(x)\| + \|T\|\eta\|y-x\|\\ &\qquad\le \eta(\|S\|+\eta)\|y-x\| + \|T\|\eta\|y-x\| \le \epsilon\|y-x\|;\end{aligned}$$

as ϵ is arbitrary, $\psi\phi$ is differentiable at x with derivative TS. **Q**

(d) It follows that if f and g are real-valued functions defined on a subset of $\mathbb{R}^r$, and $x \in \operatorname{dom} f \cap \operatorname{dom} g$ is such that $(\operatorname{grad} f)(x)$ and $(\operatorname{grad} g)(x)$ are both defined, then $\operatorname{grad}(f \times g)(x)$ is defined and equal to $f(x)\operatorname{grad} g(x) + g(x)\operatorname{grad} f(x)$. **P** Set $\phi(y) = \begin{pmatrix} f(y)\\ g(y)\end{pmatrix}$ for $y \in \operatorname{dom} f \cap \operatorname{dom} g$; then ϕ is differentiable at x with derivative the $2 \times r$ matrix $\begin{pmatrix}\operatorname{grad} f(x)\\ \operatorname{grad} g(x)\end{pmatrix}$ (262Ib). Set $\psi(z) = \zeta_1\zeta_2$ for $z = (\zeta_1,\zeta_2) \in \mathbb{R}^2$; then ψ is differentiable everywhere, with derivative the 1×2 matrix $(\,\zeta_2 \quad \zeta_1\,)$. So $f \times g = \psi\phi$ is differentiable at x with derivative

$$(g(x) \quad f(x)) \begin{pmatrix} \operatorname{grad} f(x) \\ \operatorname{grad} g(x) \end{pmatrix} = g(x) \operatorname{grad} f(x) + f(x) \operatorname{grad} g(x). \quad \mathbf{Q}$$

(e) Let D be a subset of $\mathbb{R}^r$ and $\phi : D \to \mathbb{R}^s$ any function. Set $D_0 = \{x : x \in D,\, \phi \text{ is differentiable at } x\}$. Then the derivative of ϕ, regarded as a function from D_0 to $\mathbb{R}^{rs}$, is (Lebesgue) measurable. **P** Use 262P; the point is that, writing $T(x)$ for the derivative of ϕ at x, $T(x)$ is surely a derivative of $\phi\restriction D_0$, relative to D_0, at every point of D_0. **Q** (See also 473Ya.)

(f) A function $\phi : \mathbb{R}^r \to \mathbb{R}^s$ is **smooth** if it is differentiable arbitrarily often; that is, if all its repeated partial derivatives

$$\frac{\partial^m \phi_j}{\partial \xi_{i_1} \ldots \partial \xi_{i_m}}$$

are defined and continuous everywhere on $\mathbb{R}^r$. I will write $\mathcal{D}$ for the family of real-valued functions on $\mathbb{R}^r$ which are smooth and have compact support.

473C Lipschitz functions (a) If f and g are bounded real-valued Lipschitz functions, defined on any subsets of $\mathbb{R}^r$, then $f \times g$, defined on $\operatorname{dom} f \cap \operatorname{dom} g$, is Lipschitz. **P** Let γ_f, M_f, γ_g and M_g be such that $|f(x)| \le M_f$ and $|f(x) - f(y)| \le \gamma_f \|x - y\|$ for all x, $y \in \operatorname{dom} f$, while $|g(x)| \le M_g$ and $|g(x) - g(y)| \le \gamma_g \|x - y\|$ for all x, $y \in \operatorname{dom} g$. Then for any x, $y \in \operatorname{dom} f \cap \operatorname{dom} g$,

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\le |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\ &\le (M_f \gamma_g + M_g \gamma_f)\|x - y\|. \end{aligned}$$

So $M_f \gamma_g + M_g \gamma_f$ is a Lipschitz constant for $f \times g$. **Q**

(b) Suppose that F_1, $F_2 \subseteq \mathbb{R}^r$ are closed sets with convex union C. Let $f : C \to \mathbb{R}$ be a function such that $f\restriction F_1$ and $f\restriction F_2$ are both Lipschitz. Then f is Lipschitz. **P** For each j, let γ_j be a Lipschitz constant for $f\restriction F_j$, and set $\gamma = \max(\gamma_1, \gamma_2)$, so that γ is a Lipschitz constant for both $f\restriction F_1$ and $f\restriction F_2$. Take any x, $y \in C$. If both belong to the same F_j, then $|f(x) - f(y)| \le \gamma\|x - y\|$. If $x \in F_j$ and $y \notin F_j$, then y must belong to F_{3-j}, and $(1-t)x + ty \in F_1 \cup F_2$ for every $t \in [0,1]$, because C is convex. Set $t_0 = \sup\{t : t \in [0,1],\, (1-t)x + ty \in F_j\}$, $z = (1-t_0)x + t_0 y$; then $z \in F_1 \cap F_2$, because both are closed, so

$$|f(x) - f(y)| \le |f(x) - f(z)| + |f(z) - f(y)| \le \gamma\|x - z\| + \gamma\|z - y\| = \gamma\|x - y\|.$$

As x and y are arbitrary, γ is a Lipschitz constant for f. **Q**

(c) Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ is Lipschitz. Recall that by Rademacher's theorem (262Q), $\operatorname{grad} f$ is defined almost everywhere. All the partial derivatives of f are (Lebesgue) measurable, by 473Be, so $\operatorname{grad} f$ is (Lebesgue) measurable on its domain. If γ is a Lipschitz constant for f, $\|\operatorname{grad} f(x)\| \le \gamma$ whenever $\operatorname{grad} f(x)$ is defined. **P** If $z \in \mathbb{R}^r$, then

$$\lim_{t\downarrow 0} \frac{1}{t}|f(x + tz) - f(x) - tz \,\textbf{.}\, \operatorname{grad} f(x)| = 0,$$

so

$$|z \,\textbf{.}\, \operatorname{grad} f(x)| = \lim_{t\downarrow 0} \frac{1}{t}|f(x + tz) - f(x)| \le \gamma\|z\|;$$

as z is arbitrary, $\|\operatorname{grad} f(x)\| \le \gamma$. **Q**

(d) Conversely, if $f : \mathbb{R}^r \to \mathbb{R}$ is differentiable and $\|\operatorname{grad} f(x)\| \le \gamma$ for every x, then γ is a Lipschitz constant for f. **P** Take x, $y \in \mathbb{R}^r$. Set $g(t) = f((1-t)x + ty)$ for $t \in \mathbb{R}$. The function $t \mapsto (1-t)x + ty : \mathbb{R} \to \mathbb{R}^r$ is everywhere differentiable, with constant derivative $y - x$, so by 473Bc g is differentiable, with derivative $g'(t) = (y - x)\,\textbf{.}\, \operatorname{grad} f((1-t)x + ty)$ for every t; in particular, $|g'(t)| \le \gamma\|y - x\|$ for every t. Now, by the Mean Value Theorem, there is a $t \in [0,1]$ such that $g(1) - g(0) = g'(t)$, so that $|f(y) - f(x)| = |g'(t)| \le \gamma\|y - x\|$. As x and y are arbitrary, f is γ-Lipschitz. **Q**

(e) Note that if $f \in \mathcal{D}$ then all its partial derivatives are continuous functions with compact support, so are bounded (436Ia), and f is Lipschitz as well as bounded, by (d) here.

(f)(i) If $D \subseteq \mathbb{R}^r$ is bounded and $f : D \to \mathbb{R}$ is Lipschitz, then there is a Lipschitz function $g : \mathbb{R}^r \to \mathbb{R}$, with compact support, extending f. **P** By 262Bb there is a Lipschitz function $f_1 : \mathbb{R}^r \to \mathbb{R}$ which extends f. Let $\gamma > 0$ be such that $D \subseteq B(\mathbf{0}, \gamma)$ and γ is a Lipschitz constant for f_1; set $M = |f_1(0)| + \gamma^2$; then $|f_1(x)| \le M$ for every $x \in D$, so if we set $f_2(x) = \operatorname{med}(-M, f_1(x), M)$ for $x \in \mathbb{R}^r$, f_2 is a bounded Lipschitz function extending f. Set $f_3(x) = \operatorname{med}(0, 1 + \gamma - \|x\|, 1)$ for $x \in \mathbb{R}^r$; then f_3 is a bounded Lipschitz function with compact support. By (a), $g = f_3 \times f_2$ is Lipschitz, and $g : \mathbb{R}^r \to \mathbb{R}$ is a function with compact support extending f. **Q**

(ii) It follows that if $D \subseteq \mathbb{R}^r$ is bounded and $f : D \to \mathbb{R}^s$ is Lipschitz, then there is a Lipschitz function $g : \mathbb{R}^r \to \mathbb{R}^s$, with compact support, extending f. **P** By 262Ba, we need only apply (i) to each coordinate of f. **Q**

473D Smoothing by convolution We shall need a miscellany of facts, many of them special cases of results in §§255 and 444, concerning convolutions on $\mathbb{R}^r$. Recall that I write $(f * g)(x) = \int f(y)g(x - y)\mu(dy)$ whenever f and g are real-valued functions defined almost everywhere in $\mathbb{R}^r$ and the integral is defined, and that $f * g = g * f$ (255Fb, 444Og). Now we have the following.

Lemma Suppose that f and g are Lebesgue measurable real-valued functions defined μ-almost everywhere in $\mathbb{R}^r$.

(a) If f is integrable and g is essentially bounded, then their convolution $f * g$ is defined everywhere in $\mathbb{R}^r$ and uniformly continuous, and $\|f * g\|_\infty \le \|f\|_1 \operatorname{ess\,sup} |g|$.

(b) If f is locally integrable and g is bounded and has compact support, then $f * g$ is defined everywhere in $\mathbb{R}^r$ and is continuous.

(c) If f and g are defined everywhere in $\mathbb{R}^r$ and $x \in \mathbb{R}^r \setminus (\{y : f(y) \ne 0\} + \{z : g(z) \ne 0\})$, then $(f * g)(x)$ is defined and equal to 0.

(d) If f is integrable and g is bounded, Lipschitz and defined everywhere, then $f * \operatorname{grad} g$ and $\operatorname{grad}(f * g)$ are defined everywhere and equal, where $f * \operatorname{grad} g = (f * \frac{\partial g}{\partial \xi_1}, \dots, f * \frac{\partial g}{\partial \xi_r})$. Moreover, $f * g$ is Lipschitz.

(e) If f is locally integrable, and $g \in \mathcal{D}$, then $f * g$ is defined everywhere and is smooth.

(f) If f is essentially bounded and $g \in \mathcal{D}$, then all the derivatives of $f * g$ are bounded, and $f * g$ is Lipschitz.

(g) If f is integrable and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a bounded measurable function with components $\phi_1, \dots, \phi_r$, and we write $(f * \phi)(x) = ((f * \phi_1)(x), \dots, (f * \phi_r)(x))$, then $\|(f * \phi)(x)\| \le \|f\|_1 \sup_{y \in \mathbb{R}^r} \|\phi(y)\|$ for every $x \in \mathbb{R}^r$.

proof (a) See 255K.

(b) Suppose that $g(y) = 0$ when $\|y\| \ge n$. Given $x \in \mathbb{R}^r$, set $\tilde f = f \times \chi B(x, n + 1)$. Then $\tilde f * g$ is defined everywhere and continuous, by (a), while $(f * g)(z) = (\tilde f * g)(z)$ whenever $z \in B(x, 1)$; so $f * g$ is defined everywhere in $B(x, 1)$ and is continuous at x.

(c) We have only to note that $f(y)g(x - y) = 0$ for every y.

(d) Let γ be a Lipschitz constant for g. We know that $\operatorname{grad} g$ is defined almost everywhere, is measurable, and that $\|\operatorname{grad} g(x)\| \le \gamma$ whenever it is defined (473Cc); so $(f * \operatorname{grad} g)(x)$ is defined for every x, by (a) here. Fix $x \in \mathbb{R}^r$. If y, $z \in \mathbb{R}^r$ set

$$\theta(y, z) = \frac{1}{\|z\|}\bigl(g(x - y + z) - g(x - y) - z \,\boldsymbol{.}\, \operatorname{grad} g(x - y)\bigr)$$

whenever this is defined. Then $|\theta(y, z)| \le 2\gamma$ whenever it is defined. Now suppose that $\langle z_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathbb{R}^r \setminus \{\mathbf{0}\}$ converging to $\mathbf{0}$. Then $\lim_{n \to \infty} \theta(y, z_n) = 0$ whenever $\operatorname{grad} g(x - y)$ is defined, which almost everywhere. So $\lim_{n \to \infty} \int f(y)\theta(y, z_n)\mu(dy) = 0$, by Lebesgue's Dominated Convergence Theorem. But this means that

$$\frac{1}{\|z_n\|}\bigl((f * g)(x + z_n) - (f * g)(x) - ((f * \operatorname{grad} g)(x)) \,\boldsymbol{.}\, z_n\bigr) \to 0$$

as $n \to \infty$. As $\langle z_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\operatorname{grad}(f * g)(x)$ is defined and is equal to $(f * \operatorname{grad} g)(x)$.

Now $\operatorname{grad} g$ is bounded, because g is Lipschitz, so $\operatorname{grad}(f * g) = f * \operatorname{grad} g$ also is bounded, by (a), and $f * g$ must be Lipschitz (473Cd).

(e) By (b), $f * g$ is defined everywhere and is continuous. Now, for any $i \le r$, $\frac{\partial}{\partial \xi_i}(f * g) = f * \frac{\partial g}{\partial \xi_i}$ everywhere. **P** Let $n \in \mathbb{N}$ be such that $g(y) = 0$ if $\|y\| \ge n$. Given $x \in \mathbb{R}^r$, set $\tilde f = f \times \chi B(x, n + 1)$. Then $(f * g)(z) = (\tilde f * g)(z)$ for every $z \in B(x, 1)$, so that

$$\frac{\partial(f*g)}{\partial\xi_i}(x)=\frac{\partial(\tilde f*g)}{\partial\xi_i}(x)=(\tilde f*\frac{\partial g}{\partial\xi_i})(x)$$

(by (d))

$$=(f*\frac{\partial g}{\partial\xi_i})(x)$$

(because of course $\frac{\partial g}{\partial\xi_i}$ is also zero outside $B(\mathbf{0},n)$). **Q** Inducing on k,

$$\frac{\partial^k}{\partial\xi_{i_1}\ldots\partial\xi_{i_k}}(f*g)(x)=(f*\frac{\partial^k g}{\partial\xi_{i_1}\ldots\partial\xi_{i_k}})(x)$$

for every $x\in\mathbb{R}^r$ and every $i_1,\ldots,i_k$; so we have the result.

(f) The point is just that all the partial derivatives of g, being smooth functions with compact support, are integrable, and that

$$|\frac{\partial}{\partial\xi_i}(f*g)(x)|=|(f*\frac{\partial g}{\partial\xi_i})(x)|\le\|f\|_\infty\|\frac{\partial g}{\partial\xi_i}\|_1$$

for every x and every $i\le r$. Inducing on the order of D, we see that $D(f*g)=f*Dg$ and $\|D(f*g)\|_\infty\le\|f\|_\infty\|Dg\|_1$, so that $D(f*g)$ is bounded, for any partial differential operator D. In particular, $\operatorname{grad}(f*g)$ is bounded, so that $f*g$ is Lipschitz, by 473Cd.

(g) If x, $z\in\mathbb{R}^r$, then

$$\begin{aligned}z\mathbin{.}(f*\phi)(x)&=\sum_{i=1}^r\zeta_i(f*\phi_i)(x)=\int f(y)\sum_{i=1}^r\zeta_i\phi_i(x-y)\mu(dy)\\&\le\int|f(y)||\sum_{i=1}^r\zeta_i\phi_i(x-y)|\mu(dy)\\&\le\int|f(y)|\|z\|\|\phi(x-y)\|\mu(dy)\le\|z\|\|f\|_1\sup_{y\in\mathbb{R}^r}\|\phi(y)\|.\end{aligned}$$

As z is arbitrary, $\|(f*\phi)(x)\|\le\|f\|_1\sup_{y\in\mathbb{R}^r}\|\phi(y)\|$.

473E Lemma (a) Define $h:\mathbb{R}\to[0,1]$ by setting $h(t)=\exp(\frac{1}{t^2-1})$ for $|t|<1$, 0 for $|t|\ge1$. Then h is smooth, and $h'(t)\le0$ for $t\ge0$.

(b) For $n\ge1$, define $\tilde h_n:\mathbb{R}^r\to\mathbb{R}$ by setting

$$\alpha_n=\int h((n+1)^2\|x\|^2)\mu(dx),\quad \tilde h_n(x)=\frac{1}{\alpha_n}h((n+1)^2\|x\|^2)$$

for every $x\in\mathbb{R}^r$. Then $\tilde h_n\in\mathcal{D}$, $\tilde h_n(x)\ge0$ for every x, $\tilde h_n(x)=0$ if $\|x\|\ge\frac{1}{n+1}$, and $\int\tilde h_n d\mu=1$.

(c) If $f\in\mathcal{L}^0(\mu)$, then $\lim_{n\to\infty}(f*\tilde h_n)(x)=f(x)$ for every $x\in\operatorname{dom}f$ at which f is continuous.

(d) If $f:\mathbb{R}^r\to\mathbb{R}$ is uniformly continuous (in particular, if it is either Lipschitz or a continuous function with compact support), then $\lim_{n\to\infty}\|f-f*\tilde h_n\|_\infty=0$.

(e) If $f\in\mathcal{L}^0(\mu)$ is locally integrable, then $f(x)=\lim_{n\to\infty}(f*\tilde h_n)(x)$ for μ-almost every $x\in\mathbb{R}^r$.

(f) If $f\in\mathcal{L}^p(\mu)$, where $1\le p<\infty$, then $\lim_{n\to\infty}\|f-f*\tilde h_n\|_p=0$.

proof (a) Set $h_0(t)=\exp(-\frac{1}{t})$ for $t>0$, 0 for $t\le0$. A simple induction on n shows that the nth derivative $h_0^{(n)}$ of h_0 is of the form

$$\begin{aligned}h_0^{(n)}(t)&=q_n(\frac{1}{t})\exp(-\frac{1}{t})\text{ for }t>0\\&=0\text{ for }t\le0,\end{aligned}$$

where each q_n is a polynomial of degree $2n$; the inductive hypothesis depends on the fact that $\lim_{s\to\infty} q(s)e^{-s} = 0$ for every polynomial q. So h_0 is smooth. Now $h(t) = h_0(1-t^2)$ so h also is smooth. If $0 \le t < 1$ then

$$h'(t) = -\exp(\tfrac{1}{t^2-1}) \cdot \tfrac{2t}{(t^2-1)^2} < 0;$$

if $t > 1$ then $h'(t) = 0$; since h' is continuous, $h'(t) \le 0$ for every $t \ge 0$.

(b) We need only observe that

$$x \mapsto (n+1)^2\|x\|^2 = (n+1)^2\sum_{i=1}^r \xi_i^2$$

is smooth and that the composition of smooth functions is smooth (using 473Bc).

(c) If f is continuous at x and $\epsilon > 0$, let $n_0 \in \mathbb{N}$ be such that $|f(y) - f(x)| \le \epsilon$ whenever $y \in \operatorname{dom} f$ and $\|y - x\| \le \frac{1}{n_0+1}$. Then for any $n \ge n_0$,

$$\begin{aligned}|(f * \tilde{h}_n)(x) - f(x)| &= |\int f(x-y)\tilde{h}_n(y)\mu(dy) - \int f(x)\tilde{h}_n(y)\mu(dy)| \\ &\le \int |f(x-y) - f(x)|\tilde{h}_n(y)\mu(dy) \le \int \epsilon\tilde{h}_n(y)\mu(dy) = \epsilon.\end{aligned}$$

As ϵ is arbitrary, we have the result.

(d) Repeat the argument of (c), but 'uniformly in x'; that is, given $\epsilon > 0$, take n_0 such that $|f(y) - f(x)| \le \epsilon$ whenever x, $y \in \mathbb{R}^r$ and $\|y - x\| \le \frac{1}{n_0+1}$, and see that $|(f * \tilde{h}_n)(x) - f(x)| \le \epsilon$ for every $n \ge n_0$ and every x.

(e) We know from 472Db or 261E that, for almost every $x \in \mathbb{R}^r$,

$$\lim_{\delta\downarrow 0} \frac{1}{\mu B(x,\delta)} \int_{B(x,\delta)} |f(y) - f(x)|\mu(dy) = 0.$$

Take any such x. Set $\gamma = f(x)$, Set $g(y) = |f(y) - \gamma|$ for every $y \in \operatorname{dom} f$. Let $\epsilon > 0$. Then there is some $\delta > 0$ such that $\frac{q(t)}{\beta_r t^r} \le \epsilon$ whenever $0 < t \le \delta$, where

$$q(t) = \int_{B(x,t)} g\,d\mu = \int_0^t \int_{\partial B(x,s)} g(y)\nu(dy)dt$$

by 265G, so $q'(t) = \int_{\partial B(y,t)} g\,d\nu$ for almost every $t \in [0,\delta]$, by 222E. If $n + 1 \ge \frac{1}{\delta}$, then

$$\begin{aligned}(g * \tilde{h}_n)(x) &= \int g(y)\tilde{h}_n(x-y)\mu(dy) = \int_{B(x,\delta)} g(y)\tilde{h}_n(x-y)\mu(dy) \\ &= \frac{1}{\alpha_n}\int_0^\delta \int_{\partial B(x,t)} g(y)h((n+1)^2t^2)\nu(dy)dt\end{aligned}$$

(265G again)

$$\begin{aligned}&= \frac{1}{\alpha_n}\int_0^\delta h((n+1)^2t^2)q'(t)dt \\ &= -\frac{1}{\alpha_n}\int_0^\delta 2(n+1)^2th'((n+1)^2t^2)q(t)dt\end{aligned}$$

(integrating by parts (225F), because $q(0) = h((n+1)^2\delta^2) = 0$ and both q and h are absolutely continuous)

$$\le -\frac{\epsilon}{\alpha_n}\int_0^\delta 2(n+1)^2th'((n+1)^2t^2)\beta_r t^r dt$$

(because $0 \le q(t) \le \epsilon\beta_r t^r$ and $h'((n+1)^2t^2) \le 0$ for $0 \le t \le \delta$)

$$= \epsilon$$

(applying the same calculations with $\chi\mathbb{R}^r$ in place of g). But now, since $(\gamma\chi\mathbb{R}^r * \tilde{h}_n)(x) = \gamma$ for every n,

$$|(f * \tilde{h}_n)(x) - \gamma| = |\int (f(y) - \gamma)\tilde{h}_n(x-y)\mu(dy)| \le \int |f(y) - \gamma|\tilde{h}_n(x-y)\mu(dy) \le \epsilon$$

whenever $n+1 \ge \frac{1}{\delta}$. As ϵ is arbitrary, $f(x) = \gamma = \lim_{n\to\infty}(f * \tilde{h}_n)(x)$; and this is true for μ-almost every x.

(f) Apply 444T to the indefinite-integral measure $\tilde{h}_n\mu$ over μ defined by $\tilde{h}_n$; use 444Pa for the identification of $(\tilde{h}_n\mu) * f$ with $\tilde{h}_n * f = f * \tilde{h}_n$.

473F Lemma For any measure space (X,Σ,λ) and any non-negative $f_1,\dots,f_k \in \mathcal{L}^0(\lambda)$,

$$\int \prod_{i=1}^k f_i^{1/k} d\lambda \le \prod_{i=1}^k \bigl(\int f_i d\lambda\bigr)^{1/k}.$$

proof Induce on k. Note that we can suppose that every f_i is integrable; for if any $\int f_i$ is zero, then $f_i = 0$ a.e. and the result is trivial; and if all the $\int f_i$ are greater than zero and any of them is infinite, the result is again trivial.

The induction starts with the trivial case $k = 1$. For the inductive step to $k \ge 2$, we have

$$\int \prod_{i=1}^k f_i^{1/k} d\lambda \le \| \prod_{i=1}^{k-1} f_i^{1/k} \|_{k/(k-1)} \| f_k^{1/k} \|_k$$

(by Hölder's inequality, 244E)

$$= \bigl(\int \prod_{i=1}^{k-1} f_i^{1/(k-1)} d\lambda\bigr)^{(k-1)/k} \bigl(\int f_k d\lambda\bigr)^{1/k}$$

$$\le \bigl(\prod_{i=1}^{k-1} (\int f_i d\lambda)^{1/(k-1)}\bigr)^{(k-1)/k} \bigl(\int f_k d\lambda\bigr)^{1/k}$$

(by the inductive hypothesis)

$$= \prod_{i=1}^k (\int f_i d\lambda)^{1/k},$$

as required.

473G Lemma Let (X,Σ,λ) be a σ-finite measure space and $k \ge 2$ an integer. Write λ_k for the product measure on X^k. For $x = (\xi_1,\dots,\xi_k) \in X^k$, $t \in X$ and $1 \le i \le k$ set $S_i(x,t) = (\xi'_1,\dots,\xi'_k)$ where $\xi'_i = t$ and $\xi'_j = \xi_j$ for $j \ne i$. Then if $h \in \mathcal{L}^1(\lambda_k)$ is non-negative, and we set $h_i(x) = \int h(S_i(x,t))\lambda(dt)$ whenever this is defined in $\mathbb{R}$, we have

$$\int (\prod_{i=1}^k h_i)^{1/(k-1)} d\lambda_k \le (\int h\, d\lambda_k)^{k/(k-1)}.$$

proof Induce on k.

(a) If $k = 2$, we have

$$\begin{aligned}\int h_1 \times h_2 d\lambda_k &= \iint \bigl(\int h(\tau_1,\xi_2)d\tau_1\bigr)\bigl(\int h(\xi_1,\tau_2)d\tau_2\bigr)d\xi_1 d\xi_2 \\ &= \iiiint h(\tau_1,\xi_2)h(\xi_1,\tau_2)d\tau_1 d\tau_2 d\xi_1 d\xi_2 \\ &= \iint h(\tau_1,\xi_2)d\tau_1 d\xi_2 \cdot \iint h(\xi_1,\tau_2)d\tau_2 d\xi_1 = \bigl(\int h\, d\lambda_2\bigr)^2\end{aligned}$$

by Fubini's theorem (252B) used repeatedly, because (by 253D) $(\tau_1,\tau_2,\xi_1,\xi_2) \mapsto h(\xi_1,\tau_2)h(\tau_1,\xi_2)$ is λ_4-integrable. (See 251W for a sketch of the manipulations needed to apply 252B, as stated, to the integrals above.)

(b) For the inductive step to $k \ge 3$, argue as follows. For $y \in X^{k-1}$, set $g(y) = \int h(y,t)dt$ whenever this is defined in $\mathbb{R}$, identifying X^k with $X^{k-1} \times X$, so that $g(y) = h_k(y,t)$ whenever either is defined. If $1 \le i < k$, we can consider $S_i(y,t)$ for $y \in X^{k-1}$ and $t \in X$, and we have

$$\int g(S_i(y,t))dt = \iint h(S_i(y,t),u)du dt = \int h_i(y,t)dt$$

for almost every $y \in X^{k-1}$. So

$$\int(\prod_{i=1}^{k}h_i)^{1/(k-1)}d\lambda_k = \iint(\prod_{i=1}^{k-1}h_i(y,t))^{1/(k-1)}g(y)^{1/(k-1)}dt\,\lambda_{k-1}(dy)$$

$$= \int g(y)^{1/(k-1)}\int(\prod_{i=1}^{k-1}h_i(y,t))^{1/(k-1)}dt\,\lambda_{k-1}(dy)$$

$$\le \int g(y)^{1/(k-1)}\prod_{i=1}^{k-1}\bigl(\int h_i(y,t)dt\bigr)^{1/(k-1)}\lambda_{k-1}(dy)$$

(473F)

$$= \int g(y)^{1/(k-1)}\prod_{i=1}^{k-1}g_i(y)^{1/(k-1)}\lambda_{k-1}(dy)$$

(where g_i is defined from g in the same way as h_i is defined from h)

$$\le \bigl(\int g(y)\lambda_{k-1}(dy)\bigr)^{1/(k-1)}\bigl(\int\prod_{i=1}^{k-1}g_i(y)^{1/(k-2)}\lambda_{k-1}(dy)\bigr)^{(k-2)/(k-1)}$$

(by Hölder's inequality again, this time with $\frac{1}{k-1}+\frac{k-2}{k-1}=1$)

$$\le \bigl(\int g(y)\lambda_{k-1}(dy)\bigr)^{1/(k-1)}\cdot\int g(y)\lambda_{k-1}(dy)$$

(by the inductive hypothesis)

$$= \bigl(\int g(y)\lambda_{k-1}(dy)\bigr)^{k/(k-1)} = \bigl(\int h(x)\lambda_k(dx)\bigr)^{k/(k-1)},$$

and the induction proceeds.

473H Gagliardo-Nirenberg-Sobolev inequality Suppose that $f:\mathbb{R}^r\to\mathbb{R}$ is a Lipschitz function with compact support. Then $\|f\|_{r/(r-1)}\le\int\|\operatorname{grad}f\|d\mu$.

proof By 473Cc, $\operatorname{grad}f$ is measurable and bounded, so $\|\operatorname{grad}f\|$ also is; since it must have compact support, it is integrable.

For $1\le i\le r$, $x=(\xi_1,\dots,\xi_r)\in\mathbb{R}^r$ and $t\in\mathbb{R}$ write $S_i(x,t)=(\xi'_1,\dots,\xi'_r)$ where $\xi'_i=t$ and $\xi'_j=\xi_j$ for $j\ne i$. Set $h_i(x)=\int_{-\infty}^{\infty}\|\operatorname{grad}f(S_i(x,t))\|dt$ when this is defined, which will be the case for almost every x. Now, whenever $h_i(x)$ is defined,

$$|f(x)|=|f(S_i(x,\xi_i))|=|\int_{-\infty}^{\xi_i}\tfrac{\partial}{\partial t}f(S_i(x,t))dt|\le h_i(x).$$

(Use 225E and the fact that a Lipschitz function on any bounded interval in $\mathbb{R}$ is absolutely continuous.) So $|f|\le_{\text{a.e.}}h_i$ for every $i\le r$. Accordingly

$$\int|f(x)|^{r/(r-1)}\mu(dx)\le\int\prod_{i=1}^{r}h_i(x)^{1/(r-1)}\mu(dx)\le\bigl(\int\|\operatorname{grad}f(x)\|\mu(dx)\bigr)^{r/(r-1)}$$

by 473G. Raising both sides to the power $(r-1)/r$ we have the result.

473I Lemma For any Lipschitz function $f:B(\mathbf{0},1)\to\mathbb{R}$,

$$\int_{B(\mathbf{0},1)}|f|^{r/(r-1)}d\mu\le\bigl(2^{r+4}\sqrt{r}\int_{B(\mathbf{0},1)}\|\operatorname{grad}f\|+|f|d\mu\bigr)^{r/(r-1)}.$$

proof (a) Set $g(x)=\max(0,2\|x\|^2-1)$ for $x\in B(\mathbf{0},1)$. Then $\operatorname{grad}g$ is defined at every point x such that $\|x\|<1$ and $\|x\|\ne\frac{1}{\sqrt{2}}$, and at all such points $\frac{\partial g}{\partial\xi_i}$ is either 0 or $4\xi_i$ for each i, so that $\|\operatorname{grad}g(x)\|\le4\|x\|\le4$. Hence (or otherwise) g is Lipschitz. So $f_1=f\times g$ is Lipschitz (473Ca).

By Rademacher's theorem again, $\operatorname{grad}f_1$ is defined almost everywhere in $B(\mathbf{0},1)$. Now

$$\int_{B(\mathbf{0},1)} \|\operatorname{grad} f_1\| d\mu = \int_{B(\mathbf{0},1)} \|f(x)\operatorname{grad} g(x) + g(x)\operatorname{grad} f(x)\|\mu(dx)$$

(473Bd)

$$\leq \int_{B(\mathbf{0},1)} 4|f| + \|\operatorname{grad} f\| d\mu.$$

(b) It will be convenient to have an elementary fact out in the open. Set $\phi(x) = \frac{x}{\|x\|^2}$ for $x \in \mathbb{R}^r \setminus \{\mathbf{0}\}$; note that $\phi^2(x) = x$. Then $\phi\restriction\{x : \|x\| \geq \delta\}$ is Lipschitz, for any $\delta > 0$. **P** If $\|x\| = \alpha \geq \delta$, $\|y\| = \beta \geq \delta$, then we have

$$\|\phi(x) - \phi(y)\|^2 = \frac{1}{\alpha^4}\|x\|^2 - \frac{2}{\alpha^2\beta^2} x \centerdot y + \frac{1}{\beta^4}\|y\|^2$$
$$= \frac{1}{\alpha^2\beta^2}\big(\|y\|^2 - 2x \centerdot y + \|x\|^2\big) \leq \frac{1}{\delta^4}\|x - y\|^2,$$

so $\frac{1}{\delta^2}$ is a Lipschitz constant for $\phi\restriction\mathbb{R}^r \setminus B(\mathbf{0}, \delta)$. **Q**

(c) Set $f_2(x) = f(x)$ if $\|x\| \leq 1$, $f_1\phi(x)$ if $\|x\| \geq 1$. Then f_2 is well-defined (because $f_1(x) = f(x)$ if $\|x\| = 1$), is zero outside $B(\mathbf{0}, \sqrt{2})$ (because $g(x) = 0$ if $\|x\| \leq \frac{1}{\sqrt{2}}$), and is Lipschitz. **P** By 473Cb, it will be enough to show that $f_2\restriction F$ is Lipschitz, where $F = \{x : \|x\| \geq 1\}$. But (b) shows that $\phi\restriction F$ is 1-Lipschitz, so any Lipschitz constant for f_1 is also a Lipschitz constant for $f_2\restriction F$. **Q**

If $\|x\| > 1$, then, for any $i \leq r$,

$$\frac{\partial f_2}{\partial \xi_i}(x) = \sum_{j=1}^r \frac{\partial f_1}{\partial \xi_j}(\phi(x)) \cdot \frac{\partial}{\partial \xi_i}\Big(\frac{\xi_j}{\|x\|^2}\Big) = \frac{\partial f_1}{\partial \xi_i}(\phi(x)) \cdot \frac{1}{\|x\|^2} - 2\sum_{j=1}^r \frac{\partial f_1}{\partial \xi_j}(\phi(x)) \cdot \frac{\xi_i\xi_j}{\|x\|^4}$$
$$= \frac{\partial f_1}{\partial \xi_i}(\phi(x)) \cdot \frac{1}{\|x\|^2} - \frac{2\xi_i}{\|x\|^4} x \centerdot \operatorname{grad} f(\phi(x))$$

wherever the right-hand side is defined, that is, wherever all the partial derivatives $\frac{\partial f_1}{\partial \xi_j}(\phi(x))$ are defined. But $H = B(\mathbf{0}, 1) \setminus \operatorname{dom}(\operatorname{grad} f_1)$ is negligible, and does not meet $\{x : \|x\| < \frac{1}{\sqrt{2}}\}$, so $\phi\restriction H$ is Lipschitz and $\phi[H] = \phi^{-1}[H]$ is negligible (262D); while $\operatorname{grad} f_1(\phi(x))$ is defined whenever $\|x\| > 1$ and $x \notin \phi^{-1}[H]$. So the formula here is valid for almost every $x \in F$, and

$$\Big|\frac{\partial f_2}{\partial \xi_i}(x)\Big| \leq \|\operatorname{grad} f_1(\phi(x))\| \cdot \frac{1}{\|x\|^2} + \frac{2|\xi_i|}{\|x\|^4}\|\operatorname{grad} f_1(\phi(x))\|\|x\|$$
$$= \|\operatorname{grad} f_1(\phi(x))\|\frac{\|x\| + 2|\xi_i|}{\|x\|^3} \leq 3\|\operatorname{grad} f_1(\phi(x))\|$$

for almost every $x \in F$. But (since we know that $\operatorname{grad} f_2$ is defined almost everywhere, by Rademacher's theorem, as usual) we have

$$\|\operatorname{grad} f_2(x)\| \leq 3\sqrt{r}\|\operatorname{grad} f_1(\phi(x))\|$$

for almost every $x \in F$.

(d) We are now in a position to estimate

$$\int_F \|\operatorname{grad} f_2\| d\mu = \int_{B(\mathbf{0},\sqrt{2})} \|\operatorname{grad} f_2\| d\mu - \int_{B(\mathbf{0},1)} \|\operatorname{grad} f_2\| d\mu$$

(because $f_2(x) = 0$ if $\|x\| \geq \sqrt{2}$)

$$= \int_1^{\sqrt{2}} \int_{\partial B(\mathbf{0},t)} \|\operatorname{grad} f_2(x)\|\nu(dx)dt$$

(265G, as usual)

$$\leq 3\sqrt{r}\int_1^{\sqrt{2}}\int_{\partial B(\mathbf{0},t)}\|\operatorname{grad} f_1(\tfrac{1}{t^2}x)\|\nu(dx)dt$$

(by (b) above)

$$\leq 3\sqrt{r}\int_1^{\sqrt{2}}\int_{\partial B(\mathbf{0},1/t)} t^{2r-2}\|\operatorname{grad} f_1(y)\|\nu(dy)dt$$

substituting $x = t^2y$ in the inner integral; the point being that as the function $y \mapsto t^2y$ changes all distances by a scalar multiple t^2, it must transform Hausdorff $(r-1)$-dimensional measure by a multiple t^{2r-2}. But now, substituting $s = \frac{1}{t}$ in the outer integral, we have

$$\begin{aligned}\int_F \|\operatorname{grad} f_2\|d\mu &\leq 3\sqrt{r}\int_{1/\sqrt{2}}^{1}\frac{1}{s^{2r}}\int_{\partial B(\mathbf{0},s)}\|\operatorname{grad} f_1(y)\|\nu(dy)ds \\ &\leq 2^r\cdot 3\sqrt{r}\int_{1/\sqrt{2}}^{1}\int_{\partial B(\mathbf{0},s)}\|\operatorname{grad} f_1(y)\|\nu(dy)ds \\ &= 2^r\cdot 3\sqrt{r}\int_{B(\mathbf{0},1)}\|\operatorname{grad} f_1\|d\mu \\ &\leq 2^{r+2}\sqrt{r}\int_{B(\mathbf{0},1)} 4|f| + \|\operatorname{grad} f\|d\mu\end{aligned}$$

by (a) above.

(e) Accordingly

$$\begin{aligned}\int_{\mathbb{R}^r}\|\operatorname{grad} f_2\|d\mu &= \int_{B(\mathbf{0},1)}\|\operatorname{grad} f\|d\mu + \int_F \|\operatorname{grad} f_2\|d\mu \\ &\leq 2^{r+4}\sqrt{r}\int_{B(\mathbf{0},1)}|f| + \|\operatorname{grad} f\|d\mu.\end{aligned}$$

But now we can apply 473H to see that

$$\begin{aligned}\int_{B(\mathbf{0},1)}|f|^{r/(r-1)}d\mu &\leq \int |f_2|^{r/r-1}d\mu \leq (\int \|\operatorname{grad} f_2\|d\mu)^{r/(r-1)} \\ &\leq \big(2^{r+4}\sqrt{r}\int_{B(\mathbf{0},1)}|f| + \|\operatorname{grad} f\|d\mu\big)^{r/(r-1)},\end{aligned}$$

as claimed.

473J Lemma Let $f:\mathbb{R}^r\to\mathbb{R}$ be a Lipschitz function. Then

$$\int_{B(y,\delta)}|f(x)-f(z)|\mu(dx) \leq \frac{2^r}{r}\delta^r\int_{B(y,\delta)}\|\operatorname{grad} f(x)\|\|x-z\|^{1-r}\mu(dx)$$

whenever $y\in\mathbb{R}^r$, $\delta>0$ and $z\in B(y,\delta)$.

proof (a) To begin with, suppose that f is smooth. In this case, for any x, $z\in B(y,\delta)$,

$$\begin{aligned}|f(x)-f(z)| &= \Big|\int_0^1 \frac{d}{dt}f(z+t(x-z))dt\Big| \\ &= \Big|\int_0^1 (x-z)\,.\,\operatorname{grad} f(z+t(x-z))dt\Big| \\ &\leq \|x-z\|\int_0^1 \|\operatorname{grad} f(z+t(x-z))\|dt.\end{aligned}$$

So, for $\eta>0$,

$$\int_{B(y,\delta)\cap\partial B(z,\eta)}|f(x)-f(z)|\nu(dx)$$
$$\le \eta\int_0^1\int_{B(y,\delta)\cap\partial B(z,\eta)}\|\operatorname{grad} f(z+t(x-z))\|\nu(dx)dt$$

(grad f is continuous and bounded, and the subspace measure induced by ν on $\partial B(z,\eta)$ is a (quasi-)Radon measure (471E, 471Dh), so its product with Lebesgue measure also is (417T), and there is no difficulty with the change in order of integration)

$$\le \eta\int_0^1\frac{1}{t^{r-1}}\int_{B(y,\delta)\cap\partial B(z,t\eta)}\|\operatorname{grad} f(w)\|\nu(dw)dt$$

(because if $\phi(x)=z+t(x-z)$, then $\nu\phi^{-1}[E]=\frac{1}{t^{r-1}}\nu E$ whenever ν measures E and $t>0$, while $\phi(x)\in B(y,\delta)$ whenever $x\in B(y,\delta)$)

$$=\eta^r\int_0^1\int_{B(y,\delta)\cap\partial B(z,t\eta)}\|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\nu(dw)dt$$
$$=\eta^{r-1}\int_0^\eta\int_{B(y,\delta)\cap\partial B(z,s)}\|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\nu(dw)ds$$

(substituting $s=t\eta$)

$$=\eta^{r-1}\int_{B(y,\delta)\cap B(z,\eta)}\|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\mu(dw).$$

So

$$\begin{aligned}\int_{B(y,\delta)}|f(x)-f(z)|\mu(dx)&=\int_0^{2\delta}\int_{B(y,\delta)\cap\partial B(z,\eta)}|f(x)-f(z)|\nu(dx)d\eta\\&\le\int_0^{2\delta}\eta^{r-1}\int_{B(y,\delta)\cap B(z,\eta)}\|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\mu(dw)d\eta\\&\le\int_0^{2\delta}\eta^{r-1}\int_{B(y,\delta)}\|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\mu(dw)d\eta\\&=\frac{2^r}{r}\delta^r\int_{B(y,\delta)}\|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\mu(dw).\end{aligned}$$

(b) Now turn to the general case in which f is not necessarily differentiable everywhere, but is known to be Lipschitz and bounded. We need to know that $\int_{B(y,\delta)}\|x-z\|^{1-r}\mu(dx)$ is finite; this is because

$$\begin{aligned}\int_{B(y,\delta)}\|x-z\|^{1-r}\mu(dx)\le\int_{B(z,2\delta)}\|x-z\|^{1-r}\mu(dx)&=\int_0^{2\delta}t^{1-r}\nu(\partial B(z,t))dt\\&=\int_0^{2\delta}t^{1-r}r\beta_r t^{r-1}dt=2\delta r\beta_r.\end{aligned}$$

Take the sequence $\langle\tilde h_n\rangle_{n\in\mathbb{N}}$ from 473E. Then $\langle f*\tilde h_n\rangle_{n\in\mathbb{N}}$ converges uniformly to f (473Ed), while $\langle\operatorname{grad}(f*\tilde h_n)\rangle_{n\in\mathbb{N}}=\langle\tilde h_n*\operatorname{grad} f\rangle_{n\in\mathbb{N}}$ (473Dd) is uniformly bounded (473Cc, 473Dg) and converges almost everywhere to grad f (473Ee). But this means that, setting $f_n=f*\tilde h_n$,

$$\begin{aligned}\int_{B(y,\delta)}|f(x)-f(z)|\mu(dx)&=\lim_{n\to\infty}\int_{B(y,\delta)}|f_n(x)-f_n(z)|\mu(dx)\\&\le\lim_{n\to\infty}\frac{2^r}{r}\delta^r\int_{B(y,\delta)}\|\operatorname{grad} f_n(x)\|\|x-z\|^{1-r}\mu(dx)\end{aligned}$$

(because every f_n is smooth, by 473De)

$$= \frac{2^r}{r}\delta^r \int_{B(y,\delta)} \|\operatorname{grad} f(x)\|\|x - z\|^{1-r}\mu(dx)$$

by Lebesgue's Dominated Convergence Theorem.

(c) Finally, if f is not bounded on the whole of $\mathbb{R}^r$, it is surely bounded on $B(y, \delta)$, so we can apply (b) to the function $x \mapsto \operatorname{med}(-M, f(x), M)$ for a suitable $M \geq 0$ to get the result as stated.

473K Poincaré's inequality for balls Let $B \subseteq \mathbb{R}^r$ be a non-trivial closed ball, and $f : B \to \mathbb{R}$ a Lipschitz function. Set $\gamma = \frac{1}{\mu B}\int_B f d\mu$. Then

$$\bigl(\int_B |f - \gamma|^{r/(r-1)}d\mu\bigr)^{(r-1)/r} \leq c\int_B \|\operatorname{grad} f\|d\mu,$$

where $c = 2^{r+4}\sqrt{r}(1 + 2^{r+1})$.

proof (a) To begin with (down to the end of (b)) suppose that B is the unit ball $B(\mathbf{0}, 1)$. Then, for any $x \in B$,

$$\begin{aligned} |f(x) - \gamma| &= \frac{1}{\mu B}\Bigl|\int_B f(x) - f(z)\mu(dz)\Bigr| \\ &\leq \frac{1}{\mu B}\int_B |f(x) - f(z)|\mu(dz) \\ &\leq \frac{2^r}{r} \cdot \frac{1}{\mu B}\int_B \|\operatorname{grad} f(z)\|\|x - z\|^{1-r}\mu(dz), \end{aligned}$$

by 473J. Also, for any $z \in B$,

$$\begin{aligned} \int_{B(z,2)} \|x - z\|^{1-r}\mu(dx) &= \int_0^2 \int_{\partial B(z,t)} \|x - z\|^{1-r}\nu(dx)dt \\ &= \int_0^2 t^{1-r}\nu(\partial B(z, t))dt = \int_0^2 t^{1-r}t^{r-1}\nu S_{r-1}dt = 2r\beta_r. \end{aligned}$$

So

$$\begin{aligned} \int_B |f(x) - \gamma|\mu(dx) &\leq \frac{2^r}{r} \cdot \frac{1}{\mu B}\int_B \int_B \|\operatorname{grad} f(z)\|\|x - z\|^{1-r}\mu(dz)\mu(dx) \\ &= \frac{2^r}{r\beta_r}\int_B \int_B \|\operatorname{grad} f(z)\|\|x - z\|^{1-r}\mu(dx)\mu(dz) \\ &\leq \frac{2^r}{r\beta_r}\int_B \|\operatorname{grad} f(z)\| \int_{B(z,2)} \|x - z\|^{1-r}\mu(dx)\mu(dz) \\ &\leq 2^{r+1}\int_B \|\operatorname{grad} f(z)\|\mu(dz). \end{aligned}$$

(b) Now apply 473I to $g = f(x) - \gamma$. We have

$$\begin{aligned} \int_B |f - \gamma|^{r/(r-1)}d\mu &\leq \bigl(2^{r+4}\sqrt{r}\int_B \|\operatorname{grad} f\| + |g|d\mu\bigr)^{r/(r-1)} \\ &\leq \bigl(2^{r+4}\sqrt{r}(1 + 2^{r+1})\int_B \|\operatorname{grad} f\|d\mu\bigr)^{r/(r-1)} \end{aligned}$$

(by (a))

$$= \bigl(c\int_B \|\operatorname{grad} f\|d\mu\bigr)^{r/(r-1)}.$$

(c) For the general case, express B as $B(y, \delta)$, and set $h(x) = f(y + \delta x)$ for $x \in B(\mathbf{0}, 1)$. Then $\operatorname{grad} h(x) = \delta \operatorname{grad} f(y + \delta x)$ for almost every $x \in B(\mathbf{0}, 1)$. Now

$$\int_{B(\mathbf{0},1)} h\,d\mu = \frac{1}{\delta^r}\int_{B(y,\delta)} f d\mu,$$

so

$$\frac{1}{\mu B(\mathbf{0},1)}\int_{B(\mathbf{0},1)} h\,d\mu = \frac{1}{\mu B(y,\delta)}\int_{B(y,\delta)} f d\mu = \gamma.$$

We therefore have

$$\int_{B(y,\delta)} |f-\gamma|^{r/(r-1)} d\mu = \delta^r \int_{B(\mathbf{0},1)} |h-\gamma|^{r/(r-1)} d\mu$$

$$\le \delta^r \bigl(c\int_{B(\mathbf{0},1)} \|\operatorname{grad} h\| d\mu\bigr)^{r/(r-1)}$$

(by (a)-(b) above)

$$= \delta^r \bigl(\frac{\delta c}{\delta^r}\int_{B(y,\delta)} \|\operatorname{grad} f\| d\mu\bigr)^{r/(r-1)}$$

$$= \bigl(c\int_{B(y,\delta)} \|\operatorname{grad} f\| d\mu\bigr)^{r/(r-1)}.$$

Raising both sides to the power $(r-1)/r$ we have the result as stated.

Remark As will be plain from the way in which the proof here is constructed, there is no suggestion that the formula offered for c gives anything near the best possible value.

473L Corollary Let $B \subseteq \mathbb{R}^r$ be a non-trivial closed ball, and $f : B \to [0,1]$ a Lipschitz function. Set

$$F_0 = \{x : x \in B,\ f(x) \le \tfrac{1}{4}\}, \quad F_1 = \{x : x \in B,\ f(x) \ge \tfrac{3}{4}\}.$$

Then

$$\bigl(\min(\mu F_0, \mu F_1)\bigr)^{(r-1)/r} \le 4c\int_B \|\operatorname{grad} f\| d\mu,$$

where $c = 2^{r+4}\sqrt{r}(1+2^{r+1})$.

proof Setting $\gamma = \frac{1}{\mu B}\int_B f d\mu$,

$$\int_B |f-\gamma|^{r/(r-1)} d\mu \ge \frac{1}{4^{r/(r-1)}}\mu F_0 \text{ if } \gamma \ge \frac{1}{2},$$

$$\ge \frac{1}{4^{r/(r-1)}}\mu F_1 \text{ if } \gamma \le \frac{1}{2}.$$

So 473K tells us that

$$\frac{1}{4}\bigl(\min(\mu F_0, \mu F_1)\bigr)^{(r-1)/r} \le c\int_B \|\operatorname{grad} f\| d\mu,$$

as required.

473M The case $r = 1$ The general rubric for this section declares that r is taken to be at least 2, which is clearly necessary for the formula in 473K to be appropriate. For the sake of an application in the next section, however, I mention the elementary corresponding result when $r = 1$. In this case, B is just a closed interval, and $\operatorname{grad} f$ is the ordinary derivative of f; interpreting $(\int_B |f-\gamma|^{r/(r-1)})^{(r-1)/r}$ as $\|f \times \chi B - \gamma\chi B\|_{r/(r-1)}$, it is natural to look at

$$\|f \times \chi B - \gamma\chi B\|_\infty = \sup_{x\in B} |f(x)-\gamma| \le \sup_{x,y\in B} |f(x)-f(y)| \le \int_B |f'| d\mu,$$

giving a version of 473K for $r = 1$. We see that the formula for c remains valid in the case $r = 1$, with a good deal to spare. As for 473L, if $\int_B |f'| < \frac{1}{2}$ then at least one of F_0, F_1 must be empty.

473X Basic exercises (a) Set $f(x) = \max(0, -\ln\|x\|)$, $f_k(x) = \min(k, f(x))$ for $x \in \mathbb{R}^2 \setminus \{0\}$, $k \in \mathbb{N}$. Show that $\lim_{k\to\infty} \|f - f_k\|_2 = \lim_{k\to\infty} \|\operatorname{grad} f - \operatorname{grad} f_k\|_1 = 0$, so that all the inequalities 473H-473L are valid for f.

(b) Let $k \in [1,r]$ be an integer, and set $m = \frac{(r-1)!}{(k-1)!(r-k)!}$. Let $e_1, \dots, e_r$ be the standard orthonormal basis of $\mathbb{R}^r$ and $\mathcal{J}$ the family of subsets of $\{1,\dots,r\}$ with k members. For $J \in \mathcal{J}$ let V_J be the linear span of $\{e_i : i \in J\}$, $\pi_J : \mathbb{R}^r \to V_J$ the orthogonal projection and ν_J the normalized k-dimensional Hausdorff measure on V_J. Show that if $A \subseteq \mathbb{R}^r$ then $(\mu^* A)^m \le \prod_{J\in\mathcal{J}} \nu_J^* \pi_J[A]$. (*Hint*: start with $A \subseteq [0,1]^r$ and note that $([0,1]^r)^m$ can be identified with $\prod_{J\in\mathcal{J}}[0,1]^J$.)

473Y Further exercises (a) Let $D \subseteq \mathbb{R}^r$ be any set and $\phi : D \to \mathbb{R}^s$ any function. Show that $D_0 = \{x : x \in D,\ \phi$ is differentiable at $x\}$ is a Borel subset of $\mathbb{R}^r$, and that the derivative of ϕ is a Borel measurable function. (Compare 225J.)

473 Notes and comments The point of all the inequalities 473H-473L is that they bound some measure of variance of a function f by the integral of $\|\operatorname{grad} f\|$. If $r = 2$, indeed, we are looking at $\|f\|_2$ (473H) or $\int_B |f|^2$ (473I) or something essentially equal to the variance of probability theory (473K). In higher dimensions we need to look at $\|\ \|_{r/(r-1)}$ in place of $\|\ \|_2$, and when $r = 1$ we can interpret the inequalities in terms of the supremum norm $\|\ \|_\infty$ (473M). In all cases we want to develop inequalities which will enable us to discuss a function in terms of its first derivative. In one dimension, this is the familiar Fundamental Theorem of Calculus (Chapter 22). We find there a straightforward criterion ('absolute continuity') to determine whether a given function of one variable is an indefinite integral, and that if so it is the indefinite integral of its own derivative. Even in two dimensions, this simplicity disappears. The essential problem is that a function can be the indefinite integral of an integrable gradient function without being bounded (473Xa). The principal results of this section are stated for Lipschitz functions, but in fact they apply much more widely. The argument suggested in 473Xa involves approximating the unbounded function f by Lipschitz functions f_k in a sharp enough sense to make it possible to read off all the inequalities for f from the corresponding inequalities for the f_k. This idea leads naturally to the concept of 'Sobolev space', which I leave on one side for the moment; see Evans & Gariepy 92, chap. 4, for details.

474 The distributional perimeter

The next step is a dramatic excursion, defining (for appropriate sets E) a perimeter *measure* for which a version of the Divergence Theorem is true (474E). I begin the section with elementary notes on the divergence of a vector field (474B-474C). I then define 'locally finite perimeter' (474D), 'perimeter measure' and 'outward normal' (474F) and 'reduced boundary' (474G). The definitions rely on the Riesz representation theorem, and we have to work very hard to relate them to any geometrically natural idea of 'boundary'. Even half-spaces (474I) demand some attention. From Poincaré's inequality (473K) we can prove isoperimetric inequalities for perimeter measures (474L). With some effort we can locate the reduced boundary as a subset of the topological boundary (474Xc), and obtain asymptotic inequalities on the perimeter measures of small balls (474N). With much more effort we can find a geometric description of outward normal functions in terms of 'Federer exterior normals' (474R), and get a tight asymptotic description of the perimeter measures of small balls (474S). I end with the Compactness Theorem for sets of bounded perimeter (474T).

474A Notation I had better repeat some of the notation from §473. $r \ge 2$ is a fixed integer. μ is Lebesgue measure on $\mathbb{R}^r$, and $\beta_r = \mu B(\mathbf{0},1)$ is the volume of the unit ball. $S_{r-1} = \partial B(\mathbf{0},1)$ is the unit sphere. ν is normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$. We shall sometimes need to look at Lebesgue measure on $\mathbb{R}^{r-1}$, which I will denote μ_{r-1}. As in §473, I will use Greek letters to represent coordinates, so that $x = (\xi_1,\dots,\xi_r)$ for $x \in \mathbb{R}^r$, etc., and β_r will be the r-dimensional volume of the unit ball in $\mathbb{R}^r$.

$\mathcal{D}$ is the set of smooth functions $f : \mathbb{R}^r \to \mathbb{R}$ with compact support; $\mathcal{D}_r$ the set of smooth functions $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support, that is, the set of functions $\phi = (\phi_1,\dots,\phi_r) : \mathbb{R}^r \to \mathbb{R}^r$ such that $\phi_i \in \mathcal{D}$ for every i.

I continue to use the sequence $\langle \tilde h_n \rangle_{n\in\mathbb{N}}$ from 473E; these functions all belong to $\mathcal{D}$, are non-negative everywhere and zero outside $B(\mathbf{0}, \frac{1}{n+1})$, are even, and have integral 1.

474B The divergence of a vector field (a) For a function ϕ from a subset of $\mathbb{R}^r$ to $\mathbb{R}^r$, write $\operatorname{div}\phi = \sum_{i=1}^r \frac{\partial\phi_i}{\partial\xi_i}$, where $\phi = (\phi_1,\dots,\phi_r)$; for definiteness, let us take the domain of $\operatorname{div}\phi$ to be the set of points at which ϕ is differentiable (in the strict sense of 262Fa). Note that $\operatorname{div}\phi \in \mathcal{D}$ for every $\phi \in \mathcal{D}_r$. We need the following elementary facts.

(b) If $f:\mathbb{R}^r\to\mathbb{R}$ and $\phi:\mathbb{R}^r\to\mathbb{R}^r$ are functions, then $\operatorname{div}(f\times\phi)=\phi\,.\operatorname{grad}f+f\times\operatorname{div}\phi$ at any point at which f and ϕ are both differentiable. (Use 473Bc; compare 473Bd.)

(c) If $\phi:\mathbb{R}^r\to\mathbb{R}^r$ is a Lipschitz function with compact support, then $\int\operatorname{div}\phi\,d\mu=0$. **P** $\operatorname{div}\phi$ is defined almost everywhere (by Rademacher's theorem, 262Q), measurable (473Be), bounded (473Cc) and with compact support, so

$$\int\operatorname{div}\phi\,d\mu=\sum_{i=1}^r\int\frac{\partial\phi_i}{\partial\xi_i}d\mu$$

is defined in $\mathbb{R}$. For each $i\le r$, Fubini's theorem tells us that we can replace integration with respect to μ by a repeated integral, in which the inner integral is

$$\int_{-\infty}^{\infty}\frac{\partial\phi_i}{\partial\xi_i}(\xi_1,\dots,\xi_r)d\xi_i=0$$

because $\phi_i(\xi_1,\dots,\xi_r)=0$ whenever $|\xi_i|$ is large enough. So $\int\frac{\partial\phi_i}{\partial\xi_i}d\mu$ also is zero. Summing over i, we have the result. **Q**

(d) If $\phi:\mathbb{R}^r\to\mathbb{R}^r$ and $f:\mathbb{R}^r\to\mathbb{R}$ are Lipschitz functions, one of which has compact support, then $f\times\phi$ is Lipschitz. **P** Take $n\in\mathbb{N}$ such that $f(x)\phi(x)=\mathbf{0}$ for $\|x\|>n$, and $\gamma\ge0$ such that $|f(x)-f(y)|\le\gamma\|x-y\|$ and $\|\phi(x)-\phi(y)\|\le\gamma\|x-y\|$ for all x, $y\in\mathbb{R}^r$, while also $|f(x)|\le\gamma$ whenever $\|x\|\le n+1$ and $\|\phi(x)\|\le\gamma$ whenever $\|x\|\le n+1$. If x, $y\in\mathbb{R}^r$ then

— if $\|x\|\le n+1$ and $\|y\|\le n+1$,

$$\|f(x)\phi(x)-f(y)\phi(y)\|\le|f(x)|\|\phi(x)-\phi(y)\|+\|\phi(y)\||f(x)-f(y)|\le2\gamma^2\|x-y\|;$$

— if $\|x\|\le n$ and $\|y\|>n+1$,

$$|f(x)\phi(x)-f(y)\phi(y)\|=|f(x)|\|\phi(x)\|\le\gamma^2\le\gamma^2\|x-y\|;$$

— if $\|x\|>n$ and $\|y\|>n$, $|f(x)\phi(x)-f(y)\phi(y)\|=0$.

So $2\gamma^2$ is a Lipschitz constant for $f\times\phi$. **Q**

It follows that

$$\int\phi\,.\operatorname{grad}f\,d\mu+\int f\times\operatorname{div}\phi\,d\mu=0.$$

P f and ϕ and $f\times\phi$ are all differentiable almost everywhere. So

$$\int\phi\,.\operatorname{grad}f\,d\mu+\int f\times\operatorname{div}\phi\,d\mu=\int\operatorname{div}(f\times\phi)d\mu=0$$

by (b) and (c) above. **Q**

(e) If $f\in\mathcal{L}^\infty(\mu)$, $g\in\mathcal{L}^1(\mu)$ is even (that is, $g(-x)$ is defined and equal to $g(x)$ for every $x\in\operatorname{dom}g$), and $\phi:\mathbb{R}^r\to\mathbb{R}^r$ is a Lipschitz function with compact support, then $\int(f*g)\times\operatorname{div}\phi=\int f\times\operatorname{div}(g*\phi)$, where $g*\phi=(g*\phi_1,\dots,g*\phi_r)$. **P** For each i,

$$\int(f*g)\times\frac{\partial\phi_i}{\partial\xi_i}d\mu=\iint f(x)g(y)\frac{\partial\phi_i}{\partial\xi_i}(x+y)\mu(dy)\mu(dx)$$

(255G/444Od)

$$=\iint f(x)g(-y)\frac{\partial\phi_i}{\partial\xi_i}(x+y)\mu(dy)\mu(dx)$$

(because g is even)

$$=\int f\times(g*\frac{\partial\phi_i}{\partial\xi_i})d\mu=\int f\times\frac{\partial}{\partial\xi_i}(g*\phi_i)d\mu$$

as in 473Dd. Now take the sum over i of both sides. **Q**

474C Invariance under isometries: Proposition Suppose that $T:\mathbb{R}^r\to\mathbb{R}^r$ is an isometry, and that ϕ is a function from a subset of $\mathbb{R}^r$ to $\mathbb{R}^r$. Then

$$\operatorname{div}(T^{-1}\phi T)=(\operatorname{div}\phi)T.$$

proof Set $z = T(\mathbf{0})$. By 4A4Jb, the isometry $x \mapsto T(x) - z$ is linear and preserves inner products, so there is an orthogonal matrix S such that $T(x) = z + S(x)$ for every $x \in \mathbb{R}^r$. Now suppose that $x \in \mathbb{R}^r$ is such that $(\operatorname{div}\phi)(T(x))$ is defined. Then $T(y) - T(x) - S(y-x) = 0$ for every y, so T is differentiable at x, with derivative S, and ϕT is differentiable at x, with derivative DS, where D is the derivative of ϕ at $T(x)$, by 473Bc. Also $T^{-1}(y) = S^{-1}(y-z)$ for every y, so T^{-1} is differentiable at $\phi(T(x))$ with derivative S^{-1}, and $T^{-1}\phi T$ is differentiable at x, with derivative $S^{-1}DS$. Now if D is $\langle\delta_{ij}\rangle_{1\le i,j\le r}$ and S is $\langle\sigma_{ij}\rangle_{1\le i,j\le r}$ and $S^{-1}DS$ is $\langle\tau_{ij}\rangle_{1\le i,j\le r}$, then S^{-1} is the transpose $\langle\sigma_{ji}\rangle_{1\le i,j\le r}$ of S, because S is orthogonal, so

$$\begin{aligned}\operatorname{div}(T^{-1}\phi T)(x) &= \sum_{i=1}^r \tau_{ii} = \sum_{i=1}^r\sum_{j=1}^r \sigma_{ji}\sum_{k=1}^r \delta_{jk}\sigma_{ki}\\ &= \sum_{j=1}^r\sum_{k=1}^r \delta_{jk}\sum_{i=1}^r \sigma_{ji}\sigma_{ki} = \sum_{j=1}^r \delta_{jj} = \operatorname{div}\phi(T(x))\end{aligned}$$

because $\sum_{i=1}^r \sigma_{ji}\sigma_{jk} = 1$ if $j = k$ and 0 otherwise. If $\operatorname{div}(T^{-1}\phi T)(x)$ is defined, then (because T^{-1} also is an isometry)

$$(\operatorname{div}\phi)(T(x)) = \operatorname{div}(TT^{-1}\phi TT^{-1})(T(x)) = \operatorname{div}(T^{-1}\phi T)(T^{-1}T(x)) = \operatorname{div}(T^{-1}\phi T)(x).$$

So the functions $\operatorname{div}(T^{-1}\phi T)$ and $(\operatorname{div}\phi)T$ are identical.

474D Locally finite perimeter: Definition Let $E \subseteq \mathbb{R}^r$ be a Lebesgue measurable set. Its **perimeter** $\operatorname{per}E$ is

$$\sup\{|\int_E \operatorname{div}\phi\,d\mu| : \phi : \mathbb{R}^r \to B(\mathbf{0},1) \text{ is a Lipschitz function with compact support}\}$$

(allowing ∞). A set $E \subseteq \mathbb{R}^r$ has **locally finite perimeter** if it is Lebesgue measurable and

$$\sup\{|\int_E \operatorname{div}\phi\,d\mu| : \phi : \mathbb{R}^r \to \mathbb{R}^r \text{ is a Lipschitz function, } \|\phi\| \le \chi B(\mathbf{0},n)\}$$

is finite for every $n \in \mathbb{N}$. Of course a Lebesgue measurable set with finite perimeter also has locally finite perimiter.

474E Theorem Suppose that $E \subseteq \mathbb{R}^r$ has locally finite perimeter.

(i) There are a Radon measure λ_E^∂ on $\mathbb{R}^r$ and a Borel measurable function $\psi : \mathbb{R}^r \to S_{r-1}$ such that

$$\int_E \operatorname{div}\phi\,d\mu = \int \phi\,.\,\psi\,d\lambda_E^\partial$$

for every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support.

(ii) This formula uniquely determines λ_E^∂, which can also be defined by saying that

$$\lambda_E^\partial(G) = \sup\{|\int_E \operatorname{div}\phi\,d\mu| : \phi : \mathbb{R}^r \to \mathbb{R}^r \text{ is Lipschitz, } \|\phi\| \le \chi G\}$$

whenever $G \subseteq \mathbb{R}^r$ is open.

(iii) If $\hat\psi$ is another function defined λ_E^∂-a.e. and satisfying the formula in (i), then $\hat\psi$ and ψ are equal λ_E^∂-almost everywhere.

proof (a)(i) For each $l \in \mathbb{N}$, set

$$\gamma_l = \sup\{|\int_E \operatorname{div}\phi\,d\mu| : \phi : \mathbb{R}^r \to \mathbb{R}^r \text{ is Lipschitz, } \|\phi\| \le \chi B(\mathbf{0},l)\}.$$

If $f : \mathbb{R}^r \to \mathbb{R}$ is a Lipschitz function and $f(x) = 0$ for $\|x\| \ge l$, then $|\int_E \frac{\partial f}{\partial\xi_i}d\mu| \le \gamma_l\|f\|_\infty$ for every $i \le r$. **P** It is enough to consider the case $\|f\|_\infty = 1$, since the result is trivial if $\|f\|_\infty = 0$, and otherwise we can look at an appropriate scalar multiple of f. In this case, set $\phi(x) = f(x)e_i$ for every x, where e_i is the unit vector $(0,\dots,0,1,0,\dots,0)$ with a 1 in the ith place. Then ϕ is Lipschitz and $\|\phi\| = |f| \le \chi B(\mathbf{0},l)$, so

$$|\int_E \frac{\partial f}{\partial\xi_i}d\mu| = |\int_E \operatorname{div}\phi\,d\mu| \le \gamma_l. \text{ } \mathbf{Q}$$

(ii) Write C_k for the space of continuous functions with compact support from $\mathbb{R}^r$ to $\mathbb{R}$. By 473Dc and 473De, $f * \tilde h_n \in \mathcal{D}$ for every $f \in C_k$ and $n \in \mathbb{N}$. Now the point is that

$$L_i(f) = \lim_{n\to\infty}\int_E \frac{\partial}{\partial\xi_i}(f * \tilde h_n)d\mu$$

is defined whenever $f \in C_k$ and $i \le r$. **P** Applying 473Ed, we see that $\|f - f * \tilde{h}_n\|_\infty \to 0$ as $n \to \infty$. Let l be such that $f(x) = 0$ for $\|x\| \ge l$. Then $\|(f * \tilde{h}_m) - (f * \tilde{h}_n)\|_\infty \to 0$ as m, $n \to \infty$, while all the $f * \tilde{h}_m$ are zero outside $B(\mathbf{0}, l+1)$ (473Dc), so that

$$\Bigl|\int_E \frac{\partial}{\partial \xi_i}(f * \tilde{h}_m) d\mu - \int_E \frac{\partial}{\partial \xi_i}(f * \tilde{h}_n) d\mu\Bigr| \le \gamma_{l+1}\|(f * \tilde{h}_m) - (f * \tilde{h}_n)\|_\infty \to 0$$

as m, $n \to \infty$. Thus $\langle\int_E \frac{\partial}{\partial \xi_i}(f * \tilde{h}_n) d\mu\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence and must have a limit. **Q**

(iii) If $f \in C_k$ is Lipschitz and zero outside $B(\mathbf{0}, l)$, then

$$|\int_E \frac{\partial f}{\partial \xi_i} d\mu - \int_E \frac{\partial}{\partial \xi_i}(f * \tilde{h}_n) d\mu| \le \gamma_{l+1}\|f - f * \tilde{h}_n\|_\infty \to 0$$

as $n \to \infty$, and $L_i(f) = \int_E \frac{\partial f}{\partial \xi_i} d\mu$. Consequently $|L_i(f)| \le \gamma_l \|f\|_\infty$.

(b) Because all the functionals $f \mapsto \int_E \frac{\partial}{\partial \xi_i}(f * \tilde{h}_n) d\mu$ are linear, L_i is linear. Moreover, by the last remark in (a-iii), it is order-bounded when regarded as a linear functional on the Riesz space C_k, so is expressible as a difference $L_i^+ - L_i^-$ of positive linear functionals (356B).

By the Riesz Representation Theorem (436J), we have Radon measures λ_i^+, λ_i^- on $\mathbb{R}^r$ such that $L_i^+(f) = \int f d\lambda_i^+$, $L_i^-(f) = \int f d\lambda_i^-$ for every $f \in C_k$. Let $\hat{\lambda}$ be the sum $\sum_{i=1}^r \lambda_i^+ + \lambda_i^-$, so that $\hat{\lambda}$ is a Radon measure (416De) and every λ_i^+, λ_i^- is an indefinite-integral measure over $\hat{\lambda}$ (416Sb).

For each $i \le r$, let g_i^+, g_i^- be Radon-Nikodým derivatives of λ_i^+, λ_i^- with respect to $\hat{\lambda}$. Adjusting them on a $\hat{\lambda}$-negligible set if necessary, we may suppose that they are all bounded non-negative Borel measurable functions from $\mathbb{R}^r$ to $\mathbb{R}$. (Recall from 256C that $\hat{\lambda}$ must be the completion of its restriction to the Borel σ-algebra.) Set $g_i = g_i^+ - g_i^-$ for each i. Then

$$\begin{aligned}\int_E \frac{\partial f}{\partial \xi_i} d\mu &= L_i^+(f) - L_i^-(f) = \int f d\lambda_i^+ - \int f d\lambda_i^- \\ &= \int f \times g_i^+ d\hat{\lambda} - \int f \times g_i^- d\hat{\lambda} = \int f \times g_i d\hat{\lambda}\end{aligned}$$

for every Lipschitz function f with compact support (235K). Set $g = \sqrt{\sum_{i=1}^r g_i^2}$. For $i \le r$, set $\psi_i(x) = \frac{g_i(x)}{g(x)}$ when $g(x) \ne 0$, $\frac{1}{\sqrt{r}}$ when $g(x) = 0$, so that $\psi = (\psi_1, \dots, \psi_r) : \mathbb{R}^r \to S_{r-1}$ is Borel measurable. Let λ_E^∂ be the indefinite-integral measure over $\hat{\lambda}$ defined by g; then λ_E^∂ is a Radon measure on $\mathbb{R}^r$ (256E/416Sa).

(c) Now take any Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support. Express it as $(\phi_1, \dots, \phi_r)$ where $\phi_i : \mathbb{R}^r \to \mathbb{R}$ is a Lipschitz function with compact support for each i. Then

$$\begin{aligned}\int_E \operatorname{div}\phi\, d\mu &= \sum_{i=1}^r \int_E \frac{\partial \phi_i}{\partial \xi_i} d\mu = \sum_{i=1}^r L_i(\phi_i) \\ &= \sum_{i=1}^r L_i^+(\phi_i) - \sum_{i=1}^r L_i^-(\phi_i) = \sum_{i=1}^r \int \phi_i d\hat{\lambda}_i^+ - \sum_{i=1}^r \int \phi_i d\hat{\lambda}_i^- \\ &= \sum_{i=1}^r \int \phi_i \times g_i^+ d\hat{\lambda} - \sum_{i=1}^r \int \phi_i \times g_i^- d\hat{\lambda}\end{aligned}$$

(by 235K again)

$$= \sum_{i=1}^r \int \phi_i \times g_i d\hat{\lambda} = \sum_{i=1}^r \int \phi_i \times \psi_i d\lambda_E^\partial$$

(235K once more, because $\psi_i \times g = g_i$)

$$= \int \phi \centerdot \psi \, d\lambda_E^\partial.$$

So we have λ_E^∂ and ψ satisfying (i).

(d) Now suppose that $G \subseteq \mathbb{R}^r$ is open. If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function with compact support and $\|\phi\| \le \chi G$, then

$$|\int_E \operatorname{div} \phi \, d\mu| = |\int \phi \,.\, \psi \, d\lambda_E^\partial| \le \int \|\phi\| d\lambda_E^\partial \le \lambda_E^\partial(G).$$

On the other hand, if $\gamma < \lambda_E^\partial(G)$, let $G_0 \subseteq G$ be a bounded open set such that $\gamma < \lambda_E^\partial(G_0)$, and set $\epsilon = \frac{1}{3}(\lambda_E^\partial(G_0) - \gamma)$. Let $K \subseteq G_0$ be a compact set such that $\lambda_E^\partial(G_0 \setminus K) \le \epsilon$. Let $\delta > 0$ be such that $\|x - y\| \ge 2\delta$ whenever $y \in K$ and $x \in \mathbb{R}^r \setminus G_0$, and set $H = \{x : \inf_{y \in K} \|x - y\| < \delta\}$. Now there are $f_1, \dots , f_r \in C_k$ such that

$$\sum_{i=1}^r f_i^2 \le \chi H, \quad \int \sum_{i=1}^r f_i \times \psi_i d\lambda_E^\partial \ge \gamma.$$

P For each $i \le r$, we can find a sequence $\langle g_{mi} \rangle_{m \in \mathbb{N}}$ in C_k such that $\int |g_{mi} - (\psi_i \times \chi K)| d\lambda_E^\partial \le 2^{-m}$ for every $m \in \mathbb{N}$ (416I); multiplying the g_{mi} by a function which takes the value 1 on K and 0 outside H if necessary, we can suppose that $g_{mi}(x) = 0$ for $x \notin H$. Set

$$f_{mi} = \frac{g_{mi}}{\max(1, \sqrt{\sum_{j=1}^r g_{mj}^2})} \in C_k$$

for each m and i. Now $\lim_{m \to \infty} f_{mi}(x) = \psi_i(x)$ for every $i \le r$ whenever $\lim_{m \to \infty} g_{mi}(x) = \psi_i(x)$ for every $i \le r$, which is the case for λ_E^∂-almost every $x \in K$. Also $\sum_{i=1}^r f_{mi}^2 \le \chi H$ for every m, so $|\sum_{i=1}^r f_{mi} \times \psi_i| \le \chi H$ for every m, while

$$\lim_{m \to \infty} \sum_{i=1}^r \int_K f_{mi} \times \psi_i d\lambda_E^\partial = \sum_{i=1}^r \int_K \psi_i^2 d\lambda_E^\partial = \lambda_E^\partial(K).$$

At the same time,

$$|\sum_{i=1}^r \int_{\mathbb{R}^r \setminus K} f_{mi} \times \psi_i d\lambda_E^\partial| \le \lambda_E^\partial(H \setminus K) \le \epsilon$$

for every m, so

$$\sum_{i=1}^r \int f_{mi} \times \psi_i d\lambda_E^\partial \ge \lambda_E^\partial(G_0) - 3\epsilon = \gamma$$

for all m large enough, and we may take $f_i = f_{mi}$ for such an m. **Q**

Now, for $n \in \mathbb{N}$, set

$$\phi_n = (f_1 * \tilde{h}_n, \dots , f_r * \tilde{h}_n) \in \mathcal{D}_r.$$

For all n large enough, we shall have $\|x - y\| \ge \frac{1}{n+1}$ for every $x \in \mathbb{R}^r \setminus G_0$ and $y \in H$, so that $\phi_n(x) = 0$ if $x \notin G_0$. By 473Dg,

$$\|\phi_n(x)\| \le \sup_{y \in \mathbb{R}} \sqrt{\sum_{i=1}^r f_i(y)^2} \le 1$$

for every x and n, so that $\|\phi_n\| \le \chi G_0$ for all n large enough. Next, $\lim_{n \to \infty} \phi_n(x) = (f_1(x), \dots , f_r(x))$ for every $x \in \mathbb{R}^r$ (473Ed), so

$$\int_E \operatorname{div} \phi_n d\mu = \int \phi_n \,.\, \psi \, d\lambda_E^\partial \to \int \sum_{i=1}^r f_i \times \psi_i d\lambda_E^\partial \ge \gamma$$

as $n \to \infty$, by Lebesgue's Dominated Convergence Theorem. As γ is arbitrary,

$$\begin{aligned} \lambda_E^\partial(G) &\le \sup\{\int_E \operatorname{div} \phi \, d\mu : \phi \in \mathcal{D}_r, \, \|\phi\| \le \chi G\} \\ &\le \sup\{\int_E \operatorname{div} \phi \, d\mu : \phi \text{ is Lipschitz}, \, \|\phi\| \le \chi G\} \end{aligned}$$

and we have equality.

(e) Thus λ_E^∂ must satisfy (ii). By 416Eb, it is uniquely defined. Now suppose that $\hat{\psi}$ is another function from a λ_E^∂-conegligible set to $\mathbb{R}^r$ and satisfies (i). Then

$$\int \phi \,.\, \psi \, d\lambda_E^\partial = \int \phi \,.\, \hat{\psi} \, d\lambda_E^\partial$$

for every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support. Take any $i \le r$ and any compact set $K \subseteq \mathbb{R}^r$. For $m \in \mathbb{N}$, set $f_m(x) = \max(0, 1 - 2^m \inf_{y \in K} \|y - x\|)$ for $x \in \mathbb{R}^r$, so that $\langle f_m \rangle_{m \in \mathbb{N}}$ is a sequence of Lipschitz functions with compact support and $\lim_{m \to \infty} f_m = \chi K$. Set

$$\phi_m = (0, \dots, f_m, \dots, 0),$$

where the non-zero term is in the ith position. Then

$$\int_K \psi_i d\lambda_E^\partial = \lim_{m\to\infty} \int f_m \times \psi_i d\lambda_E^\partial = \lim_{m\to\infty} \int \phi_m \centerdot \psi \, d\lambda_E^\partial$$
$$= \lim_{m\to\infty} \int \phi_m \centerdot \hat{\psi} \, d\lambda_E^\partial = \int_K \hat{\psi}_i d\lambda_E^\partial.$$

By the Monotone Class Theorem (136C), or otherwise, $\int_F \hat{\psi}_i d\lambda_E^\partial = \int_F \psi_i d\lambda_E^\partial$ for every bounded Borel set F, so that $\hat{\psi}_i = \psi_i$ λ_E^∂-a.e.; as i is arbitrary, $\psi = \hat{\psi}$ λ_E^∂-a.e. This completes the proof.

474F Definitions In the context of 474E, I will call λ_E^∂ the **perimeter measure** of E, and if ψ is a function from a λ_E^∂-conegligible subset of $\mathbb{R}^r$ to S_{r-1} which has the property in (i) of the theorem, I will call it an **outward-normal** function for E.

The words 'perimeter' and 'outward normal' are intended to suggest geometric interpretations; much of this section and the next will be devoted to validating this suggestion.

Observe that if E has locally finite perimeter, then $\operatorname{per} E = \lambda_E^\partial(\mathbb{R}^r)$. The definitions in 474D-474E also make it clear that if E, $F \subseteq \mathbb{R}$ are Lebesgue measurable and $\mu(E \triangle F) = 0$, then $\operatorname{per} E = \operatorname{per} F$ and E has locally finite perimeter iff F has; and in this case $\lambda_E^\partial = \lambda_F^\partial$ and an outward-normal function for E is an outward-normal function for F.

474G The reduced boundary Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter; let λ_E^∂ be its perimeter measure and ψ an outward-normal function for E. The **reduced boundary** $\partial^\$ E$ is the set of those $y \in \mathbb{R}^r$ such that, for some $z \in S_{r-1}$,

$$\lim_{\delta\downarrow 0} \frac{1}{\lambda_E^\partial B(y,\delta)} \int_{B(y,\delta)} \|\psi(x) - z\| \, \lambda_E^\partial(dx) = 0.$$

(When requiring that the limit be defined, I mean to insist that $\lambda_E^\partial B(y,\delta)$ should be non-zero for every $\delta > 0$, that is, that y belongs to the support of λ_E^∂. **Warning!** Some authors use the phrase 'reduced boundary' for a slightly larger set.) Note that, writing $\psi = (\psi_1, \dots, \psi_r)$ and $z = (\zeta_1, \dots, \zeta_r)$, we must have

$$\zeta_i = \lim_{\delta\downarrow 0} \frac{1}{\lambda_E^\partial B(y,\delta)} \int_{B(y,\delta)} \psi_i d\lambda_E^\partial,$$

so that z is uniquely defined; call it $\psi_E(y)$. Of course $\partial^\$ E$ and ψ_E are determined entirely by the set E, because λ_E^∂ is uniquely determined and ψ is determined up to a λ_E^∂-negligible set (474E).

By Besicovitch's Density Theorem (472Db),

$$\lim_{\delta\downarrow 0} \frac{1}{\lambda_E^\partial B(x,\delta)} \int_{B(x,\delta)} |\psi_i(x) - \psi_i(y)| \lambda_E^\partial(dx) = 0$$

for every $i \le r$, for λ_E^∂-almost every $y \in \mathbb{R}^r$; and for any such y, $\psi_E(y)$ is defined and equal to $\psi(y)$. Thus $\partial^\$ E$ is λ_E^∂-conegligible and ψ_E is an outward-normal function for E. I will call $\psi_E : \partial^\$ E \to S_{r-1}$ the **canonical outward-normal function** of E.

Once again, we see that if E, $F \subseteq \mathbb{R}^r$ are sets with locally finite perimeter and $E \triangle F$ is Lebesgue negligible, then they have the same reduced boundary and the same canonical outward-normal function.

474H Invariance under isometries: Proposition Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Let λ_E^∂ be its perimeter measure, and ψ_E its canonical outward-normal function. If $T : \mathbb{R}^r \to \mathbb{R}^r$ is any isometry, then $T[E]$ has locally finite perimeter, $\lambda_{T[E]}^\partial$ is the image measure $\lambda_E^\partial T^{-1}$, the reduced boundary $\partial^\$ T[E]$ is $T[\partial^\$ E]$, and the canonical outward-normal function of $T[E]$ is $S\psi_E T^{-1}$, where S is the derivative of T.

proof (a) As noted in 474C, the derivative of T is constant, and is an orthogonal matrix. Suppose that $n \in \mathbb{N}$. Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function such that $\|\phi\| \le \chi B(\mathbf{0}, n)$. Then

$$|\int_{T[E]}\operatorname{div}\phi\,d\mu| = |\int_E(\operatorname{div}\phi)T\,d\mu|$$

(263D, because $|\det S| = 1$)

$$= |\int_E \operatorname{div}(T^{-1}\phi T)d\mu|$$

(474C)

$$= |\int_E \operatorname{div}(S^{-1}\phi T)d\mu|$$

(because $S^{-1}\phi T$ and $T^{-1}\phi T$ differ by a constant, and must have the same derivative)

$$\le \lambda^\partial_E(T^{-1}[B(\mathbf{0},n)])$$

because $S^{-1}\phi T$ is a Lipschitz function and

$$\|S^{-1}\phi T\| = \|\phi T\| \le \chi T^{-1}[B(\mathbf{0},n)].$$

Since $T^{-1}[B(\mathbf{0},n)]$ is bounded, $\lambda^\partial_E(T^{-1}[B(\mathbf{0},n)])$ is finite for every n, and $T[E]$ has locally finite perimeter.

(b) We can therefore speak of its perimeter measure $\lambda^\partial_{T[E]}$. Let $G \subseteq \mathbb{R}^r$ be an open set. If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function and $\|\phi\| \le \chi T[G]$, then

$$|\int_{T[E]}\operatorname{div}\phi\,d\mu| = |\int_E \operatorname{div}(S^{-1}\phi T)d\mu| \le \lambda^\partial_E(G)$$

because $S^{-1}\phi T$ is a Lipschitz function dominated by χG. As ϕ is arbitrary, $\lambda^\partial_{T[E]}(T[G]) \le \lambda^\partial_E(G)$. Applying the same argument in reverse, with T^{-1} in the place of T, we see that $\lambda^\partial_E(G) \le \lambda^\partial_{T[E]}(T[G])$, so the two are equal. This means that the Radon measures $\lambda^\partial_{T[E]}$ and $\lambda^\partial_E T^{-1}$ (418I) agree on open sets, and must be identical (416Eb again).

(c) Now consider $S\psi_E T^{-1}$. Since ψ_E is defined λ^∂_E-almost everywhere and takes values in S_{r-1}, $\psi_E T^{-1}$ and $S\psi_E T^{-1}$ are defined $\lambda^\partial_{T[E]}$-almost everywhere and take values in S_{r-1}. If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function with compact support,

$$\int_{T[E]}\operatorname{div}\phi\,d\mu = \int_E(\operatorname{div}\phi)T\,d\mu = \int_E \operatorname{div}(T^{-1}\phi T)d\mu$$

$$= \int_E \operatorname{div}(S^{-1}\phi T)d\mu = \int (S^{-1}\phi T)\boldsymbol{.}\psi_E\,d\lambda^\partial_E$$

$$= \int (\phi T)\boldsymbol{.}(S\psi_E)d\lambda^\partial_E$$

(because S is orthogonal)

$$= \int \phi\boldsymbol{.}(S\psi_E T^{-1})d(\lambda^\partial_E T^{-1})$$

(235G)

$$= \int \phi\boldsymbol{.}(S\psi_E T^{-1})d\lambda^\partial_{T[E]}.$$

Accordingly $S\psi_E T^{-1}$ is an outward-normal function for $T[E]$. Write $\psi_{T[E]}$ for the canonical outward-normal function of $T[E]$.

(d) Take $y \in \mathbb{R}$ and consider

$$\frac{1}{\lambda^\partial_{T[E]}B(y,\delta)}\int_{B(y,\delta)}\|S\psi_E T^{-1}(x) - S\psi_E T^{-1}(y)\|\lambda^\partial_{T[E]}(dx)$$

$$= \frac{1}{\lambda^\partial_E B(T^{-1}(y),\delta)}\int_{B(T^{-1}(y),\delta)}\|S\psi_E(x) - S\psi_E T^{-1}(y)\|\lambda^\partial_E(dx)$$

$$= \frac{1}{\lambda^\partial_E B(T^{-1}(y),\delta)}\int_{B(T^{-1}(y),\delta)}\|\psi_E(x) - \psi_E T^{-1}(y)\|\lambda^\partial_E(dx)$$

for any $\delta > 0$ for which

$$\lambda^{\partial}_{T[E]}B(y,\delta) = \lambda^{\partial}_{E}T^{-1}[B(y,\delta)] = \lambda^{\partial}_{E}B(T^{-1}(y),\delta)$$

is non-zero. We see that

$$\lim_{\delta\downarrow 0}\frac{1}{\lambda^{\partial}_{T[E]}B(y,\delta)}\int_{B(y,\delta)}\|S\psi_E T^{-1}(x) - S\psi_E T^{-1}(y)\|\lambda^{\partial}_{T[E]}(dx)$$

is defined and equal to 0 whenever

$$\frac{1}{\lambda^{\partial}_{E}B(T^{-1}(y),\delta)}\int_{B(T^{-1}(y),\delta)}\|\psi_E(x) - \psi_E T^{-1}(y)\|\lambda^{\partial}_{E}(dx)$$

is defined and equal to 0, that is, $T^{-1}(y) \in \partial^{\$}E$. In this case, $y \in \partial^{\$}T[E]$ and $S\psi_E T^{-1}(y) = \psi_{T[E]}(y)$. So $\partial^{\$}T[E] \supseteq T[\partial^{\$}E]$ and $S\psi_E T^{-1}$ extends $\psi_{T[E]}$.

Applying the argument to T^{-1}, we see that $S^{-1}\psi_{T[E]}T$ extends ψ_E, that is, $\psi_{T[E]}$ extends $S\psi_E T^{-1}$. So $S\psi_E T^{-1}$ is exactly the canonical outward-normal function of $T[E]$, and its domain $T[\partial^{\$}E]$ is $\partial^{\$}T[E]$.

474I Half-spaces It will be useful, and perhaps instructive, to check the most elementary special case.

Proposition Let $H \subseteq \mathbb{R}^r$ be a half-space $\{x : x\,.\,v \le \alpha\}$, where $v \in S^{r-1}$. Then H has locally finite perimeter; its perimeter measure λ^{∂}_{H} is defined by saying

$$\lambda^{\partial}_{H}(F) = \nu(F \cap \partial H)$$

whenever $F \subseteq \mathbb{R}^r$ is such that ν measures $F \cap \partial H$, and the constant function with value v is an outward-normal function for H.

proof (a) Suppose, to begin with, that v is the unit vector $(0,\dots,0,1)$ and that $\alpha = 0$, so that $H = \{x : \xi_r \le 0\}$. Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function with compact support. Then for any $i < r$

$$\int_H \frac{\partial\phi_i}{\partial\xi_i}\mu(dx) = 0$$

because we can regard this as a multiple integral in which the inner integral is with respect to ξ_i and is therefore always zero. On the other hand, integrating with respect to the rth coordinate first,

$$\begin{aligned}\int_H \frac{\partial\phi_r}{\partial\xi_r}\mu(dx) &= \int_{\mathbb{R}^{r-1}}\int_{-\infty}^{0}\frac{\partial\phi_r}{\partial\xi_r}(z,t)dt\,\mu_{r-1}(dz)\\ &= \int_{\mathbb{R}^{r-1}}\phi_r(z,0)\mu_{r-1}(dz) = \int_{\partial H}\phi_r(x)\nu(dx)\end{aligned}$$

(identifying ν on $\mathbb{R}^{r-1}\times\{0\}$ with μ_{r-1} on $\mathbb{R}^{r-1}$)

$$= \int_{\partial H}\phi\,.\,v\,d\nu = \int \phi\,.\,v\,d\lambda$$

where λ is the indefinite-integral measure over ν defined by the function $\chi(\partial H)$. Note that (by 234La) λ can also be regarded as $\nu_{\partial H}\iota^{-1}$, where $\nu_{\partial H}$ is the subspace measure on ∂H and $\iota : \partial H \to \mathbb{R}^r$ is the identity map. Now $\nu_{\partial H}$ can be identified with Lebesgue measure on $\mathbb{R}^{r-1}$, by 265B or otherwise, so in particular is a Radon measure, and λ also is a Radon measure, by 418I again or otherwise.

This means that λ and the constant function with value v satisfy the conditions of 474E, and must be the perimeter measure of H and an outward-normal function.

(b) For the general case, let S be an orthogonal matrix such that $S(0,\dots,0,1) = v$, and set $T(x) = S(x) + \alpha v$ for every x, so that $H = T[\{x : \xi_r \le 0\}]$. By 474H, the perimeter measure of H is λT^{-1} and the constant function with value v is an outward-normal function for H. Now the Radon measure $\lambda^{\partial}_{H} = \lambda T^{-1}$ is defined by saying that

$$\lambda^{\partial}_{H}F = \lambda T^{-1}[F] = \nu(T^{-1}[F] \cap \{x : \xi_r = 0\}) = \nu(F \cap T[\{x : \xi_r = 0\}]) = \nu(F \cap \partial H)$$

whenever $\nu(F \cap \partial H)$ is defined, because ν (being a scalar multiple of a Hausdorff measure) must be invariant under the isometry T.

474J Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Let λ_E^∂ be the perimeter measure of E, and ψ_E its canonical outward-normal function. Then $\mathbb{R}^r \setminus E$ also has locally finite perimeter; its perimeter measure is λ_E^∂, its reduced boundary is $\partial^\$ E$, and its canonical outward-normal function is $-\psi_E$.

proof Of course $\mathbb{R}^r \setminus E$ is Lebesgue measurable. By 474Bc,

$$\int_{\mathbb{R}^r \setminus E} \operatorname{div} \phi \, d\mu = -\int_E \operatorname{div} \phi \, d\mu = \int \phi \mathbin{.} (-\psi_E) \, d\lambda_E^\partial$$

for every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support. The uniqueness assertions in 474E tell us that $\mathbb{R}^r \setminus E$ has locally finite perimeter, that its perimeter measure is λ_E^∂, and that $-\psi_E$ is an outward-normal function for $\mathbb{R}^r \setminus E$. Referring to the definition of 'reduced boundary' in 474G, we see at once that $\partial^\$(\mathbb{R}^r \setminus E) = \partial^\$ E$ and that $\psi_{\mathbb{R}^r \setminus E} = -\psi_E$.

474K Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter; let λ_E^∂ be its perimeter measure, and ψ an outward-normal function for E. Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function with compact support, and $g \in \mathcal{D}$ an even function. Then

$$\int \phi \mathbin{.} \operatorname{grad}(g * \chi E) d\mu + \int (g * \phi) \mathbin{.} \psi \, d\lambda_E^\partial = 0.$$

proof

$$\int \phi \mathbin{.} \operatorname{grad}(g * \chi E) d\mu = -\int (g * \chi E) \times \operatorname{div} \phi \, d\mu$$

(474Bd, using 473Dd to see that $g * \chi E$ is Lipschitz)

$$= -\int \chi E \times \operatorname{div}(g * \phi) d\mu$$

(474Be)

$$= -\int (g * \phi) \mathbin{.} \psi \, d\lambda_E^\partial$$

(because $g * \phi$ is smooth and has compact support, so is Lipschitz), as required.

474L Two isoperimetric inequalities: Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and λ_E^∂ its perimeter measure.

(a) If E is bounded, then $(\mu E)^{(r-1)/r} \le \operatorname{per} E$.

(b) If $B \subseteq \mathbb{R}^r$ is a closed ball, then

$$\min(\mu(B \cap E), \mu(B \setminus E))^{(r-1)/r} \le 2c\lambda_E^\partial(\operatorname{int} B),$$

where $c = 2^{r+4}\sqrt{r}(1 + 2^{r+1})$.

proof (a) Let $\epsilon > 0$. By 473Ef, there is an $n \in \mathbb{N}$ such that $\|f - \chi E\|_{r/(r-1)} \le \epsilon$, where $f = \chi E * \tilde{h}_n$. Note that f is smooth (473De again) and has compact support, because E is bounded. Let $\eta > 0$ be such that

$$\int \|\operatorname{grad} f\| d\mu \le \int \frac{\|\operatorname{grad} f\|^2}{\sqrt{\eta + \|\operatorname{grad} f\|^2}} \, d\mu + \epsilon,$$

and set $\phi = \dfrac{\operatorname{grad} f}{\sqrt{\eta + \|\operatorname{grad} f\|^2}}$. Then $\phi \in \mathcal{D}_r$ and $\|\phi(x)\| \le 1$ for every $x \in \mathbb{R}^r$. Now we can estimate

$$\int \|\operatorname{grad} f\| d\mu \le \int \phi \mathbin{.} \operatorname{grad} f \, d\mu + \epsilon$$

$$= -\int (\tilde{h}_n * \phi) \mathbin{.} \psi \, d\lambda_E^\partial + \epsilon$$

(where ψ is an outward-normal function for E, by 474K)

$$\le \int \|\tilde{h}_n * \phi\| d\lambda_E^\partial + \epsilon \le \operatorname{per} E + \epsilon$$

because $\|(\tilde{h}_n * \phi)(x)\| \le 1$ for every $x \in \mathbb{R}^r$, by 473Dg. Accordingly

$$(\mu E)^{(r-1)/r} = \|\chi E\|_{r/(r-1)} \le \|f\|_{r/(r-1)} + \epsilon \le \int \|\operatorname{grad} f\| d\mu + \epsilon$$

(473H)

$$\le \operatorname{per} E + 2\epsilon.$$

As ϵ is arbitrary, we have the result.

(b)(i) Set $\alpha = \min(\mu(B\cap E), \mu(B\setminus E))$. If $\alpha = 0$, the result is trivial; so suppose that $\alpha > 0$. Take any $\epsilon \in \,]0,\alpha]$. Let B_1 be a closed ball, with the same centre as B and strictly smaller non-zero radius, such that $\mu(B\setminus B_1) \le \epsilon$; then $\alpha - \epsilon \le \min(\mu(B_1\cap E), \mu(B_1\setminus E))$. For $f \in \mathcal{L}^{r/(r-1)}(\mu)$ set

$$\gamma_0(f) = \frac{1}{\mu B_1}\int_{B_1} f d\mu, \quad \gamma_1(f) = \|(f\times\chi B_1) - \gamma_0(f)\chi B_1\|_{r/(r-1)};$$

then both γ_0 and γ_1 are continuous functions on $\mathcal{L}^{r/(r-1)}(\mu)$ if we give it its usual pseudometric $(f,g)\mapsto \|f-g\|_{r/(r-1)}$. Now $\gamma_1(\chi(E\cap B)) \ge \frac{1}{2}(\alpha-\epsilon)^{(r-1)/r}$. **P** We have

$$\gamma_0(\chi(E\cap B)) = \frac{\mu(B_1\cap E)}{\mu B_1},$$

$$\gamma_1(\chi(E\cap B))^{r/(r-1)} = \int_{B_1} |\chi(E\cap B) - \gamma_0(\chi(E\cap B))|^{r/(r-1)}$$

$$= \mu(B_1\cap E)\big(1 - \frac{\mu(B_1\cap E)}{\mu B_1}\big)^{r/(r-1)} + \mu(B_1\setminus E)\big(\frac{\mu(B_1\cap E)}{\mu B_1}\big)^{r/(r-1)}$$

$$= \mu(B_1\cap E)\big(\frac{\mu(B_1\setminus E)}{\mu B_1}\big)^{r/(r-1)} + \mu(B_1\setminus E)\big(\frac{\mu(B_1\cap E)}{\mu B_1}\big)^{r/(r-1)}.$$

Either $\mu(B_1\cap E) \ge \frac{1}{2}\mu B_1$ or $\mu(B_1\setminus E) \ge \frac{1}{2}\mu B_1$; suppose the former. Then

$$\gamma_1(\chi(E\cap B))^{r/(r-1)} \ge \frac{1}{2^{r/(r-1)}}\mu(B_1\setminus E) \ge \frac{1}{2^{r/(r-1)}}(\alpha-\epsilon)$$

and $\gamma_1(\chi(E\cap B)) \ge \frac{1}{2}(\alpha-\epsilon)^{(r-1)/r}$. Exchanging $B_1\cap E$ and $B_1\setminus E$ we have the same result if $\mu(B_1\cap E) \ge \frac{1}{2}\mu B_1$. **Q**

(ii) Express B as $B(y,\delta)$ and B_1 as $B(y,\delta_1)$. Take $n_0 \ge \frac{2}{\delta-\delta_1}$. Because γ_1 is $\| \,\|_{r/(r-1)}$-continuous, there is an $n \ge n_0$ such that $\gamma_1(f) \ge \frac{1}{2}(\alpha-\epsilon)^{(r-1)/r} - \epsilon$, where $f = \tilde{h}_n * \chi(E\cap B)$ (473Ef); as in part (a) of the proof, $f \in \mathcal{D}$. Let $\eta > 0$ be such that

$$\int_{B_1} \frac{\|\operatorname{grad} f\|^2}{\sqrt{\eta + \|\operatorname{grad} f\|^2}} d\mu \ge \int_{B_1} \|\operatorname{grad} f\| d\mu - \epsilon.$$

Let $m \ge n_0$ be such that $\int \phi \,\boldsymbol{.}\, \operatorname{grad} f\, d\mu \ge \int_{B_1} \|\operatorname{grad} f\| d\mu - 2\epsilon$, where

$$\phi = \tilde{h}_m * \Big(\frac{\operatorname{grad} f}{\sqrt{\eta + \|\operatorname{grad} f\|^2}} \times \chi B_1\Big).$$

Note that $\phi(x) = 0$ if $\|x - y\| \ge \frac{1}{2}(\delta+\delta_1)$, so that $(\tilde{h}_n * \phi)(x) = 0$ if $x \notin \operatorname{int} B$. By 473Dg, $\|\phi(x)\| \le 1$ for every x and $\|(\tilde{h}_n * \phi)(x)\| \le 1$ for every x, so $\|\tilde{h}_n * \phi\| \le \chi(\operatorname{int} B)$.

Now we have

$$\int \phi \,\boldsymbol{.}\, \operatorname{grad} f\, d\mu = \int \phi \,\boldsymbol{.}\, \operatorname{grad}(\tilde{h}_n * \chi(E\cap B))\, d\mu$$

$$= -\int (\tilde{h}_n * \chi(E\cap B)) \times \operatorname{div}\phi\, d\mu$$

(474Bd)

$$= -\int_{E\cap B} \operatorname{div}(\tilde{h}_n * \phi) d\mu$$

(474Be)

$$= -\int_E \operatorname{div}(\tilde{h}_n * \phi) d\mu \le \lambda_E^\partial(\operatorname{int} B)$$

(474E).

(iii) Accordingly

$$\frac{1}{2}(\alpha - \epsilon)^{(r-1)/r} - \epsilon \le \gamma_1(f) \le c \int_{B_1} \| \operatorname{grad} f \| d\mu$$

(473K)

$$\le c\Bigl(\int \phi \,\boldsymbol{.}\, \operatorname{grad} f \, d\mu + 2\epsilon\Bigr) \le c(\lambda_E^\partial(\operatorname{int} B) + 2\epsilon).$$

As ϵ is arbitrary, $\alpha^{(r-1)/r} \le 2c\lambda_E^\partial(\operatorname{int} B)$, as claimed.

474M Lemma Suppose that $E \subseteq \mathbb{R}^r$ has locally finite perimeter, with perimeter measure λ_E^∂ and an outward-normal function ψ. Then for any $y \in \mathbb{R}^r$ and any Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$,

$$\int_{E \cap B(y,\delta)} \operatorname{div} \phi \, d\mu = \int_{B(y,\delta)} \phi \,\boldsymbol{.}\, \psi \, d\lambda_E^\partial + \int_{E \cap \partial B(y,\delta)} \phi(x) \,\boldsymbol{.}\, \frac{1}{\delta}(x-y) \, \nu(dx)$$

for almost every $\delta > 0$.

proof (a) For $t > 0$, set

$$w(t) = \int_{E \cap \partial B(y,t)} \phi(x) \,\boldsymbol{.}\, \frac{1}{t}(x-y) \, \nu(dx)$$

when this is defined. By 265G, applied to functions of the form

$$x \mapsto \begin{cases} \phi(x) \,\boldsymbol{.}\, \frac{x-y}{\|x-y\|} & \text{if } x \in E \text{ and } 0 < \|x-y\| \le \alpha \\ 0 & \text{otherwise,} \end{cases}$$

w is defined almost everywhere in $]0, \infty[$ and is measurable (for Lebesgue measure on $\mathbb{R}$).

Let $\delta > 0$ be any point in the Lebesgue set of w (223D). Then

$$\lim_{t \downarrow 0} \frac{1}{t} \int_\delta^{\delta+t} |w(s) - w(\delta)| ds \le 2 \lim_{t \downarrow 0} \frac{1}{2t} \int_{\delta - t}^{\delta+t} |w(s) - w(\delta)| = 0.$$

Let $\epsilon > 0$. Then there is an $\eta > 0$ such that

$$\frac{1}{\eta} \int_\delta^{\delta+\eta} |w(s) - w(\delta)| ds \le \epsilon, \quad \int_{B(y,\delta+\eta) \setminus B(y,\delta)} \|\phi\| d\lambda_E^\partial \le \epsilon,$$

$$\int_{B(y,\delta+\eta) \setminus B(y,\delta)} \| \operatorname{div} \phi \| d\mu \le \epsilon.$$

(b) Set

$$\begin{aligned} g(x) &= 1 \text{ if } \|x-y\| \le \delta, \\ &= 1 - \frac{1}{\eta}(\|x-y\| - \delta) \text{ if } \delta \le \|x-y\| \le \delta + \eta, \\ &= 0 \text{ if } \|x-y\| \ge \delta + \eta. \end{aligned}$$

Then g is continuous, and $\operatorname{grad} g(x) = \mathbf{0}$ if $\|x-y\| < \delta$ or $\|x-y\| > \delta + \eta$; while if $\delta < \|x-y\| < \delta + \eta$, $\operatorname{grad} g(x) = -\frac{x-y}{\eta\|x-y\|}$. This means that

$$\begin{aligned} \int_E \phi \,\boldsymbol{.}\, \operatorname{grad} g \, d\mu &= -\frac{1}{\eta} \int_\delta^{\delta+\eta} \int_{E \cap \partial B(y,t)} \frac{1}{t}(x-y) \,\boldsymbol{.}\, \phi(x) \, \nu(dx) dt \\ &= -\frac{1}{\eta} \int_\delta^{\delta+\eta} w(t) dt. \end{aligned}$$

By the choice of η,

$$|\int_E \phi\,\boldsymbol{.}\,\mathrm{grad}\,g\,d\mu + w(\delta)| \le \epsilon.$$

(c) By 474E and 474Bb we have

$$\int (g\times\phi)\,\boldsymbol{.}\,\psi\,d\lambda_E^\partial = \int_E \mathrm{div}(g\times\phi)d\mu$$

(of course $g\times\phi$ is Lipschitz, by 473Ca and 262Ba)

$$= \int_E \phi\,\boldsymbol{.}\,\mathrm{grad}\,g\,d\mu + \int_E g\times\mathrm{div}\,\phi\,d\mu.$$

Next, by the choice of η,

$$|\int((g\times\phi)\,\boldsymbol{.}\,\psi\,d\lambda_E^\partial - \int_{B(y,\delta)}\phi\,\boldsymbol{.}\,\psi\,d\lambda_E^\partial| \le \int_{B(y,\delta+\eta)\setminus B(y,\delta)}\|\phi\|d\lambda_E^\partial \le \epsilon,$$

while

$$|\int_E \phi\,\boldsymbol{.}\,\mathrm{grad}\,g\,d\mu + \int_{E\cap\partial B(y,\delta)}\phi(x)\,\boldsymbol{.}\,\tfrac{1}{\delta}(x-y)\nu(dx)| = |\int_E \phi\,\boldsymbol{.}\,\mathrm{grad}\,g\,d\mu + w(\delta)| \le \epsilon$$

and

$$|\int_E g\times\mathrm{div}\,\phi\,d\mu - \int_{E\cap B(y,\delta)}\mathrm{div}\,\phi\,d\mu| \le \int_{B(y,\delta+\eta)\setminus B(y,\delta)}\|\mathrm{div}\,\phi\|d\mu \le \epsilon.$$

Putting these together, we have

$$|\int_{E\cap B(y,\delta)}\mathrm{div}\,\phi\,d\mu - \int_{B(y,\delta)}\phi\,\boldsymbol{.}\,\psi\,d\lambda_E^\partial - \int_{E\cap\partial B(y,\delta)}\phi(x)\,\boldsymbol{.}\,\tfrac{1}{\delta}(x-y)\nu(dx)| \le 3\epsilon.$$

As ϵ is arbitrary, this gives the result.

474N Lemma Let $E\subseteq\mathbb{R}^r$ be a set with locally finite perimeter, and λ_E^∂ its perimeter measure. Then, for any $y\in\partial^\$E$,

(i) $\liminf_{\delta\downarrow 0}\frac{\mu(B(y,\delta)\cap E)}{\delta^r} \ge \frac{1}{(3r)^r}$;

(ii) $\liminf_{\delta\downarrow 0}\frac{\mu(B(y,\delta)\setminus E)}{\delta^r} \ge \frac{1}{(3r)^r}$;

(iii) $\liminf_{\delta\downarrow 0}\frac{\lambda_E^\partial B(y,\delta)}{\delta^{r-1}} \ge \frac{1}{2c(3r)^{r-1}}$,

where $c = 2^{r+4}\sqrt{r}(1+2^{r+1})$;

(iv) $\limsup_{\delta\downarrow 0}\frac{\lambda_E^\partial B(y,\delta)}{\delta^{r-1}} \le 4\pi\beta_{r-2}$.

proof (a) Let ψ_E be the canonical outward-normal function of E (474G). Take $y\in\partial^\$E$. Set

$$\Phi = \{\phi : \phi \text{ is a Lipschitz function from } \mathbb{R}^r \text{ to } B(\mathbf{0},1)\}.$$

Because the space $L^1(\mu)$ is separable in its usual (norm) topology (244I), so is $\{(\mathrm{div}\,\phi\times\chi B(y,1))^\bullet : \phi\in\Phi\}$ (4A2P(a-iv)), and there must be a countable set $\Phi_0\subseteq\Phi$ such that

whenever $\phi\in\Phi$ and $m\in\mathbb{N}$ there is a $\hat\phi\in\Phi_0$ such that $\int_{B(y,1)}|\mathrm{div}\,\phi - \mathrm{div}\,\hat\phi|d\mu \le 2^{-m}$.

Now, for each $\phi\in\Phi_0$,

$$|\int_{E\cap B(y,\delta)}\mathrm{div}\,\phi\,d\mu| = |\int_{B(y,\delta)}\phi\,\boldsymbol{.}\,\psi_E\,d\lambda_E^\partial + \int_{E\cap\partial B(y,\delta)}\phi(x)\,\boldsymbol{.}\,\tfrac{1}{\delta}(x-y)\nu(dx)|$$
$$\le \lambda_E^\partial B(y,\delta) + \nu(E\cap\partial B(y,\delta))$$

for almost every $\delta>0$, by 474M. But this means that, for almost every $\delta\in\,]0,1]$,

$$\begin{aligned}\operatorname{per}(E\cap B(y,\delta))&=\sup_{\phi\in\Phi}|\int_{E\cap B(y,\delta)}\operatorname{div}\phi\,d\mu|\\&=\sup_{\phi\in\Phi_0}|\int_{E\cap B(y,\delta)}\operatorname{div}\phi\,d\mu|\le\lambda^\partial_E B(y,\delta)+\nu(E\cap\partial B(y,\delta)).\end{aligned}$$

(b) It follows that, for some $\delta_0>0$,

$$\operatorname{per}(E\cap B(y,\delta))\le 3\nu(E\cap\partial B(y,\delta))$$

for almost every $\delta\in\,]0,\delta_0]$. **P** Applying 474M with $\phi(x)=\psi_E(y)$ for every x, we have

$$0=\int_{B(y,\delta)}\psi_E(y)\,\mathbf{.}\,\psi_E(x)\lambda^\partial_E(dx)+\int_{E\cap\partial B(y,\delta)}\psi_E(y)\,\mathbf{.}\,\tfrac{1}{\delta}(x-y)\nu(dx)$$

for almost every $\delta\in[0,1]$. But by the definition of $\psi_E(y)$,

$$\lim_{\delta\downarrow 0}\tfrac{1}{\lambda^\partial_E B(y,\delta)}\int_{B(y,\delta)}\psi_E(y)\,\mathbf{.}\,\psi_E\,d\lambda^\partial_E=\lim_{\delta\downarrow 0}\tfrac{1}{\lambda^\partial_E B(y,\delta)}\int_{B(y,\delta)}\psi_E(y)\,\mathbf{.}\,\psi_E(y)\,d\lambda^\partial_E=1.$$

So there is some $\delta_0>0$ such that, for almost every $\delta\in\,]0,\delta_0]$,

$$\begin{aligned}\lambda^\partial_E B(y,\delta)&\le 2\int_{B(y,\delta)}\psi_E(y)\,\mathbf{.}\,\psi_E\,d\lambda^\partial_E\\&=-2\int_{E\cap\partial B(y,\delta)}\psi_E(y)\,\mathbf{.}\,\tfrac{1}{\delta}(x-y)\nu(dx)\le 2\nu(E\cap\partial B(y,\delta)).\end{aligned}\tag{†}$$

But this means that, for almost every such δ,

$$\operatorname{per}(E\cap B(y,\delta))\le\lambda^\partial_E B(y,\delta)+\nu(E\cap\partial B(y,\delta))\le 3\nu(E\cap\partial B(y,\delta)).\ \mathbf{Q}$$

(c) Set $g(t)=\mu(E\cap B(y,t))$ for $t\ge 0$. By 265G, $g(t)=\int_0^t\nu(E\cap\partial B(y,s))ds$ for every t, so g is absolutely continuous on $[0,1]$ and $g'(t)=\nu(E\cap\partial B(y,t))$ for almost every t. Now we can estimate

$$\begin{aligned}g(t)^{(r-1)/r}&=\mu(E\cap B(y,t))^{(r-1)/r}\le\operatorname{per}(E\cap B(y,t))\\&\le 3\nu(E\cap\partial B(y,t))=3g'(t)\end{aligned}\tag{474La}$$

for almost every $t\in[0,\delta_0]$. So

$$\tfrac{d}{dt}\bigl(g(t)^{1/r}\bigr)=\tfrac{1}{r}g(t)^{(1-r)/r}g'(t)\ge\tfrac{1}{3r}$$

for almost every $t\in[0,\delta_0]$; since $t\mapsto g(t)^{1/r}$ is non-decreasing, $g(t)^{1/r}\ge\dfrac{t}{3r}$ (222C) and $g(t)\ge(3r)^{-r}t^r$ for every $t\in[0,\delta_0]$.

(d) Accordingly

$$\liminf_{\delta\downarrow 0}\tfrac{\mu(B(y,\delta)\cap E)}{\delta^r}\ge\inf_{0<\delta\le\delta_0}\tfrac{\mu(B(y,\delta)\cap E)}{\delta^r}\ge\tfrac{1}{(3r)^r}.$$

This proves (i).

(e) Because $\lambda^\partial_{\mathbb{R}^r\setminus E}=\lambda^\partial_E$ and $-\psi_E$ is the canonical outward-normal function of $\mathbb{R}^r\setminus E$ (474J), y also belongs to $\partial^{\$}(\mathbb{R}^r\setminus E)$, so the second formula of this lemma follows from the first.

(f) By 474Lb,

$$\lambda^\partial_E B(y,\delta)\ge\tfrac{1}{2c}\min\bigl(\mu(B(y,\delta)\cap E),\mu(B(y,\delta)\setminus E)\bigr)^{(r-1)/r}$$

for every $\delta\ge 0$. So

$$\liminf_{\delta\downarrow 0}\frac{\lambda^\partial_E B(y,\delta)}{\delta^{r-1}} \ge \frac{1}{2c}\liminf_{\delta\downarrow 0}\min\bigl(\frac{\mu(B(y,\delta)\cap E)}{\delta^r},\frac{\mu(B(y,\delta)\setminus E)}{\delta^r}\bigr)^{(r-1)/r}$$

$$\ge \frac{1}{2c}\min\bigl(\liminf_{\delta\downarrow 0}\frac{\mu(B(y,\delta)\cap E)}{\delta^r},\liminf_{\delta\downarrow 0}\frac{\mu(B(y,\delta)\setminus E)}{\delta^r}\bigr)^{(r-1)/r}$$

$$\ge \frac{1}{2c}\bigl(\frac{1}{(3r)^r}\bigr)^{(r-1)/r} = \frac{1}{2c(3r)^{r-1}}.$$

Thus (iii) is true.

(g) Returning to the inequality (†) in the proof of (b) above, we have a $\delta_0 > 0$ such that

$$\lambda^\partial_E B(y,\delta) \le 2\nu(E\cap\partial B(y,\delta)) \le 2\nu(\partial B(y,\delta)) = 4\pi\beta_{r-2}\delta^{r-1}$$

(265F) for almost every $\delta \in \,]0,\delta_0]$. But this means that, for any $\delta\in[0,\delta_0[$,

$$\lambda^\partial_E B(y,\delta) \le \inf_{t>\delta}\lambda^\partial_E B(y,t) \le \inf_{t>\delta} 4\pi\beta_{r-2}t^{r-1} = 4\pi\beta_{r-2}\delta^{r-1},$$

and (iv) is true.

474O Definition Let $A\subseteq\mathbb{R}^r$ be any set, and $y\in\mathbb{R}^r$. A **Federer exterior normal to** A **at** y is a $v\in S_{r-1}$ such that,

$$\lim_{\delta\downarrow 0}\frac{\mu^*((H\triangle A)\cap B(y,\delta))}{\mu B(y,\delta)} = 0,$$

where H is the half-space $\{x : (x-y)\,\boldsymbol{.}\,v \le 0\}$.

474P Lemma If $A\subseteq\mathbb{R}^r$ and $y\in\mathbb{R}^r$, there can be at most one Federer exterior normal to A at y.

proof Suppose that v, $v'\in S_{r-1}$ are two Federer exterior normals to E at y. Set

$$H = \{x : (x-y)\,\boldsymbol{.}\,v \le 0\},\quad H' = \{x : (x-y)\,\boldsymbol{.}\,v' \le 0\}.$$

Then

$$\lim_{\delta\downarrow 0}\frac{\mu((H\triangle H')\cap B(y,\delta))}{\mu B(y,\delta)} \le \lim_{\delta\downarrow 0}\frac{\mu^*((H\triangle A)\cap B(y,\delta))}{\mu B(y,\delta)} + \lim_{\delta\downarrow 0}\frac{\mu^*((H'\triangle A)\cap B(y,\delta))}{\mu B(y,\delta)} = 0.$$

But for any $\delta > 0$,

$$(H\triangle H')\cap B(y,\delta) = y + \delta((H_0\triangle H_0')\cap B(\mathbf{0},1)),$$

where

$$H_0 = \{x : x\,\boldsymbol{.}\,v \le 0\},\quad H_0' = \{x : x\,\boldsymbol{.}\,v' \le 0\}.$$

So

$$0 = \lim_{\delta\downarrow 0}\frac{\mu((H\triangle H')\cap B(y,\delta))}{\mu B(y,\delta)} = \frac{\mu((H_0\triangle H_0')\cap B(\mathbf{0},1))}{\mu B(\mathbf{0},1)} = \frac{\mu((H_0\triangle H_0')\cap B(\mathbf{0},n))}{\mu B(\mathbf{0},n)}$$

for every $n\ge 1$, and $\mu(H_0\triangle H_0') = 0$. Since μ is strictly positive, and H_0 and H_0' are both the closures of their interiors, they must be identical; and it follows that $v = v'$.

474Q Lemma Set $c' = 2^{r+3}\sqrt{r-1}(1+2^r)$. Suppose that c^*, ϵ and δ are such that

$$c^*\ge 0,\quad \delta > 0,\quad 0<\epsilon<\frac{1}{\sqrt{2}},\quad c^*\epsilon^3 < \tfrac{1}{4}\beta_{r-1},\quad 4c'\epsilon \le \tfrac{1}{8}\beta_{r-1}.$$

Set $V_\delta = \{z : z\in\mathbb{R}^{r-1},\ \|z\|\le\delta\}$ and $C_\delta = V_\delta\times[-\delta,\delta]$, regarded as a cylinder in $\mathbb{R}^r$. Let $f\in\mathcal{D}$ be such that

$$\int_{C_\delta}\|\operatorname{grad}_{r-1} f\| + \max(\frac{\partial f}{\partial\xi_r},0)d\mu \le c^*\epsilon^3\delta^{r-1},$$

where $\operatorname{grad}_{r-1} f = (\frac{\partial f}{\partial\xi_1},\dots,\frac{\partial f}{\partial\xi_{r-1}},0)$. Set

$$F = \{x : x\in C_\delta,\ f(x)\ge\frac{3}{4}\},\quad F' = \{x : x\in C_\delta,\ f(x)\le\frac{1}{4}\}.$$

and for $\gamma \in \mathbb{R}$ set $H_\gamma = \{x : x \in \mathbb{R}^r,\ \xi_r \le \gamma\}$. Then there is a $\gamma \in \mathbb{R}$ such that

$$\mu(F \triangle (H_\gamma \cap C_\delta)) \le 9\mu(C_\delta \setminus (F \cup F')) + (c^*\beta_{r-1} + 16c')\epsilon\delta^r.$$

proof (a) For $t \in [-\delta, \delta]$ set

$$f_t(z) = f(z,t) \text{ for } z \in \mathbb{R}^{r-1},$$

$$F_t = \{z : z \in V_\delta,\ f_t(z) \ge \frac{3}{4}\}, \quad F'_t = \{z : z \in V_\delta,\ f_t(z) \le \frac{1}{4}\};$$

set

$$\gamma = \sup\bigl(\{-\delta\} \cup \{t : t \in [-\delta,\delta],\ \mu_{r-1}F_t \ge \frac{3}{4}\mu_{r-1}V_\delta\}\bigr),$$

$$G = \{t : t \in [-\delta,\delta],\ \int_{V_\delta} \|\operatorname{grad} f_t\| d\mu_{r-1} \ge \epsilon^2\delta^{r-2}\}.$$

Note that $(\operatorname{grad}_{r-1} f)(z,t) = ((\operatorname{grad} f_t)(z), 0)$, so we have

$$\int_{-\delta}^{\delta} \int_{V_\delta} \|\operatorname{grad} f_t\| d\mu_{r-1} dt = \int_{C_\delta} \|\operatorname{grad}_{r-1} f\| d\mu \le c^*\epsilon^3\delta^{r-1}$$

and $\mu_1 G \le c^*\epsilon\delta$, where μ_1 is Lebesgue measure on $\mathbb{R}$.

(b) If $t \in [-\delta,\delta] \setminus G$, then

$$\min(\mu_{r-1}F'_t, \mu_{r-1}F_t) \le 4c'\epsilon\delta^{r-1}.$$

P If $r > 2$,

$$\min\bigl(\mu_{r-1}F'_t, \mu_{r-1}F_t)\bigr)^{(r-2)/(r-1)} \le 4c' \int_{V_\delta} \|\operatorname{grad} f_t\| d\mu_{r-1}$$

(473L)

$$\le 4c'\epsilon^2\delta^{r-2}$$

because $t \notin G$, so that

$$\begin{aligned}\min(\mu_{r-1}F't, \mu_{r-1}F_t) &\le (4c'\epsilon^2)^{(r-1)/(r-2)}\delta^{r-1}\\ &\le 4c'\epsilon\delta^{r-1}\end{aligned}$$

because $4c' \ge 1$ and $\frac{2(r-1)}{r-2} \ge 1$ and $\epsilon \le 1$. If $r = 2$, then

$$\int_{V_\delta} \|\operatorname{grad} f_t\| d\mu_{r-1} \le \epsilon^2 < \frac{1}{2},$$

so at least one of F'_t, F_t is empty, as noted in 473M, and $\min(\mu_{r-1}F'_t, \mu_{r-1}F_t) = 0$. **Q**

(c) If $-\delta \le s < t \le \delta$, then

$$\int_{-\delta}^{\delta} \max(\frac{\partial f}{\partial \xi_r}(z,\xi), 0) d\xi \ge \int_s^t \frac{\partial f}{\partial \xi_r}(z,\xi) d\xi = f(z,t) - f(z,s) \ge \frac{1}{2}$$

for every $z \in F'_s \cap F_t$. Accordingly

$$\frac{1}{2}\mu_{r-1}(F'_s \cap F_t) \le \int_{C_\delta} \max(\frac{\partial f}{\partial \xi_r}, 0) d\mu \le c^*\epsilon^3\delta^{r-1} < \frac{1}{4}\beta_{r-1}\delta^{r-1} = \frac{1}{4}\mu_{r-1}V_\delta$$

and $\mu_{r-1}(F'_s \cap F_t) < \frac{1}{2}\mu_{r-1}V_\delta$. It follows that if $-\delta \le s < \gamma$, so that there is a $t > s$ such that $\mu_{r-1}F_t \ge \frac{3}{4}\mu_{r-1}V_\delta$, then $\mu_{r-1}F'_s < \frac{3}{4}\mu_{r-1}V_\delta$.

(d) Now

$$\mu((F \triangle (H_\gamma \cap C_\delta)) \le 9\mu(C_\delta \setminus (F \cup F')) + \epsilon\delta^r(c^*\beta_{r-1} + 16c').$$

P Set

$$\tilde{G} = \{t : -\delta \le t \le \delta,\ \mu_{r-1}(F_t \cup F'_t) \le \frac{7}{8}\mu_{r-1}V_\delta\},$$

$$\hat{G}=\{t:-\delta\le t\le\delta,\ \mu_{r-1}F_t\le 4c'\epsilon\delta^{r-1}\},$$

$$\hat{G}'=\{t:-\delta\le t\le\delta,\ \mu_{r-1}F'_t\le 4c'\epsilon\delta^{r-1}\}.$$

Then

$$\frac{1}{8}\mu_{r-1}V_\delta\cdot\mu_1\tilde{G}\le\mu(C_\delta\setminus(F\cup F')),$$

$$\mu(F\cap(V_\delta\times\hat{G}))\le 8c'\epsilon\delta^r,$$

$$\mu(F'\cap(V_\delta\times\hat{G}'))\le 8c'\epsilon\delta^r.$$

So if we set

$$\begin{aligned}W=(C_\delta\setminus(F\cup F'))&\cup(V_\delta\times(\tilde{G}\cup G\cup\{\gamma\}))\\&\cup(F\cap(V_\delta\times\hat{G}))\cup(F'\cap(V_\delta\times\hat{G}')),\end{aligned}$$

we shall have

$$\begin{aligned}\mu W&\le\mu(C_\delta\setminus(F\cup F'))+\mu_1\tilde{G}\cdot\mu_{r-1}V_\delta+\mu_1 G\cdot\mu_{r-1}V_\delta+16c'\epsilon\delta^r\\&\le 9\mu(C_\delta\setminus(F\cup F'))+c^*\epsilon\delta\mu_{r-1}V_\delta+16c'\epsilon\delta^r\\&=9\mu(C_\delta\setminus(F\cup F'))+\epsilon\delta^r(c^*\beta_{r-1}+16c')\end{aligned}$$

(using the estimate of $\mu_1 G$ in (a)).

? Suppose, if possible, that there is a point $(z,t)\in(F\triangle(H_\gamma\cap C_\delta))\setminus W$. Since $t\notin G$, (b) tells us that

$$\min(\mu_{r-1}F'_t,\mu_{r-1}F_t)\le 4c'\epsilon\delta^{r-1}\le\frac{1}{8}\mu_{r-1}V_\delta.$$

So $t\in\hat{G}\cup\hat{G}'$. Also, since $t\notin\tilde{G}$, $\mu_{r-1}F_t+\mu_{r-1}F'_t\ge\frac{7}{8}\mu_{r-1}V_\delta$; so (since $t\ne\gamma$) either $\mu_{r-1}F_t\ge\frac{3}{4}\mu_{r-1}V_\delta$ and $t<\gamma$, or $\mu_{r-1}F'_t\ge\frac{3}{4}\mu_{r-1}V_\delta$ and $t>\gamma$ (by (c)). Now

$$\begin{aligned}t<\gamma&\Longrightarrow\mu_{r-1}F_t\ge\frac{3}{4}\mu_{r-1}V_\delta\\&\Longrightarrow\mu_{r-1}F'_t\le 4c'\epsilon\delta^{r-1}\\&\Longrightarrow t\in\hat{G}'\\&\Longrightarrow(z,t)\notin F'\end{aligned}$$

(because $(z,t)\notin F'\cap(V_\delta\times\hat{G}')$)

$$\Longrightarrow(z,t)\in F$$

(because $(z,t)\notin C_\delta\setminus(F\cup F')$)

$$\Longrightarrow(z,t)\in F\cap H_\gamma,$$

which is impossible. And similarly

$$\begin{aligned}t>\gamma&\Longrightarrow\mu_{r-1}F'\ge\frac{3}{4}\mu_{r-1}V_\delta\\&\Longrightarrow\mu_{r-1}F_t\le 4c'\epsilon\delta^{r-1}\\&\Longrightarrow t\in\hat{G}\\&\Longrightarrow(z,t)\notin F\\&\Longrightarrow(z,t)\notin F\cup H_\gamma,\end{aligned}$$

which is equally impossible. **X**

Thus $F\triangle(H_\gamma\cap C_\delta)\subseteq W$ has measure at most

$$9\mu(C_\delta\setminus(F\cup F'))+\epsilon\delta^r(c^*\beta_{r-1}+16c'),$$

as claimed. **Q**

474R Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, ψ_E its canonical outward-normal function, and y any point of its reduced boundary $\partial^{\$}E$. Then $\psi_E(y)$ is the Federer exterior normal to E at y.

proof Write λ_E^∂ for the perimeter measure of E, as usual.

(a) To begin with (down to the end of (c-ii) below) suppose that $y = \mathbf{0}$ and that $\psi_E(y) = (0, \ldots, 0, 1) = v$ say. Set

$$c = 2^{r+4}\sqrt{r}(1 + 2^{r+1}), \quad c' = 2^{r+3}\sqrt{r-1}(1 + 2^r),$$

$$c_1 = 1 + \max\bigl(4\pi\beta_{r-2}, 2c(3r)^{r-1}\bigr),$$

(counting β_0 as 1, if $r = 2$),

$$c^* = \sqrt{2}(2\sqrt{2})^{r-1}c_1, \quad c_1^* = 10 + \frac{1}{2}(c^* + \frac{16c'}{\beta_{r-1}}).$$

As in 474Q, set

$$V_\delta = \{z : z \in \mathbb{R}^{r-1}, \|z\| \le \delta\}, \quad C_\delta = V_\delta \times [-\delta, \delta]. \quad H_\gamma = \{x : \xi_r \le \gamma\}$$

for $\delta > 0$ and $\gamma \in \mathbb{R}$, and $\operatorname{grad}_{r-1} f = (\frac{\partial f}{\partial \xi_1}, \ldots, \frac{\partial f}{\partial \xi_{r-1}}, 0)$ for $f \in \mathcal{D}$.

(b)(i) Take any $\epsilon > 0$ such that

$$\epsilon < \frac{1}{\sqrt{2}}, \quad c^*\epsilon^3 < \frac{1}{4}\beta_{r-1}, \quad 2^{r+1}c'\epsilon < \frac{1}{8}\beta_{r-1}.$$

Then there is a $\delta_0 \in \left]0, 1\right]$ such that

$$\frac{1}{\lambda_E^\partial B(\mathbf{0},\delta)}\int_{B(\mathbf{0},\delta)} \|\psi_E(x) - v\|\lambda_E^\partial(dx) \le \epsilon^3,$$

$$\frac{1}{c_1}\delta^{r-1} \le \lambda_E^\partial B(\mathbf{0}, \delta) \le c_1\delta^{r-1}$$

for every $\delta \in \left]0, 2\delta_0\sqrt{2}\right]$ (using 474N(iii) and 474N(iv) for the inequalities bounding $\lambda_E^\partial B(\mathbf{0}, \delta)$).

(ii) Suppose that $0 < \delta \le \delta_0$. Note first that

$$\begin{aligned}\int_{C_{2\delta}} \|v - \psi_E\| d\lambda_E^\partial &\le \int_{B(\mathbf{0},2\delta\sqrt{2})} \|v - \psi_E\| d\lambda_E^\partial \le \epsilon^3\lambda_E^\partial B(\mathbf{0}, 2\delta\sqrt{2}) \\ &\le c_1\epsilon^3(2\delta\sqrt{2})^{r-1} = \frac{c^*}{\sqrt{2}}\epsilon^3\delta^{r-1}.\end{aligned}$$

(iii) $\lim_{n\to\infty} \tilde{h}_n * \chi E =_{\text{a.e.}} \chi E$ (473Ee), so there is an $n \ge \frac{1}{\delta}$ such that $\int_{C_\delta} |\tilde{h}_n * \chi E - \chi E| d\mu \le \frac{1}{4}\epsilon\mu C_\delta$. Setting

$$f = \tilde{h}_n * \chi E, \quad F = \{x : x \in C_\delta, f(x) \ge \tfrac{3}{4}\}, \quad F' = \{x : x \in C_\delta, f(x) \le \tfrac{1}{4}\},$$

we have $f \in \mathcal{D}$ (473De once more) and

$$\mu(C_\delta \setminus (F \cup F')) \le \epsilon\mu C_\delta, \quad \mu(F \triangle (E \cap C_\delta)) \le \epsilon\mu C_\delta.$$

(iv)

$$\int_{C_\delta} \|\operatorname{grad}_{r-1} f\| + \max(\frac{\partial f}{\partial \xi_r}, 0) d\mu \le c^*\epsilon^3\delta^{r-1}.$$

P? Suppose, if possible, otherwise. Note that because $\mu C_\delta = 2\beta_{r-1}\delta^r$, $\lim_{\delta'\uparrow\delta} \mu C_{\delta'} = \mu C_\delta$, so there is some $\delta' < \delta$ such that

$$\int_{C_{\delta'}} \|\operatorname{grad}_{r-1} f\| + \max(\frac{\partial f}{\partial \xi_r}, 0) d\mu > c^*\epsilon^3\delta^{r-1}.$$

For $1 \le i \le r$ and $x \in \mathbb{R}^r$, set

$$\theta_i(x) = \frac{\frac{\partial f}{\partial \xi_i}(x)}{\|\operatorname{grad}_{r-1} f(x)\|} \text{ if } i < r,\ x \in C_{\delta'} \text{ and } \operatorname{grad}_{r-1}(x) \neq 0,$$
$$= 1 \text{ if } i = r,\ x \in C_{\delta'} \text{ and } \frac{\partial f}{\partial \xi_r}(x) \geq 0,$$
$$= 0 \text{ otherwise.}$$

Then all the θ_i are μ-integrable. Setting $\theta = (\theta_1, \dots, \theta_r)$,

$$\int \theta \,\boldsymbol{.}\, \operatorname{grad} f d\mu = \int_{C_{\delta'}} \|\operatorname{grad}_{r-1} f\| + \max(\frac{\partial f}{\partial \xi_r}, 0) d\mu > c^* \epsilon^3 \delta^{r-1}.$$

By 473Ef, $\langle \|\theta_i - \theta_i * \tilde{h}_k\|_1 \rangle_{k\in\mathbb{N}} \to 0$ for each i; since $\operatorname{grad} f$ is bounded,

$$\int (\tilde{h}_k * \theta) \,\boldsymbol{.}\, \operatorname{grad} f \, d\mu > c^* \epsilon^3 \delta^{r-1}$$

for any k large enough. If we ensure also that $\frac{1}{k+1} \leq \delta - \delta'$, and set $\phi = \tilde{h}_k * \theta$, we shall get a function $\phi \in \mathcal{D}$, with $\|\phi(x)\| \leq \sqrt{2}\chi C_\delta$ for every x (by 473Dc and 473Dg), such that

$$\int \phi \,\boldsymbol{.}\, \operatorname{grad} f d\mu > c^* \epsilon^3 \delta^{r-1}.$$

Moreover, referring to the definition of $*$ in 473Dd and 473Dg,

$$(\tilde{h}_n * \phi)(x) \,\boldsymbol{.}\, v = (\tilde{h}_n * (\tilde{h}_k * \theta_r))(x) \geq 0$$

for every x, because $\tilde{h}_n$, $\tilde{h}_k$ and θ_r are all non-negative.

Now

$$c^* \epsilon^3 \delta^{r-1} < \int \phi \,\boldsymbol{.}\, \operatorname{grad} f d\mu = \int \phi \,\boldsymbol{.}\, \operatorname{grad}(\tilde{h}_n * \chi E) d\mu$$
$$= -\int (\tilde{h}_n * \phi) \,\boldsymbol{.}\, \psi_E \, d\lambda^\partial_E$$

(474K)

$$\leq \int (\tilde{h}_n * \phi) \,\boldsymbol{.}\, (v - \psi_E) \, d\lambda^\partial_E \leq \sqrt{2} \int_{C_{2\delta}} \|v - \psi_E\| \, d\lambda^\partial_E$$

(because $\|(\tilde{h}_n * \phi)(x)\| \leq \sqrt{2}$ for every x, by 473Dg again, and $(\tilde{h}_n * \phi)(x) = 0$ if $x \notin C_\delta + C_{1/(n+1)} \subseteq C_{2\delta}$)

$$\leq c^* \epsilon^3 \delta^{r-1};$$

which is absurd. **XQ**

(v) By 474Q, there is a $\gamma \in \mathbb{R}$ such that

$$\mu(F \triangle (H_\gamma \cap C_\delta)) \leq 9\mu(C_\delta \setminus (F \cup F')) + (c^* \beta_{r-1} + 16c')\epsilon \delta^r$$
$$\leq 9\epsilon\mu C_\delta + \frac{1}{2\beta_{r-1}}(c^* \beta_{r-1} + 16c')\epsilon\mu C_\delta = (c_1^* - 1)\epsilon\mu C_\delta,$$

and

$$\mu((E \triangle H_\gamma) \cap C_\delta) \leq \mu(F \triangle (E \cap C_\delta)) + \mu(F \triangle (H_\gamma \cap C_\delta)) \leq c_1^* \epsilon \mu C_\delta.$$

(vi) As ϵ is arbitrary, we see that

$$\lim_{\delta\downarrow 0} \inf_{\gamma\in\mathbb{R}} \frac{1}{\mu C_\delta} \mu((E \triangle H_\gamma) \cap C_\delta) = 0.$$

(c) Again take $\epsilon \in \,]0, 1]$.

(i) By (b) above and 474N(i)-(ii) there is a $\delta_1 > 0$ such that whenever $0 < \delta \leq \delta_1$ then

$$\mu(B(\mathbf{0}, \delta) \cap E) \geq \frac{1}{2(3r)^r \beta_r} \mu B(\mathbf{0}, \delta),$$

$$\mu(B(\mathbf{0},\delta)\setminus E)\ge\frac{1}{2(3r)^r\beta_r}\mu B(\mathbf{0},\delta)$$

and there is a $\gamma\in\mathbb{R}$ such that

$$\mu((E\triangle H_\gamma)\cap C_\delta)<\min(\epsilon,\frac{\epsilon^r}{4\beta_{r-1}(3r)^r})\mu C_\delta.$$

In this case, $|\gamma|\le\epsilon\delta$. **P?** Suppose, if possible, that $\gamma<-\epsilon\delta$. Then

$$\begin{aligned}\mu(B(\mathbf{0},\epsilon\delta)\cap E)&\le\mu(E\cap C_\delta\setminus H_\gamma)\\&<\frac{\epsilon^r}{4\beta_{r-1}(3r)^r}\mu C_\delta=\frac{1}{2\beta_r(3r)^r}\mu B(\mathbf{0},\epsilon\delta)\end{aligned}$$

which is impossible. **X** In the same way, **?** if $\gamma>\epsilon\delta$,

$$\begin{aligned}\mu(B(\mathbf{0},\epsilon\delta)\setminus E)&\le\mu((C_\delta\setminus H_\gamma)\setminus E)\\&<\frac{\epsilon^r}{4\beta_{r-1}(3r)^r}\mu C_\delta=\frac{1}{2\beta_r(3r)^r}\mu B(\mathbf{0},\epsilon\delta).\ \mathbf{XQ}\end{aligned}$$

(ii) It follows that

$$\begin{aligned}\mu((E\triangle H_0)\cap C_\delta)&\le\mu((E\triangle H_\gamma)\cap C_\delta)+\mu((H_\gamma\triangle H_0)\cap C_\delta)\\&\le\epsilon\mu C_\delta+\epsilon\delta\mu_{r-1}V_\delta=\frac{3}{2}\epsilon\mu C_\delta.\end{aligned}$$

As ϵ is arbitrary,

$$\lim_{\delta\downarrow 0}\frac{\mu((E\triangle H_0)\cap C_\delta)}{\mu C_\delta}=0,$$

and

$$\lim_{\delta\downarrow 0}\frac{\mu((E\triangle H_0)\cap B_\delta)}{\mu B_\delta}\le\frac{2\beta_{r-1}}{\beta_r}\lim_{\delta\downarrow 0}\frac{\mu((E\triangle H_0)\cap C_\delta)}{\mu C_\delta}=0.$$

(d) Thus v is a Federer exterior normal to E at $\mathbf{0}$ if $\psi_E(\mathbf{0})=v$. For the general case, let S be an orthogonal matrix such that $S\psi_E(y)=v$, and set $T(x)=S(x-y)$ for every x. The point is of course that

$$\mathbf{0}=T(y)\in T[\partial^\$E]=\partial^\$T[E],\quad v=S\psi_ET^{-1}(\mathbf{0})=\psi_{T[E]}(\mathbf{0})$$

(474H). So if we set

$$H=\{x:(x-y)\mathbf{.}\psi_E(y)\le 0\}=\{x:T(x)\mathbf{.}v\le 0\}=T^{-1}[H_0],$$

then

$$\frac{\mu((H\triangle E)\cap B(y,\delta))}{\mu B(y,\delta)}=\frac{\mu((H_0\triangle T[E])\cap B(\mathbf{0},\delta))}{\mu B(\mathbf{0},\delta)}\to 0$$

as $\delta\downarrow 0$, and $\psi_E(y)$ is a Federer exterior normal to E at y, as required.

474S Corollary Let $E\subseteq\mathbb{R}^r$ be a set with locally finite perimeter, and λ_E^∂ its perimeter measure. Let y be any point of the reduced boundary of E. Then

$$\lim_{\delta\downarrow 0}\frac{\lambda_E^\partial B(y,\delta)}{\beta_{r-1}\delta^{r-1}}=1.$$

proof (a) Set $v=\psi_E(y)$ and $H=\{x:(x-y)\mathbf{.}v\le 0\}$, as in 474R. Now

$$\int_{H\cap\partial B(y,\delta)}v\mathbf{.}\frac{1}{\delta}(x-y)\nu(dx)=-\beta_{r-1}\delta^{r-1}$$

for almost every $\delta>0$. **P** Set $\phi(x)=v$ for every $x\in\mathbb{R}^r$. By 474I, ϕ is an outward-normal function for H, so 474M tells us that, for almost every $\delta>0$,

$$\int_{H\cap\partial B(y,\delta)} v\mathbin{.}\frac{1}{\delta}(x-y)\,\nu(dx)=\int_{H\cap B(y,\delta)}\operatorname{div}\phi\,d\mu-\int_{B(y,\delta)} v\mathbin{.}v\,d\lambda^\partial_H$$

$$=-\lambda^\partial_H B(y,\delta)=-\nu(B(y,\delta)\cap\partial H)$$

(using the identification of λ^∂_H in 474I)

$$=-\beta_{r-1}\delta^{r-1}$$

(identifying ν on the hyperplane ∂H with Lebesgue measure on $\mathbb{R}^{r-1}$, as usual). **Q**

(b) Now, given $\epsilon>0$, there is a $\delta_0>0$ such that whenever $0<\delta\le\delta_0$ there is an η such that $\delta\le\eta\le\delta(1+\epsilon)$ and $|\lambda^\partial_E B(y,\eta)-\beta_{r-1}\eta^{r-1}|\le\epsilon\eta^{r-1}$. **P** Let $\zeta>0$ be such that

$$\zeta(1+\frac{5\pi}{r}\beta_{r-2})(1+\epsilon)^r\le\epsilon^2.$$

By 474N(iv) and 474R and the definition of ψ_E, there is a $\delta_0>0$ such that

$$\lambda^\partial_E B(y,\delta)\le 5\pi\beta_{r-2}\delta^{r-1},$$

$$\mu((E\triangle H)\cap B(y,\delta))\le\zeta\delta^r,$$

$$\int_{B(y,\delta)}\|\psi_E(x)-v\|\lambda^\partial_E(dx)\le\zeta\lambda^\partial_E B(y,\delta)$$

whenever $0<\delta\le(1+\epsilon)\delta_0$. Take $0<\delta\le\delta_0$. Then, for almost every $\eta>0$, we have

$$\int_{B(y,\eta)} v\mathbin{.}\psi_E(x)\,\lambda^\partial_E(dx)+\int_{E\cap\partial B(y,\eta)} v\mathbin{.}\frac{1}{\eta}(x-y)\,\nu(dx)=0$$

by 474M, applied with ϕ the constant function with value v. Putting this together with (a), we see that, for almost every $\eta\in\,]0,(1+\epsilon)\delta_0]$,

$$|\lambda^\partial_E B(y,\eta)-\beta_{r-1}\eta^{r-1}|=|\int_{B(y,\eta)} v\mathbin{.}v\,d\lambda^\partial_E-\beta_{r-1}\eta^{r-1}|$$

$$\le|\int_{B(y,\eta)} v\mathbin{.}(v-\psi_E)\,d\lambda^\partial_E|+|\int_{B(y,\eta)} v\mathbin{.}\psi_E\,d\lambda^\partial_E-\beta_{r-1}\eta^{r-1}|$$

$$\le\int_{B(y,\eta)}\|\psi_E-v\|d\lambda^\partial_E$$

$$+|\int_{B(y,\eta)} v\mathbin{.}\psi_E(x)\,\lambda^\partial_E(dx)+\int_{H\cap\partial B(y,\eta)} v\mathbin{.}\frac{1}{\eta}(x-y)\,\nu(dx)|$$

(using (a) above)

$$\le\zeta\lambda^\partial_E B(y,\eta)$$

$$+|\int_{H\cap\partial B(y,\eta)} v\mathbin{.}\frac{1}{\eta}(x-y)\,\nu(dx)-\int_{E\cap\partial B(y,\eta)} v\mathbin{.}\frac{1}{\eta}(x-y)\,\nu(dx)|$$

$$\le 5\pi\beta_{r-2}\zeta\eta^{r-1}+\nu((E\triangle H)\cap\partial B(y,\eta)).$$

Integrating with respect to η, we have

$$\int_0^{\delta(1+\epsilon)}|\lambda^\partial_E B(y,\eta)-\beta_{r-1}\eta^{r-1}|d\eta\le\frac{5\pi}{r}\beta_{r-2}\zeta\delta^r(1+\epsilon)^r+\mu((E\triangle H)\cap B(y,\delta(1+\epsilon)))$$

(using 265G, as usual)

$$\le\frac{5\pi}{r}\beta_{r-2}\zeta\delta^r(1+\epsilon)^r+\zeta\delta^r(1+\epsilon)^r\le\epsilon^2\delta^r$$

by the choice of ζ. But this means that there must be some $\eta\in[\delta,\delta(1+\epsilon)]$ such that

$$|\lambda_E^\partial B(y,\eta)-\beta_{r-1}\eta^{r-1}|\le\epsilon\delta^{r-1}\le\epsilon\eta^{r-1}.\ \mathbf{Q}$$

(c) Now we see that

$$\lambda_E^\partial B(y,\delta)\le\lambda_E^\partial B(y,\eta)\le(\beta_{r-1}+\epsilon)\eta^{r-1}\le(\beta_{r-1}+\epsilon)(1+\epsilon)^{r-1}\delta^{r-1}.$$

But by the same argument we have an $\hat\eta\in[\frac{\delta}{1+\epsilon},\delta]$ such that $|\lambda_E^\partial B(y,\hat\eta)-\beta_{r-1}\hat\eta^{r-1}|\le\epsilon\hat\eta^{r-1}$, so that

$$\lambda_E^\partial B(y,\delta)\ge\lambda_E^\partial B(y,\hat\eta)\ge(\beta_{r-1}-\epsilon)\hat\eta^{r-1}\ge(\beta_{r-1}-\epsilon)(1+\epsilon)^{1-r}\delta^{r-1}.$$

Thus, for every $\delta\in\,]0,\delta_0]$,

$$(\beta_{r-1}-\epsilon)(1+\epsilon)^{1-r}\delta^{r-1}\le\lambda_E^\partial B(y,\delta)\le(\beta_{r-1}+\epsilon)(1+\epsilon)^{r-1}\delta^{r-1}.$$

As ϵ is arbitrary,

$$\lim_{\delta\downarrow 0}\frac{\lambda_E^\partial B(y,\delta)}{\delta^{r-1}}=\beta_{r-1},$$

as claimed.

474T The Compactness Theorem Let Σ be the algebra of Lebesgue measurable subsets of $\mathbb{R}^r$, and give it the topology $\mathfrak{T}_m$ of convergence in measure defined by the pseudometrics $\rho_H(E,F)=\mu((E\triangle F)\cap H)$ for measurable sets H of finite measure (cf. §§245 and 323). Then

(a) $\operatorname{per}:\Sigma\to[0,\infty]$ is lower semi-continuous;

(b) for any $\gamma<\infty$, $\{E:E\in\Sigma,\ \operatorname{per}E\le\gamma\}$ is compact.

proof (a) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be any $\mathfrak{T}_m$-convergent sequence in Σ with limit $E\in\Sigma$. If $\phi:\mathbb{R}^r\to B(\mathbf{0},1)$ is a Lipschitz function with compact support, then $\operatorname{div}\phi$ is integrable, so $F\mapsto\int_F\operatorname{div}\phi\,d\mu$ is truly continuous (225A), and

$$|\int_E\operatorname{div}\phi\,d\mu|=\lim_{n\to\infty}|\int_{E_n}\operatorname{div}\phi\,d\mu|\le\sup_{n\in\mathbb{N}}\operatorname{per}E_n.$$

As ϕ is arbitrary, $\operatorname{per}E\le\sup_{n\in\mathbb{N}}\operatorname{per}E_n$. This means that $\{E:\operatorname{per}E\le\gamma\}$ is sequentially closed, therefore closed (4A2Ld), for any γ, and per is lower semi-continuous.

(b) Let us say that a 'dyadic cube' is a set expressible in the form $\prod_{1\le i\le r}[2^{-n}k_i,2^{-n}(k_i+1)[$ where $n,k_1,\dots,k_r\in\mathbb{Z}$. Set $\mathcal{A}=\{E:\operatorname{per}E\le\gamma\}$.

(i) For $E\in\mathcal{A}$, $n\in\mathbb{N}$ and $\epsilon\in\,]0,1]$ let $G(E,n,\epsilon)$ be the union of all the dyadic cubes D with side length 2^{-n} such that $\epsilon\mu D\le\mu(E\cap D)\le(1-\epsilon)\mu D$. Then $\mu G(E,n,\epsilon)\le\frac{c_1}{2^n\epsilon}\gamma$, where $c_1=2^{r+5}(1+2^{r+1})(1+\sqrt{r})^{r+1}$.

$\mathbf{P}$ Express $G(E,n,\epsilon)$ as a disjoint union $\bigcup_{i\in I}D_i$ where each D_i is a dyadic cube of side length 2^{-n} and $\min(\mu(D_i\cap E),\mu(D_i\setminus E))\ge\epsilon\mu D_i$. Let x_i be the centre of D_i and B_i the ball $B(x_i,2^{-n-1}\sqrt{r})$, so that $D_i\subseteq B_i$ and $\mu B_i=\beta_r(\frac{1}{2}\sqrt{r})^r\mu D_i$. For any $x\in\mathbb{R}^r$, the ball $B(x,2^{-n-1}\sqrt{r})$ is included in a closed cube with side length $2^{-n}\sqrt{r}$, so can contain at the very most $(1+\sqrt{r})^r$ different x_i, because different x_i differ by at least 2^{-n} in some coordinate. Turning this round, $\sum_{i\in I}\chi B_i\le(1+\sqrt{r})^r\chi(\mathbb{R}^r)$.

Set $c=2^{r+4}\sqrt{r}(1+2^{r+1})$. Then, for each $i\in I$,

$$\begin{aligned}2c\lambda_E^\partial B_i&\ge\min(\mu(B_i\cap E),\mu(B_i\setminus E))^{(r-1)/r}\\ &\ge\min(\mu(D_i\cap E),\mu(D_i\setminus E))^{(r-1)/r}\ge(\epsilon\mu D_i)^{(r-1)/r}\ge 2^{-n(r-1)}\epsilon.\end{aligned}\tag{474Lb}$$

So

$$\begin{aligned}\mu G(E,n,\epsilon)=2^{-nr}\#(I)&\le\frac{2c}{2^n\epsilon}\sum_{i\in I}\lambda_E^\partial B_i\\ &\le\frac{2c(1+\sqrt{r})^r}{2^n\epsilon}\lambda_E^\partial(\mathbb{R}^r)\le\frac{c_1}{2^n\epsilon}\gamma.\ \mathbf{Q}\end{aligned}$$

(ii) Now let $\langle E_n\rangle_{n\in\mathbb{N}}$ be any sequence in $\mathcal{A}$. Then we can find a subsequence $\langle E'_n\rangle_{n\in\mathbb{N}}$ such that whenever $n\in\mathbb{N}$, D is a dyadic cube of side length 2^{-n} meeting $B(\mathbf{0},n)$, and i, $j\ge n$, then $|\mu(D\cap E'_i)-\mu(D\cap E'_j)|\le\frac{1}{(n+1)^{r+2}}\mu D$. Now

$$\mu((E'_n\triangle E'_{n+1})\cap B(\mathbf{0},n))\le\tfrac{3\beta_r(n+\sqrt{r})^r}{(n+1)^{r+2}}+2^{-n}(n+1)^{r+1}c_1\gamma$$

whenever $n\ge 1$. **P** Let $\mathcal{E}$ be the set of dyadic cubes of side length 2^{-n} meeting $B(\mathbf{0},n)$; then every member of $\mathcal{E}$ is included in $B(\mathbf{0},n+2^{-n}\sqrt{r})$, so $\mu(\bigcup\mathcal{E})\le\beta_r(n+\sqrt{r})^r$. Let $\mathcal{E}_1$ be the collection of those dyadic cubes of side length 2^{-n} included in $G(E'_n,n,\frac{1}{(n+1)^{r+2}})$. If $D\in\mathcal{E}\setminus\mathcal{E}_1$, either $\mu(E'_n\cap D)\le\frac{1}{(n+1)^{r+2}}\mu D$ and $\mu(E'_{n+1}\cap D)\le\frac{2}{(n+1)^{r+2}}\mu D$ and $\mu((E'_n\triangle E'_{n+1})\cap D)\le\frac{3}{(n+1)^{r+2}}\mu D$, or $\mu(D\setminus E'_n)\le\frac{1}{(n+1)^{r+2}}\mu D$ and $\mu(D\setminus E'_{n+1})\le\frac{2}{(n+1)^{r+2}}\mu D$ and $\mu((E'_n\triangle E'_{n+1})\cap D)\le\frac{3}{(n+1)^{r+2}}\mu D$. So

$$\begin{aligned}\mu((E'_n\triangle E'_{n+1})\cap B(\mathbf{0},n))&\le\sum_{D\in\mathcal{E}}\mu((E'_n\triangle E'_{n+1})\cap D)\\&\le\sum_{D\in\mathcal{E}\setminus\mathcal{E}_1}\mu((E'_n\triangle E'_{n+1})\cap D)+\mu(\bigcup\mathcal{E}_1)\\&\le\tfrac{3}{(n+1)^{r+2}}\mu(\bigcup\mathcal{E})+\mu G(E'_n,n,\tfrac{1}{(n+1)^{r+2}})\\&\le\tfrac{3\beta_r(n+\sqrt{r})^r}{(n+1)^{r+2}}+2^{-n}(n+1)^{r+2}c_1\gamma,\end{aligned}$$

as claimed. **Q**

(iii) This means that $\sum_{i=0}^\infty\mu((E'_i\triangle E'_{i+1})\cap B(\mathbf{0},n))$ is finite for each $n\in\mathbb{N}$, so that if we set $E=\bigcup_{i\in\mathbb{N}}\bigcap_{j\ge i}E'_j$, then

$$\mu((E\triangle E'_i)\cap B(\mathbf{0},n))\le\textstyle\sum_{j=i}^\infty\mu((E'_j\triangle E'_{j+1})\cap B(\mathbf{0},n))\to 0$$

as $i\to\infty$ for every $n\in\mathbb{N}$. It follows that $\lim_{i\to\infty}\rho_H(E,E'_i)=0$ whenever $\mu H<\infty$ (see the proofs of 245Eb and 323Gb). Thus we have a subsequence $\langle E'_i\rangle_{i\in\mathbb{N}}$ of the original sequence $\langle E_i\rangle_{i\in\mathbb{N}}$ which is convergent for the topology $\mathfrak{T}_m$ of convergence in measure. By (a), its limit belongs to $\mathcal{A}$. But since $\mathfrak{T}_m$ is pseudometrizable (245Eb/323Gb), this is enough to show that A is compact for $\mathfrak{T}_m$ (4A2Lf).

474X Basic exercises (a) Show that for any $E\subseteq\mathbb{R}^r$ with locally finite perimeter, its reduced boundary is a Borel set and its canonical outward-normal function is Borel measurable.

>(b) Show that if $E\subseteq\mathbb{R}^r$ has finite perimeter then either E or its complement has finite measure.

(c)(i) Show that if $E\subseteq\mathbb{R}^r$ has locally finite perimeter, then $\partial^\$E\subseteq\partial E$. (*Hint*: 474N(i)-(ii).) (ii) Show that if $H\subseteq\mathbb{R}^r$ is a half-space, as in 474I, then $\partial^\$E=\partial E$.

(d) Let $E\subseteq\mathbb{R}^r$ be a set with locally finite perimeter, and $y\in\partial^\$E$. Show that $\lim_{\delta\downarrow 0}\frac{\mu(E\cap B(y,\delta))}{\mu B(y,\delta)}=\frac{1}{2}$.

(e) In the proof of 474S, use 265E to show that

$$\int_{H\cap\partial B(y,\delta)}v\centerdot\tfrac{1}{\delta}(x-y)\nu(dx)=-\beta_{r-1}\delta^{r-1}$$

for *every* $\delta>0$.

474Y Further exercises (a) In 474E, explain how to interpret the pair $(\psi,\lambda_E^\partial)$ as a vector measure (definition: 394O[2]) $\theta_E:\mathcal{B}\to\mathbb{R}^r$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}^r$, in such a way that we have $\int_E\operatorname{div}\phi\,d\mu=\int\phi\centerdot d\theta_E$ for Lipschitz functions ϕ with compact support.

[2]Formerly 393O.

474 Notes and comments When we come to the Divergence Theorem itself in the next section, it will be nothing but a repetition of Theorem 474E with the perimeter measure and the outward-normal function properly identified. The idea of the indirect approach here is to start by defining the pair $(\psi_E, \lambda_E^\partial)$ as a kind of 'distributional derivative' of the set E. I take the space to match the details with the language of the rest of this treatise, but really 474E amounts to nothing more than the Riesz representation theorem; since the functional $\phi \mapsto \int_E \operatorname{div} \phi \, d\mu$ is linear, and we restrict attention to sets E for which it is continuous in an appropriate sense (and can therefore be extended to arbitrary continuous functions ϕ with compact support), it must be representable by a (vector) measure, as in 474Ya. For the process to be interesting, we have to be able to identify at least some of the appropriate sets E with their perimeter measures and outward-normal functions. Half-spaces are straightforward enough (474I), and 474R tells us what the outward-normal functions have to be; but for a proper description of the family of sets with locally finite perimeter we must wait until the next section. I see no quick way to show from the results here that (for instance) the union of two sets with finite perimeter again has finite perimeter. And I notice that I have not even shown that balls have finite perimeters. After 475M things should be much clearer.

I have tried to find the shortest path to the Divergence Theorem itself, and have not attempted to give 'best' results in the intermediate material. In particular, in the isoperimetric inequality 474La, I show only that the measure of a set E is controlled by the magnitude of its perimeter measure. Simple scaling arguments show that if there is any such control, then it must be of the form $\gamma(\mu E)^{(r-1)/r} \le \operatorname{per} E$; the identification of the best constant γ as $r\beta_r^{1/r}$, giving equality for balls, is the real prize, to which I shall come in 476H. Similarly, there is a dramatic jump from the crude estimates in 474N to the exact limits in 474Xd and 474S. When we say that a set E has a Federer exterior normal at a point y, we are clearly saying that there is an 'approximate' tangent plane at that point, as measured by ordinary volume μ. 474S strengthens this by saying that, when measured by the perimeter measure, the boundary of E looks like a hyperplane through y with normalised $(r-1)$-dimensional measure. In 475G below we shall come to a partial explanation of this.

The laborious arguments of 474C and 474H are doing no more than establish the geometric invariance of the concepts here, which ought, one would think, to be obvious. The trouble is that I have given definitions of inner product and divergence and Lebesgue measure in terms of the standard coordinate system of $\mathbb{R}^r$. If these were not invariant under isometries they would be far less interesting. But even if we are confident that there must be a result corresponding to 474H, I think a little thought is required to identify the exact formulae involved in the transformation.

I leave the Compactness Theorem (474T) to the end of the section because it is off the line I have chosen to the Divergence Theorem (though it can be used to make the proof of 474R more transparent; see Evans & Gariepy 92, 5.7.2). I have expressed 474T in terms of the topology of convergence in measure on the algebra of Lebesgue measurable sets. But since the perimeter of a measurable set E is not altered if we change E by a negligible set (474F), 'perimeter' can equally well be regarded as a function defined on the measure algebra, in which case 474T becomes a theorem about the usual topology of the measure algebra of Lebesgue measure, as described in §323.

475 The essential boundary

The principal aim of this section is to translate Theorem 474E into geometric terms. We have already identified the Federer exterior normal as an outward-normal function (474R), so we need to find a description of perimeter measures. Most remarkably, these turn out, in every case considered in 474E, to be just normalized Hausdorff measures (475G). This description needs the concept of 'essential boundary' (475B). In order to complete the programme, we need to be able to determine which sets have 'locally finite perimeter'; there is a natural criterion in the same language (475L). We now have all the machinery for a direct statement of the Divergence Theorem (for Lipschitz functions) which depends on nothing more advanced than the definition of Hausdorff measure (475N). (The definitions, at least, of 'Federer exterior normal' and 'essential boundary' are elementary.)

This concludes the main work of the first part of this chapter. But since we are now within reach of a reasonably direct proof of a fundamental fact about the $(r-1)$-dimensional Hausdorff measure of the boundaries of subsets of $\mathbb{R}^r$ (475Q), I continue up to Cauchy's Perimeter Theorem and the Isoperimetric Theorem for convex sets (475S, 475T).

475A Notation As in the last two sections, r will be an integer (greater than or equal to 2, unless explicitly permitted to take the value 1). μ will be Lebesgue measure on $\mathbb{R}^r$; I will sometimes write μ_{r-1} for Lebesgue measure on $\mathbb{R}^{r-1}$ and μ_1 for Lebesgue measure on $\mathbb{R}$. $\beta_r = \mu B(\mathbf{0}, 1)$ will be the measure of the unit ball in $\mathbb{R}^r$, and $S_{r-1} = \partial B(\mathbf{0}, 1)$ will be the unit sphere. ν will be normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$

(265A), that is, $\nu = 2^{-r+1}\beta_{r-1}\mu_{H,r-1}$, where $\mu_{H,r-1}$ is $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$. Recall that $\nu S_{r-1} = r\beta_r$ (265H). I will take it for granted that $x \in \mathbb{R}^r$ has coordinates $(\xi_1, \ldots, \xi_r)$.

If $E \subseteq \mathbb{R}^r$ has locally finite perimeter (474D), λ_E^∂ will be its perimeter measure (474F), $\partial^\$ E$ its reduced boundary (474G) and ψ_E its canonical outward-normal function (474G).

475B The essential boundary (In this paragraph I allow $r = 1$.) Let $A \subseteq \mathbb{R}^r$ be any set. The **essential closure** of A is the set

$$\mathrm{cl}^*A = \{x : \limsup_{\delta\downarrow 0} \tfrac{\mu^*(B(x,\delta)\cap A)}{\mu B(x,\delta)} > 0\}$$

(see 266B). Similarly, the **essential interior** of A is the set

$$\mathrm{int}^*A = \{x : \liminf_{\delta\downarrow 0} \tfrac{\mu_*(B(x,\delta)\cap A)}{\mu B(x,\delta)} = 1\}.$$

(If A is Lebesgue measurable, this is the lower Lebesgue density of A, as defined in 341E; see also 223Yf.) Finally, the **essential boundary** ∂^*A of A is the difference $\mathrm{cl}^*A \setminus \mathrm{int}^*A$.

Note that if $E \subseteq \mathbb{R}^r$ is Lebesgue measurable then $\mathbb{R}^r \setminus \partial^*E$ is the Lebesgue set of the function χE, as defined in 261E.

475C Lemma (In this lemma I allow $r = 1$.) Let A, $A' \subseteq \mathbb{R}^r$.

(a)

$$\mathrm{int}\,A \subseteq \mathrm{int}^*A \subseteq \mathrm{cl}^*A \subseteq \overline{A}, \quad \partial^*A \subseteq \partial A,$$

$$\mathrm{cl}^*A = \mathbb{R}^r \setminus \mathrm{int}^*(\mathbb{R}^r \setminus A), \quad \partial^*(\mathbb{R}^r \setminus A) = \partial^*A.$$

(b) If $A \setminus A'$ is negligible, then $\mathrm{cl}^*A \subseteq \mathrm{cl}^*A'$ and $\mathrm{int}^*A \subseteq \mathrm{int}^*A'$; in particular, if A itself is negligible, cl^*A, int^*A and ∂^*A are all empty.

(c) int^*A, cl^*A and ∂^*A are Borel sets.

(d) $\mathrm{cl}^*(A \cup A') = \mathrm{cl}^*A \cup \mathrm{cl}^*A'$ and $\mathrm{int}^*(A \cap A') = \mathrm{int}^*A \cap \mathrm{int}^*A'$, so $\partial^*(A \cup A')$, $\partial^*(A \cap A')$ and $\partial^*(A \triangle A')$ are all included in $\partial^*A \cup \partial^*A'$.

(e) $\mathrm{cl}^*A \cap \mathrm{int}^*A' \subseteq \mathrm{cl}^*(A \cap A')$, $\partial^*A \cap \mathrm{int}^*A' \subseteq \partial^*(A \cap A')$ and $\partial^*A \setminus \mathrm{cl}^*A' \subseteq \partial^*(A \cup A')$.

(f) $\partial^*(A \cap A') \subseteq (\mathrm{cl}^*A' \cap \partial A) \cup (\partial^*A' \cap \mathrm{int}\,A)$.

(g) If $E \subseteq \mathbb{R}^r$ is Lebesgue measurable, then $E \triangle \mathrm{int}^*E$, $E \triangle \mathrm{cl}^*E$ and ∂^*E are Lebesgue negligible.

(h) A is Lebesgue measurable iff ∂^*A is Lebesgue negligible.

proof (a) It is obvious that

$$\mathrm{int}\,A \subseteq \mathrm{int}^*A \subseteq \mathrm{cl}^*A \subseteq \overline{A},$$

so that $\partial^*A \subseteq \partial A$. Since

$$\tfrac{\mu^*(B(x,\delta)\cap A)}{\mu B(x,\delta)} + \tfrac{\mu_*(B(x,\delta)\setminus A)}{\mu B(x,\delta)} = 1$$

for every $x \in \mathbb{R}^r$ and every $\delta > 0$ (413Ec), $\mathbb{R}^r \setminus \mathrm{int}^*A = \mathrm{cl}^*(\mathbb{R}^r \setminus A)$. It follows that

$$\begin{aligned}\partial^*(\mathbb{R}^r \setminus A) &= \mathrm{cl}^*(\mathbb{R}^r \setminus A) \triangle \mathrm{int}^*(\mathbb{R}^r \setminus A)\\ &= (\mathbb{R}^r \setminus \mathrm{int}^*A) \triangle (\mathbb{R}^r \setminus \mathrm{cl}^*A) = \mathrm{int}^*A \triangle \mathrm{cl}^*A = \partial^*A.\end{aligned}$$

(b) If $A \setminus A'$ is negligible, then

$$\mu_*(B(x,\delta) \cap A) \le \mu_*(B(x,\delta) \cap A'), \quad \mu^*(B(x,\delta) \cap A) \le \mu^*(B(x,\delta) \cap A')$$

for all x and δ, so $\mathrm{int}^*A \subseteq \mathrm{int}^*A'$ and $\mathrm{cl}^*A \subseteq \mathrm{cl}^*A'$.

(c) The point is just that $(x, \delta) \mapsto \mu^*(A \cap B(x, \delta))$ is continuous. **P** For any x, $y \in \mathbb{R}^r$ and δ, $\eta > 0$ we have

$$\begin{aligned}|\mu^*(A \cap B(y, \eta)) - \mu^*(A \cap B(x, \delta))| &\le \mu(B(y, \eta) \triangle B(x, \delta))\\ &= 2\mu(B(x, \delta) \cup B(y, \eta)) - \mu B(x, \delta) - \mu B(y, \eta)\\ &\le \beta_r\bigl(2(\max(\delta, \eta) + \|x - y\|)^r - \delta^r - \eta^r\bigr) \to 0\end{aligned}$$

as $(y,\eta)\to(x,\delta)$. **Q** So

$$x\mapsto\limsup_{\delta\downarrow 0}\frac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)}=\inf_{\alpha\in\mathbb{Q},\alpha>0}\sup_{\gamma\in\mathbb{Q},0<\gamma\le\alpha}\frac{1}{\beta_r\gamma^r}\mu^*(A\cap B(x,\gamma))$$

is Borel measurable, and

$$\mathrm{cl}^*A=\{x:\limsup_{\delta\downarrow 0}\frac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)}>0\}$$

is a Borel set.

Accordingly $\mathrm{int}^*A=\mathbb{R}^r\setminus\mathrm{cl}^*(\mathbb{R}^r\setminus A)$ and ∂^*A are also Borel sets.

(d) For any $x\in\mathbb{R}^r$,

$$\begin{aligned}\limsup_{\delta\downarrow 0}\frac{\mu^*((A\cup A')\cap B(x,\delta))}{\mu B(x,\delta)}&\le\limsup_{\delta\downarrow 0}\frac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)}+\frac{\mu^*(A'\cap B(x,\delta))}{\mu B(x,\delta)}\\&\le\limsup_{\delta\downarrow 0}\frac{\mu(A\cap B(x,\delta))}{\mu B(x,\delta)}+\limsup_{\delta\downarrow 0}\frac{\mu(A'\cap B(x,\delta))}{\mu B(x,\delta)},\end{aligned}$$

so $\mathrm{cl}^*(A\cup A')\subseteq\mathrm{cl}^*A\cup\mathrm{cl}^*A'$. By (b), $\mathrm{cl}^*A\cup\mathrm{cl}^*A'\subseteq\mathrm{cl}^*(A\cup A')$, so we have equality. Accordingly

$$\mathrm{int}^*(A\cap A')=\mathbb{R}^r\setminus\mathrm{cl}^*((\mathbb{R}^r\setminus A)\cup(\mathbb{R}^r\setminus A'))=\mathrm{int}^*A\cap\mathrm{int}^*A'.$$

Since $\mathrm{int}^*(A\cup A')\supseteq\mathrm{int}^*A\cup\mathrm{int}^*A'$, $\partial^*(A\cup A')\subseteq\partial^*A\cup\partial^*A'$. Now

$$\partial^*(A\cap A')=\partial^*(\mathbb{R}^r\setminus(A\cap A'))\subseteq\partial^*(\mathbb{R}^r\setminus A)\cup\partial^*(\mathbb{R}^r\setminus A'))=\partial^*A\cup\partial^*A'$$

and

$$\partial^*(A\triangle A')\subseteq\partial^*(A\cap(\mathbb{R}^r\setminus A'))\cup\partial^*(A'\cap(\mathbb{R}^r\setminus A))\subseteq\partial^*A\cup\partial^*A'.$$

(e) If $x\in\mathrm{cl}^*A\cap\mathrm{int}^*A'$, then $x\notin\mathrm{cl}^*(\mathbb{R}^r\setminus A')$ so $x\notin\mathrm{cl}^*(A\setminus A')$. But from (d) we know that $x\in\mathrm{cl}^*(A\cap A')\cup\mathrm{cl}^*(A\setminus A')$, so $x\in\mathrm{cl}^*(A\cap A')$.

Now

$$\begin{aligned}\partial^*A\cap\mathrm{int}^*A'&=(\mathrm{cl}^*A\cap\mathrm{int}^*A')\setminus\mathrm{int}^*A\\&\subseteq\mathrm{cl}^*(A\cap A')\setminus\mathrm{int}^*(A\cap A')=\partial^*(A\cap A'),\end{aligned}$$

$$\begin{aligned}\partial^*A\setminus\mathrm{cl}^*A'&=\partial^*(\mathbb{R}^r\setminus A)\cap\mathrm{int}^*(\mathbb{R}^r\setminus A')\\&\subseteq\partial^*((\mathbb{R}^r\setminus A)\cap(\mathbb{R}^r\setminus A'))\\&=\partial^*(\mathbb{R}^r\setminus(A\cup A'))=\partial^*(A\cup A').\end{aligned}$$

(f) If $x\in\partial^*(A\cap A')\cap\partial A$, then of course $x\in\mathrm{cl}^*(A\cap A')\subseteq\mathrm{cl}^*A$, so $x\in\mathrm{cl}^*A\cap\partial A$. If $x\in\partial^*(A\cap A')\setminus\partial A$, then surely $x\in\overline{A}$, so $x\in\mathrm{int}\,A$. But this means that

$$\mu_*(B(x,\delta)\cap A')=\mu_*(B(x,\delta)\cap A\cap A'),\quad\mu^*(B(x,\delta)\cap A')=\mu^*(B(x,\delta)\cap A\cap A')$$

for all δ small enough, so $x\in\partial^*A'$ and $x\in\partial^*A'\cap\mathrm{int}\,A$.

(g) Applying 472Da to μ and χE, we see that

$$\begin{aligned}\partial^*E&\subseteq(E\triangle\mathrm{cl}^*E)\cup(E\triangle\mathrm{int}^*E)\\&\subseteq\{x:\chi E(x)\ne\lim_{\delta\downarrow 0}\frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)}\}\end{aligned}$$

are all μ-negligible.

(h) If A is Lebesgue measurable then (g) tells us that ∂^*A is negligible. If ∂^*A is negligible, let E be a measurable envelope of A. Then $\mu(E\cap B(x,\delta))=\mu^*(A\cap B(x,\delta))$ for all x and δ, so $\mathrm{cl}^*E=\mathrm{cl}^*A$. Similarly, if F is a measurable envelope of $\mathbb{R}^r\setminus A$, then $\mathrm{cl}^*F=\mathrm{cl}^*(\mathbb{R}^r\setminus A)=\mathbb{R}^r\setminus\mathrm{int}^*A$ (using (a)). Now (g) tells us that

$$\mu(E\cap F)=\mu(\mathrm{cl}^*E\cap\mathrm{cl}^*F)=\mu(\mathrm{cl}^*A\setminus\mathrm{int}^*A)=0.$$

But now $A\setminus E$ and $E\setminus A\subseteq(E\cap F)\cup((\mathbb{R}^r\setminus A)\setminus F)$ are Lebesgue negligible, so A is Lebesgue measurable.

475D Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $\partial^{\$}E$ its reduced boundary. Then $\partial^{\$}E \subseteq \partial^*E$ and $\nu(\partial^*E \setminus \partial^{\$}E) = 0$.

proof (a) By 474N(i), $\partial^{\$}E \subseteq \mathrm{cl}^*E$; by 474N(ii), $\partial^{\$}E \cap \mathrm{int}^*E = \emptyset$; so $\partial^{\$}E \subseteq \partial^*E$.

(b) For any $y \in \partial^*E$,

$$\limsup_{\delta\downarrow 0} \frac{1}{\delta^{r-1}} \lambda^{\partial}_E B(y,\delta) > 0.$$

P We have an $\epsilon \in \left]0, \frac{1}{2}\right]$ such that

$$\liminf_{\delta\downarrow 0} \frac{\mu(E\cap B(y,\delta))}{\mu B(y,\delta)} < 1-\epsilon, \quad \limsup_{\delta\downarrow 0} \frac{\mu(E\cap B(y,\delta))}{\mu B(y,\delta)} > \epsilon.$$

Since the function $\delta \mapsto \frac{\mu(E\cap B(y,\delta))}{\mu B(y,\delta)}$ is continuous, there is a sequence $\langle \delta_n \rangle_{n\in\mathbb{N}}$ in $]0,\infty[$ such that $\lim_{n\to\infty} \delta_n = 0$ and

$$\epsilon\mu B(y,\delta_n) \le \mu(E \cap B(y,\delta_n)) \le (1-\epsilon)\mu B(y,\delta_n)$$

for every n. Now from 474Lb we have

$$\begin{aligned}(\epsilon\beta_r)^{(r-1)/r}\delta_n^{r-1} &= (\epsilon\mu B(y,\delta_n))^{(r-1)/r} \\ &\le \min(\mu(B(y,\delta_n) \cap E), \mu(B(y,\delta_n) \setminus E))^{(r-1)/r} \le 2c\lambda^{\partial}_E B(y,\delta_n)\end{aligned}$$

for every n, where $c > 0$ is the constant there. But this means that

$$\limsup_{\delta\downarrow 0} \frac{1}{\delta^{r-1}} \lambda^{\partial}_E B(y,\delta) \ge \limsup_{n\to\infty} \frac{1}{\delta_n^{r-1}} \lambda^{\partial}_E B(y,\delta_n) \ge \frac{1}{2c}(\epsilon\beta_r)^{(r-1)/r} > 0. \quad \mathbf{Q}$$

(c) Let $\epsilon > 0$. Set

$$F_\epsilon = \{y : y \in \mathbb{R}^r \setminus \partial^{\$}E, \limsup_{\delta\downarrow 0} \frac{1}{\delta^{r-1}} \lambda^{\partial}_E B(y,\delta) > \epsilon\}.$$

Because $\partial^{\$}E$ is λ^{∂}_E-conegligible (474G), $\lambda^{\partial}_E F_\epsilon = 0$. So there is an open set $G \supseteq F_\epsilon$ such that $\lambda^{\partial}_E G \le \epsilon^2$ (256Bb/412Wb). Let $\delta > 0$. Let $\mathcal{I}$ be the family of all those non-singleton closed balls $B \subseteq G$ such that $\operatorname{diam} B \le \delta$ and $\lambda^{\partial}_E B \ge 2^{-r+1}\epsilon(\operatorname{diam} B)^{r-1}$. Then every point of F_ϵ is the centre of arbitrarily small members of $\mathcal{I}$. By Besicovitch's Covering Lemma (472B), there is a family $\langle \mathcal{I}_k \rangle_{k<5^r}$ of disjoint countable subsets of $\mathcal{I}$ such that $\mathcal{I}^* = \bigcup_{k<5^r} \bigcup \mathcal{I}_k$ covers F_ϵ. Now

$$\sum_{B\in\mathcal{I}^*} (\operatorname{diam} B)^{r-1} \le \sum_{k<5^r} \sum_{B\in\mathcal{I}_k} \frac{2^{r-1}}{\epsilon} \lambda^{\partial}_E B \le \frac{5^r 2^{r-1}}{\epsilon} \lambda^{\partial}_E G \le 5^r 2^{r-1}\epsilon.$$

As δ is arbitrary, $\mu^*_{H,r-1} F_\epsilon$ is at most $5^r 2^{r-1}\epsilon$ (264Fb/471Dc) and $\nu^* F_\epsilon \le 5^r \beta_{r-1}\epsilon$. As ϵ is arbitrary,

$$\partial^*E \setminus \partial^{\$}E \subseteq \{y : y \in \mathbb{R}^r \setminus \partial^{\$}E, \limsup_{\delta\downarrow 0} \frac{1}{\delta^{r-1}} \lambda^{\partial}_E B(y,\delta) > 0\}$$

is ν-negligible, as claimed.

475E Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter.
(a) If $A \subseteq \partial^{\$}E$, then $\nu^*A \le (\lambda^{\partial}_E)^*A$.
(b) If $A \subseteq \mathbb{R}^r$ and $\nu A = 0$, then $\lambda^{\partial}_E A = 0$.

proof (a) Given ϵ, $\delta > 0$ let $\mathcal{I}$ be the family of non-trivial closed balls $B \subseteq \mathbb{R}^r$ of diameter at most δ such that $\beta_{r-1}(\frac{1}{2}\operatorname{diam} B)^{r-1} \le (1+\epsilon)\lambda^{\partial}_E B$. By 474S, every point of A is the centre of arbitrarily small members of $\mathcal{I}$. By 472Cb, there is a countable family $\mathcal{I}_1 \subseteq I$ such that $A \subseteq \bigcup \mathcal{I}_1$ and $\sum_{B\in\mathcal{I}_1} \lambda^{\partial}_E B \le (\lambda^{\partial}_E)^*A + \epsilon$. But this means that

$$\sum_{B\in\mathcal{I}_1} (\operatorname{diam} B)^{r-1} \le (1+\epsilon)\frac{2^{r-1}}{\beta_{r-1}}((\lambda^{\partial}_E)^*A + \epsilon).$$

As δ is arbitrary,

$$\nu^*A = \frac{\beta_{r-1}}{2^{r-1}} \mu^*_{H,r-1} A \le (1+\epsilon)((\lambda^{\partial}_E)^*A + \epsilon).$$

As ϵ is arbitrary, we have the result.

(b) For $n \in \mathbb{N}$, set

$$A_n = \{x : x \in A,\ \lambda_E^\partial B(x,\delta) \le 2\beta_{r-1}\delta^{r-1} \text{ whenever } 0 < \delta \le 2^{-n}\}.$$

Now, given $\epsilon > 0$, there is a sequence $\langle D_i\rangle_{i\in\mathbb{N}}$ of sets covering A_n such that $\operatorname{diam} D_i \le 2^{-n}$ for every i and $\sum_{i=0}^{\infty}(\operatorname{diam} D_i)^{r-1} \le \epsilon$. Passing over the trivial case $A_n = \emptyset$, we may suppose that for each $i \in \mathbb{N}$ there is an $x_i \in A_n \cap D_i$, so that $D_i \subseteq B(x_i, \operatorname{diam} D_i)$ and

$$\begin{aligned}(\lambda_E^\partial)^* A_n &\le \sum_{i=0}^{\infty}(\lambda_E^\partial)^* D_i \le \sum_{i=0}^{\infty}\lambda_E^\partial B(x_i, \operatorname{diam} D_i)\\ &\le \sum_{i=0}^{\infty} 2\beta_{r-1}(\operatorname{diam} D_i)^{r-1} \le 2\beta_{r-1}\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\lambda_E^\partial A_n = 0$. And this is true for every n. As $\bigcup_{n\in\mathbb{N}} A_n \supseteq A \cap \partial^{\$}E$ (474S again), $A \setminus \bigcup_{n\in\mathbb{N}} A_n$ is λ_E^∂-negligible (474G), and so is A.

475F Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $\epsilon > 0$. Then λ_E^∂ is inner regular with respect to the family $\mathcal{E} = \{F : F \subseteq \mathbb{R}^r \text{ is Borel},\ \lambda_E^\partial F \le (1+\epsilon)\nu F\}$.

proof (a) We need some elementary bits of geometry.

(i) If $x \in \mathbb{R}^r$, $\delta > 0$, $\alpha \ge 0$ and $v \in S_{r-1}$, then

$$\mu\{z : z \in B(x,\delta),\ |(z-x)\centerdot v| \le \alpha\} \le 2\alpha\beta_{r-1}\delta^{r-1}.$$

P Translating and rotating, if necessary, we can reduce to the case $x = \mathbf{0}$, $v = (0,\dots,1)$. In this case we are looking at

$$\{z : \|z\| \le \delta,\ |\zeta_r| \le \alpha\} \subseteq \{u : u \in \mathbb{R}^{r-1},\ \|u\| \le \delta\} \times [-\alpha,\alpha]$$

which has measure $2\alpha\beta_{r-1}\delta^{r-1}$. **Q**

(ii) If $x \in \mathbb{R}^r$, $\delta > 0$, $0 < \eta \le \frac{1}{2}$, $v \in S_{r-1}$, $H = \{z : z\centerdot v \le \alpha\}$ and

$$|\mu(H \cap B(x,\delta)) - \frac{1}{2}\mu B(x,\delta)| < 2^{-r+1}\beta_{r-1}\eta\delta^r,$$

then $|x\centerdot v - \alpha| \le \eta\delta$. **P** Again translating and rotating if necessary, we may suppose that $x = \mathbf{0}$ and $v = (0,\dots,1)$. Set $H_0 = \{z : \zeta_r \le 0\}$. **?** If $\alpha > \eta\delta$, then $H \cap B(\mathbf{0},\delta)$ includes

$$(H_0 \cap B(\mathbf{0},\delta)) \cup (\{u : u \in \mathbb{R}^{r-1},\ \|u\| \le \frac{1}{2}\delta\} \times [0,\alpha'])$$

where $\alpha' = \min(|\alpha|, \frac{\sqrt{3}}{2}\delta) > \eta\delta)$, so

$$\begin{aligned}\mu(H \cap B(\mathbf{0},\delta)) - \frac{1}{2}\mu B(\mathbf{0},\delta) &= \mu(H \cap B(\mathbf{0},\delta)) - \mu(H_0 \cap B(\mathbf{0},\delta))\\ &\ge 2^{-r+1}\beta_{r-1}\delta^{r-1}\alpha' > 2^{-r+1}\beta_{r-1}\delta^r\eta,\end{aligned}$$

contrary to hypothesis. **X** Similarly, **?** if $\alpha < -\eta\delta$, then $H \cap B(\mathbf{0},\delta)$ is included in

$$H_0 \cap B(\mathbf{0},\delta) \setminus (\{u : u \in \mathbb{R}^{r-1},\ \|u\| \le \frac{1}{2}\delta\} \times]\alpha, 0]),$$

so

$$\begin{aligned}\frac{1}{2}\mu B(\mathbf{0},\delta) - \mu(H \cap B(\mathbf{0},\delta)) &= \mu(H_0 \cap B(\mathbf{0},\delta)) - \mu(H \cap B(\mathbf{0},\delta))\\ &\ge 2^{-r+1}\beta_{r-1}\delta^{r-1}\alpha' > 2^{-r+1}\beta_{r-1}\delta^r\eta,\end{aligned}$$

which is equally impossible. **X** So $|\alpha| \le \eta\delta$. **Q**

(b) Let F be such that $\lambda_E^\partial F > 0$. Let η, $\zeta > 0$ be such that

$$\eta < 1, \quad \tfrac{(1+\eta)^2}{(1-\eta)^{r-1}} \le 1+\epsilon, \quad 2(1+2^r)\beta_r\zeta < 2^{-r}\beta_{r-1}\eta.$$

Because $\partial^{\$}E$ is λ^{∂}_E-conegligible (474G again), $\lambda^{\partial}_E(F \cap \partial^{\$}E) > 0$. Because λ^{∂}_E is a Radon measure (474E) and $\psi_E : \partial^{\$}E \to S_{r-1}$ is $\mathrm{dom}(\lambda^{\partial}_E)$-measurable (474E(i), 474G), there is a compact set $K_1 \subseteq F \cap \partial^{\$}E$ such that $\lambda^{\partial}_E K_1 > 0$ and $\psi_E \restriction K_1$ is continuous, by Lusin's theorem (418J). For $x \in \partial^{\$}E$, set $H_x = \{z : (z - x) \centerdot \psi_E(x) \le 0\}$. The function

$$(x, \delta) \mapsto \mu((E \triangle H_x) \cap B(x, \delta)) : K_1 \times]0, \infty[\to \mathbb{R}$$

is Borel measurable. **P** Take a Borel set E' such that $\mu(E \triangle E') = 0$. Then

$$\{(x, \delta, z) : x \in K_1,\ z \in (E' \triangle H_x) \cap B(x, \delta)\}$$

is a Borel set in $\mathbb{R}^r \times]0, \infty[\times \mathbb{R}^r$, so its sectional measure is a Borel measurable function, by 252P. **Q**

For each $x \in K_1$,

$$\lim_{n\to\infty} \frac{\mu((E \triangle H_x) \cap B(x, 2^{-n}))}{\mu B(x, 2^{-n})} = 0$$

(474R). So there is an $n_0 \in \mathbb{N}$ such that $\lambda^{\partial}_E F_1 > 0$, where F_1 is the Borel set

$$\{x : x \in K_1,\ \mu((E \triangle H_x) \cap B(x, 2^{-n})) \le \zeta \mu B(x, 2^{-n}) \text{ for every } n \ge n_0\}.$$

Let $K_2 \subseteq F_1$ be a compact set such that $\lambda^{\partial}_E K_2 > 0$.

For each $n \in \mathbb{N}$, the function

$$x \mapsto \lambda^{\partial}_E B(x, 2^{-n})) = \lambda^{\partial}_E (x + B(\mathbf{0}, 2^{-n}))$$

is Borel measurable (444Fe). Let $y \in K_2$ be such that $\lambda^{\partial}_E(K_2 \cap B(y, \delta)) > 0$ for every $\delta > 0$ (cf. 411Nd). Set $v = \psi_E(y)$. Let $n > n_0$ be so large that $2\beta_{r-1}\|\psi_E(x) - v\| \le \beta_r \zeta$ whenever $x \in K_1$ and $\|x - y\| \le 2^{-n}$. Set $K_3 = K_2 \cap B(y, 2^{-n-1})$, so that $\lambda^{\partial}_E K_3 > 0$.

(c) We have $|(x - z) \centerdot v| \le \eta \|x - z\|$ whenever $x, z \in K_3$. **P** If $x = z$ this is trivial. Otherwise, let $k \ge n$ be such that $2^{-k-1} \le \|x - z\| \le 2^{-k}$, and set $\delta = 2^{-k}$. Set

$$H'_x = \{w : (w - x) \centerdot v \le 0\}, \quad H'_z = \{w : (w - z) \centerdot v \le 0\}.$$

Since $|(w - x) \centerdot v - (w - x) \centerdot \psi_E(x)| \le 2\delta \|\psi_E(x) - v\|$ whenever $w \in B(x, 2\delta)$,

$$(H_x \triangle H'_x) \cap B(x, 2\delta) \subseteq \{w : w \in B(x, 2\delta),\ |(w - x) \centerdot v| \le 2\delta \|\psi_E(x) - v\|\}$$

has measure at most

$$4\delta \|\psi_E(x) - v\| \beta_{r-1} (2\delta)^{r-1} \le 2\delta \beta_r \zeta (2\delta)^{r-1} = \zeta \mu B(x, 2\delta),$$

using (a-i) for the first inequality. So

$$\begin{aligned}\mu((E \triangle H'_x) \cap B(x, 2\delta)) &\le \mu((E \triangle H_x) \cap B(x, 2\delta)) + \mu((H_x \triangle H'_x) \cap B(x, 2\delta)) \\ &\le 2\zeta \mu B(x, 2\delta)\end{aligned}$$

because $k > n_0$ and $x \in F_1$. Similarly, $\mu((E \triangle H'_z) \cap B(z, \delta)) \le 2\zeta \mu B(z, \delta)$. Now observe that because $\|x - z\| \le \delta$, $B(z, \delta) \subseteq B(x, 2\delta)$,

$$\mu((E \triangle H'_x) \cap B(z, \delta)) \le 2\zeta \mu B(x, 2\delta) = 2^{r+1} \zeta \mu B(z, \delta),$$

and

$$\begin{aligned}\mu((H'_x \triangle H'_z) \cap B(z, \delta)) &\le \mu((E \triangle H'_x) \cap B(z, \delta)) + \mu((E \triangle H'_z) \cap B(z, \delta)) \\ &\le (2 + 2^{r+1}) \zeta \mu B(z, \delta).\end{aligned}$$

Since $\mu(H'_z \cap B(z, \delta)) = \frac{1}{2} \mu B(z, \delta)$,

$$|\mu(H'_x \cap B(z, \delta)) - \frac{1}{2} \mu B(z, \delta)| \le 2(1 + 2^r) \zeta \mu B(z, \delta) < 2^{-r} \beta_{r-1} \eta \delta^r,$$

and (using (a-ii) above)

$$|(x - z) \centerdot v| \le \frac{1}{2} \eta \delta \le \eta \|x - v\|. \text{ **Q**}$$

(d) Let V be the hyperplane $\{w : w \centerdot v = 0\}$, and let $T : K_3 \to V$ be the orthogonal projection, that is, $Tx = x - (x \centerdot v)v$ for every $x \in K_3$. Then (c) tells us that if $x, z \in K_3$,

$$\|Tx - Tz\| \geq \|x - z\| - |(x - z) . v| \geq (1 - \eta)\|x - z\|.$$

Because $\eta < 1$, T is injective. Consider the compact set $T[K_3]$. The inverse T^{-1} of T is $\frac{1}{1-\eta}$-Lipschitz, and $\nu K_3 > 0$ (by 475Eb), so $\nu(T[K_3]) \geq (1 - \eta)^{r-1}\nu K_3 > 0$ (264G/471J). Let $G \supseteq T[K_3]$ be an open set such that $\nu(G \cap V) \leq (1 + \eta)\nu(T[K_3])$. (I am using the fact that the subspace measure ν_V induced by ν on V is a copy of Lebesgue measure on $\mathbb{R}^{r-1}$, so is a Radon measure.) Let $\mathcal{I}$ be the family of non-trivial closed balls $B \subseteq G$ such that $(1 - \eta)^{r-1}\lambda^\partial_E T^{-1}[B] \leq (1 + \eta)\nu(B \cap V)$. Then every point of $T[K_3]$ is the centre of arbitrarily small members of $\mathcal{I}$. **P** If $x \in K_3$ and $\delta_0 > 0$, there is a $\delta \in]0, \delta_0]$ such that $B(Tx, \delta) \subseteq G$ and $\lambda^\partial_E B(x, \delta) \leq (1 + \eta)\beta_{r-1}\delta^{r-1}$ (474S once more). Now consider $B = B(Tx, (1 - \eta)\delta)$. Then $T^{-1}[B] \subseteq B(x, \delta)$, so

$$\lambda^\partial_E T^{-1}[B] \leq \lambda^\partial_E B(x, \delta) \leq (1 + \eta)\beta_{r-1}\delta^{r-1} = \tfrac{1+\eta}{(1-\eta)^{r-1}}\nu(B \cap V). \textbf{ Q}$$

By 261B/472Ca, applied in $V \cong \mathbb{R}^{r-1}$, there is a countable disjoint family $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\nu(T[K_3] \setminus \bigcup \mathcal{I}_0) = 0$.

Now $\nu(K_3 \setminus \bigcup_{B \in \mathcal{I}_0} T^{-1}[B]) = 0$, because T^{-1} is Lipschitz, so $\lambda^\partial_E(K_3 \setminus \bigcup_{B \in \mathcal{I}_0} T^{-1}[B]) = 0$ (475Eb again), and

$$\begin{aligned}\lambda^\partial_E K_3 &\leq \sum_{B \in \mathcal{I}_0} \lambda^\partial_E T^{-1}[B] \leq \tfrac{1+\eta}{(1-\eta)^{r-1}} \sum_{B \in \mathcal{I}_0} \nu(B \cap V)\\ &\leq \tfrac{1+\eta}{(1-\eta)^{r-1}}\nu(G \cap V) \leq \tfrac{(1+\eta)^2}{(1-\eta)^{r-1}}\nu(T[K_3]) \leq \tfrac{(1+\eta)^2}{(1-\eta)^{r-1}}\nu K_3\end{aligned}$$

(because T is 1-Lipschitz)

$$\leq (1 + \epsilon)\nu K_3$$

by the choice of η. Thus $K_3 \in \mathcal{E}$.

(e) This shows that every λ^∂_E-non-negligible set measured by λ^∂_E includes a λ^∂_E-non-negligible member of $\mathcal{E}$. As $\mathcal{E}$ is closed under disjoint unions, λ^∂_E is inner regular with respect to $\mathcal{E}$ (412Aa).

475G Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Then $\lambda^\partial_E = \nu \llcorner \partial^* E$, that is, for $F \subseteq \mathbb{R}^r$, $\lambda^\partial_E F = \nu(F \cap \partial^* E)$ whenever either is defined.

proof (a) Suppose first that F is a Borel set included in the reduced boundary $\partial^\$ E$ of E. Then $\nu F \leq \lambda^\partial_E F$, by 475Ea. On the other hand, for any $\epsilon > 0$ and $\gamma < \lambda^\partial_E F$, there is an $F_1 \subseteq F$ such that

$$\gamma \leq \lambda^\partial_E F_1 \leq (1 + \epsilon)\nu F_1 \leq (1 + \epsilon)\nu F,$$

by 475F; so we must have $\lambda^\partial_E F = \nu F$.

(b) Now suppose that F is measured by λ^∂_E. Because $\partial^\$ E$ is λ^∂_E-conegligible, and λ^∂_E is a σ-finite Radon measure, there is a Borel set $F' \subseteq F \cap \partial^\$ E$ such that $\lambda^\partial_E(F \setminus F') = 0$. Now $\nu F' = \lambda^\partial_E F'$, by (a), and $\nu(F \cap \partial^\$ E \setminus F') = 0$, by 475Ea, and $\nu(\partial^* E \setminus \partial^\$ E) = 0$, by 475D; so $\nu(F \cap \partial^* E)$ is defined and equal to $\lambda^\partial_E F' = \lambda^\partial_E F$.

(c) Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence of compact subsets of $\partial^\$ E$ such that $\bigcup_{n \in \mathbb{N}} K_n$ is λ^∂_E-conegligible. By (b), $\nu(\partial^* E \setminus \bigcup_{n \in \mathbb{N}} K_n) = 0$, while $\nu K_n = \lambda^\partial_E K_n$ is finite for each n. For each n, the subspace measure ν_{K_n} on K_n is a multiple of Hausdorff $(r-1)$-dimensional measure on K_n (471E), so is a Radon measure (471Dh, 471F), as is $(\lambda^\partial_E)_{K_n}$; since, by (b), ν_{K_n} and $(\lambda^\partial_E)_{K_n}$ agree on the Borel subsets of K_n, they are actually identical. So if $F \subseteq \mathbb{R}^r$ is such that ν measures $F \cap \partial^* E$, λ^∂_E will measure $F \cap K_n$ for every n, and therefore will measure F; so that in this case also $\lambda^\partial_E F = \nu(F \cap \partial^* E)$.

475H Proposition Let $V \subseteq \mathbb{R}^r$ be a hyperplane, and $T : \mathbb{R}^r \to V$ the orthogonal projection. Suppose that $A \subseteq \mathbb{R}^r$ is such that νA is defined and finite, and for $u \in V$ set

$$\begin{aligned} f(u) &= \#(A \cap T^{-1}[\{u\}]) \text{ if this is finite,}\\ &= \infty \text{ otherwise.}\end{aligned}$$

Then $\int_V f(u)\nu(du)$ is defined and at most νA.

proof (a) Because ν is invariant under isometries, we can suppose that $V = \{x : \xi_r = 0\}$, so that $Tx = (\xi_1, \dots, \xi_{r-1}, 0)$ for $x = (\xi_1, \dots, \xi_r)$. For m, $n \in \mathbb{N}$ and $u \in V$ set

$$f_{mn}(u) = \#(\{k : k \in \mathbb{Z},\ |k| \le 4^m,\ A \cap (\{u\} \times [2^{-m}k, 2^{-m}(k+1-2^{-n})[) \ne \emptyset\});$$

so that $f(u) = \lim_{m\to\infty} \lim_{n\to\infty} f_{mn}(u)$ for every $u \in V$.

(b) Suppose for the moment that A is actually a Borel set. Then $T[A \cap (\mathbb{R}^r \times [\alpha, \beta[)]$ is always analytic (423Eb, 423Bb), therefore measured by ν (432A), and every f_{mn} is measurable. Next, given $\epsilon > 0$ and m, $n \in \mathbb{N}$, there is a sequence $\langle F_i \rangle_{i\in\mathbb{N}}$ of closed sets of diameter at most 2^{-m-n}, covering A, such that $\sum_{i=0}^{\infty} 2^{-r+1}\beta_{r-1}(\operatorname{diam} F_i)^{r-1} \le \nu A + \epsilon$. Now each $T[F_i]$ is a compact set of diameter at most $\operatorname{diam} F_i$, so $\nu(T[F_i]) \le 2^{-r+1}\beta_{r-1}(\operatorname{diam} F_i)^{r-1}$ (264H); and if we set $g = \sum_{i=0}^{\infty} \chi T[F_i]$, $f_{mn} \le g$ everywhere on V, so $\int f_{mn} d\nu \le \int g\, d\nu \le \nu A + \epsilon$. As ϵ is arbitary, $\int f_{mn} d\nu \le \nu A$. Accordingly

$$\int f d\nu = \lim_{m\to\infty} \lim_{n\to\infty} \int f_{mn} d\nu$$

(because the limits are monotonic)

$$\le \nu A + \epsilon.$$

As ϵ is arbitrary, $\int f d\nu \le \nu A$.

(c) In general, there are Borel sets E, F such that $E \setminus F \subseteq A \subseteq E$ and $\nu F = 0$, by 264Fc/471Db. By (b),

$$\int \#(E \cap T^{-1}[\{u\}])\nu(du) \le \nu E, \quad \int \#(F \cap T^{-1}[\{u\}])\nu(du) \le \nu F = 0,$$

so $\int \#(A \cap T^{-1}[\{u\}])\nu(du)$ is defined and equal to $\int \#(E \cap T^{-1}[\{u\}])\nu(du) \le \nu A$.

475I Lemma (In this lemma I allow $r = 1$.) Let $\mathcal{K}$ be the family of compact sets $K \subseteq \mathbb{R}^r$ such that $K = \mathrm{cl}^*K$. Then μ is inner regular with respect to $\mathcal{K}$.

proof (a) Write $\mathcal{D}$ for the set of dyadic (half-open) cubes in $\mathbb{R}^r$, that is, sets expressible in the form $\prod_{i<r}[2^{-m}k_i, 2^{-m}(k_i+1)[$ where m, $k_0, \dots, k_{r-1} \in \mathbb{Z}$. For $m \in \mathbb{N}$ and $x \in \mathbb{R}^r$ write $C(x, m)$ for the dyadic cube with side of length 2^{-m} which contains x. Then, for any $A \subseteq \mathbb{R}^r$,

$$\lim_{m\to\infty} 2^{mr}\mu_*(A \cap C(x, m)) = 1$$

for every $x \in \mathrm{int}^*A$. **P** $C(x, m) \subseteq B(x, 2^{-m}\sqrt{r})$ for each m, so

$$2^{mr}\mu^*(C(x, m) \setminus A) \le \beta_r r^{r/2} \frac{\mu^*(B(x,2^{-m}\sqrt{r})\setminus A)}{\mu B(x,2^{-m}\sqrt{r})} \to 0$$

as $m \to \infty$. **Q**

(b) Now, given a Lebesgue measurable set $E \subseteq \mathbb{R}^r$ and $\gamma < \mu E$, choose $\langle E_n \rangle_{n\in\mathbb{N}}$, $\langle \gamma_n \rangle_{n\in\mathbb{N}}$, $\langle m_n \rangle_{n\in\mathbb{N}}$ and $\langle K_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. Start with $E_0 = E$ and $\gamma_0 = \gamma$. Given that $\gamma_n < \mu E_n$, let $K_n \subseteq E_n$ be a compact set of measure greater than γ_n. Now, by (a), there is an $m_n \ge n$ such that $\mu^* E_{n+1} > \gamma_n$, where

$$E_{n+1} = \{x : x \in K_n,\ \mu(K_n \cap C(x, m_n)) \ge \frac{2}{3}\mu C(x, m_n)\};$$

note that E_{n+1} is of the form $K_n \cap \bigcup \mathcal{D}_0$ for some set $\mathcal{D}_0$ of half-open cubes of side 2^{-m_n}, so that E_{n+1} is measurable and

$$\mu(E_{n+1} \cap C(x, m_n)) = \mu(K_n \cap C(x, m_n)) \ge \frac{2}{3}\mu C(x, m_n)$$

for every $x \in E_{n+1}$. Now set

$$\gamma_{n+1} = \max(\gamma_n, \mu E_{n+1} - \frac{1}{3} \cdot 2^{-m_n r}),$$

and continue.

At the end of the induction, set $K = \bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} E_n$. Then K is compact, $K \subseteq E$ and

$$\mu K = \lim_{n\to\infty} \mu K_n = \lim_{n\to\infty} \gamma_n \ge \gamma.$$

If $x \in K$ and $n \in \mathbb{N}$, then

$$\begin{aligned}\mu(K\cap B(x,2^{-m_n}\sqrt{r}))&\ge\mu(K\cap C(x,m_n))\ge\mu(E_{n+1}\cap C(x,m_n))-\mu E_{n+1}+\mu K\\&\ge\frac{2}{3}\mu C(x,m_n)-\mu E_{n+1}+\gamma_{n+1}\\&\ge\frac{1}{3}\cdot 2^{-m_nr}=\frac{1}{3\beta_rr^{r/2}}\mu B(x,2^{-m_n}\sqrt{r}),\end{aligned}$$

so

$$\limsup_{\delta\downarrow 0}\tfrac{\mu(K\cap B(x,\delta))}{\mu B(x,\delta)}\ge\tfrac{1}{3\beta_rr^{r/2}}>0$$

and $x\in\mathrm{cl}^*K$.

(c) Thus $K\subseteq\mathrm{cl}^*K$. Since certainly $\mathrm{cl}^*K\subseteq\overline{K}=K$, we have $K=\mathrm{cl}^*K$.

475J Lemma Let E be a Lebesgue measurable subset of $\mathbb{R}^r$, identified with $\mathbb{R}^{r-1}\times\mathbb{R}$. For $u\in\mathbb{R}^{r-1}$, set $G_u=\{t:(u,t)\in\mathrm{int}^*E\}$, $H_u=\{t:(u,t)\in\mathrm{int}^*(\mathbb{R}^r\setminus E)\}$ and $D_u=\{t:(u,t)\in\partial^*E\}$, so that G_u, H_u and D_u are disjoint and cover $\mathbb{R}$.

(a) There is a μ_{r-1}-conegligible set $Z\subseteq\mathbb{R}^{r-1}$ such that whenever $u\in Z$, $t<t'$ in $\mathbb{R}$, $t\in G_u$ and $t'\in H_u$, there is an $s\in D_u\cap\,]t,t'[$.

(b) There is a μ_{r-1}-conegligible set $Z_1\subseteq\mathbb{R}^{r-1}$ such that whenever $u\in Z_1$, t, $t'\in\mathbb{R}$, $t\in G_u$ and $t'\in H_u$, there is a member of D_u between t and t'.

(c) If E has locally finite perimeter, there is a conegligible set $Z_2\subseteq Z_1$ such that, for every $u\in Z_2$, $D_u\cap[-n,n]$ is finite for every $n\in\mathbb{N}$, G_u and H_u are open, and $D_u=\partial G_u=\partial H_u$, so that the constituent intervals of $\mathbb{R}\setminus D_u$ lie alternately in G_u and H_u.

proof (a)(i) For $q\in\mathbb{Q}$, set $f_q(u)=\sup(G_u\cap\,]-\infty,q[)$, taking $-\infty$ for $\sup\emptyset$. Then $f_q:\mathbb{R}^{r-1}\to[-\infty,q]$ is Lebesgue measurable. **P** For $\alpha<q$,

$$\{u:f_q(u)>\alpha\}=\{u:\text{there is some }t\in\,]\alpha,q[\text{ such that }(u,t)\in\mathrm{int}^*E\}.$$

Because int^*E is a Borel set (475Cc), $\{u:f_q(u)>\alpha\}$ is analytic (423Eb, 423Bc), therefore measurable (432A again). **Q**

(ii) For any $q\in\mathbb{Q}$, $W_q=\{u:f_q(u)<q,\ f_q(u)\in G_u\}$ is negligible. **P?** Suppose, if possible, otherwise. Let $n\in\mathbb{N}$ be such that $\{u:u\in W_q,\ f_q(u)>-n\}$ is not negligible. If we think of $[-\infty,q]$ as a compact metrizable space, 418J again tells us that there is a Borel set $F\subseteq\mathbb{R}^{r-1}$ such that $f_q\restriction F$ is continuous and $F_1=\{u:u\in F\cap W_q,\ f_q(u)>-n\}$ is not negligible. Note that F_1 is measurable, being the projection of the Borel set $\{(u,f_q(u)):u\in F,\ -n<f_q(u)<q\}\cap\mathrm{int}^*E$. By 475I, there is a non-negligible compact set $K\subseteq F_1$ such that $K\subseteq\mathrm{cl}^*K$, interpreting cl^*K in $\mathbb{R}^{r-1}$. Because $g\restriction K$ is continuous, it attains its maximum at $u\in K$ say.

Set $x=(u,f_q(u))$. Then, whenever $0<\delta\le\min(n+f_q(u),q-f_q(u))$,

$$B(x,2\delta)\setminus\mathrm{int}^*E\supseteq\{(w,t):w\in K\cap V(u,\delta),\ |t-f_q(u)|\le\delta,\ f_q(w)<t<q\}$$

(writing $V(u,\delta)$ for $\{w:w\in\mathbb{R}^{r-1},\ \|w-u\|\le\delta\}$)

$$\supseteq\{(w,t):w\in K\cap V(u,\delta),\ f_q(u)<t<f_q(u)+\delta\}$$

because $f_q(w)\le f_q(u)$ for every $w\in K$. So, for such δ,

$$\begin{aligned}\mu(B(x,2\delta)\setminus\mathrm{int}^*E)&\ge\delta\mu_{r-1}(K\cap V(u,\delta))\\&=\tfrac{\beta_{r-1}}{2^r\beta_r}\cdot\tfrac{\mu_{r-1}(K\cap V(u,\delta))}{\mu_{r-1}V(u,\delta)}\cdot\mu B(x,2\delta),\end{aligned}$$

and

$$\begin{aligned}\limsup_{\delta\downarrow 0}\tfrac{\mu(B(x,\delta)\setminus E)}{\mu B(x,\delta)}&=\limsup_{\delta\downarrow 0}\tfrac{\mu(B(x,2\delta)\setminus\mathrm{int}^*E)}{\mu B(x,2\delta)}\\&\ge\tfrac{\beta_{r-1}}{2^r\beta_r}\limsup_{\delta\downarrow 0}\tfrac{\mu_{r-1}(K\cap V(u,\delta))}{\mu_{r-1}V(u,\delta)}>0\end{aligned}$$

because $u \in \mathrm{cl}^*K$; but $x = (u, f_q(u))$ is supposed to belong to int^*E. **XQ**

(iii) Similarly, setting

$$f'_q(u) = \inf(H_u \cap]u, \infty[), \quad W'_q = \{u : q < f'_q(u) < \infty,\ f'_q(u) \in H_u\},$$

every W'_q is negligible. So $Z = \mathbb{R}^{r-1} \setminus \bigcup_{q\in\mathbb{Q}}(W_q \cup W'_q)$ is μ_{r-1}-conegligible.

Now if $u \in Z$, $t \in G_u$ and $t' \in H_u$, where $t < t'$, there is some $s \in]t, t'[\cap D_u$. **P?** Suppose, if possible, otherwise. Since, by hypothesis, neither t nor t' belongs to D_u, $D_u \cap [t, t'] = \emptyset$. Note that, because $u \in Z$,

$$s = \inf(G_u \cap]s, \infty[) \text{ for every } s \in G_u, \quad s = \sup(H_u \cap]-\infty, s[) \text{ for every } s \in H_u.$$

Choose $\langle s_n\rangle_{n\in\mathbb{N}}$, $\langle s'_n\rangle_{n\in\mathbb{N}}$ and $\langle \delta_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Set $s_0 = t$, $s'_0 = t'$. Given that $t \le s_n < s'_n \le t'$ and $s_n \in G_u$ and $s'_n \in H_u$, where n is even, set $s'_{n+1} = \sup(G_u \cap [s_n, s'_n])$. Then either $s'_{n+1} = s'_n$, so $s'_{n+1} \in H_u$, or $s'_{n+1} < s'_n$ and $\left]s'_{n+1}, s'_n\right] \cap G_u = \emptyset$, so $s'_{n+1} \notin G_u$ and again $s'_{n+1} \in H_u$. Let $\delta_n > 0$ be such that $\delta_n \le 2^{-n}$ and $\mu(E \cap B((u, s'_{n+1}), \delta_n)) < \frac{1}{2}\beta_r\delta_n^r$. Because the function $s \mapsto \mu(E \cap B((u, s), \delta_n))$ is continuous (443C, or otherwise), there is an $s_{n+1} \in G_u \cap \left[s_n, s'_{n+1}\right[$ such that $\mu(E \cap B((u, s), \delta_n)) \le \frac{1}{2}\beta_r\delta_n^r$ whenever $s \in [s_{n+1}, s'_{n+1}]$.

This is the inductive step from even n. If n is odd and $s_n \in G_u$, $s'_n \in H_u$ and $s_n < s'_n$, set $s_{n+1} = \inf(H_u \cap [s_n, s'_n])$. This time we find that s_{n+1} must belong to G_u. Let $\delta_n \in]0, 2^{-n}]$ be such that $\mu(E \cap B((u, s_{n+1}), \delta_n)) > \frac{1}{2}\beta_r\delta_n^r$, and let $s'_{n+1} \in H_u \cap]s_{n+1}, s'_n]$ be such that $\mu(E \cap B((u, s), \delta_n)) \ge \frac{1}{2}\beta_r\delta_n^r$ whenever $s \in [s_{n+1}, s'_{n+1}]$.

The construction provides us with a non-increasing sequence $\langle [s_n, s'_n]\rangle_{n\in\mathbb{N}}$ of closed intervals, so there must be some s in their intersection. In this case

$$\mu(E \cap B((u, s), \delta_n) \le \tfrac{1}{2}\beta_r\delta_n^r \text{ if } n \text{ is even},$$

$$\mu(E \cap B((u, s), \delta_n) \ge \tfrac{1}{2}\beta_r\delta_n^r \text{ if } n \text{ is odd}.$$

Since $\lim_{n\to\infty} \delta_n = 0$,

$$\liminf_{\delta\downarrow 0} \tfrac{\mu(E\cap B((u,s),\delta))}{\mu B((u,s),\delta)} \le \frac{1}{2}, \quad \limsup_{\delta\downarrow 0} \tfrac{\mu(E\cap B((u,s),\delta))}{\mu B((u,s),\delta)} \ge \frac{1}{2},$$

and $s \in D_u$, while of course $t \le s \le t'$. **XQ**

(iv) Thus the conegligible set Z has the property required by (a).

(b) Applying (a) to $\mathbb{R}^r \setminus E$, there is a conegligible set $Z' \subseteq \mathbb{R}^{r-1}$ such that if $u \in Z'$, $t \in H_u$, $t' \in G_u$ and $t < t'$, then D_u meets $]t, t'[$. So we can use $Z \cap Z'$.

(c) Now suppose that E has locally finite perimeter. We know that $\nu(\partial^*E \cap B(\mathbf{0}, n)) = \lambda^\partial_E B(\mathbf{0}, n)$ is finite for every $n \in \mathbb{N}$ (475G). By 475H,

$$\int_{\|u\|\le n} \#(D_u \cap [-n, n])\mu_{r-1}(du) \le \nu(\partial^*E \cap B(\mathbf{0}, 2n)) < \infty$$

for every $n \in \mathbb{N}$; but this means that $D_u \cap [-n, n]$ must be finite for almost every u such that $\|u\| \le n$, for every n, and therefore that

$$Z'_1 = \{u : u \in Z_1,\ D_u \cap [-n, n] \text{ is finite for every } n\}$$

is conegligible. For $u \in Z'_1$, $\mathbb{R} \setminus D_u$ is an open set, so is made up of a disjoint sequence of intervals with endpoints in $D_u \cup \{-\infty, \infty\}$ (see 2A2I); and because $u \in Z_1$, each of these intervals is included in either G_u or H_u. Now

$$A = \{u : \text{there are successive constituent intervals of } \mathbb{R} \setminus D_u \text{ both included in } G_u\}$$

is negligible. **P?** Otherwise, there are rationals $q < q'$ such that

$$F = \{u :]q, q'[\setminus G_u \text{ contains exactly one point}\}$$

is not negligible. Note that, writing $T : \mathbb{R}^r \to \mathbb{R}^{r-1}$ for the orthogonal projection,

$$\begin{aligned} F = {} & T[(\mathbb{R}^{r-1} \times]q, q'[) \setminus \mathrm{int}^*E] \\ & \setminus \bigcup_{q''\in\mathbb{Q}, q<q''<q'} T[(\mathbb{R}^{r-1} \times]q, q''[) \setminus \mathrm{int}^*E] \cap T[(\mathbb{R}^{r-1} \times]q'', q'[) \setminus \mathrm{int}^*E] \end{aligned}$$

is measurable. Take any $u \in F \cap \mathrm{int}^*F$, and let t be the unique point in $]q, q'[\setminus G_u$. Then whenever $0 < \delta \le \min(t - q, q' - t)$ we shall have

$$\mu(B((u, t), \delta) \setminus E) = \mu(B((u, t), \delta) \setminus \mathrm{int}^*E) \le 2\delta\mu_{r-1}(V(u, \delta) \setminus F),$$

because if $w \in F$ then $(w,s) \in \text{int}^*E$ for almost every $s \in [t-\delta, t+\delta]$. So

$$\limsup_{\delta\downarrow 0} \frac{\mu(B((u,t),\delta)\setminus E)}{\mu B((u,t),\delta)} \le \frac{2\beta_{r-1}}{\beta_r} \limsup_{\delta\downarrow 0} \frac{\mu_{r-1}(V(u,\delta)\setminus F)}{\mu_{r-1}V(u,\delta)} = 0,$$

and $(u,t) \in \text{int}^*E$. **XQ**

Similarly,

$$A' = \{u : \text{there are successive constituent intervals of } \mathbb{R}\setminus D_u \text{ both included in } H_u\}$$

is negligible. So $Z_2 = Z_1' \setminus (A \cup A')$ is a conegligible set of the kind we need.

475K Lemma Suppose that $h : \mathbb{R}^r \to [-1,1]$ is a Lipschitz function with compact support; let $n \in \mathbb{N}$ be such that $h(x) = 0$ for $\|x\| \ge n$. Suppose that $E \subseteq \mathbb{R}^r$ is a Lebesgue measurable set. Then

$$|\int_E \frac{\partial h}{\partial \xi_r} d\mu| \le 2\Bigl(\beta_{r-1}n^{r-1} + \nu(\partial^*E \cap B(\mathbf{0},n))\Bigr).$$

proof By Rademacher's theorem (262Q), $\frac{\partial h}{\partial \xi_r}$ is defined almost everywhere; as it is measurable and bounded, and is zero outside $B(\mathbf{0},n)$, the integral is well-defined. If $\nu(\partial^*E \cap B(\mathbf{0},n))$ is infinite, the result is trivial; so henceforth let us suppose that $\nu(\partial^*E \cap B(\mathbf{0},n)) < \infty$. Identify $\mathbb{R}^r$ with $\mathbb{R}^{r-1}\times\mathbb{R}$. For $u \in \mathbb{R}^{r-1}$, set

$$\begin{aligned} f(u) &= \#(\{t : (u,t) \in \partial^*E \cap B(\mathbf{0},n)\}) \text{ if this is finite,} \\ &= \infty \text{ otherwise.} \end{aligned}$$

By 475H, $\int f d\mu_{r-1} \le \nu(\partial^*E \cap B(\mathbf{0},n))$. By 475Jb, there is a μ_{r-1}-conegligible set $Z \subseteq \mathbb{R}^{r-1}$ such that $f(u)$ is finite for every $u \in Z$ and

whenever $u \in Z$ and $(u,t) \in \text{int}^*E$ and $(u,t') \in \mathbb{R}^r \setminus \text{cl}^*E$, there is an s lying between t and t' such that $(u,s) \in \partial^*E$.

For $u \in Z$, set $D_u' = \{t : (u,t) \in B(\mathbf{0},n) \setminus \partial^*E\}$, and define $g_u : D_u' \to \{0,1\}$ by setting

$$\begin{aligned} g_u(t) &= 1 \text{ if } (u,t) \in B(\mathbf{0},n) \cap \text{int}^*E, \\ &= 0 \text{ if } (u,t) \in B(\mathbf{0},n) \setminus \text{cl}^*E. \end{aligned}$$

Now if t, $t' \in D_u'$ and $g(t) \ne g(t')$, there is a point s between t and t' such that $(u,s) \in \partial^*E$; so the variation $\text{Var}_{D_u'} g_u$ of g_u (224A) is at most $f(u)$. Setting $h_u(t) = h(u,t)$ for $u \in \mathbb{R}^{r-1}$ and $t \in \mathbb{R}$, we now have

$$\int_{-\infty}^{\infty} \frac{\partial h}{\partial \xi_r}(u,t)\chi(B(\mathbf{0},n) \cap \text{int}^*E)(u,t)dt = \int_{D_u'} h_u'(t) g_u(t) dt$$

(because $h_u'(t) = 0$ if $(u,t) \notin B(\mathbf{0},n)$)

$$\le (1 + \text{Var}_{D_u'} g_u) \sup_{a\le b} |\int_a^b h_u'(t)|dt$$

(by 224J, recalling that D_u' is either empty or a bounded interval with finitely many points deleted)

$$\le (1 + f(u)) \sup_{a\le b} |h_u(b) - h_u(a)|$$

(because h_u is Lipschitz, therefore absolutely continuous on any bounded interval)

$$\le 2(1 + f(u)).$$

Integrating over u, we now have

$$|\int_E \frac{\partial h}{\partial \xi_r} d\mu| = |\int_{B(\mathbf{0},n)\cap\text{int}^*E} \frac{\partial h}{\partial \xi_r} d\mu|$$

(475Cg)

$$= \int_{V(\mathbf{0},n)} \int_{-\infty}^{\infty} \frac{\partial h}{\partial \xi_r}(u,t)\chi(B(\mathbf{0},n) \cap \text{int}^*E)(u,t)dt\, \mu_{r-1}(du)$$

(where $V(\mathbf{0},n) = \{u : u \in \mathbb{R}^{r-1},\ \|u\| \le n\}$)

$$\leq \int_{V(\mathbf{0},n)} 2(1+f(u))\mu_{r-1}(du)$$
$$\leq 2\bigl(\beta_{r-1}n^{r-1} + \nu(\partial^* E \cap B(\mathbf{0},n))\bigr).$$

475L Theorem Suppose that $E \subseteq \mathbb{R}^r$. Then E has locally finite perimeter iff $\nu(\partial^* E \cap B(\mathbf{0},n))$ is finite for every $n \in \mathbb{N}$.

proof If E has locally finite perimeter, then $\nu(\partial^* E \cap B(\mathbf{0},n)) = \lambda_E^{\partial} B(\mathbf{0},n)$ is finite for every n, by 475G. So let us suppose that $\nu(\partial^* E \cap B(\mathbf{0},n))$ is finite for every $n \in \mathbb{N}$. Then $\mu(\partial^* E \cap B(\mathbf{0},n)) = 0$ for every n (471L), $\mu(\partial^* E) = 0$ and E is Lebesgue measurable (475Ch).

If $n \in \mathbb{N}$, then

$$\sup\{|\int_E \operatorname{div}\phi\, d\mu| : \phi : \mathbb{R}^r \to \mathbb{R}^r \text{ is Lipschitz, } \|\phi\| \leq \chi B(\mathbf{0},n)\}$$
$$\leq 2r\bigl(\beta_{r-1}n^{r-1} + \nu(\partial^* E \cap B(\mathbf{0},n))\bigr)$$

is finite. **P** If $\phi = (\phi_1, \dots, \phi_r) : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function and $\|\phi\| \leq \chi B(\mathbf{0},n)$, then 475K tells us that

$$|\int_E \frac{\partial \phi_r}{\partial \xi_r} d\mu| \leq 2\Bigl(\beta_{r-1}n^{r-1} + \nu(\partial^* E \cap B(\mathbf{0},n))\Bigr).$$

But of course it is equally true that

$$|\int_E \frac{\partial \phi_j}{\partial \xi_j} d\mu| \leq 2\Bigl(\beta_{r-1}n^{r-1} + \nu(\partial^* E \cap B(\mathbf{0},n))\Bigr)$$

for every other $j \leq r$; adding, we have the result. **Q**

Since n is arbitrary, E has locally finite perimeter.

475M Corollary (a) The family of sets with locally finite perimeter is a subalgebra of the algebra of Lebesgue measurable subsets of $\mathbb{R}^r$.

(b) A set $E \subseteq \mathbb{R}^r$ is Lebesgue measurable and has finite perimeter iff $\nu(\partial^* E) < \infty$, and in this case $\nu(\partial^* E)$ is the perimeter of E.

(c) If $E \subseteq \mathbb{R}^r$ has finite measure, then $\operatorname{per} E = \liminf_{\alpha\to\infty} \operatorname{per}(E \cap B(\mathbf{0},\alpha))$.

proof (a) Recall that the definition in 474D insists that sets with locally finite perimeter should be Lebesgue measurable. If $E \subseteq \mathbb{R}^r$ has locally finite perimeter, then so has $\mathbb{R}^r \setminus E$, by 474J. If E, $F \subseteq \mathbb{R}^r$ have locally finite perimeter, then

$$\nu(\partial^*(E \cup F) \cap B(\mathbf{0},n)) \leq \nu(\partial^* E \cap B(\mathbf{0},n)) + \nu(\partial^* F \cap B(\mathbf{0},n))$$

is finite for every $n \in \mathbb{N}$, by 475L and 475Cd; by 475L in the other direction, $E \cup F$ has locally finite perimeter.

(b) If E is Lebesgue measurable and has finite perimeter (on the definition in 474D), then $\nu(\partial^* E) = \lambda_E^{\partial}\mathbb{R}^r$ is the perimeter of E, by 475G. If $\nu(\partial^* E) < \infty$, then $\mu(\partial^* E) = 0$ and E is measurable (471L and 475Ch); now E has locally finite perimeter, by 475L, and 475G again tells us that $\nu(\partial^* E) = \lambda_E^{\partial}\mathbb{R}^r$ is the perimeter of E.

(c) Now suppose that $\mu E < \infty$. For any $\alpha \geq 0$,

$$\partial^*(E \cap B(\mathbf{0},\alpha)) \subseteq (\partial^* E \cap B(\mathbf{0},\alpha)) \cup (\mathrm{cl}^* E \cap \partial B(\mathbf{0},\alpha)) \subseteq \partial^* E \cup (\mathrm{cl}^* E \cap \partial B(\mathbf{0},\alpha))$$

by 475Cf. Now we know also that

$$\int_0^\infty \nu(\mathrm{cl}^* E \cap \partial B(\mathbf{0},\alpha)) d\alpha = \mu(\mathrm{cl}^* E) = \mu E < \infty$$

(265G), so $\liminf_{\alpha\to\infty} \nu(\mathrm{cl}^* E \cap \partial B(\mathbf{0},\alpha)) = 0$. This means that

$$\liminf_{\alpha\to\infty} \operatorname{per}(E \cap B(\mathbf{0},\alpha)) = \liminf_{\alpha\to\infty} \nu(\partial^*(E \cap B(\mathbf{0},\alpha)))$$
$$\leq \liminf_{\alpha\to\infty} \nu(\partial^* E) + \nu(\mathrm{cl}^* E \cap \partial B(\mathbf{0},\alpha)) = \nu(\partial^* E).$$

In the other direction,

$$\partial^* E \cap \operatorname{int} B(\mathbf{0},\alpha) = \partial^* E \cap \operatorname{int}^* B(\mathbf{0},\alpha) \subseteq \partial^*(E \cap B(\mathbf{0},\alpha))$$

for every α, by 475Ce, so

$$\begin{aligned}\operatorname{per} E = \nu(\partial^* E) &= \lim_{\alpha\to\infty} \nu(\partial^* E \cap \operatorname{int} B(\mathbf{0},\alpha)) \\ &\le \liminf_{\alpha\to\infty} \nu(\partial^*(E\cap B(\mathbf{0},\alpha))) = \liminf_{\alpha\to\infty} \operatorname{per}(E\cap B(\mathbf{0},\alpha))\end{aligned}$$

and we must have equality.

Remark See 475Xk.

475N The Divergence Theorem Let $E \subseteq \mathbb{R}^r$ be such that $\nu(\partial^* E \cap B(\mathbf{0}, n))$ is finite for every $n \in \mathbb{N}$.
(a) E is Lebesgue measurable.
(b) For ν-almost every $x \in \partial^* E$, there is a Federer exterior normal v_x of E at x.
(c) For every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support,

$$\int_E \operatorname{div} \phi \, d\mu = \int_{\partial^* E} \phi(x) \,.\, v_x \, \nu(dx).$$

proof By 475L, E has locally finite perimeter, and in particular is Lebesgue measurable. By 474R, there is a Federer exterior normal $v_x = \psi_E(x)$ of E at x for every $x \in \partial^{\$} E$; by 475D, ν-almost every point in $\partial^* E$ is of this kind. By 474E-474F,

$$\int_E \operatorname{div} \phi \, d\mu = \int \phi \,.\, \psi_E \, d\lambda^\partial_E = \int_{\partial^{\$} E} \phi \,.\, \psi_E \, d\lambda^\partial_E,$$

and this is also equal to $\int_{\partial^* E} \phi \,.\, \psi_E \, d\lambda^\partial_E$, because $\partial^{\$} E \subseteq \partial^* E$ and $\partial^{\$} E$ is λ^∂_E-conegligible. But λ^∂_E and ν induce the same subspace measures on $\partial^* E$, by 475G, so

$$\int_E \operatorname{div} \phi \, d\mu = \int_{\partial^* E} \phi \,.\, \psi_E \, d\nu = \int_{\partial^* E} \phi(x) \,.\, v_x \, \nu(dx),$$

as claimed.

475O At the price of some technicalities which are themselves instructive, we can now proceed to some basic properties of the essential boundary.

Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and ψ_E its canonical outward-normal function. Let v be the unit vector $(0, \ldots, 0, 1)$. Identify $\mathbb{R}^r$ with $\mathbb{R}^{r-1} \times \mathbb{R}$. Then we have sequences $\langle F_n \rangle_{n\in\mathbb{N}}$, $\langle g_n \rangle_{n\in\mathbb{N}}$ and $\langle g'_n \rangle_{n\in\mathbb{N}}$ such that

(i) for each $n \in \mathbb{N}$, F_n is a Lebesgue measurable subset of $\mathbb{R}^{r-1}$, and g_n, $g'_n : F_n \to [-\infty, \infty]$ are Lebesgue measurable functions such that $g_n(u) < g'_n(u)$ for every $u \in F_n$;
(ii) if m, $n \in \mathbb{N}$ are distinct and $u \in F_m \cap F_n$, then $[g_m(u), g'_m(u)] \cap [g_n(u), g'_n(u)] = \emptyset$;
(iii) $\sum_{n=0}^\infty \int_{F_n} g'_n - g_n d\mu_{r-1} = \mu E$;
(iv) for any continuous function $h : \mathbb{R}^r \to \mathbb{R}$ with compact support,

$$\int_{\partial^* E} h(x) v \,.\, \psi_E(x) \, \nu(dx) = \sum_{n=0}^\infty \int_{F_n} h(u, g'_n(u)) - h(u, g_n(u)) \mu_{r-1}(du),$$

where we interpret $h(u, \infty)$ and $h(u, -\infty)$ as 0 if necessary;
(v) for μ_{r-1}-almost every $u \in \mathbb{R}^{r-1}$,

$$\begin{aligned}\{t : (u,t) \in \partial^* E\} = \{g_n(u) &: n \in \mathbb{N},\, u \in F_n,\, g_n(u) \ne -\infty\} \\ &\cup \{g'_n(u) : n \in \mathbb{N},\, u \in F_n,\, g'_n(u) \ne \infty\}.\end{aligned}$$

proof (a) Take a conegligible set $Z_2 \subseteq \mathbb{R}^{r-1}$ as in 475Jc. Let $Z \subseteq Z_2$ be a conegligible Borel set. For $u \in Z$ set $D_u = \{t : (u,t) \in \partial^* E\}$.

(b) For each $q \in \mathbb{Q}$, set $F'_q = \{u : u \in Z,\, (u,q) \in \operatorname{int}^* E\}$, so that F'_q is a Borel set, and for $u \in F'_q$ set

$$f_q(u) = \sup(D_u \cap \,]-\infty, q[), \quad f'_q(u) = \inf(D_u \cap \,]-\infty, q[),$$

allowing $-\infty$ as $\sup\emptyset$ and ∞ as $\inf\emptyset$. Observe that (because $D_u \cap [q-1, q+1]$ is finite) $f_q(u) < q < f'_q(u)$. Now f_q and f'_q are measurable, by the argument of part (a-i) of the proof of 475J. Enumerate $\mathbb{Q}$ as $\langle q_n \rangle_{n\in\mathbb{N}}$, and set

$$F_n = F'_{q_n} \setminus \bigcup_{m<n} \{u : u \in F'_{q_m},\, f_{q_m}(u) < q_n < f'_{q_m}(u)\}$$

for $n \in \mathbb{N}$. Set $g_n(u) = f_{q_n}(u)$, $g'_n(u) = f'_{q_n}(u)$ for $u \in F_n$.

The effect of this construction is to ensure that, for any $u \in Z$, $u \in F_n$ iff q_n is the first rational lying in one of those constituent intervals I of $\mathbb{R} \setminus D_u$ such that $\{u\} \times I \subseteq \operatorname{int}^* E$, and that now $g_n(u)$ and $g'_n(u)$ are the endpoints of that interval, allowing $\pm\infty$ as endpoints.

(c) Now let us look at the items (i)-(v) of the statement of this lemma. We have already achieved (i). If $u \in F_m \cap F_n$, then, in the language of 475J, $]g_m(u), g'_m(u)[$ is one of the constituent intervals of G_u, and $]g_n(u), g'_n(u)[$ is another; since these must be separated by one of the constituent intervals of H_u, their closures are disjoint. Thus (ii) is true. For any $u \in Z$,

$$\sum_{n \in \mathbb{N}, u \in F_n} g'_n(u) - g_n(u) = \mu_1\{t : (u,t) \in \operatorname{int}^* E\},$$

so

$$\sum_{n=0}^{\infty} \int_{F_n} g'_n - g_n d\mu_{r-1} = \int_Z \mu_1\{t : (u,t) \in \operatorname{int}^* E\} \mu_{r-1}(du) = \mu(\operatorname{int}^* E) = \mu E.$$

So (iii) is true. Also, for $u \in Z$,

$$\begin{aligned} \{t : (u,t) \in \partial^* E\} &= D_u \\ &= \{g_n(u) : n \in \mathbb{N},\, u \in F_n,\, g_n(u) \neq -\infty\} \\ &\qquad \cup \{g'_n(u) : n \in \mathbb{N},\, u \in F_n,\, g'_n(u) \neq \infty\}, \end{aligned}$$

so (v) is true.

(d) As for (iv), suppose first that $h : \mathbb{R}^r \to \mathbb{R}$ is a Lipschitz function with compact support. Set $\phi(x) = (0, \dots, h(x))$ for $x \in \mathbb{R}^r$. Then

$$\int_{\partial^* E} h(x) v \centerdot \psi_E(x)\, \nu(dx) = \int h(x) v \centerdot \psi_E(x)\, \lambda^\partial_E(dx)$$

(475G)

$$= \int \phi \centerdot \psi_E \, d\lambda^\partial_E = \int_E \operatorname{div} \phi \, d\mu$$

(474E)

$$\begin{aligned} &= \int_E \frac{\partial h}{\partial \xi_r} d\mu = \int_{\operatorname{int}^* E} \frac{\partial h}{\partial \xi_r} d\mu \\ &= \int \sum_{n \in \mathbb{N}, u \in F_n} \int_{g_n(u)}^{g'_n(u)} \frac{\partial h}{\partial \xi_r}(u,t) dt \, \mu_{r-1}(du) \\ &= \int \sum_{n \in \mathbb{N}, u \in F_n} h(u, g'_n(u)) - h(u, g_n(u)) \mu_{r-1}(du) \end{aligned}$$

(with the convention that $h(u, \pm\infty) = \lim_{t \to \pm\infty} h(u,t) = 0$)

$$= \sum_{n=0}^{\infty} \int_{F_n} h(u, g'_n(u)) - h(u, g_n(u)) \mu_{r-1}(du)$$

because if $|h| \le m\chi B(\mathbf{0}, m)$ then

$$\begin{aligned} \int \sum_{n \in \mathbb{N}, u \in F_n} &|h(u, g'_n(u)) - h(u, g_n(u))| \mu_{r-1}(du) \\ &\le 2m \int \#(\{t : (u,t) \in B(\mathbf{0}, m) \cap \partial^* E\}) \mu_{r-1}(du) \\ &\le 2m\, \nu(B(\mathbf{0}, m) \cap \partial^* E) \end{aligned}$$

(475H) is finite.

For a general continuous function h of compact support, consider the convolutions $h_k = h * \tilde{h}_k$ for large k, where $\tilde{h}_k$ is defined in 473E. If $|h| \le m\chi B(\mathbf{0}, m)$ then $|h_k| \le m\chi B(\mathbf{0}, m+1)$ for every k, so that

$$\sum_{n \in \mathbb{N}, u \in F_n} |h_k(g'_n(u)) - h_k(g_n(u))| \le 2m\#(\{t : (u,t) \in B(\mathbf{0}, m+1) \cap \partial^* E\})$$

for every $u\in\mathbb{R}^r$, $k\in\mathbb{N}$. Since $h_k\to h$ uniformly (473Ed),

$$\begin{aligned}\int_{\partial^*E}h(x)v\,.\,\psi_E(x)\,\nu(dx)&=\lim_{k\to\infty}\int_{\partial^*E}h_k(x)v\,.\,\psi_E(x)\,\nu(dx)\\&=\lim_{k\to\infty}\int\sum_{n\in\mathbb{N},u\in F_n}h_k(u,g'_n(u))-h_k(u,g_n(u))\mu_{r-1}(du)\\&=\int\sum_{n\in\mathbb{N},u\in F_n}h(u,g'_n(u))-h(u,g_n(u))\mu_{r-1}(du)\\&=\sum_{n=0}^{\infty}\int_{F_n}h(u,g'_n(u))-h(u,g_n(u))\mu_{r-1}(du),\end{aligned}$$

as required.

475P Lemma Let $v\in S_{r-1}$ be any unit vector, and $V\subseteq\mathbb{R}^r$ the hyperplane $\{x:x\,.\,v=0\}$. Let $T:\mathbb{R}^r\to V$ be the orthogonal projection. If $E\subseteq\mathbb{R}^r$ is any set with locally finite perimeter and canonical outward-normal function ψ_E, then

$$\int_{\partial^*E}|v\,.\,\psi_E|d\nu=\int_V\#(\partial^*E\cap T^{-1}[\{u\}])\nu(du),$$

interpreting $\#(\partial^*E\cap T^{-1}[\{u\}])$ as ∞ if $\partial^*E\cap T^{-1}[\{u\}]$ is infinite.

proof As usual, we may suppose that the structure (E,v) is rotated until $v=(0,\dots,1)$, so that we can identify $T(\xi_1,\dots,\xi_r)$ with $(\xi_1,\dots,\xi_{r-1})\in\mathbb{R}^{r-1}$, and $\int_V\#(\partial^*E\cap T^{-1}[\{u\}])\nu(du)$ with $\int_{\mathbb{R}^{r-1}}\#(D_u)\mu_{r-1}(du)$, where $D_u=\{t:(u,t)\in\partial^*E\}$. For each $m\in\mathbb{N}$ and $u\in\mathbb{R}^{r-1}$, set $D_u^{(m)}=\{t:(u,t)\in\partial^*E,\ \|(u,t)\|<m\}$; note that $\int\#(D_u^{(m)})\mu_{r-1}(du)\le\nu(\partial^*E\cap B(\mathbf{0},m))$ is defined and finite (475H). It follows that the integral $\int_V\#(\partial^*E\cap T^{-1}[\{u\}])\nu(du)$ is defined in $[0,\infty]$.

(a) Write Φ for the set of continuous functions $h:\mathbb{R}^r\to[-1,1]$ with compact support. Then

$$\int_{\partial^*E}|v\,.\,\psi_E|d\nu=\sup_{h\in\Phi}\int_{\partial^*E}h(x)v\,.\,\psi_E(x)\,\nu(dx).$$

P Of course

$$\int_{\partial^*E}|v\,.\,\psi_E|d\nu\ge\int_{\partial^*E}h(x)v\,.\,\psi_E(x)\,\nu(dx)$$

for any $h\in\Phi$. On the other hand, if

$$\gamma<\int_{\partial^*E}|v\,.\,\psi_E|d\nu=\int|v\,.\,\psi_E|d\lambda_E^\partial$$

(475G), then, because λ_E^∂ is a Radon measure, there is an $n\in\mathbb{N}$ such that

$$\gamma<\int_{B(\mathbf{0},n)}|v\,.\,\psi_E|d\lambda_E^\partial=\int h_0(x)v\,.\,\psi_E(x)\,\lambda_E^\partial(dx)$$

where

$$\begin{aligned}h_0(x)&=\frac{v\,.\,\psi_E(x)}{|v\,.\,\psi_E(x)|}\text{ if }x\in B(\mathbf{0},n)\text{ and }v\,.\,\psi_E(x)\ne0,\\&=0\text{ otherwise.}\end{aligned}$$

Again because λ_E^∂ is a Radon measure, there is a continuous function $h_1:\mathbb{R}^r\to\mathbb{R}$ with compact support such that

$$\int|h_1-h_0|d\lambda_E^\partial\le\int_{B(\mathbf{0},n)}|v\,.\,\psi_E|d\lambda_E^\partial-\gamma$$

(416I); of course we may suppose that h_1, like h, takes values in $[-1,1]$, so that $h_1\in\Phi$. Now

$$\int_{\partial^*E}h_1(x)v\,.\,\psi_E(x)\nu(dx)=\int h_1(x)v\,.\,\psi_E(x)\,\lambda_E^\partial(dx)\ge\gamma.$$

As γ is arbitrary,

$$\sup_{h\in\Phi}\int h(x)v\,.\,\psi_E(x)\,\nu(dx)=\int|v\,.\,\psi_E|d\nu$$

as required. **Q**

(b) Now take $\langle F_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ and $\langle g'_n\rangle_{n\in\mathbb{N}}$ as in 475O. Then

$$\int \#(D_u)\mu_{r-1}(du) = \sup_{h\in\Phi}\sum_{n=0}^{\infty}\int_{F_n} h(u,g'_n(u)) - h(u,g_n(u))\mu_{r-1}(du).$$

(As in 475O(iv), interpret $h(u,\pm\infty)$ as 0 if necessary.) **P** By 475O(ii), $g'_m(u) \neq g_n(u)$ whenever m, $n \in \mathbb{N}$ and $u \in F_m \cap F_n$, while for almost all $u \in \mathbb{R}^{r-1}$

$$D_u = (\{g_n(u) : n \in \mathbb{N},\, u \in F_n\} \setminus \{-\infty\}) \cup (\{g'_n(u) : n \in \mathbb{N},\, u \in F_n\} \setminus \{\infty\}).$$

So if $h \in \Phi$,

$$\begin{aligned}\sum_{n\in\mathbb{N},u\in F_n} h(u,g'_n(u)) - h(u,g_n(u)) &\le \#(\{g_n(u) : u \in F_n,\, g_n(u) \neq -\infty\}) \\ &\qquad + \#(\{g'_n(u) : u \in F_n,\, g'_n(u) \neq \infty\}) \\ &= \#(D_u)\end{aligned}$$

for almost every $u \in \mathbb{R}^{r-1}$, and

$$\sum_{n=0}^{\infty}\int_{F_n} h(u,g'_n(u)) - h(u,g_n(u))\mu_{r-1}(du) \le \int \#(D_u)\mu_{r-1}(du).$$

In the other direction, given $\gamma < \int \#(D_u)\mu_{r-1}(du)$, there is an $m \in \mathbb{N}$ such that $\gamma < \int \#(D_u^{(m)})\mu_{r-1}(du)$. Setting

$$H_n = \{u : u \in F_n,\, g_n(u) \in D_u^{(m)}\}, \quad H'_n = \{u : u \in F_n,\, g'_n(u) \in D_u^{(m)}\}$$

for $n \in \mathbb{N}$, we have

$$\gamma < \sum_{n=0}^{\infty}\mu_{r-1}H_n + \sum_{n=0}^{\infty}\mu_{r-1}H'_n \le \int \#(D_u^{(m)}\mu_{r-1}(du) < \infty.$$

By 418J once more, we can find $n \in \mathbb{N}$ and compact sets $K_i \subseteq H_i$, $K'_i \subseteq H'_i$ such that $g_i\restriction K_i$ and $g'_i\restriction K'_i$ are continuous for every i and

$$\begin{aligned}\gamma \le \sum_{i=0}^{n}(\mu_{r-1}K_i + \mu_{r-1}K'_i) - \sum_{i=n+1}^{\infty}(\mu_{r-1}H_i + \mu_{r-1}H'_i) \\ - \sum_{i=0}^{n}(\mu_{r-1}(H_i \setminus K_i) + \mu_{r-1}(H'_i \setminus K'_i)).\end{aligned}$$

Set

$$K = \bigcup_{i\le n}\{(u,g_i(u)) : u \in K_i\}, \quad K' = \bigcup_{i\le n}\{(u,g'_i(u)) : u \in K'_i\},$$

so that K and K' are disjoint compact subsets of $\operatorname{int} B(\mathbf{0},m)$. Let $h : \mathbb{R}^r \to \mathbb{R}$ be a continuous function such that

$$\begin{aligned}h(x) &= 1 \text{ for } x \in K', \\ &= -1 \text{ for } x \in K, \\ &= 0 \text{ for } x \in \mathbb{R}^r \setminus \operatorname{int} B(\mathbf{0},m)\end{aligned}$$

(4A2F(d-ix)); we can suppose that $-1 \le h(x) \le 1$ for every x. Then $h \in \Phi$, and

$$\begin{aligned}\sum_{i=0}^{\infty}\int_{F_i} &h(u,g'_i(u)) - h(u,g_i(u))\mu_{r-1}(du) \\ &= \sum_{i=0}^{\infty}\int_{H'_i} h(u,g'_i(u))\mu_{r-1}(du) - \sum_{i=0}^{\infty}\int_{H_i} h(u,g_i(u))\mu_{r-1}(du)\end{aligned}$$

(because h is zero outside $\operatorname{int} B(\mathbf{0},m)$)

$$\begin{aligned}&\ge \sum_{i=0}^{n}\int_{K'_i} h(u,g'_i(u))\mu_{r-1}(du) - \sum_{i=0}^{n}\int_{K_i} h(u,g_i(u))\mu_{r-1}(du) \\ &\qquad - \sum_{i=0}^{n}(\mu(H'_i \setminus K'_i) + \mu(H_i \setminus K_i)) - \sum_{i=n+1}^{\infty}(\mu H'_i + \mu H_i) \\ &\ge \gamma.\end{aligned}$$

This shows that

$$\int \#(D_u)\mu_{r-1}(du) \le \sup_{h\in\Phi}\sum_{n=0}^{\infty}\int_{F_n} h(u,g'_n(u)) - h(u,g_n(u))\mu_{r-1}(du),$$

and we have equality. **Q**

(c) Putting (a) and (b) together with 475O(iv), we have the result.

475Q Theorem (a) Let $E \subseteq \mathbb{R}^r$ be a set with finite perimeter. For $v \in S_{r-1}$ write V_v for $\{x : x\mathbin{\bullet} v = 0\}$, and let $T_v : \mathbb{R}^r \to V_v$ be the orthogonal projection. Then

$$\begin{aligned}\operatorname{per} E = \nu(\partial^* E) &= \frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\int_{V_v} \#(\partial^* E \cap T_v^{-1}[\{u\}])\nu(du)\nu(dv)\\ &= \lim_{\delta\downarrow 0}\frac{1}{2\beta_{r-1}\delta}\int_{S_{r-1}} \mu(E\triangle(E+\delta v))\nu(dv).\end{aligned}$$

(b) Suppose that $E \subseteq \mathbb{R}^r$ is Lebesgue measurable. Set

$$\gamma = \sup_{x\in\mathbb{R}^r\setminus\{0\}}\frac{1}{\|x\|}\mu(E\triangle(E+x)).$$

Then $\gamma \le \operatorname{per} E \le \frac{r\beta_r\gamma}{2\beta_{r-1}}$.

proof (a)(i) I start with an elementary fact: there is a constant c such that $\int |w\mathbin{\bullet} v|\lambda^{\partial}_{S_{r-1}}(dv) = c$ for every $w \in S_{r-1}$; this is because whenever w, $w' \in S_{r-1}$ there is an orthogonal linear transformation taking w to w', and this transformation is an automorphism of the structure $(\mathbb{R}^r, \nu, S_{r-1}, \lambda^{\partial}_{S_{r-1}})$ (474H). (In (iii) below I will come to the calculation of c.)

(ii) Now, writing ψ_E for the canonical outward-normal function of E, we have

$$\int_{S_{r-1}}\int_{V_v}\#(\partial^* E \cap T_v^{-1}[\{u\}])\nu(du)\nu(dv) = \int_{S_{r-1}}\int_{\partial^* E}|\psi_E(x)\mathbin{\bullet} v|\nu(dx)\nu(dv)$$

(475P)

$$= \iint |\psi_E(x)\mathbin{\bullet} v|\lambda^{\partial}_E(dx)\lambda^{\partial}_{S_{r-1}}(dv)$$

(by 235K, recalling that $\lambda^{\partial}_{S_{r-1}}$ and λ^{∂}_E are just indefinite-integral measures over ν, while S_{r-1} and $\partial^* E$ are Borel sets)

$$= \iint |\psi_E(x)\mathbin{\bullet} v|\lambda^{\partial}_{S_{r-1}}(dv)\lambda^{\partial}_E(dx)$$

(because $\mathbin{\bullet}$ is continuous, so $(x, v) \mapsto \psi_E(x)\mathbin{\bullet} v$ is measurable, while $\lambda^{\partial}_{S_{r-1}}$ and λ^{∂}_E are totally finite, so we can use Fubini's theorem)

$$= c\operatorname{per} E$$

(by (i) above)

$$= c\nu(\partial^* E).$$

(iii) We have still to identify the constant c. But observe that the argument above applies whenever $\nu(\partial^* E)$ is finite, and in particular applies to $E = B(\mathbf{0}, 1)$. In this case, $\partial^* B(\mathbf{0}, 1) = S_{r-1}$, and for any $v \in S_{r-1}$, $u \in V_v$ we have

$$\begin{aligned}\#(\partial^* B(\mathbf{0},1) \cap T^{-1}[\{u\}]) &= 2 \text{ if } \|u\| < 1,\\ &= 1 \text{ if } \|u\| = 1,\\ &= 0 \text{ if } \|u\| > 1.\end{aligned}$$

Since we can identify ν on V_v as a copy of Lebesgue measure μ_{r-1},

$$\int_{V_v}\#(\partial^*B(\mathbf{0},1)\cap T^{-1}[\{u\}])\nu(du)=2\nu\{u:u\in V_v,\ \|u\|<1\}=2\beta_{r-1}.$$

This is true for every $v\in V_u$, so from (ii) we get

$$2\beta_{r-1}\nu S_{r-1}=\int_{S_{r-1}}\int_{V_v}\#(\partial^*E\cap T_v^{-1}[\{u\}])\nu(du)\nu(dv)=c\nu S_{r-1},$$

and $c=2\beta_{r-1}$. Substituting this into the result of (ii), we get

$$\operatorname{per}E=\nu(\partial^*E)=\frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\int_{V_v}\#(\partial^*E\cap T_v^{-1}[\{u\}])\nu(du)\nu(dv).$$

(iv) Before continuing with the main argument, it will help to set out another elementary fact, this time about translates of certain simple subsets of $\mathbb{R}$. Suppose that (G,H,D) is a partition of $\mathbb{R}$ such that G and H are open, $D=\partial G=\partial H$ is the common boundary of G and H, and D is locally finite, that is, $D\cap[-n,n]$ is finite for every $n\in\mathbb{N}$. Then

$$\sup_{\delta>0}\frac{1}{\delta}\mu_1(G\triangle(G+\delta))=\lim_{\delta\downarrow 0}\frac{1}{\delta}\mu_1(G\triangle(G+\delta))=\#(D)$$

if you will allow me to identify '∞' with the cardinal ω. **P** If we look at the components of G, these are intervals with endpoints in D; and because $\partial G=\partial H$, distinct components of G have disjoint closures. Set $f(\delta)=\mu_1(G\triangle(G+\delta))$ for $\delta>0$. If D is infinite, then for any $n\in\mathbb{N}$ we have disjoint bounded components $I_0,\dots,I_n$ of G; for any δ small enough, $G\cap(I_j+\delta)\subseteq I_j$ for every $j\le n$ (because D is locally finite); so that

$$f(\delta)\ge\sum_{j=0}^{n}\mu(I_j\triangle(I_j+\delta))=2(n+1)\delta,$$

and $\liminf_{\delta\downarrow 0}\frac{1}{\delta}f(\delta)\ge 2(n+1)$. As n is arbitrary,

$$\sup_{\delta>0}\frac{1}{\delta}f(\delta)=\lim_{\delta\downarrow 0}\frac{1}{\delta}f(\delta)=\infty.$$

If D is empty, then G is either empty or $\mathbb{R}$, and

$$\sup_{\delta>0}\frac{1}{\delta}f(\delta)=\lim_{\delta\downarrow 0}\frac{1}{\delta}f(\delta)=0.$$

If D is finite and not empty, let $I_0,\dots,I_n$ be the components of G. Then, for all small $\delta>0$, we have

$$f(\delta)=\sum_{j=0}^{n}\mu(I_j\triangle(I_j+\delta))=2(n+1)\delta=\delta\#(D),$$

so again

$$\sup_{\delta>0}\frac{1}{\delta}f(\delta)=\lim_{\delta\downarrow 0}\frac{1}{\delta}f(\delta)=\#(D).\ \mathbf{Q}$$

(v) Returning to the proof in hand, we find that if $v\in S_{r-1}$ is such that $\int_{V_v}\#(\partial^*E\cap T_v^{-1}[\{u\}])\nu(du)$ is finite, then the integral is equal to

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\mu(E\triangle(E+\delta v))=\sup_{\delta>0}\frac{1}{\delta}\mu(E\triangle(E+\delta v)).$$

P It is enough to consider the case in which v is the unit vector $(0,\dots,0,1)$, so that we can identify V_v with $\mathbb{R}^{r-1}$ and $\mathbb{R}^r$ with $\mathbb{R}^{r-1}\times\mathbb{R}$, as in 475J; in this case, $E\cap T_v^{-1}[\{u\}]$ turns into $E[\{u\}]$. Let $Z_2\subseteq\mathbb{R}^{r-1}$ and G_u, H_u and D_u, for $u\in\mathbb{R}^{r-1}$, be as in 475Jc. In this case, for any $\delta>0$,

$$\begin{aligned}\mu(E\triangle(E+\delta v))&=\mu(\operatorname{int}^*E\triangle(\operatorname{int}^*E+\delta v))\\&=\int_{\mathbb{R}^{r-1}}\mu_1((\operatorname{int}^*E\triangle(\operatorname{int}^*E+\delta v))[\{u\}])\mu_{r-1}(du)\\&=\int_{\mathbb{R}^{r-1}}\mu_1(G_u\triangle(G_u+\delta))\mu_{r-1}(du)\\&=\int_{Z_2}\mu_1(G_u\triangle(G_u+\delta))\mu_{r-1}(du)\end{aligned}$$

because Z_2 is conegligible. By (iv),

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\mu_1(G_u\triangle(G_u+\delta))=\sup_{\delta>0}\frac{1}{\delta}\mu_1(G_u\triangle(G_u+\delta))=\#(D_u)$$

for any $u \in Z_2$. Applying Lebesgue's Dominated Convergence Theorem to arbitrary sequences $\langle\delta_n\rangle_{n\in\mathbb{N}} \downarrow 0$, we see that

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\mu(E\triangle(E+\delta v)) = \int_{Z_2}\#(D_u)\nu(du) = \int_{\mathbb{R}^{r-1}}\#(\partial^*E\cap T_v^{-1}[\{u\}])\nu(du),$$

as required. To see that

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\mu(E\triangle(E+\delta v)) = \sup_{\delta>0}\frac{1}{\delta}\mu(E\triangle(E+\delta v)),$$

set $g(\delta)=\mu(E\triangle(E+\delta v))$ for $\delta>0$. Then for δ, $\delta'>0$ we have

$$\begin{aligned}g(\delta+\delta') &= \mu(E\triangle(E+(\delta+\delta')v)) \le \mu(E\triangle(E+\delta v)) + \mu((E+\delta v)\triangle(E+(\delta+\delta' v)))\\ &= g(\delta)+\mu(\delta v+(E\triangle(E+\delta' v))) = g(\delta)+g(\delta').\end{aligned}$$

Consequently $g(\delta)\le ng(\frac{1}{n}\delta)$ whenever $\delta>0$ and $n\ge 1$, so

$$\frac{1}{\delta}g(\delta) \le \liminf_{n\to\infty}\frac{n}{\delta}g(\tfrac{1}{n}\delta) = \lim_{\delta'\downarrow 0}\frac{1}{\delta'}g(\delta')$$

for every $\delta>0$. **Q**

(vi) Putting (v) together with (i)-(iii) above,

$$\operatorname{per}E = \frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\lim_{\delta\downarrow 0}\frac{1}{\delta}\mu(E\triangle(E+\delta v))\nu(dv).$$

To see that we can exchange the limit and the integral, observe that we can again use the dominated convergence theorem, because

$$\int_{S_{r-1}}\sup_{\delta>0}\frac{1}{\delta}\mu(E\triangle(E+\delta v)) = 2\beta_{r-1}\operatorname{per}E$$

is finite. So

$$\operatorname{per}E = \lim_{\delta\downarrow 0}\frac{1}{2\beta_{r-1}\delta}\int_{S_{r-1}}\mu(E\triangle(E+\delta v))\nu(dv).$$

(b)(i) $\gamma\le\operatorname{per}E$. **P** We can suppose that E has finite perimeter.

(α) To begin with, suppose that $x=(0,\dots,0,\delta)$ where $\delta>0$. As in part (a-v) of this proof, set $v=(0,\dots,0,1)$, identify $\mathbb{R}^r$ with $\mathbb{R}^{r-1}\times\mathbb{R}$, and define G_u, $D_u\subseteq\mathbb{R}$, for $u\in\mathbb{R}^{r-1}$, and $Z_2\subseteq\mathbb{R}^{r-1}$ as in 475Jc. Suppose that $u\in Z_2$. Then $G_u=(\operatorname{int}^*E)[\{u\}]$ is an open set and its constituent intervals have endpoints in $D_u=(\partial^*E)[\{u\}]$. It follows that for any t in

$$G_u\triangle(G_u+\delta) = (\operatorname{int}^*E\triangle(\operatorname{int}^*E+\delta v))[\{u\}],$$

there must be an $s\in D_u\cap[t-\delta,\delta]$, and $t\in D_u+[0,\delta]$. Accordingly

$$(\operatorname{int}^*E\triangle(\operatorname{int}^*E+\delta v))\cap(Z_2\times\mathbb{R}) \subseteq \partial^*E+[0,\delta v],$$

writing $[0,\delta v]$ for $\{tv : t\in[0,\delta]\}$. So

$$\mu(E\triangle(E+\delta v)) = \mu(\operatorname{int}^*E\triangle(\operatorname{int}^*E+\delta v)) \le \mu^*(\partial^*E+[0,\delta v]).$$

Take any $\epsilon>0$. We have a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of sets, all of diameter at most ϵ, covering ∂^*E, and such that $2^{-r+1}\beta_{r-1}\sum_{n=0}^\infty(\operatorname{diam}A_n)^{r-1}\le\epsilon+\nu(\partial^*E)$; we can suppose that every A_n is closed. Taking $T:\mathbb{R}^r\to\mathbb{R}^{r-1}$ to be the natural projection, $T[A_n]$ is compact and has diameter at most $\operatorname{diam}A_n$, so that $\mu_{r-1}T[A_n]\le 2^{-r+1}\beta_{r-1}(\operatorname{diam}A_n)^{r-1}$ (264H again). For each $n\in\mathbb{N}$, the vertical sections of $A_n+[0,\delta v]$ have diameter at most $\epsilon+\delta$. So

$$\mu(A_n+[0,\delta v]) \le 2^{-r+1}\beta_{r-1}(\operatorname{diam}A_n)^{r-1}(\epsilon+\delta).$$

Consequently,

$$\begin{aligned}\mu(E\triangle(E+\delta v)) &\le \mu^*(\partial^*E+[0,\delta v]) \le \sum_{n=0}^\infty\mu(A_n+[0,\delta v])\\ &\le \sum_{n=0}^\infty 2^{-r+1}\beta_{r-1}(\operatorname{diam}A_n)^{r-1}(\epsilon+\delta) \le (\epsilon+\delta)(\epsilon+\nu(\partial^*E)).\end{aligned}$$

As ϵ is arbitrary,

$$\mu(E\triangle(E+x)) = \mu(E\triangle(E+\delta v)) \le \delta\nu(\partial^*E) = \|x\|\nu(\partial^*E).$$

(β) Of course the same must be true for all other non-zero $x \in \mathbb{R}^r$, so $\gamma \le \nu(\partial^*E) = \operatorname{per} E$. **Q**

(ii) For the other inequality, we need look only at the case in which γ is finite.

(α) In this case, E has finite perimeter. **P** Let $\phi : \mathbb{R}^r \to B(\mathbf{0}, 1)$ be a Lipschitz function with compact support. Take i such that $1 \le i \le r$, and consider

$$\Bigl|\int_E \frac{\partial\phi_i}{\partial\xi_i} d\mu\Bigr| = \Bigl|\int_E \lim_{n\to\infty} 2^n(\phi_i(x+2^{-n}e_i) - \phi_i(x))\mu(dx)\Bigr|$$

(where $\phi = (\phi_1, \dots , \phi_r)$)

$$= \Bigl|\lim_{n\to\infty} \int_E 2^n(\phi_i(x+2^{-n}e_i) - \phi_i(x))\mu(dx)\Bigr|$$

(by Lebesgue's Dominated Convergence Theorem, because ϕ_i is Lipschitz and has bounded support)

$$= \lim_{n\to\infty} 2^n\Bigl|\int_E (\phi_i(x+2^{-n}e_i) - \phi_i(x))\mu(dx)\Bigr|$$

$$= \lim_{n\to\infty} 2^n\Bigl|\int_{E+2^{-n}e_i} \phi_i d\mu - \int_E \phi_i d\mu\Bigr|$$

$$\le \limsup_{n\to\infty} 2^n\mu((E+2^{-n}e_i)\triangle E)$$

(because $\|\phi_i\|_\infty \le 1$)

$$\le \gamma.$$

Summing over i, $|\int_E \operatorname{div}\phi\, d\mu| \le r\gamma$. As ϕ is arbitrary, $\operatorname{per} E \le r\gamma$ is finite. **Q**

(β) By (a), we have

$$\operatorname{per} E = \lim_{\delta\downarrow 0} \frac{1}{2\beta_{r-1}\delta} \int_{S_{r-1}} \mu(E\triangle(E+\delta v))\nu(dv)$$

$$\le \frac{1}{2\beta_{r-1}}\gamma\nu S_{r-1} = \frac{r\beta_r\gamma}{2\beta_{r-1}},$$

as required.

475R Convex sets in $\mathbb{R}^r$ For the next result it will help to have some elementary facts about convex sets in finite-dimensional spaces out in the open.

Lemma (In this lemma I allow $r = 1$.) Let $C \subseteq \mathbb{R}^r$ be a convex set.

(a) If $x \in C$ and $y \in \operatorname{int} C$, then $ty + (1-t)x \in \operatorname{int} C$ for every $t \in \,]0, 1]$.
(b) $\overline{C}$ and $\operatorname{int} C$ are convex.
(c) If $\operatorname{int} C \ne \emptyset$ then $\overline{C} = \overline{\operatorname{int} C}$.
(d) If $\operatorname{int} C = \emptyset$ then C lies within some hyperplane.
(e) $\operatorname{int} \overline{C} = \operatorname{int} C$.

proof (a) Setting $\phi(z) = x + t(z-x)$ for $z \in \mathbb{R}^r$, $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a homeomorphism and $\phi[C] \subseteq C$, so

$$\phi(y) \in \phi[\operatorname{int} C] = \operatorname{int} \phi[C] \subseteq \operatorname{int} C.$$

(b) It follows at once from (a) that $\operatorname{int} C$ is convex; $\overline{C}$ is convex because $(x, y) \mapsto tx + (1-t)y$ is continuous for every $t \in [0, 1]$.

(c) From (a) we see also that if $\operatorname{int} C \ne \emptyset$ then $C \subseteq \overline{\operatorname{int} C}$, so that $\overline{C} \subseteq \overline{\operatorname{int} C}$ and $\overline{C} = \overline{\operatorname{int} C}$.

(d) It is enough to consider the case in which $\mathbf{0} \in C$, since if $C = \emptyset$ the result is trivial. **?** If $x_1, \dots , x_r$ are linearly independent elements of C, set $x = \frac{1}{r+1}\sum_{i=1}^r x_i$; then

$$x + \sum_{i=1}^{r} \alpha_i x_i = \sum_{i=1}^{r} (\alpha_i + \frac{1}{r+1}) x_i \in C$$

whenever $\sum_{i=1}^{r} |\alpha_i| \le \frac{1}{r+1}$. Also, writing $e_1, \dots, e_r$ for the standard orthonormal basis of $\mathbb{R}^r$, we can express e_i as $\sum_{j=1}^{r} \alpha_{ij} x_j$ for each j; setting $M = (r+1)\max_{i\le r} \sum_{j=1}^{r} |\alpha_{ij}|$, we have $x \pm \frac{1}{M} e_i \in C$ for every $i \le r$, so that $x + y \in C$ whenever $\|y\| \le \frac{1}{M\sqrt{r}}$, and $x \in \operatorname{int} C$. **X**

So the linear subspace of $\mathbb{R}^r$ spanned by C has dimension at most $r-1$.

(e) If $\operatorname{int} C = \mathbb{R}^r$ the result is trivial. If $\operatorname{int} C$ is empty, then (d) shows that C is included in a hyperplane, so that $\operatorname{int} \overline{C}$ is empty. Otherwise, if $x \in \mathbb{R}^r \setminus \operatorname{int} C$, there is a non-zero $e \in \mathbb{R}^r$ such that $e\cdot y \le e\cdot x$ for every $y \in \operatorname{int} C$ (4A4Db, or otherwise). Now, by (c), $e\cdot y \le e\cdot x$ for every $y \in \overline{C}$, so $x \notin \operatorname{int} \overline{C}$. This shows that $\operatorname{int} \overline{C} \subseteq \operatorname{int} C$, so that the two are equal.

475S Corollary: Cauchy's Perimeter Theorem Let $C \subseteq \mathbb{R}^r$ be a bounded convex set with non-empty interior. For $v \in S_{r-1}$ write V_v for $\{x : x\cdot v = 0\}$, and let $T_v : \mathbb{R}^r \to V_v$ be the orthogonal projection. Then

$$\nu(\partial C) = \frac{1}{\beta_{r-1}} \int_{S_{r-1}} \nu(T_v[C]) \nu(dv).$$

proof (a) The first thing to note is that $\partial^* C = \partial C$. **P** Of course $\partial^* C \subseteq \partial C$ (475Ca). If $x \in \partial C$, there is a half-space V containing x and disjoint from $\operatorname{int} C$ (4A4Db again, because $\operatorname{int} C$ is convex), so that

$$\limsup_{\delta\downarrow 0} \frac{\mu(B(x,\delta)\setminus C)}{\mu B(x,\delta)} \ge \limsup_{\delta\downarrow 0} \frac{\mu(B(x,\delta)\setminus \operatorname{int} V)}{\mu B(x,\delta)} = \frac{1}{2},$$

and $x \notin \operatorname{int}^* C$. On the other hand, if $x_0 \in \operatorname{int} C$ and $\eta > 0$ are such that $B(x_0, \eta) \subseteq \operatorname{int} C$, then for any $\delta \in \,]0,1]$ we can set $t = \frac{\delta}{\eta + \|x_0 - x\|}$, and then

$$B(x + t(x_0 - x), t\eta) \subseteq B(x,\delta) \cap ((1-t)x + tB(x_0,\eta)) \subseteq B(x,\delta) \cap C,$$

so that

$$\mu(B(x,\delta) \cap C) \ge \beta_r t^r \eta^r = \big(\frac{\eta}{\|x_0 - x\| + \eta}\big)^r \mu B(x,\delta);$$

as δ is arbitrary, $x \in \operatorname{cl}^* C$. This shows that $\partial C \subseteq \partial^* C$ so that $\partial^* C = \partial C$. **Q**

(b) We have a function $\phi : \mathbb{R}^r \to \overline{C}$ defined by taking $\phi(x)$ to be the unique point of C closest to x, for every $x \in \mathbb{R}^r$ (3A5Md). This function is 1-Lipschitz. **P** Take any x, $y \in \mathbb{R}^r$ and set $e = \phi(x) - \phi(y)$. We know that $\phi(x) - \epsilon e \in \overline{C}$, so that $\|x - \phi(x) - \epsilon e\| \ge \|x + \phi(x)\|$, for $0 \le \epsilon \le 1$; it follows that $(x - \phi(x))\cdot e \ge 0$. Similarly, $(y - \phi(y))\cdot(-e) \ge 0$. Accordingly $(x-y)\cdot e \ge e\cdot e$ and $\|x - y\| \ge \|e\|$. As x and y are arbitrary, ϕ is 1-Lipschitz. **Q**

Now suppose that $C' \supseteq C$ is a closed bounded convex set. Then $\nu(\partial C') \ge \nu(\partial C)$. **P** Let ϕ be the function defined just above. By 264G/471J again, $\nu^*(\phi[\partial C']) \le \nu(\partial C')$. But if $x \in \partial C$, there is an $e \in \mathbb{R}^r \setminus \{0\}$ such that $x\cdot e \ge y\cdot e$ for every $y \in C$. Then $\phi(x + \alpha e) = x$ for every $\alpha \ge 0$. Because C' is closed and bounded, and $x \in C \subseteq C'$, there is a greatest $\alpha \ge 0$ such that $x + \alpha e \in C'$, and in this case $x + \alpha e \in \partial C'$; since $\phi(x + \alpha e) = x$, $x \in \phi[\partial C']$. As x is arbitrary, $\partial C \subseteq \phi[\partial C']$, and

$$\nu(\partial C) \le \nu^*(\phi[\partial C']) \le \nu(\partial C'). \textbf{ Q}$$

Since we can certainly find a closed convex set $C' \supseteq C$ such that $\nu(\partial C')$ is finite (e.g., any sufficiently large ball or cube), $\nu(\partial C) < \infty$. It follows at once that $\mu(\partial C) = 0$ (471L once more).

(c) The argument so far applies, of course, to every $r \ge 1$ and every bounded convex set with non-empty interior in $\mathbb{R}^r$. Moving to the intended case $r \ge 2$, and fixing $v \in S_{r-1}$ for the moment, we see that, because T_v is an open map (if we give V_v its subspace topology), $T_v[C]$ is again a bounded convex set with non-empty (relative) interior. Since the subspace measure induced by ν on V_v is just a copy of Lebesgue measure, (b) tells us that $\nu T_v[C] = \nu(\operatorname{int}_{V_v} T_v[C])$, where here I write $\operatorname{int}_{V_v} T_v[C]$ for the interior of $T_v[C]$ in the subspace topology of V_v. Now the point is that $\operatorname{int}_{V_v} T_v[C] \subseteq T_v[\operatorname{int} C]$. **P** $\operatorname{int} C$ (taken in $\mathbb{R}^r$) is dense in C (475Rc), so $W = T_v[\operatorname{int} C]$ is a relatively open convex set which is dense in $T_v[C]$; now $W = \operatorname{int}_{V_v} \overline{W}$ (475Re, applied in $V_v \cong \mathbb{R}^{r-1}$), so $W \supseteq \operatorname{int}_{V_v} T_v[C]$. **Q**

It follows that $\#(\partial C\cap T_v^{-1}[\{u\}])=2$ for every $u\in\mathrm{int}_{V_v}T_v[C]$. **P** $T_v^{-1}[\{u\}]$ is a straight line meeting $\mathrm{int}\,C$ in y_0 say. Because $\overline{C}$ is a bounded convex set, it meets $T_v^{-1}[\{u\}]$ in a bounded convex set, which must be a non-trivial closed line segment with endpoints y_1, y_2 say. Now certainly neither y_1 nor y_2 can be in the interior of C. Moreover, the open line segments between y_1 and y_0, and between y_2 and y_0, are covered by $\mathrm{int}\,C$, by 475Ra; so $T_v^{-1}[\{u\}]\cap\partial C=\{y_1,y_2\}$ has just two members. **Q**

(d) This is true for every $v\in S_{r-1}$. But this means that we can apply 475Q to see that

$$\begin{aligned}\nu(\partial C)=\nu(\partial^*C)&=\frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\int_{V_v}\#(\partial^*C\cap T_v^{-1}[\{u\}])\nu(du)\nu(dv)\\&=\frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\int_{T_v[C]}\#(\partial^*C\cap T_v^{-1}[\{u\}])\nu(du)\nu(dv)\\&=\frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\int_{\mathrm{int}_{V_v}T_v[C]}\#(\partial^*C\cap T_v^{-1}[\{u\}])\nu(du)\nu(dv)\\&=\frac{1}{\beta_{r-1}}\int_{S_{r-1}}\nu(\mathrm{int}_{V_v}T_v[C])\nu(dv)=\frac{1}{\beta_{r-1}}\int_{S_{r-1}}\nu(T_v[C])\nu(dv),\end{aligned}$$

as required.

475T Corollary: the Convex Isoperimetric Theorem If $C\subseteq\mathbb{R}^r$ is a bounded convex set, then $\nu(\partial C)\le r\beta_r(\frac{1}{2}\operatorname{diam}C)^{r-1}$.

proof (a) If C is included in some $(r-1)$-dimensional affine subspace, then

$$\nu(\partial C)=\nu\overline{C}\le\beta_{r-1}(\frac{1}{2}\operatorname{diam}C)^{r-1}$$

by 264H once more. For completeness, I should check that $\beta_{r-1}\le r\beta_r$. **P** Comparing 265F with 265H, or working from the formulae in 252Q, we have $r\beta_r=2\pi\beta_{r-2}$. On the other hand, by the argument of 252Q,

$$\beta_{r-1}=\beta_{r-2}\int_{-\pi/2}^{\pi/2}\cos^{r-1}t\,dt\le\pi\beta_{r-2},$$

so (not coincidentally) we have a factor of two to spare. **Q**

(b) Otherwise, C has non-empty interior (475Rd), and for any orthogonal projection T of $\mathbb{R}^r$ onto an $(r-1)$-dimensional linear subspace, $\operatorname{diam}T[C]\le\operatorname{diam}C$, so $\nu(T[C])\le\beta_{r-1}(\frac{1}{2}\operatorname{diam}C)^{r-1}$. Now 475S tells us that

$$\nu(\partial C)\le(\frac{1}{2}\operatorname{diam}C)^{r-1}\nu S_{r-1}=r\beta_r(\frac{1}{2}\operatorname{diam}C)^{r-1}.$$

Remark Compare 476H below.

475X Basic exercises (a) Show that if $C\subseteq\mathbb{R}^r$ is convex, then either $\mu C=0$ and $\partial^*C=\emptyset$, or $\partial^*C=\partial C$.

(b) Let A, $A'\subseteq\mathbb{R}^r$ be any sets. Show that

$$(\partial^*A\cap\mathrm{int}^*A')\cup(\partial^*A'\cap\mathrm{int}^*A)\subseteq\partial^*(A\cap A')\subseteq(\partial^*A\cap\mathrm{cl}^*A')\cup(\partial^*A'\cap\mathrm{cl}^*A).$$

(c) Let $A\subseteq\mathbb{R}^r$ be any set, and B a non-trivial closed ball. Show that

$$\partial^*(A\cap B)\triangle((B\cap\partial^*A)\cup(A\cap\partial B))\subseteq A\cap\partial B\setminus\partial^*A.$$

>(d) Let E, $F\subseteq\mathbb{R}^r$ be measurable sets, and v the Federer exterior normal to E at $x\in\mathrm{int}^*F$. Show that v is the Federer exterior normal to $E\cap F$ at x.

(e) Let $\mathfrak{T}$ be the density topology on $\mathbb{R}^r$ (414P) defined from lower Lebesgue density (341E). Show that, for any $A\subseteq\mathbb{R}^r$, $A\cup\mathrm{cl}^*A$ is the $\mathfrak{T}$-closure of A and int^*A is the $\mathfrak{T}$-interior of the $\mathfrak{T}$-closure of A.

(f) Let $A\subseteq\mathbb{R}^r$ be any set. Show that $A\setminus\mathrm{cl}^*A$ and $\mathrm{int}^*A\setminus A$ are Lebesgue negligible.

>(g) Let $E \subseteq \mathbb{R}^r$ be such that $\nu(\partial^* E)$ and μE are both finite. Show that, taking v_x to be the Federer exterior normal to E at any point x where this is defined,

$$\int_E \operatorname{div} \phi \, d\mu = \int_{\partial^* E} \phi(x) \centerdot v_x \, \nu(dx)$$

for every bounded Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$.

>(h) Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence of measurable subsets of $\mathbb{R}^r$ such that (i) there is a measurable set E such that $\lim_{n\to\infty} \mu((E_n \triangle E) \cap B(\mathbf{0}, m)) = 0$ for every $m \in \mathbb{N}$ (ii) $\sup_{n\in\mathbb{N}} \nu(\partial^* E_n \cap B(\mathbf{0}, m))$ is finite for every $m \in \mathbb{N}$. Show that E has locally finite perimeter. (*Hint*: $\int_E \operatorname{div} \phi \, d\mu = \lim_{n\to\infty} \int_{E_n} \operatorname{div} \phi \, d\mu$ for every Lipschitz function ϕ with compact support.)

(i) Give an example of bounded convex sets E and F such that $\partial^\$(E \cup F) \not\subseteq \partial^\$ E \cup \partial^\$ F$.

(j)(i) Show that if A, $B \subseteq \mathbb{R}^r$ then $\partial^*(A \cap B) \cap \partial^*(A \cup B) \subseteq \partial^* A \cap \partial^* B$. (ii) Show that if E, $F \subseteq \mathbb{R}$ are Lebesgue measurable, then $\operatorname{per}(E \cap F) + \operatorname{per}(E \cup F) \le \operatorname{per} E + \operatorname{per} F$.

>(k) Let $E \subseteq \mathbb{R}^r$ be a set with finite Lebesgue measure and finite perimeter. (i) Show that if $H \subseteq \mathbb{R}^r$ is a half-space, then $\operatorname{per}(E \cap H) \le \operatorname{per} E$. (*Hint*: 475Ja.) (ii) Show that if $C \subseteq \mathbb{R}^r$ is convex, then $\operatorname{per}(E \cap C) \le \operatorname{per} E$. (*Hint*: by the Hahn-Banach theorem, C is a limit of polytopes; use 474Ta.) (iii) Show that in 475Mc we have $\operatorname{per} E = \lim_{\alpha\to\infty} \operatorname{per}(E \cap B(\mathbf{0}, \alpha))$.

(l) Let $E \subseteq \mathbb{R}^r$ be a set with finite measure and finite perimeter, and $f : \mathbb{R}^r \to \mathbb{R}$ a Lipschitz function. Show that for any unit vector $v \in \mathbb{R}^r$, $|\int_E v \centerdot \operatorname{grad} f \, d\mu| \le \|f\|_\infty \operatorname{per} E$.

(m) For measurable $E \subseteq \mathbb{R}^r$ set $p(E) = \sup_{x \in \mathbb{R}^r \setminus \{0\}} \frac{1}{\|x\|} \mu(E \triangle (E + x))$. (i) Show that for any measurable E, $p(E) = \limsup_{x\to\mathbf{0}} \frac{1}{\|x\|} \mu(E \triangle (E + x))$. (ii) Show that for every $\epsilon > 0$ there is an $E \subseteq \mathbb{R}^r$ such that $\operatorname{per} E = 1$ and $p(E) \ge 1 - \epsilon$. (iii) Show that if $E \subseteq \mathbb{R}^r$ is a non-trivial ball then $\operatorname{per} E = \frac{r\beta_r}{2\beta_{r-1}} p(E)$. (iv) Show that if $E \subseteq \mathbb{R}^r$ is a cube then $\operatorname{per} E = \sqrt{r} p(E)$. (*Hint*: 475Yf.)

(n) Suppose that $E \subseteq \mathbb{R}^r$ is a bounded set with finite perimeter, and ϕ, $\psi : \mathbb{R}^r \to \mathbb{R}$ two differentiable functions such that $\operatorname{grad} \phi$ and $\operatorname{grad} \psi$ are Lipschitz. Show that

$$\int_E \phi \times \nabla^2 \psi - \psi \times \nabla^2 \phi \, d\mu = \int_{\partial^* E} (\phi \times \operatorname{grad} \psi - \psi \times \operatorname{grad} \phi) \centerdot v_x \, \nu(dx)$$

where v_x is the Federer exterior normal to E at x when this is defined. (This is **Green's second identity**.)

475Y Further exercises (a) Show that if $A \subseteq \mathbb{R}^r$ is Lebesgue negligible, then there is a Borel set $E \subseteq \mathbb{R}^r$ such that $A \subseteq \partial^* E$.

(b) Let (X, ρ) be a metric space and μ a strictly positive locally finite topological measure on X. Show that we can define operations cl*, int* and ∂^* on $\mathcal{P}X$ for which parts (a)-(f) of 475C will be true.

(c) Let B be a ball in $\mathbb{R}^r$ with centre y, and v, v' two unit vectors in $\mathbb{R}^r$. Set

$$H = \{x : x \in \mathbb{R}^r, (x - y) \centerdot v \le 0\}, \quad H' = \{x : x \in \mathbb{R}^r, (x - y) \centerdot v' \le 0\}.$$

Show that $\mu((H \triangle H') \cap B) = \frac{1}{\pi} \arccos(v \centerdot v') \mu B$.

(d) Show that μ is inner regular with respect to the family of compact sets $K \subseteq \mathbb{R}^r$ such that $\limsup_{\delta\downarrow 0} \frac{\mu(B(x,\delta)\cap K)}{\mu B(x,\delta)} \ge \frac{1}{2}$ for every $x \in K$.

(e) Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of functions from $\mathbb{R}^r$ to $\mathbb{R}$ which is uniformly bounded and **uniformly Lipschitz** in the sense that there is some $\gamma \ge 0$ such that every f_n is γ-Lipschitz. Suppose that $f = \lim_{n\to\infty} f_n$ is defined everywhere in $\mathbb{R}^r$. (i) Show that if $E \subseteq \mathbb{R}^r$ has finite measure, then $\int_E z \centerdot \operatorname{grad} f d\mu = \lim_{n\to\infty} \int_E z \centerdot \operatorname{grad} f_n d\mu$ for every $z \in \mathbb{R}^r$. (*Hint*: look at E of finite perimeter first.) (ii) Show that for any convex function $\phi : \mathbb{R}^r \to [0, \infty[$, $\int \phi(\operatorname{grad} f) d\mu \le \liminf_{n\to\infty} \int \phi(\operatorname{grad} f_n) d\mu$.

(f) Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Show that

$$\sup_{x\in\mathbb{R}^r\setminus\{0\}} \frac{1}{\|x\|}\mu(E\triangle(E+x)) = \sup_{\|v\|=1}\int_{\partial^* E}|v\,.\,v_x|\nu(dx),$$

where v_x is the Federer exterior normal of E at x when this is defined.

(g) Let $E \subseteq \mathbb{R}^r$ be Lebesgue measurable. (i) Show that int^*E is an $\mathrm{F}_{\sigma\delta}$ $(=\mathbf{\Pi}^0_3)$ set, that is, is expressible as the intersection of a sequence of F_σ sets. (ii)(cf. Andretta & Camerlo 13) Show that if E is not negligible and cl^*E has empty interior, then int^*E is not $\mathrm{G}_{\delta\sigma}$ $(=\mathbf{\Sigma}^0_3)$, that is, cannot be expressed as the union of sequence of G_δ sets.

475 Notes and comments The successful identification of the distributionally-defined notion of 'perimeter', as described in §474, with the geometrically accessible concept of Hausdorff measure of an appropriate boundary, is of course the key to any proper understanding of the results of the last section as well as this one. The very word 'perimeter' would be unfair if the perimeter of $E \cup F$ were unrelated to the perimeters of E and F; and from this point of view the reduced boundary is less suitable than the essential boundary (475Cd, 475Xi). If we re-examine 474M, we see that it is saying, in effect, that for many balls B the boundary $\partial^*(E\cap B)$ is nearly $(B\cap\partial^*E)\cup(E\cap\partial B)$, and that an outward-normal function for $E \cap B$ can be assembled from outward-normal functions for E and B. But looking at 475Xc-475Xd we see that this is entirely natural; we need only ensure that $\nu(F\cap\partial B) = 0$ for a μ-negligible set F defined from E; and the 'almost every δ' in the statement of 474M is fully enough to arrange this. On the other hand, 475Xg seems to be very hard to prove without using the identification between $\nu(\partial^*E)$ and $\operatorname{per} E$.

Concerning 475Q, I ought to emphasize that it is *not* generally true that

$$\nu F = \frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\int_{V_v}\#(F\cap T_v^{-1}[\{u\}])\nu(du)\nu(dv)$$

even for $r = 2$ and compact sets F with $\nu F < \infty$. We are here approaching one of the many fundamental concepts of geometric measure theory which I am ignoring. The key word is 'rectifiability'; for 'rectifiable' sets a wide variety of concepts of k-dimensional measure coincide, including the integral-geometric form above, and ∂^*E is rectifiable whenever E has locally finite perimeter (Evans & Gariepy 92, 5.7.3). For the general theory of rectifiable sets, see the last quarter of Mattila 95, or Chapter 3 of Federer 69.

I have already noted that the largest volumes for sets of given diameter or perimeter are provided by balls (see 264H and the notes to §474). The isoperimetric theorem for convex sets (475T) is of the same form: once again, the best constant (here, the largest perimeter for a convex set of given diameter, or the smallest diameter for a convex set of given perimeter) is provided by balls.

475Qb gives an alternative characterization of 'set of finite perimeter', with bounds on the perimeter which are sometimes useful.

476 Concentration of measure

Among the myriad special properties of Lebesgue measure, a particularly interesting one is 'concentration of measure'. For a set of given measure in the plane, it is natural to feel that it is most 'concentrated' if it is a disk. There are many ways of defining 'concentration', and I examine three of them in this section (476F, 476G and 476H); all lead us to Euclidean balls as the 'most concentrated' shapes. On the sphere the same criteria lead us to caps (476K, 476Xe).

All the main theorems of this section will be based on the fact that semi-continuous functions on compact spaces attain their bounds. The compact spaces in question will be spaces of subsets, and I begin with some general facts concerning the topologies introduced in 4A2T (476A-476B). The particular geometric properties of Euclidean space which make all these results possible are described in 476D-476E, where I describe concentrating operators based on reflections. The actual theorems 476F-476H and 476K can now almost be mass-produced.

476A Proposition Let X be a topological space, $\mathcal{C}$ the family of closed subsets of X, $\mathcal{K} \subseteq \mathcal{C}$ the family of closed compact sets and μ a topological measure on X.

(a) Suppose that μ is inner regular with respect to the closed sets.

(i) If μ is outer regular with respect to the open sets (for instance, if μ is totally finite) then $\mu\restriction\mathcal{C} : \mathcal{C} \to \mathbb{R}$ is upper semi-continuous with respect to the Vietoris topology on $\mathcal{C}$.

(ii) If μ is locally finite then $\mu\restriction\mathcal{K}$ is upper semi-continuous with respect to the Vietoris topology.

(iii) If f is a non-negative μ-integrable real-valued function then $F \mapsto \int_F f d\mu : \mathcal{C} \to \mathbb{R}$ is upper semi-continuous with respect to the Vietoris topology.

(b) Suppose that μ is tight.

(i) If μ is totally finite then $\mu\restriction\mathcal{C}$ is upper semi-continuous with respect to the Fell topology on $\mathcal{C}$.

(ii) If f is a non-negative μ-integrable real-valued function then $F \mapsto \int_F f d\mu : \mathcal{C} \to \mathbb{R}$ is upper semi-continuous with respect to the Fell topology.

(c) Suppose that X is metrizable, and that ρ is a metric on X defining its topology; let $\tilde{\rho}$ be the Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$.

(i) If μ is totally finite, then $\mu\restriction\mathcal{C} \setminus \{\emptyset\}$ is upper semi-continuous with respect to $\tilde{\rho}$.

(ii) If μ is locally finite, then $\mu\restriction\mathcal{K} \setminus \{\emptyset\}$ is upper semi-continuous with respect to $\tilde{\rho}$.

(iii) If f is a non-negative μ-integrable real-valued function, then $F \mapsto \int_F f d\mu : \mathcal{C} \setminus \{\emptyset\} \to \mathbb{R}$ is upper semi-continuous with respect to $\tilde{\rho}$.

proof (a)(i) Suppose that $F \in \mathcal{C}$ and that $\mu F < \alpha$. Because μ is outer regular with respect to the open sets, there is an open set $G \supseteq F$ such that $\mu G < \alpha$. Now $\mathcal{V} = \{E : E \in \mathcal{C}, E \subseteq G\}$ is an open set for the Vietoris topology containing F, and $\mu E < \alpha$ for every $E \in \mathcal{V}$. As F and α are arbitrary, $\mu\restriction\mathcal{C}$ is upper semi-continuous for the Vietoris topology.

(ii) Given that $K \in \mathcal{K}$ and $\mu K < \alpha$, then, because μ is locally finite, there is an open set G of finite measure including K (cf. 411Ga). Now there is a closed set $F \subseteq G \setminus K$ such that $\mu F > \mu G - \alpha$, so that $\mathcal{V} = \{L : L \in \mathcal{K}, L \subseteq G \setminus F\}$ is a relatively open subset of $\mathcal{K}$ for the Vietoris topology containing K, and $\mu L < \alpha$ for every $L \in \mathcal{V}$.

(iii) Apply (i) to the indefinite-integral measure over μ defined by f; by 412Q this is still inner regular with respect to the closed sets.

(b) If $F \in \mathcal{C}$ and $\mu F < \alpha$, let $K \subseteq X \setminus F$ be a compact set such that $\mu K > \mu X - \alpha$. Then $\mathcal{V} = \{E : E \in \mathcal{C}, E \cap K = \emptyset\}$ is a neighbourhood of F and $\mu E < \alpha$ for every $E \in \mathcal{V}$. This proves (i). Now (ii) follows as in (a-iii) above.

(c)(i) If $F \in \mathcal{C} \setminus \{\emptyset\}$ and $\mu F < \alpha$, then for each $n \in \mathbb{N}$ set $F_n = \{x : \rho(x, F) \le 2^{-n}\}$. Since $\langle F_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed sets with intersection F, and μ is totally finite, there is an n such that $\mu F_n < \alpha$. If now we take $E \in \mathcal{C} \setminus \{\emptyset\}$ such that $\tilde{\rho}(E, F) \le 2^{-n}$, then $E \subseteq F_n$ so $\mu E < \alpha$. As F and α are arbitrary, $\mu\restriction\mathcal{C} \setminus \{\emptyset\}$ is upper semi-continuous.

(ii) If $K \in \mathcal{K} \setminus \{\emptyset\}$ and $\mu K < \alpha$, let $G \supseteq K$ be an open set of finite measure, as in (a-ii) above. The function $x \mapsto \rho(x, X \setminus G)$ is continuous and strictly positive on K, so has a non-zero lower bound on K, and there is some $m \in \mathbb{N}$ such that $\rho(x, X \setminus G) > 2^{-m}$ for every $x \in K$. If, as in (i) just above, we set $F_n = \{x : \rho(x, K) \le 2^{-n}\}$ for each n, $F_m \subseteq G$ has finite measure. So, as in (i), we have an $n \ge m$ such that $\mu F_n < \alpha$, and we can continue as before.

(iii) Once again this follows at once from (i).

476B Lemma Let (X, ρ) be a metric space, and $\mathcal{C}$ the family of closed subsets of X, with its Fell topology. For $\epsilon > 0$, set $U(A, \epsilon) = \{x : x \in X, \rho(x, A) < \epsilon\}$ if $A \subseteq X$ is not empty; set $U(\emptyset, \epsilon) = \emptyset$. Then for any τ-additive topological measure μ on X, the function

$$(F, \epsilon) \mapsto \mu U(F, \epsilon) : \mathcal{C} \times]0, \infty[\to [0, \infty]$$

is lower semi-continuous.

proof Set $Q = \{(F, \epsilon) : F \in \mathcal{C}, \epsilon > 0, \mu U(F, \epsilon) > \gamma\}$, where $\gamma \in \mathbb{R}$. Take any $(F_0, \epsilon_0) \in Q$. Note first that $\mu U(F_0, \epsilon_0) = \sup_{\epsilon<\epsilon_0} \mu U(F, \epsilon)$, so there is a $\delta \in \left]0, \frac{1}{2}\epsilon_0\right[$ such that $\mu U(F_0, \epsilon_0 - 2\delta) > \gamma$. Next, $\{U(x, \epsilon_0 - 2\delta) : x \in F_0\}$ is an open cover of $U(F_0, \epsilon_0 - 2\delta)$; because μ is τ-additive, there is a finite set $I \subseteq F_0$ such that $\mu(\bigcup_{x\in I} U(x, \epsilon_0 - 2\delta)) > \gamma$. Consider

$$\mathcal{V} = \{F : F \in \mathcal{C}, F \cap U(x, \delta) \ne \emptyset \text{ for every } x \in I\}.$$

By the definition of the Fell topology, $\mathcal{V}$ is open. So $\mathcal{V} \times]\epsilon_0 - \delta, \infty[$ is an open neighbourhood of (F_0, ϵ) in $\mathcal{C} \times \mathbb{R}$. If $F \in \mathcal{V}$ and $\epsilon > \epsilon_0 - \delta$, then

$$U(F, \epsilon) \supseteq \bigcup_{x\in I} U(x, \epsilon - \delta) \supseteq \bigcup_{x\in I} U(x, \epsilon_0 - 2\delta)$$

has measure greater than γ and $(F, \epsilon) \in Q$. As (F_0, ϵ_0) is arbitrary, Q is open; as γ is arbitrary, $(F, \epsilon) \mapsto \mu U(F, \epsilon)$ is lower semi-continuous.

Remark Recall that all 'ordinary' topological measures on metric spaces are τ-additive; see 438J.

476C Proposition Let (X,ρ) be a non-empty compact metric space, and suppose that its isometry group G acts transitively on X. Then X has a unique G-invariant Radon probability measure μ, which is strictly positive.

proof By 441G, G, with its topology of pointwise convergence, is a compact topological group, and the action of G on X is continuous. So 443Ud gives the result.

476D Concentration by partial reflection The following construction will be used repeatedly in the rest of the section. Let X be an inner product space. (In this section, X will be usually be $\mathbb{R}^r$, but in 493G below it will be helpful to be able to speak of abstract Hilbert spaces.) For any unit vector $e \in X$ and any $\alpha \in \mathbb{R}$, write $R_{e\alpha} : X \to X$ for the reflection in the hyperplane $V_{e\alpha} = \{x : x \in X, (x|e) = \alpha\}$, so that $R_{e\alpha}(x) = x + 2(\alpha - (x|e))e$ for every $x \in X$. Next, for any $A \subseteq X$, we can define a set $\psi_{e\alpha}(A)$ by setting

$$\begin{aligned}\psi_{e\alpha}(A) = \{x : x \in A, (x|e) \ge \alpha\} &\cup \{x : x \in A, (x|e) < \alpha, R_{e\alpha}(x) \in A\} \\ &\cup \{x : x \in \mathbb{R}^r \setminus A, (x|e) \ge \alpha, R_{e\alpha}(x) \in A\} \\ = (W \cap (A \cup R_{e\alpha}[A])) &\cup (A \cap R_{e\alpha}[A]),\end{aligned}$$

where W is the half-space $\{x : (x|e) \ge \alpha\}$. Geometrically, we construct $\psi_{e\alpha}(A)$ by moving those points of A on the 'wrong' side of the hyperplane $V_{e\alpha}$ to their reflections, provided those points are not already occupied. We have the following facts.

(a) For non-empty $A \subseteq X$ and $\epsilon > 0$, set $U(A,\epsilon) = \{x : \rho(x,A) < \epsilon\}$, where ρ is the standard metric on X. Now $U(\psi_{e\alpha}(A),\epsilon) \subseteq \psi_{e\alpha}(U(A,\epsilon))$. **P** Take $x \in U(\psi_{e\alpha}(A),\epsilon)$. Then there is a $y \in \psi_{e\alpha}(A)$ such that $\|x - y\| < \epsilon$.

case 1 Suppose $(x|e) \ge \alpha$. If $x \in U(A,\epsilon)$ then certainly $x \in \psi_{e\alpha}(U(A,\epsilon))$. Otherwise, because $\|x - y\| < \epsilon$, $y \notin A$, so $R_{e\alpha}(y) \in A$. But $\|R_{e\alpha}(x) - R_{e\alpha}(y)\| = \|x - y\| < \epsilon$, so $R_{e\alpha}(x) \in U(A,\epsilon)$ and $x \in \psi_{e\alpha}(U(A,\epsilon))$.

case 2a Suppose $(x|e) < \alpha$, $(y|e) \ge \alpha$. Then

$$\|R_{e\alpha}(x) - y\| = \|x - R_{e\alpha}(y)\| \le \|R_{e\alpha}(x) - R_{e\alpha}(y)\| = \|x - y\| < \epsilon.$$

At least one of y, $R_{e\alpha}(y)$ belongs to A, so both x and $R_{e\alpha}(x)$ belong to $U(A,\epsilon)$ and $x \in \psi_{e\alpha}(U(A,\epsilon))$.

case 2b Suppose $(x|e) < \alpha$ and $(y|e) < \alpha$. In this case, both y and $R_{e\alpha}(y)$ belong to A, so both x and $R_{e\alpha}(x)$ belong to $U(A,\epsilon)$ and again $x \in \psi_{e\alpha}(U(A,\epsilon))$.

Thus $x \in \psi_{e\alpha}(U(A,\epsilon))$ in all cases; as x is arbitrary, we have the result. **Q**

(b) If $F \subseteq X$ is closed, then $\psi_{e\alpha}(F)$ is closed. **P** Use the second formula for $\psi_{e\alpha}(F)$. **Q**

476E Lemma Let X be an inner product space, $e \in X$ a unit vector and $\alpha \in \mathbb{R}$. Let $R = R_{e\alpha} : X \to X$ be the reflection operator, and $\psi = \psi_{e\alpha} : \mathcal{P}X \to \mathcal{P}X$ the associated transformation, as described in 476D. For $x \in A \subseteq X$, define

$$\begin{aligned}\phi_A(x) &= x \text{ if } (x|e) \ge \alpha, \\ &= x \text{ if } (x|e) < \alpha \text{ and } R(x) \in A, \\ &= R(x) \text{ if } (x|e) < \alpha \text{ and } R(x) \notin A.\end{aligned}$$

Let ν be a topological measure on X which is R-invariant, that is, ν coincides with the image measure νR^{-1}.

(a) For any $A \subseteq X$, $\phi_A : A \to \psi(A)$ is a bijection. If $\alpha < 0$, then $\|\phi_A(x)\| \le \|x\|$ for every $x \in A$, with $\|\phi_A(x)\| < \|x\|$ iff $(x|e) < \alpha$ and $R(x) \notin A$.

(b) If $E \subseteq X$ is measured by ν, then $\psi(E)$ is measurable and $\nu\psi(E) = \nu E$; moreover, ϕ_E is a measure space isomorphism for the subspace measures on E and $\psi(E)$.

(c) If $\alpha < 0$ and $E \subseteq X$ is measurable, then $\int_E \|x\|\nu(dx) \ge \int_{\psi(E)} \|x\|\nu(dx)$, with equality iff $\{x : x \in E, (x|e) < \alpha, R(x) \notin E\}$ is negligible.

(d) Suppose that X is separable. Let λ be the c.l.d. product measure of ν with itself on $X \times X$. If $E \subseteq X$ is measurable, then

$$\int_{E\times E} \|x - y\|\lambda(d(x,y)) \ge \int_{\psi(E)\times\psi(E)} \|x - y\|\lambda(d(x,y)).$$

(e) Now suppose that $X = \mathbb{R}^r$. Then $\nu(\partial^*\psi(A)) \le \nu(\partial^* A)$ for every $A \subseteq \mathbb{R}^r$, where $\partial^* A$ is the essential boundary of A (definition: 475B).

proof (a) That $\phi_A : A \to \psi(A)$ is a bijection is immediate from the definitions of ψ and ϕ_A. If $\alpha < 0$, then for any $x \in A$ either $\phi_A(x) = x$ or $(x|e) < \alpha$ and $R(x) \notin A$. In the latter case

$$\|\phi_A(x)\|^2 = \|R(x)\|^2 = \|x + 2\gamma e\|^2$$

(where $\gamma = \alpha - (x|e) > 0$)

$$= \|x\|^2 + 4\gamma(x|e) + 4\gamma^2 = \|x\|^2 + 4\gamma\alpha < \|x\|^2,$$

so $\|\phi_A(x)\| < \|x\|$.

(b) If we set

$$E_1 = \{x : x \in E,\, (x|e) \ge \alpha\},$$

$$E_2 = \{x : x \in E, (x|e) < \alpha,\, R_{e\alpha}(x) \in E\},$$

$$E_3 = \{x : x \in E,\, (x|e) < \alpha,\, R_{e\alpha}(x) \notin E\},$$

$$E_4 = \{x : x \in \mathbb{R}^r \setminus E,\, (x|e) > \alpha,\, R_{e\alpha}(x) \in E\},$$

then E_1, E_2, E_3 and E_4 are disjoint measurable sets, $E = E_1 \cup E_2 \cup E_3$, $\psi(E) = E_1 \cup E_2 \cup E_4$ and $\phi_E{\restriction}E_3 = R{\restriction}E_3$ is a measure space isomorphism for the subspace measures on E_3 and E_4.

(c) By (a),

$$\int_E \|x\|\nu(dx) \ge \int_E \|\phi_E(x)\|\nu(dx) = \int_{\psi(E)} \|x\|\nu(dx)$$

by 235Gc, because $\phi_E : E \to \psi(E)$ is inverse-measure-preserving, with equality only when

$$\{x : \|x\| > \|\phi_E(x)\|\} = \{x : x \in E,\, (x|e) < \alpha,\, R(x) \notin E\}$$

is negligible.

(d) Note first that if Λ is the domain of λ then Λ includes the Borel algebra of $X \times X$ (because X is second-countable, so this is just the σ-algebra generated by products of Borel sets, by 4A3G); so that $(x, y) \mapsto \|x - y\|$ is Λ-measurable, and the integrals are defined in $[0, \infty]$. Now consider the sets

$$W_1 = \{(x, y) : x \in E,\, y \in E,\, R(x) \notin E,\, R(y) \in E,\, (x|e) < \alpha,\, (y|e) < \alpha\},$$

$$W_1' = \{(x, y) : x \in E,\, y \in E,\, R(x) \notin E,\, R(y) \in E,\, (x|e) < \alpha,\, (y|e) > \alpha\},$$

$$W_2 = \{(x, y) : x \in E,\, y \in E,\, R(x) \in E,\, R(y) \notin E,\, (x|e) < \alpha,\, (y|e) < \alpha\},$$

$$W_2' = \{(x, y) : x \in E,\, y \in E,\, R(x) \in E,\, R(y) \notin E,\, (x|e) > \alpha,\, (y|e) < \alpha\}.$$

Then $(x, y) \mapsto (x, R(y)) : W_1' \to W_1$ is a measure space isomorphism for the subspace measures induced on W_1 and W_1' by λ, so

$$\begin{aligned}\int_{W_1} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x, y)) &= \int_{W_1} \|R(x) - y\|\lambda(d(x, y)) \\ &= \int_{W_1'} \|R(x) - R(y)\|\lambda(d(x, y)) \\ &= \int_{W_1'} \|x - y\|\lambda(d(x, y)).\end{aligned}$$

Similarly,

$$\int_{W_1'} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x,y)) = \int_{W_1'} \|R(x) - y\|\lambda(d(x,y))$$
$$= \int_{W_1} \|R(x) - R(y)\|\lambda(d(x,y))$$
$$= \int_{W_1} \|x - y\|\lambda(d(x,y)).$$

So we get

$$\int_{W_1 \cup W_1'} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x,y)) = \int_{W_1 \cup W_1'} \|x - y\|\lambda(d(x,y)).$$

In the same way, $(x,y) \mapsto (R(x),y)$ is an isomorphism of the subspace measures on W_2 and W_2', and we have

$$\int_{W_2 \cup W_2'} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x,y)) = \int_{W_2 \cup W_2'} \|x - y\|\lambda(d(x,y)).$$

On the other hand, for all $(x,y) \in (E \times E) \setminus (W_1 \cup W_1' \cup W_2 \cup W_2')$, we have $\|\phi_E(x) - \phi_E(y)\| \le \|x - y\|$. (Either x and y are both left fixed by ϕ, or both are moved, or one is on the reflecting hyperplane, or one is moved to the same side of the reflecting hyperplane as the other.) So we get

$$\int_{E \times E} \|x - y\|\lambda(d(x,y)) \ge \int_{E \times E} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x,y))$$
$$= \int_{\psi(E) \times \psi(E)} \|x - y\|\lambda(d(x,y))$$

because $(x,y) \mapsto (\phi_E(x), \phi_E(y))$ is an inverse-measure-preserving transformation for the subspace measures on $E \times E$ and $\psi(E) \times \psi(E)$.

(e)(i) Because R is both an isometry and a measure space automorphism, $\mathrm{cl}^*R[A] = R[\mathrm{cl}^*A]$ and $\mathrm{int}^*R[A] = R[\mathrm{int}^*A]$, where cl^*A and int^*A are the essential closure and the essential interior of A, as in 475B. Recall that cl^*A, int^*A and ∂^*A are all Borel sets (475Cc), so that ∂^*A and $\partial^*\psi(A)$ are measurable.

(ii) Suppose that $x \cdot e = \alpha$. Then $x \in \partial^*\psi(A)$ iff $x \in \partial^*A$. **P** It is easy to check that $B(x,\delta) \cap \psi(A) = \psi(B(x,\delta) \cap A)$ for any $\delta > 0$, so that $\nu^*(B(x,\delta) \cap \psi(A)) = \nu^*(B(x,\delta) \cap A)$ for every $\delta > 0$ and $x \in \mathrm{cl}^*A$ iff $x \in \mathrm{cl}^*\psi(A)$. If $x = R(x) \in \mathrm{int}^*A$, then $x \in \mathrm{int}^*R[A]$ so $x \in \mathrm{int}^*(A \cap R[A])$ (475Cd) and $x \in \mathrm{int}^*\psi(A)$. If $x \in \mathrm{int}^*\psi(A)$ then $x \in \mathrm{int}^*(R[\psi(A)] \cap \psi(A)) \subseteq \mathrm{int}^*A$. **Q**

(iii) If $x \in \partial^*\psi(A) \setminus \partial^*A$ then $R(x) \in \partial^*A \setminus \partial^*\psi(A)$. **P** By (ii), $x \cdot e \ne \alpha$.

case 1 Suppose that $x \cdot e > \alpha$. Setting $\delta = x \cdot e - \alpha$, we see that $U(x,\delta) \cap \psi(A) = U(x,\delta) \cap (A \cup R[A])$, while $U(R(x),\delta) \cap \psi(A) = U(R(x),\delta) \cap A \cap R[A]$. Since $x \notin \mathrm{int}^*\psi(A)$, $x \notin \mathrm{int}^*(A \cup R[A])$ and $x \notin \mathrm{int}^*A$; since x also does not belong to ∂^*A, $x \notin \mathrm{cl}^*A$. However, $x \in \mathrm{cl}^*(A \cup R[A]) = \mathrm{cl}^*A \cup \mathrm{cl}^*R[A]$ (475Cd), so $x \in \mathrm{cl}^*R[A]$ and $R(x) \in \mathrm{cl}^*A$. Next, $x \notin \mathrm{int}^*R[A]$, so $R(x) \notin \mathrm{int}^*A$ and $R(x) \in \partial^*A$. Since $x \notin \mathrm{cl}^*A$, $R(x) \notin \mathrm{cl}^*R[A]$ and $R(x) \notin \mathrm{cl}^*\psi(A)$; so $R(x) \in \partial^*A \setminus \partial^*\psi(A)$.

case 2 Suppose that $x \cdot e < \alpha$. This time, set $\delta = \alpha - x \cdot e$, so that $U(x,\delta) \cap \psi(A) = U(x,\delta) \cap A \cap R[A]$ and $U(R(x),\delta) \cap \psi(A) = U(R(x),\delta) \cap (A \cup R[A])$. As $x \in \mathrm{cl}^*\psi(A)$, $x \in \mathrm{cl}^*(A \cap R[A])$ and $R(x) \in \mathrm{cl}^*A$. Also $x \in \mathrm{cl}^*A$; as $x \notin \partial^*A$, $x \in \mathrm{int}^*A$, $R(x) \in \mathrm{int}^*R[A]$ and $R(x) \in \mathrm{int}^*\psi(A)$, so that $R(x) \notin \partial^*\psi(A)$. Finally, we know that $x \in \mathrm{int}^*A$ but $x \notin \mathrm{int}^*(A \cap R[A])$ (because $x \notin \mathrm{int}^*\psi(A)$; it follows that $x \notin \mathrm{int}^*R[A]$ so $R(x) \notin \mathrm{int}^*A$ and $R(x) \in \partial^*A \setminus \partial^*\psi(A)$.

Thus all possibilities are covered and we have the result. **Q**

(iv) What this means is that if we set $E = \partial^*\psi(A) \setminus \partial^*A$ then $R[E] \subseteq \partial^*A \setminus \partial^*\psi(A)$. So

$$\nu\partial^*\psi(A) = \nu E + \nu(\partial^*\psi(A) \cap \partial^*A) = \nu R[E] + \nu(\partial^*\psi(A) \cap \partial^*A) \le \nu\partial^*A,$$

as required in (e). This ends the proof of the lemma.

476F Theorem Let $r \ge 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$. For non-empty $A \subseteq \mathbb{R}^r$ and $\epsilon > 0$, write $U(A,\epsilon)$ for $\{x : \rho(x,A) < \epsilon\}$, where ρ is the Euclidean metric on $\mathbb{R}^r$. If μ^*A is finite, then $\mu U(A,\epsilon) \ge \mu U(B_A,\epsilon)$, where B_A is the closed ball with centre $\mathbf{0}$ and measure μ^*A.

proof (a) To begin with, suppose that A is bounded. Set $\gamma = \mu^*A$ and $\beta = \mu U(A,\epsilon)$. If $\gamma = 0$ then (because $A \ne \emptyset$)

$$\mu U(A,\epsilon) \geq \mu U(\{\mathbf{0}\},\epsilon) = \mu U(B_A,\epsilon),$$

and we can stop. So let us suppose henceforth that $\gamma > 0$. Let $M \geq 0$ be such that $A \subseteq B(\mathbf{0}, M)$, and consider the family

$$\mathcal{F} = \{F : F \in \mathcal{C},\ F \subseteq B(\mathbf{0},M),\ \mu F \geq \gamma,\ \mu U(F,\epsilon) \leq \beta\},$$

where $\mathcal{C}$ is the family of closed subsets of $\mathbb{R}^r$ with its Fell topology. Because $U(\overline{A},\epsilon) = U(A,\epsilon)$, $\overline{A} \in \mathcal{F}$ and $\mathcal{F}$ is non-empty. By the definition of the Fell topology, $\{F : F \subseteq B(\mathbf{0},M)\}$ is closed; by 476A(b-ii) (applied to the functional $F \mapsto \int_F \chi B(\mathbf{0},M)d\mu$) and 476B, $\mathcal{F}$ is closed in $\mathcal{C}$, therefore compact, by 4A2T(b-iii). Next, the function

$$F \mapsto \int_F \max(0, M - \|x\|)\mu(dx) : \mathcal{C} \to [0,\infty[$$

is upper semi-continuous, by 476A(b-ii) again. It therefore attains its supremum on $\mathcal{F}$ at some $F_0 \in \mathcal{F}$ (4A2Gl). Let $F_1 \subseteq F_0$ be a closed self-supporting set of the same measure as F_0; then $U(F_1,\epsilon) \subseteq U(F_0,\epsilon)$ and $\mu F_1 = \mu F_0$, so $F_1 \in \mathcal{F}$; also

$$\int_{F_1} M - \|x\|\mu(dx) = \int_{F_0} M - \|x\|\mu(dx) \geq \int_F M - \|x\|\mu(dx)$$

for every $F \in \mathcal{F}$.

(b) Now F_1 is a ball with centre $\mathbf{0}$. **P?** Suppose, if possible, otherwise. Then there are $x_1 \in F_1$ and $x_0 \in \mathbb{R}^r \setminus F_1$ such that $\|x_0\| < \|x_1\|$. Set $e = \frac{1}{\|x_0 - x_1\|}(x_0 - x_1)$, so that e is a unit vector, and $\alpha = \frac{1}{2}e\,{\bf .}\,(x_0 + x_1)$; then

$$\alpha = \frac{1}{2\|x_0-x_1\|}(x_0 - x_1)\,{\bf .}\,(x_0 + x_1) = \frac{1}{2\|x_0-x_1\|}(\|x_0\|^2 - \|x_1\|^2) < 0.$$

Define $R = R_{e\alpha} : \mathbb{R}^r \to \mathbb{R}^r$ and $\psi = \psi_{e\alpha}$ as in 476D. Set $F = \psi(F_1)$. Then F is closed (476Db) and $\mu F = \mu F_1 \geq \mu^* A$ (476Eb). Also $U(F,\epsilon) \subseteq \psi(U(F_1,\epsilon))$ (476Da), so

$$\mu U(F,\epsilon) \leq \mu(\psi(U(F_1,\epsilon)) = \mu U(F_1,\epsilon) \leq \beta$$

and $F \in \mathcal{F}$. It follows that $\int_F M - \|x\|\mu(dx) \leq \int_{F_1} M - \|x\|\mu(dx)$; as $\mu F = \mu F_1$, $\int_F \|x\|\mu(dx) \geq \int_{F_1} \|x\|\mu(dx)$. By 476Ec, $G = \{x : x \in F_1,\ x\,{\bf .}\,e < \alpha,\ R(x) \notin F_1\}$ is negligible. But G contains x_1 and is relatively open in F_1, and F_1 is supposed to be self-supporting; so this is impossible. **XQ**

(c) Since $\mu F_1 \geq \gamma$, $F_1 \supseteq B_A$, and

$$\mu U(B_A,\epsilon) \leq \mu U(F_1,\epsilon) \leq \beta = \mu U(A,\epsilon).$$

So we have the required result for bounded A. In general, given an unbounded set A of finite measure, let δ be the radius of B_A; then

$$\begin{aligned}\mu U(B_A,\epsilon) &= \mu B(\mathbf{0},\delta+\epsilon) = \sup_{\alpha<\delta} \mu B(\mathbf{0},\alpha+\epsilon)\\ &\leq \sup_{A' \subseteq A \text{ is bounded}} \mu U(B_{A'},\epsilon) \leq \sup_{A' \subseteq A \text{ is bounded}} \mu U(A',\epsilon) = \mu U(A,\epsilon)\end{aligned}$$

because $\{U(A',\epsilon) : A' \subseteq A \text{ is bounded}\}$ is an upwards-directed family of open sets with union $U(A,\epsilon)$, and μ is τ-additive. So the theorem is true for unbounded A as well.

476G Theorem Let $r \geq 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$; write λ for the product measure on $\mathbb{R}^r \times \mathbb{R}^r$. For any measurable set $E \subseteq \mathbb{R}^r$ of finite measure, write B_E for the closed ball with centre $\mathbf{0}$ and the same measure as E. Then

$$\int_{E\times E} \|x-y\|\lambda(d(x,y)) \geq \int_{B_E\times B_E} \|x-y\|\lambda(d(x,y)).$$

proof (a) Suppose for the moment that E is compact and not empty, and that $\epsilon > 0$. Let $M \geq 0$ be such that $\|x\| \leq M$ for every $x \in E$. For a non-empty set $A \subseteq \mathbb{R}^r$ set $U(A,\epsilon) = \{x : \rho(x,A) < \epsilon\}$, where ρ is Euclidean distance on $\mathbb{R}^r$. Set $\beta = \int_{U(E,\epsilon)\times U(E,\epsilon)} \|x-y\|\lambda(d(x,y))$. Let $\mathcal{F}$ be the family of non-empty closed subsets F of the ball $B(\mathbf{0},M) = \{x : \|x\| \leq M\}$ such that $\mu F \geq \mu E$ and $\int_{U(F,\epsilon)\times U(F,\epsilon)} \|x-y\|\lambda(d(x,y)) \leq \beta$. Then $\mathcal{F}$ is compact for the Fell topology on the family $\mathcal{C}$ of closed subsets of $\mathbb{R}^r$. **P** We know from 4A2T(b-iii) that $\mathcal{C}$ is compact, and from 476A(b-ii) that $\{F : \int_F \chi B(\mathbf{0},M)d\mu \geq \mu E\}$ is closed; also $\{F : F \subseteq B(\mathbf{0},M)\}$ is closed. Let σ be the metric on $\mathbb{R}^r \times \mathbb{R}^r$ defined by setting $\sigma((x,y),(x',y')) = \max(\|x-x'\|,\|y-y'\|)$, and ν the indefinite-integral measure over λ defined by the function $(x,y) \mapsto \|x-y\|$. Then

$$U(F,\epsilon)\times U(F,\epsilon)=\{(x,y):\sigma((x,y),F\times F)<\epsilon\}=U(F\times F,\epsilon;\sigma)$$

for $F\in\mathcal{C}$ and $\epsilon>0$. Now, writing $\mathcal{C}_2$ for the family of closed sets in $\mathbb{R}^r\times\mathbb{R}^r$ with its Fell topology, we know that

$F\mapsto F\times F:\mathcal{C}\to\mathcal{C}_2$ is continuous, by 4A2Td,

$E\mapsto \nu U(E,\epsilon;\sigma):\mathcal{C}_2\to\mathbb{R}$ is lower semi-continuous, by 476B;

so $F\mapsto\nu(U(F,\epsilon)\times U(F,\epsilon))$ is lower semi-continuous, and $\{F:\int_{U(F,\epsilon)\times U(F,\epsilon)}\|x-y\|\lambda(d(x,y))\le\beta\}$ is closed. Putting these together, $\mathcal{F}$ is a closed subset of $\mathcal{C}$ and is compact. **Q**

(b) Since $E\in\mathcal{F}$, $\mathcal{F}$ is not empty. By 476A(b-ii), there is an $F_0\in\mathcal{F}$ such that $\int_{F_0}M-\|x\|\mu(dx)\ge\int_F M-\|x\|\mu(dx)$ for every $F\in\mathcal{F}$. Let $F_1\subseteq F_0$ be a closed self-supporting set of the same measure; then $U(F_1,\epsilon)\subseteq U(F_0,\epsilon)$, so $\int_{U(F_1,\epsilon)\times U(F_1,\epsilon)}\|x-y\|\lambda(d(x,y))\le\beta$ and $F_1\in\mathcal{F}$; also

$$\int_{F_1}M-\|x\|\mu(dx)=\int_{F_0}M-\|x\|\mu(dx)\ge\int_F M-\|x\|\mu(dx)$$

for every $F\in\mathcal{F}$.

Now F_1 is a ball with centre $\mathbf{0}$. **P?** Suppose, if possible, otherwise. Then (just as in the proof of 476G) there are $x_1\in F_1$ and $x_0\in\mathbb{R}^r\setminus F_1$ such that $\|x_1\|>\|x_0\|$. Set $e=\frac{1}{\|x_0-x_1\|}(x_0-x_1)$ and $\alpha=\frac{1}{2}e\cdot(x_0+x_1)<0$. Define $R=R_{e\alpha}:\mathbb{R}^r\to\mathbb{R}^r$ and $\psi=\psi_{e\alpha}$ as in 476D. Set $F=\psi(F_1)$. Then F is closed and $\mu F=\mu F_1\ge\mu E$ and $U(F,\epsilon)\subseteq\psi(U(F_1,\epsilon))$. So

$$\begin{aligned}\int_{U(F,\epsilon)\times U(F,\epsilon)}\|x-y\|\lambda(d(x,y))&\le\int_{\psi(U(F_1,\epsilon))\times\psi(U(F_1,\epsilon))}\|x-y\|\lambda(d(x,y))\\&\le\int_{U(F_1,\epsilon)\times U(F_1,\epsilon)}\|x-y\|\lambda(d(x,y))\\&\le\beta.\end{aligned}$$

(476Ed)

This means that $F\in\mathcal{F}$. Accordingly $\int_{F_1}M-\|x\|\mu(dx)\ge\int_F M-\|x\|\mu(dx)$; since $\mu F=\mu F_1$, $\int_{F_1}\|x\|\mu(dx)\le\int_F\|x\|\mu(dx)$. By 476Ec, $G=\{x:x\in F_1,\,x\cdot e<\alpha,\,R(x)\notin F_1\}$ must be negligible. But G contains x_1 and is relatively open in F_1, and F_1 is supposed to be self-supporting; so this is impossible. **XQ**

(c) Since $\mu F_1\ge\mu E$, $F_1\supseteq B_E$, and

$$\begin{aligned}\int_{B_E\times B_E}\|x-y\|\lambda(d(x,y))&\le\int_{U(F_1,\epsilon)\times U(F_1,\epsilon)}\|x-y\|\lambda(d(x,y))\\&\le\beta=\int_{U(E,\epsilon)\times U(E,\epsilon)}\|x-y\|\lambda(d(x,y)).\end{aligned}$$

At this point, recall that ϵ was arbitrary. Since E is compact,

$$\begin{aligned}\int_{E\times E}\|x-y\|\lambda(d(x,y))&=\inf_{\epsilon>0}\int_{U(E,\epsilon)\times U(E,\epsilon)}\|x-y\|\lambda(d(x,y))\\&\ge\int_{B_E\times B_E}\|x-y\|\lambda(d(x,y)).\end{aligned}$$

(d) Thus the result is proved for non-empty compact sets E. In general, given a measurable set E of finite measure, then if E is negligible the result is trivial; and otherwise, writing δ for the radius of B_E,

$$\begin{aligned}\int_{B_E\times B_E}\|x-y\|\lambda(d(x,y))&=\sup_{\alpha<\delta}\int_{B(\mathbf{0},\alpha)\times B(\mathbf{0},\alpha)}\|x-y\|\lambda(d(x,y))\\&\le\sup_{K\subseteq E\text{ is compact}}\int_{B_K\times B_K}\|x-y\|\lambda(d(x,y))\\&\le\sup_{K\subseteq E\text{ is compact}}\int_{K\times K}\|x-y\|\lambda(d(x,y))\\&=\int_{E\times E}\|x-y\|\lambda(d(x,y)),\end{aligned}$$

so the proof is complete.

476H The Isoperimetric Theorem Let $r \geq 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$. If $E \subseteq \mathbb{R}^r$ is a measurable set of finite measure, then $\operatorname{per} E \geq \operatorname{per} B_E$, where B_E is the closed ball with centre $\mathbf{0}$ and the same measure as E, while $\operatorname{per} E$ is the perimeter of E as defined in 474D.

proof (a) Suppose to begin with that $E \subseteq B(\mathbf{0}, M)$, where $M \geq 0$, and that $\operatorname{per} E < \infty$. Let $\mathcal{F}$ be the family of measurable sets $F \subseteq \mathbb{R}^r$ such that $F \setminus B(\mathbf{0}, M)$ is negligible, $\mu F \geq \mu E$ and $\operatorname{per} F \leq \operatorname{per} E$, with the topology of convergence in measure (474T). Then

$$\begin{aligned}\mathcal{F} = \{F : \operatorname{per} F \leq \operatorname{per} E,\ &\mu(F \cap B(\mathbf{0}, M)) \geq \mu E,\\ &\mu(F \cap B(\mathbf{0}, \alpha)) \leq \mu(F \cap B(\mathbf{0}, M)) \text{ for every } \alpha \geq 0\}\end{aligned}$$

is a closed subset of $\{F : \operatorname{per} F \leq \operatorname{per} E\}$, which is compact (474Tb), so $\mathcal{F}$ is compact. For $F \in \mathcal{F}$ set $h(F) = \int_F \|x\| \mu(dx)$; then $|h(F) - h(F')| \leq M\mu((F \triangle F') \cap B(\mathbf{0}, M))$ for all F, $F' \in \mathcal{F}$, so h is continuous. There is therefore an $F_0 \in \mathcal{F}$ such that $h(F_0) \leq h(F)$ for every $F \in \mathcal{F}$. Set $F_1 = \operatorname{cl}^* F_0$, so that $F_1 \triangle F_0$ is negligible (475Cg), $\operatorname{per} F_1 = \operatorname{per} F_0$ (474F), $F_1 \in \mathcal{F}$, $F_1 \subseteq B(\mathbf{0}, M)$ and $h(F_1) = h(F_0)$.

(b) Writing $\delta = \sup_{x \in F_1} \|x\|$, we have $U(\mathbf{0}, \delta) \subseteq F_1$. **P?** Otherwise, there are $x_0 \in \mathbb{R}^r \setminus F_1$ and $x_1 \in F_1$ such that $\|x_0\| < \|x_1\|$. Set $e = \frac{1}{\|x_0 - x_1\|}(x_0 - x_1)$, $\alpha = \frac{1}{2} e \cdot (x_0 + x_1) < 0$, $R = R_{e\alpha} : \mathbb{R}^r \to \mathbb{R}^r$ and $\psi = \psi_{e\alpha}$. Set $F = \psi(F_1)$ and let $\phi = \phi_{F_1} : F_1 \to F$ be the function described in 476E. Then $\|\phi(x)\| \leq \|x\|$ for every $x \in F_1$ (476Ea). In particular, $F = \phi[F_1] \subseteq B(\mathbf{0}, M)$. Now F is measurable and $\mu F = \mu F_1 \geq \mu E$, by 476Eb. Also, writing ν for normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$,

$$\operatorname{per} F = \nu(\partial^* F) \leq \nu(\partial^* F_1) = \operatorname{per} F_1 \leq \operatorname{per} E,$$

by 475Mb and 476Ee. So $F \in \mathcal{F}$, and

$$\int_F \|x\| \mu(dx) \geq \int_{F_0} \|x\| \mu(dx) = \int_{F_1} \|x\| \mu(dx).$$

By 476Ec, $G = \{x : x \in F_1,\ x \cdot e < \alpha,\ R(x) \notin F_1\}$ is negligible. But now consider $G \cap U(x_1, \eta)$ for small $\eta > 0$. Since x_1 belongs to $F_1 = \operatorname{cl}^* F_0 = \operatorname{cl}^* F_1$, but x_0 does not,

$$\limsup_{\eta \downarrow 0} \frac{\mu(F_1 \cap B(x_1, \eta))}{\mu B(x_1, \eta)} > 0 = \limsup_{\eta \downarrow 0} \frac{\mu(F_1 \cap B(x_0, \eta))}{\mu B(x_0, \eta)}.$$

There must therefore be some $\eta > 0$ such that $\eta < \frac{1}{2}\|x_1 - x_0\|$ and $\mu(F_1 \cap B(x_0, \eta)) < \mu(F_1 \cap B(x_1, \eta))$. In this case, however, $G \supseteq F_1 \cap B(x_1, \eta) \setminus R[F_1 \cap B(x_0, \eta)]$ has measure at least $\mu(F_1 \cap B(x_1, \eta)) - \mu(F_1 \cap B(x_0, \eta)) > 0$, which is impossible. **XQ**

(c) Thus $U(\mathbf{0}, \delta) \subseteq F_1 \subseteq B(\mathbf{0}, \delta)$ and $\operatorname{per} F_1 = \operatorname{per} B(\mathbf{0}, \delta)$. Since $\mu F_1 \geq \mu E$, the radius of B_E is at most δ and

$$\operatorname{per} B_E \leq \operatorname{per} B(\mathbf{0}, \delta) = \operatorname{per} F_1 \leq \operatorname{per} E.$$

(d) Thus the result is proved when E is bounded and has finite perimeter. Of course it is trivial when E has infinite perimeter. Now suppose that E is any measurable set with finite measure and finite perimeter. Set $E_\alpha = E \cap B(\mathbf{0}, \alpha)$ for $\alpha \geq 0$; then $\operatorname{per} E = \liminf_{\alpha \to \infty} \operatorname{per} E_\alpha$ (475Mc, 475Xk). By (a)-(c), $\operatorname{per} E_\alpha \geq \operatorname{per} B_{E_\alpha}$; since $\operatorname{per} B_{E_\alpha} \to \operatorname{per} B_E$ as $\alpha \to \infty$, $\operatorname{per} E \geq \operatorname{per} B_E$ in this case also.

476I Spheres in inner product spaces For the rest of the section I will use the following notation. Let X be a (real) inner product space. Then S_X will be the unit sphere $\{x : x \in X,\ \|x\| = 1\}$. Let H_X be the isometry group of S_X with its topology of pointwise convergence (441G).

A **cap** in S_X will be a set of the form $\{x : x \in S_X,\ (x|e) \geq \alpha\}$ where $e \in S_X$ and $-1 \leq \alpha \leq 1$.

When X is finite-dimensional, it is isomorphic, as inner product space, to $\mathbb{R}^r$, where $r = \dim X$ (4A4Je). If $r \geq 1$, S_X is non-empty and compact, so has a unique H_X-invariant Radon probability measure ν_X, which is strictly positive (476C). If $r \geq 1$ is an integer, we know that the $(r-1)$-dimensional Hausdorff measure of the sphere $S_{\mathbb{R}^r}$ is finite and non-zero (265F). Since Hausdorff measures are invariant under isometries (264G, 471J), and are quasi-Radon measures when totally finite (471Dh), $(r-1)$-dimensional Hausdorff measure on $S_{\mathbb{R}^r}$ is a multiple of the normalized invariant measure $\nu_{\mathbb{R}^r}$, by 476C. The same is therefore true in any r-dimensional inner product space.

476J Lemma Let X be a real inner product space and $f \in H_X$. Then $(f(x)|f(y)) = (x|y)$ for all x, $y \in S_X$. Consequently $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ whenever x, $y \in S_X$ and α, $\beta \in \mathbb{R}$ are such that $\alpha x + \beta y \in S_X$.

proof (a) We have

$$\rho(x,y)^2 = (x-y|x-y) = (x|x) - 2(x|y) + (y|y) = 2 - 2(x|y),$$

so

$$(x|y) = 1 - \frac{1}{2}\rho(x,y)^2 = 1 - \frac{1}{2}\rho(f(x),f(y))^2 = (f(x)|f(y)).$$

(b)

$$\begin{aligned}
&\|f(\alpha x+\beta y) - \alpha f(x) - \beta f(y)\|^2 \\
&\quad= (f(\alpha x+\beta y) - \alpha f(x) - \beta f(y)|f(\alpha x+\beta y) - \alpha f(x) - \beta f(y)) \\
&\quad= 1 + \alpha^2 + \beta^2 - 2\alpha(f(\alpha x+\beta y)|f(x)) - 2\beta(f(\alpha x+\beta y)|f(y)) + 2\alpha\beta(f(x)|f(y)) \\
&\quad= 1 + \alpha^2 + \beta^2 - 2\alpha(\alpha x+\beta y|x) - 2\beta(\alpha x+\beta y|y) + 2\alpha\beta(x|y) \\
&\quad= \|(\alpha x+\beta y) - \alpha x - \beta y\|^2 = 0.
\end{aligned}$$

476K I give a theorem on concentration of measure on the sphere corresponding to 476F.

Theorem Let X be a finite-dimensional inner product space of dimension at least 2, S_X its unit sphere and ν_X the invariant Radon probability measure on S_X. For a non-empty set $A \subseteq S_X$ and $\epsilon > 0$, write $U(A,\epsilon) = \{x : \rho(x,A) < \epsilon\}$, where ρ is the usual metric of X. Then there is a cap $C \subseteq S_X$ such that $\nu_X C = \nu_X^* A$, and $\nu_X(S_X \cap U(A,\epsilon)) \ge \nu_X(S_X \cap U(C,\epsilon))$ for any such C.

proof In order to apply the results of 476D-476E directly, and simplify some of the formulae slightly, it will be helpful to write ν for the Radon measure on X defined by setting $\nu E = \nu_X(E \cap S_X)$ whenever this is defined. By 214Cd, ν^* agrees with ν_X^* on $\mathcal{P}S_X$.

(a) The first step is to check that there is a cap C of S_X such that $\nu C = \nu^* A$. **P** Take any $e_0 \in S_X$, and set $C_\alpha = \{x : x \in S_X,\ (x|e_0) \ge \alpha\}$ for $\alpha \in [-1,1]$. νC_α is defined for every $\alpha \in \mathbb{R}$ because every C_α is closed and ν is a topological measure. Now examine the formulae of 265F. We can identify X with $\mathbb{R}^{r+1}$ where $r+1 = \dim X$; do this in such a way that e_0 corresponds to the unit vector $(0,\dots,0,1)$. We have a parametrization $\phi_r : D_r \to S_X$, where D_r is a Borel subset of $\mathbb{R}^r$ with interior $]-\pi,\pi[\times]0,\pi[^{r-1}$ and ϕ_r is differentiable with continuous derivative. Moreover, if $x = (\xi_1,\dots,\xi_r) \in D_r$, then $\phi_r(x) \centerdot e_0 = \cos\xi_r$, and the Jacobian J_r of ϕ_r is bounded by 1 and never zero on $\operatorname{int} D_r$. Finally, the boundary ∂D_r is negligible. What this means is that $\nu_r C_\alpha = \int_{E_\alpha} J_r d\mu_r$, where μ_r is Lebesgue measure on $\mathbb{R}^r$, ν_r is normalized Hausdorff r-dimensional measure on $\mathbb{R}^{r+1}$, and $E_\alpha = \{x : x \in D_r,\ \cos\xi_r \ge \alpha\}$. So if $-1 \le \alpha \le \beta \le 1$ then

$$\nu_r C_\alpha - \nu_r C_\beta \le \mu_r(E_\alpha \setminus E_\beta) \le 2\pi^{r-1}(\arccos\alpha - \arccos\beta);$$

because arccos is continuous, so is $\alpha \mapsto \nu_r C_\alpha$. Also, if $\alpha < \beta$, then $E_\alpha \setminus E_\beta$ is non-negligible, so $\int_{E_\alpha \setminus E_\beta} J_r d\mu_r \ne 0$ and $\nu_r C_\alpha > \nu_r C_\beta$.

This shows that $\alpha \mapsto \nu_r C_\alpha$ is continuous and strictly decreasing; since ν_r is just a multiple of ν on S_X, the same is true of $\alpha \mapsto \nu C_\alpha$.

Since $\nu C_{-1} = \nu S_X = 1$ and $\nu C_1 = \nu\{e_0\} = 0$, the Intermediate Value Theorem tells us that there is a unique α such that $\nu C_\alpha = \nu^* A$, and we can set $C = C_\alpha$. **Q**

(b) Now take any non-empty set $A \subseteq S_X$ and any $\epsilon > 0$, and set $\gamma = \nu^* A$, $\beta = \nu U(A,\epsilon)$. Let C be a cap such that $\nu^* A = \nu C$; let e_0 be the centre of C. Consider the family

$$\mathcal{F} = \{F : F \in \mathcal{C},\ F \subseteq S_X,\ \nu F \ge \gamma,\ \nu U(F,\epsilon) \le \beta\},$$

where $\mathcal{C}$ is the family of closed subsets of X with its Fell topology. Because $U(\overline{A},\epsilon) = U(A,\epsilon)$, $\overline{A} \in \mathcal{F}$ and $\mathcal{F}$ is non-empty. By 476A(b-i) and 476B, $\mathcal{F}$ is closed in $\mathcal{C}$, therefore compact, by 4A2T(b-iii) as usual. Next, the function

$$F \mapsto \int_F \max(0, 1 + (x|e_0))\nu(dx) : \mathcal{C} \to [0,\infty[$$

is upper semi-continuous, by 476A(b-ii). It therefore attains its supremum on $\mathcal{F}$ at some $F_0 \in \mathcal{F}$. Let $F_1 \subseteq F_0$ be a self-supporting closed set with the same measure as F_0; then $F_1 \in \mathcal{F}$ and $\int_F (1+(x|e_0))\nu(dx) \ge \int_{F_1}(1+(x|e_0))\nu(dx)$ for every $F \in \mathcal{F}$.

(c) F_1 is a cap with centre e_0. **P?** Otherwise, there are $x_0 \in S_X \setminus F_1$ and $x_1 \in F_1$ such that $(x_0|e_0) > (x_1|e_0)$. Set $e = \frac{1}{\|x_0-x_1\|}(x_0 - x_1)$. Then $e \in S_X$ and $(e|e_0) > 0$. Set $R = R_{e0}$ and $\psi = \psi_{e0}$ as defined in 476D; write F for $\psi(F_1)$. Note that $(x_0 + x_1|x_0 - x_1) = \|x_0\|^2 - \|x_1\|^2 = 0$, so $(x_0 + x_1|e) = 0$ and $R(x_0) = x_1$. Also $R[S_X] = S_X$, so ν is R-invariant, because ν is a multiple of Hausdorff $(r-1)$-dimensional measure on S_X and must be invariant under isometries of S_X.

We have $\nu F = \nu F_1 \ge \gamma$, by 476Eb, and

$$\nu U(F,\epsilon) \le \nu\psi(U(F_1,\epsilon)) = \nu U(F_1,\epsilon) \le \nu U(F_0,\epsilon) \le \beta$$

by 476Da. So $F \in \mathcal{F}$. But consider the standard bijection $\phi = \phi_{F_1} : F_1 \to F$ as defined in 476E. We have

$$\int_{F_1}(1 + (\phi(x)|e_0))\nu(dx) = \int_F(1 + (x|e_0))\nu(dx) \le \int_{F_1}(1 + (x|e_0))\nu(dx).$$

If we examine the definition of ϕ, we see that $\phi(x) \ne x$ only when $(x|e) < 0$ and $\phi(x) = R(x)$, so that in this case $\phi(x) - x$ is a positive multiple of e and $(\phi(x)|e_0) > (x|e_0)$. So $G = \{x : x \in F_1,\ (x|e) < 0,\ R(x) \notin F_1\}$ must be ν-negligible. But G includes a relative neighbourhood of x_1 in F_1 and F_1 is supposed to be self-supporting for ν, so this is impossible. **XQ**

(e) Now $\nu^*A = \gamma \le \nu F_1$, so $C \subseteq F_1$ and

$$\nu U(A,\epsilon) = \beta \ge \nu U(F_1,\epsilon) \ge \nu U(C,\epsilon),$$

as claimed.

476L Corollary For any $\epsilon > 0$, there is an $r_0 \ge 1$ such that whenever X is a finite-dimensional inner product space of dimension at least r_0, A_1, $A_2 \subseteq S_X$ and $\min(\nu_X^*A_1, \nu_X^*A_2) \ge \epsilon$, then there are $x \in A_1$, $y \in A_2$ such that $\|x - y\| \le \epsilon$.

proof Take $r_0 \ge 2$ such that $r_0\epsilon^3 > 2$. Suppose that $\dim X = r \ge r_0$. Fix $e_0 \in S_X$. We need an estimate of $\nu_X C_{\epsilon/2}$, where $C_{\epsilon/2} = \{x : x \in S_X,\ (x|e_0) \ge \epsilon/2\}$ as in 476K. To get this, let $e_1, \dots, e_{r-1}$ be such that $e_0, \dots, e_{r-1}$ is an orthonormal basis of X (4A4Kc). For each $i < r$, there is an $f \in H_X$ such that $f(e_i) = e_0$, so that $(x|e_i) = (f(x)|e_0)$ for every x (476J), and

$$\int (x|e_i)^2\nu_X(dx) = \int (f(x)|e_0)^2\nu_X(dx) = \int (x|e_0)^2\nu_X(dx),$$

because $f : S_X \to S_X$ is inverse-measure-preserving for ν_X.

Accordingly

$$\begin{aligned}\nu_X C_{\epsilon/2} &= \frac{1}{2}\nu_X\{x : x \in S_X,\ |(x|e_0)| \ge \epsilon/2\} \le \frac{2}{\epsilon^2}\int_{S_X}(x|e_0)^2\nu_X(dx)\\ &< r\epsilon\int_{S_X}(x|e_0)^2\nu_X(dx) = \epsilon\sum_{i=0}^{r-1}\int_{S_X}(x|e_i)^2\nu_X(dx)\\ &= \epsilon\int_{S_X}\sum_{i=0}^{r-1}(x|e_i)^2\nu_X(dx) = \epsilon \le \nu_X^*A_1.\end{aligned}$$

So, taking C to be the cap of S_X with centre e_0 and measure ν^*A_1, $C = C_\alpha$ where $\alpha < \frac{1}{2}\epsilon$, and

$$\nu_X(S_X \cap U(A_1, \tfrac{1}{2}\epsilon)) \ge \nu_X(S_X \cap U(C_\alpha, \tfrac{1}{2}\epsilon)) \ge \nu_X C_{\alpha-\epsilon/2} > \frac{1}{2}.$$

Similarly, $\nu_X(S_X \cap U(A_2, \frac{1}{2}\epsilon)) > \frac{1}{2}$ and there must be some $z \in S_X \cap U(A_1, \frac{1}{2}\epsilon) \cap U(A_2, \frac{1}{2}\epsilon)$. Take $x \in A_1$ and $y \in A_2$ such that $\|x - z\| < \frac{1}{2}\epsilon$ and $\|y - z\| < \frac{1}{2}\epsilon$; then $\|x - y\| \le \epsilon$, as required.

476X Basic exercises (a) Let X be a topological space, $\mathcal{C}$ the set of closed subsets of X, μ a topological measure on X and f a μ-integrable real-valued function; set $\phi(F) = \int_F f\,d\mu$ for $F \in \mathcal{C}$. (i) Show that if *either* μ is inner regular with respect to the closed sets and $\mathcal{C}$ is given its Vietoris topology *or* μ is tight and $\mathcal{C}$ is given its Fell topology, then ϕ is Borel measurable. (ii) Show that if X is metrizable and $\mathcal{C} \setminus \{\emptyset\}$ is given an appropriate Hausdorff metric, then $\phi\restriction\mathcal{C} \setminus \{\emptyset\}$ is Borel measurable.

(b) In the context of 476D, show that $\operatorname{diam}\psi_{e\alpha}(A) \le \operatorname{diam} A$ for all A, e and α.

>(c) Find an argument along the lines of those in 476F and 476G to prove 264H. (*Hint*: 476Xb.)

>(d) Let X be an inner product space and S_X its unit sphere. Show that every isometry $f : S_X \to S_X$ extends uniquely to an isometry $T_f : X \to X$ which is a linear operator. (*Hint*: first check the cases in which $\dim X \le 2$.) Show that f is surjective iff T_f is, so that we have a natural isomorphism between the isometry group of S_X and the group of invertible isometric linear operators. Show that this isomorphism is a homeomorphism for the topologies of pointwise convergence.

(e) Let X be a finite-dimensional inner product space, ν_X the invariant Radon probability measure on the sphere S_X, and $E \in \operatorname{dom}\nu_X$; let $C \subseteq S_X$ be a cap with the same measure as E, and let λ be the product measure of ν_X with itself on $S_X \times S_X$. Show that $\int_{C\times C}\|x-y\|\lambda(d(x,y)) \le \int_{E\times E}\|x-y\|\lambda(d(x,y))$.

(f) Let X be a finite-dimensional inner product space and ν_X the invariant Radon probability measure on the sphere S_X. (i) Without appealing to the formulae in §265, show that $\nu_X(S_X \cap H) = 0$ whenever $H \subseteq X$ is a proper affine subspace. (*Hint*: induce on $\dim H$.) (ii) Use this to prove that if $e \in S_X$ then $\alpha \mapsto \nu_X\{x : (x|e) \ge \alpha\}$ is continuous.

476Y Further exercises (a) Let X be a compact metric space and G its isometry group. Suppose that $H \subseteq G$ is a subgroup such that the action of H on X is transitive. Show that X has a unique H-invariant Radon probability measure which is also G-invariant.

(b) Let $r \ge 1$ be an integer, and $g \in C_0(\mathbb{R}^r)$ a non-negative γ-Lipschitz function, where $\gamma \ge 0$. Let $\phi : \mathbb{R}^r \to [0,\infty[$ be a convex function. Let F be the set of non-negative γ-Lipschitz functions $f \in C(\mathbb{R}^r)$ such that f has the same decreasing rearrangement as g with respect to Lebesgue measure μ on $\mathbb{R}^r$ (§373), $\sup_{\|x\|\ge n}|f(x)| \le \sup_{\|x\|\ge n}|g(x)|$ for every $n \in \mathbb{N}$ and $\int \phi(\operatorname{grad} f)d\mu \le \int \phi(\operatorname{grad} g)d\mu$. (i) Show that F is compact for the topology of pointwise convergence. (*Hint*: 475Ye.) (ii) Show that there is a $g^* \in F$ such that $g^*(x) = g^*(y)$ whenever $\|x\| = \|y\|$. (*Hint*: parts (a) and (b-i) of the proof of 479V.)

476 Notes and comments The main theorems here (476F-476H, 476K), like 264H, are all 'classical'; they go back to the roots of geometric measure theory, and the contribution of the twentieth century was to extend the classes of sets for which balls or caps provide the bounding examples. It is very striking that they can all be proved with the same tools (see 476Xc). Of course I should remark that the Compactness Theorem (474T) lies at a much deeper level than the rest of the ideas here. (The proof of 474T relies on the distributional definition of 'perimeter' in 474D, while the arguments of 476Ee and 476H work with the Hausdorff measures of essential boundaries; so that we can join these ideas together only after proving all the principal theorems of §§472-475.) So while 'Steiner symmetrization' (264H) and 'concentration by partial reflection' (476D) are natural companions, 476H is essentially harder than the other results.

In all the theorems here, as in 264H, I have been content to show that a ball or a cap is an optimum for whatever inequality is being considered. I have not examined the question of whether, and in what sense, the optimum is unique. It seems that this requires deeper analysis.

477 Brownian motion

I presented §455 with an extraordinary omission: the leading example of a Lévy process, and the inspiration for the whole project, was relegated to an anonymous example (455Xg). In this section I will take the subject up again. The theorem that the sum of independent normally distributed random variables is again normally distributed (274B), when translated into the language of this volume, tells us that we have a family $\langle\lambda_t\rangle_{t>0}$ of centered normal distributions such that $\lambda_{s+t} = \lambda_s * \lambda_t$ for all s, $t > 0$. Consequently we have a corresponding example of a Lévy process on $\mathbb{R}$, and this is the process which we call 'Brownian motion' (477A). This is special in innumerable ways, but one of them is central: we can represent it in such a way that sample paths are continuous (477B), that is, as a Radon measure on the space of continuous paths starting at 0. In this form, it also appears as a limit, for the narrow topology, of interpolations of random walks (477C).

For the geometric ideas of §479, we need Brownian motion in three dimensions; the r-dimensional theory of 477D-477G gives no new difficulties. The simplest expression of Brownian motion in $\mathbb{R}^r$ is just to take a product measure

(477Da), but in order to apply the results of §455, and match the construction with the ideas of §456, a fair bit of explanation is necessary. The geometric properties of Brownian motion begin with the invariant transformations of 477E. As for all Lévy processes, we have a strong Markov property, and Theorem 455U translates easily into the new formulation (477G), as does the theory of hitting times (477I). I conclude with a classic result on maximal values which will be useful later (477J), and with proofs that almost all Brownian paths are nowhere differentiable (477K) and have zero two-dimensional Hausdorff measure (477L).

477A Brownian motion: Theorem There are a probability space (Ω,Σ,μ) and a family $\langle X_t\rangle_{t\ge0}$ of real-valued random variables on Ω such that

(i) $X_0=0$ almost everywhere;

(ii) whenever $0\le s<t$ then X_t-X_s is normally distributed with expectation 0 and variance $t-s$;

(iii) $\langle X_t\rangle_{t\ge0}$ has independent increments.

First proof In 455P, take $U=\mathbb{R}$ and λ_t, for $t>0$, to be the distribution of a normal random variable with expectation 0 and variance t; that is, the distribution with probability density function $x\mapsto\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$. By 272T[3], $\lambda_{s+t}=\lambda_s*\lambda_t$ for all s, $t>0$. If $\epsilon>0$, then

$$\lim_{t\downarrow0}\lambda_t\,]-\epsilon,\epsilon[\,=\lim_{t\downarrow0}\lambda_1]-\tfrac{\epsilon}{\sqrt t},\tfrac{\epsilon}{\sqrt t}[\,=1,$$

so $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455P. Accordingly we have a probability measure $\hat\mu$ on $\Omega=\mathbb{R}^{[0,\infty[}$ for which, setting $X_t(\omega)=\omega(t)$, $\langle X_t\rangle_{t\ge0}$ has the required properties, as noted in 455Q-455R.

Second proof Let μ_L be Lebesgue measure on $\mathbb{R}$, and for $t\ge0$ set $u_t=\chi[0,t]^\bullet$ in $L^2(\mu_L)$, so that $(u_s|u_t)=\min(s,t)$ for s, $t\ge0$. By 456C, there is a centered Gaussian distribution μ on $\mathbb{R}^{[0,\infty[}$ with covariance matrix $\langle\min(s,t)\rangle_{s,t\ge0}$. Set $X_t(x)=x(t)$ for $x\in\mathbb{R}^{[0,\infty[}$. Then X_0 has expectation and variance both 0, that is, $X_0=0$ a.e. If $0\le s<t$, then X_t-X_s is a linear combination of X_s and X_t, so is normally distributed with expectation 0, and its variance is

$$\mathbb{E}(X_t-X_s)^2=\mathbb{E}(X_t)^2-2\mathbb{E}(X_t\times X_s)+\mathbb{E}(X_s)^2=t-2s+s=t-s.$$

If $0\le t_0<\ldots<t_n$ and $Y_i=X_{t_{i+1}}-X_{t_i}$ for $i<n$, then $\langle Y_i\rangle_{i<n}$ has a centered Gaussian distribution, by 456Ba. Also, if $i<j<n$, then

$$\begin{aligned}\mathbb{E}(Y_i\times Y_j)&=\mathbb{E}(X_{t_{i+1}}\times X_{t_{j+1}})-\mathbb{E}(X_{t_{i+1}}\times X_{t_j})-\mathbb{E}(X_{t_i}\times X_{t_{j+1}})+\mathbb{E}(X_{t_i}\times X_{t_j})\\&=t_{i+1}-t_{i+1}-t_i+t_i=0.\end{aligned}$$

So 456E assures us that $\langle Y_i\rangle_{i<n}$ is independent.

Thus $\langle X_t\rangle_{t\ge0}$ satisfies the conditions required.

477B These constructions of Brownian motion are sufficient to show that there is a process, satisfying the defining conditions (i)-(iii), which can be studied with the tools of measure theory. From 455H we see that we have a Radon measure on the space of càllàl functions representing the process, and from 455P that we have the option of moving to the càdlàg functions, with a corresponding description of the strong Markov property in terms of inverse-measure-preserving functions, as in 455U. But there is no hint yet of the most important property of Brownian motion, that 'sample paths are continuous'. With some simple inequalities from Chapter 27 and the ideas of 454Q-454S, we can find a proof of this, as follows.

Theorem Let $\langle X_t\rangle_{t\ge0}$ be as in 477A, and $\hat\mu$ the distribution of the process $\langle X_t\rangle_{t\ge0}$. Let $C([0,\infty[)_0$ be the set of continuous functions $\omega:[0,\infty[\,\to\mathbb{R}$ such that $\omega(0)=0$. Then $C([0,\infty[)_0$ has full outer measure for $\hat\mu$, and the subspace measure μ_W on $C([0,\infty[)_0$ induced by $\hat\mu$ is a Radon measure when $C([0,\infty[)_0$ is given the topology $\mathfrak{T}_c$ of uniform convergence on compact sets.

proof (a) The main part of the argument here (down to the end of (e)) is devoted to showing that $\hat\mu^*C([0,\infty[)=1$; the result will then follow easily from 454Sb.

(b) ? Suppose, if possible, that $\hat\mu^*C([0,\infty[)<1$. Then there is a non-negligible Baire set $H\subseteq\mathbb{R}^{[0,\infty[}\setminus C([0,\infty[)$. There is a countable set $D\subseteq[0,\infty[$ such that H is determined by coordinates in D (4A3Nb); we may suppose that D includes $\mathbb{Q}\cap[0,\infty[$.

[3]Formerly 272S.

(c) (The key.) Let q, q' be rational numbers such that $0 \le q < q'$, and $\epsilon > 0$. Then

$$\Pr(\sup_{t\in D\cap[q,q']}|X_t - X_q| > \epsilon) \le \frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}}e^{-\epsilon^2/18(q'-q)}.$$

P If $q = t_0 < t_1 < \ldots < t_n = q'$, set $Y_i = X_{t_i} - X_{t_{i-1}}$ for $1 \le i \le n$, so that $X_{t_m} - X_q = \sum_{i=1}^m Y_i$ for $1 \le m \le n$, and $Y_1, \ldots, Y_n$ are independent. By Etemadi's lemma (272V[4]),

$$\begin{aligned}\Pr(\sup_{i\le n}|X_{t_i} - X_q| > \epsilon) &\le 3\max_{i\le n}\Pr(|X_{t_i} - X_q| > \tfrac{1}{3}\epsilon)\\ &= 3\max_{1\le i\le n}\Pr(\frac{1}{\sqrt{t_i-q}}|X_{t_i} - X_q| > \frac{\epsilon}{3\sqrt{t_i-q}})\\ &= 6\max_{1\le i\le n}\frac{1}{\sqrt{2\pi}}\int_{\epsilon/3\sqrt{t_i-q}}^{\infty}e^{-x^2/2}dx\end{aligned}$$

(because $\frac{1}{\sqrt{t_i-q}}(X_{t_i} - X_q)$ is standard normal)

$$\begin{aligned}&= \frac{6}{\sqrt{2\pi}}\int_{\epsilon/3\sqrt{q'-q}}^{\infty}e^{-x^2/2}dx\\ &\le \frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}}e^{-\epsilon^2/18(q'-q)}\end{aligned}$$

by 274Ma. Thus if $I \subseteq [q, q']$ is any finite set containing q and q',

$$\Pr(\sup_{t\in I}|X_t - X_q| > \epsilon) \le \frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}}e^{-\epsilon^2/18(q'-q)}.$$

Taking $\langle I_n\rangle_{n\in\mathbb{N}}$ to be a non-decreasing sequence of finite sets with union $D\cap[q, q']$, starting from $I_0 = \{q, q'\}$, we get

$$\begin{aligned}\Pr(\sup_{t\in D\cap[q,q']}|X_t - X_q| > \epsilon) &= \lim_{n\to\infty}\Pr(\sup_{t\in I_n}|X_t - X_q| > \epsilon)\\ &\le \frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}}e^{-\epsilon^2/18(q'-q)},\end{aligned}$$

as required. **Q**

(d) If $\epsilon > 0$ and $n \ge 1$, then

$$\begin{aligned}&\Pr(\text{there are } t,\, u \in D\cap[0, n] \text{ such that } |t - u| \le \frac{1}{n^2} \text{ and } |X_t - X_u| > 3\epsilon)\\ &\qquad\le \frac{18n^2}{\epsilon\sqrt{2\pi}}e^{-n^2\epsilon^2/18}.\end{aligned}$$

P Divide $[0, n]$ into n^3 intervals $[q_i, q_{i+1}]$ of length $1/n^2$. For each of these,

$$\Pr(\sup_{t\in D\cap[q_i,q_{i+1}]}|X_t - X_{q_i}| > \epsilon) \le \frac{18}{n\epsilon\sqrt{2\pi}}e^{-n^2\epsilon^2/18}.$$

So

$$\Pr(\text{there are } i < n^3,\, t \in D\cap[q_i, q_{i+1}] \text{ such that } |X_t - X_{q_i}| > \epsilon)$$

is at most $\frac{18n^2}{\epsilon\sqrt{2\pi}}e^{-n^2\epsilon^2/18}$.

But if t, $u \in [0, n]$ and $|t - u| \le 1/n^2$ and $|X_t - X_u| > 3\epsilon$, there must be an $i < n^3$ such that both t and u belong to $[q_i, q_{i+2}]$, so that either there is a $t' \in D\cap[q_i, q_{i+1}]$ such that $|X_{t'} - X_{q_i}| > \epsilon$ or there is a $t' \in D\cap[q_{i+1}, q_{i+2}]$ such that $|X_{t'} - X_{q_{i+1}}| > \epsilon$. So

$$\begin{aligned}&\Pr(\text{there are } t,\, u \in D\cap[0, n] \text{ such that } |t - u| \le \frac{1}{n^2} \text{ and } |X_t - X_u| > 3\epsilon)\\ &\le \Pr(\text{there are } i < n^3,\, t \in D\cap[q_i, q_{i+1}] \text{ such that } \text{ and } |X_t - X_{q_i}| > \epsilon)\\ &\le \frac{18n^2}{\epsilon\sqrt{2\pi}}e^{-n^2\epsilon^2/18},\end{aligned}$$

[4]Formerly 272U.

as required. **Q**

(e) So if we take $G_{\epsilon n}$ to be the Baire set

$$\{\omega : \omega \in \mathbb{R}^{[0,\infty[}, \text{ there are } t,\, u \in D \cap [0,n] \text{ such that } |t-u| \le \frac{1}{n^2} \text{ and } |\omega(t)-\omega(u)| > 3\epsilon\},$$

we have

$$\hat{\mu} G_{\epsilon n} \le \frac{18n^2}{\epsilon\sqrt{2\pi}} e^{-n^2\epsilon^2/18},$$

and $\lim_{n\to\infty} \hat{\mu} G_{\epsilon n} = 0$. We can therefore find a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\sum_{k=1}^{\infty} \hat{\mu}(G_{1/k,n_k}) < \hat{\mu}H$, so that there is an $\omega \in H \setminus \bigcup_{k\ge 1} G_{1/k,n_k}$.

What this means is that if $k \ge 1$ and t, $u \in D \cap [0,n_k]$ are such that $|t-u| \le \frac{1}{n_k^2}$, then $|\omega(t)-\omega(u)| \le \frac{3}{k}$. Since $n_k \to \infty$ as $k \to \infty$, there is a continuous function $\omega' : [0,\infty[\to \mathbb{R}$ such that $\omega'\restriction D = \omega\restriction D$. But H is determined by coordinates in D, so ω' belongs to $H \cap C([0,\infty[)$, which is supposed to be empty. **X**

(f) Thus $\hat{\mu}^* C([0,\infty[) = 1$. Since $\hat{\mu}\{\omega : \omega(0) = 0\} = 1$, $C([0,\infty[) \setminus C([0,\infty[)_0$ is $\hat{\mu}$-negligible and $C([0,\infty[)_0$ is of full outer measure for $\hat{\mu}$. By 454Sb, the subspace measure $\hat{\mu}_C$ on $C([0,\infty[)$ induced by $\hat{\mu}$ is a Radon measure for $\mathfrak{T}_c$; now $C([0,\infty[)_0$ is $\hat{\mu}_C$-conegligible. The subspace measure μ_W on $C([0,\infty[)_0$ induced by $\hat{\mu}$ is also the subspace measure induced by $\hat{\mu}_C$, so is a Radon measure for the topology on $C([0,\infty[)_0$ induced by $\mathfrak{T}_c$.

Remark We can put this together with the ideas of 455H. Following the First Proof of 477A, and using 455Pc, we see that there is a unique Radon measure $\tilde{\mu}$ on $\mathbb{R}^{[0,\infty[}$ (for the topology $\mathfrak{T}_p$ of pointwise convergence) extending $\hat{\mu}$. The identity map $\iota : C([0,\infty[)_0 \to \mathbb{R}^{[0,\infty[}$ is continuous for $\mathfrak{T}_c$ and $\mathfrak{T}_p$, so the image measure $\mu_W\iota^{-1}$ is a Radon measure on $\mathbb{R}^{[0,\infty[}$ (418I). If $E \subseteq \mathbb{R}^{[0,\infty[}$ is a Baire set, then

$$\mu_W\iota^{-1}[E] = \mu_W(E \cap C([0,\infty[)_0) = \hat{\mu}E,$$

so $\mu_W\iota^{-1}$ agrees with $\tilde{\mu}$ on Baire sets, and the two must be equal. Now $C([0,\infty[)_0$ is $\tilde{\mu}$-conegligible, just because its complement has empty inverse image under ι. So μ_W is also the subspace measure on $C([0,\infty[)_0$ induced by $\tilde{\mu}$.

Equally, since of course $C([0,\infty[)_0$ is a subspace of the set C_{dlg} of càdlàg functions from $[0,\infty[$ to $\mathbb{R}$, μ_W is the subspace measure induced by the measure $\ddot{\mu}$ of Theorem 455O.

***477C** I star the next theorem because it is very hard work and will not be relied on later. Nevertheless I think the statement, at least, should be part of your general picture of Brownian motion.

Theorem For $\alpha > 0$, define $f_\alpha : \mathbb{R}^{\mathbb{N}} \to \Omega = C([0,\infty[)_0$ by setting $f_\alpha(z)(t) = \sqrt{\alpha}(\sum_{i<n} z(i) + \frac{1}{\alpha}(t-n\alpha)z(n))$ when $z \in \mathbb{R}^{\mathbb{N}}$, $n \in \mathbb{N}$ and $n\alpha \le t \le (n+1)\alpha$. Give Ω its topology $\mathfrak{T}_c$ of uniform convergence on compact sets, and $\mathbb{R}^{\mathbb{N}}$ its product topology; then f_α is continuous. For a Radon probability measure ν on $\mathbb{R}$, let $\mu_{\nu\alpha}$ be the image Radon measure $\nu^{\mathbb{N}} f_\alpha^{-1}$ on Ω, where $\nu^{\mathbb{N}}$ is the product measure on $\mathbb{R}^{\mathbb{N}}$. Let μ_W be the Radon measure of 477B, and U a neighbourhood of μ_W in the space $P_{\mathrm{R}}(\Omega)$ of Radon probability measures on Ω for the narrow topology (437Jd). Then there is a $\delta > 0$ such that $\mu_{\nu\alpha} \in U$ whenever $\alpha \in]0,\delta]$ and ν is a Radon probability measure on $\mathbb{R}$ with mean $0 = \int x\,\nu(dx)$ and variance $1 = \int x^2\nu(dx)$ and

$$\int_{\{x:|x|\ge\delta/\sqrt{\alpha}\}} x^2\nu(dx) \le \delta. \qquad (\dagger)$$

Remark The idea is that, for a given α and ν, we consider a random walk with independent identically distributed steps, with expectation 0 and variance α, at time intervals of α, and then interpolate to get a continuous function on $[0,\infty[$; and that if the step-lengths are small the result should look like Brownian motion. Moreover, this ought not to depend on the distribution ν; but in order to apply the Central Limit Theorem in a sufficiently uniform way, we need the extra regularity condition (†). On first reading you may well prefer to fix on a particular distribution ν with mean 0 and expectation 1 (e.g., the distribution which gives measure $\frac{1}{2}$ to each of $\{1\}$ and $\{-1\}$), so that (†) is satisfied whenever α is small enough compared with δ.

proof For $\delta > 0$ I will write $Q(\delta)$ for the set of pairs (ν,α) such that ν is a Radon probability measure on $\mathbb{R}$ with mean 0 and variance 1, $0 < \alpha \le \delta$ and $\int_{\{x:|x|\ge\delta/\sqrt{\alpha}\}} x^2\nu(dx) \le \delta$. Note that $Q(\delta') \subseteq Q(\delta)$ when $\delta' \le \delta$.

(a)(i) If γ, $\epsilon > 0$ there is a $\delta > 0$ such that whenever $(\nu,\alpha) \in Q(\delta)$, s, $t \ge 0$ are multiples of α such that $t-s \ge \gamma$, and $I \subseteq \mathbb{R}$ is an interval (open, closed or half-open), then

$$|\mu_{\nu\alpha}\{\omega:\omega\in\Omega,\,\omega(t)-\omega(s)\in I\}-\frac{1}{\sqrt{2\pi(t-s)}}\int_I e^{-x^2/2(t-s)}dx|\le\epsilon.$$

P For $\delta>0$, $x\in\mathbb{R}$ set $\psi_\delta(x)=x^2$ if $|x|>\delta$, 0 if $|x|\le\delta$. Let $\eta>0$ be such that whenever $Y_1,\dots,Y_k$ are independent random variables with finite variance and zero expectation, $\sum_{i=1}^k\operatorname{Var}(Y_i)=1$ and $\sum_{i=1}^k\mathbb{E}(\psi_\eta(Y_i))\le\eta$, then

$$|\Pr(\sum_{i=1}^k Y_i\le\beta)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\beta e^{-x^2/2}dx|\le\frac{\epsilon}{2}$$

for every $\beta\in\mathbb{R}$ (274F); observe that in this case

$$|\Pr(\sum_{i=1}^k Y_i<\beta)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\beta e^{-x^2/2}dx|$$
$$=\lim_{\beta'\uparrow\beta}|\Pr(\sum_{i=1}^k Y_i\le\beta')-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\beta'} e^{-x^2/2}dx|\le\frac{\epsilon}{2}$$

for every $\beta\in\mathbb{R}$, so that

$$|\Pr(\sum_{i=1}^k Y_i\in J)-\frac{1}{\sqrt{2\pi}}\int_J e^{-x^2/2}dx|\le\epsilon$$

for every interval $J\subseteq\mathbb{R}$.

Set $\delta=\min(\eta,\eta\sqrt{\gamma})$. If $I\subseteq\mathbb{R}$ is an interval, $(\nu,\alpha)\in Q(\delta)$ and s, t are multiples of α such that $t-s\ge\gamma$, set $j=\frac{s}{\alpha}$, $k=\frac{t-s}{\alpha}$ and $J=\sqrt{t-s}I=\sqrt{\frac{k}{\alpha}}I$. Then

$$\mu_{\nu\alpha}\{\omega:\omega(t)-\omega(s)\in I\}=\nu^{\mathbb{N}}\{z:f_\alpha(z)(t)-f_\alpha(z)(s)\in I\}$$
$$=\nu^{\mathbb{N}}\{z:\sqrt{\alpha}\sum_{i=j}^{j+k-1}z(i)\in I\}=\Pr(\sum_{i=0}^{k-1}Y_i\in J)$$

where $Y_i(z)=\frac{1}{\sqrt{k}}z(j+i)$. For each i, the mean and variance of Y_i are 0 and $\frac{1}{k}$, because the mean and expectation of ν are 0 and 1. Next,

$$\sum_{i=0}^{k-1}\mathbb{E}(\psi_\eta(Y_i))=k\int_{\{x:|x|>\eta\sqrt{k}\}}\frac{1}{k}x^2\nu(dx)\le\int_{\{x:|x|>\eta\sqrt{\gamma/\alpha}\}}x^2\nu(dx)$$
$$\le\int_{\{x:|x|>\delta/\sqrt{\alpha}\}}x^2\nu(dx)\le\delta\le\eta,$$

so by the choice of η,

$$|\mu_{\nu\alpha}\{\omega:\omega\in\Omega,\,\omega(t)-\omega(s)\in I\}-\frac{1}{\sqrt{2\pi(t-s)}}\int_I e^{-x^2/2(t-s)}dx|$$
$$=|\Pr(\sum_{i=0}^{k-1}Y_i\in J)-\frac{1}{\sqrt{2\pi}}\int_J e^{-x^2/2}dx|\le\epsilon.\ \mathbf{Q}$$

(ii) If γ, $\epsilon>0$, there is a $\delta>0$ such that

$$\mu_{\nu\alpha}\{\omega:\operatorname{diam}(\omega[\,[\beta,\beta+\gamma]\,])>12\epsilon\}\le\frac{3\sqrt{\gamma}}{\epsilon}e^{-\epsilon^2/2\gamma}$$

whenever $(\nu,\alpha)\in Q(\delta)$ and $\beta\ge0$. **P** Let $\eta>0$ be such that

$$6(\eta+\frac{\sqrt{\gamma+\eta}}{\epsilon\sqrt{2\pi}}e^{-\epsilon^2/2(\gamma+\eta)})\le\frac{3\sqrt{\gamma}}{\epsilon}e^{-\epsilon^2/2\gamma},$$

and let $\delta_0>0$ be such that

$$|\mu_{\nu\alpha}\{\omega:\omega\in\Omega,\,\omega(t)-\omega(s)\in I\}-\frac{1}{\sqrt{2\pi(t-s)}}\int_I e^{-x^2/2(t-s)}dx|\le\eta$$

whenever $I \subseteq \mathbb{R}$ is an interval, $(\nu, \alpha) \in Q(\delta_0)$ and s and t are multiples of α such that $t - s \geq \frac{1}{4}\gamma$. Set $\delta = \min(\frac{1}{4}\gamma, \frac{1}{2}\eta, \delta_0)$.

Fix $(\nu, \alpha) \in Q(\delta)$. Applying the last formula with $I = [-\epsilon, \epsilon]$ and then taking complements,

$$\begin{aligned}\mu_{\nu\alpha}\{\omega : |\omega(t) - \omega(s)| > \epsilon\} &\leq \eta + \frac{2}{\sqrt{2\pi(t-s)}} \int_\epsilon^\infty e^{-x^2/2(t-s)} dx \\ &= \eta + \frac{2}{\sqrt{2\pi}} \int_{\epsilon/\sqrt{t-s}}^\infty e^{-x^2/2} dx \\ &\leq \eta + \frac{2}{\sqrt{2\pi}} \int_{\epsilon/\sqrt{\gamma+\eta}}^\infty e^{-x^2/2} dx \leq \eta + \frac{\sqrt{\gamma+\eta}}{\epsilon} e^{-\epsilon^2/2(\gamma+\eta)}\end{aligned}$$

whenever s, t are multiples of α such that $\frac{1}{4}\gamma \leq t - s \leq \gamma + \eta$, using 274Ma for the last step, as in part (c) of the proof of 477B. Now if s, t are multiples of α such that $s \leq t \leq \gamma + \eta$, either $t - s \geq \frac{1}{4}\gamma$ and

$$\mu_{\nu\alpha}\{\omega : |\omega(t) - \omega(s)| > 2\epsilon\} \leq \eta + \frac{\sqrt{\gamma+\eta}}{\epsilon} e^{-\epsilon^2/2(\gamma+\eta)},$$

or $t \leq s + \frac{1}{4}\gamma$ and there is a multiple u of α such that $t + \frac{1}{4}\gamma \leq u \leq t + \frac{1}{2}\gamma$, in which case

$$\begin{aligned}\mu_{\nu\alpha}\{\omega : |\omega(t) - \omega(s)| > 2\epsilon\} &\leq \mu_{\nu\alpha}\{\omega : |\omega(u) - \omega(s)| > \epsilon\} + \mu_{\nu\alpha}\{\omega : |\omega(u) - \omega(t)| > \epsilon\} \\ &\leq 2(\eta + \frac{\sqrt{\gamma+\eta}}{\epsilon} e^{-\epsilon^2/2(\gamma+\eta)}).\end{aligned}$$

Let j, k be such that $\beta - \frac{1}{2}\eta < j\alpha \leq \beta$ and $\beta + \gamma \leq k\alpha < \beta + \gamma + \frac{1}{2}\eta$. We have

$$\begin{aligned}&\mu_{\nu\alpha}\{\omega : \operatorname{diam}(\omega[\,[\beta, \beta + \gamma]\,]) > 12\epsilon\} \\ &\quad= \nu^{\mathbb{N}}\{z : \operatorname{diam}(f_\alpha(z)[\,[\beta, \beta + \gamma]\,]) > 12\epsilon\} \\ &\quad\leq \nu^{\mathbb{N}}\{z : \sup_{t \in [\beta, \beta+\gamma]} |f_\alpha(z)(t) - f_\alpha(z)(j\alpha)| > 6\epsilon\} \\ &\quad\leq \nu^{\mathbb{N}}\{z : \text{there is an } l \text{ such that } j < l \leq k \text{ and } |f_\alpha(z)(l\alpha) - f_\alpha(z)(j\alpha)| > 6\epsilon\}\end{aligned}$$

(because $f_\alpha(z)$ is linear between its determining values at multiples of α)

$$\begin{aligned}&\quad= \nu^{\mathbb{N}}\{z : \text{there is an } l \text{ such that } j < l \leq k \text{ and } |\sum_{i=j}^{l-1} z(i)| > \frac{6\epsilon}{\sqrt{\alpha}}\} \\ &\quad\leq 3 \sup_{j<l\leq k} \nu^{\mathbb{N}}\{z : |\sum_{i=j}^{l-1} z(i)| > \frac{2\epsilon}{\sqrt{\alpha}}\}\end{aligned}$$

(Etemadi's lemma, 272V)

$$\begin{aligned}&\quad= 3 \sup_{j<l\leq k} \mu_{\nu\alpha}\{\omega : |\omega(l\alpha) - \omega(j\alpha)| > 2\epsilon\} \\ &\quad\leq 6(\eta + \frac{\sqrt{\gamma+\eta}}{\epsilon} e^{-\epsilon^2/2(\gamma+\eta)}) \leq \frac{3\sqrt{\gamma}}{\epsilon} e^{-\epsilon^2/2\gamma},\end{aligned}$$

as required. **Q**

(iii) If γ, $\epsilon > 0$ there is a $\delta > 0$ such that

$$\mu_{\nu\alpha}\{\omega : \text{there are } s,\, t \in [0, \gamma] \text{ such that } |t - s| \leq \delta \text{ and } |\omega(t) - \omega(s)| > \epsilon\} \leq \epsilon$$

whenever $(\nu, \alpha) \in Q(\delta)$. **P** Set $\eta = \epsilon/12$, and let $k \geq 1$ be such that $\frac{6\gamma k}{\eta} e^{-k^2\eta^2/2} \leq \epsilon$; set $m = \lfloor 2k^2\gamma \rfloor$. By (ii), there is a $\delta \in \left]0, \frac{1}{2k^2}\right]$ such that

$$\mu_{\nu\alpha}\{\omega : \operatorname{diam}(\omega[\,[\beta, \beta + \frac{1}{k^2}]\,]) > 12\eta\} \leq \frac{3}{k\eta} e^{-k^2\eta^2/2}$$

whenever $(\nu, \alpha) \in Q(\delta)$ and $\beta \geq 0$. Now, for such ν and α,

$$\begin{aligned}\mu_{\nu\alpha}\{\omega : \text{there are } s,\, t\in[0,\gamma] \text{ such that } |t-s|\le\delta \text{ and } |\omega(t)-\omega(s)|>\epsilon\}\\ \le \mu_{\nu\alpha}(\bigcup_{i<m}\{\omega : \operatorname{diam}(\omega[\,[\tfrac{i}{2k^2},\tfrac{i+2}{2k^2}]\,]>12\eta\})\\ \le \frac{3m}{k\eta}e^{-k^2\eta^2/2}\le\frac{6\gamma k}{\eta}e^{-k^2\eta^2/2}\le\epsilon,\end{aligned}$$

as required. **Q**

(b) Suppose that $0=t_0<t_1<\ldots<t_n$ and that $E_0,\ldots,E_{n-1}$ are intervals in $\mathbb{R}$; set $E=\{\omega:\omega\in\Omega,\ \omega(t_{i+1})-\omega(t_i)\in E_i \text{ for } i<n\}$. Then for every $\epsilon>0$ there is a $\delta>0$ such that $\mu_W E\le 3\epsilon+\mu_{\nu\alpha}E$ whenever $(\nu,\alpha)\in Q(\delta)$. **P** Of course

$$\mu_W E=\prod_{i<n}\frac{1}{\sqrt{2\pi(t_{i+1}-t_i)}}\int_{E_i}e^{-x^2/2(t_{i+1}-t_i)}dx.$$

For $\eta>0$ and $i<n$, let $F_{i\eta}$ be the interval $\{x:[x-2\eta,x+2\eta]\subseteq E_i\}$. Set $\gamma=\frac{1}{2}\min_{i<n}(t_{i+1}-t_i)$; let $\eta\in\,]0,\gamma]$ be such that

$$\prod_{i<n}\frac{1}{\sqrt{2\pi\gamma_i}}\int_{F_{i\eta}}e^{-x^2/2\gamma_i}dx\ge\mu_W E-\epsilon$$

whenever $|\gamma_i-(t_{i+1}-t_i)|\le\eta$ for every $i<n$. Next, by (a-i) and (a-iii), there is a $\delta\in\left]0,\frac{1}{2}\eta\right]$ such that

$$\begin{aligned}\prod_{i<n}\frac{1}{\sqrt{2\pi(s_{i+1}-s_i)}}\int_{F_{i\eta}}e^{-x^2/2(s_{i+1}-s_i)}dx\\ \le\epsilon+\prod_{i<n}\mu_{\nu\alpha}\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta}\},\end{aligned}$$

$$\begin{aligned}\mu_{\nu\alpha}\{\omega : \text{there are } s,\, t\in[0,t_n+\eta]\\ \text{such that } |s-t|\le\delta \text{ and } |\omega(s)-\omega(t)|>\eta\}\le\epsilon\end{aligned}$$

whenever $(\nu,\alpha)\in Q(\delta)$ and $s_0,\ldots,s_n$ are multiples of α such that $s_{i+1}-s_i\ge\gamma$ for every $i\le n$. Take any $(\nu,\alpha)\in Q(\delta)$, and for each $i\le n$ let s_i be a multiple of α such that $t_i\le s_i\le t_i+\alpha$. Then

$$\begin{aligned}\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta} \text{ for every } i<n\}\setminus E\\ =\bigcup_{i<n}\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta},\ \omega(t_{i+1})-\omega(t_i)\notin E_i\}\\ \subseteq\bigcup_{i<n}\{\omega:|(\omega(s_{i+1})-\omega(s_i))-(\omega(t_{i+1}-\omega(t_i))|>2\eta\}\\ \subseteq\bigcup_{i\le n}\{\omega:|\omega(s_i)-\omega(t_i)|>\eta\}\\ \subseteq\{\omega : \text{there are } s,\, t\in[0,t_n+\eta]\\ \text{such that } |s-t|\le\delta \text{ and } |\omega(s_i)-\omega(t_i)|>\eta\},\end{aligned}$$

so

$$\mu_{\nu\alpha}\{\omega:\omega(s_{i+1})-\omega(s_i)\in F_{i\eta} \text{ for every } i<n\}\le\epsilon+\mu_{\nu\alpha}E.$$

Next, if $s_i=k_i\alpha$ for each i,

$$\begin{aligned}\mu_{\nu\alpha}\{\omega : \omega(s_{i+1}) - \omega(s_i) \in F_{i\eta} \text{ for every } i < n\}\\ &= \nu^{\mathbb{N}}\{z : f_\alpha(z)(s_{i+1}) - f_\alpha(z)(s_i) \in F_{i\eta} \text{ for every } i < n\}\\ &= \nu^{\mathbb{N}}\{z : \sqrt{\alpha}\sum_{j=k_i}^{k_{i+1}-1} z(j) \in F_{i\eta} \text{ for every } i < n\}\\ &= \prod_{i<n}\nu^{\mathbb{N}}\{z : \sqrt{\alpha}\sum_{j=k_i}^{k_{i-1}} z(j) \in F_{i\eta}\}\\ &= \prod_{i<n}\mu_{\nu\alpha}\{\omega : \omega(s_{i+1}) - \omega(s_i) \in F_{i\eta}\}.\end{aligned}$$

So

$$\mu_W E \le \epsilon + \prod_{i<n}\frac{1}{\sqrt{2\pi(s_{i+1}-s_i)}}\int_{F_{i\eta}} e^{-x^2/2(s_{i+1}-s_i)}dx$$

(because $|(s_{i+1}-s_i)-(t_{i+1}-t_i)| \le \alpha \le \eta$ for $i < n$)

$$\le 2\epsilon + \prod_{i<n}\mu_{\nu\alpha}\{\omega : \omega(s_{i+1}) - \omega(s_i) \in F_{i\eta}\}$$

(because $s_{i+1} - s_i \ge \gamma$ for every $i < n$)

$$= 2\epsilon + \mu_{\nu\alpha}\{\omega : \omega(s_{i+1}) - \omega(s_i) \in F_{i\eta} \text{ for every } i < n\}$$

$$\le 3\epsilon + \mu_{\nu\alpha}E,$$

as required. **Q**

(c)(i) For $k \in \mathbb{N}$ let $\delta_k > 0$ be such that $\mu_{\nu\alpha}G_k \le 2^{-k}$ whenever $(\nu, \alpha) \in Q(\delta_k)$, where

$$G_k = \{\omega : \text{there are } s,\, t \in [0, k] \text{ such that } |t - s| \le \delta_k \text{ and } |\omega(t) - \omega(s)| > 2^{-k}\};$$

such exists by (a-iii) above. For k, $n \in \mathbb{N}$ set $H_{kn} = \bigcup_{i\le n} G_{k+i}$. If $k \in \mathbb{N}$ and $\langle\omega'_n\rangle_{n\in\mathbb{N}}$ is a sequence such that $\omega'_n \in \Omega \setminus H_{kn}$ for every $n \in \mathbb{N}$, $\{\omega'_n : n \in \mathbb{N}\}$ is relatively compact in Ω. **P** If $\gamma \ge 0$ and $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $2^{-k-n} \le \epsilon$ and $k + n \ge \gamma$; now for $m \ge n$, $\omega'_m \notin G_{k+n}$ so $|\omega'_m(t) - \omega'_m(s)| \le \epsilon$ whenever s, $t \in [0, \gamma]$ and $|s - t| \le \delta_{k+n}$. Of course there is a $\delta \in]0, \delta_{k+n}]$ such that $|\omega'_m(s) - \omega'_m(t)| \le \epsilon$ whenever $m < k + n$ and s, $t \in [0, \gamma]$ are such that $|s - t| \le \delta$. Since $\omega'_n(0) = 0$ for every n, the conditions of 4A2U(e-ii) are satisfied, and $\{\omega'_n : n \in \mathbb{N}\}$ is relatively compact in $C([0, \infty[)$, therefore in its closed subset Ω. **Q**

Now if we have a compact set $K \subseteq \Omega$, an open set $G \subseteq \Omega$ including K, and $k \in \mathbb{N}$, there are an $n \in \mathbb{N}$ and a finite set $I \subseteq [0, \infty[$ such that $\omega' \in G \cup H_{kn}$ whenever $\omega \in K$, $\omega' \in \Omega$ and $|\omega'(s) - \omega(s)| \le 2^{-n}$ for every $s \in I$. **P** **?** Otherwise, let $\langle q_i\rangle_{i\in\mathbb{N}}$ enumerate $\mathbb{Q} \cap [0, \infty[$. For each $n \in \mathbb{N}$ we have $\omega_n \in K$ and $\omega'_n \in \Omega \setminus (G \cup H_{kn})$ such that $|\omega'_n(q_i) - \omega_n(q_i)| \le 2^{-n}$ for every $i \le n$. Since the topology $\mathfrak{T}_c$ on Ω is metrizable (4A2U(e-i)), and both $\{\omega_n : n \in \mathbb{N}\}$ and $\{\omega'_n : n \in \mathbb{N}\}$ are relatively compact, there is a strictly increasing sequence $\langle n_i\rangle_{i\in\mathbb{N}}$ such that $\omega = \lim_{i\to\infty}\omega_{n_i}$ and $\omega' = \lim_{i\to\infty}\omega'_{n_i}$ are both defined (use 4A2Lf twice). Since $|\omega'(q) - \omega(q)| = \lim_{i\to\infty}|\omega'_{n_i}(q) - \omega_{n_i}(q)|$ is zero for every $q \in \mathbb{Q} \cap [0, \infty[$, $\omega = \omega'$; but $\omega \in K$ and $\omega' \notin G$, so this is impossible. **XQ**

(ii) Suppose that $G \subseteq \Omega$ is open and $\gamma < \mu_W E$. Then there is a $\delta > 0$ such that $\mu_{\nu\alpha}G > \gamma$ whenever $(\nu, \alpha) \in Q_\delta$. **P** Let $K \subseteq G$ be a compact set such that $\mu_W K > \gamma$. Let $k \in \mathbb{N}$, $\epsilon > 0$ be such that $\mu_W K \ge \gamma + \epsilon + 2^{-k+1}$. By (i), there are an $n \in \mathbb{N}$ and a finite set $I \subseteq [0, \infty[$ such that $\omega' \in G \cup H_{kn}$ whenever $\omega' \in \Omega$, $\omega \in K$ and $|\omega'(t) - \omega(t)| \le 2^{-n}$ for every $t \in I$; of course we can suppose that $0 \in I$ and that $\#(I) \ge 2$. Enumerate I in increasing order as $\langle t_i\rangle_{i\le m}$. For $z \in \mathbb{Z}^m$, set

$$E_z = \{\omega : \omega \in \Omega,\, \lfloor 2^n m(\omega(t_{i+1} - \omega(t_i))\rfloor = z(i) \text{ for every } i < m\};$$

set $D = \{z : z \in \mathbb{Z}^m,\, E_z \cap K \ne \emptyset\}$ and $F = \bigcup_{z\in D} E_z$. If $z \in D$ and $\omega' \in E_z$, there is an $\omega \in K \cap E_z$, in which case

$$|(\omega'(t_{i+1}) - \omega'(t_i)) - (\omega(t_{i+1} - \omega(t_i))| \le \frac{2^{-n}}{m} \text{ for every } i < m,$$

$$|\omega'(t_i) - \omega(t_i)| \le 2^{-n} \text{ for every } i \le m$$

and $\omega' \in G \cup H_{kn}$. Thus $F \subseteq G \cup H_{kn}$. As K is compact, $\{\omega(t_i) : \omega \in K\}$ is bounded for every i and D is finite. By (b) there is a $\delta > 0$ such that $\delta \le \delta_{k+i}$ for every $i \le n$ and

$$\mu_W E_z \le \frac{\epsilon}{1+\#(D)} + \mu_{\nu\alpha} E_z$$

whenever $z \in D$ and $(\nu, \alpha) \in Q(\delta)$. Now, for such ν and α,

$$\begin{aligned}\epsilon + 2^{-k+1} + \gamma \le \mu_W K \le \mu_W F = \sum_{z\in D} \mu_W E_z \le \epsilon + \sum_{z\in D} \mu_{\nu\alpha} E_z \\ = \epsilon + \mu_{\nu\alpha} F \le \epsilon + \mu_{\nu\alpha} G + \sum_{i=0}^{n} \mu_{\nu\alpha} G_{k+i} \\ \le \epsilon + \mu_{\nu\alpha} G + \sum_{i=0}^{n} 2^{-k-i} < \epsilon + 2^{-k+1} + \mu_{\nu\alpha} G\end{aligned}$$

and $\mu_{\nu\alpha} G > \gamma$, as required. **Q**

(iii) So if U is a neighbourhood of μ_W for the narrow topology on $P_{\mathrm{R}}(\Omega)$, there is a $\delta > 0$ such that $\mu_{\nu\alpha} \in U$ whenever $(\nu, \alpha) \in Q(\delta)$. **P** There are open sets $G_0, \dots , G_n$ and $\gamma_0, \dots , \gamma_n$ such that $\gamma_i < \mu_W G_i$ for each $i < n$ and U includes $\{\mu : \mu \in P_{\mathrm{R}}(\Omega),\ \mu G_i > \gamma_i \text{ for every } i < n\}$. But from (ii) we see that for each $i \le n$ there will be a $\delta'_i > 0$ such that $\mu_{\nu\alpha} G_i > \gamma_i$ for every i whenever $(\nu, \alpha) \in Q(\delta'_i)$; so setting $\delta = \min_{i\le n} \delta'_i$ we get the result. **Q**

And this is just the conclusion declared in the statement of the theorem, rephrased in the language developed in the course of the proof.

477D Multidimensional Brownian motion In §§478-479 we shall need the theory of Brownian motion in r-dimensional space. I sketch the relevant details. Fix an integer $r \ge 1$.

(a) Let μ_{W1} be the Radon probability measure on $\Omega_1 = C([0, \infty[)_0$ described in 477B; I will call it **one-dimensional Wiener measure**. We can identify the power Ω_1^r with $\Omega = C([0, \infty[\,; \mathbb{R}^r)_0$, the space of continuous functions $\omega : [0, \infty[\, \to \mathbb{R}^r$ such that $\omega(0) = 0$, with the topology of uniform convergence on compact sets; note that Ω_1 is Polish (4A2U(e-i)), so Ω_1^r also is. Because Ω_1 is separable and metrizable, the c.l.d. product measure μ_{W1}^r measures every Borel set (4A3Dc, 4A3E), while it is inner regular with respect to the compact sets (412Sb), so it is a Radon measure. I will say that $\mu_W = \mu_{W1}^r$, interpreted as a measure on $C([0, \infty[\,; \mathbb{R}^r)_0$, is ***r*-dimensional Wiener measure**.

As observed in 477B, μ_{W1} is the subspace measure on Ω_1 induced by the distribution $\hat\mu$ of the process $\langle X_t\rangle_{t\ge0}$ in 477A. So μ_W here, regarded as a measure on $C([0, \infty[)_0^r$, is the subspace measure induced by $\hat\mu^r$ on $(\mathbb{R}^{[0,\infty[})^r \cong \mathbb{R}^{[0,\infty[\times r}$ (254La).

(b) For $\omega \in \Omega$, $t \ge 0$ and $i < r$, set $X_t^{(i)}(\omega) = \omega(t)(i)$. Then $\langle X_t^{(i)}\rangle_{t\ge0,i<r}$ is a centered Gaussian process, with covariance matrix

$$\begin{aligned}\mathbb{E}(X_s^{(i)} \times X_t^{(j)}) &= 0 \text{ if } i \ne j,\\ &= \min(s, t) \text{ if } i = j.\end{aligned}$$

P Taking μ, $\hat\mu$ and $\hat\mu^r$ as in (a), $\hat\mu^r$, like $\hat\mu$, is a centered Gaussian distribution (456Be); but it is easy to check from the formula in 454J(i) that $\hat\mu^r$ can be identified with the distribution of the family $\langle X_t^{(i)}\rangle_{t\ge0,i<r}$. So $\langle X_t^{(i)}\rangle_{t\ge0,i<r}$ is a centered Gaussian process. As for the covariance matrix, if $i \ne j$ then $X_s^{(i)}$ and $X_t^{(j)}$ are determined by different factors in the product $\Omega = \Omega_1^r$, so must be independent; while if $i = j$ then $(X_s^{(i)}, X_t^{(i)})$ have the same joint distribution as (X_s, X_t) in 477A. **Q**

(c) We shall need a variety of characterizations of the Radon measure μ_W.

(i) μ_W is the only Radon probability measure on Ω such that the process $\langle X_t^{(i)}\rangle_{t\ge0,i<r}$ described in (b) is a Gaussian process with the covariance matrix there. **P** Suppose ν is another measure with these properties. The distribution of $\langle X_t^{(i)}\rangle_{t\ge0,i<r}$ (with respect to ν) must be a centered Gaussian process on $\mathbb{R}^{r\times[0,\infty[} \cong (\mathbb{R}^{[0,\infty[})^r$, and

because it has the same covariance matrix it must be equal to $\hat{\mu}^r$, by 456Bb. But this says just that $\langle X_t^{(i)}\rangle_{t\ge 0,i<r}$ has the same joint distribution with respect to μ_W and ν. By 454N, $\nu=\mu_W$. **Q**

(ii) Another way of looking at the family $\langle X_t^{(i)}\rangle_{i<r,t\ge 0}$ is to write $X_t(\omega)=\omega(t)$ for $t\ge 0$, so that $\langle X_t\rangle_{t\ge 0}$ is now a family of $\mathbb{R}^r$-valued random variables defined on Ω. We can describe its distribution in terms matching those of 455Q and 477A, which become

(i) $X_0=0$ everywhere (on Ω, that is);

(ii) whenever $0\le s<t$ then $\frac{1}{\sqrt{t-s}}(X_t-X_s)$ has the standard Gaussian distribution μ_G^r (that is, $\omega\mapsto\frac{1}{\sqrt{t-s}}(\omega(t)-\omega(s))$ is inverse-measure-preserving for μ_W and μ_G^r);

(iii) whenever $0\le t_1<\ldots<t_n$, then $X_{t_2}-X_{t_1},\ldots,X_{t_n}-X_{t_{n-1}}$ are independent (that is, taking T_i to be the σ-algebra $\{\{\omega:\omega(t_{i+1})-\omega(t_i)\in E\}:E\subseteq\mathbb{R}^r$ is a Borel set$\}$, $\mathrm{T}_1,\ldots,\mathrm{T}_{n-1}$ are independent).

Note that these properties also determine the Radon measure μ_W. **P** Once again, suppose ν is a Radon probability measure on Ω for which (ii) and (iii) are true. We wish to show that μ_W and ν give the same distribution to $\langle X_t^{(i)}\rangle_{i<r,t\ge 0}$. If $0=t_0<t_1<\ldots<t_n$, we know that μ_W and ν give the same distribution to each of the differences $Y_j=X_{t_{j+1}}-X_{t_j}$ (or, if you prefer, to each of the families $\langle Y_j^{(i)}\rangle_{i<r}$, where $Y_j^{(i)}=X_{t_{j+1}}^{(i)}-X_{t_j}^{(i)}$); moreover, if Σ_j is the σ-algebra generated by $\{Y_j^{(i)}:i<r\}$ for each j, then μ_W and ν agree that $\langle\Sigma_j\rangle_{j<n}$ is independent. So $\mu_W E=\nu E$ whenever E is of the form $\bigcap_{j<n}E_j$ where $E_j\in\Sigma_j$ for each $j<n$. By the Monotone Class Theorem, μ_W and ν agree on the σ-algebra Σ generated by sets of this type, which is the σ-algebra generated by $\{Y_j^{(i)}:i<r,\ j<n\}$. But as $X_{t_j}=\sum_{i<j}Y_i$ for every $j\le n$, every $X_{t_j}^{(i)}$ is Σ-measurable, and μ_W and ν give the same distribution to $\langle X_{t_j}^{(i)}\rangle_{i<r,j\le n}$. As this is true whenever $0=t_0<\ldots<t_n$, μ_W and ν give the same distribution to the whole family $\langle X_t^{(i)}\rangle_{i<r,t\ge 0}$, and must be equal. **Q**

(d) In order to apply Theorem 455U, we need to go a little deeper, in order to relate the product-measure definition of μ_W to the construction in 455P. I will use the ideas of part (b) of the proof of 455R. Consider the process $\langle X_t^{(i)}\rangle_{t\ge 0,i<r}$ and the associated distribution $\hat{\mu}^r$ on $(\mathbb{R}^{[0,\infty[})^r\cong(\mathbb{R}^r)^{[0,\infty[}$. Setting $X_t=\langle X_t^{(i)}\rangle_{i<r}$, $\langle X_t\rangle_{t\ge 0}$ is an $\mathbb{R}^r$-valued process satisfying the conditions of 455Q with $U=\mathbb{R}^r$. **P** $X_0=0$ a.e. because every $X_0^{(i)}$ is zero a.e. If $0\le s<t$ then $X_t-X_s=\langle X_t^{(i)}-X_s^{(i)}\rangle_{i<r}$ has the same distribution as X_{t-s} because $X_t^{(i)}-X_s^{(i)}$ has the same distribution as $X_{t-s}^{(i)}$ for each i and $\langle X_t^{(i)}-X_s^{(i)}\rangle_{i<r}$, $\langle X_{t-s}^{(i)}\rangle_{i<r}$ are both independent. If $0\le t_0<t_1<\ldots<t_n$ then $\langle X_{t_{j+1}}^{(i)}-X_{t_j}^{(i)}\rangle_{i<r,j<n}$ is independent so $\langle X_{t_{j+1}}-X_{t_j}\rangle_{j<n}$ is independent (using 272K, or otherwise). Finally, when $t\downarrow 0$, $X_t\to 0$ in measure because $X_t^{(i)}\to 0$ in measure for each i. **Q**

For $t>0$, let λ_t be the distribution of X_t. Then λ_t is the centered Gaussian distribution on $\mathbb{R}^r$ with covariance matrix $\langle\sigma_{ij}\rangle_{i,j<r}$ where $\sigma_{ij}=t$ if $i=j$ and 0 if $i\ne j$ (456Ba, with $T(\omega)=\omega(t)$ for $\omega\in\mathbb{R}^{[0,\infty[\times r}\cong(\mathbb{R}^r)^{[0,\infty[}$). By 455R, the process of 455P can be applied to $\langle\lambda_t\rangle_{t>0}$ to give us a measure $\hat{\nu}$ on $(\mathbb{R}^r)^{[0,\infty[}$, the completion of a Baire measure, such that

$$\begin{aligned}\hat{\nu}\{\omega:\omega(t_i)\in F_i\text{ for every }i\le n\}&=\Pr(X_{t_i}\in F_i\text{ for every }i\le n)\\&=\hat{\mu}^r\{\omega:\omega(t_i)\in F_i\text{ for every }i\le n\}\end{aligned}$$

whenever $F_0,\ldots,F_n\subseteq\mathbb{R}^r$ are Borel sets and $t_0,\ldots,t_n\in[0,\infty[$. Since sets of this kind generate the Baire σ-algebra of $(\mathbb{R}^r)^{[0,\infty[}$, $\hat{\nu}$ must be equal to $\hat{\mu}^r$, that is, $\hat{\mu}^r$ is the result of applying 455P to $\langle\lambda_t\rangle_{t>0}$.

By 455H, $\hat{\nu}$ has a unique extension to a measure $\tilde{\nu}$ on $(\mathbb{R}^r)^{[0,\infty[}$ which is a Radon measure for the product topology. But if we write $\iota:C([0,\infty[;\mathbb{R}^r)_0\to(\mathbb{R}^r)^{[0,\infty[}$ for the identity map, the image measure $\mu_W\iota^{-1}$ is a Radon measure on $(\mathbb{R}^r)^{[0,\infty[}$ for the product topology and extends $\hat{\mu}^r$, so must be equal to $\tilde{\nu}$. Thus $\tilde{\nu}C([0,\infty[;\mathbb{R}^r)_0=1$. Of course $C([0,\infty[;\mathbb{R}^r)_0$ is included in the space of càllàl functions from $[0,\infty[$ to $\mathbb{R}^r$, so that we have a strengthening of the results in §455. Similarly, writing $\ddot{\nu}$ for the subspace measure induced by $\hat{\nu}$ or $\hat{\mu}^r$ on the space C_{dlg} of càdlàg functions from $[0,\infty[$ to $\mathbb{R}^r$, μ_W is the subspace measure on $C([0,\infty[;\mathbb{R}^r)_0$ induced by $\ddot{\nu}$.

By 4A3Wa, every Baire subset of C_{dlg} is the intersection of C_{dlg} with a Baire subset of $(\mathbb{R}^r)^{[0,\infty[}$, and is therefore measured by $\ddot{\nu}$. In particular, $C([0,\infty[;\mathbb{R}^r)$ and $C([0,\infty[;\mathbb{R}^r)_0$ are measured by $\ddot{\nu}$ (4A3Wd).

477E Invariant transformations of Wiener measure: Proposition Let $r\ge 1$ be an integer, and μ_W Wiener measure on $\Omega=C([0,\infty[;\mathbb{R}^r)_0$. Let $\hat{\mu}^r$ be the product measure on $(\mathbb{R}^{[0,\infty[})^r$ as described in 477D.

(a) Suppose that $f : (\mathbb{R}^{[0,\infty[})^r \to (\mathbb{R}^{[0,\infty[})^r$ is inverse-measure-preserving for $\hat{\mu}^r$, and that $\Omega_0 \subseteq \Omega$ is a μ_W-conegligible set such that $f[\Omega_0] \subseteq \Omega_0$. Then $f{\restriction}\Omega_0$ is inverse-measure-preserving for the subspace measure induced by μ_W on Ω_0.

(b) Suppose that $\hat{T} : \mathbb{R}^{r\times[0,\infty[} \to \mathbb{R}^{r\times[0,\infty[}$ is a linear operator such that, for i, $j < r$ and s, $t \ge 0$,

$$\int (\hat{T}\omega)(i,s)(\hat{T}\omega)(j,t)\hat{\mu}^r(d\omega) = \min(s,t) \text{ if } i = j,$$
$$= 0 \text{ if } i \ne j.$$

Then, identifying $\mathbb{R}^{r\times[0,\infty[}$ with $(\mathbb{R}^{[0,\infty[})^r$, $\hat{T}$ is inverse-measure-preserving for $\hat{\mu}^r$.

(c) Suppose that $t \ge 0$. Define $S_t : \Omega \to \Omega$ by setting $(S_t\omega)(s) = \omega(s+t) - \omega(s)$ for $s \ge 0$ and $\omega \in \Omega$. Then S_t is inverse-measure-preserving for μ_W.

(d) Let $T : \mathbb{R}^r \to \mathbb{R}^r$ be an orthogonal transformation. Define $\tilde{T} : \Omega \to \Omega$ by setting $(\tilde{T}\omega)(t) = T(\omega(t))$ for $t \ge 0$ and $\omega \in \Omega$. Then $\tilde{T}$ is an automorphism of (Ω, μ_W).

(e) Suppose that $\alpha > 0$. Define $U_\alpha : \Omega \to \Omega$ by setting $U_\alpha(\omega)(t) = \frac{1}{\sqrt{\alpha}}\omega(\alpha t)$ for $t \ge 0$ and $\omega \in \Omega$. Then U_α is an automorphism of (Ω, μ_W).

(f) Set

$$\Omega_0 = \{\omega : \omega \in \Omega,\ \lim_{t\to\infty} \frac{1}{t}\omega(t) = 0\},$$

and define $R : \Omega_0 \to \Omega_0$ by setting

$$(R\omega)(t) = t\omega(\frac{1}{t}) \text{ if } t > 0,$$
$$= 0 \text{ if } t = 0.$$

Then Ω_0 is μ_W-conegligible and R is an automorphism of Ω_0 with its subspace measure.

(g) Suppose that $1 \le r' \le r$, and that $\mu_W^{(r')}$ is Wiener measure on $C([0,\infty[\,;\mathbb{R}^{r'})_0$. Define $P : \Omega \to C([0,\infty[\,;\mathbb{R}^{r'})_0$ by setting $(P\omega)(t)(i) = \omega(t)(i)$ for $t \ge 0$, $i < r'$ and $\omega \in \Omega$. Then $\mu_W^{(r')}$ is the image measure $\mu_W P^{-1}$.

proof The following arguments will unscrupulously identify $C([0,\infty[\,;\mathbb{R}^r)_0$ with $C([0,\infty[)_0^r$, and $\mathbb{R}^{r\times[0,\infty[}$ with $(\mathbb{R}^r)^{[0,\infty[}$ and $(\mathbb{R}^{[0,\infty[})^r$.

(a) Because μ_W is the subspace measure on Ω induced by $\hat{\mu}^r$ (477Da), the subspace measure ν on Ω_0 induced by μ_W is also the subspace measure on Ω_0 induced by $\hat{\mu}^r$ (214Ce). If $E \subseteq \Omega_0$ is measured by ν, there is an $F \in \operatorname{dom}\hat{\mu}^r$ such that $E = F \cap \Omega_0$, and now

$$\nu E = \hat{\mu}^r F = \hat{\mu}^r f^{-1}[F] = \nu(\Omega_0 \cap f^{-1}[F]) = \nu(f{\restriction}\Omega_0)^{-1}[E].$$

As E is arbitrary, $f{\restriction}\Omega_0$ is inverse-measure-preserving for ν.

(b) By 456Ba, $\hat{\mu}^r\hat{T}^{-1}$ is a centered Gaussian distribution on $\mathbb{R}^{r\times[0,\infty[}$. The hypothesis asserts that its covariance matrix is the same as that of $\hat{\mu}^r$ (477Db), so that $\hat{\mu}^r = \hat{\mu}^r\hat{T}^{-1}$ (456Bb), that is, $\hat{T}$ is inverse-measure-preserving for $\hat{\mu}^r$.

(c) Define $\hat{S}_t : \mathbb{R}^{r\times[0,\infty[} \to \mathbb{R}^{r\times[0,\infty[}$ by setting $(\hat{S}_t\omega)(i,s) = \omega(i,s+t) - \omega(i,s)$, this time for $\omega \in \mathbb{R}^{r\times[0,\infty[}$, $i < r$ and $s \ge 0$. Then $\hat{S}_t$ is linear, and for s, $u \in [0,\infty[$, i, $j < r$

$$\begin{aligned}\int (\hat{S}_t\omega)(i,s)(\hat{S}_t\omega)(j,u)\mu^r(d\omega)&\\ &= \int (\omega(i,s+t) - \omega(t))(\omega(j,u+t) - \omega(j,t))\hat{\mu}^r(d\omega)\\ &= 0 \text{ if } i \ne j,\\ &= \min(s+t,u+t) - \min(s+t,t)\\ &\qquad - \min(t,u+t) + \min(t,t) = \min(s,u) \text{ if } i = j.\end{aligned}$$

By (b), $\hat{S}_t$ is inverse-measure-preserving for $\hat{\mu}^r$. Now $\hat{S}_t[\Omega] \subseteq \Omega$, so $S_t = \hat{S}_t{\restriction}\Omega$ is inverse-measure-preserving for μW, by (a).

(d) If we define $\hat{T} : (\mathbb{R}^r)^{[0,\infty[} \to (\mathbb{R}^r)^{[0,\infty[}$ by setting $(\hat{T})(\omega)(t) = T(\omega(t))$ for $x \in (\mathbb{R}^r)^{[0,\infty[}$ and $t \ge 0$, then $\hat{T}$ is linear. Suppose that T is defined by the matrix $\langle \alpha_{ij}\rangle_{i,j<r}$. For $\omega \in \mathbb{R}^{r\times[0,\infty[}$, $t \in [0,\infty[$ and $i < r$,

$$(\hat{T}\omega)(i,t) = \textstyle\sum_{k=0}^{r-1} \alpha_{ik}\omega(k,t).$$

So, for i, $j < r$ and s, $t \ge 0$,

$$\begin{aligned}\int (\hat{T}\omega)(i,s)(\hat{T}\omega)(j,t)\hat{\mu}^r(d\omega) &= \sum_{k=0}^{r-1}\sum_{l=0}^{r-1} \alpha_{ik}\alpha_{jl}\int \omega(k,s)\omega(l,t)\hat{\mu}^r(d\omega)\\ &= \sum_{k=0}^{r-1} \alpha_{ik}\alpha_{jk}\min(s,t)\\ &= \min(s,t) \text{ if } i = j,\\ &= 0 \text{ if } i \ne j.\end{aligned}$$

So $\hat{T}$ is $\hat{\mu}^r$-inverse-measure-preserving. If we think of $\hat{T}$ as operating on $(\mathbb{R}^r)^{[0,\infty[}$, then $\hat{T}(\omega) = T\omega$ is continuous for every $\omega \in C([0,\infty[\,;\mathbb{R}^r)$, so $\hat{T}[\Omega] \subseteq \Omega$ and $\tilde{T} = \hat{T}\restriction\Omega$ is μ_W-inverse-measure-preserving.

Now the same argument applies to T^{-1}, so $\tilde{T}^{-1} = (T^{-1})^{\sim}$ also is inverse-measure-preserving, and $\tilde{T}$ is an automorphism of (Ω, μ_W).

(e) This time, we have $\hat{U}_\alpha : \mathbb{R}^{r\times[0,\infty[} \to \mathbb{R}^{r\times[0,\infty[}$ defined by the formula $(\hat{U}_\alpha\omega)(i,t) = \frac{1}{\sqrt{\alpha}}\omega(i,\alpha t)$ for $i < r$, $t \ge 0$ and $\omega \in \mathbb{R}^{r\times[0,\infty[}$. Once again, $\hat{U}_\alpha$ is linear. This time,

$$\begin{aligned}\int (\hat{U}_\alpha\omega)(i,s)(\hat{U}_\alpha\omega)(j,t)\hat{\mu}^r(d\omega) &= \frac{1}{\alpha}\int \omega(i,\alpha s)\omega(j,\alpha t)\hat{\mu}^r(d\omega)\\ &= 0 \text{ if } i \ne j,\\ &= \frac{1}{\alpha}\min(\alpha s, \alpha t) = \min(s,t) \text{ if } i = j.\end{aligned}$$

As before, it follows that $\hat{U}_\alpha$ is $\hat{\mu}^r$-inverse-measure-preserving, so that $U_\alpha = \hat{U}_\alpha\restriction\Omega$ is μ_W-inverse-measure-preserving. In the same way as in (d), $U_\alpha^{-1} = U_{1/\alpha}$ is μ_W-inverse-measure-preserving, so U_α is an automorphism of (Ω, μ_W).

(f) Define $\hat{R} : \mathbb{R}^{r\times[0,\infty[} \to \mathbb{R}^{r\times[0,\infty[}$ by setting

$$\begin{aligned}\hat{R}(\omega)(i,t) &= t\omega(i,\tfrac{1}{t}) \text{ if } i < r \text{ and } t > 0,\\ &= \omega(i,0) \text{ if } i < r \text{ and } t = 0.\end{aligned}$$

Then, if i, $j < r$ and s, $t > 0$,

$$\begin{aligned}\int (\hat{R}\omega)(i,s)(\hat{R}\omega)(j,t)\hat{\mu}^r(d\omega) &= st\int \omega(i,\tfrac{1}{s})\omega(j,\tfrac{1}{t})\hat{\mu}^r(d\omega)\\ &= 0 \text{ if } i \ne j,\\ &= st\min(\tfrac{1}{s},\tfrac{1}{t}) = \min(s,t) \text{ if } i = j.\end{aligned}$$

If $s = 0$, then $(\hat{R}\omega)(i,s) = \omega(i,s) = 0$ for almost every ω, so that $\int(\hat{R}\omega)(i,s)(\hat{R}\omega)(j,t)\hat{\mu}^r(d\omega) = 0$; and similarly if $t = 0$. So $\hat{R}$ is $\hat{\mu}^r$-inverse-measure-preserving.

At this point I think we need a new argument. Consider the set

$$E = \{\omega : \omega \in (\mathbb{R}^r)^{[0,\infty[}, \lim_{q\in\mathbb{Q},q\downarrow 0}\omega(q) = 0\}.$$

Then E is a Baire set in $(\mathbb{R}^r)^{[0,\infty[} \cong \mathbb{R}^{r\times[0,\infty[}$. Since $E \supseteq \Omega$, $\hat{\mu}^r E = 1$. Consequently $\hat{\mu}^r\hat{R}^{-1}[E] = 1$. But, for $\omega \in (\mathbb{R}^r)^{[0,\infty[}$,

$$\omega \in \hat{R}^{-1}[E] \iff 0 = \lim_{q\in\mathbb{Q},q\downarrow 0}(\hat{R}\omega)(q) = \lim_{q\in\mathbb{Q},q\downarrow 0} q\omega(\tfrac{1}{q}) = \lim_{q\in\mathbb{Q},q\to\infty}\tfrac{1}{q}\omega(q).$$

So

$$\Omega_0 = \{\omega : \omega \in \Omega,\ \lim_{t\to\infty} \frac{1}{t}\omega(t) = 0\} = \{\omega : \omega \in \Omega,\ \lim_{q\in\mathbb{Q},q\to\infty} \frac{1}{q}\omega(q) = 0\}$$

(because every member of Ω is continuous)

$$= \Omega \cap \hat{R}^{-1}[E]$$

is μ_W-conegligible. Next, for $\omega \in \Omega_0$, $\hat{R}\omega$ is continuous on $]0,\infty[$ and

$$0 = \omega(0) = \lim_{t\to\infty} \frac{1}{t}\omega(t) = \lim_{t\downarrow 0} \omega(t)$$

$$= (\hat{R}\omega)(0) = \lim_{t\to\infty} (\hat{R}\omega)(\frac{1}{t}) = \lim_{t\downarrow 0} t(\hat{R}\omega)(\frac{1}{t})$$

$$= \lim_{t\downarrow 0} (\hat{R}\omega)(t) = \lim_{t\to\infty} \frac{1}{t}(\hat{R}\omega)(t),$$

so $\hat{R}\omega \in \Omega_0$. By (a), $R = \hat{R}\restriction\Omega_0$ is inverse-measure-preserving for the subspace measure $\nu = (\mu_W)_{\Omega_0}$; being an involution, it is an automorphism of (Ω_0, ν).

(g) If we identify μ_W and $\mu_W^{(r')}$ with μ_{W1}^r and $\mu_{W1}^{r'}$, as in 477Da, this is elementary.

477F Proposition Let $r \ge 1$ be an integer. Then Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$ is strictly positive for the topology $\mathfrak{T}_c$ of uniform convergence on compact sets.

proof (a) Let $\mathcal{I}$ be a partition of $[0,\infty[$ into bounded intervals (open, closed or half-open). As usual, set $X_t(\omega) = \omega(t)$ for $t \in [0,\infty[$ and $\omega \in (\mathbb{R}^r)^{[0,\infty[}$. Define $\langle Y_t\rangle_{t\ge 0}$, $\langle Z_t\rangle_{t\ge 0}$ as follows. If $t \in I \in \mathcal{I}$, $a = \inf I$ and $b = \sup I$, then

$$Y_t = X_a + \tfrac{t-a}{b-a}(X_b - X_a) \text{ if } a < b,$$

$$= X_a = X_t = X_b \text{ if } a = b,$$

$$Z_t = X_t - Y_t.$$

(i) With respect to the centered Gaussian distribution $\hat{\mu}^r$ on $(\mathbb{R}^{[0,\infty[})^r \cong (\mathbb{R}^r)^{[0,\infty[}$, $(\langle Y_t\rangle_{t\ge 0}, \langle Z_t\rangle_{t\ge 0})$ is a centered Gaussian process. **P** The map $\omega \mapsto (\langle Y_t(\omega)\rangle_{t\ge 0}, \langle Z_t(\omega)\rangle_{t\ge 0})$ is linear and continuous, so we can apply 456Ba (strictly speaking, we apply this to the family $(\langle Y_t^{(i)}(\omega)\rangle_{i<r,t\ge 0}, \langle Z_t^{(i)}(\omega)\rangle_{i<r,t\ge 0})$ regarded as linear operators from $\mathbb{R}^{r\times[0,\infty[}$ to its square). **Q**

(ii) If s, $t \in [0,\infty[$ then $\mathbb{E}(Y_s \times Y_t) = \mathbb{E}(X_s \times Y_t)$. **P** Let I, J be the members of $\mathcal{I}$ containing s, t respectively; set $a = \min I$, $b = \max I$, $c = \min J$ and $d = \max J$.

case 1 If $a = b$ then $Y_s = X_s$ and we can stop.

case 2 If $a < b \le c$, then

$$\mathbb{E}(X_a \times X_c) = \mathbb{E}(X_a \times X_d) = a, \quad \mathbb{E}(X_a \times Y_t) = a,$$

$$\mathbb{E}(X_b \times X_c) = \mathbb{E}(X_b \times X_d) = b, \quad \mathbb{E}(X_b \times Y_t) = b,$$

$$\mathbb{E}(X_s \times X_c) = \mathbb{E}(X_s \times X_d) = s, \quad \mathbb{E}(X_s \times Y_t) = s,$$

$$\mathbb{E}(Y_s \times Y_t) = a + \tfrac{s-a}{b-a}(b - a) = s = \mathbb{E}(X_s \times Y_t).$$

case 3 If $d \le a < b$, then

$$c = \mathbb{E}(X_s \times X_c) = \mathbb{E}(X_a \times X_c) = \mathbb{E}(X_b \times X_c) = \mathbb{E}(Y_s \times X_c),$$

$$d = \mathbb{E}(X_s \times X_d) = \mathbb{E}(X_a \times X_d) = \mathbb{E}(X_b \times X_d) = \mathbb{E}(Y_s \times X_d);$$

since Y_t is a convex combination of X_c and X_d, $\mathbb{E}(Y_s \times Y_t) = \mathbb{E}(X_s \times Y_t)$.

case 4 If $a = c < b = d$, then

$$\mathbb{E}(X_a \times Y_t) = \mathbb{E}(X_a \times X_a) + \frac{t-a}{b-a}\,\mathbb{E}(X_a \times (X_b - X_a)) = a,$$

$$\mathbb{E}(X_b \times Y_t) = \mathbb{E}(X_b \times X_a) + \frac{t-a}{b-a}\,\mathbb{E}(X_b \times (X_b - X_a)) = a + \frac{t-a}{b-a}(b-a) = t,$$

$$\mathbb{E}(X_s \times Y_t) = \mathbb{E}(X_s \times X_a) + \frac{t-a}{b-a}\,\mathbb{E}(X_s \times (X_b - X_a)) = a + \frac{t-a}{b-a}(s-a),$$

$$\begin{aligned}\mathbb{E}(Y_s \times Y_t) &= \mathbb{E}(X_a \times Y_t) + \frac{s-a}{b-a}\,\mathbb{E}((X_b - X_a) \times Y_t)\\ &= a + \frac{s-a}{b-a}(t-a) = \mathbb{E}(X_s \times Y_t). \quad \mathbf{Q}\end{aligned}$$

(iii) Accordingly $\mathbb{E}(Z_s \times Y_t) = 0$ for all s, $t \ge 0$. It follows that if Σ_1, Σ_2 are the σ-algebras of subsets of $(\mathbb{R}^r)^{[0,\infty[}$ defined by $\{Y_t : t \ge 0\}$ and $\{Z_t : t \ge 0\}$ respectively, Σ_1 and Σ_2 are $\hat{\mu}^r$-independent (456Eb).

(b) Observe next that if $x_1, \dots, x_n \in \mathbb{R}^r$, $0 < t_1 \le \dots \le t_n$ and $\delta > 0$, then $\hat{\mu}^r\{\omega : \|\omega(t_i) - x_i\| \le \delta \text{ for } 1 \le i \le n\}$ is greater than 0. **P** Set $x_0 = 0$ in $\mathbb{R}^r$ and $t_0 = 0$. For each $i < n$, the distribution of $\frac{1}{\sqrt{t_{i+1}-t_i}}(X_{t_{i+1}} - X_{t_i})$ is the standard Gaussian distribution μ_G^r on $\mathbb{R}^r$, which has strictly positive probability density function with respect to Lebesgue measure. So

$$\begin{aligned}\Pr(\|(X_{t_{i+1}} - X_{t_i}) - (x_{i+1} - x_i)\| \le \frac{1}{n}\delta) &= \mu_G^r\{x : \|\sqrt{t_{i+1} - t_i}x - x_{i+1} + x_i\| \le \frac{1}{n}\delta\}\\ &> 0.\end{aligned}$$

Next, $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent, so

$$\begin{aligned}0 &< \prod_{i<n} \Pr(\|(X_{t_{i+1}} - X_{t_i}) - (x_{i+1} - x_i)\| \le \frac{1}{n}\delta)\\ &= \Pr(\|(X_{t_{i+1}} - X_{t_i}) - (x_{i+1} - x_i)\| \le \frac{1}{n}\delta \text{ for every } i < n)\\ &\le \Pr(\|X_{t_i} - x_i\| \le \delta \text{ for every } i \le n). \quad \mathbf{Q}\end{aligned}$$

(c) Let $G \subseteq \Omega$ be a non-empty $\mathfrak{T}_c$-open set. Then there are $\omega_0 \in \Omega$, $m \in \mathbb{N}$ and $\delta > 0$ such that G includes

$$V = \{\omega : \omega \in \Omega,\ \|\omega(t) - \omega_0(t)\| \le 6\delta \text{ for every } t \in [0, m]\}.$$

For $n \in \mathbb{N}$, let F_n be

$$\begin{aligned}\{\omega : \omega \in (\mathbb{R}^r)^{[0,\infty[}, \|\omega(q) - \omega(q')\| \le \delta\\ \text{whenever } q,\ q' \in \mathbb{Q} \cap [0, m] \text{ and } |q - q'| \le 2^{-n}\}.\end{aligned}$$

Then $\Omega \subseteq \bigcup_{n\in\mathbb{N}} F_n$ so there is an $n \ge 0$ such that $\omega_0 \in F_n$ and $\hat{\mu}^r F_n > 0$. Let $\mathcal{I}$ be $\{[2^{-n}k, 2^{-n}(k+1)[: k \in \mathbb{N}\}$, and let $\langle Y_t\rangle_{t\ge0}$, $\langle Z_t\rangle_{t\ge0}$ be the families of random variables defined from $\mathcal{I}$ by the method of (a) above, with corresponding independent σ-algebras Σ_1, Σ_2. If $\omega \in F_n$, then $\|Z_t(\omega)\| \le \delta$ for every $t \in \mathbb{Q} \cap [0, m]$. **P** If $t \in [a, b[\in \mathcal{I}$, then $b - a = 2^{-n}$ so $\|X_t(\omega) - X_a(\omega)\|$, $\|X_t(\omega) - X_b(\omega)\|$ and therefore $\|Z_t(\omega)\| = \|X_t(\omega) - Y_t(\omega)\|$ are all at most δ. **Q** Set

$$F = \{\omega : \omega \in (\mathbb{R}^r)^{[0,\infty[},\ \|Z_t(\omega)\| \le \delta \text{ for every } t \in \mathbb{Q} \cap [0, m]\};$$

then $F \in \Sigma_2$ and $\hat{\mu}^r F > 0$.

Next, set

$$E = \{\omega : \omega \in (\mathbb{R}^r)^{[0,\infty[},\ \|\omega(2^{-n}k) - \omega_0(2^{-n}k)\| \le \delta \text{ for } 1 \le k \le 2^n m\}.$$

By (b), $\hat{\mu}^r E > 0$. But since $Y_{2^{-n}k}(\omega) = X_{2^{-n}k}(\omega) = \omega(2^{-n}k)$ whenever $k \le 2^n m$, $E \in \Sigma_1$. Accordingly $\hat{\mu}^r(E \cap F) = \hat{\mu}^r E \cdot \hat{\mu}^r F > 0$. But $E \cap F \cap \Omega \subseteq V$. **P** If $\omega \in E \cap F \cap \Omega$, then $t \mapsto X_t(\omega)$, $t \mapsto Y_t(\omega)$ and $t \mapsto Z_t(\omega)$ are all continuous, so $\|Z_t(\omega)\| \le \delta$ for every $t \in [0, m]$. If $t \in [0, m]$, let $k < 2^n m$ be such that $2^{-n}k \le t \le 2^{-n}(k+1)$. Then

$$\begin{aligned}\|\omega(t)-\omega_0(t)\| &\le \|\omega(t)-\omega(2^{-n}k)\|+\|\omega(2^{-n}k)-\omega_0(2^{-n}k)\|+\|\omega_0(2^{-n}k)-\omega_0(t)\|\\ &\le \|Z_t(\omega)\|+\|Y_t(\omega)-Y_{2^{-n}k}(\omega)\|+2\delta\\ &\le \delta+\|\omega(2^{-n}(k+1))-\omega(2^{-n}k)\|+2\delta\\ &\le \|\omega(2^{-n}(k+1))-\omega_0(2^{-n}(k+1))\|+\|\omega_0(2^{-n}(k+1))-\omega_0(2^{-n}k)\|\\ &\qquad\qquad+\|\omega_0(2^{-n}k)-\omega(2^{-n}k)\|+3\delta\\ &\le 6\delta.\end{aligned}$$

As t is arbitrary, $\omega\in V$. **Q**

Accordingly

$$\mu_W G\ge\mu_W(E\cap F\cap\Omega)=\hat\mu^r(E\cap F)>0.$$

As G is arbitrary, μ_W is strictly positive.

477G The strong Markov property With the identification in 477Dd, we are ready for one of the most important properties of Brownian motion.

Theorem Suppose that $r\ge 1$, μ_W is Wiener measure on $\Omega=C([0,\infty[\,;\mathbb{R}^r)_0$ and Σ is its domain. For $t\ge 0$ let Σ_t be

$$\{F:F\in\Sigma,\ \omega'\in F\text{ whenever }\omega\in F,\ \omega'\in\Omega\text{ and }\omega'\restriction[0,t]=\omega\restriction[0,t]\},$$

$$\Sigma_t^+=\bigcap_{s>t}\Sigma_s,$$

and let $\tau:\Omega\to[0,\infty]$ be a stopping time adapted to the family $\langle\Sigma_t^+\rangle_{t\ge0}$. Define $\phi_\tau:\Omega\times\Omega\to\Omega$ by saying that

$$\begin{aligned}\phi_\tau(\omega,\omega')(t)&=\omega(t)\text{ if }t\le\tau(\omega),\\ &=\omega(\tau(\omega))+\omega'(t-\tau(\omega))\text{ if }t\ge\tau(\omega).\end{aligned}$$

Then ϕ_τ is inverse-measure-preserving for $\mu_W\times\mu_W$ and μ_W.

proof (a) At this point I apply the general theory of §455 in something like its full strength. As in 477Dd, let $\langle\lambda_t\rangle_{t>0}$ be the standard family of Gaussian distributions on $\mathbb{R}^r$, $\ddot\nu$ the corresponding measure on the space C_{dlg} of càdlàg functions from $[0,\infty[$ to $\mathbb{R}^r$, and $\ddot\Sigma$ its domain; then $\ddot\nu\Omega=1$ and μ_W is the subspace measure on Ω induced by $\ddot\nu$. As in 455U, let $\ddot\Sigma_t$ be

$$\{F:F\in\ddot\Sigma,\ \omega'\in F\text{ whenever }\omega\in F,\ \omega'\in C_{\mathrm{dlg}}\text{ and }\omega'\restriction[0,t]=\omega\restriction[0,t]\},$$

and $\ddot\Sigma_t^+=\bigcap_{s>t}\ddot\Sigma_s$, for $t>0$.

(b) For $\omega\in C_{\mathrm{dlg}}$ set

$$\ddot\tau(\omega)=\inf\{t:\text{there is an }\omega'\in\Omega\text{ such that }\omega'\restriction[0,t]=\omega\restriction[0,t]\text{ and }\tau(\omega')\le t\},$$

counting $\inf\emptyset$ as ∞. Then $\ddot\tau$ is a stopping time adapted to $\langle\ddot\Sigma_t^+\rangle_{t\ge0}$. **P** For $t\ge0$, set

$$F_t=\{\omega:\omega\in\Omega,\ \tau(\omega)<t\}\in\Sigma_t$$

(455Lb), and

$$F'_t=\{\omega:\omega\in C_{\mathrm{dlg}},\text{ there is an }\omega'\in F_t\text{ such that }\omega\restriction[0,t]=\omega'\restriction[0,t]\}.$$

Since $F_t\in\Sigma_t$, $F'_t\cap\Omega=F_t$; as Ω is $\ddot\nu$-conegligible and $\ddot\nu$ is complete, $F'_t\in\ddot\Sigma$; now of course $F'_t\in\ddot\Sigma_t$. If $\omega\in F'_t$, let $\omega'\in F_t$ be such that $\omega\restriction[0,t]=\omega'\restriction[0,t]$; then ω' witnesses that $\ddot\tau(\omega)\le\tau(\omega')<t$. If $\omega\in C_{\mathrm{dlg}}$ and $\ddot\tau(\omega)<t$, let $q<t$ and $\omega'\in\Omega$ be such that q is rational, $\omega\restriction[0,q]=\omega'\restriction[0,q]$ and $\tau(\omega')<q$; then

$$\omega\in F'_q\in\ddot\Sigma_q\subseteq\ddot\Sigma_t.$$

This shows that

$$\{\omega:\omega\in C_{\mathrm{dlg}},\ \ddot\tau(\omega)<t\}=\bigcup_{q\in[0,t]\cap\mathbb{Q}}F'_q\in\ddot\Sigma_t.$$

By 455Lb in the other direction, $\ddot\tau$ is a stopping time adapted to $\langle\ddot\Sigma_t^+\rangle_{t\ge0}$. **Q**

(c) By 455U, $\ddot\phi:C_{\mathrm{dlg}}\times C_{\mathrm{dlg}}\to C_{\mathrm{dlg}}$ is inverse-measure-preserving for $\ddot\nu\times\ddot\nu$ and $\ddot\nu$, where

$$\ddot{\phi}(\omega,\omega')(t) = \omega(t) \text{ if } t < \ddot{\tau}(\omega),$$
$$= \omega(\ddot{\tau}(\omega)) + \omega'(t - \ddot{\tau}(\omega)) \text{ if } t \ge \ddot{\tau}(\omega).$$

Now $\phi_\tau = \ddot{\phi}\restriction\Omega\times\Omega$ and $\mu_W \times \mu_W$ is the subspace measure on $\Omega\times\Omega$ induced by $\ddot{\nu}\times\ddot{\nu}$ (251Q), so ϕ_τ also is inverse-measure-preserving.

477H Some families of σ-algebras The σ-algebras considered in Theorem 477G can be looked at in other ways which are sometimes useful.

Proposition Let $r \ge 1$ be an integer, μ_W r-dimensional Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$ and Σ its domain. Set $X_t^{(i)}(\omega) = \omega(t)(i)$ for $t \ge 0$ and $i < r$. For $I \subseteq [0,\infty[$, let T_I be the σ-algebra of subsets of Ω generated by $\{X_s^{(i)} - X_t^{(i)} : s,\, t \in I,\, i < r\}$, and $\hat{\mathrm{T}}_I$ the σ-algebra $\{E\triangle F : E \in \mathrm{T}_I,\, \mu_W F = 0\}$.

(a) $\mathrm{T}_{[0,\infty[}$ is the Borel σ-algebra of Ω either for the topology of pointwise convergence inherited from $(\mathbb{R}^r)^{[0,\infty[}$ or $\mathbb{R}^{r\times[0,\infty[}$, or for the topology of uniform convergence on compact sets.

(b) If $\mathcal{I}$ is a family of subsets of $[0,\infty[$ such that for all distinct $I,\ J \in \mathcal{I}$ either $\sup I \le \inf J$ or $\sup J \le \inf I$ (counting $\inf\emptyset$ as ∞ and $\sup\emptyset$ as 0), then $\langle\hat{\mathrm{T}}_I\rangle_{I\in\mathcal{I}}$ is an independent family of σ-algebras.

(c) For $t \ge 0$, let Σ_t be the σ-algebra of sets $F \in \Sigma$ such that $\omega' \in F$ whenever $\omega \in F$, $\omega' \in \Omega$ and $\omega'\restriction[0,t] = \omega\restriction[0,t]$, and $\Sigma_t^+ = \bigcap_{s>t}\Sigma_s$. Write $\hat{\mathrm{T}}_{[0,t]}^+$ for $\bigcap_{s>t}\hat{\mathrm{T}}_{[0,s]}$. Then, for any $t \ge 0$,

$$\mathrm{T}_{[0,t]} \subseteq \Sigma_t \subseteq \Sigma_t^+ \subseteq \hat{\mathrm{T}}_{[0,t]}^+ = \hat{\mathrm{T}}_{[0,t]} = \hat{\mathrm{T}}_{[0,t[}.$$

(d) On the tail σ-algebra $\bigcap_{t\ge0}\hat{\mathrm{T}}_{[t,\infty[}$, μ_W takes only the values 0 and 1.

proof (a) Write $\mathcal{B}(\Omega,\mathfrak{T}_p)$, $\mathcal{B}(\Omega,\mathfrak{T}_c)$ for the Borel algebras under the topologies $\mathfrak{T}_p$, $\mathfrak{T}_c$ of pointwise convergence and uniform convergence on compact sets. Then $\mathrm{T}_{[0,\infty[} \subseteq \mathcal{B}(\Omega,\mathfrak{T}_p)$ because the functionals $X_t^{(i)}$ are all $\mathfrak{T}_p$-continuous, and $\mathcal{B}(\Omega,\mathfrak{T}_p) \subseteq \mathcal{B}(\Omega,\mathfrak{T}_c)$ because $\mathfrak{T}_p \subseteq \mathfrak{T}_c$.

Now $\mathrm{T}_{[0,\infty[}$ includes a base for $\mathfrak{T}_c$. **P** Suppose that $\omega \in \Omega$, $n \in \mathbb{N}$ and $V = \{\omega' : \omega' \in \Omega,\ \sup_{t\in[0,n]}\|\omega'(t) - \omega(t)\| < 2^{-n}\}$. Then (because every member of Ω is continuous)

$$V = \bigcup_{m\in\mathbb{N}}\bigcap_{q\in\mathbb{Q}\cap[0,n]}\{\omega' : \sum_{i=0}^{r-1}|X_q^{(i)}(\omega') - X_q^{(i)}(\omega)|^2 \le 2^{-2n} - 2^{-m}\}$$

belongs to $\mathrm{T}_{[0,\infty[}$; but such sets form a base for $\mathfrak{T}_c$. **Q**

Since $\mathfrak{T}_c$ is separable and metrizable, $\mathcal{B}(\Omega,\mathfrak{T}_c) \subseteq \mathrm{T}_{[0,\infty[}$ (4A3Da).

(b)(i) Suppose to begin with that $\mathcal{I}$ is finite and every member of $\mathcal{I}$ is finite. If we enumerate $\{0\}\cup\bigcup\mathcal{I}$ in ascending order as $\langle t_j\rangle_{j\le n}$, $\langle X_{t_{j+1}}^{(i)} - X_{t_j}^{(i)}\rangle_{j<n,i<r}$ is an independent family of real-valued random variables. Taking $J_I = \{t_j : j < n,\ t_j \in I,\ t_{j+1} \in I\}$ for $I \in \mathcal{I}$, $\{X_{t_{j+1}}^{(i)} - X_{t_j}^{(i)} : j \in J_I\}$ generates T_I for each $I \in \mathcal{I}$, because of the separation property of the members of $\mathcal{I}$, and $\langle J_I\rangle_{I\in\mathcal{I}}$ is disjoint. By 272K, $\langle\mathrm{T}_I\rangle_{I\in\mathcal{I}}$ is independent.

(ii) Now suppose only that $\mathcal{I}$ is finite and not empty. For $I \in \mathcal{I}$, set $\mathrm{T}'_I = \bigcup_{J\subseteq I\text{ is finite}}\mathrm{T}_J$ for $I \in \mathcal{I}$; then T'_I is an algebra of sets, and T_I is the σ-algebra generated by T'_I. If $E_I \in \mathrm{T}'_I$ for $I \in \mathcal{I}$, there are $J_I \in [I]^{<\omega}$ such that $E_I \in \mathrm{T}_{J_I}$ for $I \in \mathcal{I}$, so $\mu_W(\bigcap_{I\in\mathcal{I}}E_I) = \prod_{I\in\mathcal{I}}\mu_W(E_I)$, by (α). Inducing on n, and using the Monotone Class Theorem for the inductive step, we see that $\mu_W(\bigcap_{I\in\mathcal{I}}E_I) = \prod_{I\in\mathcal{I}}\mu_W(E_I)$ whenever $E_I \in \mathrm{T}_I$ for every $I \in \mathcal{I}$ and $\#(\{I : E_I \notin \mathrm{T}'_I\}) \le n$. At the end of the induction, with $n = \#(\mathcal{I})$, we have $\mu_W(\bigcap_{I\in\mathcal{I}}E_I) = \prod_{I\in\mathcal{I}}\mu_W(E_I)$ whenever $E_I \in \mathrm{T}_I$ for every $I \in \mathcal{I}$; that is, $\langle\mathrm{T}_I\rangle_{I\in\mathcal{I}}$ is independent.

(iii) Thus $\langle\mathrm{T}_I\rangle_{I\in\mathcal{J}}$ is independent for every non-empty finite $\mathcal{J} \subseteq \mathcal{I}$, and $\langle\mathrm{T}_I\rangle_{I\in\mathcal{I}}$ is independent (272Bb).

(c)(i) If $s,\ s' \le t$ and $i < r$ then $X_s^{(i)}$, $X_{s'}^{(i)}$ and $X_s^{(i)} - X_{s'}^{(i)}$ are Σ_t-measurable, so $\mathrm{T}_{[0,t]} \subseteq \Sigma_t$. Of course $\Sigma_t \subseteq \Sigma_t^+$.

(ii) $\Sigma_t \subseteq \hat{\mathrm{T}}_{[0,t]}$. **P** Suppose that $F \in \Sigma_t$. Set $D = [0,t]\cap(\mathbb{Q}\cup\{t\})$, and set $g(\omega) = \omega\restriction D$ for $\omega \in \Omega$; then $g : \Omega \to (\mathbb{R}^r)^D$ is continuous (when Ω is given the topology of pointwise convergence inherited from $(\mathbb{R}^r)^{[0,\infty[}$, for definiteness), and $F = g^{-1}[g[F]]$. Now the Borel σ-algebra of $(\mathbb{R}^r)^D \cong \mathbb{R}^{r\times D}$ is the σ-algebra generated by the functionals $\omega \mapsto \omega(t)(i) : (\mathbb{R}^r)^D \to \mathbb{R}$ for $t \in D$ and $i < r$ (4A3D(c-i)), and for such t and i, $\omega \mapsto g(\omega)(t)(i) = X_t^{(i)}(\omega)$ is $\mathrm{T}_{[0,t]}$-measurable; so g is $\mathrm{T}_{[0,t]}$-measurable. Now there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of compact subsets of F such that $\sup_{n\in\mathbb{N}}\mu_W K_n = \mu_W F$. In this case, $g[K_n] \subseteq (\mathbb{R}^r)^D$ is compact and $K'_n = g^{-1}[g[K_n]]$ belongs to $\mathrm{T}_{[0,t]}$, for each n. So $F' = \bigcup_{n\in\mathbb{N}}K'_n$ belongs to $\mathrm{T}_{[0,t]}$, and $\mu_W(F\setminus F') = 0$.

Similarly, applying the same argument to $\Omega\setminus F$, we have an $F''\in\mathrm{T}_{[0,t]}$ such that $F''\supseteq F$ and $\mu_W(F''\setminus F)=0$. So $F\in\hat{\mathrm{T}}_{[0,t]}$. **Q**

Consequently $\Sigma_t^+\subseteq\hat{\mathrm{T}}^+_{[0,t]}$.

(iii)(*α*) Let $\mathcal{A}$ be the family of those sets $G\in\Sigma$ such that χG has a conditional expectation on $\hat{\mathrm{T}}^+_{[0,t]}$ which is $\mathrm{T}_{[0,t]}$-measurable. Then $\mathcal{A}$ is a Dynkin class (definition: 136A). If $E\in\mathrm{T}_{[0,t]}$ and $F\in\mathrm{T}_{[s,\infty[}$ where $s>t$, then $(\mu_WF)\chi\Omega$ is a conditional expectation of χF on $\hat{\mathrm{T}}^+_{[0,t]}$, because $\hat{\mathrm{T}}^+_{[0,t]}\subseteq\hat{\mathrm{T}}_{[0,s]}$ and $\mathrm{T}_{[s,\infty[}$ are independent. As $E\in\hat{\mathrm{T}}^+_{[0,t]}$, $(\mu_WF)\chi E$ is a conditional expectation of $\chi(E\cap F)$ on $\hat{\mathrm{T}}^+_{[0,t]}$ (233Eg), and $E\cap F\in\mathcal{A}$. Since $\mathcal{E}=\{E\cap F:E\in\Sigma_t,\,F\in\bigcup_{s>t}\mathrm{T}_{[s,\infty[}\}$ is closed under finite intersections, the Monotone Class Theorem, in the form 136B, shows that $\mathcal{A}$ includes the σ-algebra T generated by $\mathcal{E}$; note that T includes $\mathrm{T}_{[0,t]}\cup\mathrm{T}_{[s,\infty[}$ whenever $s>t$. Now $X_u^{(i)}-X_s^{(i)}$ is T-measurable whenever $0\le s\le u$. **P** If $u\le t$, $X_u^{(i)}-X_s^{(i)}$ is $\mathrm{T}_{[0,t]}$-measurable, therefore T-measurable; if $t<s$, $X_u^{(i)}-X_s^{(i)}$ is $\mathrm{T}_{[s,\infty[}$-measurable, therefore T-measurable. If $s\le t<u$, let $\langle t_n\rangle_{n\in\mathbb{N}}$ be a sequence in $]t,u]$ with limit t. Then

$$X_u^{(i)}-X_s^{(i)}=\lim_{n\to\infty}(X_u^{(i)}-X_{t_n}^{(i)})+(X_t^{(i)}-X_s^{(i)})$$

is T-measurable. **Q**

(*β*) This means that T includes $\mathrm{T}_{[0,\infty[}$. It follows that $\mathcal{A}=\Sigma$, because for any $G\in\Sigma$ there is a $G'\in\mathrm{T}_{[0,\infty[}$ such that $G\triangle G'$ is negligible, and now χG and $\chi G'$ have the same conditional expectations. So $\hat{\mathrm{T}}^+_{[0,t]}=\hat{\mathrm{T}}_{[0,t]}$. **P** Of course $\hat{\mathrm{T}}^+_{[0,t]}\supseteq\hat{\mathrm{T}}_{[0,t]}$. If $H\in\hat{\mathrm{T}}^+_{[0,t]}$ there is a $\mathrm{T}_{[0,t]}$-measurable function g which is a conditional expectation of χH on $\hat{\mathrm{T}}^+_{[0,t]}$. But in this case $g=_{\text{a.e.}}\chi H$, so, setting $E=\{\omega:g(\omega)=1\}\in\mathrm{T}_{[0,t]}$, $E\triangle H$ is negligible and $H\in\hat{\mathrm{T}}_{[0,t]}$. Thus $\hat{\mathrm{T}}^+_{[0,t]}\subseteq\hat{\mathrm{T}}_{[0,t]}$. **Q**

(*γ*) Observe next that $X_t^{(i)}$ is $\mathrm{T}_{[0,t[}$-measurable for $i<r$. **P** If $t=0$ then $X_t^{(i)}$ is the constant function with value 0. Otherwise, there is a strictly increasing sequence $\langle s_n\rangle_{n\in\mathbb{N}}$ in $[0,t[$ with limit t, so that $X_t^{(i)}=\lim_{n\to\infty}X_{s_n}^{(i)}$ is the limit of a sequence of $\mathrm{T}_{[0,t[}$-measurable functions and is itself $\mathrm{T}_{[0,t[}$-measurable. **Q** But this means that $\mathrm{T}_{[0,t]}=\mathrm{T}_{[0,t[}$, so $\hat{\mathrm{T}}^+_{[0,t]}\subseteq\hat{\mathrm{T}}_{[0,t]}=\hat{\mathrm{T}}_{[0,t[}$. In the other direction, of course $\mathrm{T}_{[0,t[}\subseteq\mathrm{T}^+_{[0,t]}$ and $\hat{\mathrm{T}}_{[0,t[}\subseteq\hat{\mathrm{T}}^+_{[0,t]}$, so we have equality.

(d) Set $\mathrm{T}'=\bigcup_{t\ge0}\mathrm{T}_{[0,t]}$. If $E\in\bigcap_{t\ge0}\hat{\mathrm{T}}_{[t,\infty[}$, then $\mu_W(E\cap F)=\mu_WE\cdot\mu_WF$ for every $F\in\mathrm{T}'$. By the Monotone Class Theorem again, $\mu_W(E\cap F)=\mu_WE\cdot\mu_WF$ for every F in the σ-algebra generated by T', which is $\mathcal{B}(\Omega)$, by (a). Now $\mu_W\llcorner E$ (definition: 234M[5]) and $(\mu_WE)\mu_W$ are Radon measures on Ω (416S) which agree on $\mathcal{B}(\Omega)$, so must be identical. In particular,

$$\mu_WE=(\mu_W\llcorner E)(E)=(\mu_WE)^2$$

and μ_WE must be either 0 or 1.

477I Hitting times In 455M I introduced 'hitting times'. I give a paragraph now to these in the special case of Brownian motion; such stopping times will dominate the applications of the theory in §§478-479. Take $r\ge1$, and let μ_W be Wiener measure on $\Omega=C([0,\infty[\,;\mathbb{R}^r)_0$ and Σ its domain; for $t\ge0$ define Σ_t^+ and $\mathrm{T}_{[0,t]}$ as in 477G and 477H. Give Ω its topology of uniform convergence on compact sets.

(a) Suppose that $A\subseteq\mathbb{R}^r$. For $\omega\in\Omega$ set $\tau(\omega)=\inf\{t:t\in[0,\infty[,\,\omega(t)\in A\}$, counting $\inf\emptyset$ as ∞. I will call τ the **Brownian hitting time** to A, or the **Brownian exit time** from $\mathbb{R}^r\setminus A$. I will say that the **Brownian hitting probability** of A, or the **Brownian exit probability** of $\mathbb{R}^r\setminus A$, is $\mathrm{hp}(A)=\mu_W\{\omega:\tau(\omega)<\infty\}$ if this is defined. More generally, I will write

$$\mathrm{hp}^*(A)=\mu_W^*\{\omega:\tau(\omega)<\infty\}=\mu_W^*\{\omega:\omega^{-1}[A]\ne\emptyset\},$$

the **outer Brownian hitting probability**, for any $A\subseteq\mathbb{R}^r$.

(b) If $A\subseteq\mathbb{R}^r$ is analytic, the Brownian hitting time to A is a stopping time adapted to the family $\langle\Sigma_t^+\rangle_{t\ge0}$. **P** Let C_{dlg} be the space of càdlàg functions from $[0,\infty[$ to $\mathbb{R}^r$, and define $\ddot\Sigma$ as in the proof of 477G; let $\ddot\tau$ be the

[5] Formerly 234E.

hitting time on C_{dlg} defined by A. By 455Ma, $\ddot{\tau}$ is $\ddot{\Sigma}$-measurable, so $\tau = \ddot{\tau}\restriction\Omega$ is Σ-measurable. Now (as in 455Mb) $\{\omega : \tau(\omega) < t\} \in \Sigma_t$ for every t, so τ is adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$. **Q**

In particular, there is a well-defined Brownian hitting probability of A.

(c) Let $F \subseteq \mathbb{R}^r$ be a closed set, and τ the Brownian hitting time to F.

(i) If $\tau(\omega) < \infty$, then

$$\tau(\omega) = \inf \omega^{-1}[F] = \min \omega^{-1}[F]$$

because ω is continuous. If $0 \notin F$ and $\tau(\omega) < \infty$, then $\omega(\tau(\omega)) \in \partial F$.

(ii) τ is lower semi-continuous. **P** For any $t \in [0,\infty[$,

$$\{\omega : \tau(\omega) > t\} = \{\omega : \omega(s) \notin F \text{ for every } s \le t\}$$

is open in Ω. **Q**

(iii) τ is adapted to $\langle\mathrm{T}_{[0,t]}\rangle_{t\ge 0}$. **P** Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of open sets including F such that $F = \bigcap_{n\in\mathbb{N}} \overline{G}_n$. Then, for $\omega \in \Omega$ and $t > 0$,

$$\tau(\omega) \le t \iff \omega[\,[0,t]\,] \cap F \ne \emptyset$$

(because ω is continuous)

$$\iff \omega[\,[0,t]\,] \cap G_n \ne \emptyset \text{ for every } n \in \mathbb{N}$$

(because $\omega[\,[0,t]\,]$ is compact)

$$\iff \text{ for every } n \in \mathbb{N} \text{ there is a rational } q \le t \text{ such that } \omega(q) \in G_n.$$

So

$$\{\omega : \tau(\omega) \le t\} = \bigcap_{n\in\mathbb{N}} \bigcup_{q\in\mathbb{Q}\cap[0,t]} \{\omega : \omega(q) \in G_n\} \in \mathrm{T}_{[0,t]}.$$

Of course $\{\omega : \tau(\omega) = 0\}$ is either Ω (if $0 \in F$) or $\emptyset$ (if $0 \notin F$), so belongs to $\mathrm{T}_{[0,0]}$. **Q**

In the language of 477G, we have $\mathrm{T}_{[0,t]} \subseteq \Sigma_t$ for every $t \ge 0$ (477Hc), so τ must also be adapted to $\langle\Sigma_t\rangle_{t\ge 0}$.

(d) If $A \subseteq \mathbb{R}^r$ is any set, then

$$\mathrm{hp}^*(A) = \min\{\mathrm{hp}(B) : B \supseteq A \text{ is an analytic set}\} = \min\{\mathrm{hp}(E) : E \supseteq A \text{ is a } \mathrm{G}_\delta \text{ set}\}.$$

P Of course

$$\begin{aligned}\mathrm{hp}^*(A) &\le \inf\{\mathrm{hp}(B) : B \supseteq A \text{ is an analytic set}\}\\ &= \min\{\mathrm{hp}(B) : B \supseteq A \text{ is an analytic set}\}\\ &\le \inf\{\mathrm{hp}(E) : E \supseteq A \text{ is a } \mathrm{G}_\delta \text{ set}\} = \min\{\mathrm{hp}(E) : E \supseteq A \text{ is a } \mathrm{G}_\delta \text{ set}\}\end{aligned}$$

just because hp^* is an order-preserving function. If $\gamma > \mathrm{hp}^*(A)$, there is a compact $K \subseteq \Omega$ such that $\omega^{-1}[A] = \emptyset$ for every $\omega \in K$ and $\mu_W K \ge 1 - \gamma$. Now $F = \{\omega(t) : \omega \in K,\, t \in [0,\infty[\}$ is a K_σ set not meeting A, so $E = \mathbb{R}^r \setminus F$ is a G_δ set including A. Since $\omega^{-1}[E]$ is empty for every $\omega \in K$, $\mathrm{hp}(E) \le \mu_W(\Omega \setminus K) \le \gamma$. As γ is arbitrary,

$$\inf\{\mathrm{hp}(E) : E \supseteq A \text{ is a } \mathrm{G}_\delta \text{ set}\} \le \mathrm{hp}^*(A)$$

and we have equality throughout. **Q**

(e) If $A \subseteq \mathbb{R}^r$ is analytic, then $\mathrm{hp}(A) = \sup\{\mathrm{hp}(K) : K \subseteq A \text{ is compact}\}$. **P** Suppose that $\gamma < \mathrm{hp}(A)$. Set $E = \{(\omega,t) : \omega \in \Omega,\, t \ge 0,\, \omega(t) \in A\}$. Then E is analytic and $\mathrm{hp}(A) = \mu_W \pi_1[E]$, where $\pi_1(\omega,t) = \omega$ for $(\omega,t) \in E$. Let λ be the subspace measure $(\mu_W)_{\pi_1[E]}$. By 433D, there is a Radon measure λ' on E such that $\lambda = \lambda'\pi_1^{-1}$. Then $\lambda' E = \mathrm{hp}(A) > \gamma$, so there is a compact set $L \subseteq E$ such that $\lambda' L \ge \gamma$. Set $K = \{\omega(t) : (\omega,t) \in L\}$; then $K \subseteq \mathbb{R}^r$ is compact, and

$$\mathrm{hp}(K) = \mu_W\{\omega : \omega(t) \in K \text{ for some } t \ge 0\} \ge \mu_W \pi_1[L] \ge \lambda' L \ge \gamma.$$

As γ is arbitrary, $\mathrm{hp}(A) \le \sup\{\mathrm{hp}(K) : K \subseteq A \text{ is compact}\}$; the reverse inequality is trivial. **Q**

Remark 477Id-477Ie are characteristic of Choquet capacities (432J-432L); see 478Xe below.

477J As an example of the use of 477G, I give a classical result on one-dimensional Brownian motion.

Proposition Let μ_W be Wiener measure on $\Omega = C([0,\infty[)_0$. Set $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$. Then

$$\Pr(\max_{s\le t} X_s \ge \alpha) = 2\Pr(X_t \ge \alpha) = \frac{2}{\sqrt{2\pi}}\int_{\alpha/\sqrt{t}}^{\infty} e^{-u^2/2}du$$

whenever $t > 0$ and $\alpha \ge 0$.

proof Let τ be the Brownian hitting time to $F = \{x : x \in \mathbb{R},\, x \ge \alpha\}$; because F is closed, τ is a stopping time adapted to $\langle \Sigma_t\rangle_{t\ge0}$, as in 477Ic. Let $\phi_\tau : \Omega\times\Omega \to \Omega$ be the corresponding inverse-measure-preserving function as in 477G, and set $E = \{\omega : \tau(\omega) < t\}$. Note that as $\omega(\tau(\omega)) = \alpha$ whenever $\tau(\omega)$ is finite, $\Pr(\tau = t) \le \Pr(X_t = \alpha) = 0$, and

$$\mu_W E = \Pr(\tau \le t) = \Pr(\max_{s\le t} X_s \ge \alpha).$$

Now

$$\Pr(X_t \ge \alpha) = \mu_W\{\omega : \omega(t) \ge \alpha\} = \mu_W^2\{(\omega,\omega') : \phi_\tau(\omega,\omega')(t) \ge \alpha\}$$
$$= \mu_W^2\{(\omega,\omega') : \tau(\omega) \le t,\ \phi_\tau(\omega,\omega')(t) \ge \alpha\}$$

(because if $\tau(\omega) > t$ then $\phi_\tau(\omega,\omega')(t) = \omega(t) < \alpha$)

$$= \mu_W^2\{(\omega,\omega') : \tau(\omega) < t,\ \phi_\tau(\omega,\omega')(t) \ge \alpha\}$$

(because $\{\omega : \tau(\omega) = t\}$ is negligible)

$$= \mu_W^2\{(\omega,\omega') : \tau(\omega) < t,\ \omega(\tau(\omega)) + \omega'(t-\tau(\omega)) \ge \alpha\}$$
$$= \mu_W^2\{(\omega,\omega') : \tau(\omega) < t,\ \omega'(t-\tau(\omega)) \ge 0\}$$
$$= \int_E \mu_W\{\omega' : \omega'(t-\tau(\omega)) \ge 0\}\mu_W(d\omega)$$
$$= \frac{1}{2}\mu_W E = \frac{1}{2}\Pr(\max_{s\le t} X_s \ge \alpha).$$

To compute the value, observe that X_t has the same distribution as $\sqrt{t}Z$ where Z is a standard normal random variable, so that

$$\Pr(X_t \ge \alpha) = \Pr(Z \ge \tfrac{\alpha}{\sqrt{t}}) = \frac{1}{\sqrt{2\pi}}\int_{\alpha/\sqrt{t}}^{\infty} e^{-u^2/2}du.$$

477K Typical Brownian paths A vast amount is known concerning the nature of 'typical' members of Ω; that is to say, a great many interesting μ_W-conegligible sets have been found. Here I will give only a couple of basic results; the first because it is essential to any picture of Brownian motion, and the second because it is relevant to a question in §479. Others are in 478M, 478Yi and 479R.

Proposition Let μ_W be one-dimensional Wiener measure on $\Omega = C([0,\infty[)_0$. Then μ_W-almost every element of Ω is nowhere differentiable.

proof Note first that if $\eta > 0$ and Z is a standard normal random variable, then $\Pr(|Z| \le \eta) \le \eta$, because the maximum value of the probability density function of Z is $\frac{1}{\sqrt{2\pi}} \le \frac{1}{2}$. For $m, n, k \in \mathbb{N}$, set

$$F_m = \{\omega : \omega \in \Omega \text{ and there is a } t \in [0,m[\text{ such that } \limsup_{s\downarrow 0} \tfrac{|\omega(s)-\omega(t)|}{s-t} < m\},$$

$$E_{mnk} = \{\omega : \omega \in \Omega,\ |\omega(2^{-n}(k+2)) - \omega(2^{-n}(k+1))| \le 3\cdot 2^{-n}m,$$
$$|\omega(2^{-n}(k+3)) - \omega(2^{-n}(k+2))| \le 5\cdot 2^{-n}m,$$
$$|\omega(2^{-n}(k+4)) - \omega(2^{-n}(k+3))| \le 7\cdot 2^{-n}m\},$$

$$E_{mn} = \bigcup_{k<2^n m} E_{mnk}.$$

Now we can estimate the measure of E_{mnk}, because for any α, $t \ge 0$, $2^{n/2}(X_{t+2^{-n}} - X_t)$ has a standard normal distribution (taking $X_t(\omega) = \omega(t)$, as usual), so

$$\Pr(|X_{t+2^{-n}} - X_t| \le \alpha) \le 2^{n/2}\alpha;$$

since E_{mnk} is the intersection of three independent sets of this type,

$$\mu_W E_{mnk} \le 2^{n/2} \cdot 3 \cdot 2^{-n}m \cdot 2^{n/2} \cdot 5 \cdot 2^{-n}m \cdot 2^{n/2} \cdot 7 \cdot 2^{-n}m = 105\,m^3 2^{-3n/2}.$$

Accordingly

$$\mu_W E_{mn} \le \sum_{k<2^n m} \mu_W E_{mnk} \le 105\,m^4 2^{-n/2}.$$

Next, observe that $F_m \subseteq \bigcup_{l\in\mathbb{N}} \bigcap_{n\ge l} E_{mn}$. **P** If $\omega \in F_m$, let $t \in [0, m[$ be such that $\limsup_{s\downarrow 0} \frac{|\omega(s)-\omega(t)|}{s-t} < m$, and $l \in \mathbb{N}$ such that $|\omega(s) - \omega(t)| \le m(s-t)$ whenever $t < s \le t + 4 \cdot 2^{-l}$. Take any $n \ge l$. Then there is a $k < 2^n m$ such that $2^{-n}k \le t < 2^{-n}(k+1)$. In this case,

$$|\omega(2^{-n}(k+j)) - \omega(t)| \le 2^{-n}jm$$

for $1 \le j \le 4$,

$$|\omega(2^{-n}(k+j+1)) - \omega(2^{-n}(k+j))| \le (2j+1)2^{-n}m$$

for $1 \le j \le 3$, and $\omega \in E_{mnk} \subseteq E_{mn}$. **Q**

Since

$$\mu_W(\bigcup_{l\in\mathbb{N}} \bigcap_{n\ge l} E_{mn}) \le \liminf_{n\to\infty} \mu_W E_{mn} = 0,$$

F_m is negligible. So $F = \bigcup_{m\in\mathbb{N}} F_m$ is negligible. But F includes any member of ω which is differentiable at any point of $]0, \infty[$, and more. So almost every path is nowhere differentiable.

477L Theorem Let $r \ge 1$ be an integer, and μ_W Wiener measure on $\Omega = C([0, \infty[; \mathbb{R}^r)_0$; for $s > 0$ let μ_{Hs} be s-dimensional Hausdorff measure on $\mathbb{R}^r$.

(a) (Taylor 53) $\{\omega(t) : t \in [0, \infty[\}$ is μ_{H2}-negligible for μ_W-almost every ω.

(b) Now suppose that $r \ge 2$. For $\omega \in \Omega$, let F_ω be the compact set $\{\omega(t) : t \in [0, 1]\}$. Then for μ_W-almost every $\omega \in \Omega$, $\mu_{Hs} F_\omega = \infty$ for every $s \in]0, 2[$.

proof (a)(i) For $0 \le s \le t$ and $\omega \in \Omega$ set $K_{st}(\omega) = \{\omega(u) : s \le u \le t\}$ and $d_{st}(\omega) = \operatorname{diam} K_{st}(\omega)$. Note that $d_{st} : \Omega \to [0, \infty[$ is continuous (for the topology of uniform convergence on compact sets, of course).

(α) If $0 \le s \le t$ then $\mathbb{E}(d_{st}^2) \le 8r(t-s)$. **P** As $\langle X_{u+s} - X_s\rangle_{u\ge 0}$ and $\langle X_u\rangle_{u\ge 0}$ have the same distribution, d_{st} has the same distribution as $d_{0,t-s}$, and we may suppose that $s = 0$. [If you prefer: if $S_s : \Omega \to \Omega$ is the shift operator of 477Ec, $K_{st}(\omega) = \omega(s) + K_{0,t-s}(S_s\omega)$, so $d_{st}(\omega) = d_{0,t-s}(S_s\omega)$, while S_s is inverse-measure-preserving.] In this case,

$$d_{0t}(\omega)^2 \le 4 \max_{s\in[0,t]} \|\omega(s)\|^2 \le 4\sum_{j=0}^{r-1} \max_{s\in[0,t]} \omega(s)(j)^2.$$

For each $j < r$,

$$\begin{aligned}\int \max_{s\in[0,t]} \omega(s)(j)^2 \mu_W(d\omega) &= \int_0^\infty \mu_W\{\omega : \max_{s\in[0,t]} \omega(s)(j)^2 \ge \beta\} d\beta \\ &\le \int_0^\infty \mu_W\{\omega : \max_{s\in[0,t]} \omega(s)(j) \ge \sqrt{\beta}\} \\ &\qquad + \mu_W\{\omega : \min_{s\in[0,t]} \omega(s)(j) \le -\sqrt{\beta}\} d\beta \\ &= 2\int_0^\infty \mu_W\{\omega : \max_{s\in[0,t]} \omega(s)(j) \ge \sqrt{\beta}\} d\beta\end{aligned}$$

(because μ_W is invariant under reflections in $\mathbb{R}^r$, see 477Ed)

$$= 4\int_0^\infty \mu_W\{\omega : \omega(t)(j) \ge \sqrt{\beta}\} d\beta$$

(by 477J, applied to the jth coordinate projection of Ω onto $C([0, \infty[)_0$, which is inverse-measure-preserving, by 477Da or 477Ed and 477Eg)

$$= 2\int_0^\infty \mu_W\{\omega : \omega(t)(j)^2 \ge \beta\}d\beta$$

(again because μ_W is symmetric)

$$= 2\int_\Omega \omega(t)(j)^2 \mu_W(d\omega) = 2\mathbb{E}(tZ^2)$$

(where Z is a standard normal random variable)

$$= 2t.$$

Summing,

$$\mathbb{E}(d_{0t}^2) \le 4\sum_{j=0}^{r-1} \int_\Omega \max_{s\in[0,t]} \omega(s)(j)^2 \mu_W(d\omega) \le 8rt. \text{ } \mathbf{Q}$$

(β) For any $\epsilon > 0$, $\Pr(d_{01} \le \epsilon) > 0$. **P** $\{\omega : d_{01}(\omega) \le \epsilon\}$ is a neighbourhood of 0 for the topology of uniform convergence on compact sets, so has non-zero measure, by 477F. **Q**

(ii) For a non-empty finite set $I \subseteq [0,\infty[$ and $\omega \in \Omega$ set

$$g_I(\omega) = \sum_{j=0}^{n-1} d_{t_{j-1},t_j}(\omega)^2$$

where $\langle t_j \rangle_{j\le n}$ enumerates I in increasing order. For $0 \le s \le t$ and $\omega \in \Omega$ set

$$h_{st}(\omega) = \inf_{\{s,t\}\subseteq I\subseteq[s,t] \text{ is finite}} g_I(\omega);$$

then h_{st} is $\mathrm{T}_{[s,t]}$-measurable, in the language of 477H. **P** The point is that $I \mapsto g_I(\omega)$ is a continuous function of the members of I, at least if we restrict attention to sets I of a fixed size. So if D is any countable dense subset of $[s,t]$ containing s and t,

$$h_{st} = \inf_{\{s,t\}\subseteq I\subseteq D \text{ is finite}} g_I.$$

On the other hand, if $I \subseteq D$ is enumerated as $\langle t_i \rangle_{i\le n}$,

$$g_I(\omega) = \sum_{i=0}^{n-1} \max_{u,u'\in D\cap[t_i,t_{i+1}]} \|\omega(u) - \omega(u')\|^2,$$

so g_I is $\mathrm{T}_{[s,t]}$-measurable. **Q**

(iii) We need the following facts about the h_{st}.

(α) If $0 \le s \le t$, then the distribution of h_{st} is the same as the distribution of $h_{0,t-s}$, again because $\langle X_{s+u} - X_s \rangle_{u\ge0}$ has the same distribution as $\langle X_u \rangle_{u\ge0}$. [In the language suggested in the proof of (i-α), we have $g_{s+I}(\omega) = g_I(S_s(\omega))$ for any $\omega \in \Omega$ and non-empty finite $I \subseteq [0,\infty[$, so $h_{st}(\omega) = h_{0,t-s}(S_s\omega)$.]

(β) h_{0t} has finite expectation. **P** $h_{0t} \le g_{\{0,t\}} = d_{0t}^2$, so we can use (i). **Q**

(γ) $h_{su} \le h_{st} + h_{tu}$ if $s \le t \le u$. **P** If $\{s,t\} \subseteq I \subseteq [s,t]$ and $\{t,u\} \subseteq J \subseteq [t,u]$ then $\{s,u\} \subseteq I \cup J \subseteq [s,u]$ and $g_{I\cup J} = g_I + g_J$. **Q**

(δ) If $s \le t \le u$ then h_{st} and h_{tu} are independent, because $\mathrm{T}_{[s,t]}$ and $\mathrm{T}_{[t,u]}$ are independent (477H(b-i)).

(ϵ) The distribution of h_{0t} is the same as the distribution of th_{01} whenever $t \ge 0$. **P** The case $t = 0$ is trivial. For $t > 0$, define $U_t : \Omega \to \Omega$ by saying that $U_t(\omega)(s) = \frac{1}{\sqrt{t}}\omega(ts)$, as in 477Ee. Then

$$K_{su}(U_t(\omega)) = \frac{1}{\sqrt{t}}K_{ts,tu}(\omega), \quad d_{su}(U_t(\omega)) = \frac{1}{\sqrt{t}}d_{ts,tu}(\omega),$$

$$g_I(U_t(\omega)) = \frac{1}{t}g_{tI}(\omega), \quad h_{su}(U_t(\omega)) = \frac{1}{t}h_{ts,tu}(\omega),$$

whenever $s \le u$, $\{s,u\} \subseteq I \subseteq [s,u]$ and $\omega \in \Omega$, and

$$\mu_W\{\omega : th_{01}(\omega) \ge \alpha\} = \mu_W\{\omega : th_{01}(U_t(\omega)) \ge \alpha\}$$

(because U_t is an automorphism of (Ω, μ_W))

$$= \mu_W\{\omega : h_{0t}(\omega) \ge \alpha\}$$

for every $\alpha \in \mathbb{R}$. **Q**

(ζ) Consequently

$$\mathbb{E}(h_{st}) = \mathbb{E}(h_{0,t-s}) = (t-s)\mathbb{E}(h_{01})$$

whenever $s \le t$, and

$$\mathbb{E}(h_{st}) + \mathbb{E}(h_{tu}) = \mathbb{E}(h_{su})$$

whenever $s \le t \le u$. Since $h_{st} + h_{tu} \ge h_{su}$, by (iii), we must have $h_{st} + h_{tu} =_{\text{a.e.}} h_{su}$.

(η) For any $\eta > 0$, $\Pr(h_{01} \le 4\eta^2) > 0$. **P** By (i-$\beta$), $\Pr(d_{01} \le 2\eta) > 0$, and $h_{01} \le g_{\{0,1\}} = d_{01}^2$. **Q**

(iv) For $t \ge 0$ let ϕ_t be the characteristic function of h_{0t}, that is, $\phi_t(\alpha) = \mathbb{E}(\exp(i\alpha h_{0t}))$ for $\alpha \in \mathbb{R}$ (285Ab). Working through the facts listed above, we see that

$$\phi_t(1) = \mathbb{E}(\exp(ih_{0t})) = \mathbb{E}(\exp(ith_{01}))$$

(by (iii-ϵ))

$$= \phi_1(t),$$

$$\phi_1(s)\phi_1(t) = \phi_s(1)\phi_t(1) = \mathbb{E}(\exp(ih_{0s}))\mathbb{E}(\exp(ih_{0t}))$$

$$= \mathbb{E}(\exp(ih_{0s}))\mathbb{E}(\exp(ih_{s,s+t}))$$

(by (iii-α))

$$= \mathbb{E}(\exp(ih_{0s})\exp(ih_{s,s+t}))$$

(because h_{0s} and $h_{s,s+t}$ are independent, by (c-iv))

$$= \mathbb{E}(\exp(i(h_{0s} + h_{s,s+t}))) = \mathbb{E}(ih_{0,s+t})$$

(by (iii-ζ))

$$= \phi_1(s+t),$$

for all s, $t \ge 0$; while ϕ_1 is differentiable, because h_{01} has finite expectation ((iii-α) above and 285Fd). It follows that there is a $\gamma \in \mathbb{R}$ such that $\phi_1(t) = e^{i\gamma t}$ for every $t \in \mathbb{R}$. **P** Set $\gamma = \frac{1}{i}\phi_1'(0) = \mathbb{E}(h_{01})$ (285Fd) and $\psi(t) = e^{-i\gamma t}\phi_1(t)$ for $t \in \mathbb{R}$. If $t > 0$, then

$$\phi_1'(t) = \lim_{s\downarrow 0}\frac{1}{s}(\phi_1(t+s) - \phi_1(t)) = \phi_1(t)\lim_{s\downarrow 0}\frac{1}{s}(\phi_1(s) - 1) = i\phi_1(t)\gamma$$

and $\psi'(t) = 0$. Since ψ is continuous on $[0,\infty[$ (285Fb), it must be constant, and $\phi_1(t) = e^{i\gamma t}\psi(0) = e^{i\gamma t}$ for every $t \ge 0$. As for negative t, we have

$$\phi_1(t) = \overline{\phi_1(-t)} = \overline{e^{-i\gamma t}} = e^{i\gamma t}$$

for $t \le 0$, by 285Fc. **Q**

(v) Thus we see that h_{01} has the same characteristic function as the distribution concentrated at γ, and this must therefore be the distribution of h_{01} (285M); that is, $h_{01} =_{\text{a.e.}} \gamma$. Now (iii-$\eta$) tells us that $\gamma = 0$.

Since h_{0t} has the same distribution as th_{01}, $h_{0t} =_{\text{a.e.}} 0$ for every $t \ge 0$. But now observe that if $t \ge 0$, $\omega \in \Omega$ and $h_{0t}(\omega) = 0$, then for any $\eta > 0$ there is a finite $I \subseteq [0,t]$, containing 0 and t, such that $g_I(\omega) \le \eta^2$. This means that $K_{0t}(\omega)$ can be covered by finitely many sets $K_{t_j,t_{j+1}}(\omega)$ with $\sum_{j=0}^{n-1}\operatorname{diam} K_{t_j,t_{j+1}}(\omega)^2 \le \eta^2$. All the diameters here must of course be less than or equal to η. As η is arbitrary, $\mu_{H2}K_{0t}(\omega) = 0$.

For each $t \ge 0$, this is true for almost every ω. But this means that, for almost every ω, $\mu_{H2}K_{0n}(\omega) = 0$ for every n, and $\mu_{H2}\{\omega(t) : t \ge 0\} = 0$, as claimed.

(b)(i) To begin with, take a fixed $s \in\,]0,2[$. Let μ_{L1} be Lebesgue measure on $[0,1]$. For each $\omega \in \Omega$, let ζ_ω be the image measure $\mu_{L1}(\omega\restriction[0,1])^{-1}$ on F_ω. Then

$$\int_\Omega\int_{F_\omega}\int_{F_\omega}\frac{1}{\|x-y\|^s}\zeta_\omega(dx)\zeta_\omega(dy)\mu_W(d\omega)$$
$$=\int_\Omega\int_0^1\int_0^1\frac{1}{\|\omega(t)-\omega(u)\|^s}dt\,du\,\mu_W(d\omega)$$
$$=\int_0^1\int_0^1\int_\Omega\frac{1}{\|\omega(t)-\omega(u)\|^s}\mu_W(d\omega)\,dt\,du$$

(of course $(\omega,t,u)\mapsto\frac{1}{\|\omega(t)-\omega(u)\|^s}$ is continuous and non-negative, so there is no difficulty with the change in order of integration)

$$=2\int_0^1\int_u^1\int_\Omega\frac{1}{\|\omega(t)-\omega(u)\|^s}\mu_W(d\omega)\,dt\,du$$
$$=2\int_0^1\int_u^1\int_\Omega\frac{1}{\|\omega(t-u)\|^s}\mu_W(d\omega)\,dt\,du$$

(because X_t-X_u has the same distribution as X_{t-u}, as in (a-i-α))

$$=2\int_0^1\int_0^{1-u}\int_\Omega\frac{1}{\|\omega(t)\|^s}\mu_W(d\omega)\,dt\,du\le 2\int_0^1\int_\Omega\frac{1}{\|\omega(t)\|^s}\mu_W(d\omega)\,dt$$
$$=2\int_0^1\int_{\mathbb{R}^r}\frac{1}{(\sqrt{2\pi t})^r}\frac{1}{\|x\|^s}e^{-\|x\|^2/2t}\mu(dx)\,dt$$

(here μ is Lebesgue measure on $\mathbb{R}^r$)

$$=\frac{2}{(\sqrt{2\pi})^r}\int_0^1\frac{1}{t^{r/2}}\int_0^\infty\frac{1}{\alpha^s}\cdot r\beta_r\alpha^{r-1}e^{-\alpha^2/2t}d\alpha\,dt$$
$$=\frac{2r\beta_r}{(\sqrt{2\pi})^r}\int_0^1\frac{1}{t^{r/2}}\int_0^\infty\frac{(\beta\sqrt{t})^{r-1}}{(\beta\sqrt{t})^s}e^{-\beta^2/2}\sqrt{t}d\beta\,dt$$
$$=\frac{2r\beta_r}{(\sqrt{2\pi})^r}\int_0^1\frac{1}{t^{s/2}}dt\int_0^\infty\beta^{r-s-1}e^{-\beta^2/2}d\beta<\infty$$

because $\frac{s}{2}<1$ and $r-s>0$. So $\int_{F_\omega}\int_{F_\omega}\frac{1}{\|x-y\|^s}\zeta_\omega(dx)\zeta_\omega(dy)$ is finite for almost every ω. Since ζ_ω is always a probability measure with support included in F_ω, $\mu_{Hs}F_\omega=\infty$ for all such ω (471Tb).

(ii) Setting $s_n=2-2^{-n}$ for each n, we see that, for almost every $\omega\in\Omega$, $\mu_{Hs_n}F_\omega=\infty$ for every n. But for any such ω, $\mu_{Hs}F_\omega=\infty$ for every $s\in\,]0,2[$, by 471L.

477X Basic exercises (a) Use 272Yc[6] to simplify the formulae in the proof of 477B.

(b) Let μ_W be Wiener measure on $\Omega=C([0,\infty[)_0$, and set $X_t(\omega)=\omega(t)$ for $t\ge 0$ and $\omega\in\Omega$. Let f be a real-valued tempered function on $\mathbb{R}$ (definition: 284D). For $x\in\mathbb{R}$ and $0<t<b$, let $\nu_x^{(t,b)}$ be the distribution of a normally distributed random variable with mean x and variance $b-t$, so that $g(x,t)=\int f(y)\nu_x^{(t,b)}(dy)$ can be regarded as the expectation of $f(X_b)$ given that $X_t=x$. (i) Show that g satisfies the **backwards heat equation** $2\frac{\partial g}{\partial t}+\frac{\partial^2 g}{\partial x^2}=0$. (ii) Interpret this in terms of the disintegration $\nu_x^{(t,b)}=\int\nu_z^{(u,b)}\nu_x^{(t,u)}(dz)$ as $u\downarrow t$.

(c)(i) Show that the measure $\hat\mu^r$ of 477Da can be constructed directly by applying 455A with $(X_t,\mathcal{B}_t)=(\mathbb{R}^r,\mathcal{B}(\mathbb{R}^r))$ for every $t\ge 0$ and suitable Gaussian distributions $\nu_x^{(s,t)}$ on $\mathbb{R}^r$. (ii) Show that the measure $\hat\mu^r$ can be constructed by applying 455A to $T=r\times[0,\infty[$ with its lexicographic ordering and suitable Gaussian distributions $\nu_x^{(s,t)}$ on $\mathbb{R}$.

(d) Let $r\ge 1$ be an integer. (i) Show that there is a centered Gaussian process $\langle Y_t\rangle_{t\in[0,1]}=\langle Y_t^{(i)}\rangle_{t\in[0,1],i<r}$ such that $\mathbb{E}(Y_s^{(i)}\times Y_t^{(j)})=0$ if $i\ne j$, $\min(s,t)-st$ otherwise. (ii) Show that if $\langle X_t\rangle_{t\ge 0}$ is ordinary r-dimensional

[6] Formerly 272Ye.

Brownian motion, then $\langle Y_t\rangle_{t\in[0,1]}$ has the same distribution as $\langle X_t - tX_1\rangle_{t\in[0,1]}$. (iii) Show that the process $\langle Y_t\rangle_{t\in[0,1]}$ (the **Brownian bridge**) can be represented by a Radon probability measure μ_{bridge} on the space $C([0,1];\mathbb{R}^r)_{00}$ of continuous functions from $[0,1]$ to $\mathbb{R}^r$ taking the value 0 at both ends of the interval. (iv) For $\omega\in C([0,1];\mathbb{R}^r)_{00}$ define $\overleftrightarrow{\omega}\in C([0,1];\mathbb{R}^r)_{00}$ by setting $\overleftrightarrow{\omega}(t)=\omega(1-t)$ for $t\in[0,1]$. Show that $\omega\mapsto\overleftrightarrow{\omega}$ is an automorphism of $(C([0,1];\mathbb{R}^r)_{00},\mu_{\text{bridge}})$.

(e) Let (Ω,Σ,μ) be a complete probability space and $\langle X_t\rangle_{t\ge0}$ a family of real-valued random variables on Ω with independent increments. For $I\subseteq[0,\infty[$ let T_I be the σ-algebra generated by $\{X_s-X_t : s,\,t\in I\}$. Let $\mathcal{I}$ be a family of subsets of $[0,\infty[$ such that for all distinct $I,\,J\in\mathcal{I}$ either $\sup I\le\inf J$ or $\sup J\le\inf I$. Show that $\langle\mathrm{T}_I\rangle_{I\in\mathcal{I}}$ is an independent family of σ-algebras.

(f) Suppose that $H\subseteq\mathbb{R}^r$ is an F_σ set and that $\tau:\Omega\to[0,\infty]$ is the Brownian hitting time to H, as defined in 477I. Show that τ is Borel measurable.

(g) Let μ_W be one-dimensional Wiener measure, and τ the hitting time to $\{1\}$. Show that the distribution of τ has probability density function $x\mapsto\frac{1}{x\sqrt{2\pi x}}e^{-1/2x}$ for $x>0$.

(h) Let μ_W be one-dimensional Wiener measure on $\Omega=C([0,\infty[)_0$. Show that, for μ_W-almost every $\omega\in\Omega$, the total variation $\mathrm{Var}_{[s,t]}(\omega)$ is infinite whenever $0\le s<t$.

477Y Further exercises (a) Write D_n for $\{2^{-n}i : i\in\mathbb{N}\}$ and $D=\bigcup_{n\in\mathbb{N}}D_n$, the set of dyadic rationals in $[0,\infty[$. For $d\in D$ define $f_d\in C([0,\infty[)$ as follows. If $n\in\mathbb{N}$, $f_n(t)=0$ if $t\le n$, $t-n$ if $n\le t\le n+1$, 1 if $t\ge n+1$. If $d=2^{-n}k$ where $n\ge1$ and $k\in\mathbb{N}$ is odd, $f_d(t)=\max(0,\frac{1}{\sqrt{2^{n+1}}}(1-2^n|t-d|))$ for $t\ge0$. Now let $\langle Z_d\rangle_{d\in D}$ be an independent family of standard normal distributions, and set $\omega_n(t)=\sum_{d\in D_n}f_d(t)Z_d$ for $t\ge0$, so that each ω_n is a random continuous function on $[0,\infty[$. Show that for any $n\in\mathbb{N}$ and $\epsilon>0$,

$$\Pr(\sup_{t\in[0,n]}|\omega_{n+1}(t)-\omega_n(t)|\ge\epsilon)\le 2^n n\cdot\frac{2}{\sqrt{2\pi}}\int_{2\epsilon\sqrt{2^n}}^\infty e^{-x^2/2}dx,$$

and hence that $\langle\omega_n\rangle_{n\in\mathbb{N}}$ converges almost surely to a continuous function. Explain how to interpret this as a construction of Wiener measure on $\Omega=C([0,\infty[)_0$, as the image measure $\mu_G^Dg^{-1}$ where $g:\mathbb{R}^D\to\Omega$ is almost continuous (for the topology $\mathfrak{T}_c$ on Ω) and μ_G^D is the product of copies of the standard normal distribution μ_G.

(b) Fix $p\in\,]0,1[\,\setminus\{\frac12\}$. (i) For $\alpha\in\mathbb{R}$ set $h_\alpha(t)=|t-\alpha|^{p-\frac12}-|t|^{p-\frac12}$ when this is defined. Show that $h_\alpha\in\mathcal{L}^2(\mu_L)$, where μ_L is Lebesgue measure, and that $\|h_\alpha\|_2^2=|\alpha|^{2p}\|h_1\|_2^2$ and $\|h_\alpha-h_\beta\|_2=\|h_{\alpha-\beta}\|_2$ for all $\alpha,\,\beta\in\mathbb{R}$. (ii) Show that there is a centered Gaussian process $\langle X_\alpha\rangle_{\alpha\in\mathbb{R}}$ such that $\mathbb{E}(X_\alpha\times X_\beta)=|\alpha|^{2p}+|\beta|^{2p}-|\alpha-\beta|^{2p}$ for all $\alpha,\,\beta\in\mathbb{R}$. (iii) Show that such a process can be represented by a Radon measure on $C(\mathbb{R})$. (*Hint*: 477Ya.) (This is **fractional Brownian motion**.)

(c) Let μ_W be Wiener measure on $\Omega=C([0,\infty[\,;\mathbb{R}^r)_0$, where $r\ge1$, and set $X_t(\omega)=\omega(t)$ for $t\ge0$ and $\omega\in\Omega$. Let f be a real-valued tempered function on $\mathbb{R}^r$ (definition: 284Wa). For $x\in\mathbb{R}^r$ and $0<t<b$, let $\nu_x^{(t,b)}$ be the distribution of $x+X_{b-t}$, so that $g(x,t)=\int f(y)\nu_x^{(t,b)}(dy)$ can be regarded as the expectation of $f(X_b)$ given that $X_t=x$. Show that g satisfies the backwards heat equation $2\frac{\partial g}{\partial t}+\sum_{i=0}^{r-1}\frac{\partial^2 g}{\partial\xi_i^2}=0$.

(d) Let $r\ge1$ be an integer, and ν a Radon probability measure on $\mathbb{R}^r$ such that $x\mapsto a\,.\,x$ has expectation 0 and variance $\|a\|^2$ for every $a\in\mathbb{R}^r$. Let Ω be $C([0,\infty[\,;\mathbb{R}^r)_0$, and for $\alpha>0$ define $f_\alpha:(\mathbb{R}^r)^{\mathbb{N}}\to\Omega$ by setting $f_\alpha(x)(t)=\sqrt{\alpha}(\sum_{i<n}z(i)+\frac{1}{\alpha}(t-n\alpha)z(n))$ when $z\in(\mathbb{R}^r)^{\mathbb{N}}$, $n\in\mathbb{N}$ and $n\alpha\le t\le(n+1)\alpha$; let μ_α be the image Radon measure $\nu^{\mathbb{N}}f_\alpha^{-1}$ on Ω. Show that Wiener measure μ_W is the limit $\lim_{\alpha\downarrow0}\mu_\alpha$ for the narrow topology.

(e) Let μ_W be Wiener measure on $C([0,\infty[)_0$, and $\gamma>\frac12$. Show that $\lim_{t\to\infty}\frac{1}{t^\gamma}\omega(t)=\lim_{t\downarrow0}\frac{1}{t^{1-\gamma}}\omega(t)=0$ for μ_W-almost every ω.

(f) Write out a proof of 477G which works directly from the Gaussian-distribution characterization of Wiener measure, without appealing to results from §455 other than 455L. (I think you will need to start with stopping times taking finitely and countably many values, as in 455C; but you will find great simplifications.)

(g) Let $\hat{\mu}$ be the Gaussian distribution on $\mathbb{R}^{[0,\infty[}$ corresponding to Brownian motion, as in 477A. For $t \ge 0$ let Σ_t be the family of Baire subsets of $\mathbb{R}^{[0,\infty[}$ determined by coordinates in $[0,t]$, and $\Sigma_t^+ = \bigcap_{s>t} \Sigma_s$. For $\omega \in \mathbb{R}^{[0,\infty[}$ set $\tau(\omega) = \inf\{q : q \in \mathbb{Q},\, \omega(q) \ge 1\}$, counting $\inf\emptyset$ as ∞. Show that τ is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$. For ω, $\omega' \in \mathbb{R}^{[0,\infty[}$ define $\phi_\tau(\omega,\omega') \in \mathbb{R}^{[0,\infty[}$ by setting $\phi_\tau(\omega,\omega')(t) = \omega(t)$ if $t \le \tau(\omega)$, $\omega(\tau(\omega)) + \omega'(t-\tau(\omega))$ if $t > \tau(\omega)$. Show that ϕ_τ is not inverse-measure-preserving for the product measure $\hat{\mu}^2$ and $\hat{\mu}$. (*Hint*: show that $\{\omega : \tau(\omega) \in D\}$ is negligible for every countable set D.)

(h) Let μ_W be one-dimensional Wiener measure on $\Omega = C([0,\infty[)_0$. (i) Show by induction on k that Pr(there are $t_0 < t_1 < \ldots < t_k \le t$ such that $X_{t_j} = (-1)^j$ for every $j \le k$) = Pr(there is an $s \le t$ such that $X_s = 2k+1$) for any $t \ge 0$. (*Hint*: 477J.) (ii) For $\omega \in \Omega$, $k \in \mathbb{N}$ define $\tau_k(\omega)$ by saying that $\tau_0(\omega) = \inf\{t : |\omega(t)| = 1\}$, $\tau_{k+1}(\omega) = \inf\{t : t > \tau_k(\omega),\, \omega(t) = -\omega(\tau_k(\omega))\}$. Show that τ_k is a stopping time adapted to the family $\langle\Sigma_t\rangle_{t\ge0}$ of 477H, and is finite μ_W-a.e. (iii) Set $E_k = \{\omega : \tau_k(\omega) \le 1 < \tau_{k+1}(\omega)\}$, $p_k = \mu_W E_k$, $F_k = \{\omega :$ there is an $s \le 1$ such that $\omega(s) = 2k+1\}$, $q_k = \mu_W F_k$. Show that

$$q_k = \frac{1}{2}p_k + \sum\nolimits_{j=k+1}^{\infty} p_j = \frac{2}{\sqrt{2\pi}}\int_{2k+1}^{\infty} e^{-x^2/2}dx.$$

(iv) Show that $\Pr(\tau_0 \le 1) = \sum_{k=0}^{\infty} p_k = 2\sum_{k=0}^{\infty}(-1)^k q_k$.

(i) Let μ_W be Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$, where $r \ge 1$. Show that, for μ_W-almost every $\omega \in \Omega$, $\{\frac{\omega(t)}{\|\omega(t)\|} : t \ge t_0,\, \omega(t) \ne 0\}$ is dense in $\partial B(\mathbf{0},1)$ for every $t_0 \ge 0$.

(j)(i) Show that, for any $r \ge 1$, the topology $\mathfrak{T}_c$ of uniform convergence on compact sets is a complete linear space topology on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$. (ii) Show that Wiener measure on Ω is a centered Gaussian measure in the sense of 466N. (*Hint*: 466Ye.)

477 Notes and comments The 'first proof' of 477A calls on a result which is twenty-five pages into §455, and you will probably be glad to be assured that all it really needs is 455A, fragments from the proof of 455P, and part (a) of the proof of 455R. So it is not enormously harder than the 'second proof', based on the elementary theory of Gaussian processes.

With this theorem, we have two routes to the first target: setting up a measure space with a family of random variables representing Brownian motion. I repeat that this is a secondary issue. Brownian motion begins with the family of joint distributions of finite indexed sets $(X_{t_0},\ldots,X_{t_n})$ satisfying the properties listed in 477A. It is one of the triumphs of Kolmogorov's theory of probability that these distributions can be represented by a family of real-valued functions on a set with a countably additive measure; but they would still be of the highest importance if they could not. In order to show that it can be done, we can use either the time-dependent approach based on conditional expectations, as in 455A, 455P and the 'first proof' of 477A, or the timeless Gaussian-distribution approach through 456C, as in the 'second proof' of 477A. Both, of course, depend on Kolmogorov's theorem 454D. They have different advantages, and it will be very useful to be able to call on the intuitions of both. The 'first proof' leads us naturally into the theory of Lévy processes, in which other families of distributions replace normal distributions.

To get to the continuity of sample paths, we need to do quite a bit more, and the proof of 477B is one way of filling the gap. At this point it becomes tempting to abandon both proofs of 477A and start again with the method of 477Ya, the 'Lévy-Ciesielski' construction, *not* using Kolmogorov's theorem. But if we do this, we shall have to devise a new argument to prove the strong Markov property 477G, rather than quoting 455U. Of course the special properties of Gaussian processes mean that a direct proof of 477G is still quite a bit easier than the general results of §455 (477Yf). I make no claim that one approach is 'better' than another; they all throw light on the result.

What I here call 'Wiener measure' (477B) is a particular realization of Brownian motion. It is so convincing that it is tempting to regard it as 'the' real basis of Brownian motion. I do not mean to assert this in any way which will bind me in future. But (as a measure theorist, rather than a probabilist) I think that the specific measures of 477B and 477D are worth as much attention as any. One reason for not insisting that the space $C([0,\infty[)_0$ is the only right place to start is that we may at any moment wish to move to something smaller, as in 477Ef. The approach here gives a very direct language in which to express theorems of the form 'almost every Brownian path is …' (477K, 477L), and every such theorem carries an implicit suggestion that we could move to a conegligible set and a subspace measure.

In 477C I sketch an alternative characterization of one-dimensional Wiener measure. Five pages seem to be rather a lot for a proof of something which surely has to be true, if we can get the hypotheses right; but I do not see a genuinely shorter route, and I think in fact that the indigestibility of the argument as presented is due to compression more than to pedantry. At least I have tried to put the key step into part (a-i) of the proof. We have to use the Central Limit Theorem; we have to use a finite-approximation version of it, rather than a limit version; the ideas of this proof do not demand Lindeberg's formulation, but this is what we have to hand in Volume 2; and if we are going to consider interpolations for general random walks, we need something to force a sufficient degree of equicontinuity, and (†) is what comes naturally from the result in 274F. It is surely obvious that I have been half-hearted in the generality of the theorem as given. There can be no reason for insisting on steps being at uniform time intervals, or on stationary processes, or even on variances being exactly correct, provided that everything averages out nicely in the limit. The idea does require that steps be independent, but after that we just need hypotheses adequate for the application of the Central Limit Theorem.

Clearly we can also look for r-dimensional versions of the theorem. I have not done so because they would inevitably demand vector-valued versions of the Central Limit Theorem, and while a combination of the ideas of §§274 and 456 would take us a long way, it would not belong to this section. However I give 477Yd as an example which can be dealt with without much general theory.

Already in the elementary results 477Eb-477Eg we see that Wiener measure is a remarkable construction. It is a general principle that the more symmetries an object has, the more important it is; this one has a surprising symmetry (477Ef), which is even better. I take it as confirmation that we have a good representation, that all these symmetries can be represented by actual inverse-measure-preserving functions, rather than leaving them as manipulations of distributions.

477F is a natural result, and a further confirmation that in $C([0,\infty[\,;\mathbb{R}^r)_0$ we have got hold of an appropriate space of functions. The proof I give depends on an aspect of the structure developed in 477Ya.

The next really important result is the 'strong Markov property' (477G). This is clearly a central property of Brownian motion. It may not be quite so clear what the formulation here is trying to say. As in 477E, I am expressing the result in terms of an inverse-measure-preserving function. This makes no sense unless we have a probability space Ω in which we can put two elements ω, ω' together to form a third; so we are more or less forced to look at a space of paths. But not all spaces of paths will do. For an indication of what can happen if we work with the wrong realization, see 477Yg.

In 477H we have two kinds of zero-one law. One, 477Hd, is explicit; the tail σ-algebra $\bigcap_{t\ge 0}\hat{\mathrm{T}}_{[t,\infty[}$ behaves like the tail of an independent sequence of σ-algebras (272O). But the formula '$\bigcap_{s>t}\Sigma_s\subseteq\hat{\mathrm{T}}_{[0,t]}$' in 477Hc can be thought of as a relative zero-one law. There are many events (e.g., $\{\omega:\liminf_{s\downarrow t}\frac{1}{s-t}\omega(s)\ge 0\}$) which belong to $\bigcap_{s>t}\Sigma_s$ and not to $\mathrm{T}_{[0,t]}$ or Σ_t, but all collapse to events in $\mathrm{T}_{[0,t]}$ if we rearrange them on appropriate negligible sets. This is really a special case of 455T.

The formulae in the first application of the strong Markov property (477J) demand a bit of concentrated attention, but I think that the key step at the end of the proof (moving from μ_W^2 to $\int\ldots d\mu_W$) faithfully represents the intuition: once we have reached the level α, we have an even chance of rising farther. For the discrete case, 272Yc is a version of the same idea. From the distribution of the hitting time to $\{\alpha\}$ we can deduce the distribution of the hitting time to $\{-1,1\}$ (477Yh); but I do not know of a corresponding exact result for the hitting time to the unit circle for two-dimensional Brownian motion.

I expect you have been shown a continuous function which is nowhere differentiable. In 477K we see that 'almost every' function is of this type. What a hundred and fifty years ago seemed to be an exotic counter-example now presents itself as a representative of the typical case. The very crude estimates in the proof of 477K are supposed to furnish a straightforward proof of the result, without asking for anything which might lead to refinements. Of course there is much more to be said, starting with 477Xh. In 477L we have an interesting result which will be useful in §479, when I return to geometric properties of Brownian paths.

478 Harmonic functions

In this section and the next I will attempt an introduction to potential theory. This is an enormous subject and my choice of results is necessarily somewhat arbitrary. My principal aim is to give the most elementary properties of Newtonian capacity, which will appear in §479. It seems that this necessarily involves a basic understanding of harmonic and superharmonic functions. I approach these by the 'probabilistic' route, using Brownian motion as described in §477.

The first few paragraphs, down to 478J, do not in fact involve Brownian motion; they rely on multidimensional advanced calculus and on the Divergence Theorem. (The latter is applied only to continuously differentiable functions and domains of very simple types, so we need far less than the quoted result in 475N.) Defining 'harmonic function' in terms of average values over concentric spherical shells (478B), the first step is to identify this with the definition in terms of the Laplacian differential operator (478E). An essential result is a formula for a harmonic function inside a sphere in terms of its values on the boundary and the 'Poisson kernel' (478Ib), and we also need to understand the effects of smoothing by convolution with appropriate functions (478J, following 473D-473E). I turn to Brownian motion with Dynkin's formula (478K), relating the expected value of $f(X_\tau)$ for a stopped Brownian process X_τ to an integral in terms of $\nabla^2 f$. This is already enough to deal with the asymptotic behaviour of Brownian paths, which depends in a striking way on the dimension of the space (478M).

We can now approach Dirichlet's problem. If we have a bounded open set $G \subseteq \mathbb{R}^r$, there is a family $\langle \mu_x \rangle_{x \in G}$ of probability measures such that whenever $f : \overline{G} \to \mathbb{R}$ is continuous and $f \restriction G$ is harmonic, then $f(x) = \int f d\mu_x$ for every $x \in G$ (478Pc). So this family of 'harmonic measures' gives a formula continuously extending a function on ∂G to a harmonic function on G, whenever such an extension exists (478S). The method used expresses μ_x in terms of the distribution of points at which Brownian paths starting at x strike ∂G, and relies on Dynkin's formula through Theorem 478O. The strong Markov property of Brownian motion now enables us to relate harmonic measures associated with different sets (478R).

478A Notation $r \ge 1$ will be an integer; if you find it easier to focus on one dimensionality at a time, you should start with $r = 3$, because $r = 1$ is too easy and $r = 2$ is exceptional. μ will be Lebesgue measure on $\mathbb{R}^r$, and $\| \, \|$ the Euclidean norm on $\mathbb{R}^r$; ν will be normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$. In the elementary case $r = 1$, ν will be counting measure on $\mathbb{R}$.

β_r will be the volume of the unit ball in $\mathbb{R}^r$, that is,

$$\beta_r = \frac{1}{k!}\pi^k \text{ if } r = 2k \text{ is even,}$$
$$= \frac{2^{2k+1}k!}{(2k+1)!}\pi^k \text{ if } r = 2k+1 \text{ is odd.}$$

Recall that

$$\nu(\partial B(\mathbf{0},1)) = r\beta_r = \frac{2}{(k-1)!}\pi^k \text{ if } r = 2k \text{ is even,}$$
$$= \frac{2^{2k+1}k!}{(2k)!}\pi^k \text{ if } r = 2k+1 \text{ is odd}$$

(265F/265H).

In the formulae below, there are repeated expressions of the form $\frac{1}{\|x-y\|^{r-1}}$, $\frac{1}{\|x-y\|^{r-2}}$; in these, it will often be convenient to interpret $\frac{1}{0}$ as ∞, so that we have $[0,\infty]$-valued functions defined everywhere.

It will be convenient to do some calculations in the one-point compactification $\mathbb{R}^r \cup \{\infty\}$ of $\mathbb{R}^r$ (3A3O). For a set $A \subseteq \mathbb{R}^r$, write $\overline{A}^\infty$ and $\partial^\infty A$ for its closure and boundary taken in $\mathbb{R}^r \cup \{\infty\}$; that is,

$$\overline{A}^\infty = \overline{A}, \quad \partial^\infty A = \partial A$$

if A is bounded, and

$$\overline{A}^\infty = \overline{A} \cup \{\infty\}, \quad \partial^\infty A = \partial A \cup \{\infty\}$$

if A is unbounded. Note that $\overline{A}^\infty$ and $\partial^\infty A$ are always compact. In this context I will take $x + \infty = \infty$ for every $x \in \mathbb{R}^r$.

μ_W will be r-dimensional Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$, the space of continuous functions ω from $[0,\infty[$ to $\mathbb{R}^r$ such that $\omega(0) = 0$ (477D), endowed with the topology of uniform convergence on compact sets; Σ will be the domain of μ_W. The probabilistic notations $\mathbb{E}$ and $\Pr$ will always be with respect to μ_W or some directly associated probability. μ_W^2 will be the product measure on $\Omega \times \Omega$. I will write $X_t(\omega) = \omega(t)$ for $t \in [0,\infty[$ and $\omega \in \Omega$, and if $\tau : \Omega \to [0,\infty]$ is a function, I will write $X_\tau(\omega) = \omega(\tau(\omega))$ whenever $\omega \in \Omega$ and $\tau(\omega)$ is finite.

As in 477Hc, I will write Σ_t for the σ-algebra of sets $F \in \Sigma$ such that $\omega' \in F$ whenever $\omega \in F$, $\omega' \in \Omega$ and $\omega' \restriction [0,t] = \omega \restriction [0,t]$, and $\Sigma_t^+ = \bigcap_{s>t} \Sigma_s$. $\mathrm{T}_{[0,t]}$ will be the σ-algebra of subsets of Ω generated by $\{X_s : s \le t\}$.

478B Harmonic and superharmonic functions Let $G \subseteq \mathbb{R}^r$ be an open set and $f : G \to [-\infty,\infty]$ a function.

(a) f is **superharmonic** if $\frac{1}{\nu(\partial B(x,\delta))} \int_{\partial B(x,\delta)} f d\nu$ is defined in $[-\infty,\infty]$ and less than or equal to $f(x)$ whenever $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$.

(b) f is **subharmonic** if $-f$ is superharmonic, that is, $\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} f d\nu$ is defined in $[-\infty,\infty]$ and greater than or equal to $f(x)$ whenever $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$.

(c) f is **harmonic** if it is both superharmonic and subharmonic, that is, $\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} f d\nu$ is defined and equal to $f(x)$ whenever $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$.

478C Elementary facts Let $G \subseteq \mathbb{R}^r$ be an open set.

(a) If $f : G \to [-\infty,\infty]$ is a function, then f is superharmonic iff $-f$ is subharmonic.

(b) If f, $g : G \to [-\infty,\infty[$ are superharmonic functions, then $f+g$ is superharmonic. **P** If $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$, then $\int_{\partial B(x,\delta)} f d\nu$ and $\int_{\partial B(x,\delta)} g\, d\nu$ are defined in $[-\infty,\infty[$, so $\int_{\partial B(x,\delta)} f+g\, d\nu$ is defined and is

$$\int_{\partial B(x,\delta)} f d\nu + \int_{\partial B(x,\delta)} g\, d\nu \le \nu(\partial B(x,\delta))\,(f(x)+g(x)). \quad \mathbf{Q}$$

(c) If f, $g : G \to [-\infty,\infty]$ are superharmonic functions, then $f \wedge g$ is superharmonic. **P** If $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$, then $\int_{\partial B(x,\delta)} f d\nu$ and $\int_{\partial B(x,\delta)} g\, d\nu$ are defined in $[-\infty,\infty]$, so $\int_{\partial B(x,\delta)} f \wedge g\, d\nu$ is defined and is at most

$$\min\bigl(\int_{\partial B(x,\delta)} f d\nu, \int_{\partial B(x,\delta)} g\, d\nu\bigr) \le \nu(\partial B(x,\delta))\,\min(f(x),g(x)). \quad \mathbf{Q}$$

(d) Let $f : G \to \mathbb{R}$ be a harmonic function which is locally integrable with respect to Lebesgue measure on G (that is, every point of G has a neighbourhood V such that $\int_V f d\mu$ is defined and finite). Then

$$f(x) = \frac{1}{\mu B(x,\delta)}\int_{B(x,\delta)} f d\mu$$

whenever $x \in G$, $\delta > 0$ and $B(x,\delta) \subseteq G$. **P** By 265G,

$$\int_{B(x,\delta)} f d\mu = \int_0^t \int_{\partial B(x,t)} f d\nu\, dt = \int_0^t \nu(\partial B(x,\delta)) f(x) dt = \beta_r \delta^r f(x). \quad \mathbf{Q}$$

So f is continuous. **P** If $x \in G$, take $\delta > 0$ such that $B(x,2\delta) \subseteq G$, and set $f_1(y) = f(y)$ for $y \in B(x,2\delta)$, 0 for $y \in \mathbb{R}^r \setminus B(x,2\delta)$. Set $g = \frac{1}{\mu B(\mathbf{0},\delta)}\chi B(\mathbf{0},\delta)$. Then f_1 is integrable, so the convolution $f_1 * g$ is continuous (444Rc). Also, for any $y \in B(x,\delta)$,

$$\begin{aligned}(f_1 * g)(y) &= \int f_1(z) g(y-z)\mu(dz) = \int_{B(y,\delta)} \frac{f_1(z)}{\mu B(y,\delta)}\mu(dz)\\ &= \frac{1}{\mu B(y,\delta)}\int_{B(y,\delta)} f(z)\mu(dz) = f(y),\end{aligned}$$

so f is continuous at x. **Q**

478D Maximal principle One of the fundamental properties of harmonic functions will hardly be used in the exposition here, but I had better give it a suitably prominent place.

Proposition Let $G \subseteq \mathbb{R}^r$ be a non-empty open set. Suppose that $g : \overline{G}^\infty \to \,]-\infty,\infty]$ is lower semi-continuous, $g(y) \ge 0$ for every $y \in \partial^\infty G$, and $g \restriction G$ is superharmonic. Then $g(x) \ge 0$ for every $x \in G$.

proof ? Otherwise, set $\gamma = \inf_{x\in G} g(x) = \inf\{g(y) : y \in \overline{G}^\infty\}$. Because $\overline{G}^\infty$ is compact and g is lower semi-continuous, $K = \{x : x \in G,\ g(x) = \gamma\}$ is non-empty and compact (4A2B(d-viii)). Let $x \in K$ be such that $\|x\|$ is maximal, and $\delta > 0$ such that $B(x,\delta) \subseteq G$. Then $\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} g\, d\nu \le g(x)$. But $g(y) \ge g(x)$ for every $y \in \partial B(x,\delta)$ and

$$\{y : y \in \partial B(x,\delta),\ g(y) > g(x)\} \supseteq \{y : y \in \partial B(x,\delta),\ (y-x)\,.\,x \ge 0\}$$

is not ν-negligible, so this is impossible. **X**

478E Theorem Let $G \subseteq \mathbb{R}^r$ be an open set and $f : G \to \mathbb{R}$ a function with continuous second derivative. Write $\nabla^2 f$ for its Laplacian $\operatorname{div}\operatorname{grad} f = \sum_{i=1}^{r} \frac{\partial^2 f}{\partial \xi_i^2}$.

(a) f is superharmonic iff $\nabla^2 f \le 0$ everywhere in G.
(b) f is subharmonic iff $\nabla^2 f \ge 0$ everywhere in G.
(c) f is harmonic iff $\nabla^2 f = 0$ everywhere in G.

proof (a)(i) For $x \in G$ set

$$R_x = \rho(x, \mathbb{R}^r \setminus G) = \inf_{y \in \mathbb{R}^r \setminus G} \|x - y\|,$$

counting $\inf \emptyset$ as ∞; for $0 < \gamma < R_x$ set

$$g_x(\gamma) = \frac{1}{\gamma^{r-1}} \int_{\partial B(x,\gamma)} f(y)\nu(dy) = \int_{\partial B(\mathbf{0},1)} f(x + \gamma z)\nu(dz).$$

Because f is continuously differentiable, $g_x'(\gamma)$ is defined and equal to $\int_{\partial B(\mathbf{0},1)} \frac{\partial}{\partial \gamma} f(x + \gamma z)\nu(dz)$ for $\gamma \in]0, R_x[$.

Set $\phi = \operatorname{grad} f$, so that $\nabla^2 f = \operatorname{div} \phi$. Each ball $B(x,\gamma)$ has finite perimeter; its essential boundary is its ordinary boundary; the Federer exterior normal v_y at y is $\frac{1}{\gamma}(y - x)$; and if $y = x + \gamma z$, where $\|z\| = 1$, then $\phi(y)\,\mathbf{.}\,v_y$ is $\frac{\partial}{\partial \gamma} f(x + \gamma z)$. So the Divergence Theorem (475N) tells us that

$$\begin{aligned}\int_{B(x,\gamma)} \nabla^2 f d\mu &= \int_{\partial B(x,\gamma)} \phi(y)\,\mathbf{.}\,v_y \nu(dy) \\ &= \gamma^{r-1} \int_{\partial B(\mathbf{0},1)} \frac{\partial}{\partial \gamma} f(x + \gamma z)\nu(dz) = \gamma^{r-1} g_x'(\gamma).\end{aligned}$$

(ii) If $\nabla^2 f \le 0$ everywhere in G, and $B(x,\gamma) \subseteq G$, then $g_x'(t) \le 0$ for $0 < t \le \gamma$, so

$$g_x(\gamma) \le \lim_{t \downarrow 0} g_x(t) = r\beta_r f(x);$$

as x and γ are arbitrary, f is superharmonic.

(iii) If f is superharmonic, and $x \in G$, then

$$g_x(\gamma) \le r\beta_r f(x) = \lim_{t \downarrow 0} g_x(t)$$

for every $\gamma \in]0, R_x[$. So there must be arbitrarily small $\gamma > 0$ such that $g_x'(\gamma) \le 0$ and $\int_{B(x,\gamma)} \nabla^2 f d\mu \le 0$; as $\nabla^2 f$ is continuous, $(\nabla^2 f)(x) \le 0$.

(b)-(c) are now immediate.

478F Basic examples (a) For any y, $z \in \mathbb{R}^r$,

$$x \mapsto \frac{1}{\|x-z\|^{r-2}}, \quad x \mapsto \frac{(x-z)\,\mathbf{.}\,y}{\|x-z\|^r},$$

$$x \mapsto \frac{\|y-z\|^2 - \|x-y\|^2}{\|x-z\|^r} = 2\frac{(x-z)\,\mathbf{.}\,(y-z)}{\|x-z\|^r} - \frac{1}{\|x-z\|^{r-2}}$$

are harmonic on $\mathbb{R}^r \setminus \{z\}$.

(b) For any $z \in \mathbb{R}^2$,

$$x \mapsto \ln \|x - z\|$$

is harmonic on $\mathbb{R}^2 \setminus \{z\}$.

proof The Laplacians are easiest to calculate when $z = 0$, of course, but in any case you only have to get the algebra right to apply 478Ec.

Remark The function $x \mapsto \frac{\|y-z\|^2 - \|x-y\|^2}{\|x-z\|^r}$ is the **Poisson kernel**; see 478I below.

478G We shall need a pair of exact integrals involving the functions here, with an easy corollary.

Lemma (a) $\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|x-z\|^{r-2}}\nu(dz)=\frac{1}{\max(\delta,\|x\|)^{r-2}}$ whenever $x\in\mathbb{R}^r$ and $\delta>0$.

(b) $\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{|\delta^2-\|x\|^2|}{\|x-z\|^r}\nu(dz)=\frac{1}{\max(\delta,\|x\|)^{r-2}}$ whenever $x\in\mathbb{R}^r$, $\delta>0$ and $\|x\|\ne\delta$.

(c) $\int_{B(\mathbf{0},\delta)}\frac{1}{\|x-z\|^{r-2}}\mu(dz)\le\frac{1}{2}r\beta_r\delta^2$ whenever $x\in\mathbb{R}^r$ and $\delta>0$.

proof (a)(i) The first thing to note is that there is a function $g:[0,\infty[\,\setminus\{\delta\}\to[0,\infty[$ such that $g(\|x\|)=\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|x-z\|^{r-2}}\nu(dz)$ whenever $\|x\|\ne\delta$. **P** If $\|x\|=\|y\|$, then there is an orthogonal transformation $T:\mathbb{R}^r\to\mathbb{R}^r$ such that $Tx=y$, so that

$$\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|y-z\|^{r-2}}\nu(dz)=\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|y-Tz\|^{r-2}}\nu(dz)$$

(because T is an automorphism of $(\mathbb{R}^r,B(\mathbf{0},\delta),\nu)$)

$$=\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|Tx-Tz\|^{r-2}}\nu(dz)=\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|x-z\|^{r-2}}\nu(dz).\ \mathbf{Q}$$

(ii) Now suppose that $0<\gamma<\delta$. Then

$$\begin{aligned}g(\gamma)&=\frac{1}{\nu(\partial B(\mathbf{0},\gamma))}\int_{\partial B(\mathbf{0},\gamma)}g(\gamma)\nu(dx)\\&=\frac{1}{\nu(\partial B(\mathbf{0},\gamma))}\int_{\partial B(\mathbf{0},\gamma)}\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|x-z\|^{r-2}}\nu(dz)\nu(dx)\\&=\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\nu(\partial B(\mathbf{0},\gamma))}\int_{\partial B(\mathbf{0},\gamma)}\frac{1}{\|x-z\|^{r-2}}\nu(dx)\nu(dz)\\&=\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|z\|^{r-2}}\nu(dz)\end{aligned}$$

(because the function $x\mapsto\frac{1}{\|x-z\|^2}$ is harmonic in $\mathbb{R}^r\setminus\{z\}$, by 478Fa)

$$=\frac{1}{\delta^{r-2}}.$$

(iii) Next, if $\gamma>\delta$,

$$g(\gamma)=\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\nu(\partial B(\mathbf{0},\gamma))}\int_{\partial B(\mathbf{0},\gamma)}\frac{1}{\|x-z\|^{r-2}}\nu(dx)\nu(dz)$$

(as in (ii))

$$=\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\gamma^{r-2}}\nu(dz)$$

(by (ii), with γ and δ interchanged)

$$=\frac{1}{\gamma^{r-2}}.$$

(iv) So we have the result if $\|x\|\ne\delta$. If $\|x\|=\delta$ and $r\ge2$, set $x_n=(1+2^{-n})x$ for each $n\in\mathbb{N}$. If $z\in\partial B(\mathbf{0},\delta)$, $\langle\|x_n-z\|\rangle_{n\in\mathbb{N}}$ is a decreasing sequence with limit $\|x-z\|$, so $\langle\frac{1}{\|x_n-z\|^{r-2}}\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with limit $\frac{1}{\|x-z\|^{r-2}}$. By B.Levi's theorem,

$$\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|x-z\|^{r-2}}\nu(dz)=\lim_{n\to\infty}\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|x_n-z\|^{r-2}}\nu(dz)$$
$$=\lim_{n\to\infty}\frac{1}{\|x_n\|^{r-2}}=\frac{1}{\|x\|^{r-2}}=\frac{1}{\delta^{r-2}}.$$

Finally, if $r=1$ and $\|x\|=|x|=\delta$, we are just trying to take the average of $|x-\delta|$ and $|x-(-\delta)|$, which will be $\delta=\frac{1}{\delta^{r-2}}$.

(b) We can follow the same general line.

(i) Define $f:\mathbb{R}^r\setminus\partial B(\mathbf{0},\delta)\to\mathbb{R}$ by setting $f(x)=\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{\|x\|^2-\delta^2}{\|x-z\|^r}\nu(dz)$ when $\|x\|\ne\delta$. Then f is harmonic. **P** If x and $\gamma>0$ are such that $B(x,\gamma)\subseteq\mathbb{R}^r\setminus\partial B(\mathbf{0},\delta)$, then

$$\frac{1}{\nu(\partial B(x,\gamma))}\int_{\partial B(x,\gamma)}f(y)\nu(dy)$$
$$=\frac{1}{\nu(\partial B(x,\gamma))}\int_{\partial B(x,\gamma)}\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{\|y\|^2-\delta^2}{\|y-z\|^r}\nu(dz)\nu(dy)$$
$$=\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\nu(\partial B(x,\gamma))}\int_{\partial B(x,\gamma)}\frac{\|y\|^2-\delta^2}{\|y-z\|^r}\nu(dy)\nu(dz)$$
$$=\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{\|x\|^2-\delta^2}{\|x-z\|^r}\nu(dz)$$

(because the functions $y\mapsto\frac{\|y\|^2-\delta^2}{\|y-z\|^r}$ are harmonic when $\|z\|=\delta$, by 478Fa)

$$=f(x).\mathbf{Q}$$

Since f is smooth, $\nabla^2 f=0$ everywhere off $\partial B(\mathbf{0},\delta)$, by 478Ec.

(ii) As before, we have a function $h:[0,\infty[\setminus\{\delta\}\to[0,\infty[$ such that $f(x)=h(\|x\|)$ whenever $\|x\|\ne\delta$. If $0<\gamma<\delta$ then

$$h(\gamma)=\frac{1}{\nu(\partial B(\mathbf{0},\gamma))}\int_{\partial B(\mathbf{0},\gamma)}h(\gamma)\nu(dy)=\frac{1}{\nu(\partial B(\mathbf{0},\gamma))}\int_{\partial B(\mathbf{0},\gamma)}f(y)\nu(dy)$$
$$=f(0)=\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{-\delta^2}{\|z\|^r}\nu(dz)=-\frac{1}{\delta^{r-2}},$$

and

$$\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{|\delta^2-\|x\|^2|}{\|x-z\|^r}\nu(dz)=-h(\gamma)=\frac{1}{\delta^{r-2}}$$

if $\|x\|=\gamma<\delta$.

(iii) For $\gamma>\delta$ I start with an elementary estimate. If $\|x\|=\gamma>\delta$ then $\frac{\|x\|^2-\delta^2}{\|x-z\|^r}$ lies between $\frac{\gamma^2-\delta^2}{(\gamma+\delta)^r}$ and $\frac{\gamma^2-\delta^2}{(\gamma-\delta)^r}$ for every $z\in\partial B(x,\delta)$, so that $\gamma^{r-2}f(x)$ lies between $\frac{1-(\delta/\gamma)^2}{(1+(\delta/\gamma))^r}$ and $\frac{1-(\delta/\gamma)^2}{(1-(\delta/\gamma))^r}$, and is approximately 1 if γ is large.

(iv) Now we can use the Divergence Theorem again, as follows. If $\delta<\gamma<\beta$ consider the region $E=B(\mathbf{0},\beta)\setminus B(\mathbf{0},\gamma)$ and the function $\phi=\operatorname{grad}f$. As f is smooth, ϕ is defined everywhere off $\partial B(\mathbf{0},\delta)$, and $\phi(x)=\frac{h'(\|x\|)}{\|x\|}x$ at every $x\in\mathbb{R}^r\setminus\partial B(\mathbf{0},\delta)$. The essential boundary of E is $\partial E=\partial B(\mathbf{0},\gamma)\cup\partial B(\mathbf{0},\beta)$; the Federer exterior normal at $x\in\partial B(\mathbf{0},\gamma)$ is $v_x=-\frac{1}{\gamma}x$ and at $x\in\partial B(\mathbf{0},\beta)$ it is $v_x=\frac{1}{\beta}x$; and $\operatorname{div}\phi=\nabla^2 f$ is zero everywhere on E. So 475N tells us that

$$\begin{aligned}0&=\int_{\partial B(\mathbf{0},\beta)}\phi(x)\textbf{.}v_x\nu(dx)+\int_{\partial B(\mathbf{0},\gamma)}\phi(x)\textbf{.}v_x\nu(dx)\\&=\int_{\partial B(\mathbf{0},\beta)}h'(\beta)\nu(dx)-\int_{\partial B(\mathbf{0},\gamma)}h'(\gamma)\nu(dx)\\&=r\beta_r\beta^{r-1}h'(\beta)-r\beta_r\gamma^{r-1}h'(\gamma).\end{aligned}$$

This shows that $h'(\gamma)$ is inversely proportional to γ^{r-1}.

(v) If $r\ge 3$, there are α, $\beta\in\mathbb{R}$ such that $h(\gamma)=\alpha+\frac{\beta}{\gamma^{r-2}}$ for every $\gamma>\delta$. But since (iii) shows us that $\lim_{\gamma\to\infty}\gamma^{r-2}h(\gamma)=1$, we must have $h(\gamma)=\frac{1}{\gamma^{r-2}}$ for $\gamma>\delta$, as declared. If $r=2$, then we can express h in the form $h(\gamma)=\alpha+\beta\ln\gamma$; this time, $\lim_{\gamma\to\infty}h(\gamma)=1$, so once more $h(\gamma)=1=\frac{1}{\gamma^{r-2}}$ for every γ.

(vi) Finally, if $r=1$ and $|x|>\delta$, then, as in (a-iv) above, $\frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)}\frac{|\delta^2-\|x\|^2|}{\|x-z\|^r}\nu(dz)$ is the average of $\frac{x^2-\delta^2}{|x-\delta|}=|x+\delta|$ and $\frac{x^2-\delta^2}{|x+\delta|}=|x-\delta|$, so is $|x|=\frac{1}{|x|^{r-2}}$.

(c) This follows easily from (a);

$$\begin{aligned}\int_{B(\mathbf{0},\delta)}\frac{1}{\|x-z\|^{r-2}}\mu(dz)&=\int_0^\delta\int_{\partial B(\mathbf{0},t)}\frac{1}{\|x-z\|^{r-2}}\nu(dz)dt\\&=\int_0^\delta\frac{r\beta_r t^{r-1}}{\max(t,\|x\|)^{r-2}}dt\le\int_0^\delta r\beta_r t\,dt=\frac{1}{2}r\beta_r\delta^2.\end{aligned}$$

478H Corollary If $r\ge 2$, then $x\mapsto\frac{1}{\|x-z\|^{r-2}}:\mathbb{R}^r\to[0,\infty]$ is superharmonic for any $z\in\mathbb{R}^r$.

proof If $\delta>0$ and $x\in\mathbb{R}^r$,

$$\begin{aligned}\frac{1}{\nu\partial B(x,\delta)}\int_{\partial B(x,\delta)}\frac{1}{\|y-z\|^{r-2}}\nu(dy)&=\frac{1}{\nu\partial B(\mathbf{0},\delta)}\int_{\partial B(\mathbf{0},\delta)}\frac{1}{\|y+x-z\|^{r-2}}\nu(dy)\\&=\frac{1}{\max(\delta,\|x-z\|)^{r-2}}\le\frac{1}{\|x-z\|^{r-2}}.\end{aligned}$$

478I The Poisson kernel gives a basic method of building and describing harmonic functions.

Theorem Suppose that $y\in\mathbb{R}^r$ and $\delta>0$; let $S=\partial B(y,\delta)$ be the sphere with centre y and radius δ.

(a) Let ζ be a totally finite Radon measure on S, and define f on $\mathbb{R}^r\setminus S$ by setting

$$f(x)=\frac{1}{r\beta_r\delta}\int_S\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\zeta(dz)$$

for $x\in\mathbb{R}^r\setminus S$. Then f is continuous and harmonic.

(b) Let $g:S\to\mathbb{R}$ be a ν_S-integrable function, where ν_S is the subspace measure on S induced by ν, and define $f:\mathbb{R}^r\to\mathbb{R}$ by setting

$$\begin{aligned}f(x)&=\frac{1}{r\beta_r\delta}\int_S g(z)\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)\text{ if }x\in\mathbb{R}^r\setminus S,\\&=g(x)\text{ if }x\in S.\end{aligned}$$

(i) f is continuous and harmonic in $\mathbb{R}^r\setminus S$.

(ii) If $r\ge 2$, then

$$\liminf_{z\in S,z\to z_0}g(x)=\liminf_{x\to z_0}f(x),\quad\limsup_{x\to z_0}f(x)=\limsup_{z\in S,z\to z_0}g(x)$$

for every $z_0\in S$.

(iii) f is continuous at any point of S where g is continuous, and if g is lower semi-continuous then f also is.
(iv) $\sup_{x\in\mathbb{R}^r} f(x)=\sup_{z\in S} g(z)$ and $\inf_{x\in\mathbb{R}^r} f(x)=\inf_{z\in S} g(z)$.

proof (a) f is continuous just because $x\mapsto\frac{\delta^2-\|x-y\|^2}{\|x-z\|^r}$ is continuous for each $z\in S$ and uniformly bounded for $z\in S$ and x running over any compact set not meeting S.

Suppose that $\|x-y\|<\delta$ and $\eta>0$ is such that $B(x,\eta)\cap S=\emptyset$. Then

$$\begin{aligned}\frac{1}{\nu(\partial B(x,\eta))}\int_{\partial B(x,\eta)}f d\nu&=\frac{1}{\nu(\partial B(x,\eta))}\int_{\partial B(x,\eta)}\frac{1}{r\beta_r\delta}\int_S\frac{\delta^2-\|w-y\|^2}{\|w-z\|^r}\zeta(dz)\nu(dw)\\&=\frac{1}{r\beta_r\delta}\int_S\frac{1}{\nu(\partial B(x,\eta))}\int_{\partial B(x,\eta)}\frac{\delta^2-\|w-y\|^2}{\|w-z\|^r}\nu(dw)\zeta(dz)\\&=\frac{1}{r\beta_r\delta}\int_S\frac{\delta^2-\|x-y\|^2}{\|x-z\|^r}\zeta(dz)\end{aligned}$$

(because $w\mapsto\frac{\delta^2-\|w-y\|^2}{\|w-z\|^r}$ is harmonic on $\mathbb{R}^r\setminus\{z\}$ whenever $z\in S$, by 478Fa)

$$=f(x).$$

As x and η are arbitrary, f is harmonic on $\operatorname{int}B(y,\delta)$. Similarly, it is harmonic on $\mathbb{R}^r\setminus B(y,\delta)$.

(b)(i) Applying (a) to the indefinite-integral measures over the subspace measure ν_S defined by the positive and negative parts of g, we see that f is continuous and harmonic in $\mathbb{R}^r\setminus S$.

(ii)($\boldsymbol{\alpha}$) If $x\notin S$,

$$\begin{aligned}\int_S\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)&=\int_{\partial B(\mathbf{0},\delta)}\frac{|\delta^2-\|x-y\|^2|}{\|x-y-z\|^r}\nu(dz)\\&=\frac{\nu(\partial B(\mathbf{0},\delta))}{\max(\delta,\|x-y\|)^{r-2}}\end{aligned}$$

by 478Gb. In particular, if x is close to, but not on, the sphere S, $\int_S\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)$ is approximately $\frac{\nu(\partial B(\mathbf{0},\delta)}{\delta^{r-2}}=r\beta_r\delta$.

($\boldsymbol{\beta}$) Set $M=\int_S|g|d\nu$, and take $z_0\in S$; set $\gamma=\limsup_{x\in S,x\to z_0}g(x)$. If $\gamma=\infty$ then certainly $\limsup_{x\to z_0}f(x)\le\gamma$. Otherwise, take $\eta>0$. Let $\alpha_0\in\,]0,\delta[$ be such that

$$|\frac{1}{r\beta_r\delta}\int_S\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)-1|\le\eta$$

whenever $0<|\delta-\|x-y\||\le\alpha_0$, and $g(z)\le\gamma+\eta$ whenever $z\in S$ and $0<\|z-z_0\|\le2\alpha_0$. Let $\alpha\in\,]0,\alpha_0]$ be such that $(2\delta+\alpha_0)(M+|\gamma|)\alpha\nu S\le 2^r r\beta_r\delta\alpha_0^r\eta$.

If $\|x-z_0\|\le\alpha$ and $\|x-y\|\ne\delta$, then $|\delta-\|x-y\||\le\|x-z_0\|\le\alpha_0$ and $|\delta^2-\|x-y\|^2|\le\|x-z_0\|(2\delta+\alpha_0)$, so

$$\begin{aligned}f(x)-\gamma&\le\eta|\gamma|+\frac{1}{r\beta_r\delta}\int_S(g(z)-\gamma)\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)\\&\le\eta|\gamma|+\frac{1}{r\beta_r\delta}\int_{S\cap B(z_0,2\alpha_0)}(g(z)-\gamma)\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)\\&\qquad+\frac{1}{r\beta_r\delta}\int_{S\setminus B(z_0,2\alpha_0)}(|g(z)|+|\gamma|)\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)\\&\le\eta|\gamma|+\frac{\eta}{r\beta_r\delta}\int_S\frac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz)\\&\qquad+\frac{1}{r\beta_r\delta}\int_{S\setminus B(z_0,2\alpha_0)}(|g(z)|+|\gamma|)\frac{\|x-z_0\|(2\delta+\alpha_0)}{2^r\alpha_0^r}\nu(dz)\end{aligned}$$

(because $r\ge2$, so $\nu\{z_0\}=0$)

$$\leq \eta|\gamma| + \eta(1+\eta) + (M+|\gamma|)\tfrac{\alpha(2\delta+\alpha_0)}{2^r r\beta_r\delta\alpha_0^r}\nu S$$
$$\leq (|\gamma|+1+\eta+1)\eta.$$

Also, of course, $f(x)-\gamma = g(x)-\gamma \leq \eta$ if $0<\|x-z_0\|\leq\alpha$ and $\|x-y\|=\delta$. As η is arbitrary, $\limsup_{x\to z_0} f(x)\leq\gamma$. In the other direction, $\limsup_{x\to z_0} f(x)\geq\gamma$ just because f extends g.

(γ) Similarly, or applying (β) to $-g$, $\liminf_{x\to z_0} f(x) = \liminf_{x\in S,x\to z_0} g(x)$.

(iii)(α) If $r\geq 2$, it follows at once from (ii) that if g is continuous at $z_0\in S$, so is f, and that if g is lower semi-continuous (so that $g(z_0)\leq\liminf_{x\in S,x\to z_0} g(x)$ for every $z_0\in S$) then f also is lower semi-continuous.

(β) If $r=1$, then $S=\{y-\delta,y+\delta\}$ and $\nu\{y-\delta\}=\nu\{y+\delta\}=1$, so

$$f(x) = \frac{1}{2\delta}\Big(\tfrac{|\delta^2-(x-y)^2|}{|x-(y-\delta)|}g(y-\delta) + \tfrac{|\delta^2-(x-y)^2|}{|x-(y+\delta)|}g(y+\delta)\Big)$$
$$= \frac{1}{2\delta}\big(|x-(y+\delta)|g(y-\delta) + |x-(y-\delta)|g(y+\delta)\big)$$

for $x\in\mathbb{R}\setminus S$, and $\lim_{x\mapsto y\pm\delta} f(x) = g(y\pm\delta)$, so f is continuous.

(iv) If $g(z)\leq\alpha<\infty$ for every $z\in S$, then

$$f(x) = \frac{1}{r\beta_r\delta}\int_S g(z)\tfrac{|\delta^2-\|x-y\|^2|}{\|x-z\|^r}\nu(dz) \leq \frac{\alpha}{r\beta_r\delta}\int_{\partial B(\mathbf{0},\delta)}\tfrac{|\delta^2-\|x-y\|^2|}{\|x-y-z\|^r}\nu(dz)$$
$$= \frac{\alpha}{r\beta_r\delta}\cdot\frac{\nu(\partial B(\mathbf{0},\delta))}{\max(\delta,\|x-y\|)^{r-2}} = \frac{\alpha\delta^{r-2}}{\max(\delta,\|x-y\|)^{r-2}} \leq \alpha$$

for every $x\in\mathbb{R}^r\setminus S$. So $\sup_{x\in\mathbb{R}^r} f(x) = \sup_{z\in S} g(z)$; similarly, $\inf_{x\in\mathbb{R}^r} f(x) = \inf_{z\in S} g(z)$.

478J Convolutions and smoothing: Proposition (a) Suppose that $f:\mathbb{R}^r\to[0,\infty]$ is Lebesgue measurable, and $G\subseteq\mathbb{R}^r$ an open set such that $f{\restriction}G$ is superharmonic. Let $h:\mathbb{R}^r\to[0,\infty]$ be a Lebesgue integrable function, and $f*h$ the convolution of f and h. If $H\subseteq G$ is an open set such that $H-\{z:h(z)\neq 0\}\subseteq G$, then $(f*h){\restriction}H$ is superharmonic.

(b) Suppose, in (a), that $h(y)=h(z)$ whenever $\|y\|=\|z\|$ and that $\int_{\mathbb{R}^r} h\,d\mu\leq 1$. If $x\in G$ and $\gamma>0$ are such that $B(x,\gamma)\subseteq G$ and $h(y)=0$ whenever $\|y\|\geq\gamma$, then $(f*h)(x)\leq f(x)$.

(c) Let $f:\mathbb{R}^r\to[0,\infty]$ be a lower semi-continuous function, and $\langle\tilde h_n\rangle_{n\in\mathbb{N}}$ the sequence of 473E. If $G\subseteq\mathbb{R}^r$ is an open set such that $f{\restriction}G$ is superharmonic, then $f(x)=\lim_{n\to\infty}(f*\tilde h_n)(x)$ for every $x\in G$.

proof (a) If $x\in H$ and $\delta>0$ are such that $B(x,\delta)\subseteq H$, then

$$\int_{\partial B(x,\delta)}(f*h)(y)\nu(dy) = \int_{\partial B(x,\delta)}\int_{\mathbb{R}^r} f(y-z)h(z)\mu(dz)\nu(dy)$$
$$= \int_{\mathbb{R}^r} h(z)\int_{\partial B(x,\delta)} f(y-z)\nu(dy)\mu(dz)$$
$$= \int_{\mathbb{R}^r} h(z)\int_{\partial B(x-z,\delta)} f(y)\nu(dy)\mu(dz)$$
$$\leq r\beta_r\delta^{r-1}\int_{\mathbb{R}^r} h(z)f(x-z)\mu(dz)$$

(because if $h(z)\neq 0$ then $B(x-z,\delta)=B(x,\delta)-z$ is included in G)

$$= \nu(\partial B(x,\delta))\cdot(f*h)(x).$$

(b) Let $g:[0,\infty[\,\to[0,\infty]$ be such that $h(y)=g(\|y\|)$ for every y. Then

$$(f*h)(x) = \int_{\mathbb{R}^r} f(y)h(x-y)\mu(dy) = \int_0^\gamma\int_{\partial B(x,t)} f(y)g(t)\nu(dy)dt$$
$$\leq \int_0^\gamma r\beta_r t^{r-1}f(x)g(t)dt = f(x)\int_{\mathbb{R}^r} h\,d\mu \leq f(x).$$

(c) By (b), $(f*\tilde h_n)(x)\le f(x)$ for every sufficiently large n, so $\limsup_{n\to\infty}(f*\tilde h_n)(x)\le f(x)$. In the other direction, if $x\in G$ and $\alpha<f(x)$, there is a $\delta>0$ such that $B(x,\delta)\subseteq G$ and $f(y)\ge\alpha$ for every $y\in B(x,\delta)$. Now there is an $m\in\mathbb{N}$ such that $\tilde h_n(y)=0$ whenever $n\ge m$ and $\|y\|\ge\delta$; so that

$$(f*\tilde h_n)(x)=\int f(y)\tilde h_n(x-y)\mu(dy)=\int_{B(x,\delta)}f(y)\tilde h_n(x-y)\mu(dy)$$
$$\ge\alpha\int_{B(x,\delta)}\tilde h_n(x-y)\mu(dy)=\alpha$$

whenever $n\ge m$. As α is arbitrary, $f(x)=\lim_{n\to\infty}(f*\tilde h_n)(x)$.

478K Dynkin's formula: Lemma Let μ_W be r-dimensional Wiener measure on $\Omega=C([0,\infty[\,;\mathbb{R}^r)_0$; set $X_t(\omega)=\omega(t)$ for $\omega\in\Omega$ and $t\ge0$. Let $f:\mathbb{R}^r\to\mathbb{R}$ be a three-times-differentiable function such that f and its first three derivatives are continuous and bounded.

(a) $\mathbb{E}(f(X_t))=f(0)+\frac{1}{2}\mathbb{E}(\int_0^t(\nabla^2 f)(X_s)ds)$ for every $t\ge0$.

(b) If $\tau:\Omega\to[0,\infty[$ is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$ and $\mathbb{E}(\tau)$ is finite, then

$$\mathbb{E}(f(X_\tau))=f(0)+\frac{1}{2}\mathbb{E}(\int_0^\tau(\nabla^2 f)(X_s)ds).$$

proof (a)(i) We need a special case of the multidimensional Taylor's theorem. If $f:\mathbb{R}^r\to\mathbb{R}$ is three times differentiable and $x=(\xi_1,\dots,\xi_r)$, $y=(\eta_1,\dots,\eta_r)\in\mathbb{R}^r$, then there is a z in the line segment $[x,y]$ such that

$$f(y)=f(x)+\sum_{i=1}^r(\eta_i-\xi_i)\frac{\partial f}{\partial\xi_i}(x)+\frac{1}{2}\sum_{i=1}^r\sum_{j=1}^r(\eta_i-\xi_i)(\eta_j-\xi_j)\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}(x)$$
$$+\frac{1}{6}\sum_{i=1}^r\sum_{j=1}^r\sum_{k=1}^r(\eta_i-\xi_i)(\eta_j-\xi_j)(\eta_k-\xi_k)\frac{\partial^3 f}{\partial\xi_i\partial\xi_j\partial\xi_k}(z).$$

P Set $g(\beta)=f(\beta y+(1-\beta)x)$ for $\beta\in\mathbb{R}$. Then g is three times differentiable, with

$$g'(\beta)=\sum_{k=1}^r(\eta_k-\xi_k)\frac{\partial f}{\partial\xi_k}(\beta y+(1-\beta)x),$$
$$g''(\beta)=\sum_{j=1}^r\sum_{k=1}^r(\eta_j-\xi_j)(\eta_k-\xi_k)\frac{\partial^2 f}{\partial\xi_j\partial\xi_k}(\beta y+(1-\beta)x),$$
$$g'''(\beta)=\sum_{i=1}^r\sum_{j=1}^r\sum_{k=1}^r(\eta_i-\xi_i)(\eta_j-\xi_j)(\eta_k-\xi_k)\frac{\partial^3 f}{\partial\xi_i\partial\xi_j\partial\xi_k}(\beta y+(1-\beta)x).$$

Now by Taylor's theorem with remainder, in one dimension, there is a $\beta\in\,]0,1[$ such that

$$g(1)=g(0)+g'(0)+\frac{1}{2}g''(0)+\frac{1}{6}g'''(\beta)$$

and all we have to do is to set $z=\beta y+(1-\beta)x$ and substitute in the values for $g(1),\dots,g'''(\beta)$. **Q**

(ii) Let $M\ge0$ be such that $\|\frac{\partial^3 f}{\partial\xi_i\partial\xi_j\partial\xi_k}\|_\infty\le M$ whenever $1\le i,\,j,\,k\le r$. Let K be $\mathbb{E}((\sum_{i=1}^r|Z_i|)^3)$ when $Z_1,\dots,Z_r$ are independent real-valued random variables with standard normal distribution. (To see that this is finite, observe that

$$\mathbb{E}((\sum_{i=1}^r|Z_i|)^3)\le\mathbb{E}(r^3\max_{i\le r}|Z_i|^3)\le r^3\,\mathbb{E}(\sum_{i=1}^r|Z_i|^3)=r^4\,\mathbb{E}(|Z|^3)$$

(where Z is a random variable with standard normal distribution)

$$=\frac{2r^4}{\sqrt{2\pi}}\int_0^\infty t^3e^{-t^2/2}dt<\infty.)$$

For any x, $y \in \mathbb{R}^r$ we have

$$\Bigl|f(y) - f(x) - \sum_{i=1}^{r}(\eta_i - \xi_i)\frac{\partial f}{\partial \xi_i}(x) + \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}(\eta_i - \xi_i)(\eta_j - \xi_j)\frac{\partial^2 f}{\partial \xi_i \partial \xi_j}(x)\Bigr| \le \frac{M}{6}(\sum_{i=1}^{r}|\eta_i - \xi_i|)^3.$$

If $0 \le s \le t$ and $\omega \in \Omega$, then

$$\begin{aligned}\Bigl|f(\omega(t)) - f(\omega(s)) - \sum_{i=1}^{r}\frac{\partial f}{\partial \xi_i}(\omega(s))(\omega_i(t) - \omega_i(s)) \\ - \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\frac{\partial^2 f}{\partial \xi_i \partial \xi_j}(\omega(s))(\omega_i(t) - \omega_i(s))(\omega_j(t) - \omega_j(s))\Bigr| \\ \le \frac{M}{6}(\sum_{i=1}^{r}|\omega_i(t) - \omega_i(s)|)^3,\end{aligned}$$

writing $\omega_1, \ldots, \omega_r \in C([0, \infty[)_0$ for the coordinates of $\omega \in \Omega$. Integrating with respect to ω, we have

$$\begin{aligned}|\mathbb{E}(f(X_t) - &f(X_s) - \tfrac{1}{2}(t-s)(\nabla^2 f)(X_s))| \\ &= \Bigl|\mathbb{E}(f(X_t) - f(X_s)) - \sum_{i=1}^{r}\mathbb{E}(\frac{\partial f}{\partial \xi_i}(X_s))\mathbb{E}(X_t^{(i)} - X_s^{(i)}) \\ &\quad - \frac{1}{2}\sum_{i=1}^{r}\mathbb{E}(\frac{\partial^2 f}{\partial \xi_i^2}(X_s))\mathbb{E}(X_t^{(i)} - X_s^{(i)})^2 \\ &\quad - \frac{1}{2}\sum_{i=1}^{r}\sum_{j\ne i}\mathbb{E}(\frac{\partial^2 f}{\partial \xi_i \partial \xi_j})(X_s)\mathbb{E}(X_t^{(i)} - X_s^{(i)})\mathbb{E}(X_t^{(j)} - X_s^{(j)})\Bigr|\end{aligned}$$

(writing $X_t^{(i)}(\omega) = \omega_i(t)$ for $1 \le i \le r$, and recalling that $\mathbb{E}(X_t^{(i)} - X_s^{(i)}) = 0$ for every i, while $\mathbb{E}(X_t^{(i)} - X_s^{(i)})^2 = t - s$)

$$\begin{aligned}&= \Bigl|\mathbb{E}(f(X_t) - f(X_s)) - \sum_{i=1}^{r}\mathbb{E}((X_t^{(i)} - X_s^{(i)})\frac{\partial f}{\partial \xi_i}(X_s)) \\ &\quad - \frac{1}{2}\sum_{i=1}^{r}\mathbb{E}((X_t^{(i)} - X_s^{(i)})^2\frac{\partial^2 f}{\partial \xi_i^2}(X_s)) \\ &\quad - \frac{1}{2}\sum_{i=1}^{r}\sum_{j\ne i}\mathbb{E}((X_t^{(i)} - X_s^{(i)})(X_t^{(j)} - X_s^{(j)})\frac{\partial^2 f}{\partial \xi_i \partial \xi_j}(X_s))\Bigr|\end{aligned}$$

(because for any $i \le r$ the random variables $\frac{\partial f}{\partial \xi_i}(X_s)$ and $X_t^{(i)} - X_s^{(i)}$ are independent, while for any distinct i, $j \le r$ the random variables $\frac{\partial^2 f}{\partial \xi_i \partial \xi_j}(X_s)$, $X_t^{(i)} - X_s^{(i)}$ and $X_t^{(j)} - X_s^{(j)}$ are independent)

$$\begin{aligned}&= \Bigl|\mathbb{E}\bigl(f(X_t) - f(X_s) - \sum_{i=1}^{r}(X_t^{(i)} - X_s^{(i)})\frac{\partial f}{\partial \xi_i}(X_s) \\ &\quad - \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}(X_t^{(i)} - X_s^{(i)})(X_t^{(j)} - X_s^{(j)})\frac{\partial^2 f}{\partial \xi_i \partial \xi_j}(X_s)\bigr)\Bigr| \\ &\le \frac{M}{6}\mathbb{E}((\sum_{i=1}^{r}|X_t^{(i)} - X_s^{(i)}|)^3) \le \frac{MK}{6}(t-s)^{3/2}\end{aligned}$$

because the $X_t^{(i)} - X_s^{(i)}$ are independent random variables all with the same distribution as $\sqrt{t-s}\,Z$ where Z is standard normal.

(iii) Now fix $t \geq 0$ and $n \geq 1$; set $s_k = \frac{k}{n}t$ for $k \leq n$. Set

$$g_n(\omega) = \sum_{k=0}^{n-1}(s_{k+1} - s_k)(\nabla^2 f)(\omega(s_k))$$

for $\omega \in \Omega$. Then

$$\begin{aligned}|\int \big(f(\omega(t)) - f(0) - \tfrac{1}{2}g_n(\omega)\big)\mu_W(d\omega)| \\ &= |\sum_{k=0}^{n-1}\mathbb{E}(f(X_{s_{k+1}}) - f(X_{s_k}) - \tfrac{1}{2}(s_{k+1} - s_k)(\nabla^2 f)(X_{s_k}))| \\ &\leq \sum_{k=0}^{n-1}\frac{MK}{6}(\frac{t}{n})^{3/2} = \frac{MKt\sqrt{t}}{6\sqrt{n}}.\end{aligned}$$

On the other hand,

$$\lim_{n\to\infty} g_n(\omega) = \int_0^t (\nabla^2 f)(\omega(s))ds$$

for every ω (the Riemann integral $\int_0^t (\nabla^2 f)(\omega(s))ds$ is defined because $s \mapsto (\nabla^2 f)(\omega(s))$ is continuous), and $|g_n(\omega)| \leq t\|\nabla^2 f\|_\infty < \infty$ for every ω, so by Lebesgue's Dominated Convergence Theorem

$$\begin{aligned}\mathbb{E}(f(X_t) - f(0)) &= \frac{1}{2}\lim_{n\to\infty}\int g_n(\omega)\mu_W(d\omega) = \frac{1}{2}\int \lim_{n\to\infty} g_n(\omega)\mu_W(d\omega) \\ &= \frac{1}{2}\int\int_0^t (\nabla^2 f)(\omega(s))ds\,\mu_W(d\omega) = \frac{1}{2}\mathbb{E}\Big(\int_0^t (\nabla^2 f)(X_s)ds\Big)\end{aligned}$$

as claimed.

(b)(i) Consider first the case in which τ takes values in a finite set $I \subseteq [0,\infty[$. In this case we can induce on $\#(I)$. If $I = \{t_0\}$ then

$$\mathbb{E}(f(X_\tau)) = \mathbb{E}(f(X_{t_0})) = f(0) + \tfrac{1}{2}\mathbb{E}(\int_0^{t_0}(\nabla^2 f)(X_s)ds) = f(0) + \tfrac{1}{2}\mathbb{E}(\int_0^\tau (\nabla^2 f)(X_s)ds)$$

by (a). For the inductive step to $\#(I) > 1$, set $t_0 = \min I$, $E = \{\omega : \tau(\omega) = t_0\}$ and

$$\begin{aligned}\phi(\omega,\omega')(t) &= \omega(t) \text{ if } t \leq t_0, \\ &= \omega(t_0) + \omega'(t - t_0) \text{ if } t \geq t_0,\end{aligned}$$

for ω, $\omega' \in \Omega$, so that ϕ is inverse-measure-preserving (477G). Set

$$\sigma_\omega(\omega') = \tau(\phi(\omega,\omega')) - t_0$$

for ω, $\omega' \in \Omega$. If $\omega \in \Omega \setminus E$, σ_ω is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\geq 0}$, taking fewer than $\#(I)$ values. **P** Suppose that $t > 0$ and $F = \{\omega' : \sigma_\omega(\omega') < t\}$. If $\omega' \in F$, $\tilde\omega' \in \Omega$ and $\tilde\omega'\restriction[0,t] = \omega'\restriction[0,t]$, then $\tau(\phi(\omega,\omega')) < t + t_0$, while $\phi(\omega,\tilde\omega')(s) = \phi(\omega,\omega')(s)$ whenever $s \leq t + t_0$; so that

$$\sigma_\omega(\tilde\omega') + t_0 = \tau(\phi(\omega,\tilde\omega')) < t + t_0$$

and $\tilde\omega' \in F$. Thus $F \in \Sigma_t$; as t is arbitrary, σ_ω is adapted to $\langle\Sigma_t^+\rangle_{t\geq 0}$ (455Lb). Also every value of σ_ω belongs to $\{t - t_0 : t \in I, t > t_0\}$ which is smaller than I. **Q**

Writing $\int \ldots d\omega$ and $\int \ldots d\omega'$ for integration with respect to μ_W,

$$\begin{aligned}&\mathbb{E}\Big(\int_0^\tau (\nabla^2 f)(X_s)ds\Big) \\ &\quad= \int_\Omega\int_0^{\tau(\omega)}(\nabla^2 f)(\omega(s))ds\,d\omega \\ &\quad= \int_\Omega\int_\Omega\int_0^{\tau(\phi(\omega,\omega'))}(\nabla^2 f)(\phi(\omega,\omega')(s))ds\,d\omega' d\omega\end{aligned}$$

$$= \int_E \int_\Omega \int_0^{t_0} (\nabla^2 f)(\omega(s)) ds\, d\omega' d\omega$$

$$+ \int_{\Omega\setminus E} \int_\Omega \int_0^{t_0+\sigma_\omega(\omega')} (\nabla^2 f)(\phi(\omega,\omega')(s)) ds\, d\omega' d\omega$$

$$= \int_E \int_0^{t_0} (\nabla^2 f)(\omega(s)) ds\, d\omega + \int_{\Omega\setminus E} \int_\Omega \int_0^{t_0} (\nabla^2 f)(\phi(\omega,\omega')(s)) ds\, d\omega' d\omega$$

$$+ \int_{\Omega\setminus E} \int_\Omega \int_{t_0}^{t_0+\sigma_\omega(\omega')} (\nabla^2 f)(\phi(\omega,\omega')(s)) ds\, d\omega' d\omega$$

$$= \int_\Omega \int_0^{t_0} (\nabla^2 f)(\omega(s)) ds\, d\omega$$

$$+ \int_{\Omega\setminus E} \int_\Omega \int_{t_0}^{t_0+\sigma_\omega(\omega')} (\nabla^2 f)(\omega(t_0) + \omega'(s-t_0)) ds\, d\omega' d\omega$$

$$= \int_\Omega \int_0^{t_0} (\nabla^2 f)(\omega(s)) ds\, d\omega + \int_{\Omega\setminus E} \int_\Omega \int_0^{\sigma_\omega(\omega')} (\nabla^2 f)(\omega(t_0) + \omega'(s)) ds\, d\omega' d\omega$$

$$= 2\int_\Omega f(\omega(t_0)) - f(0) d\omega$$

$$+ 2\int_{\Omega\setminus E} \int_\Omega f(\omega(t_0) + \omega'(\sigma_\omega(\omega'))) - f(\omega(t_0)) d\omega' d\omega$$

(applying the inductive hypothesis to the function $x \mapsto f(\omega(t_0)+x)$ and the stopping time σ_ω)

$$= 2\int_E f(\omega(t_0)) - f(0) d\omega + 2\int_{\Omega\setminus E} \int_\Omega f(\omega(t_0) + \omega'(\sigma_\omega(\omega'))) - f(0) d\omega' d\omega$$

$$= 2\int_E \int_\Omega f\big(\phi(\omega,\omega')(\tau(\phi(\omega,\omega')))\big) - f(0) d\omega' d\omega$$

$$+ 2\int_{\Omega\setminus E} \int_\Omega f\big(\phi(\omega,\omega')(\tau(\phi(\omega,\omega')))\big) - f(0) d\omega' d\omega$$

$$= 2\int_\Omega \int_\Omega f\big(\phi(\omega,\omega')(\tau(\phi(\omega,\omega')))\big) - f(0) d\omega' d\omega$$

$$= 2\int_\Omega f(\omega(\tau(\omega))) - f(0) d\omega = 2\big(\mathbb{E}(f(X_\tau)) - f(0)\big).$$

Turning this around, we have the formula we want, so the induction proceeds.

(ii) Now suppose that every value of τ belongs to an infinite set of the form $\{t_n : n \in \mathbb{N}\} \cup \{\infty\}$ where $\langle t_n\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence in $[0,\infty[$. In this case, for $n \in \mathbb{N}$, define

$$\tau_n(\omega) = \min(\tau(\omega), t_n)$$

for $\omega \in \Omega$, so that τ_n takes values in the finite set $\{t_0, \dots, t_n\}$, and

$$\{\omega : \tau_n(\omega) < t\} = \{\omega : \tau(\omega) < t\} \in \Sigma_t \text{ if } t \le t_n,$$
$$= \Omega \in \Sigma_t \text{ if } t > t_n.$$

Now τ is finite a.e., so $\tau =_{\text{a.e.}} \lim_{n\to\infty} \tau_n$; it follows that

$$f(X_\tau(\omega)) = f(\omega(\tau(\omega))) = \lim_{n\to\infty} f(\omega(\tau_n(\omega))) = \lim_{n\to\infty} f(X_{\tau_n}(\omega))$$

for almost every ω; because f is bounded,

$$\mathbb{E}(f(X_\tau)) = \lim_{n\to\infty} \mathbb{E}(f(X_{\tau_n})).$$

On the other side,

$$\int_0^{\tau(\omega)} (\nabla^2 f)(\omega(s)) ds = \lim_{n\to\infty} \int_0^{\tau_n(\omega)} (\nabla^2 f)(\omega(s)) ds$$

for almost every ω. At this point, recall that we are supposing that τ has finite expectation and that $\nabla^2 f$ is bounded. So

$$|\int_0^{\tau_n(\omega)}(\nabla^2 f)(\omega(s))ds| \le \int_0^{\tau(\omega)}|(\nabla^2 f)(\omega(s))ds| \le \|\nabla^2 f\|_\infty \tau(\omega)$$

for every ω, and the dominated convergence theorem assures us that

$$\begin{aligned}\mathbb{E}\Bigl(\int_0^\tau(\nabla^2 f)(X_s)ds\Bigr) &= \int_\Omega\int_0^{\tau(\omega)}(\nabla^2 f)(\omega(s))ds\,d\omega \\ &= \lim_{n\to\infty}\int_\Omega\int_0^{\tau_n(\omega)}(\nabla^2 f)(\omega(s))ds\,d\omega \\ &= \lim_{n\to\infty}\mathbb{E}\Bigl(\int_0^{\tau_n}(\nabla^2 f)(X_s)ds\Bigr).\end{aligned}$$

Accordingly

$$\mathbb{E}(f(X_\tau)) = \lim_{n\to\infty}\mathbb{E}(f(X_{\tau_n})) = \lim_{n\to\infty} f(0) + \tfrac{1}{2}\mathbb{E}\Bigl(\int_0^{\tau_n}(\nabla^2 f)(X_s)ds\Bigr)$$

(by (i))

$$= f(0) + \tfrac{1}{2}\mathbb{E}\Bigl(\int_0^\tau(\nabla^2 f)(X_s)ds\Bigr),$$

as required.

(iii) Suppose just that τ has finite expectation. This time, for $n \in \mathbb{N}$, define a stopping time τ_n by saying that

$$\begin{aligned}\tau_n(\omega) &= 2^{-n}k \text{ if } k \ge 1 \text{ and } 2^{-n}(k-1) \le \tau(\omega) < 2^{-n}k, \\ &= \infty \text{ if } \tau(\omega) = \infty.\end{aligned}$$

If $t > 0$, set $t' = 2^{-n}k$ where $2^{-n}k < t \le 2^{-n}(k+1)$; then

$$\{\omega : \tau_n(\omega) < t\} = \{\omega : \tau(\omega) < t'\} \in \Sigma_{t'} \subseteq \Sigma_t.$$

So τ_n is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$; as $\tau_n \le 2^{-n} + \tau$, $\mathbb{E}(\tau_n) < \infty$. Again we have $\tau(\omega) = \lim_{n\to\infty}\tau_n(\omega)$ for every ω. The arguments of (ii) now tell us that, as before,

$$f(X_\tau(\omega)) = \lim_{n\to\infty} f(X_{\tau_n}(\omega))$$

(because f is continuous),

$$\int_0^{\tau(\omega)}(\nabla^2 f)(\omega(s))ds = \lim_{n\to\infty}\int_0^{\tau_n(\omega)}(\nabla^2 f)(\omega(s))ds$$

for almost every ω, so that

$$\mathbb{E}(f(X_\tau)) = \lim_{n\to\infty}\mathbb{E}(f(X_{\tau_n})),$$

$$\mathbb{E}(\int_0^\tau(\nabla^2 f)(X_s)ds) = \lim_{n\to\infty}\mathbb{E}(\int_0^{\tau_n}(\nabla^2 f)(X_s)ds).$$

(This time, of course, we need to check that

$$|\int_0^{\tau_n(\omega)}(\nabla^2 f)(\omega(s))ds| \le \|\nabla^2 f\|_\infty \tau_0(\omega)$$

for almost every ω, to confirm that we have dominated convergence.) So once again the desired formula can be got by taking the limit of a sequence of equalities we already know.

478L Theorem Let μ_W be r-dimensional Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^r)_0$, $f : \mathbb{R}^r \to [0,\infty]$ a lower semi-continuous superharmonic function, and $\tau : \Omega \to [0,\infty]$ a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$. Set $H = \{\omega : \omega \in \Omega, \tau(\omega) < \infty\}$. Then

$$f(x) \ge \int_H f(x + \omega(\tau(\omega)))\mu_W(d\omega)$$

for every $x \in \mathbb{R}^r$.

proof (a) To begin with, suppose that f is real-valued and bounded. Let $\langle\tilde{h}_m\rangle_{m\in\mathbb{N}}$ be the sequence of 473E/478J, and for $m \in \mathbb{N}$ set $f_m = f * \tilde{h}_m$. Then each f_m is non-negative, smooth with bounded derivatives of all orders

(473De) and superharmonic (478Ja), so $\nabla^2 f_m \le 0$ (478Ea). Set $\tau_n(\omega) = \min(n, \tau(\omega))$ for $n \in \mathbb{N}$ and $\omega \in \Omega$; then each τ_n is a stopping time, adapted to $\langle \Sigma_t^+ \rangle_{t \ge 0}$, with finite expectation. In the language of 478K,

$$\mathbb{E}(f_m(x + X_{\tau_n})) = f_m(x) + \tfrac{1}{2}\mathbb{E}\Big(\int_0^{\tau_n} (\nabla^2 f_m)(x + X_s) ds\Big) \le f_m(x)$$

whenever m, $n \in \mathbb{N}$. Letting $n \to \infty$,

$$\int_H f_m(x + \omega(\tau(\omega)))\mu_W(d\omega) = \lim_{n\to\infty} \int_H f_m(x + \omega(\tau_n(\omega)))\mu_W(\omega)$$

(because f_m, and every $\omega \in \Omega$, are continuous, and $\tau(\omega) = \lim_{n\to\infty} \tau_n(\omega)$ for every ω)

$$\le \liminf_{n\to\infty} \int_{\mathbb{R}^r} f_m(x + \omega(\tau_n(\omega)))\mu_W(d\omega)$$

(because f_m is non-negative)

$$\le f_m(x)$$

by Fatou's Lemma. Now $f = \lim_{m\to\infty} f_m$ (478Jc), so

$$\begin{aligned}\int_H f(x + \omega(\tau(\omega)))\mu_W(\omega) &\le \liminf_{m\to\infty} \int_H f_m(x + \omega(\tau(\omega)))\mu_W(\omega)\\ &\le \liminf_{m\to\infty} f_m(x) = f(x),\end{aligned}$$

which is what we need to know.

(b) For the general case, set $g_k = f \wedge k\chi\mathbb{R}^r$ for each $k \in \mathbb{N}$. Then g_k is non-negative, lower semi-continuous, superharmonic (478Cc) and bounded. So

$$\begin{aligned}\int_H f(x + \omega(\tau(\omega)))\mu_W(d\omega) &= \lim_{k\to\infty} \int_H g_k(x + \omega(\tau(\omega)))\mu_W(d\omega)\\ &\le \lim_{k\to\infty} g_k(x) = f(x).\end{aligned}$$

478M Proposition (a) If $r = 1$, then $\{\omega(t) : t \ge 0\} = \mathbb{R}$ for almost every $\omega \in \Omega$.
(b) If $r \le 2$, then $\{\omega(t) : t \ge 0\}$ is dense in $\mathbb{R}^2$ for almost every $\omega \in \Omega$.
(c) If $r \ge 2$, then for every $z \in \mathbb{R}^2$, $z \notin \{\omega(t) : t > 0\}$ for almost every $\omega \in \Omega$.
(d) If $r \ge 3$, then $\lim_{t\to\infty} \|\omega(t)\| = \infty$ for almost every $\omega \in \Omega$.

proof (a) Suppose that α, $\beta > 0$ and that τ is the Brownian exit time from $]-\alpha, \beta[$; then τ is a stopping time adapted to $\langle \Sigma_t \rangle_{t \ge 0}$ (477Ic). Now τ is almost everywhere finite and $\Pr(X_\tau = \beta) = \frac{\alpha}{\alpha+\beta}$. **P** Since $\Pr(|X_t| \le \max(\alpha, \beta)) \to 0$ as $t \to \infty$, τ is finite a.e., and $\Pr(X_\tau = \beta) + \Pr(X_\tau = -\alpha) = 1$. Set $\tau_n(\omega) = \min(n, \tau(\omega))$ for each n, and $f(x) = x$ for $x \in \mathbb{R}$. Then 478K tells us that

$$\mathbb{E}(X_{\tau_n}) = \mathbb{E}(f(X_{\tau_n})) = f(0) + \tfrac{1}{2}\mathbb{E}(\int_0^{\tau_n} (\nabla^2 f)(X_s) ds) = f(0) = 0.$$

Since $\langle X_{\tau_n} \rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence converging almost everywhere to X_τ,

$$\beta \Pr(X_\tau = \beta) - \alpha \Pr(X_\tau = -\alpha) = \mathbb{E}(X_\tau) = \lim_{n\to\infty} \mathbb{E}(X_{\tau_n}) = 0,$$

and $\Pr(X_\tau = \beta) = \frac{\alpha}{\alpha+\beta}$. **Q**

Letting $\alpha \to \infty$, we see that $\Pr(\exists\, t \ge 0,\ X_t = \beta) = 1$. Similarly, $-\alpha$ lies on almost every sample path.

Thus almost every sample path must pass through every point of $\mathbb{Z}$; since sample paths are continuous, they almost all cover $\mathbb{R}$.

(b) For $r = 1$ this is covered by (a); take $r = 2$. Suppose that $z \in \mathbb{R}^2$ and that $\delta > 0$. Then almost every sample path meets $B(z, \delta)$. **P** If $\delta \ge \|z\|$ this is trivial. Otherwise, take $R > \|z\|$ and let τ be the Brownian exit time from $G = \operatorname{int} B(z, R) \setminus B(z, \delta)$. We have $\Pr(\|X_t\| \le R + \|z\|) \to 0$ as $t \to \infty$ (because $\Pr(\|X_t\| \le \alpha) \le \Pr(|Z| \le \frac{\alpha}{\sqrt{t}})$

where Z is a standard normal random variable), so τ is finite a.e. Once again, set $\tau_n(\omega) = \min(n, \tau(\omega))$ for $n \in \mathbb{N}$ and $\omega \in \Omega$; this time, take $f(x) = \ln \|x - z\|$ for $x \in \mathbb{R}^2 \setminus \{z\}$. Then

$$\mathbb{E}(f(X_{\tau_n})) = f(0) = \ln \|z\|$$

(use 478Fb), so

$$\begin{aligned} \ln R \cdot \Pr(X_\tau \in \partial B(z,R)) + \ln\delta \cdot \Pr(X_\tau \in \partial B(z,\delta)) \\ = \mathbb{E}(f(X_\tau)) = \lim_{n\to\infty} \mathbb{E}(f(X_{\tau_n})) = \ln \|z\| \end{aligned}$$

and

$$\Pr(X_\tau \in \partial B(z,\delta)) = \tfrac{\ln R - \ln\|z\|}{\ln R - \ln\delta}.$$

Letting $R \to \infty$, we see that $\Pr(\exists\, t \geq 0,\, \omega(t) \in B(z,\delta)) = 1$; that is, almost every sample path meets $B(z,\delta)$. **Q**

Letting $B(z,\delta)$ run over a sequence of balls constituting a network for the topology of $\mathbb{R}^2$, we see that almost every path meets every non-empty open set and is dense in $\mathbb{R}^2$.

(c)(i) Consider first the case $r = 2$.

(α) Suppose that $z \neq 0$. In this case, take δ, R such that $0 < \delta < \|z\| < R$ and let τ be the Brownian exit time from $G = \operatorname{int} B(z,R) \setminus B(z,\delta)$, as in the proof of (b). As before, we have

$$\Pr(X_\tau \in \partial B(z,\delta)) = \tfrac{\ln R - \ln\|z\|}{\ln R - \ln\delta}.$$

This time, looking at the limit as $\delta \downarrow 0$, we see that

$$\{\omega : \text{there is a } t \geq 0 \text{ such that } \omega(t) = z \text{ but } \|\omega(s) - z\| < R \text{ for every } s \leq t\}$$

is negligible. Taking the union of these sets over large integer R, we see that

$$\{\omega : \text{there is a } t \geq 0 \text{ such that } \omega(t) = z\}$$

is negligible, as required.

(β) As for $z = 0$, take any $\epsilon > 0$. Then

$$\begin{aligned} &\mu_W\{\omega : \text{there is some } t \geq \epsilon \text{ such that } \omega(t) = 0\} \\ &\quad = \mu_W^2\{(\omega,\omega') : \text{ there is some } t \geq 0 \text{ such that } \omega'(t) = -\omega(\epsilon)\} \\ &\quad = \mu_W^2\{(\omega,\omega') : \omega(\epsilon) \neq 0 \text{ and there is some } t \geq 0 \text{ such that } \omega'(t) = -\omega(\epsilon)\} \end{aligned}$$

(because the distribution of X_ϵ is atomless, so $\{\omega : \omega(\epsilon) = 0\}$ is negligible)

$$= 0$$

by (α). Taking the union over rational $\epsilon > 0$, $\{\omega :$ there is some $t > 0$ such that $\omega(t) = 0\}$ is negligible.

(ii) If $r > 2$, set $Tx = (\xi_1, \xi_2)$ for $x = (\xi_1, \dots, \xi_r) \in \mathbb{R}^r$. Then $\omega \mapsto T\omega : C([0,\infty[\,;\mathbb{R}^r)_0 \to C([0,\infty[\,;\mathbb{R}^2)_0$ is inverse-measure-preserving for r-dimensional Wiener measure μ_{Wr} on $C([0,\infty[\,;\mathbb{R}^r)_0$ and two-dimensional Wiener measure μ_{W2} on $C([0,\infty[\,;\mathbb{R}^2)_0$, by 477D(c-i) or otherwise. So

$$\{\omega : z \in \omega[\,]0,\infty[\,]\} \subseteq \{\omega : Tz \in (T\omega)[\,]0,\infty[\,]\}$$

is negligible.

(d)(i) Fix $\gamma \in [0,\infty[$ and $\epsilon > 0$ for the moment. Set $g(x) = \int \frac{1}{\|y\|^{r-2}} \tilde{h}_0(x-y)\mu(dy)$ for $x \in \mathbb{R}^r$, where $\tilde{h}_0$ is the function of 473E; then g is smooth (473De), strictly positive and superharmonic (478Ja). In addition, we have the following.

(α) All the derivatives of g are bounded. **P** As shown in the proof of 473De, $\frac{\partial g}{\partial\xi_i}(x) = \int \frac{1}{\|y\|^{r-2}} \frac{\partial}{\partial\xi_i} \tilde{h}_0(x - y)\mu(dy)$ for $1 \leq i \leq r$ and $x \in \mathbb{R}^r$. Inducing on the order of D, and using 478Gc at the last step, we see that

$$\begin{aligned}(Dg)(x) &= \int \frac{1}{\|y\|^{r-2}}(D\tilde{h}_0)(x-y)\mu(dy) = \int \frac{1}{\|x-y\|^{r-2}}(D\tilde{h}_0)(y)\mu(dy)\\ &= \int_{B(\mathbf{0},1)} \frac{1}{\|x-y\|^{r-2}}(D\tilde{h}_0)(y)\mu(dy)\\ &\le \|D\tilde{h}_0\|_\infty \int_{B(\mathbf{0},1)} \frac{1}{\|x-y\|^{r-2}}\mu(dy) \le \frac{1}{2}r\beta_r\|D\tilde{h}_0\|_\infty\end{aligned}$$

for any partial differential operator D and any $x \in \mathbb{R}^r$. **Q**

(β) $\lim_{\|x\|\to\infty} g(x) = 0$, because

$$g(x) \le \|\tilde{h}_0\|_\infty \int_{B(\mathbf{0},1)} \frac{1}{\|x-y\|^{r-2}}\mu(dy) \le \|\tilde{h}_0\|_\infty \frac{\beta_r}{(\|x\|-1)^{r-2}}$$

whenever $\|x\| > 1$.

(γ) $g(x) = g(y)$ whenever $\|x\| = \|y\|$. **P** Let $T : \mathbb{R}^r \to \mathbb{R}^r$ be an orthogonal transformation such that $Tx = y$. Then

$$g(x) = \int \frac{1}{\|x-z\|^{r-2}}\tilde{h}_0(z)\mu(dz) = \int \frac{1}{\|T(x-z)\|^{r-2}}\tilde{h}_0(Tz)\mu(dz)$$

(because $\tilde{h}_0 T = \tilde{h}_0$)

$$= \int \frac{1}{\|y-Tz\|^{r-2}}\tilde{h}_0(Tz)\mu(dz) = \int \frac{1}{\|y-z\|^{r-2}}\tilde{h}_0(z)\mu(dz)$$

(because T is an automorphism of $(\mathbb{R}^r, \mu)$)

$$= g(y).$$ **Q**

(ii) Let $\beta > 0$ be the common value of $g(y)$ for $\|y\| = \gamma$. Take $x \in \mathbb{R}^r$ such that $\|x\| > \gamma$, and $n \in \mathbb{N}$. Define

$$\tau(\omega) = \min(\{n\} \cup \{t : \|x + \omega(t)\| \le \gamma\})$$

for $\omega \in \Omega$. Then τ is a bounded stopping time adapted to $\langle \Sigma_t \rangle_{t \ge 0}$, so

$$\begin{aligned}\beta \Pr(\tau < n) &\le \mathbb{E}(g(x + X_\tau))\\ &= g(x) + \frac{1}{2}\mathbb{E}(\int_0^\tau (\nabla^2 g)(x + X_s)ds) \le g(x).\end{aligned}$$

Letting $n \to \infty$, we see that

$$\mu_W\{\omega : \|x + \omega(t)\| \le \gamma \text{ for some } t \ge 0\} \le \frac{1}{\beta}g(x).$$

(iii) Now let $n > \gamma$ be an integer such that $\frac{1}{\beta}g(x) \le \epsilon$ whenever $\|x\| \ge n$. As in (a) and (b-i) above, $\lim_{t\to\infty} \Pr(\|X_t\| \le n) = 0$; take $m \in \mathbb{N}$ such that $\Pr(\|X_m\| \le n) \le \epsilon$. Let σ be the stopping time with constant value m, with $\phi_\sigma : \Omega \times \Omega \to \Omega$ the corresponding inverse-measure-preserving function (477G). Set $F = \{\omega : \|\omega(m)\| > n\}$. Now

$$\begin{aligned}&\Pr(\|X_t\| \le \gamma \text{ for some } t \ge m)\\ &\qquad = \mu_W^2\{(\omega, \omega') : \|\phi_\sigma(\omega, \omega')(t)\| \le \gamma \text{ for some } t \ge m\}\end{aligned}$$

(where μ_W^2 is the product measure on $\Omega \times \Omega$)

$$\begin{aligned}
&= \mu_W^2\{(\omega,\omega') : \|\omega(m)+\omega'(t-m)\| \le \gamma \text{ for some } t \ge m\}\\
&\le \mu_W^2\{(\omega,\omega') : \|\omega(m)\| \le n \text{ or } \|\omega(m)\| \ge n\\
&\qquad\qquad \text{and } \|\omega(m)+\omega'(t)\| \le \gamma \text{ for some } t \ge 0\}\\
&\le \mu\{\omega : \|\omega(m)\| \le n\}\\
&\qquad + \int_F \mu_W\{\omega' : \|\omega(m)+\omega'(t)\| \le \gamma \text{ for some } t \ge 0\}\mu_W(d\omega)\\
&\le \epsilon + \int_F \frac{1}{\beta} g(\omega(m))\mu_W(d\omega) \le \epsilon + \epsilon\mu_W F \le 2\epsilon.
\end{aligned}$$

As ϵ is arbitrary, $\Pr(\liminf_{t\to\infty}\|X_t\| < \gamma) = 0$; as γ is arbitrary, $\Pr(\lim_{t\to\infty}\|X_t\| = \infty) = 1$.

Remark In 479R I will show that there is a surprising difference between the cases $r = 3$ and $r \ge 4$.

478N Wandering paths Let $G \subseteq \mathbb{R}^r$ be an open set, and for $x \in G$ set

$$F_x(G) = \{\omega : \text{either } \tau_x(\omega) < \infty \text{ or } \lim_{t\to\infty}\|\omega(t)\| = \infty\}$$

where τ_x is the Brownian exit time from $G - x$. I will say that G has **few wandering paths** if $F_x(G)$ is conegligible for every $x \in G$. In this case we can be sure that, if $x \in G$, then for almost every ω *either* $\lim_{t\to\infty}\|\omega(t)\| = \infty$ *or* $\omega(t) \notin G - x$ for some t. So we can speak of $X_{\tau_x}(\omega) = \omega(\tau_x(\omega))$, taking this to be ∞ if $\omega \in F_x(G)$ and $\tau_x(\omega) = \infty$; and ω will be continuous on $[0, \tau_x(\omega)]$ for every $\omega \in F_x(G)$. We find that $X_{\tau_x} : \Omega \to \partial^\infty(G - x)$ is Borel measurable. **P** τ_x is the Brownian hitting time to the closed set $\mathbb{R}^r \setminus (G - x)$, so is a stopping time adapted to $\langle \mathrm{T}_{[0,t]}\rangle_{t\ge 0}$ (477Ic). Let $\mathcal{B}(\Omega)$ be the Borel σ-algebra of Ω for the topology of uniform convergence on compact sets; then $\mathrm{T}_{[0,t]} \subseteq \mathcal{B}(\Omega)$ for every $t \ge 0$. The function

$$(t,\omega) \mapsto X_t(\omega) : [0,\infty[\, \times \Omega \to \mathbb{R}^r$$

is continuous, therefore $\mathcal{B}([0,\infty[)\widehat{\otimes}\mathcal{B}(\Omega)$-measurable (4A3D(c-i)); so X_{τ_x} is $\mathcal{B}(\Omega)$-measurable (455Ld). **Q**

From 478M, we see that if $r \ge 3$ then any open set in $\mathbb{R}^r$ will have few wandering paths, while if $r \le 2$ then G will have few wandering paths whenever it is not dense in $\mathbb{R}^r$. Note that if $G \subseteq \mathbb{R}^r$ is open, H is a component of G, and $x \in H$, then the exit times from $H - x$ and $G - x$ are the same, just because sample paths are continuous, and $F_x(G) = F_x(H)$. It follows at once that if G has more than one component then it has few wandering paths.

478O Theorem Let $G \subseteq \mathbb{R}^r$ be an open set with few wandering paths and $f : \overline{G}^\infty \to \mathbb{R}$ a bounded lower semi-continuous function such that $f{\restriction}G$ is superharmonic. Take $x \in G$ and let $\tau : \Omega \to [0,\infty]$ be the Brownian exit time from $G - x$ (477Ia). Then $f(x) \ge \mathbb{E}(f(x + X_\tau))$.

proof It will be enough to deal with the case $f \ge 0$.

(a) Extend f to a function $\tilde f : \mathbb{R}^r \cup \{\infty\} \to \mathbb{R}$ by setting $\tilde f(x) = 0$ for $x \notin \overline{G}^\infty$. Since f is bounded, so is $\tilde f$. Let $\langle \tilde h_n\rangle_{n\in\mathbb{N}}$ be the sequence of 473E/478Jc, and for $n \in \mathbb{N}$ set $f_n = (\tilde f{\restriction}\mathbb{R}^r) * \tilde h_n$. Also, for $n \in \mathbb{N}$, set

$$G_n = \{y : y \in G,\ \|y\| < n,\ \rho(y, \mathbb{R}^r \setminus G) > \tfrac{1}{n+1}\}$$

(interpreting $\rho(y,\emptyset)$ as ∞ if $G = \mathbb{R}^r$), and let τ_n be the Brownian exit time from $G_n - x$.

(b) For $y \in G$, $f_n(y) \le f(y)$ for all sufficiently large n and $f(y) = \lim_{n\to\infty} f_n(y)$ (478Jb). Also $f_n{\restriction}G_n$ is superharmonic (478Ja). Each f_n is smooth with bounded derivatives of all orders (473De), and $(\nabla^2 f_n)(y) \le 0$ for $y \in G_n$ (478Ea).

If $m \ge n$,

$$\mathbb{E}(f_m(x + X_{\tau_n})) = f_m(x) + \tfrac{1}{2}\mathbb{E}(\textstyle\int_0^{\tau_n} (\nabla^2 f_m)(x + X_s)ds) \le f_m(x)$$

(478K). Consequently

$$\begin{aligned}
\mathbb{E}(f(x + X_{\tau_n})) &\le \liminf_{m\to\infty} \mathbb{E}(f_m(x + X_{\tau_n}))\\
&\le \liminf_{m\to\infty} f_m(x) = f(x).
\end{aligned}$$

(c) For every $\omega\in\Omega$, $\langle\tau_n(\omega)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with limit $\tau(\omega)$. **P** Since $\tau_n(\omega)\le\tau_{n+1}(\omega)\le\tau(\omega)$ for every n, $t=\lim_{n\in\mathbb{N}}\tau_n(\omega)$ is defined in $[0,\infty]$. If $t=\infty$ then surely $t=\tau(\omega)$. Otherwise, $\omega(t)=\lim_{n\to\infty}\omega(\tau_n(\omega))\notin G-x$, so again $t=\tau(\omega)$. **Q**

Consequently

$$f(x+\omega(\tau(\omega)))\le\liminf_{n\to\infty}f(x+\omega(\tau_n(\omega)))$$

for almost every ω. **P** In the language of 478N, we can suppose that $\omega\in F_x(G)$, so that $\omega(\tau(\omega))=\lim_{n\to\infty}\omega(\tau_n(\omega))$ in $\mathbb{R}^r\cup\{\infty\}$, and we can use the fact that f is lower semi-continuous. **Q** So

$$\mathbb{E}(f(x+X_\tau))\le\liminf_{n\to\infty}\mathbb{E}(f(x+X_{\tau_n}))\le f(x)$$

as required.

478P Harmonic measures (a) Let $A\subseteq\mathbb{R}^r$ be an analytic set and $x\in\mathbb{R}^r$. Let $\tau:\Omega\to[0,\infty]$ be the Brownian hitting time to $A-x$ (477I). Then τ is Σ-measurable, where Σ is the domain of μ_W (455Ma). Setting $H=\{\omega:\tau(\omega)<\infty\}$, $X_\tau:H\to\mathbb{R}^r$ is Σ-measurable. **P** By 4A3Wc, $(\omega,t)\mapsto\omega(t)$ is $\Sigma\widehat{\otimes}\mathcal{B}([0,\infty[)$-measurable, while $\omega\mapsto(\omega,\tau(\omega))$ is $(\Sigma,\Sigma\widehat{\otimes}\mathcal{B}([0,\infty[))$-measurable. So $\omega\mapsto\omega(\tau(\omega))$ is Σ-measurable on H. **Q**

Consider the function $\omega\mapsto x+\omega(\tau(\omega)):H\to\mathbb{R}^r$. This induces a Radon image measure μ_x on $\mathbb{R}^r$ defined by saying that

$$\mu_xF=\mu_W\{\omega:\omega\in H,\,x+\omega(\tau(\omega))\in F\}=\Pr(x+X_\tau\in F)$$

whenever this is defined. Because every $\omega\in\Omega$ is continuous, $X_\tau(\omega)\in\partial(A-x)$ for every $\omega\in H$, and ∂A is conegligible for μ_x. I will call μ_x the **harmonic measure for arrivals in** A **from** x. Of course $\mu_x\mathbb{R}^r$ is the Brownian hitting probability of A.

Note that if $F\subseteq\mathbb{R}^r$ is closed and $x\in\mathbb{R}^r\setminus F$, then the Brownian hitting time to $F-x$ is the same as the Brownian hitting time to $\partial F-x$, because all paths are continuous, so that the harmonic measure for arrivals in F from x coincides with the harmonic measure for arrivals in ∂F from x.

(b) We now have an easy corollary of 478L. Let $A\subseteq\mathbb{R}^r$ be an analytic set, $x\in\mathbb{R}^r$, and μ_x the harmonic measure for arrivals in A from x. If $f:\mathbb{R}^r\to[0,\infty]$ is a lower semi-continuous superharmonic function, $f(x)\ge\int fd\mu_x$. **P** Let τ be the Brownian hitting time to $A-x$, and $H=\{\omega:\tau(\omega)<\infty\}$. Then

$$\int fd\mu_x=\int_H f(x+\omega(\tau(\omega)))d\mu_W$$

(because μ_x is the image measure of the subspace measure $(\mu_W)_H$ under $\omega\mapsto x+\omega(\tau(\omega))$)

$$\le f(x)$$

by 478L. **Q**

(c) We can re-interpret 478O in this language. Let $G\subseteq\mathbb{R}^r$ be an open set with few wandering paths, and $x\in G$. Let μ_x be the harmonic measure for arrivals in $\mathbb{R}^r\setminus G$ from x. In this case, taking τ to be the Brownian exit time from $G-x$ and $H=\{\omega:\tau(\omega)<\infty\}$, we know that $\lim_{t\to\infty}\|\omega(t)\|=\infty$ for almost every $\omega\in\Omega\setminus H$. If $f:\partial^\infty G\to[-\infty,\infty]$ is a function, then

$$\mathbb{E}(f(x+X_\tau))=\int_H f(x+X_\tau(\omega))\mu_W(d\omega)+f(\infty)\mu_W(\Omega\setminus H)$$

(counting $f(\infty)$ as zero if G is bounded)

$$=\int fd\mu_x+f(\infty)(1-\mu_x\mathbb{R}^r)$$

if either integral is defined in $[-\infty,\infty]$ (235J[7]). In particular, if $f:\overline{G}^\infty\to\mathbb{R}$ is a bounded lower semi-continuous function and $f\restriction G$ is superharmonic, then $f(x)\ge\int fd\mu_x+f(\infty)(1-\mu_x\mathbb{R}^r)$, by 478O. Similarly, if $f:\overline{G}^\infty\to\mathbb{R}$ is continuous and $f\restriction G$ is harmonic, then $f(x)=\int fd\mu_x+f(\infty)(1-\mu_x\mathbb{R}^r)$ for every $x\in G$.

[7]Formerly 235L.

(d) Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of analytic subsets of $\mathbb{R}^r$, with union A. For $x\in\mathbb{R}^r$, let $\mu_x^{(n)}$, μ_x be the harmonic measures for arrivals in A_n, A respectively from x. Then μ_x is the limit $\lim_{n\to\infty}\mu_x^{(n)}$ for the narrow topology on the space of totally finite Radon measures on $\mathbb{R}^r$ (437Jd). **P** Let τ_n, τ be the Brownian hitting times for A_n-x, $A-x$ respectively. Then $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with limit τ. Since every $\omega\in\Omega$ is continuous, $X_\tau(\omega)=\lim_{n\to\infty}X_{\tau_n}(\omega)$ whenever $\tau(\omega)<\infty$. Set

$$H_n=\{\omega:\tau_n(\omega)<\infty\} \text{ for each } n,\ H=\{\omega:\tau(\omega)<\infty\}=\bigcup_{n\in\mathbb{N}}H_n.$$

If $f\in C_b(\mathbb{R}^r)$, then $f(x+X_\tau(\omega))=\lim_{n\to\infty}f(x+X_{\tau_n}(\omega))$ for every $\omega\in H$, so

$$\int f d\mu_x=\int_H f(x+X_\tau)d\mu_W=\lim_{n\to\infty}\int_{H_n}f(x+X_{\tau_n})d\mu_W=\lim_{n\to\infty}\int f d\mu_x^{(n)}.$$

As f is arbitrary, $\mu_x=\lim_{n\to\infty}\mu_x^{(n)}$ (437Kc). **Q**

478Q It is generally difficult to find formulae describing harmonic measures. Theorem 478I, however, gives us a technique for an important special case.

Proposition Let S be the sphere $\partial B(y,\delta)$, where $y\in\mathbb{R}^r$ and $\delta>0$. For $x\in\mathbb{R}^r\setminus S$, let ζ_x be the indefinite-integral measure over ν defined by the function

$$\begin{aligned} z&\mapsto\frac{|\delta^2-\|x-y\|^2|}{r\beta_r\delta\|x-z\|^r} \text{ if } z\in S,\\ &\mapsto 0 \text{ if } z\in\mathbb{R}^r\setminus S.\end{aligned}$$

(a) If $x\in\operatorname{int}B(y,\delta)$, then the harmonic measure μ_x for arrivals in S from x is ζ_x.

(b) In particular, the harmonic measure μ_y for arrivals in S from y is $\frac{1}{\nu S}\nu\llcorner S$.

(c) Suppose that $r\ge 2$. If $x\in\mathbb{R}^r\setminus B(y,\delta)$, then the harmonic measure μ_x for arrivals in S from x is ζ_x. In particular, $\mu_x\mathbb{R}^r=\frac{\delta^{r-2}}{\|x-y\|^{r-2}}$.

proof (a) If $g\in C_b(\mathbb{R}^r)$, then 478Ib tells us that we have a continuous function f_g which extends g, is harmonic on $\mathbb{R}^r\setminus S$ and is such that $f_g(x)=\int g\,d\zeta_x$. Now $G=\operatorname{int}B(y,\delta)$ is bounded, so it has few wandering paths (478N) and the harmonic measure μ_x is defined, with $f_g(x)=\int f_g d\mu_x$, by 478Pc. But this means that

$$\int g\,d\mu_x=\int f_g d\mu_x=f_g(x)=\int g\,d\zeta_x.$$

As g is arbitrary, $\mu_x=\zeta_x$ (415I), as claimed.

(b) If $x=y$ then

$$\frac{\delta^2-\|x-y\|^2}{r\beta_r\delta\|x-z\|^r}=\frac{\delta^2}{r\beta_r\delta^{r+1}}=\frac{1}{\nu S}$$

if $z\in S$. Since $\nu\llcorner S$ is the indefinite-integral measure over ν defined by χS (234M[8]), we have the result.

(c)(i) To see that ζ_x is the harmonic measure, we can use the same argument as in (a), with decorations. If $g\in C_b(\mathbb{R}^r)$, then 478Ib gives us a bounded continuous function f_g, harmonic on $H=\mathbb{R}^r\setminus B(y,\delta)$, such that f_g agrees with g on S, and $f_g(x)=\int g\,d\zeta_x$. S is conegligible for both μ_x and ζ_x.

(α) If $r\ge 3$, then

$$\limsup_{\|x\|\to\infty}|f_g(x)|\le\frac{\nu S}{r\beta_r\delta}\limsup_{\|x\|\to\infty}\frac{\|x-y\|^2-\delta^2}{(\|x\|-\delta-\|y\|)^r}=0.$$

So setting $f_g(\infty)=0$, $f_g:\overline{H}^\infty\to\mathbb{R}$ is continuous and bounded, and harmonic on H; so that

$$\int g\,d\zeta_x=f_g(x)=\int f_g d\mu_x+f_g(\infty)(1-\mu_x\mathbb{R}^r)=\int g d\mu_x.$$

As in (a), we conclude that $\zeta_x=\mu_x$.

(β) If $r=2$, then by 478Mb we see that almost every $\omega\in\Omega$ takes values in $B(y,\delta)-x$; so $\mu_x\mathbb{R}^r=1$. Set $\underline{f}_g(x)=f_g(x)$ for $x\in\mathbb{R}^2$, $\underline{f}_g(\infty)=\liminf_{\|x\|\to\infty}f_g(x)$. Then $\underline{f}_g$ is lower semi-continuous on $\overline{H}^\infty$ and harmonic on H, so

[8]Formerly 234E.

$$\int g\,d\zeta_x = f_g(x) = \underline{f}_g(x) \geq \int \underline{f}_g d\mu_x + \underline{f}_g(\infty)(1-\mu_x\mathbb{R}^r) = \int g\,d\mu_x.$$

Applying the same argument to $-g$, we see that $\int g\,d\zeta_x \leq \int g\,d\mu_x$, so in fact the integrals are equal, and we have the result in this case also.

(ii) Now

$$\mu_x\mathbb{R}^r = \zeta_x S = \int_S \frac{\|x-y\|^2-\delta^2}{r\beta_r\delta\|x-z\|^r}\nu(dz)$$

$$= \int_{\partial B(\mathbf{0},\delta)} \frac{\|x-y\|^2-\delta^2}{r\beta_r\delta\|x-y-z\|^r}\nu(dz) = \frac{\nu(\partial B(\mathbf{0},\delta))}{r\beta_r\delta\|x-y\|^{r-2}}$$

(478Gb)

$$= \frac{\delta^{r-2}}{\|x-y\|^{r-2}}.$$

478R Theorem Let A, $B \subseteq \mathbb{R}^r$ be analytic sets with $A \subseteq B$. For $x \in \mathbb{R}^r$, let $\mu_x^{(A)}$, $\mu_x^{(B)}$ be the harmonic measures for arrivals in A, B respectively from x. Then, for any $x \in \mathbb{R}^r$, $\langle\mu_y^{(A)}\rangle_{y\in\mathbb{R}^r}$ is a disintegration of $\mu_x^{(A)}$ over $\mu_x^{(B)}$.

proof (a) Let τ be the Brownian hitting time to $B-x$, and τ' the hitting time to $A-x$; then $\tau(\omega) \leq \tau'(\omega)$ for every ω. If $\tau(\omega) < \infty$, set $f(\omega) = x+\omega(\tau(\omega))$, so that $\mu_x^{(A)}$ is the image measure $(\mu_W)_H f^{-1}$, where $H = \{\omega : \tau(\omega) < \infty\}$. Define $\phi_\tau : \Omega \times \Omega \to \Omega$ as in 477G, so that

$$\phi_\tau(\omega,\omega')(t) = \omega(t) \text{ if } t \leq \tau(\omega),$$
$$= \omega(\tau(\omega)) + \omega'(t-\tau(\omega)) \text{ if } t \geq \tau(\omega).$$

Then we have $x+\phi_\tau(\omega,\omega')(t) \in A$ iff $t \geq \tau(\omega)$ and $f(\omega)+\omega'(t-\tau(\omega)) \in A$; so if we write σ_ω for the Brownian hitting time to $A-f(\omega)$ when $\omega \in H$, $\tau'(\phi_\tau(\omega,\omega')) = \tau(\omega)+\sigma_\omega(\omega')$.

(b) Now suppose that $E \subseteq \mathbb{R}^r$ is a Borel set. Then

$$\begin{aligned}
\mu_x^{(A)}(E) &= \mu_W\{\omega : \tau'(\omega) < \infty,\ x+\omega(\tau'(\omega)) \in E\}\\
&= \mu_W^2\{(\omega,\omega') : \tau'(\phi_\tau(\omega,\omega')) < \infty,\ x+\phi_\tau(\omega,\omega')(\tau'(\phi_\tau(\omega,\omega'))) \in E\}\\
&= \mu_W^2\{(\omega,\omega') : \tau(\omega) < \infty,\ \sigma_\omega(\omega') < \infty,\ f(\omega)+\omega'(\sigma_\omega(\omega')) \in E\}\\
&= \int_H \mu_W\{\omega' : \sigma_\omega(\omega') < \infty,\ f(\omega)+\omega'(\sigma_\omega(\omega')) \in E\}\mu_W(d\omega)\\
&= \int_H \mu_{f(\omega)}^{(A)}(E)\mu_W(d\omega) = \int \mu_y^{(A)}(E)\mu_x^{(B)}(dy).
\end{aligned}$$

The definition in 452E demands that this formula should be valid whenever E is measured by $\mu_x^{(A)}$; but in general there will be Borel sets E', E'' such that $E' \subseteq E \subseteq E''$ and $\mu_x^{(A)}(E') = \mu_x^{(A)}(E) = \mu_x^{(A)}(E'')$, in which case we must have $\mu_y^{(A)}(E') = \mu_y^{(A)}(E) = \mu_y^{(A)}(E'')$ for $\mu_x^{(B)}$-almost every y, and again $\mu_x^{(A)}(E) = \int \mu_y^{(A)}(E)\mu_x^{(B)}(dy)$.

478S Corollary Let $A \subseteq \mathbb{R}^r$ be an analytic set, and $f : \partial A \to \mathbb{R}$ a bounded universally measurable function. For $x \in \mathbb{R}^r \setminus \overline{A}$ set $g(x) = \int f d\mu_x$, where μ_x is the harmonic measure for arrivals in A from x. Then g is harmonic.

proof Suppose that $\delta > 0$ is such that $B(x,\delta) \cap \overline{A} = \emptyset$, and set $S = \partial B(x,\delta) = \partial(\mathbb{R}^r \setminus \operatorname{int} B(x,\delta))$. Then the harmonic measure for arrivals in $\mathbb{R}^r \setminus \operatorname{int} B(x,\delta)$ from x is $\frac{1}{\nu S}\nu \llcorner S$ (478Qb). So

$$g(x) = \int f d\mu_x = \int_S \frac{1}{\nu S}\int f d\mu_y \nu(dy)$$

(478R, 452F)

$$= \frac{1}{\nu S}\int_S g(y)\nu(dy).$$

As x and δ are arbitrary, g is harmonic.

478T Corollary Let $A \subseteq \mathbb{R}^r$ be an analytic set, and for $x \in \mathbb{R}^r$ let μ_x be the harmonic measure for arrivals in A from x. Then $x \mapsto \mu_x$ is continuous on $\mathbb{R}^r \setminus \overline{A}$ for the total variation metric on the set of totally finite Radon measures on $\mathbb{R}^r$ (definition: 437Qa).

proof Take any $y \in \mathbb{R}^r \setminus \overline{A}$. Let $\delta > 0$ be such that $B(y,\delta) \cap A = \emptyset$, and set $S = \partial B(y,\delta)$. For $x \in \operatorname{int} B(y,\delta)$, let ζ_x be the harmonic measure for arrivals in $\mathbb{R}^r \setminus B(y,\delta)$ from x, so that ζ_x is the indefinite-integral measure over ν defined by the function

$$\begin{aligned} z &\mapsto \tfrac{\delta^2-\|x-y\|^2}{r\beta_r\delta\|x-z\|^r} \text{ if } z \in S, \\ &\mapsto 0 \text{ if } z \in \mathbb{R}^r \setminus S \end{aligned}$$

(478Qa). Then, for any $x \in \operatorname{int} B(y,\delta)$, $\langle \mu_z \rangle_{z \in \mathbb{R}^r}$ is a disintegration of μ_x over ζ_x. So if $E \subseteq \mathbb{R}^r$ is a Borel set,

$$\mu_x E = \int \mu_z(E)\,\zeta_x(dz) = \tfrac{1}{r\beta_r\delta}\int_S \mu_z(E)\,\tfrac{\delta^2-\|x-y\|^2}{\|x-z\|^r}\nu(dz).$$

But this means that

$$|\mu_x(E) - \mu_y(E)| \le \tfrac{\nu S}{r\beta_r\delta}\sup_{z\in S}\Bigl|\tfrac{\delta^2-\|x-y\|^2}{\|x-z\|^r} - \tfrac{\delta^2}{\|y-z\|^r}\Bigr|;$$

as E is arbitrary, the distance from μ_x to μ_y is at most

$$\delta^{r-2}\sup_{z\in S}\Bigl|\tfrac{\delta^2-\|x-y\|^2}{\|x-z\|^r} - \tfrac{\delta^2}{\|y-z\|^r}\Bigr|,$$

which is small if x is close to y.

478U A variation on the technique of 478R enables us to say something about Brownian paths starting from a point in the essential closure of a set.

Proposition Suppose that $A \subseteq \mathbb{R}^r$ and that 0 belongs to the essential closure cl^*A of A as defined in 475B. Then the outer Brownian hitting probability $\mathrm{hp}^*(A)$ of A (477Ia) is 1.

proof (a) Take that $\alpha \in \,]0,1[$ such that $\frac{1-\alpha^2}{(1+\alpha)^r} = \frac{1}{2}$. Suppose that $E \subseteq \mathbb{R}^r$ is analytic, and that $0 < \delta_0 < \ldots < \delta_n$ are such that $\delta_i \le \alpha\delta_{i+1}$ for $i < n$. For $i \le n$, let τ_i be the Brownian hitting time to $S_i = \partial B(\mathbf{0},\delta_i)$. Then

$$\mu_W\{\omega : \omega(\tau_i(\omega)) \notin E \text{ for every } i \le n\} \le \textstyle\prod_{i=0}^n(1 - \frac{\nu(E\cap S_i)}{2\nu S_i}).$$

P Induce on n. If $n = 0$, then

$$\mu_W\{\omega : \omega(\tau_0(\omega)) \notin E\} = 1 - \mu_0^{(S_i)}(E)$$

(where $\mu_0^{(S_i)}$ is the harmonic measure for arrivals in S_i from 0)

$$= 1 - \tfrac{\nu(E\cap S_i)}{\nu S_i} \le 1 - \tfrac{\nu(E\cap S_i)}{2\nu S_i}$$

(478Qb). For the inductive step to $n+1 \ge 1$, let $\phi : \Omega \times \Omega \to \Omega$ be the inverse-measure-preserving function corresponding to the stopping time τ_n as in 477G; when $\tau_n(\omega)$ is finite, set $y(\omega) = \omega(\tau_n(\omega)) \in S_n$. Since $\tau_i(\omega) < \tau_{i+1}(\omega)$ whenever $i \le n$ and $\tau_i(\omega)$ is finite,

$$\tau_i(\phi(\omega,\omega')) = \tau_i(\omega)$$

for $i \le n$ and ω, $\omega' \in \Omega$. As for $\tau_{n+1}(\phi(\omega,\omega'))$, this is infinite if $\tau_n(\omega) = \infty$, and otherwise is $\sigma_{y(\omega)}(\omega')$, where σ_y is the Brownian hitting time of $S_{n+1} - y$. Now if $y \in S_n$, then

$$\mu_y^{(S_{n+1})}(E) = \int_{E\cap S_{n+1}} \tfrac{\delta_{n+1}^2-\delta_n^2}{r\beta_r\delta_{n+1}\|x-y\|^r}\nu(dx)$$

(478Qa)

$$\geq \frac{\delta_{n+1}^2-\delta_n^2}{r\beta_r\delta_{n+1}(\delta_{n+1}+\delta_n)^r}\nu(E\cap S_{n+1})$$
$$\geq \frac{1-\alpha^2}{r\beta_r\delta_{n+1}^{r-1}(1+\alpha)^r}\nu(E\cap S_{n+1}) = \frac{\nu(E\cap S_{n+1})}{2\nu S_{n+1}}.$$

Consequently

$$\mu_W\{\omega : \omega(\tau_i(\omega)) \notin E \text{ for every } i \leq n+1\}$$
$$= (\mu_W \times \mu_W)\{(\omega,\omega') : \phi(\omega,\omega')(\tau_i(\phi(\omega,\omega'))) \notin E \text{ for every } i \leq n+1\}$$
$$= (\mu_W \times \mu_W)\{(\omega,\omega') : \omega(\tau_i(\omega)) \notin E \text{ for every } i \leq n,\ \omega'(\sigma_{y(\omega)}(\omega')) \notin E\}$$
$$= \int_V \mu_W\{\omega' : \omega'(\sigma_{y(\omega)}(\omega')) \notin E\}\mu_W(d\omega)$$

(setting $V = \{\omega : \omega(\tau_i(\omega)) \notin E \text{ for every } i \leq n\}$)

$$\leq \mu_W V \cdot \sup_{y\in S_n}(1-\mu_y^{(S_{n+1})}E)$$
$$\leq \mu_W V \cdot (1-\frac{\nu(E\cap S_{n+1})}{2\nu S_{n+1}}) \leq \prod_{i=0}^{n+1}(1-\frac{\nu(E\cap S_i)}{2\nu S_i})$$

by the inductive hypothesis. So the induction continues. **Q**

(b) In particular, under the conditions of (a), $\operatorname{hp}(E) \geq 1-\prod_{i=0}^n(1-\frac{\nu(E\cap S_i)}{2\nu S_i})$. Now suppose that $A \subseteq \mathbb{R}^r$ and that $0 \in \operatorname{cl}^*A$. Let $E \supseteq A$ be an analytic set such that $\operatorname{hp}(E) = \operatorname{hp}^*(A)$ (477Id). Then $0 \in \operatorname{cl}^*E$; set $\gamma = \frac{1}{3}\limsup_{\delta\downarrow 0}\frac{\mu(E\cap B(\mathbf{0},\delta))}{\mu B(\mathbf{0},\delta)} > 0$. For any $\delta > 0$, there is a $\delta' \in \,]0,\delta]$ such that $\frac{\nu(E\cap\partial B(\mathbf{0},\delta'))}{\nu\partial B(\mathbf{0},\delta')} \geq 2\gamma$. **P** Let $\beta \in \,]0,\delta]$ be such that $\mu(E\cap B(\mathbf{0},\beta)) \geq 2\gamma\mu B(\mathbf{0},\beta)$. Then

$$\int_0^\beta \nu(E\cap\partial B(\mathbf{0},t))dt \geq 2\gamma\int_0^\beta \nu\partial B(\mathbf{0},t)dt,$$

so there must be a $\delta' \in \,]0,\beta]$ such that $\nu(E\cap\partial B(\mathbf{0},\delta')) \geq 2\gamma\nu\partial B(\mathbf{0},\delta')$. **Q**

We can therefore find, for any $n \in \mathbb{N}$, $0 < \delta_0 < \ldots < \delta_n$ such that $\delta_i \leq \alpha\delta_{i+1}$ for every $i < n$ (where α is chosen as in (a) above) and $\nu(E\cap\partial B(\mathbf{0},\delta_i)) \geq 2\gamma\nu\partial B(\mathbf{0},\delta_i)$ for every i. As noted at the beginning of this part of the proof, it follows that $\operatorname{hp}(E) \geq 1-(1-\gamma)^{n+1}$. As this is true for every $n \in \mathbb{N}$, $\operatorname{hp}(E) = 1$, so $\operatorname{hp}^*(A) = 1$, as claimed.

***478V Theorem** (a) Let $G \subseteq \mathbb{R}^r$ be an open set with few wandering paths and $f : \overline{G}^\infty \to \mathbb{R}$ a continuous function such that $f\restriction G$ is harmonic. For $x \in \mathbb{R}^r$ let $\tau_x : \Omega \to [0,\infty]$ be the Brownian exit time from $G-x$. Set

$$g_{\tau_x}(\omega) = f(x+\omega(\tau_x(\omega))) \text{ if } \tau_x(\omega) < \infty,$$
$$= f(\infty) \text{ if } \lim_{t\to\infty}\|\omega(t)\| = \infty \text{ and } \tau_x(\omega) = \infty.$$

Then $f(x) = \mathbb{E}(g_{\tau_x})$.

(b) Now suppose that σ is a stopping time adapted to $\langle\Sigma_t\rangle_{t\geq 0}$ such that $\sigma(\omega) \leq \tau_x(\omega)$ for every ω. Set

$$g_\sigma(\omega) = g_{\tau_x}(\omega) \text{ if } \sigma(\omega) = \tau_x(\omega) = \infty,$$
$$= f(x+\omega(\sigma(\omega))) \text{ otherwise.}$$

As in 455Lc, set $\Sigma_\sigma = \{E : E \in \operatorname{dom}\mu_W,\ E\cap\{\omega : \sigma(\omega) \leq t\} \in \Sigma_t \text{ for every } t \geq 0\}$. Then g_σ is a conditional expectation of g_{τ_x} on Σ_σ.

proof (a)(i) Of course if $x \notin G$ then $\tau_x(\omega) = 0$ and $g_{\tau_x}(\omega) = f(x)$ for every ω and the result is trivial. So we can suppose that $x \in G$. Note next that if there is any ω such that $\lim_{t\to\infty}\|\omega(t)\| = \infty$ and $\tau_x(\omega) = \infty$, then G must be unbounded, so $f(\infty)$ will be defined. Because G has few wandering paths, g_{τ_x} is defined almost everywhere.

(ii) Let $m \in \mathbb{N}$ be such that $\rho(x,\mathbb{R}^r\setminus G) > \frac{1}{m+1}$ and $\|x\| < m$; for $n \geq m$, set $G_n = \{y : \|y\| < n,\ \rho(y,\mathbb{R}^r\setminus G) > \frac{1}{n+1}\}$, let τ'_{xn} be the Brownian exit time from $G_n - x$ and set $\tau_{xn}(\omega) = \min(n,\tau'_{xn}(\omega))$ for every ω. Note that by 477I(c-i), $x+\omega(\tau_{xn}(\omega)) \in \overline{G}_n$ for every ω.

By 477I(c-iii) and 455L(c-v), τ_{xn} is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge 0}$. Let $\langle\tilde h_n\rangle_{n\in\mathbb{N}}$ be the sequence of 473E, and for $n\ge m$ set $f_n=\tilde f*\tilde h_n$, where $\tilde f$ is the extension of $f{\restriction}\overline{G}$ to $\mathbb{R}^r$ which takes the value 0 on $\mathbb{R}^r\setminus\overline{G}$. Then f_n is smooth and bounded. By 478Jb (applied in turn to the functions $\tilde f+M\chi\mathbb{R}^r$ and $-\tilde f+M\chi\mathbb{R}^r$ where $M=\sup_{y\in G}|f(y)|$, which of course are both superharmonic on G), f_n agrees with f on G_n, so that $f_n{\restriction}G_n$ is harmonic and $\nabla^2 f_n$ is zero on G_n (478Ec). Also, because both f_n and f are continuous, they agree on $\overline{G}_n$ and $f(x+\omega(\tau_{xn}(\omega)))=f_n(x+\omega(\tau_{xn}(\omega)))$ for every ω.

If $n\ge m$, $\omega\in\Omega$ and $0\le s<\tau_{xn}(\omega)$, then $x+\omega(s)\in G_n$ so $(\nabla^2 f_n)(x+\omega(s))=0$. Dynkin's formula (478K), applied to the function $y\mapsto f_n(x+y)$, therefore tells us that $f(x)=f_n(x)=\int f_n(x+\omega(\tau_{xn}(\omega)))\mu_W(d\omega)$.

(iii) If $\omega\in\Omega$ and $t<\tau_x(\omega)$, then the compact set $x+\omega[\,[0,t]\,]$ is included in the open set G and there is an $n\ge\max(m,t)$ such that it is included in G_n. So $\lim_{n\to\infty}\tau_{xn}(\omega)=\tau_x(\omega)$ and, because f is continuous on $\overline{G}^\infty$,

$$g_{\tau_x}(\omega)=\lim_{n\to\infty}f(x+\omega(\tau_{xn}(\omega)))=\lim_{n\to\infty}f_n(x+\omega(\tau_{xn}(\omega)))$$

for almost every ω. Since $\|f_n\|_\infty\le\|f\|_\infty<\infty$ for every n, Lebesgue's Dominated Convergence Theorem tells us that

$$\mathbb{E}(g_{\tau_x})=\lim_{n\to\infty}\int f_n(x+\omega(\tau_{xn}(\omega)))\mu_W(d\omega)=f(x),$$

as required.

(b)(i) If ω_0, $\omega_1\in\Omega$, $\sigma(\omega_0)=t$ and $\omega_1{\restriction}[0,t]=\omega_0{\restriction}[0,t]$, then $\sigma(\omega_1)=t$. **P** The set $\{\omega:\sigma(\omega)\le t\}$ belongs to Σ_t; as it contains ω_0, it contains ω_1, and $\sigma(\omega_1)\le t$. But now ω_0 agrees with ω_1 on $[0,\sigma(\omega_1)]$, so $\sigma(\omega_1)\ge\sigma(\omega)=t$. **Q**

If $H\in\Sigma_\sigma$ then $\omega_0\in H$ iff $\omega_1\in H$. **P** For every $t\ge 0$, $H\cap\{\omega:\sigma(\omega)\le t\}$ belongs to Σ_t, so contains ω_0 iff it contains ω_1. **Q**

(ii) Of course $E_\infty=\{\omega:\sigma(\omega)=\infty\}$ belongs to Σ_σ, because it has empty intersection with every set $\{\omega:\sigma(\omega)\le t\}$.

(iii) g_σ is Σ_σ-measurable. **P** For $n\in\mathbb{N}$, $\omega\in\Omega$ and $t\ge 0$, set

$$\begin{aligned}h_n(t,\omega)&=f(\omega(2^{-n}\lfloor 2^n t\rfloor))\text{ if }\omega(2^{-n}\lfloor 2^n t\rfloor)\in G,\\&=0\text{ otherwise.}\end{aligned}$$

Then h_n is $(\mathcal{B}([0,\infty[)\times\Sigma)$-measurable, so if we set $h(t,\omega)=\lim_{n\to\infty}h_n(t,\omega)$ when this is defined, h also will be $(\mathcal{B}([0,\infty[)\times\Sigma)$-measurable, and $\omega\mapsto h(\sigma(\omega),\omega)$ is Σ-measurable. Now, because $\sigma\le\tau$, $g_\sigma(\omega)=h(\sigma(\omega),\omega)$ for almost every $\omega\in\Omega\setminus E_\infty$; because μ_W is complete, g_σ is Σ-measurable. But now observe that if $t\ge 0$ and $\alpha\in\mathbb{R}$, $\{\omega:\sigma(\omega)\le t,\ g_\sigma(\omega)\ge\alpha\}$ belongs to Σ and is determined by coordinates less than or equal to t, so belongs to Σ_t. As t is arbitrary, $\{\omega:g_\sigma(\omega)\ge\alpha\}\in\Sigma_\sigma$; as α is arbitrary, g_σ is Σ_σ-measurable. **Q**

(iv) As in 477G, define $\phi_\sigma:\Omega\times\Omega\to\Omega$ by saying that

$$\begin{aligned}\phi_\sigma(\omega,\omega')(t)&=\omega(t)\text{ if }t\le\sigma(\omega),\\&=\omega(\sigma(\omega))+\omega'(t-\sigma(\omega))\text{ if }t\ge\sigma(\omega).\end{aligned}$$

Then 477G tells us that ϕ_σ is inverse-measure-preserving.

(v) $\tau_x(\phi_\sigma(\omega,\omega'))=\sigma(\omega)+\tau_{x+\omega(\sigma(\omega))}(\omega')$ for all ω, $\omega'\in\Omega$. **P** If $\sigma(\omega)=\infty$ then $\phi_\sigma(\omega,\omega')=\omega$ and

$$\tau_x(\phi_\sigma(\omega,\omega'))=\tau_x(\omega)=\sigma(\omega).$$

If $\sigma(\omega)=\tau_x(\omega)$ is finite then $\omega(\sigma(\omega))\notin G-x$ and $\phi_\sigma(\omega,\omega'){\restriction}[0,\tau_x(\omega)]=\omega{\restriction}[0,\tau_x(\omega)]$, so

$$\tau_x(\phi_\sigma(\omega,\omega'))=\tau_x(\omega)=\sigma(\omega)=\sigma(\omega)+\tau_{x+\omega(\sigma(\omega))}(\omega').$$

If $\sigma(\omega)<\tau_x(\omega)$ then $\omega(t)=\phi_\sigma(\omega,\omega')(t)$ belongs to $G-x$ for every $t\le\sigma(\omega))$ and

$$\begin{aligned}\tau_x(\phi_\sigma(\omega,\omega'))&=\inf\{t:t\ge\sigma(\omega),\ \omega(\sigma(\omega))+\omega'(t-\sigma(\omega))\notin G-x\}\\&=\sigma(\omega)+\inf\{t:\omega'(t)\notin G-x-\omega(\sigma(\omega))\}=\sigma(\omega)+\tau_{x+\omega(\sigma(\omega))}(\omega').\ \textbf{Q}\end{aligned}$$

Consequently, if $\omega\in\Omega$, $\sigma(\omega)<\infty$ and $y=\omega(\sigma(\omega))$,

$$\phi_\sigma(\omega,\omega')(\tau_x(\phi_\sigma(\omega,\omega')))=\phi_\sigma(\omega,\omega')(\sigma(\omega)+\tau_{x+y}(\omega'))=y+\omega'(\tau_{x+y}(\omega'))$$

whenever either $\tau_x(\phi_\sigma(\omega,\omega'))$ or $\tau_{x+y}(\omega')$ is finite,

$$g_{\tau_x}(\phi_\sigma(\omega,\omega')) = f(x+\phi_\sigma(\omega,\omega')(\tau_x(\phi_\sigma(\omega,\omega')))) = f(x+y+\omega'(\tau_{x+y}(\omega')))$$

for almost every ω'.

(vi) If $H \in \Sigma_\sigma$ then $\phi_\sigma^{-1}[H] = H \times \Omega$. **P** If ω, $\omega' \in \Omega$ then $\phi_\sigma(\omega,\omega')\restriction[0,\sigma(\omega)] = \omega\restriction[0,\sigma(\omega)]$, so by (i) above $\phi_\sigma(\omega,\omega') \in H$ iff $\omega \in H$. **Q**

If $H \cap E_\infty = \emptyset$, we now have

$$\begin{aligned}\int_H g_{\tau_x} &= \int_{\phi_\sigma^{-1}[H]} g_{\tau_x}(\phi_\sigma(\omega,\omega'))d(\omega,\omega')\\ &= \int_H\int f(x+\omega(\sigma(\omega))+\omega'(\tau_{x+\omega(\sigma(\omega))}(\omega)))d\omega' d\omega\\ &= \int_H f(x+\omega(\sigma(\omega)))d\omega\end{aligned}$$

(by (a) above)

$$= \int_H g_\sigma.$$

Of course we also have $\int_H g_{\tau_x} = \int_H g_\sigma$ if $H \subseteq E_\infty$. So $\int_H g_\sigma = \int_H g_{\tau_x}$ for every $H \in \Sigma_\sigma$, and g_σ is a conditional expectation of g_{τ_x} on Σ_σ.

478X Basic exercises (a) Let $G \subseteq \mathbb{R}^r$ be an open set, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of superharmonic functions from G to $[0,\infty]$. Show that $\liminf_{n\to\infty} f_n$ is superharmonic.

(b) Let $G \subseteq \mathbb{R}^r$ be an open set, and $f : G \to \mathbb{R}$ a continuous harmonic function. Show that f is smooth. (*Hint*: put 478I and 478D together.)

(c) Let $G \subseteq \mathbb{R}^r$ be an open set, and $f : G \to [0,\infty]$ a lower semi-continuous superharmonic function. Show that there is are sequences $\langle G_n\rangle_{n\in\mathbb{N}}$, $\langle f_n\rangle_{n\in\mathbb{N}}$ such that (i) $\langle G_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of open sets with union G (ii) for each $n \in \mathbb{N}$, $f_n : G_n \to [0,\infty[$ is a bounded smooth superharmonic function and $f_n \le f\restriction G_n$ (iii) $f(x) = \lim_{n\to\infty} f_n(x)$ for every $x \in G$.

>(d) Let $\langle X_t\rangle_{t\ge0}$ be Brownian motion in $\mathbb{R}^r$, and $\delta > 0$. Let τ be the Brownian hitting time to $\{x : \|x\| \ge \delta\}$. Show that $\mathbb{E}(\tau) = \frac{\delta^2}{r}$. (*Hint*: in 478K, take $f(x) = \|x\|^2$.)

(e) Show that $\mathrm{hp}^* : \mathcal{P}\mathbb{R}^r \to [0,1]$ is an outer regular Choquet capacity (definition: 432J) iff $r \ge 3$. (*Hint*: if $r \ge 3$, μ_W is inner regular with respect to $\{K : K \subseteq \Omega, \lim_{t\to\infty}\inf_{\omega\in K}\|\omega(t)\| = \infty\}$.)

(f) Show that an open subset of $\mathbb{R}$ has few wandering paths iff it is not $\mathbb{R}$ itself.

(g) Suppose $r = 2$. (i) Show that if $x \in \mathbb{R}^2 \setminus \{0\}$, then the Brownian exit time from $\mathbb{R}^2 \setminus \{x\}$ is infinite a.e. (*Hint*: use the method of part (b-ii) of the proof of 478M to show that if $R > \|x\|$ and $\delta > 0$ is small enough then most sample paths meet $\partial B(x,R)$ before they meet $B(x,\delta)$.) (ii) Show that if $G \subseteq \mathbb{R}^2$ is an open set with countable complement then G does not have few wandering paths.

(h) Suppose that $r \ge 3$. Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of analytic sets in $\mathbb{R}^r$ such that A_0 is bounded and $\bigcap_{n\in\mathbb{N}}\bar A_n = \bigcap_{n\in\mathbb{N}} A_n$, and $x \in \mathbb{R}^r$. Let $\mu_x^{(n)}$, μ_x be the harmonic measures for arrivals in A_n, $\bigcap_{m\in\mathbb{N}} A_m$ from x. Show that $\mu_x = \lim_{n\to\infty}\mu_x^{(n)}$ for the narrow topology on the set of totally finite Radon measures on $\mathbb{R}^r$. (*Hint*: 478Xe, 478Pd.)

(i) Let $A \subseteq \mathbb{R}$ be an analytic set, $x \in \mathbb{R}$ and μ_x the harmonic measure for arrivals in A from x. For $y \in \mathbb{R}$ let δ_y be the Dirac measure on $\mathbb{R}$ concentrated at y. Show that (i) if A is empty, then μ_x is the zero measure; (ii) if $A \ne \emptyset$ but $A \cap [x,\infty[\, = \emptyset$ then $\mu_x = \delta_{\sup A}$; (iii) if $A \ne \emptyset$ but $A \cap\,]-\infty,x] = \emptyset$ then $\mu_x = \delta_{\inf A}$; (iv) if A meets both $]-\infty,x]$ and $[x,\infty[$, and $y = \sup(A \cap\,]-\infty,x])$, $z = \inf(A \cap [x,\infty[)$, then $\mu_x = \delta_x$ if $y = z = x$, and otherwise $\mu_x = \frac{z-x}{z-y}\delta_y + \frac{x-y}{z-y}\delta_z$.

(j) Prove 478Qb by a symmetry argument not involving the calculations of 478I.

>(k) Let $G \subseteq \mathbb{R}^r$ be an open set, and for $x \in G$ let μ_x be the harmonic measure for arrivals in $\mathbb{R}^r \setminus G$ from x. Show that for any bounded universally measurable function $f : \partial G \to \mathbb{R}$, the function $x \mapsto \int f d\mu_x : G \to \mathbb{R}$ is continuous and harmonic.

(l) (i) Suppose that $r = 2$, and that x, y, $z \in \mathbb{R}^r$ are such that $\|x\| < 1 = \|z\|$ and $y = 0$. Identify $\mathbb{R}^2$ with $\mathbb{C}$, and express x, z as $\gamma e^{i\theta}$ and e^{it} respectively. Show that, in the language of 272Yg, $\frac{\|y-z\|^2-\|x-y\|^2}{\|x-z\|^r} = A_\gamma(\theta - t)$. (ii) Compare 478I(b-iii) with 272Yg(iii).

478Y Further exercises (a)(i) Show that there is a function $f : \mathbb{R} \to \mathbb{Q}$ which is 'harmonic' in the sense of 478B, but is not continuous. (*Hint*: take f to be a linear operator when $\mathbb{R}$ is regarded as a linear space over $\mathbb{Q}$.) (ii) Show that if the continuum hypothesis is true, there is a surjective function $f : \mathbb{R}^2 \to \{0, 1\}$ which is 'harmonic' in the sense of 478B.

(b) Let $G \subseteq \mathbb{R}^r$ be a connected open set, and $f : G \to [0, \infty]$ a superharmonic Lebesgue measurable function which is not everywhere infinite. Show that f is locally integrable.

(c) Let $G \subseteq \mathbb{R}^2$ be an open set, and $f : G \to \mathbb{C}$ a function which is analytic when regarded as a function of a complex variable. Show that $\mathcal{R}\mathrm{e}\, f$ is harmonic. (*Hint*: The non-trivial part is the theorem that f has continuous second partial derivatives.)

(d) Define $\psi : \mathbb{R}^r \setminus \{0\} \to \mathbb{R}^r$ by setting $\psi(x) = \frac{x}{\|x\|^2}$. For a $[-\infty, \infty]$-valued function f defined on a subset of $\mathbb{R}^r$, set $f^*(x) = \frac{1}{\|x\|^{r-2}} f(\psi(x))$ for $x \in \psi^{-1}[\mathrm{dom}\, f] \setminus \{0\}$. (This is the **Kelvin transform** of f relative to the sphere $\partial B(\mathbf{0}, 1)$.) (i) Show that if f is real-valued and twice continuously differentiable, then $(\nabla^2 f^*)(x) = \frac{1}{\|x\|^{r+2}}(\nabla^2 f)(\psi(x))$ for $x \in \mathrm{dom}\, f^*$. (ii) Show that if $\mathrm{dom}\, f$ is open and f is non-negative, lower semi-continuous and superharmonic, then f^* is superharmonic.

(e) Let $G \subseteq \mathbb{R}^2$ be an open set. Show that G has few wandering paths iff there is an $x \in \mathbb{R}^2$ such that $\mathrm{hp}((\mathbb{R}^2 \setminus (G \cup \{x\})) - x) > 0$.

(f) Show that 478K remains true if we replace 'three-times-differentiable function such that f and its first three derivatives are continuous and bounded' with 'twice-differentiable function such that f and its first two derivatives are continuous and bounded'.

(g) Let $f : \mathbb{R}^r \to \mathbb{R}$ be a twice-differentiable function such that f and its first two derivatives are continuous and bounded. Show that

$$\lim_{\delta \downarrow 0} \frac{1}{\delta^2}\Bigl(f(x) - \frac{1}{\mu B(x,\delta)}\int_{B(x,\delta)} f d\mu\Bigr) = -\frac{(\nabla^2 f)(x)}{2(r+2)}$$

for every $x \in \mathbb{R}^r$.

(h) Let $G \subseteq \mathbb{R}^r$ be an open set and $f : G \to \mathbb{R}$ a continuous function such that

$$\lim_{\delta \downarrow 0} \frac{1}{\delta^2}\Bigl(f(x) - \frac{1}{\mu B(x,\delta)}\int_{B(x,\delta)} f d\mu\Bigr) = 0$$

for every $x \in G$. Show that f is harmonic.

(i) Let $f : \mathbb{R}^r \to [0, \infty]$ be a lower semi-continuous superharmonic function. Show that $f\omega$ is continuous for μ_W-almost every $\omega \in \Omega$.

(j) Suppose that $A \subseteq \mathbb{R}^r$. Show that $x \mapsto \mathrm{hp}^*(A - x)$ is lower semi-continuous at every point of $\mathbb{R}^r \setminus A$, and continuous at every point of $\mathbb{R}^r \setminus \overline{A}$.

(k) Suppose that $A \subseteq \mathbb{R}^r$ is such that $\inf_{\delta > 0} \mathrm{hp}^*(A \cap B(\mathbf{0}, \delta)) > 0$. Show that $\mathrm{hp}^*(A) = 1$.

(l) Let μ_W be three-dimensional Wiener measure on $\Omega = C([0,\infty[\,;\mathbb{R}^3)_0$, and e a unit vector in $\mathbb{R}^3$. Set $Y_t(\omega) = \frac{1}{\|\omega(t)-e\|}$ for $\omega \in \Omega$ and $t \in [0,\infty[$ such that $\omega(t) \ne e$. (i) Show that if $R > 1$ then

$$\begin{aligned}\mathbb{E}(Y_t) &= \frac{1}{(\sqrt{2\pi t})^3}\int \frac{\exp(-\|x\|^2/2t)}{\|x-e\|}\mu(dx)\\ &\le \frac{1}{(\sqrt{2\pi t})^3}\int_{B(\mathbf{0},1)}\frac{1}{\|x\|}\mu(dx) + \frac{1}{(\sqrt{2\pi t})^3}\int_{B(e,R)\setminus B(e,1)} e^{-\|x\|^2/2t}\mu(dx) + \frac{1}{R} \to \frac{1}{R}\end{aligned}$$

as $t \to \infty$, so that $\lim_{t\to\infty}\mathbb{E}(Y_t) = 0$ and $\langle Y_t\rangle_{t\ge0}$ is not a martingale. (ii) Show that if $n \ge 1$ and τ_n is the Brownian hitting time to $B(e,2^{-n})$, then $\langle Y_{t\wedge\tau_n}\rangle_{t\ge0}$ is a martingale, where $t\wedge\tau_n$ is the stopping time $\omega \mapsto \min(t,\tau_n(\omega))$. (iii) Show that $\langle\tau_n(\omega)\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence with limit ∞ for almost every ω. ($\langle Y_t\rangle_{t\ge0}$ is a 'local martingale'.)

478 Notes and comments I find that books are still being published on the subject of potential theory which ignore Brownian motion. To my eye, Newtonian potential, at least (and this is generally acknowledged to be the core of the subject), is an essentially geometric concept, and random walks are an indispensable tool for understanding it. So I am giving these priority, at the cost of myself ignoring Green's functions.

The definitions in 478B are already unconventional; most authors take it for granted that harmonic functions should be finite-valued and continuous (see 478Cd). All the work of this section refers to measurable functions. But there are things which can be said about non-measurable functions satisfying the definitions here (478Ya). Let me draw your attention to 478Fa and 478H. If we want to say that $x \mapsto \frac{1}{\|x\|^{r-2}}$ is harmonic, we have to be careful not to define it at 0. If (for $r \ge 3$) we allow $\frac{1}{0^{r-2}} = \infty$, we get a superharmonic function. If (for $r = 1$) we allow $\frac{1}{0^{-1}} = 0$, we get a subharmonic function. The slightly paradoxical phenomenon of 478Yl is another manifestation of this.

I hope that using the operations ${}^{-\infty}$ and ∂^∞ does not make things more difficult. The point is that by compactifying $\mathbb{R}^r$ we get an efficient way of talking about $\lim_{\|x\|\to\infty} f(x)$ when we need to. This is particularly effective for Brownian paths, since in three and more dimensions almost all paths go off to infinity (478Md). In two dimensions the situation is more complex (478Mb-478Mc), and we have to consider the possibility that a path ω in an open set may be 'wandering', in the sense that it neither strikes the boundary nor goes to infinity, and $\lim_{t\to\infty} f(\omega(t))$ may fail to exist even for the best-behaved functions f. Of course this already happens in one dimension, but only when $G = \mathbb{R}$, and classical potential theory (though not, I think, Brownian potential theory) is nearly trivial in the one-dimensional case.

Many readers will also find that setting $r = 3$ and $r\beta_r = 4\pi$ will make the formulae easier to digest. I allow for variations in r partly in order to cover the case $r = 2$ (in this section, though not in the next, many of the ideas translate directly into the one- and two-dimensional cases), and partly because it is not always easy to guess at a formula for $r \ge 4$ from the formula for $r = 3$. There is little extra work to be done, given that §§472-475 cover the general case.

I call 478K a 'lemma' because I have made no attempt to look for weakest adequate hypotheses; of course we don't really need third derivatives (478Yf). The 'theorems' are 478L and 478O, where the hypotheses seem to mark natural boundaries of the arguments given. In 478O I use a language which is both unusual and slightly contorted, in order to do as much as possible without splitting the cases $r \le 2$ from the rest. Of course any result involving the notion of 'few wandering paths' really has two forms; one when $r \ge 3$, so that there is no restriction on the open set and we are genuinely making use of the one-point compactification of $\mathbb{R}^r$, and one when $r \le 2$, in which essentially all our paths are bounded.

Theorem 478O leads directly to a solution of Dirichlet's problem, in the sense that, for an open set G with few wandering paths, we have a family of measures enabling us to calculate the values within G of a continuous function on $\overline{G}^\infty$ which satisfies Laplace's equation inside G (478Pc). We do not get a satisfactory existence theorem; we can use harmonic measures to generate many harmonic functions on G (478S), but we do not get good information on their behaviour near ∂G, and are left guessing at which continuous functions on $\partial^\infty G$ will be extended continuously. The method does, however, make it clear that what matters is the geometry of the boundary; we need to know whether, starting from a point near the boundary, a random walk will hit the boundary soon. So at least we can see from 478M that (if $r \ge 2$) an isolated point of ∂G will be at worst an irrelevant distraction. For $r \ge 3$ the next section will give some useful information (479P *et seq.*), though I shall not have space for a proper analysis.

The idea of 478Vb is that we have a particularly dramatic kind of martingale. Writing S for the set of stopping times $\sigma \le \tau$, it is easy to see that the family $\langle g_\sigma \rangle_{\sigma \in S}$ is a martingale in the sense that $\Sigma_\sigma \subseteq \Sigma_{\sigma'}$ and g_σ is a conditional expectation of $g_{\sigma'}$ on Σ_σ whenever $\sigma \le \sigma'$ in S.

479 Newtonian capacity

I end the chapter with a sketch of fragments of the theory of Newtonian capacity. I introduce equilibrium measures as integrals of harmonic measures (479B); this gives a quick definition of capacity (479C), with a substantial number of basic properties (479D, 479E), including its extendability to a Choquet capacity (479Ed). I give sufficient fragments of the theory of Newtonian potentials (479F, 479J) and harmonic analysis (479G, 479I) to support the classical definitions of capacity and equilibrium measures in terms of potential and energy (479K, 479N). The method demands some Fourier analysis extending that of Chapter 28 (479H). 479P is a portmanteau theorem on generalized equilibrium measures and potentials with an exact description of the latter in terms of outer Brownian hitting probabilities. I continue with notes on capacity and Hausdorff measure (479Q), self-intersecting Brownian paths (479R) and an example of a discontinuous equilibrium potential (479S). Yet another definition of capacity, for compact sets, can be formulated in terms of gradients of potential functions (479U); this leads to a simple inequality relating capacity to Lebesgue measure (479V). The section ends with an alternative description of capacity in terms of a measure on the family of closed subsets of $\mathbb{R}^r$ (479W).

479A Notation In this section, unless otherwise stated, r will be a fixed integer greater than or equal to 3. As in §478, μ will be Lebesgue measure on $\mathbb{R}^r$, and β_r the measure of the unit ball $B(\mathbf{0},1)$; ν will be normalized $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$, so that $\nu(\partial B(\mathbf{0},1)) = r\beta_r$.

Recall that if ζ is a measure on a space X, and $E \in \operatorname{dom}\zeta$, then $\zeta \llcorner E$ is defined by saying that $(\zeta \llcorner E)(F) = \zeta(E \cap F)$ whenever $F \subseteq X$ and ζ measures $E \cap F$ (234M[9]). If ζ is a Radon measure, so is $\zeta \llcorner E$ (416S).

As in §478, Ω will be $C([0,\infty[\,;\mathbb{R}^r)_0$, with the topology of uniform convergence on compact sets; μ_W will be Wiener measure on Ω. Recall that the Brownian hitting probability $\mathrm{hp}(D)$ of a set $D \subseteq \mathbb{R}^r$ is $\mu_W\{\omega : \omega^{-1}[D] \ne \emptyset\}$ if this is defined, and that for any $D \subseteq \mathbb{R}^r$ the outer Brownian hitting probability is $\mathrm{hp}^*(D) = \mu_W^*\{\omega : \omega^{-1}[D] \ne \emptyset\}$ (477Ia).

If $x \in \mathbb{R}^r$ and $A \subseteq \mathbb{R}^r$ is an analytic set, $\mu_x^{(A)}$ will be the harmonic measure for arrivals in A from x (478P); note that $\mu_x^{(A)}(\mathbb{R}^r) = \mu_x^{(A)}(\partial A) = \mathrm{hp}(A - x)$.

I will write ρ_{tv} for the total variation metric on the space $M_{\mathrm{R}}^+(\mathbb{R}^r)$ of totally finite Radon measures on $\mathbb{R}^r$, so that

$$\rho_{\mathrm{tv}}(\lambda,\zeta) = \sup\nolimits_{E,F \subseteq \mathbb{R}^r \text{ are Borel}} \lambda E - \zeta E - \lambda F + \zeta F$$

for λ, $\zeta \in M_{\mathrm{R}}^+(\mathbb{R}^r)$ (437Qa).

479B Theorem Let $A \subseteq \mathbb{R}^r$ be a bounded analytic set.

(i) There is a Radon measure λ_A on $\mathbb{R}^r$, with support included in ∂A, defined by saying that $\langle \frac{1}{r\beta_r\gamma}\mu_x^{(A)} \rangle_{x \in \partial B(\mathbf{0},\gamma)}$ is a disintegration of λ_A over the subspace measure $\nu_{\partial B(\mathbf{0},\gamma)}$ whenever $\gamma > 0$ and $\overline{A} \subseteq \operatorname{int} B(\mathbf{0},\gamma)$.

(ii) λ_A is the limit $\lim_{\|x\|\to\infty} \|x\|^{r-2}\mu_x^{(A)}$ for the total variation metric on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

proof (a) Suppose that $\gamma > 0$ is such that $\overline{A} \subseteq \operatorname{int} B(\mathbf{0},\gamma)$. By 478T, $x \mapsto \mu_x^{(A)}(E) : \partial B(\mathbf{0},\gamma) \to [0,\infty[$ is continuous for every Borel set $E \subseteq \mathbb{R}^r$, and $\mu_x^{(A)}(\mathbb{R}^r \setminus \overline{A}) = 0$ for every x, so $\{\mu_x^{(A)} : x \in \partial B(\mathbf{0},\gamma)\}$ is uniformly tight. By 452Da, we have a unique totally finite Radon measure ζ_γ such that $\langle \frac{1}{r\beta_r\gamma}\mu_x^{(A)} \rangle_{x \in \partial B(\mathbf{0},\gamma)}$ is a disintegration of ζ_γ over the subspace measure $\nu_{\partial B(\mathbf{0},\gamma)}$. Since $\mathbb{R}^r \setminus \partial A$ is $\mu_x^{(A)}$-negligible for every $x \in \partial B(\mathbf{0},\gamma)$ (478Pa), it is also ζ_γ-negligible.

(b) Now suppose that $\overline{A} \subseteq \operatorname{int} B(\mathbf{0},\gamma)$ and that $\|x\| = M\gamma$, where $M > 1$. Then 478R tells us that $\langle \mu_y^{(A)} \rangle_{y \in \mathbb{R}^r}$ is a disintegration of $\mu_x^{(A)}$ over $\mu_x^{(B(\mathbf{0},\gamma))}$. So, for any Borel set $E \subseteq \mathbb{R}^r$,

[9] Formerly 234E.

$$|\zeta_\gamma E-\|x\|^{r-2}\mu_x^{(A)}E| = \frac{1}{r\beta_r\gamma}\Bigl|\int_S \mu_y^{(A)}E\,\nu(dy)-\|x\|^{r-2}\int_{\mathbb{R}^r}\mu_y^{(A)}E\,\mu_x^{(B(\mathbf{0},\gamma))}(dy)\Bigr|$$

(where $S=\partial B(\mathbf{0},\gamma)$)

$$= \frac{1}{r\beta_r\gamma}\Bigl|\int_S \mu_y^{(A)}E\,\nu(dy)-\|x\|^{r-2}\int_S\mu_y^{(A)}E\,\mu_x^{(S)}(dy)\Bigr|$$

(478Pa)

$$= \frac{1}{r\beta_r\gamma}\Bigl|\int_S \mu_y^{(A)}E\,\nu(dy)-\|x\|^{r-2}\int_S\frac{\|x\|^2-\gamma^2}{r\beta_r\gamma\|x-y\|^r}\mu_y^{(A)}E\,\nu(dy)\Bigr|$$

(478Q)

$$\le \frac{1}{r\beta_r\gamma}\int_S|1-\|x\|^{r-2}\frac{\|x\|^2-\gamma^2}{\|x-y\|^r}|\mu_y^{(A)}E\nu(dy)$$

$$\le \frac{\nu S}{r\beta_r\gamma}\sup_{y\in S}|1-\|x\|^{r-2}\frac{\|x\|^2-\gamma^2}{\|x-y\|^r}|$$

$$\le \gamma^{r-2}\sup_{y\in S}(|1-\frac{\|x\|^r}{\|x-y\|^r}|+\frac{\gamma^2\|x\|^{r-2}}{\|x-y\|^r})$$

$$= \gamma^{r-2}\bigl(|\frac{M^r\gamma^r}{(M\gamma-\gamma)^r}-1|+\frac{\gamma^rM^{r-2}}{\gamma^r(M-1)^r}\bigr)$$

$$= \gamma^{r-2}\bigl(|\frac{M^{r-2}}{(M-1)^r}-1|+\frac{M^r}{(M-1)^r}\bigr).$$

(c) Accordingly

$$\rho_{\mathrm{tv}}(\zeta_\gamma,\|x\|^{r-2}\mu_x^{(A)})\le 2\gamma^{r-2}\bigl(|\frac{M^r}{(M-1)^r}-1|+\frac{M^{r-2}}{(M-1)^r}\bigr)$$

whenever $\gamma>0$, $\overline{A}\subseteq\operatorname{int}B(\mathbf{0},\gamma)$ and $\|x\|=M\gamma>\gamma$; so that

$$\zeta_\gamma=\lim_{\|x\|\to\infty}\|x\|^{r-2}\mu_x^{(A)}$$

for the total variation metric whenever $\gamma>0$ and $\overline{A}\subseteq\operatorname{int}B(\mathbf{0},\gamma)$. We can therefore write λ_A for the limit, and both (i) and (ii) will be true, since I have already checked that $\operatorname{supp}(\zeta_\gamma)\subseteq\partial A$ for all large γ.

479C Definitions (a)(i) In the context of 479B, I will call λ_A the **equilibrium measure** of the bounded analytic set A, and $\lambda_A\mathbb{R}^r=\lambda_A(\partial A)$ the **Newtonian capacity** $\operatorname{cap}A$ of A.

(ii) For any $D\subseteq\mathbb{R}^r$, its **Choquet-Newton capacity** will be

$$c(D)=\inf_{G\supseteq D\text{ is open}}\sup_{K\subseteq G\text{ is compact}}\operatorname{cap}K.$$

(I will confirm in 479Ed below that c is in fact a capacity as defined in §432.) Sets with zero Choquet-Newton capacity are called **polar**.

(b) If ζ is a Radon measure on $\mathbb{R}^r$, the **Newtonian potential** associated with ζ is the function $W_\zeta:\mathbb{R}^r\to[0,\infty]$ defined by the formula

$$W_\zeta(x)=\int_{\mathbb{R}^r}\frac{1}{\|y-x\|^{r-2}}\zeta(dy)$$

for $x\in\mathbb{R}^r$. The **energy** of ζ is now

$$\operatorname{energy}(\zeta)=\int W_\zeta d\zeta=\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}\zeta(dy)\zeta(dx).$$

If A is a bounded analytic subset of $\mathbb{R}^r$, the potential $\tilde W_A=W_{\lambda_A}$ is the **equilibrium potential** of A.

(In 479P below I will describe constructions of equilibrium measures and potentials for arbitrary subsets D of $\mathbb{R}^r$ such that $c(D)$ is finite.)

(c) If ζ is a Radon measure on $\mathbb{R}^r$, I will write U_ζ for the **$(r-1)$-potential** of ζ, defined by saying that $U_\zeta(x)=\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-1}}\zeta(dy)\in[0,\infty]$ for $x\in\mathbb{R}^r$.

479D The machinery in Theorem 479B gives an efficient method of approaching several fundamental properties of equilibrium measures. I start with some elementary calculations.

Proposition (a) For any $\gamma > 0$ and $z \in \mathbb{R}^r$, the Newtonian capacity of $B(z,\gamma)$ is γ^{r-2}, the equilibrium measure of $B(z,\gamma)$ is $\frac{1}{r\beta_r\gamma}\nu\llcorner\partial B(z,\gamma)$, and the equilibrium potential of $B(z,\gamma)$ is given by

$$\tilde{W}_{B(z,\gamma)}(x) = \min(1, \frac{\gamma^{r-2}}{\|x-z\|^{r-2}})$$

for every $x \in \mathbb{R}^r$.

(b) Let $A \subseteq \mathbb{R}^r$ be a bounded analytic set with equilibrium measure λ_A and equilibrium potential $\tilde{W}_A$.

(i) $\tilde{W}_A(x) \le 1$ for every $x \in \mathbb{R}^r$.

(ii) If $B \subseteq A$ is another analytic set, $\tilde{W}_B \le \tilde{W}_A$.

(iii) $\tilde{W}_A(x) = 1$ for every $x \in \operatorname{int} A$.

(c) Let A, $B \subseteq \mathbb{R}^r$ be bounded analytic sets.

(i) Defining $+$ and $\le$ as in 234G[10] and 234P, $\lambda_{A\cup B} \le \lambda_A + \lambda_B$.

(ii) $\lambda_A B \le \operatorname{cap} B$.

proof (a) For $x \in \mathbb{R}^r \setminus B(z,\gamma)$, $\mu_x^{(B(z,\gamma))}$ is the indefinite-integral measure over $\nu\llcorner\partial B(z,\gamma)$ defined by the function $y \mapsto \frac{\|x-z\|^2-\gamma^2}{r\beta_r\gamma\|x-y\|^r}$ (478Qc). So $\|x\|^{r-2}\mu_x^{(B(z,\gamma))}$ is the indefinite-integral measure defined by

$$y \mapsto f_x(y) = \frac{\|x\|^{r-2}(\|x-z\|^2-\gamma^2)}{r\beta_r\gamma\|x-y\|^r}.$$

As $\|x\| \to \infty$, $f_x(y) \to \frac{1}{r\beta_r\gamma}$ uniformly for $y \in \partial B(z,\gamma)$, so $\lambda_{B(\mathbf{0},\gamma)} = \lim_{\|x\|\to\infty} \|x\|^{r-2}\mu_x^{(B(z,\gamma))}$ is $\frac{1}{r\beta_r\gamma}\nu\llcorner\partial B(z,\gamma)$. Consequently the capacity of $B(z,\gamma)$ is $\frac{1}{r\beta_r\gamma}\nu(\partial B(z,\gamma)) = \gamma^{r-2}$, and the equilibrium potential is

$$\tilde{W}_{B(z,\gamma)}(x) = \frac{1}{r\beta_r\gamma}\int_{\partial B(z,\gamma)} \frac{1}{\|y-x\|^{r-2}}\nu(dy)$$

$$= \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)} \frac{1}{\|y+z-x\|^{r-2}}\nu(dy) = \frac{1}{r\beta_r\gamma}\cdot\frac{\nu(\partial B(\mathbf{0},\gamma))}{\max(\gamma,\|x-z\|)^{r-2}}$$

(478Ga)

$$= \min(1, \frac{\gamma^{r-2}}{\|x-z\|^{r-2}}).$$

(b)(i) Let $\gamma > 0$ be such that $\overline{A} \subseteq \operatorname{int} B(\mathbf{0},\gamma)$. Then

$$\tilde{W}_A(x) = \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}}\lambda_A(dy) = \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}}\mu_z^{(A)}(dy)\nu(dz)$$

(452F)

$$\le \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)} \frac{1}{\|x-z\|^{r-2}}\nu(dz)$$

(478Pb, 478H)

$$= \frac{\nu(\partial B(\mathbf{0},\gamma))}{r\beta_r\gamma}\frac{1}{\max(\gamma,\|x\|)^{r-2}}$$

(478Ga)

$$\le 1.$$

(ii) Let $\gamma > 0$ be such that $\overline{A} \subseteq \operatorname{int} B(\mathbf{0},\gamma)$. Then

[10] Formerly 112Xe.

$$\begin{aligned}\tilde{W}_B(x) &= \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}}\lambda_B(dy)\\ &= \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}\mu_z^{(B)}(dy)\nu(dz)\\ &= \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}\mu_w^{(B)}(dy)\mu_z^{(A)}(dw)\nu(dz)\end{aligned}$$

(because $\langle\mu_w^{(B)}\rangle_{w\in\mathbb{R}^r}$ is a disintegration of $\mu_z^{(B)}$ over $\mu_z^{(A)}$ for every z, by 478R)

$$\le \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r}\frac{1}{\|x-w\|^{r-2}}\mu_z^{(A)}(dw)\nu(dz)$$

(by 478Pb, because $y\mapsto\frac{1}{\|x-y\|^{r-2}}$ is continuous and superharmonic)

$$= \tilde{W}_A(x).$$

(iii) If $x\in\operatorname{int}A$, there is a $\gamma>0$ such that $B(x,\gamma)\subseteq A$; now, putting (a) and (ii) above together,

$$\tilde{W}_A(x)\ge\tilde{W}_{B(x,\gamma)}(x)=1.$$

Since we know from (i) that $\tilde{W}_A(x)\le 1$, we have equality.

(c)(i) Suppose that $K\subseteq\mathbb{R}^r$ is compact and that $x\in\mathbb{R}^r$. Let τ_A, τ_B and $\tau_{A\cup B}$ be the Brownian hitting times to $A-x$, $B-x$ and $(A\cup B)-x$ respectively. Then $\tau_{A\cup B}=\tau_A\wedge\tau_B$. Now

$$\begin{aligned}\mu_x^{(A\cup B)}(K) &= \mu_W\{\omega:\tau_{A\cup B}(\omega)<\infty,\ x+\omega(\tau_{A\cup B}(\omega))\in K\}\\ &\le \mu_W\{\omega:\tau_{A\cup B}(\omega)=\tau_A(\omega)<\infty,\ x+\omega(\tau_A(\omega))\in K\}\\ &\qquad\qquad + \mu_W\{\omega:\tau_{A\cup B}(\omega)=\tau_B(\omega)<\infty,\ x+\omega(\tau_B(\omega))\in K\}\\ &\le \mu_x^{(A)}(K)+\mu_x^{(B)}(K).\end{aligned}$$

Multiplying by $\|x\|^{r-2}$ and letting $\|x\|\to\infty$,

$$\lambda_{A\cup B}(K)\le\lambda_A K+\lambda_B K=(\lambda_A+\lambda_B)(K)$$

for every K, which is the criterion of 416E(a-ii).

(ii) For any $x\in\mathbb{R}^r$,

$$\mu_x^{(A)}(B)=\mu_W\{\omega:\tau_A(\omega)<\infty,\ x+\omega(\tau_A(\omega))\in B\}$$

(where $\tau_A(\omega)$ is the Brownian hitting time to $A-x$)

$$\le\mu_W\{\omega:\omega^{-1}[B-x]\ne\emptyset\}=\mu_x^{(B)}(\mathbb{R}^r).$$

Multiplying by $\|x\|^{r-2}$ and taking the limit as $\|x\|\to\infty$, $\lambda_A B\le\operatorname{cap}B$.

479E Theorem (a) Newtonian capacity cap is submodular (definition: 432Jc).

(b) Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of analytic subsets of $\mathbb{R}^r$ with bounded union A.

(i) The equilibrium measure λ_A is the limit $\lim_{n\to\infty}\lambda_{A_n}$ for the narrow topology on the space $M_{\mathrm{R}}^+(\mathbb{R}^r)$ of totally finite Radon measures on $\mathbb{R}^r$.

(ii) $\operatorname{cap}A=\lim_{n\to\infty}\operatorname{cap}A_n$.

(iii) The equilibrium potential $\tilde{W}_A$ is $\lim_{n\to\infty}\tilde{W}_{A_n}=\sup_{n\in\mathbb{N}}\tilde{W}_{A_n}$.

(c) Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of bounded analytic subsets of $\mathbb{R}^r$ such that $\bigcap_{n\in\mathbb{N}}A_n=\bigcap_{n\in\mathbb{N}}\overline{A_n}=A$ say.

(i) λ_A is the limit $\lim_{n\to\infty}\lambda_{A_n}$ for the narrow topology on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

(ii) $\operatorname{cap}A=\lim_{n\to\infty}\operatorname{cap}A_n$.

(d)(i) Choquet-Newton capacity $c:\mathcal{P}\mathbb{R}^r\to[0,\infty]$ is the unique outer regular Choquet capacity on $\mathbb{R}^r$ extending cap.

(ii) c is submodular.

(iii) $c(A) = \sup\{\operatorname{cap} K : K \subseteq A \text{ is compact}\}$ for every analytic set $A \subseteq \mathbb{R}^r$.

proof (a) Let A, $B \subseteq \mathbb{R}^r$ be bounded analytic sets. If $x \in \mathbb{R}^r$, then

$$\operatorname{hp}((A \cup B) - x) + \operatorname{hp}((A \cap B) - x) \le \operatorname{hp}(A - x) + \operatorname{hp}(B - x).$$

P For $C \subseteq \mathbb{R}^r$ set

$$H_C = \{\omega : \omega \in \Omega, \text{ there is some } t \ge 0 \text{ such that } x + \omega(t) \in C\},$$

so that if C is an analytic set, $\operatorname{hp}(C - x) = \mu_W H_C$. Then

$$H_{A\cup B} = H_A \cup H_B, \quad H_{A\cap B} \subseteq H_A \cap H_B,$$

so

$$\begin{aligned}
\operatorname{hp}((A \cup B) - x) + \operatorname{hp}((A \cap B) - x) &= \mu_W H_{A\cup B} + \mu_W H_{A\cap B} \\
&\le \mu_W(H_A \cup H_B) + \mu_W(H_A \cap H_B) \\
&= \mu_W H_A + \mu_W H_B \\
&= \operatorname{hp}(A - x) + \operatorname{hp}(B - x). \text{ **Q**}
\end{aligned}$$

Consequently

$$\begin{aligned}
\operatorname{cap}(A \cup B) + \operatorname{cap}(A \cap B) &= \lambda_{A\cup B}(\mathbb{R}^r) + \lambda_{A\cap B}(\mathbb{R}^r) \\
&= \lim_{\|x\|\to\infty} \|x\|^{r-2}(\operatorname{hp}((A \cup B) - x) + \operatorname{hp}((A \cap B) - x) \\
&\le \lim_{\|x\|\to\infty} \|x\|^{r-2}(\operatorname{hp}(A - x) + \operatorname{hp}(B - x)) \\
&= \operatorname{cap} A + \operatorname{cap} B.
\end{aligned}$$

As A and B are arbitrary, cap is submodular.

(b)(i) Let $f : \mathbb{R}^r \to \mathbb{R}$ be any bounded continuous function. For any $x \in \mathbb{R}^r$, $\int f d\mu_x^{(A)} = \lim_{n\to\infty} \int f d\mu_x^{(A_n)}$. **P** Let τ, τ_n be the Brownian hitting times to $A - x$, $A_n - x$ respectively. Observe that $\langle \tau_n(\omega)\rangle_{n\in\mathbb{N}}$ is non-increasing and

$$\tau(\omega) = \inf\{t : x + \omega(t) \in \textstyle\bigcup_{n\in\mathbb{N}} A_n\} = \lim_{n\to\infty} \tau_n(\omega)$$

for every $\omega \in \Omega$. Set $H = \{\omega : \tau(\omega) < \infty\}$, $H_n = \{\omega : \tau_n(\omega) < \infty\}$. Then $\langle H_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with union H, and for $\omega \in H$

$$f(x + \tau(\omega)) = \lim_{n\to\infty} f(x + \tau_n(\omega))$$

because f and ω are continuous. Accordingly

$$\begin{aligned}
\int f d\mu_x^{(A)} &= \int_H f(x + \omega(\tau(\omega))) \\
&= \lim_{n\to\infty} \int_{H_n} f(x + \omega(\tau_n(\omega))) = \lim_{n\to\infty} \int f d\mu_x^{(A_n)}. \text{ **Q**}
\end{aligned}$$

Taking $\gamma > 0$ such that $\overline{A} \subseteq \operatorname{int} B(\mathbf{0}, \gamma)$,

$$\int f d\lambda_A = \frac{1}{r\beta_r\gamma} \int_{\partial B(\mathbf{0},\gamma)} \int f d\mu_x^{(A)} \nu(dx)$$

(452F)

$$= \frac{1}{r\beta_r\gamma} \int_{\partial B(\mathbf{0},\gamma)} \lim_{n\to\infty} \int f d\mu_x^{(A_n)} \nu(dx)$$

$$= \lim_{n\to\infty} \frac{1}{r\beta_r\gamma} \int_{\partial B(\mathbf{0},\gamma)} \int f d\mu_x^{(A_n)} \nu(dx) = \lim_{n\to\infty} \int f d\lambda_{A_n}.$$

As f is arbitrary, $\lambda_A = \lim_{n\to\infty} \lambda_{A_n}$ for the narrow topology (437Kc).

(ii) Taking $f = \chi\mathbb{R}^r$ in (i), we see that $\operatorname{cap} A = \lim_{n\to\infty} \operatorname{cap} A_n$.

(iii) For any $x \in \mathbb{R}^r$,

$$\tilde{W}_A(x) = \int \frac{1}{\|y-x\|^{r-2}} \lambda_A(dy) \le \liminf_{n\to\infty} \int \frac{1}{\|y-x\|^{r-2}} \lambda_{A_n}(dy)$$

because $y \mapsto \frac{1}{\|y-x\|^{r-2}}$ is non-negative and continuous (437Jg). As $\tilde{W}_{A_n}(x) \le \tilde{W}_A(x)$ for every n (479D(b-ii)), $\tilde{W}_A(x) = \lim_{n\to\infty} \tilde{W}_{A_n}(x) = \sup_{n\in\mathbb{N}} \tilde{W}_{A_n}(x)$.

(c) Most of the ideas of (b) still work. Again take $f \in C_b(\mathbb{R}^r)$. Then $\int f d\mu_x^{(A)} = \lim_{n\to\infty} \int f d\mu_x^{(A_n)}$ for any $x \in \mathbb{R}^r$. **P** As before, let τ, τ_n be the Brownian hitting times to $A - x$, $A_n - x$ respectively. This time, $\langle \tau_n \rangle_{n\in\mathbb{N}}$ is non-decreasing. Let Ω' be the conegligible subset of Ω consisting of those functions ω such that $\lim_{t\to\infty} \|\omega(t)\| = \infty$. If $\omega \in \Omega'$ and $t = \lim_{n\to\infty} \tau_n(\omega)$ is finite, then for every $n \in \mathbb{N}$ there is a $t_n \le t + 2^{-n}$ such that $x + \omega(t_n) \in A_n$. Let $s \in [0, t]$ be a cluster point of $\langle t_n \rangle_{n\in\mathbb{N}}$; then $x + \omega(s)$ is a cluster point of $\langle x + \omega(t_n) \rangle_{n\in\mathbb{N}}$, so belongs to $\bigcap_{n\in\mathbb{N}} \overline{A}_n = A$, and $\tau(\omega) \le s \le t$. Since $\tau(\omega) \ge \tau(\omega_n)$ for every n, we have $\tau(\omega) = \lim_{n\to\infty} \tau(\omega_n)$.

Setting $H = \{\omega : \tau(\omega) < \infty\}$ and $H_n = \{\omega : \tau_n(\omega) < \infty\}$, $\langle H_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with intersection H, and for $\omega \in H$

$$f(x + \tau(\omega)) = \lim_{n\to\infty} f(x + \tau_n(\omega)).$$

So once again

$$\begin{aligned}\int f d\mu_x^{(A)} &= \int_H f(x + \omega(\tau(\omega)))\mu_W(d\omega) \\ &= \lim_{n\to\infty} \int_{H_n} f(x + \omega(\tau_n(\omega)))\mu_W(d\omega) = \lim_{n\to\infty} \int f d\mu_x^{(A_n)}. \textbf{ Q}\end{aligned}$$

The rest of the argument follows (b-i) and (b-ii) unchanged.

(d)(i) I seek to apply 432Lb.

(α) Let $\mathcal{K}$ be the family of compact subsets of $\mathbb{R}^r$ and set $c_1 = \operatorname{cap} \restriction \mathcal{K}$. By (a), c_1 is submodular. If $G \subseteq \mathbb{R}^r$ is a bounded open set, then it is expressible as the union of a non-decreasing sequence of compact sets, so by (b-ii) we have $\operatorname{cap} G = \sup\{c_1(L) : L \in \mathcal{K}, L \subseteq G\}$; and if $K \in \mathcal{K}$, there is a non-increasing sequence $\langle G_n \rangle_{n\in\mathbb{N}}$ of bounded open sets such that $K = \bigcap_{n\in\mathbb{N}} G_n = \bigcap_{n\in\mathbb{N}} \overline{G}_n$, and now $c_1(K) = \lim_{n\to\infty} \operatorname{cap} G_n$, by (c-ii). But this means that

$$c_1(K) \le \inf_{G\supseteq K \text{ is open}} \sup_{L\subseteq G \text{ is compact}} c_1(L) \le \inf_{n\in\mathbb{N}} \operatorname{cap} G_n = c_1(K).$$

So all the conditions of 432Lb are satisfied, and c, as defined in 479C(a-ii), is the unique extension of c_1 to an outer regular Choquet capacity on $\mathbb{R}^r$.

(β) Now $c(A) = \operatorname{cap} A$ for every bounded analytic set $A \subseteq \mathbb{R}^r$. **P**

$$\begin{aligned} c(A) &= \sup_{K\subseteq A \text{ is compact}} c(K) \\ &= \sup_{K\subseteq A \text{ is compact}} \operatorname{cap} K \le \operatorname{cap} A \le \inf_{G\supseteq A \text{ is bounded and open}} \operatorname{cap} G \\ &= \inf_{G\supseteq A \text{ is open}} \sup_{L\subseteq G \text{ is compact}} c(L) = c(A). \textbf{ Q}\end{aligned}$$

(432K)

So c extends cap, as claimed, and must be the unique outer regular Choquet capacity doing so.

(ii)-(iii) By 432Lb, c is submodular; and (iii) is covered by the argument in (i-β).

479F I now wish to describe an entirely different characterization of the capacity and equilibrium measure of a compact set, which demands a substantial investment in harmonic analysis (down to 479I) and an excursion into Fourier analysis (479H). I begin with general remarks about Newtonian potentials.

Theorem Let ζ be a totally finite Radon measure on $\mathbb{R}^r$, and set $G = \mathbb{R}^r \setminus \operatorname{supp} \zeta$, where $\operatorname{supp} \zeta$ is the support of ζ (411Nd). Let W_ζ be the Newtonian potential associated with ζ.

(a) $W_\zeta : \mathbb{R}^r \to [0,\infty]$ is lower semi-continuous, and $W_\zeta \restriction G : G \to [0,\infty[$ is continuous.

(b) W_ζ is superharmonic, and $W_\zeta \restriction G$ is harmonic.

(c) W_ζ is locally μ-integrable; in particular, it is finite μ-a.e.

(d) If ζ has compact support, then $\zeta\mathbb{R}^r = \lim_{\|x\|\to\infty} \|x\|^{r-2} W_\zeta(x)$.

(e) If $W_\zeta \restriction \operatorname{supp}\zeta$ is continuous then W_ζ is continuous.

(f) If K is a compact set such that $W_\zeta \restriction K$ is continuous and finite-valued then $W_{\zeta \llcorner K}$ is continuous.

(g) If W_ζ is finite ζ-a.e. and $f : \mathbb{R}^r \to [0,\infty]$ is a lower semi-continuous superharmonic function such that $f \ge W_\zeta$ ζ-a.e., then $f \ge W_\zeta$.

(h) If ζ' is another Radon measure on $\mathbb{R}^r$ and $\zeta' \le \zeta$, then $W_{\zeta'} \le W_\zeta$ and $\operatorname{energy}(\zeta') \le \operatorname{energy}(\zeta)$.

proof (a) If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{R}^r$ with limit x, then $\frac{1}{\|y-x\|^{r-2}} = \lim_{n\to\infty} \frac{1}{\|y-x_n\|^{r-2}}$ for every y (counting $\frac{1}{0}$ as ∞, as usual), so that $W_\zeta(x) \le \liminf_{n\to\infty} W_\zeta(x_n)$, by Fatou's Lemma. As x and $\langle x_n\rangle_{n\in\mathbb{N}}$ are arbitrary, W_ζ is lower semi-continuous.

If $x \in G$, then $\frac{1}{\|y-x_n\|^{r-2}} \le \frac{2}{\rho(x,\operatorname{supp}\zeta)^{r-2}}$ for all n large enough and all $y \in \operatorname{supp}\zeta$, so Lebesgue's Dominated Convergence Theorem tells us that $W_\zeta(x) = \lim_{n\to\infty} W_\zeta(x_n)$ and that W_ζ is continuous at x, as well as finite-valued there.

(b) If $x \in \mathbb{R}^r$ and $\delta > 0$, then

$$\begin{aligned}\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} W_\zeta d\nu &= \frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)}\int_{\mathbb{R}^r} \frac{1}{\|z-y\|^{r-2}}\zeta(dz)\nu(dy)\\ &= \int_{\mathbb{R}^r} \frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} \frac{1}{\|z-y\|^{r-2}}\nu(dy)\zeta(dz)\\ &= \int_{\mathbb{R}^r} \frac{1}{\nu(\partial B(\mathbf{0},\delta))}\int_{\partial B(\mathbf{0},\delta)} \frac{1}{\|z-x-y\|^{r-2}}\nu(dy)\zeta(dz)\\ &\ge \int_{\mathbb{R}^r} \frac{1}{\|z-x\|^{r-2}}\zeta(dz) = W_\zeta(x)\end{aligned}$$

(478Ga) with equality if $B(x,\delta)$ does not meet $\operatorname{supp}\zeta$, since then $z - x \notin B(\mathbf{0},\delta)$ and

$$\frac{1}{\nu(\partial B(x,\delta))}\int_{\partial B(x,\delta)} \frac{1}{\|z-x-y\|^{r-2}}\nu(dy) = \frac{1}{\|z-x\|^{r-2}}$$

for ζ-almost every z.

(c) For any $\gamma > 0$ and $y \in \mathbb{R}^r$, $\int_{B(\mathbf{0},\gamma)} \frac{1}{\|y-x\|^{r-2}}\mu(dx) \le \frac{1}{2}r\beta_r\gamma^2$ (478Gc), so

$$\begin{aligned}\int_{B(\mathbf{0},\gamma)} W_\zeta d\mu &= \int_{B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r} \frac{1}{\|y-x\|^{r-2}}\zeta(dy)\mu(dx)\\ &= \int_{\mathbb{R}^r}\int_{B(\mathbf{0},\gamma)} \frac{1}{\|y-x\|^{r-2}}\mu(dx)\zeta(dy) \le \frac{1}{2}r\beta_r\gamma^2\zeta\mathbb{R}^r\end{aligned}$$

is finite.

(d) If ζ has compact support, there is an $\gamma > 0$ such that $\operatorname{supp}\zeta \subseteq B(\mathbf{0},\gamma)$. In this case, for $\|x\| > \gamma$, we have

$$\frac{\|x\|^{r-2}}{(\|x\|+\gamma)^{r-2}}\zeta\mathbb{R}^r \le \int_{\mathbb{R}^r} \frac{\|x\|^{r-2}}{\|x-y\|^{r-2}}\zeta(dy) = W_\zeta(x)\|x\|^{r-2} \le \frac{\|x\|^{r-2}}{(\|x\|-\gamma)^{r-2}}\zeta\mathbb{R}^r$$

so all the terms converge to $\zeta\mathbb{R}^r$ as $\|x\| \to \infty$.

(e) Since W_ζ is lower semi-continuous, it will be enough to show that $H = \{x : W_\zeta(x) < \gamma\}$ is open for every $\gamma \in \mathbb{R}$. Take $x_0 \in H$. If $x_0 \notin \operatorname{supp}\zeta$ then W_ζ is continuous at x_0, by 479Fa, and H is certainly a neighbourhood of x_0. If $x_0 \in \operatorname{supp}\zeta$ take $\eta \in]0, 2^{-r}(\gamma - W_\zeta(x_0))[$. Because $W_\zeta(x_0)$ is finite, $\zeta\{x_0\} = 0$; because $W_\zeta \restriction \operatorname{supp}\zeta$ is continuous, there is a $\delta > 0$ such that $\int_{B(x_0,\delta)} \frac{1}{\|x_0-y\|^{r-2}}\zeta(dy) \le \eta$ and $|W_\zeta(x) - W_\zeta(x_0)| \le \eta$ whenever $x \in B(x_0,\delta) \cap \operatorname{supp}\zeta$. Let $\delta' \in]0,\delta[$ be such that

$$\Bigl|\frac{1}{\|x-y\|^{r-2}} - \frac{1}{\|x_0-y\|^{r-2}}\Bigr| \le \frac{\eta}{\zeta\mathbb{R}^r}$$

whenever $\|x-x_0\|\le\delta'$ and $\|y-x_0\|\ge\delta$; then

$$|\int_{\mathbb{R}^r\setminus B(x_0,\delta)}\frac{1}{\|x-y\|^{r-2}}\zeta(dy)-\int_{\mathbb{R}^r\setminus B(x_0,\delta)}\frac{1}{\|x_0-y\|^{r-2}}\zeta(dy)|\le\eta$$

whenever $x\in B(x_0,\delta')$.

Take $x\in B(x_0,\frac{1}{2}\delta')$, and let $z\in B(x_0,\delta)\cap\operatorname{supp}\zeta$ be such that $\|x-z\|=\rho(x,B(x_0,\delta)\cap\operatorname{supp}\zeta)$. We have $\|x-z\|\le\|x-x_0\|$ so $\|z-x_0\|\le 2\|x-x_0\|\le\delta'$. If $y\in B(x_0,\delta)\cap\operatorname{supp}\zeta$, then $\|y-z\|\le\|x-y\|+\|x-z\|\le 2\|x-y\|$; so

$$\begin{aligned}\int_{B(x_0,\delta)}\frac{1}{\|x-y\|^{r-2}}\zeta(dy)&\le 2^{r-2}\int_{B(x_0,\delta)}\frac{1}{\|z-y\|^{r-2}}\zeta(dy)\\&=2^{r-2}(W_\zeta(z)-\int_{\mathbb{R}^r\setminus B(x_0,\delta)}\frac{1}{\|z-y\|^{r-2}}\zeta(dy))\\&\le 2^{r-2}(2\eta+W_\zeta(x_0)-\int_{\mathbb{R}^r\setminus B(x_0,\delta)}\frac{1}{\|x_0-y\|^{r-2}}\zeta(dy))\\&=2^{r-1}\eta+2\int_{B(x_0,\delta)}\frac{1}{\|x_0-y\|^{r-2}}\zeta(dy)\\&\le 2^{r-1}\eta+2\eta\le(2^r-1)\eta,\end{aligned}$$

$$\begin{aligned}W_\zeta(x)&\le(2^r-1)\eta+\int_{\mathbb{R}^r\setminus B(x_0,\delta)}\frac{1}{\|x-y\|^{r-2}}\zeta(dy)\\&\le 2^r\eta+\int_{\mathbb{R}^r\setminus B(x_0,\delta)}\frac{1}{\|x_0-y\|^{r-2}}\zeta(dy)\le 2^r\eta+W_\zeta(x_0)<\gamma.\end{aligned}$$

Thus $B(x_0,\frac{1}{2}\delta')\subseteq H$ and again H is a neighbourhood of x_0. As x_0 is arbitrary, H is open; as γ is arbitrary, W_ζ is continuous.

(f) Setting $H=\mathbb{R}^r\setminus K$, $\zeta=\zeta\llcorner K+\zeta\llcorner H$, so $W_\zeta=W_{\zeta\llcorner K}+W_{\zeta\llcorner H}$ (234Hc[11]). Now $W_{\zeta\llcorner K}$ and $W_{\zeta\llcorner H}$ are both lower semi-continuous and non-negative, so if $W_\zeta\restriction K$ is continuous and finite-valued then $W_{\zeta\llcorner K}\restriction K$ is continuous (4A2B(d-ix)). Since $\operatorname{supp}(\zeta\llcorner K)\subseteq K$, (e) tells us that $W_{\zeta\llcorner K}$ is continuous.

(g) ? Suppose that $f(x_0)<W_\zeta(x_0)$. Since $\{x:f(x)\ge W_\zeta(x),\ W_\zeta(x)<\infty\}$ is ζ-conegligible, and W_ζ is measurable therefore ζ-almost continuous (418J), there is a compact set K such that $W_\zeta(x)<\infty$ and $W_\zeta(x)\le f(x)$ for every $x\in K$, $W_\zeta\restriction K$ is continuous and $\int_K\frac{1}{\|x_0-y\|^{r-2}}\zeta(dy)>f(x_0)$. Set $\zeta'=\zeta\llcorner K$. By (f), $W_{\zeta'}$ is continuous; $W_{\zeta'}(x_0)>f(x_0)$; and $f(x)\ge W_\zeta(x)\ge W_{\zeta'}(x)$ for every $x\in K\supseteq\operatorname{supp}\zeta'$.

Set $g=f-W_{\zeta'}$ and $\alpha=\inf_{x\in\mathbb{R}^r}g(x)<0$. Because f is lower semi-continuous and $W_{\zeta'}$ is continuous, g is lower semi-continuous; because ζ' has compact support, $\lim_{\|x\|\to\infty}W_{\zeta'}(x)=0$ ((d) above) and $\liminf_{\|x\|\to\infty}g(x)\ge 0$; so $L=\{x:g(x)=\alpha\}$ is non-empty and compact. Note that L is disjoint from K. Let $x_1\in L$ be a point of maximum norm. Then $x_1\notin K$, while $W_{\zeta'}\restriction\mathbb{R}^r\setminus K$ is harmonic ((b) above). Let $\delta>0$ be such that $B(x_1,\delta)\cap K=\emptyset$. Then we have

$$\frac{1}{\nu(\partial B(x_1,\delta))}\int_{\partial B(x_1,\delta)}g\,d\nu>\alpha$$

because $g(x)\ge\alpha$ for every x and $g(x)>\alpha$ whenever $x\ne x_1$ and $(x-x_1)\cdot x_1\ge 0$. But we also have

$$\begin{aligned}\frac{1}{\nu(\partial B(x_1,\delta))}\int_{\partial B(x_1,\delta)}g\,d\nu&=\frac{1}{\nu(\partial B(x_1,\delta))}\int_{\partial B(x_1,\delta)}f\,d\nu-\frac{1}{\nu(\partial B(x_1,\delta))}\int_{\partial B(x_1,\delta)}W_{\zeta'}\,d\nu\\&\le f(x_1)-W_{\zeta'}(x_1)=\alpha,\end{aligned}$$

which is impossible. **X**

(h) By 234Qc, $W_{\zeta'}\le W_\zeta$; so

$$\text{energy}(\zeta')=\int W_{\zeta'}d\zeta'\le\int W_\zeta d\zeta'\le\int W_\zeta d\zeta=\text{energy}(\zeta).$$

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[11] Formerly 212Xh.

479G At this point I embark on an extended parenthesis, down to 479I, covering some essential material from harmonic analysis and Fourier analysis. The methods here apply equally to the cases $r = 1$ and $r = 2$.

Lemma (In this result, r may be any integer greater than or equal to 1.) For $\alpha \in \mathbb{R}$, set $k_\alpha(x) = \frac{1}{\|x\|^\alpha}$ for $x \in \mathbb{R}^r \setminus \{0\}$. If $\alpha < r$, $\beta < r$ and $\alpha + \beta > r$, then $k_{\alpha+\beta-r}$ is a constant multiple of the convolution $k_\alpha * k_\beta$ (definition: 255E, 444O).

proof (a) First note that

$$\int_{B(\mathbf{0},1)} k_\alpha(x)\mu(dx) = r\beta_r \int_0^1 \frac{t^{r-1}}{t^\alpha} dt = \frac{r\beta_r}{r-\alpha}$$

is finite. Consequently k_α is expressible as the sum of an integrable function and a bounded function, and in particular is locally integrable.

(b) For $x \in \mathbb{R}^r$, set $f(x) = \int_{\mathbb{R}^r} k_\alpha(x-y)k_\beta(x-y)\mu(dy) \in [0,\infty]$. If $e \in \mathbb{R}^r$ is a unit vector, then $f(e)$ is finite. **P** For any $y \in \mathbb{R}^r$, at least one of $\|e-y\|$, $\|e+y\|$ is greater than or equal to 1, so $k_\alpha(e-y)k_\beta(e+y) \le k_\alpha(e-y)+k_\beta(e+y)$. Consequently

$$\int_{B(\mathbf{0},2)} k_\alpha(e-y)k_\alpha(e+y)\mu(dy) \le \int_{B(\mathbf{0},2)} k_\alpha(e-y) + k_\alpha(e+y)\mu(dy)$$

is finite. On the other hand, if $\|y\| \ge 2$, $\|e-y\|$ and $\|e-y\|$ are both at least $\frac{1}{2}\|y\|$, so

$$\begin{aligned}\int_{\mathbb{R}^r\setminus B(\mathbf{0},2)} k_\alpha(e-y)k_\beta(e+y)\mu(dy) &\le \int_{\mathbb{R}^r\setminus B(\mathbf{0},2)} \frac{2^\alpha}{\|y\|^\alpha}\cdot\frac{2^\beta}{\|y\|^\beta}\mu(dy)\\ &= 2^{\alpha+\beta}r\beta_r\int_2^\infty \frac{t^{r-1}}{t^{\alpha+\beta}}dt = \frac{2^r r\beta_r}{\alpha+\beta-r}\end{aligned}$$

is finite. Putting these together, $f(e) = \int_{\mathbb{R}^r} k_\alpha(e-y)k_\beta(e-y)\mu(dy)$ is finite. **Q**

(c) If e, $e' \in \mathbb{R}^r$ are unit vectors, then $f(e) = f(e')$. **P** Let $T : \mathbb{R}^r \to \mathbb{R}^r$ be an orthogonal transformation such that $Te = e'$. Then

$$f(e') = \int_{\mathbb{R}^r} k_\alpha(Te-y)k_\beta(Te-y)\mu(dy) = \int_{\mathbb{R}^r} k_\alpha(Te-Ty)k_\beta(Te-Ty)\mu(dy)$$

(because T is an automorphism of $(\mathbb{R}^r,\mu)$)

$$= \int_{\mathbb{R}^r} k_\alpha(e-y)k_\beta(e-y)\mu(dy)$$

(because $k_\alpha(x)$, $k_\beta(x)$ are functions of $\|x\|$)

$$= f(e). \text{ **Q**}$$

Let c be the constant value of $f(e)$ for $\|e\| = 1$.

(d) If $x \in \mathbb{R}^r \setminus \{0\}$, $f(x) = \frac{c}{\|x\|^{\alpha+\beta-r}}$. **P** Set $t = \|x\|$, $e = \frac{1}{t}x$. Then

$$f(x) = \int_{\mathbb{R}^r} k_\alpha(te-y)k_\beta(te-y)\mu(dy) = \int_{\mathbb{R}^r} t^r k_\alpha(te-tz)k_\beta(te-tz)\mu(dz)$$

(substituting $y = tz$)

$$= t^{r-\alpha-\beta}\int_{\mathbb{R}^r} k_\alpha(e-z)k_\beta(e-z)\mu(dz) = \frac{c}{\|x\|^{\alpha+\beta-r}}. \text{ **Q**}$$

(e) If $x \in \mathbb{R}^r \setminus \{0\}$, $(k_\alpha * k_\beta)(x) = 2^{\alpha+\beta-r}ck_{\alpha+\beta-r}(x)$. **P** Set $z = \frac{1}{2}x$. Then

$$\begin{aligned}(k_\alpha * k_\beta)(x) &= \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^\alpha\|y\|^\beta}\mu(dy) = \int_{\mathbb{R}^r} \frac{1}{\|x-y-z\|^\alpha\|y+z\|^\beta}\mu(dy)\\ &= \int_{\mathbb{R}^r} \frac{1}{\|z-y\|^\alpha\|z+y\|^\beta}\mu(dy) = \frac{c}{\|z\|^{\alpha+\beta-r}} = 2^{\alpha+\beta-r}ck_{\alpha+\beta-r}(x). \text{ **Q**}\end{aligned}$$

(f) Of course it is of no importance what happens at 0, but for completeness: $k_{\alpha+\beta-r}$ is declared to be undefined there, and $\int_{\mathbb{R}^r} \frac{1}{\|y\|^\alpha \|y\|^\beta} \mu(dy)$ is infinite for any α and β, so $k_\alpha * k_\beta$ also is undefined at 0 on the convention of 255E or 444O. Thus we have $k_\alpha * k_\beta = 2^{\alpha+\beta-r} c k_{\alpha+\beta-r}$ in the strict sense.

Remark The functions k_α are called **Riesz kernels**. It will be helpful later to have a name for the constant arising here in a special case. If $r \geq 3$, I will take $c_r > 0$ to be the constant such that $c_r k_{r-2} = k_{r-1} * k_{r-1}$.

479H Now for some Fourier analysis which wasn't quite reached in Chapter 28. In the following, I will define the Fourier transform $\hat{f}$ and the inverse Fourier transform $\check{f}$, for μ-measurable complex-valued functions f defined μ-almost everywhere in $\mathbb{R}^r$, as in 283W and 284W, and $\hat{\zeta}$, for a totally finite Radon measure ζ on $\mathbb{R}^r$, by the formula offered in 285Ya for probability measures. The convolution $\zeta * f$ of a measure and a function will be defined as in 444H. Thus the basic formulae are

$$\hat{f}(y) = \frac{1}{(\sqrt{2\pi})^r} \int_{\mathbb{R}^r} e^{-iy \cdot x} f(x) \mu(dx), \quad \check{f}(y) = \frac{1}{(\sqrt{2\pi})^r} \int_{\mathbb{R}^r} e^{iy \cdot x} f(x) \mu(dx)$$

for μ-integrable f,

$$\hat{\zeta}(y) = \frac{1}{(\sqrt{2\pi})^r} \int_{\mathbb{R}^r} e^{-iy \cdot x} \zeta(dx), \quad (\zeta * f)(x) = \int_{\mathbb{R}^r} f(x-y) \zeta(dy).$$

Theorem (In this result, r may be any integer greater than or equal to 1.) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$.

(a) If $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, then $\zeta * f$ is μ-integrable and $(\zeta * f)^\wedge = (\sqrt{2\pi})^r \hat{\zeta} \times \hat{f}$.

(b) If ζ has compact support and $h : \mathbb{R}^r \to \mathbb{C}$ is a rapidly decreasing test function (284Wa), then $\zeta * h$ and $h \times \hat{\zeta}$ are rapidly decreasing test functions.

(c) Suppose that f is a tempered function on $\mathbb{R}^r$ (284Wa). If *either* ζ has compact support *or* f is expressible as the sum of a μ-integrable function and a bounded function, then $\zeta * f$ is defined μ-almost everywhere and is a tempered function.

(d) Suppose that f, g are tempered functions on $\mathbb{R}^r$ such that g represents the Fourier transform of f (284Wd). If *either* ζ has compact support *or* f is expressible as the sum of a bounded function and a μ-integrable function, then $(\sqrt{2\pi})^r \hat{\zeta} \times g$ represents the Fourier transform of $\zeta * f$.

proof (a)(i) To begin with, suppose that f is real-valued and non-negative. As in §444, I will write $f\mu$ for the indefinite-integral measure defined by f over μ. By 444K, $\zeta * f$ is μ-integrable and $(\zeta * f)\mu = \zeta * f\mu$.

As the formula used here for $\hat{\zeta}$ does not quite match that of 445C, whatever parametrization we use for the characters of the topological group $\mathbb{R}^r$, I had better not try to quote Chapter 44 when discussing Fourier transforms. Going back to first principles,

$$\begin{aligned}
(\zeta * f)^\wedge(y) &= \frac{1}{(\sqrt{2\pi})^r} \int e^{-iy \cdot x} (\zeta * f)(x) \mu(dx) = \frac{1}{(\sqrt{2\pi})^r} \int e^{-iy \cdot x} (\zeta * f)\mu(dx) \\
&= \frac{1}{(\sqrt{2\pi})^r} \int e^{-iy \cdot x} (\zeta * f\mu)(dx) = \frac{1}{(\sqrt{2\pi})^r} \iint e^{-iy \cdot (x+z)} \zeta(dz)(f\mu)(dx) \\
&\qquad \text{(444C)} \\
&= \frac{1}{(\sqrt{2\pi})^r} \int e^{-iy \cdot z} \zeta(dz) \int e^{-iy \cdot x} (f\mu)(dx) \\
&= \hat{\zeta}(y) \int e^{-iy \cdot x} f(x) \mu(dx) = (\sqrt{2\pi})^r \hat{\zeta}(y) \hat{f}(y)
\end{aligned}$$

for every $y \in \mathbb{R}^r$.

(ii) For general integrable complex-valued functions f, apply (i) to the positive and negative parts of the real and imaginary parts of f.

(b)(i) Because h is continuous, so is $\zeta * h$ (444Ib). If $j < r$, then, as in 123D,

$$\Bigl(\frac{\partial}{\partial \xi_j}(\zeta * h)\Bigr)(x) = \frac{\partial}{\partial \xi_j} \int h(x-y) \zeta(dy) = \int \frac{\partial}{\partial \xi_j} h(x-y) \zeta(dy) = \Bigl(\zeta * \frac{\partial h}{\partial \xi_j}\Bigr)(x)$$

because $\frac{\partial h}{\partial \xi_j}$ is bounded. Since $\frac{\partial h}{\partial \xi_j}$ is again a rapidly decreasing test function, we can repeat this process to see that $\zeta * h$ is smooth. Next, let $\gamma > 0$ be such that the support of ζ is included in $B(\mathbf{0},\gamma)$. If $k \in \mathbb{N}$, then $M = \sup_{x\in\mathbb{R}^r}(\gamma^k + \|x\|^k)|h(x)|$ is finite, and

$$\|x\|^k|h(y)| \le (\|y\| + \gamma)^k|h(y)| \le 2^k M$$

whenever $\|x - y\| \le \gamma$. So

$$\|x\|^k|(\zeta * h)(x)| \le \zeta\mathbb{R}^r \cdot \|x\|^k \sup_{\|y-x\|\le\gamma} |h(y)| \le 2^k M\zeta\mathbb{R}^r$$

for every $x \in \mathbb{R}^r$. Applying this to all the partial derivatives of h, we see that $\zeta * h$ is a rapidly decreasing test function.

(ii) Again suppose that $j < r$. Because ζ has compact support, $\int \|x\|\zeta(dx)$ is finite, so $\frac{\partial}{\partial \eta_j}\hat{\zeta}(y)$ is defined and equal to $-i\int \xi_j e^{-iy\cdot x}\zeta(dx)$ for every $y \in \mathbb{R}^r$ (cf. 285Fd). More generally, whenever $j_1,\dots,j_m < r$,

$$\frac{\partial^m}{\partial\eta_{j_1}\dots\partial\eta_{j_m}}\hat{\zeta}(y) = (-i)^m\int \xi_{j_1}\dots\xi_{j_m}e^{-iy\cdot x}\zeta(dx),$$

so all the partial derivatives of $\hat{\zeta}$ are defined everywhere and bounded. It follows that $\hat{\zeta}$ is smooth and $h \times \hat{\zeta}$ is a rapidly decreasing test function.

(c)(i) To begin with, suppose that f is real and non-negative, and that ζ has compact support. Set $f_n = f \times \chi B(\mathbf{0}, n)$ for each $n \in \mathbb{N}$. Then f_n is integrable, so $\zeta * f_n$ is defined μ-a.e.; also $\langle \zeta * f_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, and $(\zeta * f)(x) = \sup_{n\in\mathbb{N}}(\zeta * f_n)(x)$ whenever the latter is defined and finite.

Let $\gamma \ge 1$ be such that the support of ζ is included in $B(\mathbf{0},\gamma)$, and let $k \in \mathbb{N}$ be such that $\int_{\mathbb{R}^r} \frac{1}{1+\|x\|^k} f(x)\mu(dx)$ is finite. If $y \in B(\mathbf{0},\gamma)$ and $x \in \mathbb{R}^r$, then $\|x\| \le 2\max(\gamma, \|x + y\|)$, so $\frac{1}{1+\|x+y\|^k} \le \frac{M}{1+\|x\|^k}$, where $M = 1 + 2^k\gamma^k$. It follows that

$$\begin{aligned}
\int_{\mathbb{R}^r} \frac{1}{1+\|x\|^k}(\zeta * f_n)(x)\mu(dx) &= \int_{\mathbb{R}^r}\int_{B(\mathbf{0},\gamma)} \frac{1}{1+\|x\|^k} f_n(x-y)\zeta(dy)\mu(dx)\\
&= \int_{B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r} \frac{1}{1+\|x\|^k} f_n(x-y)\mu(dx)\zeta(dy)\\
&= \int_{B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r} \frac{1}{1+\|x+y\|^k} f_n(x)\mu(dx)\zeta(dy)\\
&\le M\int_{B(\mathbf{0},\gamma)}\int_{\mathbb{R}^r} \frac{1}{1+\|x\|^k} f(x)\mu(dx)\zeta(dy)
\end{aligned}$$

for every $n \in \mathbb{N}$. Consequently $\int_{\mathbb{R}^r} \frac{1}{1+\|x\|^k}(\zeta * f)(x)\mu(dx)$ is defined and finite.

(ii) Now suppose that f is expressible as $f_1 + f_\infty$, where f_1 is μ-integrable, f_∞ is bounded and both are real-valued and non-negative. Adjusting f_1 and f_2 on a μ-negligible set if necessary, we can suppose that f_∞ is Borel measurable and defined everywhere on $\mathbb{R}^r$. By (a), $\zeta * f_1$ is defined and μ-integrable. Next, $\zeta * f_\infty$ is defined everywhere, is bounded, and is Borel measurable (444Ia). So $\zeta * f =_{\text{a.e.}} \zeta * f_1 + \zeta * f_\infty$ is the sum of a μ-integrable function and a bounded Borel measurable function, and is tempered.

(iii) These arguments deal with the case in which $f \ge 0$. For the general case, apply (i) or (ii) to the four parts of f, as in (a-ii).

(d)(i) Suppose to begin with that ζ has compact support. Let h be a rapidly decreasing test function. Set $\overleftrightarrow{h}(x) = h(-x)$ for every $x \in \mathbb{R}^r$. Then $\overleftrightarrow{h}$ is a rapidly decreasing test function, and

$$(\zeta * \overleftrightarrow{h})(-x) = \int \overleftrightarrow{h}(-x - y)\zeta(dy) = \int h(x + y)\zeta(dy)$$

for every $x \in \mathbb{R}^r$. Accordingly

$$\int(\zeta*f)\times h\,d\mu=\iint h(x)f(x-y)\zeta(dy)\mu(dx)$$
$$=\iint h(x)f(x-y)\mu(dx)\zeta(dy)$$

(because $\zeta*|f|$ is tempered, so $\iint|h(x)f(x-y)|\zeta(dy)\mu(dx)=\int|h|\times(\zeta*|f|)d\mu$ is finite)

$$=\iint h(x+y)f(x)\mu(dx)\zeta(dy)$$
$$=\iint h(x+y)f(x)\zeta(dy)\mu(dx)$$

(because $\iint|h(x+y)f(x)|\mu(dx)\zeta(dy)=\iint|h(x)f(x-y)|\mu(dx)\zeta(dy)$ is finite)

$$=\int f\times(\zeta*\overset{\leftrightarrow}{h})^{\leftrightarrow}d\mu=\int g\times((\zeta*\overset{\leftrightarrow}{h})^{\leftrightarrow})^{\vee}d\mu$$

(because $\zeta*\overset{\leftrightarrow}{h}$ and $(\zeta*\overset{\leftrightarrow}{h})^{\leftrightarrow}$ are rapidly decreasing test functions, by (b))

$$=\int g\times(\zeta*\overset{\leftrightarrow}{h})^{\wedge}d\mu=(\sqrt{2\pi})^r\int g\times\hat{\zeta}\times(\overset{\leftrightarrow}{h})^{\wedge}d\mu$$

(by (a))

$$=(\sqrt{2\pi})^r\int g\times\hat{\zeta}\times\check{h}\,d\mu.$$

As h is arbitrary, $(\sqrt{2\pi})^r g\times\hat{\zeta}$ represents the Fourier transform of $\zeta*f$.

(ii) Now suppose that f is expressible as f_1+f_∞ where f_1 is μ-integrable and f_∞ is bounded. By (c), $\zeta*|f|$ is defined almost everywhere and is a tempered function. Set $\zeta_n=\zeta\llcorner B(\mathbf{0},n)$ for each n. Then $(\sqrt{2\pi})^r g\times\hat{\zeta}_n$ represents the Fourier transform of ζ_n*f, for each n. Now $\langle\hat{\zeta}_n\rangle_{n\in\mathbb{N}}$ converges uniformly to $\hat{\zeta}$, and $\langle\zeta_n*f\rangle_{n\in\mathbb{N}}$ converges to $\zeta*f$ at every point at which $\zeta*|f|$ is defined and finite, which is μ-almost everywhere. So if h is a rapidly decreasing test function,

$$\int h\times(\sqrt{2\pi})^r g\times\hat{\zeta}=\lim_{n\to\infty}\int h\times(\sqrt{2\pi})^r g\times\hat{\zeta}_n$$

(the convergence is dominated by the integrable function $(\sqrt{2\pi})^r\zeta\mathbb{R}^r\cdot|h\times g|$)

$$=\lim_{n\to\infty}\int\hat{h}\times(\zeta_n*f)=\int\hat{h}\times(\zeta*f)$$

(this convergence being dominated by the integrable function $|\hat{h}|\times(\zeta*|f|)$). As h is arbitrary, $(\sqrt{2\pi})^r g\times\hat{\zeta}$ represents the Fourier transform of $\zeta*f$.

479I Proposition (In this result, r may be any integer greater than or equal to 1.)

(a) Suppose that $0<\alpha<r$.

(i) There is a tempered function representing the Fourier transform of k_α.

(ii) There is a measurable function g_0, defined almost everywhere on $[0,\infty[$, such that $y\mapsto g_0(\|y\|)$ represents the Fourier transform of k_α.

(iii) In (ii),

$$2^{\alpha/2}\Gamma(\tfrac{\alpha}{2})\int_0^\infty t^{r-1}g_0(t)e^{-\epsilon t^2}dt=2^{(r-\alpha)/2}\Gamma(\tfrac{r-\alpha}{2})\int_0^\infty t^{\alpha-1}e^{-\epsilon t^2}dt$$

for every $\epsilon>0$.

(iv) $2^{\alpha/2}\Gamma(\frac{\alpha}{2})g_0(t)=2^{(r-\alpha)/2}\Gamma(\frac{r-\alpha}{2})t^{\alpha-r}$ for almost every $t>0$.

(v) $2^{(r-\alpha)/2}\Gamma(\frac{r-\alpha}{2})k_{r-\alpha}$ represents the Fourier transform of $2^{\alpha/2}\Gamma(\frac{\alpha}{2})k_\alpha$.

(b) Suppose that ζ_1, ζ_2 are totally finite Radon measures on $\mathbb{R}^r$, and $0<\alpha<r$. If $\zeta_1*k_\alpha=\zeta_2*k_\alpha$ μ-a.e., then $\zeta_1=\zeta_2$.

proof (a)(i) Set $\beta=\frac{1}{2}(\alpha+r)$. Then k_β is expressible as f_1+f_2 where f_1 is integrable and f_2 is square-integrable. **P**

$$\int_{B(\mathbf{0},1)} k_\beta d\mu = r\beta_r\int_0^1 \frac{t^{r-1}}{t^\beta}dt$$

is finite because $\beta<r$;

$$\int_{\mathbb{R}^r\setminus B(\mathbf{0},1)} k_\beta^2 d\mu = r\beta_r\int_1^\infty \frac{t^{r-1}}{t^{2\beta}}dt$$

is finite because $2\beta>r$. So we can take $f_1=k_\alpha\times\chi B(\mathbf{0},1)$ and $f_2=k_\alpha-f_1$. **Q**

479G tells us that there is a constant c such that

$$k_\alpha = ck_\beta * k_\beta = c(f_1*f_1+2f_1*f_2+f_2*f_2).$$

Now f_1*f_1 is integrable and f_1*f_2 is square-integrable (444Ra), so both have Fourier transforms represented by tempered functions; while the continuous function f_2*f_2 also has a Fourier transform represented by an integrable function (284Wi). Assembling these, k_α has a Fourier transform represented by a tempered function.

(ii) We can therefore represent the Fourier transform of k_α by the function g, where

$$g(y)=\lim_{n\to\infty}\frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\,.\,x}e^{-\|x\|^2/n}k_\alpha(x)\mu(dx)$$

is defined μ-almost everywhere (284M/284Wg). Now suppose that $T:\mathbb{R}^r\to\mathbb{R}^r$ is any orthogonal transformation, and that $y\in\operatorname{dom}g$. Then

$$\begin{aligned}g(y)&=\lim_{n\to\infty}\frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\,.\,x}e^{-\|x\|^2/n}k_\alpha(x)\mu(dx)\\&=\lim_{n\to\infty}\frac{1}{(\sqrt{2\pi})^r}\int e^{-iy\,.\,T^\top x}e^{-\|T^\top x\|^2/n}k_\alpha(T^\top x)\mu(dx)\end{aligned}$$

(because the transpose $T^\top$ of T acts as an automorphism of $(\mathbb{R}^r,\mu)$)

$$=\lim_{n\to\infty}\frac{1}{(\sqrt{2\pi})^r}\int e^{-iTy\,.\,x}e^{-\|x\|^2/n}k_\alpha(x)\mu(dx),$$

and $g(Ty)$ is defined and equal to $g(y)$. So we can set $g_0(t)=g(y)$ whenever $y\in\operatorname{dom}g$ and $\|y\|=t$, and we shall have $y\mapsto g_0(\|y\|)$ representing the Fourier transform of k_α.

(iii) If $\epsilon>0$, then $x\mapsto e^{-\epsilon\|x\|^2}$ is a rapidly decreasing test function, and its Fourier transform is the function $x\mapsto\frac{1}{(\sqrt{2\epsilon})^r}e^{-\|x\|^2/4\epsilon}$ (283N/283Wi[12]). We therefore have

$$\int g_0(\|y\|)e^{-\epsilon\|y\|^2}\mu(dy)=\frac{1}{(\sqrt{2\epsilon})^r}\int k_\alpha(x)e^{-\|x\|^2/4\epsilon}\mu(dx),$$

that is,

$$r\beta_r\int_0^\infty t^{r-1}g_0(t)e^{-\epsilon t^2}dt=\frac{r\beta_r}{(\sqrt{2\epsilon})^r}\int_0^\infty\frac{t^{r-1}}{t^\alpha}e^{-t^2/4\epsilon}dt;$$

simplifying,

$$\begin{aligned}\int_0^\infty t^{r-1}g_0(t)e^{-\epsilon t^2}dt&=\frac{1}{(\sqrt{2\epsilon})^r}\int_0^\infty t^{r-1-\alpha}e^{-t^2/4\epsilon}dt\\&=\frac{2^{r-\alpha}}{2\cdot 2^{r/2}\epsilon^{\alpha/2}}\int_0^\infty u^{(r-\alpha-2)/2}e^{-u}du\end{aligned}$$

(substituting $u=t^2/4\epsilon$)

$$=\frac{2^{r-\alpha}}{2\cdot 2^{r/2}\epsilon^{\alpha/2}}\Gamma(\frac{r-\alpha}{2}).$$

On the other hand,

[12] Formerly 283We.

$$\int_0^\infty t^{\alpha-1}e^{-\epsilon t^2}dt=\int_0^\infty \frac{(\sqrt{u})^{\alpha-2}}{2\epsilon^{\alpha/2}}e^{-u}du=\frac{1}{2\epsilon^{\alpha/2}}\Gamma(\frac{\alpha}{2}).$$

Putting these together,

$$2^{\alpha/2}\Gamma(\tfrac{\alpha}{2})\int_0^\infty t^{r-1}g_0(t)e^{-\epsilon t^2}dt=2^{(r-\alpha)/2}\Gamma(\tfrac{r-\alpha}{2})\int_0^\infty t^{(r-1)-(r-\alpha)}e^{-\epsilon t^2}dt$$

for every $\epsilon>0$.

(iv) Set

$$g_1(t)=t^{r-1}e^{-t^2}\bigl(2^{\alpha/2}\Gamma(\tfrac{\alpha}{2})g_0(t)-2^{(r-\alpha)/2}\Gamma(\tfrac{r-\alpha}{2})t^{\alpha-r}\bigr)$$

for $t>0$. Then g_1 is integrable and $\int_0^\infty g_1(t)e^{-\epsilon t^2}dt=0$ for every $\epsilon\ge 0$. It follows that $g_1=0$ a.e. **P** Consider the linear span A of the functions $t\mapsto e^{-\epsilon t^2}$ for $\epsilon\ge 0$. This is a subalgebra of $C_b([0,\infty[)$ containing the constant functions and separating the points of $[0,\infty[$. It follows that for every $\gamma\ge 0$, $\delta>0$ and $h\in C_b([0,\infty[)$, there is an $f\in A$ such that $|f(t)-h(t)|\le\delta$ for $t\in[0,\gamma]$ and $\|f\|_\infty\le\|h\|_\infty$ (281E). Since $\int_0^\infty g_1\times f=0$, we must have

$$|\int_0^\infty g_1\times h|\le\delta\|g_1\|_1+2\|h\|_\infty\int_\gamma^\infty|g_1(t)|dt.$$

As δ and γ are arbitrary, $\int_0^\infty g_1\times h=0$; as h is arbitrary, $\int_0^a g_1=0$ for every $a\ge 0$, and g_1 must be zero almost everywhere (222D). **Q**

Accordingly $2^{\alpha/2}\Gamma(\frac{\alpha}{2})g_0(t)=2^{(r-\alpha)/2}\Gamma(\frac{r-\alpha}{2})t^{\alpha-r}$ for almost every $t\ge 0$.

(v) Now

$$y\mapsto 2^{(r-\alpha)/2}\Gamma(\tfrac{r-\alpha}{2})k_{r-\alpha}(y)=_{\text{a.e.}}2^{\alpha/2}\Gamma(\tfrac{\alpha}{2})g_0(\|y\|)$$

represents the Fourier transform of $2^{\alpha/2}\Gamma(\frac{\alpha}{2})k_\alpha$.

(b) By (a), the Fourier transform of k_α is represented by a tempered function g which is non-zero μ-a.e. As k_α is the sum of an integrable function and a bounded function, 479Hd tells us that the Fourier transform of ζ_1*k_α is represented by $(\sqrt{2\pi})^r\hat{\zeta}_1\times g$; and similarly for ζ_2. As $\zeta_1*k_\alpha=_{\text{a.e.}}\zeta_2*k_\alpha$, $\hat{\zeta}_1\times g=_{\text{a.e.}}\hat{\zeta}_2\times g$ (284Ib) and $\hat{\zeta}_1=_{\text{a.e.}}\hat{\zeta}_2$. Since $\hat{\zeta}_1$ and $\hat{\zeta}_2$ are both continuous (285Fb), they are equal everywhere; in particular,

$$\zeta_1\mathbb{R}^r=\hat{\zeta}_1(0)=\hat{\zeta}_2(0)=\zeta_2\mathbb{R}^r.$$

If $\zeta_1=\zeta_2$ is the zero measure, we can stop. Otherwise, they can be expressed as $\gamma\zeta_1'$ and $\gamma\zeta_2'$ where ζ_1' and ζ_2' are probability measures and $\gamma>0$. In this case, ζ_1' and ζ_2' have the same characteristic function (285D) and must be equal (285M); so $\zeta_1=\zeta_2$, as claimed.

Remark The functions $\zeta*k_\alpha$ are called **Riesz potentials**.

479J Now I return to the study of Newtonian potential when $r\ge 3$.

Lemma (a) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$. Let U_ζ be the $(r-1)$-potential of ζ and W_ζ the Newtonian potential of ζ; let k_{r-1} and k_{r-2} be the Riesz kernels. Then $U_\zeta=_{\text{a.e.}}\zeta*k_{r-1}$ and $W_\zeta=_{\text{a.e.}}\zeta*k_{r-2}$.

(b) Let ζ, ζ_1 and ζ_2 be totally finite Radon measures on $\mathbb{R}^r$.

(i) $\int_{\mathbb{R}^r}W_{\zeta_1}d\zeta_2=\int_{\mathbb{R}^r}W_{\zeta_2}d\zeta_1=\frac{1}{c_r}\int_{\mathbb{R}^r}U_{\zeta_1}\times U_{\zeta_2}d\mu$, where c_r is the constant of 479G.

(ii) The energy energy(ζ) of ζ is $\frac{1}{c_r}\|U_\zeta\|_2^2$, counting $\|U_\zeta\|_2$ as ∞ if $U_\zeta\notin\mathcal{L}^2(\mu)$.

(iii) If $\zeta=\zeta_1+\zeta_2$ then $U_\zeta=U_{\zeta_1}+U_{\zeta_2}$ and $W_\zeta=W_{\zeta_1}+W_{\zeta_2}$; similarly, $U_{\alpha\zeta}=\alpha U_\zeta$ and $W_{\alpha\zeta}=\alpha W_\zeta$ for $\alpha\ge 0$.

(iv) If $U_{\zeta_1}=U_{\zeta_2}$ μ-a.e., then $\zeta_1=\zeta_2$.

(v) If $W_{\zeta_1}=W_{\zeta_2}$ μ-a.e., then $\zeta_1=\zeta_2$.

(vi) $\zeta\mathbb{R}^r=\lim_{\gamma\to\infty}\frac{1}{r\beta_r\gamma}\int_{\partial B(0,\gamma)}W_\zeta d\nu$.

(c) Let $M_{\mathrm{R}}^+(\mathbb{R}^r)$ be the set of totally finite Radon measures on $\mathbb{R}^r$, with its narrow topology. Then energy : $M_{\mathrm{R}}^+(\mathbb{R}^r)\to[0,\infty]$ is lower semi-continuous.

proof (a) As k_{r-1} and k_{r-2} are both expressible as sums of integrable functions and bounded functions, $\zeta*k_{r-1}$ and $\zeta*k_{r-2}$ are both defined a.e. (479Hc); and now we have only to read the definitions to see that U_ζ and W_ζ are these convolutions with the technical adjustment that they are permitted to take the value ∞.

(b)(i) For any x, $y\in\mathbb{R}^r$,

$$\begin{aligned}\frac{1}{\|x-y\|^{r-2}}&=k_{r-2}(x-y)=\frac{1}{c_r}(k_{r-1}*k_{r-1})(x-y)\\&=\frac{1}{c_r}\int_{\mathbb{R}^r}\frac{1}{\|x-y-z\|^{r-1}\|z\|^{r-1}}\mu(dz)\\&=\frac{1}{c_r}\int_{\mathbb{R}^r}\frac{1}{\|x-z\|^{r-1}\|z-y\|^{r-1}}\mu(dz)=\frac{1}{c_r}\int_{\mathbb{R}^r}\frac{1}{\|x-z\|^{r-1}\|y-z\|^{r-1}}\mu(dz).\end{aligned}$$

So

$$\begin{aligned}\int_{\mathbb{R}^r}W_{\zeta_1}d\zeta_2&=\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}\zeta_1(dx)\zeta_2(dy)\\&=\frac{1}{c_r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-z\|^{r-1}\|y-z\|^{r-1}}\mu(dz)\zeta_1(dx)\zeta_2(dy)\\&=\frac{1}{c_r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-z\|^{r-1}\|y-z\|^{r-1}}\zeta_1(dx)\zeta_2(dy)\mu(dz)\\&=\frac{1}{c_r}\int_{\mathbb{R}^r}U_{\zeta_1}(z)U_{\zeta_2}(z)\mu(dz)=\int_{\mathbb{R}^r}U_{\zeta_1}\times U_{\zeta_2}d\mu.\end{aligned}$$

Hence (or otherwise)

$$\int_{\mathbb{R}^r}W_{\zeta_2}d\zeta_1=\frac{1}{c_r}\int_{\mathbb{R}^r}U_{\zeta_2}\times U_{\zeta_1}d\mu=\int_{\mathbb{R}^r}W_{\zeta_1}d\zeta_2.$$

(ii) Take $\zeta_1=\zeta_2=\zeta$ in (i).

(iii) This is immediate from 234Hc.

(iv)-(v) Put (a) and 479Ib together.

(vi) For any $\gamma>0$,

$$\frac{1}{r\beta_r\gamma}\int_{\partial B(0,\gamma)}W_\zeta d\nu=\int W_\zeta d\lambda_{B(0,\gamma)}$$

(479Da)

$$=\int\tilde{W}_{B(0,\gamma)}d\zeta$$

((i) above)

$$=\int\min(1,\frac{\gamma^{r-2}}{\|x-z\|^{r-2}})\zeta(dx)$$

(479Da again)

$$\to\zeta\mathbb{R}^r$$

as $\gamma\to\infty$.

(c) The map $\zeta\mapsto\zeta\times\zeta:M_{\mathrm{R}}^+(\mathbb{R}^r)\to M_{\mathrm{R}}^+(\mathbb{R}^r\times\mathbb{R}^r)$ is continuous, by 437Ma. Next, the function $\lambda\mapsto\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}\lambda(d(x,y)):M_{\mathrm{R}}^+(\mathbb{R}^r\times\mathbb{R}^r)\to[0,\infty]$ is lower semi-continuous, by 437Jg again. So energy is the composition of a lower semi-continuous function with a continuous function, and is lower semi-continuous (4A2B(d-ii)).

479K Lemma Let $K\subseteq\mathbb{R}^r$ be a compact set, with equilibrium measure λ_K. Then $\lambda_K K=\operatorname{cap}K=\operatorname{energy}(\lambda_K)$, and if ζ is any Radon measure on $\mathbb{R}^r$ such that $\zeta K\ge\operatorname{cap}K\ge\operatorname{energy}(\zeta)$, $\zeta=\lambda_K$.

proof (a) We know that $\lambda_K K=\lambda_K\mathbb{R}^r=\operatorname{cap}K$ (479C(a-i)). So if K has zero capacity then λ_K is the zero measure and $\operatorname{energy}(\lambda_K)=0$; also the only Radon measure on $\mathbb{R}^r$ with zero energy is λ_K, and we can stop. So henceforth let us suppose that $\operatorname{cap}K>0$.

Set

$$e = \inf\{\text{energy}(\zeta) : \zeta \text{ is a Radon probability measure on } \mathbb{R}^r \text{ such that } \zeta K \geq \operatorname{cap} K\}.$$

Because $\tilde{W}_K(x) \leq 1$ for every $x \in \mathbb{R}^r$ (479D(b-i)),

$$e \leq \text{energy}(\lambda_K) = \int \tilde{W}_K d\lambda_K \leq \lambda_K \mathbb{R}^r = \operatorname{cap} K$$

is finite.

(b) Consider the set Q of Radon measures ζ on $\mathbb{R}^r$ such that $\zeta K = \zeta\mathbb{R}^r = \operatorname{cap} K$. With its narrow topology, Q is homeomorphic to the set of Radon measures on K of magnitude $\operatorname{cap} K$, which is compact (437R(f-ii)). Since energy : $Q \to [0,\infty]$ is lower semi-continuous (479Jc), there is a $\lambda \in Q$ with energy e (4A2B(d-viii)).

In fact there is exactly one such member of Q. **P** Suppose that ζ is any other member of Q with energy e. Write u_ζ for the equivalence class of U_ζ in L^2. Then $\frac{1}{2}(\zeta+\lambda)$ belongs to Q and $U_{\frac{1}{2}(\zeta+\lambda)} = \frac{1}{2}(U_\zeta + U_\lambda)$ (479J(b-iii)). So, defining c_r as in 479G,

$$e + \frac{1}{c_r}\|u_\zeta - u_\lambda\|_2^2 \leq \text{energy}(\frac{1}{2}(\zeta+\lambda)) + \frac{1}{4c_r}(u_\zeta - u_\lambda|u_\zeta - u_\lambda)$$

$$= \frac{1}{4c_r}(u_\zeta + u_\lambda|u_\zeta + u_\lambda) + \frac{1}{4c_r}(u_\zeta - u_\lambda|u_\zeta - u_\lambda)$$

(479J(b-ii))

$$= \frac{1}{2c_r}(\|u_\zeta\|_2^2 + \|u_\lambda\|_2^2) = e.$$

It follows that $\|u_\zeta - u_\lambda\|_2 = 0$ and $U_\zeta =_{\text{a.e.}} U_\lambda$. Consequently $\zeta = \lambda$ (479J(b-iv)). **Q**

(c)(i) If ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, then $\int W_\zeta d\lambda \geq \frac{e\,\zeta K}{\operatorname{cap} K}$. **P** If $\zeta K = 0$ this is trivial. Otherwise, set $\zeta' = \frac{\operatorname{cap} K}{\zeta K}\zeta \llcorner K$. Then ζ' has finite energy (479Fh) and belongs to Q, so for any $\alpha \in [0,1]$ we have $\alpha\zeta' + (1-\alpha)\lambda \in Q$, and

$$\begin{aligned} c_r e &\leq c_r \,\text{energy}(\alpha\zeta' + (1-\alpha)\lambda) = \|\alpha u_{\zeta'} + (1-\alpha)u_\lambda\|_2^2 \\ &= \alpha^2\|u_{\zeta'}\|_2^2 + 2\alpha(1-\alpha)(u_{\zeta'}|u_\lambda) + (1-\alpha)^2\|u_\lambda\|_2^2 \\ &= \alpha^2\|u_{\zeta'}\|_2^2 + 2\alpha(1-\alpha)(u_{\zeta'}|u_\lambda) + (1-\alpha)^2 c_r e \\ &= c_r e + 2\alpha((u_{\zeta'}|u_\lambda) - c_r e) + \alpha^2(\|u_{\zeta'}\|_2^2 - 2(u_{\zeta'}|u_\lambda) + c_r e). \end{aligned}$$

It follows that $(u_{\zeta'}|u_\lambda) - c_r e \geq 0$ and

$$\int W_\zeta d\lambda \geq \int W_{\zeta\llcorner K} d\lambda$$

(479Fh)

$$= \frac{\zeta K}{\operatorname{cap} K}\int W_{\zeta'} d\lambda = \frac{\zeta K}{c_r \operatorname{cap} K}\int U_{\zeta'} \times U_\lambda d\mu$$

(479J(b-i))

$$= \frac{\zeta K}{c_r \operatorname{cap} K}(u_{\zeta'}|u_\zeta) \geq \frac{\zeta K}{c_r \operatorname{cap} K} c_r e = \frac{e\,\zeta K}{\operatorname{cap} K},$$

as claimed. **Q**

(ii) If ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, then $W_\lambda(x) \geq \frac{e}{\operatorname{cap} K}$ for ζ-almost every $x \in K$. **P?** Otherwise, set $E = \{x : x \in K,\, W_\lambda(x) < \frac{e}{\operatorname{cap} K}\}$, and consider $\zeta' = \zeta \llcorner E$. Then

$$\int W_{\zeta'} d\lambda = \int W_\lambda d\zeta'$$

(479J(b-i))

$$< \frac{e}{\operatorname{cap} K}\zeta' E \le \frac{e\zeta' K}{\operatorname{cap} K},$$

contradicting (i). **XQ**

(iii) $W_\lambda(x) = \frac{e}{\operatorname{cap} K}$ for λ-almost every $x \in K$. **P** Since λ has finite energy, (ii) tells us that $W_\lambda(x) \ge \frac{e}{\operatorname{cap} K}$ for λ-almost every $x \in K$. Since

$$\int_K W_\lambda d\lambda \le \int W_\lambda d\lambda = e = \frac{e}{\operatorname{cap} K}\lambda K,$$

we must have $W_\lambda(x) = \frac{e}{\operatorname{cap} K}$ for λ-almost every $x \in K$. **Q**

(iv) Since $\lambda K = \lambda\mathbb{R}^r$, 479Fg, with f the constant function with value $\frac{e}{\operatorname{cap} K}$, tells us that $W_\lambda(x) \le \frac{e}{\operatorname{cap} K}$ for every $x \in \mathbb{R}^r$.

(d) For $x \in \mathbb{R}^r \setminus K$, $W_\lambda(x) \le \frac{e}{\operatorname{cap} K}\operatorname{hp}(K - x)$. **P** Set $G = \mathbb{R}^r \setminus K$, and let τ be the Brownian exit time from $G - x$. Define $f : \overline{G}^\infty \to [0,1]$ by setting

$$\begin{aligned} f(y) &= 0 \text{ if } y \in \partial G = \partial K, \\ &= \frac{e}{\operatorname{cap} K} - W_\lambda(y) \text{ if } y \in G, \\ &= \frac{e}{\operatorname{cap} K} \text{ if } y = \infty. \end{aligned}$$

Because $W_\lambda{\restriction}G$ is continuous and harmonic (479Fa), so is $f{\restriction}G$. Because λ has compact support, $\lim_{y\to\infty} W_\lambda(y) = 0$ (479Fd), so f is continuous at ∞; because $W_\lambda(y) \le \frac{e}{\operatorname{cap} K}$ for every y, f is lower semi-continuous. So

$$\frac{e}{\operatorname{cap} K} - W_\lambda(x) = f(x) \ge \mathbb{E}(f(x + X_\tau))$$

(478O, because $r \ge 3$ and $\mathbb{R}^r$ has few wandering paths)

$$= \frac{e}{\operatorname{cap} K}\Pr(\tau = \infty) = \frac{e}{\operatorname{cap} K}(1 - \Pr(\tau < \infty)).$$

Thus $W_\lambda(x)$ is at most $\frac{e}{\operatorname{cap} K}\Pr(\tau < \infty)$. But $\Pr(\tau < \infty)$ is just the Brownian hitting probability $\operatorname{hp}(K - x)$. **Q**

(e) $e = \operatorname{energy}(\lambda_K)$. **P**

$$e \le \operatorname{energy}(\lambda_K) \le \operatorname{cap} K$$

((a) above)

$$= \lambda K = \lambda\mathbb{R}^r = \lim_{\|x\|\to\infty} \|x\|^{r-2} W_\lambda(x)$$

(479Fd)

$$\le \frac{e}{\operatorname{cap} K}\liminf_{\|x\|\to\infty} \|x\|^{r-2}\operatorname{hp}(K - x)$$

(by (d) of this proof)

$$= \frac{e}{\operatorname{cap} K}\cdot \operatorname{cap} K$$

(479B(ii))

$$= e. \textbf{ Q}$$

(f) From this we see at once that $\lambda = \lambda_K$ and $\operatorname{cap} K = \operatorname{energy}(\lambda_K)$. Now suppose that ζ is a Radon measure on $\mathbb{R}^r$ such that $\zeta K \ge \operatorname{cap} K \ge \operatorname{energy}(\zeta)$. Set $\zeta' = \frac{\operatorname{cap} K}{\zeta K}\zeta \llcorner K$; then $\zeta' \in Q$, while $\zeta' \le \zeta$, so

$$e = \operatorname{cap} K \geq \operatorname{energy}(\zeta) \geq \operatorname{energy}(\zeta')$$

by 479Fh. It follows that $\zeta' = \lambda$ and $\lambda \leq \zeta$. Accordingly $W_\lambda \leq W_\zeta$ (479Fh again),

$$\operatorname{energy}(\zeta) = \int W_\zeta d\zeta \geq \int W_\lambda d\zeta \geq \int_K W_\lambda d\zeta \geq \zeta K$$

((c-ii) above)

$$\geq \operatorname{cap} K \geq \operatorname{energy}(\zeta),$$

and we have equality throughout. Since λ is non-zero and the kernel $(x, y) \mapsto \frac{1}{\|x-y\|^{r-2}}$ is strictly positive, W_λ is strictly positive. It follows that $\zeta(\mathbb{R}^r \setminus K) = 0$ and $\zeta \in Q$; consequently $\zeta = \lambda = \lambda_K$, as required.

479L I shall wish later to quote a couple of the facts which appeared in the course of the proof above, and I think it will be safer to list them now.

Corollary Let $K \subseteq \mathbb{R}^r$ be a compact set with equilibrium potential $\tilde{W}_K$.

(a) If ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, then $\tilde{W}_K(x) = 1$ for ζ-almost every $x \in K$.

(b) If ζ is a Radon measure on $\mathbb{R}^r$ such that $W_\zeta \leq 1$ everywhere on K, $\zeta K \leq \operatorname{cap} K$.

(c) $\tilde{W}_K(x) \leq \operatorname{hp}(K - x)$ for every $x \in \mathbb{R}^r \setminus K$.

proof (a)(i) Suppose first that $\operatorname{cap} K > 0$. Working through the proof of 479K, we discover, in parts (e)-(f) of the proof, that $e = \operatorname{cap} K$ and $\lambda = \lambda_K$, so we just have to put (c-ii) of the proof together with 479D(b-i).

(ii) If $\operatorname{cap} K = 0$, let B be a non-trivial closed ball disjoint from A, and consider $L = K \cup B$. Then $\operatorname{cap} B = \operatorname{cap} L$ (479Ea) and $\lambda_L K = 0$, by 479D(c-ii), so

$$\lambda_L B = \lambda_L L = \operatorname{cap} L = \operatorname{energy}(\lambda_L) = \operatorname{cap} B$$

and $\lambda_L = \lambda_B$ (479K). Now $\tilde{W}_L = 1$ ζ-a.e. on L, while

$$\tilde{W}_L(x) = \tilde{W}_B(x) < 1$$

for every $x \in \mathbb{R}^r \setminus B$ (479Da), and in particular for every $x \in K$; so K must be ζ-negligible.

(b) Set $\zeta' = \zeta \llcorner K$; then $W_{\zeta'} \leq W_\zeta$, so $\operatorname{energy}(\zeta') = \int_K W_{\zeta'} d\zeta' \leq \zeta' K$ is finite. By (a), $\tilde{W}_K \geq 1$ ζ'-a.e., so

$$\zeta K = \zeta' K \leq \int \tilde{W}_K d\zeta' = \int W_{\zeta'} d\lambda_K$$

(479J(b-i))

$$\leq \lambda_K \mathbb{R}^r = \operatorname{cap} K.$$

(c) If $\operatorname{cap} K = 0$ then λ_K is the zero measure and the result is trivial. Otherwise, again look at the proof of 479K; in part (d), we saw that $W_\lambda(x) \leq \frac{e}{\operatorname{cap} K} \operatorname{hp}(K - x)$; but we now know that $e = \operatorname{cap} K$ and $\lambda = \lambda_K$, so we get $\tilde{W}_K(x) \leq \operatorname{hp}(K - x)$, as claimed.

479M In 479Ed we saw that there is a natural extension of Newtonian capacity to a Choquet capacity defined on every subset of $\mathbb{R}^r$. However the importance of Newtonian capacity lies as much in the equilibrium measures and potentials as in the simple quantity of capacity itself, and the methods of 479B-479E do not seem to yield these by any direct method. With the new ideas of 479K-479L, we can now approach the problem of defining equilibrium measures for unbounded analytic sets of finite capacity.

Lemma Let $A \subseteq \mathbb{R}^r$ be an analytic set with finite Choquet-Newton capacity $c(A)$.

(a) $\lim_{\gamma \to \infty} c(A \setminus B(\mathbf{0}, \gamma)) = 0$.

(b) $\lambda_A = \lim_{\gamma \to \infty} \lambda_{A \cap B(\mathbf{0}, \gamma)}$ is defined for the total variation metric on the space $M_{\mathrm{R}}^+(\mathbb{R}^r)$ of totally finite Radon measures on $\mathbb{R}^r$.

(c)(i) $\lambda_A \mathbb{R}^r = c(A)$.

(ii) $\operatorname{supp}(\lambda_A) \subseteq \partial A$.

(iii) If $B \subseteq \mathbb{R}^r$ is another analytic set such that $c(B) < \infty$, then $\lambda_{A\cup B} \le \lambda_A + \lambda_B$.

(d)(i) $\tilde{W}_A = W_{\lambda_A}$ is the limit $\lim_{\gamma\to\infty} \tilde{W}_{A\cap B(\mathbf{0},\gamma)} = \sup_{\gamma\ge 0} \tilde{W}_{A\cap B(\mathbf{0},\gamma)}$.

(ii) $\tilde{W}_A(x) \le 1$ for every $x \in \mathbb{R}^r$.

(iii) If ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, $\tilde{W}_A(x) = 1$ for ζ-almost every $x \in A$.

(iv) energy$(\lambda_A) = c(A)$.

proof (a) ? Otherwise, set

$$\alpha = \lim_{\gamma\to\infty} c(A \setminus B(\mathbf{0},\gamma)) = \inf_{\gamma>0} c(A \setminus B(\mathbf{0},\gamma)) > 0.$$

Set $\epsilon = \frac{1}{8}\alpha$ and $\delta = \frac{3}{2}\sqrt{\alpha}$. Let γ be such that $c(A \setminus B(\mathbf{0},\gamma)) \le \alpha + \epsilon$, and let $K \subseteq A \setminus B(\mathbf{0},\gamma)$ be a compact set such that $\operatorname{cap} K \ge \alpha - \epsilon$ (479E(d-iii)). Let γ' be such that $K \subseteq B(\mathbf{0},\gamma')$, and let $L \subseteq A \setminus B(\mathbf{0},\gamma'+\delta)$ be a compact set such that $\operatorname{cap} L \ge \alpha - \epsilon$.

Set $\zeta = \frac{2}{3}(\lambda_K + \lambda_L)$. Then $W_\zeta = \frac{2}{3}(\tilde{W}_K + \tilde{W}_L)$. If $x \in K$, then $\|x - y\| \ge \delta$ for every $y \in L$, so

$$\tilde{W}_L(x) \le \tfrac{1}{\delta^2}\lambda_L L \le \tfrac{\alpha+\epsilon}{\delta^2} = \frac{1}{2};$$

similarly, $\tilde{W}_K(x) \le \frac{1}{2}$ for every $x \in L$. So $W_\zeta(x) \le 1$ for every $x \in K \cup L$, and therefore for every $x \in \mathbb{R}^r$, by 479Fg. But this means that

$$c(A \setminus B(\mathbf{0},\gamma)) \ge \operatorname{cap}(K \cup L) \ge \zeta(K \cup L)$$

(479Lb)

$$= \frac{2}{3}(\operatorname{cap} K + \operatorname{cap} L) \ge \frac{4}{3}(\alpha - \epsilon) > \alpha + \epsilon,$$

which is impossible. **X**

(b) For $\gamma \ge 0$ set $\alpha_\gamma = c(A \setminus B(\mathbf{0},\gamma))$ and $\zeta_\gamma = \lambda_{A\cap B(\mathbf{0},\gamma)}$. If $0 \le \gamma \le \gamma'$ and $E \subseteq \mathbb{R}^r$ is Borel, then $|\zeta_\gamma E - \zeta_{\gamma'} E| \le \alpha_\gamma$. **P**

$$\zeta_{\gamma'} E \le \zeta_\gamma E + \lambda_{E\cap B(\mathbf{0},\gamma')\setminus B(\mathbf{0},\gamma)}(K)$$

(479D(c-i))

$$\le \zeta_\gamma E + c(A \cap B(\mathbf{0},\gamma') \setminus B(\mathbf{0},\gamma)) \le \zeta_\gamma E + c(A \setminus B(\mathbf{0},\gamma)) = \zeta_\gamma E + \alpha_\gamma.$$

On the other side we now have

$$\begin{aligned}\zeta_\gamma E &= c(A \cap B(\mathbf{0},\gamma)) - \zeta_\gamma(\mathbb{R}^r \setminus E)\\ &\le c(A \cap B(\mathbf{0},\gamma')) - \zeta_{\gamma'}(\mathbb{R}^r \setminus E) + \alpha_\gamma = \zeta_{\gamma'} E + \alpha_\gamma.\end{aligned}$$

So $|\zeta_\gamma E - \zeta_{\gamma'} E| \le \alpha_\gamma$. **Q** It follows at once that $\rho_{\mathrm{tv}}(\zeta_\gamma, \zeta_{\gamma'}) \le 2\alpha_\gamma$.

Since $\lim_{\gamma\to\infty} \alpha_\gamma = 0$, by (a), $\langle \zeta_n \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for ρ_{tv}. As noted in 437Q(a-iii), $M_{\mathrm{R}}^+(\mathbb{R}^r)$ is complete, so $\lim_{\gamma\to\infty} \lambda_{A\cap B(\mathbf{0},\gamma)} = \lim_{n\to\infty} \zeta_n$ is defined, and we have our measure λ_A.

(c)(i) Now

$$\lambda_A \mathbb{R}^r = \lim_{n\to\infty} \lambda_{A\cap B(\mathbf{0},n)}(\mathbb{R}^r) = \lim_{n\to\infty} c(A \cap B(\mathbf{0},n)) = c(A).$$

(ii) For any $\gamma \ge 0$,

$$\lambda_{A\cap B(\mathbf{0},\gamma)}(\mathbb{R}^r \setminus \partial A) \le \lambda_{A\cap B(\mathbf{0},\gamma)}(\partial B(\mathbf{0},\gamma))$$

(because the support of $\lambda_{A\cap B(\mathbf{0},\gamma)}$ is included in $\partial(A \cap B(\mathbf{0},\gamma)) \subseteq \partial A \cup \partial B(\mathbf{0},\gamma)$)

$$\begin{aligned}&\le |\lambda_A(\partial B(\mathbf{0},\gamma)) - \lambda_{A\cap B(\mathbf{0},\gamma)}(\partial B(\mathbf{0},\gamma))| + \lambda_A(\partial B(\mathbf{0},\gamma))\\ &\le \rho_{\mathrm{tv}}(\lambda_A, \lambda_{A\cap B(\mathbf{0},\gamma)}) + \lambda_A(\partial B(\mathbf{0},\gamma)).\end{aligned}$$

So

$$\lambda_A(\mathbb{R}^r \setminus \partial A) = \lim_{\gamma\to\infty} \lambda_{A\cap B(\mathbf{0},\gamma)}(\mathbb{R}^r \setminus \partial A)$$
$$\le \lim_{\gamma\to\infty} \rho_{\mathrm{tv}}(\lambda_A, \lambda_{A\cap B(\mathbf{0},\gamma)}) + \lim_{\gamma\to\infty} \lambda_A(\partial B(\mathbf{0},\gamma)) = 0.$$

(iii) For any compact set $K \subseteq \mathbb{R}^r$,

$$\lambda_{A\cup B}(K) = \lim_{\gamma\to\infty} \lambda_{(A\cup B)\cap B(\mathbf{0},\gamma)}(K) \le \lim_{\gamma\to\infty} \lambda_{A\cap B(\mathbf{0},\gamma)}(K) + \lambda_{B\cap B(\mathbf{0},\gamma)}(K)$$

(479D(c-i))

$$= \lambda_A(K) + \lambda_B(K) = (\lambda_A + \lambda_B)(K).$$

By 416Ea, $\lambda_{A\cup B} \le \lambda_A + \lambda_B$.

(d)(i) By 479D(b-ii), the supremum and the limit are the same. Suppose that $x \in \mathbb{R}^r$ and $\epsilon > 0$. Start with $\gamma > \|x\|$. Since $\tilde W_{A\cap B(\mathbf{0},\gamma)}(x)$ is finite, there is a $\delta \in \,]0, \gamma - \|x\|[$ such that $\int_{B(\mathbf{0},\delta)} \frac{1}{\|x-y\|^{r-2}} \lambda_{A\cap B(\mathbf{0},\gamma)}(dy) \le \epsilon$. If $\gamma' \ge \gamma \ge 0$, then

$$\lambda_{A\cap B(\mathbf{0},\gamma')} \le \lambda_{A\cap B(\mathbf{0},\gamma)} + \lambda_{A\cap B(\mathbf{0},\gamma')\setminus B(\mathbf{0},\gamma)},$$

so

$$\int_{B(\mathbf{0},\delta)} \frac{1}{\|x-y\|^{r-2}} \lambda_{A\cap B(\mathbf{0},\gamma')}(dy) \le \int_{B(\mathbf{0},\delta)} \frac{1}{\|x-y\|^{r-2}} \lambda_{A\cap B(\mathbf{0},\gamma)}(dy)$$
$$+ \int_{B(\mathbf{0},\delta)} \frac{1}{\|x-y\|^{r-2}} \lambda_{A\cap B(\mathbf{0},\gamma')\setminus B(\mathbf{0},\gamma)}(dy)$$

(234Hc, 234Qc)

$$= \int_{B(\mathbf{0},\delta)} \frac{1}{\|x-y\|^{r-2}} \lambda_{A\cap B(\mathbf{0},\gamma)}(dy)$$

(because $\operatorname{int} B(\mathbf{0},\gamma)$ is $\lambda_{A\cap B(\mathbf{0},\gamma')\setminus B(\mathbf{0},\gamma)}$-negligible)

$$\le \epsilon.$$

So, setting $M = \frac{1}{\delta^{r-2}}$,

$$|\tilde W_{A\cap B(\mathbf{0},\gamma')}(x) - \int \min(M, \frac{1}{\|x-y\|^{r-2}}) \lambda_{A\cap B(\mathbf{0},\gamma')}(dy)| \le \epsilon.$$

Using (c-ii) and (c-iii) to apply the same argument with A in place of $A \cap B(\mathbf{0},\gamma')$, we get

$$|\tilde W_A(x) - \int \min(M, \frac{1}{\|x-y\|^{r-2}}) \lambda_A(dy)| \le \epsilon.$$

On the other hand,

$$\int \min(M, \frac{1}{\|x-y\|^{r-2}}) \lambda_A(dy) = \lim_{\gamma'\to\infty} \int \min(M, \frac{1}{\|x-y\|^{r-2}}) \lambda_{A\cap B(\mathbf{0},\gamma')}(dy)$$

(437Q(a-ii)), so

$$\limsup_{\gamma'\to\infty} |\tilde W_{A\cap B(\mathbf{0},\gamma')}(x) - \tilde W_A(x)| \le 2\epsilon.$$

As ϵ is arbitrary, $W_A(x) = \lim_{\gamma'\to\infty} W_{A\cap B(\mathbf{0},\gamma')}(x)$, as claimed.

(ii) It follows at once that $\tilde W_A \le 1$ everywhere.

(iii) Write $E = \{x : x \in A, \tilde W_A(x) < 1\}$, and let ζ be a Radon measure on $\mathbb{R}^r$ of finite energy. **?** If $\zeta E > 0$, there is a compact set $K \subseteq E$ such that $\zeta K > 0$. Now there is a $\gamma > 0$ such that $K \subseteq B(\mathbf{0},\gamma)$, in which case

$$\tilde W_K(x) \le \tilde W_{A\cap B(\mathbf{0},\gamma)}(x) < 1$$

for every $x \in K$, and $\zeta K = 0$, by 479La. **X** So $\zeta E = 0$, as required.

(iv) By (ii) and (c-i),

$$\mathrm{energy}(\lambda_A) = \int \tilde{W}_A d\lambda_A \le \lambda_A \mathbb{R}^r = c(A).$$

In the other direction, for any $\gamma \ge 0$,

$$\mathrm{energy}(\lambda_A) = \int \tilde{W}_A d\lambda_A \ge \int \tilde{W}_{A\cap B(\mathbf{0},\gamma)} d\lambda_A = \int \tilde{W}_A d\lambda_{A\cap B(\mathbf{0},\gamma)}$$

(479J(b-i))

$$\ge \int \tilde{W}_{A\cap B(\mathbf{0},\gamma)} d\lambda_{A\cap B(\mathbf{0},\gamma)} = c(A\cap B(\mathbf{0},\gamma));$$

taking the limit as $\gamma \to \infty$, $\mathrm{energy}(\lambda_A) \ge c(A)$ and we have equality.

479N We are ready to match the definitions in 479C to some alternative definitions of capacity.

Theorem Let $A \subseteq \mathbb{R}^r$ be an analytic set with finite Choquet-Newton capacity $c(A)$.

(a) Writing W_ζ for the Newtonian potential of a Radon measure ζ on $\mathbb{R}^r$,

$$c(A) = \sup\{\zeta A : \zeta \text{ is a Radon measure on } \mathbb{R}^r,\ W_\zeta(x) \le 1 \text{ for every } x \in \mathbb{R}^r\};$$

if A is closed, the supremum is attained.

(b) $c(A) = \inf\{\mathrm{energy}(\zeta) : \zeta$ is a Radon measure on $\mathbb{R}^r$, $\zeta A \ge c(A)\}$; if A is closed, the infimum is attained.

(c) If $A \ne \emptyset$, $c(A) = \sup\{\frac{1}{\mathrm{energy}(\zeta)} : \zeta$ is a Radon measure on $\mathbb{R}^r$ such that $\zeta A = 1\}$, counting $\frac{1}{\infty}$ as zero; if A is closed, the supremum is attained.

proof Note first that if there is a Radon measure ζ on $\mathbb{R}^r$, with finite energy, such that $\zeta A > 0$, then $c(A) > 0$. **P** By 479M(d-iii), $\tilde{W}_A = 1$ ζ-a.e. on A. So $\tilde{W}_A$ cannot be identically 0, and $0 < \lambda_A \mathbb{R}^r = c(A)$, by 479M(c-i). **Q**

(a)(i) We know from 479E(d-i) and 479D(b-i) that

$$c(A) = \sup\{\mathrm{cap}\, K : K \subseteq A \text{ is compact}\} = \sup\{\lambda_K K : K \subseteq A \text{ is compact}\}$$
$$= \sup\{\lambda_K A : K \subseteq A \text{ is compact}\} \le \sup\{\zeta A : W_\zeta \le \chi\mathbb{R}^r\}.$$

(ii) If ζ is a Radon measure on $\mathbb{R}^r$ and $W_\zeta \le \chi\mathbb{R}^r$, then

$$\zeta A = \sup_{K\subseteq A \text{ is compact}} \zeta K \le \sup_{K\subseteq A \text{ is compact}} \mathrm{cap}\, K$$

(479Lb)

$$= c(A).$$

Thus $\sup\{\zeta A : W_\zeta \le \chi\mathbb{R}^r\} \le c(A)$ and we have equality.

(iii) If A is closed, then by 479M(c-ii)

$$\lambda_A A = \lambda_A(\partial A) = \lambda_A \mathbb{R}^r = c(A)$$

so λ_A witnesses that the supremum is attained.

(b)(i) ? Suppose, if possible, that there is a Radon measure ζ on $\mathbb{R}^r$ such that $\zeta A \ge c(A) > \mathrm{energy}(\zeta)$. Let $\alpha \in \,]0,1[$ be such that $\alpha^4 c(A) \ge \mathrm{energy}(\zeta)$. Since

$$\zeta A = \sup\{\zeta K : K \subseteq A \text{ is compact}\}, \quad c(A) = \sup\{\mathrm{cap}\, K : K \subseteq A \text{ is compact}\},$$

there is a compact $K \subseteq A$ such that $\zeta K \ge \alpha\zeta A$ and $\mathrm{cap}\, K > \alpha^2 c(A)$. Set $\zeta' = \frac{\mathrm{cap}\, K}{\zeta K}\zeta$. Then

$$\mathrm{energy}(\zeta') = \big(\frac{\mathrm{cap}\, K}{\zeta K}\big)^2 \mathrm{energy}(\zeta) \le \big(\frac{c(A)}{\alpha\zeta A}\big)^2 \alpha^4 c(A)$$
$$\le \alpha^2 c(A) < \mathrm{cap}\, K = \zeta' K;$$

which is impossible, by 479K. **X**

So $c(A) \le \inf\{\text{energy}(\zeta) : \zeta A \ge c(A)\}$.

(ii) Take any $\epsilon > 0$. Then there is a compact set $K \subseteq A$ such that $(1+\epsilon)\operatorname{cap} K \ge c(A)$. Set $\zeta = (1+\epsilon)\lambda_K$; then

$$\zeta A \ge c(A), \quad \text{energy}(\zeta) = (1+\epsilon)^2 \text{energy}(\lambda_K) = (1+\epsilon)^2 \operatorname{cap} K \le (1+\epsilon)^2 c(A).$$

As ϵ is arbitrary, $c(A) \ge \inf\{\text{energy}(\zeta) : \zeta A \ge c(A)\}$ and we have equality.

(iii) If A is closed, then

$$\lambda_A A = \lambda_A(\partial A) = \lambda_A \mathbb{R}^r = c(A)$$

by 479M(c-i) and (c-ii), while $\text{energy}(\lambda_A) = c(A)$ by 479M(d-iv). So λ_A witnesses that $c(A) = \min\{\text{energy}(\zeta) : \zeta A \ge c(A)\}$.

(c)(i) Suppose that ζ is a Radon measure on $\mathbb{R}^r$ such that $\zeta A = 1$. If $\text{energy}(\zeta) = \infty$ then of course $\frac{1}{\text{energy}(\zeta)} \le c(A)$. Otherwise, $c(A) > 0$, as remarked at the beginning of this part of the proof. Set $\zeta' = c(A)\zeta$. By (b),

$$c(A) \le \text{energy}(\zeta') = c(A)^2 \text{energy}(\zeta),$$

so $\frac{1}{\text{energy}(\zeta)} \le c(A)$.

Thus $\sup\{\frac{1}{\text{energy}(\zeta)} : \zeta A = \zeta\mathbb{R}^r = 1\} \le c(A)$.

(ii) If $c(A) = 0$ then the supremum is attained by any Radon measure ζ such that $\zeta A = 1$, so we can stop. If $c(A) > 0$, then for any $\alpha \in \,]0,1[$ there is a compact set $K \subseteq A$ such that $\operatorname{cap} K \ge \alpha c(A)$. Set $\zeta = \frac{1}{\operatorname{cap} K}\lambda_K$; then

$$\zeta K = \zeta\mathbb{R}^r = \zeta A = 1$$

and

$$\frac{1}{\text{energy}(\zeta)} = \frac{(\operatorname{cap} K)^2}{\text{energy}(\lambda_K)} = \operatorname{cap} K \ge \alpha c(A).$$

As α is arbitrary, $c(A) \le \sup\{\frac{1}{\text{energy}(\zeta)} : \zeta A = \zeta\mathbb{R}^r = 1\}$ and we have equality.

(iii) If A is closed and $c(A) > 0$, then $\zeta = \frac{1}{c(A)}\lambda_A$ witnesses that the supremum is attained, as in (b) above.

479O Polar sets To make the final step, to arbitrary sets with finite Choquet-Newton capacity, we seem to need an alternative description of polar sets.

Proposition For a set $D \subseteq \mathbb{R}^r$, the following are equiveridical:

(i) D is polar, that is, $c(D) = 0$;

(ii) there is a totally finite Radon measure ζ on $\mathbb{R}^r$ such that its Newtonian potential W_ζ is infinite at every point of D;

(iii) there is an analytic set $E \supseteq D$ such that $\zeta E = 0$ whenever ζ is a Radon measure on $\mathbb{R}^r$ with finite energy.

proof (i)⇒(ii) If (i) is true, then for each $n \in \mathbb{N}$ there is a bounded open set $G_n \supseteq D \cap B(\mathbf{0}, n)$ such that $c(G_n) \le 2^{-n}$. Try $\zeta = \sum_{n=0}^{\infty} \lambda_{G_n}$, defining the sum as in 234G. Then $\zeta\mathbb{R}^r = \sum_{n=0}^{\infty} c(G_n)$ is finite, and $W_\zeta = \sum_{n=0}^{\infty} \tilde{W}_{G_n}$ (234Hc). If $x \in D \cap B(\mathbf{0}, n)$, then $\tilde{W}_{G_m}(x) = 1$ for every $m \ge n$ (479D(b-iii)), so $W_\zeta(x) = \infty$. Thus ζ witnesses that (ii) is true.

(ii)⇒(iii) Suppose that λ is a totally finite Radon measure such that $W_\lambda(x) = \infty$ for every $x \in D$. Set $E = \{x : W_\lambda(x) = \infty\}$; then E is a G_δ set, because W_λ is lower semi-continuous (479Fa). **?** If there is a Radon measure ζ on $\mathbb{R}^r$, with finite energy, such that $\zeta E > 0$, let $K \subseteq E$ be a compact set such that $\zeta K > 0$. Set $\zeta_1 = \frac{1}{\zeta K}\zeta \llcorner K$; then ζ_1 has finite energy and $\zeta_1 K = 1$, so $\operatorname{cap} K \ge \frac{1}{\text{energy}(\zeta_1)} > 0$, by 479Nc.

Let $G \supseteq K$ be a bounded open set; set $\lambda_1 = \lambda \llcorner G$ and $\lambda_2 = \lambda \llcorner (\mathbb{R}^r \setminus G)$, so that $\lambda = \lambda_1 + \lambda_2$ and $W_\lambda = W_{\lambda_1} + W_{\lambda_2}$ (234Hc). Since $W_{\lambda_2}(x)$ is finite for $x \in G$ (479Fa), $W_{\lambda_1}(x) = \infty$ for every $x \in K$. Let $\epsilon > 0$ be such that

$\epsilon\lambda_1\mathbb{R}^r < \operatorname{cap} K$. Then ϵW_{λ_1} is a lower semi-continuous superharmonic function greater than or equal to $\tilde{W}_K$ on $K \supseteq \operatorname{supp}(\lambda_K)$, so $\epsilon W_{\lambda_1} \ge \tilde{W}_K$ everywhere (479Fg). But this means that

$$\epsilon\lambda_1\mathbb{R}^r = \epsilon \lim_{\|x\|\to\infty} \|x\|^{r-2} W_{\lambda_1}(x)$$

(479Fd)

$$\ge \lim_{\|x\|\to\infty} \|x\|^{r-2}\tilde{W}_K(x) = \lambda_K\mathbb{R}^r = \operatorname{cap} K > \epsilon\lambda_1\mathbb{R}^r,$$

which is absurd. **X**

So E witnesses that (iii) is true.

(iii)⇒(i) Suppose that $E \supseteq D$ is analytic and that $\zeta E = 0$ whenever energy(ζ) is finite. If $K \subseteq E$ is compact and ζ is a Radon probability measure on $\mathbb{R}^r$ such that $\zeta K = 1$, then energy(ζ) must be infinite; by 479Nc, $\operatorname{cap} K = 0$. As K is arbitrary, $c(E) = 0$ and $c(D) = 0$.

479P At last I come to my final extension of the notions of equilibrium measure and potential, together with a direct expression of the latter in terms of Brownian hitting probabilities.

Theorem Let $D \subseteq \mathbb{R}^r$ be a set with finite Choquet-Newton capacity $c(D)$.

(a) There is a totally finite Radon measure λ_D on $\mathbb{R}^r$ such that $\lambda_D = \lambda_A$, as defined in 479Mb, whenever $A \supseteq D$ is analytic and $c(A) = c(D)$.

(b) Write $\tilde{W}_D = W_{\lambda_D}$ for the equilibrium potential corresponding to the equilibrium measure λ_D. Then $\tilde{W}_D(x) = \operatorname{hp}^*((D \setminus \{x\}) - x)$ for every $x \in \mathbb{R}^r$.

(c)(i)(α) $\lambda_D\mathbb{R}^r = c(D)$;

(β) if ζ is any Radon measure on $\mathbb{R}^r$ with finite energy, $\tilde{W}_D(x) = 1$ for ζ-almost every $x \in D$;

(γ) energy(λ_D) $= c(D)$;

(δ) if $D' \subseteq D$ and $c(D') = c(D)$, then $\lambda_{D'} = \lambda_D$.

(ii) $\operatorname{supp}(\lambda_D) \subseteq \partial D$.

(iii) For any $D' \subseteq \mathbb{R}^r$ such that $c(D') < \infty$,

(α) $\lambda_D^*(D') \le c(D')$;

(β) $\lambda_{D\cup D'} \le \lambda_D + \lambda_{D'}$;

(γ) $\tilde{W}_{D\cap D'} + \tilde{W}_{D\cup D'} \le \tilde{W}_D + \tilde{W}_{D'}$;

(δ) $\rho_{\mathrm{tv}}(\lambda_D, \lambda_{D'}) \le 2c(D \triangle D')$.

(iv) If $\langle D_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of sets with union D, then

(α) $\tilde{W}_D = \lim_{n\to\infty} \tilde{W}_{D_n} = \sup_{n\in\mathbb{N}} \tilde{W}_{D_n}$;

(β) $\langle\lambda_{D_n}\rangle_{n\in\mathbb{N}} \to \lambda_D$ for the narrow topology on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

(v) $c(D) = \inf\{\zeta\mathbb{R}^r : \zeta \text{ is a Radon measure on } \mathbb{R}^r,\ W_\zeta \ge \chi D\}$

$= \inf\{\text{energy}(\zeta) : \zeta \text{ is a Radon measure on } \mathbb{R}^r,\ W_\zeta \ge \chi D\}$.

(vi) Writing cl^*D for the essential closure of D, $c(\mathrm{cl}^*D) \le c(D)$ and $\tilde{W}_{\mathrm{cl}^*D} \le \tilde{W}_D$.

(vii) Suppose that $f : D \to \mathbb{R}^r$ is γ-Lipschitz, where $\gamma \ge 0$. Then $c(f[D]) \le \gamma^{r-2}c(D)$.

proof (a)(i) If A, $B \subseteq \mathbb{R}^r$ are analytic sets, $c(B) < \infty$ and $A \subseteq B$, then

$$\tilde{W}_A = \sup_{n\in\mathbb{N}} \tilde{W}_{A\cap B(\mathbf{0},n)}$$

(479M(d-i))

$$\le \sup_{n\in\mathbb{N}} \tilde{W}_{B\cap B(\mathbf{0},n)}$$

(479D(b-ii))

$$= \tilde{W}_B.$$

If $c(A) = c(B)$, then $\lambda_A = \lambda_B$. **P**

$$c(A) = \text{energy}(\lambda_A)$$

(479M(d-iv))

$$= \int \tilde{W}_A d\lambda_A \le \int \tilde{W}_B d\lambda_A = \int \tilde{W}_A d\lambda_B$$

(479J(b-i))

$$\le \int \tilde{W}_B d\lambda_A = \text{energy}(\lambda_B) = c(B) = \lambda_B \mathbb{R}^r$$

(479M(c-i)). So we must have equality throughout, and $\tilde{W}_A = \tilde{W}_B$ λ_B-a.e. By 479Fg, $\tilde{W}_A \ge \tilde{W}_B$ everywhere and

$$W_{\lambda_B} = \tilde{W}_B = \tilde{W}_A = W_{\lambda_A}.$$

By 479J(b-v), $\lambda_B = \lambda_A$. **Q**

(ii) Now consider the given set D. By 479E(d-i), there is an analytic set $A \supseteq D$ such that $c(A) = c(D)$. If B is another such set, then $c(A \cap B) = c(A) = c(B)$, so $\lambda_{A \cap B} = \lambda_A = \lambda_B$. We therefore have a common measure which we can take to be λ_D. Of course this agrees with 479Mb if D itself is analytic, and with 479B if D is bounded and analytic.

(b) Write $h_D(x)$ for $\text{hp}^*((D \setminus \{x\}) - x)$.

(i) To begin with, suppose that $D = K$ is compact and that $x \notin K$, so that $h_D(x) = h_K(x) = \text{hp}(K - x)$.

(α) $h_K(x) \ge \tilde{W}_K(x)$. **P** 479Lc. **Q**

(β) In fact $h_K(x) = \tilde{W}_K(x)$. **P** Let $\epsilon > 0$. Set $E = \{y : y \in K,\ \tilde{W}_K(y) < 1\}$. Because $\tilde{W}_K$ is lower semi-continuous, E is an F_σ set, therefore analytic; by 479M(d-iii), E satisfies condition (iii) of 479O, and is polar. By 479O(ii), there is a totally finite Radon measure ζ on $\mathbb{R}^r$ such that $W_\zeta(y) = \infty$ for every $y \in E$. Let H be a bounded open set, including K, such that $x \notin \overline{H}$; set $\zeta_1 = \zeta \llcorner H$ and $\zeta_2 = \zeta \llcorner (\mathbb{R}^r \setminus H)$. Then $\zeta = \zeta_1 + \zeta_2$, so $W_\zeta = W_{\zeta_1} + W_{\zeta_2}$ (479J(b-iii)). Since H is open and ζ_2-negligible, $W_{\zeta_2}(y)$ is finite for every $y \in K$ (479Fa), and $W_{\zeta_1}(y) = \infty$ for every $y \in E$; while $W_{\zeta_1}(x)$ is finite because the support of ζ_1 is included in $\overline{H}$.

There is therefore an $\eta > 0$ such that $\eta W_{\zeta_1}(x) \le \epsilon$. Consider $\lambda = \lambda_K + \eta\zeta_1$. We have $W_\lambda(y) \ge 1$ for every $y \in K$, while W_λ is superharmonic and lower semi-continuous (479Fa, 479Fb); as the support of λ is included in the compact set $\overline{H}$, $\lim_{\|y\|\to\infty} W_\lambda(y) = 0$ (479Fd). Consequently

$$h_K(x) = \mu_x^{(K)}(K) \le \int W_\lambda d\mu_x^{(K)} \le W_\lambda(x)$$

(478Pc, with $G = \mathbb{R}^r \setminus K$)

$$\le \tilde{W}_K(x) + \epsilon.$$

As ϵ is arbitrary, $h_K(x) \le \tilde{W}_K(x)$ and we have equality. **Q**

(ii) If $D = A$ is analytic, note that $\text{cap}\{x\} = 0$ (479Da, or otherwise), so $c(A \setminus \{x\}) = c(A)$, because c is monotonic and submodular, therefore subadditive (479E(d-ii)). Now we know that

$$h_A(x) = \sup\{\text{hp}(K - x) : K \subseteq A \setminus \{x\} \text{ is compact}\}$$

(477Ie) and

$$c(A \setminus \{x\}) = \sup\{\text{cap}\, K : K \subseteq A \setminus \{x\} \text{ is compact}\}$$

(479E(d-iii)). So there is a non-decreasing sequence $\langle K_n \rangle_{n\in\mathbb{N}}$ of compact subsets of $A \setminus \{x\}$ such that

$$h_A(x) = \sup\nolimits_{n\in\mathbb{N}} \text{hp}(K_n - x), \quad c(A \setminus \{x\}) = \sup\nolimits_{n\in\mathbb{N}} \text{cap}\, K_n.$$

Set $E = \bigcup_{n\in\mathbb{N}} K_n$; then $E \subseteq A$ and $c(E) = c(A)$, so $\lambda_E = \lambda_A$ ((a-i) above) and $\tilde{W}_E = \tilde{W}_A$. Accordingly

$$h_A(x) = \sup_{n\in\mathbb{N}} \text{hp}(K_n - x) = \sup_{n\in\mathbb{N}} \tilde{W}_{K_n}(x)$$

((a-i) above)

$$= \sup_{m,n\in\mathbb{N}} \tilde{W}_{K_n\cap B(\mathbf{0},m)}(x) = \sup_{m\in\mathbb{N}} \tilde{W}_{E\cap B(\mathbf{0},m)}(x)$$

(apply 479E(b-iii) twice)

$$= \tilde{W}_E(x) = \tilde{W}_A(x).$$

(iii) For the general case, note first that $h_D \le \tilde{W}_D$. **P** There is a G$_\delta$ set $E \supseteq D$ such that $c(E) = c(D)$, so $\lambda_E = \lambda_D$. Now, for any $x \in \mathbb{R}^r$,

$$h_D(x) \le h_E(x) = \tilde{W}_E(x) = \tilde{W}_D(x),$$

using (ii) for the central equality. **Q**

Equally, $h_D \ge \tilde{W}_D$. **P** If $x \in \mathbb{R}^r$, there is a G$_\delta$ set $H \supseteq (D \setminus \{x\}) - x$ such that

$$h_D(x) = \mathrm{hp}^*((D \setminus \{x\}) - x) = \mathrm{hp}\, H$$

(477Id). Set $A = (H + x) \cup \{x\}$; then $A \supseteq D$ and

$$h_D(x) = h_A(x) = \tilde{W}_A(x) \ge \tilde{W}_{A\cap E}(x) = \tilde{W}_D(x),$$

using (a-i) again for the inequality. **Q**

So $h_D = \tilde{W}_D$, as claimed.

(c) Fix an analytic set $A \supseteq D$ such that $c(A) = c(D)$; replacing A by $A \cap \overline{D}$ if necessary, we may suppose that $A \subseteq \overline{D}$. We have $\lambda_D = \lambda_A$ and $\tilde{W}_D = \tilde{W}_A$.

(i)(*α*)

$$\lambda_D\mathbb{R}^r = \lambda_A\mathbb{R}^r = c(A) = c(D)$$

by 479M(c-i).

(*β*)-(*γ*) 479M(d-iii) tells us that $\tilde{W}_D(x) = \tilde{W}_A(x) = 1$ for ζ-almost every $x \in A$, and therefore for ζ-almost every $x \in D$. At the same time,

$$\mathrm{energy}(\lambda_D) = \mathrm{energy}(\lambda_A) = c(A) = c(D)$$

by 479M(d-iv).

(*δ*) Of course $A \supseteq D'$ and $c(A) = c(D')$, so $\lambda_{D'} = \lambda_A = \lambda_D$.

(ii) $\overline{A} = \overline{D}$ and $\mathrm{int}\, A \supseteq \mathrm{int}\, D$, so $\partial A \subseteq \partial D$ and

$$\lambda_D(\mathbb{R}^r \setminus \partial D) = \lambda_A(\mathbb{R}^r \setminus \partial D) \le \lambda_A(\mathbb{R}^r \setminus \partial A) = 0$$

by 479M(c-ii). As ∂D is closed, it includes $\mathrm{supp}(\lambda_D)$.

(iii) Let $A' \supseteq D'$ be an analytic set such that $c(A') = c(D')$.

(*α*)

$$\lambda_D^*(D') \le \lambda_D(A') = \lambda_A(A') \le \sup_{m\in\mathbb{N}} \lambda_{A\cap B(\mathbf{0},m)}(A')$$

(479Mb)

$$= \sup_{m,n\in\mathbb{N}} \lambda_{A\cap B(\mathbf{0},m)}(A' \cap B(\mathbf{0},n)) \le \sup_{n\in\mathbb{N}} c(A' \cap B(\mathbf{0},n))$$

(479D(c-ii))

$$= c(A')$$

(because c is a capacity)

$$= c(D').$$

(*β*) Because c is subadditive, we know that $c(D \cup D')$ is finite. Let $B \supseteq D \cup D'$ be an analytic set such that $c(B) = c(D \cup D')$. Then

$$\lambda_{D\cup D'} = \lambda_{B\cap(A\cup A')} \le \lambda_{B\cap A} + \lambda_{B\cap A'}$$

(479M(c-iii))

$$= \lambda_D + \lambda_{D'}.$$

(γ) This is immediate from (b) and the general fact that $\zeta^*(U\cap V)+\zeta^*(U\cup V)\le\zeta^*U+\zeta^*V$ for any measure ζ and any sets U and V (132Xk).

(δ) As usual, it will be enough to show that $|\lambda_D E - \lambda_{D'}E| \le c(D\triangle D')$ for every Borel set $E\subseteq\mathbb{R}^r$; by symmetry, all we need to check is that $\lambda_{D'}E \le \lambda_D E + c(D\triangle D')$ for every Borel set E. **P**

case 1 Suppose that D and D' are both bounded Borel sets. Take $x\in\mathbb{R}^r$, and let τ, $\tau':\Omega\to[0,\infty]$ be the Brownian arrival times to $D-x$, $D'-x$ respectively. Then

$$\begin{aligned}\mu_x^{(D')}E &= \mu_W\{\omega : \tau'(\omega)<\infty,\, x+\omega(\tau'(\omega))\in E\}\\ &\le \mu_W\{\omega : \tau(\omega)<\infty,\, x+\omega(\tau(\omega))\in E\} + \mu_W\{\omega : \tau(\omega)\ne\tau'(\omega)\}\\ &\le \mu_x^{(D)}E + \mu_W\{\omega : \text{there is some } t\ge 0 \text{ such that } x+\omega(t)\in D\triangle D'\}\\ &= \mu_x^{(D)}E + \mu_x^{(D\triangle D')}\mathbb{R}^r.\end{aligned}$$

So

$$\begin{aligned}\lambda_{D'}E &= \lim_{\|x\|\to\infty}\|x\|^{r-2}\mu_x^{(D')}E\\ &\le \lim_{\|x\|\to\infty}\|x\|^{r-2}\mu_x^{(D)}E + \lim_{\|x\|\to\infty}\|x\|^{r-2}\mu_x^{(D\triangle D')}\mathbb{R}^r = \lambda_D E + c(D\triangle D').\end{aligned}$$

case 2 Suppose that D, D' are Borel sets, not necessarity bounded. Set $D_n = D\cap B(\mathbf{0},n)$, $D'_n = D'\cap B(\mathbf{0},n)$. Then

$$\lambda_{D'}E = \lim_{n\to\infty}\lambda_{D'_n}E$$

(479Mb)

$$\le \lim_{n\to\infty}\lambda_{D_n}E + \lim_{n\to\infty}c(D_n\triangle D'_n)$$

(by case 1)

$$= \lambda_D E + \lim_{n\to\infty}c((D\triangle D')\cap B(\mathbf{0},n)) = \lambda_D E + c(D\triangle D')$$

because c is a capacity.

case 3 In general, let $G\supseteq D$, $G'\supseteq D'$ and $H\supseteq D\triangle D'$ be G_δ sets such that $c(G)=c(D)$, $c(G')=c(D')$ and $c(H)=c(D\triangle D')$. Set

$$G_1 = G\cap(G'\cup H),\quad G'_1 = G'\cap(G\cup H);$$

these are Borel sets, while $D\subseteq G_1\subseteq G$, $D'\subseteq G'_1\subseteq G'$ and $G_1\triangle G_2\subseteq H$. So

$$\lambda_{D'}E = \lambda_{G'_1}E \le \lambda_{G_1}E + c(G\triangle G_1)$$

(by case 2)

$$\le \lambda_D E + c(H) = \lambda_D E + c(D\triangle D')$$

and we have the result in this case also. So we're done. **Q**

(iv)(α) This follows immediately from (b) above.

(β) Consider first the case in which every D_n is analytic. Returning to the proof of 479M, or putting 479Ma together with (iii-δ) here, we see that for any m, $n\in\mathbb{N}$ we shall have

$$\rho_{\mathrm{tv}}(\lambda_{D_n},\lambda_{D_n\cap B(\mathbf{0},m)})\le 2c(D_n\setminus B(\mathbf{0},m))\le 2c(D\setminus B(\mathbf{0},m))=2\alpha_m$$

say, and that $\lim_{m\to\infty}\alpha_m=0$. So if $G\subseteq\mathbb{R}^r$ is any open set,

$$\lambda_D G=\lim_{m\to\infty}\lambda_{D\cap B(\mathbf{0},m)}G\le\lim_{m\to\infty}\liminf_{n\to\infty}\lambda_{D_n\cap B(\mathbf{0},m)}G$$

(479E(c-i))

$$\le\lim_{m\to\infty}\liminf_{n\to\infty}\lambda_{D_n}G+2\alpha_m=\liminf_{n\to\infty}\lambda_{D_n}G.$$

Since we know also that

$$\lambda_D\mathbb{R}^r=c(D)=\lim_{n\to\infty}c(D_n)=\lim_{n\to\infty}\lambda_{D_n}\mathbb{R}^r,$$

$\langle\lambda_{D_n}\rangle_{n\in\mathbb{N}}\to\lambda_D$ for the narrow topology.

For the general case, take analytic sets $A_n\supseteq D_n$, $A\supseteq D$ such that $c(A_n)=c(D_n)$ for every n and $c(A)=c(D)$. Set $A'_n=A\cap\bigcap_{m\ge n}A_m$ for each n, $A'=\bigcup_{n\in\mathbb{N}}A_n$; then

$$\langle\lambda_{D_n}\rangle_{n\in\mathbb{N}}=\langle\lambda_{A'_n}\rangle_{n\in\mathbb{N}}\to\lambda_{A'}=\lambda_D$$

for the narrow topology.

(v) Let Q be the set of Radon measures ζ on $\mathbb{R}^r$ such that $W_\zeta\ge\chi D$.

(α) I show first that $\inf_{\zeta\in Q}\zeta\mathbb{R}^r$ and $\inf_{\zeta\in Q}\mathrm{energy}(\zeta)\}$ are both less than or equal to $c(D)$. **P** Let $\epsilon>0$. Because c is outer regular (479E(d-i)), there is an open set $G\supseteq D$ such that $c(G)\le c(D)+\epsilon$. Set $\zeta=\lambda_G$. Then

$$W_\zeta=\tilde W_G\ge\chi G\ge\chi D$$

(479D(b-iii)), so $\zeta\in Q$, while

$$\zeta\mathbb{R}^r=\mathrm{energy}(\zeta)=c(G)\le c(D)+\epsilon.\ \mathbf{Q}$$

(β) Now suppose that $\zeta\in Q$. Then $c(D)\le\min(\zeta\mathbb{R}^r,\mathrm{energy}(\zeta))$. **P** Take any $\gamma<c(D)$ and $\epsilon>0$. Let $A\supseteq D$ be an analytic set such that $c(A)=c(D)$; replacing A by $\{x:x\in A,\ W_\zeta(x)\ge 1\}$ if necessary, we can suppose that $W_\zeta\ge\chi A$. For each $n\in\mathbb{N}$, let ζ_n be the totally finite measure $(1+\epsilon)\zeta\llcorner B(\mathbf{0},n)$. Then $\langle W_{\zeta_n}\rangle_{n\in\mathbb{N}}$ is non-decreasing and has supremum $(1+\epsilon)W_\zeta$ (479M(d-i)), so $A=\bigcup_{n\in\mathbb{N}}A_n$, where $A_n=\{x:x\in A,\ W_{\zeta_n}(x)\ge 1\}$. There are an $n\in\mathbb{N}$ such that $c(A_n)>\gamma$ and a compact $K\subseteq A_n$ such that $\mathrm{cap}\,K\ge\gamma$ (432K). Now $W_{\zeta_n}\ge\tilde W_K$ λ_K-a.e., so $W_{\zeta_n}\ge\tilde W_K$ everywhere (479Fg) and

$$\gamma\le\mathrm{cap}\,K=\int\tilde W_K d\lambda_K\le\int W_{\zeta_n}d\lambda_K=\int\tilde W_K d\zeta_n$$

(479J(b-i))

$$\le\int W_{\zeta_n}d\zeta_n\le(1+\epsilon)\int W_\zeta d\zeta_n\le(1+\epsilon)^2\int W_\zeta d\zeta=(1+\epsilon)^2\,\mathrm{energy}(\zeta).$$

Moreover, 479J(c-vi), applied to ζ_n and λ_K, tells us that

$$\zeta\mathbb{R}^r\ge\zeta_n\mathbb{R}^r\ge\lambda_K\mathbb{R}^r=\mathrm{cap}\,K\ge\gamma.$$

As γ and ϵ are arbitrary, $c(D)\le\min(\mathrm{energy}(\zeta),\zeta\mathbb{R}^r)$, as claimed. **Q**

(γ) Putting these together, we see that $c(D)=\inf_{\zeta\in Q}\zeta\mathbb{R}^r=\inf_{\zeta\in Q}\mathrm{energy}(\zeta)$.

(vi) If $x\in\mathrm{cl}^*A$, then $0\in\mathrm{cl}^*((A\setminus\{x\})-x)$ and $\tilde W_A(x)=\mathrm{hp}^*((A\setminus\{x\})-x)=1$ for every $x\in\mathrm{cl}^*E$, by 478U and (b) above. Now

$$c(\mathrm{cl}^*D)\le c(\mathrm{cl}^*A)\le\mathrm{energy}(\lambda_A)$$

((v) above)

$$=c(A)$$

(479M(d-iv))

$$= c(D).$$

(vii)(α) Consider first the case $D = A$, so that $f[D] = f[A]$ is analytic. We can suppose that $c(f[A]) > 0$, in which case $A \neq \emptyset$ and $\gamma > 0$. Take any $\epsilon > 0$. By 479Nc there is a Radon measure ζ on $\mathbb{R}^r$ such that $\zeta f[A] = 1$ and $c(f[A]) \le \frac{1+\epsilon}{\text{energy}(\zeta)}$. Applying 433D to the subspace measure $\zeta_{f[A]}$, we see that there is a Radon probability measure ζ' on A such that $\zeta_{f[A]}$ is the image measure $\zeta' f^{-1}$; let λ be the Radon probability measure on $\mathbb{R}^r$ extending ζ'. Then

$$\begin{aligned}\text{energy}(\zeta) &= \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}}\zeta(dx)\zeta(dy) \ge \int_{f[A]}\int_{f[A]} \frac{1}{\|x-y\|^{r-2}}\zeta(dx)\zeta(dy)\\ &= \int_A\int_{f[A]} \frac{1}{\|x-f(v)\|^{r-2}}\zeta(dx)\zeta'(dv) = \int_A\int_A \frac{1}{\|f(u)-f(v)\|^{r-2}}\zeta'(du)\zeta'(dv)\end{aligned}$$

(applying 235J[13] twice)

$$\begin{aligned}&\ge \int_A\int_A \frac{1}{\gamma^{r-2}\|u-v\|^{r-2}}\zeta'(du)\zeta'(dv)\\ &= \frac{1}{\gamma^{r-2}}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \frac{1}{\|u-v\|^{r-2}}\lambda(du)\lambda(dv) = \frac{1}{\gamma^{r-2}}\,\text{energy}(\lambda).\end{aligned}$$

By 479Nc in the other direction,

$$c(A) \ge \frac{1}{\text{energy}(\lambda)} \ge \frac{1}{\gamma^{r-2}\,\text{energy}(\zeta)} \ge \frac{1}{(1+\epsilon)\gamma^{r-2}}c(f[A]).$$

As ϵ is arbitrary, $c(f[A]) \le \gamma^{r-2}c(A)$.

(β) In general, since $f : D \to \mathbb{R}^r$ is certainly uniformly continuous, it has a continuous extension $g : \overline{D} \to \mathbb{R}^r$ (3A4G), which is still γ-Lipschitz. Now (α) tells us that

$$c(f[D]) \le c(g[A]) \le \gamma^{r-2}c(A) = \gamma^{r-2}c(D),$$

as required.

479Q Hausdorff measure: Theorem For $s \in \,]0,\infty[$ let μ_{Hs} be Hausdorff s-dimensional measure on $\mathbb{R}^r$. Let D be any subset of $\mathbb{R}^r$.

(a) If the Choquet-Newton capacity $c(D)$ is non-zero, then $\mu^*_{H,r-2}D = \infty$.

(b) If $s > r-2$ and $\mu^*_{Hs}D > 0$, then $c(D) > 0$.

proof (a) Let $E \supseteq D$ be a G_δ set such that $\mu_{H,r-2}E = \mu^*_{H,r-2}D$ (471Db). Then $c(E) > 0$. Let $K \subseteq E$ be a compact set such that $\operatorname{cap} K > 0$. Then

$$\operatorname{cap} K = \int_K\int_K \frac{1}{\|x-y\|^{r-2}}\lambda_K(dx)\lambda_K(dy)$$

is finite and not 0; applying 471Tb to the subspace measure on K,

$$\infty = \mu_{H,r-2}K = \mu_{H,r-2}E = \mu^*_{H,r-2}D.$$

(b) Let $E \supseteq D$ be a G_δ set such that $c(E) = c(D)$ (479E(d-i)). Then $\mu_{Hs}E > 0$. By 471Ta, there is a non-zero topological measure ζ_0 on E such that $\int_E\int_E \frac{1}{\|x-y\|^{r-2}}\zeta_0(dx)\zeta_0(dy)$ is finite. Let $K \subseteq E$ be a compact set such that $\zeta_0 K > 0$, and let ζ be the Radon measure on $\mathbb{R}^r$ such that $\zeta H = \zeta_0(K \cap H)$ for every Borel set $H \subseteq \mathbb{R}^r$; then

$$\text{energy}(\zeta) = \int_K\int_K \frac{1}{\|x-y\|^{r-2}}\zeta_0(dx)\zeta_0(dy)$$

is finite, while K is ζ-conegligible. By 479Nc (or, more directly, by the first remark in the proof of 479N), $\operatorname{cap} K > 0$, so that $c(D) = c(E) \ge \operatorname{cap} K > 0$.

[13] Formerly 235L.

479R I come to the promised difference between Brownian motion in $\mathbb{R}^3$ and in higher dimensions, following 478M.

Proposition (a) Suppose that $r = 3$. Then almost every $\omega \in \Omega$ is not injective.

(b) If $r \geq 4$, then almost every $\omega \in \Omega$ is injective.

proof In this proof, I will take $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$ and $t \geq 0$.

(a)(i) For $\omega \in \Omega$ set $F_\omega = \{\omega(t) : t \in [0,1]\}$. Then $\operatorname{cap} F_\omega > 0$ for μ_W-almost every ω. **P** By 477Lb, $\mu_{H,3/2} F_\omega = \infty$ for almost every ω. For any such ω, there is a non-zero Radon measure ζ_0 on F_ω such that $\int_{F_\omega}\int_{F_\omega} \frac{1}{\|x-y\|}\zeta_0(dx)\zeta_0(dy)$ is finite (471Ta). Let ζ be the Radon measure on $\mathbb{R}^r$, extending ζ_0, for which F_ω is conegligible. Then $\zeta(F_\omega) > 0$ and $\operatorname{energy}(\zeta) < \infty$. (This is where we need to know that $r = 3$.) So $\operatorname{cap} F_\omega > 0$ (479Nc). **Q**

(ii) Consider $E_0 = \{\omega : \text{there are } s \leq 1 \text{ and } t \geq 2 \text{ such that } \omega(s) = \omega(t)\}$. (This is an F_σ set, so is measurable.) Take τ to be the stopping time with constant value 2 and $\phi_\tau : \Omega \times \Omega \to \Omega$ the corresponding inverse-measure-preserving function as in 477G; set $H = \{\omega : \omega \in \Omega, \, \omega(2) \notin F_\omega\}$. Then

$$\begin{aligned}\mu_W E_0 &= \int_\Omega \mu_W\{\omega' : \phi_\tau(\omega, \omega') \in E_0\}\mu_W(d\omega)\\ &= \int_\Omega \mu_W\{\omega' : \text{ there is some } t \geq 0 \text{ such that } \omega(2) + \omega'(t) \in F_\omega\}\mu_W(d\omega)\\ &= \mu_W(\Omega \setminus H) + \int_H \tilde{W}_{F_\omega}(\omega(2))\mu_W(d\omega)\end{aligned}$$

(479Pb)

$$> 0$$

because $\tilde{W}_{F_\omega}(\omega(2)) > 0$ whenever $\omega \in H$ and $\operatorname{cap} F_\omega > 0$, which is so for almost every $\omega \in H$.

(iii) Now, setting

$$E_n = \{\omega : \text{there are } s \in [n, n+1] \text{ and } t \geq n+2 \text{ such that } \omega(s) = \omega(t)\},$$

we have $\mu_W E_n = \mu_W E_0$ for every n, because $\langle X_{s+n} - X_n\rangle_{s\geq 0}$ has the same distribution as $\langle X_s\rangle_{s\geq 0}$. So if $E = \bigcap_{n\in\mathbb{N}}\bigcup_{m\geq n} E_m$, $\mu_W E > 0$. But E belongs to the tail σ-algebra $\bigcap_{t\geq 0} \mathrm{T}_{[t,\infty[}$, so has measure either 0 or 1 (477Hd), and must be conegligible. Since every $\omega \in E$ is self-intersecting, we see that almost every Brownian path is self-intersecting.

(b)(i) Suppose that q, $q' \in \mathbb{Q}$ are such that $0 \leq q < q'$. This time, set $F_\omega = \{\omega(t) : t \in [0, q]\}$. For almost every ω, F_ω has zero two-dimensional Hausdorff measure (477La), so has zero $(r-2)$-dimensional Hausdorff measure (because $r \geq 4$), and therefore has zero capacity (479Qa). Also

$$\mu_W\{\omega : \omega(q') \in F_\omega\} = (\mu_W \times \mu_W)\{(\omega, \omega') : \omega'(q'-q) \in F_\omega - \omega(q)\} = 0$$

because the distribution of $X_{q'-q}$ is absolutely continuous with respect to Lebesgue measure and $\mu F_\omega = 0$ for μ_W-almost every ω. But this means that

$$\begin{aligned}&\mu_W\{\omega : \text{there is a } t \geq q' \text{ such that } \omega(t) \in F_\omega\}\\ &\qquad= (\mu_W \times \mu_W)\{(\omega, \omega') : \text{there is a } t \geq 0 \text{ such that } \omega'(t) \in F_\omega - \omega(q')\}\\ &\qquad= \int_\Omega \tilde{W}_{F_\omega}(\omega(q'))\mu(d\omega) = 0,\end{aligned}$$

that is,

$$\{\omega : \text{there are } s \leq q, \, t \geq q' \text{ such that } \omega(s) = \omega(t)\}$$

is negligible. As q and q' are arbitrary, almost every sample path is injective.

479S A famous classical problem concerned, in effect, the continuity of potential functions, in particular the continuity of functions of the form $\tilde{W}_K$. I think that even with the modern theory as sketched above, this is not quite trivial, so I spell out an example.

Example Suppose that $e \in \mathbb{R}^r$ is a unit vector. Then there is a sequence $\langle \delta_n \rangle_{n\in\mathbb{N}}$ of strictly positive real numbers such that the equilibrium potential $\tilde{W}_K$ is discontinuous at e whenever $K \subseteq B(\mathbf{0},1)$ is compact, $e \in \overline{\operatorname{int} K}$ and $\|x - te\| \le \delta_n$ whenever $n \in \mathbb{N}$, $t \in [1-2^{-n},1]$, $x \in K$ and $\|x\| = t$.

proof For $n \in \mathbb{N}$, let K_n be the line segment $\{te : 1-2^{-n} \le t \le 1-2^{-n-1}\}$. Then the one-dimensional Hausdorff measure of K_n is finite, so $\operatorname{cap} K_n = 0$ (479Qa). By 479E(c-ii), $\lim_{\delta\downarrow 0} \operatorname{cap}(K_n + B(\mathbf{0},\delta)) = 0$; let $\delta_n \in \left]0, 2^{-n-2}\right[$ be such that $\operatorname{cap}(K_n + B(\mathbf{0},\delta_n)) \le 2^{-3n-6}$. Setting $L_n = K_n + B(\mathbf{0},\delta_n)$, the distance from e to L_n is at least 2^{-n-2}. By 479Pb,

$$\operatorname{hp}(L_n - e) = \tilde{W}_{L_n}(e) \le 4^{n+2}\lambda_{L_n}(\mathbb{R}^r) = 4^{n+2} \operatorname{cap} L_n \le 2^{-n-2}.$$

Suppose that $K \subseteq B(\mathbf{0},1)$ is compact, $e \in \overline{\operatorname{int} K}$ and $\|x - te\| \le \delta_n$ whenever $n \in \mathbb{N}$, $t \in [1-2^{-n},1]$, $x \in K$ and $\|x\| = t$. Then $K \subseteq \bigcup_{n\in\mathbb{N}} L_n \cup \{e\}$. Using the full strength of 479Pb,

$$\tilde{W}_K(e) = \operatorname{hp}((K \setminus \{e\}) - e) \le \operatorname{hp}(\textstyle\bigcup_{n\in\mathbb{N}} L_n - e) \le \sum_{n=0}^{\infty} \operatorname{hp}(L_n - e) \le \frac{1}{2}.$$

On the other hand, $\tilde{W}_K(x) = 1$ for every $x \in \operatorname{int} K$ (479D(b-iii)), so $\tilde{W}_K$ is not continuous at e.

***479T** This concludes the main argument of the section, which you may feel is quite enough. However, there is an important alternative method of calculating the capacity of a compact set, based on gradients of potential functions (479U), and a couple of further results are reasonably accessible (479V-479W) which reflect other concerns of this volume.

Lemma (a) If $g : \mathbb{R}^r \to \mathbb{R}$ is a smooth function with compact support,

$$\int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}} \nabla^2 g \, d\mu = -r(r-2)\beta_r g(x)$$

for every $x \in \mathbb{R}^r$.

(b) Let g, $h : \mathbb{R}^r \to \mathbb{R}$ be smooth functions with compact support. Then

$$\int_{\mathbb{R}^r} h \times \nabla^2 g \, d\mu = \int_{\mathbb{R}^r} g \times \nabla^2 h = -\int_{\mathbb{R}^r} \operatorname{grad} h \,\boldsymbol{.}\, \operatorname{grad} g \, d\mu.$$

(c) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$, and $W_\zeta : \mathbb{R}^r \to [0,\infty]$ the associated Newtonian potential. Then $\int_{\mathbb{R}^r} W_\zeta \times \nabla^2 g \, d\mu = -r(r-2)\beta_r \int_{\mathbb{R}^r} g \, d\zeta$ for every smooth function $g : \mathbb{R}^r \to \mathbb{R}$ with compact support.

(d) Let ζ be a totally finite Radon measure on $\mathbb{R}^r$ such that W_ζ is finite-valued everywhere and Lipschitz. Then $\int_{\mathbb{R}^r} \operatorname{grad} f \,\boldsymbol{.}\, \operatorname{grad} W_\zeta d\mu = r(r-2)\beta_r \int_{\mathbb{R}^r} f \, d\zeta$ for every Lipschitz function $f : \mathbb{R}^r \to \mathbb{R}$ with compact support.

(e) Let $K \subseteq \mathbb{R}^r$ be a compact set, and $\epsilon > 0$. Then there is a Radon measure ζ on $\mathbb{R}^r$, with support included in $K + B(\mathbf{0},\epsilon)$, such that W_ζ is a smooth function with compact support, $W_\zeta \ge \chi K$, $\zeta\mathbb{R}^r \le \operatorname{cap} K + \epsilon$ and

$$\int_{\mathbb{R}^r} \|\operatorname{grad} W_\zeta\|^2 d\mu = r(r-2)\beta_r \operatorname{energy}(\zeta) \le r(r-2)\beta_r \zeta\mathbb{R}^r.$$

proof (a)(i) Consider first the case $x = 0$. Setting $f(y) = \frac{1}{\|y\|^{r-2}}$ for $y \ne 0$, we have $\operatorname{grad} f(y) = -\frac{r-2}{\|y\|^r} y$ and $(\nabla^2 f)(y) = 0$ for $y \ne 0$ (478Fa); also f is locally integrable, by 478Ga. So $\int_{\mathbb{R}^r} f \times \nabla^2 g \, d\mu$ is well-defined.

Let $R > 0$ be such that g is zero outside $B(\mathbf{0},R)$, and set $M = \|\operatorname{grad} g\|_\infty$; take $\epsilon \in \left]0, R\right[$. Then

$$\int_{\mathbb{R}^r \setminus B(\mathbf{0},\epsilon)} f \times \nabla^2 g \, d\mu = \int_{B(\mathbf{0},R)\setminus B(\mathbf{0},\epsilon)} f \times \nabla^2 g - g \times \nabla^2 f \, d\mu$$
$$= \int_{B(\mathbf{0},R)\setminus B(\mathbf{0},\epsilon)} \operatorname{div}(f \times \operatorname{grad} g - g \times \operatorname{grad} f) d\mu$$

(use 474Bb)

$$= \int_{\partial B(\mathbf{0},R)} (\frac{1}{\|y\|^{r-2}} \operatorname{grad} g(y) + \frac{(r-2)g(y)}{\|y\|^r} y) \,\boldsymbol{.}\, \frac{y}{\|y\|} \nu(dy)$$
$$- \int_{\partial B(\mathbf{0},\epsilon)} (\frac{1}{\|y\|^{r-2}} \operatorname{grad} g(y) + \frac{(r-2)g(y)}{\|y\|^r} y) \,\boldsymbol{.}\, \frac{y}{\|y\|} \nu(dy)$$

(475Nc)

$$= -\int_{\partial B(\mathbf{0},\epsilon)} \Bigl(\frac{1}{\|y\|^{r-1}} y \,\boldsymbol{.}\, \operatorname{grad} g(y) + \frac{(r-2)g(y)}{\|y\|^{r-1}}\Bigr)\nu(dy)$$
$$= -\frac{1}{\epsilon^{r-1}} \int_{\partial B(\mathbf{0},\epsilon)} (y \,\boldsymbol{.}\, \operatorname{grad} g(y) + (r-2)g(y))\nu(dy).$$

Now we have

$$|\textstyle\int_{\partial B(\mathbf{0},\epsilon)} y \,\boldsymbol{.}\, \operatorname{grad} g(y)\nu(dy)| \le \epsilon M \nu(\partial B(\mathbf{0},\epsilon)) \le r\beta_r \epsilon^r M,$$

so

$$|\int_{\mathbb{R}^r \setminus B(\mathbf{0},\epsilon)} f \times \nabla^2 g\, d\mu + r(r-2)\beta_r g(0)|$$
$$\le r\beta_r \epsilon M + \frac{1}{\epsilon^{r-1}} |r\beta_r \epsilon^{r-1}(r-2)g(0) - \int_{\partial B(\mathbf{0},\epsilon)} (r-2)g(y)\nu(dy)|$$
$$\le r\beta_r \epsilon M + \frac{r-2}{\epsilon^{r-1}} \int_{\partial B(\mathbf{0},\epsilon)} |g(0) - g(y)|\nu(dy)$$
$$\le r\beta_r \epsilon M + r(r-2)\beta_r \sup_{y \in \partial B(\mathbf{0},\epsilon)} |g(0) - g(y)| \to 0$$

as $\epsilon \downarrow 0$; that is,

$$\textstyle\int_{\mathbb{R}^r} f \times \nabla^2 g\, d\mu = -r(r-2)\beta_r g(0).$$

(ii) For the general case, apply (i) to the function $y \mapsto g(x+y)$.

(b) Take $R > 0$ so large that both g and h are zero outside $B(\mathbf{0},R)$, and $M \ge \max(\|\nabla^2 g\|_\infty, \|\nabla^2 h\|_\infty)$.

(i) We have

$$\int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}} |(\nabla^2 g)(x)(\nabla^2 h)(y)| \mu(dx)\mu(dy)$$
$$\le M^2 \int_{B(\mathbf{0},R)} \int_{B(\mathbf{0},R)} \frac{1}{\|x-y\|^{r-2}} \mu(dx)\mu(dy) \le M^2 \int_{B(\mathbf{0},R)} \frac{1}{2} r\beta_r R^2 \mu(dy)$$

(478Gc)

$$< \infty.$$

So

$$-r(r-2)\beta_r \int_{\mathbb{R}^r} h \times \nabla^2 g\, d\mu = \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}} (\nabla^2 h)(x)(\nabla^2 g)(y)\mu(dx)\mu(dy)$$

(by (a))

$$= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \frac{1}{\|x-y\|^{r-2}} (\nabla^2 g)(y)(\nabla^2 h)(x)\mu(dy)\mu(dx)$$
$$= -r(r-2)\beta_r \int_{\mathbb{R}^r} g \times \nabla^2 h\, d\mu.$$

Thus $\int_{\mathbb{R}^r} g \times \nabla^2 h\, d\mu = \int_{\mathbb{R}^r} h \times \nabla^2 g\, d\mu$.

(ii) By 473Bd, $\operatorname{grad}(g \times h) = g \times \operatorname{grad} h + h \times \operatorname{grad} g$, so 474Bb tells us that $\nabla^2(g \times h) = 2 \operatorname{grad} g \,\boldsymbol{.}\, \operatorname{grad} h + g \times \nabla^2 h + h \times \nabla^2 g$, and

$$\int_{\mathbb{R}^r} \nabla^2(g \times h) d\mu = \int_{B(\mathbf{0},R)} \nabla^2(g \times h) d\mu$$
$$= \int_{\partial B(\mathbf{0},R)} \operatorname{grad}(g \times h) \,\boldsymbol{.}\, \frac{x}{\|x\|} \nu(dx) = 0,$$

so

$$\int_{\mathbb{R}^r}\operatorname{grad}g\,\boldsymbol{.}\operatorname{grad}h\,d\mu=-\frac{1}{2}\int_{\mathbb{R}^r}g\times\nabla^2 h+h\times\nabla^2 g\,d\mu=-\int_{\mathbb{R}^r}g\times\nabla^2 h\,d\mu.$$

(c) If $g(x)=0$ for $\|x\|\ge R$ and $|(\nabla^2 g)(x)|\le M$ for every x, then

$$\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}|(\nabla^2 g)(x)|\mu(dx)\le M\int_{B(\mathbf{0},R)}\frac{1}{\|x-y\|^{r-2}}\mu(dx)\le\frac{1}{2}Mr\beta_r R^2$$

for every y (478Gc again). We can therefore apply (a) and integrate with respect to ζ to see that

$$\begin{aligned}-r(r-2)\beta_r\int_{\mathbb{R}^r}g\,d\zeta&=\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}(\nabla^2 g)(x)\mu(dx)\zeta(dy)\\&=\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\frac{1}{\|x-y\|^{r-2}}(\nabla^2 g)(x)\zeta(dy)\mu(dx)\\&=\int_{\mathbb{R}^r}W_\zeta(x)(\nabla^2 g)(x)\mu(dx),\end{aligned}$$

as required.

(d) Let $\langle\tilde h_n\rangle_{n\in\mathbb{N}}$ be the smoothing sequence of 473E.

(i) Suppose to begin with that f is smooth. For $n\in\mathbb{N}$ set $g_n=\tilde h_n*W_\zeta$. As W_ζ is continuous, $\lim_{n\to\infty}g_n=W_\zeta$ (473Ec); as $\|W_\zeta\|_\infty\le 1$, $\|g_n\|_\infty\le 1$ for every n (473Da). Because f has compact support,

$$\lim_{n\to\infty}\int_{\mathbb{R}^r}g_n\times\nabla^2 f\,d\mu=\int_{\mathbb{R}^r}W_\zeta\times\nabla^2 f\,d\mu$$

by the dominated convergence theorem. Next, $\operatorname{grad}g_n=\tilde h_n*\operatorname{grad}W_\zeta$ for each n (473Dd). As $\operatorname{grad}W_\zeta$ is essentially bounded (473Cc), all its coordinates are locally integrable, so $\operatorname{grad}W_\zeta=_{\text{a.e.}}\lim_{n\to\infty}\operatorname{grad}g_n$ (473Ee). We therefore have

$$\begin{aligned}\int_{\mathbb{R}^r}\operatorname{grad}f\,\boldsymbol{.}\operatorname{grad}W_\zeta d\mu&=\lim_{n\to\infty}\int_{\mathbb{R}^r}\operatorname{grad}f\,\boldsymbol{.}\operatorname{grad}g_n d\mu\\&=-\lim_{n\to\infty}\int_{\mathbb{R}^r}g_n\times\nabla^2 f\,d\mu\end{aligned}$$

((b) above)

$$=-\int_{\mathbb{R}^r}W_\zeta\times\nabla^2 f\,d\mu=r(r-2)\beta_r\int_{\mathbb{R}^r}f\,d\zeta$$

by (c).

(ii) For the general case, smooth on the other side, setting $f_n=\tilde h_n*f$ for every n. This time, $f_n\to f$ uniformly (473Ed), so $\int_{\mathbb{R}^r}f d\zeta=\lim_{n\to\infty}\int_{\mathbb{R}^r}f_n d\zeta$. On the other hand, if f is M-Lipschitz, $\operatorname{grad}f_n=\tilde h_n*\operatorname{grad}f$ converges μ-a.e. to $\operatorname{grad}f$, and $\|\operatorname{grad}f_n\|_\infty$ is at most M for every n; also there is a bounded set outside which all the f_n and $\operatorname{grad}f_n$ are zero, and $\|\operatorname{grad}W_\zeta\|$ is bounded. So

$$\int_{\mathbb{R}^r}\operatorname{grad}f\,\boldsymbol{.}\operatorname{grad}W_\zeta d\mu=\lim_{n\to\infty}\int_{\mathbb{R}^r}\operatorname{grad}f_n\,\boldsymbol{.}\operatorname{grad}W_\zeta d\mu.$$

Applying (i) to each f_n and taking the limit, we get the equality we seek.

(e)(i) There is a compact set $L\subseteq K+B(\mathbf{0},\frac{\epsilon}{2})$ such that $K\subseteq\operatorname{int}L$ and $\operatorname{cap}L\le\operatorname{cap}K+\epsilon$ (479Ed). Let $n\in\mathbb{N}$ be such that $\frac{1}{n+1}\le\frac{\epsilon}{2}$ and $K+B(\mathbf{0},\frac{1}{n+1})\subseteq L$. Set $h=\lambda_L*\tilde h_n$, where $\tilde h_n$ is the function of 473E, as before; let $\zeta=h\mu$ be the corresponding indefinite-integral measure over μ. Because $\tilde h_n$ is zero outside $B(\mathbf{0},\frac{1}{n+1})$ and the support of λ_L is included in L, the support of ζ is included in $L+B(\mathbf{0},\frac{1}{n+1})\subseteq K+B(\mathbf{0},\epsilon)$.

(ii) By 444Pa, we have

$$W_\zeta=\zeta*k_{r-2}=(h\mu)*k_{r-2}=h*k_{r-2}$$

where k_{r-2} is the Riesz kernel (479G). Now $W_\zeta=\tilde W_L*\tilde h_n$. **P** For $m\in\mathbb{N}$, set $f_m=k_{r-2}\times\chi B(\mathbf{0},m)$, so that f_m is μ-integrable. Observe that

$$\tilde{W}_L(x) = \int_{\mathbb{R}^r} k_{r-2}(x-y)\lambda_L(dy) = \int_{\mathbb{R}^r} \lim_{m\to\infty} f_m(x-y)\lambda_L(dy)$$
$$= \lim_{m\to\infty}\int_{\mathbb{R}^r} f_m(x-y)\lambda_L(dy) = \lim_{m\to\infty}(\lambda_L * f_m)(x)$$

for each x; moreover, because $\langle f_m\rangle_{m\in\mathbb{N}}$ is non-decreasing, so is $\langle\lambda_L * f_m\rangle_{m\in\mathbb{N}}$. For each m,

$$h * f_m = (h\mu) * f_m = (\lambda_L * \tilde{h}_n)\mu * f_m = (\lambda_L * \tilde{h}_n\mu) * f_m$$

(444K)

$$= \lambda_L * (\tilde{h}_n\mu * f_m)$$

(444Ic)

$$= \lambda_L * (\tilde{h}_n * f_m) = \lambda_L * (f_m * \tilde{h}_n) = \lambda_L * (f_m\mu * \tilde{h}_n)$$
$$= (\lambda_L * f_m\mu) * \tilde{h}_n = (\lambda_L * f_m)\mu * \tilde{h}_n = (\lambda_L * f_m) * \tilde{h}_n.$$

Now, for each x,

$$W_\zeta(x) = \int_{\mathbb{R}^r} h(x-y)k_{r-2}(y)\mu(dy) = \lim_{m\to\infty}\int_{\mathbb{R}^r} h(x-y)f_m(y)\mu(dy)$$
$$= \lim_{m\to\infty}(h * f_m)(x) = \lim_{m\to\infty}((\lambda_L * f_m) * \tilde{h}_n)(x)$$
$$= \lim_{m\to\infty}\int_{\mathbb{R}^r}(\lambda_L * f_m)(y)\tilde{h}_n(x-y)\mu(dy)$$
$$= \int_{\mathbb{R}^r}\lim_{m\to\infty}(\lambda_L * f_m)(y)\tilde{h}_n(x-y)\mu(dy)$$
$$= \int_{\mathbb{R}^r}\tilde{W}_L(y)\tilde{h}_n(x-y)\mu(dy) = (\tilde{W}_L * \tilde{h}_n)(x). \text{ }\mathbf{Q}$$

(iii) Since $\tilde{W}_L(x) = 1$ whenever $x \in \operatorname{int} L$ (479D(b-iii)), and $x + y \in \operatorname{int} L$ whenever $x \in K$ and $\tilde{h}_n(y) \neq 0$, $W_\zeta(x) = 1$ for every $x \in K$. Because both $\tilde{W}_L$ and $\tilde{h}_n$ have compact support, so does W_ζ; because $\tilde{h}_n$ is smooth, so is W_ζ (473De).

(iv) Now

$$\int_{\mathbb{R}^r}\|\operatorname{grad} W_\zeta\|^2 d\mu = -\int_{\mathbb{R}^r} W_\zeta \times \nabla^2 W_\zeta\, d\mu$$

((b) above)

$$= r(r-2)\beta_r\int_{\mathbb{R}^r} W_\zeta\, d\zeta$$

((c) above)

$$= r(r-2)\beta_r \operatorname{energy}(\zeta) \le r(r-2)\beta_r\zeta\mathbb{R}^r$$

because $\|W_\zeta\|_\infty \le \|\tilde{W}_L\|_\infty\|\tilde{h}_n\|_1 \le 1$.

(v) Finally,

$$\zeta\mathbb{R}^r = (h\mu)\mathbb{R}^r = (\lambda_L * \tilde{h}_n\mu)\mathbb{R}^r = \lambda_L\mathbb{R}^r \cdot (\tilde{h}_n\mu)\mathbb{R}^r$$
$$= \lambda_L\mathbb{R}^r = \operatorname{cap} L \le \epsilon + \operatorname{cap} K.$$

***479U Theorem** Let $K \subseteq \mathbb{R}^r$ be compact, and let Φ be the set of Lipschitz functions $g : \mathbb{R}^r \to \mathbb{R}$ such that $g(x) \ge 1$ for every $x \in K$ and $\lim_{\|x\|\to\infty} g(x) = 0$. Then

$$r(r-2)\beta_r \operatorname{cap} K = \inf\{\int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu : g \in \Phi \text{ is smooth and has compact support}\}$$
$$= \inf\{\int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu : g \in \Phi\}.$$

proof (a) By 479Te,

$$\inf\{\int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu : g \in \Phi \text{ is smooth and has compact support}\} \le r(r-2)\beta_r \operatorname{cap} K.$$

(b) Now suppose that $g \in \Phi$ is a smooth function with compact support. Then $r(r-2)\beta_r \operatorname{cap} K \le \int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu$. **P** Take any $\epsilon \in]0,1[$. Then there is a $\delta > 0$ such that $g(x) \ge 1-\epsilon$ for every $x \in K + B(\mathbf{0},\delta)$. By 479Te, there is a Radon measure ζ on $\mathbb{R}^r$, with support included in $K + B(\mathbf{0},\delta)$, such that W_ζ is smooth and has compact support, $W_\zeta \ge \chi K$, $\zeta\mathbb{R}^r \le \operatorname{cap} K + \epsilon$ and

$$\int_{\mathbb{R}^r} \|\operatorname{grad} W_\zeta\|^2 d\mu = r(r-2)\beta_r \operatorname{energy}(\zeta) \le r(r-2)\beta_r \zeta\mathbb{R}^r.$$

In this case,

$$\int_{\mathbb{R}^r} \operatorname{grad} g \,.\, \operatorname{grad} W_\zeta d\mu = r(r-2)\beta_r \int_{\mathbb{R}^r} g\, d\zeta$$

(479Td)

$$\ge (1-\epsilon) r(r-2)\beta_r \zeta\mathbb{R}^r \ge (1-\epsilon)\int_{\mathbb{R}^r} \|\operatorname{grad} W_\zeta\|^2.$$

Setting $v = (1-\epsilon)\operatorname{grad} W_\zeta$, we have

$$\int_{\mathbb{R}^r} v \,.\, \operatorname{grad} g d\mu \ge \int_{\mathbb{R}^r} \|v\|^2 d\mu.$$

But this means that

$$\int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu = 2\int_{\mathbb{R}^r} v \,.\, \operatorname{grad} g d\mu - \int_{\mathbb{R}^r} \|v\|^2 d\mu + \int_{\mathbb{R}^r} \|v - \operatorname{grad} g\|^2 d\mu$$
$$\ge \int_{\mathbb{R}^r} \|v\|^2 d\mu \ge (1-\epsilon)^2 \int_{\mathbb{R}^r} \|\operatorname{grad} W_\zeta\|^2 d\mu$$
$$= (1-\epsilon)^2 r(r-2)\beta_r \int_{\mathbb{R}^r} W_\zeta d\zeta \ge (1-\epsilon)^2 r(r-2)\beta_r \int_{\mathbb{R}^r} \tilde{W}_K d\zeta$$

(because $W_\zeta \ge \tilde{W}_K$ on K, so $W_\zeta \ge \tilde{W}_K$ everywhere, by 479Fg)

$$= (1-\epsilon)^2 r(r-2)\beta_r \int_{\mathbb{R}^r} W_\zeta d\lambda_K$$

(479J(b-i))

$$\ge (1-\epsilon)^2 r(r-2)\beta_r \int_{\mathbb{R}^r} \tilde{W}_K d\lambda_K = (1-\epsilon)^2 r(r-2)\beta_r \operatorname{cap} K.$$

As ϵ is arbitrary, $r(r-2)\beta_r \operatorname{cap} K \le \int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu$. **Q**

(c) If $g \in \Phi$ has compact support, then $\int_{\mathbb{R}^r} \|\operatorname{grad} g\|^2 d\mu \ge r(r-2)\beta_r \operatorname{cap} K$. **P** Let $R > 0$ be such that g is zero outside $B(\mathbf{0},R)$. Let $M \ge 0$ be such that g is M-Lipschitz; then $\|\operatorname{grad} g(x)\| \le M$ for every $x \in \operatorname{dom} \operatorname{grad} g$ (473Cc). Take any $\epsilon > 0$. As in 479T, let $\langle \tilde{h}_n \rangle_{n\in\mathbb{N}}$ be the smoothing sequence of 473E. For $n \in \mathbb{N}$, set $g_n = (1+\epsilon)\tilde{h}_n * g$. Then $\operatorname{grad} g_n = (1+\epsilon)\tilde{h}_n * \operatorname{grad} g$ (473Dd) and $\|\operatorname{grad} g_n\|_\infty \le M(1+\epsilon)$ (473Da). In the limit, $(1+\epsilon)\operatorname{grad} g =_{\text{a.e.}} \lim_{n\to\infty} \operatorname{grad} g_n$ (473Ee).

There is an $m \in \mathbb{N}$ such that $(1+\epsilon)g(x) \ge 1$ for every $x \in K + B(\mathbf{0}, \frac{1}{m+1})$; now if $n \ge m$,

$$g_n(x) \ge (1+\epsilon)\inf_{\|y\|\le 1/(n+1)} g(x-y) \ge 1$$

for every $x \in K$. So

$$\begin{aligned}(1+\epsilon)^2\int_{\mathbb{R}^r}\|\operatorname{grad} g\|^2 d\mu &= (1+\epsilon)^2\int_{B(\mathbf{0},R+1)}\|\operatorname{grad} g\|^2 d\mu\\ &= \lim_{n\to\infty}\int_{B(\mathbf{0},R+1)}\|\operatorname{grad} g_n\|^2 d\mu\end{aligned}$$

(by the dominated convergence theorem)

$$= \lim_{n\to\infty}\int_{\mathbb{R}^r}\|\operatorname{grad} g_n\|^2 d\mu$$

(because every g_n is zero outside $B(\mathbf{0},R+1)$)

$$\ge r(r-2)\beta_r \operatorname{cap} K$$

(applying (b) to g_n for $n\ge m$). As ϵ is arbitrary, $r(r-2)\beta_r\operatorname{cap}K\le\int_{\mathbb{R}^r}\|\operatorname{grad} g\|^2 d\mu$. **Q**

(d) If $g\in\Phi$, then $\int_{\mathbb{R}^r}\|\operatorname{grad} g\|^2 d\mu\ge r(r-2)\beta_r\operatorname{cap}K$. **P** Let $\epsilon>0$. Set $g_1(x)=\max(0,(1+\epsilon)g(x)-\epsilon)$ for $x\in\mathbb{R}^r$. Then $g_1\in\Phi$ has compact support, and $\|g_1(x)-g_1(y)\|\le(1+\epsilon)\|g(x)-g(y)\|$ for all x, $y\in\mathbb{R}^r$, so $\|\operatorname{grad} g_1(x)\|\le(1+\epsilon)\|\operatorname{grad} g(x)\|$ whenever both gradients are defined. Accordingly

$$(1+\epsilon)^2\int_{\mathbb{R}^r}\|\operatorname{grad} g\|^2 d\mu\ge\int_{\mathbb{R}^r}\|\operatorname{grad} g_1\|^2 d\mu\ge r(r-2)\beta_r\operatorname{cap}K$$

by (c). As ϵ is arbitrary, we have the result. **Q**

(e) Putting (a) and (d) together, the theorem is proved.

***479V** We are ready for another theorem along the lines of 476H, this time relating capacity and Lebesgue measure.

Theorem Let $D\subseteq\mathbb{R}^r$ be a set of finite outer Lebesgue measure, and B_D the closed ball with centre 0 and the same outer measure as D. Then the Choquet-Newton capacity $c(D)$ of D is at least $\operatorname{cap}B_D$.

proof (a) We need an elementary fact about gradients. Suppose that f, $g:\mathbb{R}^r\to\mathbb{R}$ and $x\in\mathbb{R}^r$ are such that $\operatorname{grad} f$, $\operatorname{grad} g$, $\operatorname{grad}(f\vee g)$ and $\operatorname{grad}(f\wedge g)$ are all defined at x. Then $\{\operatorname{grad}(f\vee g)(x),\operatorname{grad}(f\wedge g)(x)\}=\{\operatorname{grad} f(x),\operatorname{grad} g(x)\}$. **P** (i) If $f(x)>g(x)$ then (because f and g are both continuous at x) we have $\operatorname{grad}(f\vee g)(x)=\operatorname{grad} f(x)$, $\operatorname{grad}(f\wedge g)(x)=\operatorname{grad} g(x)$ and the result is immediate. (ii) The same argument applies if $f(x)<g(x)$. (iii) If $f(x)=g(x)$, consider $h=|f-g|=(f\vee g)-(f\wedge g)$. Then $\operatorname{grad} h(x)$ is defined, and $h(x)=0\le h(y)$ for every y. So all the partial derivatives of h have to be zero at x, and $\operatorname{grad} h(x)=0$, that is, $\lim_{y\to x}\frac{1}{\|y-x\|}h(y)=0$. It follows at once that $\operatorname{grad} f(x)=\operatorname{grad} g(x)$, and therefore both are equal to $\operatorname{grad}(f\vee g)(x)$ and $\operatorname{grad}(f\wedge g)(x)$. So again we have the result. **Q**

(b) Now for a further clause to add to Lemma 476E. Suppose that $e\in S_{r-1}=\partial B(\mathbf{0},1)$ and $\alpha\in\mathbb{R}$; let $R=R_{e\alpha}$ be the reflection in the plane $\{x:x\centerdot e=\alpha\}$, and $\psi=\psi_{e\alpha}:\mathcal{P}\mathbb{R}^r\to\mathcal{P}\mathbb{R}^r$ the partial-reflection operator of 476D-476E, that is,

$$\psi(D)=(W\cap(D\cup R[D]))\cup(W'\cap D\cap R[D])$$

for $D\subseteq\mathbb{R}^r$, where $W=\{x:x\centerdot e\ge\alpha\}$ and $W'=\{x:x\centerdot e\le\alpha\}$. Then $c(\psi(D))\le c(D)$ for every $D\subseteq\mathbb{R}^r$.

P(i) Suppose first that $D=K$ is compact. Take any $\gamma>r(r-2)\beta_r\operatorname{cap}K$. By 479U, there is a Lipschitz function $f:\mathbb{R}^r\to\mathbb{R}$ such that $f(x)\ge1$ for every $x\in K$, $\lim_{\|x\|\to\infty}f(x)=0$ and $\int_{\mathbb{R}^r}\|\operatorname{grad} f\|^2 d\mu\le\gamma$. Set $g=fR$. Of course g is Lipschitz and $\int_{\mathbb{R}^r}\|\operatorname{grad} g\|^2 d\mu=\int_{\mathbb{R}^r}\|\operatorname{grad} f\|^2 d\mu$. Now $f\vee g$ and $f\wedge g$ are also Lipschitz, so for almost every $x\in\mathbb{R}^r$ all the gradients $\operatorname{grad} f(x)$, $\operatorname{grad} g(x)$, $\operatorname{grad}(f\vee g)(x)$ and $\operatorname{grad}(f\wedge g)(x)$ are defined; by (a), $\|\operatorname{grad} f\|^2+\|\operatorname{grad} g\|^2=_{\text{a.e.}}\|\operatorname{grad}(f\vee g)\|^2+\|\operatorname{grad}(f\wedge g)\|^2$.

Now consider the function h defined by saying that

$$\begin{aligned}h(x)&=(f\vee g)(x)\text{ if }x\in W,\\ &=(f\wedge g)(x)\text{ if }x\in W'.\end{aligned}$$

h is Lipschitz and $\lim_{\|x\|\to\infty}h(x)=0$; also $h(x)\ge1$ for every $x\in\psi(K)$. For $x\in W\setminus W'$, $h(x)=(f\vee g)(x)$ and $h(Rx)=(f\wedge g)(x)$, so $\{\operatorname{grad} h(x),\operatorname{grad}(hR)(x)\}=\{\operatorname{grad}(f\vee g)(x),\operatorname{grad}(f\wedge g)(x)\}$ if the gradients are defined; for

$x \in W' \setminus W$, $h(x) = (f \wedge g)(x)$ and $h(Rx) = (f \vee g)(x)$, so $\{\operatorname{grad} h(x), \operatorname{grad}(hR)(x)\} = \{\operatorname{grad}(f \vee g)(x), \operatorname{grad}(f \wedge g)(x)\}$ if the gradients are defined. Accordingly

$$\begin{aligned}\|\operatorname{grad} h\|^2 + \|\operatorname{grad}(hR)\|^2 &=_{\text{a.e.}} \|\operatorname{grad}(f \vee g)\|^2 + \|\operatorname{grad}(f \wedge g)\|^2 \\ &=_{\text{a.e.}} \|\operatorname{grad} f\|^2 + \|\operatorname{grad} g\|^2.\end{aligned}$$

By 479U again,

$$\begin{aligned}r(r-2)\beta_r \operatorname{cap}(\psi(K)) \le \int_{\mathbb{R}^r} \|\operatorname{grad} h\|^2 d\mu &= \frac{1}{2}\int_{\mathbb{R}^r} (\|\operatorname{grad} h\|^2 + \|\operatorname{grad}(hR)\|^2) d\mu \\ &= \frac{1}{2}\int_{\mathbb{R}^r} (\|\operatorname{grad} f\|^2 + \|\operatorname{grad} g\|^2) d\mu \le \gamma.\end{aligned}$$

As γ is arbitrary, $\operatorname{cap}(\psi(K)) \le \operatorname{cap} K$.

(ii) Now suppose that $D = G$ is open. Then there is a non-decreasing sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of compact sets with union G, and $\langle \psi(K_n)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with union $\psi(G)$. So

$$c(\psi(G)) = \sup_{n\in\mathbb{N}} \operatorname{cap}(\psi(K_n)) \le \sup_{n\in\mathbb{N}} \operatorname{cap} K_n = c(G).$$

(iii) Finally, for arbitrary $D \subseteq \mathbb{R}^r$, take any $\gamma > c(D)$. Then there is an open set G such that $D \subseteq G$ and $c(G) \le \gamma$ (because c is outer regular, see 479E(d-i)). In this case, $\psi(D) \subseteq \psi(G)$, so

$$c(\psi(D)) \le c(\psi(G)) \le c(G) \le \gamma.$$

As γ is arbitrary, $c(\psi(D)) \le c(D)$ and we are done. **Q**

(c) Now suppose that E is a bounded Lebesgue measurable subset of $\mathbb{R}^r$ with finite perimeter.

(i) Let $M \ge 0$ be such that $E \subseteq B(\mathbf{0}, M)$. Consider

$$\begin{aligned}\mathcal{E} = \{F : F \subseteq B(\mathbf{0}, M) &\text{ is Lebesgue measurable,} \\ &\mu F = \mu E,\ \operatorname{per} F \le \operatorname{per} E,\ c(F) \le c(E)\}.\end{aligned}$$

Then $\mathcal{E}$ is compact for the topology $\mathfrak{T}_m$ of convergence in measure as described in 474T. **P** By 474T,

$$\mathcal{E}_1 = \{F : F \text{ is Lebesgue measurable, } \operatorname{per} F \le \operatorname{per} E\}$$

is compact. So if $\langle F_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{E}$, it has a subsequence $\langle F'_n\rangle_{n\in\mathbb{N}}$ which is $\mathfrak{T}_m$-convergent to $F \in \mathcal{E}_1$ say (4A2Le; recall that, as noted in the proof of 474T, $\mathfrak{T}_m$ is pseudometrizable). Taking a further subsequence if necessary, we can suppose that $\mu((F \triangle F'_n) \cap B(\mathbf{0}, M)) \le 2^{-n}$ for every $n \in \mathbb{N}$. Set $F' = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} F'_n$. Because every F'_n is included in $B(\mathbf{0}, M)$, F' is a $\mathfrak{T}_m$-limit of $\langle F'_n\rangle_{n\in\mathbb{N}}$. So $\mu F' = \lim_{n\to\infty} \mu F'_n = \mu E$, and

$$c(F') = \lim_{m\to\infty} c(\bigcap_{n\ge m} F'_n) \le c(E).$$

Finally, $F' \triangle F$ is negligible, so $\partial^* F' = \partial^* F$ and $\operatorname{per}(F') = \operatorname{per} F \le \operatorname{per} E$. Thus $F' \in \mathcal{E}$. As $\langle F_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{E}$ is relatively compact, by 4A2Le in the opposite direction. **Q**

(ii) Because $B(\mathbf{0}, M)$ is bounded, the function $F \mapsto \int_F \|x\|\mu(dx) : \mathcal{E} \to [0, \infty[$ is continuous, and must attain its infimum at H say. Let B_E be the ball with centre 0 and the same measure as E. Then $B_E \subseteq \operatorname{cl}^* H$. **P** (Compare part (b) of the proof of 476H.) **?** Otherwise, take $z \in B_E \setminus \operatorname{cl}^* H$. Then

$$\lim_{\delta>0} \frac{\mu(B(z,\delta)\setminus H)}{\mu B(z,\delta)} = 1, \quad \lim_{\delta>0} \frac{\mu(B(z,\delta)\setminus B_E)}{\mu B(z,\delta)} \le \frac{1}{2},$$

so there is a $\delta > 0$ such that $\mu(B(z,\delta) \setminus B_E) < \mu(B(z,\delta) \setminus H)$ and $\mu(B_E \setminus H) > 0$. Because

$$\mu(\operatorname{cl}^* H) = \mu H = \mu E = \mu B_E,$$

$\operatorname{cl}^* H \setminus B_E$ is also non-negligible. Take $x_1 \in \operatorname{cl}^* H \setminus B_E$ and $x_0 \in B_E \setminus \operatorname{cl}^* H$. Then $\delta_0 = \|x_1\| - \|x_0\|$ is greater than 0. Since

$$\limsup_{\delta\downarrow 0} \frac{\mu(H\cap B(x_1,\delta))}{\mu B(x_1,\delta)} > 0 = \lim_{\delta\downarrow 0} \frac{\mu(H\cap B(x_0,\delta))}{\mu B(x_0,\delta)},$$

there is a $\delta \in \left]0, \frac{1}{2}\delta_0\right[$ such that $\mu(H \cap B(x_1,\delta)) > \mu(H \cap B(x_0,\delta))$. Now let e be the unit vector $\frac{1}{\|x_0-x_1\|}(x_0 - x_1)$, and set $\alpha = e \cdot \frac{1}{2}(x_0 + x_1)$. Consider the reflection $R = R_{e\alpha}$ and the operator $\psi = \psi_{e\alpha}$; set $H_1 = \psi(H)$ and let $\phi =$

$\phi_{H_1} : H \to H_1$ be the function of 476E. As $\alpha < 0$, $\|\phi(x)\| \le \|x\|$ for every $x \in H$; moreover, $R[B(x_1, \delta)] = B(x_0, \delta)$, so

$$\{x : \|\phi(x)\| < \|x\|\} \supseteq \{x : x \in B(x_1, \delta) \cap H,\ Rx \notin H\}$$

is not negligible. So $\int_{H_1} \|x\|\mu(dx) < \int_H \|x\|\mu(dx)$. On the other hand, we surely have $H_1 \subseteq B(\mathbf{0}, M)$, $\mu H_1 = \mu H = \mu E$ and $\operatorname{per} H_1 \le \operatorname{per} H \le \operatorname{per} E$ (476Ee); and, finally, $c(H_1) \le c(H) \le c(E)$, by (b) of this proof. Thus $H_1 \in \mathcal{E}$ and the functional $F \mapsto \int_F \|x\|\mu(dx)$ is not minimized at H. **XQ**

(iii) Accordingly

$$\operatorname{cap} B_E \le c(\mathrm{cl}^* H) \le c(H)$$

(479P(c-vi))

$$\le c(E).$$

(d) Thus $\operatorname{cap} B_E \le c(E)$ whenever $E \subseteq \mathbb{R}^r$ is Lebesgue measurable, bounded and has finite perimeter. Consequently $\operatorname{cap} B_K \le \operatorname{cap} K$ for every compact set $K \subseteq \mathbb{R}^r$. **P** If $\epsilon > 0$, there is an open set $G \supseteq K$ such that $c(G) \le \operatorname{cap} K + \epsilon$. Now there is a set E, a finite union of balls, such that $K \subseteq E \subseteq G$. In this case, E has finite perimeter and is bounded, while of course $B_E \supseteq B_K$. So

$$\operatorname{cap} B_K \le \operatorname{cap} B_E \le c(E) \le c(G) \le \operatorname{cap} K + \epsilon.$$

As ϵ is arbitrary, $\operatorname{cap} B_K \le \operatorname{cap} K$. **Q**

It follows that $\operatorname{cap} B_E \le c(E)$ for every measurable set $E \subseteq \mathbb{R}^r$ of finite measure. **P** If $K \subseteq E$ is compact, then $\operatorname{cap} B_K \le \operatorname{cap} K \le c(E)$. But as $\mu E = \sup\{\mu K : K \subseteq E$ is compact$\}$, $\operatorname{diam} B_E = \sup\{\operatorname{diam} B_K : K \subseteq E$ is compact$\}$; because capacity is a continuous function of radius (479Da),

$$\begin{aligned} \operatorname{cap} B_E) &= \sup\{\operatorname{cap} B_K : K \subseteq E \text{ is compact}\} \\ &\le \sup\{\operatorname{cap} K : K \subseteq E \text{ is compact }\} \le c(E). \ \mathbf{Q} \end{aligned}$$

Finally, if D is any set of finite outer measure, there is a G_δ set $E \supseteq D$ such that $c(E) = c(D)$ and $\mu E = \mu^* D$, so that

$$\operatorname{cap} B_D = \operatorname{cap} B_E \le c(E) = c(D),$$

and we have the general result claimed.

***479W** I conclude with an alternative representation of Choquet-Newton capacity c in terms of a measure on the space of closed subsets of $\mathbb{R}^r$.

Theorem Let $\mathcal{C}^+$ be the family of non-empty closed subsets of $\mathbb{R}^r$, with its Fell topology (4A2T). Then there is a unique Radon measure θ on $\mathcal{C}^+$ such that $\theta^*\{C : C \in \mathcal{C}^+,\ D \cap C \ne \emptyset\}$ is the Choquet-Newton capacity $c(D)$ of D for every $D \subseteq \mathbb{R}^r$.

proof (a) Recall that the Fell topology on $\mathcal{C} = \mathcal{C}^+ \cup \{\emptyset\}$ is compact (4A2T(b-iii)) and metrizable (4A2Tf), so $\mathcal{C}^+$ is locally compact and Polish. For $D \subseteq \mathbb{R}^r$, set $\Psi D = \{C : C \in \mathcal{C}^+,\ C \cap D \ne \emptyset\}$. Of course $\Psi(\bigcup \mathcal{A}) = \bigcup_{D \in \mathcal{A}} \Psi D$ for every family $\mathcal{A}$ of subsets of $\mathbb{R}^r$.

(b) Let Ω' be the set of those $\omega \in \Omega$ such that $\lim_{t\to\infty} \|\omega(t)\| = \infty$; because $r \ge 3$, Ω' is conegligible in Ω (478Md). If $\omega \in \Omega'$, then $\omega[\,[0,\infty[\,]$ is closed. For $x \in \mathbb{R}^r$ and $\omega \in \Omega'$, set $h_x(\omega) = x + \omega[\,[0,\infty[\,] \in \mathcal{C}^+$. Then $h_x : \Omega' \to \mathcal{C}^+$ is Borel measurable. **P** (α) If $G \subseteq \mathbb{R}^r$ is open, then

$$\{\omega : \omega \in \Omega',\ h_x(\omega) \cap G \ne \emptyset\} = \bigcup_{t \ge 0} \{\omega : x + \omega(t) \in G\}$$

is relatively open in Ω'. (β) If $K \subseteq \mathbb{R}^r$ is compact,

$$\{(\omega, t) : x + \omega(t) \in K\}$$

is closed in $\Omega \times [0, \infty[$, so its projection $\{\omega : x + \omega[\,[0,\infty[\,] \cap K \ne \emptyset\}$ is F_σ, and $\{\omega : \omega \in \Omega',\ h_x(\omega) \cap K = \emptyset\}$ is a G_δ set in Ω'. (γ) Because $\mathcal{C}^+$ is hereditarily Lindelöf, this is enough to prove that h_x is Borel measurable (4A3Db). **Q**

(c) Let T be the ring of subsets of $\mathcal{C}^+$ generated by sets of the form ΨE where $E \subseteq \mathbb{R}^r$ is bounded and is either compact or open. Then we have an additive functional $\phi : \mathrm{T} \to [0, \infty[$ such that $\phi(\Psi K) = \operatorname{cap} K$ for every compact set $K \subseteq \mathbb{R}^r$. **P** For $x \in \mathbb{R}^r$ let $h_x : \Omega' \to \mathcal{C}^+$ be as in (b). Then we have a corresponding scaled Radon image measure $\phi_x = \|x\|^{r-2}(\mu_W)_{\Omega'}h_x^{-1}$ on $\mathcal{C}^+$ (418I), defined by setting $\phi_x H = \|x\|^{r-2}\mu_W\{\omega : x + \omega[\,[0,\infty[\,] \in H\}$ whenever this is defined. If $E \subseteq \mathbb{R}^r$ is either compact or open, then

$$\{\omega : x + \omega[\,[0,\infty[\,] \cap E \neq \emptyset\}$$

is F_σ or open, respectively, so $\phi_x(\Psi E)$ is defined; accordingly $\phi_x H$ is defined for every $H \in \mathrm{T}$. If $\gamma > 0$, $E \subseteq B(\mathbf{0},\gamma)$ and $\|x\| > \gamma$, then

$$\phi_x^*(\Psi E) \le \phi_x(\Psi(B(\mathbf{0},\gamma))) = \|x\|^{r-2}\operatorname{hp}(B(\mathbf{0},\gamma) - x) = \gamma^{r-2}$$

(478Qc). So $\limsup_{\|x\|\to\infty} \phi_x H$ is finite for every $H \in \mathrm{T}$. Take an ultrafilter $\mathcal{F}$ on $\mathbb{R}^r$ containing $\mathbb{R}^r \setminus B(\mathbf{0},\gamma)$ for every $\gamma > 0$; then $\phi H = \lim_{x\to\mathcal{F}} \phi_x H$ is defined in $[0,\infty[$ for every $H \in \mathrm{T}$, and ϕ is additive. If $E \subseteq \mathbb{R}^r$ is bounded and either compact or open, then

$$\phi(\Psi E) = \lim_{x\to\mathcal{F}} \|x\|^{r-2}\operatorname{hp}(E - x) = c(E)$$

by 479B(ii). **Q**

(d) ϕ is inner regular with respect to the compact sets, in the sense that $\phi H = \sup\{\phi L : L \in \mathrm{T}$ is compact, $L \subseteq H\}$ for every $H \in \mathrm{T}$. **P** Note first that the set

$$\mathcal{H} = \{H : H \in \mathrm{T},\ \phi H = \sup\{\phi L : L \in \mathrm{T} \text{ is compact, } L \subseteq H\}\}$$

is a sublattice of T. Suppose that E, $H \subseteq \mathbb{R}^r$ are bounded sets which are either compact or open, and $\epsilon > 0$. Then there are a compact $K \subseteq E$ and a bounded open $G \supseteq H$ such that $c(E) \le \epsilon + \operatorname{cap} K$ and $c(G) \le \epsilon + c(H)$ (479E). Now ΨK is a closed subset of $\mathcal{C}$ included in $\mathcal{C}^+$, so is a compact subset of $\mathcal{C}^+$, while ΨG is open; thus $L = \Psi K \setminus \Psi G$ is a compact subset of $\Psi E \setminus \Psi H$, and of course $L \in \mathrm{T}$. Now

$$\begin{aligned}\phi(\Psi E \setminus \Psi H) &\le \phi L + \phi(\Psi E \setminus \Psi K) + \phi(\Psi G \setminus \Psi H)\\ &= \phi L + \phi(\Psi E) - \phi(\Psi K) + \phi(\Psi G) - \phi(\Psi H)\\ &= \phi L + c(E) - \operatorname{cap} K + c(G) - c(H) \le \phi L + 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\Psi E \setminus \Psi H$ belongs to $\mathcal{H}$.

Since any member of T is expressible as a finite union of finite intersections of sets of this kind, $\mathrm{T} \subseteq \mathcal{H}$, as required. **Q**

(e) Let $\mathcal{L}$ be the family of compact subsets of $\mathcal{C}^+$. If $L \in \mathcal{L}$, it is a closed subset of $\mathcal{C}$ not containing $\emptyset$, so there must be a compact set $K \subseteq \mathbb{R}^r$ such that $L \subseteq \Psi K$. Thus every member of $\mathcal{L}$ is covered by a member of T, and we have a functional $\phi_1 : \mathcal{L} \to [0,\infty[$ defined by setting $\phi_1 L = \inf\{\phi E : E \in \mathrm{T},\ L \subseteq E\}$ for $L \in \mathcal{L}$.

I seek to apply 413I. Of course $\emptyset \in \mathcal{L}$ and $\mathcal{L}$ is closed under finite disjoint unions and countable intersections; moreover, if $\langle L_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{L}$ with empty intersection, one of the L_n must be empty, so $\inf_{n\in\mathbb{N}} \phi_1 L_n = 0$. Now turn to condition (α) of the theorem:

$$\phi_1 L_1 = \phi_1 L_0 + \sup\{\phi_1 L : L \in \mathcal{L},\ L \subseteq L_1 \setminus L_0\} \text{ whenever } L_0,\ L_1 \in \mathcal{L} \text{ and } L_0 \subseteq L_1.$$

(i) If L_0, $L \in \mathcal{L}$ are disjoint, then $\phi_1(L_0 \cup L) \ge \phi_1 L_0 + \phi_1 L$. **P** The topology $\mathfrak{S}$ of $\mathcal{C}^+$ is generated by sets of the form ΨG, where $G \subseteq \mathbb{R}^r$ is open, and by sets of the form $\mathcal{C}^+ \setminus \Psi K$, where $K \subseteq \mathbb{R}^r$ is compact. It is therefore generated by

$$\{\Psi G \setminus \Psi K : G \subseteq \mathbb{R}^r \text{ is bounded and open, } K \subseteq \mathbb{R}^r \text{ is compact}\} \subseteq \mathrm{T}.$$

So $\mathfrak{S} \cap \mathrm{T}$ is a base for $\mathfrak{S}$ and disjoint compact sets in $\mathcal{C}^+$ can be separated by members of T (4A2F(h-i)); let $E_0 \in \mathrm{T}$ be such that $L_0 \subseteq E_0 \subseteq \mathcal{C}^+ \setminus L$. Now if $E \in \mathrm{T}$ and $E \supseteq L_0 \cup L$,

$$\phi E = \phi(E \cap E_0) + \phi(E \setminus E_0) \ge \phi_1 L_0 + \phi_1 L;$$

as E is arbitrary, $\phi_1(L_0 \cup L) \ge \phi_1 L_0 + \phi_1 L$. **Q**

(ii) If L_0, $L_1 \in \mathcal{L}$, $L_0 \subseteq L_1$ and $\epsilon > 0$, there is an $L \in \mathcal{L}$ such that $L \subseteq L_1 \cup L_0$ and $\phi_1 L_1 \le \phi_1 L_0 + \phi_1 L + 3\epsilon$. **P** Let E_0, $E_1 \in \mathrm{T}$ be such that $L_0 \subseteq E_0$, $L_1 \subseteq E_1$ and $\phi E_0 \le \phi_1 L_0 + \epsilon$. By (d), there is an $L' \in \mathcal{L} \cap \mathrm{T}$ such that $L' \subseteq E_1 \setminus E_0$ and $\phi L' \ge \phi(E_1 \setminus E_0) - \epsilon$. Set $L = L' \cap L_1$. Then $L \in \mathcal{L}$ and $L \subseteq L_1 \setminus L_0$. Let $E \in \mathrm{T}$ be such that $L \subseteq E$ and $\phi E \le \phi_1 L + \epsilon$. Then $L_1 \subseteq E_0 \cup E \cup ((E_1 \setminus E_0) \setminus L')$, so

$$\phi_1 L_1 \le \phi E_0 + \phi E + \phi(E_1 \setminus E_0) - \phi L' \le \phi_1 L_1 + \phi L + 3\epsilon,$$

as required. **Q**

Putting this together with (i), the final condition of 413I is satisfied.

(f) We therefore have a complete locally determined measure θ on $\mathcal{C}^+$ extending ϕ_1 and inner regular with respect to $\mathcal{L}$. For $E \subseteq \mathcal{C}^+$, θ measures E iff θ measures $E \cap L$ for every $L \in \mathcal{L}$ (412Ja); so θ measures all closed subsets of $\mathcal{C}^+$, and is a topological measure. Of course θ is inner regular with respect to the compact sets. If $C \in \mathcal{C}^+$, there is a bounded open set $G \subseteq \mathbb{R}^r$ meeting C, and now ΨG is an open set containing C and included in the compact set $\Psi\overline{G}$; accordingly

$$\theta(\Psi G) \le \theta(\Psi\overline{G}) = \phi_1(\Psi\overline{G}) = \phi(\Psi\overline{G}) = \operatorname{cap}\overline{G}$$

is finite. Thus θ is locally finite and is a Radon measure.

(g) As in (f), we have

$$\theta(\Psi K) = \phi_1(\Psi K) = \phi(\Psi K) = \operatorname{cap} K$$

for every compact $K \subseteq \mathbb{R}^r$. Next, $\theta(\Psi G) = c(G)$ for every open $G \subseteq \mathbb{R}^r$. **P** If G is bounded,

$$\begin{aligned}\theta(\Psi G) &= \sup\{\theta L : L \subseteq \Psi G \text{ is compact}\}\\ &= \sup\{\phi_1 L : L \subseteq \Psi G \text{ is compact}\} \le \phi(\Psi G) = c(G)\\ &= \sup\{\operatorname{cap} K : K \subseteq G \text{ is compact}\}\\ &= \sup\{\theta(\Psi K) : K \subseteq G \text{ is compact}\} \le \theta(\Psi G).\end{aligned}$$

If G is unbounded, then there is a non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of bounded open sets with union G, so

$$\theta(\Psi G) = \theta(\bigcup_{n\in\mathbb{N}} \Psi G_n) = \sup_{n\in\mathbb{N}} \theta(\Psi G_n) = \sup_{n\in\mathbb{N}} c(G_n) = c(G). \text{ } \mathbf{Q}$$

(h) Now suppose that $D \subseteq \mathbb{R}^r$ is any bounded set. We have

$$\theta^*(\Psi D) \le \inf\{\theta(\Psi G) : G \supseteq D \text{ is open}\} = \inf\{c(G) : G \supseteq D \text{ is open}\} = c(D).$$

? Suppose, if possible, that $\theta^*(\Psi D) < c(D)$. Let $G \supseteq D$ be a bounded open set. Then there is a compact $L \subseteq \Psi G \setminus \Psi D$ such that $\theta L > \theta(\Psi G) - c(D)$. Set $F = \bigcup_{C\in L} C$; then F is closed (4A2T(e-iii)) and disjoint from D, so $G \setminus F$ is open, $D \subseteq G \setminus F$ and $\Psi(G \setminus F)$ is disjoint from L. But this means that

$$c(D) \le c(G \setminus F) = \theta(\Psi(G \setminus F)) \le \theta(\Psi G) - \theta L < c(D),$$

which is absurd. **X** So $\theta^*(\Psi D) = c(D)$.

If $D \subseteq \mathbb{R}^r$ is any set, then it is expressible as the union of a non-decreasing sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of bounded sets, so

$$c(D) = \lim_{n\to\infty} c(D_n) = \lim_{n\to\infty} \theta^*(\Psi D_n) = \theta^*(\bigcup_{n\in\mathbb{N}} \Psi D_n) = \theta^*(\Psi D).$$

Thus θ has all the properties declared.

(i) To see that θ is unique, consider the base $\mathcal{V}$ for the topology of $\mathcal{C}^+$ consisting of sets of the form $\bigcap_{i\in I} \Psi G_i \setminus \Psi K$ where $\langle G_i\rangle_{i\in I}$ is a non-empty finite family of bounded open sets in $\mathbb{R}^r$ and $K \subseteq \mathbb{R}^r$ is compact. The conditions that θ must satisfy determine its value on any set of the form $\Psi(G \cup K) = \Psi G \cup \Psi K$ where $G \subseteq \mathbb{R}^r$ is open and $K \subseteq \mathbb{R}^r$ is compact, and therefore determine its values on $\mathcal{V}$. By 415H(iv), θ is fixed by these.

479X Basic exercises (a) Let ζ be a Radon measure on $\mathbb{R}^r$. Show that

$$\zeta\mathbb{R}^r = \lim_{\gamma\to\infty} \frac{2}{r\beta_r\gamma^2}\int_{B(\mathbf{0},\gamma)} W_\zeta d\mu.$$

>(b)(i) Show directly from 479B-479C that Choquet-Newton capacity c is invariant under isometries of $\mathbb{R}^r$. (ii) Show that $c(\alpha D) = \alpha^{r-2}c(D)$ whenever $\alpha \ge 0$ and $D \subseteq \mathbb{R}^r$.

(c) Suppose that ζ_1 and ζ_2 are totally finite Radon measures on $\mathbb{R}^r$. Show that $W_{\zeta_1*\zeta_2} = \zeta_1 * W_{\zeta_2} = \zeta_2 * W_{\zeta_1}$.

(d) Show that there is a closed set $F \subseteq \mathbb{R}^r$ such that $\operatorname{hp}(F) < 1$ but $c(F) = \infty$. (*Hint*: look at the proof of 479Ma.)

(e) Let $K \subseteq \mathbb{R}^r$ be compact. Show that $\operatorname{int}\{x : \tilde{W}_K(x) < 1\}$ is the unbounded component of $\mathbb{R}^r \setminus \operatorname{supp}\lambda_K$. (*Hint*: setting $L = \operatorname{supp}\lambda_K$, show that $\operatorname{cap}(\operatorname{supp}\lambda_K) = \operatorname{cap} K$ so that $\tilde{W}_K = \tilde{W}_L$.)

(f) Let $A \subseteq \mathbb{R}^r$ be an analytic set such that $c(A) < \infty$. Show that $\tilde{W}_A = \sup\{W_\zeta : \zeta$ is a Radon measure on $\mathbb{R}^r$, A is ζ-conegligible, $W_\zeta \le 1\}$.

>(g) Show that there is a universally negligible set $D \subseteq B(\mathbf{0}, 1)$ such that $c(D) = 1$. (*Hint*: use the ideas of 439F to find D such that $\{\|x\| : x \in D\}$ is universally negligible and $x \mapsto \|x\| : D \to [0, 1]$ is injective, but $\nu^*\{\frac{x}{\|x\|} : x \in D$, $\|x\| \ge 1 - \delta\} = r\beta_r$ for every $\delta \in]0, 1[$; compute $c(D)$ with the aid of 479D, 479P(c-iii-α) and 479P(c-vii).)

(h) Suppose that $D \subseteq D' \subseteq \mathbb{R}^r$ and $c(D') < \infty$. Show that $\int f d\lambda_D \le \int f d\lambda_{D'}$ for every lower semi-continuous superharmonic $f : \mathbb{R}^r \to [0, \infty]$.

(i) Let $D \subseteq \mathbb{R}^r$ be a bounded set. Show that $c(D) = \lim_{\|x\| \to \infty} \|x\|^{r-2} \operatorname{hp}^*(D - x)$. (*Hint*: 477Id.)

(j) Let $T : \mathbb{R}^r \to \mathbb{R}^r$ be an isometry, and $D \subseteq \mathbb{R}^r$ a set such that $c(D) < \infty$. Show that $\tilde{W}_{T[D]}Tx) = \tilde{W}_D(x)$ for every $x \in \mathbb{R}^r$ and that $\lambda_{T[D]}$ is the image measure $\lambda_D T^{-1}$.

(k) Let $D \subseteq \mathbb{R}^r$ be a set such that $c(D) < \infty$, and $\alpha > 0$. Show that $\tilde{W}_{\alpha D}(x) = \tilde{W}_D(\frac{1}{\alpha}x)$ for every $x \in \mathbb{R}^r$, and that $\lambda_{\alpha D} = \alpha^{r-2}\lambda_D T^{-1}$, where $T(x) = \alpha x$ for $x \in \mathbb{R}^r$.

(l) Show that $c(D) = \inf\{\zeta\mathbb{R}^r : \zeta$ is a Radon measure on $\mathbb{R}^r$, $W_\zeta \ge \chi D\} = \inf\{\operatorname{energy}(\zeta) : \zeta$ is a Radon measure on $\mathbb{R}^r$, $W_\zeta \ge \chi D\}$ for any $D \subseteq \mathbb{R}^r$.

(m) Let $K \subseteq \mathbb{R}^r$ be a compact set, with complement G, and Φ the set of continuous harmonic functions $f : G \to [0, 1]$ such that $\lim_{\|x\| \to \infty} f(x) = 0$. Show that $\tilde{W}_K \restriction G$ is the greatest element of Φ. (*Hint*: 479Pb, 478Pc.)

(n)(i) Show that if G is a convex open set then $\operatorname{hp}(G - x) = 1$ for every $x \in \overline{G}$. (ii) Show that if $D \subseteq \mathbb{R}^r$ is a convex bounded set with non-empty interior, then $\tilde{W}_D$ is continuous.

(o) Show that if D, $D' \subseteq \mathbb{R}^r$ and $c(D \cup D') < \infty$, then $\tilde{W}_{D \cap D'} + \tilde{W}_{D \cup D'} \le \tilde{W}_D + \tilde{W}_{D'}$.

(p) Let $D \subseteq \mathbb{R}^r$ be a set such that $c(D) < \infty$, and set $\tilde{D} = \{x : \tilde{W}_D(x) = 1\}$. Show that (i) $D \setminus \tilde{D}$ is polar (ii) $\lambda_{\tilde{D}} = \lambda_D$. (*Hint*: reduce to the case in which $D = A$ is analytic; use 479Fg to show that $\tilde{W}_{\tilde{A}} \le \tilde{W}_A$; use 479J(b-v).)

(q) Let $\mathcal{A}$ be the set of subsets of $\mathbb{R}^r$ with finite Choquet-Newton capacity, and ρ the pseudometric $(D, D') \mapsto 2c(D \cup D') - c(D) - c(D')$ (432Xj). (i) Show that $\|U_{\lambda_D} - U_{\lambda_{D'}}\|_2^2 \le 2c_r\rho(D, D')$ for D, $D' \in \mathcal{A}$. (ii) Show that $\rho(D, D') = 0$ iff $\lambda_D = \lambda_{D'}$.

(r) Suppose that $D \subseteq \mathbb{R}^r$. Show that the following are equiveridical: (i) D is polar; (ii) there is some $x \in \mathbb{R}^r$ such that $\operatorname{hp}(D - x) = 0$; (iii) $\operatorname{hp}((D \setminus \{x\}) - x) = 0$ for every $x \in \mathbb{R}^r$.

(s) Suppose that $\langle D_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of subsets of $\mathbb{R}^r$ such that $\inf_{n \in \mathbb{N}} c(D_n)$ is finite and $\bigcap_{n \in \mathbb{N}} D_n = \bigcap_{n \in \mathbb{N}} \overline{D_n} = F$ say. Show that $\langle \lambda_{D_n} \rangle_{n \in \mathbb{N}} \to \lambda_D$ for the narrow topology, and that $\langle c(D_n) \rangle_{n \in \mathbb{N}} \to c(F)$. (Compare 479Ye.)

(t) For $\omega \in \Omega$ set $\tau(\omega) = \sup\{t : \|\omega(t)\| \le 1\}$. (i) Show that $\tau : \Omega \to [0, \infty]$ is measurable. (ii) Show that if $r \le 2$ then $\tau = \infty$ a.e. (iii) Show that if $r \ge 3$ then τ is not a stopping time. (iv) Show that if $3 \le r \le 4$ then τ is finite a.e., but has infinite expectation. (v) Show that if $r \ge 5$ then τ has finite expectation. (*Hint*: show that if $r \ge 2$ then

$$\Pr(\tau \ge t) = \frac{1}{(\sqrt{2\pi})^r} \int e^{-\|x\|^2/2} \frac{1}{\max(1, (\sqrt{t})^{r-2}\|x\|^{r-2})} dx.)$$

479Y Further exercises (a)(i) Show that there is an open set $G \subseteq B(\mathbf{0}, 1)$, dense in $B(\mathbf{0}, 1)$, such that $c(G) < 1$. (ii) Show that $\operatorname{cap}(\operatorname{supp} \lambda_G) = 1$.

(b) In 479G, suppose that $0 < \alpha < r$, $0 < \beta < r$ and $\alpha + \beta > r$. Show that

$$k_{\alpha+\beta-r} = \frac{\Gamma(r - \frac{\alpha+\beta}{2})\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}{(\sqrt{\pi})^r \Gamma(\frac{\alpha+\beta-r}{2})\Gamma(\frac{r-\alpha}{2})\Gamma(\frac{r-\beta}{2})} k_\alpha * k_\beta.$$

(c) Let $A \subseteq \mathbb{R}^r$ be an analytic set with $c(A) < \infty$. (i) Show that for every $\gamma > 0$ there is a Radon measure ζ_γ on $\mathbb{R}^r$ such that $\langle \frac{1}{r\beta_r\gamma}\mu_x^{(A)}\rangle_{x\in\partial B(\mathbf{0},\gamma)}$ is a disintegration of ζ_γ over the subspace measure $\nu_{\partial B(\mathbf{0},\gamma)}$. (ii) Show that $\lim_{\gamma\to\infty}\zeta_\gamma = \lambda_A$ for the total variation metric on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

(d) Set $c'(D) = \sup\{\zeta^*D : \zeta$ is a Radon measure on $\mathbb{R}^r$ such that $W_\zeta \le 1$ everywhere$\}$ for $D \subseteq \mathbb{R}^r$. Show that c' is a Choquet capacity on $\mathbb{R}^r$, extending Newtonian capacity for compact sets, which is different from Choquet-Newton capacity.

(e) Let $\langle D_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of subsets of $\mathbb{R}^r$ with finite Choquet-Newton capacity. For each $n \in \mathbb{N}$, set $\tilde{D}_n = \{x : \tilde{W}_{D_n}(x) = 1\}$, and set $A = \bigcap_{n\in\mathbb{N}}\tilde{D}_n$. Show that $c(A) = \inf_{n\in\mathbb{N}} c(D_n)$ and that λ_A is the limit of $\langle\lambda_{D_n}\rangle_{n\in\mathbb{N}}$ for the narrow topology on $M_{\mathrm{R}}^+(\mathbb{R}^r)$.

(f) Let $\mathcal{A}$, ρ be as in 479Xq, and let $(\bar{\mathcal{A}}, \bar{\rho})$ be the corresponding metric space, identifying members of $\mathcal{A}$ which are zero distance apart. Show that $\bar{\mathcal{A}}$ is complete.

(g) Let $E \subseteq \mathbb{R}^r$ be a set of finite Lebesgue measure, and B_E the ball with centre 0 and the same measure as E. Show that $\operatorname{energy}(\mu \llcorner E) \le \operatorname{energy}(\mu \llcorner B_E)$.

(h) Prove 479V from 479U and 476Yb.

(i)(i) Show that c is alternating of all orders, that is,

$$\sum_{J\subseteq I,\#(J)\text{ is even}} c(D \cup \bigcup_{i\in J} D_i) \le \sum_{J\subseteq I,\#(J)\text{ is odd}} c(D \cup \bigcup_{i\in J} D_i)$$

whenever I is a non-empty finite set, $\langle D_i\rangle_{i\in I}$ is a family of subsets of $\mathbb{R}^r$ and D is another subset of $\mathbb{R}^r$. (Cf. 132Yf.) (ii) Show that if $c(D \cup \bigcup_{i\in I} D_i) < \infty$, then

$$\sum_{J\subseteq I,\#(J)\text{ is even}} \tilde{W}_{D\cup\bigcup_{i\in J} D_i} \le \sum_{J\subseteq I,\#(J)\text{ is odd}} \tilde{W}_{D\cup\bigcup_{i\in J} D_i}.$$

(j) Let us say that if X is a Polish space, a set $A \subseteq X$ is **projectively universally measurable** if $W[A]$ is universally measurable whenever Y is a Polish space and $W \subseteq X \times Y$ is analytic. Show that we can replace the word 'analytic' by the phrase 'projectively universally measurable' in all the theorems of this section.

(k) Suppose that $A \subseteq \mathbb{R}^r$ is analytic and non-empty, and $x \in \mathbb{R}^r$ is such that $\rho(x, A) = \delta > 0$. Show that $\operatorname{energy}(\mu_x^{(A)}) \le \frac{1}{\delta^{r-2}}$.

(l) Show that if $D \subseteq \mathbb{R}^r$ and $c(D) < \infty$, then $c(\{x : \tilde{W}_D(x) \ge \gamma\}) \le \frac{1}{\gamma}c(D)$ for every $\gamma > 0$.

(m) For a set $D \subseteq \mathbb{R}^r$ with $c(D) < \infty$, set $\mathrm{cl}_{\mathrm{cap}}D = \{x : \tilde{W}_D(x) = 1\}$. Show that $c(D) = c(\mathrm{cl}_{\mathrm{cap}}D) = c(D \cup \mathrm{cl}_{\mathrm{cap}}D)$.

479 Notes and comments Newtonian potential is another of the great concepts of mathematics, and is one of the points at which physical problems and intuitions have stimulated and illuminated the development of analysis. As with all the best ideas of mathematics, there is more than one route to it, and any proper understanding of it must include a matching of the different approaches. In the exposition here I start with a description of equilibrium measures in terms of harmonic measures (479B), themselves defined in 478P in terms of Brownian motion. We are led quickly to definitions of capacity and equilibrium potential (479C), with some elementary properties (479D). Moreover, some very striking further results (479E, 479W) are already accessible.

However we are still rather far from the original physical concept of 'capacity' of a conductor. If you have ever studied electrostatics, the ideas here may recall some basic physical principles. The kernel $x \mapsto \frac{1}{\|x\|^{r-2}}$ represents the potential energy field of a point mass or charge; the potential W_ζ represents the field due to a mass or charge with distribution ζ. The capacity of a set K is the largest charge that can be put on K without raising the potential of any point above 1 (479Na), and the infimum of the charges which raise the potential of every point of K to 1 (479P(c-v)). The result that λ_K is supported by ∂K (479B(i)) corresponds to the principle that the charge on a conductor always collects on the surface of the conductor; 479D(b-iii) corresponds to the principle that there is no electric field inside a conductor.

At the same time, the equilibrium measure is supposed to be the (unique) distribution of the charge, which on physical grounds ought to be the distribution with least energy, as in 479K. To reach these ideas, it seems that we need to know various non-trivial facts from classical analysis, which I set out in 479G-479I. The deepest of these is in 479Ib: for the Riesz kernels k_α, the convolution $\zeta * k_\alpha$ determines the totally finite Radon measure ζ. I do not know of any way of establishing this except through the Fourier analysis of 479H and the detailed calculations of 479G and 479Ia.

The ideas here are connected in so many ways that there is no clear flow to the logic, and we are more than usually in danger of using circular arguments. In my style of exposition, this complexity manifests itself in an exceptional density of detailed back-references; I hope that these will enable you to check the proofs effectively. On a larger scale, the laborious zigzag progression from the original notion of capacity of compact sets, as in 479K and 479U, through bounded analytic sets (479B, 479E) and general analytic sets (479M, 479N) to arbitrary sets (479P), displays a choice of path to which there are surely many alternatives.

Of course we cannot expect all the properties of Newtonian capacity to have recognizable forms in such a general context as that of 479P (see 479Xg, which shows that we cannot hope to replicate the ideas of 479Na-479Nc), but the elementary results if 479D mostly extend (479Pc). More importantly, we have a quite new characterization of equilibrium potentials (479Pb). With these techniques available, we can learn a good deal more about Brownian motion. 479R is a curious and striking fact to go with 477K, 477L and 478M. It is not a surprise that capacity and Hausdorff dimension should be linked, but it is notable that the phase change is at dimension $r-2$ (479Q); this goes naturally with 479P(c-vii). I know of no such dramatic difference between four and five dimensions, but for some purposes 479Xt marks a significant change.

My treatment is an unconventional one, so perhaps I should indicate points where you should expect other authors to diverge from it. While the notions of Newtonian capacity, equilibrium measure and equilibrium potential are solidly established for compact sets in $\mathbb{R}^r$ (at least up to scalar factors, and for $r \ge 3$), the extension to general bounded analytic sets is not I think standard. (I try to signal this by writing $c(A)$ in place of $\operatorname{cap} A$, after 479E, for sets which are not guaranteed to be compact, even when the definition in 479C(a-i) is applicable. The fact that this step gives very little extra trouble is a demonstration of the power of the Brownian-motion approach.) The further extension of Newtonian capacity, defined on compact sets, to a Choquet capacity, defined on every subset of $\mathbb{R}^r$ (479Ed), is surely not standard, which is why I give the extension a different name. (While Choquet certainly considered the capacity which I here call 'Choquet-Newton capacity', I fear that the phrase has no real historical justification; but I hope it will convey some of the right ideas.) You may have noticed that I give essentially nothing concerning differential equations, which have traditionally been one of the central concerns of potential theory; there are hints in 479Xm and 479T.

A weakness in the formulae of 479B is that they are not self-evidently translation-invariant. Of course it is easy to show that in fact we have an isometry-invariant construction (479Xb), and this can also be seen from the descriptions of capacity and equilibrium potentials in 479N and 479Pb. Because the capacity c is countably subadditive, it is easy to build a dense open subset G of $\mathbb{R}^r$ such that $c(G)$ is finite (see 479Ya), and for such a set we cannot ask that λ_G should be describable as a limit of $\|x\|^{r-2}\mu_x^{(G)}$ as $\|x\| \to \infty$. But if we start from 479B(i) rather than 479B(ii), we do have an averaged form, with

$$\lambda_A = \lim_{\gamma\to\infty} \frac{1}{r\beta_r\gamma}\int_{\partial B(\mathbf{0},\gamma)} \mu_x^{(A)}\nu(dx)$$

whenever A is an analytic set and $c(A) < \infty$ (479Yc; see also 479J(b-vi) and 479Xa).

The factor $(\sqrt{2\pi})^r$ in 479H repeats that of 283Wg[14]. The appearance of $\sqrt{2\pi}$ in 283M, but not 445G, is proof that the conventions of Chapter 28 are not reconcilable with those of §445.

In 479O I describe one of the important notions of 'small' set in Euclidean space, to go with 'negligible' and 'meager'. I have no space to deal with it properly here, but the applications in the proofs of 479P, 479R and 479S will give an idea of its uses; another is in 479Xm. As another example of the logical complexity of the patterns here, consider the problem of either proving 479Pb without 479O, or extending 479O to cover 479Xr without passing through a version of 479P.

Quite a lot of the work here is caused by the need to accommodate discontinuous equilibrium potentials (479S). This has been an important theme in general potential theory. 479Pb shows that the problem is essentially geometric: if a compact set K has a sufficiently narrow spike at e, then a Brownian path starting at e can easily fail to enter K

[14]Formerly 283Wj.

again.

As I have written the theory out, 479T-479U are rather separate from the rest, being closer in spirit to the work of §§473-475. They explore some more of the basic principles of potential theory. Note, in particular, the formula of 479Ta, which amounts to saying that (under the right conditions) a function g is a multiple of $k_{r-2} * \nabla^2 g$; of course this can be thought of as a method of finding a particular solution of the equation $\nabla^2 g = f$; equally, it gives an approach to the problem of expressing a given g as a potential W_ζ. From 479Tb and 479Tc we see that in the sense of distributional derivatives we can think of $r(r-2)\beta_r\zeta$ as representing the Laplacian $-\nabla^2 W_\zeta$; recall that as W_ζ is superharmonic (479Fb), we expect $\nabla^2 W_\zeta$ to be negative (478E).

I give a bit of space to 479V because it links the material here to that of §476, and this book is about such linkages, and because it supports my thesis that capacity is a geometrical concept. 479W is characteristic of Choquet capacities which are alternating of all orders (479Yi). I spell it out here because it calls on the Fell topology, which is important elsewhere in this volume.

It is natural to ask which of the ideas here applying to analytic sets can be extended to wider classes. If you look back to where analytic sets first entered the discussion, in the theory of hitting times (455M), you will see that we needed a class of universally measurable sets which would be invariant under various operations, notably projections (479Yj). In Volume 5 we shall meet axiom systems in which there are various interesting possibilities.

This section is firmly directed at Euclidean space of three or more dimensions. The harmonic and Fourier analysis of 479G-479I applies unchanged to dimensions 1 and 2; so does 479Tb. On the line, Brownian hitting probabilities are trivial; in the plane, they are very different from hitting probabilities in higher dimensions, but still of considerable interest. Theorems 479B, 479E and 479W still work, but 'capacity', if defined by the formulation of 479Ca, is bounded by 1. The geometric nature of the results changes dramatically, and 479I cannot be applied in the same way, since we no longer have $0 < r-2$.

Chapter 48

Gauge integrals

For the penultimate chapter of this volume I turn to a completely different approach to integration which has been developed in the last fifty years, following KURZWEIL 57 and HENSTOCK 63. This depends for its inspiration on a formulation of the Riemann integral (see 481Xe), and leads in particular to some remarkable extensions of the Lebesgue integral (§§483-484). While (in my view) it remains peripheral to the most important parts of measure theory, it has deservedly attracted a good deal of interest in recent years, and is entitled to a place here.

From the very beginning, in the definitions of §122, I have presented the Lebesgue integral in terms of almost-everywhere approximations by simple functions. Because the integral $\int \lim_{n\to\infty} f_n$ of a limit is *not* always the limit $\lim_{n\to\infty}\int f_n$ of the integrals, we are forced, from the start, to constrain ourselves by the ordering, and to work with monotone or dominated sequences. This almost automatically leads us to an 'absolute' integral, in which $|f|$ is integrable whenever f is, whether we start from measures (as in Chapter 11) or from linear functionals (as in §436). For four volumes now I have been happily developing the concepts and intuitions appropriate to such integrals. But if we return to one of the foundation stones of Lebesgue's theory, the Fundamental Theorem of Calculus, we find that it is easy to construct a differentiable function f such that the absolute value $|f'|$ of its derivative is not integrable (483Xd). It was observed very early (PERRON 1914) that the Lebesgue integral can be extended to integrate the derivative of any function which is differentiable everywhere. The achievement of HENSTOCK 63 was to find a formulation of this extension which was conceptually transparent enough to lend itself to a general theory, fragments of which I will present here.

The first step is to set out the essential structures on which the theory depends (§481), with a first attempt at a classification scheme. (One of the most interesting features of the Kurzweil-Henstock approach is that we have an extraordinary degree of freedom in describing our integrals, and apart from the Henstock integral itself it is not clear that we have yet found the right canonical forms to use.) In §482 I give a handful of general theorems showing what kinds of result can be expected and what difficulties arise. In §483, I work through the principal properties of the Henstock integral on the real line, showing, in particular, that it coincides with the Perron and special Denjoy integrals. Finally, in §484, I look at a very striking integral on $\mathbb{R}^r$, due to W.F.Pfeffer.

481 Tagged partitions

I devote this section to establishing some terminology (481A-481B, 481E-481G) and describing a variety of examples (481I-481Q), some of which will be elaborated later. The clearest, simplest and most important example is surely Henstock's integral on a closed bounded interval (481J), so I recommend turning immediately to that paragraph and keeping it in mind while studying the notation here. It may also help you to make sense of the definitions here if you glance at the statements of some of the results in §482; in this section I give only the formula defining gauge integrals (481C), with some elementary examples of its use (481Xb-481Xh).

481A Tagged partitions and Riemann sums The common idea underlying all the constructions of this chapter is the following. We have a set X and a functional ν defined on some family $\mathcal{C}$ of subsets of X. We seek to define an integral $\int f d\nu$, for functions f with domain X, as a limit of *finite* **Riemann sums** $\sum_{i=0}^{n} f(x_i)\nu C_i$, where $x_i \in X$ and $C_i \in \mathcal{C}$ for $i \le n$. There is no strict reason, at this stage, to forbid repetitions in the string $(x_0, C_0), \dots, (x_n, C_n)$, but also little to be gained from allowing them, and it will simplify some of the formulae below if I say from the outset that a **tagged partition** on X will be a finite subset $\boldsymbol{t}$ of $X \times \mathcal{P}X$.

So one necessary element of the definition will be a declaration of which tagged partitions $\{(x_0, C_0), \dots, (x_n, C_n)\}$ will be employed, in terms, for instance, of which sets C_i are permitted, whether they are allowed to overlap at their boundaries, whether they are required to cover the space, and whether each **tag** x_i is required to belong to the corresponding C_i. The next element of the definition will be a description of a filter $\mathcal{F}$ on the set T of tagged partitions, so that the integral will be the limit (when it exists) of the sums along the filter, as in 481C below.

In the formulations studied in this chapter, the C_i will generally be disjoint, but this is not absolutely essential, and it is occasionally convenient to allow them to overlap in 'small' sets, as in 481Ya. In some cases, we can restrict attention to families for which the C_i are non-empty and have union X, so that $\{C_0, \dots, C_n\}$ is a partition of X in the strict sense.

481B Notation Let me immediately introduce notations which will be in general use throughout the chapter.

(a) First, a shorthand to describe a particular class of sets of tagged partitions. If X is a set, a **straightforward set of tagged partitions** on X is a set of the form

$$T = \{\boldsymbol{t} : \boldsymbol{t} \in [Q]^{<\omega},\ C \cap C' = \emptyset \text{ whenever } (x, C),\ (x', C') \text{ are distinct members of } \boldsymbol{t}\}$$

where $Q \subseteq X \times \mathcal{P}X$; I will say that T is **generated** by Q. In this case, of course, Q can be recovered from T, since $Q = \bigcup T$. Note that no control is imposed on the tags at this point. It remains theoretically possible that a pair $(x, \emptyset)$ should belong to Q, though in many applications this will be excluded in one way or another.

(b) If X is a set and $\boldsymbol{t} \subseteq X \times \mathcal{P}X$ is a tagged partition, I write

$$W_{\boldsymbol{t}} = \bigcup\{C : (x, C) \in \boldsymbol{t}\}.$$

(c) If X is a set, $\mathcal{C}$ is a family of subsets of X, f and ν are real-valued functions, and $\boldsymbol{t} \in [X \times \mathcal{C}]^{<\omega}$ is a tagged partition, then

$$S_{\boldsymbol{t}}(f, \nu) = \sum_{(x,C)\in\boldsymbol{t}} f(x)\nu C$$

whenever $\boldsymbol{t} \subseteq \operatorname{dom} f \times \operatorname{dom} \nu$.

481C Proposition Let X be a set, $\mathcal{C}$ a family of subsets of X, $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a non-empty set of tagged partitions and $\mathcal{F}$ a filter on T. For real-valued functions f and ν, set

$$I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}} S_{\boldsymbol{t}}(f, \nu)$$

if this is defined in $\mathbb{R}$.

(a) I_ν is a linear functional defined on a linear subspace of $\mathbb{R}^X$.

(b) Now suppose that $\nu C \ge 0$ for every $C \in \mathcal{C}$. Then

(i) I_ν is a positive linear functional (definition: 351F);

(ii) if f, $g : X \to \mathbb{R}$ are such that $|f| \le g$ and $I_\nu(g)$ is defined and equal to 0, then $I_\nu(f)$ is defined and equal to 0.

proof (a) We have only to observe that if f, g are real-valued functions and $\alpha \in \mathbb{R}$, then

$$S_{\boldsymbol{t}}(f + g, \nu) = S_{\boldsymbol{t}}(f, \nu) + S_{\boldsymbol{t}}(g, \nu), \quad S_{\boldsymbol{t}}(\alpha f, \nu) = \alpha S_{\boldsymbol{t}}(f, \nu)$$

whenever the right-hand sides are defined, and apply 2A3Sf.

(b) If $g \ge 0$ in $\mathbb{R}^X$, that is, $g(x) \ge 0$ for every $x \in X$, then $S_{\boldsymbol{t}}(g, \nu) \ge 0$ for every $\boldsymbol{t} \in T$, so the limit $I_\nu(g)$, if defined, will also be non-negative. Next, if $|f| \le g$, then $|S_{\boldsymbol{t}}(f, \nu)| \le S_{\boldsymbol{t}}(g, \nu)$ for every $\boldsymbol{t}$, so if $I_\nu(g) = 0$ then $I_\nu(f)$ also is zero.

481D Remarks (a) Functionals $I_\nu = \lim_{\boldsymbol{t}\to\mathcal{F}} S_{\boldsymbol{t}}(.,\nu)$, as described above, are called **gauge integrals**.

(b) In fact even greater generality is possible at this point. There is no reason why f and ν should take real values. All we actually need is an interpretation of sums of products $f(x) \times \nu C$ in a space in which we can define limits. So for any linear spaces U, V and W with a bilinear functional $\phi : U \times V \to W$ (253A) and a Hausdorff linear space topology on W, we can set out to construct an integral of a function $f : X \to U$ with respect to a functional $\nu : \mathcal{C} \to V$ as a limit of sums $S_{\boldsymbol{t}}(f, \nu) = \sum_{(x,C)\in\boldsymbol{t}} \phi(f(x), \nu C)$ in W. I will not go farther along this path here. But it is worth noting that the constructions of this chapter lead the way to interesting vector integrals of many types.

(c) An extension which is, however, sometimes useful is to allow ν to be undefined (or take values outside $\mathbb{R}$, such as $\pm\infty$) on part of $\mathcal{C}$. In this case, set $\mathcal{C}_0 = \nu^{-1}[\mathbb{R}]$. Provided that $T \cap [X \times \mathcal{C}_0]^{<\omega}$ belongs to $\mathcal{F}$, we can still define I_ν, and 481C will still be true.

481E Gauges The most useful method (so far) of defining filters on sets of tagged partitions is the following.

(a) If X is a set, a **gauge** on X is a subset δ of $X \times \mathcal{P}X$. For a gauge δ, a tagged partition $\boldsymbol{t}$ is **δ-fine** if $\boldsymbol{t} \subseteq \delta$. Now, for a set Δ of gauges and a non-empty set T of tagged partitions, we can seek to define a filter $\mathcal{F}$ on T as the filter generated by sets of the form $T_\delta = \{\boldsymbol{t} : \boldsymbol{t} \in T \text{ is } \delta\text{-fine}\}$ as δ runs over Δ. Of course we shall need to establish that T and Δ are compatible in the sense that $\{T_\delta : \delta \in \Delta\}$ has the finite intersection property; this will ensure that there is indeed a filter containing every T_δ (4A1Ia).

In nearly all cases, Δ will be non-empty and downwards-directed (that is, for any δ_1, $\delta_2 \in \Delta$ there will be a $\delta \in \Delta$ such that $\delta \subseteq \delta_1 \cap \delta_2$); in this case, we shall need only to establish that T_δ is non-empty for every $\delta \in \Delta$. Note that the filter on T generated by $\{T_\delta : \delta \in \Delta\}$ depends only on T and the filter on $X \times \mathcal{P}X$ generated by Δ.

(b) The most important gauges (so far) are 'neighbourhood gauges'. If $(X, \mathfrak{T})$ is a topological space, a **neighbourhood gauge** on X is a set expressible in the form $\delta = \{(x, C) : x \in X, C \subseteq G_x\}$ where $\langle G_x \rangle_{x \in X}$ is a family of open sets such that $x \in G_x$ for every $x \in X$. It is useful to note (i) that the family $\langle G_x \rangle_{x \in X}$ can be recovered from δ, since $G_x = \bigcup\{A : (x, A) \in \delta\}$ (ii) that $\delta_1 \cap \delta_2$ is a neighbourhood gauge whenever δ_1 and δ_2 are. When (X, ρ) is a metric space, we can define a neighbourhood gauge δ_h from any function $h : X \to]0, \infty[$, setting

$$\delta_h = \{(x, C) : x \in X, C \subseteq X, \rho(y, x) < h(x) \text{ for every } y \in C\}.$$

The set of gauges expressible in this form is coinitial with the set of all neighbourhood gauges and therefore defines the same filter on any compatible set T of tagged partitions. Specializing yet further, we can restrict attention to constant functions h, obtaining the **uniform metric gauges**

$$\delta_\eta = \{(x, C) : x \in X, C \subseteq X, \rho(x, y) < \eta \text{ for every } y \in C\}$$

for $\eta > 0$, used in the Riemann integral (481I). (The use of the letter 'δ' to represent a gauge has descended from its traditional appearance in the definition of the Riemann integral.)

(c) If X is a set and $\Delta \subseteq \mathcal{P}(X \times \mathcal{P}X)$ is a family of gauges on X, I will say that Δ is **countably full** if whenever $\langle \delta_n \rangle_{n \in \mathbb{N}}$ is a sequence in Δ, and $\phi : X \to \mathbb{N}$ is a function, then there is a $\delta \in \Delta$ such that $(x, C) \in \delta_{\phi(x)}$ whenever $(x, C) \in \delta$. I will say that Δ is **full** if whenever $\langle \delta_x \rangle_{x \in X}$ is a family in Δ, then there is a $\delta \in \Delta$ such that $(x, C) \in \delta_x$ whenever $(x, C) \in \delta$.

Of course a full set of gauges is countably full. Observe that if $(X, \mathfrak{T})$ is any topological space, the set of all neighbourhood gauges on X is full.

481F Residual sets The versatility and power of the methods being introduced here derives from the insistence on taking *finite* sums $\sum_{(x,C) \in \boldsymbol{t}} f(x) \nu C$, so that all questions about convergence are concentrated in the final limit $\lim_{\boldsymbol{t} \to \mathcal{F}} S_{\boldsymbol{t}}(f, \nu)$. Since in a given Riemann sum we can look at only finitely many sets C of finite measure, we cannot insist, even when ν is Lebesgue measure on $\mathbb{R}$, that $W_{\boldsymbol{t}}$ should always be X. There are many other cases in which it is impossible or inappropriate to insist that $W_{\boldsymbol{t}} = X$ for every tagged partition in T. We shall therefore need to add something to the definition of the filter $\mathcal{F}$ on T beyond what is possible in the language of 481E. In the examples below, the extra condition will always be of the following form. There will be a collection $\mathfrak{R}$ of **residual families** $\mathcal{R} \subseteq \mathcal{P}X$. It will help to have a phrase corresponding to the phrase 'δ-fine': if $\mathcal{R} \subseteq \mathcal{P}X$, and $\boldsymbol{t}$ is a tagged partition on X, I will say that $\boldsymbol{t}$ is **$\mathcal{R}$-filling** if $X \setminus W_{\boldsymbol{t}} \in \mathcal{R}$. Now, given a family $\mathfrak{R}$ of residual sets, and a family Δ of gauges on X, we can seek to define a filter $\mathcal{F}(T, \Delta, \mathfrak{R})$ on T as that generated by sets of the form T_δ, for $\delta \in \Delta$, and $T'_{\mathcal{R}}$, for $\mathcal{R} \in \mathfrak{R}$, where

$$T'_{\mathcal{R}} = \{\boldsymbol{t} : \boldsymbol{t} \in T \text{ is } \mathcal{R}\text{-filling}\}.$$

When there is such a filter, that is, the family $\{T_\delta : \delta \in \Delta\} \cup \{T'_{\mathcal{R}} : \mathcal{R} \in \mathfrak{R}\}$ has the finite intersection property, I will say that T is **compatible** with Δ and $\mathfrak{R}$.

It is important here to note that we shall *not* suppose that, for a typical residual family $\mathcal{R} \in \mathfrak{R}$, subsets of members of $\mathcal{R}$ again belong to $\mathcal{R}$; there will frequently be a restriction on the 'shape' of members of $\mathcal{R}$ as well as on their size. On the other hand, it will usually be helpful to arrange that $\mathfrak{R}$ is a filter base, so that (if Δ is also downwards-directed, and neither Δ nor $\mathfrak{R}$ is empty) we need only show that $T_\delta \cap T'_{\mathcal{R}}$ is always non-empty, and $\{T_{\delta\mathcal{R}} : \delta \in \Delta, \mathcal{R} \in \mathfrak{R}\}$ will be a base for $\mathcal{F}(T, \Delta, \mathfrak{R})$.

If the filter $\mathcal{F}$ is defined as in 481Ea, with no mention of a family $\mathfrak{R}$, we can still bring the construction into the framework considered here by setting $\mathfrak{R} = \emptyset$. If it is convenient to define T in terms which do not impose any requirement on the sets $W_{\boldsymbol{t}}$, but nevertheless we wish to restrict attention to sums $S_{\boldsymbol{t}}(f, \nu)$ for which the tagged partition covers the whole space X, we can do so by setting $\mathfrak{R} = \{\{\emptyset\}\}$.

481G Subdivisions When we come to analyse the properties of integrals constructed by the method of 481C, there is an important approach which depends on the following combination of features. I will say that $(X, T, \Delta, \mathfrak{R})$ is a **tagged-partition structure allowing subdivisions**, **witnessed by** $\mathcal{C}$, if

(i) X is a set.

(ii) Δ is a non-empty downwards-directed family of gauges on X.

(iii)(α) $\mathfrak{R}$ is a non-empty downwards-directed collection of families of subsets of X, all containing $\emptyset$;

(β) for every $\mathcal{R} \in \mathfrak{R}$ there is an $\mathcal{R}' \in \mathfrak{R}$ such that $A \cup B \in \mathcal{R}$ whenever A, $B \in \mathcal{R}'$ are disjoint.

(iv) $\mathcal{C}$ is a family of subsets of X such that whenever $C, C' \in \mathcal{C}$ then $C \cap C' \in \mathcal{C}$ and $C \setminus C'$ is expressible as the union of a disjoint finite subset of $\mathcal{C}$.

(v) Whenever $\mathcal{C}_0 \subseteq \mathcal{C}$ is finite and $\mathcal{R} \in \mathfrak{R}$, there is a finite set $\mathcal{C}_1 \subseteq \mathcal{C}$, including $\mathcal{C}_0$, such that $X \setminus \bigcup \mathcal{C}_1 \in \mathcal{R}$.

(vi) $T \subseteq [X \times \mathcal{C}]^{<\omega}$ is (in the language of 481Ba) a non-empty straightforward set of tagged partitions on X.

(vii) Whenever $C \in \mathcal{C}$, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ there is a δ-fine tagged partition $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}} \subseteq C$ and $C \setminus W_{\boldsymbol{t}} \in \mathcal{R}$.

481H Remarks (a) Conditions (ii) and (iii-α) of 481G are included primarily for convenience, since starting from any Δ and $\mathfrak{R}$ we can find non-empty directed sets leading to the same filter $\mathcal{F}(T, \Delta, \mathfrak{R})$. (iii-$\beta$), on the other hand, is saying something new.

(b) It is important to note, in (vii) of 481G, that the tags of $\boldsymbol{t}$ there are *not* required to belong to the set C.

(c) All the applications below will fall into one of two classes. In one type, the residual families $\mathcal{R} \in \mathfrak{R}$ will be families of 'small' sets, in some recognisably measure-theoretic sense, and, in particular, we shall have subsets of members of any $\mathcal{R}$ belonging to $\mathcal{R}$. In the other type, (vii) of 481G will be true because we can always find $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}} = C$.

(d) The following elementary fact got left out of §136 and Chapter 31. Let $\mathfrak{A}$ be a Boolean algebra and $C \subseteq \mathfrak{A}$. Set

$$E = \{\sup C_0 : C_0 \subseteq C \text{ is finite and disjoint}\}.$$

If $c \cap c'$ and $c \setminus c'$ belong to E for all c, $c' \in C$, then E is a subring of $\mathfrak{A}$. **P** Write $\mathcal{D}$ for the family of finite disjoint subsets of C. (i) If C_0, $C_1 \in \mathcal{D}$, then for $c \in C_0$, $c' \in C_1$ there is a $D_{cc'} \in \mathcal{D}$ with supremum $c \cap c'$. Now $D = \bigcup_{c \in C_0, c' \in C_1} D_{cc'}$ belongs to $\mathcal{D}$ and has supremum $(\sup C_0) \cap (\sup C_1)$. Thus $e \cap e' \in E$ for all e, $e' \in E$. (ii) Of course $e \cup e' \in E$ whenever e, $e' \in E$ and $e \cap e' = 0$. (iii) Again suppose that C_0, $C_1 \in \mathcal{D}$. Then $c \setminus c' \in E$ for all $c \in C_0$, $c' \in C_1$. By (i), $c \setminus \sup C_1 \in E$ for every $c \in C_0$; by (ii), $(\sup C_0) \setminus (\sup C_1) \in E$. Thus $e \setminus e' \in E$ for all e, $e' \in E$. (iv) Putting (ii) and (iii) together, $e \triangle e' \in E$ for all e, $e' \in E$. (v) As $0 = \sup \emptyset$ belongs to E, E is a subring of $\mathfrak{A}$. **Q**

In particular, if $\mathcal{C} \subseteq \mathcal{P}X$ has the properties in (iv) of 481G, then

$$\mathcal{E} = \{\bigcup \mathcal{C}_0 : \mathcal{C}_0 \subseteq \mathcal{C} \text{ is finite and disjoint}\}$$

is a ring of subsets of X.

(e) Suppose that X is a set and that $\mathfrak{R} \subseteq \mathcal{P}\mathcal{P}X$ satisfies (iii) of 481G. Then for every $\mathcal{R} \in \mathfrak{R}$ there is a non-increasing sequence $\langle \mathcal{R}_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{R}$ such that $\bigcup_{i \le n} A_i \in \mathcal{R}$ whenever $A_i \in \mathcal{R}_i$ for $i \le n$ and $\langle A_i \rangle_{i \le n}$ is disjoint. **P** Take $\mathcal{R}_0 \in \mathfrak{R}$ such that $\mathcal{R}_0 \subseteq \mathcal{R}$ and $A \cup B \in \mathcal{R}$ for all disjoint A, $B \in \mathcal{R}_0$; similarly, for $n \in \mathbb{N}$, choose $\mathcal{R}_{n+1} \in \mathfrak{R}$ such that $\mathcal{R}_{n+1} \subseteq \mathcal{R}_n$ and $A \cup B \in \mathcal{R}_n$ for all disjoint A, $B \in \mathcal{R}_{n+1}$. Now, given that $A_i \in \mathcal{R}_i$ for $i \le n$ and $\langle A_i \rangle_{i \le n}$ is disjoint, we see by downwards induction on m that $\bigcup_{m < i \le n} A_i \in \mathcal{R}_m$ for each $m \le n$, so that $\bigcup_{i \le n} A_i \in \mathcal{R}$. **Q**

(f) If $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, then T is compatible with Δ and $\mathfrak{R}$ in the sense of 481F. **P** $\emptyset \in T$ so T is not empty. Take $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$. Let $\langle \mathcal{R}_i \rangle_{i \in \mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that $\bigcup_{i \le n} A_i \in \mathcal{R}$ whenever $A_i \in \mathcal{R}_i$ for $i \le n$ and $\langle A_i \rangle_{i \le n}$ is disjoint ((e) above). There is a finite set $\mathcal{C}_1 \subseteq \mathcal{C}$ such that $X \setminus \bigcup \mathcal{C}_1 \in \mathcal{R}_0$, by 481G(iv). By (d), there is a disjoint family $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $\bigcup \mathcal{C}_0 = \bigcup \mathcal{C}_1$; enumerate $\mathcal{C}_0$ as $\langle C_i \rangle_{i < n}$. For each $i < n$, there is a δ-fine $\boldsymbol{t}_i \in T$ such that $W_{\boldsymbol{t}_i} \subseteq C_i$ and $C_i \setminus W_{\boldsymbol{t}_i} \in \mathcal{R}_{i+1}$, by 481G(vii). Set $\boldsymbol{t} = \bigcup_{i < n} \boldsymbol{t}_i$; then $\boldsymbol{t} \in T$ is δ-fine, and

$$X \setminus W_{\boldsymbol{t}} = (X \setminus \bigcup \mathcal{C}_1) \cup \bigcup_{i < n} (C_i \setminus W_{\boldsymbol{t}_i}) \in \mathcal{R}$$

by the choice of $\langle \mathcal{R}_i \rangle_{i \in \mathbb{N}}$. Thus we have a δ-fine $\mathcal{R}$-filling member of T; as δ and $\mathcal{R}$ are arbitrary, T is compatible with Δ and $\mathfrak{R}$. **Q**

(g) For basic results which depend on 'subdivisions' as described in 481G(vii), see 482A-482B below. A hypothesis asserting the existence of a different sort of subdivision appears in 482G(iv).

481I I now run through some simple examples of these constructions, limiting myself for the moment to the definitions, the proofs that T is compatible with Δ and $\mathfrak{R}$, and (when appropriate) the proofs that the structures allow subdivisions.

The proper Riemann integral Fix a non-empty closed interval $X = [a,b] \subseteq \mathbb{R}$. Write $\mathcal{C}$ for the set of all intervals (open, closed or half-open, and allowing the empty set to count as an interval) included in $[a,b]$, and set $Q = \{(x,C) : C \in C,\ x \in \overline{C}\}$; let T be the straightforward set of tagged partitions generated by Q. Let Δ be the set of uniform metric gauges on X, and $\mathfrak{R} = \{\{\emptyset\}\}$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. If $a < b$, then Δ is not countably full.

proof (i), (iii) and (vi) of 481G are trivial and (ii), (iv) and (v) are elementary. As for (vii), given $\eta > 0$ and $C \in \mathcal{C}$, take a disjoint family $\langle C_i\rangle_{i\in I}$ of non-empty intervals of length less than 2η covering C, and x_i to be the midpoint of C_i for $i \in I$; then $\boldsymbol{t} = \{(x_i,C_i) : i \in I\}$ belongs to T and is δ_η-fine, in the language of 481E, and $W_{\boldsymbol{t}} = C$.

Of course (apart from the trivial case $a = b$) Δ is not countably full, since if we take δ_n to be the gauge $\{(x,C) : |x-y| < 2^{-n}$ for every $y \in C\}$ and any unbounded function $\phi : [a,b] \to \mathbb{N}$, there is no $\delta \in \Delta$ such that $(x,C) \in \delta_{\phi(x)}$ whenever $(x,C) \in \Delta$.

481J The Henstock integral on a bounded interval (Henstock 63) Take X, $\mathcal{C}$, T and $\mathfrak{R}$ as in 481I. This time, let Δ be the set of *all* neighbourhood gauges on $[a,b]$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and Δ is countably full.

proof Again, only 481G(vii) needs more than a moment's consideration. Take any $C \in \mathcal{C}$. If $C = \emptyset$, then $\boldsymbol{t} = \emptyset$ will suffice. Otherwise, set $a_0 = \inf C$, $b_0 = \sup C$ and let T_0 be the family of δ-fine partitions $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}}$ is a relatively closed initial subinterval of C, that is, is of the form $C \cap [a_0, y_{\boldsymbol{t}}]$ for some $y_{\boldsymbol{t}} \in [a_0,b_0]$. Set $A = \{y_{\boldsymbol{t}} : \boldsymbol{t} \in T_0\}$. I have to show that there is a $\boldsymbol{t} \in T_0$ such that $W_{\boldsymbol{t}} = C$, that is, that $b_0 \in A$.

Observe that there is an $\eta_0 > 0$ such that $(a_0,A) \in \delta$ whenever $A \subseteq [a,b] \cap [a_0-\eta_0, a_0+\eta_0]$, and now $\{(a_0, [a_0, a_0+\eta_0] \cap C)\}$ belongs to T_0, so $\min(a_0+\eta_0, b_0) \in A$ and A is a non-empty subset of $[a_0,b_0]$. It follows that $c = \sup A$ is defined in $[a_0,b_0]$. Let $\eta > 0$ be such that $(c,A) \in \delta$ whenever $A \subseteq [a,b] \cap [c-\eta, c+\eta]$. There is some $\boldsymbol{t} \in T_0$ such that $y_{\boldsymbol{t}} \ge c-\eta$. If $y_{\boldsymbol{t}} = b_0$, we can stop. Otherwise, set $C' = C \cap \,]y_{\boldsymbol{t}}, c+\eta]$. Then $(c,C') \in \delta$ and $c \in \overline{C'}$ and $C' \cap W_{\boldsymbol{t}}$ is empty, so $\boldsymbol{t}' = \boldsymbol{t} \cup \{(c,C')\}$ belongs to T_0 and $y_{\boldsymbol{t}'} = \min(c+\eta, b_0)$. Since $y_{\boldsymbol{t}'} \le c$, this shows that $y_{\boldsymbol{t}'} = c = b_0$ and again $b_0 \in A$, as required.

Δ is full just because it is the family of neighbourhood gauges.

481K The Henstock integral on $\mathbb{R}$ This time, set $X = \mathbb{R}$ and let $\mathcal{C}$ be the family of all bounded intervals in $\mathbb{R}$. Let T be the straightforward set of tagged partitions generated by $\{(x,C) : C \in \mathcal{C},\ x \in \overline{C}\}$. Following 481J, let Δ be the set of all neighbourhood gauges on $\mathbb{R}$. This time, set $\mathfrak{R} = \{\mathcal{R}_{ab} : a \le b \in \mathbb{R}\}$, where $\mathcal{R}_{ab} = \{\mathbb{R} \setminus [c,d] : c \le a,\ d \ge b\} \cup \{\emptyset\}$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof This time we should perhaps take a moment to look at (iii) of 481G. But all we need to note is that $\mathcal{R}_{ab} \cap \mathcal{R}_{a'b'} = \mathcal{R}_{\min(a,a'),\max(b,b')}$, and that any two members of $\mathcal{R}_{ab}$ have non-empty intersection. Conditions (i), (ii), (iv), (v) and (vi) of 481G are again elementary, so once more we are left with (vii). But this can be dealt with by exactly the same argument as in 481J.

481L The symmetric Riemann-complete integral (cf. Carrington 72, chap. 3) Again take $X = \mathbb{R}$, and $\mathcal{C}$ the set of all bounded intervals in $\mathbb{R}$. This time, take T to be the straightforward set of tagged partitions generated by the set of pairs (x,C) where $C \in \mathcal{C} \setminus \{\emptyset\}$ and x is the *midpoint* of C. As in 481K, take Δ to be the set of all neighbourhood gauges on $\mathbb{R}$; but this time take $\mathfrak{R} = \{\mathcal{R}'_a : a \ge 0\}$, where $\mathcal{R}'_a = \{\mathbb{R} \setminus [-c,c] : c \ge a\} \cup \{\emptyset\}$. Then T is compatible with Δ and $\mathfrak{R}$.

proof Take $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$. For $x \ge 0$, let $\theta(x) > 0$ be such that $(x,D) \in \delta$ whenever $D \subseteq [x-\theta(x), x+\theta(x)]$ and $(-x,D) \in \delta$ whenever $D \subseteq [-x-\theta(x), -x+\theta(x)]$. Write A for the set of those $a > 0$ such that there is a finite sequence $(a_0,\dots,a_n)$ such that $0 < a_0 < a_1 < \ldots < a_n = a$, $a_0 \le \theta(0)$ and $a_{i+1} - a_i \le 2\theta(\frac{1}{2}(a_i + a_{i+1}))$ for $i < n$.

? Suppose, if possible, that A is bounded above. Then $c = \inf([0,\infty[\setminus \overline{A})$ is defined in $[0,\infty[$. Observe that if $0 < a \le \theta(0)$, then the one-term sequence $<a>$ witnesses that $a \in A$. So $c \ge \theta(0) > 0$. Now there must be u, v such that $c < u < v < \min(c+\theta(c), 2c)$ and $]u,v[\cap A = \emptyset$; on the other hand, the interval $]2c-v, 2c-u[$ must contain a point x of A. Set $y = 2c - x$. Then we can find $a_0 < \ldots < a_n = x$ such that $0 < a_0 \le \theta(0)$ and $a_{i+1} - a_i \le 2\theta(\frac{1}{2}(a_i + a_{i+1}))$ for $i < n$; setting $a_{n+1} = y$, we see that $\langle a_i\rangle_{i\le n+1}$ witnesses that $y \in A$, though $y \in \,]u,v[$. **X**

This contradiction shows that A is unbounded above. So now suppose that $\mathcal{R} = \mathcal{R}_a$ where $a \ge 0$. Take $a_0,\dots,a_n$ such that $0 < a_0 < \ldots < a_n$, $0 < a_0 \le \theta(0)$ and $a_{i+1} - a_i \le 2\theta(\frac{1}{2}(a_i + a_{i+1}))$ for $i < n$, and $a_n \ge a$. For $i < n$, set

$x_i = \frac{1}{2}(a_i + a_{i+1})$, $C_i =]a_i, a_{i+1}]$, $x'_i = -x_i$, $C'_i = [-a_{i+1}, a_i[$. Then x_i, x'_i are the midpoints of C_i, C'_i and (by the choice of the function θ) $(x_i, C_i) \in \delta$, $(x'_i, C'_i) \in \delta$ for $i < n$. So if we set

$$\boldsymbol{t} = \{(x_i, C_i) : i < n\} \cup \{(x'_i, C'_i) : i < n\} \cup \{(0, [-a_0, a_0])\}$$

we shall obtain a δ-fine $\mathcal{R}$-filling member of T.

As Δ and $\mathfrak{R}$ are both downwards-directed, this is enough to show that T is compatible with Δ and $\mathfrak{R}$.

481M The McShane integral on an interval (McShane 73) As in 481J, take $X = [a, b]$ and let $\mathcal{C}$ be the family of subintervals of $[a, b]$. This time, take T to be the straightforward set of tagged partitions generated by $Q = X \times \mathcal{C}$, so that *no* condition is imposed relating the tags to their associated intervals. As in 481J, let Δ be the set of all neighbourhood gauges on X, and $\mathfrak{R} = \{\{\emptyset\}\}$. Proceed as before. Since the only change is that Q and T have been enlarged, $(X, T, \Delta, \mathfrak{R})$ is still a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

481N The McShane integral on a topological space (Fremlin 95) Now let $(X, \mathfrak{T}, \Sigma, \mu)$ be any effectively locally finite τ-additive topological measure space, and take $\mathcal{C} = \{E : E \in \Sigma, \mu E < \infty\}$, $Q = X \times \mathcal{C}$; let T be the straightforward set of tagged partitions generated by Q. Again let Δ be the set of all neighbourhood gauges on X. This time, define $\mathfrak{R}$ as follows. For any set $E \in \Sigma$ of finite measure and $\eta > 0$, let $\mathcal{R}_{E\eta}$ be the set $\{F : F \in \Sigma, \mu(F \cap E) \le \eta\}$, and set $\mathfrak{R} = \{\mathcal{R}_{E\eta} : \mu E < \infty, \eta > 0\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof As usual, everything is elementary except perhaps 481G(vii). But if $C \in \mathcal{C}$, $\delta \in \Delta$, $E \in \Sigma$, $\mu E < \infty$ and $\eta > 0$, take for each $x \in X$ an open set G_x containing x such that $(x, A) \in \delta$ whenever $A \subseteq G_x$. $\{G_x : x \in X\}$ is an open cover of X, so by 414Ea there is a finite family $\langle x_i \rangle_{i<n}$ in X such that $\mu(E \cap C \setminus \bigcup_{i<n} G_{x_i}) \le \eta$; setting $C_i = C \cap G_{x_i} \setminus \bigcup_{j<i} G_{x_j}$ for $i < n$, we get a δ-fine tagged partition $\boldsymbol{t} = \{(x_i, C_i) : i < n\}$ such that $C \setminus W_{\boldsymbol{t}} \in \mathcal{R}_{E\eta}$.

481O Convex partitions in $\mathbb{R}^r$ Fix $r \ge 1$. Let us say that a **convex polytope** in $\mathbb{R}^r$ is a non-empty bounded set expressible as the intersection of finitely many open or closed half-spaces; let $\mathcal{C}$ be the family of convex polytopes in $X = \mathbb{R}^r$, and T the straightforward set of tagged partitions generated by $\{(x, C) : x \in \overline{C}\}$. Let Δ be the set of neighbourhood gauges on $\mathbb{R}^r$. For $a \ge 0$, let $\mathcal{C}_a$ be the set of closed convex polytopes $C \subseteq \mathbb{R}^r$ such that, for some $b \ge a$, $B(0, b) \subseteq C \subseteq B(0, 2b)$, where $B(0, b)$ is the ordinary Euclidean ball with centre 0 and radius b; set $\mathcal{R}_a = \{\mathbb{R}^r \setminus C : C \in \mathcal{C}_a\} \cup \{\emptyset\}$, and $\mathfrak{R} = \{\mathcal{R}_a : a \ge 0\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof As usual, only 481G(vii) requires thought.

(a) We need a geometrical fact: if $C \in \mathcal{C}$, $x \in \overline{C}$ and $y \in C$, then $\alpha y + (1 - \alpha)x \in C$ for every $\alpha \in]0, 1]$. **P** The family of sets $C \subseteq \mathbb{R}^r$ with this property is closed under finite intersections and contains all half-spaces. **Q** It follows that if C_1, $C_2 \in \mathcal{C}$ are not disjoint, then $\overline{C_1 \cap C_2} = \overline{C}_1 \cap \overline{C}_2$.

(b) Write $\mathcal{D} \subseteq \mathcal{C}$ for the family of products of non-empty bounded intervals in $\mathbb{R}$. The next step is to show that if $D \in \mathcal{D}$ and $\delta \in \Delta$, then there is a δ-fine tagged partition $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}} = D$ and $\boldsymbol{t} \subseteq \mathbb{R}^r \times \mathcal{D}$. **P** Induce on r. For $r = 1$ this is just 481J again. For the inductive step to $r + 1$, suppose that $D \subseteq \mathbb{R}^{r+1}$ is a product of bounded intervals and that δ is a neighbourhood gauge on $\mathbb{R}^{r+1}$. Identifying $\mathbb{R}^{r+1}$ with $\mathbb{R}^r \times \mathbb{R}$, express D as $D' \times L$, where $D' \subseteq \mathbb{R}^r$ is a product of bounded intervals and $L \subseteq \mathbb{R}$ is a bounded interval. For $y \in \mathbb{R}^r$, $\alpha \in \mathbb{R}$ let $G(y, \alpha)$, $H(y, \alpha)$ be open sets containing y, α respectively such that $((y, \alpha), A) \in \delta$ whenever $A \subseteq G(y, \alpha) \times H(y, \alpha)$. For $y \in \mathbb{R}^r$, set $\delta_y = \{(\alpha, A) : \alpha \in \mathbb{R}, A \subseteq H(y, \alpha)\}$; then δ_y is a neighbourhood gauge on $\mathbb{R}$.

By the one-dimensional case there is a δ_y-fine tagged partition $\boldsymbol{s}_y \in T_1$ such that $W_{\boldsymbol{s}_y} = L$, where I write T_1 for the set of tagged partitions used in 481K. Set

$$\delta' = \{(y, A) : y \in \mathbb{R}^r, A \subseteq G(y, \alpha) \text{ for every } (\alpha, F) \in \boldsymbol{s}_y\}.$$

δ' is a neighbourhood gauge on $\mathbb{R}^r$. By the inductive hypothesis, there is a δ'-fine tagged partition $\boldsymbol{u} \in T_r$ such that $W_{\boldsymbol{u}} = D'$, where here T_r is the set of tagged partitions on $\mathbb{R}^r$ corresponding to the r-dimensional version of this result. Consider the family

$$\boldsymbol{t} = \{((y, \alpha), E \times F) : (y, E) \in \boldsymbol{u}, (\alpha, F) \in \boldsymbol{s}_y\}.$$

For $(y, E) \in \boldsymbol{u}$, $(\alpha, F) \in \boldsymbol{s}_y$, we have

$$y \in \overline{E}, \quad E \subseteq G(y, \alpha), \quad \alpha \in \overline{F}, \quad F \subseteq H(y, \alpha),$$

so

$$(y,\alpha)\in\overline{E\times F},\quad E\times F\subseteq G(y,\alpha)\times H(y,\alpha),$$

and $((y,\alpha),E\times F)\in\delta$. If $((y,\alpha),E\times F)$, $((y',\alpha'),E'\times F')$ are distinct members of $\boldsymbol{t}$, then either $(y,E)\ne(y',E')$ so $E\cap E'=\emptyset$ and $(E\times F)\cap(E'\times F')$ is empty, or $y=y'$ and (α,F), (α',F') are distinct members of $\boldsymbol{s}_y$, so that $F\cap F'=\emptyset$ and again $E\times F$, $E'\times F'$ are disjoint. Thus $\boldsymbol{t}$ is a δ-fine member of T_{r+1}. Finally,

$$W_{\boldsymbol{t}}=\bigcup_{(y,E)\in\boldsymbol{u}}\bigcup_{(\alpha,F)\in\boldsymbol{s}_y}E\times F=\bigcup_{(y,E)\in\boldsymbol{u}}E\times L=D'\times L=D.$$

So the induction proceeds. **Q**

(c) Now suppose that C_0 is an arbitrary member of $\mathcal{C}$ and that δ is a neighbourhood gauge on $\mathbb{R}^r$. Set

$$\delta'=\delta\cap\{(x,A):\text{either }x\in\overline{C}_0\text{ or }A\cap\overline{C}_0=\emptyset\}.$$

Then δ' is a neighbourhood gauge on $\mathbb{R}^r$, being the intersection of δ with the neighbourhood gauge associated with the family $\langle U_x\rangle_{x\in\mathbb{R}^r}$, where $U_x=\mathbb{R}^r$ if $x\in\overline{C}_0$, $\mathbb{R}^r\setminus\overline{C}_0$ otherwise. Let $D\in\mathcal{D}$ be such that $C_0\subseteq D$. By (b), there is a δ'-fine tagged partition $\boldsymbol{t}\in T$ such that $W_{\boldsymbol{t}}=D$. Set $\boldsymbol{s}=\{(x,C\cap C_0):(x,C)\in\boldsymbol{t},\,C\cap C_0\ne\emptyset\}$. Since $\boldsymbol{t}\subseteq\delta'$, $x\in\overline{C}_0$ whenever $(x,C)\in\boldsymbol{t}$ and $C\cap C_0\ne\emptyset$. By (a), $x\in\overline{C\cap C_0}$ for all such pairs (x,C); and of course $(x,C\cap C_0)\in\delta$ for every $(x,C)\in\boldsymbol{t}$. So $\boldsymbol{s}$ belongs to T, and $W_{\boldsymbol{s}}=W_{\boldsymbol{t}}\cap C_0=C_0$. As C_0 and δ are arbitrary, 481G(vii) is satisfied.

481P Box products (cf. Muldowney 87, Prop. 1) Let $\langle(X_i,\mathfrak{T}_i)\rangle_{i\in I}$ be a non-empty family of non-empty compact metrizable spaces with product $(X,\mathfrak{T})$. Set $\pi_i(x)=x(i)$ for $x\in X$ and $i\in I$. For each $i\in I$, let $\mathcal{C}_i\subseteq\mathcal{P}X_i$ be such that (α) whenever E, $E'\in\mathcal{C}_i$ then $E\cap E'\in\mathcal{C}_i$ and $E\setminus E'$ is expressible as the union of a disjoint finite subset of $\mathcal{C}_i$ (β) $\mathcal{C}_i$ includes a base for $\mathfrak{T}_i$.

Let $\mathcal{C}$ be the set of subsets of X of the form

$$C=\{X\cap\bigcap_{i\in J}\pi_i^{-1}[E_i]:J\in[I]^{<\omega},\,E_i\in\mathcal{C}_i\text{ for every }i\in J\},$$

and let T be the straightforward set of tagged partitions generated by $\{(x,C):C\in\mathcal{C},\,x\in\overline{C}\}$. Let Δ be the set of those neighbourhood gauges δ on X defined by families $\langle G_x\rangle_{x\in X}$ of open sets such that, for some countable $J\subseteq I$, every G_x is determined by coordinates in J (definition: 254M). Then $(X,T,\Delta,\{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. Δ is countably full; Δ is full iff $I'=\{i:\#(X_i)>1\}$ is countable.

proof Conditions (i), (iii) and (vi) of 481G are trivial, and (ii), (iv) and (v) are elementary; so we are left with (vii), as usual. **?** Suppose, if possible, that $C\in\mathcal{C}$ and $\delta\in\Delta$ are such that there is no δ-fine $\boldsymbol{t}\in T$ with $W_{\boldsymbol{t}}=C$. Let $\langle G_x\rangle_{x\in X}$ be the family of open sets determining δ, and $J\subseteq I$ a non-empty countable set such that G_x is determined by coordinates in J for every $x\in X$. For $i\in J$, let $\mathcal{C}'_i\subseteq\mathcal{C}_i$ be a countable base for $\mathfrak{T}_i$ (4A2P(a-iii)), and take a sequence $\langle(i_n,E_n)\rangle_{n\in\mathbb{N}}$ running over $\{(i,E):i\in J,\,E\in\mathcal{C}'_i\}$.

Write $\mathcal{D}=\{W_{\boldsymbol{t}}:\boldsymbol{t}\in T\text{ is }\delta\text{-fine}\}$. Note that if D_1, $D_2\in\mathcal{D}$ are disjoint then $D_1\cup D_2\in\mathcal{D}$. So if $D\in\mathcal{C}\setminus\mathcal{D}$ and $C\in\mathcal{C}$, there must be some $D'\in\mathcal{C}\setminus\mathcal{D}$ such that either $D'\subseteq D\cap C$ or $D'\subseteq D\setminus C$, just because $\mathcal{C}$ satisfies 481G(iv). Now choose $\langle C_n\rangle_{n\in\mathbb{N}}$ inductively so that $C_0=C$ and

$$C_n\in\mathcal{C}\setminus\mathcal{D},$$

$$\text{either }C_{n+1}\subseteq C_n\cap\pi_{i_n}^{-1}[E_n]\text{ or }C_{n+1}\subseteq C_n\setminus\pi_{i_n}^{-1}[E_n]$$

for every $n\in\mathbb{N}$. Because X, being a product of compact spaces, is compact, there is an $x\in\bigcap_{n\in\mathbb{N}}\overline{C_n}$. We know that G_x is determined by coordinates in J, so $G_x=\tilde\pi^{-1}[\tilde\pi[G_x]]$, where $\tilde\pi$ is the canonical map from X onto $Y=\prod_{i\in J}X_i$. $V=\tilde\pi[G_x]$ is open, so there must be a finite set $K\subseteq J$ and a family $\langle V_i\rangle_{i\in K}$ such that $x(i)\in V_i\in\mathfrak{T}_i$ for every $i\in K$ and $\{y:y\in Y,\,y(i)\in V_i\text{ for every }i\in K\}$ is included in V. This means that $\{z:z\in X,\,z(i)\subseteq V_i\text{ for every }i\in K\}$ is included in G_x. Now, for each $i\in K$, there is some $m\in\mathbb{N}$ such that $i=i_m$ and $x(i)\in E_m\subseteq V_i$. Because $x\in\overline{C}_{m+1}$, $\pi_{i_m}^{-1}[E_m]$ cannot be disjoint from C_{m+1}, and $C_{m+1}\subseteq\pi_{i_m}^{-1}[E_m]\subseteq\pi_i^{-1}[V_i]$.

But this means that, for any n large enough, $C_n\subseteq G_x$ and $\boldsymbol{t}=\{(x,C_n)\}$ is a δ-fine member of T with $W_{\boldsymbol{t}}=C_n$; contradicting the requirement that $C_n\notin\mathcal{D}$. **X**

This contradiction shows that 481G(vii) also is satisfied.

To see that Δ is countably full, note that if $\langle\delta_n\rangle_{n\in\mathbb{N}}$ is a sequence in Δ, we have for each $n\in\mathbb{N}$ a countable set $J_n\subseteq I$ and a family $\langle G_{nx}\rangle_{x\in X}$ of open sets, all determined by coordinates in J_n, such that $x\in G_{nx}$ and $(x,C)\in\delta_n$ whenever $x\in X$ and $C\subseteq G_{nx}$. Now, given $\phi:X\to\mathbb{N}$, set $\delta=\{(x,C):x\in X,\,C\subseteq G_{\phi(x),x}\}$, and observe that $\delta\in\Delta$ and that $(x,C)\in\delta_{\phi(x)}$ whenever $(x,C)\in\Delta$.

If I' is countable, then Δ is the set of all neighbourhood gauges on X, so is full. If I' is uncountable, then for $j \in I'$ and $x \in X$ choose a proper open subset H_{jx} of X_j containing $x(j)$ and set $G_{jx} = \{y : y \in X,\ y(j) \in H_{jx}\}$. For $j \in I'$ set $\delta_j = \{(x, C) : C \subseteq G_{jx}\} \in \Delta$. Let $\phi : X \to I'$ be any function such that $\phi[X]$ is uncountable; then there is no $\delta \in \Delta$ such that $(x, C) \in \delta_{\phi(x)}$ whenever $(x, C) \in \delta$, so $\langle \delta_{\phi(x)} \rangle_{x \in X}$ witnesses that Δ is not full.

481Q The approximately continuous Henstock integral (Gordon 94, chap. 16) Let μ be Lebesgue measure on $\mathbb{R}$. As in 481K, let $\mathcal{C}$ be the family of non-empty bounded intervals in $\mathbb{R}$, T the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$, and $\mathfrak{R} = \{\mathcal{R}_{ab} : a,\ b \in \mathbb{R}\}$, where $\mathcal{R}_{ab} = \{\mathbb{R} \setminus [c, d] : c \le a,\ d \ge b\} \cup \{\emptyset\}$ for $a,\ b \in \mathbb{R}$.

This time, define gauges as follows. Let E be the set of families $\boldsymbol{e} = \langle E_x \rangle_{x \in \mathbb{R}}$ where every E_x is a measurable set containing x such that x is a density point of E_x (definition: 223B). For $\boldsymbol{e} = \langle E_x \rangle_{x \in \mathbb{R}} \in \mathrm{E}$, set

$$\delta_{\boldsymbol{e}} = \{(x, C) : C \in \mathcal{C},\ x \in \overline{C},\ \inf C \in E_x \text{ and } \sup C \in E_x\}.$$

Set $\Delta = \{\delta_{\boldsymbol{e}} : \boldsymbol{e} \in \mathrm{E}\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and Δ is full.

proof (a) Turning to 481G, we find, as usual, that most of the conditions are satisfied for elementary reasons. Since we have here a new kind of gauge, we had better check 481G(ii); but if $\boldsymbol{e} = \langle E_x \rangle_{x \in \mathbb{R}}$ and $\boldsymbol{e}' = \langle E'_x \rangle_{x \in \mathbb{R}}$ both belong to E, so does $\boldsymbol{e} \wedge \boldsymbol{e}' = \langle E_x \cap E'_x \rangle_{x \in \mathbb{R}}$, because

$$\begin{aligned}\liminf_{\eta \downarrow 0} \frac{1}{2\eta} \mu([x - \eta, x + \eta] \cap E_x \cap E'_x) \\ \ge \lim_{\eta \downarrow 0} \frac{1}{2\eta} \bigl(\mu([x - \eta, x + \eta] \cap E_x) + \mu([x - \eta, x + \eta] \cap E'_x) - 2\eta\bigr) = 1\end{aligned}$$

for every x; and now $\delta_{\boldsymbol{e}} \cap \delta_{\boldsymbol{e}'} = \delta_{\boldsymbol{e} \wedge \boldsymbol{e}'}$ belongs to Δ. Everything else we have done before, except, of course, (vii).

(b) So take any $\delta \in \Delta$; express δ as $\delta_{\boldsymbol{e}}$, where $\boldsymbol{e} = \langle E_x \rangle_{x \in \mathbb{R}} \in \mathrm{E}$. For $x,\ y \in \mathbb{R}$, write $x \frown y$ if $x \le y$ and $E_x \cap E_y \cap [x, y] \ne \emptyset$; note that we always have $x \frown x$. For $x \in \mathbb{R}$, let $\eta_x > 0$ be such that $\mu(E_x \cap [x - \eta, x + \eta]) \ge \frac{5}{3}\eta$ whenever $0 \le \eta \le \eta_x$.

Fix $a < b$ in $\mathbb{R}$ for the moment. Say that a finite string $(x_0, \dots, x_n)$ is 'acceptable' if $a \le x_0 \frown \dots \frown x_n \le b$ and $\mu([x_0, x_n] \cap \bigcup_{i<n} E^+_{x_i}) \ge \frac{1}{2}(x_n - x_0)$, where $E^+_x = E_x \cap [x, \infty[$ for $x \in \mathbb{R}$. Observe that if $(x_0, \dots, x_m)$ and $(x_m, x_{m+1}, \dots, x_n)$ are both acceptable, so is $(x_0, \dots, x_n)$. For $x \in [a, b]$, set

$$h(x) = \sup\{x_n : (x, x_1, \dots, x_n) \text{ is acceptable}\};$$

this is defined in $[x, b]$ because the string (x) is acceptable. If $a \le x < b$, then (x, y) is acceptable whenever $y \in E_x$ and $0 \le y \le \min(b, x + \eta_x)$, so $h(x) > x$. Now choose sequences $\langle x_i \rangle_{i \in \mathbb{N}}$, $\langle n_k \rangle_{k \in \mathbb{N}}$ inductively, as follows. $n_0 = 0$ and $x_0 = a$. Given that $x_i \frown x_{i+1}$ for $i < n_k$ and that $(x_{n_j}, \dots, x_{n_k})$ is acceptable for every $j \le k$, let $n_{k+1} > n_k$, $(x_{n_k+1}, \dots, x_{n_{k+1}})$ be such that $(x_{n_k}, \dots, x_{n_{k+1}})$ is acceptable and $x_{n_{k+1}} \ge \frac{1}{2}(x_{n_k} + h(x_{n_k}))$; then $(x_{n_j}, \dots, x_{n_{k+1}})$ is acceptable for any $j \le k + 1$; continue.

At the end of the induction, set

$$c = \sup_{i \in \mathbb{N}} x_i = \sup_{k \in \mathbb{N}} x_{n_k}.$$

Then there are infinitely many i such that $x_i \frown c$. **P** For any $k \in \mathbb{N}$,

$$\begin{aligned}\mu([x_{n_k}, c] \cap \bigcup_{i \ge n_k} E^+_{x_i}) &= \lim_{l \to \infty} \mu\bigl([x_{n_k}, x_{n_l}] \cap \bigcup_{n_k \le i < n_l} E^+_{x_i}\bigr) \\ &\ge \lim_{l \to \infty} \frac{1}{2}(x_{n_l} - x_{n_k})\end{aligned}$$

(because $(x_{n_k}, \dots, x_{n_l})$ is always an acceptable string)

$$= \frac{1}{2}(c - x_{n_k}).$$

But this means that if we take k so large that $x_{n_k} \ge c - \eta_c$, so that $\mu(E_c \cap [x_{n_k}, c]) \ge \frac{2}{3}(c - x_{n_k})$, there must be some $z \in E_c \cap \bigcup_{i \ge n_k} E^+_i \cap [x_{n_k}, c]$; and if $i \ge n_k$ is such that $z \in E^+_i$, then z witnesses that $x_i \frown c$. As k is arbitrarily large, we have the result. **Q**

? If $c<b$, take any $y\in E_c$ such that $c<y\le\min(c+\eta_c,b)$. Take $k\in\mathbb{N}$ such that $x_{n_k}\ge c-\frac{1}{3}(y-c)$, and $j\ge n_k$ such that $x_j\frown c$. In this case, $(x_{n_k},x_{n_k+1},\dots,x_j,c,y)$ is an acceptable string, because

$$\mu E_c^+\cap[x_{n_k},y]\ge\mu E_c\cap[c,y]\ge\tfrac{2}{3}(y-c)\ge\tfrac{1}{2}(y-x_{n_k}).$$

But this means that $h(x_{n_k})\ge y$, so that

$$x_{n_{k+1}}\le c<\frac{1}{2}(x_{n_k}+h(x_{n_k})),$$

contrary to the choice of $x_{n_k+1},\dots,x_{n_{k+1}}$. **X**

Thus $c=b$. We therefore have a $j\in\mathbb{N}$ such that $x_j\frown b$, and $a=x_0\frown\dots\frown x_j\frown b$.

(c) Now suppose that $C\in\mathcal{C}$. Set $a=\inf C$ and $b=\sup C$. If $a=b$, then $\boldsymbol{t}=\{(a,C)\}\in T$ and $W_{\boldsymbol{t}}=C$. Otherwise, (b) tells us that we have $x_0,\dots,x_n$ such that $a=x_0\frown\dots\frown x_n=b$. Choose $a_i\in[x_{i-1},x_i]\cap E_{x_{i-1}}\cap E_{x_i}$ for $1\le i\le n$. Set $C_i=[a_i,a_{i+1}[$ for $1\le i<n$, $C_0=[a,a_1[$, $C_n=[a_n,b]$; set $I=\{i:i\le n,\ C\cap C_i\ne\emptyset\}$; and check that $(x_i,C\cap C_i)\in\delta$ for $i\in I$, so that $\boldsymbol{t}=\{(x_i,C\cap C_i):i\in I\}$ is a δ-fine member of T with $W_{\boldsymbol{t}}=C$. As C and δ are arbitrary, 481G(vii) is satisfied.

(d) Δ is full. **P** Let $\langle\delta'_x\rangle_{x\in\mathbb{R}}$ be a family in Δ. For each $x\in X$, there is a measurable set E_x such that x is a density point of E_x and $(x,C)\in\delta'_x$ whenever $C\in\mathcal{C}$, $x\in\overline{C}$ and both $\inf C$, $\sup C$ belong to E_x. Set $\boldsymbol{e}=\langle E_x\rangle_{x\in\mathbb{R}}\in\mathrm{E}$; then $(x,C)\in\delta'_x$ whenever $(x,C)\in\delta_{\boldsymbol{e}}$. **Q**

481X Basic exercises (a) Let X, $\mathcal{C}$, T and $\mathcal{F}$ be as in 481C. Show that if $f:X\to\mathbb{R}$, $\mu:\mathcal{C}\to\mathbb{R}$ and $\nu:\mathcal{C}\to\mathbb{R}$ are functions, then $I_{\mu+\nu}(f)=I_\mu(f)+I_\nu(f)$ whenever the right-hand side is defined.

>(b) Let I be any set. Set $T=\{\{(i,\{i\}):i\in J\}:J\in[I]^{<\omega}\}$, $\delta=\{(i,\{i\}):i\in I\}$, $\Delta=\{\delta\}$. For $J\in[I]^{<\omega}$ set $\mathcal{R}_J=\{I\setminus K:J\subseteq K\in[I]^{<\omega}\}\cup\{\emptyset\}$; set $\mathfrak{R}=\{\mathcal{R}_J:J\in[I]^{<\omega}\}$. Show that $(I,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $[I]^{<\omega}$, and that Δ is full. Let $\nu:[I]^{<\omega}\to\mathbb{R}$ be any additive functional. Show that, for a function $f:I\to\mathbb{R}$, $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)=\sum_{i\in I}f(i)\nu(\{i\})$ if either exists in $\mathbb{R}$.

>(c) Set $T=\{\{(n,\{n\}):n\in I\}:I\in[\mathbb{Z}]^{<\omega}\}$, $\delta=\{(n,\{n\}):n\in\mathbb{Z}\}$, $\Delta=\{\delta\}$. For $I\in[\mathbb{Z}]^{<\omega}$ and m, $n\in\mathbb{N}$ set $R_{mn}=\mathbb{Z}\setminus\{i:-m\le i\le n\}$, $\mathcal{R}'_n=\{R_{kl}:k,\,l\ge n\}\cup\{\emptyset\}$, $\mathcal{R}''_n=\{R_{kk}:k\ge n\}\cup\{\emptyset\}$, $\mathfrak{R}'=\{\mathcal{R}'_n:n\in\mathbb{N}\}$, $\mathfrak{R}''=\{\mathcal{R}''_n:n\in\mathbb{N}\}$. Show that $(\mathbb{Z},T,\Delta,\mathfrak{R}')$ and $(\mathbb{Z},T,\Delta,\mathfrak{R}'')$ are tagged-partition structures allowing subdivisions, witnessed by $[\mathbb{Z}]^{<\omega}$, and that Δ is full. Let μ be counting measure on $\mathbb{Z}$. Show that, for a function $f:\mathbb{Z}\to\mathbb{R}$, (i) $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R}')}S_{\boldsymbol{t}}(f,\nu)=\lim_{m,n\to\infty}\sum_{i=-m}^n f(i)$ if either is defined in $\mathbb{R}$ (ii) $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R}'')}S_{\boldsymbol{t}}(f,\mu)=\lim_{n\to\infty}\sum_{i=-n}^n f(i)$ if either is defined in $\mathbb{R}$.

(d) Set $X=\mathbb{N}\cup\{\infty\}$, and let T be the straightforward set of tagged partitions generated by $\{(n,\{n\}):n\in\mathbb{N}\}\cup\{(\infty,X\setminus n):n\in\mathbb{N}\}$ (interpreting a member of $\mathbb{N}$ as the set of its predecessors). For $n\in\mathbb{N}$ set $\delta_n=\{(i,\{i\}):i\in\mathbb{N}\}\cup\{(\infty,A):A\subseteq X\setminus n\}$; set $\Delta=\{\delta_n:n\in\mathbb{N}\}$. Show that $(X,T,\Delta,\{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}=[\mathbb{N}]^{<\omega}\cup\{X\setminus I:I\in[\mathbb{N}]^{<\omega}\}$, and that Δ is full. Let $h:\mathbb{N}\to\mathbb{R}$ be any function, and define $\nu:\mathcal{C}\to\mathbb{R}$ by setting $\nu I=\sum_{i\in I}h(i)$, $\nu(X\setminus I)=-\nu I$ for $I\in[\mathbb{N}]^{<\omega}$. Let $f:X\to\mathbb{R}$ be any function such that $f(\infty)=0$. Show that $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\{\{\emptyset\}\})}S_{\boldsymbol{t}}(f,\mu)=\lim_{n\to\infty}\sum_{i=0}^n f(i)h(i)$ if either is defined in $\mathbb{R}$.

>(e) Take X, T, Δ and $\mathfrak{R}$ as in 481I. Show that if μ is Lebesgue measure on $[a,b]$ then the gauge integral $I_\mu=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(.,\mu)$ is the ordinary Riemann integral $\mathrm{R}\!\!\int_a^b$ as described in 134K. (*Hint*: show first that they agree on step-functions.)

>(f) Let (X,Σ,μ) be a semi-finite measure space, and Σ^f the family of measurable sets of finite measure. Let T be the straightforward set of tagged partitions generated by $\{(x,E):x\in E\in\Sigma^f\}$. For $E\in\Sigma^f$ and $\epsilon>0$ set $\mathcal{R}_{E\epsilon}=\{F:F\in\Sigma,\ \mu(E\setminus F)\le\epsilon\}$; set $\mathfrak{R}=\{\mathcal{R}_{E\epsilon}:E\in\Sigma^f,\ \epsilon>0\}$. Let $\mathfrak{E}$ be the family of countable partitions of X into measurable sets, and set $\delta_{\mathcal{E}}=\bigcup_{E\in\mathcal{E}}\{(x,A):x\in E,\ A\subseteq E\}$ for $\mathcal{E}\in\mathfrak{E}$, $\Delta=\{\delta_{\mathcal{E}}:\mathcal{E}\in\mathfrak{E}\}$. Show that $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by Σ^f. In what circumstances is Δ full or countably full? Show that, for a function $f:X\to\mathbb{R}$, $\int fd\mu=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\mu)$ if either is defined in $\mathbb{R}$. (*Hint*: when showing that if $I_\mu(f)$ is defined then f is μ-virtually measurable, you will need 413G or something similar; compare 482E.)

(g) Let (X,Σ,μ) be a totally finite measure space, and T the straightforward set of tagged partitions generated by $\{(x,E):x\in E\in\Sigma\}$. Let $\mathfrak{E}$ be the family of finite partitions of X into measurable sets, and set $\delta_{\mathcal{E}}=\bigcup_{E\in\mathcal{E}}\{(x,A):$

$x \in E$, $A \subseteq E\}$ for $\mathcal{E} \in \mathfrak{E}$, $\Delta = \{\delta_{\mathcal{E}} : \mathcal{E} \in \mathfrak{E}\}$. Show that $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by Σ. In what circumstances is Δ full or countably full? Show that, for a function $f : X \to \mathbb{R}$, $I_\mu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$ is defined iff $f \in \mathcal{L}^\infty(\mu)$ (definition: 243A), and that then $I_\mu(f) = \int f d\mu$.

(h) Let X be a zero-dimensional compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. Let T be the straightforward set of tagged partitions generated by $\{(x, E) : x \in E \in \mathcal{E}\}$. Let Δ be the set of all neighbourhood gauges on X. Show that $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{E}$. Now let $\nu : \mathcal{E} \to \mathbb{R}$ be an additive functional, and set $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\{\{\emptyset\}\})} S_{\boldsymbol{t}}(f, \nu)$ when $f : X \to \mathbb{R}$ is such that the limit is defined. (i) Show that $I_\nu(\chi E) = \nu E$ for every $E \in \mathcal{E}$. (ii) Show that if ν is bounded then $I_\nu(f)$ is defined for every $f \in C(X)$, and is equal to $\oint f d\nu$ as defined in 363L, if we identify X with the Stone space of $\mathfrak{A}$ and $C(X)$ with $L^\infty(\mathfrak{A})$.

(i) Let X be a set, Δ a set of gauges on X, $\mathfrak{R}$ a collection of families of subsets of X, and T a set of tagged partitions on X which is compatible with Δ and $\mathfrak{R}$. Let $H \subseteq X$ be such that there is a $\tilde{\delta} \in \Delta$ such that $H \cap A = \emptyset$ whenever $x \in X \setminus H$ and $(x, A) \in \tilde{\delta}$, and set $\delta_H = \{(x, A \cap H) : x \in H, (x, A) \in \delta\}$ for $\delta \in \Delta$, $\Delta_H = \{\delta_H : \delta \in \Delta\}$, $\mathfrak{R}_H = \{\{R \cap H : R \in \mathcal{R}\} : \mathcal{R} \in \mathfrak{R}\}$, $T_H = \{\{(x, C \cap H) : (x, C) \in \boldsymbol{t}, x \in H\} : \boldsymbol{t} \in T\}$. Show that T_H is compatible with Δ_H and $\mathfrak{R}_H$.

(j) Let X be a set, Σ an algebra of subsets of X, and $\nu : \Sigma \to [0, \infty[$ an additive functional. Set $Q = \{(x, C) : x \in C \in \Sigma\}$ and let T be the straightforward set of tagged partitions generated by Q. Let $\mathbb{E}$ be the set of disjoint families $\mathcal{E} \subseteq \Sigma$ such that $\sum_{E \in \mathcal{E}} \nu E = \nu X$, and $\Delta = \{\delta_{\mathcal{E}} : \mathcal{E} \in \mathbb{E}\}$, where

$$\delta_{\mathcal{E}} = \{(x, C) : (x, C) \in Q \text{ and there is an } E \in \mathcal{E} \text{ such that } C \subseteq E\}$$

for $\mathcal{E} \in \mathbb{E}$. Set $\mathfrak{R} = \{\mathcal{R}_\epsilon : \epsilon > 0\}$ where $\mathcal{R}_\epsilon = \{E : E \in \Sigma, \nu E \le \epsilon\}$ for $\epsilon > 0$. Show that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by Σ.

481Y Further exercises (a) Suppose that $[a, b]$, $\mathcal{C}$, T and Δ are as in 481J. Let $T' \subseteq [[a, b] \times \mathcal{C}]^{<\omega}$ be the set of tagged partitions $\boldsymbol{t} = \{(x_i, [a_i, a_{i+1}]) : i < n\}$ where $a = a_0 \le x_0 \le a_1 \le x_2 \le a_2 \le \ldots \le x_{n-1} \le a_n = b$. Show that T', as well as T, is compatible with Δ in the sense of 481Ea; let $\mathcal{F}'$, $\mathcal{F}$ be the corresponding filters on T' and T. Show that if $\nu : \mathcal{C} \to \mathbb{R}$ is a functional which is additive in the sense that $\nu(C \cup C') = \nu C + \nu C'$ whenever C, C' are disjoint members of $\mathcal{C}$ with union in $\mathcal{C}$, and if $\nu\{x\} = 0$ for every $x \in [a, b]$ and $f : [a, b] \to \mathbb{R}$ is any function, then $I'_\nu(f) = I_\nu(f)$ if either is defined, where

$$I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}} S_{\boldsymbol{t}}(f, \nu), \quad I'_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}'} S_{\boldsymbol{t}}(f, \nu)$$

are the gauge integrals associated with $(T, \mathcal{F})$ and $(T', \mathcal{F}')$.

(b) Let us say that a family $\mathfrak{R}$ of residual families is 'the simple residual structure complementary to $\mathcal{H} \subseteq \mathcal{P}X$' if $\mathfrak{R} = \{\mathcal{R}_H : H \in \mathcal{H}\}$, where $\mathcal{R}_H = \{X \setminus H' : H \subseteq H' \in \mathcal{H}\} \cup \{\emptyset\}$ for each $H \in \mathcal{H}$. Suppose that, for each member i of a non-empty finite set I, $(X_i, T_i, \Delta_i, \mathfrak{R}_i)$ is a tagged-partition structure allowing subdivisions, witnessed by an upwards-directed family $\mathcal{C}_i \subseteq \mathcal{P}X_i$, where X_i is a topological space, Δ_i is the set of all neighbourhood gauges on X_i, and $\mathfrak{R}_i$ is the simple residual structure complementary to $\mathcal{C}_i$. Set $X = \prod_{i \in I} X_i$ and let Δ be the set of neighbourhood gauges on X; let $\mathcal{C}$ be $\{\prod_{i \in I} C_i : C_i \in \mathcal{C}_i$ for each $i \in I\}$, and $\mathfrak{R}$ the simple residual structure based on $\mathcal{C}$; and let T be the straightforward set of tagged partitions generated by $\{(\langle x_i \rangle_{i \in I}, \prod_{i \in I} C_i) : \{(x_i, C_i)\} \in T_i$ for every $i \in I\}$. Show that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

(c) Give an example to show that, in 481Xi, $(X, T, \Delta, \mathfrak{R})$ can be a tagged-partition structure allowing subdivisions, while $(H, T_H, \Delta_H, \mathfrak{R}_H)$ is not.

481 Notes and comments In the examples above I have tried to give an idea of the potential versatility of the ideas here. Further examples may be found in HENSTOCK 91. The goal of 481A-481F is the formula $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ (481C, 481E); the elaborate notation reflects the variety of the applications. One of these is a one-step definition of the ordinary integral (481Xf). In §483 I will show that the Henstock integral (481J-481K) properly extends the Lebesgue integral on $\mathbb{R}$. In 481Xc I show how adjusting $\mathfrak{R}$ can change the class of integrable functions; in 481Xd I show how a similar effect can sometimes be achieved by adding a point at infinity and adjusting T and Δ. As will become apparent in later sections, one of the great strengths of gauge integrals is their ability to incorporate special limiting processes. Another is the fact that we don't need to assume that the functionals ν are countably additive; see 481Xd. In the formulae of this section, I don't even ask for finite additivity; but of course the functional I_ν is likely to have a rather small domain if ν behaves too erratically.

'Gauges', as I describe them here, have moved rather briskly forward from the metric gauges δ_h (481Eb), which have sufficed for most of the gauge integrals so far described. But the generalization affects only the notation, and makes it clear why so much of the theory of the ordinary Henstock integral applies equally well to the 'approximately continuous Henstock integral' (481Q), for instance. You will observe that the sets Q of 481Ba are 'gauges' in the wide sense used here. But (as the examples of this section show clearly) we generally use them in a different way.

In ordinary measure theory, we have a fairly straightforward theory of subspaces (§214) and a rather deeper theory of product spaces (chap. 25). For gauge integrals, there are significant difficulties in the theory of subspaces, some of which will appear in the next section (see 482G-482H). For closed subspaces, something can be done, as in 481Xi; but the procedure suggested there may lose some essential element of the original tagged-partition structure (481Yc). For products of gauge integrals, we do have a reasonably satisfying version of Fubini's theorem (482M); I offer 481O and 481P as alternative approaches. However, the example of the Pfeffer integral (§484) shows that other constructions may be more effective tools for geometric measure theory.

You will note the concentration on 'neighbourhood gauges' (481Eb) in the work above. This is partly because they are 'full' in the sense of 481Ec. As will appear repeatedly in the next section, this flexibility in constructing gauges is just what one needs when proving that functions are gauge-integrable.

While I have used such phrases as 'Henstock integral', 'symmetric Riemann-complete integral' above, I have not in fact discussed integrals here, except in the exercises; in most of the examples in 481I-481Q there is no mention of any functional ν from which a gauge integral I_ν can be defined. The essence of the method is that we can set up a tagged-partition structure quite independently of any set function, and it turns out that the properties of a gauge integral depend more on this structure than on the measure involved.

482 General theory

I turn now to results which can be applied to a wide variety of tagged-partition structures. The first step is a 'Saks-Henstock' lemma (482B), a fundamental property of tagged-partition structures allowing subdivisions. In order to relate gauge integrals to the ordinary integrals treated elsewhere in this treatise, we need to know when gauge-integrable functions are measurable (482E) and when integrable functions are gauge-integrable (482F). There are significant difficulties when we come to interpret gauge integrals over subspaces, but I give a partial result in 482G. 482I, 482K and 482M are gauge-integral versions of the Fundamental Theorem of Calculus, B.Levi's theorem and Fubini's theorem, while 482H is a limit theorem of a new kind, corresponding to classical improper integrals.

Henstock's integral (481J-481K) remains the most important example and the natural test case for the ideas here; I will give the details in the next section, and you may wish to take the two sections in parallel.

482A Lemma Suppose that $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions (481G), witnessed by $\mathcal{C}\subseteq\mathcal{P}X$.

(a) Whenever $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ and E belongs to the subalgebra of $\mathcal{P}X$ generated by $\mathcal{C}$, there is a δ-fine $\boldsymbol{s}\in T$ such that $W_{\boldsymbol{s}}\subseteq E$ and $E\setminus W_{\boldsymbol{s}}\in\mathcal{R}$.

(b) Whenever $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ and $\boldsymbol{t}\in T$ is δ-fine, there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}'\in T$ including $\boldsymbol{t}$.

(c) Suppose that $f:X\to\mathbb{R}$, $\nu:\mathcal{C}\to\mathbb{R}$, $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ and $\epsilon\ge0$ are such that $|S_{\boldsymbol{t}}(f,\nu)-S_{\boldsymbol{t}'}(f,\nu)|\le\epsilon$ whenever $\boldsymbol{t},\boldsymbol{t}'\in T$ are δ-fine and $\mathcal{R}$-filling. Then

(i) $|S_{\boldsymbol{t}}(f,\nu)-S_{\boldsymbol{t}'}(f,\nu)|\le\epsilon$ whenever $\boldsymbol{t},\boldsymbol{t}'\in T$ are δ-fine and $W_{\boldsymbol{t}}=W_{\boldsymbol{t}'}$;

(ii) whenever $\boldsymbol{t}\in T$ is δ-fine, and $\delta'\in\Delta$, there is a δ'-fine $\boldsymbol{s}\in T$ such that $W_{\boldsymbol{s}}\subseteq W_{\boldsymbol{t}}$ and $|S_{\boldsymbol{s}}(f,\nu)-S_{\boldsymbol{t}}(f,\nu)|\le\epsilon$.

(d) Suppose that $f:X\to\mathbb{R}$ and $\nu:\mathcal{C}\to\mathbb{R}$ are such that $I_\nu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is defined, where $\mathcal{F}(T,\Delta,\mathfrak{R})$ is the filter described in 481F. Then for any $\epsilon>0$ there is a $\delta\in\Delta$ such that $S_{\boldsymbol{t}}(f,\nu)\le I_\nu(f)+\epsilon$ for every δ-fine $\boldsymbol{t}\in T$.

(e) Suppose that $f:X\to\mathbb{R}$ and $\nu:\mathcal{C}\to\mathbb{R}$ are such that $I_\nu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is defined. Then for any $\epsilon>0$ there is a $\delta\in\Delta$ such that $|S_{\boldsymbol{t}}(f,\nu)|\le\epsilon$ whenever $\boldsymbol{t}\in T$ is δ-fine and $W_{\boldsymbol{t}}=\emptyset$.

proof (a) By 481He, there is a non-increasing sequence $\langle\mathcal{R}_k\rangle_{k\in\mathbb{N}}$ in $\mathfrak{R}$ such that $\bigcup_{i\le k}A_i\in\mathcal{R}$ whenever $A_i\in\mathcal{R}_i$ for every $i\le k$ and $\langle A_i\rangle_{i\le k}$ is disjoint. Let $\mathcal{C}_0$ be a finite subset of $\mathcal{C}$ such that E belongs to the subalgebra of $\mathcal{P}X$ generated by $\mathcal{C}_0$, and let $\mathcal{C}_1\supseteq\mathcal{C}_0$ be a finite subset of $\mathcal{C}$ such that $X\setminus W\in\mathcal{R}_0$, where $W=\bigcup\mathcal{C}_1$ (481G(v)). Then either $E\subseteq W$ or $E\supseteq X\setminus W$. In either case, $E\cap W$ belongs to the ring generated by $\mathcal{C}$, so is expressible as $\bigcup_{i<n}C_i$ where $\langle C_i\rangle_{i<n}$ is a disjoint family in $\mathcal{C}$ (481Hd).

For each $i < n$, let $\boldsymbol{s}_i$ be a δ-fine member of T such that $W_{\boldsymbol{s}_i} \subseteq C_i$ and $C_i \setminus W_{\boldsymbol{s}_i} \in \mathcal{R}_{i+1}$ (481G(vii)). Set $\boldsymbol{s} = \bigcup_{i<n} \boldsymbol{s}_i$. Because $\langle W_{\boldsymbol{s}_i} \rangle_{i<n}$ is disjoint and T is a straightforward set of tagged partitions, $\boldsymbol{s} \in T$; $\boldsymbol{s}$ is δ-fine because every $\boldsymbol{s}_i$ is; $W_{\boldsymbol{s}} = \bigcup_{i<n} W_{\boldsymbol{s}_i}$ is included in E; and $E \setminus W_{\boldsymbol{s}}$ is either $\bigcup_{i<n} C_i \setminus W_{\boldsymbol{s}_i}$ or $(X \setminus W) \cup \bigcup_{i<n}(C_i \setminus W_{\boldsymbol{s}_i})$, and in either case belongs to $\mathcal{R}$, by the choice of $\langle \mathcal{R}_n \rangle_{n\in\mathbb{N}}$.

(b) Set $E = X \setminus W_{\boldsymbol{t}}$. By (a), there is a δ-fine $\boldsymbol{s} \in T$ such that $W_{\boldsymbol{s}} \subseteq E$ and $E \setminus W_{\boldsymbol{s}} \in \mathcal{R}$. Set $\boldsymbol{t}' = \boldsymbol{t} \cup \boldsymbol{s}$; this works.

(c)(i) As in (b), there is a δ-fine $\boldsymbol{s} \in T$ such that $W_{\boldsymbol{s}} \cap W_{\boldsymbol{t}} = \emptyset$ and $\boldsymbol{t} \cup \boldsymbol{s}$ is $\mathcal{R}$-filling. Now $W_{\boldsymbol{t}\cup\boldsymbol{s}} = W_{\boldsymbol{t}'\cup\boldsymbol{s}}$, so $\boldsymbol{t}' \cup \boldsymbol{s}$ also is $\mathcal{R}$-filling, and

$$|S_{\boldsymbol{t}}(f,\nu) - S_{\boldsymbol{t}'}(f,\nu)| = |S_{\boldsymbol{t}\cup\boldsymbol{s}}(f,\nu) - S_{\boldsymbol{t}'\cup\boldsymbol{s}}(f,\nu)| \le \epsilon.$$

(ii) Replacing δ' by a lower bound of $\{\delta, \delta'\}$ in Δ if necessary, we may suppose that $\delta' \subseteq \delta$. Enumerate $\boldsymbol{t}$ as $\langle (x_i, C_i) \rangle_{i<n}$. Let $\langle \mathcal{R}_k \rangle_{k\in\mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that $\bigcup_{i\le k} A_i \in \mathcal{R}$ whenever $\langle A_i \rangle_{i\le k}$ is disjoint and $A_i \in \mathcal{R}_i$ for every $i \le k$. For each $i < n$, let $\boldsymbol{s}_i$ be a δ'-fine member of T such that $W_{\boldsymbol{s}_i} \subseteq C_i$ and $C_i \setminus W_{\boldsymbol{s}_i} \in \mathcal{R}_{i+1}$, and set $\boldsymbol{s} = \bigcup_{i<n} \boldsymbol{s}_i$, so that $\boldsymbol{s} \in T$ is δ'-fine. By (a), there is a δ-fine $\boldsymbol{u} \in T$ such that $W_{\boldsymbol{u}} \cap W_{\boldsymbol{t}} = \emptyset$ and $X \setminus (W_{\boldsymbol{t}} \cup W_{\boldsymbol{u}}) \in \mathcal{R}_0$. Set $\boldsymbol{t}' = \boldsymbol{t} \cup \boldsymbol{u}$, $\boldsymbol{s}' = \boldsymbol{s} \cup \boldsymbol{u}$; then $\boldsymbol{t}'$ and $\boldsymbol{s}'$ are δ-fine and $\mathcal{R}$-filling, because

$$X \setminus W_{\boldsymbol{s}'} = (X \setminus (W_{\boldsymbol{t}} \cup W_{\boldsymbol{u}})) \cup \bigcup_{i<n}(C_i \setminus W_{\boldsymbol{s}_i}) \in \mathcal{R},$$

by the choice of $\langle \mathcal{R}_k \rangle_{k\in\mathbb{N}}$. So

$$|S_{\boldsymbol{t}}(f,\nu) - S_{\boldsymbol{s}}(f,\nu)| = |S_{\boldsymbol{t}'}(f,\nu) - S_{\boldsymbol{s}'}(f,\nu)| \le \epsilon,$$

as required.

(d) There are $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that $|S_{\boldsymbol{t}}(f,\nu) - I_\nu(f)| \le \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling. If $\boldsymbol{t} \in T$ is an arbitrary δ-fine tagged partition, there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}' \supseteq \boldsymbol{t}$, by (b), so

$$S_{\boldsymbol{t}}(f,\nu) \le S_{\boldsymbol{t}'}(f,\nu) \le I_\nu(f) + \epsilon,$$

as claimed.

(e) Let $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ be such that $|S_{\boldsymbol{s}}(f,\nu) - I_\nu(f)| \le \frac{1}{2}\epsilon$ whenever $\boldsymbol{s} \in T$ is δ-fine and $\mathcal{R}$-filling. If $\boldsymbol{t} \in T$ is δ-fine and $W_{\boldsymbol{t}} = \emptyset$, take any δ-fine $\mathcal{R}$-filling $\boldsymbol{s} \in T$, and consider $\boldsymbol{s}' = \boldsymbol{s} \setminus \boldsymbol{t}$, $\boldsymbol{s}'' = \boldsymbol{s} \cup \boldsymbol{t}$. Because $W_{\boldsymbol{s}} \cap W_{\boldsymbol{t}} = \emptyset$, both $\boldsymbol{s}'$ and $\boldsymbol{s}''$ belong to T; both are δ-fine; and because $W_{\boldsymbol{s}'} = W_{\boldsymbol{s}''} = W_{\boldsymbol{s}}$, both are $\mathcal{R}$-filling. So

$$|S_{\boldsymbol{t}}(f,\nu)| = |S_{\boldsymbol{s}''}(f,\nu) - S_{\boldsymbol{s}'}(f,\nu)| \le |S_{\boldsymbol{s}''}(f,\nu) - I_\nu(f)| + |S_{\boldsymbol{s}'}(f,\nu) - I_\nu(f)| \le \epsilon,$$

as required.

482B Saks-Henstock Lemma Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $f : X \to \mathbb{R}$, $\nu : \mathcal{C} \to \mathbb{R}$ functions such that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is defined. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. Then there is a unique additive functional $F : \mathcal{E} \to \mathbb{R}$ such that for every $\epsilon > 0$ there are $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that

(α) $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\nu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T$,

(β) $|F(E)| \le \epsilon$ whenever $E \in \mathcal{E} \cap \mathcal{R}$.

Moreover, $F(X) = I_\nu(f)$.

proof (a) For $E \in \mathcal{E}$, write T_E for the set of those $\boldsymbol{t} \in T$ such that, for every $(x, C) \in \boldsymbol{t}$, either $C \subseteq E$ or $C \cap E = \emptyset$. For any $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and finite $\mathcal{D} \subseteq \mathcal{E}$ there is a δ-fine $\boldsymbol{t} \in \bigcap_{E\in\mathcal{D}} T_E$ such that $E \setminus W_{\boldsymbol{t}} \in \mathcal{R}$ for every $E \in \mathcal{D}$. **P** Let $\langle \mathcal{R}_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that whenever $A_i \in \mathcal{R}_i$ for $i \le n$ and $\langle A_i \rangle_{i\le n}$ is disjoint then $\bigcup_{i\le n} A_i \in \mathcal{R}$ (481He again). Let $\mathcal{E}_0$ be the subalgebra of $\mathcal{E}$ generated by $\mathcal{D}$, and enumerate the atoms of $\mathcal{E}_0$ as $\langle E_i \rangle_{i<n}$. By 482Aa, there is for each $i < n$ a δ-fine $\boldsymbol{s}_i \in T$ such that $W_{\boldsymbol{s}_i} \subseteq E_i$ and $E_i \setminus W_{\boldsymbol{s}_i} \in \mathcal{R}_i$. Set $\boldsymbol{t} = \bigcup_{i<n} \boldsymbol{s}_i$. If $E \in \mathcal{D}$ then $E = \bigcup_{i\in J} E_i$ for some $J \subseteq n$. For any $(x, C) \in \boldsymbol{t}$, there is some $i < n$ such that $C \subseteq E_i$, so that $C \subseteq E$ if $i \in J$, $C \cap E = \emptyset$ otherwise; thus $\boldsymbol{t} \in T_E$. Moreover, $E \setminus W_{\boldsymbol{t}} = \bigcup_{i\in J}(E_i \setminus W_{\boldsymbol{s}_i})$ belongs to $\mathcal{R}$. **Q**

(b) We therefore have a filter $\mathcal{F}^*$ on T generated by sets of the form

$$T_{E\delta\mathcal{R}} = \{\boldsymbol{t} : \boldsymbol{t} \in T_E \text{ is } \delta\text{-fine, } E \setminus W_{\boldsymbol{t}} \in \mathcal{R}\}$$

as δ runs over Δ, $\mathcal{R}$ runs over $\mathfrak{R}$ and E runs over $\mathcal{E}$. For $\boldsymbol{t} \in T$, $E \subseteq X$ set $\boldsymbol{t}_E = \{(x, C) : (x, C) \in \boldsymbol{t}, C \subseteq E\}$. Now $F(E) = \lim_{\boldsymbol{t}\to\mathcal{F}^*} S_{\boldsymbol{t}_E}(f,\nu)$ is defined for every $E \in \mathcal{E}$. **P** For any $\epsilon > 0$, there are $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ such that $|I_\nu(f) - S_{\boldsymbol{t}}(f,\nu)| \le \epsilon$ for every δ-fine $\mathcal{R}$-filling $\boldsymbol{t} \in T$. Let $\mathcal{R}' \in \mathfrak{R}$ be such that $A \cup B \in \mathcal{R}$ for all disjoint A, $B \in \mathcal{R}'$. If $\boldsymbol{t}$, $\boldsymbol{t}'$ belong to $T_{E,\delta,\mathcal{R}'} = T_{X\setminus E,\delta,\mathcal{R}'}$, then set

$$\boldsymbol{s} = \{(x,C) : (x,C) \in \boldsymbol{t}',\, C \subseteq E\} \cup \{(x,C) : (x,C) \in \boldsymbol{t},\, C \cap E = \emptyset\}.$$

Then $\boldsymbol{s} \in T_E$ is δ-fine, and also $E \setminus W_{\boldsymbol{s}} = E \setminus W_{\boldsymbol{t}'}$, $(X \setminus E) \setminus W_{\boldsymbol{s}} = (X \setminus E) \setminus W_{\boldsymbol{t}}$ both belong to $\mathcal{R}'$; so their union $X \setminus W_{\boldsymbol{s}}$ belongs to $\mathcal{R}$, and $\boldsymbol{s}$ is $\mathcal{R}$-filling. Accordingly

$$\begin{aligned}|S_{\boldsymbol{t}_E}(f,\nu) - S_{\boldsymbol{t}'_E}(f,\nu)| &= |S_{\boldsymbol{t}}(f,\nu) - S_{\boldsymbol{s}}(f,\nu)| \\ &\le |S_{\boldsymbol{t}}(f,\nu) - I_\nu(f)| + |I_\nu(f) - S_{\boldsymbol{s}}(f,\nu)| \le 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, this is enough to show that $\liminf_{\boldsymbol{t}\to\mathcal{F}^*} S_{\boldsymbol{t}_E}(f,\nu) = \limsup_{\boldsymbol{t}\to\mathcal{F}^*} S_{\boldsymbol{t}_E}(f,\nu)$, so that the limit $\lim_{\boldsymbol{t}\to\mathcal{F}^*} S_{\boldsymbol{t}_E}(f,\nu)$ is defined (2A3Sf). **Q**

(c) $F(\emptyset) = 0$. **P** Let $\epsilon > 0$. By 482Ae, there is a $\delta \in \Delta$ such that $|S_{\boldsymbol{t}}(f,\nu)| \le \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $W_{\boldsymbol{t}} = \emptyset$. Since $\{\boldsymbol{t} : \boldsymbol{t}$ is δ-fine$\}$ belongs to $\mathcal{F}^*$,

$$|F(\emptyset)| = |\lim_{\boldsymbol{t}\to\mathcal{F}^*} S_{\boldsymbol{t}_\emptyset}(f,\nu)| \le \epsilon;$$

as ϵ is arbitrary, $F(\emptyset) = 0$. **Q**

If E, $E' \in \mathcal{E}$, then

$$S_{\boldsymbol{t}_{E\cup E'}}(f,\nu) + S_{\boldsymbol{t}_{E\cap E'}}(f,\nu) = S_{\boldsymbol{t}_E}(f,\nu) + S_{\boldsymbol{t}_{E'}}(f,\nu)$$

for every $\boldsymbol{t} \in T_E \cap T_{E'}$; as $T_E \cap T_{E'}$ belongs to $\mathcal{F}^*$,

$$F(E \cup E') + F(E \cap E') = F(E) + F(E').$$

Since $F(\emptyset) = 0$, $F(E \cup E') = F(E) + F(E')$ whenever $E \cap E' = \emptyset$, and F is additive.

(d) Now suppose that $\epsilon > 0$. Let $\delta \in \Delta$, $\mathcal{R}^* \in \mathfrak{R}$ be such that $|I_\nu(f) - S_{\boldsymbol{t}}(f,\nu)| \le \frac{1}{4}\epsilon$ for every δ-fine, $\mathcal{R}^*$-filling $\boldsymbol{t} \in T$. Let $\mathcal{R} \in \mathfrak{R}$ be such that $A \cup B \in \mathcal{R}^*$ for all disjoint A, $B \in \mathcal{R}$.

(i) If $\boldsymbol{t} \in T$ is δ-fine, then $|F(W_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f,\nu)| \le \frac{1}{2}\epsilon$. **P** For any $\eta > 0$, there is a δ-fine $\boldsymbol{s} \in T$ such that

$|I_\nu(f) - S_{\boldsymbol{s}}(f,\nu)| \le \eta$,

for every $(x,C) \in \boldsymbol{s}$, either $C \subseteq W_{\boldsymbol{t}}$ or $C \cap W_{\boldsymbol{t}} = \emptyset$,

$(X \setminus W_{\boldsymbol{t}}) \setminus W_{\boldsymbol{s}} \in \mathcal{R}$, $W_{\boldsymbol{t}} \setminus W_{\boldsymbol{s}} \in \mathcal{R}$,

$|F(W_{\boldsymbol{t}}) - \sum_{(x,C)\in\boldsymbol{s},C\subseteq W_{\boldsymbol{t}}} f(x)\nu C| \le \eta$

because the set of $\boldsymbol{s}$ with these properties belongs to $\mathcal{F}^*$. Now, setting $\boldsymbol{s}_1 = \{(x,C) : (x,C) \in \boldsymbol{s},\, C \subseteq W_{\boldsymbol{t}}\}$ and $\boldsymbol{t}' = \boldsymbol{t} \cup (\boldsymbol{s} \setminus \boldsymbol{s}_1)$, $\boldsymbol{t}'$ is δ-fine and $\mathcal{R}^*$-filling, like $\boldsymbol{s}$, so

$$\begin{aligned}|F(W_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f,\nu)| &\le |F(W_{\boldsymbol{t}}) - S_{\boldsymbol{s}_1}(f,\nu)| + |S_{\boldsymbol{s}_1}(f,\nu) - S_{\boldsymbol{t}}(f,\nu)| \\ &\le \eta + |S_{\boldsymbol{s}}(f,\nu) - S_{\boldsymbol{t}'}(f,\nu)| \\ &\le \eta + |S_{\boldsymbol{s}}(f,\nu) - I_\nu(f)| + |I_\nu(f) - S_{\boldsymbol{t}'}(f,\nu)| \le \eta + \frac{1}{2}\epsilon.\end{aligned}$$

As η is arbitrary we have the result. **Q**

(ii) So if $\boldsymbol{t} \in T$ is δ-fine, $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\nu C| \le \epsilon$. **P** Set $\boldsymbol{t}' = \{(x,C) : (x,C) \in \boldsymbol{t},\, F(C) \le f(x)\nu C\}$, $\boldsymbol{t}'' = \boldsymbol{t} \setminus \boldsymbol{t}''$. Then both $\boldsymbol{t}'$ and $\boldsymbol{t}''$ are δ-fine, so

$$\begin{aligned}&\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\nu C| \\ &\qquad = |\sum_{(x,C)\in\boldsymbol{t}'} F(C) - f(x)\nu C| + |\sum_{(x,C)\in\boldsymbol{t}''} F(C) - f(x)\nu C| \le \epsilon. \ \mathbf{Q}\end{aligned}$$

(iii) If $E \in \mathcal{E} \cap \mathcal{R}$, then $|F(E)| \le \epsilon$. **P** Let $\mathcal{R}' \in \mathfrak{R}$ be such that $A \cup B \in \mathcal{R}$ whenever A, $B \in \mathcal{R}'$ are disjoint. Let $\boldsymbol{t}$ be such that

$\boldsymbol{t} \in T_E$ is δ-fine,

$E \setminus W_{\boldsymbol{t}}$ and $(X \setminus E) \setminus W_{\boldsymbol{t}}$ both belong to $\mathcal{R}'$,

$|F(E) - S_{\boldsymbol{t}_E}(f,\nu)| \le \frac{1}{2}\epsilon$;

once again, the set of candidates belongs to $\mathcal{F}^*$, so is not empty. Then $\boldsymbol{t}$ and $\boldsymbol{t}_{X\setminus E}$ are both $\mathcal{R}^*$-filling and δ-fine, so

$$|F(E)| \le \frac{1}{2}\epsilon + |S_{\boldsymbol{t}_E}(f,\nu)| = \frac{1}{2}\epsilon + |S_{\boldsymbol{t}}(f,\nu) - S_{\boldsymbol{t}_{X\setminus E}}(f,\nu)| \le \epsilon. \ \mathbf{Q}$$

As ϵ is arbitrary, this shows that F has all the required properties.

(e) I have still to show that F is unique. Suppose that $F' : \mathcal{E} \to \mathbb{R}$ is another functional with the same properties, and take $E \in \mathcal{E}$ and $\epsilon > 0$. Then there are δ, $\delta' \in \Delta$ and $\mathcal{R}$, $\mathcal{R}' \in \mathfrak{R}$ such that

$\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\nu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T$,

$\sum_{(x,C)\in\boldsymbol{t}} |F'(C) - f(x)\nu C| \le \epsilon$ for every δ'-fine $\boldsymbol{t} \in T$,

$|F(R)| \le \epsilon$ whenever $R \in \mathcal{E} \cap \mathcal{R}$,

$|F'(R)| \le \epsilon$ whenever $R \in \mathcal{E} \cap \mathcal{R}'$.

Now taking $\delta'' \in \Delta$ such that $\delta'' \subseteq \delta \cap \delta'$, and $\mathcal{R}'' \in \mathfrak{R}$ such that $\mathcal{R}'' \subseteq \mathcal{R} \cap \mathcal{R}'$, there is a δ''-fine $\boldsymbol{t} \in T$ such that $E' = W_{\boldsymbol{t}}$ is included in E and $E \setminus E' \in \mathcal{R}''$. In this case

$$|F(E) - F'(E)| \le |F(E \setminus E')| + \sum_{(x,C)\in\boldsymbol{t}} |F(C) - F'(C)| + |F'(E \setminus E')|$$

(because F and F' are both additive)

$$\le 2\epsilon + \sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\nu C| + \sum_{(x,C)\in\boldsymbol{t}} |F'(C) - f(x)\nu C| \le 4\epsilon.$$

As ϵ and E are arbitrary, $F = F'$, as required.

(f) Finally, to calculate $F(X)$, take any $\epsilon > 0$. Let $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ be such that $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\nu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T$ and $|F(E)| \le \epsilon$ whenever $E \in \mathcal{E} \cap \mathcal{R}$. Let $\boldsymbol{t}$ be any δ-fine $\mathcal{R}$-filling member of T such that $|S_{\boldsymbol{t}}(f,\nu) - I_\nu(f)| \le \epsilon$. Then, because F is additive,

$$|F(X) - I_\nu(f)| \le |F(X) - F(W_{\boldsymbol{t}})| + |\sum_{(x,C)\in\boldsymbol{t}} F(C) - f(x)\nu C| + |S_{\boldsymbol{t}}(f,\nu) - I_\nu(f)|$$

$$\le 3\epsilon.$$

As ϵ is arbitrary, $F(X) = I_\nu(f)$.

482C Definition In the context of 482B, I will call the function F the **Saks-Henstock indefinite integral** of f; of course it depends on the whole structure $(X, T, \Delta, \mathfrak{R}, \mathcal{C}, f, \nu)$ and not just on (X, f, ν). You should *not* take it for granted that $F(E) = I_\nu(f \times \chi E)$ (482Ya); but see 482G.

482D The Saks-Henstock lemma characterizes the gauge integral, as follows.

Theorem Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\nu : \mathcal{C} \to \mathbb{R}$ any function. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. If $f : X \to \mathbb{R}$ is any function, then the following are equiveridical:

(i) $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is defined in $\mathbb{R}$;

(ii) there is an additive functional $F : \mathcal{E} \to \mathbb{R}$ such that

(α) for every $\epsilon > 0$ there is a $\delta \in \Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\mu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T$,

(β) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \le \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}$;

(iii) there is an additive functional $F : \mathcal{E} \to \mathbb{R}$ such that

(α) for every $\epsilon > 0$ there is a $\delta \in \Delta$ such that $|F(W_{\boldsymbol{t}}) - \sum_{(x,C)\in\boldsymbol{t}} f(x)\mu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T$,

(β) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \le \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}$.

In this case, $F(X) = I_\nu(f)$.

proof (i)$\Rightarrow$(ii) is just 482B above, and (ii)$\Rightarrow$(iii) is elementary, because $F(W_{\boldsymbol{t}}) = \sum_{(x,C)\in\boldsymbol{t}} F(C)$ whenever $F : \mathcal{E} \to \mathbb{R}$ is additive and $\boldsymbol{t} \in T$; so let us assume (iii) and seek to prove (i). Given $\epsilon > 0$, take $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that (α) and (β) of (iii) are satisfied. Let $\boldsymbol{t} \in T$ be δ-fine and $\mathcal{R}$-filling. Then

$$|F(X) - S_{\boldsymbol{t}}(f,\mu)| \le |F(X \setminus W_{\boldsymbol{t}})| + |F(W_{\boldsymbol{t}}) - \textstyle\sum_{(x,C)\in\boldsymbol{t}} f(x)\mu C| \le 2\epsilon.$$

As ϵ is arbitrary, $I_\nu(f)$ is defined and equal to $F(X)$.

482E Theorem Let (X, ρ) be a metric space and μ a complete locally determined measure on X with domain Σ. Let $\mathcal{C}$, Q, T, Δ and $\mathfrak{R}$ be such that

(i) $\mathcal{C} \subseteq \Sigma$ and μC is finite for every $C \in \mathcal{C}$;

(ii) $Q \subseteq X \times \mathcal{C}$, and for each $C \in \mathcal{C}$, $(x, C) \in Q$ for almost every $x \in C$;

(iii) T is the straightforward set of tagged partitions generated by Q;

(iv) Δ is a downwards-directed family of gauges on X containing all the uniform metric gauges;

(v) if $\delta \in \Delta$, there are a negligible set $F \subseteq X$ and a neighbourhood gauge δ_0 on X such that $\delta \supseteq \delta_0 \setminus (F \times \mathcal{P}X)$;

(vi) $\mathfrak{R}$ is a downwards-directed collection of families of subsets of X such that whenever $E \in \Sigma$, $\mu E < \infty$ and $\epsilon > 0$, there is an $\mathcal{R} \in \mathfrak{R}$ such that $\mu^*(E \cap R) \leq \epsilon$ for every $R \in \mathcal{R}$;

(vii) T is compatible with Δ and $\mathfrak{R}$.

Let $f : X \to \mathbb{R}$ be any function such that $I_\mu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$ is defined. Then f is Σ-measurable.

proof ? Suppose, if possible, otherwise.

Because μ is complete and locally determined, there are a measurable set E of non-zero finite measure and $\alpha < \beta$ in $\mathbb{R}$ such that

$$\mu^*\{x : x \in E,\ f(x) \leq \alpha\} = \mu^*\{x : x \in E,\ f(x) \geq \beta\} = \mu E$$

(413G). Let $\epsilon > 0$ be such that $(\beta - \alpha)(\mu E - 3\epsilon) > 2\epsilon$. Let $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ be such that $|S_{\boldsymbol{t}}(f, \mu) - I_\nu(f)| \leq \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling. By (v), there are a negligible set $F \subseteq X$ and a family $\langle G_x \rangle_{x \in X}$ of open sets such that $x \in G_x$ for every $x \in X$ and $\delta \supseteq \{(x, C) : x \in X \setminus F,\ C \subseteq G_x\}$. For $m \geq 1$, set

$$A_m = \{x : x \in E \setminus F,\ f(x) \leq \alpha,\ U_{1/m}(x) \subseteq G_x\},$$

writing $U_{1/m}(x)$ for $\{y : \rho(y, x) < \frac{1}{m}\}$,

$$B_m = \{x : x \in E \setminus F,\ f(x) \geq \beta,\ U_{1/m}(x) \subseteq G_x\}.$$

Then there is some $m \geq 1$ such that $\mu^* A_m \geq \mu E - \epsilon$ and $\mu^* B_m \geq \mu E - \epsilon$. By (iv), there is a $\delta' \in \Delta$ such that

$$\delta' \subseteq \delta \cap \{(x, C) : x \in C \subseteq U_{1/3m}(x)\}.$$

By (vi), there is an $\mathcal{R}' \in \mathfrak{R}$ such that $\mathcal{R}' \subseteq \mathcal{R}$ and $\mu^*(R \cap E) \leq \epsilon$ for every $R \in \mathcal{R}'$.

Let $\boldsymbol{t}$ be any δ'-fine $\mathcal{R}'$-filling member of T. Enumerate $\boldsymbol{t}$ as $\langle (x_i, C_i) \rangle_{i<n}$. Set

$$J = \{i : i < n,\ C_i \cap A_m \text{ is negligible}\}, \quad J' = \{i : i < n,\ C_i \cap B_m \text{ is negligible}\}.$$

Then

$$\mu(E \cap \bigcup_{i \in J} C_i) \leq \mu_*(E \setminus A_m) = \mu E - \mu^*(E \cap A_m) \leq \epsilon,$$

and similarly $\mu(E \cap \bigcup_{i \in J'} C_i) \leq \epsilon$. Also, because $X \setminus \bigcup_{i<n} C_i = X \setminus W_{\boldsymbol{t}}$ belongs to $\mathcal{R}'$, $\mu(E \setminus \bigcup_{i<n} C_i) \leq \epsilon$. So, setting $K = n \setminus (J \cup J')$, $\sum_{i \in K} \mu C_i \geq \mu E - 3\epsilon$.

For $i \in K$, $\mu^*(C_i \cap A_m) > 0$, while $\{x : x \in C_i,\ (x, C_i) \notin Q\}$ is negligible, by (ii), so we can find $x'_i \in C_i \cap A_m$ such that $(x'_i, C_i) \in Q$; similarly, there is an $x''_i \in C_i \cap B_m$ such that $(x''_i, C_i) \in Q$. For other $i < n$, set $x'_i = x''_i = x_i$. Now $\boldsymbol{s} = \{(x'_i, C_i) : i < n\}$ and $\boldsymbol{s}' = \{(x''_i, C_i) : i < n\}$ belong to T. Of course they are $\mathcal{R}'$-filling, therefore $\mathcal{R}$-filling, because $\boldsymbol{t}$ is. We also see that, because $(x_i, C_i) \in \delta'$, the diameter of C_i is at most $\frac{2}{3m}$ for each $i < n$, so that $C_i \subseteq G_{x'_i}$; as also $x'_i \in A_m \subseteq X \setminus F$, $(x'_i, C_i) \in \delta$, for each $i \in K$. But since surely $(x_i, C_i) \in \delta' \subseteq \delta$ for $i \in n \setminus K$, this means that $\boldsymbol{s}$ is δ-fine. Similarly, $\boldsymbol{s}'$ is δ-fine.

We must therefore have

$$|S_{\boldsymbol{s}'}(f, \mu) - S_{\boldsymbol{s}}(f, \mu)| \leq |S_{\boldsymbol{s}'}(f, \mu) - I_\mu(f)| + |S_{\boldsymbol{s}}(f, \mu) - I_\mu(f)| \leq 2\epsilon.$$

But

$$\begin{aligned} S_{\boldsymbol{s}'}(f, \mu) - S_{\boldsymbol{s}}(f, \mu) &= \sum_{i \in K} (f(x''_i) - f(x'_i)) \mu C_i \\ &\geq (\beta - \alpha) \sum_{i \in K} \mu C_i \geq (\beta - \alpha)(\mu E - 3\epsilon) > 2\epsilon \end{aligned}$$

by the choice of ϵ. **X**

So we have the result.

482F Proposition Let X, Σ, μ, $\mathfrak{T}$, T, Δ and $\mathfrak{R}$ be such that

(i) (X, Σ, μ) is a measure space;

(ii) $\mathfrak{T}$ is a topology on X such that μ is inner regular with respect to the closed sets and outer regular with respect to the open sets;

(iii) $T \subseteq [X \times \Sigma]^{<\omega}$ is a set of tagged partitions such that $C \cap C'$ is empty whenever (x, C), (x', C') are distinct members of any $\boldsymbol{t} \in T$;

(iv) Δ is a set of gauges on X containing every neighbourhood gauge on X;

(v) $\mathfrak{R}$ is a collection of families of subsets of X such that whenever $\mu E < \infty$ and $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $\mu^*(E \cap R) \leq \epsilon$ for every $R \in \mathcal{R}$;

(vi) T is compatible with Δ and $\mathfrak{R}$.

Then $I_\mu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$ is defined and equal to $\int f d\mu$ for every μ-integrable function $f : X \to \mathbb{R}$.

proof (a) It is worth noting straight away that we can replace (X, Σ, μ) by its completion $(X, \hat{\Sigma}, \hat{\mu})$. **P** We need to check that $\hat{\mu}$ is inner and outer regular. But inner regularity is 412Ha, and outer regularity is equally elementary: if $\hat{\mu}E < \gamma$, there is an $E' \in \Sigma$ such that $E \subseteq E'$ and $\mu E' = \hat{\mu}E$ (212C), and now there is an open set $G \in \Sigma$ such that $E' \subseteq G$ and $\mu G \leq \gamma$, so that $E \subseteq G$ and $\hat{\mu}G \leq \gamma$. Since we are not changing T or Δ or $\mathfrak{R}$, $I_{\hat{\mu}}(f) = I_\mu(f)$ if either is defined; while also $\int f d\mu = \int f d\hat{\mu}$ if either is defined, by 212Fb. **Q**

So let us suppose that μ is actually complete.

(b) In this case, f is measurable. Suppose to begin with that it is non-negative. Let $\epsilon > 0$. For $m \in \mathbb{Z}$, set $E_m = \{x : x \in X, (1 + \epsilon)^m \leq f(x) < (1 + \epsilon)^{m+1}\}$. Then E_m is measurable and has finite measure, so there is a measurable open set $G_m \supseteq E_m$ such that $(1 + \epsilon)^{m+1}\mu(G_m \setminus E_m) \leq 2^{-|m|}\epsilon$.

Take a set H_0 of finite measure and $\eta_0 > 0$ such that $\int_E f d\mu \leq \epsilon$ whenever $E \in \Sigma$ and $\mu(E \cap H_0) \leq 2\eta_0$ (225A); replacing H_0 by $\{x : x \in H_0, f(x) > 0\}$ if necessary, we may suppose that $H_0 \subseteq \bigcup_{m\in\mathbb{Z}} E_m$. Let $F \subseteq H_0$ be a closed set such that $\mu(H_0 \setminus F) \leq \eta_0$.

Define $\langle V_x\rangle_{x\in X}$ by setting $V_x = G_m$ if $m \in \mathbb{Z}$ and $x \in E_m$, $V_x = X \setminus F$ if $f(x) = 0$. Let $\delta \in \Delta$ be the corresponding neighbourhood gauge $\{(x, C) : x \in X, C \subseteq V_x\}$. Let $\mathcal{R} \in \mathfrak{R}$ be such that $\mu^*(R \cap H_0) \leq \eta_0$ for every $R \in \mathcal{R}$.

Suppose that $\boldsymbol{t}$ is any δ-fine $\mathcal{R}$-filling member of T. Enumerate $\boldsymbol{t}$ as $\langle(x_i, C_i)\rangle_{i<n}$. For each $m \in \mathbb{Z}$, set $J_m = \{i : i < n, x_i \in E_m\}$. Then $C_i \subseteq V_{x_i} \subseteq G_m$ for every $i \in J_m$, so

$$\begin{aligned} S_{\boldsymbol{t}}(f, \mu) &= \sum_{i<n} f(x_i)\mu C_i = \sum_{m\in\mathbb{Z}} \sum_{i\in J_m} f(x_i)\mu C_i \leq \sum_{m\in\mathbb{Z}} (1 + \epsilon)^{m+1}\mu G_m \\ &\leq (1 + \epsilon) \sum_{m\in\mathbb{Z}} (1 + \epsilon)^m \mu E_m + \sum_{m\in\mathbb{Z}} (1 + \epsilon)^{m+1}\mu(G_m \setminus E_m) \\ &\leq (1 + \epsilon) \int f d\mu + \sum_{m\in\mathbb{Z}} 2^{-|m|}\epsilon = (1 + \epsilon) \int f d\mu + 3\epsilon. \end{aligned}$$

On the other hand, set $F' = F \cap \bigcup_{i<n} C_i$. Because $X \setminus \bigcup_{i<n} C_i \in \mathcal{R}$, $\mu(H_0 \setminus F') \leq 2\eta_0$, and

$$\begin{aligned} S_{\boldsymbol{t}}(f, \mu) &\geq \sum_{m\in\mathbb{Z}} \sum_{i\in J_m} f(x_i)\mu(C_i \cap F) \\ &\geq \tfrac{1}{1+\epsilon} \sum_{m\in\mathbb{Z}} \sum_{i\in J_m} (1 + \epsilon)^{m+1}\mu(C_i \cap F) \\ &\geq \tfrac{1}{1+\epsilon} \int_{F'} f d\mu \geq \tfrac{1}{1+\epsilon}(\int f d\mu - \epsilon). \end{aligned}$$

What this means is that

$$\{\boldsymbol{t} : \tfrac{1}{1+\epsilon}(\textstyle\int f d\mu - \epsilon) \leq S_{\boldsymbol{t}}(f, \nu) \leq (1 + \epsilon)\textstyle\int f d\mu + 3\epsilon\}$$

belongs to $\mathcal{F}(T, \Delta, \mathfrak{R})$, for any $\epsilon > 0$. So $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$ is defined and equal to $\int f d\mu$.

(c) In general, f is expressible as $f^+ - f^-$ where f^+ and f^- are non-negative integrable functions, so

$$I_\nu(f) = I_\nu(f^+) - I_\nu(f^-) = \textstyle\int f d\mu$$

by 481Ca.

482G Proposition Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. Suppose that

(i) $\mathfrak{T}$ is a topology on X, and Δ is the set of neighbourhood gauges on X;

(ii) $\nu : \mathcal{C} \to \mathbb{R}$ is a function which is additive in the sense that if $C_0, \dots, C_n \in \mathcal{C}$ are disjoint and have union $C \in \mathcal{C}$, then $\nu C = \sum_{i=0}^{n} \nu C_i$;

(iii) whenever $E \in \mathcal{C}$ and $\epsilon > 0$, there are closed sets $F \subseteq E$, $F' \subseteq X \setminus E$ such that $\sum_{(x,C)\in \boldsymbol{t}} |\nu C| \le \epsilon$ whenever $\boldsymbol{t} \in T$ and $W_{\boldsymbol{t}} \cap (F \cup F') = \emptyset$;

(iv) for every $E \in \mathcal{C}$ and $x \in X$ there is a neighbourhood G of x such that if $C \in \mathcal{C}$, $C \subseteq G$ and $\{(x, C)\} \in T$, there is a finite partition $\mathcal{D}$ of C into members of $\mathcal{C}$, each either included in E or disjoint from E, such that $\{(x, D)\} \in T$ for every $D \in \mathcal{D}$;

(v) for every $C \in \mathcal{C}$ and $\mathcal{R} \in \mathfrak{R}$, there is an $\mathcal{R}' \in \mathfrak{R}$ such that $C \cap A \in \mathcal{R}$ whenever $A \in \mathcal{R}'$.

Let $f : X \to \mathbb{R}$ be a function such that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$, and $F : \mathcal{E} \to \mathbb{R}$ the Saks-Henstock indefinite integral of f. Then $I_\nu(f \times \chi E)$ is defined and equal to $F(E)$ for every $E \in \mathcal{E}$.

proof (a) Because both F and I_ν are additive, and $F(X) = I_\nu(f)$, and either E or its complement is a finite disjoint union of members of $\mathcal{C}$ (see 481Hd), it is enough to consider the case in which $E \in \mathcal{C}$.

(b) Let $\epsilon > 0$. For each $x \in X$ let G_x be an open set containing x such that whenever $C \in \mathcal{C}$, $C \subseteq G$ and $\{(x, C)\} \in T$, there is a finite partition $\mathcal{D}$ of C into members of $\mathcal{C}$ such that $\{(x, D)\} \in T$ for every $D \in \mathcal{D}$ and every member of $\mathcal{D}$ is either included in E or disjoint from E. For each $n \in \mathbb{N}$, let $F_n \subseteq E$, $F'_n \subseteq X \setminus E$ be closed sets such that $\sum_{(x,C)\in\boldsymbol{t}} |\nu C| \le \frac{2^{-n}\epsilon}{n+1}$ whenever $\boldsymbol{t} \in T$ and $W_{\boldsymbol{t}} \cap (F_n \cup F'_n) = \emptyset$; now define G'_x, for $x \in X$, by saying that

$$\begin{aligned} G'_x &= G_x \setminus F'_n \text{ if } x \in E \text{ and } n \le |f(x)| < n+1, \\ &= G_x \setminus F_n \text{ if } x \in X \setminus E \text{ and } n \le |f(x)| < n+1. \end{aligned}$$

Let $\delta_0 \in \Delta$ be the neighbourhood gauge defined by the family $\langle G'_x \rangle_{x\in X}$. Let $\delta \in \Delta$ and $\mathcal{R}_1 \in \mathfrak{R}$ be such that $\delta \subseteq \delta_0$, $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\nu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T$, and $|F(E)| \le \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}_1$. Let $\mathcal{R} \in \mathfrak{R}$ be such that $R \cap E \in \mathcal{R}_1$ whenever $R \in \mathcal{R}$.

(c) As in the proof of 482B, let T_E be the set of those $\boldsymbol{t} \in T$ such that, for each $(x, C) \in \boldsymbol{t}$, either $C \subseteq E$ or $C \cap E = \emptyset$. The key to the proof is the following fact: if $\boldsymbol{t} \in T$ is δ-fine, then there is a δ-fine $\boldsymbol{s} \in T_E$ such that $W_{\boldsymbol{s}} = W_{\boldsymbol{t}}$ and $S_{\boldsymbol{s}}(g, \nu) = S_{\boldsymbol{t}}(g, \nu)$ for every $g : X \to \mathbb{R}$. **P** For each $(x, C) \in \boldsymbol{t}$, we know that $C \subseteq G'_x \subseteq G_x$, because $\delta \subseteq \delta_0$. Let $\mathcal{D}_{(x,C)}$ be a finite partition of C into members of $\mathcal{C}$, each either included in E or disjoint from E, such that $\{(x, D)\} \in T$ for every $D \in \mathcal{D}_{(x,C)}$. Then $\boldsymbol{s} = \{(x, D) : (x, C) \in \boldsymbol{t},\ D \in \mathcal{D}_{(x,C)}\}$ belongs to T_E. Because δ is a neighbourhood gauge, $(x, D) \in \delta$ whenever $(x, C) \in \boldsymbol{t}$ and $D \in \mathcal{D}_{(x,C)}$, so $\boldsymbol{s}$ is δ-fine.

If $g : X \to \mathbb{R}$ is any function,

$$\begin{aligned} S_{\boldsymbol{s}}(g, \nu) &= \sum_{(x,C)\in\boldsymbol{t}} \sum_{D\in\mathcal{D}_{(x,C)}} g(x)\nu D \\ &= \sum_{(x,C)\in\boldsymbol{t}} g(x) \sum_{D\in\mathcal{D}_{(x,C)}} \nu D = \sum_{(x,C)\in\boldsymbol{t}} g(x)\nu C \end{aligned}$$

(because ν is additive)

$$= S_{\boldsymbol{t}}(g, \nu). \ \mathbf{Q}$$

(d) Now suppose that $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling. Let $\boldsymbol{s} \in T_E$ be as in (c), and set

$$\boldsymbol{s}^* = \{(x, D) : (x, D) \in \boldsymbol{s},\ x \in E,\ D \subseteq E\},$$

$$\boldsymbol{s}' = \{(x, D) : (x, D) \in \boldsymbol{s},\ x \notin E,\ D \subseteq E\},$$

$$\boldsymbol{s}'' = \{(x, D) : (x, D) \in \boldsymbol{s},\ x \in E,\ D \cap E = \emptyset\}.$$

Because $\boldsymbol{s} \in T_E$,

$$W_{\boldsymbol{s}^*\cup\boldsymbol{s}'} = E \cap W_{\boldsymbol{s}} = E \cap W_{\boldsymbol{t}}$$

and $E \setminus W_{\boldsymbol{s}^*\cup\boldsymbol{s}'} = E \setminus W_{\boldsymbol{t}}$ belongs to $\mathcal{R}_1$, by the choice of $\mathcal{R}$. Accordingly

$$|F(E) - S_{\boldsymbol{s}^*\cup\boldsymbol{s}'}(f, \nu)| \le |F(E) - F(W_{\boldsymbol{s}^*\cup\boldsymbol{s}'})| + |F(W_{\boldsymbol{s}^*\cup\boldsymbol{s}'}) - S_{\boldsymbol{s}^*\cup\boldsymbol{s}'}(f, \nu)| \le 2\epsilon$$

because $\boldsymbol{s}^* \cup \boldsymbol{s}' \subseteq \boldsymbol{s}$ is δ-fine.

For $n \in \mathbb{N}$ set

$$\boldsymbol{s}'_n = \{(x, D) : (x, D) \in \boldsymbol{s}',\, n \le |f(x)| < n+1\},$$

$$\boldsymbol{s}''_n = \{(x, D) : (x, D) \in \boldsymbol{s}'',\, n \le |f(x)| < n+1\}.$$

Then $W_{\boldsymbol{s}'_n} \subseteq E \setminus F_n$. **P** If $(x, D) \in \boldsymbol{s}'_n$, there is a $C \in \mathcal{C}$ such that $D \subseteq E \cap C$ and $(x, C) \in \boldsymbol{t}$, while $x \notin E$, so that $C \subseteq G'_x$ and $C \cap F_n = \emptyset$. **Q** Similarly, $W_{\boldsymbol{s}''_n} \subseteq (X \setminus E) \setminus F'_n$. Thus $W_{\boldsymbol{s}'_n \cup \boldsymbol{s}''_n}$ is disjoint from $F_n \cup F'_n$ and

$$\begin{aligned} |S_{\boldsymbol{s}'_n}(f, \nu) - S_{\boldsymbol{s}''_n}(f, \nu)| &= |\sum_{(x,D)\in \boldsymbol{s}'_n} f(x_i)\nu D - \sum_{(x,D)\in \boldsymbol{s}''_n} f(x_i)\nu D| \\ &\le \sum_{(x,D)\in \boldsymbol{s}'_n \cup \boldsymbol{s}''_n} |f(x_i)||\nu D| \\ &\le (n+1) \sum_{(x,D)\in \boldsymbol{s}'_n \cup \boldsymbol{s}''_n} |\nu D| \le 2^{-n}\epsilon \end{aligned}$$

by the choice of F_n and F'_n.

Consequently,

$$|F(E) - S_{\boldsymbol{t}}(f \times \chi E, \nu)| = |F(E) - S_{\boldsymbol{s}}(f \times \chi E, \nu)| = |F(E) - S_{\boldsymbol{s}^* \cup \boldsymbol{s}''}(f, \nu)|$$

(because $\boldsymbol{s}^* \cup \boldsymbol{s}'' = \{(x, D) : (x, D) \in \boldsymbol{s},\, x \in E\}$)

$$\le |F(E) - S_{\boldsymbol{s}^* \cup \boldsymbol{s}'}(f, \nu)| + |S_{\boldsymbol{s}'}(f, \nu) - S_{\boldsymbol{s}''}(f, \nu)|$$

(because $\boldsymbol{s}^*$, $\boldsymbol{s}'$ and $\boldsymbol{s}''$ are disjoint subsets of $\boldsymbol{s}$)

$$\le 2\epsilon + |\sum_{n=0}^{\infty} S_{\boldsymbol{s}'_n}(f, \nu) - \sum_{n=0}^{\infty} S_{\boldsymbol{s}''_n}(f, \nu)|$$

(the infinite sums are well-defined because $\boldsymbol{s}$ is finite, so that all but finitely many terms are zero)

$$\le 2\epsilon + \sum_{n=0}^{\infty} |S_{\boldsymbol{s}'_n}(f, \nu) - S_{\boldsymbol{s}''_n}(f, \nu)|$$

$$\le 2\epsilon + \sum_{n=0}^{\infty} 2^{-n}\epsilon = 4\epsilon.$$

As ϵ is arbitrary, $I_\nu(f \times \chi E)$ is defined and equal to $F(E)$, as required.

482H Proposition Suppose that X, $\mathfrak{T}$, $\mathcal{C}$, ν, T, Δ and $\mathfrak{R}$ satisfy the conditions (i)-(v) of 482G, and that $f : X \to \mathbb{R}$, $\langle H_n \rangle_{n\in\mathbb{N}}$, H and γ are such that

(vi) $\langle H_n \rangle_{n\in\mathbb{N}}$ is a sequence of open subsets of X with union H,

(vii) $I_\nu(f \times \chi H_n)$ is defined for every $n \in \mathbb{N}$,

(viii) $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} I_\nu(f \times \chi W_{\boldsymbol{t}\restriction H})$ is defined and equal to γ,

where $\boldsymbol{t}\restriction H = \{(x, C) : (x, C) \in \boldsymbol{t},\, x \in H\}$ for $\boldsymbol{t} \in T$. Then $I_\nu(f \times \chi H)$ is defined and equal to γ.

proof Let $\epsilon > 0$. For each $n \in \mathbb{N}$, let F_n be the Saks-Henstock indefinite integral of $f \times \chi H_n$. Let $\delta_n \in \Delta$ be such that

$$\begin{aligned} 2^{-n}\epsilon &\ge \sum_{(x,C)\in \boldsymbol{s}} |F_n(C) - (f \times \chi H_n)(x)\nu C| \\ &\ge |F_n(W_{\boldsymbol{s}}) - S_{\boldsymbol{s}}(f \times \chi H_n, \nu)| \end{aligned}$$

whenever $\boldsymbol{s} \in T$ is δ_n-fine. Set

$$\begin{aligned} \tilde{\delta} = \{(x, A) : x \in X \setminus H,\, A \subseteq X\} \\ \cup \bigcup_{n\in\mathbb{N}} \{(x, A) : x \in H_n \setminus \bigcup_{i<n} H_i,\, A \subseteq H_n,\, (x, A) \in \delta_n\}, \end{aligned}$$

so that $\tilde{\delta} \in \Delta$. Note that if $x \in H$ and $C \in \mathcal{C}$ and $(x, C) \in \tilde{\delta}$, then there is some $n \in \mathbb{N}$ such that $x \in H_n$ and $C \subseteq H_n$, so that

$$I_\nu(f \times \chi C) = I_\nu((f \times \chi H_n) \times \chi C) = F_n(C)$$

is defined, by 482G; this means that $I_\nu(f \times \chi W_{\boldsymbol{t} \restriction H})$ will be defined for every $\tilde{\delta}$-fine $\boldsymbol{t} \in T$. Let $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ be such that $|\gamma - I_\nu(f \times \chi W_{\boldsymbol{t} \restriction H})| \le \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling.

Let $\boldsymbol{t} \in T$ be $(\delta \cap \tilde{\delta})$-fine and $\mathcal{R}$-filling. For $n \in \mathbb{N}$, set $\boldsymbol{t}_n = \{(x, C) : (x, C) \in \boldsymbol{t},\ x \in H_n \setminus \bigcup_{i<n} H_i\}$. Then $\boldsymbol{t} \restriction H = \bigcup_{n \in \mathbb{N}} \boldsymbol{t}_n$, and $\boldsymbol{t}_n$ is δ_n-fine and $W_{\boldsymbol{t}_n} \subseteq H_n$ for every n. So

$$|\gamma - S_{\boldsymbol{t}}(f \times \chi H, \nu)| = |\gamma - \sum_{n=0}^{\infty} S_{\boldsymbol{t}_n}(f \times \chi H_n, \nu)|$$

$$\le |\gamma - I_\nu(f \times \chi W_{\boldsymbol{t} \restriction H})| + \sum_{n=0}^{\infty} |I_\nu(f \times \chi W_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f \times \chi H_n, \nu)|$$

(note that $\boldsymbol{t}_n = \emptyset$ for all but finitely many n, so that $I_\nu(f \times \chi W_{\boldsymbol{t} \restriction H}) = \sum_{n=0}^{\infty} I_\nu(f \times \chi W_{\boldsymbol{t}_n})$)

$$\le \epsilon + \sum_{n=0}^{\infty} |I_\nu(f \times \chi H_n \times \chi W_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f \times \chi H_n, \nu)|$$

(because $\boldsymbol{t}$ is δ-fine and $\mathcal{R}$-filling, while $W_{\boldsymbol{t}_n} \subseteq H_n$ for each n)

$$= \epsilon + \sum_{n=0}^{\infty} |F_n(W_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f \times \chi H_n, \nu)|$$

(by 482G)

$$\le \epsilon + \sum_{n=0}^{\infty} 2^{-n}\epsilon$$

(because every $\boldsymbol{t}_n$ is δ_n-fine)

$$= 3\epsilon.$$

As ϵ is arbitrary, $\gamma = I_\nu(f \times \chi H)$, as claimed.

Remark For applications of this result see 483Bd and 483N.

482I Integrating a derivative As will appear in the next two sections, the real strength of gauge integrals is in their power to integrate derivatives. I give an elementary general expression of this fact. In the formulae below, we can think of f as a 'derivative' of F if $\nu = \theta$ is strictly positive and additive and we rephrase condition (iii) as

'$\lim_{C \to \mathcal{G}_x} \frac{F(C)}{\nu C} = f(x)$',

where $\mathcal{G}_x$ is the filter on $\mathcal{C}$ generated by the sets $\{C : (x, C) \in \delta\}$ as δ runs over Δ.

Theorem Let X, $\mathcal{C} \subseteq \mathcal{P}X$, $\Delta \subseteq \mathcal{P}(X \times \mathcal{P}X)$, $\mathfrak{R} \subseteq \mathcal{P}\mathcal{P}X$, $T \subseteq [X \times \mathcal{C}]^{<\omega}$, $f : X \to \mathbb{R}$, $\nu : \mathcal{C} \to \mathbb{R}$, $F : \mathcal{C} \to \mathbb{R}$, $\theta : \mathcal{C} \to [0, 1]$ and $\gamma \in \mathbb{R}$ be such that

(i) T is a straightforward set of tagged partitions which is compatible with Δ and $\mathfrak{R}$,

(ii) Δ is a full set of gauges on X,

(iii) for every $x \in X$ and $\epsilon > 0$ there is a $\delta \in \Delta$ such that $|f(x)\nu C - F(C)| \le \epsilon\theta C$ whenever $(x, C) \in \delta$,

(iv) $\sum_{i=0}^{n} \theta C_i \le 1$ whenever $C_0, \dots, C_n \in \mathcal{C}$ are disjoint,

(v) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|\gamma - \sum_{C \in \mathcal{C}_0} F(C)| \le \epsilon$ whenever $\mathcal{C}_0 \subseteq \mathcal{C}$ is a finite disjoint set and $X \setminus \bigcup \mathcal{C}_0 \in \mathcal{R}$.

Then $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined and equal to γ.

proof Let $\epsilon > 0$. For each $x \in X$ let $\delta_x \in \Delta$ be such that $|f(x)\nu C - F(C)| \le \epsilon\theta C$ whenever $(x, C) \in \delta_x$. Because Δ is full, there is a $\delta \in \Delta$ such that $(x, C) \in \delta_x$ whenever $(x, C) \in \delta$. Let $\mathcal{R} \in \mathfrak{R}$ be as in (v). If $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling, then

$$|S_{\boldsymbol{t}}(f,\nu)-\gamma| \le |\gamma - \sum_{(x,C)\in\boldsymbol{t}} F(C)| + \sum_{(x,C)\in\boldsymbol{t}} |f(x)\nu C - F(C)|$$
$$\le \epsilon + \sum_{(x,C)\in\boldsymbol{t}} \epsilon\theta C$$

(because $X\setminus\bigcup_{(x,C)\in\boldsymbol{t}} C \in \mathcal{R}$, while $(x,C)\in\delta_x$ whenever $(x,C)\in\boldsymbol{t}$)

$$\le 2\epsilon$$

by condition (iv). As ϵ is arbitrary, we have the result.

482J Definition Let X be a set, $\mathcal{C}$ a family of subsets of X, $T\subseteq[X\times\mathcal{C}]^{<\omega}$ a family of tagged partitions, $\nu:\mathcal{C}\to[0,\infty[$ a function, and Δ a family of gauges on X. I will say that ν is **moderated** (with respect to T and Δ) if there are a $\delta\in\Delta$ and a function $h:X\to]0,\infty[$ such that $S_{\boldsymbol{t}}(h,\nu)\le 1$ for every δ-fine $\boldsymbol{t}\in T$.

482K B.Levi's theorem Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, such that Δ is countably full, and $\nu:\mathcal{C}\to[0,\infty[$ a function which is moderated with respect to T and Δ.

Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of functions from X to $\mathbb{R}$ with supremum $f:X\to\mathbb{R}$. If $\gamma=\lim_{n\to\infty} I_\nu(f_n)$ is defined in $\mathbb{R}$, then $I_\nu(f)$ is defined and equal to γ.

proof As in the proof of 123A, we may, replacing $\langle f_n\rangle_{n\in\mathbb{N}}$ by $\langle f_n-f_0\rangle_{n\in\mathbb{N}}$ if necessary, suppose that $f_n(x)\ge 0$ for every $n\in\mathbb{N}$ and $x\in X$.

(a) Take $\epsilon>0$. Then there is a $\delta\in\Delta$ such that $S_{\boldsymbol{t}}(f,\nu)\le\gamma+4\epsilon$ for every δ-fine $\boldsymbol{t}\in T$.

P Fix a strictly positive function $h:X\to]0,\infty[$ and a $\tilde\delta\in\Delta$ such that $S_{\boldsymbol{t}}(h,\nu)\le 1$ for every $\tilde\delta$-fine $\boldsymbol{t}\in T$. For each $n\in\mathbb{N}$ choose $\delta_n\in\Delta$ and $\mathcal{R}_n\in\mathfrak{R}$ such that $|I_\nu(f_n)-S_{\boldsymbol{t}}(f,\nu)|\le 2^{-n-1}\epsilon$ for every δ_n-fine $\mathcal{R}_n$-filling $\boldsymbol{t}\in T$. For each $x\in X$, take $r_x\in\mathbb{N}$ such that $f(x)\le f_{r_x}(x)+\epsilon h(x)$. Let $\delta\in\Delta$ be such that $(x,C)\in\tilde\delta\cap\delta_{r_x}$ for every $(x,C)\in\delta$.

Suppose that $\boldsymbol{t}\in T$ is δ-fine. Enumerate $\boldsymbol{t}$ as $\langle(x_i,C_i)\rangle_{i<n}$. Let $k\in\mathbb{N}$ be so large that $S_{\boldsymbol{t}}(f,\nu)\le S_{\boldsymbol{t}}(f_k,\nu)+\epsilon$ and $r_{x_i}\le k$ for every $i<n$. For $m\le k$, set $J_m=\{i:i<n,\ r_{x_i}=m\}$. For each $i\in J_m$, $(x_i,C_i)\in\delta_m$. By 482A(c-ii), there is a δ_k-fine $\boldsymbol{s}_m\in T$ such that $W_{\boldsymbol{s}_m}\subseteq\bigcup_{i\in J_m}C_i$ and $|S_{\boldsymbol{s}_m}(f_m,\nu)-\sum_{i\in J_m}f_m(x_i)\nu C_i|\le 2^{-m}\epsilon$.

Set $\boldsymbol{s}=\bigcup_{m\le k}\boldsymbol{s}_m$, so that $\boldsymbol{s}$ is a δ_k-fine member of T and

$$\sum_{m=0}^{k}\sum_{i\in J_m} f_m(x_i)\nu C_i \le \sum_{m=0}^{k} S_{\boldsymbol{s}_m}(f_m,\nu)+2^{-m}\epsilon \le \sum_{m=0}^{k} S_{\boldsymbol{s}_m}(f_k,\nu)+2^{-m}\epsilon$$
$$\le S_{\boldsymbol{s}}(f_k,\nu)+2\epsilon \le I_\nu(f_k)+3\epsilon$$

(because $\boldsymbol{s}$ extends to a δ_k-fine $\mathcal{R}_k$-filling member of T, by 482Ab)

$$\le\gamma+3\epsilon.$$

Now $\boldsymbol{t}$ is $\tilde\delta$-fine, so $S_{\boldsymbol{t}}(h,\nu)\le 1$. Accordingly

$$S_{\boldsymbol{t}}(f,\nu)=\sum_{i<n} f(x_i)\nu C_i=\sum_{m=0}^{k}\sum_{i\in J_m} f(x_i)\nu C_i$$
$$\le\sum_{m=0}^{k}\sum_{i\in J_m}(f_m(x_i)+\epsilon h(x_i))\nu C_i\le\gamma+3\epsilon+\epsilon S_{\boldsymbol{t}}(h,\nu)\le\gamma+4\epsilon,$$

as required. **Q**

As ϵ is arbitrary, $\limsup_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is at most γ.

(b) On the other hand, given $\epsilon>0$, there is an $n\in\mathbb{N}$ such that $I_\nu(f_n)\ge\gamma-\epsilon$. So taking $\delta_n\in\Delta$, $\mathcal{R}_n\in\mathfrak{R}$ as in (a) above,

$$S_{\boldsymbol{t}}(f,\nu)\ge S_{\boldsymbol{t}}(f_n,\nu)\ge I_\nu(f_n)-\epsilon\ge\gamma-2\epsilon$$

for every δ_n-fine $\mathcal{R}_n$-filling $\boldsymbol{t}\in T$. So $\liminf_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is at least γ, and $I_\nu(f)=\gamma$.

482L Lemma Let X be a set, $\mathcal{C}$ a family of subsets of X, Δ a countably full downwards-directed set of gauges on X, $\mathfrak{R} \subseteq \mathcal{PP}X$ a downwards-directed collection of residual families, and $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a straightforward set of tagged partitions of X compatible with Δ and $\mathfrak{R}$. Suppose further that whenever $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and $\boldsymbol{t} \in T$ is δ-fine, there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}' \in T$ including $\boldsymbol{t}$. (For instance, $(X, T, \Delta, \mathfrak{R})$ might be a tagged-partition structure allowing subdivisions, as in 482Ab.) If $\nu : \mathcal{C} \to [0, \infty[$ and $f : X \to [0, \infty[$ are such that $I_\nu(f) = 0$, and $g : X \to \mathbb{R}$ is such that $g(x) = 0$ whenever $f(x) = 0$, then $I_\nu(g) = 0$.

proof Let $\epsilon > 0$. For each $n \in \mathbb{N}$, let $\delta_n \in \Delta$, $\mathcal{R}_n \in \mathfrak{R}$ be such that $S_{\boldsymbol{t}}(f, \nu) \le 2^{-n}\epsilon$ for every δ_n-fine $\mathcal{R}_n$-filling $\boldsymbol{t} \in T$. For $x \in X$, set $\phi(x) = \min\{n : |g(x)| \le nf(x)\}$; let $\delta \in \Delta$ be such that $(x, C) \in \delta_{\phi(x)}$ whenever $(x, C) \in \delta$.

Let $\boldsymbol{t}$ be any δ-fine member of T. Then $|S_{\boldsymbol{t}}(g, \nu)| \le 2\epsilon$. **P** Enumerate $\boldsymbol{t}$ as $\langle (x_i, C_i)\rangle_{i<n}$. For each $m \in \mathbb{N}$, set $K_m = \{i : i < n,\ \phi(x_i) = m\}$; then $\{(x_i, C_i) : i \in K_m\}$ is a δ_m-fine member of T, so extends to a δ_m-fine $\mathcal{R}_m$-filling member $\boldsymbol{t}_m$ of T, and

$$\sum_{i \in K_m} |g(x_i)\nu C_i| \le m\sum_{i \in K_m} f(x_i)\nu C_i \le mS_{\boldsymbol{t}_m}(f, \nu) \le 2^{-m}m\epsilon.$$

Summing over m,

$$|S_{\boldsymbol{t}}(g, \nu)| \le \sum_{m=0}^{\infty} 2^{-m}m\epsilon = 2\epsilon. \textbf{ Q}$$

Because T is compatible with Δ and $\mathcal{R}$, this is enough to show that $I_\nu(g)$ is defined and equal to 0.

482M Fubini's theorem Suppose that, for $i = 1$ and $i = 2$, we have X_i, $\mathfrak{T}_i$, T_i, Δ_i, $\mathcal{C}_i$ and ν_i such that

(i) $(X_i, \mathfrak{T}_i)$ is a topological space;

(ii) Δ_i is the set of neighbourhood gauges on X_i;

(iii) $T_i \subseteq [X_i \times \mathcal{C}_i]^{<\omega}$ is a straightforward set of tagged partitions, compatible with Δ_i and $\{\{\emptyset\}\}$;

(iv) $\nu_i : \mathcal{C}_i \to [0, \infty[$ is a function;

(v) ν_1 is moderated with respect to T_1 and Δ_1;

(vi) whenever $\delta \in \Delta_1$ and $\boldsymbol{s} \in T_1$ is δ-fine, there is a δ-fine $\boldsymbol{s}' \in T_1$, including $\boldsymbol{s}$, such that $W_{\boldsymbol{s}'} = X_1$.

Write X for $X_1 \times X_2$; Δ for the set of neighbourhood gauges on X; $\mathcal{C}$ for $\{C \times D : C \in \mathcal{C}_1,\ D \in \mathcal{C}_2\}$; Q for $\{((x, y), C \times D) : \{(x, C)\} \in T_1,\ \{(y, D)\} \in T_2\}$; T for the straightforward set of tagged partitions generated by Q; and set $\nu(C \times D) = \nu_1 C \cdot \nu_2 D$ for $C \in \mathcal{C}_1$, $D \in \mathcal{C}_2$.

(a) T is compatible with Δ and $\{\{\emptyset\}\}$.

(b) Let I_{ν_1}, I_{ν_2} and I_ν be the gauge integrals defined by these structures as in 481C-481F. Suppose that $f : X \to \mathbb{R}$ is such that $I_\nu(f)$ is defined. Set $f_x(y) = f(x, y)$ for $x \in X_1$, $y \in X_2$. Let $g : X_1 \to \mathbb{R}$ be any function such that $g(x) = I_{\nu_2}(f_x)$ whenever this is defined. Then $I_{\nu_1}(g)$ is defined and equal to $I_\nu(f)$.

proof (a) Let $\delta \in \Delta$; we seek a tagged partition $\boldsymbol{u} \in T$ such that $W_{\boldsymbol{u}} = X$. Let $\langle V_{xy}\rangle_{(x,y)\in X}$ be the family of open sets in X defining δ; choose open sets $G_{xy} \subseteq X_1$, $H_{xy} \subseteq X_2$ such that $(x, y) \in G_{xy} \times H_{xy} \subseteq V_{xy}$ for all $x \in X_1$, $y \in X_2$. For each $x \in X_1$, let δ_x be the neighbourhood gauge on X_2 defined from the family $\langle H_{xy}\rangle_{y\in X_2}$. Then there is a δ_x-fine tagged partition $\boldsymbol{t}_x \in T_2$ such that $W_{\boldsymbol{t}_x} = X_2$. Set $G_x = X_1 \cap \bigcap_{(y,D)\in\boldsymbol{t}_x} G_{xy}$.

The family $\langle G_x\rangle_{x\in X_1}$ defines a neighbourhood gauge δ^* on X_1, and there is a δ^*-fine $\boldsymbol{s} \in T_1$ such that $W_{\boldsymbol{s}} = X_1$. Now consider

$$\boldsymbol{u} = \{((x, y), C \times D) : (x, C) \in \boldsymbol{s},\ (y, D) \in \boldsymbol{t}_x\}.$$

Then it is easy to check (just as in part (b) of the proof of 481O) that $\boldsymbol{u}$ is a δ-fine member of T with $W_{\boldsymbol{u}} = X$.

(b)(i) Set $A = \{x : x \in X_1,\ I_{\nu_2}(f_x)$ is defined$\}$. Let $h : X_1 \to \mathbb{R}$ be any function such that $h(x) = 0$ for every $x \in A$. For $x \in X_1$, set

$$h_0(x) = \inf(\{1\} \cup \{\sup_{\boldsymbol{t},\boldsymbol{t}'\in F} S_{\boldsymbol{t}}(f_x, \nu_2) - S_{\boldsymbol{t}'}(f_x, \nu_2) : F \in \mathcal{F}(T_2, \Delta_2, \{\{\emptyset\}\})\}).$$

(Thus $I_{\nu_2}(f_x)$ is defined iff $h_0(x) = 0$.) Then $I_{\nu_1}(h_0) = 0$. **P** Let $\epsilon > 0$. Then there is a $\delta \in \Delta$ such that $S_{\boldsymbol{u}}(f, \nu) - S_{\boldsymbol{u}'}(f, \nu) \le \epsilon$ whenever $\boldsymbol{u}$, $\boldsymbol{u}' \in T$ are δ-fine and $W_{\boldsymbol{u}} = W_{\boldsymbol{u}'} = X$. Define $\langle V_{xy}\rangle_{(x,y)\in X}$, $\langle G_{xy}\rangle_{(x,y)\in X}$, $\langle H_{xy}\rangle_{(x,y)\in X}$ and $\langle \delta_x\rangle_{x\in X_1}$ from δ as in (a) above. For each $x \in X_1$, we can find δ_x-fine partitions $\boldsymbol{t}_x$, $\boldsymbol{t}'_x \in T_2$ such that $W_{\boldsymbol{t}_x} = W_{\boldsymbol{t}'_x} = X_2$ and $S_{\boldsymbol{t}_x}(f_x, \nu_2) - S_{\boldsymbol{t}'_x}(f_x, \nu_2) \ge \frac{1}{2}h_0(x)$. Set $G_x = X_1 \cap \bigcap_{(y,D)\in\boldsymbol{t}_x\cup\boldsymbol{t}'_x} G_{xy}$.

Let δ^* be the neighbourhood gauge on X_1 defined from $\langle G_x\rangle_{x\in X_1}$. Let $\boldsymbol{s}$ be any δ^*-fine member of T_1 with $W_{\boldsymbol{s}} = X_1$. Set

$$\boldsymbol{u} = \{((x, y), C \times D) : (x, C) \in \boldsymbol{s},\ (y, D) \in \boldsymbol{t}_x\},$$

$$\boldsymbol{u}' = \{((x, y), C \times D) : (x, C) \in \boldsymbol{s},\ (y, D) \in \boldsymbol{t}'_x\},$$

Then $\boldsymbol{u}$ and $\boldsymbol{u}'$ are δ-fine members of T with $W_{\boldsymbol{u}} = W_{\boldsymbol{u}'} = X$, so

$$\begin{aligned} S_{\boldsymbol{s}}(h_0,\nu_1) &= \sum_{(x,C)\in\boldsymbol{s}} h_0(x)\nu_1(C) \le 2\sum_{(x,C)\in\boldsymbol{s}} (S_{\boldsymbol{t}_x}(f_x,\nu_2) - S_{\boldsymbol{t}'_x}(f_x,\nu_2))\nu_1(C) \\ &= 2\Bigl(\sum_{\substack{(x,C)\in\boldsymbol{s}\\(y,D)\in\boldsymbol{t}_x}} f(x,y)\nu_1(C)\nu_2(D) - \sum_{\substack{(x,C)\in\boldsymbol{s}\\(y,D)\in\boldsymbol{t}'_x}} f(x,y)\nu_1(C)\nu_2(D)\Bigr) \\ &= 2\bigl(S_{\boldsymbol{u}}(f,\nu) - S_{\boldsymbol{u}'}(f,\nu)\bigr) \le 2\epsilon. \end{aligned}$$

As ϵ is arbitrary, $I_{\nu_1}(h_0) = 0$. **Q**

By 482L, $I_{\nu_1}(h)$ also is zero.

(ii) Again take any $\epsilon > 0$, and let $\delta \in \Delta$ be such that $|S_{\boldsymbol{u}}(f,\nu) - I_\nu(f)| \le \epsilon$ for every δ-fine $\boldsymbol{u} \in T$ such that $W_{\boldsymbol{u}} = X$. Define $\langle V_{xy}\rangle_{(x,y)\in X}$, $\langle G_{xy}\rangle_{(x,y)\in X}$, $\langle H_{xy}\rangle_{(x,y)\in X}$ and $\langle\delta_x\rangle_{x\in X_1}$ from δ as in (a) and (i) above. Let $\tilde\delta \in \Delta_1$, $\tilde h : X_1 \to \,]0,\infty[$ be such that $S_{\boldsymbol{s}}(\tilde h,\nu_1) \le 1$ for every $\tilde\delta$-fine $\boldsymbol{s} \in T_1$.

For $x \in A$, let $\boldsymbol{t}_x \in T_2$ be δ_x-fine and such that $W_{\boldsymbol{t}_x} = X_2$ and $|S_{\boldsymbol{t}_x}(f_x,\nu_2) - I_{\nu_2}(f_x)| \le \epsilon\tilde h(x)$; for $x \in X_1 \setminus A$, let $\boldsymbol{t}_x$ be any δ_x-fine member of T_2 such that $W_{\boldsymbol{t}_x} = X_2$. Set $G_x = X_1 \cap \bigcap_{(y,D)\in\boldsymbol{t}_x} G_{xy}$ for every $x \in X_1$. Let δ^* be the neighbourhood gauge on X_1 defined by $\langle G_x\rangle_{x\in X_1}$.

Set $h_1(x) = g(x) - S_{\boldsymbol{t}_x}(f_x,\nu_2)$ for $x \in X_1 \setminus A$, 0 for $x \in A$. Then $I_{\nu_1}(|h_1|) = 0$, by (a). Let $\delta' \in \Delta_1$ be such that $\delta' \subseteq \delta^* \cap \tilde\delta$ and $S_{\boldsymbol{s}}(|h_1|,\nu_1) \le \epsilon$ for every δ'-fine $\boldsymbol{s} \in T_1$ such that $W_{\boldsymbol{s}} = X_1$.

Now suppose that $\boldsymbol{s} \in T_1$ is δ'-fine and that $W_{\boldsymbol{s}} = X_1$. Set

$$\boldsymbol{u} = \{((x,y), C\times D) : (x,C) \in \boldsymbol{s},\ (y,D) \in \boldsymbol{t}_x\},$$

so that $\boldsymbol{u} \in T$ is δ-fine and $W_{\boldsymbol{u}} = X$. Then

$$\begin{aligned} |S_{\boldsymbol{s}}(g,\nu_1) - I_\nu(f)| &\le |S_{\boldsymbol{u}}(f,\nu) - I_\nu(f)| + |S_{\boldsymbol{s}}(g,\nu_1) - S_{\boldsymbol{u}}(f,\nu)| \\ &\le \epsilon + \Bigl|\sum_{(x,C)\in\boldsymbol{s}} \bigl(g(x) - \sum_{(y,D)\in\boldsymbol{t}_x} f(x,y)\nu_2 D\bigr)\nu_1 C\Bigr| \\ &= \epsilon + \Bigl|\sum_{(x,C)\in\boldsymbol{s}} \bigl(g(x) - S_{\boldsymbol{t}_x}(f_x,\nu_2)\bigr)\nu_1 C\Bigr| \\ &\le \epsilon + \sum_{(x,C)\in\boldsymbol{s}} |h_1(x)|\nu_1 C \\ &\qquad + \sum_{\substack{(x,C)\in\boldsymbol{s}\\x\in A}} |g(x) - \sum_{(y,D)\in\boldsymbol{t}_x} f(x,y)\nu_2 D|\nu_1 C \\ &\le 2\epsilon + \sum_{\substack{(x,C)\in\boldsymbol{s}\\x\in A}} \epsilon\tilde h(x)\nu_1(C) \le 2\epsilon + \epsilon S_{\boldsymbol{s}}(\tilde h,\nu_1) \le 3\epsilon. \end{aligned}$$

As ϵ is arbitrary, $I_{\nu_1}(g)$ is defined and equal to $I_\nu(f)$, as claimed.

482X Basic exercises (a) Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\mathcal{E}$ the algebra of subsets of X generated by $\mathcal{C}$. Write $\mathfrak{I}$ for the set of pairs (f,ν) such that $f : X \to \mathbb{R}$ and $\nu : \mathcal{C} \to \mathbb{R}$ are functions and $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is defined; for $(f,\nu) \in \mathfrak{I}$, let $F_{f\nu} : \mathcal{E} \to \mathbb{R}$ be the corresponding Saks-Henstock indefinite integral. Show that $(f,\nu) \mapsto F_{f\nu}$ is bilinear in the sense that

$$F_{f+g,\nu} = F_{f\nu} + F_{g\nu}, \quad F_{\alpha f,\nu} = F_{f,\alpha\nu} = \alpha F_{f\nu}, \quad F_{f,\mu+\nu} = F_{f\mu} + F_{f\nu}$$

whenever (f,ν), (g,ν) and (f,μ) belong to $\mathfrak{I}$ and $\alpha \in \mathbb{R}$.

(b) Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\mathcal{E}$ the algebra of subsets of X generated by $\mathcal{C}$. Let $f : X \to \mathbb{R}$ and $\nu : \mathcal{C} \to [0,\infty[$ be functions such that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is defined, and let $F : \mathcal{E} \to \mathbb{R}$ be the corresponding Saks-Henstock indefinite integral. Show that F is non-negative iff $I_\nu(f^-) = 0$, where $f^-(x) = \max(0,-f(x))$ for every $x \in X$.

>(c) Let X be a zero-dimensional compact Hausdorff space, $\mathcal{E}$ the algebra of open-and-closed subsets of X, $Q = \{(x,E) : x \in E \in \mathcal{E}\}$, T the straightforward set of tagged partitions generated by Q, Δ the set of neighbourhood gauges on X and $\nu : \mathcal{E} \to \mathbb{R}$ a non-negative additive functional. Let μ be the corresponding Radon measure on X (416Qa) and I_ν the gauge integral defined by $(X, T, \Delta, \{\{\emptyset\}\})$ (cf. 481Xh). Show that, for $f : X \to \mathbb{R}$, $I_\nu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$. (*Hint*: if f is measurable but not μ-integrable, take x_0 such that f is not integrable on any neighbourhood of x_0. Given $\delta \in \Delta$, fix a δ-fine partition containing (x_0, V_0) for some V_0; now replace (x_0, V_0) by refinements $\{(x_0, V_0'), (x_1, V_1), \dots, (x_n, V_n)\}$, where $\sum_{i=1}^n f(x_i)\nu V_i$ is large, to show that $S_{\boldsymbol{t}}(f, \nu)$ cannot be controlled by δ.)

>(d) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space in which μ is inner regular with respect to the closed sets and outer regular with respect to the open sets (see 412W). Let T be the straightforward set of tagged partitions generated by $X \times \{E : \mu E < \infty\}$, Δ the set of neighbourhood gauges on X, and $\mathfrak{R} = \{\mathcal{R}_{E\eta} : \mu E < \infty,\ \eta > 0\}$, where $\mathcal{R}_{E\eta} = \{F : \mu(F \cap E) \le \eta\}$, as in N. Show that if I_μ is the associated gauge integral, and $f : X \to \mathbb{R}$ is a function, then $I_\mu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$.

(e) Let $\mathcal{C}$ be the set of non-empty subintervals of $X = [0,1]$, T the straightforward tagged-partition structure generated by $[0,1] \times \mathcal{C}$, and Δ the set of neighbourhood gauges on $[0,1]$, as in 481M. Let μ be any Radon measure on $[0,1]$, and I_μ the gauge integral defined from μ and the tagged-partition structure $([0,1], T, \Delta, \{\{\emptyset\}\})$. Show that, for any $f : [0,1] \to \mathbb{R}$, $I_\mu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$.

(f) Let $\mathcal{C}$ be the set of non-empty subintervals of $[0,1]$, T the straightforward tagged-partition structure generated by $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$, and Δ the set of neighbourhood gauges on X, as in 481J. Let $\mathcal{E}$ be the ring of subsets of $[0,1]$ generated by $\mathcal{C}$, $\nu : \mathcal{E} \to \mathbb{R}$ a bounded additive functional, and I_ν the gauge integral defined from ν and $(X, T, \Delta, \{\{\emptyset\}\})$. Show that $I_\nu(\chi\,[a,b[) = \lim_{x \uparrow a, y \uparrow b} \nu([x,y])$ whenever $0 < a < b \le 1$.

(g) Let $\mathcal{C}$ be the set of non-empty subintervals of $X = [0,1]$, T the straightforward tagged-partition structure generated by $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$, and Δ the set of uniform metric gauges on $[0,1]$, as in 481I. Let μ be the Dirac measure on $[0,1]$ concentrated at $\frac{1}{2}$, and let $I_\mu = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \{\{\emptyset\}\})} S_{\boldsymbol{t}}(\ , \mu)$ be the corresponding gauge integral. Show that $I_\mu(\chi[0,1])$ is defined but that $I_\mu(\chi[0, \frac{1}{2}])$ is not.

>(h)(i) Show that the McShane integral on an interval $[a,b]$ as described in 481M coincides with the Lebesgue integral on $[a,b]$. (ii) Show that if $(X, \mathfrak{T}, \Sigma, \mu)$ is a quasi-Radon measure space and μ is outer regular with respect to the open sets then the McShane integral as described in 481N coincides with the usual integral.

(i) Explain how the results in 481Xb-481Xc can be regarded as special cases of 482H.

(j) Let $(X, \mathfrak{T})$ be a topological space, $\mathcal{C}$ a ring of subsets of X, $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a straightforward set of tagged partitions, $\nu : \mathcal{C} \to [0, \infty[$ an additive function, and Δ the family of neighbourhood gauges on X. Suppose that there is a sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ of open sets, covering X, such that $\sup\{\nu C : C \in \mathcal{C},\ C \subseteq G_n\}$ is finite for every $n \in \mathbb{N}$. Show that ν is moderated with respect to T and Δ.

(k) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space. Let T be the straightforward tagged-partition structure generated by $\{(x, E) : \mu E < \infty,\ x \in E\}$ and Δ the set of all neighbourhood gauges on X. Show that μ is moderated with respect to T and Δ iff there is a sequence of open sets of finite measure covering X.

(l) Let $r \ge 1$ be an integer, and μ a Radon measure on $\mathbb{R}^r$. Let Q be the set of pairs (x, C) where $x \in \mathbb{R}^r$ and C is a closed ball with centre x, and T the straightforward set of tagged partitions generated by Q. Let Δ be the set of neighbourhood gauges on $\mathbb{R}^r$, and $\mathfrak{R} = \{\mathcal{R}_{E\eta} : \mu E < \infty,\ \eta > 0\}$, where $\mathcal{R}_{E\eta} = \{F : \mu(F \cap E) \le \eta\}$, as in 482Xd. (i) Show that T is compatible with Δ and $\mathfrak{R}$. (*Hint*: 472C.) (ii) Show that if I_μ is the associated gauge integral, and $f : \mathbb{R}^r \to \mathbb{R}$ is a function, then $I_\mu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$.

(m) Let $(X, T, \Delta, \mathfrak{R})$, Σ and ν be as in 481Xj, so that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by an algebra Σ of subsets of X, and $\nu : \Sigma \to [0, \infty[$ is additive. Let I_ν be the corresponding gauge integral, and $V \subseteq \mathbb{R}^X$ its domain. (i) Show that $\chi E \in V$ and $I_\nu(\chi E) = \nu E$ for every $E \in \Sigma$. (ii) Show that if $f \in \mathbb{R}^X$ then $f \in V$ iff for every $\epsilon > 0$ there is a disjoint family $\mathcal{E} \subseteq \Sigma$ such that $\sum_{E \in \mathcal{E}} \nu E = \nu X$ and $\sum_{E \in \mathcal{E}} \nu E \cdot \sup_{x,y \in E} |f(x) - f(y)| \le \epsilon$. (iii) Show that if $\langle f_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in V with supremum $f \in \mathbb{R}^X$, and $\gamma = \sup_{n \in \mathbb{N}} I_\nu(f_n)$ is finite, then $I_\nu(f)$ is defined and equal to γ. (iv) Show that I_ν extends $\oint d\nu$ as described in 363L, if we identify $L^\infty(\Sigma)$ with a space $\mathcal{L}^\infty$ of functions as in 363H. (v) Show that if Σ is a σ-algebra of sets then I_ν extends $\oint d\nu$ as described in 364Xj.

482Y Further exercises (a) Let X be the interval $[0,1]$, $\mathcal{C}$ the family of subintervals of X, Q the set $\{(x,C) : C \in \mathcal{C},\, x \in \overline{\operatorname{int} C}\}$, T the straightforward set of tagged partitions generated by Q, Δ the set of neighbourhood gauges on X, and $\mathfrak{R}$ the singleton $\{[X]^{<\omega}\}$. Show that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. For $C \in \mathcal{C}$ set $\nu C = 1$ if $0 \in \overline{\operatorname{int} C}$, 0 otherwise, and let f be $\chi\{0\}$. Show that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is defined and equal to 1. Let F be the Saks-Henstock indefinite integral of f. Show that $F(]0,1]) = 1$.

(b) Set $X = \mathbb{R}$ and let $\mathcal{C}$ be the family of non-empty bounded intervals in X; set $Q = \{(x,C) : C \in \mathcal{C},\, x = \inf C\}$, and let T be the straightforward set of tagged partitions generated by Q. Let $\mathfrak{T}$ be the Sorgenfrey right-facing topology on X, and Δ the set of neighbourhood gauges for $\mathfrak{T}$. Set $\mathcal{R}_n = \{E : E \in \Sigma,\, \mu([-n,n] \cap E) \le 2^{-n}\}$ for $n \in \mathbb{N}$, where μ is Lebesgue measure on $\mathbb{R}$ and Σ its domain, and write $\mathfrak{R} = \{\mathcal{R}_n : n \in \mathbb{N}\}$. Show that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions. Show that if $f : X \to \mathbb{R}$ is such that $I_\mu(f)$ is defined, then f is Lebesgue measurable.

(c) Give an example of X, $\mathfrak{T}$, Σ, μ, T, Δ, $\mathcal{C}$, f and C such that $(X, \mathfrak{T}, \Sigma, \mu)$ is a compact metrizable Radon probability space, Δ is the set of all neighbourhood gauges on X, $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, $f : X \to \mathbb{R}$ is a function such that $I_\mu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\{\{\emptyset\}\})} S_{\boldsymbol{t}}(f,\mu)$ is defined, $C \in \mathcal{C}$ is a closed set with negligible boundary, and $I_\mu(f \times \chi C)$ is not defined.

(d) Suppose that, for $i = 1$ and $i = 2$, we have a tagged-partition structure $(X_i, T_i, \Delta_i, \mathfrak{R}_i)$ allowing subdivisions, witnessed by a ring $\mathcal{C}_i \subseteq \mathcal{P}X_i$, where X_i is a topological space, Δ_i is the set of all neighbourhood gauges on X_i, and $\mathfrak{R}_i$ is the simple residual structure complementary to $\mathcal{C}_i$, as in 481Yb. Set $X = X_1 \times X_2$ and let Δ be the set of neighbourhood gauges on X; set $\mathcal{C} = \{C \times D : C \in \mathcal{C}_1,\, D \in \mathcal{C}_2\}$; let $\mathfrak{R}$ be the simple residual structure on X complementary to $\mathcal{C}$; and let T be the straightforward tagged-partition structure generated by $\{((x,y), C \times D) : \{(x,C)\} \in T_1,\, \{(y,D)\} \in T_2\}$. For each i, let $\nu_i : \mathcal{C}_i \to [0,\infty[$ be a function moderated with respect to T_i and Δ_i, and define $\nu : \mathcal{C} \to [0,\infty[$ by setting $\nu(C \times D) = \nu_1 C \cdot \nu_2 D$ for $C \in \mathcal{C}_1$, $D \in \mathcal{C}_2$. Show that T is compatible with Δ and $\mathfrak{R}$. Let I_{ν_1}, I_{ν_2}, I_ν be the gauge integrals defined by these structures. Suppose that $f : X \to \mathbb{R}$ is such that $I_\nu(f)$ is defined. Set $f_x(y) = f(x,y)$ for $x \in X_1$, $y \in X_2$. Let $g : X_1 \to \mathbb{R}$ be any function such that $g(x) = I_{\nu_2}(f_x)$ whenever this is defined. Show that $I_{\nu_1}(g)$ is defined and equal to $I_\nu(f)$.

482 Notes and comments In 482E, 482F and 482G the long lists of conditions reflect the variety of possible applications of these arguments. The price to be paid for the versatility of the constructions here is a theory which is rather weak in the absence of special hypotheses. As everywhere in this book, I try to set ideas out in maximal convenient generality; you may feel that in this section the generality is becoming inconvenient; but the theory of gauge integrals has not, to my eye, matured to the point that we can classify the systems here even as provisionally as I set out to classify topological measure spaces in Chapters 41 and 43.

Enthusiasts for gauge integrals offer two substantial arguments for taking them seriously, apart from the universal argument in pure mathematics, that these structures offer new patterns for our delight and new challenges to our ingenuity. First, they say, gauge integrals integrate more functions than Lebesgue-type integrals, and it is the business of a theory of integration to integrate as many functions as possible; and secondly, gauge integrals offer an easier path to the principal theorems. I have to say that I think the first argument is sounder than the second. It is quite true that the Henstock integral on $\mathbb{R}$ (481K) can be rigorously defined in fewer words, and with fewer concepts, than the Lebesgue integral. The style of Chapters 11 and 12 is supposed to be better adapted to the novice than the style of this chapter, but you will have no difficulty in putting the ideas of 481A, 481C, 481J and 481K together into an elementary definition of an integral for real functions in which the only non-trivial argument is that establishing the existence of enough tagged partitions (481J), corresponding I suppose to Proposition 114D. But the path I took in defining the integral in §122, though arduous at that point, made (I hope) the convergence theorems of §123 reasonably natural; the proof of 482K, on the other hand, makes significant demands on our technique. Furthermore, the particular clarity of the one-dimensional Henstock integral is not repeated in higher dimensions. Fubini's theorem, with exact statement and full proof, even for products of Lebesgue measures on Euclidean spaces, is a lot to expect of an undergraduate; but Lebesgue measure on $\mathbb{R}^r$ makes sense in a way that it is quite hard to repeat with gauge integrals. (For instance, Lebesgue measure is invariant under isometries; this is not particularly easy to prove – see 263A – but at least it is true; if we want a gauge integral which is invariant under isometries, then we have to use a construction such as 481O, which does not directly match any natural general definition of 'product gauge integral' along the lines of 481P, 482M or 482Yd.)

In my view, a stronger argument for taking gauge integrals seriously is their 'power', that is, their ability to provide us with integrals of many functions in consistent ways. 482E, 482F and 482I give us an idea of what to expect. If we start from a measure space (X, Σ, μ) and build a gauge integral I_μ from a set $T \subseteq [X \times \Sigma]^{<\omega}$ of tagged partitions, then we can hope that integrable functions will be gauge-integrable, with the right integrals (482F); while gauge-integrable functions will be measurable (482E). What this means is that for *non-negative* functions, the integrals will coincide. Any 'new' gauge-integrable functions f will be such that $\int f^+ = \int f^- = \infty$; the gauge integral will offer a process for cancelling the divergent parts of these integrals. On the other hand, we can hope for a large class of gauge-integrable derivatives. In the next two sections, I will explain how this works in the Henstock and Pfeffer integrals. For simple examples calling for such procedures, see the formulae of §§282 and 283; for radical applications of the idea, see MULDOWNEY 87.

Against this, gauge integrals are not effective in 'general' measure spaces, and cannot be, because there is no structure in an abstract measure space (X, Σ, μ) which allows us to cancel an infinite integral $\int f^+ = \int_F f$ against $\int f^- = \int_{X \setminus F} f$. Put another way, if a tagged-partition structure is invariant under all automorphisms of the structure (X, Σ, μ), as in 481Xf-481Xg, we cannot expect anything better than the standard integral. In order to get something new, the most important step seems to be the specification of a family $\mathcal{C}$ of 'regular' sets, preliminary to describing a set T of tagged partitions. To get a 'powerful' gauge integral, we want a fine filter on T, corresponding to a small set $\mathcal{C}$ and a large set of gauges. The residual families of 481F are generally introduced just to ensure 'compatibility' in the sense described there; as a rule, we try to keep them simple. But even if we take the set of all neighbourhood gauges, as in the Henstock integral, this is not enough unless we also sharply restrict both the family $\mathcal{C}$ and the permissible tags (482Xc-482Xe). The most successful restrictions, so far, have been 'geometric', as in 481J and 481O, and 484F below. Further limitations on admissible pairs (x, C), as in 481L and 481Q, in which $\mathcal{C}$ remains the set of intervals, but fewer tags are permitted, also lead to very interesting results.

Another limitation in the scope of gauge integrals is the difficulty they have in dealing with spaces of infinite measure. Of course we expect to have to specify a limiting procedure if we are to calculate $I_\mu(f)$ from sums $S_{\boldsymbol{t}}(f, \mu)$ which necessarily consider only sets of finite measure, and this is one of the functions of the collections $\mathfrak{R}$ of residual families. But this is not yet enough. In B.Levi's theorem (482K) we already need to suppose that our set-function ν is 'moderated' in order to determine how closely $f_n(x)$ needs to approximate each $f(x)$. The condition

$$S_{\boldsymbol{t}}(h, \nu) \le 1 \text{ for every } \delta\text{-fine } \boldsymbol{t}$$

of 482J is very close to saying that $I_\mu(h) \le 1$. But it is *not* the same as saying that μ is σ-finite; it suggests rather that X should be covered by a sequence of open sets of finite measure (482Xk).

Because gauge integrals are not absolute – that is, we can have $I_\nu(f)$ defined and finite while $I_\nu(|f|)$ is not – we are bound to have difficulties with integrals $\int_H f$, even if we interpret these in the simplest way, as $I_\nu(f \times \chi H)$, so that we do not need a theory of subspaces as developed in §214. 482G(iii)-(v) are an attempt to find a reasonably versatile sufficient set of conditions. The 'multiplier problem', for a given gauge integral I_ν, is the problem of characterizing the functions g such that $I_\nu(f \times g)$ is defined whenever $I_\nu(f)$ is defined, and even for some intensively studied integrals remains challenging. In 484L I will give an important case which is not covered by 482G.

One of the striking features of gauge integrals is that there is no need to assume that the set-function ν is countably additive. We can achieve countable additivity of the integral – in the form of B.Levi's theorem, for instance – by requiring only that the set of gauges should be 'countably full' (482K, 482L; contrast 482Xg). If we watch our definitions carefully, we can make this match the rules for Stieltjes integrals (114Xa, 482Xf). In 481Db I have already remarked on the potential use of gauge integrals in vector integration.

It is important to recognise that a value $F(E)$ of a Saks-Henstock indefinite integral (482B-482C) cannot be identified with either $I_\nu(f \times \chi E)$ or with $I_{\nu \restriction \mathcal{P}E}(f \restriction E)$, because in the formula $F(E) = \lim_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f, \nu)$ used in the proof of 482B the tags of the partitions $\boldsymbol{t}_E$ need not lie in E. (See 482Ya.) The idea of 482G is to impose sufficient conditions to ensure that the contributions of 'straddling' elements (x, C), where either $x \in E$ and $C \not\subseteq E$ or $x \notin E$ and $C \cap E \ne \emptyset$, are insignificant. To achieve this we seem to need both a regularity condition on the functional ν (condition (iii) of 482G) and a geometric condition on the set $\mathcal{C}$ underlying T (482G(iv)). As usual, the regularity condition required is closer to *outer* than to *inner* regularity, in contexts in which there is a distinction.

I am not sure that I have the 'right' version of Proposition 482E. The hypothesis there is that we have a metric space. But in the principal non-metrizable cases the result is still valid (482Xc-482Xd, 482Yb), and the same happens in 482Xl, where condition 482E(ii) is not satisfied. Proposition 482H is a 'new' limit theorem; it shows that certain improper integrals from the classical theory can be represented as gauge integrals. The hypotheses seem, from where we are standing at the moment, to be exceedingly restrictive. In the leading examples in §483, however, the central requirement 482H(viii) is satisfied for straightforward geometric reasons.

Gauge integrals insist on finite functions defined everywhere. But since we have an effective theory of negligible sets (482L), we can easily get a consistent theory of integration for functions which are defined and real-valued almost everywhere if we say that

$$I_\nu(f) = I_\nu(g) \text{ whenever } g : X \to \mathbb{R} \text{ extends } f \restriction f^{-1}[\mathbb{R}]$$

whenever $I_\nu(g) = I_\nu(g')$ for all such extensions.

483 The Henstock integral

I come now to the original gauge integral, the 'Henstock integral' for real functions. The first step is to check that the results of §482 can be applied to show that this is an extension of both the Lebesgue integral and the improper Riemann integral (483B), coinciding with the Lebesgue integral for non-negative functions (483C). It turns out that any Henstock integrable function can be approximated in a strong sense by a sequence of Lebesgue integrable functions (483G). The Henstock integral can be identified with the Perron and special Denjoy integrals (483J, 483N, 483Yh). Much of the rest of the section is concerned with indefinite Henstock integrals. Some of the results of §482 on tagged-partition structures allowing subdivisions condense into a particularly strong Saks-Henstock lemma (483F). If f is Henstock integrable, it is equal almost everywhere to the derivative of its indefinite Henstock integral (483I). Finally, indefinite Henstock integrals can be characterized as continuous ACG$_*$ functions (483R).

483A Definition The following notation will apply throughout the section. Let $\mathcal{C}$ be the family of non-empty bounded intervals in $\mathbb{R}$, and let $T \subseteq [\mathbb{R} \times \mathcal{C}]^{<\omega}$ be the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$. Let Δ be the set of all neighbourhood gauges on $\mathbb{R}$. Set $\mathfrak{R} = \{\mathcal{R}_{ab} : a \le b \in \mathbb{R}\}$, where $\mathcal{R}_{ab} = \{\mathbb{R} \setminus [c, d] : c \le a,\ d \ge b\} \cup \{\emptyset\}$. Then $(\mathbb{R}, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions (481K), so T is compatible with Δ and $\mathfrak{R}$ (481Hf). The **Henstock integral** is the gauge integral defined by the process of 481E-481F from $(\mathbb{R}, T, \Delta, \mathfrak{R})$ and one-dimensional Lebesgue measure μ. For a function $f : \mathbb{R} \to \mathbb{R}$ I will say that f is **Henstock integrable**, and that $\oint f = \gamma$, if $\lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$ is defined and equal to $\gamma \in \mathbb{R}$. For α, $\beta \in [-\infty, \infty]$ I will write $\oint_\alpha^\beta f$ for $\oint f \times \chi\,]\alpha, \beta[$ if this is defined in $\mathbb{R}$. I will use the symbol $\int$ for the ordinary integral, so that $\int f d\mu$ is the Lebesgue integral of f.

483B Tracing through the theorems of §482, we have the following.

Theorem (a) Every Henstock integrable function on $\mathbb{R}$ is Lebesgue measurable.

(b) Every Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, with the same integral.

(c) If f is Henstock integrable so is $f \times \chi C$ for every interval $C \subseteq \mathbb{R}$.

(d) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is any function, and $-\infty \le \alpha < \beta \le \infty$. Then

$$\oint_\alpha^\beta f = \lim_{a \downarrow \alpha} \oint_a^\beta f = \lim_{b \uparrow \beta} \oint_\alpha^b f = \lim_{a \downarrow \alpha, b \uparrow \beta} \oint_a^b f$$

if any of the four terms is defined in $\mathbb{R}$.

proof (a) Apply 482E.

(b) Apply 482F, referring to 134F to confirm that condition 482F(ii) is satisfied.

It will be useful to note at once that this shows that $\oint f \times \chi\{a\} = 0$ for every $f : \mathbb{R} \to \mathbb{R}$ and every $a \in \mathbb{R}$. Consequently $\oint_a^b f = \oint_a^c f + \oint_c^b f$ whenever $a \le c \le b$ and the right-hand side is defined.

(c)-(d) In the following argument, f will always be a function from $\mathbb{R}$ to itself; when f is Henstock integrable, F^{SH} will be its Saks-Henstock indefinite integral (482B-482C).

(i) The first thing to check is that the conditions of 482G are satisfied by $\mathbb{R}$, T, Δ, $\mathfrak{R}$, $\mathcal{C}$ and $\mu \restriction \mathcal{C}$. **P** 482G(i) is just 481K, and 482G(ii) is trivial. 482G(iii-α) and 482G(v) are elementary, and so is 482G(iii-β) — if you like, this is a special case of 412W(b-iii). As for 482G(iv), if $E \in \mathcal{C}$ is a singleton $\{x\}$, then whenever $(x, C) \in \mathcal{C}$ we can express C as a union of one or more of the sets $C \cap \,]-\infty, x[$, $C \cap \{x\}$ and $C \cap \,]x, \infty[$, and for any non-empty C' of these we have $\{(x, C')\} \in T$ and either $C' \subseteq E$ or $C' \cap E = \emptyset$. Otherwise, let $\eta > 0$ be half the length of E, and let δ be the uniform metric gauge $\{(x, A) : A \subseteq \,]x - \eta, x + \eta[\}$. Then if $x \in \partial E$ and $(x, C) \in T \cap \delta$, we can again express C as a union of one or more of the sets $C \cap \,]-\infty, x[$, $C \cap \{x\}$ and $C \cap \,]x, \infty[$, and these will witness that 482G(iv) is satisfied. **Q**

(ii) Now suppose that f is Henstock integrable. Then $\fint_a^b f$ is defined whenever $a \le b$ in $\mathbb{R}$. **P** 482G tells us that $\fint f \times \chi C$ is defined and equal to $F^{\text{SH}}(C)$ for every $C \in \mathcal{C}$; in particular, $\fint_a^b f = \fint f \times \chi\,]a,b[$ is defined whenever $a < b$ in $\mathbb{R}$. **Q** Note that because $\fint f \times \chi\{c\} = 0$ for every c, $\fint f \times \chi C = \fint_{\inf C}^{\sup C} f$ whenever $C \in \mathcal{C}$ is non-empty.

This proves (c) for bounded intervals; we shall come to unbounded intervals in (vii) below.

(iii) If f is Henstock integrable, then $\lim_{a\to-\infty,b\to\infty} \fint_a^b f$ is defined and equal to $\fint f$. **P** Given $\epsilon > 0$, there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F^{\text{SH}}(\mathbb{R} \setminus C)| \le \epsilon$ whenever $C \in \mathcal{C}$ and $\mathbb{R} \setminus C \in \mathcal{R}$; that is, there are $a_0 \le b_0$ such that

$$|\fint f - \fint_a^b f| = |F^{\text{SH}}(\mathbb{R} \setminus \,]a,b[)| = |F^{\text{SH}}(\mathbb{R} \setminus [a,b])| \le \epsilon$$

whenever $a \le a_0 \le b_0 \le b$. As ϵ is arbitrary, $\lim_{a\to-\infty,b\to\infty} \fint_a^b f$ is defined and equal to $\fint f$. **Q**

(iv) It follows that if f is Henstock integrable and $c \in \mathbb{R}$, $\lim_{b\to\infty} \fint_c^b f$ is defined. **P** Let $\epsilon > 0$. Then there are $a_0 \le c$, $b_0 \ge c$ such that $|\fint_a^b f - \fint f| \le \epsilon$ whenever $a \le a_0$ and $b \ge b_0$. But this means that

$$|\fint_c^b f - \fint_c^{b'} f| = |\fint_{a_0}^b f - \fint_{a_0}^{b'} f| \le 2\epsilon$$

whenever b, $b' \ge b_0$. As ϵ is arbitrary, $\lim_{b\to\infty} \int_c^b f$ is defined. **Q**

Similarly, $\lim_{a\to-\infty} \fint_a^c f$ is defined.

(v) Moreover, if f is Henstock integrable and $a < b$ in $\mathbb{R}$, then $\lim_{c\uparrow b} \fint_a^c f$ is defined and equal to $\int_a^b f$. **P** Let $\epsilon > 0$. Then there is a $\delta \in \Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}} |F^{\text{SH}}(C) - f(x)\mu C| \le \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine. Let $\eta > 0$ be such that $\eta|f(b)| \le \epsilon$ and $(b,[c,b[) \in \delta$ whenever $b - \eta \le c < b$. Then whenever $\max(a, b-\eta) \le c < b$,

$$\fint_a^b f - \fint_a^c f = F^{\text{SH}}([c,b[) \le |F^{\text{SH}}([c,b[) - f(b)\mu\,[c,b[\,| + |f(b)|\mu\,[c,b[\le 2\epsilon.$$

As ϵ is arbitrary, $\lim_{c\uparrow b} \fint_a^c f = \fint_a^b f$. **Q**

Similarly, $\lim_{c\downarrow a} \fint_c^b f$ is defined and equal to $\fint_a^b f$.

(vi) Now for a much larger step. If $-\infty \le \alpha < \beta \le \infty$ and $f : \mathbb{R} \to \mathbb{R}$ is such that $\lim_{a\downarrow\alpha,b\uparrow\beta} \fint_a^b f$ is defined and equal to γ, then $\fint_\alpha^\beta f$ is defined and equal to γ. **P** I seek to apply 482H, with $H = \,]\alpha,\beta[$ and $H_n = \,]a_n,b_n[$, where $\langle a_n\rangle_{n\in\mathbb{N}}$ is a strictly decreasing sequence with limit α, $\langle b_n\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence with limit β, and $a_0 < b_0$. We have already seen that the conditions of 482G are satisfied; 482H(vi) is elementary, and 482H(vii) is covered by (ii) above. So we are left with 482H(viii). Given $\epsilon > 0$, let $m \in \mathbb{N}$ be such that $|\gamma - \fint_a^b f| \le \epsilon$ whenever $\alpha < a \le a_m$ and $b_m \le b < \beta$. For $x \in \mathbb{R}$ let G_x be an open set, containing x, such that

$$\begin{aligned} \overline{G}_x &\subseteq H \text{ if } x \in H,\\ &\subseteq \,]-\infty, a_m[\ \text{ if } x < a_m,\\ &\subseteq \,]b_m, \infty[\ \text{ if } x > b_m, \end{aligned}$$

and let $\delta \in \Delta$ be the neighbourhood gauge corresponding to $\langle G_x\rangle_{x\in\mathbb{R}}$. Now suppose that $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}_{a_m b_m}$-filling. Then $W_{\boldsymbol{t}}$ is a closed bounded interval including $[a_m,b_m]$. Since $\langle C\rangle_{(x,C)\in\boldsymbol{t}}$ is a disjoint family of intervals, $\boldsymbol{t}$ must have an enumeration $\langle(x_i,C_i)\rangle_{i\le r}$ where $x \le y$ whenever $i \le j \le r$, $x \in C_i$ and $y \in C_j$. As $H \subseteq \mathbb{R}$ is an interval, there are $i_0 \le i_1$ such that

$$\boldsymbol{t}\restriction H = \{(x_i,C_i) : i \le r,\, x_i \in H\} = \{(x_i,C_i) : i_0 \le i \le i_1\}.$$

Because $W_{\boldsymbol{t}}$ is an interval, so is $W_{\boldsymbol{t}\restriction H} = \bigcup_{i_0\le i\le i_1} C_i$; set $a = \inf W_{\boldsymbol{t}\restriction H}$ and $b = \sup W_{\boldsymbol{t}\restriction H}$; then $\fint f \times \chi W_{\boldsymbol{t}} = \fint_a^b f$ (see (ii)). Next, $\overline{W}_{\boldsymbol{t}\restriction H} \subseteq H$ (because $\overline{G}_x \subseteq H$ if $x \in H$) and $[a_m,b_m] \subseteq W_{\boldsymbol{t}\restriction H}$ (because $[a_m,b_m] \subseteq W_{\boldsymbol{t}}$ and $\overline{G}_x \cap [a_m,b_m] = \emptyset$ for $x \notin [a_m,b_m]$). So $\alpha < a \le a_m$, $b_m \le b < \beta$, and

$$|\gamma - \fint f \times \chi W_{\boldsymbol{t}\restriction H}| = |\gamma - \fint_a^b f| \le \epsilon.$$

As ϵ is arbitrary, this shows that

$$\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} \fint f \times \chi W_{\boldsymbol{t}\restriction H} = \gamma,$$

as required by 482H(viii). So $\fint_\alpha^\beta f = \fint f \times \chi H$ is defined and equal to γ, as claimed. **Q**

(vii) We are now in a position to confirm that if f is Henstock integrable then $\not\int_c^\infty f = \lim_{b\to\infty} \not\int_c^b f$ is defined for every $c \in \mathbb{R}$. **P** By (iii) and (iv), $\lim_{a\downarrow c} \not\int_a^{c+1} f = \not\int_c^{c+1} f$ and $\lim_{b\to\infty} \not\int_{c+1}^b f$ are both defined. So

$$\begin{aligned}\lim_{a\downarrow c, b\to\infty} \not\int_a^b f &= \lim_{a\downarrow c} \not\int_a^{c+1} f + \lim_{b\to\infty} \not\int_{c+1}^b f \\ &= \not\int_c^{c+1} f + \lim_{b\to\infty} \not\int_{c+1}^b = \lim_{b\to\infty} \not\int_c^b f\end{aligned}$$

is defined, and is equal to $\not\int_c^\infty f$, by (vi). **Q**

Similarly, $\not\int_{-\infty}^c f = \lim_{a\to-\infty} \not\int_a^c f$ is defined. So $\not\int f \times \chi C$ is defined for sets C of the form $]c, \infty[$ or $]-\infty, c[$, and therefore for any unbounded interval, since the case $C = \mathbb{R}$ is immediate. So the proof of (c) is complete.

(viii) As for (d), (vi) has already given us part of it: if $\lim_{a\downarrow\alpha, b\uparrow\beta} \not\int_a^b f$ is defined, this is $\not\int_\alpha^\beta f$. In the other direction, if $\not\int_\alpha^\beta f$ is defined, set $g = f \times \chi]\alpha, \beta[$, so that g is Henstock integrable, and take any $c \in]\alpha, \beta[$. Then $\lim_{b\uparrow\beta} \not\int_c^b g$ is defined, and equal to $\not\int_c^\beta g$, by (v) if β is finite and by (vii) if $\beta = \infty$. Similarly, $\lim_{a\downarrow\alpha} \not\int_a^c g$ is defined and equal to $\not\int_\alpha^c g$. Consequently

$$\lim_{a\downarrow\alpha, b\uparrow\beta} \not\int_a^b f = \lim_{a\downarrow\alpha, b\uparrow\beta} \not\int_a^b g = \lim_{a\downarrow\alpha} \not\int_a^c g + \lim_{b\uparrow\beta} \not\int_c^b g$$

is defined, and must be equal to $\not\int_\alpha^\beta g = \not\int_\alpha^\beta f$, while also

$$\lim_{a\downarrow\alpha} \not\int_a^\beta f = \lim_{a\downarrow\alpha} \not\int_a^\beta g = \lim_{a\downarrow\alpha} \not\int_a^c g + \not\int_c^\beta g = \not\int_\alpha^c g + \not\int_c^\beta g = \not\int_\alpha^\beta g = \not\int_\alpha^\beta f,$$

and similarly

$$\lim_{b\uparrow\beta} \not\int_\alpha^b f = \not\int_\alpha^\beta f.$$

(ix) Finally, we need to consider the case in which we are told that $\lim_{a\downarrow\alpha} \not\int_a^\beta f$ is defined. Taking any $c \in]\alpha, \beta[$, we know that $\not\int_c^\beta f$ is defined, by (ii) or (vii) applied to $f \times \chi]a, \beta[$ for some $a \le c$, and equal to $\lim_{b\uparrow\beta} \not\int_c^b f$, by (v) or (vii). But this means that

$$\lim_{a\downarrow\alpha, b\uparrow\beta} \not\int_a^b f = \lim_{a\downarrow\alpha} \not\int_a^c f + \lim_{b\uparrow\beta} \not\int_c^b f$$

is defined, so (vi) and (viii) tell us that $\not\int_\alpha^\beta f$ is defined and equal to $\lim_{a\downarrow\alpha} \not\int_a^\beta f$. The same argument, suitably inverted, deals with the case in which $\lim_{b\uparrow\beta} \not\int_\alpha^b f$ is defined.

483C Corollary The Henstock and Lebesgue integrals agree on non-negative functions, in the sense that if $f : \mathbb{R} \to [0, \infty[$ then $\not\int f = \int f d\mu$ if either is defined in $\mathbb{R}$.

proof If f is Lebesgue integrable, it is Henstock integrable, with the same integral, by 483Bb. If it is Henstock integrable, then it is measurable, by 483Ba, so that $\int f d\mu$ is defined in $[0, \infty]$; but

$$\int f d\mu = \sup\{\int g\, d\mu : g \le f \text{ is a non-negative simple function}\}$$

(213B)

$$= \sup\{\not\int g : g \le f \text{ is a non-negative simple function}\} \le \not\int f$$

(481Cb) is finite, so f is Lebesgue integrable.

483D Corollary If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, then $\alpha \mapsto \not\int_{-\infty}^\alpha f : [\infty, \infty] \to \mathbb{R}$ and $(\alpha, \beta) \mapsto \not\int_\alpha^\beta f : [-\infty, \infty]^2 \to \mathbb{R}$ are continuous and bounded.

proof Let F be the indefinite Henstock integral of f. Take any $x_0 \in \mathbb{R}$ and $\epsilon > 0$. By 483Bd, there is an $\eta_1 > 0$ such that $|\not\int_{-\infty}^x f - \not\int_{-\infty}^{x_0} f| \le \epsilon$ whenever $x_0 - \eta_1 \le x \le x_0$. By 483Bd again, there is an $\eta_2 > 0$ such that $|\not\int_x^\infty f - \not\int_{x_0}^\infty f| \le \epsilon$

whenever $x_0 \le x \le x_0+\eta_2$. But this means that $|F(x)-F(x_0)| \le \epsilon$ whenever $x_0-\eta_1 \le x \le x_0+\eta_2$. As ϵ is arbitrary, F is continuous at x_0.

We know also that $\lim_{x\to\infty} F(x) = \mathop{\mathrm{I\!\!\!\oint}} f$ is defined in $\mathbb{R}$; while

$$\lim_{x\to-\infty} F(x) = \mathop{\mathrm{I\!\!\!\oint}} f - \lim_{x\to-\infty} \mathop{\mathrm{I\!\!\!\oint}}_x^\infty f = 0$$

is also defined, by 483Bd once more. So F is continuous at $\pm\infty$.

Now writing $G(\alpha,\beta) = \mathop{\mathrm{I\!\!\!\oint}}_\alpha^\beta f$, we have $G(\alpha,\beta) = F(\beta)-F(\alpha)$ if $\alpha \le \beta$ and zero if $\beta \le \alpha$. So G also is continuous. F and G are bounded because $[-\infty,\infty]$ is compact.

483E Definition If $f:\mathbb{R}\to\mathbb{R}$ is Henstock integrable, then its **indefinite Henstock integral** is the function $F:\mathbb{R}\to\mathbb{R}$ defined by saying that $F(x) = \mathop{\mathrm{I\!\!\!\oint}}_{-\infty}^x f$ for every $x\in\mathbb{R}$.

483F In the present context, the Saks-Henstock lemma can be sharpened, as follows.

Theorem Let $f:\mathbb{R}\to\mathbb{R}$ and $F:\mathbb{R}\to\mathbb{R}$ be functions. Then the following are equiveridical:

(i) f is Henstock integrable and F is its indefinite Henstock integral;

(ii)(α) F is continuous,

(β) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x)$ is defined in $\mathbb{R}$,

(γ) for every $\epsilon>0$ there are a gauge $\delta\in\Delta$ and a non-decreasing function $\phi:\mathbb{R}\to[0,\epsilon]$ such that $|f(x)(b-a)-F(b)+F(a)| \le \phi(b)-\phi(a)$ whenever $a\le x\le b$ in $\mathbb{R}$ and $(x,[a,b])\in\delta$.

proof (i)$\Rightarrow$(ii) (α)-(β) are covered by 483D. As for (γ), 482G tells us that we can identify the Saks-Henstock indefinite integral of f with $E \mapsto \mathop{\mathrm{I\!\!\!\oint}} f\times\chi E : \mathcal{E}\to\mathbb{R}$, where $\mathcal{E}$ is the algebra generated by $\mathcal{C}$. Let $\epsilon>0$. Then there is a $\delta\in\Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}} |f(x)\mu C - \mathop{\mathrm{I\!\!\!\oint}} f\times\chi C| \le \epsilon$ for every δ-fine $\boldsymbol{t}\in T$. Set

$$\phi(a) = \sup_{\boldsymbol{t}\in T\cap\delta} \sum_{(x,C)\in\boldsymbol{t}, C\subseteq\,]-\infty,a]} |f(x)\mu C - \mathop{\mathrm{I\!\!\!\oint}} f\times\chi C|,$$

so that $\phi:\mathbb{R}\to[0,\epsilon]$ is a non-decreasing function. Now suppose that $a\le y\le b$ and that $(y,[a,b])\in\delta$. In this case, whenever $\boldsymbol{t}\in T\cap\delta$, $\boldsymbol{s} = \{(x,C) : (x,C)\in\boldsymbol{t},\ C\subseteq\,]-\infty,a]\}\cup\{(y,]a,b[)\}$ also belongs to $T\cap\delta$. Now $F(b)-F(a) = \mathop{\mathrm{I\!\!\!\oint}}_a^b f$, so

$$\begin{aligned}\phi(b) &\ge \sum_{(x,C)\in\boldsymbol{s}} |f(x)\mu C - \mathop{\mathrm{I\!\!\!\oint}} f\times\chi C| \\ &= \sum_{(x,C)\in\boldsymbol{t}, C\subseteq]-\infty,a]} |f(x)\mu C - \mathop{\mathrm{I\!\!\!\oint}} f\times\chi C| + |f(y)(b-a)-F(b)+F(a)|.\end{aligned}$$

As $\boldsymbol{t}$ is arbitrary,

$$\phi(b) \ge \phi(a) + |f(y)(b-a)-F(b)+F(a)|,$$

that is, $|f(y)(b-a)-F(b)+F(a)| \le \phi(b)-\phi(a)$, as called for by ($\gamma$).

(ii)$\Rightarrow$(i) Assume (ii). Set $\gamma = \lim_{x\to\infty} F(x)$. Let $\epsilon>0$. Let $a\le b$ be such that $|F(x)|\le\epsilon$ for every $x\le a$ and $|F(x)-\gamma|\le\epsilon$ for every $x\ge b$. Let $\delta\in\Delta$, $\phi:\mathbb{R}\to[0,\epsilon]$ be such that ϕ is non-decreasing and $|f(x)(\beta-\alpha)-F(\beta)+F(\alpha)| \le \phi(\beta)-\phi(\alpha)$ whenever $\alpha\le x\le\beta$ and $(x,[\alpha,\beta])\in\delta$. Let $\delta'\in\Delta$ be such that $(x,\overline{A})\in\delta$ whenever $(x,A)\in\delta'$. For $C\in\mathcal{C}$ set $\lambda C = F(\sup C)-F(\inf C)$, $\nu C = \phi(\sup C)-\phi(\inf C)$; then if $(x,C)\in\delta'$, $(x,[\inf C,\sup C])\in\delta$, so $|f(x)\mu C-\lambda C| \le \nu C$. Note that λ and ν are both additive in the sense that $\lambda(C\cup C') = \lambda C+\lambda C'$, $\nu(C\cup C') = \nu C+\nu C'$ whenever C, C' are disjoint members of $\mathcal{C}$ such that $C\cup C'\in\mathcal{C}$ (cf. 482G(iii-α)).

Let $\boldsymbol{t}\in T$ be δ'-fine and $\mathcal{R}_{ab}$-filling. Then $W_{\boldsymbol{t}}$ is of the form $[c,d]$ where $c\le a$ and $b\le d$. So

$$\begin{aligned}|S_{\boldsymbol{t}}(f,\mu)-\gamma| &\le 2\epsilon + |S_{\boldsymbol{t}}(f,\mu)-F(d)+F(c)| = 2\epsilon + |S_{\boldsymbol{t}}(f,\mu)-\lambda[c,d]| \\ &= 2\epsilon + |\sum_{(x,C)\in\boldsymbol{t}} f(x)\mu C - \lambda C| \le 2\epsilon + \sum_{(x,C)\in\boldsymbol{t}} \nu C \\ &= 2\epsilon + \nu[c,d] \le 3\epsilon.\end{aligned}$$

As ϵ is arbitrary, f is Henstock integrable, with integral γ.

I still have to check that F is the indefinite integral of f. Set $F_1(x) = \text{⨏}_{-\infty}^{x} f$ for $x \in \mathbb{R}$, and $G = F - F_1$. Then (ii) applies equally to the pair (f, F_1), because (i)$\Rightarrow$(ii). So, given $\epsilon > 0$, we have δ, $\delta_1 \in \Delta$ and non-decreasing functions ϕ, $\phi_1 : \mathbb{R} \to [0, \epsilon]$ such that

$|f(x)(b-a) - F(b) + F(a)| \le \phi(b) - \phi(a)$ whenever $a \le x \le b$ in $\mathbb{R}$ and $(x, [a,b]) \in \delta$,

$|f(x)(b-a) - F_1(b) + F_1(a)| \le \phi_1(b) - \phi_1(a)$ whenever $a \le x \le b$ in $\mathbb{R}$ and $(x, [a,b]) \in \delta_1$.

Putting these together,

$|G(b) - G(a)| \le \psi(b) - \psi(a)$ whenever $a \le x \le b$ in $\mathbb{R}$ and $(x, [a,b]) \in \delta \cap \delta_1$,

where $\psi = \phi + \phi_1$. But if $a \le b$ in $\mathbb{R}$, there are $a_0 \le x_0 \le a_1 \le x_1 \le \ldots \le x_{n-1} \le a_n$ such that $a = a_0$, $a_n = b$ and $(x_i, [a_i, a_{i+1}]) \in \delta$ for $i < n$ (481J), so that

$$\begin{aligned}|G(b) - G(a)| &\le \sum_{i=0}^{n-1} |G(a_{i+1}) - G(a_i)| \\ &\le \sum_{i=0}^{n-1} \psi(a_{i+1}) - \psi(a_i) = \psi(b) - \psi(a) \le 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, G is constant. As $\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} F_1(x) = 0$, $F = F_1$, as required.

483G Theorem Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function. Then there is a countable cover $\mathcal{K}$ of $\mathbb{R}$ by compact sets such that $f \times \chi K$ is Lebesgue integrable for every $K \in \mathcal{K}$.

proof (a) For $n \in \mathbb{N}$ set $E_n = \{x : |x| \le n, |f(x)| \le n\}$. By 483Ba, f is Lebesgue measurable, so for each $n \in \mathbb{N}$ we can find a compact set $K_n \subseteq E_n$ such that $\mu(E_n \setminus K_n) \le 2^{-n}$; f is Lebesgue integrable over K_n, and $Y = \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} K_n$ is Lebesgue negligible.

Let F be the indefinite Henstock integral of f, and take a gauge $\delta_0 \in \Delta$ and a non-decreasing function $\phi : \mathbb{R} \to [0,1]$ such that $|f(x)(b-a) - F(b) + F(a)| \le \phi(b) - \phi(a)$ whenever $a \le x \le b$ and $(x, [a,b]) \in \delta_0$ (483F). Because $\text{⨏} |f| \times \chi Y = \int_Y |f| d\mu = 0$ (483Bb), there is a $\delta_1 \in \Delta$ such that $S_{\boldsymbol{t}}(|f| \times \chi Y, \mu) \le 1$ whenever $\boldsymbol{t} \in T$ is δ_1-fine (482Ad). For $C \in \mathcal{C}$, set $\lambda C = F(\sup C) - F(\inf C)$, $\nu C = \phi(\sup C) - \phi(\inf C)$. Set

$$D_n = \{x : x \in E_n \cap Y,\ (x, [a,b]) \in \delta_0 \cap \delta_1 \text{ whenever } x - 2^{-n} \le a \le x \le b \le x + 2^{-n}\},$$

so that $\bigcup_{n\in\mathbb{N}} D_n = Y$; set $K'_n = \overline{D}_n$, so that K'_n is compact and $\bigcup_{n\in\mathbb{N}} K'_n \supseteq Y$.

(b) The point is that $f \times \chi K'_n$ is Lebesgue integrable for each n. **P** For $k \in \mathbb{N}$, let A_k be the set of points $x \in K'_n$ such that $\left]x, x + 2^{-k}\right] \cap K'_n = \emptyset$; then A_k is finite, because $K'_n \subseteq [-n, n]$ is bounded. Similarly, if $A'_k = \{x : x \in K'_n,\ \left[x - 2^{-k}, x\right[\cap K'_n = \emptyset\}$, A'_k is finite. Set $B = K'_n \setminus \bigcup_{k\in\mathbb{N}} (A_k \cup A'_k)$, so that $K'_n \setminus B$ is countable.

Set

$$\delta = \delta_0 \cap \delta_1 \cap \{(x, A) : x \in \mathbb{R},\ A \subseteq \left]x - 2^{-n-1}, x + 2^{-n-1}\right[\},$$

so that $\delta \in \Delta$. Note that if $C \in \mathcal{C}$, $x \in B \cap \overline{C}$, $(x, C) \in \delta$ and $\mu C > 0$, then $\operatorname{int} C$ meets K'_n (because there are points of K'_n arbitrarily close to x on both sides) so $\operatorname{int} C$ meets D_n; and if $y \in D_n \cap \operatorname{int} C$ then $(y, C) \in \delta_0 \cap \delta_1$, because $\operatorname{diam} C \le 2^{-n}$. This means that if $\boldsymbol{t} \in T$ is δ-fine and $\boldsymbol{t} \subseteq B \times \mathcal{C}$, then there is a $\delta_0 \cap \delta_1$-fine $\boldsymbol{s} \in T$ such that $\boldsymbol{s} \subseteq D_n \times \mathcal{C}$, $W_{\boldsymbol{s}} \subseteq W_{\boldsymbol{t}}$ and whenever $(x, C) \in \boldsymbol{t}$ and C is not a singleton, there is a y such that $(y, C) \in \boldsymbol{s}$. Accordingly

$$\begin{aligned}\sum_{(x,C)\in\boldsymbol{t}} |\lambda C| \le \sum_{(y,C)\in\boldsymbol{s}} |\lambda C| &\le \sum_{(y,C)\in\boldsymbol{s}} |f(y)\mu C - \lambda C| + \sum_{(y,C)\in\boldsymbol{s}} |f(y)|\mu C \\ &\le \sum_{(y,C)\in\boldsymbol{s}} \nu C + S_{\boldsymbol{s}}(|f| \times \chi Y, \mu) \le 2.\end{aligned}$$

But this means that if $\boldsymbol{t} \in T$ is δ-fine,

$$S_{\boldsymbol{t}}(|f \times \chi B|, \mu) = \sum_{(x,C)\in\boldsymbol{t}\restriction B} |f(x)\mu C|$$

(where $\boldsymbol{t}\restriction B = \boldsymbol{t} \cap (B \times \mathcal{C})$)

$$\leq \sum_{(x,C)\in\boldsymbol{t}\restriction B} |f(x)\mu C - \lambda C| + \sum_{(x,C)\in\boldsymbol{t}\restriction B} |\lambda C|$$
$$\leq \sum_{(x,C)\in\boldsymbol{t}\restriction B} \nu C + 2 \leq 3.$$

It follows that if g is a μ-simple function and $0 \leq g \leq |f \times \chi B|$,

$$\int g\,d\mu = \text{H}\!\!\int g \leq \sup_{\boldsymbol{t}\in T \text{ is } \delta\text{-fine}} S_{\boldsymbol{t}}(g,\mu)$$
$$\leq \sup_{\boldsymbol{t}\in T \text{ is } \delta\text{-fine}} S_{\boldsymbol{t}}(|f \times \chi B|,\mu) \leq 3,$$

and $|f \times \chi B|$ is μ-integrable, by 213B, so $f \times \chi B$ is μ-integrable, by 122P. As $K'_n \setminus B$ is countable, therefore negligible, $f \times \chi K'_n$ is μ-integrable. **Q**

(c) So if we set $\mathcal{K} = \{K_n : n \in \mathbb{N}\} \cup \{K'_n : n \in \mathbb{N}\}$, we have a suitable family.

483H Upper and lower derivates: Definition Let $F : \mathbb{R} \to \mathbb{R}$ be any function. For $x \in \mathbb{R}$, set

$$\bar{D}F(x) = \limsup_{y\to x} \tfrac{F(y)-F(x)}{y-x}, \quad \underline{D}F(x) = \liminf_{y\to x} \tfrac{F(y)-F(x)}{y-x}$$

in $[-\infty,\infty]$, that is, $\bar{D}F(x) = \max(\bar{D}^+F(x), \bar{D}^-F(x))$ and $\underline{D}F(x) = \min(\underline{D}^+F(x), \underline{D}^-F(x))$ as defined in 222J.

483I Theorem Suppose that $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, and F is its indefinite Henstock integral. Then $F'(x)$ is defined and equal to $f(x)$ for almost every $x \in \mathbb{R}$.

proof For $n \in \mathbb{N}$, set $A_n = \{x : |x| \leq n, \bar{D}F(x) > f(x) + 2^{-n}\}$. Then $\mu^* A_n \leq 2^{-n+1}$. **P** Let $\delta \in \Delta$ and $\phi : \mathbb{R} \to [0, 4^{-n}]$ be such that ϕ is non-decreasing and $|f(x)(b-a) - F(b) + F(a)| \leq \phi(b) - \phi(a)$ whenever $a \leq x \leq b$ and $(x,[a,b]) \in \delta$ (483F). Let $\mathcal{I}$ be the set of non-trivial closed intervals $[a,b] \subseteq \mathbb{R}$ such that, for some $x \in [a,b] \cap A_n$, $(x,[a,b]) \in \delta$ and $\frac{F(b)-F(a)}{b-a} \geq f(x) + 2^{-n}$. By Vitali's theorem (221A) we can find a countable disjoint family $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $A_n \setminus \bigcup\mathcal{I}_0$ is negligible; so we have a finite family $\mathcal{I}_1 \subseteq \mathcal{I}_0$ such that $\mu^*(A_n \setminus \bigcup\mathcal{I}_1) \leq 2^{-n}$. Enumerate $\mathcal{I}_1$ as $\langle [a_i,b_i]\rangle_{i<m}$, and for each $i < m$ take $x_i \in [a_i,b_i] \cap A_n$ such that $(x_i,[a_i,b_i]) \in \delta$ and $F(b_i) - F(a_i) \geq (b_i - a_i)(f(x_i) + 2^{-n})$. Then

$$\phi(b_i) - \phi(a_i) \geq |f(x_i)(b_i - a_i) - F(b_i) + F(a_i)| \geq 2^{-n}(b_i - a_i)$$

for each $i < m$, so

$$\mu(\bigcup\mathcal{I}_1) = \textstyle\sum_{i<m} b_i - a_i \leq 2^n \sum_{i<m} \phi(b_i) - \phi(a_i) \leq 2^{-n},$$

and $\mu^* A_n \leq 2^{-n+1}$. **Q**

Accordingly $\{x : \bar{D}F(x) > f(x)\} = \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} A_n$ is negligible. Similarly, or applying the argument to $-f$, $\{x : \underline{D}F(x) < f(x)\}$ is negligible. So $\bar{D}F \leq_{\text{a.e.}} f \leq_{\text{a.e.}} \underline{D}F$. Since $\underline{D}F \leq \bar{D}F$ everywhere, $\bar{D}F =_{\text{a.e.}} \underline{D}F =_{\text{a.e.}} f$. But $F'(x) = f(x)$ whenever $\bar{D}F(x) = \underline{D}F(x) = f(x)$, so we have the result.

483J Theorem Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the following are equiveridical:

(i) f is Henstock integrable;

(ii) for every $\epsilon > 0$ there are functions F_1, $F_2 : \mathbb{R} \to \mathbb{R}$, with finite limits at both $-\infty$ and ∞, such that $\bar{D}F_1(x) \leq f(x) \leq \underline{D}F_2(x)$ and $0 \leq F_2(x) - F_1(x) \leq \epsilon$ for every $x \in \mathbb{R}$.

proof (i)$\Rightarrow$(ii) Suppose that f is Henstock integrable and that $\epsilon > 0$. Let F be the indefinite Henstock integral of f. Let $\delta \in \Delta$, $\phi : \mathbb{R} \to [0, \frac{1}{2}\epsilon]$ be such that ϕ is non-decreasing and $|f(x)(b-a) - F(b) + F(a)| \leq \phi(b) - \phi(a)$ whenever $a \leq x \leq b$ and $(x,[a,b]) \in \delta$ (483F). Set $F_1 = F - \phi$, $F_2 = F + \phi$; then $F_1(x) \leq F_2(x) \leq F_1(x) + \epsilon$ for every $x \in \mathbb{R}$, and the limits at $\pm\infty$ are defined because F and ϕ both have limits at both ends. If $x \in \mathbb{R}$, there is an $\eta > 0$ such that $(x,A) \in \delta$ whenever $A \subseteq [x-\eta, x+\eta]$. So if $x - \eta \leq a \leq x \leq b \leq x + \eta$ and $a < b$,

$$|\tfrac{F(b)-F(a)}{b-a} - f(x)| \leq \tfrac{\phi(b)-\phi(a)}{b-a},$$

and

$$\frac{F_1(b)-F_1(a)}{b-a} \le f(x) \le \frac{F_2(b)-F_2(a)}{b-a}.$$

In particular, this is true whenever $x-\eta \le a < x = b$ or $x = a < b \le x+\eta$. So $\overline{D}F_1(x) \le f(x) \le \underline{D}F_2(x)$. As x is arbitrary, we have a suitable pair F_1, F_2.

(ii)⇒(i) Suppose that (ii) is true. Take any $\epsilon > 0$. Let F_1, $F_2 : \mathbb{R} \to \mathbb{R}$ be as in the statement of (ii).

(α) We need to know that $F_2 - F_1$ is non-decreasing. **P** Set $G = F_2 - F_1$. Then

$$\begin{aligned}\liminf_{y\to x} \frac{G(y)-G(x)}{y-x} &= \liminf_{y\to x} \frac{F_2(y)-F_2(x)}{y-x} - \frac{F_1(y)-F_1(x)}{y-x} \\ &\ge \liminf_{y\to x} \frac{F_2(y)-F_2(x)}{y-x} - \limsup_{y\to x} \frac{F_1(y)-F_1(x)}{y-x}\end{aligned}$$

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$$= \underline{D}F_2(x) - \overline{D}F_1(x) \ge 0$$

for any $x \in \mathbb{R}$. **?** If $a < b$ and $G(a) > G(b)$, set $\gamma = \frac{G(a)-G(b)}{2(b-a)}$, and choose $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively as follows. $a_0 = a$ and $b_0 = b$. Given that $a_n < b_n$ and $G(a_n) - G(b_n) > \gamma(b_n - a_n)$, set $c = \frac{1}{2}(a_n + b_n)$; then either $G(a_n) - G(c) > \gamma(c - a_n)$ or $G(c) - G(b_n) \ge \gamma(b_n - c)$; in the former case, take $a_{n+1} = a_n$ and $b_{n+1} = c$; in the latter, take $a_{n+1} = c$ and $b_{n+1} = b_n$. Set $x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Then for each n, either $G(a_n) - G(x) > \gamma(x - a_n)$ or $G(x) - G(b_n) > \gamma(b_n - x)$. In either case, we have a y such that $0 < |y-x| \le 2^{-n}(b-a)$ and $\frac{G(y)-G(x)}{y-x} < -\gamma$. So $\underline{D}G(x) \le -\gamma < 0$, which is impossible. **X**

Thus G is non-decreasing, as required. **Q**

(β) Let $a \le b$ be such that $|F_1(x) - F_1(a)| \le \epsilon$ whenever $x \le a$ and $|F_1(x) - F_1(b)| \le \epsilon$ whenever $x \ge b$. Let $h : \mathbb{R} \to \,]0,\infty[$ be a strictly positive integrable function such that $\int h\,d\mu \le \epsilon$. Then $\mathcal{I}\!\!\!\int h \le \epsilon$, by 483Bb, so there is a $\delta_0 \in \Delta$ such that $S_{\boldsymbol{t}}(h,\mu) \le 2\epsilon$ for every δ_0-fine $\boldsymbol{t} \in T$ (482Ad). For $x \in \mathbb{R}$ let $\eta_x > 0$ be such that

$$\frac{F_1(y)-F_1(x)}{y-x} \le f(x) + h(x), \quad \frac{F_2(y)-F_2(x)}{y-x} \ge f(x) - h(x)$$

whenever $0 < |y-x| \le 2\eta_x$; set $\delta = \{(x,A) : (x,A) \in \delta_0,\ A \subseteq \,]x-\eta_x, x+\eta_x[\}$, so that $\delta \in \Delta$. Note that if $x \in \mathbb{R}$ and $x - \eta_x \le \alpha \le x \le \beta \le x + \eta_x$, then

$$F_1(\beta) - F_1(x) \le (\beta - x)(f(x) + h(x)), \quad F_1(x) - F_1(\alpha) \le (x - \alpha)(f(x) + h(x)),$$

so that $F_1(\beta) - F_1(\alpha) \le (\beta - \alpha)(f(x) + h(x))$; and similarly $F_2(\beta) - F_2(\alpha) \ge (\beta - \alpha)(f(x) - h(x))$.

For $C \in \mathcal{C}$, set

$$\lambda_1 C = F_1(\sup C) - F_1(\inf C), \quad \lambda_2 C = F_2(\sup C) - F_2(\inf C).$$

Then if $C \in \mathcal{C}$, $x \in \overline{C}$ and $(x,C) \in \delta$,

$$\lambda_1 C \le (f(x) + h(x))\mu C, \quad \lambda_2 C \ge (f(x) - h(x))\mu C.$$

Suppose that $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}_{ab}$-filling. Then $W_{\boldsymbol{t}} = [\alpha, \beta]$ for some $\alpha \le a$ and $\beta \ge b$, so that

$$\begin{aligned}S_{\boldsymbol{t}}(f,\mu) &= \sum_{(x,C)\in\boldsymbol{t}} f(x)\mu C \le \sum_{(x,C)\in\boldsymbol{t}} \lambda_2 C + h(x)\mu C = \lambda_2[\alpha,\beta] + S_{\boldsymbol{t}}(h,\mu) \\ &\le F_2(\beta) - F_2(\alpha) + 2\epsilon \le F_1(\beta) - F_1(\alpha) + 3\epsilon \le F_1(b) - F_1(a) + 5\epsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}S_{\boldsymbol{t}}(f,\mu) &= \sum_{(x,C)\in\boldsymbol{t}} f(x)\mu C \ge \sum_{(x,C)\in\boldsymbol{t}} \lambda_1 C - h(x)\mu C \\ &= \lambda_1[\alpha,\beta] - S_{\boldsymbol{t}}(h,\mu) \ge F_1(\beta) - F_1(\alpha) - 2\epsilon \ge F_1(b) - F_1(a) - 4\epsilon.\end{aligned}$$

But this means that if $\boldsymbol{t}$, $\boldsymbol{t}'$ are two δ-fine $\mathcal{R}_{ab}$-filling members of T, $|S_{\boldsymbol{t}}(f,\mu) - S_{\boldsymbol{t}'}(f,\mu)| \le 9\epsilon$. As ϵ is arbitrary,

$$\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\mu) = \mathcal{H}\!\!\int f$$

is defined.

Remark The formulation (ii) above is a version of the method of integration described by Perron 1914.

483K Proposition Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function, and F its indefinite Henstock integral. Then $F[E]$ is Lebesgue negligible for every Lebesgue negligible set $E \subseteq \mathbb{R}$.

proof Let $\epsilon > 0$. By 483C and 482Ad, as usual, together with 483F, there are a $\delta \in \Delta$ and a non-decreasing $\phi : \mathbb{R} \to [0,\epsilon]$ such that

$$S_{\boldsymbol{t}}(|f| \times \chi E,\mu) \le \epsilon, \quad |f(x)(b-a) - F(b) + F(a)| \le \phi(b) - \phi(a)$$

whenever $\boldsymbol{t} \in T$ is δ-fine, $a \le x \le b$ and $(x,[a,b]) \in \delta$. For $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, set

$$E_{ni} = \{x : x \in E \cap [2^{-n}i, 2^{-n}(i+1)[\,, (x,A) \in \delta \text{ whenever } A \subseteq [x-2^{-n}, x+2^{-n}]\}.$$

Set $J_n = \{i : i \in \mathbb{Z}, -4^n < i \le 4^n, E_{ni} \ne \emptyset\}$. Observe that

$$E = \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \bigcup_{i\in J_m} E_{mi}.$$

For $i \in J_n$, take x_{ni}, $y_{ni} \in E_{ni}$ such that $x_{ni} \le y_{ni}$ and

$$\min(F(x_{ni}), F(y_{ni})) \le \inf F[E_{ni}] + 4^{-n}\epsilon,$$

$$\max(F(x_{ni}), F(y_{ni})) \ge \sup F[E_{ni}] - 4^{-n}\epsilon,$$

so that $\mu^* F[E_{ni}] \le |F(y_{ni}) - F(x_{ni})| + 2^{-2n+1}\epsilon$. Now, for each $i \in J_n$, $(x_{ni}, [x_{ni}, y_{ni}]) \in \delta$, while $[x_{ni}, y_{ni}] \subseteq [2^{-n}i, 2^{-n}(i+1)[$, so $\boldsymbol{t} = \{(x_{ni}, [x_{ni}, y_{ni}]) : i \in J_n\}$ is a δ-fine member of T, and

$$\begin{aligned}
\mu^* F[\bigcup_{i\in J_n} E_{ni}] &\le \sum_{i\in J_n} \mu^* F[E_{ni}] \le \sum_{i\in J_n} 2^{-2n+1}\epsilon + |F(y_{ni}) - F(x_{ni})| \\
&\le 4\epsilon + \sum_{i\in J_n} |f(x_{ni})(y_{ni} - x_{ni})| + \sum_{i\in J_n} \phi(y_{ni}) - \phi(x_{ni}) \\
&\le 4\epsilon + S_{\boldsymbol{t}}(|f| \times \chi E,\mu) + \epsilon \le 6\epsilon.
\end{aligned}$$

Since this is true for every $n \in \mathbb{N}$,

$$\begin{aligned}
\mu^* F[E] &= \mu^* F[\bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \bigcup_{i\in J_m} E_i] \\
&= \mu^*(\bigcup_{n\in\mathbb{N}} F[\bigcap_{m\ge n} \bigcup_{i\in J_m} E_i]) = \sup_{n\in\mathbb{N}} \mu^* F[\bigcap_{m\ge n} \bigcup_{i\in J_m} E_i]
\end{aligned}$$

(132Ae)

$$\le 6\epsilon.$$

As ϵ is arbitrary, $F[E]$ is negligible, as claimed.

Remark Compare 225M.

483L Definition If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, I write $\|f\|_H$ for $\sup_{a\le b} |\mathcal{H}\!\!\int_a^b f|$. It is elementary to check that this is a seminorm on the linear space of all Henstock integrable functions. (It is finite-valued by 483D.)

483M Proposition (a) If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, then $|\mathcal{H}\!\!\int f| \le \|f\|_H$, and $\|f\|_H = 0$ iff $f = 0$ a.e.

(b) Write $\mathcal{HL}^1$ for the linear space of all Henstock integrable real-valued functions on $\mathbb{R}$, and HL^1 for $\{f^\bullet : f \in \mathcal{HL}^1\} \subseteq L^0(\mu)$ (§241). If we write $\|f^\bullet\|_H = \|f\|_H$ for every $f \in \mathcal{HL}^1$, then HL^1 is a normed space. The ordinary space $L^1(\mu)$ of equivalence classes of Lebesgue integrable functions is a linear subspace of HL^1, and $\|u\|_H \le \|u\|_1$ for every $u \in L^1(\mu)$.

(c) We have a linear operator $T : HL^1 \to C_b(\mathbb{R})$ defined by saying that $T(f^\bullet)$ is the indefinite Henstock integral of f for every $f \in \mathcal{HL}^1$, and $\|T\| = 1$.

proof (a) Of course

$$|\,\mathop{\mathrm{I\!\!\!\int}} f| = \lim_{a\to-\infty,b\to\infty} |\,\mathop{\mathrm{I\!\!\!\int}}_a^b f| \le \|f\|_H$$

(using 483Bd). Let F be the indefinite Henstock integral of f, so that $F(b)-F(a)=\mathop{\mathrm{I\!\!\!\int}}_a^b f$ whenever $a\le b$. If $f=0$ a.e., then $F(x)=\int_{-\infty}^x f d\mu = 0$ for every x, by 483Bb, so $\|f\|_H=0$. If $\|f\|_H=0$, then F is constant, so $f=F'=0$ a.e., by 483I.

(b) That HL^1 is a normed space follows immediately from (a). (Compare the definitions of the norms $\|\,\|_p$ on L^p, for $1\le p\le\infty$, in §§242-244.) By 483Bb, $L^1(\mu)\subseteq HL^1$, and

$$\|u\|_H \le \|u^+\|_H + \|u^-\|_H = \|u^+\|_1 + \|u^-\|_1 = \|u\|_1$$

for every $u\in L^1(\mu)$, writing u^+ and u^- for the positive and negative parts of u, as in Chapter 24.

(c) If f, $g\in\mathcal{HL}^1$ and $f^\bullet=g^\bullet$, then f and g have the same indefinite Henstock integral, by 483Bb or otherwise; so T is defined as a function from HL^1 to $\mathbb{R}^{\mathbb{R}}$. By 483F, Tu is continuous and bounded for every $u\in HL^1$, and by 481Ca T is linear. If $f\in\mathcal{HL}^1$ and $Tf^\bullet=F$, then $\|f\|_H=\sup_{x,y\in\mathbb{R}}|F(y)-F(x)|$; since $\lim_{x\to-\infty}F(x)=0$, $\|f\|_H\ge\|F\|_\infty$; as f is arbitrary, $\|T\|\le 1$. On the other hand, for any non-negative Lebesgue integrable function f, $\|Tf^\bullet\|_\infty=\|f\|_1=\|f\|_H$, so $\|T\|=1$.

483N Proposition Suppose that $\langle I_m\rangle_{m\in M}$ is a disjoint family of open intervals in $\mathbb{R}$ with union G, and that $f:\mathbb{R}\to\mathbb{R}$ is a function such that $f_m=f\times\chi I_m$ is Henstock integrable for every $m\in M$. If $\sum_{m\in M}\|f_m\|_H<\infty$, then $f\times\chi G$ is Henstock integrable, and $\mathop{\mathrm{I\!\!\!\int}} f\times\chi G=\sum_{m\in M}\mathop{\mathrm{I\!\!\!\int}} f_m$.

proof I seek to apply 482H again. We have already seen, in the proof of 483Bc, that the conditions of 482G are satisfied by $\mathbb{R}$, T, Δ, $\mathfrak{R}$, $\mathcal{C}$, $\mathfrak{T}$ and μ. Of course $G=\bigcup_{m\in M}I_m$ is the union of a sequence of open sets over which f is Henstock integrable. So we have only to check 482H(viii).

Set

$$\delta_0=\bigcup_{m\in M}\{(x,A):x\in I_m,\,A\subseteq I_m\}\cup\{(x,A):x\in\mathbb{R}\setminus G,\,A\subseteq\mathbb{R}\},$$

so that $\delta_0\in\Delta$. For each $m\in M$ let F_m^{SH} be the Saks-Henstock indefinite integral of f_m. Let $\epsilon>0$. Then there is a finite set $M_0\subseteq M$ such that $\sum_{m\in M\setminus M_0}\|f_m\|_H\le\epsilon$. Next, there must be $\delta_1\in\Delta$ and $\mathcal{R}\in\mathfrak{R}$ such that

$$\sum_{m\in M_0}|\,\mathop{\mathrm{I\!\!\!\int}} f_m - S_{\boldsymbol{t}}(f_m,\mu)|\le\epsilon$$

for every δ_1-fine $\mathcal{R}$-filling $\boldsymbol{t}\in T$, and $\delta_2\in\Delta$ such that $\sum_{m\in M_0}|S_{\boldsymbol{t}}(f_m,\mu)-F_m^{\text{SH}}(W_{\boldsymbol{t}})|\le\epsilon$ for every δ_2-fine $\boldsymbol{t}\in T$.

Now let $\boldsymbol{t}\in T$ be $(\delta_0\cap\delta_1\cap\delta_2)$-fine and $\mathcal{R}$-filling. For each $m\in M$ set $\boldsymbol{t}_m=\boldsymbol{t}\restriction I_m$, so that $\boldsymbol{t}\restriction G=\bigcup_{m\in M}\boldsymbol{t}_m$. Because $W_{\boldsymbol{t}}$ is an interval, each $W_{\boldsymbol{t}_m}$ must be an interval, as in part (c)-(d)(vi) of the proof of 483B, and $W_{\boldsymbol{t}_m}$ is a subinterval of I_m because $\boldsymbol{t}$ is δ_0-fine. So (using 482G)

$$|F_m^{\text{SH}}(W_{\boldsymbol{t}_m})|=|\,\mathop{\mathrm{I\!\!\!\int}} f_m\times\chi W_{\boldsymbol{t}_m}|\le\|f_m\|_H.$$

Also

$$\begin{aligned}\sum_{m\in M_0}|\,\mathop{\mathrm{I\!\!\!\int}} f_m-\mathop{\mathrm{I\!\!\!\int}} f\times\chi W_{\boldsymbol{t}_m}| &\le \sum_{m\in M_0}|\,\mathop{\mathrm{I\!\!\!\int}} f_m - S_{\boldsymbol{t}}(f_m,\mu)|+\sum_{m\in M_0}|S_{\boldsymbol{t}}(f_m,\mu)-F_m^{\text{SH}}(W_{\boldsymbol{t}_m}|\\ &\le 2\epsilon.\end{aligned}$$

On the other hand,

$$\sum_{m\in M\setminus M_0}|\,\mathop{\mathrm{I\!\!\!\int}} f_m-\mathop{\mathrm{I\!\!\!\int}} f\times\chi W_{\boldsymbol{t}_m}|\le 2\sum_{m\in M\setminus M_0}\|f_m\|_H\le 2\epsilon.$$

Putting these together,

$$|\,\mathop{\mathrm{I\!\!\!\int}} f\times\chi W_{\boldsymbol{t}\restriction G}-\sum_{m\in M}\mathop{\mathrm{I\!\!\!\int}} f_m|=|\sum_{m\in M}\mathop{\mathrm{I\!\!\!\int}} f\times\chi W_{\boldsymbol{t}_m}-\sum_{m\in M}\mathop{\mathrm{I\!\!\!\int}} f_m|$$

(because $\boldsymbol{t}$ is finite, so all but finitely many terms in the sum $\sum_{m\in M}f\times\chi W_{\boldsymbol{t}_m}$ are zero)

$$\le\sum_{m\in M}|\,\mathop{\mathrm{I\!\!\!\int}} f\times\chi W_{\boldsymbol{t}_m}-\mathop{\mathrm{I\!\!\!\int}} f_m|\le 4\epsilon.$$

As ϵ is arbitrary, condition 482H(viii) is satisfied, with

$$\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} \oint f \times \chi W_{\boldsymbol{t}\restriction G} = \sum_{m\in M} \oint f_m,$$

and 482H gives the result we seek.

483O Definitions (a) For any real-valued function F, write $\omega(F)$ for $\sup_{x,y\in\operatorname{dom}F} |F(x)-F(y)|$, the **oscillation** of F. (Interpret $\sup\emptyset$ as 0, so that $\omega(\emptyset)=0$.)

(b) Let $F:\mathbb{R}\to\mathbb{R}$ be a function. For $A\subseteq\mathbb{R}$, we say that F is $\mathbf{AC}_*$ **on** A if for every $\epsilon>0$ there is an $\eta>0$ such that $\sum_{I\in\mathcal{I}}\omega(F\restriction\overline{I})\le\epsilon$ whenever $\mathcal{I}$ is a disjoint family of open intervals with endpoints in A and $\sum_{I\in\mathcal{I}}\mu I\le\eta$. Note that whether F is AC_* on A is *not* determined by $F\restriction A$, since it depends on the behaviour of F on intervals with endpoints in A.

(c) Finally, F is $\mathbf{ACG}_*$ if it is continuous and there is a countable family $\mathcal{A}$ of sets, covering $\mathbb{R}$, such that F is AC_* on every member of $\mathcal{A}$.

483P Elementary results (a)(i) If F, $G:\mathbb{R}\to\mathbb{R}$ are functions and $A\subseteq B\subseteq\mathbb{R}$, then $\omega(F+G\restriction A)\le\omega(F\restriction A)+\omega(G\restriction A)$ and $\omega(F\restriction A)\le\omega(F\restriction B)$.

(ii) If F is the indefinite Henstock integral of $f:\mathbb{R}\to\mathbb{R}$ and $C\subseteq\mathbb{R}$ is an interval, then $\|f\times\chi C\|_H=\omega(F\restriction C)$.

(iii) If $F:\mathbb{R}\to\mathbb{R}$ is continuous, then $(a,b)\mapsto\omega(F\restriction[a,b]):\mathbb{R}^2\to\mathbb{R}$ is continuous, and $\omega(F\restriction\overline{A})=\omega(F\restriction A)$ for every set $A\subseteq\mathbb{R}$.

(b)(i) If $F:\mathbb{R}\to\mathbb{R}$ is AC_* on $A\subseteq\mathbb{R}$, it is AC_* on every subset of A.

(ii) If $F:\mathbb{R}\to\mathbb{R}$ is continuous and is AC_* on $A\subseteq\mathbb{R}$, it is AC_* on $\overline{A}$. **P** Let $\epsilon>0$. Let $\eta>0$ be such that $\sum_{I\in\mathcal{I}}\omega(F\restriction\overline{I})\le\epsilon$ whenever $\mathcal{I}$ is a disjoint family of open intervals with endpoints in A and $\sum_{I\in\mathcal{I}}\mu I\le\eta$. Let $\mathcal{I}$ be a disjoint family of open intervals with endpoints in $\overline{A}$ and $\sum_{I\in\mathcal{I}}\mu I\le\frac{1}{2}\eta$. Let $\mathcal{I}_0\subseteq\mathcal{I}$ be a non-empty finite set; then we can enumerate $\mathcal{I}_0$ as $\langle\,]a_i,b_i[\,\rangle_{i\le n}$ where $a_0,b_0,\dots,a_n,b_n\in\overline{A}$ and $a_0\le b_0\le a_1\le b_1\le\dots\le a_n\le b_n$. Because $(a,b)\mapsto\omega(F\restriction[a,b])$ is continuous, as noted in (a-ii) above, we can find $a_0',\dots,b_n'\in A$ such that $a_0'\le b_0'\le a_1'\le\dots\le a_n'\le b_n'$, $\sum_{i=0}^n b_i'-a_i'\le\eta$, and $\sum_{i=0}^n|\omega(F\restriction[a_i',b_i'])-\omega(F\restriction[a_i,b_i])|\le\epsilon$; so that

$$\sum_{I\in\mathcal{I}_0}\omega(F\restriction\overline{I})\le\sum_{i=0}^n\omega(F\restriction[a_i',b_i'])\le2\epsilon.$$

As $\mathcal{I}_0$ is arbitrary, $\sum_{I\in\mathcal{I}}\omega(F\restriction\overline{I})\le2\epsilon$; as ϵ is arbitrary, F is AC_* on $\overline{A}$. **Q**

483Q Lemma Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous function, and $K\subseteq\mathbb{R}$ a non-empty compact set such that F is AC_* on K. Write $\mathcal{I}$ for the family of non-empty bounded open intervals, disjoint from K, with endpoints in K.

(a) $\sum_{I\in\mathcal{I}}\omega(F\restriction I)$ is finite.

(b) Write a^* for $\inf K=\min K$. Then there is a Lebesgue integrable function $g:\mathbb{R}\to\mathbb{R}$, zero off K, such that

$$F(x)-F(a^*)=\int_{a^*}^x g+\sum_{J\in\mathcal{I},J\subseteq[a^*,x]}F(\sup J)-F(\inf J)$$

for every $x\in K$.

proof (a) Let $\eta>0$ be such that $\sum_{I\in\mathcal{J}}\omega(F\restriction\overline{I})\le1$ whenever $\mathcal{J}$ is a disjoint family of open intervals with endpoints in K and $\sum_{I\in\mathcal{J}}\mu I\le\eta$. Let m_0, $m_1\in\mathbb{Z}$ be such that $K\subseteq[m_0\eta,m_1\eta]$. For integers m between m_0 and m_1, let $\mathcal{I}_m$ be the set of intervals in $\mathcal{I}$ included in $]m\eta,(m+1)\eta[$. Then $\sum_{I\in\mathcal{I}_m}\mu I\le\eta$, so $\sum_{I\in\mathcal{I}_m}\omega(F\restriction I)\le1$ for each m. Also every member of $\mathcal{J}=\mathcal{I}\setminus\bigcup_{m_0\le m<m_1}\mathcal{I}_m$ contains $m\eta$ for some m between m_0 and m_1, so $\#(\mathcal{J})\le m_1-m_0$. Accordingly

$$\begin{aligned}\sum_{I\in\mathcal{I}}\omega(F\restriction\overline{I})&\le\sum_{I\in\mathcal{J}}\omega(F\restriction\overline{I})+\sum_{m=m_0}^{m_1-1}\sum_{I\in\mathcal{I}_m}\omega(F\restriction\overline{I})\\&\le\sum_{I\in\mathcal{J}}\omega(F\restriction\overline{I})+m_1-m_0<\infty\end{aligned}$$

because F is continuous, therefore bounded on every bounded interval.

(b)(i) Set $b^*=\sup K=\max K$. Define $G:[a^*,b^*]\to\mathbb{R}$ by setting

$$G(x) = F(x) \text{ if } x \in K,$$
$$= \frac{F(b)(x-a)+F(a)(b-x)}{b-a} \text{ if } x \in \,]a,b[\, \in \mathcal{I}.$$

Then G is absolutely continuous. **P** G is continuous because F is. Let $\epsilon > 0$. Let $\eta_1 > 0$ be such that $\sum_{I\in\mathcal{J}} \omega(F\restriction\overline{I}) \le \epsilon$ whenever $\mathcal{J}$ is a disjoint family of open intervals with endpoints in K and $\sum_{I\in\mathcal{J}} \mu I \le \eta_1$. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be a finite set such that $\sum_{I\in\mathcal{I}\setminus\mathcal{I}_0} \omega(F\restriction\overline{I}) \le \epsilon$, and take $M > 0$ such that $|F(b) - F(a)| \le M(b-a)$ whenever $]a,b[\, \in \mathcal{I}_0$; set $\eta = \min(\eta_1, \frac{\epsilon}{M}) > 0$.

Let $\mathcal{J}^*$ be the set of non-empty open subintervals J of $[a^*,b^*]$ such that either $J \cap K = \emptyset$ or both endpoints of J belong to K. Let $\mathcal{J} \subseteq \mathcal{J}^*$ be a disjoint family such that $\sum_{I\in\mathcal{J}} \mu I \le \eta$. Set $\mathcal{J}' = \{J : J \in \mathcal{J},\, J \cap K = \emptyset\}$. Then

$$\sum_{J\in\mathcal{J}\setminus\mathcal{J}'} |G(\sup J) - G(\inf J)| = \sum_{J\in\mathcal{J}\setminus\mathcal{J}'} |F(\sup J) - F(\inf J)|$$
$$\le \sum_{J\in\mathcal{J}\setminus\mathcal{J}'} \omega(F\restriction\overline{J}) \le \epsilon.$$

On the other hand,

$$\sum_{J\in\mathcal{J}'} |G(\sup J) - G(\inf J)| = \sum_{I\in\mathcal{I}_0} \sum_{\substack{J\in\mathcal{J}\\ J\subseteq I}} |G(\sup J) - G(\inf J)|$$
$$+ \sum_{I\in\mathcal{I}\setminus\mathcal{I}_0} \sum_{\substack{J\in\mathcal{J}\\ J\subseteq I}} |G(\sup J) - G(\inf J)|$$
$$\le M \sum_{I\in\mathcal{I}_0} \sum_{\substack{J\in\mathcal{J}\\ J\subseteq I}} \mu J + \sum_{I\in\mathcal{I}\setminus\mathcal{I}_0} |F(\sup I) - F(\inf I)|$$

(because G is monotonic on $\overline{I}$ for each $I \in \mathcal{I}$)

$$\le M\eta + \epsilon \le 2\epsilon,$$

so $\sum_{J\in\mathcal{J}} |G(\sup J) - G(\inf J)| \le 3\epsilon$.

Generally, if J is any non-empty open subinterval of $[a^*,b^*]$, we can split it into at most three intervals belonging to $\mathcal{J}^*$. So if $\mathcal{J}$ is any disjoint family of non-empty open subintervals of $[a^*,b^*]$ with $\sum_{J\in\mathcal{J}} \mu J \le \eta$, we can find a family $\tilde{\mathcal{J}} \subseteq \mathcal{J}^*$ with $\sum_{J\in\tilde{\mathcal{J}}} \mu J = \sum_{J\in\mathcal{J}} \mu J$ and $\sum_{J\in\tilde{\mathcal{J}}} |G(\sup J) - G(\inf J)| \ge \sum_{J\in\mathcal{J}} |G(\sup J) - G(\inf J)|$. But this means that $\sum_{J\in\mathcal{J}} |G(\sup J) - G(\inf J)| \le 3\epsilon$. As ϵ is arbitrary, G is absolutely continuous. **Q**

(ii) By 225E, G' is Lebesgue integrable and $G(x) = G(a^*) + \int_{a^*}^x G'$ for every $x \in [a^*,b^*]$. Set $g(x) = G'(x)$ when $x \in K$ and $G'(x)$ is defined, 0 for other $x \in \mathbb{R}$, so that $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable. Now take any $x \in K$. Then

$$F(x) = G(x) = G(a^*) + \int_{a^*}^x G' = F(a^*) + \int_{a^*}^x g + \int_{[a^*,x]\setminus K} G'$$
$$= F(a^*) + \int_{a^*}^x g + \sum_{\substack{I\in\mathcal{I}\\ I\subseteq[a^*,x]}} \int_I G'$$

(because $\mathcal{I}$ is a disjoint countable family of measurable sets, and $\bigcup_{I\in\mathcal{I},I\subseteq[a^*,x]} I = [a^*,x] \setminus K$)

$$= F(a^*) + \int_{a^*}^x g + \sum_{\substack{I\in\mathcal{I}\\ I\subseteq[a^*,x]}} G(\sup I) - G(\inf I)$$

(note that this sum is absolutely summable)

$$= F(a^*) + \int_{a^*}^x g + \sum_{\substack{I\in\mathcal{I}\\ I\subseteq[a^*,x]}} F(\sup I) - F(\inf I)$$

as required.

483R Theorem Let $F:\mathbb{R}\to\mathbb{R}$ be a function. Then F is an indefinite Henstock integral iff it is ACG$_*$, $\lim_{x\to-\infty}F(x)=0$ and $\lim_{x\to\infty}F(x)$ is defined in $\mathbb{R}$.

proof (a) Suppose that F is the indefinite Henstock integral of $f:\mathbb{R}\to\mathbb{R}$.

(i) By 483F, F is continuous, with limit zero at $-\infty$ and finite at ∞. So I have just to show that there is a sequence of sets, covering $\mathbb{R}$, on each of which F is AC$_*$. Recall that there is a sequence $\langle K_m\rangle_{m\in\mathbb{N}}$ of compact sets, covering $\mathbb{R}$, such that $f\times K_m$ is Lebesgue integrable for every $m\in\mathbb{N}$ (483G). By the arguments of (i)$\Rightarrow$(ii) in the proof of 483J, there is a function $F_2\ge F$ such that $\underline{D}F_2\ge f$ and F_2-F is non-decreasing and takes values between 0 and 1. For $n\in\mathbb{N}$, $j\in\mathbb{Z}$ set

$$I_{nj}=[2^{-n}j,2^{-n}(j+1)],$$

$$B_{nj}=\{x:x\in I_{nj},\ \tfrac{F_2(y)-F_2(x)}{y-x}\ge -n\text{ whenever }y\in I_{nj}\setminus\{x\}\}.$$

Observe that $\bigcup_{n\in\mathbb{N},j\in\mathbb{Z}}B_{nj}=\mathbb{R}$, so that $\{B_{nj}\cap K_m:m,\,n\in\mathbb{N},\,j\in\mathbb{Z}\}$ is a countable cover of $\mathbb{R}$. It will therefore be enough to show that F is AC$_*$ on every $B_{nj}\cap K_m$.

(ii) Fix m, $n\in\mathbb{N}$ and $j\in\mathbb{Z}$, and set $A=B_{nj}\cap K_m$. If $A=\emptyset$, then of course F is AC$_*$ on A; suppose that A is not empty. Set $G(x)=F_2(x)+nx$ for $x\in I_{nj}$, so that $F=G-(F_2-F)-H$ on I_{nj}, where $H(x)=nx$. Whenever a, $b\in B_{nj}$ and $a\le x\le b$, then $G(a)\le G(x)\le G(b)$, because $x\in I_{nj}$. So if $a_0,b_0,a_1,b_1,\dots,a_k,b_k\in A$ and $a_0\le b_0\le a_1\le b_1\le\dots\le a_k\le b_k$,

$$\textstyle\sum_{i=0}^k\omega(G\restriction[a_i,b_i])=\sum_{i=0}^k G(b_i)-G(a_i)\le G(b_k)-G(a_0)\le\omega(G\restriction I_{nj}),$$

and

$$\begin{aligned}\sum_{i=0}^k\omega(F\restriction[a_i,b_i])&\le\sum_{i=0}^k\omega(G\restriction[a_i,b_i])+\sum_{i=0}^k\omega(F_2-F\restriction[a_i,b_i])+\sum_{i=0}^k\omega(H\restriction[a_i,b_i])\\&\le\omega(G\restriction I_{nj})+\omega(F_2-F\restriction I_{nj})+\omega(H\restriction I_{nj})\end{aligned}$$

(because F_2-F and H are monotonic)

$$\begin{aligned}&\le\omega(F\restriction I_{nj})+2\omega(F_2-F\restriction I_{nj})+2\omega(H\restriction I_{nj})\\&\le\omega(F\restriction I_{nj})+2(1+n\mu I_{nj})=M\end{aligned}$$

say, which is finite, because F is bounded.

(iii) By 2A2I (or 4A2Rj), the open set $\mathbb{R}\setminus\overline{A}$ is expressible as a countable union of disjoint non-empty open intervals. Two of these are unbounded; let $\mathcal{I}$ be the set consisting of the rest, so that $\overline{A}\cup\bigcup\mathcal{I}=[a^*,b^*]$ is a closed interval included in I_{nj}. If I, I' are distinct members of $\mathcal{I}$ and $\inf I\le\inf I'$, then $\sup I\le\inf I'$, because $I\cap I'=\emptyset$, and there must be a point of $\overline{A}$ in the interval $[\sup I,\inf I']$; so in fact there must be a point of A in this interval, since A does not meet either I or I'. It follows that $\sum_{I\in\mathcal{I}}\omega(F\restriction I)\le M$. **P** If $\mathcal{I}_0\subseteq\mathcal{I}$ is finite and non-empty, we can enumerate it as $\langle I_i\rangle_{i\le k}$ where $\sup I_i\le\inf I_{i'}$ whenever $i<i'\le k$. We can find $a_0,\dots,a_{k+1}\in A$ such that $a_0\le\inf I_0$, $\sup I_i\le a_{i+1}\le\inf I_{i+1}$ for every $i<k$, and $\sup I_k\le a_{k+1}$; so that

$$\textstyle\sum_{I\in\mathcal{I}_0}\omega(F\restriction I)=\sum_{i=0}^k\omega(F\restriction I_i)\le\sum_{i=0}^k\omega(F\restriction[a_i,a_{i+1}])\le M.$$

As $\mathcal{I}_0$ is arbitrary, this gives the result. **Q**

(iv) Whenever $a^*\le x\le y\le b^*$,

$$\textstyle|F(y)-F(x)|\le\int_{\overline{A}\cap[x,y]}|f|d\mu+\sum_{I\in\mathcal{I}}\omega(F\restriction I\cap\,]x,y[).$$

P For each $I\in\mathcal{I}$,

$$\|f\times\chi(I\cap\,]x,y[)\|_H=\omega(F\restriction I\cap\,]x,y[)\le\omega(F\restriction I)$$

(483Pa). So

$$\textstyle\sum_{I\in\mathcal{I}}\|f\times\chi(I\cap\,]x,y[)\|_H\le\sum_{I\in\mathcal{I}}\omega(F\restriction I)$$

is finite. Writing $H=\,]x,y[\,\cap\bigcup\mathcal{I}$, $\oint f\times\chi H$ is defined and equal to $\sum_{I\in\mathcal{I}}\oint f\times\chi(I\cap\,]x,y[)$, by 483N. On the other hand,

$$]x,y[\setminus H\subseteq\overline{A}\subseteq K_m,$$

so $f\times\chi(]x,y[\setminus H)$ is Lebesgue integrable. Accordingly

$$\begin{aligned}|F(y)-F(x)|&=|\mathrm{H}\!\!\!\!\int f\times\chi\,]x,y[\,|\\&\le|\mathrm{H}\!\!\!\!\int f\times\chi H|+|\int f\times\chi(]x,y[\setminus H)d\mu|\\&\le\sum_{I\in\mathcal{I}}|\mathrm{H}\!\!\!\!\int f\times\chi(I\cap\,]x,y[)|+\int_{\overline{A}\cap[x,y]}|f|d\mu\\&\le\sum_{I\in\mathcal{I}}\omega(F\restriction I\cap\,]x,y[)+\int_{\overline{A}\cap[x,y]}|f|d\mu,\end{aligned}$$

as claimed. **Q**

(v) So if $a^*\le a\le b\le b^*$,

$$\omega(F\restriction[a,b])\le\textstyle\sum_{I\in\mathcal{I}}\omega(F\restriction I\cap[a,b])+\int_{\overline{A}\cap[a,b]}|f|d\mu.$$

P We have only to observe that if $a\le x\le y\le b$, then $\omega(F\restriction I\cap\,]x,y[)\le\omega(F\restriction I\cap[a,b])$ for every $I\in\mathcal{I}$, and $\int_{\overline{A}\cap]x,y[}|f|d\mu\le\int_{\overline{A}\cap[a,b]}|f|d\mu$. **Q**

(vi) Now let $\epsilon>0$. Setting $\tilde F(x)=\int_{a^*}^x|f|\times\chi\overline{A}\,d\mu$ for $x\in[a^*,b^*]$, $\tilde F$ is absolutely continuous (225E), and there is an $\eta_0>0$ such that $\sum_{i=0}^k\tilde F(b_i)-\tilde F(a_i)\le\epsilon$ whenever $a^*\le a_0\le b_0\le a_1\le b_1\le\ldots\le a_k\le b_k\le b^*$ and $\sum_{i=0}^k b_i-a_i\le\eta_0$. Take $\mathcal{I}_0\subseteq\mathcal{I}$ to be a finite set such that $\sum_{I\in\mathcal{I}\setminus\mathcal{I}_0}\omega(F\restriction I)\le\epsilon$, and let $\eta>0$ be such that $\eta\le\eta_0$ and $\eta<\operatorname{diam}I$ for every $I\in\mathcal{I}_0$.

Suppose that $a_0,b_0,\ldots,a_k,b_k\in A$ are such that $a_0\le b_0\le\ldots\le a_k\le b_k$ and $\sum_{i=0}^k b_i-a_i\le\eta$. Then no member of $\mathcal{I}_0$ can be included in any interval $[a_i,b_i]$, and therefore, because no a_i or b_i can belong to $\bigcup\mathcal{I}$, no member of $\mathcal{I}_0$ meets any $[a_i,b_i]$. Also, of course, $a^*\le a_0$ and $b_k\le b^*$. We therefore have

$$\begin{aligned}\sum_{i=0}^k\omega(F\restriction[a_i,b_i])&\le\sum_{i=0}^k\Bigl(\sum_{I\in\mathcal{I}}\omega(F\restriction I\cap[a_i,b_i])+\int_{\overline{A}\cap[a_i,b_i]}|f|d\mu\Bigr)\\&=\sum_{i=0}^k\sum_{I\in\mathcal{I}\setminus\mathcal{I}_0}\omega(F\restriction I\cap[a_i,b_i])+\sum_{i=0}^k\tilde F(b_i)-\tilde F(a_i)\\&\le\sum_{i=0}^k\sum_{\substack{I\in\mathcal{I}\setminus\mathcal{I}_0\\ I\subseteq[a_i,b_i]}}\omega(F\restriction I)+\epsilon\end{aligned}$$

(because if $I\in\mathcal{I}$ meets $[a_i,b_i]$, it is included in it)

$$\le\sum_{I\in\mathcal{I}\setminus\mathcal{I}_0}\omega(F\restriction I)+\epsilon\le2\epsilon.$$

As ϵ is arbitrary, F is AC$_*$ on A. This completes the proof that F is ACG$_*$ and therefore satisfies the conditions given.

(b) Now suppose that F satisfies the conditions. Set $F(-\infty)=0$ and $F(\infty)=\lim_{x\to\infty}F(x)$, so that $F:[-\infty,\infty]\to\mathbb{R}$ is continuous. For $x\in\mathbb{R}$, set $f(x)=F'(x)$ if this is defined, 0 otherwise. Let $\mathcal{J}$ be the family of all non-empty intervals $C\subseteq\mathbb{R}$ such that $\mathrm{H}\!\!\!\!\int f\times\chi C$ is defined and equal to $F(\sup C)-F(\inf C)$, and let $\mathcal{I}$ be the set of non-empty open intervals I such that every non-empty subinterval of I belongs to $\mathcal{J}$. I seek to show that $\mathbb{R}$ belongs to $\mathcal{I}$.

(i) Of course singleton intervals belong to $\mathcal{J}$. If C_1, $C_2\in\mathcal{J}$ are disjoint and $C=C_1\cup C_2$ is an interval, then

$$\begin{aligned}\mathrm{H}\!\!\!\!\int f\times\chi C&=\mathrm{H}\!\!\!\!\int f\times\chi C_1+\mathrm{H}\!\!\!\!\int f\times\chi C_2\\&=F(\sup C_1)-F(\inf C_1)+F(\sup C_2)-F(\inf C_2)\\&=F(\sup C)-F(\inf C)\end{aligned}$$

and $C \in \mathcal{J}$. If I_1, $I_2 \in \mathcal{I}$ and $I_1 \cap I_2$ is non-empty, then $I_1 \cup I_2$ is an interval; also any subinterval C of $I_1 \cup I_2$ is either included in one of the I_j or is expressible as a disjoint union $C_1 \cup C_2$ where C_j is a subinterval of I_j for each j; so $C \in \mathcal{J}$ and $I \in \mathcal{I}$. If I_1, $I_2 \in \mathcal{I}$ and $\sup I_1 = \inf I_2$, then $I = I_1 \cup I_2 \cup \{\sup I_1\} \in \mathcal{I}$, because any subinterval of I is expressible as the disjoint union of at most three intervals in $\mathcal{J}$.

(ii) If $\mathcal{I}_0 \subseteq \mathcal{I}$ is non-empty and upwards-directed, then $\bigcup \mathcal{I}_0 \in \mathcal{I}$. **P** This is a consequence of 483Bd. If we take a non-empty open subinterval J of $\bigcup \mathcal{I}_0$ and express it as $]\alpha, \beta[$, where $-\infty \le \alpha < \beta \le \infty$, then whenever $\alpha < a < b < \beta$ there are members of $\mathcal{I}_0$ containing a and b, and therefore a member of $\mathcal{I}_0$ containing both, so that $[a, b] \in \mathcal{J}$. Accordingly $\int_a^b f$ is defined and equal to $F(b) - F(a)$. Since F is continuous, $\lim_{a \downarrow \alpha, b \uparrow \beta} \int_a^b f$ is defined and equal to $F(\beta) - F(\alpha)$; by 483Bd, $\int_\alpha^\beta f$ is defined and equal to $F(\beta) - F(\alpha)$, so that $J \in \mathcal{J}$. I wrote this out for open intervals, for convenience; but any non-empty subinterval of $\bigcup \mathcal{I}_0$ is either a singleton or expressible as an open interval with at most two points added, so belongs to $\mathcal{J}$. Accordingly $\bigcup \mathcal{I}_0 \in \mathcal{I}$. **Q**

(iii) It follows that every member of $\mathcal{I}$ is included in a maximal member of $\mathcal{I}$. Let $\mathcal{I}^*$ be the set of maximal members of $\mathcal{I}$. By (i), these are all disjoint, so no endpoint of any member of $\mathcal{I}^*$ can belong to $\bigcup \mathcal{I}^*$.

? Suppose, if possible, that $\mathbb{R} \notin \mathcal{I}$. Then $\bigcup \mathcal{I} = \bigcup \mathcal{I}^*$ cannot be $\mathbb{R}$, and $V = \mathbb{R} \setminus \bigcup \mathcal{I}$ is a non-empty closed set. By (i), no two distinct members of $\mathcal{I}^*$ can share a boundary point, so V has no isolated points.

We are supposing that F is ACG$_*$, so there is a countable family $\mathcal{A}$ of sets, covering $\mathbb{R}$, such that F is AC$_*$ on A for every $A \in \mathcal{A}$. By Baire's theorem (3A3G or 4A2Ma), applied to the locally compact Polish space V, $V \setminus \overline{A}$ cannot be dense in V for every $A \in \mathcal{A}$, so there are an $A \in \mathcal{A}$ and a bounded open interval $\tilde{J}$ such that $\emptyset \ne V \cap \tilde{J} \subseteq \overline{A}$. Set $K = \overline{\tilde{J} \cap A}$; by 483Pb, F is AC$_*$ on K, and $V \cap \tilde{J} \subseteq K$. Because V has no isolated points, $V \cap \tilde{J}$ is infinite, so, setting $a^* = \min K$ and $b^* = \max K$, $V \cap]a^*, b^*[$ is non-empty.

Let $\mathcal{I}_0$ be the family of non-empty bounded open intervals, disjoint from K, with endpoints in K. By 483Q, $\sum_{I \in \mathcal{I}_0} \omega(F \restriction I)$ is finite, and there is a Lebesgue integrable function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 0$ for $x \in \mathbb{R} \setminus K$ and

$$F(x) = F(a^*) + \int_{a^*}^x g + \sum_{I \in \mathcal{I}_0, I \subseteq [a^*, x]} F(\sup I) - F(\inf I)$$

for $x \in K$. Since every member of $\mathcal{I}_0$ is disjoint from V, it is included in some member of $\mathcal{I}^*$ and belongs to $\mathcal{J}$, so $F(\sup I) - F(\inf I) = \int f \times \chi I$ for every $I \in \mathcal{I}_0$. If $I \in \mathcal{I}_0$, then

$$\begin{aligned} \|f \times \chi I\|_H &= \sup_{C \in \mathcal{C}} |\int f \times \chi I \times \chi C| = \sup_{C \in \mathcal{C}, C \subseteq I} |\int f \times \chi C| \\ &= \sup_{C \in \mathcal{C}, C \subseteq I} |F(\sup C) - F(\inf C)| = \omega(F \restriction I) \end{aligned}$$

because every non-empty subinterval of I belongs to $\mathcal{J}$. So $\sum_{I \in \mathcal{I}_0} \|f \times \chi I\|_H$ is finite, and $f \times \chi H$ is Henstock integrable, where $H = \bigcup \mathcal{I}_0$, by 483N. Moreover, if $x \in K$, then

$$\begin{aligned} \int_{a^*}^x f \times \chi H &= \int f \times \chi(\bigcup \{I : I \in \mathcal{I}_0,\, I \subseteq [a^*, x]\}) \\ &= \sum_{I \in \mathcal{I}_0, I \subseteq [a^*, x]} \int f \times \chi I = \sum_{I \in \mathcal{I}_0, I \subseteq [a^*, x]} F(\sup I) - F(\inf I). \end{aligned}$$

But this means that if $y \in [a^*, b^*]$, and $x = \max(K \cap [a^*, y])$, so that $]x, y[\subseteq H$, then

$$\begin{aligned} F(y) &= F(x) + F(y) - F(x) \\ &= F(a^*) + \int_{a^*}^x g + \sum_{I \in \mathcal{I}_0, I \subseteq [a^*, x]} (F(\sup I) - F(\inf I)) + \int f \times \chi]x, y[\\ &= F(a^*) + \int g \times \chi(K \cap [a^*, y[) + \int_{a^*}^x f \times \chi H + \int f \times \chi]x, y[\\ &= F(a^*) + \int_{-\infty}^y h \end{aligned}$$

where $h = f \times \chi H + g \times \chi K$ is Henstock integrable because $f \times \chi H$ is Henstock integrable and $g \times \chi K$ is Lebesgue integrable.

Accordingly, if C is any non-empty subinterval of $[a^*, b^*]$, $F(\sup C) - F(\inf C) = \int h \times \chi C$. But we also know that $\frac{d}{dy} \int_{-\infty}^y h = h(y)$ for almost every y, by 483I. So $F'(y)$ is defined and equal to $h(y)$ for almost every $y \in [a^*, b^*]$, and $h=f$ a.e. on $[a^*,b^*]$. This means that $F(\sup C)-F(\inf C)=\fint f\times\chi C$ for any non-empty subinterval C of $[a^*,b^*]$, and $]a^*,b^*[\,\in\mathcal{I}$. But $]a^*,b^*[$ meets V, so this is impossible. $\mathbf{X}$

(iv) This contradiction shows that $\mathbb{R}\in\mathcal{I}$ and that F is an indefinite Henstock integral, as required.

483X Basic exercises >(a) Let $[a,b]\subseteq\mathbb{R}$ be a non-empty closed interval, and let I_μ be the gauge integral on $[a,b]$ defined from Lebesgue measure and the tagged-partition structure defined in 481J. Show that, for $f:\mathbb{R}\to\mathbb{R}$, $I_\mu(f\restriction[a,b])=\fint f\times\chi[a,b]$ if either is defined.

>(b) Extract ideas from the proofs of 482G and 482H to give a direct proof of 483B(c)-(d).

(c) Set $f(0)=0$ and $f(x)=\frac{\sin x}{x}$ for other real x. Show that f is Henstock integrable, and that $\fint_0^\infty f=\frac{\pi}{2}$. (*Hint*: 283Da.)

(d) Set $f(x)=\dfrac{1}{x}\cos(\dfrac{1}{x^2})$ for $0<x\le 1$, 0 for other real x. Show that f is Henstock integrable but not Lebesgue integrable. (*Hint*: by considering $\dfrac{d}{dx}(x^2\sin\dfrac{1}{x^2})$, show that $\lim_{a\downarrow 0}\int_a^1 f$ is defined.)

(e) Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function. Show that there is a finitely additive functional $\lambda:\mathcal{P}\mathbb{R}\to\mathbb{R}$ such that for every $\epsilon>0$ there are a gauge $\delta\in\Delta$ and a Radon measure ν on $\mathbb{R}$ such that $\nu\mathbb{R}\le\epsilon$ and $|S_{\boldsymbol{t}}(f,\mu)-\lambda W_{\boldsymbol{t}}|\le\nu W_{\boldsymbol{t}}$ for every δ-fine $\boldsymbol{t}\in T$.

>(f) Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function. Show that $\bigcup\{G:G\subseteq\mathbb{R}$ is open, f is Lebesgue integrable over $G\}$ is dense. (*Hint*: 483G.)

>(g) Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function and F its indefinite Henstock integral. Show that f is Lebesgue integrable iff F is of bounded variation on $\mathbb{R}$. (*Hint*: 224I.)

>(h) Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\lim_{x\to-\infty}F(x)=0$, $\lim_{x\to\infty}F(x)$ is defined in $\mathbb{R}$, and $F'(x)$ is defined for all but countably many $x\in\mathbb{R}$. Show that F is the indefinite Henstock integral of any function $f:\mathbb{R}\to\mathbb{R}$ extending F'. (*Hint*: in 483J, take F_1 and F_2 differing from F by saltus functions.)

(i) Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function, and $\langle I_n\rangle_{n\in\mathbb{N}}$ a disjoint sequence of intervals in $\mathbb{R}$. Show that $\lim_{n\to\infty}\|f\times\chi I_n\|_H=0$.

(j) Show that HL^1 is not a Banach space. (*Hint*: there is a continuous function which is nowhere differentiable (477K).)

>(k) Use 483N to replace part of the proof of 483Bd.

(l) Let $F:\mathbb{R}\to\mathbb{R}$ be such that $\bar{D}F$ and $\underline{D}F$ are both finite everywhere. Show that $F(b)-F(a)=\fint\bar{D}F\times\chi[a,b]$ whenever $a\le b$ in $\mathbb{R}$. (*Hint*: F is AC_* on $\{x:|F(y)-F(x)|\le n|y-x|$ whenever $|y-x|\le 2^{-n}\}$.)

(m) For integers $r\ge 1$, write $\mathcal{C}_r$ for the family of subsets of $\mathbb{R}^r$ of the form $\prod_{i<r}C_i$ where $C_i\subseteq\mathbb{R}$ is a bounded interval for each $i<r$. Set $Q_r=\{(x,C):C\in\mathcal{C}_r,\ x\in\overline{C}\}$; let T_r be the straightforward set of tagged partitions generated by Q_r, Δ_r the set of neighbourhood gauges on $\mathbb{R}^r$, and $\mathfrak{R}_r=\{\mathcal{R}_C:C\in\mathcal{C}_r\}$ where $\mathcal{R}_C=\{\mathbb{R}^r\setminus C':C\subseteq C'\in\mathcal{C}_r\}\cup\{\emptyset\}$ for $C\in\mathcal{C}_r$. Let ν_r be the restriction of r-dimensional Lebesgue measure to $\mathcal{C}_r$. (i) Show that $(\mathbb{R}^r,T_r,\Delta_r,\mathfrak{R}_r)$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}_r$. (ii) For a function $f:\mathbb{R}^r\to\mathbb{R}$ write $\fint f(x)dx$ for the gauge integral $I_{\nu_r}(f)$ associated with this structure when it is defined. Show that if r, $s\ge 1$ are integers, $f:\mathbb{R}^{r+s}\to\mathbb{R}$ has compact support and $\fint f(z)dz$ is defined, then, identifying $\mathbb{R}^{r+s}$ with $\mathbb{R}^r\times\mathbb{R}^s$, $\fint g(x)dx$ is defined and equal to $\fint f(x,y)d(x,y)$ whenever $g:\mathbb{R}^r\to\mathbb{R}$ is such that $g(x)=\fint f(x,y)dy$ for every $x\in\mathbb{R}^r$ for which this is defined.

483Y Further exercises (a) Let us say that a **Lebesgue measurable neighbourhood gauge** on $\mathbb{R}$ is a neighbourhood gauge of the form $\{(x,A):x\in\mathbb{R},\ A\subseteq\,]x-\eta_x,x+\eta_x[\}$ where $x\mapsto\eta_x$ is a Lebesgue measurable function from $\mathbb{R}$ to $]0,\infty[$. Let $\tilde{\Delta}$ be the set of Lebesgue measurable neighbourhood gauges. Show that the gauge integral defined by the tagged-partition structure $(\mathbb{R},T,\tilde{\Delta},\mathfrak{R})$ and μ is the Henstock integral.

(b) Show that if $\Delta_0 \subseteq \Delta$ is any set of cardinal at most $\mathfrak{c}$, then the gauge integral defined by $(\mathbb{R}, T, \Delta_0, \mathfrak{R})$ and μ does not extend the Lebesgue integral, so is not the Henstock integral.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function with indefinite Henstock integral F, and ν a totally finite Radon measure on $\mathbb{R}$. Set $G(x) = \nu\,]-\infty, x]$ for $x \in \mathbb{R}$. Show that $f \times G$ is Henstock integrable, with indefinite Henstock integral H, where $H(x) = F(x)G(x) - \int_{]-\infty,x]} F\,d\nu$ for $x \in \mathbb{R}$.

(d) Let ν be any Radon measure on $\mathbb{R}$, I_ν the gauge integral defined from ν and the tagged-partition structure of 481K and this section, and $f : \mathbb{R} \to \mathbb{R}$ a function.

(i) Show that if $I_\nu(f)$ is defined, then f is $\operatorname{dom}\nu$-measurable.

(ii) Show that if $\int f d\nu$ is defined in $\mathbb{R}$, then $I_\nu(f)$ is defined and equal to $\int f d\nu$.

(iii) Show that if $\alpha \in\,]-\infty, \infty]$ then $I_\nu(f \times \chi\,]-\infty, \alpha[) = \lim_{\beta\uparrow\alpha} I_\nu(f \times \chi\,]-\infty, \beta[)$ if either is defined in $\mathbb{R}$.

(iv) Suppose that $I_\nu(f)$ is defined. (α) Let F^{SH} be the Saks-Henstock indefinite integral of f with respect to ν. Show that for any $\epsilon > 0$ there are a Radon measure ζ on $\mathbb{R}$ and a $\delta \in \Delta$ such that $\zeta\mathbb{R} \le \delta$ and $|F^{\text{SH}}(W_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f, \nu)| \le \zeta W_{\boldsymbol{t}}$ whenever $\boldsymbol{t} \in T$ is δ-fine. (β) Show that there is a countable cover of $\mathbb{R}$ by compact sets K such that $\int_K |f| d\nu < \infty$.

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function, and $G : \mathbb{R} \to \mathbb{R}$ a function of bounded variation. Show that $f \times G$ is Henstock integrable, and

$$\oint f \times G \le (\lim_{x\to\infty} |G(x)| + \operatorname{Var}_{\mathbb{R}} G) \sup_{x\in\mathbb{R}} |\oint_{-\infty}^{x} f|.$$

(Compare 224J.)

(f) Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function and $g : \mathbb{R} \to \mathbb{R}$ a Lebesgue integrable function; let F and G be their indefinite (Henstock) integrals. Show that $\oint f \times G + \int g \times F\,d\mu$ is defined and equal to $\lim_{x\to\infty} F(x)G(x)$.

(g) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that f is Lebesgue integrable iff $f \times g$ is Henstock integrable for every bounded continuous function $g : \mathbb{R} \to \mathbb{R}$.

(h) Let U be a linear subspace of $\mathbb{R}^{\mathbb{R}}$ and $\phi : U \to \mathbb{R}$ a linear functional such that (i) $f \in U$ and $\phi f = \int f d\mu$ for every Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$ (ii) $f \times \chi C \in U$ whenever $f \in U$ and $C \in \mathcal{C}$ (iii) whenever $f \in \mathbb{R}^{\mathbb{R}}$ and $\mathcal{I}$ is a disjoint family of non-empty open intervals such that $f \times \chi I \in U$ for every $I \in \mathcal{I}$ and $\sum_{I\in\mathcal{I}} \sup_{C\subseteq I, C\in\mathcal{C}} |\phi(f \times \chi C)| < \infty$, then $f \times \chi(\bigcup\mathcal{I}) \in U$ and $\phi(f \times \chi(\bigcup\mathcal{I})) = \sum_{I\in\mathcal{I}} \phi(f \times \chi I)$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is any Henstock integrable function, then $f \in U$ and $\phi(f) = \oint f$. (*Hint*: use the argument of part (b) of the proof of 483R.)

(i) (Bongiorno Piazza & Preiss 00) Let $\mathcal{C}$ be the set of non-empty subintervals of a closed interval $[a, b] \subseteq \mathbb{R}$, and T the straightforward set of tagged partitions generated by $[a, b] \times \mathcal{C}$. Let $\Delta_{[a,b]}$ be the set of neighbourhood gauges on $[a, b]$. For $\alpha \ge 0$ set

$$T_\alpha = \{\boldsymbol{t} : \boldsymbol{t} \in T,\ \sum_{(x,C)\in\boldsymbol{t}} \rho(x, C) \le \alpha\},$$

writing $\rho(x, C) = \inf_{y\in C} |x - y|$ as usual. Show that T_α is compatible with $\Delta_{[a,b]}$ and $\mathfrak{R} = \{\{\emptyset\}\}$ in the sense of 481F. Show that if I_α is the gauge integral defined from $[a, b]$, T_α, $\Delta_{[a,b]}$, $\mathfrak{R}$ and Lebesgue measure, then I_α extends the ordinary Lebesgue integral and $I_\alpha(F') = F(b) - F(a)$ whenever $F : [a, b] \to \mathbb{R}$ is differentiable relative to its domain.

(j) Let V be a Banach space and $f : \mathbb{R} \to V$ a function. For $\boldsymbol{t} \in T$, set $S_{\boldsymbol{t}}(f, \mu) = \sum_{(x,C)\in\boldsymbol{t}} \mu C \cdot f(x)$. We say that f is **Henstock integrable**, with **Henstock integral** $v = \oint f \in V$, if $v = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \mu)$.

(i) Show that the set $\mathcal{HL}^1_V$ of Henstock integrable functions from $\mathbb{R}$ to V is a linear subspace of $V^{\mathbb{R}}$ including the space $\mathcal{L}^1_V$ of Bochner integrable functions (253Yf), and that $\oint : \mathcal{HL}^1_V \to V$ is a linear operator extending the Bochner integral.

(ii) Show that if $f : \mathbb{R} \to V$ is Henstock integrable, so is $f \times \chi C$ for every interval $C \subseteq \mathbb{R}$, and that $(a, b) \mapsto \oint f \times \chi\,]a, b[$ is continuous. Set $\|f\|_H = \sup_{C\in\mathcal{C}} \|\oint f \times \chi C\|$.

(iii) Show that if $\mathcal{I}$ is a disjoint family of open intervals in $\mathbb{R}$, and $f : \mathbb{R} \to V$ is such that $f \times \chi I \in \mathcal{HL}^1_V$ for every $I \in \mathcal{I}$ and $\sum_{I\in\mathcal{I}} \|f \times \chi I\|_H$ is finite, then $\oint f \times \chi(\bigcup\mathcal{I})$ is defined and equal to $\sum_{I\in\mathcal{I}} \oint f \times \chi I$.

(iv) Define $f : \mathbb{R} \to \ell^\infty([0, 1])$ by setting $f(x) = \chi([0, 1] \cap\,]-\infty, x])$ for $x \in \mathbb{R}$. Show that f is Henstock integrable, but that if $F(x) = \oint f \times \chi\,]-\infty, x[$ for $x \in \mathbb{R}$, then $\lim_{y\to x} \frac{1}{y-x}(F(y) - F(x))$ is not defined in $\ell^\infty([0, 1])$ for any $x \in [0, 1]$.

(k) Find a function $f:\mathbb{R}^2\to\mathbb{R}$, with compact support, such that $⨏ f$ is defined in the sense of 483Xm, but $⨏ fT$ is not defined, where $T(x,y)=\frac{1}{\sqrt{2}}(x+y,x-y)$ for x, $y\in\mathbb{R}$.

(l) Show that for a function $g:\mathbb{R}\to\mathbb{R}$ the following are equiveridical: (i) there is a function $h:\mathbb{R}\to\mathbb{R}$, of bounded variation, such that $g=_{\text{a.e.}}h$ (ii) g is a **multiplier** for the Henstock integral, that is, $f\times g$ is Henstock integrable for every Henstock integrable $f:\mathbb{R}\to\mathbb{R}$.

483 Notes and comments I hope that the brief account here (largely taken from Gordon 94) will give an idea of the extraordinary power of gauge integrals. While what I am calling the 'Henstock integral', regarded as a linear functional on a space of real functions, was constructed long ago by Perron and Denjoy, the gauge integral approach makes it far more accessible, and gives clear pathways to corresponding Stieltjes and vector integrals (483Yd, 483Yj).

Starting from our position in the fourth volume of a book on measure theory, it is natural to try to describe the Henstock integral in terms of the Lebesgue integral, as in 483C (they agree on non-negative functions) and 483Yh (offering an extension process to generate the Henstock integral from the Lebesgue integral); on the way, we see that Henstock integrable functions are necessarily Lebesgue integrable over many intervals (483Xf). Alternatively, we can set out to understand indefinite Henstock integrals and their derivatives, just as Lebesgue integrable functions can be characterized as almost everywhere equal to derivatives of absolutely continuous functions (222E, 225E), because if f is Henstock integrable then it is equal almost everywhere to the derivative of its indefinite integral (483I). Any differentiable function (indeed, any continuous function differentiable except on a countable set) is an indefinite Henstock integral (483Xh). Recall that the Cantor function (134H) is continuous and differentiable almost everywhere but is not an indefinite integral, so we have to look for a characterization which can exclude such cases. For this we have to work quite hard, but we find that 'ACG_* functions' are the appropriate class (483R).

Gauge integrals are good at integrating derivatives (see 483Xh), but bad at integrating over subspaces. Even to show that $f\times\chi\,[0,\infty[$ is Henstock integrable whenever f is (483Bc) involves us in some unexpected manoeuvres. I give an argument which is designed to show off the general theory of §482, and I recommend you to look for short cuts (483Xb), but any method must depend on careful examination of the exact classes $\mathcal{C}$ and $\mathfrak{R}$ chosen for the definition of the integral. We do however have a new kind of convergence theorem in 482H and 483N.

One of the incidental strengths of the Henstock integral is that it includes the improper Riemann integral (483Bd, 483Xc); so that, for instance, Carleson's theorem (286U) can be written in the form

$$\hat{f}(y)=\frac{1}{\sqrt{2\pi}}⨏ e^{-ixy}f(x)dx \text{ for almost every } y \text{ if } f:\mathbb{R}\to\mathbb{C} \text{ is square-integrable.}$$

But to represent the many expressions of the type $\lim_{a\to\infty}\int_{-a}^{a}f$ in §283 (e.g., 283F, 283I and 283L) directly in the form $I_\mu(f)$ we need to change $\mathfrak{R}$, as in 481L or 481Xc.

484 The Pfeffer integral

I give brief notes on what seems at present to be the most interesting of the multi-dimensional versions of the Henstock integral, leading to Pfeffer's Divergence Theorem (484N).

484A Notation This section will depend heavily on Chapter 47, and will use much of the same notation. $r\ge 2$ will be a fixed integer, and μ will be Lebesgue measure on $\mathbb{R}^r$, while μ_{r-1} is Lebesgue measure on $\mathbb{R}^{r-1}$. As in §§473-475, let ν be 'normalized' $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$, as described in §265; that is, $\nu=2^{-r+1}\beta_{r-1}\mu_{H,r-1}$, where $\mu_{H,r-1}$ is $(r-1)$-dimensional Hausdorff measure on $\mathbb{R}^r$ as described in §264, and

$$\begin{aligned}\beta_{r-1}&=\frac{2^{2k}k!\pi^{k-1}}{(2k)!} \text{ if } r=2k \text{ is even,}\\ &=\frac{\pi^k}{k!} \text{ if } r=2k+1 \text{ is odd}\end{aligned}$$

is the Lebesgue measure of a ball of radius 1 in $\mathbb{R}^{r-1}$ (264I). For this section only, let us say that a subset of $\mathbb{R}^r$ is **thin** if it is of the form $\bigcup_{n\in\mathbb{N}}A_n$ where ν^*A_n is finite for every n. Note that every thin set is μ-negligible (471L). For $A\subseteq\mathbb{R}^r$, write ∂A for its ordinary topological boundary. If $x\in\mathbb{R}^r$ and $\epsilon>0$, $B(x,\epsilon)$ will be the closed ball $\{y:\|y-x\|\le\epsilon\}$.

I will use the term **dyadic cube** for sets of the form $\prod_{i<r}[2^{-m}k_i, 2^{-m}(k_i+1)[$ where $m, k_0, \dots, k_{r-1} \in \mathbb{Z}$; write $\mathcal{D}$ for the set of dyadic cubes in $\mathbb{R}^r$. Note that if $D, D' \in \mathcal{D}$, either $D \subseteq D'$ or $D' \subseteq D$ or $D \cap D' = \emptyset$; so if $\mathcal{D}_0 \subseteq \mathcal{D}$, the maximal members of $\mathcal{D}_0$ are disjoint.

It will be helpful to have an abbreviation for the following expression: set

$$\alpha^* = \min(\frac{1}{r^{r/2}}, \frac{2^{r-2}}{r\beta_r^{(r-1)/r}}).$$

(As will become apparent, the actual value of this constant is of no importance; but the strict logic of the arguments below depends on α^* being small enough.)

As in §475, I write int^*A, cl^*A and ∂^*A for the essential interior, essential closure and essential boundary of a set $A \subseteq \mathbb{R}^r$ (475B). Recall that a set $A \subseteq \mathbb{R}^r$ has finite perimeter in the sense of 474D iff $\nu(\partial^*A)$ is finite, and then

$$\nu(\partial^*A) = \lambda_A^\partial(\mathbb{R}^r) = \operatorname{per} A$$

is the perimeter of A (475M); we shall also need to remember that A is necessarily Lebesgue measurable.

$\mathcal{C}$ will be the family of subsets of $\mathbb{R}^r$ with locally finite perimeter, and $\mathcal{V}$ the family of bounded sets in $\mathcal{C}$, that is, the family of bounded sets with finite perimeter. Note that $\mathcal{C}$ is an algebra of subsets of $\mathbb{R}^r$ (475Ma), and that $\mathcal{V}$ is an ideal in $\mathcal{C}$.

484B Theorem (Tamanini & Giacomelli 89) Let $E \subseteq \mathbb{R}^r$ be a Lebesgue measurable set of finite measure and perimeter, and $\epsilon > 0$. Then there is a Lebesgue measurable set $G \subseteq E$ such that $\operatorname{per} G \le \operatorname{per} E$, $\mu(E \setminus G) \le \epsilon$ and $\operatorname{cl}^*G = \overline{G}$.

proof (Pfeffer 91b) **(a)** Set $\alpha = \frac{1}{\epsilon}\operatorname{per} E$. For measurable sets $G \subseteq E$ set $q(G) = \operatorname{per} G - \alpha\mu G$. Then there is a self-supporting measurable set $G \subseteq E$ such that $q(G) \le q(G')$ whenever $G' \subseteq E$ is measurable.

P Write Σ for the family of Lebesgue measurable subsets of $\mathbb{R}^r$; give Σ the topology of convergence in measure defined by the pseudometrics $\rho_H(G, G') = \mu((G \triangle G') \cap H)$ for measurable sets H of finite measure (cf. 474T). Extend q to Σ by setting $q(G) = \operatorname{per}(E \cap G) - \alpha\mu(E \cap G)$ for every $G \in \Sigma$. Because $\operatorname{per} : \Sigma \to [0, \infty]$ is lower semi-continuous for the topology of convergence in measure (474Ta), and $G \mapsto E \cap G$, $G \mapsto \mu(E \cap G)$ are continuous, $q : \Sigma \to [0, \infty[$ is lower semi-continuous (4A2Bd). Next, $\mathcal{K} = \{G : G \in \Sigma, \operatorname{per} G \le \operatorname{per} E\}$ is compact (474Tb), while $\mathcal{L} = \{G : \mu(G \setminus E) = 0\}$ is closed, so there is a $G_0 \in \mathcal{L} \cap \mathcal{K}$ such that $q(G_0) = \inf_{G \in \mathcal{L} \cap \mathcal{K}} q(G)$ (4A2Gl). Since $G_0 \in \mathcal{L}$, $\operatorname{per}(G_0 \cap E) = \operatorname{per}(G_0)$ and $\mu(G_0 \cap E) = \mu G_0$, so we may suppose that $G_0 \subseteq E$. Moreover, there is a self-supporting set $G \subseteq G_0$ such that $G_0 \setminus G$ is negligible (414F), and we still have $q(G) = q(G_0)$. Of course $q(G) \le q(E)$, just because $E \in \mathcal{L} \cap \mathcal{K}$.

? If there is a measurable set $G' \subseteq E$ such that $q(G') < q(G)$, then

$$\operatorname{per} G' = q(G') + \alpha\mu G' \le q(E) + \alpha\mu E = \operatorname{per} E,$$

so $G' \in \mathcal{K}$; but this means that $G' \in \mathcal{L} \cap \mathcal{K}$ and $q(G) = q(G_0) \le q(G')$. **X** So G has the required properties. **Q**

(b) Since $q(G) \le q(E)$, we must have

$$\operatorname{per} G + \alpha\mu(E \setminus G) = q(G) + \alpha\mu E \le q(E) + \alpha\mu E = \operatorname{per} E = \alpha\epsilon.$$

So $\mu(E \setminus G) \le \epsilon$.

(c) Next, $\overline{G} \subseteq \operatorname{cl}^*G$. **P** Let $x \in \overline{G}$. For every $t > 0$, set $U_t = \{y : \|y - x\| < t\}$; then

$$\begin{aligned}\operatorname{per}(G \cap U_t) + \operatorname{per}(G \setminus U_t) &= \nu(\partial^*(G \cap U_t)) + \nu(\partial^*(G \setminus U_t)) \\ &\le \nu(\partial^*G \cap U_t) + \nu(\operatorname{cl}^*G \cap \partial U_t) \\ &\qquad + \nu(\partial^*G \setminus U_t) + \nu(\operatorname{cl}^*G \cap \partial U_t)\end{aligned}$$

(475Cf, because $\partial(\mathbb{R}^r \setminus U_t) = \partial U_t$))

$$= \nu(\partial^*G) + 2\nu(\operatorname{cl}^*G \cap \partial U_t) = \operatorname{per} G + 2\nu(G \cap \partial U_t)$$

for almost every $t > 0$, because

$$\int_0^\infty \nu((G \triangle \operatorname{cl}^*G) \cap \partial U_t)dt = \mu(G \triangle \operatorname{cl}^*G) = 0$$

(265G). So, for almost every t,

$$\mu(G \cap U_t)^{(r-1)/r} \le \operatorname{per}(G \cap U_t)$$

(474La)

$$\begin{aligned} &\le \operatorname{per} G + 2\nu(G \cap \partial U_t) - \operatorname{per}(G \setminus U_t) \\ &= q(G) + \alpha\mu(G \cap U_t) + 2\nu(G \cap \partial U_t) - q(G \setminus U_t) \\ &\le \alpha\mu(G \cap U_t) + 2\nu(G \cap \partial U_t) \end{aligned}$$

because $q(G)$ is minimal.

For $t > 0$, set

$$g(t) = \mu(G \cap U_t) = \int_0^t \nu(G \cap \partial U_s)ds,$$

so that

$$g'(t) = \nu(G \cap \partial U_t) \ge \frac{1}{2}(g(t)^{(r-1)/r} - \alpha g(t))$$

for almost every t. Because G is self-supporting and U_t is open and $G \cap U_t \ne \emptyset$, $g(t) > 0$ for every $t > 0$; and $\lim_{t\downarrow 0} g(t) = 0$.

Set

$$h(t) = \frac{d}{dt}g(t)^{1/r} = \frac{g'(t)}{rg(t)^{(r-1)/r}} \ge \frac{1}{2r}(1 - \alpha g(t)^{1/r})$$

for almost every t. Then

$$\limsup_{t\downarrow 0} \frac{g(t)^{1/r}}{t} = \limsup_{t\downarrow 0} \frac{1}{t}\int_0^t h \ge \frac{1}{2r},$$

and

$$\limsup_{t\downarrow 0} \frac{\mu(G \cap B(x,t)))}{\mu B(x,t))} = \limsup_{t\downarrow 0} \frac{g(t)}{\beta_r t^r} > 0.$$

Thus $x \in \mathrm{cl}^* G$. As x is arbitrary, $\overline{G} \subseteq \mathrm{cl}^* G$. **Q**

Since we certainly have $\mathrm{cl}^* G \subseteq \overline{G}$, this G serves.

484C Lemma Let $E \in \mathcal{V}$ and $l \in \mathbb{N}$ be such that $\max(\operatorname{per} E, \operatorname{diam} E) \le l$. Then E is expressible as $\bigcup_{i<n} E_i$ where $\langle E_i \rangle_{i<n}$ is disjoint, $\operatorname{per} E_i \le 1$ for each $i < n$ and n is at most $2^r(l+1)^r(4r(2l^2+1))^{r/(r-1)} + 2^{r+1}l^2$.

proof For $D \in \mathcal{D}$, write $\mathcal{D}_D$ for $\{D' : D' \in \mathcal{D},\ D' \subseteq D,\ \operatorname{diam} D' = \frac{1}{2}\operatorname{diam} D\}$, the family of the 2^r dyadic subcubes of D at the next level down.

(a) If $l \le 1$ the result is trivial, so let us suppose that $l \ge 2$. Let $m \in \mathbb{N}$ be minimal subject to $4r(2l^2+1) \le 2^{m(r-1)}$, so that $2^m \le 2(4r(2l^2+1))^{1/(r-1)}$. Then we can cover E by a family $\mathcal{L}_0$ of dyadic cubes of side 2^{-m} with

$$\#(\mathcal{L}_0) \le (2^m l + 1)^r \le 2^{mr}(l+1)^r \le 2^r(l+1)^r(4r(2l^2+1))^{r/(r-1)}.$$

(b) Let $\mathcal{L}_1$ be the set of those $D \in \mathcal{D}$ such that $\lfloor 2\nu(D' \cap \partial^* E)\rfloor < \lfloor 2\nu(D \cap \partial^* E)\rfloor$ for every $D' \in \mathcal{D}_D$. Then $\#(\mathcal{L}_1) \le 2l^2$. **P** For $k \ge 1$, set

$$\mathcal{L}_1^{(k)} = \{F : D \in \mathcal{L}_1,\ \lfloor 2\nu(D \cap \partial^* E)\rfloor = k\}.$$

If D, $D' \in \mathcal{L}_1^{(k)}$ are distinct, neither can be included in the other, so they are disjoint. Accordingly $k\#(\mathcal{L}_1^{(k)}) \le 2\nu(\partial^* E) \le 2l$ and $\#(\mathcal{L}_1^{(k)}) \le 2l$. Since $\mathcal{L}_1 = \bigcup_{1 \le k \le l} \mathcal{L}_1^{(k)}$, $\#(\mathcal{L}_1) \le 2l^2$. **Q**

(c) For $D \in \mathcal{D}$, set $\tilde{D} = D \setminus \bigcup\{D' : D' \in \mathcal{L}_1,\ D' \subseteq D\}$. Then $\nu(\tilde{D} \cap \partial^* E) \le \frac{1}{2}$. **P?** Otherwise, set $j = \lfloor 2\nu(D \cap \partial^* E)\rfloor \ge 1$, and choose $\langle D_i \rangle_{i\in\mathbb{N}}$ inductively, as follows. $D_0 = D$. Given that $D_i \in \mathcal{D}$, $D_i \subseteq D$ and $\lfloor 2\nu(D_i \cap \partial^* E)\rfloor = j$, $\nu((D \setminus D_i) \cap \partial^* E) < \frac{1}{2}$ and $D_i \cap \tilde{D}$ is non-empty, so $D_i \notin \mathcal{L}_1$ and there must be a $D_{i+1} \in \mathcal{D}_{D_i}$ such that $\lfloor 2\nu(D_{i+1} \cap \partial^* E)\rfloor = j$. Continue. This gives us a strictly decreasing sequence $\langle D_i \rangle_{i\in\mathbb{N}}$ in $\mathcal{D}$ such that $\nu(D_i \cap \partial^* E) \ge \frac{j}{2}$ for every i. But (because $\operatorname{per} E$ is finite) this means that, writing x for the unique member of $\bigcap_{i\in\mathbb{N}} \overline{D}_i$, $\nu\{x\} \ge \frac{j}{2}$, which is absurd. **XQ**

(d) Set

$$\mathcal{L}_1' = \{D : D \in \mathcal{L}_1 \text{ is included in some member of } \mathcal{L}_0\},$$

$$\mathcal{L}_2 = \mathcal{L}_0 \cup \bigcup\{\mathcal{D}_D : D \in \mathcal{L}_1'\}, \quad \mathcal{K} = \{\tilde{D} : D \in \mathcal{L}_2\}.$$

(i)

$$\begin{aligned}\#(\mathcal{K}) \le \#(\mathcal{L}_2) &\le \#(\mathcal{L}_0) + 2^r\#(\mathcal{L}_1)\\ &\le 2^r(l+1)^r(4r(2l^2+1))^{r/(r-1)} + 2^{r+1}l^2.\end{aligned}$$

(ii) $\bigcup\mathcal{K} \supseteq E$. **P** If $x \in E$, there is a smallest member D of $\mathcal{L}_2$ containing it, because certainly $x \in \bigcup\mathcal{L}_0$. But now x cannot belong to any member of $\mathcal{L}_1$ included in D, so $x \in \tilde{D}$. **Q**

(iii) $\mathcal{K}$ is disjoint. **P** If D_1, $D_2 \in \mathcal{L}_2$ are disjoint, then of course $\tilde{D}_1 \cap \tilde{D}_2 = \emptyset$. If $D_1 \subset D_2$, then $D_1 \notin \mathcal{L}_0$, so there is a $D \in \mathcal{L}_1'$ such that $D_1 \in \mathcal{D}_D$; in this case $D \subseteq D_2$ so $\tilde{D}_2 \subseteq D_2 \setminus D$ is disjoint from D_1. **Q**

(iv) $\operatorname{per}(D \cap E) \le 1$ for every $D \in \mathcal{K}$. **P** Take $D_0 \in \mathcal{L}_2$ such that $D = \tilde{D}_0$; then

$$\nu(\partial D) \le \nu(\partial D_0) + \sum_{D' \in \mathcal{L}_1'} \nu(\partial D') \le 2r(2l^2+1)2^{-m(r-1)} \le \frac{1}{2}$$

by the choice of m. So

$$\operatorname{per}(D \cap E) \le \nu(\partial D) + \nu(D \cap \partial^* E) \le \frac{1}{2} + \frac{1}{2} = 1$$

by 475Cf. **Q**

(e) So if we take $\langle E_i\rangle_{i<n}$ to be an enumeration of $\{E \cap D : D \in \mathcal{K}\}$, we shall have the required result.

484D Definitions The gauge integrals of this section will be based on the following residual families. Let H be the family of strictly positive sequences $\eta = \langle\eta(i)\rangle_{i\in\mathbb{N}}$ in $\mathbb{R}$. For $\eta \in \mathrm{H}$, write $\mathcal{M}_\eta$ for the set of disjoint sequences $\langle E_i\rangle_{i\in\mathbb{N}}$ of measurable subsets of $\mathbb{R}^r$ such that $\mu E_i \le \eta(i)$ and $\operatorname{per} E_i \le 1$ for every $i \in \mathbb{N}$, and E_i is empty for all but finitely many i. For $\eta \in \mathrm{H}$ and $V \in \mathcal{V}$ set

$$\mathcal{R}_\eta = \{\bigcup_{i\in\mathbb{N}} E_i : \langle E_i\rangle_{i\in\mathbb{N}} \in \mathcal{M}_\eta\} \subseteq \mathcal{C}, \quad \mathcal{R}_\eta^{(V)} = \{R : R \subseteq \mathbb{R}^r,\, R \cap V \in \mathcal{R}_\eta\};$$

finally, set $\mathfrak{R} = \{\mathcal{R}_\eta^{(V)} : V \in \mathcal{V},\, \eta \in \mathrm{H}\}$.

484E Lemma (a)(i) For every $\mathcal{R} \in \mathfrak{R}$, there is an $\eta \in \mathrm{H}$ such that $\mathcal{R}_\eta \subseteq \mathcal{R}$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $C \in \mathcal{C}$, there is an $\mathcal{R}' \in \mathfrak{R}$ such that $C \cap R \in \mathcal{R}$ whenever $R \in \mathcal{R}'$.

(b)(i) If $\eta \in \mathrm{H}$ and $\gamma \ge 0$, there is an $\epsilon > 0$ such that $R \in \mathcal{R}_\eta$ whenever $\mu R \le \epsilon$, $\operatorname{diam} R \le \gamma$ and $\operatorname{per} R \le \gamma$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $\gamma \ge 0$, there is an $\epsilon > 0$ such that $R \in \mathcal{R}$ whenever $\mu R \le \epsilon$ and $\operatorname{per} R \le \gamma$.

(c) If $\mathcal{R} \in \mathfrak{R}$ there is an $\mathcal{R}' \in \mathfrak{R}$ such that $R \cup R' \in \mathcal{R}$ whenever R, $R' \in \mathcal{R}'$ and $R \cap R' = \emptyset$.

(d)(i) If $\eta \in \mathrm{H}$ and $A \subseteq \mathbb{R}^r$ is a thin set, then there is a set $\mathcal{D}_0 \subseteq \mathcal{D}$ such that every point of A belongs to the interior of $\bigcup\mathcal{D}_1$ for some finite $\mathcal{D}_1 \subseteq \mathcal{D}_0$, and $\bigcup\mathcal{D}_1 \in \mathcal{R}_\eta$ for every finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $A \subseteq \mathbb{R}^r$ is a thin set, then there is a set $\mathcal{D}_0 \subseteq \mathcal{D}$ such that every point of A belongs to the interior of $\bigcup\mathcal{D}_1$ for some finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$, and $\bigcup\mathcal{D}_1 \in \mathcal{R}$ for every finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$.

proof (a)(i) Express $\mathcal{R}$ as $\mathcal{R}_{\eta'}^{(V)}$ where $\eta' \in \mathrm{H}$ and $V \in \mathcal{V}$. Let $l \in \mathbb{N}$ be such that $\max(\operatorname{diam} V, 1 + \operatorname{per} V) \le l$, and take $n \ge 2^r(l+1)^r(4r(2l^2+1))^{r/(r-1)} + 2^{r+1}l^2$. Set $\eta(i) = \min\{\eta'(j) : ni \le j < n(i+1)\}$ for every $i \in \mathbb{N}$, so that $\eta \in \mathrm{H}$. If $R \in \mathcal{R}_\eta$, express it as $\bigcup_{i\in\mathbb{N}} E_i$ where $\langle E_i\rangle_{i\in\mathbb{N}} \in \mathcal{M}_\eta$. For each i, $\max(\operatorname{diam}(E_i \cap V), \operatorname{per}(E_i \cap V)) \le l$, so by 484C we can express $E_i \cap V$ as $\bigcup_{ni\le j<n(i+1)} E_j'$, where $\langle E_j'\rangle_{ni\le j<n(i+1)}$ is disjoint and $\operatorname{per} E_j' \le 1$ for each j. Now

$$\mu E_j' \le \mu E_i \le \eta(i) \le \eta'(j)$$

for $ni \le j < n(i+1)$. Also $\{j : E_j' \ne \emptyset\}$ is finite because $\{j : E_j = \emptyset\}$ is finite. So $\langle E_j'\rangle_{j\in\mathbb{N}} \in \mathcal{M}_{\eta'}$ and $R \cap V = \bigcup_{j\in\mathbb{N}} E_j'$ belongs to $\mathcal{R}_{\eta'}$, that is, $R \in \mathcal{R}$. As R is arbitrary, $\mathcal{R}_\eta \subseteq \mathcal{R}$.

(ii) Express $\mathcal{R}$ as $\mathcal{R}_\eta^{(V)}$, where $V \in \mathcal{V}$ and $\eta \in \mathrm{H}$. By (i), there is an $\eta' \in \mathrm{H}$ such that $\mathcal{R}_{\eta'} \subseteq \mathcal{R}_\eta^{(C\cap V)}$. Set $\mathcal{R}' = \mathcal{R}_{\eta'}^{(V)} \in \mathfrak{R}$. If $R \in \mathcal{R}'$, then $R \cap V \in \mathcal{R}_{\eta'}$, so $C \cap R \cap V \in \mathcal{R}_\eta$ and $C \cap R \in \mathcal{R}$.

(b)(i) Take $l \ge \gamma$ and $n \ge 2^r(l+1)^r(4r(2l^2+1))^{r/(r-1)} + 2^{r+1}l^2$, and set $\epsilon = \min_{i<n} \eta(i)$. If $\mu R \le \epsilon$, $\operatorname{diam} R \le l$ and $\operatorname{per} R \le l$, then R is expressible as $\bigcup_{i<n} E_i$ where $\langle E_i \rangle_{i<n}$ is disjoint and $\operatorname{per} E_i \le 1$ for each $i < n$. Since $\mu E_i \le \mu R \le \eta(i)$ for each i, $R \in \mathcal{R}_\eta$.

(ii) Express $\mathcal{R}$ as $\mathcal{R}_\eta^{(V)}$. By (i), there is an $\epsilon > 0$ such that $R \in \mathcal{R}_\eta$ whenever $\operatorname{diam} R \le \operatorname{diam} V$, $\operatorname{per} R \le \gamma + \operatorname{per} V$ and $\mu R \le \epsilon$; and this ϵ serves.

(c) Express $\mathcal{R}$ as $\mathcal{R}_\eta^{(V)}$. Set $\eta'(i) = \min(\eta(2i), \eta(2i+1))$ for every i; then $\eta' \in \mathrm{H}$ and if $\langle E_i \rangle_{i\in\mathbb{N}}$, $\langle E'_i \rangle_{i\in\mathbb{N}}$ belong to $\mathcal{M}_{\eta'}$ and have disjoint unions, then $(E_0, E'_0, E_1, E'_1, \dots) \in \mathcal{M}_\eta$; this is enough to show that $R \cup R' \in \mathcal{R}_\eta$ whenever R, $R' \in \mathcal{R}_{\eta'}$ are disjoint, so that $R \cup R' \in \mathcal{R}$ whenever R, $R' \in \mathcal{R}_{\eta'}^{(V)}$ are disjoint.

(d)(i)(α) We can express A as $\bigcup_{i\in\mathbb{N}} A_i$ where $\mu^*_{H,r-1} A_i < 2^{-r}/2r$ for every i. **P** Because A is thin, it is the union of a sequence of sets of finite outer measure for ν, and therefore for $\mu_{H,r-1}$. On each of these the subspace measure is atomless (471E, 471Dg), so that the set can be dissected into finitely many sets of measure less than $2^{-r}/2r$ (215D). **Q**

(β) For each $i \in \mathbb{N}$, we can cover A_i by a sequence $\langle A_{ij} \rangle_{j\in\mathbb{N}}$ of sets such that $\operatorname{diam} A_{ij} < \eta(i)$ for every j and $\sum_{j=0}^\infty (\operatorname{diam} A_{ij})^{r-1} < 2^{-r}/2r$; enlarging the A_{ij} slightly if need be, we can suppose that they are all open. Now we can cover each A_{ij} by 2^r cubes $D_{ijk} \in \mathcal{D}$ in such a way that the side length of each D_{ijk} is at most the diameter of A_{ij}.

Setting $\mathcal{D}_0 = \{D_{ijk} : i, j \in \mathbb{N}, k < 2^r\}$, we see that every point of A belongs to an open set A_{ij} which is covered by a finite subset of $\mathcal{D}_0$. If $\mathcal{D}_1 \subseteq \mathcal{D}_0$ is finite, let $\mathcal{D}'_1$ be the family of maximal elements of $\mathcal{D}_1$, so that $\mathcal{D}'_1$ is disjoint and $\bigcup \mathcal{D}'_1 = \bigcup \mathcal{D}_1$. Express $\mathcal{D}'_1$ as $\{D_{ijk} : (i,j,k) \in I\}$ where $I \subseteq \mathbb{N} \times \mathbb{N} \times 2^r$ is finite and $\langle D_{ijk} \rangle_{(i,j,k)\in I}$ is disjoint. Set $I_i = \{(j,k) : (i,j,k) \in I\}$ and $E_i = \bigcup_{(j,k)\in I_i} D_{ijk}$ for $i \in \mathbb{N}$. Then $\langle E_i \rangle_{i\in\mathbb{N}}$ is disjoint, $E_i = \emptyset$ for all but finitely many i, and for each $i \in \mathbb{N}$

$$\begin{aligned}\mu E_i &\le \sum_{j=0}^\infty \sum_{k=0}^{2^r-1} \mu D_{ijk} \le \sum_{j=0}^\infty 2^r (\operatorname{diam} A_{ij})^r \\ &\le 2^r \sum_{j=0}^\infty \eta(i)(\operatorname{diam} A_{ij})^{r-1} \le \eta(i),\end{aligned}$$

$$\begin{aligned}\nu(\partial E_i) &\le \sum_{(j,k)\in I_i} \nu(\partial D_{ijk}) \le \sum_{j=0}^\infty \sum_{k=0}^{2^r-1} \nu(\partial D_{ijk}) \\ &\le \sum_{j=0}^\infty 2^r \cdot 2r(\operatorname{diam} A_{ij})^{r-1} \le 1.\end{aligned}$$

So $\bigcup \mathcal{D}_1 = \bigcup_{i\in\mathbb{N}} E_i$ belongs to $\mathcal{R}_\eta$.

(ii) By (a-i), there is an $\eta \in \mathrm{H}$ such that $\mathcal{R}_\eta \subseteq \mathcal{R}$. By (i) here, there is a $\mathcal{D}_0 \subseteq \mathcal{D}$ such that every point of A belongs to the interior of $\bigcup \mathcal{D}_1$ for some finite $\mathcal{D}_1 \subseteq \mathcal{D}_0$, and $\bigcup \mathcal{D}_1 \in \mathcal{R}_\eta \subseteq \mathcal{R}$ for every finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$.

484F A family of tagged-partition structures For $\alpha > 0$, let $\mathcal{C}_\alpha$ be the family of those $C \in \mathcal{V}$ such that $\mu C \ge \alpha(\operatorname{diam} C)^r$ and $\alpha \operatorname{per} C \le (\operatorname{diam} C)^{r-1}$, and let T_α be the straightforward set of tagged partitions generated by the set

$$\{(x, C) : C \in \mathcal{C}_\alpha,\ x \in \operatorname{cl}^* C\}.$$

Let Θ be the set of functions $\theta : \mathbb{R}^r \to [0, \infty[$ such that $\{x : \theta(x) = 0\}$ is thin (definition: 484A), and set $\Delta = \{\delta_\theta : \theta \in \Theta\}$, where $\delta_\theta = \{(x, A) : x \in \mathbb{R}^r,\ \theta(x) > 0,\ \|y - x\| < \theta(x)$ for every $y \in A\}$.

Then whenever $0 < \alpha < \alpha^*$, $(\mathbb{R}^r, T_\alpha, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof (a) We had better look again at all the conditions in 481G.

(i) and (vi) really are trivial. (iv) and (v) are true because $\mathcal{C}$ is actually an algebra of sets and $\emptyset \in \mathcal{R}$ for every $\mathcal{R} \in \mathfrak{R}$.

For (ii), we have to observe that the union of two thin sets is thin, so that $\theta \wedge \theta' \in \Theta$ for all θ, $\theta' \in \Theta$; since $\delta_\theta \cap \delta_{\theta'} = \delta_{\theta \wedge \theta'}$, this is all we need.

(iii)(α) follows from 484Ea: given V, $V' \in \mathcal{V}$ and η, $\eta' \in \mathrm{H}$, take $\tilde{\eta}$, $\tilde{\eta}' \in \mathrm{H}$ such that $\mathcal{R}_{\tilde{\eta}} \subseteq \mathcal{R}_{\eta}^{(V)}$ and $\mathcal{R}_{\tilde{\eta}'} \subseteq \mathcal{R}_{\eta'}^{(V')}$. Then $V \cup V' \in \mathcal{V}$ and $\mathcal{R}_{\eta}^{(V)} \cap \mathcal{R}_{\eta'}^{(V')} \supseteq \mathcal{R}_{\tilde{\eta}\wedge\tilde{\eta}'}^{(V\cup V')}$. ($\beta$) is just 484Ec.

(b) Now let us turn to 481G(vii).

(i) Fix $C \in \mathcal{C}$, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$. Express δ as δ_θ where $\theta \in \Theta$. Let $\mathcal{R}' \in \mathfrak{R}$ be such that $A \cup A' \in \mathcal{R}$ whenever A, $A' \in \mathcal{R}'$ are disjoint. Express $\mathcal{R}'$ as $\mathcal{R}_\eta^{(V)}$ where $V \in \mathcal{V}$ and $\eta \in \mathrm{H}$. By 484E(b-ii), there is an $\epsilon > 0$ such that $R \in \mathcal{R}'$ whenever $\operatorname{per} R \le 2\operatorname{per}(C \cap V)$ and $\mu R \le \epsilon$.

By 484B, there is an $E \subseteq C \cap V$ such that $\operatorname{per} E \le \operatorname{per}(C \cap V)$, $\mu((C \cap V) \setminus E) \le \epsilon$ and $\mathrm{cl}^*E = \overline{E}$. In this case,

$$\operatorname{per}(C \cap V \setminus E) \le \operatorname{per}(C \cap V) + \operatorname{per} E \le 2\operatorname{per}(C \cap V),$$

so $C \cap V \setminus E \in \mathcal{R}'$, by the choice of ϵ. By 484E(a-ii), there is an $\mathcal{R}'' \in \mathfrak{R}$ such that $E \cap R \in \mathcal{R}'$ whenever $R \in \mathcal{R}''$.

Now consider

$$A = \{x : \theta(x) = 0\} \cup \partial^*E \cup \{x : \limsup_{\zeta\downarrow 0} \sup_{x\in G, 0<\operatorname{diam} G \le \zeta} \frac{\nu^*(G\cap\partial^*E)}{(\operatorname{diam} G)^{r-1}} > 0\}.$$

Because $\operatorname{per} E < \infty$ and

$$\{x : x \in \mathbb{R}^r \setminus \partial^*E,\ \limsup_{\zeta\downarrow 0} \sup_{x\in G, 0<\operatorname{diam} G \le \zeta} \frac{\nu^*(G\cap\partial^*E)}{(\operatorname{diam} G)^{r-1}} > 0\}$$

is ν-negligible (471Pc), A is thin. By 484E(d-ii), there is a set $\mathcal{D}_0 \subseteq \mathcal{D}$ such that

$$A \subseteq \bigcup\{\operatorname{int}(\bigcup\mathcal{D}_1) : \mathcal{D}_1 \in [\mathcal{D}_0]^{<\omega}\}, \quad \bigcup\mathcal{D}_1 \in \mathcal{R}'' \text{ for every } \mathcal{D}_1 \in [\mathcal{D}_0]^{<\omega}.$$

(ii) Write T' for the set of those δ-fine $\boldsymbol{t} \in T_\alpha$ such that every member of $\boldsymbol{t}$ is of the form $(x, D \cap E)$ for some $D \in \mathcal{D}$. If $x \in \operatorname{int}^*E \setminus A$, there is an $h(x) > 0$ such that $h(x) < \theta(x)$ and $\{(x, D \cap E)\} \in T'$ whenever $D \in \mathcal{D}$, $x \in \overline{D}$ and $\operatorname{diam} D \le h(x)$. **P** Let $\epsilon_1 > 0$ be such that $r^{r/2}\epsilon_1 \le \frac{1}{2}$ and

$$\alpha \le \frac{1-r^{r/2}\epsilon_1}{r^{r/2}}, \quad \frac{1}{\alpha} \ge (2r + r^{r/2}\epsilon_1)\cdot\frac{1}{2^{r-1}}\Bigl(\frac{\beta_r}{1-r^{r/2}\epsilon_1}\Bigr)^{(r-1)/r};$$

this is where we need to know that $\alpha < \alpha^*$. Because $x \notin A$, $\theta(x) > 0$; let $h(x) \in \,]0, \theta(x)[$ be such that (α) $\nu^*(D \cap \partial^*E) \le \epsilon_1(\operatorname{diam} D)^{r-1}$ whenever $x \in \overline{D}$ and $\operatorname{diam} D \le h(x)$ (β) $\mu(B(x,t) \setminus E) \le \epsilon_1 t^r$ whenever $0 \le t \le h(x)$. Now suppose that $D \in \mathcal{D}$ and $x \in \overline{D}$ and $\operatorname{diam} D \le h(x)$. Then, writing γ for the side length of D,

$$\begin{aligned}\mu(D \cap E) &\ge \mu D - \mu(B(x, \operatorname{diam} D) \setminus E) \ge \gamma^r - \epsilon_1(\operatorname{diam} D)^r\\ &= \gamma^r(1 - r^{r/2}\epsilon_1) \ge \alpha\gamma^r r^{r/2} = \alpha(\operatorname{diam} D)^r \ge \alpha\operatorname{diam}(D \cap E)^r.\end{aligned}$$

Using 264H, we see also that

$$\operatorname{diam}(D \cap E) \ge \Bigl(\frac{2^r}{\beta_r}\mu(D \cap E)\Bigr)^{1/r} \ge 2\gamma\Bigl(\frac{1-r^{r/2}\epsilon_1}{\beta_r}\Bigr)^{1/r}.$$

Next,

$$\operatorname{per}(D \cap E) \le \nu(\partial D) + \nu(D \cap \partial^*E)$$

(475Cf)

$$\begin{aligned}&\le 2r\gamma^{r-1} + \epsilon_1(\operatorname{diam} D)^{r-1} \le \gamma^{r-1}(2r + r^{r/2}\epsilon_1)\\ &\le (2r + r^{r/2}\epsilon_1)\cdot\frac{1}{2^{r-1}}\Bigl(\frac{\beta_r}{1-\epsilon_1 r^{r/2}}\Bigr)^{(r-1)/r}\operatorname{diam}(D \cap E)^{r-1}\\ &\le \frac{1}{\alpha}\operatorname{diam}(D \cap E)^{r-1}.\end{aligned}$$

So $D \cap E \in \mathcal{C}_\alpha$. Also, for every $s > 0$, there is a $D' \in \mathcal{D}$ such that $D' \subseteq D$ and $x \in \overline{D'}$ and $\operatorname{diam} D' \le s$. In this case

$$\mu(B(x, \operatorname{diam} D') \setminus E) \le \epsilon_1(\operatorname{diam} D')^r = \epsilon_1 r^{r/2}\mu D' \le \frac{1}{2}\mu D',$$

so

$$\mu(D\cap E\cap B(x,\operatorname{diam}D'))\ge\mu D'-\mu(B(x,\operatorname{diam}D')\setminus E)$$
$$\ge\frac{1}{2}\mu D'=\frac{1}{2\beta_r r^{r/2}}\mu B(x,\operatorname{diam}D').$$

As $\operatorname{diam}D'$ is arbitrarily small, $x\in\mathrm{cl}^*(D\cap E)$ and $\boldsymbol{t}=\{(x,D\cap E)\}\in T_\alpha$. Finally, since $\operatorname{diam}D<\theta(x)$, $(x,D\cap E)\in\delta$ and $\boldsymbol{t}\in T'$. **Q**

(iii) Let $\mathcal{H}$ be the set of those $H\subseteq\mathbb{R}^r$ such that $W_{\boldsymbol{t}}\subseteq E\cap H\subseteq W_{\boldsymbol{t}}\cup\bigcup\mathcal{D}_1$ for some $\boldsymbol{t}\in T'$ and finite $\mathcal{D}_1\subseteq\mathcal{D}_0$. Then $H\cup H'\in\mathcal{H}$ whenever H, $H'\in\mathcal{H}$ are disjoint. If $\langle D_n\rangle_{n\in\mathbb{N}}$ is any strictly decreasing sequence in $\mathcal{D}$, then some D_n belongs to $\mathcal{H}$. **P** Let x be the unique point of $\bigcap_{n\in\mathbb{N}}\overline{D_n}$.

case 1 If $x\in A$, then there is a finite subset $\mathcal{D}_1$ of $\mathcal{D}_0$ whose union is a neighbourhood of x, and therefore includes D_n for some n; so $\boldsymbol{t}=\emptyset$ and $\mathcal{D}_1$ witness that $D_n\in\mathcal{H}$.

case 2 If $x\in E\setminus A$, then $x\in\mathrm{int}^*E$, so $h(x)>0$ and there is some $n\in\mathbb{N}$ such that $\operatorname{diam}D_n\le h(x)$. In this case $\boldsymbol{t}=\{(x,D_n\cap E)\}$ belongs to T', by the choice of $h(x)$, so that $\boldsymbol{t}$ and $\emptyset$ witness that $D_n\in\mathcal{H}$.

case 3 Finally, if $x\notin E\cup A$, then $x\notin\mathrm{cl}^*E$ so $x\notin\overline{E}$ and there is some n such that $D_n\cap E=\emptyset$, in which case $\boldsymbol{t}=\mathcal{D}_1=\emptyset$ witness that $D_n\in\mathcal{H}$. **Q**

(iv) In fact $\mathbb{R}^r\in\mathcal{H}$. **P?** Otherwise, because $E\subseteq V$ is bounded, it can be covered by a finite disjoint family in $\mathcal{D}$, and there must be some $D_0\in\mathcal{D}\setminus\mathcal{H}$. Now we can find $\langle D_n\rangle_{n\ge1}$ in $\mathcal{D}\setminus\mathcal{H}$ such that $D_n\subseteq D_{n-1}$ and $\operatorname{diam}D_n=\frac{1}{2}\operatorname{diam}D_{n-1}$ for every n. But this contradicts (iii).**XQ**

(v) We therefore have a $\boldsymbol{t}\in T'$ and a finite set $\mathcal{D}_1\subseteq\mathcal{D}_0$ such that $W_{\boldsymbol{t}}\subseteq E\subseteq W_{\boldsymbol{t}}\cup\bigcup\mathcal{D}_1$. Now we can find $\boldsymbol{t}'\subseteq\boldsymbol{t}$ and $\mathcal{D}_1'\subseteq\mathcal{D}_1$ such that $W_{\boldsymbol{t}'}\cap\bigcup\mathcal{D}_1'=\emptyset$ and $E\subseteq W_{\boldsymbol{t}'}\cup\bigcup\mathcal{D}_1'$. **P** Express $\boldsymbol{t}$ as $\langle(x_i,D_i\cap E)\rangle_{i\in I}$ where $D_i\in\mathcal{D}$ for each i. Then $E\subseteq\bigcup_{i\in I}D_i\cup\bigcup\mathcal{D}_1$. Set $\mathcal{D}_1'=\{D:D\in\mathcal{D}_1,\ D\not\subseteq D_i$ for every $i\in I\}$, $J=\{i:i\in I,\ D_i\not\subseteq D$ for every $D\in\mathcal{D}_1'\}$, $\boldsymbol{t}'=\{(x_i,D_i\cap E):i\in J\}$. **Q**

By the choice of $\mathcal{D}_0$, $\bigcup\mathcal{D}_1'\in\mathcal{R}''$; by the choice of $\mathcal{R}''$, $E\setminus W_{\boldsymbol{t}'}=E\cap\bigcup\mathcal{D}_1'$ belongs to $\mathcal{R}'$. But we know also that $C\cap V\setminus E\in\mathcal{R}'$, that is, $C\setminus E\in\mathcal{R}'$, because $\mathcal{R}'=\mathcal{R}_\eta^{(V)}$. By the choice of $\mathcal{R}'$, $C\setminus W_{\boldsymbol{t}'}\in\mathcal{R}$. And $\boldsymbol{t}'\in T'$ is a δ-fine member of T_α. As C, δ and $\mathcal{R}$ are arbitrary, 481G(vii) is satisfied, and the proof is complete.

484G The Pfeffer integral (a) For $\alpha\in\,]0,\alpha^*[$, write I_α for the linear functional defined by setting

$$I_\alpha(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T_\alpha,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\mu)$$

whenever $f:\mathbb{R}^r\to\mathbb{R}$ is such that the limit is defined. (See 481F for the notation $\mathcal{F}(T_\alpha,\Delta,\mathfrak{R})$.) Then if $0<\beta\le\alpha<\alpha^*$ and $I_\beta(f)$ is defined, so is $I_\alpha(f)$, and the two are equal. **P** All we have to observe is that $\mathcal{C}_\alpha\subseteq\mathcal{C}_\beta$ so that $T_\alpha\subseteq T_\beta$, while $\mathcal{F}(T_\alpha,\Delta,\mathfrak{R})$ is just $\{A\cap T_\alpha:A\in\mathcal{F}(T_\beta,\Delta,\mathfrak{R})\}$. **Q**

(b) Let $f:\mathbb{R}^r\to\mathbb{R}$ be a function. I will say that it is **Pfeffer integrable**, with **Pfeffer integral** $\mathrm{P}\!\!\!\int f$, if

$$\mathrm{P}\!\!\!\int f=\lim_{\alpha\downarrow0}I_\alpha(f)$$

is defined; that is to say, if $I_\alpha(f)$ is defined whenever $0<\alpha<\alpha^*$.

484H The first step is to work through the results of §482 to see which ideas apply directly to the limit integral $\mathrm{P}\!\!\!\int$.

Proposition (a) The domain of $\mathrm{P}\!\!\!\int$ is a linear space of functions, and $\mathrm{P}\!\!\!\int$ is a positive linear functional.

(b) If f, $g:\mathbb{R}^r\to\mathbb{R}$ are such that $|f|\le g$ and $\mathrm{P}\!\!\!\int g=0$, then $\mathrm{P}\!\!\!\int f$ is defined and equal to 0.

(c) If $f:\mathbb{R}^r\to\mathbb{R}$ is Pfeffer integrable, then there is a unique additive functional $F:\mathcal{C}\to\mathbb{R}$ such that whenever $\epsilon>0$ and $0<\alpha<\alpha^*$ there are $\delta\in\Delta$ and $\mathcal{R}\in\mathfrak{R}$ such that

$$\sum_{(x,C)\in\boldsymbol{t}}|F(C)-f(x)\mu C|\le\epsilon\text{ for every }\delta\text{-fine }\boldsymbol{t}\in T_\alpha,$$

$$|F(E)|\le\epsilon\text{ whenever }E\in\mathcal{C}\cap\mathcal{R}.$$

Moreover, $F(\mathbb{R}^r)=\mathrm{P}\!\!\!\int f$.

(d) Every Pfeffer integrable function is Lebesgue measurable.

(e) Every Lebesgue integrable function is Pfeffer integrable, with the same integral.

(f) A non-negative function is Pfeffer integrable iff it is Lebesgue integrable.

proof (a)-(b) Immediate from 481C.

(c) For each $\alpha \in \,]0,\alpha^*[$ let F_α be the Saks-Henstock indefinite integral corresponding to the the structure $(\mathbb{R}^r, T_\alpha, \Delta, \mathfrak{R}, \mu)$. Then all the F_α coincide. **P** Suppose that $0 < \beta \le \alpha < \alpha^*$. Then, for any $\epsilon > 0$, there are $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ such that

$$\sum_{(x,C)\in\boldsymbol{t}} |F_\beta(C) - f(x)\mu C| \le \epsilon \text{ for every } \delta\text{-fine } \boldsymbol{t} \in T_\beta,$$

$$|F_\beta(E)| \le \epsilon \text{ whenever } E \in \mathcal{R}.$$

Since $T_\alpha \subseteq T_\beta$, this means that

$$\sum_{(x,C)\in\boldsymbol{t}} |F_\beta(C) - f(x)\mu C| \le \epsilon \text{ for every } \delta\text{-fine } \boldsymbol{t} \in T_\alpha.$$

And this works for any $\epsilon > 0$. By the uniqueness assertion in 482B, F_β must be exactly the same as F_α. **Q**

So we have a single functional F; and 482B also tells us that

$$F(\mathbb{R}^r) = I_\alpha(f) = \operatorname{Pf}\!\!\int f$$

for every α.

(d) In fact if there is any α such that $I_\alpha(f)$ is defined, f must be Lebesgue measurable. **P** We have only to check that the conditions of 482E are satisfied by μ, $\mathcal{C}_\alpha$, $\{(x,C) : C \in \mathcal{C}_\alpha,\ x \in \mathrm{cl}^*C\}$, T_α, Δ and $\mathfrak{R}$. (i), (iii) and (v) are built into the definitions above, and (iv) and (vii) are covered by 484F. 482E(ii) is true because $C \setminus \mathrm{cl}^*C$ is negligible for every $C \in \mathcal{C}$ (475Cg).

As for 482E(vi), this is true because if $\mu E < \infty$ and $\epsilon > 0$, there are $n \in \mathbb{N}$ and $\eta \in \mathrm{H}$ such that $\mu(E\setminus B(\mathbf{0},n)) \le \frac{1}{2}\epsilon$ and $\sum_{i=0}^\infty \eta(i) \le \frac{1}{2}\epsilon$, so that

$$\mu(E \cap R) \le \mu(E \setminus B(\mathbf{0},n)) + \mu(R \cap B(\mathbf{0},n)) \le \epsilon$$

for every $R \in \mathcal{R}_\eta^{(B(\mathbf{0},n))}$. **Q**

(e) This time, we have to check that the conditions of 482F are satisfied by T_α, Δ and $\mathfrak{R}$ whenever $0 < \alpha < \alpha^*$. **P** Of course μ is inner regular with respect to the closed sets and outer regular with respect to the open sets (134F). Condition 482F(v) just repeats 482E(v), verified in (d) above. **Q**

(f) If $f \ge 0$ is integrable in the ordinary sense, then it is Pfeffer integrable, by (e). If it is Pfeffer integrable, then it is measurable; but also $\int g\,d\mu = \operatorname{Pf}\!\!\int g \le \operatorname{Pf}\!\!\int f$ for every simple function $g \le f$, so f is integrable (213B).

484I Definition If $f : \mathbb{R}^r \to \mathbb{R}$ is Pfeffer integrable, I will call the function $F : \mathcal{C} \to \mathbb{R}$ defined in 484Hc the **Saks-Henstock indefinite integral** of f.

484J In fact 484Hc characterizes the Pfeffer integral, just as the Saks-Henstock lemma can be used to define general gauge integrals based on tagged-partition structures allowing subdivisions.

Proposition Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ and $F : \mathcal{C} \to \mathbb{R}$ are such that

(i) F is additive,

(ii) whenever $0 < \alpha < \alpha^*$ and $\epsilon > 0$ there is a $\delta \in \Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\mu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T_\alpha$,

(iii) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \le \epsilon$ for every $E \in \mathcal{C} \cap \mathcal{R}$.

Then f is Pfeffer integrable and F is the Saks-Henstock indefinite integral of f.

proof By 482D, the gauge integral $I_\alpha(f)$ is defined and equal to $F(\mathbb{R}^r)$ for every $\alpha \in \,]0,\alpha^*[$. So f is Pfeffer integrable. Now 484Hc tells us that F must be its Saks-Henstock indefinite integral.

484K Lemma Suppose that $\alpha > 0$ and $0 < \alpha' < \alpha\min(\frac{1}{2}, 2^{r-1}(\frac{\alpha}{2\beta_r})^{(r-1)/r})$. If $E \in \mathcal{C}$ is such that $E \subseteq \mathrm{cl}^*E$, then there is a $\delta \in \Delta$ such that $\{(x, C \cap E)\} \in T_{\alpha'}$ whenever $(x,C) \in \delta$, $x \in E$ and $\{(x,C)\} \in T_\alpha$.

proof Take $\epsilon > 0$ such that $\beta_r\epsilon \le \frac{1}{2}\alpha$ and

$$\frac{1}{\alpha} + 2^{r-1}\epsilon \le \frac{2^{r-1}}{\alpha'}\Bigl(\frac{\alpha}{2\beta_r}\Bigr)^{(r-1)/r}.$$

Set

$$A = \partial^* E \cup \{x : \lim_{\zeta\downarrow 0} \sup_{x\in G, 0<\operatorname{diam} G\le\zeta} \frac{\nu^*(G\cap\partial^* E)}{(\operatorname{diam} G)^{r-1}} > 0\},$$

so that A is a thin set, as in (b-i) of the proof of 484F. (Of course $\partial^* E$ is thin because $\nu(\partial^* E \cap B(\mathbf{0}, n))$ is finite for every $n \in \mathbb{N}$.)

For $x \in E \setminus A$, we have $x \in \operatorname{int}^* E$ (because $E \subseteq \operatorname{cl}^* E$), so there is a $\theta(x) > 0$ such that

$$\mu(B(x,\zeta)\setminus E) \le \epsilon\mu B(x,\zeta), \quad \nu(\partial^* E \cap B(x,\zeta)) \le \epsilon(2\zeta)^{r-1}$$

whenever $0 < \zeta \le 2\theta(x)$. If we set $\theta(x) = 0$ for $x \in E \cap A$ and $\theta(x) = 1$ for $x \in \mathbb{R}^r \setminus E$, then $\theta \in \Theta$ and $\delta_\theta \in \Delta$.

Now suppose that $x \in E$, $(x, C) \in \delta_\theta$ and $\{(x, C)\} \in T_\alpha$, that is, that $C \in \mathcal{C}_\alpha$ and $x \in (E \cap \operatorname{cl}^* C) \setminus A$ and $\|x - y\| < \theta(x)$ for every $y \in C$. Set $\gamma = \operatorname{diam} C \le 2\theta(x)$. Then

$$\mu(C \setminus E) \le \mu(B(x,\gamma)\setminus E) \le \epsilon\mu B(x,\gamma) = \beta_r\epsilon\gamma^r,$$

so

$$\begin{aligned}\mu(C\cap E) &\ge \mu C - \beta_r\epsilon\gamma^r \ge (\alpha - \beta_r\epsilon)\gamma^r\\ &\ge \frac{1}{2}\alpha\gamma^r \ge \alpha'\gamma^r \ge \alpha' \operatorname{diam}(C\cap E)^r.\end{aligned}$$

Next,

$$\begin{aligned}\operatorname{per}(C\cap E) &= \nu(\partial^*(C\cap E)) \le \nu(B(x,\gamma)\cap(\partial^* C\cup\partial^* E))\\ &\le \nu(\partial^* C) + \nu(B(x,\gamma)\cap\partial^* E) \le \frac{1}{\alpha}\gamma^{r-1} + \epsilon(2\gamma)^{r-1} = (\frac{1}{\alpha} + 2^{r-1}\epsilon)\gamma^{r-1}.\end{aligned}$$

Moreover,

$$2^{-r}\beta_r \operatorname{diam}(C\cap E)^r \ge \mu(C\cap E) \ge \frac{1}{2}\alpha\gamma^r$$

(264H), so $\operatorname{diam}(C\cap E) \ge 2(\frac{\alpha}{2\beta_r})^{1/r}\gamma$ and

$$\operatorname{per}(C\cap E) \le (\frac{1}{\alpha} + 2^{r-1}\epsilon)\cdot\frac{1}{2^{r-1}}\Bigl(\frac{2\beta_r}{\alpha}\Bigr)^{(r-1)/r}\operatorname{diam}(C\cap E)^{r-1} \le \frac{1}{\alpha'}\operatorname{diam}(C\cap E)^{r-1}.$$

Putting these together, we see that $C \in \mathcal{C}_{\alpha'}$.

Finally, because $x \in \operatorname{cl}^* C \cap \operatorname{int}^* E \subseteq \operatorname{cl}^*(C\cap E)$ (475Ce), $\{(x, C\cap E)\} \in T_{\alpha'}$.

484L Proposition Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ is Pfeffer integrable, and that $F : \mathcal{C} \to \mathbb{R}$ is its Saks-Henstock indefinite integral. Then $⨏ f \times \chi E$ is defined and equal to $F(E)$ for every $E \in \mathcal{C}$.

proof (a) To begin with (down to the end of (d) below), suppose that $E \in \mathcal{C}$ is such that $\operatorname{int}^* E \subseteq E \subseteq \operatorname{cl}^* E$. For $C \in \mathcal{C}$ set $F_1(C) = F(C \cap E)$. I seek to show that F_1 satisfies the conditions of 484J.

Of course $F_1 : \mathcal{C} \to \mathbb{R}$ is additive. If $\epsilon > 0$, there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(G)| \le \epsilon$ whenever $G \in \mathcal{R}$, by 484H; now there is an $\mathcal{R}' \in \mathfrak{R}$ such that $G \cap E \in \mathcal{R}$ for every $G \in \mathcal{R}'$ (484E(a-ii)), so that $|F_1(G)| \le \epsilon$ for every $G \in \mathcal{C} \cap \mathcal{R}'$. Thus F_1 satisfies (iii) of 484J.

(b) Take $\alpha \in]0, \alpha^*[$ and $\epsilon > 0$. Take α' such that $0 < \alpha' < \alpha\min(\frac{1}{2}, 2^{r-1}(\frac{\alpha}{2\beta_r})^{(r-1)/r})$. Applying 484K to E and its complement, and appealing to the definition of F, we see that there is a $\delta \in \Delta$ such that

(α) $\{(x, C\cap E)\} \in T_{\alpha'}$ whenever $(x, C) \in \delta$, $x \in E$ and $\{(x, C)\} \in T_\alpha$,

(β) $\{(x, C\setminus E)\} \in T_{\alpha'}$ whenever $(x, C) \in \delta$, $x \in \mathbb{R}^r \setminus E$ and $\{(x, C)\} \in T_\alpha$,

(γ) $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\mu C| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T_{\alpha'}$.

(For (α), we need to know that $E \subseteq \operatorname{cl}^* E$ and for (β) we need $\operatorname{int}^* E \subseteq E$.) Next, choose for each $n \in \mathbb{N}$ closed sets $H_n \subseteq E$, $H'_n \subseteq \mathbb{R}^r \setminus E$ such that $\mu(E \setminus H_n) \le 2^{-n}\epsilon$ and $\mu((\mathbb{R}^r \setminus E)\setminus H'_n) \le 2^{-n}\epsilon$. Define $\theta : \mathbb{R}^r \to]0, \infty[$ by setting

$$\begin{aligned}\theta(x) &= \min(1, \frac{1}{2}\rho(x, H'_n)) \text{ if } x \in E \text{ and } n \le |f(x)| < n+1,\\ &= \min(1, \frac{1}{2}\rho(x, H_n)) \text{ if } x \in \mathbb{R}^r \setminus E \text{ and } n \le |f(x)| < n+1,\end{aligned}$$

writing $\rho(x,A)=\inf_{y\in A}\|x-y\|$ if $A\subseteq\mathbb{R}^r$ is non-empty, ∞ if $A=\emptyset$. Then $\delta_\theta\in\Delta$. Let $\mathcal{R}'\in\mathfrak{R}$ be such that $A\cap E\in\mathcal{R}$ whenever $A\in\mathcal{R}'$.

(c) Write f_E for $f\times\chi E$. Then $\sum_{(x,C)\in\boldsymbol{t}}|F_1(C)-f_E(x)\mu(C)|\le 11\epsilon$ whenever $\boldsymbol{t}\in T_\alpha$ is $(\delta\cap\delta_\theta)$-fine. **P** Set

$$\boldsymbol{t}'=\{(x,C\cap E):(x,C)\in\boldsymbol{t},\,x\in E\}.$$

By clause (α) of the choice of δ, $\boldsymbol{t}'\in T_{\alpha'}$, and of course it is δ-fine. So

$$\sum_{(x,C)\in\boldsymbol{t},x\in E}|F(C\cap E)-f(x)\mu(C\cap E)|\le\epsilon$$

by clause (γ) of the choice of δ. Next,

$$\sum_{(x,C)\in\boldsymbol{t},x\in E}|f(x)\mu(C\setminus E)|=\sum_{n=0}^{\infty}\sum_{\substack{(x,C)\in\boldsymbol{t},x\in E\\ n\le|f(x)|<n+1}}|f(x)|\mu(C\setminus E)$$

$$\le\sum_{n=0}^{\infty}(n+1)\mu((\mathbb{R}^r\setminus E)\setminus H_n')$$

(because $\operatorname{diam}C\le\theta(x)$, so $C\cap H_n'=\emptyset$ whenever $(x,C)\in\boldsymbol{t}$, $x\in E$ and $n\le|f(x)|<n+1$)

$$\le\sum_{n=0}^{\infty}2^{-n}(n+1)\epsilon=4\epsilon,$$

and

$$\sum_{(x,C)\in\boldsymbol{t},x\in E}|F(C\cap E)-f(x)\mu C|\le 5\epsilon.$$

Similarly,

$$\sum_{(x,C)\in\boldsymbol{t},x\notin E}|F(C\setminus E)-f(x)\mu C)|\le 5\epsilon.$$

But as

$$\sum_{(x,C)\in\boldsymbol{t},x\notin E}|F(C)-f(x)\mu C|\le\epsilon$$

(because surely $\boldsymbol{t}$ itself belongs to $T_{\alpha'}$), we have

$$\sum_{(x,C)\in\boldsymbol{t},x\notin E}|F(C\cap E)|\le 6\epsilon.$$

Putting these together,

$$\sum_{(x,C)\in\boldsymbol{t}}|F_1(C)-f_E(x)\mu C|=\sum_{\substack{(x,C)\in\boldsymbol{t}\\ x\in E}}|F(C\cap E)-f(x)\mu C|+\sum_{\substack{(x,C)\in\boldsymbol{t}\\ x\notin E}}|F(C\cap E)|$$

$$\le 5\epsilon+6\epsilon=11\epsilon.\ \mathbf{Q}$$

(d) As α and ϵ are arbitrary, condition (ii) of 484J is satisfied by F_1 and f_E, so $⨎ f_E=F_1(\mathbb{R}^r)=F(E)$.

(e) This completes the proof when $\operatorname{int}^*E\subseteq E\subseteq\operatorname{cl}^*E$. For a general set $E\in\mathcal{C}$, set $E_1=(E\cup\operatorname{int}^*E)\cap\operatorname{cl}^*E$. Then $E\triangle E_1$ is negligible, so

$$\operatorname{int}^*E_1=\operatorname{int}^*E\subseteq E_1\subseteq\operatorname{cl}^*E=\operatorname{cl}^*E_1.$$

Also $⨎ f\times\chi(E\setminus E_1)=\int f\times\chi(E\setminus E_1)d\mu$, $⨎ f\times\chi(E_1\setminus E)=\int f\times\chi(E\setminus E_1)d\mu$ are both zero, and

$$⨎ f\times\chi E=⨎ f\times\chi E_1=F(E_1)=F(E).$$

(To see that $F(E_1)=F(E)$, note that $E\setminus E_1$ and $E_1\setminus E$, being negligible sets, have empty essential boundary and zero perimeter, so belong to every member of $\mathfrak{R}$, by 484E(b-ii), or otherwise.)

484M Lemma Let G, $H\in\mathcal{C}$ be disjoint and $\phi:\mathbb{R}^r\to\mathbb{R}^r$ a continuous function. If *either* $G\cup H$ is bounded *or* ϕ has compact support,

$$\int_{\partial^*(G\cup H)}\phi\cdot\psi_{G\cup H}\,d\nu=\int_{\partial^*G}\phi\cdot\psi_G\,d\nu+\int_{\partial^*H}\phi\cdot\psi_G\,d\nu,$$

where ψ_G, ψ_H and $\psi_{G\cup H}$ are the canonical outward-normal functions (474G).

proof (a) Suppose first that ϕ is a Lipschitz function with compact support. Then 475N tells us that

$$\int_{\partial^*(G\cup H)} \phi \centerdot \psi_{G\cup H}\, d\nu = \int_{G\cup H} \operatorname{div}\phi\, d\mu = \int_G \operatorname{div}\phi\, d\mu + \int_H \operatorname{div}\phi\, d\mu$$
$$= \int_{\partial^* G} \phi \centerdot \psi_G\, d\nu + \int_{\partial^* H} \phi \centerdot \psi_G\, d\nu.$$

(Recall from 474R that we can identify canonical outward-normal functions with Federer exterior normals, as in the statement of 475N.)

(b) Now suppose that ϕ is a continuous function with compact support. Let $\langle \tilde h_n\rangle_{n\in\mathbb{N}}$ be the smoothing sequence of 473E. Then all the functions $\phi * \tilde h_n$ are Lipschitz and $\langle \phi * \tilde h_n\rangle_{n\in\mathbb{N}}$ converges uniformly to ϕ (473Df, 473Ed). So

$$\int_{\partial^*(G\cup H)} \phi \centerdot \psi_{G\cup H}\, d\nu = \lim_{n\to\infty} \int_{\partial^*(G\cup H)} (\phi * \tilde h_n) \centerdot \psi_{G\cup H}\, d\nu$$
$$= \lim_{n\to\infty} \int_{\partial^* G} (\phi * \tilde h_n) \centerdot \psi_G\, d\nu + \lim_{n\to\infty} \int_{\partial^* H} (\phi * \tilde h_n) \centerdot \psi_H\, d\nu$$
$$= \int_{\partial^* G} \phi \centerdot \psi_G\, d\nu + \int_{\partial^* H} \phi \centerdot \psi_G\, d\nu.$$

(c) If, on the other hand, ϕ is continuous and G and H are bounded, then we can find a continuous function $\tilde\phi$ with compact support agreeing with ϕ on $\overline{G\cup H}$ (4A2G(e-i), or otherwise); applying (b) to $\tilde\phi$, we get the required result for ϕ.

484N Pfeffer's Divergence Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ a continuous function with compact support such that $\{x : x \in \mathbb{R}^r,\ \phi \text{ is not differentiable at } x\}$ is thin. Let v_x be the Federer exterior normal to E at any point x where the normal exists. Then $f\!\!\!\!\int \operatorname{div}\phi \times \chi E$ is defined and equal to $\int_{\partial^* E} \phi(x) \centerdot v_x \nu(dx)$.

proof (a) Let n be such that $\phi(x) = \mathbf{0}$ for $\|x\| \ge n$. For $C \in \mathcal{C}$, set $F(C) = \int_{\partial^* C} \phi \centerdot \psi_C \nu(dx)$, where ψ_C is the canonical outward-normal function; recall that $\psi_E(x) = v_x$ for ν-almost every $x \in \partial^* E$ (474R, 475D). By 484M, F is additive.

(b) If $0 < \alpha < \alpha^*$ and $\epsilon > 0$ and $x \in \mathbb{R}^r$ is such that ϕ is differentiable at x, there is a $\gamma > 0$ such that $|F(C) - \operatorname{div}\phi(x)\mu C| \le \epsilon\mu C$ whenever $C \in \mathcal{C}_\alpha$, $x \in \overline{C}$ and $\operatorname{diam} C \le \gamma$. **P** Let T be the derivative of ϕ at x. Let $\gamma > 0$ be such that $\|\phi(y) - \phi(x) - T(y-x)\| \le \alpha^2\epsilon\|y-x\|$ whenever $\|y-x\| \le \gamma$. Let $\tilde\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function with compact support such that $\tilde\phi(y) = \phi(x) + (y-x)$ whenever $\|y-x\| \le \gamma$ (473Cf). If $C \in \mathcal{C}_\alpha$ has diameter at most γ and $x \in \overline{C}$, then

$$|F(C) - \operatorname{div}\phi(x)\mu C| = \Bigl|\int_{\partial^* C} \phi \centerdot \psi_C \nu(dx) - \int_C \operatorname{div}\tilde\phi\, d\mu\Bigr|$$

(because T is the derivative of $\tilde\phi$ everywhere on $B(x,\gamma)$, so $\operatorname{div}\tilde\phi(y) = \operatorname{div}\phi(x)$ for every $y \in C$)

$$= \Bigl|\int_{\partial^* C} (\phi - \tilde\phi) \centerdot \psi_C \nu(dx)\Bigr|$$

(applying the Divergence Theorem 475N to $\tilde\phi$)

$$\le \nu(\partial^* C) \sup_{y\in C} \|\phi(y) - \tilde\phi(y)\|$$
$$\le \alpha^2\epsilon \operatorname{diam} C \operatorname{per} C \le \alpha\epsilon(\operatorname{diam} C)^r \le \epsilon\mu C$$

because $C \in \mathcal{C}_\alpha$. **Q**

(c) If $\epsilon > 0$ and $\alpha \in \,]0,\alpha^*[$, there is a $\delta \in \Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - \operatorname{div}\phi(x)\mu C| \le \epsilon$ whenever $\boldsymbol{t} \in T_\alpha$ is δ-fine. **P** Let $\zeta > 0$ be such that $\zeta\mu B(\mathbf{0}, n+2) \le \epsilon$. Set $A = \{x : x \in \mathbb{R}^r,\ \phi \text{ is not differentiable at } x\}$, and for

$x \in \mathbb{R}^r \setminus A$ let $\theta(x) \in \left]0, \frac{1}{2}\right]$ be such that $|F(C) - \operatorname{div}\phi(x)\mu C| \le \zeta\mu C$ whenever $C \in \mathcal{C}_\alpha$, $x \in \overline{C}$ and $\operatorname{diam} C \le \theta(x)$; for $x \in A$ set $\theta(x) = 0$. Now suppose that $\boldsymbol{t} \in T_\alpha$ is δ_θ-fine. Then

$$\sum_{(x,C)\in\boldsymbol{t}} |F(C) - \operatorname{div}\phi(x)\mu C| = \sum_{\substack{(x,C)\in\boldsymbol{t} \\ x\in B(\mathbf{0},n+1)}} |F(C) - \operatorname{div}\phi(x)\mu C|$$

(because $\operatorname{diam} C \le 1$ whenever $(x, C) \in \boldsymbol{t}$, so if $\|x\| > n + 1$ then $F(C) = \operatorname{div}\phi(x) = 0$)

$$\le \sum_{\substack{(x,C)\in\boldsymbol{t} \\ x\in B(\mathbf{0},n+1)}} \zeta\mu C \le \zeta\mu B(\mathbf{0}, n + 2) \le \epsilon. \text{ } \mathbf{Q}$$

(d) Because ϕ is a continuous function with compact support, it is uniformly continuous (apply 4A2Jf to each of the coordinates of ϕ). For $\zeta > 0$, let $\gamma(\zeta) > 0$ be such that $\|\phi(x) - \phi(y)\| \le \zeta$ whenever $\|x - y\| \le \gamma(\zeta)$.

If $C \in \mathcal{C}$ and $\operatorname{per} C \le 1$ and $\mu C \le \zeta\gamma(\zeta)$, where $\zeta > 0$, then $|F(C)| \le r\zeta(2\|\phi\|_\infty + \frac{1}{2})$, writing $\|\phi\|_\infty$ for $\sup_{x\in\mathbb{R}^r} \|\phi(x)\|$. $\mathbf{P}$ For $1 \le i \le r$, let $\phi_i : \mathbb{R}^r \to \mathbb{R}$ be the ith component of ϕ, and v_i the ith unit vector $(0, \dots , 1, \dots , 0)$; write

$$\alpha_i = \int_{\partial^* C} \phi_i(x)(v_i|\psi_E(x))\nu(dx),$$

so that $F(C) = \sum_{i=1}^r \alpha_i$. I start by examining α_r. By 475O, we have sequences $\langle H_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ and $\langle g'_n\rangle_{n\in\mathbb{N}}$ such that

(i) for each $n \in \mathbb{N}$, H_n is a Lebesgue measurable subset of $\mathbb{R}^{r-1}$, and g_n, $g'_n : H_n \to [-\infty, \infty]$ are Lebesgue measurable functions such that $g_n(u) < g'_n(u)$ for every $u \in H_n$;

(ii) if m, $n \in \mathbb{N}$ then $g_m(u) \ne g'_n(u)$ for every $u \in H_m \cap H_n$;

(iii) $\sum_{n=0}^\infty \int_{H_n} g'_n - g_n d\mu_{r-1} = \mu C$;

(iv)

$$\alpha_r = \sum_{n=0}^\infty \int_{H_n} \phi_r(u, g'_n(u)) - \phi_r(u, g_n(u))\mu_{r-1}(du),$$

where we interpret $\phi_r(u, \infty)$ and $\phi_r(u, -\infty)$ as 0 if necessary;

(v) for μ_{r-1}-almost every $u \in \mathbb{R}^{r-1}$,

$$\{t : (u, t) \in \partial^* C\} = \{g_n(u) : n \in \mathbb{N}, u \in H_n, g_n(u) \ne -\infty\} \cup \{g'_n(u) : n \in \mathbb{N}, u \in H_n, g'_n(u) \ne \infty\}.$$

From (iii) we see that g'_n and g_n are both finite almost everywhere on H_n, for every n. Consequently, by (v) and 475H,

$$2\sum_{n=0}^\infty \mu_{r-1} H_n = \int \#(\{t : (u, t) \in \partial^* C\})\mu_{r-1}(du) \le \nu(\partial^* C) \le 1.$$

For each n, set

$$H'_n = \{u : u \in H_n, g'_n(u) - g_n(u) > \gamma(\zeta)\}.$$

Then $\gamma(\zeta) \sum_{n=0}^\infty \mu H'_n \le \mu C$ so $\sum_{n=0}^\infty \mu H'_n \le \zeta$ and

$$|\sum_{n=0}^\infty \int_{H'_n} \phi_r(u, g'_n(u)) - \phi_r(u, g_n(u))\mu_{r-1}(du)| \le 2\|\phi\|_\infty \sum_{n=0}^\infty \mu H'_n \le 2\zeta\|\phi\|_\infty.$$

On the other hand, for $n \in \mathbb{N}$ and $u \in H_n \setminus H'_n$, $|\phi_r(u, g'_n(u)) - \phi_r(u, g_n(u))| \le \zeta$, so

$$|\sum_{n=0}^\infty \int_{H_n\setminus H'_n} \phi_r(u, g'_n(u)) - \phi_r(u, g_n(u))\mu_{r-1}(du)| \le \sum_{n=0}^\infty \zeta\mu_{r-1}(H_n \setminus H'_n) \le \frac{1}{2}\zeta.$$

Putting these together,

$$\alpha_r \le 2\zeta\|\phi\|_\infty + \frac{1}{2}\zeta.$$

But of course the same arguments apply to all the α_i, so

$$|F(C)| \le \sum_{i=1}^{r} |\alpha_i| \le r\zeta(2\|\phi\|_\infty + \tfrac{1}{2}),$$

as claimed. **Q**

(e) If $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \le \epsilon$ for every $E \in \mathcal{C} \cap \mathcal{R}$. **P** Let $\langle \epsilon_i \rangle_{i\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $r(2\|\phi\|_\infty + \frac{1}{2})\sum_{i=0}^{\infty} \epsilon_i \le \epsilon$. For each $i \in \mathbb{N}$, set $\eta(i) = \epsilon_i\gamma(\epsilon_i) > 0$. Set $V = B(\mathbf{0}, n+1)$, and take any $E \in \mathcal{C} \cap \mathcal{R}_\eta^{(V)}$. Then $F(E \setminus V) = 0$, so $F(E) = F(E \cap V)$, while $E \cap V \in \mathcal{R}_\eta$. Express $E \cap V$ as $\bigcup_{i\le n} E_i$, where $\langle E_i \rangle_{i\le n}$ is disjoint, $\operatorname{per} E_i \le 1$ and $\mu E_i \le \eta(i)$ for each i. Then $|F(E_i)| \le r\epsilon_i(2\|\phi\|_\infty + \frac{1}{2})$ for each i, by (d), so

$$|F(E)| = |F(E \cap V)| \le \sum_{i=0}^{n} |F(E_i)| \le \epsilon,$$

as required. **Q**

(f) By 484J, $\operatorname{div}\phi$ is Pfeffer integrable. Moreover, by the uniqueness assertion in 484Hc, its Saks-Henstock indefinite integral is just the function F here. By 484L, $F(E) = \int\!\!\!\!\!\!\mathrm{f}\ \operatorname{div}\phi \times \chi E$ for every $E \in \mathcal{C}$, as required.

484O Differentiating the indefinite integral: Theorem Let $f : \mathbb{R}^r \to \mathbb{R}$ be a Pfeffer integrable function, and F its Saks-Henstock indefinite integral. Then whenever $0 < \alpha < \alpha^*$,

$$\begin{aligned} f(x) &= \lim_{\zeta\downarrow 0} \sup\{\frac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \le \zeta\} \\ &= \lim_{\zeta\downarrow 0} \inf\{\frac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \le \zeta\} \end{aligned}$$

for μ-almost every $x \in \mathbb{R}^r$.

proof (a) It will be useful to know the following: if $C \in \mathcal{C}_\alpha$, $\operatorname{diam} C > 0$, $x \in \overline{C}$ and $\epsilon > 0$, then for any sufficiently small $\zeta > 0$, $C \cup B(x,\zeta) \in \mathcal{C}_{\alpha/2}$ and $|F(C \cup B(x,\zeta)) - F(C)| \le \epsilon$. **P** Let $\mathcal{R} \in \mathfrak{R}$ be such that $|F(R)| \le \epsilon$ whenever $R \in \mathcal{C} \cap \mathcal{R}$, let $\mathcal{R}' \in \mathfrak{R}$ be such that $(\mathbb{R}^r \setminus C) \cap R \in \mathcal{R}$ whenever $R \in \mathcal{R}'$ (484E(a-ii)), and let $\eta \in \mathrm{H}$ be such that $\mathcal{R}_\eta \subseteq \mathcal{R}'$ (484E(a-i)). Then for all sufficiently small $\zeta > 0$, we shall have $\operatorname{per} B(x,\zeta) \le 1$ and $\mu B(x,\zeta) \le \eta(0)$, so that $B(x,\zeta) \in \mathcal{R}_\eta$, $B(x,\zeta) \setminus C \in \mathcal{R}$ and

$$|F(C \cup B(x,\zeta)) - F(C)| = |F(B(x,\zeta) \setminus C)| \le \epsilon.$$

Next, for all sufficiently small $\zeta > 0$,

$$\begin{aligned} \mu(C \cup B(x,\zeta)) &\ge \mu C \ge \alpha(\operatorname{diam} C)^r \\ &\ge \frac{\alpha}{2}(\zeta + \operatorname{diam} C)^r \ge \frac{\alpha}{2}\operatorname{diam}(C \cup B(x,\zeta))^r \end{aligned}$$

(because $x \in \overline{C}$) and

$$\begin{aligned} \operatorname{per}(C \cup B(x,\zeta)) &\le \operatorname{per} C + \operatorname{per} B(x,\zeta) \le \frac{1}{\alpha}(\operatorname{diam} C)^{r-1} + \operatorname{per} B(x,\zeta) \\ &\le \frac{2}{\alpha}(\operatorname{diam} C)^{r-1} \le \frac{2}{\alpha}\operatorname{diam}(C \cup B(x,\zeta))^{r-1}, \end{aligned}$$

so that $C \cup B(x,\zeta) \in \mathcal{C}_{\alpha/2}$. **Q**

(b) For $x \in \mathbb{R}$, set

$$g(x) = \lim_{\zeta\downarrow 0} \sup\{\frac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ \operatorname{diam} C \le \zeta\}.$$

? Suppose, if possible, that there are rational numbers $q < q'$ and $n \in \mathbb{N}$ such that $A = \{x : \|x\| \le n,\ f(x) \le q < q' < g(x)\}$ is not μ-negligible. Set

$$\epsilon = \frac{(q'-q)\alpha}{4\beta_r}\mu^* A > 0.$$

Let $\theta \in \Theta$ be such that

$$\sum_{(x,C)\in \boldsymbol{t}} |F(C) - f(x)\mu C| \le \epsilon$$

for every δ_θ-fine $\boldsymbol{t} \in T_{\alpha/2}$. Let $\mathcal{I}$ be the family of all balls $B(x,\zeta)$ where $x \in A$, $0 < \zeta \le \theta(x)$ and there is a $C \in \mathcal{C}_\alpha$ such that $x \in \overline{C}$, $\operatorname{diam} C = \zeta$ and $\frac{F(C)}{\mu C} > q'$. Then every member of $A_1 = A \setminus \theta^{-1}[\{0\}]$ is the centre of arbitrarily small members of $\mathcal{I}$, so by 472C there is a countable disjoint family $\mathcal{J}_0 \subseteq \mathcal{I}$ such that

$$\mu(\bigcup \mathcal{J}_0) > \frac{1}{2}\mu^* A_1 = \frac{1}{2}\mu^* A.$$

There is therefore a finite family $\mathcal{J}_1 \subseteq \mathcal{J}_0$ such that $\mu(\bigcup \mathcal{J}_1) > \frac{1}{2}\mu^* A$; enumerate $\mathcal{J}_1$ as $\langle B(x_i,\zeta_i)\rangle_{i\le n}$ where, for each $i \le n$, $x_i \in A$, $0 < \zeta_i \le \theta_i(x)$ and there is a $C_i \in \mathcal{C}_\alpha$ such that $x_i \in \overline{C_i}$, $\operatorname{diam} C_i = \zeta_i$ and $F(C_i) > q'\mu C_i$. By (a), we can enlarge C_i by adding a sufficiently small ball around x_i to form a $C_i' \in \mathcal{C}_{\alpha/2}$ such that $x_i \in \operatorname{int} C_i'$, $C_i' \subseteq B(x_i,\zeta_i)$ and $F(C_i') \ge q'\mu C_i'$.

Consider $\boldsymbol{t} = \{(x_i, C_i') : i \le n\}$. Then, because the balls $B(x_i,\zeta_i)$ are disjoint, and $x_i \in \operatorname{int} C_i' \subseteq \mathrm{cl}^* C_i'$ for every i, $\boldsymbol{t}$ is a δ_θ-fine member of $T_{\alpha/2}$. So $\sum_{i=0}^n F(C_i') \le \epsilon + \sum_{i=0}^n f(x_i)\mu C_i'$. But as $F(C_i') \ge q'\mu C_i'$ and $f(x_i) \le q$ for every i, this means that $(q'-q)\sum_{i=0}^n \mu C_i' \le \epsilon$.

But now remember that $\operatorname{diam} C_i' \ge \operatorname{diam} C_i = \zeta_i$ and that $C_i' \in \mathcal{C}_{\alpha/2}$ for each i. This means that

$$\mu C_i' \ge \frac{\alpha}{2}\zeta_i^r \ge \frac{\alpha}{2\beta_r}\mu B(x_i,\zeta_i)$$

for each i, and

$$\begin{aligned}\epsilon \ge (q'-q)\sum_{i=0}^n \mu C_i' &\ge \tfrac{(q'-q)\alpha}{2\beta_r}\sum_{i=0}^n \mu B(x_i,\zeta_i)\\ &> \tfrac{(q'-q)\alpha}{4\beta_r}\mu^* A = \epsilon\end{aligned}$$

which is absurd. **X**

(c) Since q, q' and n are arbitrary, this means that $g \le_{\text{a.e.}} f$. Similarly (or applying (b) to $-f$ and $-F$)

$$f(x) \le \lim_{\zeta\downarrow 0} \inf\{\tfrac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\, x \in \overline{C},\, 0 < \operatorname{diam} C \le \zeta\}$$

for almost all x, as required.

484P Lemma Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be an injective Lipschitz function, and H the set of points at which it is differentiable; for $x \in H$, write $T(x)$ for the derivative of ϕ at x and $J(x)$ for $|\det T(x)|$. Then, for μ-almost every $x \in \mathbb{R}^r$,

$$\begin{aligned}J(x) &= \lim_{\zeta\downarrow 0}\sup\{\tfrac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\, x \in \overline{C},\, 0 < \operatorname{diam} C \le \zeta\}\\ &= \lim_{\zeta\downarrow 0}\inf\{\tfrac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\, x \in \overline{C},\, 0 < \operatorname{diam} C \le \zeta\}\end{aligned}$$

for every $\alpha > 0$.

proof By Rademacher's theorem (262Q), H is conegligible. Let H' be the Lebesgue set of J, so that H' also is conegligible (261E). Take any $x \in H'$ and $\epsilon > 0$. Then there is a $\zeta_0 > 0$ such that $\int_{B(x,\zeta)} |J(y) - J(x)|\mu(dy) \le \epsilon\mu B(x,\zeta)$ for every $\zeta \in [0,\zeta_0]$. Now suppose that $C \in \mathcal{C}_\alpha$, $x \in \overline{C}$ and $0 < \operatorname{diam} C \le \zeta_0$. Then $\mu(C \setminus H) = 0$ so $\mu\phi[C \setminus H] = 0$ (262D), and

$$\mu\phi[C] = \mu\phi[C \cap H] = \int_{C\cap H} J\,d\mu$$

(263D(iv)). So

$$\begin{aligned}|\mu\phi[C] - J(x)\mu C| = |\int_{C\cap H} J\,d\mu - J(x)\mu(C\cap H)| &\le \int_{C\cap H} |J(y) - J(x)| d\mu\\ &\le \int_{B(x,\operatorname{diam} C)} |J(y) - J(x)| d\mu \le \epsilon\mu B(x, \operatorname{diam} C)\\ &= \beta_r\epsilon(\operatorname{diam} C)^r \le \frac{\beta_r\epsilon}{\alpha}\mu C.\end{aligned}$$

Thus $|\frac{\mu\phi[C]}{\mu C}-J(x)|\le\frac{\beta_r}{\alpha}\epsilon$ whenever $C\in\mathcal{C}_\alpha$, $x\in\overline{C}$ and $0<\operatorname{diam}C\le\zeta_0$; as ϵ is arbitrary,

$$\begin{aligned}J(x)&=\lim_{\zeta\downarrow 0}\sup\{\frac{\mu\phi[C]}{\mu C}:C\in\mathcal{C}_\alpha,\,x\in\overline{C},\,0<\operatorname{diam}C\le\zeta\}\\&=\lim_{\zeta\downarrow 0}\inf\{\frac{\mu\phi[C]}{\mu C}:C\in\mathcal{C}_\alpha,\,x\in\overline{C},\,0<\operatorname{diam}C\le\zeta\}.\end{aligned}$$

And this is true for μ-almost every x.

484Q Definition If (X,ρ) and (Y,σ) are metric spaces, a function $\phi:X\to Y$ is a **lipeomorphism** if it is bijective and both ϕ and ϕ^{-1} are Lipschitz. Of course a lipeomorphism is a homeomorphism.

484R Lemma Let $\phi:\mathbb{R}^r\to\mathbb{R}^r$ be a lipeomorphism.

(a) For any set $A\subseteq\mathbb{R}^r$,

$$\operatorname{cl}^*(\phi[A])=\phi[\operatorname{cl}^*A],\quad \operatorname{int}^*(\phi[A])=\phi[\operatorname{int}^*A],\quad \partial^*(\phi[A])=\phi[\partial^*A].$$

(b) $\phi[C]\in\mathcal{C}$ for every $C\in\mathcal{C}$, and $\phi[V]\in\mathcal{V}$ for every $V\in\mathcal{V}$.

(c) For any $\alpha>0$ there is an $\alpha'\in\,]0,\alpha]$ such that $\phi[C]\in\mathcal{C}_{\alpha'}$ for every $C\in\mathcal{C}_\alpha$ and $\{(\phi(x),\phi[C]):(x,C)\in\boldsymbol{t}\}$ belongs to $T_{\alpha'}$ for every $\boldsymbol{t}\in T_\alpha$.

(d) For any $\mathcal{R}\in\mathfrak{R}$ there is an $\mathcal{R}'\in\mathfrak{R}$ such that $\phi[R]\in\mathcal{R}$ for every $R\in\mathcal{R}'$.

(e) $\theta\phi:\mathbb{R}^r\to[0,\infty[$ belongs to Θ for every $\theta\in\Theta$.

proof Let γ be so large that it is a Lipschitz constant for both ϕ and ϕ^{-1}. Observe that in this case

$$\phi^{-1}[B(\phi(x),\frac{\zeta}{\gamma})]\subseteq B(x,\zeta),\quad \phi[B(x,\zeta)]\supseteq B(\phi(x),\frac{\zeta}{\gamma})$$

for every $x\in\mathbb{R}^r$ and $\zeta\ge 0$, while

$$\mu^*A=\mu^*\phi^{-1}[\phi[A]]\le\gamma^r\mu^*\phi[A],\quad \nu^*A\le\gamma^{r-1}\nu^*\phi[A]$$

for every $A\subseteq\mathbb{R}^r$ (264G/471J).

(a) If $A\subseteq\mathbb{R}^r$ and $x\in\operatorname{cl}^*A$, set

$$\epsilon=\frac{1}{2}\limsup_{\zeta\downarrow 0}\tfrac{\mu^*(B(x,\zeta)\cap A)}{\mu B(x,\zeta)}>0.$$

Take any $\zeta_0>0$. Then there is a ζ such that $0<\zeta\le\zeta_0$ and $\mu^*(B(x,\frac{\zeta}{\gamma})\cap A)\ge\epsilon\mu B(x,\frac{\zeta}{\gamma})$, so that

$$\begin{aligned}\mu^*(B(\phi(x),\zeta)\cap\phi[A])&\ge\mu^*\phi[B(x,\frac{\zeta}{\gamma})\cap A]\ge\frac{1}{\gamma^r}\mu^*(B(x,\frac{\zeta}{\gamma})\cap A)\\&\ge\tfrac{\epsilon}{\gamma^r}\mu B(x,\frac{\zeta}{\gamma})=\tfrac{\epsilon}{\gamma^{2r}}\mu B(x,\zeta).\end{aligned}$$

As ζ_0 is arbitrary,

$$\limsup_{\zeta\downarrow 0}\tfrac{\mu^*(B(\phi(x),\zeta)\cap\phi[A]}{\mu B(\phi(x),\zeta)}\ge\tfrac{\epsilon}{\gamma^{2r}}>0,$$

and $\phi(x)\in\operatorname{cl}^*(\phi[A])$.

This shows that $\phi[\operatorname{cl}^*A]\subseteq\operatorname{cl}^*(\phi[A])$. The same argument applies to ϕ^{-1} and $\phi[A]$, so that $\phi[\operatorname{cl}^*A]$ must be equal to $\operatorname{cl}^*(\phi[A])$. Taking complements, $\phi[\operatorname{int}^*A]=\operatorname{int}^*(\phi[A])$, so that $\phi[\partial^*A]=\partial^*(\phi[A])$.

(b) Take $C\in\mathcal{C}$. Then, for any $n\in\mathbb{N}$, $\phi^{-1}[B(\mathbf{0},n)]$ is bounded, so is included in $B(\mathbf{0},m)$ for some m. Now

$$\begin{aligned}\nu(\partial^*\phi[C]\cap B(\mathbf{0},n))&=\nu(\phi[\partial^*C]\cap B(\mathbf{0},n))\\&\le\nu(\phi[\partial^*C\cap B(\mathbf{0},m)])\le\gamma^{r-1}\nu(\partial^*C\cap B(\mathbf{0},m))\end{aligned}$$

is finite. This shows that $\phi[C]$ has locally finite perimeter and belongs to $\mathcal{C}$. Since $\phi[V]$ is bounded whenever V is bounded, $\phi[V]\in\mathcal{V}$ whenever $V\in\mathcal{V}$.

(c) Set $\alpha'=\gamma^{-2r}\alpha$. Note that as γ^2 is a Lipschitz constant for the identity map, $\gamma\ge 1$, and $\alpha'\le\min(\alpha,\gamma^{2-2r}\alpha)$. If $C\in\mathcal{C}_\alpha$, then

$$\mu\phi[C] \geq \frac{1}{\gamma^r}\mu C \geq \frac{\alpha}{\gamma^r}(\operatorname{diam} C)^r \geq \alpha \operatorname{diam} \phi[C])^r \geq \alpha'(\operatorname{diam} \phi[C])^r,$$

$$\begin{aligned}\operatorname{per} \phi[C] &= \nu(\partial^*(\phi[C])) = \nu(\phi[\partial^* C]) \leq \gamma^{r-1}\nu(\partial^* C)\\ &\leq \frac{\gamma^{r-1}}{\alpha}(\operatorname{diam} C)^{r-1} \leq \frac{\gamma^{r-1}}{\alpha}(\gamma \operatorname{diam} \phi[C])^{r-1} \leq \frac{1}{\alpha'}(\operatorname{diam} \phi[C])^{r-1}.\end{aligned}$$

So $C \in \mathcal{C}_{\alpha'}$.

If now $\boldsymbol{t} \in T_\alpha$, then, for any $(x, C) \in \boldsymbol{t}$, $\phi[C] \in \mathcal{C}_{\alpha'}$ and $\phi(x) \in \operatorname{cl}^*\phi[C]$; also, because ϕ is injective, $\langle \phi[C]\rangle_{(x,C)\in\boldsymbol{t}}$ is disjoint, so $\{(\phi(x), \phi[C]) : (x, C) \in \boldsymbol{t}\} \in T_{\alpha'}$.

(d) Express $\mathcal{R}$ as $\mathcal{R}_\eta^{(V)}$ where $V \in \mathcal{V}$ and $\eta \in \mathrm{H}$, so that $R \in \mathcal{R}$ whenever $R \cap V \in \mathcal{R}$. By 484Ec and 481He, there is a sequence $\langle \mathcal{Q}_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{R}$ such that $\bigcup_{i\leq n} A_i \in \mathcal{R}$ whenever $n \in \mathbb{N}$, $\langle A_i\rangle_{i\leq n}$ is disjoint and $A_i \in \mathcal{Q}_i$ for every i. By 484E(b-ii), there is an $\eta' \in \mathrm{H}$ such that $R \in \mathcal{Q}_i$ whenever $i \in \mathbb{N}$ and R is such that $\mu R \leq \gamma^r\eta'(i)$ and $\operatorname{per} R \leq \gamma^{r-1}$. Try $\mathcal{R}' = \mathcal{R}_{\eta'}^{(\phi^{-1}[V])} \in \mathfrak{R}$. If $R \in \mathcal{R}'$, we can express $R \cap \phi^{-1}[V]$ as $\bigcup_{i\leq n} E_i$ where $\operatorname{per} E_i \leq 1$ and $\mu E_i \leq \eta'_i$ for each $i \leq n$, and $\langle E_i\rangle_{i\leq n}$ is disjoint. So $\phi[R] \cap V = \bigcup_{i\leq n} \phi[E_i]$ and $\langle \phi[E_i]\rangle_{i\leq n}$ is disjoint. Now, for each i,

$$\mu\phi[E_i] \leq \gamma^r \mu E_i \leq \gamma^r \eta'(i), \quad \operatorname{per} \phi[E_i] \leq \gamma^{r-1} \operatorname{per} E_i \leq \gamma^{r-1},$$

so $\phi[E_i] \in \mathcal{Q}_i$. By the choice of $\langle \mathcal{Q}_i\rangle_{i\in\mathbb{N}}$, $\phi[R] \cap V \in \mathcal{R}$ and $\phi[R] \in \mathcal{R}$. So $\mathcal{R}'$ has the property we need.

(e) We have only to observe that if A is the thin set $\theta^{-1}[\{0\}]$, then $(\theta\phi)^{-1}[\{0\}] = \phi^{-1}[A]$ is also thin. **P** If $A = \bigcup_{n\in\mathbb{N}} A_n$ where $\nu^* A_n$ is finite for every n, then $\phi^{-1}[A] = \bigcup_{n\in\mathbb{N}} \phi^{-1}[A_n]$, while $\nu^*\phi^{-1}[A_n] \leq \gamma^{r-1}\nu^* A_n$ is finite for every $n \in \mathbb{N}$. **Q**

484S Theorem Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a lipeomorphism. Let H be the set of points at which ϕ is differentiable. For $x \in H$, write $T(x)$ for the derivative of ϕ at x; set $J(x) = |\det T(x)|$ for $x \in H$, 0 for $x \in \mathbb{R}^r \setminus H$. Then, for any function $f : \mathbb{R}^r \to \mathbb{R}^r$,

$$\text{Pf}\!\!\int f = \text{Pf}\!\!\int J \times f\phi$$

if either is defined in $\mathbb{R}$.

proof (a) Let $H' \subseteq H$ be a conegligible set such that

$$\begin{aligned}J(x) &= \lim_{\zeta\downarrow 0} \sup\{\frac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\, x \in \overline{C},\, 0 < \operatorname{diam} C \leq \zeta\}\\ &= \lim_{\zeta\downarrow 0} \inf\{\frac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\, x \in \overline{C},\, 0 < \operatorname{diam} C \leq \zeta\}\end{aligned}$$

for every $\alpha > 0$ and every $x \in H'$ (484P). To begin with (down to the end of (c)), suppose that f is Pfeffer integrable and that $f\phi(x) = 0$ for every $x \in \mathbb{R}^r \setminus H'$. Let F be the Saks-Henstock indefinite integral of f, and define $G : \mathcal{C} \to \mathbb{R}$ by setting $G(C) = F(\phi[C])$ for every $C \in \mathcal{C}$ (using 484Rb to see that this is well-defined).

(b) G and $J \times f\phi$ satisfy the conditions of 484J.

P(i) Of course G is additive, because F is.

(ii) Suppose that $0 < \alpha < \alpha^*$ and $\epsilon > 0$. Let $\alpha' \in \,]0, \alpha^*[$ be such that $\{(\phi(x), \phi[C]) : (x, C) \in \boldsymbol{t}\} \in T_{\alpha'}$ whenever $\boldsymbol{t} \in T_\alpha$ (484Rc). Let $\theta_1 \in \Theta$ be such that $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\mu C| \leq \frac{1}{2}\epsilon$ for every δ_{θ_1}-fine $\boldsymbol{t} \in T_{\alpha'}$. Let $\theta_2 : \mathbb{R}^r \to \,]0, 1]$ be such that whenever $x \in H'$, $n \leq \|x\| + |f\phi(x)| < n + 1$, $C \in \mathcal{C}_{\alpha'}$, $x \in \overline{C}$ and $\operatorname{diam} C \leq 2\theta_2(x)$ then

$$|\mu\phi[C] - J(x)\mu C| \leq \frac{\epsilon\mu C}{2^{n+2}\beta_r(n+2)^r(n+1)}.$$

Set $\theta(x) = \min(\frac{1}{\gamma}\theta_1\phi(x), \theta_2(x))$ for $x \in \mathbb{R}^r$, where $\gamma > 0$ is a Lipschitz constant for ϕ, so that $\theta \in \Theta$ (484Re).

If $\boldsymbol{t} \in T_\alpha$ is δ_θ-fine, set $\boldsymbol{t}' = \{(\phi(x), \phi[C]) : (x, C) \in \boldsymbol{t}\}$. Then $\boldsymbol{t}' \in T_{\alpha'}$, by the choice of α'. If $(x, C) \in \boldsymbol{t}$, then $\theta(x) > 0$ so $\theta_1\phi(x) > 0$; also, for any $y \in \phi[C]$,

$$\|\phi(x) - y\| \leq \gamma\|x - \phi^{-1}(y)\| < \gamma\theta(x) \leq \theta_1\phi(x).$$

This shows that $\boldsymbol{t}'$ is δ_{θ_1}-fine. We therefore have

$$\begin{aligned}\sum_{(x,C)\in\boldsymbol{t}}|G(C)-J(x)f(\phi(x))\mu C| &\le \sum_{(x,C)\in\boldsymbol{t}}|F(\phi[C])-f(\phi(x))\mu\phi[C]|\\&\qquad+\sum_{(x,C)\in\boldsymbol{t}}|f(\phi(x))||\mu\phi[C]-J(x)\mu C|\\&\le \sum_{(x,C)\in\boldsymbol{t}'}|F(C)-f(x)\mu C|\\&\qquad+\sum_{\substack{(x,C)\in\boldsymbol{t}\\ x\in H'}}|f(\phi(x))||\mu\phi[C]-J(x)\mu C|\end{aligned}$$

(because if $x\notin H'$ then $f(\phi(x))=0$)

$$\le \frac{1}{2}\epsilon+\sum_{n=0}^{\infty}\sum_{\substack{(x,C)\in\boldsymbol{t},x\in H'\\ n\le\|x\|+|f(\phi(x))|<n+1}}\tfrac{(n+1)\epsilon\mu C}{2^{n+2}\beta_r(n+2)^r(n+1)}$$

$$\le \frac{1}{2}\epsilon+\sum_{n=0}^{\infty}\tfrac{\epsilon\mu B(\mathbf{0},n+2)}{2^{n+2}\beta_r(n+2)^r}$$

(remembering that $\theta_2(x)\le 1$, so $C\subseteq B(\mathbf{0},n+2)$ whenever $(x,C)\in\boldsymbol{t}$ and $\|x\|<n+1$)

$$=\epsilon.$$

As $\boldsymbol{t}$ is arbitrary, this shows that G and $J\times f\phi$ satisfy (ii) of 484J.

(iii) Given $\epsilon>0$, there is an $\mathcal{R}\in\mathfrak{R}$ such that $|F(C)|\le\epsilon$ for every $C\in\mathcal{C}\cap\mathcal{R}$. Now by 484Rd there is an $\mathcal{R}'\in\mathfrak{R}$ such that $\phi[R]\in\mathcal{R}$ for every $R\in\mathcal{R}'$, so that $|G(C)|\le\epsilon$ for every $C\in\mathcal{C}\cap\mathcal{R}'$. Thus G satisfies (iii) of 484J. **Q**

(c) This shows that $J\times f\phi$ is Pfeffer integrable, with Saks-Henstock indefinite integral G; so, in particular,

$$\mathrm{P}\!\!\int f\times\phi=G(\mathbb{R}^r)=F(\mathbb{R}^r)=\mathrm{P}\!\!\int f.$$

(d) Now suppose that f is an arbitrary Pfeffer integrable function. In this case set $f_1=f\times\chi\phi[H']$. Because $\mathbb{R}^r\setminus H'$ is μ-negligible, so is $\phi[\mathbb{R}^r\setminus H']$, and $f_1=f$ μ-a.e. Also, of course, $f\phi=f_1\phi$ μ-a.e. Because the Pfeffer integral extends the Lebesgue integral (484He),

$$\mathrm{P}\!\!\int J\times f\phi=\mathrm{P}\!\!\int J\times f_1\phi=\mathrm{P}\!\!\int f_1=\mathrm{P}\!\!\int f.$$

(e) All this has been on the assumption that f is Pfeffer integrable. If $g=J\times f\phi$ is Pfeffer integrable, consider $\tilde{J}\times g\phi^{-1}$, where $\tilde{J}(x)=|\det\tilde{T}(x)|$ whenever the derivative $\tilde{T}(x)$ of ϕ^{-1} at x is defined, and otherwise is zero. Now

$$\tilde{J}(x)g(\phi^{-1}(x))=\tilde{J}(x)\cdot J(\phi^{-1}(x))f(x)$$

for every x. But, for μ-almost every x,

$$\tilde{J}(x)J(\phi^{-1}(x))=|\det\tilde{T}(x)||\det T(\phi^{-1}(x))|=|\det\tilde{T}(x)T(\phi^{-1}(x))|=1$$

because $\tilde{T}(x)T(\phi^{-1}(x))$ is (whenever it is defined) the derivative at $\phi^{-1}(x)$ of the identity function $\phi^{-1}\phi$, by 473Bc. (I see that we need to know that $\{x:\phi$ is differentiable at $\phi^{-1}(x)\}=\phi[H]$ is conegligible.) So $\tilde{J}\times g\phi^{-1}=f$ μ-a.e., and f is Pfeffer integrable. This completes the proof.

484X Basic exercises >(a) Show that for every $\mathcal{R}\in\mathfrak{R}$ there are $\eta\in\mathrm{H}$ and $n\in\mathbb{N}$ such that $\mathcal{R}_\eta^{(B(\mathbf{0},n))}\subseteq\mathcal{R}$.

>(b) (Pfeffer 91a) For $\alpha>0$ let $\mathcal{C}'_\alpha$ be the family of bounded Lebesgue measurable sets C such that $\mu C\ge\alpha\operatorname{diam}C\operatorname{per}C$. Show that $\mathcal{C}_{\sqrt{\alpha}}\subseteq\mathcal{C}'_\alpha\subseteq\mathcal{C}_{\min(\alpha,\alpha^r)}$. (*Hint*: 474La.)

>(c) For $\alpha>0$, let $\mathcal{C}''_\alpha$ be the family of bounded convex sets $C\subseteq\mathbb{R}^r$ such that $\mu C\ge\alpha(\operatorname{diam}C)^r$. Show that if $0<\alpha<1/2r$ then $\mathcal{C}''_\alpha\subseteq\mathcal{C}_\alpha$ (*hint*: 475T) and $T_\alpha\cap[\mathbb{R}^r\times\mathcal{C}''_\alpha]^{<\omega}$ is compatible with Δ and $\mathfrak{R}$. (*Hint*: use the argument of 484F, but in part (b) take $C=\mathbb{R}^r$, $E=V$ a union of members of $\mathcal{D}$.)

(d) Describe a suitable filter $\mathcal{F}$ to express the Pfeffer integral directly in the form considered in 481C.

(e) Let $f : \mathbb{R}^r \to \mathbb{R}$ be a Pfeffer integrable function. Show that there is some $n \in \mathbb{N}$ such that $\int_{\mathbb{R}^r \setminus B(\mathbf{0},n)} |f| d\mu$ is finite.

(f) (Here take $r = 2$.) Let $\langle \delta_n \rangle_{n\in\mathbb{N}}$ be a strictly decreasing summable sequence in $]0, 1]$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by saying that $f(x) = \frac{(-1)^n}{(n+1)(\delta_n^2 - \delta_{n+1}^2)}$ if $n \in \mathbb{N}$ and $\delta_{n+1} \le \|x\| < \delta_n$, 0 otherwise. Show that $\lim_{\delta\downarrow 0} \int_{\mathbb{R}^2 \setminus B(\mathbf{0},\delta)} f d\mu$ is defined, but that f is not Pfeffer integrable. (*Hint*: 484J.)

(g) (Again take $r = 2$.) Show that there are a Lebesgue integrable $f_1 : \mathbb{R} \to \mathbb{R}$ and a Henstock integrable $f_2 : \mathbb{R} \to \mathbb{R}$, both with bounded support, such that $(\xi_1, \xi_2) \mapsto f_1(\xi_1)f_2(\xi_2) : \mathbb{R}^2 \to \mathbb{R}$ is not Pfeffer integrable.

(h) Let $E \subseteq \mathbb{R}^r$ be a bounded set with finite perimeter, and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ a differentiable function. Let v_x be the Federer exterior normal to E at any point x where the normal exists. Show that $\mathop{\int\!\!\!\!\!\!\mathrm{f}} \operatorname{div} \phi \times \chi E$ is defined and equal to $\int_{\partial^* E} \phi(x) \boldsymbol{.} v_x \nu(dx)$.

(i) Show that there is a Lipschitz function $f : \mathbb{R}^r \to [0, 1]$ such that $\mathbb{R}^r \setminus \operatorname{dom} f'$ is not thin. (*Hint*: there is a Lipschitz function $f : \mathbb{R} \to [0, 1]$ not differentiable at any point of the Cantor set.)

484Y Further exercises (a) Let $E \subseteq \mathbb{R}^r$ be any Lebesgue measurable set, and $\epsilon > 0$. Show that there is a Lebesgue measurable set $G \subseteq E$ such that $\operatorname{per} G \le \operatorname{per} E$, $\mu(E \setminus G) \le \epsilon$ and $\mathrm{cl}^* G = \overline{G}$.

(b) Give an example of a compact set $K \subseteq \mathbb{R}^2$ with zero one-dimensional Hausdorff measure such that whenever $\theta : K \to]0, \infty[$ is a strictly positive function, and $\gamma \in \mathbb{R}$, there is a disjoint family $\langle B(x_i, \zeta_i) \rangle_{i\le n}$ of balls such that $x_i \in K$ and $\zeta_i \le \theta(x_i)$ for every i, while $\operatorname{per}(\bigcup_{i\le n} B(x_i, \zeta_i)) \ge \gamma$.

484 Notes and comments Listing the properties of the Pfeffer integral as developed above, we have

expected relations with Lebesgue measure and integration (484Hd-484Hf);
Saks-Henstock indefinite integrals (484H-484J);
integration over suitable subsets (484L);
a divergence theorem (484N);
a density theorem (484O);
a change-of-variable theorem for lipeomorphisms (484S).

The results on indefinite integrals and integration over subsets are restricted in comparison with what we have for the Lebesgue integral, since we can deal only with sets with locally finite perimeter; and 484S is similarly narrower in scope than 263D(v). Pfeffer's Divergence Theorem, on the other hand, certainly applies to many functions ϕ for which $\operatorname{div} \phi$ is not Lebesgue integrable, though it does not entirely cover 475N (see 484Xi). In comparison with the one-dimensional case, the Pfeffer integral does not share the most basic property of the special Denjoy integral (483Bd, 484Xf), but 484N is a step towards the Perron integral (483J). 484O is a satisfactory rendering of the idea in 483I, and even for Lebesgue integrable functions adds something to 261C. Throughout, I have written on the assumption that $r \ge 2$. It would be possible to work through the same arguments with $r = 1$, but in this case we should find that 'thin' sets became countable, therefore easily controllable by neighbourhood gauges, making the methods here inappropriate.

The whole point of 'gauge integrals' is that we have an enormous amount of freedom within the framework of §§481-482. There is a corresponding difficulty in making definitive choices. The essential ideology of the Pfeffer integral is that we take an intersection of a family of gauge integrals, each determined by a family $\mathcal{C}_\alpha$ of sets which are 'Saks regular' in the sense that their measures, perimeters and diameters are linked (compare 484Xb). Shrinking $\mathcal{C}_\alpha$ and T_α, while leaving Δ and $\mathfrak{R}$ unchanged, of course leads to a more 'powerful' integral (supposing, at least, that we do not go so far that T_α is no longer compatible with Δ and $\mathfrak{R}$), so that Pfeffer's Divergence Theorem will remain true. One possibility is to turn to convex sets (484Xc), though we could not then expect invariance under lipeomorphisms.

The family $\Delta = \{\delta_\theta : \theta \in \Theta\}$ of gauges is designed to permit the exclusion of tags from thin sets; apart from this refinement, we are looking at neighbourhood gauges, just as with the Henstock integral. This feature, or something like it, seems to be essential when we come to the identification $F(E) = \mathop{\int\!\!\!\!\!\!\mathrm{f}} f \times \chi E$ in 484L, which is demanded by the formula in the target theorem 484N. In order to make our families T_α compatible in the sense of 481F, we are then forced to allow non-trivial residual families; with some effort (484C, 484Ed), we can get tagged-partition structures allowing subdivisions (484F). Note that this is one of the cases in which our residual families $\mathcal{R}_\eta^{(V)}$ are defined by 'shape' as well as by 'size'. In the indefinite-integral characterization of the Pfeffer integral (484J), we certainly cannot demand 'for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \le \epsilon$ whenever $E \in \mathcal{C}$ is included in a member of $\mathcal{R}$', since all small balls belong to $\mathcal{R}$, and we should immediately be driven to the Lebesgue integral. However I use the construction $\mathcal{R}_\eta^{(V)} = \{R : R \cap V \in \mathcal{R}_\eta\}$ (484D) as a quick method of eliminating any difficulties at infinity (484Xe). We do not of course need to look at arbitrary sets $V \in \mathcal{V}$ here (484Xa).

Observe that 484B can be thought of as a refinement of 475I. As usual, the elaborate formula in the statement of 484C is there only to emphasize that we have a bound depending only on l and r. Note that 484S depends much more on the fact that the Pfeffer integral can be characterized in the language of 484J, than on the exact choices made in forming $\mathfrak{R}$ and the $\mathcal{C}_\alpha$. For a discussion of integrals *defined* by Saks-Henstock lemmas, see PFEFFER 01.

It would be agreeable to be able to think of the Pfeffer integral as a product in some sense, so we naturally look for Fubini-type theorems. I give 484Xg to indicate one of the obstacles.

Chapter 49

Further topics

I conclude the volume with notes on six almost unconnected special topics. In §491 I look at equidistributed sequences and the ideal $\mathcal{Z}$ of sets with asymptotic density zero. I give the principal theorems on the existence of equidistributed sequences in abstract topological measure spaces, and examine the way in which an equidistributed sequence can induce an embedding of a measure algebra in the quotient algebra $\mathcal{P}\mathbb{N}/\mathcal{Z}$. The next three sections are linked. In §492 I present some forms of 'concentration of measure' which echo ideas from §476 in combinatorial, rather than geometric, contexts, with theorems of Talagrand and Maurey on product measures and the Haar measure of a permutation group. In §493 I show how the ideas of §§449, 476 and 492 can be put together in the theory of 'extremely amenable' topological groups. Some of the important examples of extremely amenable groups are full groups of measure-preserving automorphisms of measure algebras, previously treated in §383; these are the subject of §494, where I look also at some striking algebraic properties of these groups. In §495, I move on to Poisson point processes, with notes on disintegrations and some special cases in which they can be represented by Radon measures. In §496, I revisit the Maharam submeasures of Chapter 39, showing that various ideas from the present volume can be applied in this more general context. In §497, I give a version of Tao's proof of Szemerédi's theorem on arithmetic progressions, based on a deep analysis of relative independence, as introduced in §458. Finally, in §498 I give a pair of simple, but perhaps surprising, results on subsets of sets of positive measure in product spaces.

491 Equidistributed sequences

In many of the most important topological probability spaces, starting with Lebesgue measure (491Xg), there are sequences which are equidistributed in the sense that, in the limit, they spend the right proportion of their time in each part of the space (491Yf). I give the basic results on existence of equidistributed sequences in 491E-491H, 491Q and 491R. Investigating such sequences, we are led to some interesting properties of the asymptotic density ideal $\mathcal{Z}$ and the quotient algebra $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ (491A, 491I-491K, 491P). For 'effectively regular' measures (491L-491M), equidistributed sequences lead to embeddings of measure algebras in $\mathfrak{Z}$ (491N).

491A The asymptotic density ideal (a) If I is a subset of $\mathbb{N}$, its **upper asymptotic density** is $d^*(I) = \limsup_{n\to\infty} \frac{1}{n}(I\cap n)$, and its **asymptotic density** is $d(I) = \lim_{n\to\infty} \frac{1}{n}\#(I\cap n)$ if this is defined. It is easy to check that d^* is a submeasure on $\mathcal{P}\mathbb{N}$ (definition: 392A), so that

$$\mathcal{Z} = \{I : I \subseteq \mathbb{N},\, d^*(I) = 0\} = \{I : I \subseteq \mathbb{N},\, d(I) = 0\}$$

is an ideal, the **asymptotic density ideal**.

(b) Note that

$$\mathcal{Z} = \{I : I \subseteq \mathbb{N},\, \lim_{n\to\infty} 2^{-n}\#(I\cap 2^{n+1}\setminus 2^n) = 0\}.$$

P If $I \subseteq \mathbb{N}$ and $d^*(I) = 0$, then

$$2^{-n}\#(I\cap 2^{n+1}\setminus 2^n) \le 2\cdot 2^{-n-1}\#(I\cap 2^{n+1}) \to 0$$

as $n\to\infty$. In the other direction, if $\lim_{n\to\infty} 2^{-n}\#(I\cap 2^{n+1}\setminus 2^n) = 0$, then for any $\epsilon > 0$ there is an $m\in\mathbb{N}$ such that $\#(I\cap 2^{k+1}\setminus 2^k) \le 2^k\epsilon$ for every $k \ge m$. In this case, for $n \ge 2^m$, take k_n such that $2^{k_n} \le n < 2^{k_n+1}$, and see that

$$\frac{1}{n}\#(I\cap n) \le 2^{-k_n}(\#(I\cap 2^m) + \textstyle\sum_{k=m}^{k_n} 2^k\epsilon) \le 2^{-k_n}\#(I\cap 2^m) + 2\epsilon \to 2\epsilon$$

as $n\to\infty$, and $d^*(I) \le 2\epsilon$; as ϵ is arbitrary, $I\in\mathcal{Z}$. **Q**

(c) Writing $\mathcal{D}$ for the domain of d,

$$\begin{aligned}\mathcal{D} &= \{I : I\subseteq\mathbb{N},\, \limsup_{n\to\infty}\frac{1}{n}\#(I\cap n) = \liminf_{n\to\infty}\frac{1}{n}\#(I\cap n)\}\\ &= \{I : I\subseteq\mathbb{N},\, d^*(I) = 1 - d^*(\mathbb{N}\setminus I)\},\end{aligned}$$

$$\mathbb{N}\in\mathcal{D},\quad \text{if } I,\, J\in\mathcal{D} \text{ and } I\subseteq J \text{ then } J\setminus I\in\mathcal{D},$$

$$\text{if } I,\, J \in \mathcal{D} \text{ and } I \cap J = \emptyset \text{ then } I \cup J \in \mathcal{D} \text{ and } d(I \cup J) = d(I) + d(J).$$

It follows that if $\mathcal{I} \subseteq \mathcal{D}$ and $I \cap J \in \mathcal{I}$ for all $I,\, J \in \mathcal{I}$, then the subalgebra of $\mathcal{P}\mathbb{N}$ generated by $\mathcal{I}$ is included in $\mathcal{D}$ (313Ga). But note that $\mathcal{D}$ itself is *not* a subalgebra of $\mathcal{P}\mathbb{N}$ (491Xa).

(d) The following elementary fact will be useful. If $\langle l_n \rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ such that $\lim_{n\to\infty} l_{n+1}/l_n = 1$, and $I \subseteq \mathbb{R}$, then

$$d^*(I) \le \limsup_{n\to\infty} \frac{1}{l_{n+1}-l_n} \#(I \cap l_{n+1} \setminus l_n).$$

P Set $\gamma = \limsup_{n\to\infty} \frac{1}{l_{n+1}-l_n} \#(I \cap l_{n+1} \setminus l_n)$, and take $\epsilon > 0$. Let n_0 be such that $\#(I \cap l_{n+1} \setminus l_n) \le (\gamma+\epsilon)(l_{n+1}-l_n)$ and $l_{n+1} - l_n \le \epsilon l_n$ for every $n \ge n_0$, and write M for $\#(I \cap l_{n_0})$. If $m > l_{n_0}$, take k such that $l_k \le m < l_{k+1}$; then

$$\begin{aligned}\#(I \cap m) &\le M + \sum_{n=n_0}^{k-1} \#(I \cap l_{n+1} \setminus l_n) + (m - l_k)\\ &\le M + \sum_{n=n_0}^{k-1} (\gamma+\epsilon)(l_{n+1} - l_n) + l_{k+1} - l_k \le M + m(\gamma+\epsilon) + \epsilon m,\end{aligned}$$

so

$$\frac{1}{m}\#(I \cap m) \le \frac{M}{m} + \gamma + 2\epsilon.$$

Accordingly $d^*(I) \le \gamma + 2\epsilon$; as ϵ is arbitrary, $d^*(I) \le \gamma$. **Q**

***(e)** The following remark will not be used directly in this section, but is one of the fundamental properties of the ideal $\mathcal{Z}$. If $\langle I_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{Z}$, there is an $I \in \mathcal{Z}$ such that $I_n \setminus I$ is finite for every n. **P** Set $J_n = \bigcup_{j\le n} I_j$ for each n, so that $\langle J_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{Z}$. Let $\langle l_n \rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that, for each n, $\#(J_n \cap k) \le 2^{-n}k$ for every $k \ge l_n$. Set $I = \bigcup_{n\in\mathbb{N}} J_n \setminus l_n$. Then $I_n \setminus I \subseteq l_n$ is finite for each n. Also, if $n \in \mathbb{N}$ and $l_n \le k < l_{n+1}$,

$$\#(I \cap k) \le \#(J_n \cap k) \le 2^{-n}k,$$

so $I \in \mathcal{Z}$. **Q**

491B Equidistributed sequences Let X be a topological space and μ a probability measure on X. I say that a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in X is **(asymptotically) equidistributed** if $\mu F \ge d^*(\{i : x_i \in F\})$ for every measurable closed set $F \subseteq X$; equivalently, if $\mu G \le \liminf_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\ x_i \in G\})$ for every measurable open set $G \subseteq X$.

Remark Equidistributed sequences are often called **uniformly distributed**. Traditionally, such sequences have been defined in terms of their action on continuous functions, as in 491Cf. I have adopted the definition here in order to deal both with Radon measures on spaces which are not completely regular (so that we cannot identify the measure with an integral) and with Baire measures (so that there may be closed sets which are not measurable). Note that we cannot demand that the sets $\{i : x_i \in F\}$ should have well-defined densities (491Xi).

491C I work through a list of basic facts. The technical details (if we do not specialize immediately to metrizable or compact spaces) are not quite transparent, so I set them out carefully.

Proposition Let X be a topological space, μ a probability measure on X and $\langle x_n \rangle_{n\in\mathbb{N}}$ a sequence in X.

(a) $\langle x_n \rangle_{n\in\mathbb{N}}$ is equidistributed iff $\int f d\mu \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$ for every measurable bounded lower semi-continuous function $f : X \to \mathbb{R}$.

(b) If μ measures every zero set and $\langle x_n \rangle_{n\in\mathbb{N}}$ is equidistributed, then $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i) = \int f d\mu$ for every $f \in C_b(X)$.

(c) Suppose that μ measures every zero set in X. If $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i) = \int f d\mu$ for every $f \in C_b(X)$, then $d^*(\{n : x_n \in F\}) \le \mu F$ for every zero set $F \subseteq X$.

(d) Suppose that X is normal and that μ measures every zero set and is inner regular with respect to the closed sets. If $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i) = \int f d\mu$ for every $f \in C_b(X)$, then $\langle x_n \rangle_{n\in\mathbb{N}}$ is equidistributed.

(e) Suppose that μ is τ-additive and there is a base $\mathcal{G}$ for the topology of X, consisting of measurable sets and closed under finite unions, such that $\mu G \le \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \le n,\, x_i \in G\})$ for every $G \in \mathcal{G}$. Then $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

(f) Suppose that X is completely regular and that μ measures every zero set and is τ-additive. Then $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed iff the limit $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$ is defined and equal to $\int f d\mu$ for every $f \in C_b(X)$.

(g) Suppose that X is metrizable and that μ is a topological measure. Then $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed iff the limit $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$ is defined and equal to $\int f d\mu$ for every $f \in C_b(X)$.

(h) Suppose that X is compact, Hausdorff and zero-dimensional, and that μ is a Radon measure on X. Then $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed iff $d(\{n : x_n \in G\}) = \mu G$ for every open-and-closed subset G of X.

proof (a)(i) Suppose that $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed. Let $f : X \to [0,1]$ be a measurable lower semi-continuous function and $k \ge 1$. For each $j < k$ set $G_j = \{x : f(x) > \frac{j}{k}\}$. Then

$$\liminf_{n\to\infty} \frac{1}{n+1}\sum_{j=1}^{n} \chi G_j(x_i) = \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \le n,\, x_i \in G_j\}) \ge \mu G_j$$

because $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed and G_j is a measurable open set. Also $f - \frac{1}{k}\chi X \le \frac{1}{k}\sum_{j=1}^{k} \chi G_j \le f$, so

$$\begin{aligned}\int f d\mu - \frac{1}{k} &\le \frac{1}{k}\sum_{j=1}^{k} \mu G_j \le \frac{1}{k}\sum_{j=1}^{k} \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \chi G_j(x_i) \\ &\le \frac{1}{k}\liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n}\sum_{j=1}^{k} \chi G_j(x_i) \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i).\end{aligned}$$

As k is arbitrary, $\int f d\mu \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$.

The argument above depended on f taking values in $[0,1]$. But multiplying by an appropriate positive scalar we see that $\int f d\mu \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$ for every bounded measurable lower semi-continuous $f : X \to [0,\infty[$, and adding a multiple of χX we see that the same formula is valid for all bounded measurable lower semi-continuous $f : X \to \mathbb{R}$.

(ii) Conversely, if $\int f d\mu \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$ for every bounded measurable lower semi-continuous $f : X \to \mathbb{R}$, and $G \subseteq X$ is a measurable open set, then χG is lower semi-continuous, so $\mu G \le \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \le n,\, x_i \in G\})$. As G is arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

(b) Apply (a) to the lower semi-continuous functions f and (Recall that if μ measures every zero set, then every bounded continuous real-valued function is integrable, by 4A3L.)

(c) Let $F \subseteq X$ be a zero set, and $\epsilon > 0$. Then there is a continuous $f : X \to \mathbb{R}$ such that $F = f^{-1}[\{0\}]$. Let $\delta > 0$ be such that $\mu\{x : 0 < |f(x)| \le \delta\} \le \epsilon$, and set $g = (\chi X - \frac{1}{\delta}|f|)^+$. Then $g : X \to [0,1]$ is continuous and $\chi F \le g$, so

$$\begin{aligned}d^*(\{n : x_n \in F\}) &\le \limsup_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} g(x_i) \\ &= \int g\, d\mu \le \mu\{x : |f(x)| \le \delta\} \le \mu F + \epsilon.\end{aligned}$$

As ϵ and F are arbitrary, we have the result.

(d) Let $F \subseteq X$ be a measurable closed set and $\epsilon > 0$. Because μ is inner regular with respect to the closed sets, there is a measurable closed set $F' \subseteq X \setminus F$ such that $\mu F' \ge \mu(X \setminus F) - \epsilon$. Because X is normal, there is a continuous function $f : X \to [0,1]$ such that $\chi F \le f \le \chi(X \setminus F')$. Now

$$d^*(\{n : x_n \in F\}) \le \limsup_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i) = \int f d\mu \le \mu(X \setminus F') \le \mu F + \epsilon.$$

As F and ϵ are arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

(e) Let $G \subseteq X$ be a measurable open set, and $\epsilon > 0$. Then $\mathcal{H} = \{H : H \in \mathcal{G},\, H \subseteq G\}$ is upwards-directed and has union G; since μ is τ-additive, there is an $H \in \mathcal{H}$ such that $\mu H \ge \mu G - \epsilon$. Now

$$\mu G \le \epsilon + \mu H \le \epsilon + \liminf_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\, x_i \in H\})$$
$$\le \epsilon + \liminf_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\, x_i \in G\});$$

as ϵ and G are arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

(f)(i) If $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed then (b) tells us that $\int f d\mu = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i)$ for every $f \in C_b(X)$. (ii) Suppose that $\int f d\mu = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i)$ for every $f \in C_b(X)$. If $G \subseteq X$ is a cozero set, we can apply (c) to its complement to see that $\mu G \le \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \le n,\, x_i \in G\})$. So applying (e) with $\mathcal{G}$ the family of cozero sets we see that $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

(g) Because every closed set is a zero set, this follows at once from (b) and (c).

(h) If $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed and $G \subseteq X$ is open-and-closed, then $d^*(\{n : x_n \in G\}) \le \mu G$ because G is closed and $d^*(\{n : x_n \notin G\}) \le 1 - \mu G$ because G is open; so $d(\{n : x_n \in G\}) = \mu G$. If the condition is satisfied, then (e) tells us that $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

491D The next lemma provides a useful general criterion for the existence of equidistributed sequences.

Lemma Let X be a topological space and μ a probability measure on X. Suppose that there is a sequence $\langle \nu_n\rangle_{n\in\mathbb{N}}$ of point-supported probability measures on X such that $\mu F \ge \limsup_{n\to\infty} \nu_n F$ for every measurable closed set $F \subseteq X$. Then μ has an equidistributed sequence.

proof For each $n \in \mathbb{N}$, let $q_n : X \to [0,1]$ be such that $\nu_n E = \sum_{x\in E} q_n(x)$ for every $E \subseteq X$. Let $q'_n : X \to [0,1]$ be such that $\sum_{x\in X} q'_n(x) = 1$, $K_n = \{x : q'_n(x) > 0\}$ is finite, $q'_n(x)$ is rational for every x, and $\sum_{x\in X} |q_n(x) - q'_n(x)| \le 2^{-n}$; then $\mu F \ge \limsup_{n\to\infty} \nu'_n F$ for every measurable closed F, where ν'_n is defined from q'_n. For each n, let $s_n \ge 1$ be such that $r_n(x) = q'_n(x)s_n$ is an integer for every $x \in K_n$. Let $\langle x_{ni}\rangle_{i<s_n}$ be a family in K_n such that $\#(\{i : i < s_n,\, x_{ni} = x\}) = r_n(x)$ for each $x \in K_n$; then $\nu'_n E = \frac{1}{s_n}\#(\{i : i < s_n,\, x_{ni} \in E\})$ for every $E \subseteq X$.

Let $\langle m_k\rangle_{k\in\mathbb{N}}$ be such that $s_{k+1} \le 2^{-k}\sum_{j=0}^k m_j s_j$ for each k. Set $l_0 = 0$. Given l_n, take the largest k such that $\sum_{j=0}^{k-1} m_j s_j \le l_n$; set $l_{n+1} = l_n + s_k$ and $x_i = x_{k,i-l_n}$ for $l_n \le i < l_{n+1}$; continue. By the choice of the m_k, $l_{n+1}/l_n \to 1$ as $n \to \infty$. For any $E \subseteq X$, $\#(\{i : l_n \le i < l_{n+1},\, x_i \in E\}) = \#(\{j : j < s_k,\, x_{kj} \in E\})$ whenever $\sum_{j=0}^{k-1} m_j s_j \le l_n < \sum_{j=0}^k m_j s_j$. So for any measurable closed set $F \subseteq X$,

$$d^*(\{i : x_i \in F\}) \le \limsup_{k\to\infty} \frac{1}{s_k}\#(\{j : j < s_k,\, x_{kj} \in F\})$$

(491Ad)

$$= \limsup_{k\to\infty} \nu'_k F \le \mu F.$$

As F is arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ is an equidistributed sequence for μ.

491E Proposition (a)(i) Suppose that X and Y are topological spaces, μ a probability measure on X and $f : X \to Y$ a continuous function. If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X which is equidistributed with respect to μ, then $\langle f(x_n)\rangle_{n\in\mathbb{N}}$ is equidistributed with respect to the image measure μf^{-1}.

(ii) Suppose that (X,μ) and (Y,ν) are topological probability spaces and $f : X \to Y$ is a continuous inverse-measure-preserving function. If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X which is equidistributed with respect to μ, then $\langle f(x_n)\rangle_{n\in\mathbb{N}}$ is equidistributed with respect to ν.

(b) Let X be a topological space and μ a probability measure on X, and suppose that X has a countable network consisting of sets measured by μ. Let λ be the ordinary product measure on $X^{\mathbb{N}}$. Then λ-almost every sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $X^{\mathbb{N}}$ is μ-equidistributed.

proof (a)(i) Let $F \subseteq Y$ be a closed set which is measured by μf^{-1}. Then $f^{-1}[F]$ is a closed set in X measured by μ. So

$$d^*(\{n : f(x_n) \in F\}) = d^*(\{n : x_n \in f^{-1}[F]\}) \le \mu f^{-1}[F].$$

(ii) Replace 'μf^{-1}' above by 'ν'.

(b) Let $\mathcal{A}$ be a countable network for the given topology $\mathfrak{S}$ of X consisting of measurable sets, and let $\mathcal{E}$ be the countable subalgebra of $\mathcal{P}X$ generated by $\mathcal{A}$. Let $\mathfrak{T} \supseteq \mathfrak{S}$ be the second-countable topology generated by $\mathcal{E}$; then μ is a τ-additive topological measure with respect to $\mathfrak{T}$, and $\mathcal{E}$ is a base for $\mathfrak{T}$ closed under finite unions. If $E \in \mathcal{E}$, then $d(\{n : x_n \in E\}) = \mu E$ for λ-almost every sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X, by the strong law of large numbers (273J). So

$$d(\{n : x_n \in E\}) = \mu E \text{ for every } E \in \mathcal{E}$$

for λ-almost every $\langle x_n\rangle_{n\in\mathbb{N}}$. Now 491Ce tells us that any such sequence is equidistributed with respect to $\mathfrak{T}$ and therefore with respect to $\mathfrak{S}$.

491F Theorem Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces, each of which has an equidistributed sequence. If $\#(I) \le \mathfrak{c}$, then the τ-additive product measure λ on $X = \prod_{i\in I} X_i$ (definition: 417G) has an equidistributed sequence.

proof (a) For the time being (down to the end of (f)), let us suppose that every μ_i is inner regular with respect to the Borel sets. (This will make it possible to use the theorems of §417, in particular 417H and 417J.)

The formulae of some of the arguments below will be simplified if we immediately re-index the family $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ as $\langle(X_A, \mathfrak{T}_A, \Sigma_A, \mu_A)\rangle_{A\in\mathcal{A}}$ where $\mathcal{A} \subseteq \mathcal{P}\mathbb{N}$. For each $A \in \mathcal{A}$, let $\langle t_{An}\rangle_{n\in\mathbb{N}}$ be an equidistributed sequence in X_A; for $n \in \mathbb{N}$, let ν_{An} be the point-supported measure on X_A defined by setting $\nu_{An}E = \frac{1}{n+1}\#(\{i : i \le n, t_{Ai} \in E\})$ for $E \subseteq X_A$. For each finite set $\mathcal{I} \subseteq \mathcal{A}$, set $Y_{\mathcal{I}} = \prod_{A\in\mathcal{I}} X_A$; set $\pi_{\mathcal{I}}(x) = x{\restriction}\mathcal{I} \in Y_{\mathcal{I}}$ for $x \in X$. Let $\lambda_{\mathcal{I}}$ be the τ-additive product of $\langle\mu_I\rangle_{A\in\mathcal{I}}$ and, for each n, let $\check\nu_{\mathcal{I}n}$ be the product of the measures $\langle\nu_{An}\rangle_{A\in\mathcal{I}}$. (Because $\mathcal{I}$ is finite, this is a point-supported probability measure. I do not say 'τ-additive product' here because I do not wish to assume that all singleton sets are Borel, so the ν_{An} may not be inner regular with respect to Borel sets.)

(b) Suppose that $\mathcal{I} \subseteq \mathcal{A}$ is finite and that $W \subseteq Y_{\mathcal{I}}$ is an open set. Then $\lambda_{\mathcal{I}}W \le \liminf_{n\to\infty} \check\nu_{\mathcal{I}n}W$. **P** Induce on $\#(\mathcal{I})$. If $\mathcal{I} = \emptyset$, $Y_{\mathcal{I}}$ is a singleton and the result is trivial. For the inductive step, if $\mathcal{I} \ne \emptyset$, take any $A \in \mathcal{I}$ and set $\mathcal{I}' = \mathcal{I} \setminus \{A\}$. Then we can identify $Y_{\mathcal{I}}$ with $Y_{\mathcal{I}'} \times X_A$, $\lambda_{\mathcal{I}}$ with the τ-additive product of $\lambda_{\mathcal{I}'}$ and μ_A (417J), and each $\check\nu_{\mathcal{I}n}$ with the product of $\check\nu_{\mathcal{I}'n}$ and ν_{An}.

Let $\mathcal{V}$ be the family of those subsets V of $Y_{\mathcal{I}}$ which are expressible as a finite union of sets of the form $U \times H$ where $U \subseteq Y_{\mathcal{I}'}$ and $H \subseteq X_A$ are open. Then $\mathcal{V}$ is a base for the topology of $Y_{\mathcal{I}}$ closed under finite unions. Let $\epsilon > 0$. Because $\lambda_{\mathcal{I}}$ is τ-additive, there is a $V \in \mathcal{V}$ such that $\lambda_{\mathcal{I}}V \ge \lambda W - \epsilon$. The function $t \mapsto \lambda_{\mathcal{I}'}V[\{t\}] : X_A \to [0,1]$ is lower semi-continuous (417Ba), so 491Ca tells us that

$$\begin{aligned}\lambda_{\mathcal{I}}V &= \int \lambda_{\mathcal{I}'}V[\{t\}]\mu_A(dt)\\ &\le \liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\lambda_{\mathcal{I}'}V[\{t_{Ai}\}] = \liminf_{n\to\infty}\int\lambda_{\mathcal{I}'}V[\{t\}]\nu_{An}(dt).\end{aligned}$$

At the same time, there are only finitely many sets of the form $V[\{t\}]$, and for each of these we have $\lambda_{\mathcal{I}'}V[\{t\}] \le \liminf_{n\to\infty}\check\nu_{\mathcal{I}'n}V[\{t\}]$, by the inductive hypothesis. So there is an $n_0 \in \mathbb{N}$ such that $\lambda_{\mathcal{I}'}V[\{t\}] \le \check\nu_{\mathcal{I}'n}V[\{t\}] + \epsilon$ for every $n \ge n_0$ and every $t \in X_A$. We must therefore have

$$\begin{aligned}\lambda_{\mathcal{I}}W \le \lambda_{\mathcal{I}}V + \epsilon &\le \liminf_{n\to\infty}\int\lambda_{\mathcal{I}'}V[\{t\}]\nu_{An}(dt) + \epsilon\\ &\le \liminf_{n\to\infty}\int\check\nu_{\mathcal{I}'n}V[\{t\}]\nu_{An}(dt) + 2\epsilon\\ &= \liminf_{n\to\infty}\check\nu_{\mathcal{I}n}V + 2\epsilon \le \liminf_{n\to\infty}\check\nu_{\mathcal{I}n}W + 2\epsilon.\end{aligned}$$

As ϵ and W are arbitrary, the induction proceeds. **Q**

(c) For $K \subseteq m \in \mathbb{N}$, set $\mathcal{A}_{mK} = \{A : A \in \mathcal{A},\ A \cap m = K\}$ and $Z_{mK} = \prod_{A\in\mathcal{A}_{mK}} X_A$. (If $\mathcal{A}_{mK} = \emptyset$ then $Z_{mK} = \{\emptyset\}$.) Then for each $m \in \mathbb{N}$ we can identify X with the finite product $\prod_{K\subseteq m} Z_{mK}$. For $K \subseteq m \in \mathbb{N}$ and $n \in \mathbb{N}$, define $z_{mKn} \in Z_{mK}$ by setting $z_{mKn}(A) = t_{An}$ for $A \in \mathcal{A}_{mK}$; let $\tilde\nu_{mKn}$ be the point-supported measure on Z_{mK} defined by setting $\tilde\nu_{mKn}W = \frac{1}{n+1}\#(\{i : i \le n,\ z_{mKi} \in W\})$ for each $W \subseteq Z_{mK}$. For $n \in \mathbb{N}$ let $\tilde\nu_n$ be the measure on X which is the product of the measures $\tilde\nu_{nKn}$ for $K \subseteq n$; this too is point-supported.

(d) If $\mathcal{I} \subseteq \mathcal{A}$ is finite, there is an $m \in \mathbb{N}$ such that $\check\nu_{\mathcal{I}n} = \tilde\nu_n\pi_{\mathcal{I}}^{-1}$ for every $n \ge m$. **P** Let m be such that $A \cap m \ne A' \cap m$ for all distinct A, $A' \in \mathcal{I}$. If $n \ge m$, then $\tilde\nu_n$ is the product of the $\tilde\nu_{nKn}$ for $K \subseteq n$. Now $\pi_{\mathcal{I}}$, interpreted as a function from $\prod_{K\subseteq n} Z_{nK}$ onto $Y_{\mathcal{I}}$, is of the form $\pi_{\mathcal{I}}(\langle z_K\rangle_{K\subseteq n}) = \langle z_{A\cap n}(A)\rangle_{A\in\mathcal{I}}$, so the image measure $\tilde{\nu}_n\pi_{\mathcal{I}}^{-1}$ is the product of the family $\langle\tilde{\nu}_{n,A\cap n,n}\hat{\pi}_A^{-1}\rangle_{A\in\mathcal{I}}$, writing $\hat{\pi}_A(z) = z(A)$ when $A\cap n = K$ and $z\in Z_{nK}$ (254H, or otherwise). But, looking back at the definitions,

$$\begin{aligned}\tilde{\nu}_{n,A\cap n,n}\hat{\pi}_A^{-1}[E] &= \frac{1}{n+1}\#(\{i : i\le n,\ z_{n,A\cap n,i}\in\hat{\pi}_A^{-1}[E]\})\\ &= \frac{1}{n+1}\#(\{i : i\le n,\ z_{n,A\cap n,i}(A)\in E\})\\ &= \frac{1}{n+1}\#(\{i : i\le n,\ t_{Ai}\in E\}) = \nu_{An}E\end{aligned}$$

for every $E\subseteq X_A$. So $\tilde{\nu}_n\pi_{\mathcal{I}}^{-1}$ is the product of $\langle\nu_{An}\rangle_{A\in\mathcal{I}}$, which is $\check{\nu}_{\mathcal{I}n}$. **Q**

(e) Let $\mathcal{W}$ be the family of those open sets $W\subseteq X$ expressible in the form $\pi_{\mathcal{I}}^{-1}[W']$ for some finite $\mathcal{I}\subseteq\mathcal{A}$ and some open $W'\subseteq Y_{\mathcal{I}}$. If $W\in\mathcal{W}$, then $\lambda W\le\liminf_{n\in\mathbb{N}}\tilde{\nu}_nW$. **P** Take $\mathcal{I}\in[\mathcal{A}]^{<\omega}$ and an open $W'\subseteq Y_{\mathcal{I}}$ such that $W=\pi_{\mathcal{I}}^{-1}[W']$. Then

$$\lambda W = \lambda_{\mathcal{I}}W'$$

(417K)

$$\le\liminf_{n\to\infty}\check{\nu}_{\mathcal{I}n}W'$$

(by (b) above)

$$=\liminf_{n\to\infty}\tilde{\nu}_n\pi_{\mathcal{I}}^{-1}[W']$$

(by (d))

$$=\liminf_{n\to\infty}\tilde{\nu}_nW.\ \mathbf{Q}$$

(f) If now $F\subseteq X$ is any closed set and $\epsilon>0$, then (because $\mathcal{W}$ is a base for the topology of X closed under finite unions) there is a $W\in\mathcal{W}$ such that $W\subseteq X\setminus F$ and $\lambda W\ge 1-\lambda F+\epsilon$. In this case

$$\limsup_{n\to\infty}\tilde{\nu}_nF\le 1-\liminf_{n\to\infty}\tilde{\nu}_nW\le 1-\lambda W\le\lambda F+\epsilon.$$

As ϵ is arbitrary, $\limsup_{n\to\infty}\tilde{\nu}_nF\le\lambda F$; as F is arbitrary, 491D tells us that there is an equidistributed sequence in X.

(g) All this was on the assumption that every μ_i is inner regular with respect to the Borel sets. For the superficially more general case enunciated, given only that each μ_i is a τ-additive topological measure with an equidistributed sequence, let μ_i' be the restriction of μ_i to the Borel σ-algebra of X_i for each $i\in I$. Each μ_i' is still τ-additive and equidistributed sequences for μ_i are of course equidistributed for μ_i'. If we take λ' to be the τ-additive product of $\langle\mu_i'\rangle_{i\in I}$, then it must agree with λ on the open sets of X and therefore on the closed sets, and an equidistributed sequence for λ' will be an equidistributed sequence for λ. This completes the proof.

491G Corollary The usual measure of $\{0,1\}^{\mathfrak{c}}$ has an equidistributed sequence.

proof The usual measure of $\{0,1\}$ of course has an equidistributed sequence (just set $x_i=0$ for even i, $x_i=1$ for odd i), so 491F gives the result at once.

491H Theorem (Veech 71) Any separable compact Hausdorff topological group has an equidistributed sequence for its Haar probability measure.

proof Let X be a separable compact Hausdorff topological group. Recall that X has exactly one Haar probability measure μ, which is both a left Haar measure and a right Haar measure (442Ic).

(a) We need some elementary facts about convolutions.

(i) If ν_1 and ν_2 are point-supported probability measures on X, then $\nu_1*\nu_2$ is point-supported. **P** If $\nu_1E=\sum_{x\in E}q_1(x)$ and $\nu_2E=\sum_{x\in E}q_2(x)$ for every $E\subseteq X$, then

$$(\nu_1*\nu_2)(E)=(\nu_1\times\nu_2)\{(x,y):xy\in E\}=\sum_{xy\in E}q_1(x)q_2(y)=\sum_{z\in E}q(z)$$

where $q(z)=\sum_{x\in X}q_1(x)q_2(x^{-1}z)$ for $z\in X$. **Q**

(ii) Let ν, λ be Radon probability measures on X. Suppose that $f\in C(X)$, $\alpha\in\mathbb{R}$ and $\epsilon>0$ are such that $|\int f(yxz)\nu(dx)-\alpha|\le\epsilon$ for every y, $z\in X$. Then $|\int f(yxz)(\lambda*\nu)(dx)-\alpha|\le\epsilon$ and $|\int f(yxz)(\lambda*\nu)(dx)-\alpha|\le\epsilon$ for every y, $z\in X$. **P**

$$|\int f(yxz)(\lambda*\nu)(dx)-\alpha|$$
$$=|\iint f(ywxz)\nu(dx)\lambda(dw)-\alpha|$$

(444C)

$$\le\int|\int f(ywxz)\nu(dx)-\alpha|\lambda(dw)\le\int\epsilon\lambda(dw)=\epsilon,$$

$$|\int f(yxz)(\nu*\lambda)(dx)-\alpha|$$
$$=|\iint f(yxwz)\nu(dx)\lambda(dw)-\alpha|$$
$$\le\int|\int f(yxwz)\nu(dx)-\alpha|\lambda(dw)\le\int\epsilon\lambda(dw)=\epsilon.\ \mathbf{Q}$$

(b) Let $A\subseteq X$ be a countable dense set. Let N be the set of point-supported probability measures ν on X which are defined by functions q such that $\{x:q(x)>0\}$ is a finite subset of A and $q(x)$ is rational for every x. Then N is countable. Now, for every $f\in C(X)$ and $\epsilon>0$, there is a $\nu\in$ N such that $|\int f(yxz)\nu(dx)-\int fd\mu|\le\epsilon$ for all y, $z\in X$. **P** Because X is compact, f is uniformly continuous for the right uniformity of X (4A2Jf), so there is a neighbourhood U of the identity e such that $|f(x')-f(x)|\le\frac{1}{2}\epsilon$ whenever $x'x^{-1}\in U$. Next, again because X is compact, there is a neighbourhood V of e such that $yxy^{-1}\in U$ whenever $x\in V$ and $y\in X$ (4A5Ej). Because A is dense, $V^{-1}x\cap A\ne\emptyset$ for every $x\in X$, that is, $VA=X$; once more because X is compact, there are $x_0,\dots,x_n\in A$ such that $X=\bigcup_{i\le n}Vx_i$. Set $E_i=Vx_i\setminus\bigcup_{j<i}Vx_j$ for each $i\le n$. Let $\alpha_0,\dots,\alpha_n\in[0,1]\cap\mathbb{Q}$ be such that $\sum_{i=0}^n\alpha_i=1$ and $\|f\|_\infty\sum_{i=0}^n|\alpha_i-\mu E_i|\le\frac{1}{2}\epsilon$, and define $\nu\in$ N by setting $\nu E=\sum\{\alpha_i:i\le n,\ x_i\in E\}$ for every $E\subseteq X$.

Let y, $z\in X$. If $i\le n$ and $x\in E_i$, then $xx_i^{-1}\in V$ so $(yxz)(yx_iz)^{-1}=yxx_i^{-1}y^{-1}\in U$ and $|f(yxz)-f(yx_iz)|\le\frac{1}{2}\epsilon$. Accordingly

$$|\int f(yxz)\nu(dx)-\int f(x)\mu(dx)|$$
$$=|\sum_{i=0}^n\alpha_if(yx_iz)-\int f(yxz)\mu(dx)|$$

(because X is unimodular)

$$\le\sum_{i=0}^n|\alpha_if(yx_iz)-\int_{E_i}f(yxz)\mu(dx)|$$
$$\le\sum_{i=0}^n|\alpha_i-\mu E_i||f(yx_iz)|+\sum_{i=0}^n|f(yx_iz)\mu E_i-\int_{E_i}f(yxz)\mu(dx)|$$
$$\le\|f\|_\infty\sum_{i=0}^n|\alpha_i-\mu E_i|+\sum_{i=0}^n\int_{E_i}|f(yxz)-f(yx_iz)|\mu(dx)$$
$$\le\frac{\epsilon}{2}+\sum_{i=0}^n\frac{\epsilon}{2}\mu E_i=\epsilon.\ \mathbf{Q}$$

(c) Let $\langle\nu_n\rangle_{n\in\mathbb{N}}$ be a sequence running over N, and set $\lambda_n=\nu_0*\nu_1*\ldots*\nu_n$ for each n. (Recall from 444B that convolution is associative.) Then each λ_n is a point-supported probability measure on X, by (a-i). Also

$\lim_{n\to\infty}\int f d\lambda_n = \int f d\mu$ for every $f \in C(X)$. **P** If $f \in C(X)$ and $\epsilon > 0$, then (b) tells us that there is an $m \in \mathbb{N}$ such that $|\int f(yxz)\nu_m(dx) - \int f d\mu| \le \epsilon$ for all y, $z \in X$. For any $n \ge m$, λ_n is of the form $\lambda' * \nu_m * \lambda''$. By (a-ii), used in both parts successively, $|\int f d\lambda_n - \int f d\mu| \le \epsilon$. As ϵ is arbitrary, we have the result. **Q**

(d) If $F \subseteq X$ is closed, then

$$\begin{aligned}\mu F &= \inf\{\int f d\mu : \chi F \le f \in C(X)\}\\ &= \inf_{\chi F\le f}\limsup_{n\to\infty}\int f d\lambda_n \ge \limsup_{n\to\infty}\lambda_n F.\end{aligned}$$

By 491D, μ has an equidistributed sequence.

491I The quotient $\mathcal{P}\mathbb{N}/\mathcal{Z}$ I now return to the asymptotic density ideal $\mathcal{Z}$, moving towards a striking relationship between the corresponding quotient algebra and equidistributed sequences. Since $\mathcal{Z} \lhd \mathcal{P}\mathbb{N}$, we can form the quotient algebra $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$. The functionals d and d^* descend naturally to $\mathfrak{Z}$ if we set

$$\bar d^*(I^\bullet) = d^*(I), \quad \bar d(I^\bullet) = d(I) \text{ whenever } d(I) \text{ is defined.}$$

(a) $\bar d^*$ is a strictly positive submeasure on $\mathfrak{Z}$. **P** $\bar d^*$ is a submeasure on $\mathfrak{Z}$ because d^* is a submeasure on $\mathcal{P}\mathbb{N}$. $\bar d^*$ is strictly positive because $\mathcal{Z} \supseteq \{I : d^*(I) = 0\}$. **Q**

(b) Let $\bar\rho$ be the metric on $\mathfrak{Z}$ defined by saying that $\bar\rho(a,b) = \bar d^*(a \triangle b)$ for all a, $b \in \mathfrak{Z}$. Under ρ, the Boolean operations $\cup$, $\cap$, $\triangle$ and $\setminus$ and the function $\bar d^* : \mathfrak{Z} \to [0,1]$ are uniformly continuous (392Hb[1]), and $\mathfrak{Z}$ is complete. **P** Let $\langle c_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{Z}$ such that $\bar\rho(c_{n+1}, c_n) \le 2^{-n}$ for every $n \in \mathbb{N}$; then $\bar\rho(c_r, c_i) \le 2^{-i+1}$ for $i \le r$. For each $n \in \mathbb{N}$ choose $C_n \subseteq \mathbb{N}$ such that $C_n^\bullet = c_n$; then $d^*(C_r \triangle C_i) \le 2^{-i+1}$ for $i \le r$. Choose a strictly increasing sequence $\langle k_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $k_{n+1} \ge 2k_n$ for every n and, for each $n \in \mathbb{N}$,

$$\frac{1}{m}\#((C_n \triangle C_i) \cap m) \le 2^{-i+2} \text{ whenever } i \le n,\ m \ge k_n.$$

Set $C = \bigcup_{n\in\mathbb{N}} C_n \cap k_{n+1} \setminus k_n$, and $c = C^\bullet \in \mathfrak{Z}$. If $n \in \mathbb{N}$ and $m \ge k_{n+1}$, then take $r > n$ such that $k_r \le m < k_{r+1}$; in this case $k_i \le 2^{i-r}m$ for $i \le r$, so

$$\begin{aligned}\#((C\triangle C_n)\cap m) &\le k_n + \sum_{i=n}^{r-1}\#((C\triangle C_n)\cap k_{i+1}\setminus k_i) + \#((C\triangle C_n)\cap m\setminus k_r)\\ &= k_n + \sum_{i=n}^{r-1}\#((C_i\triangle C_n)\cap k_{i+1}\setminus k_i) + \#((C_r\triangle C_n)\cap m\setminus k_r)\\ &\le k_n + \sum_{i=n+1}^{r-1}\#((C_i\triangle C_n)\cap k_{i+1}) + \#((C_r\triangle C_n)\cap m)\\ &\le k_n + \sum_{i=n+1}^{r-1} 2^{-n+2}k_{i+1} + 2^{-n+2}m\\ &\le k_n + \sum_{i=n+1}^{r-1} 2^{-n+2}2^{i+1-r}m + 2^{-n+2}m\\ &\le k_n + 2^{-n+3}m + 2^{-n+2}m.\end{aligned}$$

But this means that

$$\bar\rho(c, c_n) = d^*(C\triangle C_n) \le \lim_{m\to\infty}\frac{k_n}{m} + 2^{-n+3} + 2^{-n+2} \le 2^{-n+4}$$

for every n, and $\langle c_n\rangle_{n\in\mathbb{N}}$ converges to c in $\mathfrak{Z}$. **Q**

[1]Formerly 393Bb.

***(c)** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{Z}$, there is an $a \in \mathfrak{Z}$ such that $a \subseteq a_n$ for every n and $\bar{d}^*(a) = \inf_{n\in\mathbb{N}} \bar{d}^*(a_n)$. **P** For each $n \in \mathbb{N}$, choose $I_n \subseteq \mathbb{N}$ such that $I_n^\bullet = a_n$; replacing I_n by $\bigcap_{j\le n} I_j$ if necessary, we can arrange that $I_{n+1} \subseteq I_n$ for every n. Set $\gamma = \inf_{n\in\mathbb{N}} \bar{d}^*(a_n) = \inf_{n\in\mathbb{N}} d^*(I_n)$. Let $\langle k_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $\#(I_n \cap k_n) \ge (\gamma - 2^{-n})k_n$ for every n. Set $I = \bigcup_{n\in\mathbb{N}} I_n \cap k_n$ and $a = I^\bullet \in \mathfrak{Z}$. Then $\#(I \cap k_n) \ge (\gamma - 2^{-n})k_n$ for every n, so $\bar{d}^*(a) = d^*(I) \ge \gamma$. Also $I \setminus I_n \subseteq k_n$ is finite, so $a \subseteq a_n$, for every n. Of course it follows at once that $\bar{d}^*(a) = \gamma$ exactly, as required. **Q**

***(d)** $\bar{d}^*$ is a Maharam submeasure on $\mathfrak{Z}$. (Immediate from (c).)

491J Lemma Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ such that $\lim_{n\to\infty} \bar{d}^*(a_n) + \bar{d}^*(1 \setminus a_n) = 1$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is topologically convergent to $a \in \mathfrak{Z}$; $a = \sup_{n\in\mathbb{N}} a_n$ in $\mathfrak{Z}$ and $d^*(a) + d^*(1 \setminus a) = 1$.

proof (a) The point is that if $m \le n$ then $\bar{d}^*(a_n \setminus a_m) \le \bar{d}^*(a_n) + \bar{d}^*(1 \setminus a_m) - 1$. **P** Let I, $J \subseteq \mathbb{N}$ be such that $I^\bullet = a_m$ and $J^\bullet = a_n$. For any $k \ge 1$,

$$\frac{1}{k}\#(k \cap J) + \frac{1}{k}\#(k \setminus I) = \frac{1}{k}\#(k \cap J \setminus I) + \frac{1}{k}\#(k \setminus (I \setminus J)),$$

so

$$\begin{aligned}
d^*(J) + d^*(\mathbb{N} \setminus I) &= \limsup_{k\to\infty} \frac{1}{k}\#(k \cap J) + \limsup_{k\to\infty} \frac{1}{k}\#(k \setminus I) \\
&\ge \limsup_{k\to\infty} \frac{1}{k}\#(k \cap J) + \frac{1}{k}\#(k \setminus I) \\
&= \limsup_{k\to\infty} \frac{1}{k}\#(k \cap J \setminus I) + \frac{1}{k}\#(k \setminus (I \setminus J)) \\
&\ge \limsup_{k\to\infty} \frac{1}{k}\#(k \cap J \setminus I) + \liminf_{k\to\infty} \frac{1}{k}\#(k \setminus (I \setminus J)) = d^*(J \setminus I) + 1
\end{aligned}$$

because $a_m \subseteq a_n$, so $I \setminus J \in \mathcal{Z}$. But this means that

$$\bar{d}^*(a_n \setminus a_m) = d^*(J \setminus I) \le d^*(J) + d^*(\mathbb{N} \setminus I) - 1 = \bar{d}^*(a_n) + \bar{d}^*(1 \setminus a_m) - 1. \quad \mathbf{Q}$$

(b) Accordingly

$$\begin{aligned}
\limsup_{m\to\infty} \sup_{n\ge m} \bar{\rho}(a_m, a_n) &= \limsup_{m\to\infty} \sup_{n\ge m} \bar{d}^*(a_n \setminus a_m) \\
&\le \limsup_{m\to\infty} \sup_{n\ge m} \bar{d}^*(a_n) + \bar{d}^*(1 \setminus a_m) - 1 \\
&= \limsup_{m\to\infty} \sup_{n\ge m} \bar{d}^*(a_n) - \bar{d}^*(a_m)
\end{aligned}$$

(because $\lim_{m\to\infty} \bar{d}^*(a_m) + \bar{d}^*(1 \setminus a_m) = 1$)

$$= 0,$$

and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{Z}$.

(c) Because $\mathfrak{Z}$ is complete, $a = \lim_{n\to\infty} a_n$ is defined in $\mathfrak{Z}$. For each $m \in \mathbb{N}$, $a_m \setminus a = \lim_{n\to\infty} a_m \setminus a_n = 0$ (because $\setminus$ is continuous), so $a_m \subseteq a$; thus a is an upper bound of $\{a_n : n \in \mathbb{N}\}$. If b is any upper bound of $\{a_n : n \in \mathbb{N}\}$, then $a \setminus b = \lim_{n\to\infty} a_n \setminus b = 0$; so $a = \sup_{n\in\mathbb{N}} a_n$. Finally,

$$\bar{d}^*(a) + \bar{d}^*(1 \setminus a) = \lim_{n\to\infty} \bar{d}^*(a_n) + \bar{d}^*(1 \setminus a_n) = 1.$$

491K Corollary Set $D = \operatorname{dom} \bar{d} = \{I^\bullet : I \subseteq \mathbb{N},\ d(I) \text{ is defined}\} \subseteq \mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$.

(a) If $I \subseteq \mathbb{N}$ and $I^\bullet \in D$ then $d(I)$ is defined.

(b) $D = \{a : a \in \mathfrak{Z},\ \bar{d}^*(a) + \bar{d}^*(1 \setminus a) = 1\}$; if $a \in D$ then $1 \setminus a \in D$; if a, $b \in D$ and $a \cap b = 0$, then $a \cup b \in D$ and $\bar{d}(a \cup b) = \bar{d}(a) + \bar{d}(b)$; if a, $b \in D$ and $a \subseteq b$ then $b \setminus a \in D$ and $\bar{d}(b \setminus a) = \bar{d}(b) - \bar{d}(a)$.

(c) D is a topologically closed subset of $\mathfrak{Z}$.

(d) If $A \subseteq D$ is upwards-directed, then $\sup A$ is defined in $\mathfrak{Z}$ and belongs to D; moreover there is a sequence in A with the same supremum as A, and $\sup A$ belongs to the topological closure of A.

(e) Let $\mathfrak{B} \subseteq D$ be a subalgebra of $\mathfrak{Z}$. Then the following are equiveridical:

(i) $\mathfrak{B}$ is topologically closed in $\mathfrak{Z}$;

(ii) $\mathfrak{B}$ is order-closed in $\mathfrak{Z}$;

(iii) setting $\bar{\nu} = \bar{d}^* \restriction \mathfrak{B} = \bar{d} \restriction \mathfrak{B}$, $(\mathfrak{B}, \bar{\nu})$ is a probability algebra.

In this case, $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$.

(f) If $I \subseteq D$ is closed under either $\cap$ or $\cup$, then the topologically closed subalgebra of $\mathfrak{Z}$ generated by I, which is also the order-closed subalgebra of $\mathfrak{Z}$ generated by I, is included in D.

proof (a) There is a $J \subseteq \mathbb{N}$ such that $d(J)$ is defined and $I \triangle J \in \mathcal{Z}$. But in this case $d(I \triangle J) = 0$, so $d(I \cup J) = d(J) + d(I \setminus J)$ is defined; also $d(J \setminus I) = 0$, so $d(I) = d(I \cup J) - d(J \setminus I)$ is defined.

(b) These facts all follow directly from the corresponding results concerning $\mathcal{P}\mathbb{N}$ and d (491Ac).

(c) All we have to know is that $a \mapsto \bar{d}^*(a)$, $a \mapsto 1 \setminus a$ are continuous; so that $\{a : \bar{d}^*(a) + \bar{d}^*(1 \setminus a) = 1\}$ is closed.

(d) Because A is upwards-directed, and $\bar{d}^*$ is a non-decreasing functional on $\mathfrak{Z}$, there is a non-decreasing sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in A such that $\lim_{n\to\infty} \bar{d}^*(a_n) = \sup_{a\in A} \bar{d}^*(a) = \gamma$ say. By 491J, $b = \lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n$ is defined in $\mathfrak{Z}$ and belongs to D. If $a \in A$ and $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\bar{d}^*(a_n) \geq \gamma - \epsilon$. Let $a' \in A$ be a common upper bound of a and a_n. Then

$$\bar{d}^*(a \setminus b) \leq \bar{d}^*(a' \setminus a_n) = \bar{d}^*(a') - \bar{d}^*(a_n) \leq \gamma - \bar{d}^*(a_n) \leq \epsilon.$$

As ϵ is arbitrary, $a \subseteq b$; as a is arbitrary, b is an upper bound of A; as $b = \sup_{n\in\mathbb{N}} a_n$, b must be the supremum of A.

(e)(i)$\Rightarrow$(ii) Suppose that $\mathfrak{B}$ is topologically closed. If $A \subseteq \mathfrak{B}$ is a non-empty upwards-directed subset with supremum $b \in \mathfrak{Z}$, then (d) tells us that $b \in \overline{A} \subseteq \mathfrak{B}$. It follows that $\mathfrak{B}$ is order-closed in $\mathfrak{Z}$ (313E(a-i)).

(ii)$\Rightarrow$(iii) Suppose that $\mathfrak{B}$ is order-closed in $\mathfrak{Z}$. If $A \subseteq \mathfrak{B}$ is non-empty, then $A' = \{a_0 \cup \ldots \cup a_n : a_0, \ldots, a_n \in A\}$ is non-empty and upwards-directed, so has a supremum in $\mathfrak{Z}$, which must belong to $\mathfrak{B}$, and must be the least upper bound of A in $\mathfrak{B}$. Thus $\mathfrak{B}$ is Dedekind (σ-)complete. Now let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{B}$ and set $b_n = \sup_{i\leq n} a_i$ for each n. Then $\langle b_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in D so has a limit and supremum $b \in D$, and $b \in \mathfrak{B}$. Also $\bar{d}^*(b_n) = \sum_{i=0}^n \bar{d}^*(a_i)$ for each n (induce on n), so

$$\bar{\nu}b = \bar{d}^*(b) = \lim_{n\to\infty} \bar{d}^*(b_n) = \sum_{i=0}^\infty \bar{d}^*(a_i) = \sum_{i=0}^\infty \bar{\nu}a_i.$$

Since certainly $\bar{\nu}0 = 0$, $\bar{\nu}1 = 1$ and $\bar{\nu}b > 0$ whenever $b \in \mathfrak{B} \setminus \{0\}$, $(\mathfrak{B}, \bar{\nu})$ is a probability algebra.

(iii)$\Rightarrow$(i) Suppose that $(\mathfrak{B}, \bar{\nu})$ is a probability algebra. Then it is complete under its measure metric (323Gc), which agrees on $\mathfrak{B}$ with the metric $\bar{\rho}$ of $\mathfrak{Z}$; so $\mathfrak{B}$ must be topologically closed in $\mathfrak{Z}$.

We see also that $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$. **P** (Compare 323H.) If $A \subseteq \mathfrak{B}$ is non-empty and downwards-directed and has infimum 0 in $\mathfrak{B}$, and $b \in \mathfrak{Z}$ is any lower bound of A in $\mathfrak{Z}$, then

$$\bar{d}^*(b) \leq \inf_{a\in A} \bar{d}^*(a) = \inf_{a\in A} \bar{\nu}a = 0$$

(321F), so $b = 0$. Thus $\inf A = 0$ in $\mathfrak{Z}$. As A is arbitrary, this is enough to show that the identity map from $\mathfrak{B}$ to $\mathfrak{Z}$ is order-continuous (313Lb), that is, that $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$. **Q**

(f) Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{Z}$ generated by I. If I is closed under $\cap$, then (b), (d) and 313Gc tell us that $\mathfrak{B} \subseteq D$. If I is closed under $\cup$, then $I' = \{1 \setminus a : a \in I\}$ is a subset of D closed under $\cap$, while $\mathfrak{B}$ is the order-closed subalgebra generated by I', so again $\mathfrak{B} \subseteq D$. By (e), $\mathfrak{B}$ is in either case topologically closed. So we see that the topologically closed subalgebra generated by I is included in D; by (e) again, it is equal to $\mathfrak{B}$.

491L Effectively regular measures The examples 491Xf and 491Yc show that the definition in 491B is drawn a little too wide for comfort, and allows some uninteresting pathologies. These do not arise in the measure spaces we care most about, and the following definitions provide a fire-break. Let (X, Σ, μ) be a measure space, and $\mathfrak{T}$ a topology on X.

(a) I will say that a measurable subset K of X of finite measure is **regularly enveloped** if for every $\epsilon > 0$ there are an open measurable set G and a closed measurable set F such that $K \subseteq G \subseteq F$ and $\mu(F \setminus K) \leq \epsilon$.

(b) Note that the family $\mathcal{K}$ of regularly enveloped measurable sets of finite measure is closed under finite unions and countable intersections. **P** (i) If K_1, $K_2 \in \mathcal{K}$ and $*$ is either $\cup$ or $\cap$, let $\epsilon > 0$. Take measurable open sets G_1, G_2 and measurable closed sets F_1, F_2 such that $K_i \subseteq G_i \subseteq F_i$ and $\mu(F_i \setminus K_i) \leq \frac{1}{2}\epsilon$ for both i. Then $G_1 * G_2$ is a measurable open set, $F_1 * F_2$ is a measurable closed set, $K_1 * K_2 \subseteq G_1 * G_2 \subseteq F_1 * F_2$ and $\mu((F_1 * F_2) \setminus (K_1 * K_2)) \le \epsilon$. As ϵ is arbitrary, $K_1 * K_2 \in \mathcal{K}$. (ii) If $\langle K_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with intersection K and $\epsilon > 0$, let $n \in \mathbb{N}$ be such that $\mu K_n < \mu K + \epsilon$. Then we can find a measurable open set G and a measurable closed set F such that $K_n \subseteq G \subseteq F$ and $\mu F \le \mu K + \epsilon$. As ϵ is arbitrary, $K \in \mathcal{K}$. Together with (i), this is enough to show that $\mathcal{K}$ is closed under countable intersections. **Q**

(c) Now I say that μ is **effectively regular** if it is inner regular with respect to the regularly enveloped sets of finite measure.

491M Examples (a) Any totally finite Radon measure is effectively regular. **P** Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a totally finite Radon measure space. If $K \subseteq X$ is compact and $\epsilon > 0$, let $L \subseteq X \setminus K$ be a compact set such that $\mu L \ge 1 - \mu K + \epsilon$. Let G, H be disjoint open sets including K, L respectively (4A2F(h-i)). Then $K \subseteq G \subseteq X \setminus H$, G is open, $X \setminus H$ is closed, both G and $X \setminus H$ are measurable, and $\mu((X \setminus H) \setminus K) \le \epsilon$. This shows that every compact set is regularly enveloped, and μ is effectively regular. **Q**

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space such that $\mathfrak{T}$ is a regular topology. Then μ is effectively regular. **P** Let $E \in \Sigma$ and take $\gamma < \mu E$. Choose sequences $\langle E_n \rangle_{n\in\mathbb{N}}$ and $\langle G_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. $E_0 \subseteq E$ is to be any measurable set such that $\gamma < \mu E_0 < \infty$. Given that $\mu E_n > \gamma$, let G be an open set of finite measure such that $\mu(E_n \cap G) > \gamma$ (414Ea), and $F \subseteq G \setminus E_n$ a closed set such that $\mu F \ge \mu(G \setminus E_n) - 2^{-n}$. Let $\mathcal{H}$ be the family of open sets H such that $\overline{H} \subseteq G \setminus F$. Then $\mathcal{H}$ is upwards-directed and covers E_n (because $\mathfrak{T}$ is regular), so there is a $G_n \in \mathcal{H}$ such that $\mu(E_n \cap G_n) > \gamma$ (414Ea again). Now $\mu(\overline{G}_n \setminus E_n) \le 2^{-n}$. Set $E_{n+1} = E_n \cap G_n$, and continue.

At the end of the induction, set $K = \bigcap_{n\in\mathbb{N}} E_n$. For each n, $K \subseteq G_n \subseteq \overline{G}_n$ and

$$\lim_{n\to\infty} \mu(\overline{G}_n \setminus K) \le \lim_{n\to\infty} 2^{-n} + \mu(E_n \setminus K) = 0,$$

so K is regularly enveloped. At the same time, $K \subseteq E$ and $\mu K \ge \gamma$. As E and γ are arbitrary, μ is effectively regular. **Q**

(c) Any totally finite Baire measure is effectively regular. **P** Let μ be a totally finite Baire measure on a topological space X. If $F \subseteq X$ is a zero set, let $f : X \to \mathbb{R}$ be a continuous function such that $F = f^{-1}[\{0\}]$. For each $n \in \mathbb{N}$, set $G_n = \{x : |f(x)| < 2^{-n}\}$, $F_n = \{x : |f(x)| \le 2^{-n}\}$; then G_n is a measurable open set, F_n is a measurable closed set, $F \subseteq G_n \subseteq F_n$ for every n and $\lim_{n\to\infty} \mu F_n = \mu F$ (because μ is totally finite). This shows that every zero set is regularly enveloped; as μ is inner regular with respect to the zero sets (412D), μ is effectively regular. **Q**

(d) A totally finite completion regular topological measure is effectively regular. (As in (c), all zero sets are regularly enveloped.)

491N Theorem Let X be a topological space and μ an effectively regular probability measure on X, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Suppose that $\langle x_n \rangle_{n\in\mathbb{N}}$ is an equidistributed sequence in X. Then we have a unique order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ such that $\pi G^\bullet \subseteq \{n : x_n \in G\}^\bullet$ for every measurable open set $G \subseteq X$, and $\bar{d}^*(\pi a) = \bar{\mu} a$ for every $a \in \mathfrak{A}$.

proof (a) Define $\theta : \mathcal{P}X \to \mathfrak{Z}$ by setting $\theta A = \{n : x_n \in A\}^\bullet$ for $A \subseteq X$; then θ is a Boolean homomorphism. If $F \subseteq X$ is closed and measurable, then $\bar{d}^*(\theta F) \le \mu F$, because $\langle x_n \rangle_{n\in\mathbb{N}}$ is equidistributed. Write $\mathcal{K}$ for the family of regularly enveloped measurable sets.

If $K \in \mathcal{K}$, then $\pi_0 K = \inf\{\theta G : K \subseteq G \in \Sigma \cap \mathfrak{T}\}$ is defined in $\mathfrak{Z}$, $\bar{d}^*(\pi_0 K) = \mu K$ and $\pi_0 K \in D$ as defined in 491K. **P** For each $n \in \mathbb{N}$, let $G_n, F_n \in \Sigma$ be such that $K \subseteq G_n \subseteq F_n$, G_n is open, F_n is closed and $\mu(F_n \setminus K) \le 2^{-n}$. Set $H_n = X \setminus \bigcap_{i\le n} G_i$. Then

$$\begin{aligned}\bar{d}^*(\theta H_n) + \bar{d}^*(1 \setminus \theta H_n) &\le \bar{d}^*(\theta H_n) + \bar{d}^*(\theta(\bigcap_{i\le n} F_i)) \le \mu H_n + \mu(\bigcap_{i\le n} F_i)\\ &\le \mu(X \setminus K) + \mu F_n \le 1 + 2^{-n}.\end{aligned}$$

Also $\langle \theta H_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{Z}$. By 491J, $a = \lim_{n\to\infty} \theta H_n = \sup_{n\in\mathbb{N}} \theta H_n$ is defined in $\mathfrak{Z}$ and belongs to D. Set

$$b = 1 \setminus a = \lim_{n\to\infty} 1 \setminus \theta H_n = \lim_{n\to\infty} \theta(\bigcap_{i\le n} G_i),$$

so that b also belongs to D. If $K \subseteq G \in \Sigma \cap \mathfrak{T}$, then

$$b \setminus \theta G = \lim_{n\to\infty} \theta(\bigcap_{i\le n} G_i) \setminus \theta G = \lim_{n\to\infty} \theta(\bigcap_{i\le n} G_i \setminus G),$$

and

$$\begin{aligned}\bar d^*(b \setminus \theta G) &= \lim_{n\to\infty} \bar d^*(\theta(\bigcap_{i\le n} G_i \setminus G))\\ &\le \lim_{n\to\infty} \bar d^*(\theta(\bigcap_{i\le n} F_i \setminus G)) \le \lim_{n\to\infty} \mu(\bigcap_{i\le n} F_i \setminus G) = 0.\end{aligned}$$

This shows that $b \subseteq \theta G$ whenever $K \subseteq G \in \Sigma \cap \mathfrak{T}$. On the other hand, any lower bound of $\{\theta G : K \subseteq G \in \Sigma \cap \mathfrak{T}\}$ is also a lower bound of $\{\theta(\bigcap_{i\le n} G_i) : n \in \mathbb{N}\}$, so is included in b. Thus $b = \inf\{\theta G : K \subseteq G \in \Sigma \cap \mathfrak{T}\}$ and $\pi_0(K) = b$ is defined.

To compute $\bar d^*(b)$, observe first that $b \subseteq 1 \setminus \theta H_n \subseteq \theta F_n$ for every n, so

$$\bar d^*(b) \le \inf_{n\in\mathbb{N}} \bar d^*(\theta F_n) \le \inf_{n\in\mathbb{N}} \mu F_n = \mu K.$$

On the other hand,

$$\bar d^*(\theta(\bigcap_{i\le n} G_i)) \ge 1 - \bar d^*(\theta H_n) \ge 1 - \mu H_n \ge \mu K$$

for every n, so

$$\bar d^*(b) = \lim_{n\to\infty} \bar d^*(\theta(\bigcap_{i\le n} G_i)) \ge \mu K.$$

Accordingly $\bar d^*(b) = \mu K$, and $\pi_0 K$ has the required properties. **Q**

(b) If K, $L \in \mathcal{K}$, then $\pi_0(K \cap L) = \pi_0 K \cap \pi_0 L$. **P** We know that $K \cap L \in \mathcal{K}$ (491Lb). Now

$$\begin{aligned}\pi_0 K \cap \pi_0 L &= \inf\{\theta G : K \subseteq G \in \mathfrak{T}\} \cap \inf\{\theta H : L \subseteq H \in \mathfrak{T}\}\\ &= \inf\{\theta G \cap \theta H : K \subseteq G \in \mathfrak{T},\, L \subseteq H \in \mathfrak{T}\}\\ &= \inf\{\theta(G \cap H) : K \subseteq G \in \mathfrak{T},\, L \subseteq H \in \mathfrak{T}\} \supseteq \pi_0(K \cap L).\end{aligned}$$

Now suppose that $U \supseteq K \cap L$ is a measurable open set and $\epsilon > 0$. Let G, G' be measurable open sets and F, F' measurable closed sets such that $K \subseteq G \subseteq F$, $L \subseteq G' \subseteq F'$, $\mu(F \setminus K) \le \epsilon$ and $\mu(F' \setminus L) \le \epsilon$. Then

$$\begin{aligned}\bar d^*(\pi_0 K \cap \pi_0 L \setminus \theta U) &\le \bar d^*(\theta G \cap \theta G' \setminus \theta U) = \bar d^*(\theta(G \cap G' \setminus U))\\ &\le \bar d^*(\theta(F \cap F' \setminus U)) \le \mu(F \cap F' \setminus U) \le 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\pi_0 K \cap \pi_0 L \subseteq \theta U$; as U is arbitrary, $\pi_0 K \cap \pi_0 L \subseteq \pi_0(K \cap L)$. **Q**

This means that $\{\pi_0 K : K \subseteq X$ is a regularly embedded measurable set$\}$ is a subset of D closed under $\cap$. By 491Kf, the topologically closed subalgebra $\mathfrak{B}$ of $\mathfrak{Z}$ generated by this family is included in D; by 491Ke, $\mathfrak{B}$ is order-closed and regularly embedded in $\mathfrak{Z}$, and $(\mathfrak{B}, \bar d^*{\restriction}\mathfrak{B})$ is a probability algebra.

(c) Now observe that if we set $Q = \{K^\bullet : K \in \mathcal{K}\} \subseteq \mathfrak{A}$, we have a function $\pi : Q \to \mathfrak{B}$ defined by setting $\pi K^\bullet = \pi_0 K$ whenever $K \in \mathcal{K}$. **P** Suppose that K, $L \in \mathcal{K}$ and $\mu(K \triangle L) = 0$. Then

$$\bar d^*(\pi_0 K \triangle \pi_0 L) = \bar d^*(\pi_0 K) + \bar d^*(\pi_0 L) - 2\bar d^*(\pi_0 K \cap \pi_0 L)$$

(because $\pi_0 K$ and $\pi_0 L$ belong to $\mathfrak{B} \subseteq D$)

$$\begin{aligned}&= \bar d^*(\pi_0 K) + \bar d^*(\pi_0 L) - 2\bar d^*(\pi_0(K \cap L))\\ &= \mu K + \mu L - 2\mu(K \cap L) = 0.\end{aligned}$$

So $\pi_0 K = \pi_0 L$ and either can be used to define $\pi K^\bullet$. **Q** Next, the same formulae show that $\pi : Q \to \mathfrak{B}$ is an isometry when Q is given the measure metric of $\mathfrak{A}$, since if K, L belong to $\mathcal{K}$,

$$\bar\rho(\pi K^\bullet, \pi L^\bullet) = \bar d^*(\pi_0 K \triangle \pi_0 L) = \mu K + \mu L - 2\mu(K \cap L) = \mu(K \triangle L) = \bar\mu(K^\bullet \triangle L^\bullet).$$

As Q is dense in $\mathfrak{A}$ (412N), there is a unique extension of π to an isometry from $\mathfrak{A}$ to $\mathfrak{B}$.

(d) Because

$$\pi(K^\bullet \cap L^\bullet) = \pi(K \cap L)^\bullet = \pi_0(K \cap L) = \pi_0 K \cap \pi_0 L = \pi K^\bullet \cap \pi L^\bullet$$

for all K, $L \in \mathcal{K}$, $\pi(a \cap a') = \pi a \cap \pi a'$ for all a, $a' \in \mathfrak{A}$. It follows that π is a Boolean homomorphism. **P** The point is that $\bar{d}^*(\pi a) = \bar{\mu} a$ for every $a \in Q$, and therefore for every $a \in \mathfrak{A}$. Now if $a \in \mathfrak{A}$, $\pi(1 \setminus a)$ must be disjoint from πa (since certainly $\pi 0 = 0$), and has the same measure as $1 \setminus \pi a$ (remember that we know that $(\mathfrak{B}, \bar{d}^* \restriction \mathfrak{B})$ is a measure algebra), so must be equal to $1 \setminus \pi a$. By 312H, π is a Boolean homomorphism. **Q**

By 324G, π is order-continuous when regarded as a function from $\mathfrak{A}$ to $\mathfrak{B}$. Because $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$, π is order-continuous when regarded as a function from $\mathfrak{A}$ to $\mathfrak{Z}$.

(e) Let $G \in \Sigma \cap \mathfrak{T}$. For any $\epsilon > 0$, there is a $K \in \mathcal{K}$ such that $K \subseteq G$ and $\mu(G \setminus K) \le \epsilon$. In this case, $\pi K^\bullet = \pi_0 K \subseteq \theta G$. So

$$\bar{d}^*(\pi G^\bullet \setminus \theta G) \le \bar{d}^*(\pi G^\bullet \setminus \pi K^\bullet) = \bar{\mu}(G^\bullet \setminus K^\bullet) = \mu(G \setminus K) \le \epsilon.$$

As ϵ is arbitrary, $\pi G^\bullet \subseteq \theta G$.

(f) This shows that we have a homomorphism π with the required properties. To see that π is unique, suppose that $\pi' : \mathfrak{A} \to \mathfrak{Z}$ is any homomorphism of the same kind. In this case

$$\bar{d}^*(1 \setminus \pi' a) = \bar{d}^*(\pi'(1 \setminus a)) = \bar{\mu}(1 \setminus a) = 1 - \bar{\mu} a = 1 - \bar{d}^*(\pi' a),$$

so $\pi' a \in D$, for every $a \in \mathfrak{A}$. If $K \in \mathcal{K}$, then $\pi' K^\bullet \subseteq \theta G$ whenever $K \subseteq G \in \Sigma \cap \mathfrak{T}$, so $\pi' K^\bullet \subseteq \pi_0 K = \pi K^\bullet$. As both $\pi K^\bullet$ and $\pi' K^\bullet$ belong to D,

$$\bar{d}^*(\pi K^\bullet \setminus \pi' K^\bullet) = \bar{d}^*(\pi K^\bullet) - \bar{d}^*(\pi' K^\bullet) = \mu K - \mu K = 0,$$

and $\pi K^\bullet = \pi' K^\bullet$. As $\{K^\bullet : K \in \mathcal{K}\}$ is topologically dense in $\mathfrak{A}$, and both π and π' are continuous, they must be equal.

491O Proposition Let X be a topological space and μ an effectively regular probability measure on X which measures every zero set, and suppose that $\langle x_n \rangle_{n \in \mathbb{N}}$ is an equidistributed sequence in X. Let $\mathfrak{A}$ be the measure algebra of μ and $\pi : \mathfrak{A} \to \mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ the regular embedding described in 491N; let $T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{Z})$ be the corresponding order-continuous Banach algebra embedding (363F). Let $S : \ell^\infty(X) \to \ell^\infty$ be the Riesz homomorphism defined by setting $(Sf)(n) = f(x_n)$ for $f \in \ell^\infty(X)$ and $n \in \mathbb{N}$, and $R : \ell^\infty \to L^\infty(\mathfrak{Z})$ the Riesz homomorphism corresponding to the Boolean homomorphism $I \mapsto I^\bullet : \mathcal{P}\mathbb{N} \to \mathfrak{Z}$. For $f \in \mathcal{L}^\infty(\mu)$ let $f^\bullet$ be the corresponding member of $L^\infty(\mu) \cong L^\infty(\mathfrak{A})$ (363I). Then $T_\pi(f^\bullet) = RSf$ for every $f \in C_b(X)$.

proof To begin with, suppose that $f : X \to [0, 1]$ is continuous and $k \ge 1$. For each $i \le k$ set $G_i = \{x : f(x) > \frac{i}{k}\}$, $F_i = \{x : f(x) \ge \frac{i}{k}\}$. Then $\frac{1}{k}\sum_{i=1}^k \chi F_i \le f \le \frac{1}{k}\sum_{i=0}^k \chi G_i$. So

$$\frac{1}{k}\sum\nolimits_{i=1}^k \chi(\pi F_i^\bullet) \le T_\pi f^\bullet \le \frac{1}{k}\sum\nolimits_{i=0}^k \chi(\pi G_i^\bullet),$$

$$\frac{1}{k}\sum\nolimits_{i=1}^k \chi(\theta F_i) \le RSf \le \frac{1}{k}\sum\nolimits_{i=0}^k \chi(\theta G_i)$$

where $\theta : \mathcal{P}X \to \mathfrak{Z}$ is the Boolean homomorphism described in 491N, because $RS : \ell^\infty \to L^\infty(\mathfrak{Z})$ is the Riesz homomorphism corresponding to θ (see 363Fa, 363Fg). Now 491N tells us that $\pi G^\bullet \subseteq \theta G$ for every cozero set $G \subseteq X$, so

$$\begin{aligned} T_\pi f^\bullet &\le \frac{1}{k}\sum_{i=0}^k \chi(\pi G_i^\bullet) \le \frac{1}{k}\sum_{i=0}^k \chi(\theta G_i) \\ &\le \frac{1}{k}\sum_{i=0}^k \chi(\theta F_i) = \frac{1}{k}e + \sum_{i=1}^k \chi(\theta F_i) \le \frac{1}{k}e + RSf \end{aligned}$$

where e is the standard order unit of the M-space $L^\infty(\mathfrak{Z})$. But looking at complements we see that we must have $\pi F^\bullet \supseteq \theta F$ for every zero set $F \subseteq X$, so

$$RSf \le \frac{1}{k}\sum_{i=0}^{k}\chi(\theta G_i) \le \frac{1}{k}\sum_{i=0}^{k}\chi(\theta F_i)$$
$$\le \frac{1}{k}\sum_{i=0}^{k}\chi(\pi F_i^\bullet) = \frac{1}{k}e + \sum_{i=1}^{k}\chi(\pi F_i^\bullet) \le \frac{1}{k}e + T_\pi f^\bullet.$$

This means that $|T_\pi f^\bullet - RSf| \le \frac{1}{k}e$ for every $k \ge 1$, so that $T_\pi f^\bullet = RSf$. This is true whenever $f \in C_b(X)$ takes values in $[0,1]$; as all the operators here are linear, it is true for every $f \in C_b(X)$.

491P Proposition Any probability algebra $(\mathfrak{A},\bar\mu)$ of cardinal at most $\mathfrak{c}$ can be regularly embedded as a subalgebra of $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ in such a way that $\bar\mu$ is identified with the restriction of the submeasure $\bar d^*$ to the image of $\mathfrak{A}$.

proof The usual measure of $\{0,1\}^{\mathfrak{c}}$ is a totally finite Radon measure, so is effectively regular (491Ma). It has an equidistributed sequence (491G), so its measure algebra $(\mathfrak{B}_{\mathfrak{c}},\bar\nu_{\mathfrak{c}})$ can be regularly embedded in $\mathfrak{Z}$ in a way which matches $\bar\nu_{\mathfrak{c}}$ with $\bar d^*$ (491N). Now if $(\mathfrak{A},\bar\mu)$ is any probability algebra of cardinal at most $\mathfrak{c}$, it can be regularly embedded (by a measure-preserving homomorphism) in $(\mathfrak{B}_{\mathfrak{c}},\bar\nu_{\mathfrak{c}})$ (332N), and therefore in $(\mathfrak{Z},\bar d^*)$. This completes the proof.

491Q Corollary Every Radon probability measure on $\{0,1\}^{\mathfrak{c}}$ has an equidistributed sequence.

proof Let μ be a Radon probability measure on $\{0,1\}^{\mathfrak{c}}$, and $(\mathfrak{A},\bar\mu)$ its measure algebra. By 491P, there is a measure-preserving embedding $\pi:\mathfrak{A}\to\mathfrak{Z}$, and $\pi[\mathfrak{A}]\subseteq D$ as defined in 491K. For $\xi<\mathfrak{c}$ let $a_\xi\in\mathfrak{A}$ be the equivalence class of $\{x : x(\xi)=1\}$, and let $I_\xi\subseteq\mathbb{N}$ be such that $I_\xi^\bullet = \pi a_\xi$ in $\mathfrak{Z}$. Define $x_n(\xi)$, for $n\in\mathbb{N}$ and $\xi<\mathfrak{c}$, by setting $x_n(\xi)=1$ if $n\in I_\xi$, 0 otherwise. Now suppose that $E\subseteq\{0,1\}^{\mathfrak{c}}$ is a basic open set of the form $\{x : x(\xi)=1$ for $\xi\in K$, 0 for $\xi\in L\}$, where K, $L\subseteq\mathfrak{c}$ are finite. Set $b=\pi E^\bullet$ in $\mathfrak{Z}$,

$$I = \{n : x_n\in E\} = \mathbb{N}\cap\bigcap_{\xi\in K} I_\xi \setminus \bigcup_{\xi\in L} I_\xi.$$

Then

$$b = \pi E^\bullet = \pi(\inf_{\xi\in K} a_\xi \setminus \sup_{\xi\in L} a_\xi)$$
$$= \inf_{\xi\in K}\pi a_\xi \setminus \sup_{\xi\in L}\pi a_\xi = \inf_{\xi\in K} I_\xi^\bullet \setminus \sup_{\xi\in L} I_\xi^\bullet = I^\bullet.$$

Since $b\in D$, $d(I)$ is defined and is equal to $\bar d^*(b) = \bar\mu E^\bullet = \mu E$.

If we now take E to be an open-and-closed subset of $\{0,1\}^{\mathfrak{c}}$, it can be expressed as a disjoint union of finitely many basic open sets of the type just considered; because d is additive on disjoint sets, $d(\{n : x_n\in E\})$ is defined and equal to μE. But this is enough to ensure that $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed, by 491Ch.

491R In this section I have been looking at probability measures with equidistributed sequences. A standard line of investigation is to ask which of our ordinary constructions, applied to such measures, lead to others of the same kind, as in 491Ea and 491F. We find that the language developed here enables us to express another result of this type.

Proposition Let X be a topological space, μ an effectively regular topological probability measure on X which has an equidistributed sequence, and ν a probability measure on X which is an indefinite-integral measure over μ. Then ν has an equidistributed sequence.

proof Let $\mathcal{K}$ be the family of regularly enveloped measurable sets.

(a) Consider first the case in which ν has Radon-Nikodým derivative of the form $\frac{1}{\mu K}\chi K$ for some $K\in\mathcal{K}$ of non-zero measure. For each $m\in\mathbb{N}$, we have an open set $G_m\supseteq K$ such that $\mu(\overline{G}_m\setminus K)\le 2^{-m}$; of course we can arrange that $G_{m+1}\subseteq G_m$ for each m. Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be an equidistributed sequence for μ. Then there is an $I\subseteq\mathbb{N}$ such that $d(I)=\mu K$ and $\{n : n\in I,\, x_n\notin G_m\}$ is finite for every m. **P** For each $m\in\mathbb{N}$, set $I_m=\{n : x_n\in G_m\}$. We know that $\liminf_{n\to\infty}\frac{1}{n}\#(I_m\cap n)\ge\mu G_m\ge\mu K$ for each m, so we can find a strictly increasing sequence $\langle k_m\rangle_{m\in\mathbb{N}}$ such that $\frac{1}{n}\#(I_m\cap n)\ge\mu K-2^{-m}$ whenever $m\in\mathbb{N}$ and $n>k_m$. Set $I=\bigcup_{m\in\mathbb{N}} I_m\cap k_{m+1}$. If $k_m<n\le k_{m+1}$,

$$\frac{1}{n}\#(I\cap n)\ge\frac{1}{n}\#(I_m\cap n)\ge\mu K-2^{-m};$$

so $\liminf_{n\to\infty}\frac{1}{n}\#(I\cap n)\ge\mu K$. On the other hand, for any $m\in\mathbb{N}$,

$$\{n:n\in I,\,x_n\notin F_m\}\subseteq I\setminus I_m\subseteq k_{m+1}$$

is finite, so

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\#(I\cap n)&\le\limsup_{n\to\infty}\frac{1}{n}\#(\{i:i<n,\,x_i\in F_m\})\\&\le\mu F_m\le\mu K+2^{-m}.\end{aligned}$$

Accordingly $\limsup_{n\to\infty}\frac{1}{n}\#(I\cap n)\le\mu K$ and $d(I)$ is defined and equal to μK. **Q**

Let $\langle j_n\rangle_{n\in\mathbb{N}}$ be the increasing enumeration of I, and set $y_n=x_{j_n}$ for each n. Then $\langle y_n\rangle_{n\in\mathbb{N}}$ is equidistributed for ν. **P** Note first that

$$\lim_{n\to\infty}\frac{n}{j_n}=\lim_{n\to\infty}\frac{1}{j_n}\#(I\cap j_n)=\mu K.$$

Let $F\subseteq X$ be closed. Then $\nu F=\frac{\mu(F\cap K)}{\mu K}$. On the other hand, for any $m\in\mathbb{N}$,

$$\begin{aligned}d^*(\{n:y_n\in F\})&=\limsup_{n\to\infty}\frac{1}{n}\#(\{i:i<n,\,x_{j_i}\in F\})\\&=\limsup_{n\to\infty}\frac{1}{n}\#(\{i:i<j_n,\,i\in I,\,x_i\in F\})\\&=\limsup_{n\to\infty}\frac{1}{n}\#(\{i:i<j_n,\,i\in I,\,x_i\in F\cap G_m\})\end{aligned}$$

(because $\{i:i\in I,\,x_i\notin G_m\}$ is finite)

$$\begin{aligned}&\le\limsup_{n\to\infty}\frac{j_n}{n}\frac{1}{j_n}\#(\{i:i<j_n,\,x_i\in F\cap F_m\})\\&=\frac{1}{\mu K}\limsup_{n\to\infty}\frac{1}{j_n}\#(\{i:i<j_n,\,x_i\in F\cap F_m\})\\&\le\frac{1}{\mu K}\limsup_{n\to\infty}\frac{1}{n}\#(\{i:i<n,\,x_i\in F\cap F_m\})\\&\le\frac{1}{\mu K}\mu(F\cap F_m)\le\frac{1}{\mu K}(\mu(F\cap K)+2^{-m})=\nu F+\frac{1}{2^m\mu K};\end{aligned}$$

as m is arbitrary, $d^*(\{n:y_n\in F\})\le\nu F$; as F is arbitrary, $\langle y_n\rangle_{n\in\mathbb{N}}$ is equidistributed for ν. **Q**

(b) Now turn to the general case. Let f be a Radon-Nikodým derivative of ν; we may suppose that f is measurable and non-negative. Then there is a sequence $\langle K_m\rangle_{m\in\mathbb{N}}$ in $\mathcal{K}$ such that $f=_{\text{a.e.}}\sum_{m=0}^\infty\frac{1}{m+1}\chi K_m$. **P** Choose f_m, K_m inductively, as follows. $f_0=f$. Given that $f_m\ge0$ is measurable, set $E_m=\{x:f_m(x)\ge\frac{1}{m+1}\}$ and let $K_m\in\mathcal{K}$ be such that $K_m\subseteq E_m$ and $\mu(E_m\setminus K_m)\le2^{-m}$; set $f_{m+1}=f_m-\frac{1}{m+1}\chi K_m$. Then $\langle f_m\rangle_{m\in\mathbb{N}}$ is non-increasing; set $g=\lim_{m\to\infty}f_m$. **?** If g is not zero almost everywhere, let $r\in\mathbb{N}$ be such that $\mu E>2^{-r+1}$ where $E=\{x:g(x)\ge\frac{1}{r+1}\}$. Then $E\subseteq E_m$ for every $m\ge r$, so $\mu(E\setminus K_m)\le2^{-m}$ for every $m\ge r$ and $F=E\cap\bigcap_{m\ge r}K_m$ is not empty. Take $x\in F$; then $f_{m+1}(x)\le f_m(x)-\frac{1}{m+1}$ for every $m\ge r$, which is impossible. **X** So $g=0$ a.e. and $f=_{\text{a.e.}}\sum_{m=0}^\infty\frac{1}{m+1}\chi K_m$. **Q**

By (a), we have for each m a sequence $\langle y_{mn}\rangle_{n\in\mathbb{N}}$ in X such that

$$\mu(F\cap K_m)\ge\mu K_m\cdot\limsup_{n\to\infty}\frac{1}{n}\#(\{i:i<n,\,y_{mi}\in F\})$$

for every closed $F\subseteq X$. For $n\in\mathbb{N}$, let ν_n be the point-supported measure on X defined by setting

$$\nu_nA=\sum_{m=0}^\infty\frac{\mu K_m}{(n+1)(m+1)}\#(\{i:i\le n,\,y_{mi}\in A\})$$

for $A\subseteq X$; because $\sum_{m=0}^\infty\frac{\mu K_m}{m+1}=\int fd\mu=1$, ν_n is a probability measure. If $F\subseteq X$ is closed,

$$\limsup_{n\to\infty}\nu_n F \le \sum_{m=0}^{\infty}\frac{\mu K_m}{m+1}\limsup_{n\to\infty}\frac{1}{n+1}\#(\{i : i \le n,\, y_{mi} \in F\})$$

(because $\sum_{m=0}^{\infty}\frac{\mu K_m}{m+1} < \infty$)

$$\le \sum_{m=0}^{\infty}\frac{1}{m+1}\mu(F\cap K_m) = \int_F f d\mu = \nu F.$$

So 491D tells us that there is an equidistributed sequence for ν, as required.

491S The asymptotic density filter Corresponding to the asymptotic density ideal, of course we have a filter. It is not surprising that convergence along this filter, in the sense of 2A3Sb, should be interesting and sometimes important.

(a) Set

$$\mathcal{F}_d = \{\mathbb{N}\setminus I : I \in \mathcal{Z}\} = \{I : I \subseteq \mathbb{N},\, \lim_{n\to\infty}\frac{1}{n}\#(I\cap n) = 1\}.$$

Then $\mathcal{F}_d$ is a filter on $\mathbb{N}$, the **(asymptotic) density filter**.

(b) For a bounded sequence $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{C}$, $\lim_{n\to\mathcal{F}_d}\alpha_n = 0$ iff $\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}|\alpha_k| = 0$. **P** Set $M = \sup_{k\in\mathbb{N}}|\alpha_k|$, and for $\epsilon > 0$ set $I_\epsilon = \{n : |\alpha_n| \le \epsilon\}$. Then, for any $n \ge 1$,

$$\frac{\epsilon}{n+1}\#((n+1)\setminus I_\epsilon) \le \frac{1}{n+1}\sum_{k=0}^{n}|\alpha_k| \le \epsilon + \frac{M}{n+1}\#((n+1)\setminus I_\epsilon).$$

So if $\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}|\alpha_k| = 0$, then $\lim_{n\to\infty}\frac{1}{n+1}\#((n+1)\setminus I_\epsilon) = 0$, that is, $\mathbb{N}\setminus I_\epsilon \in \mathcal{Z}$ and $I_\epsilon \in \mathcal{F}_d$; as ϵ is arbitrary, $\lim_{n\to\mathcal{F}_d}\alpha_n = 0$. While if $\lim_{n\to\mathcal{F}_d}\alpha_n = 0$ then $\mathbb{N}\setminus I_\epsilon \in \mathcal{Z}$ and $\limsup_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}|\alpha_k| \le \epsilon$; again, ϵ is arbitrary, so $\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}|\alpha_k| = 0$. **Q**

(c) For any $m \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, $A + m \in \mathcal{F}_d$ iff $A \in \mathcal{F}_d$. **P** For any $n \ge m$, $\#(n\cap(A+m)) = \#((n-m)\cap A)$, so

$$d(A+m) = \lim_{n\to\infty}\frac{1}{n}\#(n\cap(A+m)) = \lim_{n\to\infty}\frac{1}{n}\#(n\cap A) = d(A)$$

if either $d(A+m)$ or $d(A)$ is defined, in particular, if either $A+m$ or A belongs to $\mathcal{F}_d$. **Q** Hence, or otherwise, for any (real or complex) sequence $\langle\alpha_n\rangle_{n\in\mathbb{N}}$, $\lim_{n\to\mathcal{F}_d}\alpha_n = \lim_{n\to\mathcal{F}_d}\alpha_{m+n}$ if either is defined.

491X Basic exercises (a) (i) Show that if I, $J \in \mathcal{D} = \operatorname{dom} d$ as defined in 491A, then $I\cup J \in \mathcal{D}$ iff $I\cap J \in \mathcal{D}$ iff $I\setminus J \in \mathcal{D}$ iff $I\triangle J \in \mathcal{D}$. (ii) Show that if $\mathcal{E} \subseteq \mathcal{D}$ is an algebra of sets, then $d\restriction\mathcal{E}$ is additive. (iii) Find I, $J \in \mathcal{D}$ such that $I\cap J \notin \mathcal{D}$.

>(b) Suppose that $I \subseteq \mathbb{N}$ and that $f : \mathbb{N}\to\mathbb{N}$ is strictly increasing. Show that $d^*(f[I]) \le d^*(I)d^*(f[\mathbb{N}])$, with equality if either I or $f[\mathbb{N}]$ has asymptotic density.

(c) Show that if $A \subseteq \mathbb{N}$ and $0 \le \alpha \le d^*(A)$ there is a $B \subseteq A$ such that $d^*(B) = \alpha$ and $d^*(A\setminus B) = d^*(A) - \alpha$.

(d)(i) Let $I \subseteq \mathbb{N}$ be such that $d(J) = d^*(J\cap I) + d^*(J\setminus I)$ for every $J \subseteq \mathbb{N}$ such that $d(J)$ is defined. Show that either $I \in \mathcal{Z}$ or $\mathbb{N}\setminus I \in \mathcal{Z}$. (ii) Show that for every $\epsilon > 0$ there is an $I \subseteq \mathbb{N}$ such that $d^*(I) = \epsilon$ but $d(J) = 1$ whenever $J \supseteq I$ and $d(J)$ is defined.

(e) Let (X,Σ,μ) be a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ. For $x \in X$, set $I_x = \{n : n \in \mathbb{N},\, x \in E_n\}$. Show that $\int d^*(I_x)\mu(dx) \ge \liminf_{n\to\infty}\mu E_n$.

(f) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a compact Radon probability space. Take any point ∞ not belonging to X, and give $X\cup\{\infty\}$ the topology generated by $\{G\cup\{\infty\}:G\in\mathfrak{T}\}$. Show that $X\cup\{\infty\}$ is compact and that the image measure μ_∞ of μ under the identity map from X to $X\cup\{\infty\}$ is a quasi-Radon measure, inner regular with respect to the compact sets. Show that if we set $x_n=\infty$ for every n, then $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed for μ_∞.

>(g)(i) Show that a sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ in $[0,1]$ is equidistributed with respect to Lebesgue measure iff $\lim_{n\to\infty}\frac{1}{n+1}\#(\{i:i\le n,\ t_i\le\beta\})=\beta$ for every $\beta\in[0,1]$. (ii) Show that if $\alpha\in\mathbb{R}$ is irrational then the sequence $\langle{<}n\alpha{>}\rangle_{n\in\mathbb{N}}$ of fractional parts of multiples of α is equidistributed in $[0,1]$ with respect to Lebesgue measure. (*Hint*: 281Yi.) (iii) Show that a function $f:[0,1]\to\mathbb{R}$ is Riemann integrable iff $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)$ is defined in $\mathbb{R}$ for every sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $[0,1]$ which is equidistributed for Lebesgue measure.

(h) Show that the usual measure on the split interval (419L) has an equidistributed sequence.

>(i) Show that if X is a Hausdorff space and $f:\mathbb{N}\to X$ is injective, then there is an open set $G\subseteq X$ such that $f^{-1}[G]$ does not have asymptotic density. (*Hint*: show that if $A\subseteq\mathbb{N}$ and $d^*(A)>\gamma$, there is an $n\in\mathbb{N}$ such that for every $m\ge n$ there is an open set $G\supseteq f[m\setminus n]$ such that $d^*(A\setminus f^{-1}[G])>\gamma$. You may prefer to tackle the case of metrizable X first.)

(j) Let X be a metrizable space, and μ a quasi-Radon probability measure on X. (i) Show that there is an equidistributed sequence for μ. (ii) Show that if the support of μ is not compact, and $\langle x_n\rangle_{n\in\mathbb{N}}$ is an equidistributed sequence for μ, then there is a continuous integrable function $f:X\to\mathbb{R}$ such that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)=\infty$.

(k) Let $\phi:\mathfrak{c}\to\mathcal{P}\mathbb{N}$ be an injective function. For each $n\in\mathbb{N}$ let λ_n be the uniform probability measure on $\mathcal{P}(\mathcal{P}n)$, giving measure 2^{-2^n} to each singleton. Define $\psi_n:\mathcal{P}(\mathcal{P}n)\to\{0,1\}^{\mathfrak{c}}$ by setting $\psi_n(\mathcal{I})(\xi)=1$ if $\phi(\xi)\cap n\in\mathcal{I}$, 0 otherwise, and let ν_n be the image measure $\lambda_n\psi_n^{-1}$. Show that ν_nE is the usual measure of E whenever $E\subseteq\{0,1\}^{\mathfrak{c}}$ is determined by coordinates in a finite set on which the map $\xi\mapsto\phi(\xi)\cap n$ is injective. Use this with 491D to prove 491G.

>(l)(i) Let Z be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, with its usual measure. Show that there is no equidistributed sequence in Z. (*Hint*: meager sets in Z have negligible closures.) (ii) Show that Dieudonné's measure on ω_1 (411Q) has no equidistributed sequence. (iii) Show that if $\#(I)>\mathfrak{c}$ then the usual measure on $\{0,1\}^I$ has no equidistributed sequence. (*Hint*: if $\langle x_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\{0,1\}^I$, there is an infinite $J\subseteq I$ such that $\langle x_n(\eta)\rangle_{n\in\mathbb{N}}=\langle x_n(\xi)\rangle_{n\in\mathbb{N}}$ for all η, $\xi\in J$.) (iv) Show that if X is a topological group with a Haar probability measure μ, and X is not separable, then μ has no equidistributed sequence. (*Hint*: use 443D to show that every separable subset is negligible.)

(m) Let X be a compact Hausdorff abelian topological group and μ its Haar probability measure. Show that a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X is equidistributed for μ iff $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n\chi(x_i)=0$ for every non-trivial character $\chi:X\to S^1$. (*Hint*: 281G.)

(n)(i) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{Z}=\mathcal{P}\mathbb{N}/\mathcal{Z}$. Show that there is an $a\in\mathfrak{Z}$ such that $a_n\subseteq a$ for every $n\in\mathbb{N}$ and $\bar d^*(a)=\sup_{n\in\mathbb{N}}\bar d^*(a_n)$. (ii) Show that $\mathfrak{Z}$ is not Dedekind σ-complete. (*Hint*: 393Bc[2].)

(o) Let $\mathfrak{Z}$, $\bar d^*$ and D be as in 491K. Show that if $a\in D\setminus\{0\}$ and $\mathfrak{Z}_a$ is the principal ideal of $\mathfrak{Z}$ generated by a, then $(\mathfrak{Z}_a,\bar d^*\restriction\mathfrak{Z}_a)$ is isomorphic, up to a scalar multiple of the submeasure, to $(\mathfrak{Z},\bar d^*)$.

(p) Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X. Show that μ is effectively regular iff whenever $E\in\Sigma$, $\mu E<\infty$ and $\epsilon>0$ there are a measurable open set G and a measurable closed set $F\supseteq G$ such that $\mu(F\setminus E)+\mu(E\setminus G)\le\epsilon$.

(q) Let X be a normal topological space and μ a topological measure on X which is inner regular with respect to the closed sets and effectively locally finite. Show that μ is effectively regular.

(r) Let X be a topological space and μ an effectively regular measure on X. (i) Show that the completion and c.l.d. version of μ are also effectively regular. (ii) Show that if $Y\subseteq X$ then the subspace measure is again effectively regular. (iii) Show that any totally finite indefinite-integral measure over μ is effectively regular.

[2]Formerly 392Hc.

(s)(i) Let X_1, X_2 be topological spaces with effectively regular measures μ_1, μ_2. Show that the c.l.d. product measure on $X_1 \times X_2$ is effectively regular with respect to the product topology. (*Hint*: 412R.) (ii) Let $\langle X_i \rangle_{i\in I}$ be a family of topological spaces and μ_i an effectively regular probability measure on X_i for each i. Show that the product probability measure on $\prod_{i\in I} X_i$ is effectively regular.

(t) Give $[0,1]$ the topology $\mathfrak{T}$ generated by the usual topology and $\{[0,1]\setminus A : A \subseteq \mathbb{Q}\}$. Let μ_L be Lebesgue measure on $[0,1]$, and Σ its domain. For $E \in \Sigma$ set $\mu E = \mu_L E + \#(E\cap\mathbb{Q})$ if $E\cap\mathbb{Q}$ is finite, ∞ otherwise. Show that μ is a σ-finite quasi-Radon measure with respect to the topology $\mathfrak{T}$, but is not effectively regular.

(u) Let $\mathfrak{A}$ be a countable Boolean algebra and ν a finitely additive functional on $\mathfrak{A}$ such that $\nu 1 = 1$. Show that there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathcal{P}\mathbb{N}$ such that $d(\pi a)$ is defined and equal to νa for every $a \in \mathfrak{A}$ (i) using 491Xc (ii) using 392H[3], 491P and 341Xc.

(v) Let X be a dyadic space. (i) Show that there is a Radon probability measure on X with support X. (ii) Show that the following are equiveridical: (α) $w(X) \le \mathfrak{c}$; (β) every Radon probability measure on X has an equidistributed sequence; (γ) X is separable. (*Hint*: 4A2Dd, 418L.)

(w)(i) Give an example of a Radon probability space (X,μ) with a closed conegligible set $F \subseteq X$ such that μ has an equidistributed sequence but the subspace measure μ_F does not. (*Hint*: the Stone space of the measure algebra of Lebesgue measure embeds into $\{0,1\}^{\mathfrak{c}}$.) (ii) Show that if X is a topological space with an effectively regular topological probability measure μ which has an equidistributed sequence, and $G \subseteq X$ is a non-negligible open set, then the normalized subspace measure $\dfrac{1}{\mu G}\mu_G$ has an equidistributed sequence. (*Hint*: 491R.)

(x) Let X be a topological space. A sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in X is called **statistically convergent** to $x \in X$ if $d(\{n : x_n \in G\}) = 1$ for every open set G containing x. (i) Show that if X is first-countable then $\langle x_n \rangle_{n\in\mathbb{N}}$ is statistically convergent to x iff there is a set $I \subseteq \mathbb{N}$ such that $d(I) = 1$ and $\langle x_n \rangle_{n\in I}$ converges to x in the sense that $\{n : n \in I,\ x_n \notin G\}$ is finite for every open set G containing x. (ii) Show that a bounded sequence $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$ is statistically convergent to α iff $\lim_{n\to\infty} \dfrac{1}{n}\sum_{i=0}^{n-1} |\alpha_i - \alpha| = 0$.

491Y Further exercises (a) Show that every subset A of $\mathbb{N}$ is expressible in the form $I_A \triangle J_A$ where $d(I_A) = d(J_A) = \frac{1}{2}$ (i) by a direct construction, with $A \mapsto I_A$ a continuous function (ii) using 443D.

(b) Let $\mathfrak{A}$ be a Boolean algebra, and $\nu : \mathfrak{A} \to [0,\infty]$ a submeasure. Show that ν is uniformly exhaustive iff whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$, there is a set $I \subseteq \mathbb{N}$ such that $d^*(I) > 0$ and $\inf_{i\in I\cap n} a_i \ne 0$ for every $n \in \mathbb{N}$.

(c) Find a topological space X with a τ-additive probability measure μ on X, a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in X and a base $\mathcal{G}$ for the topology of X, consisting of measurable sets and closed under finite intersections, such that $\mu G \le \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \le n,\ x_i \in G\})$ for every $G \in \mathcal{G}$ but $\langle x_n \rangle_{n\in\mathbb{N}}$ is not equidistributed. (*Hint*: take $\#(X) = 4$.)

(d) Let X be a compact Hausdorff space on which every Radon probability measure has an equidistributed sequence. Show that the cylindrical σ-algebra of $C(X)$ is the σ-algebra generated by sets of the form $\{f : f \in C(X),\ f(x) > \alpha\}$ where $x \in X$ and $\alpha \in \mathbb{R}$. (*Hint*: 436J, 491Cb.)

(e) Give $\omega_1 + 1$ and $[0,1]$ their usual compact Hausdorff topologies. Let $\langle t_i \rangle_{i\in\mathbb{N}}$ be a sequence in $[0,1]$ which is equidistributed for Lebesgue measure μ_L, and set $Q = \{t_i : i \in \mathbb{N}\}$, $X = (\omega_1 \times ([0,1]\setminus Q)) \cup (\{\omega_1\} \times Q)$, with the subspace topology inherited from $(\omega_1 + 1) \times [0,1]$. (i) Set $F = \{\omega_1\} \times Q$. Show that F is a closed Baire set in the completely regular Hausdorff space X. (ii) Show that if $f \in C_b(X)$ then there are a $g_f \in C([0,1])$ and a $\zeta < \omega_1$ such that such that $f(\xi,t) = g_f(t)$ whenever $(\xi,t) \in X$ and $\zeta \le \xi \le \omega_1$. (iii) Show that there is a Baire measure μ on X such that $\int f d\mu = \int g_f d\mu_L$ for every $f \in C_b(X)$. (iv) Show that $\mu F = 0$. (v) Show that $\int f d\mu = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(\omega_1, t_i)$ for every $f \in C_b(X)$, but that $\langle (\omega_1, t_i) \rangle_{i\in\mathbb{N}}$ is not equidistributed with respect to μ.

[3]Formerly 393B.

(f) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a topological measure space. Let $\mathcal{E}$ be the Jordan algebra of X (411Yc). (i) Suppose that μ is a complete probability measure on X and $\langle x_n\rangle_{n\in\mathbb{N}}$ an equidistributed sequence in X. Show that the asymptotic density $d(\{n : x_n \in E\})$ is defined and equal to μE for every $E \in \mathcal{E}$. (ii) Suppose that μ is a probability measure on X and that $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X such that $d(\{n : x_n \in E\})$ is defined and equal to μE for every $E \in \mathcal{E}$. Show that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i) = \int f d\mu$ for every $f \in C_b(X)$.

(g) Show that a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $[0,1]$ is equidistributed for Lebesgue measure iff there is some $r_0 \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n x_i^r = \frac{1}{r+1}$ for every $r \ge r_0$.

(h) Let Z, μ, $X = Z \times \{0,1\}$ and ν be as described in 439K, so that μ is a Radon probability measure on the compact metrizable space Z, X has a compact Hausdorff topology finer than the product topology, and ν is a measure on Z extending μ. (i) Show that if $f \in C(X)$, then $\{t : t \in Z,\ f(t,0) \ne f(t,1)\}$ is countable. (*Hint*: 4A2F(h-vii).) (ii) Show that $\int f(t,0)\mu(dt) = \int f(t,1)\nu(dt)$ for every $f \in C(X)$. (iii) Let λ be the measure νg^{-1} on Z, where $g(t,0) = g(t,1) = (t,1)$ for $t \in Z$. Show that there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $Z \times \{0\}$ such that $\int f d\lambda = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i)$ for every $f \in C(X)$, but that $\langle x_n\rangle_{n\in\mathbb{N}}$ is not λ-equidistributed.

(i)(i) Show that a Radon probability measure on an extremally disconnected compact Hausdorff space has an equidistributed sequence iff it is point-supported. (*Hint*: see the hint for 326Yo.) (ii) Show that there is a separable compact Hausdorff space with a Radon probability measure which has no equidistributed sequence.

(j) Show that there is a countable dense set $D \subseteq [0,1]^{\mathfrak{c}}$ such that no sequence in D is equidistributed for the usual measure on $[0,1]^{\mathfrak{c}}$.

(k) Let $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ and $\bar d^* : \mathfrak{Z} \to [0,1]$ be as in 491I. Show that $\bar d^*$ is **order-continuous on the left** in the sense that whenever $A \subseteq \mathfrak{Z}$ is non-empty and upwards-directed and has a supremum $c \in \mathfrak{Z}$, then $\bar d^*(c) = \sup_{a\in A} \bar d^*(a)$.

(l)(i) Show that $\mathfrak{Z}$ is weakly (σ,∞)-distributive. (ii) Show that $\mathfrak{Z} \cong \mathfrak{Z}^{\mathbb{N}}$. (iii) Show that $\mathfrak{Z}$ has the σ-interpolation property, but is not Dedekind σ-complete. (iv) Show that $\mathfrak{Z}$ has many involutions in the sense of 382O.

(m) Let (X,ρ) be a separable metric space and μ a Borel probability measure on X. (i) Show that there is an equidistributed sequence in X. (ii) Show that if $\langle x_n\rangle_{n\in\mathbb{N}}$ is an equidistributed sequence in X, and $\langle y_n\rangle_{n\in\mathbb{N}}$ is a sequence in X such that $\lim_{n\to\infty}\rho(x_n,y_n) = 0$, then $\langle y_n\rangle_{n\in\mathbb{N}}$ is equidistributed. (iii) Show that if $f : X \to \mathbb{R}$ is a bounded function, then $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i) - f(y_i) = 0$ for all equidistributed sequences $\langle x_n\rangle_{n\in\mathbb{N}}$, $\langle y_n\rangle_{n\in\mathbb{N}}$ in X iff $\{x : f$ is continuous at $x\}$ is conegligible, and in this case $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i) = \int f d\mu$ for every equidistributed sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X. (*Hint*: $\{x : f$ is continuous at $x\} = \bigcap_{m\in\mathbb{N}} G_m$, where $G_m = \bigcup\{G : G \subseteq X$ is open, $\sup_{x,y\in G}|f(x) - f(y)| \le 2^{-m}\}$.) Compare 491Xg.

(n) Let $(X,\mathfrak{T})$ be a topological space, μ a probability measure on X, and $\phi : X \to X$ an inverse-measure-preserving function. (i) Suppose that $\mathfrak{T}$ has a countable network consisting of measurable sets, and that ϕ is ergodic. Show that $\langle \phi^n(x)\rangle_{n\in\mathbb{N}}$ is equidistributed for almost every $x \in X$. (*Hint*: 372Qb.) (ii) Suppose that μ is *either* inner regular with respect to the closed sets *or* effectively regular, and that $\{x : \langle\phi^n(x)\rangle_{n\in\mathbb{N}}$ is equidistributed$\}$ is not negligible. Show that ϕ is ergodic.

(o) Let $\langle X_\xi\rangle_{\xi<\mathfrak{c}}$ be a family of topological spaces with countable networks consisting of Borel sets, and μ a τ-additive topological probability measure on $X = \prod_{\xi<\mathfrak{c}} X_\xi$. Show that μ has an equidistributed sequence.

(p)(i) Show that there is a family $\langle a_\xi\rangle_{\xi<\mathfrak{c}}$ in $\mathfrak{Z}$ such that $\inf_{\xi\in I} a_\xi = 0$ and $\sup_{\xi\in I} a_\xi = 1$ for every infinite $I \subseteq \mathfrak{c}$. (ii) Show that if $B \subseteq \mathfrak{Z}\setminus\{0\}$ has cardinal less than $\mathfrak{c}$ then there is an $a \in \mathfrak{Z}$ such that $b \cap a$ and $b \setminus a$ are non-zero for every $b \in B$.

(q) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a τ-additive topological probability space. A sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X is **completely equidistributed** if, for every $r \ge 1$, the sequence $\langle\langle x_{n+i}\rangle_{i<r}\rangle_{n\in\mathbb{N}}$ is equidistributed for some (therefore any) τ-additive extension of the c.l.d. product measure μ^r on X^r. (i) Show that if there is an equidistributed sequence in X, then there is a completely equidistributed sequence in X. (ii) Show that if $\mathfrak{T}$ is second-countable, then $\mu^{\mathbb{N}}$-almost every sequence in X is completely equidistributed. (iii) Show that if X has two disjoint open sets of non-zero measure, then no sequence which is well-distributed in the sense of 281Ym can be completely equidistributed.

(r) Suppose, in 491O, that μ is a topological measure. Show that $T_\pi f^\bullet \le RSf$ for every bounded lower semi-continuous $f : X \to \mathbb{R}$.

(s) (M.Elekes) Let (X, Σ, μ) be a σ-finite measure space and $\langle E_n \rangle_{n\in\mathbb{N}}$ a sequence in Σ such that $\bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} E_m$ is conegligible. Show that there is an $I \in \mathcal{Z}$ such that $\bigcap_{n\in\mathbb{N}} \bigcup_{m\in I\setminus n} E_m$ is conegligible.

491Z Problem It is known that for almost every $x > 1$ the sequence $\langle <x^n> \rangle_{n\in\mathbb{N}}$ of fractional parts of powers of x is equidistributed for Lebesgue measure on $[0, 1]$ (Kuipers & Niederreiter 74, p. 35). But is $\langle <(\frac{3}{2})^n> \rangle_{n\in\mathbb{N}}$ equidistributed?

491 Notes and comments The notations d^*, d (491A) are standard, and usefully suggestive. But coming from measure theory we have to remember that d^*, although a submeasure, is not an outer measure, the domain of d is not an algebra of sets (491Xa), and d and d^* are related by only one of the formulae we expect to connect a measure with an outer measure (491Ac, 491Xd). The limits $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^n f(x_i)$ are the **Cesàro means** of the sequences $\langle f(x_n) \rangle_{n\in\mathbb{N}}$. The delicacy of the arguments here arises from the fact that the family of (bounded) sequences with Cesàro means, although a norm-closed linear subspace of ℓ^∞, is neither a sublattice nor a subalgebra. When we turn to the quotient algebra $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$, we find ourselves with a natural submeasure to which we can apply ideas from §392 to good effect (491I; see also 491Yk and 491Yl). What is striking is that equidistributed sequences induce regular embeddings of measure algebras in $\mathfrak{Z}$ which can be thought of as measure-preserving (491N).

Most authors have been content to define an 'equidistributed sequence' to be one such that the integrals of bounded continuous functions are correctly specified (491Cf, 491Cg); that is, that the point-supported measures $\frac{1}{n+1} \sum_{i=0}^n \delta_{x_i}$ converge to μ in the vague topology on an appropriate class of measures (437J). I am going outside this territory in order to cover some ideas I find interesting. 491Yh shows that it makes a difference; there are Borel measures on compact Hausdorff spaces which have sequences which give the correct Cesàro means for continuous functions, but lie within negligible closed sets; and the same can happen with Baire measures (491Ye). It seems to be difficult, in general, to determine whether a topological probability space – even a compact Radon probability space – has an equidistributed sequence. In the proofs of 491D-491G I have tried to collect the principal techniques for showing that spaces do have equidistributed sequences. In the other direction, it is obviously impossible for a space to have an equidistributed sequence if every separable subspace is negligible (491Xl). For an example of a separable compact Hausdorff space with a Radon measure which does not have an equidistributed sequence, we seem to have to go deeper (491Yi).

491Z is a famous problem. It is not clear that it is a problem in measure theory, and there is no reason to suppose that any of the ideas of this treatise beyond 491Xg(i) are relevant. I mention it because I think everyone should know that it is there.

492 Combinatorial concentration of measure

'Concentration of measure' takes its most dramatic forms in the geometrically defined notions of concentration explored in §476. But the phenomenon is observable in many other contexts, if we can devise the right abstract geometries to capture it. In this section I present one of Talagrand's theorems on the concentration of measure in product spaces, using the Hamming metric (492D), and Maurey's theorem on concentration of measure in permutation groups (492H).

492A Lemma Let (X, Σ, μ) be a totally finite measure space, $\alpha < \beta$ in $\mathbb{R}$, $\phi : [\alpha, \beta] \to \mathbb{R}$ a convex function, and $f : X \to [\alpha, \beta]$ a Σ-measurable function. Then

$$\int \phi(f(x))\mu(dx) \le \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \int f d\mu + \frac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha} \mu X.$$

proof If $t \in [\alpha, \beta]$ then $t = \frac{t-\alpha}{\beta-\alpha}\beta + \frac{\beta-t}{\beta-\alpha}\alpha$, so

$$\phi(t) \le \frac{t-\alpha}{\beta-\alpha}\phi(\beta) + \frac{\beta-t}{\beta-\alpha}\phi(\alpha) = \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}t + \frac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha}.$$

Accordingly

$$\phi(f(x)) \le \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}f(x) + \frac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha}$$

for every $x \in X$; integrating with respect to x, we have the result.

492B Corollary Let (X, Σ, μ) be a probability space and $f : X \to [\alpha, 1]$ a measurable function, where $0 < \alpha \le 1$. Then $\int \frac{1}{f} d\mu \cdot \int f d\mu \le \frac{(1+\alpha)^2}{4\alpha}$.

proof Set $\gamma = \int f d\mu$, so that $\alpha \le \gamma \le 1$. By 492A, with $\phi(t) = \frac{1}{t}$,

$$\int \frac{1}{f} d\mu \le \frac{\gamma}{1-\alpha}(1 - \frac{1}{\alpha}) + \frac{1}{1-\alpha}(\frac{1}{\alpha} - \alpha) = \frac{1+\alpha-\gamma}{\alpha}.$$

Now $\frac{1+\alpha-\gamma}{\alpha} \cdot \gamma$ takes its maximum value $\frac{(1+\alpha)^2}{4\alpha}$ when $\gamma = \frac{1+\alpha}{2}$, so this is also the maximum possible value for $\int \frac{1}{f} \int f$.

492C Lemma $\frac{1}{2}(1 + \cosh t) \le e^{t^2/4}$ for every $t \in \mathbb{R}$.

proof For $k \ge 1$, $4^k k! \le 2(2k)!$ (induce on k), so

$$1 + \cosh t = 2 + \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} \le 2 + \sum_{k=1}^{\infty} \frac{2t^{2k}}{4^k k!} = 2e^{t^2/4}.$$

492D Theorem (Talagrand 95) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i<n}$ be a non-empty finite family of probability spaces with product (X, Λ, λ). Let ρ be the **normalized Hamming metric** on X defined by setting $\rho(x, y) = \frac{1}{n}\#(\{i : i < n, x(i) \ne y(i)\})$ for x, $y \in X$. If $W \in \Lambda$ and $\lambda W > 0$, then

$$\overline{\int} e^{t\rho(x,W)} \lambda(dx) \le \frac{1}{\lambda W} e^{t^2/4n}$$

for every $t \ge 0$.

proof The formulae below will go much more smoothly if we work with the simple Hamming metric $\sigma(x, y) = \#(\{i : x(i) \ne y(i)\})$ instead of ρ. In this case, we can make sense of the case $n = 0$, and this will be useful. In terms of σ, our target is to prove that if $W \in \Lambda$ and $\lambda W > 0$, then

$$\overline{\int} e^{t\sigma(x,W)} \lambda(dx) \le \frac{1}{\lambda W} e^{nt^2/4}$$

for every $t \ge 0$.

(a) To begin with, suppose that every $X_i = Z = \{0, 1\}^{\mathbb{N}}$, every μ_i is a Borel measure, and W is compact. Note that in this case λ is a Radon measure (because the X_i are compact and metrizable), and

$$\{x : \sigma(x, W) \le m\} = \bigcup_{I \subseteq n, \#(I) \le m} \{x : \exists\, y \in W,\, x{\restriction}n \setminus I = y{\restriction}n \setminus I\}$$

is compact for every m, so the function $x \mapsto \sigma(x, W)$ is measurable.

Induce on n. If $n = 0$ we must have $W = X = \{\emptyset\}$ and $\sigma(x, W) = 0$ for every x, so the result is trivial. For the inductive step to $n \ge 1$, we have $W \subseteq X \times X_n$, where $X = \prod_{i<n} X_i$, and we are looking at $\iint e^{t\sigma((x,\xi),W)} \lambda(dx)\mu_n(d\xi)$. Now, setting $V_\xi = \{x : (x, \xi) \in W\}$ for $\xi \in X_n$,

$$V = \bigcup_{\xi \in X_n} V_\xi = \{x : \exists\, \xi \in X_n,\, (x, \xi) \in W\},$$

we have

$$\sigma((x, \xi), W) \le \min(\sigma(x, V_\xi), 1 + \sigma(x, V))$$

for all x and ξ, counting $\sigma(x, \emptyset)$ as ∞ if V_ξ is empty. So, for any $\xi \in X_n$,

$$\begin{aligned}\int e^{t\sigma((x,\xi),W)} \lambda(dx) &\le \min(\int e^{t\sigma(x,V_\xi)} \lambda(dx), e^t \int e^{t\sigma(x,V)} \lambda(dx)) \\ &\le e^{nt^2/4} \min(\frac{1}{\lambda V_\xi}, \frac{e^t}{\lambda V})\end{aligned}$$

by the inductive hypothesis, counting $\min(\frac{1}{0},\frac{e^t}{\lambda V})$ as $\frac{e^t}{\lambda V}$.

It follows that if we set $f(\xi)=\max(e^{-t},\frac{\lambda V_\xi}{\lambda V})$ for $\xi\in X_n$,

$$\begin{aligned}(\lambda\times\mu_n)(W)\int e^{t\sigma((x,\xi),W)}\lambda(dx)\mu_n(d\xi)&\\ \le\int\lambda V_\xi\mu_n(d\xi)\cdot e^{nt^2/4}\int\min(\frac{1}{\lambda V_\xi},\frac{e^t}{\lambda V})\mu_n(d\xi)&\\ =e^{nt^2/4}\int\frac{\lambda V_\xi}{\lambda V}\mu_n(d\xi)\cdot\int\min(e^t,\frac{\lambda V}{\lambda V_\xi})\mu_n(d\xi)&\\ \le e^{nt^2/4}\int f(\xi)\mu_n(d\xi)\cdot\int\frac{1}{f(\xi)}\mu_n(d\xi)&\\ \le e^{nt^2/4}\cdot\tfrac{(1+e^{-t})^2}{4e^{-t}}&\end{aligned}$$

(492B)

$$=\frac{1}{2}e^{nt^2/4}(1+\cosh t)\le e^{(n+1)t^2/4}$$

by 492C, and

$$\int e^{t\sigma((x,\xi),W)}\lambda(dx)\mu_n(d\xi)\le\tfrac{1}{(\lambda\times\mu_n)(W)}e^{(n+1)t^2/4},$$

so the induction continues.

(b) Now turn to the general case. If $W\in\Lambda$, there is a $W_1\subseteq W$ such that $W_1\in\widehat{\bigotimes}_{i<n}\Sigma_i$ and $\lambda W_1=\lambda W$ (251Wf). There must be countably-generated σ-subalgebras Σ'_i of Σ_i such that $W_1\in\widehat{\bigotimes}_{i<n}\Sigma'_i$. For each $i<n$, let $\langle E_{ik}\rangle_{k\in\mathbb{N}}$ be a sequence in Σ_i generating Σ'_i, and let $h_i:X_i\to Z$ be the corresponding Marczewski functional, so that $h_i(\xi)=\langle\chi E_{ik}(\xi)\rangle_{k\in\mathbb{N}}$ for $\xi\in X_i$. Let μ'_i be the Borel measure on Z defined by setting $\mu'_iF=\mu_ih_i^{-1}[F]$ for every Borel set $F\subseteq Z$, and let ν be the product of the measures μ'_i on $Y=Z^n$. If we set $h(x)=\langle h_i(x(i))\rangle_{i<n}$ for $x\in X$, then $h:X\to Y$ is inverse-measure-preserving for λ and ν (254H). Moreover, by the choice of the E_{ik}, $W_1=h^{-1}[V]$ for some Borel set $V\subseteq Y$.

Because Y is a compact metrizable space, ν is the completion of a Borel measure and is a Radon measure (433Cb). For each $I\subseteq n$, write ν_I for the product measure on Z^I, and set

$$V_I=\{u:u\in Z^{n\setminus I},\,\nu_I\{v:v\in Z^I,\,(u,v)\in V\}>0\},$$

$$V'_I=\{y:y\in Y,\,y{\restriction}n\setminus I\in V_I\}.$$

Then $\nu(V\setminus V'_I)=0$ for every $I\subseteq n$, so if we set $V'=\bigcap_{I\subseteq n}V'_I$ then $\nu V'=\nu V$. (Of course $V'\subseteq V'_\emptyset=V$.)

Take any $\gamma\in\,]0,\lambda W[\,=\,]0,\nu V'[$. Let $K\subseteq V'$ be a compact set such that $\nu K\ge\gamma$. Set $g(y)=e^{t\sigma(y,K)}$ for $y\in Y$, where I write σ for the Hamming metric on Y (regarded as a product of n factor spaces). Then $g:Y\to\mathbb{R}$ is Borel measurable and $gh:X\to\mathbb{R}$ is Λ-measurable. Also, for any $x\in X$, $\sigma(x,W)\le\sigma(h(x),K)$. **P** Take $y\in K$ such that $\sigma(h(x),y)=\sigma(h(x),K)$, and set

$$I=\{i:h(x)(i))\ne y(i)\},\quad u=h(x){\restriction}n\setminus I=y{\restriction}n\setminus I.$$

Because $y\in V'$, $u\in V_I$ and $\nu_IH>0$, where $H=\{v:v\in Z^I,\,(u,v)\in V\}$. But if we write λ_I for the product measure on $\prod_{i\in I}X_i$, and $h_I(z)=\langle h_i(z(i))\rangle_{i\in I}$ for $z\in\prod_{i\in I}X_i$, then h_I is inverse-measure-preserving for λ_I and ν_I; in particular, $h_I^{-1}[H]$ is non-empty. This means that we can find an $x'\in X$ such that $x'{\restriction}n\setminus I=x{\restriction}n\setminus I$ and $x'{\restriction}I\in h_I^{-1}[H]$. In this case, $h(x')\in V$, so $x'\in W_1\subseteq W$, and

$$\sigma(x,W)\le\sigma(x,x')\le\#(I)=\sigma(h(x),K).\ \mathbf{Q}$$

Accordingly

$$e^{t\sigma(x,W)}\le e^{t\sigma(h(x),K)}=g(h(x))$$

for every $x\in X$, and

$$\overline{\int} e^{t\sigma(x,W)}\lambda(dx) \le \int gh\,d\lambda = \int g\,d\nu$$

(because g is ν-integrable and h is inverse-measure-preserving, see 235G[4])

$$\le \frac{1}{\nu K}e^{nt^2/4}$$

(by (a))

$$\le \frac{1}{\gamma}e^{nt^2/4}.$$

As γ is arbitrary,

$$\overline{\int} e^{t\sigma(x,W)}\lambda(dx) \le \frac{1}{\lambda W}e^{nt^2/4},$$

as claimed.

492E Corollary Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i<n}$ be a non-empty finite family of probability spaces with product (X,Λ,λ).

(a) Let ρ be the normalized Hamming metric on X. If $W\in\Lambda$ and $\lambda W>0$, then

$$\lambda^*\{x:\rho(x,W)\ge\gamma\} \le \frac{1}{\lambda W}e^{-n\gamma^2}$$

for every $\gamma\ge 0$.

(b) If W, $W'\in\Lambda$ and $\gamma>0$ are such that $e^{-n\gamma^2}<\lambda W\cdot\lambda W'$ then there are $x\in W$, $x'\in W'$ such that $\#(\{i:i<n,\ x(i)\ne x'(i)\})<n\gamma$.

proof (a) Set $t=2n\gamma$. By 492D, there is a measurable function $f:X\to\mathbb{R}$ such that $f(x)\ge e^{t\rho(x,W)}$ for every $x\in X$ and $\int fd\lambda \le \frac{1}{\lambda W}e^{t^2/4n}$. So

$$\lambda^*\{x:\rho(x,W)\ge\gamma\} \le \lambda\{x:f(x)\ge e^{t\gamma}\} \le e^{-t\gamma}\int fd\lambda$$

$$\le \frac{1}{\lambda W}e^{-t\gamma+t^2/4n} = \frac{1}{\lambda W}e^{-n\gamma^2}.$$

(b) By (a), $\lambda^*\{x:\rho(x,W')\ge\gamma\}<\lambda W$, so there must be an $x\in W$ such that $\rho(x,W')<\gamma$.

492F The next theorem concerns concentration of measure in permutation groups. I approach this through a general result about slowly-varying martingales (492G).

Lemma $e^t\le t+e^{t^2}$ for every $t\in\mathbb{R}$.

proof If $t\ge 1$ then $t\le t^2$ so

$$e^t\le e^{t^2}\le t+e^{t^2}.$$

If $0\le t\le 1$ then

$$e^t = 1+t+\sum_{n=2}^\infty \frac{t^n}{n!} \le 1+t+\sum_{k=1}^\infty\Bigl(\frac{1}{(2k)!}+\frac{1}{(2k+1)!}\Bigr)t^{2k}$$

$$\le 1+t+\sum_{k=1}^\infty\frac{t^{2k}}{k!} = t+e^{t^2}.$$

If $t\le 0$ then

$$e^t = 1+t+\sum_{n=2}^\infty \frac{t^n}{n!} \le 1+t+\sum_{k=1}^\infty\frac{t^{2k}}{(2k)!}$$

$$\le 1+t+\sum_{k=1}^\infty\frac{t^{2k}}{k!} = t+e^{t^2}.$$

[4] Formerly 235I.

492G Lemma (Milman & Schechtman 86) Let (X,Σ,μ) be a probability space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a martingale on X. Suppose that $f_n\in\mathcal{L}^\infty(\mu)$ for every n, and that $\alpha_n\ge\operatorname{ess\,sup}|f_n-f_{n-1}|$ for $n\ge 1$. Then for any $n\ge 1$ and $\gamma\ge 0$,

$$\Pr(f_n-f_0\ge\gamma)\le\exp(-\gamma^2/4\sum_{i=1}^n\alpha_i^2),$$

at least if $\sum_{i=1}^n\alpha_i^2>0$.

proof Let $\langle\Sigma_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of σ-subalgebras of Σ to which $\langle f_n\rangle_{n\in\mathbb{N}}$ is a adapted. Extending the functions f_n if necessary, we may suppose that they are all defined on the whole of X.

(a) I show first that

$$\mathbb{E}(\exp(\lambda(f_n-f_0)))\le\exp(\lambda^2\sum_{i=1}^n\alpha_i^2)$$

for any $n\ge 0$ and any $\lambda>0$. **P** Induce on n. For $n=0$, interpreting $\sum_{i=1}^0$ as 0, this is trivial. For the inductive step to $n+1$, set $g=f_n-f_{n-1}$ and let g_1, g_2 be conditional expectations of $\exp(\lambda g)$ and $\exp(\lambda^2g^2)$ on Σ_{n-1}. Because $|g|\le\alpha_n$ a.e., $\exp(\lambda^2g^2)\le\exp(\lambda^2\alpha_n^2)$ a.e. and $g_2\le\exp(\lambda^2\alpha_n^2)$ a.e. Because $\exp(\lambda g)\le\lambda g+\exp(\lambda^2g^2)$ wherever g is defined (492F), and 0 is a conditional expectation of g on Σ_{n-1}, $g_1\le g_2\le\exp(\lambda^2\alpha_n^2)$ a.e.

Now observe that $f_{n-1}-f_0$ is Σ_{n-1}-measurable, so that $\exp(\lambda(f_{n-1}-f_0))\times g_1$ is a conditional expectation of $\exp(\lambda(f_{n-1}-f_0))\times\exp(\lambda g)=\exp(\lambda(f_n-f_0))$ on Σ_{n-1} (233Eg). Accordingly

$$\begin{aligned}\mathbb{E}(\exp(\lambda(f_n-f_0)))&=\mathbb{E}(\exp(\lambda(f_{n-1}-f_0))\times g_1)\\&\le\operatorname{ess\,sup}|g_1|\cdot\mathbb{E}(\exp(\lambda(f_{n-1}-f_0)))\\&\le\exp(\lambda^2\alpha_n^2)\exp(\lambda^2\sum_{i=1}^{n-1}\alpha_i^2)\end{aligned}$$

(by the inductive hypothesis)

$$=\exp(\lambda^2\sum_{i=1}^n\alpha_i^2)$$

and the induction continues. **Q**

(b) Now take $n\ge 1$ and $\gamma\ge 0$. Set $\lambda=\gamma/2\sum_{i=1}^n\alpha_i^2$. Then

$$\begin{aligned}\Pr(f_n-f_0\ge\gamma)&=\Pr(\exp(\lambda(f_n-f_0))\ge e^{\lambda\gamma})\\&\le e^{-\lambda\gamma}\mathbb{E}(\exp(\lambda(f_n-f_0)))\le e^{-\lambda\gamma}\exp(\lambda^2\sum_{i=1}^n\alpha_i^2)\end{aligned}$$

(by (a) above)

$$=e^{-\lambda\gamma/2}=\exp(-\gamma^2/4\sum_{i=1}^n\alpha_i^2)$$

as claimed.

492H Theorem (Maurey 79) Let X be a non-empty finite set and G the group of all permutations of X with its discrete topology. For π, $\phi\in G$ set

$$\rho(\pi,\phi)=\frac{\#(\{x:x\in X,\pi(x)\ne\phi(x)\})}{\#(X)}.$$

Then ρ is a metric on G. Give G its Haar probability measure, and let $f:G\to\mathbb{R}$ be a 1-Lipschitz function. Then

$$\Pr(f-\mathbb{E}(f)\ge\gamma)\le\exp(-\frac{\gamma^2\#(X)}{16})$$

for any $\gamma\ge 0$.

proof We may suppose that $X=n=\{0,\dots,n-1\}$ where $n=\#(X)$. For $m\le n$, $p:m\to n$ set $A_p=\{\pi:\pi\in G,\ \pi\restriction m=p\}$, and let Σ_m be the subalgebra of $\mathcal{P}G$ generated by $\{A_p:p\in n^m\}$. Then

$$\{\emptyset, G\} = \Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \subseteq \Sigma_{n-1} = \Sigma_n = \mathcal{P}G;$$

for $m > n$ set $\Sigma_m = \mathcal{P}G$. For each m let f_m be the (unique) conditional expectation of f on Σ_m, so that

$$f_m(\pi) = \frac{1}{\#(A_p)} \sum_{\phi \in A_p} f(\phi)$$

whenever $\pi \in G$ and $p = \pi \restriction m$. Now we find that $|f_m(\pi) - f_{m-1}(\pi)| \le \frac{2}{n}$ for every $m \ge 1$ and $\pi \in G$. **P** If $m > n$ this is trivial. Otherwise, set $p = \pi \restriction m-1$ and $k = \pi(m-1)$. Set $J = p[m-1] = \{\pi(i) : i < m-1\}$, and for $j \in n \setminus J$ let $p_j = p^\frown{<}j{>}$ be that function from m to n which extends p and takes the value j at $m-1$; let α_j be the common value of $f_m(\phi)$ for $\phi \in A_{p_j}$, so that $f_m(\pi) = \alpha_k$. Now, for each $j \in n \setminus (J \cup \{k\})$, the function $\phi \mapsto (\overleftarrow{j\,k})\phi$ is a bijection from A_{p_k} to A_{p_j}, where $(\overleftarrow{j\,k}) \in G$ is the transposition which exchanges j and k. But this means that

$$\begin{aligned}
|\alpha_j - \alpha_k| &= \Bigl|\frac{1}{(n-m)!} \sum_{\phi \in A_{p_j}} f(\phi) - \frac{1}{(n-m)!} \sum_{\phi \in A_{p_k}} f(\phi)\Bigr| \\
&= \frac{1}{(n-m)!} \Bigl| \sum_{\phi \in A_{p_j}} f(\phi) - f((\overleftarrow{j\,k})\phi)\Bigr| \\
&\le \sup_{\phi \in A_{p_j}} \bigl|f(\phi) - f((\overleftarrow{j\,k})\phi)\bigr| \le \frac{2}{n}
\end{aligned}$$

because f is 1-Lipschitz and $\rho(\phi, (\overleftarrow{j\,k})\phi) = \frac{2}{n}$ for every ϕ. And this is true for every $j \in n \setminus (J \cup \{k\})$.

Accordingly

$$\begin{aligned}
|f_{m-1}(\pi) - f_m(\pi)| &= \Bigl|\frac{1}{(n-m+1)!} \sum_{\phi \in A_p} f(\phi) - \alpha_k\Bigr| = \Bigl|\frac{1}{n-m+1} \sum_{j \in n \setminus J} \alpha_j - \alpha_k\Bigr| \\
&\le \frac{1}{n-m+1} \sum_{j \in n \setminus J} |\alpha_j - \alpha_k| \le \frac{1}{n-m+1} \sum_{j \in n \setminus J} \frac{2}{n} = \frac{2}{n},
\end{aligned}$$

as claimed. **Q**

(b) Now observe that $f = f_{n-1}$ and that f_0 is the constant function with value $\mathbb{E}(f)$, so that

$$\Pr(f - \mathbb{E}(f) \ge \gamma) = \Pr(f_{n-1} - f_0 \ge \gamma) \le \exp(-\gamma^2/4\sum_{i=1}^{n-1}(\frac{2}{n})^2)$$

(492G)

$$\le \exp(-\frac{n\gamma^2}{16}),$$

which is what we were seeking to prove.

492I Corollary Let X be a non-empty finite set, with $\#(X) = n$, and G the group of all permutations of X. Let μ be the Haar probability measure of G when given its discrete topology. Suppose that $A \subseteq G$ and $\mu A \ge \frac{1}{2}$. Then

$$\mu\{\pi : \pi \in G, \exists\, \phi \in A, \#(\{x : x \in X, \pi(x) \ne \phi(x)\}) \le k\} \ge 1 - \exp(-\frac{k^2}{64n})$$

for every $k \le n$.

proof If $\exp(-\frac{k^2}{64n}) \ge \frac{1}{2}$, this is trivial, since the left-hand-side of the inequality is surely at least $\frac{1}{2}$. Otherwise, set $g(\pi) = \frac{1}{n} \min_{\phi \in A} \#(\{x : x \in X, \pi(x) \ne \phi(x)\})$ for $\pi \in G$, so that g is 1-Lipschitz for the metric ρ of 492H. Applying 492H to $f = -g$, we see that

$$\Pr(\mathbb{E}(g) - g \ge \frac{k}{2n}) \le \exp(-\frac{k^2}{64n}) < \frac{1}{2},$$

and there must be some $\pi \in A$ such that $\mathbb{E}(g) - g(\pi) < \frac{k}{2n}$, so that $\mathbb{E}(g) < \frac{k}{2n}$. This means that

$$\begin{aligned}\mu\{\pi : \pi \in G,\, \exists\, \phi \in A, \#(\{x : x \in X,\, \pi(x) \neq \phi(x)\}) \le k\} \\ = 1 - \mu\{\pi : \pi \in G,\, g(\pi) > \frac{k}{n}\} \\ \ge 1 - \Pr(g - \mathbb{E}(g) \ge \frac{k}{2n}) \ge 1 - \exp(-\frac{k^2}{64n}),\end{aligned}$$

applying 492H to g itself.

492X Basic exercises (a) Let (X, Σ, μ) be a probability space, and $\langle f_n \rangle_{n \in \mathbb{N}}$ a martingale on X. Suppose that $f_n \in \mathcal{L}^\infty(\mu)$ for every n, and that $\sigma = \sqrt{\sum_{n=1}^\infty \alpha_n^2}$ is finite and not zero, where $\alpha_n = \operatorname{ess\,sup} |f_n - f_{n-1}|$ for $n \ge 1$. Show that $f = \lim_{n\to\infty} f_n$ is defined a.e., and that $\Pr(f - f_0 \ge \gamma) \le \exp(-\gamma^2/4\sigma^2)$ for every $\gamma \ge 0$. (*Hint*: show first that $\|f_n\|_1 \le \|f_n\|_2 \le \sigma + \|f_0\|_2$ for every n, so that we can apply 275G.)

(b) Let (X, ρ) be a metric space and μ a topological probability measure on X. Suppose that γ, $\epsilon > 0$ are such that $\Pr(f - \mathbb{E}(f) \ge \gamma) \le \epsilon$ whenever $f : X \to [-\gamma, \gamma]$ is 1-Lipschitz. Show that if $\mu F \ge \frac{1}{2}$ then $\mu\{x : \rho(x, F) \ge 2\gamma\} \le \epsilon$.

(c) Let (X, ρ) be a metric space and μ a topological probability measure on X. Suppose that γ, $\epsilon > 0$ are such that $\mu\{x : \rho(x, F) > \gamma\} \le \epsilon$ whenever $\mu F \ge \frac{1}{2}$. Show that if $f : X \to [-1, 1]$ is a 1-Lipschitz function then $\Pr(f - \mathbb{E}(f) > 2\gamma + 2\epsilon) \le \epsilon$.

(d) Use 492G to show that if $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i<n}$ is a non-empty finite family of probability spaces with product (X, Λ, λ), and X is given its normalized Hamming metric, and $f \in \mathcal{L}^\infty(\lambda)$ is 1-Lipschitz, then $\Pr(f - \mathbb{E}(f) \ge \gamma) \le e^{-n\gamma^2/4}$ for every $\gamma \ge 0$. (*Hint*: if $\Sigma_k \subseteq \Lambda$ is the σ-algebra of subsets of X determined by coordinates in k, and f_k is a conditional expectation of f on Σ_k, then $\operatorname{ess\,sup} |f_{k+1} - f_k| \le \frac{1}{n}$.)

492 Notes and comments In metric spaces, we can say that a probability measure is 'concentrated' if every Lipschitz function f is almost constant in the sense that, for some α, the sets $\{x : |f(x) - \alpha| \ge \gamma\}$ have small measure. What is astonishing is that this does not mean that the measure itself is concentrated on a small set. In 492H, the measure is the Haar probability measure, spread as evenly as it well could be. Of course, when I say that $\{x : |f(x) - \alpha| \ge \gamma\}$ has 'small' measure, I have to let some other parameter – in 492H, the size of X – vary, while γ itself is fixed. Also the shapes of the formulae depend on which normalizations we choose (observe the effect of moving from ρ to σ in the proof of 492D). But the value of 492H is that it gives a strong bound which is independent of the particular function f, provided that it is 1-Lipschitz. This kind of concentration of measure can be described either in terms of the variation of Lipschitz functions from their means or in terms of the measures of neighbourhoods of sets of measure $\frac{1}{2}$ (492Xb-492Xc). The latter, in a more abstract context, is what is described by the concentration functions of measures on uniform spaces; there is an example of this in 493C.

The martingale method can be used to prove a version of 492E (492Xd). The method of 492D gives a better exponent ($e^{-n\gamma^2}$ in place of $e^{-n\gamma^2/4}$) and also information of a slightly different kind, in that it can be applied directly to sets W of small measure, at least provided that $\gamma > \frac{1}{\sqrt{n}}$ in 492E. We also need a little more measure theory here, since sets which are measured by product measures can be geometrically highly irregular, and our Lipschitz functions $x \mapsto \rho(x, W)$ need not be measurable.

In the proof of 492G we have an interesting application of the idea of 'martingale'. The inequality here is quite different from the standard martingale inequalities like 275D or 275F or 275Yd-275Ye. It gives a very strong inequality concerning the difference $f_n - f_0$, at the cost of correspondingly strong hypotheses on the differences $f_i - f_{i-1}$; but since we need control of $\sum_i \operatorname{ess\,sup} |f_i - f_{i-1}|^2$, not of $\sum_i \operatorname{ess\,sup} |f_i - f_{i-1}|$, there is scope for applications like 492H. What the inequality tells us is that most of the time the differences $f_i - f_{i-1}$ cancel out, just as in the Central Limit Theorem, and that once again we have a vaguely Gaussian sum $f_n - f_0$.

Concentration of measure, in many forms, has been studied intensively in the context of the geometry of normed spaces, as in Milman & Schechtman 86, from which 492F-492I are taken.

493 Extremely amenable groups

A natural variation on the idea of 'amenable group' (§449) is the concept of 'extremely amenable' group (493A). Expectedly, most of the ideas of 449C-449E can be applied to extremely amenable groups (493B); unexpectedly, we find not only that there are interesting extremely amenable groups, but that we need some of the central ideas of measure theory to study them. I give a criterion for extreme amenability of a group in terms of the existence of suitably concentrated measures (493C) before turning to three examples: measure algebras under symmetric difference (493D), L^0 spaces (493E) and isometry groups of spheres in infinite-dimensional Hilbert spaces (493G).

493A Definition Let G be a topological group. Then G is **extremely amenable** or has the **fixed point on compacta property** if every continuous action of G on a compact Hausdorff space has a fixed point.

493B Proposition (a) Let G and H be topological groups such that there is a continuous surjective homomorphism from G onto H. If G is extremely amenable, so is H.

(b) Let G be a topological group and suppose that there is a dense subset A of G such that every finite subset of A is included in an extremely amenable subgroup of G. Then G is extremely amenable.

(c) Let G be a topological group with an extremely amenable normal subgroup H such that G/H is extremely amenable. Then G is extremely amenable.

(d) The product of any family of extremely amenable topological groups is extremely amenable.

(e) Let G be a topological group. Then G is extremely amenable iff there is a point in the greatest ambit Z of G (definition: 449D) which is fixed by the action of G on Z.

(f) Let G be an extremely amenable topological group. Then every dense subgroup of G is extremely amenable.

proof We can use the same arguments as in 449C-449F, with some simplifications.

(a) As in 449Ca, let $\phi : G \to H$ be a continuous surjective homomorphism, X a non-empty compact Hausdorff space and $\bullet : H \times X \to X$ a continuous action. Let $\bullet_1$ be the continuous action of G on X defined by the formula $a\bullet_1 x = \phi(a)\bullet x$. Then any fixed point for $\bullet_1$ is a fixed point for $\bullet$.

(b) Let X be a non-empty compact Hausdorff space and $\bullet$ a continuous action of G on X. For $I \in [A]^{<\omega}$ let H_I be an extremely amenable subgroup of G including I. The restriction of the action to $H_I \times X$ is a continuous action of H_I on X, so

$$\{x : a\bullet x = x \text{ for every } a \in I\} \supseteq \{x : a\bullet x = x \text{ for every } a \in H_I\}$$

is closed and non-empty. Because X is compact, there is an $x \in X$ such that $a\bullet x = x$ for every $a \in A$. Now $\{a : a\bullet x = x\}$ includes the dense set A, so is the whole of G, and x is fixed under the action of G. As X and $\bullet$ are arbitrary, G is extremely amenable.

(c) Let X be a compact Hausdorff space and $\bullet$ a continuous action of G on X. Set $Q = \{x : x \in X, a\bullet x = x$ for every $a \in H\}$; then Q is a closed subset of X and, because H is extremely amenable, is non-empty. Next, $b\bullet x \in Q$ for every $x \in Q$ and $b \in G$. **P** If $a \in H$, then $b^{-1}ab \in H$ and

$$a\bullet(b\bullet x) = (ab)\bullet x = (bb^{-1}ab)\bullet x = b\bullet((b^{-1}ab)\bullet x) = b\bullet x.$$

As a is arbitrary, $b\bullet x \in Q$. **Q** Accordingly we have a continuous action of G on the compact Hausdorff space Q.

If $b \in G$, $a \in H$ and $x \in Q$, then $(ba)\bullet x = b\bullet x$. So we have an action of G/H on Q defined by saying that $b^\bullet\bullet x = b\bullet x$ for every $b \in G$ and $x \in Q$, and this is continuous for the quotient topology on G/H, as in the proof of 449Cc. Because G/H is extremely amenable, there is a point x of Q which is fixed under the action of G/H. So $b\bullet x = b^\bullet\bullet x = x$ for every $b \in G$, and x is fixed under the action of G. As X and $\bullet$ are arbitrary, G is extremely amenable.

(d) By (c), the product of two extremely amenable topological groups is extremely amenable, since each can be regarded as a normal subgroup of the product. It follows that the product of finitely many extremely amenable topological groups is extremely amenable. Now let $\langle G_i\rangle_{i\in I}$ be any family of extremely amenable topological groups with product G. For finite $J \subseteq I$ let H_J be the set of those $a \in G$ such that $a(i)$ is the identity in G_i for every $i \in I\setminus J$. Then H_J is isomorphic (as topological group) to $\prod_{i\in J} G_i$, so is extremely amenable. Since $\{H_J : J \in [I]^{<\omega}\}$ is an upwards-directed family of subgroups of G with dense union, (b) tells us that G is extremely amenable.

(e) Repeat the arguments of 449E(i)$\Leftrightarrow$(ii), noting that if $z_0 \in Z$ is a fixed point under the action of G on Z, then its images under the canonical maps ϕ of 449Dd will be fixed for other actions.

(f) Again, the idea is to repeat the argument of 449F(a-ii). As there, let H be a dense subgroup of G, U the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity, and V the space of bounded real-valued functions on H which are uniformly continuous for the right uniformity. As in 449F(a-ii), we have an extension operator $T : V \to U$ defined by saying that Tg is the unique continuous extension of g for every $g \in V$; and $b \bullet_l Tg = T(b \bullet_l g)$ for every $b \in H$ and $g \in V$. Now T is a Riesz homomorphism. So if $z \in Z$ is fixed by the action of G, that is, $z(a \bullet_l f) = z(f)$ for every $a \in G$ and $f \in U$, then $zT : V \to \mathbb{R}$ is a Riesz homomorphism, with $z(T\chi H) = 1$, and $z(T(b \bullet_l g)) = z(b \bullet_l Tg) = z(Tg)$ whenever $g \in V$ and $b \in H$. Thus zT is a fixed point of the greatest ambit of H, and H is extremely amenable.

493C Theorem Let G be a topological group and $\mathcal{B}$ its Borel σ-algebra. Suppose that for every $\epsilon > 0$, open neighbourhood V of the identity of G, finite set $I \subseteq G$ and finite family $\mathcal{E}$ of zero sets in G there is a finitely additive functional $\nu : \mathcal{B} \to [0,1]$ such that $\nu G = 1$ and

(i) $\nu(VF) \ge 1 - \epsilon$ whenever $F \in \mathcal{E}$ and $\nu F \ge \frac{1}{2}$,

(ii) for every $a \in I$ there is a $b \in aV$ such that $|\nu(bF) - \nu F| \le \epsilon$ for every $F \in \mathcal{E}$.

Then G is extremely amenable.

proof (a) Write P for the set of finitely additive functionals $\nu : \mathcal{B} \to [0,1]$ such that $\nu G = 1$. If V is an open neighbourhood of the identity e of G, $\epsilon > 0$, $I \in [D]^{<\omega}$ and $\mathcal{E}$ is a finite family of zero sets in G, let $A(V, \epsilon, I, \mathcal{E})$ be the set of those $\nu \in P$ satisfying (i) and (ii) above. Our hypothesis is that none of these sets $A(V, \epsilon, I, \mathcal{E})$ are empty; since $A(V, \epsilon, I, \mathcal{E}) \subseteq A(V', \epsilon', I', \mathcal{E}')$ whenever $V \subseteq V'$, $\epsilon \le \epsilon'$, $I \supseteq I'$ and $\mathcal{E} \supseteq \mathcal{E}'$, there is an ultrafilter $\mathcal{F}$ on P containing all these sets.

Let U be the space of bounded real-valued functionals on G which are uniformly continuous for the right uniformity on G. If we identify $L^\infty(\mathcal{B})$ with the space of bounded Borel measurable real-valued functions on G (363H), then U is a Riesz subspace of $L^\infty(\mathcal{B})$. For each $\nu \in P$ we have a positive linear functional $\int d\nu : L^\infty(\mathcal{B}) \to \mathbb{R}$ (363L). For $f \in U$ set $z(f) = \lim_{\nu \to \mathcal{F}} \int f d\nu$.

(b) $z : U \to \mathbb{R}$ is a Riesz homomorphism, and $z(\chi G) = 1$. **P** Of course z is a positive linear functional taking the value 1 at χG, just because all the integrals $\int d\nu$ are. Now suppose that f_0, $f_1 \in U$ and $f_0 \wedge f_1 = 0$. Take any $\epsilon > 0$. Then there is an open neighbourhood V of e such that $|f_i(x) - f_i(y)| \le \epsilon$ whenever $xy^{-1} \in V$ and $i \in \{0,1\}$. Set $F_i = \{x : f_i(x) = 0\}$, $E_i = VF_i$ for each i. Then $F_0 \cup F_1 = X$, so $\nu F_0 + \nu F_1 \ge 1$ for every $\nu \in P$, and there is a $j \in \{0,1\}$ such that $A_0 = \{\nu : \nu F_j \ge \frac{1}{2}\} \in \mathcal{F}$. Next,

$$A_1 = \{\nu\text{: if } \nu F_j \ge \frac{1}{2} \text{ then } \nu(VF_j) \ge 1 - \epsilon\}$$

belongs to $\mathcal{F}$. Accordingly $\lim_{\nu \to \mathcal{F}} \nu E_j \ge 1 - \epsilon$. As $f_j(x) \le \epsilon$ for every $x \in E_j$,

$$z(f_j) = \lim_{\nu \to \mathcal{F}} \int f_j d\nu \le \epsilon(1 + \|f_j\|_\infty).$$

This shows that $\min(z(f_0), z(f_1)) \le \epsilon(1 + \|f_1\|_\infty + \|f_2\|_\infty)$. As ϵ is arbitrary, $\min(z(f_0), z(f_1)) = 0$; as f_0 and f_1 are arbitrary, z is a Riesz homomorphism (352G(iv)). **Q**

Thus z belongs to the greatest ambit Z of G.

(c) $\lim_{\nu \to \mathcal{F}} \int (a^{-1} \bullet_l f) d\nu = \lim_{\nu \to \mathcal{F}} \int f d\nu$ for every non-negative $f \in U$ and $a \in G$. **P** Take any $\epsilon > 0$. Let V be an open neighbourhood of e such that $|f(x) - f(y)| \le \epsilon$ whenever $x \in Vy$; then

$$\|a^{-1} \bullet_l f - b^{-1} \bullet_l f\|_\infty = \sup_{x \in G} |f(ax) - f(bx)| \le \epsilon$$

whenever $b \in Va$. For $n \in \mathbb{N}$ set $F_n = \{x : x \in G,\ f(x) \ge n\epsilon\}$. Set $m = \lfloor \frac{1}{\epsilon} \|f\|_\infty \rfloor$, so that $F_n = \emptyset$ for every $n > m$. Set $\delta = \frac{1}{m+1}$,

$$\begin{aligned} A &= \{\nu :\ \text{there is a } b \in Va \text{ such that } |\nu(b^{-1}F_n) - \nu F_n| \le \delta \text{ for every } n \le m\} \\ &= \{\nu :\ \text{there is a } c \in a^{-1}V^{-1} \text{ such that } |\nu(cF_n) - \nu F_n| \le \delta \text{ for every } n \le m\} \in \mathcal{F}. \end{aligned}$$

Take any $\nu \in A$ and $b \in Va$ such that $|\nu(b^{-1}F_n) - \nu F_n| \le \delta$ for every $n \le m$. Then, setting $g = \sum_{n=1}^m \epsilon \chi F_n$, we have $g \in L^\infty(\mathcal{B})$ and $g \le f \le g + \epsilon \chi G$. Since $b^{-1} \bullet_l g$ (in the language of 4A5Cc) is just $\sum_{n=1}^m \epsilon \chi(b^{-1}F_n)$, we have

$$|\int a^{-1} \bullet_l f \, d\nu - \int f \, d\nu| \le \epsilon + |\int b^{-1} \bullet_l f \, d\nu - \int f \, d\nu| \le 3\epsilon + |\int b^{-1} \bullet_l g \, d\nu - \int g \, d\nu|$$

(because $\|b^{-1} \bullet_l g - b^{-1} \bullet_l f\|_\infty = \|g - f\|_\infty \le \epsilon$)

$$\leq 3\epsilon + \epsilon\sum_{n=1}^{m}|\nu F_n - \nu(b^{-1}F_n)| \leq 3\epsilon + m\epsilon\delta \leq 4\epsilon.$$

As $A \in \mathcal{F}$,

$$|\lim_{\nu\to\mathcal{F}} \textstyle\fint a^{-1}\bullet_l f d\nu - \fint f d\nu| \leq 5\epsilon;$$

as ϵ is arbitrary, $\lim_{\nu\to\mathcal{F}}\fint a^{-1}\bullet_l f\,d\nu = \lim_{\nu\to\mathcal{F}}\fint f d\nu$. **Q**

(d) Thus, for any $a \in G$,

$$(a\bullet z)(f) = z(a^{-1}\bullet_l f) = \lim_{\nu\to\mathcal{F}} \textstyle\fint a^{-1}\bullet_l f\,d\nu = \lim_{\nu\to\mathcal{F}} \fint f\,d\nu = z(f)$$

for every non-negative $f \in U$ and therefore for every $f \in U$, and $a\bullet z = z$. So $z \in Z$ is fixed under the action of G on Z; by 493Ba, this is enough to ensure that G is extremely amenable.

493D I turn now to examples of extremely amenable groups. The first three are groups which we have already studied for other reasons.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be an atomless measure algebra. Then $\mathfrak{A}$, with the group operation $\triangle$ and the measure-algebra topology (definition: 323A), is an extremely amenable group.

proof (a) To begin with let us suppose that $(\mathfrak{A}, \bar\mu)$ is a probability algebra; write σ for the measure metric of $\mathfrak{A}$, so that $\sigma(a, a') = \bar\mu(a \triangle a')$ for a, $a' \in \mathfrak{A}$. I seek to apply 493C.

(i) Let V be an open neighbourhood of 0 in $\mathfrak{A}$, $\epsilon \in \mathopen]0, 3\mathclose[$, $I \in [\mathfrak{A}]^{<\omega}$ and $\mathcal{E}$ a finite family of zero sets in $\mathfrak{A}$. Let $\gamma > 0$ be such that $V \supseteq \{a : \bar\mu a \leq 2\gamma\}$. Let $\mathfrak{B}_0$ be the finite subalgebra of $\mathfrak{A}$ generated by I and B_0 the set of atoms in $\mathfrak{B}_0$. Set

$$t = \frac{1}{\gamma}\ln\frac{3}{\epsilon}, \quad n = \lceil\max(t^2, \sup_{b\in B_0}\frac{1}{\bar\mu b})\rceil.$$

Because $\mathfrak{A}$ is atomless, we can split any member of $\mathfrak{A}\setminus\{0\}$ into two parts of equal measure (331C); if, starting from the disjoint set B_0, we successively split the largest elements until we have a disjoint set B with just n elements, then we shall have $\bar\mu b \leq \frac{2}{n}$ for every $b \in B$. We have a natural identification between $\{0,1\}^B$ and the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by B, matching $x \in \{0,1\}^B$ with $f(x) = \sup\{b : b \in B,\ x(b) = 1\}$. Writing ρ for the normalized Hamming metric on $\{0,1\}^B$ (492D), we have $\sigma(f(x), f(y)) \leq 2\rho(x, y)$ for all x, $y \in \{0,1\}^B$. **P** Set $J = \{b : b \in B,\ x(b) \neq y(b)\}$, so that

$$\sigma(f(x), f(y)) = \bar\mu(f(x) \triangle f(y)) = \bar\mu(\sup J) = \textstyle\sum_{b\in J}\bar\mu b \leq \frac{2}{n}\#(J) = 2\rho(x, y). \quad \mathbf{Q}$$

(ii) Let ν_B be the usual measure on $\{0,1\}^B$ and set $\lambda E = \nu_B f^{-1}[E]$ for every Borel set $E \subseteq \mathfrak{A}$. Then λ is a probability measure. Note that $f : \{0,1\}^B \to \mathfrak{B}$ is a group isomorphism if we give $\{0,1\}^B$ the addition $+_2$ corresponding to its identification with $\mathbb{Z}_2^B$, and $\mathfrak{B}$ the operation $\triangle$. Because ν_B is translation-invariant for $+_2$, its copy, the subspace measure $\lambda_{\mathfrak{B}}$ on the λ-conegligible finite set $\mathfrak{B}$, is translation-invariant for $\triangle$. But this means that $\lambda\{b \triangle d : d \in F\} = \lambda F$ whenever $b \in \mathfrak{B}$ and $F \subseteq \mathfrak{B}$, and therefore that $\lambda\{b \triangle d : d \in F\} = \lambda F$ whenever $b \in I$ and $F \in \mathcal{E}$. This shows that λ satisfies condition (ii) of 493C.

(iii) Now suppose that $F \in \mathcal{E}$ and that $\lambda F \geq \frac{1}{2}$. Set $W = f^{-1}[F]$, so that $\nu_B W \geq \frac{1}{2}$. By 492D,

$$\int e^{t\rho(x,W)}\nu_B(dx) \leq 2e^{t^2/4n} \leq 2e^{1/4} \leq 3,$$

so

$$\nu_B\{x : \rho(x, W) \geq \gamma\} = \nu_B\{x : t\rho(x, W) \geq \ln\tfrac{3}{\epsilon}\} = \nu_B\{x : e^{t\rho(x,W)} \geq \tfrac{3}{\epsilon}\} \leq \epsilon.$$

Accordingly

$$\begin{aligned}\lambda\{a \triangle d : a \in V,\ d \in F\} &\geq \lambda\{a : \sigma(a, F) \leq 2\gamma\} = \nu_B\{x : \sigma(f(x), F) \leq 2\gamma\}\\ &\geq \nu_B\{x : \sigma(f(x), f[W]) \leq 2\gamma\} \geq \nu_B\{x : \rho(x, W) \leq \gamma\}\end{aligned}$$

(because f is 2-Lipschitz)

$$\geq 1 - \epsilon.$$

So λ also satisfies (i) of 493C.

(iv) Since V, ϵ, I and $\mathcal{E}$ are arbitrary, 493C tells us that $\mathfrak{A}$ is an extremely amenable group, at least when $(\mathfrak{A}, \bar{\mu})$ is an atomless probability algebra.

(b) For the general case, observe first that if $(\mathfrak{A}, \bar{\mu})$ is atomless and totally finite then $(\mathfrak{A}, \triangle)$ is an extremely amenable group; this is trivial if $\mathfrak{A} = \{0\}$, and otherwise there is a probability measure on $\mathfrak{A}$ which induces the same topology, so we can apply (a). For a general atomless measure algebra $(\mathfrak{A}, \bar{\mu})$, set $\mathfrak{A}^f = \{c : c \in \mathfrak{A}, \bar{\mu}c < \infty\}$ and for $c \in \mathfrak{A}^f$ let $\mathfrak{A}_c$ be the principal ideal generated by c. Then $\mathfrak{A}_c$ is a subgroup of $\mathfrak{A}$ and the measure-algebra topology of $\mathfrak{A}_c$, regarded as a measure algebra in itself, is the subspace topology induced by the measure-algebra topology of $\mathfrak{A}$. So $\{\mathfrak{A}_c : c \in \mathfrak{A}^f\}$ is an upwards-directed family of extremely amenable subgroups of $\mathfrak{A}$ with union which is dense in $\mathfrak{A}$, so $\mathfrak{A}$ itself is extremely amenable, by 493Bb. This completes the proof.

493E Theorem (PESTOV 02) Let (X, Σ, μ) be an atomless measure space. Then $L^0(\mu)$, with the group operation $+$ and the topology of convergence in measure, is an extremely amenable group.

proof It will simplify some of the formulae if we move at once to the space $L^0(\mathfrak{A})$, where $(\mathfrak{A}, \bar{\mu})$ is the measure algebra of (X, Σ, μ); for the identification of $L^0(\mathfrak{A})$ with $L^0(\mu)$ see 364Ic; for a note on convergence in measure in $L^0(\mathfrak{A})$, see 367L; of course $\mathfrak{A}$ is atomless if (X, Σ, μ) is (322Bg).

(a) I seek to prove that $S(\mathfrak{A})$, with the group operation of addition and the topology of convergence in measure, is extremely amenable. As in 493D, I start with the case in which $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, and use 493C.

(i) Take an open neighbourhood V of 0 in $S(\mathfrak{A})$, an $\epsilon \in]0, 3[$, a finite set $I \subseteq S(\mathfrak{A})$ and a finite family $\mathcal{E}$ of zero sets in $S(\mathfrak{A})$. Let $\gamma > 0$ be such that $u \in V$ whenever $u \in S(\mathfrak{A})$ and $\bar{\mu}[\![u \neq 0]\!] \leq 2\gamma$. Let $\mathfrak{B}_0$ be a finite subalgebra of $\mathfrak{A}$ such that I is included in the linear subspace of $S(\mathfrak{A})$ generated by $\{\chi b : b \in \mathfrak{B}_0\}$, and B_0 the set of atoms of $\mathfrak{B}_0$. As in the proof of 493D, set

$$t = \frac{1}{\gamma} \ln \frac{3}{\epsilon}, \quad n = \lceil \max(t^2, \sup_{b \in B_0} \frac{1}{\bar{\mu}b}) \rceil,$$

and let $B \subseteq \mathfrak{A} \setminus \{0\}$ be a partition of unity with n elements, refining B_0, such that $\bar{\mu}b \leq \frac{2}{n}$ for every $b \in B$. We have a natural identification between $\mathbb{R}^B$ and the linear subspace of $S(\mathfrak{A})$ generated by $\{\chi b : b \in B\}$, matching $x \in \mathbb{R}^B$ with $f(x) = \sum_{b \in B} x(b)\chi b$, which is continuous if $\mathbb{R}^B$ is given its product topology. Writing ρ for the normalized Hamming metric on $\mathbb{R}^B$, we have

$$\bar{\mu}[\![f(x) \neq f(y)]\!] = \sum_{x(b) \neq y(b)} \bar{\mu}b \leq \frac{2}{n}\#(\{b : x(b) \neq y(b)\}) = 2\rho(x, y)$$

for all x, $y \in \mathbb{R}^B$.

(ii) Set $\beta = \sup_{v \in I} \|v\|_\infty$ (if $I = \emptyset$, take $\beta = 0$). Let $M > 0$ be so large that $(M + \beta)^n \leq (1 + \frac{1}{2}\epsilon)M^n$. On $\mathbb{R}$, write μ_L for Lebesgue measure and μ'_L for the indefinite-integral measure over μ_L defined by the function $\frac{1}{2M}\chi[-M, M]$, so that $\mu'_L E = \frac{1}{2M}\mu_L(E \cap [-M, M])$ whenever $E \subseteq \mathbb{R}$ and $E \cap [-M, M]$ is Lebesgue measurable. Let λ, λ' be the product measures on $\mathbb{R}^B$ defined from μ_L and μ'_L. Let ν be the Borel probability measure on $S(\mathfrak{A})$ defined by setting $\nu F = \lambda' f^{-1}[F]$ for every Borel set $F \subseteq S(\mathfrak{A})$.

Now $|\nu(v + F) - \nu F| \leq \epsilon$ for every $v \in I$ and Borel set $F \subseteq S(\mathfrak{A})$. **P** Because B refines B_0, v is expressible as $f(y)$ for some $y \in \mathbb{R}^B$; because $\|v\|_\infty \leq \beta$, $|y(b)| \leq \beta$ for every $b \in B$. Because $f : \mathbb{R}^B \to S(\mathfrak{A})$ is linear, $f^{-1}[v + F] = y + f^{-1}[F]$. Now

$$\begin{aligned}|\nu(v + F) - \nu F| &= |\lambda' f^{-1}[v + F] - \lambda' f^{-1}[F]| \\ &= \tfrac{1}{(2M)^n} \big|\lambda(f^{-1}[v + F] \cap [-M, M]^n) - \lambda(f^{-1}[F] \cap [-M, M]^n)\big|\end{aligned}$$

(use 253I, or otherwise)

$$
\begin{aligned}
&= \frac{1}{(2M)^n}\left|\lambda((y+f^{-1}[F])\cap[-M,M]^n)-\lambda(f^{-1}[F]\cap[-M,M]^n)\right| \\
&= \frac{1}{(2M)^n}\left|\lambda(f^{-1}[F]\cap([-M,M]^n-y))-\lambda(f^{-1}[F]\cap[-M,M]^n)\right| \\
&\le \frac{1}{(2M)^n}\lambda(([-M,M]^n-y)\triangle[-M,M]^n) \\
&= \frac{2}{(2M)^n}\lambda(([-M,M]^n-y)\setminus[-M,M]^n) \\
&\le \frac{2}{(2M)^n}\lambda(([-M-\beta,M+\beta]^n\setminus[-M,M]^n) \\
&= \frac{2}{M^n}((M+\beta)^n-M^n)\le\epsilon. \text{ } \mathbf{Q}
\end{aligned}
$$

So ν satisfies (ii) of 493C.

(iii) Now suppose that $F\in\mathcal{E}$ and $\nu F\ge\frac{1}{2}$. Set $W=f^{-1}[F]$, so that $\lambda'W\ge\frac{1}{2}$. Just as in the proof of 493D, $\int e^{t\rho(x,W)}\lambda'(dx)\le 2e^{t^2/4n}\le 3$, so

$$\lambda'\{x:\rho(x,W)\ge\gamma\}=\lambda'\{x:e^{t\rho(x,W)}\ge\tfrac{3}{\epsilon}\}\le\epsilon,$$

and

$$
\begin{aligned}
\nu\{v+u:v\in V,\,u\in F\}&\ge\nu\{w:\exists\,u\in F,\,\bar\mu[\![u\ne w]\!]\le 2\gamma\}\\
&\ge\lambda'\{x:\rho(x,W)\le\gamma\}\ge 1-\epsilon.
\end{aligned}
$$

So ν also satisfies (i) of 493C.

(iv) Since V, ϵ, I and $\mathcal{E}$ are arbitrary, 493C tells us that $S(\mathfrak{A})$ is an extremely amenable group, at least when $(\mathfrak{A},\bar\mu)$ is an atomless probability algebra.

(b) The rest of the argument is straightforward, as in 493D. First, $S(\mathfrak{A})$ is extremely amenable whenever $(\mathfrak{A},\bar\mu)$ is an atomless totally finite measure algebra. For a general atomless measure algebra $(\mathfrak{A},\bar\mu)$, set $\mathfrak{A}^f=\{c:\bar\mu c<\infty\}$. For each $c\in\mathfrak{A}^f$, let $\mathfrak{A}_c$ be the corresponding principal ideal of $\mathfrak{A}$. Then we can identify $S(\mathfrak{A}_c)$, as topological group, with the linear subspace of $L^0(\mathfrak{A})$ generated by $\{\chi a:a\in\mathfrak{A}_c\}$, and it is extremely amenable. Since $\{S(\mathfrak{A}_c):c\in\mathfrak{A}^f\}$ is an upwards-directed family of extremely amenable subgroups of $L^0(\mathfrak{A})$ with dense union in $L^0(\mathfrak{A})$, $L^0(\mathfrak{A})$ itself is extremely amenable, by 493Bb, as before.

493F Returning to the ideas of §476, we find another remarkable example of an extremely amenable topological group. I recall the notation of 476I. Let X be a (real) inner product space. S_X will be the unit sphere $\{x:x\in X,\,\|x\|=1\}$. Let H_X be the isometry group of S_X with its topology of pointwise convergence. When X is finite-dimensional, it is isomorphic, as inner product space, to $\mathbb{R}^r$, where $r=\dim X$. In this case S_X is compact, so (if $r\ge 1$) has a unique H_X-invariant Radon probability measure ν_X, which is strictly positive, and is a multiple of $(r-1)$-dimensional Hausdorff measure; also H_X is compact (441Gb), so has a unique Haar probability measure λ_X.

Lemma For any $m\in\mathbb{N}$ and any $\epsilon>0$, there is an $r(m,\epsilon)\ge 1$ such that whenever X is a finite-dimensional inner product space over $\mathbb{R}$ of dimension at least $r(m,\epsilon)$, $x_0,\dots,x_{m-1}\in S_X$, Q_1, $Q_2\subseteq H_X$ are closed sets and $\min(\lambda_XQ_1,\lambda_XQ_2)\ge\epsilon$, then there are $f_1\in Q_1$, $f_2\in Q_2$ such that $\|f_1(x_i)-f_2(x_i)\|\le\epsilon$ for every $i<m$.

proof Induce on m. For $m=0$, the result is trivial. For the inductive step to $m+1$, take $r(m+1,\epsilon)>r(m,\frac{1}{3}\epsilon)$ such that whenever $r(m+1,\epsilon)\le\dim X<\omega$ and A_1, $A_2\subseteq S_X$ and $\min(\nu_X^*A_1,\nu_X^*A_2)\ge\frac{1}{2}\epsilon$ then there are $x\in A_1$, $y\in A_2$ such that $\|x-y\|\le\frac{1}{3}\epsilon$; this is possible by 476L.

Now take any inner product space X over $\mathbb{R}$ of finite dimension $r\ge r(m+1,\epsilon)$, closed sets Q_1, $Q_2\subseteq H_X$ such that $\min(\lambda_XQ_1,\lambda_XQ_2)\ge\epsilon$, and $x_0,\dots,x_m\in S_X$. Let Y be the $(r-1)$-dimensional subspace $\{x:x\in X,\,(x|x_m)=0\}$, so that $\dim Y\ge r(m,\frac{1}{3}\epsilon)$, and for $i<m$ let $y_i\in Y$ be a unit vector such that x_i is a linear combination of y_i and x_m. Set $H'_Y=\{f:f\in H_X,\,f(x_m)=x_m\}$; then $f\mapsto f\restriction S_Y$ is a topological group isomorphism from H'_Y to H_Y. $\mathbf{P}$ (i) If $f\in H'_Y$ and $x\in S_X$, then

$$x\in S_Y\iff(x|x_m)=0\iff(f(x)|f(x_m))=0\iff f(x)\in S_Y,$$

so $f\restriction S_Y$ is a permutation of S_Y and belongs to H_Y. (ii) If $g \in H_Y$, we can define $f \in H'_Y$ by setting $f(\alpha x+\beta x_m) = \alpha g(x) + \beta x_m$ whenever $x \in S_Y$ and $\alpha^2+\beta^2 = 1$, and $f\restriction S_Y = g$. (iii) Note that H'_Y is a closed subgroup of H_X, so in itself is a compact Hausdorff topological group. Since the map $f \mapsto f\restriction S_Y : H'_Y \to H_Y$ is a bijective continuous group homomorphism between compact Hausdorff topological groups, it is a topological group isomorphism. **Q**

Let λ'_Y be the Haar probability measure of H'_Y. Then $\lambda_X Q_1 = \int \lambda'_Y(H'_Y \cap f^{-1}Q_1)\lambda_X(df)$ (443Ue), so $\lambda_X Q'_1 \ge \frac{1}{2}\epsilon$, where $Q'_1 = \{f : \lambda'_Y(H'_Y \cap f^{-1}Q_1) \ge \frac{1}{2}\epsilon\}$. Similarly, setting $Q'_2 = \{f : \lambda'_Y(H'_Y \cap f^{-1}Q_2) \ge \frac{1}{2}\epsilon\}$, $\lambda_X Q'_2 \ge \frac{1}{2}\epsilon$. Next, setting $\theta(f) = f(x_m)$ for $f \in H_X$, $\lambda_X\theta^{-1}$ is an H_X-invariant Radon probability measure on S_X (443Ub), so must be equal to ν_X. Accordingly

$$\nu_X(\theta[Q'_j]) = \lambda_X(\theta^{-1}[\theta[Q'_j]]) \ge \lambda_X Q'_j \ge \frac{1}{2}\epsilon$$

for both j.

We chose $r(m+1,\epsilon)$ so large that we can be sure that there are $z_1 \in \theta[Q'_1]$, $z_2 \in \theta[Q'_2]$ such that $\|z_1 - z_2\| \le \frac{1}{3}\epsilon$. Let $h_1 \in Q'_1$, $h_2 \in Q'_2$ be such that $h_1(x_m) = \theta(h_1) = z_1$ and $h_2(x_m) = z_2$. Let $h \in H_X$ be such that $h(z_1) = z_2$ and $\|h(x) - x\| \le \frac{1}{3}\epsilon$ for every $x \in S_X$ (4A4Jg). Set $\tilde h_2 = hh_1$, so that $\tilde h_2(x_m) = z_2$ and $\|h_1(x) - \tilde h_2(x)\| \le \frac{1}{3}\epsilon$ for every $x \in S_X$. Note that $\tilde h_2^{-1}h_2 \in H'_Y$, so that $\tilde h_2$ and h_2 belong to the same left coset of H'_Y, and

$$\lambda'_Y(H'_Y \cap \tilde h_2^{-1}Q_2) = \lambda'_Y(H'_Y \cap h_2^{-1}Q_2) \ge \frac{1}{2}\epsilon$$

by 443Qa.

At this point, recall that $\dim Y \ge r(m, \frac{1}{3}\epsilon)$, and that λ'_Y is a copy of λ_Y, the Haar probability measure on Y. So we have $g_1 \in H'_Y \cap h_1^{-1}Q_1$, $g_2 \in H'_Y \cap \tilde h_2^{-1}Q_2$ such that $\|g_1(y_i) - g_2(y_i)\| \le \frac{1}{3}\epsilon$ for every $i < m$. We have $f_1 = h_1g_1 \in Q_1$ and $f_2 = \tilde h_2 g_2 \in Q_2$. For any $i < m$,

$$\begin{aligned}\|f_1(y_i) - f_2(y_i)\| &\le \|h_1g_1(y_i) - h_1g_2(y_i)\| + \|h_1g_2(y_i) - \tilde h_2g_2(y_i)\|\\ &\le \|g_1(y_i) - g_2(y_i)\| + \frac{1}{3}\epsilon \le \frac{2}{3}\epsilon.\end{aligned}$$

Also $g_1(x_m) = g_2(x_m) = x_m$, so

$$\|f_1(x_m) - f_2(x_m)\| = \|h_1(x_m) - \tilde h_2(x_m)\| \le \frac{1}{3}\epsilon.$$

If $i < m$, then $x_i = (x_i|x_m)x_m + (x_i|y_i)y_i$, so $f_j(x_i) = (x_i|x_m)f_j(x_m) + (x_i|y_i)f_j(y_i)$ for both j (476J) and

$$\begin{aligned}\|f_1(x_i) - f_2(x_i)\| &\le \frac{1}{3}\epsilon|(x_i|x_m)| + \frac{2}{3}\epsilon|(x_i|y_i)|\\ &\le \sqrt{(\frac{1}{3}\epsilon)^2 + (\frac{2}{3}\epsilon)^2}\sqrt{(x_i|x_m)^2 + (x_i|y_i)^2} \le \epsilon.\end{aligned}$$

So f_1 and f_2 witness that the induction proceeds.

493G Theorem Let X be an infinite-dimensional inner product space over $\mathbb{R}$. Then the isometry group H_X of its unit sphere S_X, with its topology of pointwise convergence, is extremely amenable.

proof (a) Let $\mathcal{Y}$ be the family of finite-dimensional subspaces of X. For $Y \in \mathcal{Y}$, write $Y^\perp$ for the orthogonal complement of Y, so that $X = Y \oplus Y^\perp$ (4A4Jf). For $q \in H_Y$ define $\theta_Y(q) : S_X \to S_X$ by saying that $\theta_Y(q)(\alpha y+\beta z) = \alpha q(y) + \beta z$ whenever $y \in S_Y$, $z \in S_{Y^\perp}$ and $\alpha^2+\beta^2 = 1$. Then $\theta_Y : H_Y \to H_X$ is a injective group homomorphism. Also it is continuous, because $q \mapsto \alpha q(y) + \beta z$ is continuous for all relevant α, β, y and z.

If Y, $W \in \mathcal{Y}$ and $Y \subseteq W$ then $\theta_Y[H_Y] \subseteq \theta_W[H_W]$. **P** For any $q \in H_Y$ we can define $q' \in H_W$ by saying that $q'(\alpha y + \beta x) = \alpha q(y) + \beta x$ whenever $y \in S_Y$, $x \in S_{W\cap Y^\perp}$ and $\alpha^2+\beta^2 = 1$. Now $\theta_Y(q) = \theta_W(q') \in \theta_W[H_W]$. **Q**

Set $G^* = \bigcup_{Y\in\mathcal{Y}} \theta_Y[H_Y]$, so that G^* is a subgroup of H_X.

(b) Let V be an open neighbourhood of the identity in G^* (with the subspace topology inherited from the topology of pointwise convergence on H_X), $\epsilon > 0$ and $I \subseteq G^*$ a finite set. Then there is a Borel probability measure λ on G^* such that

(i) $\lambda(fQ) = \lambda Q$ for every $f \in I$ and every closed set $Q \subseteq G^*$,

(ii) $\lambda(VQ) \ge 1 - \epsilon$ whenever $Q \subseteq G^*$ is closed and $\lambda Q \ge \frac{1}{2}$.

P We may suppose that $\epsilon \le \frac{1}{2}$. Let $J \in [S_X]^{<\omega}$ and $\delta > 0$ be such that $f \in V$ whenever $f \in G^*$ and $\|f(x) - x\| \le \delta$ for every $x \in J$. We may suppose that J is non-empty; set $m = \#(J)$. Let $Y \in \mathcal{Y}$ be such that $J \subseteq Y$ and $I \subseteq \theta_Y[H_Y]$ and $\dim Y = r \ge r(m,\epsilon)$, as chosen in 493F. (This is where we need to know that X is infinite-dimensional.) Set $\lambda F = \lambda_Y \theta_Y^{-1}[F]$ for every Borel set $F \subseteq G^*$, where λ_Y is the Haar probability measure of H_Y, as before.

If $f \in I$ and $F \subseteq G^*$ is closed, then

$$\lambda(fF) = \lambda_Y \theta_Y^{-1}[fF] = \lambda_Y(\theta_Y^{-1}(f)\theta_Y^{-1}[F]) = \lambda_Y \theta_Y^{-1}[F] = \lambda F.$$

So λ satisfies condition (i).

? Suppose, if possible, that $Q_1 \subseteq G^*$ is a closed set such that $\lambda Q_1 \ge \frac{1}{2}$ and $\lambda(VQ_1) < 1 - \epsilon$; set $Q_2 = G^* \setminus VQ_1$. Then $\theta_Y^{-1}[Q_1]$ and $\theta_Y^{-1}[Q_2]$ are subsets of H_Y both of measure at least ϵ. Set $R_j = \{q : q \in H_Y,\ q^{-1} \in \theta_Y^{-1}[Q_j]\}$ for each j; because H_Y is compact, therefore unimodular,

$$\lambda_Y R_j = \lambda_Y \theta_Y^{-1}[Q_j] = \lambda Q_j \ge \epsilon$$

for both j. Because $\dim Y \ge r(m,\epsilon)$, there are $q_1 \in R_1$, $q_2 \in R_2$ such that $\|q_1(x) - q_2(x)\| \le \epsilon$ for $x \in J$. Set $f = \theta_Y(q_2^{-1}q_1)$. If $x \in J$, then

$$\|f(x) - x\| = \|q_2^{-1}q_1(x) - x\| = \|q_1(x) - q_2(x)\| \le \epsilon.$$

As this is true whenever $x \in J$ and $f \in V$. On the other hand, $\theta_Y(q_1^{-1}) \in Q_1$ and $\theta_Y(q_2^{-1}) \in Q_2$ and $f\theta_Y(q_1^{-1}) = \theta_Y(q_2^{-1})$, so $\theta_Y(q_2^{-1}) \in VQ_1 \cap Q_2$, which is impossible. **X**

Thus λ satisfies (ii). **Q**

(c) By 493C, G^* is extremely amenable. But G^* is dense in H_X. **P** If $f \in H_X$ and $I \subseteq S_X$ is finite and not empty, let Y_1 be the linear subspace of X generated by I, and let $(y_1, \dots, y_m)$ be an orthonormal basis of Y_1. Set $z_j = f(y_j)$ for each j, so that $(z_1, \dots, z_m)$ is orthonormal (476J); let Y be the linear subspace of X generated by $y_1, \dots, y_m, z_1, \dots, z_m$. Set $r = \dim Y$ and extend the orthonormal sets $(y_1, \dots, y_m)$ and $(z_1, \dots, z_m)$ to orthonormal bases $(y_1, \dots, y_r)$ and $(z_1, \dots, z_r)$ of Y. Then we have an isometric linear operator $T : Y \to Y$ defined by saying that $Ty_i = z_i$ for each i; set $q = T{\restriction}S_Y \in H_Y$. By 476J, $q(x) = f(x)$ for every $x \in I$, so $\theta_Y(q)$ agrees with f on I, while $\theta_Y(q) \in G^*$. As f and I are arbitrary, G^* is dense in G. **Q**

So 493Bb tells us that H_X is extremely amenable, and the proof is complete.

493H The following result shows why extremely amenable groups did not appear in Chapter 44.

Theorem (Veech 77) If G is a locally compact Hausdorff topological group with more than one element, it is not extremely amenable.

proof If G is compact, this is trivial, since the left action of G on itself has no fixed point; so let us assume henceforth that G is not compact.

(a) Let Z be the greatest ambit of G, $a \mapsto \hat{a} : G \to Z$ the canonical map, and U the space of bounded right-uniformly continuous real-valued functions on G. (I aim to show that the action of G on Z has no fixed point.) Take any $z^* \in Z$. Let V_0 be a compact neighbourhood of the identity e in G, and let $B_0 \subseteq G$ be a maximal set such that $V_0 b \cap V_0 c = \emptyset$ for all distinct b, $c \in B_0$. Then for any $a \in G$ there is a $b \in B_0$ such that $V_0 a \cap V_0 b \ne \emptyset$, that is, $a \in V_0^{-1}V_0B_0$. So if we set $Y_0 = \overline{\{\hat{b} : b \in B_0\}} \subseteq Z$, $\{a{\bullet}y : a \in V_0^{-1}V_0,\ y \in Y_0\}$ is a compact subset of Z including $\{\hat{a} : a \in G\}$, and is therefore the whole of Z (449Dc). Let $a_0 \in V_0^{-1}V_0$, $y_0 \in Y_0$ be such that $a_0{\bullet}y_0 = z^*$, and set $B_1 = a_0B_0$, $V_1 = a_0V_0a_0^{-1}$; then $z^* \in \overline{\{\hat{b} : b \in B_1\}}$ and $V_1b \cap V_1c = \emptyset$ for all distinct b, $c \in B_1$.

(b) Because V_1 is compact and G is not compact, there is an $a_1 \in G \setminus V_1$. Let $V_2 \subseteq V_1$ be a neighbourhood of e such that $a_1^{-1}V_2V_2^{-1}a_1 \subseteq V_1$. Then we can express B_1 as $D_0 \cup D_1 \cup D_2$ where $a_1D_i \cap V_2D_i = \emptyset$ for all i. **P** Consider $\{(b,c) : b,\ c \in B_1,\ a_1b \in V_2c\}$. Because $V_2c \cap V_2c' \subseteq V_1c \cap V_1c' = \emptyset$ for all distinct c, $c' \in B_1$, this is the graph of a function $h : D \to B_1$ for some $D \subseteq B_1$. **?** If h is not injective, we have distinct b, $c \in B_1$ and $d \in B_1$ such that a_1b and a_1c both belong to V_2d. But in this case b and c both belong to $a_1^{-1}V_2d$ and $bc^{-1} \in a_1^{-1}V_2dd^{-1}V_2^{-1}a_1 \subseteq V_1$ and $b \in V_1c$, which is impossible. **X** At the same time, if $b \in B_1$, then $a_1b \notin V_2b$ because $a_1 \notin V_2$, so $h(b) \ne b$ for every $b \in D$.

Let $D_0 \subseteq D$ be a maximal set such that $h[D_0] \cap D_0 = \emptyset$, and set $D_1 = h[D_0]$, $D_2 = B_1 \setminus (D_0 \cup D_1)$. Then $h[D_0] \cap D_0 = \emptyset$ by the choice of D_0; $h[D \cap D_1] \cap D_1 = \emptyset$ because h is injective and $D_1 \subseteq h[D \setminus D_1]$; and $h[D \cap D_2] \subseteq D_0$ because if $b \in D \cap D_2$ there must have been some reason why we did not put b into D_0, and it wasn't because $b \in h[D_0]$ or because $h(b) = b$. So $h[D_i] \cap D_i = \emptyset$ for all i, which is what was required. **Q**

(c) Since $z^* \in \overline{\{\hat{b} : b \in B_1\}}$, there must be some $j \le 2$ such that $z^* \in \overline{\{\hat{b} : b \in D_j\}}$. Now recall that the right uniformity on G, like any uniformity, can be defined by some family of pseudometrics (4A2Ja). There is therefore a pseudometric ρ on G such that $W_\epsilon = \{(a, b) : a,\, b \in G,\, \rho(a, b) \le \epsilon\}$ is a member of the right uniformity on G for every $\epsilon > 0$ and $W_1 \subseteq \{(a, b) : ab^{-1} \subseteq V_2\}$. If now we set

$$f(a) = \min(1, \rho(a, D_j)) = \min(1, \inf\{\rho(a, b) : b \in D_j\})$$

for $a \in G$, $f : G \to \mathbb{R}$ is bounded and uniformly continuous for the right uniformity, so belongs to U. On the other hand, if $b,\, c \in D_j$, then $a_1 b \notin V_2 c$, that is, $a_1 bc^{-1} \notin V_2$ and $\rho(a_1 b, c) > 1$; as c is arbitrary, $f(a_1 b) = 1$.

(d) Now $\hat{b}(f) = f(b) = 0$ for every $b \in D_j$, so $z^*(f) = 0$. On the other hand, because $z \mapsto a_1{\bullet}z$ is continuous,

$$a_1{\bullet}z^* \in \overline{\{a_1{\bullet}\hat{b} : b \in D_j\}} = \overline{\{\widehat{a_1 b} : b \in D_j\}},$$

so

$$(a_1{\bullet}z^*)(f) \ge \inf_{b \in D_j} \widehat{a_1 b}(f) = \inf_{b \in D_j} f(a_1 b) = 1,$$

and $a_1{\bullet}z^* \ne z^*$. As z^* is arbitrary, this shows that the action of G on Z has no fixed point, and G is not extremely amenable.

493X Basic exercises (a) Let G be a Hausdorff topological group, and $\widehat{G}$ its completion with respect to its bilateral uniformity. Show that G is extremely amenable iff $\widehat{G}$ is.

>(b) Let X be a set with more than one member and ρ the zero-one metric on X. Let G be the isometry group of X with the topology of pointwise convergence. Show that G is not extremely amenable. (*Hint*: give X a total ordering $\le$, and let x, y be any two points of X. For $a \in G$ set $f(a) = 1$ if $a^{-1}(x) < a^{-1}(y)$, -1 otherwise. Show that, in the language of 449D, $f \in U$. Show that if $(\overleftarrow{x\,y})$ is the transposition exchanging x and y then $(\overleftarrow{x\,y}){\bullet}_l f = -f$, while $|z(f)| = 1$ for every z in the greatest ambit of G.) (Compare 449Xh.)

(c) Show that under the conditions of 493C there is a finitely additive functional $\nu : \mathcal{B} \to [0, 1]$ such that $\nu(aF) = \nu F$ for every $a \in G$ and every zero set $F \subseteq G$, while $\nu(VF) = 1$ whenever V is a neighbourhood of the identity, F is a zero set and $\nu F > \frac{1}{2}$.

(d) Prove 493G for infinite-dimensional inner product spaces over $\mathbb{C}$.

(e) Let X be any (real or complex) inner product space. Show that the isometry group of X, with its topology of pointwise convergence, is amenable. (*Hint*: 449Cd.)

(f) Let X be a separable Hilbert space. (i) Show that the isometry group G of its unit sphere, with its topology of pointwise convergence, is a Polish group. (ii) Show that if X is infinite-dimensional, then every countable discrete group can be embedded as a closed subgroup of G, so that G is an extremely amenable Polish group with a closed subgroup which is not amenable. (Cf. 449K.)

(g) If X is a (real or complex) Hilbert space, a bounded linear operator $T : X \to X$ is **unitary** if it is an invertible isometry. Show that the set of unitary operators on X, with its strong operator topology (3A5I), is an extremely amenable topological group.

(h) Let G be a topological group carrying Haar measures. Show that it is extremely amenable iff its topology is the indiscrete topology. (*Hint*: 443L.)

493Y Further exercises (a) For a Boolean algebra $\mathfrak{A}$ and a group G with identity e, write $S(\mathfrak{A}; G)$ for the set of partitions of unity $\langle a_g\rangle_{g\in G}$ in $\mathfrak{A}$ such that $\{g : a_g \ne 0\}$ is finite. For $\langle a_g\rangle_{g\in G}$, $\langle b_g\rangle_{g\in G} \in S(A)$, write $\langle a_g\rangle_{g\in G} \cdot \langle b_g\rangle_{g\in G} = \langle c_g\rangle_{g\in G}$ where $c_g = \sup\{a_h \cap b_{h^{-1}g} : h \in G\}$ for $g \in G$. (i) Show that under this operation $S(\mathfrak{A}; G)$ is a group. (ii) Show that if we write $h\chi a$ for the member $\langle a_g\rangle_{g\in G}$ of $S(\mathfrak{A}; G)$ such that $a_h = a$ and $a_g = 0$ for other $g \in G$, then $g\chi a \cdot h\chi b = (gh)\chi(a \cap b)$, and $S(\mathfrak{A}; G)$ is generated by $\{g\chi a : g \in G,\, a \in \mathfrak{A}\}$. (iii) Show that if $\mathfrak{A} = \Sigma/\mathcal{I}$ where Σ is an algebra of subsets of a set X and $\mathcal{I}$ is an ideal of Σ, then $S(\mathfrak{A}; G)$ can be identified with a space of equivalence classes in a suitable subgroup of G^X. (iv) Devise a universal mapping theorem for the construction $S(\mathfrak{A}; G)$ which matches 361F in the case $(G, \cdot) = (\mathbb{R}, +)$. (v) Now suppose that $(\mathfrak{A}, \bar{\mu})$ is a measure algebra and that G is a topological group. Show that we have a topology on $S(\mathfrak{A}; G)$, making it a topological group, for which basic neighbourhoods of the identity $e\chi 1$ are of the form $V(c,\epsilon,U)=\{\langle a_g\rangle_{g\in G}:\bar\mu(c\cap\sup_{g\in G\setminus U}a_g)\le\epsilon\}$ with $\bar\mu c<\infty$, $\epsilon>0$ and U a neighbourhood of the identity in G. (vi) Show that if G is an amenable locally compact Hausdorff group and $(\mathfrak{A},\bar\mu)$ is an atomless measure algebra, then $S(\mathfrak{A};G)$ is extremely amenable. (*Hint*: Pestov 02.) *(vi) Explore possible constructions of spaces $L^0(\mathfrak{A};G)$. (See Hartman & Mycielski 58.)

493 Notes and comments In writing this section I have relied heavily on Pestov 99 and Pestov 02, where you may find many further examples of extremely amenable groups. It is a striking fact that while the theories of locally compact groups and extremely amenable groups are necessarily almost entirely separate (493H), both are dependent on measure theory. Curiously, what seems to have been the first non-trivial extremely amenable group to be described was found in the course of investigating the Control Measure Problem (Herer & Christensen 75).

The theory of locally compact groups has for seventy years now been a focal point for measure theory. Extremely amenable groups have not yet had such an influence. But they encourage us to look again at concentration-of-measure theorems, which are of the highest importance for quite separate reasons. In all the principal examples of this section, and again in the further example to come in §494, we need concentration of measure in product spaces (493D-493E and 494J), permutation groups (494I) or on spheres in Euclidean space (493G). 493D and 493E are special cases of a general result in Pestov 02 (493Ya(vi)) which itself extends an idea from Glasner 98. I note that 493D needs only concentration of measure in $\{0,1\}^I$, while 493E demands something rather closer to the full strength of Talagrand's theorem 492D.

I have expressed 493G as a theorem about the isometry groups of spheres in infinite-dimensional inner product spaces; of course these are isomorphic to the orthogonal groups of the whole spaces with their strong operator topologies (476Xd). Adapting the basic concentration-of-measure theorem 476K to the required lemma 493F involves an instructive application of ideas from §443.

494 Groups of measure-preserving automorphisms

I return to the study of automorphism groups of measure algebras, as in Chapter 38 of Volume 3, but this time with the intention of exploring possible topological group structures. Two topologies in particular have attracted interest, the 'weak' and 'uniform' topologies (494A). After a brief account of their basic properties (494B-494C) I begin work on the four main theorems. The first is the Halmos-Rokhlin theorem that if $(\mathfrak{A},\bar\mu)$ is the Lebesgue probability algebra the set of weakly mixing measure-preserving automorphisms of $\mathfrak{A}$ which are not mixing is comeager for the weak topology on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ (494E). This depends on a striking characterization of weakly mixing automorphisms of a probability algebra in terms of eigenvectors of the corresponding operators on the complex Hilbert space $L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ (494D, 494Xj(i)). It turns out that there is an elegant example of a weakly mixing automorphism which is not mixing which can be described in terms of a Gaussian distribution of the kind introduced in §456, so I give it here (494F).

We need a couple of preliminary results on fixed-point subalgebras (494G-494H) before approaching the other three theorems. If $(\mathfrak{A},\bar\mu)$ is an atomless probability algebra, then $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is extremely amenable under its weak topology (494L); if $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is given its uniform topology, then every group homomorphism from $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ to a Polish group is continuous (494O); finally, there is no strictly increasing sequence of subgroups with union $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ (494Q). All these results have wide-ranging extensions to full subgroups of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ subject to certain restrictions on the fixed-point subalgebras.

The work of this section will rely heavily on concepts and results from Volume 3 which have hardly been mentioned so far in the present volume. I hope that the cross-references, and the brief remarks in 494Ac-494Ad, will be adequate.

494A Definitions (Halmos 56) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ the group of measure-preserving automorphisms of $\mathfrak{A}$ (see §383). Write $\mathfrak{A}^f$ for $\{c:c\in\mathfrak{A},\,\bar\mu c<\infty\}$.

(a) I will say that the **weak topology** on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is that generated by the pseudometrics $(\pi,\phi)\mapsto\bar\mu(\pi c\,\triangle\,\phi c)$ as c runs over $\mathfrak{A}^f$.

(b) I will say that the **uniform topology** on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is that generated by the pseudometrics

$$(\pi,\phi)\mapsto\sup_{a\in\mathfrak{A}}\bar\mu(c\cap(\pi a\,\triangle\,\phi a))$$

as c runs over $\mathfrak{A}^f$.

(c) I recall some notation from Volume 3. For any Boolean algebra $\mathfrak{A}$ and $a \in \mathfrak{A}$, $\mathfrak{A}_a$ will be the principal ideal of $\mathfrak{A}$ generated by a (312D). I will generally use the symbol ι for the identity in the automorphism group $\operatorname{Aut}\mathfrak{A}$ of $\mathfrak{A}$. If $\pi \in \operatorname{Aut}\mathfrak{A}$ and $a \in \mathfrak{A}$, a supports π if $\pi d = d$ whenever $d \cap a = 0$; the support

$$\operatorname{supp}\pi = \sup\{a \,\triangle\, \pi a : a \in \mathfrak{A}\}$$

of π is the smallest member of $\mathfrak{A}$ supporting π, if this is defined (381Bb, 381Ei). A subgroup G of $\operatorname{Aut}\mathfrak{A}$ is 'full' if $\phi \in G$ whenever $\phi \in \operatorname{Aut}\mathfrak{A}$ and there are $\langle a_i\rangle_{i\in I}$, $\langle \pi_i\rangle_{i\in I}$ such that $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ and $\pi_i \in G$ and $\phi d = \pi_i d$ whenever $i \in I$ and $d \subseteq a_i$ (381Be).

If a, $b \in \mathfrak{A} \setminus \{0\}$ are disjoint and $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that $\pi a = b$, then $(\overleftarrow{a \,{}_\pi\, b}) \in \operatorname{Aut}\mathfrak{A}$ will be the exchanging involution defined by saying that

$$\begin{aligned}(\overleftarrow{a \,{}_\pi\, b})(d) &= \pi d \text{ if } d \subseteq a,\\ &= \pi^{-1}d \text{ if } d \subseteq b,\\ &= d \text{ if } d \subseteq 1 \setminus (a \cup b)\end{aligned}$$

(381R).

(d) In addition, I will repeatedly use the following ideas. Suppose that $(\mathfrak{A}, \bar\mu)$ is a probability algebra (322Aa), $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ (323I), and $L^\infty(\mathfrak{C})$ the M-space defined in §363. Then for each $a \in \mathfrak{A}$ we have a conditional expectation $u_a \in L^\infty(\mathfrak{C})$ of χa on $\mathfrak{C}$, so that $\int_c u_a = \bar\mu(a \cap c)$ for every $c \in \mathfrak{C}$ (365R).

If $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$ (331A), $a \in \mathfrak{A}$, and $v \in L^\infty(\mathfrak{C})$ is such that $0 \le v \le u_a$, there is a $b \in \mathfrak{A}$ such that $b \subseteq a$ and $v = u_b$ (apply Maharam's lemma 331B to the functional $c \mapsto \int_c v : \mathfrak{C} \to [0,1]$). Elaborating on this, we see that if $\langle v_n\rangle_{n\in\mathbb{N}}$ is a sequence in $L^\infty(\mathfrak{C})^+$ and $\sum_{i=0}^n v_i \le u_a$ for every n, there are disjoint $b_0, \ldots \subseteq a$ such that $v_i = u_{b_i}$ for every i (choose the b_i inductively).

494B Proposition Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and give $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology.

(a) $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is a topological group.

(b) $(\pi, a) \mapsto \pi a : \operatorname{Aut}_{\bar\mu}\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is continuous for the weak topology on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and the measure-algebra topology on $\mathfrak{A}$.

(c) If $(\mathfrak{A}, \bar\mu)$ is semi-finite (definition: 322Ad), $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is Hausdorff.

(d) If $(\mathfrak{A}, \bar\mu)$ is localizable (definition: 322Ae), $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is complete under its bilateral uniformity.

(e) If $(\mathfrak{A}, \bar\mu)$ is σ-finite (definition: 322Ac) and $\mathfrak{A}$ has countable Maharam type (definition: 331Fa), then $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is a Polish group.

proof (a) (Compare 441G.) Set $\rho_c(\pi, \phi) = \bar\mu(\pi c \,\triangle\, \phi c)$ for π, $\phi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $c \in \mathfrak{A}^f$; it is elementary to check that ρ_c is always a pseudometric, so 494Aa is a proper definition of a topology. If π, $\phi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $c \in \mathfrak{A}^f$, then for any π', $\phi' \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ we have

$$\begin{aligned}\rho_c(\pi'\phi', \pi\phi) = \bar\mu(\pi'\phi'c \,\triangle\, \pi\phi c) &\le \bar\mu(\pi'\phi'c \,\triangle\, \pi'\phi c) + \bar\mu(\pi'\phi c \,\triangle\, \pi\phi c)\\ &= \bar\mu(\phi'c \,\triangle\, \phi c) + \rho_{\phi c}(\pi', \pi) = \rho_c(\phi', \phi) + \rho_{\phi c}(\pi', \pi);\end{aligned}$$

as c is arbitrary, $(\pi', \phi') \mapsto \pi'\phi'$ is continuous at (π, ϕ); thus multiplication is continuous. If $\phi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $c \in \mathfrak{A}^f$, then for any $\pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$

$$\begin{aligned}\rho_c(\pi^{-1}, \phi^{-1}) = \bar\mu(\pi^{-1}c \,\triangle\, \phi^{-1}c) &= \bar\mu(c \,\triangle\, \pi\phi^{-1}c)\\ &= \bar\mu(\phi\phi^{-1}c \,\triangle\, \pi\phi^{-1}c) = \rho_{\phi^{-1}c}(\pi, \phi);\end{aligned}$$

as c is arbitrary, $\pi \mapsto \pi^{-1}$ is continuous at ϕ; thus inversion is continuous and $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is a topological group.

(b) Suppose that $\phi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$, $b \in \mathfrak{A}$ and that V is a neighbourhood of ϕb in $\mathfrak{A}$. Then there are $c \in \mathfrak{A}^f$ and $\epsilon > 0$ such that V includes $\{d : \bar\mu(c \cap (d \,\triangle\, \phi b)) \le 4\epsilon\}$. In this case, because inversion in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is continuous,

$$U = \{\pi : \pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A},\ \bar\mu(\pi^{-1}c \,\triangle\, \phi^{-1}c) \le \epsilon,\ \bar\mu(\pi(\phi^{-1}c \cap b) \,\triangle\, \phi(\phi^{-1}c \cap b)) \le \epsilon\},$$

$$V' = \{a : a \in \mathfrak{A},\ \bar\mu(\phi^{-1}c \cap (a \,\triangle\, b)) \le \epsilon\}$$

are neighbourhoods of ϕ, b respectively. If $\pi \in U$ and $a \in V'$, then

$$\begin{aligned}\bar\mu(c\cap(\pi a\,\triangle\,\phi b)) &\le \bar\mu(c\cap(\pi a\,\triangle\,\pi b))+\bar\mu(c\cap(\pi b\,\triangle\,\phi b))\\ &=\bar\mu(\pi^{-1}c\cap(a\,\triangle\,b))+\bar\mu((c\cap\pi b)\,\triangle\,(c\cap\phi b))\\ &\le\bar\mu(\pi^{-1}c\,\triangle\,\phi^{-1}c)+\bar\mu(\phi^{-1}c\cap(a\,\triangle\,b))+\bar\mu(\pi(\pi^{-1}c\cap b)\,\triangle\,\phi(\phi^{-1}c\cap b))\\ &\le\epsilon+\epsilon+\bar\mu(\pi(\pi^{-1}c\cap b)\,\triangle\,\pi(\phi^{-1}c\cap b))\\ &\qquad\qquad+\bar\mu(\pi(\phi^{-1}c\cap b)\,\triangle\,\phi(\phi^{-1}c\cap b))\\ &\le 2\epsilon+\bar\mu((\pi^{-1}c\cap b)\,\triangle\,(\phi^{-1}c\cap b))+\epsilon\\ &\le 3\epsilon+\bar\mu(\pi^{-1}c\,\triangle\,\phi^{-1}c)\le 4\epsilon,\end{aligned}$$

and $\pi a\in V$. As V, ϕ and b are arbitrary, $(\pi,a)\mapsto\pi a$ is continuous.

(c) Because $(\mathfrak{A},\bar\mu)$ is semi-finite, the measure-algebra topology on $\mathfrak{A}$ is Hausdorff (323Ga), so the product topology on $\mathfrak{A}^{\mathfrak{A}}$ is Hausdorff. Now $\pi\mapsto\langle\pi a\rangle_{a\in\mathfrak{A}}:\mathrm{Aut}_{\bar\mu}\mathfrak{A}\to\mathfrak{A}^{\mathfrak{A}}$ is injective, and by (b) it is continuous, so the topology of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ must be Hausdorff.

(d)(i) For $c\in\mathfrak{A}^f$ and $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, set $\theta_c(\pi)=\pi c$; then $\theta_c:\mathrm{Aut}_{\bar\mu}\mathfrak{A}\to\mathfrak{A}^f$ is uniformly continuous for the bilateral uniformity of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and the measure metric ρ of $\mathfrak{A}^f$ (323Ad). **P** We have $\rho(d,d')=\bar\mu(d\,\triangle\,d')$ for d, $d'\in\mathfrak{A}^f$. Let $\epsilon>0$; then $U=\{\pi:\rho_c(\pi,\iota)\le\epsilon\}$ is a neighbourhood of ι in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, and $W=\{(\pi,\phi):\phi^{-1}\pi\in U\}$ belongs to the bilateral uniformity. If $(\pi,\phi)\in W$, then

$$\rho(\theta_c(\pi),\theta_c(\phi))=\bar\mu(\pi c\,\triangle\,\phi c)=\bar\mu(\phi^{-1}\pi c\,\triangle\,c)=\rho_c(\phi^{-1}\pi,\iota)\le\epsilon;$$

as ϵ is arbitrary, θ_c is uniformly continuous. **Q**

(ii) Let $\mathcal{F}$ be a filter on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ which is Cauchy for the bilateral uniformity on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. If $c\in\mathfrak{A}^f$, the image filter $\theta_c[[\mathcal{F}]]$ is Cauchy for the measure metric on $\mathfrak{A}^f$ (4A2Ji). Because $\mathfrak{A}^f$ is complete under its measure metric (323Mc), $\theta_c[[\mathcal{F}]]$ converges to ψ_0c say for the measure metric.

If c, $d\in\mathfrak{A}^f$ and $*$ is either of the Boolean operations $\cap$, $\triangle$, then

$$\psi_0(c*d)=\lim_{\pi\to\mathcal{F}}\pi(c*d)=\lim_{\pi\to\mathcal{F}}\pi c*\pi d=\lim_{\pi\to\mathcal{F}}\pi c*\lim_{\pi\to\mathcal{F}}\pi d$$

(because $*$ is continuous for the measure metric, see 323Ma)

$$=\psi_0c*\psi_0d.$$

So $\psi_0:\mathfrak{A}^f\to\mathfrak{A}^f$ is a ring homomorphism. Next, if $c\in\mathfrak{A}^f$, then

$$\bar\mu\psi_0c=\bar\mu(\lim_{\pi\to\mathcal{F}}\pi c)=\lim_{\pi\to\mathcal{F}}\bar\mu\pi c=\bar\mu c$$

because $\bar\mu:\mathfrak{A}^f\to[0,\infty[$ is continuous (323Mb).

Now recall that $\pi\mapsto\pi^{-1}:\mathrm{Aut}_{\bar\mu}\mathfrak{A}\to\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is uniformly continuous for the bilateral uniformity (4A5Hc). So if we set $\theta'_c(\pi)=\pi^{-1}c$ for $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, we can apply the argument just above to θ' to find a ring homomorphism $\psi'_0:\mathfrak{A}^f\to\mathfrak{A}^f$ such that $\psi'_0c=\lim_{\pi\to\mathcal{F}}\pi^{-1}c$ for every $c\in\mathfrak{A}^f$. To relate ψ_0 and ψ'_0, we can argue as follows. Given $c\in\mathfrak{A}^f$,

$$\begin{aligned}\bar\mu(c\,\triangle\,\psi_0\psi'_0c)&=\bar\mu(c\,\triangle\lim_{\phi\to\mathcal{F}}\phi\psi'_0c)=\lim_{\phi\to\mathcal{F}}\bar\mu(c\,\triangle\,\phi\psi'_0c)=\lim_{\phi\to\mathcal{F}}\bar\mu(\phi^{-1}c\,\triangle\,\psi'_0c)\\ &=\bar\mu(\lim_{\phi\to\mathcal{F}}(\phi^{-1}c\,\triangle\,\psi'_0c))=\bar\mu((\lim_{\phi\to\mathcal{F}}\phi^{-1}c)\,\triangle\,\psi'_0c)=\bar\mu(\psi'_0c\,\triangle\,\psi'_0c)=0;\end{aligned}$$

as c is arbitrary, $\psi_0\psi'_0$ is the identity on $\mathfrak{A}^f$. Similarly, $\psi'_0\psi_0$ is the identity on $\mathfrak{A}^f$. Thus ψ_0, ψ'_0 are the two halves of a measure-preserving ring isomorphism of $\mathfrak{A}^f$.

If we give $\mathfrak{A}$ its measure-algebra uniformity (323Ab), then ψ_0 is uniformly continuous for the induced uniformity on $\mathfrak{A}^f$. **P** If c, d_1, $d_2\in\mathfrak{A}^f$, then

$$\bar\mu(c\cap(\psi_0d_1\,\triangle\,\psi_0d_2))=\bar\mu(\psi_0^{-1}c\cap(d_1\,\triangle\,d_2)).\ \mathbf{Q}$$

Since $\mathfrak{A}^f$ is dense in $\mathfrak{A}$ for the measure-algebra topology on $\mathfrak{A}$ (323Bb), and $\mathfrak{A}$ is complete for the measure-algebra uniformity (323Gc), there is a unique extension of ψ_0 to a uniformly continuous function $\psi:\mathfrak{A}\to\mathfrak{A}$ (3A4G). Since the Boolean operations $\triangle$, $\cap$ on $\mathfrak{A}$ are continuous for the measure-algebra topology (323Ba), ψ is a ring homomorphism. Similarly, we have a unique continuous $\psi':\mathfrak{A}\to\mathfrak{A}$ extending ψ'_0; since $\psi\psi'$ and $\psi'\psi$ are continuous functions agreeing with the identity operator ι on $\mathfrak{A}^f$, they are both ι, and $\psi \in \operatorname{Aut}\mathfrak{A}$. To see that ψ is measure-preserving, note just that if $a \in \mathfrak{A}$ then

$$\bar{\mu}\psi a = \sup\{\bar{\mu}c : c \in \mathfrak{A}^f,\ c \subseteq \psi a\} = \sup\{\bar{\mu}\psi_0 c : c \in \mathfrak{A}^f,\ \psi_0 c \subseteq \psi a\}$$

(because ψ_0 is a permutation of $\mathfrak{A}^f$)

$$= \sup\{\bar{\mu}c : c \in \mathfrak{A}^f,\ \psi c \subseteq \psi a\} = \sup\{\bar{\mu}c : c \in \mathfrak{A}^f,\ c \subseteq a\} = \bar{\mu}a.$$

Thus $\psi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$.

Finally, $\mathcal{F} \to \psi$. **P** If $c \in \mathfrak{A}^f$ and $\epsilon > 0$, there is an $F \in \mathcal{F}$ such that $\bar{\mu}(\pi c \triangle \phi c) \le \epsilon$ whenever π, $\phi \in F$. We have

$$\bar{\mu}(\pi c \triangle \psi c) = \bar{\mu}(\pi c \triangle \lim_{\phi\to\mathcal{F}} \phi c) = \lim_{\phi\to\mathcal{F}} \bar{\mu}(\pi c \triangle \phi c) \le \epsilon$$

for every $\pi \in F$. As c and ϵ are arbitrary, $\mathcal{F} \to \psi$ for the weak topology on $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$. **Q** As $\mathcal{F}$ is arbitrary, the bilateral uniformity is complete.

(e)(i) The point is that $\mathfrak{A}^f$ is separable for the measure metric. **P** Because $\mathfrak{A}$ has countable Maharam type, there is a countable subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ which τ-generates $\mathfrak{A}$; by 323J, $\mathfrak{B}$ is dense in $\mathfrak{A}$ for the measure algebra topology. Next, there is a non-decreasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with supremum 1. Set $D = \{b \cap c_n : b \in \mathfrak{B},\ n \in \mathbb{N}\}$. Then D is a countable subset of $\mathfrak{A}^f$. If $c \in \mathfrak{A}^f$ and $\epsilon > 0$, there are an $n \in \mathbb{N}$ such that $\bar{\mu}(c \setminus c_n) \le \epsilon$, and a $b \in \mathfrak{B}$ such that $\bar{\mu}(c_n \cap (c \triangle b)) \le \epsilon$. Now $d = b \cap c_n$ belongs to D, and

$$\begin{aligned}\bar{\mu}(c \triangle d) &\le \bar{\mu}(c \triangle (c \cap c_n)) + \bar{\mu}((c \cap c_n) \triangle (b \cap c_n))\\ &= \bar{\mu}(c \setminus c_n) + \bar{\mu}(c_n \cap (c \triangle b)) \le 2\epsilon.\end{aligned}$$

As c and ϵ are arbitrary, D is dense in $\mathfrak{A}^f$ and $\mathfrak{A}^f$ is separable. **Q**

(ii) Let D be a countable dense subset of $\mathfrak{A}^f$, and $\mathcal{U}$ the family of sets of the form

$$\{\pi : \pi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A},\ \bar{\mu}(d \triangle \pi d') < 2^{-n}\}$$

where d, $d' \in D$ and $n \in \mathbb{N}$. All these sets are open for the weak topology. **P** If $U = \{\pi : \bar{\mu}(d \triangle \pi d') < 2^{-n}\}$ and $\phi \in U$, set $\eta = \frac{1}{3}(2^{-n} - \bar{\mu}(d \triangle \phi d'))$. Then $V = \{\pi : \bar{\mu}(d \cap (\pi d' \triangle \phi d')) \le \eta\}$ is a neighbourhood of ϕ. If $\pi \in V$, then

$$\bar{\mu}(d \triangle \pi d') \le \bar{\mu}(d \triangle \phi d') + \bar{\mu}(\phi d' \triangle \pi d') < 2^{-n}$$

and $\pi \in U$. Thus $\phi \in \operatorname{int} U$; as ϕ is arbitrary, U is open. **Q**

(iii) In fact $\mathcal{U}$ is a subbase for the weak topology on $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$. **P** If $W \subseteq \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is open and $\phi \in W$, there are $c_0, \dots, c_n \in \mathfrak{A}^f$ and $k \in \mathbb{N}$ such that W includes $\{\pi : \bar{\mu}(\pi c_i \triangle \phi c_i) \le 2^{-k}$ for every $i \le n\}$. Let $d_0, \dots, d_n, d'_0, \dots, d'_n \in D$ be such that $\bar{\mu}(d_i \triangle c_i) < 2^{-k-2}$, $\bar{\mu}(d'_i \triangle \phi c_i) < 2^{-k-2}$ for each $i \le n$. Set $U_i = \{\pi : \bar{\mu}(d'_i \triangle \pi d_i) < 2^{-k-1}\}$; then $U_i \in \mathcal{U}$ and $\phi \in U_i$ for each $i \le n$, because

$$\bar{\mu}(d'_i \triangle \phi d_i) \le \bar{\mu}(d'_i \triangle \phi c_i) + \bar{\mu}(\phi c_i \triangle \phi d_i) = \bar{\mu}(d'_i \triangle \phi c_i) + \bar{\mu}(c_i \triangle d_i) < 2^{-k-1}.$$

If $\pi \in U_i$, then

$$\bar{\mu}(\pi c_i \triangle \phi c_i) \le \bar{\mu}(\pi c_i \triangle \pi d_i) + \bar{\mu}(\pi d_i \triangle d'_i) + \bar{\mu}(d'_i \triangle \phi c_i) \le 2^{-k},$$

so $\bigcap_{i\le n} U_i \subseteq W$. As W and ϕ are arbitrary, $\mathcal{U}$ is a subbase for the topology. **Q**

(iv) Since $\mathcal{U}$ is countable, the weak topology is second-countable (4A2Oa). Since the weak topology is a group topology, it is regular (4A5Ha, or otherwise); by (c) above it is Hausdorff; so by 4A2Pb it is separable and metrizable. Accordingly the bilateral uniformity is metrizable (4A5Q(v)); by (d) above, $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is complete under the bilateral uniformity, so its topology is Polish.

494C Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and give $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ its uniform topology.

(a) $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is a topological group.

(b) For $c \in \mathfrak{A}^f$ and $\epsilon > 0$, set

$$U(c, \epsilon) = \{\pi : \pi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A},\ \pi \text{ is supported by an } a \in \mathfrak{A} \text{ such that } \bar{\mu}(c \cap a) \le \epsilon\}.$$

Then $\{U(c, \epsilon) : c \in \mathfrak{A}^f,\ \epsilon > 0\}$ is a base of neighbourhoods of ι.

(c) The set of periodic measure-preserving automorphisms of $\mathfrak{A}$ with supports of finite measure is dense in $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$.

(d) The weak topology on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is coarser than the uniform topology.

(e) If $(\mathfrak{A},\bar\mu)$ is semi-finite, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is Hausdorff.

(f) If $(\mathfrak{A},\bar\mu)$ is localizable, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is complete under its bilateral uniformity.

(g) If $(\mathfrak{A},\bar\mu)$ is semi-finite and G is a full subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, then G is closed.

(h) If $(\mathfrak{A},\bar\mu)$ is σ-finite, then $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is metrizable.

(i) Suppose that $(\mathfrak{A},\bar\mu)$ is σ-finite and $\mathfrak{A}$ has countable Maharam type. If $D\subseteq\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is countable, then the full subgroup G of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ generated by D, with its induced topology, is a Polish group.

proof (a) For π, $\phi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and $c\in\mathfrak{A}^f$, set $\rho'_c(\pi,\phi)=\sup_{a\in\mathfrak{A}}\bar\mu(c\cap(\pi a\triangle\phi a))$; as in part (a) of the proof of 494B, it is elementary that every ρ'_c is a pseudometric, so the uniform topology $\mathfrak{T}_u$ is properly defined. If π, ϕ, π', $\phi'\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, $c\in\mathfrak{A}^f$ and $a\in\mathfrak{A}$, then

$$\begin{aligned}\bar\mu(c\cap(\pi'\phi'a\triangle\pi\phi a))&\le\bar\mu(c\cap(\pi'\phi'a\triangle\pi\phi'a))+\bar\mu(c\cap(\pi\phi'a\triangle\pi\phi a))\\&\le\rho'_c(\pi',\pi)+\bar\mu(\pi^{-1}c\cap(\phi'a\triangle\phi a))\\&\le\rho'_c(\pi',\pi)+\rho'_{\pi^{-1}c}(\phi',\phi);\end{aligned}$$

as a is arbitrary, $\rho'_c(\pi'\phi',\pi\phi)\le\rho'_c(\pi',\pi)+\rho'_{\pi^{-1}c}(\phi',\phi)$; as c is arbitrary, $(\pi',\phi')\mapsto\pi'\phi'$ is continuous at (π,ϕ); thus multiplication is continuous. If π, $\phi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, $c\in\mathfrak{A}^f$ and $a\in\mathfrak{A}$, then

$$\bar\mu(c\cap(\pi^{-1}a\triangle\phi^{-1}a))=\bar\mu(\phi c\cap(\phi\pi^{-1}a\triangle\pi\pi^{-1}a)\le\rho'_{\phi c}(\phi,\pi);$$

thus $\rho'_c(\pi^{-1},\phi^{-1})\le\rho'_{\phi c}(\pi,\phi)$, $\pi\mapsto\pi^{-1}$ is continuous at ϕ, and inversion is continuous. So once more we have a topological group.

(b)(i) If $c\in\mathfrak{A}^f$, $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and $\epsilon>0$ are such that $\rho'_c(\pi,\iota)\le\frac13\epsilon$, then $\pi\in U(c,\epsilon)$. **P** Consider $A=\{a:a\in\mathfrak{A}_c,\ a\cap\pi a=0\}$. Then

$$\bar\mu a\le\bar\mu(c\cap(\pi a\triangle a))\le\rho'_c(\pi,\iota)\le\frac{1}{3}\epsilon$$

for every $a\in A$. If $B\subseteq A$ is upwards-directed, then $b^*=\sup B$ is defined in $\mathfrak{A}$, and $\bar\mu b^*=\sup_{b\in B}\bar\mu b$ (321C). Now $\pi b^*=\sup_{b\in B}\pi b$, so $b^*\cap\pi b^*=\sup_{b\in B}b\cap\pi b=0$, and $b^*\in A$. By Zorn's Lemma, A has a maximal element a^*. Suppose that $d\in\mathfrak{A}_c$ is disjoint from $\pi^{-1}a^*\cup a^*\cup\pi a^*$. Then $a^*\cup(d\setminus\pi d)\in A$; by the maximality of a^*, $d\subseteq\pi d$ and $d=\pi d$ (because $\bar\mu d=\bar\mu\pi d<\infty$). Thus $(1\setminus c)\cup(\pi^{-1}a^*\cup a^*\cup\pi a^*)$ supports π and witnesses that $\pi\in U(c,\epsilon)$. **Q**

So every $U(c,\epsilon)$ is a $\mathfrak{T}_u$-neighbourhood of ι.

(ii) Conversely, if $c\in\mathfrak{A}^f$, $\epsilon>0$ and $\pi\in U(c,\epsilon)$, then $\rho'_c(\pi,\iota)\le\epsilon$. **P** Let $d\in\mathfrak{A}$ be such that π is supported by d and $\bar\mu(c\cap d)\le\epsilon$. Then, for any $a\in\mathfrak{A}$, $a\triangle\pi a\subseteq d$, so $\bar\mu(c\cap(a\triangle\pi a))\le\epsilon$; which is what we need to know. **Q**

So $\{U(c,\epsilon):c\in\mathfrak{A}^f,\ \epsilon>0\}$ is a base of neighbourhoods of ι for $\mathfrak{T}_u$.

(c) Take a non-empty open subset U of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and $\phi\in U$.

(i) By (b), there are a $c\in\mathfrak{A}^f$ and an $\epsilon>0$ such that $U(c,3\epsilon)U(c,3\epsilon)\subseteq U^{-1}\phi$. Now there is a $\psi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\psi^{-1}\phi\in U(c,3\epsilon)$ and ψ is supported by $e=c\cup\phi c$. **P** By 332L, applied to $\mathfrak{A}_e$ and $\phi\restriction\mathfrak{A}_c$, there is a measure-preserving automorphism $\psi_0:\mathfrak{A}_e\to\mathfrak{A}_e$ agreeing with ϕ on $\mathfrak{A}_c$; now set

$$\psi a=\psi_0(a\cap e)\cup(a\setminus e)$$

for every $a\in\mathfrak{A}$ to get $\psi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ agreeing with ϕ on $\mathfrak{A}_c$ and supported by e. As $\psi^{-1}\phi a=a$ for $a\subseteq c$, $\psi^{-1}\phi$ is supported by $1\setminus c$ and belongs to $U(c,3\epsilon)$. **Q**

(ii) By 381H, applied to $\psi\restriction\mathfrak{A}_e$, there is a partition $\langle c_m\rangle_{1\le m\le\omega}$ of unity in $\mathfrak{A}_e$ such that $\psi c_m\subseteq c_m$ for every m, $\psi\restriction\mathfrak{A}_{c_m}$ is periodic with period m for every $m\in\mathbb{N}\setminus\{0\}$, and $\psi\restriction\mathfrak{A}_{c_\omega}$ is aperiodic. Of course $\psi c_m=c_m$ for every m, just because $\bar\mu\psi c_m=\bar\mu c_m$. Let $n\ge1$ be such that $\bar\mu c_\omega\le n!\epsilon$ and $\bar\mu(\sup_{n<m<\omega}c_m)\le\epsilon$. By the Halmos-Rokhlin-Kakutani lemma (386C), applied to $\psi\restriction\mathfrak{A}_{c_\omega}$, there is a $b\subseteq c_\omega$ such that $b,\psi b,\dots,\psi^{n!-1}b$ are disjoint and $\bar\mu(c_\omega\setminus\sup_{i<n!}\psi^ib)\le\epsilon$. Note that $\bar\mu b$ is also at most ϵ.

Set

$$d=\sup_{n<m<\omega}c_m\cup(c_\omega\setminus\sup_{i<n!}\psi^ib),\qquad d'=d\cup\psi^{n!-1}b,$$

and let $\pi:\mathfrak{A}\to\mathfrak{A}$ be the measure-preserving Boolean automorphism such that

$$\begin{aligned}\pi a &= \psi a \text{ if } 1 \le m \le n \text{ and } a \subseteq c_m,\\ &= \psi a \text{ if } 0 \le i \le n! - 2 \text{ and } a \subseteq \psi^i b,\\ &= \psi^{-n!+1} a \text{ if } a \subseteq \psi^{n!-1} b,\\ &= a \text{ if } a \subseteq d \cup (1 \setminus e).\end{aligned}$$

Then

$$\begin{aligned}\pi^{n!} a = \psi^{n!} a &= a \text{ if } 1 \le m \le n \text{ and } a \subseteq c_m,\\ &= \psi^{n!}\psi^{-n!} a = a \text{ if } 0 \le i < n! \text{ and } a \subseteq \psi^i b,\\ &= a \text{ if } a \subseteq d \cup (1 \setminus e),\end{aligned}$$

so $\pi^{n!} = \iota$ and π is periodic. Since π is supported by e, and $\mathfrak{A}_e$ is Dedekind complete, π has a support of finite measure. On the other hand $\pi a = \psi a$ whenever $a \cap d' = 0$, so $\pi^{-1}\psi$ is supported by d' and belongs to $U(c, \bar\mu d') \subseteq U(c, 3\epsilon)$.

Now

$$\pi^{-1}\phi = \pi^{-1}\psi\psi^{-1}\phi \in U(c,3\epsilon)U(c,3\epsilon) \subseteq U^{-1}\phi, \quad \pi \in U;$$

as U is arbitrary, the set of periodic automorphisms with supports of finite measure is dense in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

(d) Let V be a neighbourhood of the identity ι for the weak topology $\mathfrak{T}_w$ on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Then there are $c_0, \dots, c_k \in \mathfrak{A}^f$ and $\epsilon_0, \dots, \epsilon_k > 0$ such that

$$V \supseteq \{\pi : \bar\mu(c_i \triangle \pi c_i) \le \epsilon_i \text{ for every } i \le k\}.$$

Set $c = \sup_{i \le k} c_i$, $\epsilon = \frac{1}{2}\min_{i \le k} \epsilon_i$. If $\pi \in U(c, \epsilon)$ as defined in (b), there is an $a \in \mathfrak{A}$, supporting π, such that $\bar\mu(c \cap a) \le \epsilon$. In this case, for each $i \le k$, $c_i \setminus \pi c_i \subseteq c \cap a$, so

$$\bar\mu(c_i \triangle \pi c_i) = 2\bar\mu(c_i \setminus \pi c_i) \le 2\bar\mu(c \cap a) \le \epsilon_i.$$

Thus $V \supseteq U(c, \epsilon)$ and V is a neighbourhood of ι for $\mathfrak{T}_u$. As $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is a topological group under either topology, it follows that $\mathfrak{T}_u$ is finer than $\mathfrak{T}_w$ (4A5Fb).

(e) Because $(\mathfrak{A}, \bar\mu)$ is semi-finite, the weak topology is Hausdorff (494Bc), so the uniform topology, being finer, must also be Hausdorff.

(f) Let $\mathcal{F}$ be a Cauchy filter for the $\mathfrak{T}_u$-bilateral uniformity on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Because the identity map from $(\mathrm{Aut}_{\bar\mu}\mathfrak{A}, \mathfrak{T}_u)$ to $(\mathrm{Aut}_{\bar\mu}\mathfrak{A}, \mathfrak{T}_w)$ is continuous ((d) above), it is uniformly continuous for the corresponding bilateral uniformities (4A5Hd), and $\mathcal{F}$ is Cauchy for the $\mathfrak{T}_w$-bilateral uniformity (4A2Ji). It follows that $\mathcal{F}$ has a $\mathfrak{T}_w$-limit ψ say (494Bd), in which case ψa is the limit $\lim_{\pi \to \mathcal{F}} \pi a$, for the measure-algebra topology of $\mathfrak{A}$, for every $a \in \mathfrak{A}$ (494Bb). But ψ is also the $\mathfrak{T}_u$-limit of $\mathcal{F}$. **P** Suppose that $c \in \mathfrak{A}^f$ and $\epsilon > 0$. Set $V(c, \epsilon) = \{\pi : \rho'_c(\pi, \iota) \le \epsilon\}$, where ρ'_c is defined as (a) above. Then $V(c, \epsilon)$ is a $\mathfrak{T}_u$-neighbourhood of ι, so $\{(\pi, \phi) : \phi\pi^{-1} \in V(c, \epsilon)\}$ belongs to the $\mathfrak{T}_u$-bilateral uniformity, and there is an $F \in \mathcal{F}$ such that $\phi\pi^{-1} \in V(c, \epsilon)$ whenever π, $\phi \in F$.

Now if $\phi \in F$ and $a \in \mathfrak{A}$,

$$\bar\mu(c \cap (\phi a \triangle \psi a)) = \bar\mu(c \cap (\phi a \triangle \lim_{\pi \to \mathcal{F}} \pi a)) = \lim_{\pi \to \mathcal{F}} \bar\mu(c \cap (\phi a \triangle \pi a))$$

(because $b \mapsto \bar\mu(c \cap (\phi a \triangle b))$ is continuous)

$$= \lim_{\pi \to \mathcal{F}} \bar\mu(c \cap (\phi\pi^{-1}\pi a \triangle \pi a)) \le \sup_{\pi \in F, b \in \mathfrak{A}} \bar\mu(c \cap (\phi\pi^{-1} b \triangle b)) \le \epsilon.$$

Thus $\rho'_c(\phi, \psi) \le \epsilon$ for every $\phi \in F$. As c and ϵ are arbitrary, $\mathcal{F}$ is $\mathfrak{T}_u$-convergent to ψ. **Q**

As $\mathcal{F}$ is arbitrary, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is complete for the $\mathfrak{T}_u$-bilateral uniformity.

(g)(i) Suppose that ϕ belongs to the closure of G in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Let B be the set of those $b \in \mathfrak{A}$ for which there is a $\pi \in G$ such that π and ϕ agree on the principal ideal $\mathfrak{A}_b$. Then B is order-dense in $\mathfrak{A}$. **P** Suppose that $a \in \mathfrak{A} \setminus \{0\}$. Because $(\mathfrak{A}, \bar\mu)$ is semi-finite, there is a non-zero $c \in \mathfrak{A}^f$ such that $c \subseteq a$. Take $\epsilon \in]0, \bar\mu c[$. Then there is a $\pi \in G$ such that $\pi^{-1}\phi \in U(c, \epsilon)$. Let $d \in \mathfrak{A}$ be such that d supports $\pi^{-1}\phi$ and $\bar\mu(c \cap d) \le \epsilon$. Set $b = c \setminus d$. If $b' \subseteq b$, then $\pi^{-1}\phi b' = b'$, that is, $\phi b' = \pi b'$; so π and ϕ agree on $\mathfrak{A}_b$ and $b \in B$, while $0 \ne b \subseteq a$. **Q**

(ii) There is therefore a partition $\langle b_i\rangle_{i\in I}$ of unity consisting of members of B. For each $i \in I$ take $\pi_i \in G$ such that π_i and ϕ agree on $\mathfrak{A}_{b_i}$; because G is full, $\langle(b_i, \pi_i)\rangle_{i\in I}$ witnesses that $\phi \in G$. As ϕ is arbitrary, G is closed.

(h) Let $\langle c_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}^f$ with supremum 1. Then $\{U(c_n, 2^{-n}) : n \in \mathbb{N}\}$ is a base of neighbourhoods of ι. **P** If $c \in \mathfrak{A}^f$ and $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\bar\mu(c \setminus c_n) + 2^{-n} \le \epsilon$. If $\pi \in U(c_n, 2^{-n})$, there is an $a \in \mathfrak{A}$, supporting π, such that $\bar\mu(c_n \cap a) \le 2^{-n}$; in which case $\bar\mu(c \cap a) \le \epsilon$ and $\pi \in U(c, \epsilon)$. Thus we have found an n such that $U(c_n, 2^{-n}) \subseteq U(c, \epsilon)$. **Q**

By 4A5Q, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is metrizable.

(i)(α) By (h), $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and therefore G are metrizable; the bilateral uniformity of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is therefore metrizable (4A5Q(v)). By (f), $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is complete under its bilateral uniformity; by (g), G is closed, so is complete under the induced uniformity. So there is a metric on G, inducing its topology, under which G is complete, and all I have to show is that G is separable.

(β) Since the subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ generated by D is again countable, we may suppose that D is itself a subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Let $\langle c_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}^f$ with supremum 1, and $\mathfrak{B}$ a countable subalgebra of $\mathfrak{A}$, which τ-generates $\mathfrak{A}$; by 323J again, $\mathfrak{B}$ is dense in $\mathfrak{A}$ for the measure-algebra topology. For $m, n \in \mathbb{N}$, $\pi_0, \dots, \pi_m \in D$ and $b_0, \dots, b_m \in \mathfrak{B}$, write $E(m, n, \pi_0, \dots, \pi_m, b_0, \dots, b_m)$ for

$$\{\pi : \pi \in G,\ \textstyle\sum_{i=0}^m \bar\mu(c_n \cap b_i \cap \mathrm{supp}(\pi^{-1}\pi_i)) \le 2^{-n}\}.$$

(The supports are defined because $\mathfrak{A}$ is Dedekind complete; see 381F.) Let $D' \subseteq G$ be a countable set such that $D' \cap E(m, n, \pi_0, \dots, \pi_m, b_0, \dots, b_m)$ is non-empty whenever $m, n \in \mathbb{N}$, $\pi_0, \dots, \pi_m \in D$ and $b_0, \dots, b_m \in \mathfrak{B}$ are such that $E(m, n, \pi_0, \dots, \pi_m, b_0, \dots, b_m)$ is non-empty.

Suppose that $\pi \in G$, $c \in \mathfrak{A}^f$ and $\epsilon > 0$. Let $n \in \mathbb{N}$ be such that $\bar\mu(c \setminus c_n) + 2^{-n+2} < \epsilon$. We have a family $\langle(a_j, \pi_j)\rangle_{j\in J}$ such that $\langle a_j\rangle_{j\in J}$ is a partition of unity in $\mathfrak{A}$ consisting of elements of finite measure, and, for each $j \in J$, $\pi_j \in D$ and π agrees with π_j on $\mathfrak{A}_{a_j}$ (381Ia), that is, $a_j \cap \mathrm{supp}(\pi^{-1}\pi_j) = 0$. Let $j_0, \dots, j_m \in J$ be such that $\bar\mu(c_n \setminus \sup_{i\le m} a_{j_i}) \le 2^{-n}$; for each $i \le m$, let $b_i \in \mathfrak{B}$ be such that $\bar\mu(c_n \cap (b_i \triangle a_{j_i})) \le \frac{2^{-n}}{m+1}$. In this case, $\pi \in E(m, n, \pi_{j_0}, \dots, \pi_{j_m}, b_0, \dots, b_m)$, so there is a $\tilde\pi \in D' \cap E(m, n, \pi_{j_0}, \dots, \pi_{j_m}, b_0, \dots, b_m)$. Consider $d = \mathrm{supp}(\pi^{-1}\tilde\pi)$. If we set $d_i = \mathrm{supp}(\pi^{-1}\pi_{j_i}) \cup \mathrm{supp}(\tilde\pi^{-1}\pi_{j_i})$, then π and $\tilde\pi$ both agree with π_{j_i} on $1 \setminus d_i$, so $d \subseteq d_i$. Now

$$\begin{aligned}\bar\mu(c_n \cap d) &\le \bar\mu(c_n \setminus \sup_{i\le m} b_i) + \sum_{i=0}^m \bar\mu(c_n \cap b_i \cap d)\\ &\le \bar\mu(c_n \setminus \sup_{i\le m} a_{j_i}) + \sum_{i=0}^m \bar\mu(a_{j_i} \setminus b_i) + \sum_{i=0}^m \bar\mu(c_n \cap b_i \cap d_i)\\ &\le 2^{-n} + 2^{-n} + \sum_{i=0}^m \bar\mu(c_n \cap b_i \cap \mathrm{supp}(\pi^{-1}\pi_{j_i})) + \sum_{i=0}^m \bar\mu(c_n \cap b_i \cap \mathrm{supp}(\tilde\pi^{-1}\pi_{j_i}))\\ &\le 4 \cdot 2^{-n} = 2^{-n+2},\end{aligned}$$

and $\bar\mu(c \cap d) < \epsilon$. But this means that $\pi^{-1}\tilde\pi \in U(c, \epsilon)$; as c, ϵ and π are arbitrary, D' is dense in G and G is separable.

494D Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\phi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Let $T = T_\phi : L^2_{\mathbb{C}} \to L^2_{\mathbb{C}}$ be the corresponding operator on the complex Hilbert space $L^2_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathfrak{A}, \bar\mu)$ (366M). Then the following are equiveridical:

(α) ϕ is weakly mixing (definition: 372Ob);

(β) $\inf_{k\in\mathbb{N}} |(T^k w|w)| < 1$ whenever $w \in L^2_{\mathbb{C}}$, $\|w\|_2 = 1$ and $\int w = 0$;

(γ) $\inf_{k\in\mathbb{N}} |(T^k w|w)| = 0$ whenever $w \in L^2_{\mathbb{C}}$, $\|w\|_2 = 1$ and $\int w = 0$.

proof (a) Regarding $\mathbb{Z}$, with addition and its discrete topology, as a topological group, its dual group is the circle group $S^1 = \{z : z \in \mathbb{C}, |z| = 1\}$ with multiplication and its usual topology (445Bb-445Bc); the duality being given by the functional $(k, z) \mapsto z^k : \mathbb{Z} \times S^1 \to S^1$. For $u \in L^2_{\mathbb{C}}$, define $h_u : \mathbb{Z} \to \mathbb{C}$ by setting $h_u(k) = (T^k u|u)$ for $k \in \mathbb{Z}$. Then h_u is positive definite in the sense of 445L. **P** If $\zeta_0, \dots, \zeta_n \in \mathbb{C}$ and $m_0, \dots, m_n \in \mathbb{Z}$, then

$$\sum_{j,k=0}^n \zeta_j\bar\zeta_k h_u(m_j - m_k) = \sum_{j,k=0}^n \zeta_j\bar\zeta_k (T^{m_j-m_k}u|u) = \sum_{j,k=0}^n \zeta_j\bar\zeta_k (T^{m_j}u|T^{m_k}u)$$

(366Me)

$$= (\sum_{j=0}^{n}\zeta_j T^{m_j}u|\sum_{k=0}^{n}\zeta_k T^{m_k}u) \ge 0. \text{ }\mathbf{Q}$$

By Bochner's theorem (445N), there is a Radon probability measure ν_u on S^1 such that

$$\int z^k \nu_u(dz) = h_u(k) = (T^k u|u)$$

for every $k \in \mathbb{Z}$. Note that

$$\nu_u(S^1) = \int z^0 d\nu_u = (u|u) = \|u\|_2^2.$$

(b)(i) Let $P \subseteq C(S^1;\mathbb{C})$ be the set of functions which are expressible in the form

$$p(z) = \sum_{k\in\mathbb{Z}} \zeta_k z^k \text{ for every } z \in S^1$$

where $\zeta_k \in \mathbb{C}$ for every $k \in \mathbb{Z}$ and $\{k : \zeta_k \ne 0\}$ is finite. Then P is a linear subspace of the complex Banach space $C(S^1;\mathbb{C})$, closed under multiplication. Also, if $p \in P$, then $\bar p \in P$, where $\bar p(z) = \overline{p(z)}$ for every $z \in S^1$. **P** If $p(z) = \sum_{k\in\mathbb{Z}} \zeta_k z^k$, then

$$\bar p(z) = \sum_{k\in\mathbb{Z}} \bar\zeta_k z^{-k} = \sum_{k\in\mathbb{Z}} \bar\zeta_{-k} z^k$$

for every $z \in S^1$. **Q** Of course P contains the constant function $z \mapsto z^0$ and the identity function $z \mapsto z$, so by the Stone-Weierstrass theorem (281G) P is $\|\,\|_\infty$-dense in $C(S^1;\mathbb{C})$.

(ii) For any $p \in P$ the coefficients of the corresponding expression $p(z) = \sum_{k\in\mathbb{Z}} \zeta_k z^k$ are uniquely defined, since

$$\zeta_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}p(e^{it})dt$$

for every k. So we can define u_p, for every $u \in L^2_{\mathbb{C}}$, by saying that $u_p = \sum_{k\in\mathbb{Z}} \zeta_k T^k u$. Now we have

$$(u_p|u) = \sum_{k\in\mathbb{Z}} \zeta_k (T^k u|u) = \sum_{k\in\mathbb{Z}} \zeta_k \int z^k \nu_u(dz) = \int p\, d\nu_u.$$

We also see that

$$\int u_p = (u_p|\chi 1) = \sum_{k\in\mathbb{Z}} \zeta_k (T^k u|\chi 1)$$

$$= \sum_{k\in\mathbb{Z}} \zeta_k (u|T^{-k}\chi 1) = \sum_{k\in\mathbb{Z}} \zeta_k (u|\chi 1) = p(1)\int u.$$

It is elementary to check that if $p \in P$ and $q(z) = zp(z)$ for every $z \in S^1$, then $u_q = Tu_p$. Note also that $p \mapsto u_p : P \to L^2_{\mathbb{C}}$ is linear.

(iii) For any $p \in P$ and $u \in L^2_{\mathbb{C}}$, $\|u_p\|_2 \le \|u\|_2\|p\|_\infty$. **P** If $p(z) = \sum_{k\in\mathbb{Z}} \zeta_k z^k$ for $z \in S^1$, set

$$q(z) = p(z)\overline{p(z)} = \sum_{j,k\in\mathbb{Z}} \zeta_j \bar\zeta_k z^{j-k}$$

for $z \in S^1$. Then

$$\|u_p\|_2^2 = (\sum_{j\in\mathbb{Z}} \zeta_j T^j u|\sum_{k\in\mathbb{Z}} \zeta_k T^k u) = \sum_{j,k\in\mathbb{Z}} \zeta_j \bar\zeta_k (T^j u|T^k u) = \sum_{j,k\in\mathbb{Z}} \zeta_j \bar\zeta_k (T^{j-k}u|u)$$

$$= (u_q|u) = \int q\, d\nu_u \le \|q\|_\infty \nu_u(S^1) = \|p\|_\infty^2 \|u\|_2^2. \text{ }\mathbf{Q}$$

(c) Case 1 Suppose that $\nu_u\{z\} = 0$ whenever $u \in L^2_{\mathbb{C}}$, $z \in S^1$ and $\int u = 0$.

(i) If $u \in L^2_{\mathbb{C}}$ and $\int u = 0$, then $\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} |(T^k u|u)|^2 = 0$. **P** For any $n \in \mathbb{N}$,

$$\frac{1}{n+1}\sum_{k=0}^{n}|(T^k u|u)|^2=\frac{1}{n+1}\sum_{k=0}^{n}(T^k u|u)(u|T^k u)=\frac{1}{n+1}\sum_{k=0}^{n}(T^k u|u)(T^{-k}u|u)$$
$$=\frac{1}{n+1}\sum_{k=0}^{n}\int z^k\nu_u(dz)\int w^{-k}\nu_u(dw)$$
$$=\frac{1}{n+1}\sum_{k=0}^{n}\int z^k w^{-k}\nu_u^2(d(z,w))$$

where ν_u^2 is the product measure on $(S^1)^2$. But observe that

$$|\tfrac{1}{n+1}\textstyle\sum_{k=0}^{n} z^k w^{-k}|\le 1$$

for all z, $w\in S^1$, while for $w\ne z$ we have

$$\tfrac{1}{n+1}\textstyle\sum_{k=0}^{n} z^k w^{-k}=\frac{1-(w^{-1}z)^{n+1}}{(n+1)(1-w^{-1}z)}\to 0.$$

Since

$$\nu_u^2\{(w,z):w=z\}=\int\nu_u\{z\}\nu_u(dz)=0,$$

Lebesgue's Dominated Convergence Theorem tells us that

$$\lim_{n\to\infty}\tfrac{1}{n+1}\textstyle\sum_{k=0}^{n}|(T^k u|u)|^2=\lim_{n\to\infty}\tfrac{1}{n+1}\sum_{k=0}^{n}\int z^k w^{-k}\nu_u^2(d(z,w))=0. \ \mathbf{Q}$$

(ii) Write $\mathcal{F}_d$ for the asymptotic density filter on $\mathbb{N}$ (491S). If $u\in L^2_{\mathbb{C}}$ and $\int u=0$, then $\lim_{k\to\mathcal{F}_d}|(T^k u|u)|^2=0$, by (i) above and 491Sb. It follows at once that $\lim_{k\to\mathcal{F}_d}(T^k u|u)=0$.

In fact $\lim_{k\to\mathcal{F}_d}(T^k u|v)=0$ whenever $\int u=\int v=0$. **P** We have

$$\lim_{k\to\mathcal{F}_d}(T^k u|v)+(T^k v|u)=\lim_{k\to\mathcal{F}_d}(T^k(u+v)|u+v)-(T^k u|v)-(T^k v|v)=0, \qquad (*)$$

and similarly

$$\lim_{k\to\mathcal{F}_d} i(T^k u|v)-i(T^k v|u)=\lim_{k\to\mathcal{F}_d}(T^k(iu)|v)+(T^k v|iu)=0,$$

so $\lim_{k\to\mathcal{F}_d}(T^k u|v)-(T^k v|u)=0$; adding this to (*), $\lim_{k\to\mathcal{F}}(T^k u|v)=0$. **Q**

(iii) Now take any a, $b\in\mathfrak{A}$ and set $u=\chi a-(\bar\mu a)\chi 1$, $v=\chi b-(\bar\mu b)\chi 1$. In this case, $\int u=\int v=0$ and

$$(T^k u|v)=(\chi(\phi^k a)-(\bar\mu a)\chi 1|\chi b-(\bar\mu b)\chi 1)$$
$$=\bar\mu(b\cap\phi^k a)-\bar\mu a\cdot\bar\mu b-\bar\mu(\phi^k a)\cdot\bar\mu b+\bar\mu a\cdot\bar\mu b=\bar\mu(b\cap\phi^k a)-\bar\mu a\cdot\bar\mu b$$

for every k, so $\lim_{k\to\mathcal{F}_d}\bar\mu(b\cap\phi^k a)-\bar\mu a\cdot\bar\mu b=0$ and

$$\lim_{n\to\infty}\tfrac{1}{n+1}\textstyle\sum_{k=0}^{n}|\bar\mu(b\cap\phi^k a)-\bar\mu a\cdot\bar\mu b|=0,$$

by 491Sb in the other direction. As a and b are arbitrary, ϕ is weakly mixing.

(d) Case 2 Suppose there are $u\in L^2_{\mathbb{C}}$ and $t\in\,]-\pi,\pi]$ such that $\int u=0$ and $\nu_u\{e^{it}\}>0$.

(i) For $n\in\mathbb{N}$, set set $f_n(z)=\max(0,1-2^n|z-e^{it}|)$ for $z\in S^1$. Then

$$|zf_n(z)-e^{it}f_n(z)|\le 2^{-n}\text{ if }|z-e^{it}|\le 2^{-n},$$
$$=0\text{ for other }z\in S^1.$$

Because P is $\|\,\|_\infty$-dense in $C(S^1;\mathbb{C})$, there is a $p_n\in P$ such that $\|p_n-f_n\|_\infty\le 2^{-n}$, in which case

$$|zp_n(z)-e^{it}p_n(z)|\le 3\cdot 2^{-n}$$

for every $z\in S^1$. Set $q_n(z)=zp_n(z)$ for $z\in S^1$; then

$$\|Tu_{p_n} - e^{-it}u_{p_n}\|_2 = \|u_{q_n} - e^{-it}u_{p_n}\|_2 \le \|u\|_2\|q_n - e^{-it}p_n\|_\infty$$

(by (b-iii))

$$\le 3 \cdot 2^{-n}\|u\|_2,$$

while

$$\|u_{p_n}\|_2 \le \|u\|_2\|p_n\|_\infty \le 2\|u\|_2.$$

(ii) Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Then $v = \lim_{n\to\mathcal{F}} u_{p_n}$ is defined for the weak topology of the complex Hilbert space $L^2_{\mathbb{C}}$ (4A4Ka). Also

$$(v|u) = \lim_{n\to\mathcal{F}}(u_{p_n}|u) = \lim_{n\to\infty}\int p_n d\nu_u = \lim_{n\to\infty}\int f_n d\nu_u = \nu_u\{e^{it}\} > 0$$

so $v \ne 0$. But we also have

$$\int v = (v|\chi 1) = \lim_{n\to\mathcal{F}}(u_{p_n}|\chi 1) = \lim_{n\to\mathcal{F}} p_n(1)\int u = 0,$$

and, taking limits in the weak topology on $L^2_{\mathbb{C}}$,

$$Tv = \lim_{n\to\mathcal{F}} Tu_{p_n}$$

(because T is continuous for the weak topology, see 4A4Bd)

$$= \lim_{n\to\mathcal{F}} e^{it}u_{p_n} = e^{it}v.$$

Set $w = \frac{1}{\|v\|_2}v$; then $\|w\|_2 = 1$, $\int w = 0$,

$$\inf_{k\in\mathbb{N}} |(T^k w|w)| = \inf_{k\in\mathbb{N}} |e^{ikt}(w|w)| = 1$$

and (β) is false.

(e) Putting (c) and (d) together, we see that either (α) is true or (β) is false, that is, that (β) implies (α).

(f) On the other hand, (α) implies (γ). **P** Suppose that ϕ is weakly mixing. Then

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n} |\bar\mu(b\cap\phi^k a) - \bar\mu a\cdot\bar\mu b| = 0$$

for all a, $b \in \mathfrak{A}$; by 491Sb again,

$$\lim_{k\to\mathcal{F}_d} \bar\mu(b\cap\phi^k a) - \bar\mu a\cdot\bar\mu b = 0,$$

that is,

$$\lim_{k\to\mathcal{F}_d}(T^k\chi a|\chi b) = (\chi a|\chi 1)\cdot(\chi 1|\chi b),$$

whenever a, $b \in \mathfrak{A}$. Because $(\,|\,)$ is sesquilinear,

$$\lim_{k\to\mathcal{F}_d}(T^k u|v) = (u|\chi 1)\cdot(\chi 1|v)$$

whenever u, v belong to $S_{\mathbb{C}} = S_{\mathbb{C}}(\mathfrak{A})$, the complex linear span of $\{\chi a : a \in \mathfrak{A}\}$. Because $S_{\mathbb{C}}$ is norm-dense in $L^2_{\mathbb{C}}$ (366Mb), and $\{T^k u : k \in \mathbb{N}\}$ is norm-bounded, we shall have

$$\lim_{k\to\mathcal{F}_d}(u|T^{-k}v) = \lim_{k\to\mathcal{F}_d}(T^k u|v) = (u|\chi 1)\cdot(\chi 1|v)$$

whenever $u \in S_{\mathbb{C}}$ and $v \in L^2_{\mathbb{C}}$; now $\{T^{-k}v : k \in \mathbb{N}\}$ is norm-bounded, so

$$\lim_{k\to\mathcal{F}_d}(T^k u|v) = \lim_{k\to\mathcal{F}_d}(u|T^{-k}v) = (u|\chi 1)\cdot(\chi 1|v)$$

for all u, $v \in L^2_{\mathbb{C}}$. In particular, if $\|w\|_2 = 1$ and $\int w = 0$,

$$\inf_{k\in\mathbb{N}} |(T^k w|w)| \le \lim_{k\to\mathcal{F}_d} |(T^k w|w)| = |(w|\chi 1)|^2 = 0,$$

as required. **Q**

(g) Since (γ) obviously implies (β), the three conditions are indeed equiveridical.

494E Theorem (Halmos 44, Rokhlin 48) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and give $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ its weak topology.

(a) If $\mathfrak{A}\neq\{0,1\}$, the set of mixing measure-preserving Boolean automorphisms is meager in $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$.

(b) If $\mathfrak{A}$ is atomless and homogeneous, the set of two-sided Bernouilli shifts on $(\mathfrak{A},\bar{\mu})$ (definition: 385Qb) is dense in $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$.

(c) If $\mathfrak{A}$ has countable Maharam type, the set of weakly mixing measure-preserving Boolean automorphisms is a G_δ subset of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$.

(d) If $\mathfrak{A}$ is atomless and has countable Maharam type, the set of weakly mixing measure-preserving Boolean automorphisms which are not mixing is comeager in $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, and is not empty.

proof (a) Take $a\in\mathfrak{A}\setminus\{0,1\}$. Let $\delta>0$ be such that $\bar{\mu}a>\delta+(\bar{\mu}a)^2$, and consider

$$F_n=\{\pi:\pi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A},\ \bar{\mu}(a\cap\pi^k a)\le\delta+(\bar{\mu}a)^2\text{ for every }k\ge n\}.$$

Because $\pi\mapsto\bar{\mu}(a\cap\pi^k a)$ is continuous for every k (494Bb), every F_n is closed. Because F_n cannot contain any periodic automorphism, $(\operatorname{Aut}_{\bar{\mu}}\mathfrak{A})\setminus F_n$ is dense for the uniform topology on $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ (494Cc) and therefore for the weak topology (494Cd). Accordingly $\bigcup_{n\in\mathbb{N}}F_n$ is meager; and every mixing measure-preserving automorphism belongs to $\bigcup_{n\in\mathbb{N}}F_n$.

(b) Suppose that $\phi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, $A\subseteq\mathfrak{A}$ is finite and $\epsilon>0$.

(i) By 494Cc, there is a periodic $\psi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\bar{\mu}(\phi a\,\triangle\,\psi a)\le\epsilon$ for every $a\in\mathfrak{A}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\{\psi^k a:k\in\mathbb{Z},\,a\in A\}$; then $\mathfrak{B}$ is finite (because $\{\psi^k:k\in\mathbb{Z}\}$ is finite). Let B be the set of atoms of $\mathfrak{B}$. Since $\psi[\mathfrak{B}]=\mathfrak{B}$, $\psi[B]=B$ and $\psi{\restriction}B$ is a permutation of B. Let $B_0\subseteq B$ be such that B_0 meets each orbit of $\psi{\restriction}B$ in just one point; enumerate B_0 as $\langle b_j\rangle_{j<n}$.

Let $r\ge1$ be such that $\#(B)+1\le\epsilon r$. For each $j<n$, let m_j be the size of the orbit of $\psi{\restriction}B$ containing b_j, and $p_j=\lceil r\bar{\mu}b_j\rceil-1$; set $M=\sum_{j=0}^{n-1}m_jp_j$. Because $\mathfrak{A}$ is atomless, we can find a disjoint family $\langle c_{jl}\rangle_{l<p_j}$ in $\mathfrak{A}_{b_j}$ such that $\bar{\mu}c_{jl}=\frac{1}{r}$ for every $l<p_j$. Because $\langle\psi^k b_j\rangle_{j<n,k<m_j}$ is disjoint, so is $\langle\psi^k c_{jl}\rangle_{j<n,l<p_j,k<m_j}$. Set

$$C=\{\psi^k c_{jl}:j<n,\,l<p_j,\,k<m_j\},\quad c=\sup C;$$

then

$$\bar{\mu}c=\frac{M}{r}=\frac{1}{r}\sum_{j=0}^{n-1}p_jm_j\ge\frac{1}{r}\sum_{j=0}^{n-1}m_j(r\bar{\mu}b_j-1)$$

$$=1-\frac{1}{r}\sum_{j=0}^{n-1}m_j=1-\frac{\#(B)}{r}\ge1-\epsilon.$$

We shall need to know later that

$$\frac{M}{r}=\frac{1}{r}\sum_{j=0}^{n-1}p_jm_j<\frac{1}{r}\sum_{j-0}^{n-1}rm_j\bar{\mu}b_j=1.$$

(ii) Let $f:C\to C$ be the cyclic permutation defined by setting

$$\begin{aligned}f(\psi^k c_{jl})&=\psi^{k+1}c_{jl}\text{ if }j<n,\,l<p_j,\,k\le m_j-2,\\&=c_{j,l+1}\text{ if }j<n,\,l\le p_j-2,\,k=m_j-1,\\&=c_{j+1,0}\text{ if }j\le n-2,\,l=p_j-1,\,k=m_j-1,\\&=c_{00}\text{ if }j=n-1,\,l=p_j-1,\,k=m_j-1.\end{aligned}$$

Set

$$C'=\{c:c\in C,\ f(c)\text{ and }\psi(c)\text{ are included in different members of }B\}.$$

Then $\#(C')\le n$. **P** If $c\in C'$, express it as $\psi^k c_{jl}$ where $j<n$, $l<p_j$ and $k<m_j$. We surely have $f(c)\ne\psi c$, so k must be m_j-1. In this case,

$$\psi c=\psi^{m_j}c_{jl}\subseteq\psi^{m_j}b_j=b_j,$$

so $f(c)\not\subseteq b_j$ and l must be p_j-1. Thus $c=\psi^{m_j-1}c_{j,p_j-1}$ for some $j<n$, and there are only n objects of this form. **Q**

(iii) We know that there is a two-sided Bernouilli shift π_0 on $(\mathfrak{A},\bar\mu)$ (385Sb). Now π_0 is mixing (385Se), therefore ergodic (372Qb) and aperiodic (386D). We know that $\frac{M}{r}<1$, so by 386C again there is a $d_0\in\mathfrak{A}$ such that $d_0,\pi_0 d_0,\dots,\pi_0^{M-1}d_0$ are disjoint and $\bar\mu d_0=\frac{1}{r}$. Because $\bar\mu f^i(c_{00})=\bar\mu\pi_0^i d_0=\frac{1}{r}$ for every $i<M$ and $\mathfrak{A}$ is homogeneous, there is a $\theta\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\theta(\pi_0^i d_0)=f^i(c_{00})$ for every $i<M$. Set $\pi=\theta\pi_0\theta^{-1}$; then π is a two-sided Bernouilli shift (385Sg). Now

$$\pi f^i(c_{00})=\theta\pi_0\theta^{-1}f^i(c_{00})=\theta\pi_0\pi_0^i d_0=f^{i+1}(c_{00})$$

whenever $i\le M-2$. So

$$\begin{aligned}C''&=\{c:c\in C,\ \pi c\text{ and }\psi(c)\text{ are included in different members of }B\}\\&\subseteq C'\cup\{f^{M-1}(c_{00})\}\end{aligned}$$

has at most $n+1$ members.

Because B is disjoint, $e=\sup_{b\in B}\pi b\,\triangle\,\psi b$ is disjoint from

$$\sup_{b\in B}\pi b\cap\psi b\supseteq\sup(C\setminus C'')$$

and has measure at most

$$\bar\mu(\sup C'')+\bar\mu(1\setminus c)\le\frac{n+1}{r}+\epsilon\le 2\epsilon.$$

If $a\in A$, then a is the supremum of the members of B it includes, so $\pi a\,\triangle\,\psi a\subseteq e$ and

$$\bar\mu(\pi a\,\triangle\,\phi a)\le\bar\mu(\pi a\,\triangle\,\psi a)+\bar\mu(\psi a\,\triangle\,\phi a)\le 3\epsilon.$$

(iv) Thus, given $\phi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, $A\in[\mathfrak{A}]^{<\omega}$ and $\epsilon>0$, we can find a two-sided Bernouilli shift π such that $\bar\mu(\pi a\,\triangle\,\phi a)\le 3\epsilon$ for every $a\in A$; as ϕ, A and ϵ are arbitrary, the two-sided Bernouilli shifts are dense in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

(c)(i) The point is that $L^2_{\mathbb{C}}=L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ is separable in its norm topology. **P** By 331O, there is a countable set $A\subseteq\mathfrak{A}$ which is dense for the measure-algebra topology of $\mathfrak{A}$. Let C be a countable dense subset of $\mathbb{C}$, and

$$D=\{\textstyle\sum_{j=0}^n\zeta_j\chi a_j:\zeta_0,\dots,\zeta_n\in C,\ a_0,\dots,a_n\in A\},$$

so that D is a countable subset of $L^2_{\mathbb{C}}$. Because the function

$$(\zeta_0,\dots,\zeta_n,a_0,\dots,a_n)\mapsto\textstyle\sum_{j=0}^n\zeta_j\chi a_j:\mathbb{C}^{n+1}\times\mathfrak{A}^{n+1}\to L^2_{\mathbb{C}}$$

is continuous for each n, $\overline{D}$ contains $\sum_{j=0}^n\zeta_j\chi a_j$ whenever $\zeta_0,\dots,\zeta_n\in\mathbb{C}$ and $a_0,\dots,a_n\in\mathfrak{A}$, that is, $S_{\mathbb{C}}=S_{\mathbb{C}}(\mathfrak{A})\subseteq\overline{D}$. But $S_{\mathbb{C}}$ is norm-dense in $L^2_{\mathbb{C}}$, so D also is dense and $L^2_{\mathbb{C}}$ is separable. **Q**

(ii) For $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, let $T_\pi:L^2_{\mathbb{C}}\to L^2_{\mathbb{C}}$ be the corresponding linear operator, as in 494D. We need to know that the function $\pi\mapsto T_\pi v:\mathrm{Aut}_{\bar\mu}\mathfrak{A}\to L^2_{\mathbb{C}}$ is continuous for every $v\in L^2$. **P** It is elementary to check that $a\mapsto\chi a:\mathfrak{A}\to L^2_{\mathbb{C}}$ is continuous for the measure-algebra topology on $\mathfrak{A}$, so $(\pi,a)\mapsto T_\pi\chi a=\chi\pi a$ is continuous (494Ba-494Bb), and $\pi\mapsto T_\pi\chi a$ is continuous, for every $a\in\mathfrak{A}$. Because addition and scalar multiplication are continuous on $L^2_{\mathbb{C}}$, $\pi\mapsto T_\pi v$ is continuous for every $v\in S_{\mathbb{C}}$. Now if v is any member of $L^2_{\mathbb{C}}$, $\phi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and $\epsilon>0$, there is a $v'\in S_{\mathbb{C}}$ such that $\|v-v'\|_2\le\epsilon$, in which case

$$\{\pi:\|T_\pi v-T_\phi v\|_2\le 3\epsilon\}\supseteq\{\pi:\|T_\pi v'-T_\phi v'\|\le\epsilon\}$$

is a neighbourhood of ϕ. Thus $\pi\mapsto T_\pi v$ is continuous for arbitrary $v\in L^2_{\mathbb{C}}$. **Q**

(iii) It follows from (i) that the set $V=\{v:v\in L^2_{\mathbb{C}},\ \|v\|_2=1,\ \int v=0\}$ is separable (4A2P(a-iv)). Let D' be a countable dense subset of V. For $v\in D'$, set

$$F_v=\{\pi:|(T_\pi^k v|v)|\ge\frac{1}{2}\text{ for every }k\in\mathbb{N}\}.$$

Since the maps

$$\pi\mapsto\pi^k\mapsto T_{\pi^k}v=T_\pi^k v\mapsto(T_\pi^k v|v)$$

are all continuous (494Ba and (ii) just above), F_v is closed. Consider $E=\mathrm{Aut}_{\bar\mu}\mathfrak{A}\setminus\bigcup_{v\in D'}F_v$. If $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is weakly mixing, then $(\alpha)\Rightarrow(\gamma)$ of 494D tells us that $\pi\in E$. On the other hand, if $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is not weakly mixing, $(\beta)\Rightarrow(\alpha)$ of 494D tells us that there is a $w\in V$ such that $\inf_{k\in\mathbb{N}}|(T_\pi^k w|w)|\ge 1$. Let $v\in D'$ be such that $\|v-w\|_2\le\frac{1}{4}$. Then, for any $k\in\mathbb{N}$,

$$|(T_\pi^k v|v)| \ge |(T_\pi^k w|v)| - \|T_\pi^k w - T_\pi^k v\|_2 \|v\|_2 \ge |(T_\pi^k w|v)| - \frac{1}{4}$$

$$\ge |(T_\pi^k w|w)| - \|T_\pi^k w\|_2 \|v - w\|_2 - \frac{1}{4} \ge \frac{1}{2}.$$

So $\pi \in F_v \subseteq (\mathrm{Aut}_{\bar\mu}\mathfrak{A}) \setminus E$. Thus the set of weakly mixing automorphisms is precisely E, and is a G_δ set.

(d) We know that every two-sided Bernouilli shift is weakly mixing (385Se, 372Qb), so the set E of weakly mixing automorphisms is dense, by (b) here, and G_δ, by (c), therefore comeager. By (a), the set E' of weakly mixing automorphisms which are not mixing is also comeager. By 494Be, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is a Polish space, so E' is non-empty.

494F 494Ed tells us that 'many' automorphisms of the Lebesgue probability algebra are weakly mixing but not mixing. It is another matter to give an explicit description of one. Bare-handed constructions (e.g., Chacon 69) demand ingenuity and determination. I prefer to show you an example taken from Tao l08, Lecture 12, Exercises 5 and 8, although it will take some pages in the style of this book, as it gives practice in using ideas already presented.

Example (a) There is a Radon probability measure ν on $\mathbb{R}$, zero on singletons, such that

$$\int \cos(2\pi \cdot 3^j t)\, \nu(dt) = \int \cos 2\pi t\, \nu(dt) > 0$$

for every $j \in \mathbb{N}$.

(b) Set $\sigma_{jk} = \int \cos 2\pi(k-j)t\, \nu(dt)$ for j, $k \in \mathbb{Z}$. Then there is a centered Gaussian distribution μ on $X = \mathbb{R}^{\mathbb{Z}}$ with covariance matrix $\langle \sigma_{jk} \rangle_{j,k\in\mathbb{Z}}$.

(c) Let $S : X \to X$ be the shift operator defined by saying that $(Sx)(j) = x(j+1)$ for $x \in X$ and $j \in \mathbb{Z}$. Then S is an automorphism of (X, μ).

(d) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ and $\phi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ the automorphism represented by S. Then ϕ is not mixing.

(e) ϕ is weakly mixing.

proof (a)(i) Let $\tilde\nu$ be the usual measure on $\mathcal{P}\mathbb{N}$ (254Jb, 464A). Define $h : \mathcal{P}\mathbb{N} \to \mathbb{R}$ by setting

$$h(I) = \frac{2}{3} \sum_{j\in I} 3^{-j}$$

for $I \subseteq \mathbb{N}$. Then h is continuous, so the image measure $\nu = \tilde\nu h^{-1}$ is a Radon probability measure on $\mathbb{R}$ (418I). Also h is injective, so ν, like $\tilde\nu$, is zero on singletons.

(ii) The function $t \mapsto \langle\!3t\!\rangle = 3t - \lfloor 3t \rfloor$ is inverse-measure-preserving for ν. **P** Set $\psi_0(I) = \{j : j+1 \in I\}$ for $I \subseteq \mathbb{N}$, $\psi_1(t) = \langle\!3t\!\rangle$ for $t \in \mathbb{R}$. Then $\psi_0 : \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is inverse-measure-preserving for $\tilde\nu$, because

$$\tilde\nu\{I : \psi_0(I) \cap J = K\} = \tilde\nu\{I : I \cap (J+1) = K+1\} = 2^{-\#(J)}$$

whenever $K \subseteq J \in [\mathbb{N}]^{<\omega}$. Next, for any $I \in \mathcal{P}\mathbb{N} \setminus \{\mathbb{N}, \mathbb{N} \setminus \{0\}\}$,

$$\psi_1(h(I)) = \langle 2\sum_{j\in I} 3^{-j} \rangle = 2\sum_{j\in I\setminus\{0\}} 3^{-j} = 2\sum_{j+1\in I} 3^{-j-1} = h(\psi_0(I)).$$

So $\psi_1 h =_{\text{a.e.}} h\psi_0$, and

$$\nu\psi_1^{-1} = \tilde\nu h^{-1}\psi_1^{-1} = \tilde\nu\psi_0^{-1}h^{-1} = \tilde\nu h^{-1} = \nu. \ \mathbf{Q}$$

Similarly, if we set

$$\begin{aligned} \theta(t) &= \frac{1}{3} - t \text{ if } 0 \le t \le \frac{1}{3}, \\ &= \frac{5}{3} - t \text{ if } \frac{2}{3} \le t \le 1, \\ &= t \text{ otherwise}, \end{aligned}$$

then $\theta h(I) = h(I \triangle (\mathbb{N} \setminus \{0\}))$ for every $I \subseteq \mathbb{N}$, and $\nu\theta^{-1} = \nu$.

(iii) Consequently, for any $m \in \mathbb{N}$,

$$\int \cos(2\pi \cdot 3mt)\, \nu(dt) = \int \cos(2\pi m\langle 3t \rangle)\, \nu(dt) = \int \cos 2\pi mt\, \nu(dt)$$

(235G). Inducing on j, we see that

$$\int \cos(2\pi \cdot 3^j t)\,\nu(dt) = \int \cos 2\pi t\,\nu(dt)$$

for every $j \in \mathbb{N}$.

(iv) As for $\int \cos 2\pi t\,\nu(dt)$, this is equal to $\int \cos 2\pi\theta(t)\,\nu(dt)$. Now

$$\begin{aligned}\cos 2\pi t + \cos 2\pi\theta(t) &= \cos 2\pi t + \cos 2\pi(\frac{1}{3} - t)\\ &\qquad = 2\cos\frac{\pi}{3}\cos 2\pi(t - \frac{1}{6}) > 0 \text{ if } 0 \le t \le \frac{1}{3},\\ &= \cos 2\pi t + \cos 2\pi(\frac{5}{3} - t)\\ &\qquad = 2\cos\frac{5\pi}{3}\cos 2\pi(t - \frac{5}{6}) > 0 \text{ if } \frac{2}{3} \le t \le 1;\end{aligned}$$

but $h[\mathcal{P}\mathbb{N}] \subseteq [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, so $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ is ν-conegligible, and $\cos 2\pi t + \cos 2\pi\theta(t) > 0$ for ν-almost every t. Accordingly

$$\int \cos 2\pi t\,\nu(dt) = \frac{1}{2}\int \cos 2\pi t + \cos 2\pi\theta(t)\nu(dt) > 0.$$

(b) $\sigma_{jk} = \sigma_{kj}$ for all j, $k \in \mathbb{Z}$. If $J \subseteq \mathbb{Z}$ is finite and $\langle\gamma_j\rangle_{j\in J} \in \mathbb{R}^J$, then

$$\begin{aligned}\sum_{j,k\in J}\gamma_j\gamma_k\sigma_{jk} &= \sum_{j,k\in J}\gamma_j\gamma_k\int \cos 2\pi(k-j)t\,\nu(dt)\\ &= \sum_{j,k\in J}\gamma_j\gamma_k\int \cos 2\pi kt\cos 2\pi jt + \sin 2\pi kt\sin 2\pi jt\,\nu(dt)\\ &= \int \sum_{j,k\in J}\gamma_j\gamma_k\cos 2\pi kt\cos 2\pi jt\,\nu(dt)\\ &\qquad + \int \sum_{j,k\in J}\gamma_j\gamma_k\sin 2\pi kt\sin 2\pi jt\,\nu(dt)\\ &= \int \sum_{j\in J}\gamma_j\cos 2\pi jt\sum_{k\in J}\gamma_k\cos 2\pi kt\,\nu(dt)\\ &\qquad + \int \sum_{j\in J}\gamma_j\sin 2\pi jt\sum_{k\in J}\gamma_k\sin 2\pi kt\,\nu(dt)\\ &\ge 0.\end{aligned}$$

By 456C(iv), we have a Gaussian distribution of the right kind.

(c) Of course S is linear, and $\mathbb{Z}$ is countable, so the image measure μS^{-1} is a centered Gaussian distribution (456Ba). Since

$$\begin{aligned}\int x(j)x(k)(\mu S^{-1})(dx) &= \int (Sx)(j)(Sx)(k)\mu(dx)\\ &= \int x(j+1)x(k+1)\mu(dx) = \sigma_{j+1,k+1} = \sigma_{jk}\end{aligned}$$

for all j, $k \in \mathbb{Z}$, μS^{-1} and μ have the same covariance matrix, and are equal (456Bb). Thus the bijection S is an automorphism of (X, μ).

(d) Write L^2 for $L^2(\mathfrak{A}, \bar\mu)$, and $T_\phi : L^2 \to L^2$ for the linear operator associated with the automorphism ϕ. For $k \in \mathbb{Z}$, set $f_k(x) = x(k)$ for $x \in X$ and $u_k = f_k^\bullet \in L^2$. Then $f_k S = f_{k+1}$ so $T_\phi u_k = u_{k+1}$, by 364Qd. Consider

$$\begin{aligned}(T_\phi^{3^j}u_0|u_0) &= \int u_{3^j}\times u_0 = \int x(3^j)x(0)\mu(dx)\\ &= \sigma_{3^j,0} = \int \cos(2\pi\cdot 3^j t)\nu(dt) = \int \cos 2\pi t\,\nu(dt) \ne 0,\end{aligned}$$

for every j, while

$$\int u_0 = \int x(0)\mu(dx) = 0.$$

By 372Q(a-iv), π is not mixing.

(e)(i) $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |\int e^{2\pi ikt}\nu(dt)|^2 = 0$. **P** For any $n \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{n+1}\sum_{k=0}^{n}|\int e^{2\pi ikt}\nu(dt)|^2 &= \frac{1}{n+1}\sum_{k=0}^{n}\int e^{2\pi iks}\nu(ds)\int e^{-2\pi ikt}\nu(dt)\\ &= \int \frac{1}{n+1}\sum_{k=0}^{n} e^{2\pi ik(s-t)}\nu^2(d(s,t))\end{aligned}$$

where ν^2 is the product measure on $\mathbb{R}^2$. Now, for any s, $t \in \mathbb{R}$, $|\frac{1}{n+1}\sum_{k=0}^{n} e^{2\pi ik(s-t)}| \le 1$ for every n, while if $s-t$ is not an integer,

$$\frac{1}{n+1}\sum_{k=0}^{n} e^{2\pi ik(s-t)} = \frac{1-\exp(2\pi i(n+1)(s-t))}{(n+1)(1-\exp(2\pi i(s-t)))} \to 0$$

as $n \to \infty$. As ν is zero on singletons,

$$\nu^2\{(s,t) : s-t \in \mathbb{Z}\} = \int \nu\{s : s \in t+\mathbb{Z}\}\nu(ds) = 0.$$

So

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} |\int e^{2\pi ikt}\nu(dt)|^2 = \lim_{n\to\infty} \int \frac{1}{n+1}\sum_{k=0}^{n} e^{2\pi ik(s-t)}\nu^2(d(s,t)) = 0$$

by Lebesgue's dominated convergence theorem. **Q**

Consequently, as in (c-ii) of the proof of 494D,

$$\begin{aligned}0 = \lim_{k\to\mathcal{F}_d} |\int e^{2\pi ikt}\nu(dt)|^2 &= \lim_{k\to\mathcal{F}_d}\int e^{2\pi ikt}\nu(dt)\\ &= \lim_{k\to\mathcal{F}_d} \mathcal{R}\mathrm{e}\int e^{2\pi ikt}\nu(dt) = \lim_{k\to\mathcal{F}_d}\int \cos(2\pi kt)\nu(dt).\end{aligned}$$

By 491Sc,

$$\lim_{k\to\mathcal{F}_d} \sigma_{jk} = \lim_{k\to\mathcal{F}_d} \sigma_{j,j+k} = \lim_{k\to\mathcal{F}_d} \int \cos(2\pi kt)\nu(dt) = 0.$$

(ii) Suppose that f, $g : X \to \mathbb{R}$ are functions such that, for some finite $J \subseteq \mathbb{Z}$, there are continuous bounded functions f_0, $g_0 : \mathbb{R}^J \to \mathbb{R}$ such that $f(x) = f_0(x\restriction J)$ and $g(x) = g_0(x\restriction J)$ for every $x \in \mathbb{R}^X$. Then $\lim_{n\to\mathcal{F}_d} \int fS^n \times g\,d\mu = \int f d\mu \int g\,d\mu$.

P For any $n \in \mathbb{N}$, define $R_n : X \to \mathbb{R}^{J\times\{0,1\}}$ by setting

$$R_n(x)(j,0) = x(j), \quad R_n(x)(j,1) = x(j+n)$$

for $x \in X$ and $j \in J$; then R_n is linear, so μR_n^{-1} is a centered Gaussian distribution on $\mathbb{R}^{J\times\{0,1\}}$. The covariance matrix $\sigma^{(n)}$ of μR_n^{-1} is given by

$$\begin{aligned}\sigma^{(n)}_{(j,\epsilon),(k,\epsilon')} &= \int z(j,\epsilon)z(k,\epsilon')\mu R_n^{-1}(dz) = \int (R_n x)(j,\epsilon)(R_n x)(k,\epsilon')\mu(dx)\\ &= \int x(j)x(k)\mu(dx) = \sigma_{jk} \text{ if } \epsilon = \epsilon' = 0,\\ &= \int x(j)x(k+n)\mu(dx) = \sigma_{j,k+n} \text{ if } \epsilon = 0,\ \epsilon' = 1,\\ &= \int x(j+n)x(k)\mu(dx) = \sigma_{j+n,k} = \sigma_{k,j+n} \text{ if } \epsilon = 1,\ \epsilon' = 0,\\ &= \int x(j+n)x(j+n)\mu(dx) = \sigma_{j+n,k+n} = \sigma_{jk} \text{ if } \epsilon = \epsilon' = 1\end{aligned}$$

for all j, $k \in J$. So

$$\lim_{n\to\mathcal{F}_d}\sigma^{(n)}_{(j,\epsilon),(k,\epsilon')} = \sigma_{jk} \text{ if } \epsilon=\epsilon',$$
$$= 0 \text{ if } \epsilon\ne\epsilon'.$$

Let $\tilde\mu$ be the centered Gaussian distribution $\tilde\mu$ on $\mathbb{R}^{J\times\{0,1\}}$ with covariance matrix τ where

$$\tau_{(j,\epsilon),(k,\epsilon')} = \sigma_{jk} \text{ if } \epsilon=\epsilon',$$
$$= 0 \text{ if } \epsilon\ne\epsilon'$$

for any j, $k\in J$. By 456Q, there is such a distribution and $\tilde\mu=\lim_{n\to\mathcal{F}_d}\mu R_n^{-1}$ for the narrow topology.

Next observe that, for $x\in X$ and $z\in\mathbb{R}^{J\times\{0,1\}}$,

$$\begin{aligned}R_n(x)=z &\Longrightarrow x(j+n)=z(j,1) \text{ for every } j\in J\\ &\Longleftrightarrow (S^nx)(j)=z(j,1) \text{ for every } j\in J\\ &\Longrightarrow f(S^nx)=f_0'(z),\\ R_n(x)=z &\Longrightarrow x(j)=z(j,0) \text{ for every } r<m\\ &\Longrightarrow g(x)=g_0'(z),\end{aligned}$$

where we set

$$f_0'(z)=f_0(\langle z(j,1)\rangle_{j\in J}),\quad g_0'(z)=g_0(\langle z(j,0)\rangle_{j\in J}$$

for $z\in\mathbb{R}^{J\times\{0,1\}}$. So $fS^n=f_0'R_n$, $g=g_0'R_n$,

$$\int fS^n\times g\,d\mu=\int(f_0'R_n)\times(g_0'R_n)d\mu=\int f_0'\times g_0'd(\mu R_n^{-1})$$

for every n, and

$$\lim_{n\to\mathcal{F}_d}\int fS^n\times g\,d\mu=\int f_0'\times g_0'd\tilde\mu$$

because $f_0'\times g_0'$ is a bounded continuous function (437Mb).

Since $\tau_{(j,0),(k,1)}=0$ whenever j, $k\in J$, the σ-algebras Σ_0, Σ_1 generated by coordinates in $J\times\{0\}$, $J\times\{1\}$ respectively are $\tilde\mu$-independent (456Eb). Since f_0' is Σ_0-measurable and g_0' is Σ_1-measurable,

$$\lim_{n\to\mathcal{F}_d}\int fS^n\times g\,d\mu=\int f_0'\times g_0'\,d\tilde\mu=\int f_0'd\tilde\mu\int g_0'd\tilde\mu$$

(272D, 272R)

$$\begin{aligned}&=\lim_{n\to\mathcal{F}_d}\int f_0'd(\mu R_n^{-1})\cdot\lim_{n\to\mathcal{F}_d}\int g_0'd(\mu R_n^{-1})\\ &=\lim_{n\to\mathcal{F}_d}\int f_0'R_nd\mu\int g_0'R_nd\mu\\ &=\lim_{n\to\mathcal{F}_d}\int fS^nd\mu\int g\,d\mu\\ &=\lim_{n\to\mathcal{F}_d}\int fd\mu\int g\,d\mu=\int fd\mu\cdot\int g\,d\mu,\end{aligned}$$

as required. **Q**

(iii) If F, $F'\subseteq X$ are compact, then $\lim_{n\to\mathcal{F}_d}\mu(S^{-n}[F]\cap F')=\mu F\cdot\mu F'$. **P** Let $\epsilon>0$. For $k\in\mathbb{N}$, set $J_k=\{j:j\in\mathbb{Z},\,|j|\le k\}$ and $F_k=\{x\restriction J_k:x\in F\}$. Set $f_k^{(0)}(z)=\max(0,1-2^k\rho_k(z,F_k))$ for $z\in\mathbb{R}^{J_k}$, where ρ_k is Euclidean distance in $\mathbb{R}^{J_k}$, and $f_k(x)=f_k^{(0)}(x\restriction J_k)$ for $x\in X$. Then $\langle f_k(x)\rangle_{k\in\mathbb{N}}\to\chi F(x)$ for every $x\in X$. So there is a $k\in\mathbb{N}$ such that $\int|f_k-\chi F|d\mu\le\epsilon$. Set $f=f_k$; then f is a continuous function from X to $[0,1]$, $\int|f-\chi F|\le\epsilon$, and f factors through the continuous function $f_k^{(0)}:\mathbb{R}^{J_k}\to[0,1]$.

Similarly, there is a continuous function $g:X\to[0,1]$ such that $\int|g-\chi F'|d\mu\le\epsilon$ and g factors through a continuous function on $\mathbb{R}^{J_l}$ for some l. Setting $J=J_k\cup J_l$, we see that f and g satisfy the conditions of (ii) and

$$\lim_{n\to\mathcal{F}_d}\int fS^n\times g\,d\mu=\int fd\mu\int g\,d\mu.$$

But for every $n \in \mathbb{N}$,

$$|\mu(S^{-n}[F] \cap F') - \int fS^n \times g| \le \int |fS^n - \chi S^{-n}[F]| + |g - \chi F'| d\mu$$
$$= \int |f - \chi F| + |g - \chi F'| d\mu \le 2\epsilon,$$

$$|\mu F \cdot \mu F' - \int f d\mu \int g\, d\mu| \le \int |f - \chi F| + |g - \chi F'| d\mu \le 2\epsilon,$$

so

$$\limsup_{n\to\mathcal{F}_d} |\mu(S^{-n}[F] \cap F') - \mu F \cdot \mu F'|$$
$$\le 4\epsilon + \lim_{n\to\mathcal{F}_d} |\int fS^n \times g\, d\mu - \int f d\mu \int g\, d\mu| = 4\epsilon.$$

As ϵ is arbitrary, $\lim_{n\to\mathcal{F}_d} \mu(S^{-n}[F] \cap F') = \mu F \cdot \mu F'$. **Q**

(iv) Now suppose that a, $b \in \mathfrak{A}$ and $\epsilon > 0$. Because μ is a Radon measure (454J(iii)), there are compact sets F_0, $F_1 \subseteq X$ such that $\bar\mu(a \triangle F_0^\bullet) + \bar\mu(b \triangle F_1^\bullet) \le \epsilon$. Now, for any $n \in \mathbb{N}$,

$$|\bar\mu(\phi^n a \cap b) - \mu(S^{-n}[F_0] \cap F_1)| = |\bar\mu(\phi^n a \cap b) - \bar\mu(\phi^n F_0^\bullet \cap F_1^\bullet)|$$
$$\le \bar\mu(\phi^n a \triangle \phi^n F_0^\bullet) + \bar\mu(b \triangle F_1^\bullet)$$
$$= \bar\mu(a \triangle F_0^\bullet) + \bar\mu(b \triangle F_1^\bullet) \le \epsilon,$$

$$|\bar\mu a \cdot \bar\mu b - \mu F_0 \cdot \mu F_1| \le |\bar\mu a - \mu F_0| + |\bar\mu b - \mu F_1| \le \epsilon.$$

So

$$\limsup_{n\to\mathcal{F}_d} |\bar\mu(\phi^n a \cap b) - \bar\mu a \cdot \bar\mu b|$$
$$\le 2\epsilon + \lim_{n\to\mathcal{F}_d} |\mu(S^{-n}[F_0] \cap F_1) - \mu F_0 \cdot \mu F_1| = 2\epsilon$$

by (iii). As ϵ, a and b are arbitrary, ϕ is weakly mixing (using 491Sb once more).

Remark Of course the measure ν of part (a) is Cantor measure (256Hc, 256Xk).

494G Proposition Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and G a full subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, with fixed-point subalgebra $\mathfrak{C}$ (definition: 395Ga).

(a) If $a \in \mathfrak{A}^f$ and $\pi \in G$, there is a $\phi \in G$, supported by $a \cup \pi a$, such that $\phi d = \pi d$ for every $d \subseteq a$.

(b) If $(\mathfrak{A}, \bar\mu)$ is localizable and a, $b \in \mathfrak{A}^f$, then the following are equiveridical:
  (i) there is a $\pi \in G$ such that $\pi a \subseteq b$;
  (ii) $\bar\mu(a \cap c) \le \bar\mu(b \cap c)$ for every $c \in \mathfrak{C}$.

(c) If $(\mathfrak{A}, \bar\mu)$ is localizable and a, $b \in \mathfrak{A}^f$, then the following are equiveridical:
  (i) there is a $\pi \in G$ such that $\pi a = b$;
  (ii) $\bar\mu(a \cap c) = \bar\mu(b \cap c)$ for every $c \in \mathfrak{C}$.

(d) If $(\mathfrak{A}, \bar\mu)$ is totally finite (definition: 322Ab) and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$ such that $\bar\mu(a_i \cap c) = \bar\mu(b_i \cap c)$ for every $i \in I$ and $c \in \mathfrak{C}$, there is a $\pi \in G$ such that $\pi a_i = b_i$ for every $i \in I$.

(e) If $(\mathfrak{A}, \bar\mu)$ is localizable and $H = \{\pi : \pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A},\ \pi c = c$ for every $c \in \mathfrak{C}\}$, then H is the closure of G for the weak topology of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

proof (a) Let $\langle(a_i, n_i, b_i)\rangle_{i\in I}$ be a maximal family such that
— $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}_{\pi a \setminus a} \setminus \{0\}$,
— $\langle b_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}_{a \setminus \pi a}$;
— for every $i \in I$, $n_i \in \mathbb{Z}$ and $\pi^{n_i} a_i = b_i$.

Because $\bar\mu a < \infty$, I is countable. Set

$$a' = (\pi a \setminus a) \setminus \sup_{i\in I} a_i, \quad b' = (a \setminus \pi a) \setminus \sup_{i\in I} b_i;$$

then

$$\bar\mu a' = \bar\mu\pi a - \bar\mu(a\cap\pi a) - \sum_{i\in I}\bar\mu a_i = \bar\mu a - \bar\mu(a\cap\pi a) - \sum_{i\in I}\bar\mu b_i = \bar\mu b'.$$

? If $a'\ne 0$, set $c=\sup_{n\in\mathbb{Z}}\pi^n a'$. Then $\pi c = c$, so

$$\bar\mu(c\cap b_i) = \bar\mu(c\cap\pi^{n_i}a_i) = \bar\mu(\pi^{n_i}(c\cap a_i)) = \bar\mu(c\cap a_i)$$

for every $i\in I$, and

$$\begin{aligned}\bar\mu(c\cap b') &= \bar\mu(c\cap a\setminus\pi a) - \sum_{i\in I}\bar\mu(c\cap b_i) = \bar\mu(c\cap a) - \bar\mu(c\cap a\cap\pi a) - \sum_{i\in I}\bar\mu(c\cap b_i)\\ &= \bar\mu(c\cap\pi a) - \bar\mu(c\cap a\cap\pi a) - \sum_{i\in I}\bar\mu(c\cap a_i) = \bar\mu(c\cap\pi a\setminus a) - \sum_{i\in I}\bar\mu(c\cap a_i)\\ &= \bar\mu(c\cap a') = \bar\mu a' > 0,\end{aligned}$$

and $c\cap b'\ne 0$. There is therefore an $n\in\mathbb{Z}$ such that $\pi^n a'\cap b'\ne 0$. But now, setting $d = a'\cap\pi^{-n}b'$, $d\ne 0$ and we ought to have added $(d,n,\pi^n d)$ to $\langle(a_i,n_i,b_i)\rangle_{i\in I}$. **X**

Thus $\sup_{i\in I}a_i = \pi a\setminus a$ and $\sup_{i\in I}b_i = a\setminus\pi a$. Now we can define $\phi\in\operatorname{Aut}\mathfrak{A}$ by the formula

$$\begin{aligned}\phi d &= \pi d \text{ if } d\subseteq a,\\ &= \pi^{n_i}d \text{ if } i\in I \text{ and } d\subseteq a_i,\\ &= d \text{ if } d\cap(a\cup\pi a) = 0\end{aligned}$$

(381C, because I is countable and $\mathfrak{A}$ is Dedekind σ-complete). Because G is full, $\phi\in G$; ϕ is supported by $a\cup\pi a$, and ϕ agrees with π on $\mathfrak{A}_a$, as required.

(b)(i)⇒(ii) If $\pi a\subseteq b$ and $c\in\mathfrak{C}$, then

$$\bar\mu(a\cap c) = \bar\mu\pi(a\cap c) = \bar\mu(\pi a\cap\pi c) \le \bar\mu(b\cap c).$$

(ii)⇒(i) Now suppose that $\bar\mu(a\cap c)\le\bar\mu(b\cap c)$ for every $c\in\mathfrak{C}$.

(α) Consider first the case in which $a\cap b = 0$. Let $\langle(a_i,\pi_i,b_i)\rangle_{i\in I}$ be a maximal family such that
— $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}_a\setminus\{0\}$,
— $\langle b_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}_b$,
— for every $i\in I$, $\pi_i\in G$ and $\pi_i a_i = b_i$.
Set $a' = \sup_{i\in I}a_i$, $b' = \sup_{i\in I}b_i$ and

$$c = \operatorname{upr}(b\setminus b',\mathfrak{C}) = \sup_{\pi\in G}\pi(b\setminus b')\in\mathfrak{C}$$

(395G, because $\mathfrak{A}$ is Dedekind complete).

$a\cap c = a'\cap c$. **P?** Otherwise, $a\setminus a'$ meets c, so there is a $\pi\in G$ such that $(a\setminus a')\cap\pi(b\setminus b')\ne 0$, in which case we ought to have added

$$((a\setminus a')\cap\pi(b\setminus b'),\pi^{-1},\pi^{-1}(a\setminus a')\cap b\setminus b')$$

to our family $\langle(a_i,\pi_i,b_i)\rangle_{i\in I}$. **XQ**

Now note that $1\setminus c\in\mathfrak{C}$, so

$$\begin{aligned}\bar\mu(a\setminus c) \le \bar\mu(b\setminus c) = \bar\mu(b'\setminus c) &= \sum_{i\in I}\bar\mu(b_i\setminus c)\\ &= \sum_{i\in I}\bar\mu\pi_i(a_i\setminus c) = \sum_{i\in I}\bar\mu(a_i\setminus c) = \bar\mu(a'\setminus c),\end{aligned}$$

so $a\setminus c = a'\setminus c$ and $a = a'$.

Accordingly we can define $\pi\in\operatorname{Aut}\mathfrak{A}$ by setting

$$\begin{aligned}\pi d &= \pi_i d \text{ if } i\in I \text{ and } d\subseteq a_i,\\ &= \pi_i^{-1}d \text{ if } i\in I \text{ and } d\subseteq b_i,\\ &= d \text{ if } d\subseteq 1\setminus(a\cup b')\end{aligned}$$

(381C again). Because G is full, $\pi\in G$, and

$$\pi a = \pi(\sup_{i\in I} a_i) = \sup_{i\in I} \pi a_i = \sup_{i\in I} b_i = b' \subseteq b.$$

(β) For the general case, we have

$$\bar\mu(c \cap a \setminus b) = \bar\mu(c\cap a) - \bar\mu(c\cap a\cap b) \le \bar\mu(c\cap b) - \bar\mu(c\cap a\cap b) = \bar\mu(c\cap b\setminus a)$$

for every $c \in \mathfrak{C}$, so (α) tells us that there is a $\pi_0 \in G$ such that $\pi_0(a\setminus b) \subseteq b\setminus a$. Now if we set $\pi = (\overleftarrow{a\setminus b \,{}_{\pi_0}\, \pi_0(a\setminus b)})$, $\pi \in G$ (because G is full, see 381Sd), and $\pi a \subseteq b$, as required.

(c) If $\pi \in G$ and $\pi a = b$, then $\pi a \subseteq b$ and $\pi^{-1}b \subseteq a$, so (b) tells us that $\bar\mu(a\cap c) = \bar\mu(b\cap c)$ for every $c\in\mathfrak{C}$. If $\bar\mu(a\cap c) = \bar\mu(b\cap c)$ for every $c\in\mathfrak{C}$, then (b) tells us that there is a $\pi\in G$ such that $\pi a\subseteq b$; but as $\bar\mu\pi a = \bar\mu a = \bar\mu b$, we have $\pi a = b$.

(d) Let j be any object not belonging to I and set $a_j = 1\setminus \sup_{i\in I} a_i$, $b_j = 1\setminus\sup_{i\in I} b_i$. Then

$$\bar\mu(a_j\cap c) = \bar\mu c - \sum_{i\in I}\bar\mu(a_i\cap c) = \bar\mu c - \sum_{i\in I}\bar\mu(b_i\cap c) = \bar\mu(b_j\cap c)$$

for every $c\in\mathfrak{C}$. Set $J = I\cup\{j\}$. By (c), there is for each $i\in J$ a $\pi_i\in G$ such that $\pi_i a_i = b_i$. Now $\langle a_i\rangle_{i\in J}$ and $\langle b_i\rangle_{i\in J}$ are partitions of unity in $\mathfrak{A}$, so there is a $\pi\in\operatorname{Aut}\mathfrak{A}$ such that $\pi d = \pi_i d$ whenever $i\in J$ and $d\subseteq a_j$; because G is full, $\pi\in G$, and has the property we seek.

(e)(i) If $a\in\mathfrak{A}$, then $U = \{\pi : \pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A},\ \pi a\not\subseteq a\}$ is open for the weak topology. **P** The functions

$$\pi\mapsto\pi a : \operatorname{Aut}_{\bar\mu}\mathfrak{A}\to\mathfrak{A}, \quad b\mapsto b\setminus a : \mathfrak{A}\to\mathfrak{A}$$

are continuous (494Bb and 323Ba), and $c\mapsto\bar\mu c : \mathfrak{A}\to[0,\infty]$ is lower semi-continuous (323Cb, because $\mathfrak{A}$ is semi-finite), so $\pi\mapsto\bar\mu(\pi a\setminus a)$ is lower semi-continuous (4A2B(d-ii)) and $U = \{\pi : \bar\mu(\pi a\setminus a) > 0\}$ is open. **Q**

Consequently, $\{\pi : \pi c\subseteq c$ for every $c\in\mathfrak{C}\}$ is closed. But if $\pi c\subseteq c$ for every $c\in\mathfrak{C}$, then $\pi c = c$ for every $c\in\mathfrak{C}$. So H is closed. Of course H includes G, so $\overline{G}\subseteq H$.

(ii) Suppose that $\pi\in H$ and that U is an open neighbourhood of π. Then there are $a_0,\dots,a_n\in\mathfrak{A}^f$ and $\delta > 0$ such that U includes $\{\phi : \phi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A},\ \bar\mu(\pi a_i\,\triangle\,\phi a_i)\le\delta$ for every $i\le n\}$. Set $e = \sup_{i\le n} a_i$; let $\mathfrak{B}$ be the finite subalgebra of $\mathfrak{A}_e$ generated by $\{e\cap a_i : i\le n\}$, and B the set of its atoms (definition: 316K). If $b\in B$, then $\bar\mu(\pi b\cap c) = \bar\mu(b\cap c)$ for every $c\in\mathfrak{C}$, so there is a $\phi_b\in G$ such that $\phi_b b = \pi b$, by (c) above. Equally, there is a $\phi\in G$ such that $\phi e = \pi e$. Now we can define $\psi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ by saying that

$$\begin{aligned}\psi d &= \phi_b d \text{ if } b\in B \text{ and } d\subseteq b,\\ &= \phi d \text{ if } d\subseteq 1\setminus e;\end{aligned}$$

as usual, $\psi\in G$, while $\psi b = \pi b$ for every $b\in B$. But this means that $\psi a_i = \pi a_i$ for every $i\le n$, so $\psi\in G\cap U$. As U is arbitrary, $\pi\in\overline{G}$; as π is arbitrary, G is dense in H and $H = \overline{G}$.

494H Proposition Let $\mathfrak{A}$ be a Boolean algebra, G a full subgroup of $\operatorname{Aut}\mathfrak{A}$, and $a\in\mathfrak{A}$. Set $G_a = \{\pi : \pi\in G,\ \pi$ is supported by $a\}$, $H_a = \{\pi\restriction\mathfrak{A}_a : \pi\in G_a\}$.

(a) G_a is a full subgroup of $\operatorname{Aut}\mathfrak{A}$ and H_a is a full subgroup of $\operatorname{Aut}\mathfrak{A}_a$, for every $a\in\mathfrak{A}$.

(b) Suppose that $\mathfrak{A}$ is Dedekind complete, and that the fixed-point subalgebra of G is $\mathfrak{C}$. Then the fixed-point subalgebra of H_a is $\{a\cap c : c\in\mathfrak{C}\}$.

proof (a)(i) By 381Eb and 381Eh, G_a is a subgroup of G, and $\pi\mapsto\pi\restriction\mathfrak{A}_a$ is a group homomorphism from G_a to $\operatorname{Aut}\mathfrak{A}_a$, so its image H_a is a subgroup of $\operatorname{Aut}\mathfrak{A}_a$.

(ii) Suppose that $\phi\in\operatorname{Aut}\mathfrak{A}$ and that $\langle(a_i,\pi_i)\rangle_{i\in I}$ is a family in $\mathfrak{A}\times G_a$ such that $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ and $\pi_i d = \phi d$ whenever $i\in I$ and $d\subseteq a_i$. Then $\phi\in G$, because G is full; and

$$\phi d = \sup_{i\in I}\pi_i(d\cap a_i) = \sup_{i\in I} d\cap a_i = d$$

whenever $d\cap a = 0$, so ϕ is supported by a and belongs to G_a.

(iii) Suppose that $\phi\in\operatorname{Aut}\mathfrak{A}_a$ and that $\langle(a_i,\pi_i)\rangle_{i\in I}$ is a family in $\mathfrak{A}_a\times H_a$ such that $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}_a$ and $\pi_i d = \phi d$ whenever $i\in I$ and $d\subseteq a_i$. For each $i\in I$, there is a $\pi_i'\in G_a$ such that $\pi_i = \pi_a'\restriction\mathfrak{A}_a$. Take $j\notin I$ and set $J = I\cup\{j\}$, $a_j = 1\setminus a$, $\pi_j' = \iota$; define $\psi\in\operatorname{Aut}\mathfrak{A}$ by setting $\psi d = \phi d$ for $d\subseteq a$, d for $d\subseteq 1\setminus a$. Then $\langle a_j\rangle_{j\in J}$ is a partition of unity in $\mathfrak{A}$ and $\psi d = \pi_j' d$ whenever $j\in J$ and $d\subseteq a_j$, so $\psi\in G$. Also ψ is supported by a, so $\phi = \psi\restriction\mathfrak{A}_a$ belongs to H_a. As ϕ and $\langle(a_i,\pi_i)\rangle_{i\in I}$ are arbitrary, H_a is full.

(b)(i) If $c \in \mathfrak{C}$, then $\pi(a \cap c) = \pi a \cap \pi c = a \cap c$ whenever $\pi \in G$ and $\pi a = a$, so $a \cap c$ belongs to the fixed-point subalgebra of H_a.

(ii) In the other direction, take any b in the fixed-point subalgebra of H_a. Set $c = \mathrm{upr}(b, \mathfrak{C}) = \sup_{\pi \in G} \pi b$ (395G once more). Of course $b \subseteq a \cap c$. **?** If $b \neq a \cap c$, set $e = a \cap c \setminus b$. Then there is a $\pi \in G$ such that $e_1 = e \cap \pi b \neq 0$; set $e_2 = \pi^{-1} e_1 \subseteq b$ and $\phi = (\overleftarrow{e_2 \, \pi \, e_1})$. Then $\phi \in G$ (381Sd again) and ϕ is supported by $e_1 \cup e_2 \subseteq a$, so $\phi \restriction \mathfrak{A}_a \in H_a$; but $\phi b \neq b$, so this is impossible. **X** Thus b is expressed as the intersection of a with a member of $\mathfrak{C}$, as required.

494I I take the proof of the next theorem in a series of lemmas, the first being the leading special case.

Lemma (Giordano & Pestov 92) Let $(\mathfrak{A}, \bar\mu)$ be an atomless homogeneous probability algebra (definitions: 316Kb, 316N). Then $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, with its weak topology, is extremely amenable.

proof I seek to apply 493C.

(a) Take $\epsilon > 0$, a neighbourhood V of the identity in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, a finite set $I \subseteq \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and a finite family $\mathcal{A}$ of zero sets in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Let $\delta > 0$ and $K \in [\mathfrak{A}]^{<\omega}$ be such that $\pi \in V$ whenever $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and $\bar\mu(a \triangle \pi a) \le \delta$ for every $a \in K$. Let C be the set of atoms of the finite subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by K, and D the set of atoms of the subalgebra $\mathfrak{D}$ generated by $K \cup \bigcup_{\pi \in I} \pi[K]$; set $k = \#(C)$ and $k' = \#(D)$. Let $m \in \mathbb{N}$ be so large that $2kk' \le m\delta$ and $(m\delta - 1)^2 \ge 64m \ln \frac{1}{\epsilon}$; set $r = \lfloor m\delta \rfloor$, so that $\exp(-\frac{r^2}{64m}) \le \epsilon$.

(b) For each $d \in D$ let E_d be a maximal disjoint family in $\mathfrak{A}_d$ such that $\bar\mu e = \frac{1}{m}$ for every $e \in E_d$; let E be a partition of unity in $\mathfrak{A}$, including $\bigcup_{d \in D} E_d$, such that $\bar\mu e = \frac{1}{m}$ for every $e \in E$. Let H be the group of permutations of E. Then we have a group homomorphism $\theta : H \to \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\theta(\psi) \restriction E = \psi$ for every $\psi \in H$. **P** Fix $e_0 \in E$. Then for each $e \in E$ there is a measure-preserving isomorphism $\phi_e : \mathfrak{A}_{e_0} \to \mathfrak{A}_e$, because $\mathfrak{A}$ is homogeneous (331I). For $\psi \in H_E$, E and $\langle \psi e \rangle_{e \in E}$ are partitions of unity in $\mathfrak{A}$, so we can define $\theta(\psi) \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ by the formula

$$\theta(\psi)(a) = \phi_{\psi e} \phi_e^{-1} a \text{ whenever } a \subseteq e \in E.$$

It is easy to see that $\theta(\psi)(e) = \psi e$ for every $e \in E$. If ψ, $\psi' \in H_E$, then

$$\begin{aligned}\theta(\psi\psi')(a) &= \phi_{\psi\psi' e} \phi_e^{-1} a \\ &= \phi_{\psi\psi' e} \phi_{\psi' e}^{-1} \phi_{\psi' e} \phi_e^{-1} a = \theta(\psi)\theta(\psi')(a)\end{aligned}$$

whenever $a \subseteq e \in E$; so θ is a group homomorphism. **Q**

Write G for $\theta[H]$, so that G is a subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

(c) $I \subseteq GV^{-1}$. **P** Take $\pi \in I$. For $c \in C$, set

$$E'_c = \bigcup \{E_d : d \in D,\, d \subseteq c\}, \quad E''_c = \bigcup \{E_d : d \in D,\, d \subseteq \pi c\}.$$

Since $\sup E_d \subseteq d$ and $\bar\mu(d \setminus \sup E_d) \le \frac{1}{m}$ for every $d \in D$, $\sup E'_c \subseteq c$ and $\bar\mu(c \setminus \sup E'_c) \le \frac{k'}{m}$; so $m\bar\mu c - k' \le \#(E'_c) \le m\bar\mu c$. Similarly, $\sup E''_c \subseteq \pi c$ and

$$m\bar\mu c - k' = m\bar\mu\pi c - k' \le \#(E''_c) \le m\bar\mu\pi c = m\bar\mu c.$$

Let $\tilde E'_c \subseteq E'_c$, $\tilde E''_c \subseteq E''_c$ be sets of size $\min(\#(E'_c), \#(E''_c)) \ge m\bar\mu c - k'$. Setting $c' = \sup \tilde E'_c$ and $c'' = \sup \tilde E''_c$ we have

$$c' \subseteq c, \quad \bar\mu(c \setminus c') = \frac{1}{m}(m\bar\mu c - \#(\tilde E'_c)) \le \frac{k'}{m},$$

and similarly $c'' \subseteq \pi c$ and $\bar\mu(\pi c \setminus c'') \le \dfrac{k'}{m}$.

Because $\langle \tilde E'_c \rangle_{c \in C}$ and $\langle \tilde E''_c \rangle_{c \in C}$ are both disjoint, there is a $\psi \in H$ such that $\psi[\tilde E'_c] = \tilde E''_c$ for every $c \in C$. Set $\phi = \theta(\psi)$; then $\phi \in G$ and $\phi c' = c''$ for every $c \in C$. Now this means that

$$\begin{aligned}\bar\mu(c \triangle \pi^{-1}\phi c) = \bar\mu(\pi c \triangle \phi c) &\le \bar\mu(\pi c \triangle c'') + \bar\mu(c'' \triangle \phi c) \\ &= \bar\mu(\pi c \triangle c'') + \bar\mu(c' \triangle c) \le \frac{2k'}{m}\end{aligned}$$

for every $c \in C$. Consequently

$$\bar\mu(a \triangle \pi^{-1}\phi a) \le \frac{2kk'}{m} \le \delta$$

for every $a \in K$, and $\pi^{-1}\phi \in V$. Accordingly $\pi \in \phi V^{-1} \subseteq GV^{-1}$; as π is arbitrary, $I \subseteq GV^{-1}$. **Q**

(d) I am ready to introduce the functional ν demanded by the hypotheses of 493C. Let λ be the Haar probability measure on the finite group H, and ν the image measure $\lambda\theta^{-1}$, regarded as a measure on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. If $\pi \in I$, then (c) tells us that there is a $\psi \in H$ such that $\phi = \theta(\psi)$ belongs to πV. In this case, for any $A \in \mathcal{A}$,

$$\nu(\phi A) = \lambda\theta^{-1}[\phi A] = \lambda(\psi\theta^{-1}[A]) = \lambda\theta^{-1}[A] = \nu A.$$

So ν satisfies condition (ii) of 493C.

(e) As for condition (i) of 493C, consider $W = \{\psi : \psi \in H,\ \#(\{e : e \in E,\ \psi e \neq e\}) \leq r\}$. Then $\theta[W] \subseteq V$. **P** If $\psi \in W$, then $\theta(\psi)(d) = d$ whenever $d \subseteq e \in E$ and $\psi e = e$. So $\theta(\psi)$ is supported by $b = \sup\{e : e \in E,\ \psi e \neq e\}$. Now $\bar\mu b \leq \frac{r}{m} \leq \delta$. So $\bar\mu(a \mathbin{\triangle} \phi a) \leq \bar\mu b \leq \delta$ for every $a \in \mathfrak{A}$, and $\phi \in V$. **Q**

Now suppose that $F \subseteq \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $\nu F \geq \frac{1}{2}$. Then

$$\nu(VF) = \lambda\theta^{-1}[VF] \geq \lambda(W\theta^{-1}[F])$$

(because $\theta[W] \subseteq V$)

$$\geq 1 - \exp(-\frac{r^2}{64m})$$

(by 492I, because $\lambda\theta^{-1}[F] = \nu F \geq \frac{1}{2}$)

$$\geq 1 - \epsilon.$$

So ν satisfies the first condition in 493C.

(f) As ϵ, V, I and $\mathcal{A}$ are arbitrary, $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is extremely amenable.

494J Lemma Let $(\mathfrak{C}, \bar\lambda)$ be a totally finite measure algebra, $(\mathfrak{B}, \bar\nu)$ a probability algebra, and $(\mathfrak{A}, \bar\mu)$ the localizable measure algebra free product $(\mathfrak{C}, \bar\lambda)\widehat{\otimes}(\mathfrak{B}, \bar\nu)$ (325E). Give $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology, and let G be the subgroup $\{\pi : \pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A},\ \pi(c \otimes 1) = c \otimes 1 \text{ for every } c \in \mathfrak{C}\}$. Suppose that $\mathfrak{B}$ is either finite, with all its atoms of the same measure, or homogeneous. Then G is amenable, and if either $\mathfrak{B}$ is homogeneous or $\mathfrak{C}$ is atomless, G is extremely amenable.

proof (a) Let $\mathcal{E}$ be the family of finite partitions of unity in $\mathfrak{C}$ not containing $\{0\}$. Then for any $E \in \mathcal{E}$ we have a function $\theta_E : (\operatorname{Aut}_{\bar\nu}\mathfrak{B})^E \to G$ defined by saying that

$$\theta_E(\boldsymbol{\phi})(c \otimes b) = \sup_{e \in E}(c \cap e) \otimes \phi_e b$$

whenever $\boldsymbol{\phi} = \langle\phi_e\rangle_{e \in E} \in (\operatorname{Aut}_{\bar\nu}\mathfrak{B})^E$, $c \in \mathfrak{C}$ and $b \in \mathfrak{B}$. **P** For each $e \in E$, the defining universal mapping theorem 325Da tells us that there is a unique measure-preserving Boolean homomorphism $\psi_e : \mathfrak{A} \to \mathfrak{A}$ such that $\psi_e(c \otimes 1) = c \otimes 1$ and $\psi_e(1 \otimes b) = 1 \otimes \phi_e b$ for all $b \in \mathfrak{B}$ and $c \in \mathfrak{C}$. To see that ψ_e is surjective, note that $\psi_e[\mathfrak{A}]$ must be a closed subalgebra including $\mathfrak{C} \otimes \mathfrak{B}$, which is dense (324Kb, 325D(c-i)). So $\psi_e \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$. Now $\langle e \otimes 1\rangle_{e \in E}$ is a partition of unity in $\mathfrak{A}$, and $\psi_e(e \otimes 1) = e \otimes 1$ for every e, so we have a $\pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ defined by saying that $\pi a = \sup_{e \in E}\psi_e(a \cap e)$ for every $a \in \mathfrak{A}$. Because G is full, $\pi \in G$. So we can set $\theta_E(\boldsymbol{\phi}) = \pi$. Of course π is the only automorphism satisfying the given formula for $\theta_E(\boldsymbol{\phi})$. **Q**

(b)(i) It is easy to check that if $E \in \mathcal{E}$ then θ_E is a group homomorphism from $(\operatorname{Aut}_{\bar\nu}\mathfrak{B})^E$ to G; write G_E for its set of values. Because $0 \notin E$, θ_E is injective, and G_E is a subgroup of G isomorphic to the group $(\operatorname{Aut}_{\bar\nu}\mathfrak{B})^E$. Give $\operatorname{Aut}_{\bar\nu}\mathfrak{B}$ its weak topology, $(\operatorname{Aut}_{\bar\nu}\mathfrak{B})^E$ the product topology and G the topology induced by the weak topology of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

(ii) θ_E is continuous. **P** If U is a neighbourhood of the identity in G_E, there are $a_0, \dots, a_n \in \mathfrak{A}$ and $\epsilon > 0$ such that U includes $\{\pi : \pi \in G_E,\ \bar\mu(a_i \mathbin{\triangle} \pi a_i) \leq 3\epsilon \text{ for every } i \leq n\}$. For each $i \leq n$, there is an $a_i' \in \mathfrak{C} \otimes \mathfrak{B}$ such that $\bar\mu(a_i \mathbin{\triangle} a_i') \leq \epsilon$. Let $\mathfrak{B}_0$ be a finite subalgebra of $\mathfrak{B}$ such that $a_i' \in \mathfrak{C} \otimes \mathfrak{B}_0$ for every $i \leq n$. Let $\delta > 0$ be such that $\delta\bar\lambda 1 \leq \epsilon$. Then there is a neighbourhood V of the identity in $\operatorname{Aut}_{\bar\nu}\mathfrak{B}$ such that $\bar\nu(b \mathbin{\triangle} \phi b) \leq \delta$ whenever $\phi \in V$ and $b \in \mathfrak{B}_0$. If now $\boldsymbol{\phi} = \langle\phi_e\rangle_{e \in E}$ belongs to V^E, then for each $i \leq n$ we can express a_i' as $\sup_{j \leq m_i} c_{ij} \otimes b_{ij}$ where $\langle c_{ij}\rangle_{j \leq m_i}$ is a partition of unity in $\mathfrak{C}$ and $b_{ij} \in \mathfrak{B}_0$ for every $j \leq m_i$ (315Oa). So

$$\bar\mu(a_i' \mathbin{\triangle} \theta_E(\boldsymbol{\phi})a_i') \leq \sum_{\substack{j \leq m_i \\ e \in E}} \bar\mu(((c_{ij} \cap e) \otimes b_{ij}) \mathbin{\triangle} \theta_E(\boldsymbol{\phi})((c_{ij} \cap e) \otimes b_{ij}))$$

(because $\langle(c_{ij} \cap e) \otimes b_{ij}\rangle_{j \leq m_i, e \in E}$ is a disjoint family with supremum a_i')

$$= \sum_{\substack{j \le m_i \\ e \in E}} \bar\mu(((c_{ij} \cap e) \otimes b_{ij}) \triangle ((c_{ij} \cap e) \otimes \phi_e b_{ij}))$$
$$= \sum_{\substack{j \le m_i \\ e \in E}} \bar\mu((c_{ij} \cap e) \otimes (b_{ij} \triangle \phi_e b_{ij}))$$
$$= \sum_{\substack{j \le m_i \\ e \in E}} \bar\lambda(c_{ij} \cap e) \cdot \bar\nu(b_{ij} \triangle \phi_e b_{ij}) \le \delta \sum_{\substack{j \le m_i \\ e \in E}} \bar\lambda(c_{ij} \cap e) \le \epsilon,$$

and

$$\bar\mu(a_i \triangle \theta_E(\boldsymbol{\phi})a_i) \le \bar\mu(a_i \triangle a'_i) + \bar\mu(a'_i \triangle \theta_E(\boldsymbol{\phi})a'_i) + \bar\mu(\theta_E(\boldsymbol{\phi})a'_i \triangle \theta_E(\boldsymbol{\phi})a_i) \le 3\epsilon.$$

This is true for every $i \le n$, so $\theta_E(\boldsymbol{\phi}) \in U$ whenever $\boldsymbol{\phi} \in V^E$. As U is arbitrary, θ_E is continuous. **Q**

(iii) θ_E^{-1} is continuous. **P** Let V be a neighbourhood of the identity in $\mathrm{Aut}_{\bar\nu}\,\mathfrak{B}$. Then there are $\epsilon > 0$ and $b_0, \dots, b_n \in \mathfrak{B}$ such that $\phi \in V$ whenever $\phi \in \mathrm{Aut}_{\bar\nu}\,\mathfrak{B}$ and $\bar\nu(b_i \triangle \phi b_i) \le \epsilon$ for every $i \le n$. Let $\delta > 0$ be such that $\delta \le \epsilon\bar\lambda e$ for every $e \in E$, and let U be

$$\{\pi : \pi \in G_E,\ \bar\mu((e \otimes b_i) \triangle \pi(e \otimes b_i)) \le \delta \text{ whenever } e \in E \text{ and } i \le n\}.$$

Then U is a neighbourhood of the identity in G_E. If $\boldsymbol{\phi} = \langle\phi_e\rangle_{e\in E} \in (\mathrm{Aut}_{\bar\nu}\,\mathfrak{B})^E$ is such that $\theta_E(\boldsymbol{\phi}) \in U$, then for every $e \in E$ and $i \le n$ we have

$$\bar\nu(b_i \triangle \phi_e b_i) = \frac{1}{\bar\lambda e}\bar\mu((e \otimes b_i) \triangle \theta_E(\boldsymbol{\phi})(e \otimes b_i)) \le \frac{\delta}{\bar\lambda e} \le \epsilon,$$

so $\boldsymbol{\phi} \in V^E$. As V is arbitrary, θ_E^{-1} is continuous. **Q**

(iv) Putting these together, θ_E is a topological group isomorphism.

(c) The next step is to show that $\bigcup_{E\in\mathcal{E}} G_E$ is dense in G.

(i) Note first that there is an upwards-directed family $\mathbb{D}$ of finite subalgebras $\mathfrak{D}$ of $\mathfrak{B}$ such that if $\mathfrak{D} \in \mathbb{D}$ then every atom of $\mathfrak{D}$ has the same measure, and $\bigcup\mathbb{D}$ is dense in $\mathfrak{B}$ (for the measure-algebra topology of $\mathfrak{B}$). **P** If $\mathfrak{B}$ is finite, with all its atoms of the same measure, this is trivial; take $\mathbb{D} = \{\mathfrak{B}\}$. Otherwise, because $\mathfrak{B}$ is homogeneous, $(\mathfrak{B}, \bar\nu)$ must be isomorphic to the measure algebra $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ of the usual measure on $\{0,1\}^\kappa$ for some infinite cardinal κ, and we can take $\mathbb{D}$ to be the family of subalgebras determined by finite subsets of κ. **Q**

(ii) Suppose that $\pi \in G$, $a_0, \dots, a_n \in \mathfrak{A}$ and $\epsilon > 0$. Let $\mathfrak{A}_0$ be the subalgebra of $\mathfrak{A}$ generated by $a_0, \dots, a_n$ and A the set of its atoms; let $\eta > 0$ be such that $12\eta\#(A) \le \epsilon$. Consider subalgebras of $\mathfrak{A}$ of the form $\mathfrak{C}_0 \otimes \mathfrak{D}$ where $\mathfrak{C}_0$ is a finite subalgebra of $\mathfrak{C}$ and $\mathfrak{D} \in \mathbb{D}$. This is an upwards-directed family of subalgebras, and the closure of its union includes $c \otimes b$ whenever $c \in \mathfrak{C}$ and $b \in \mathfrak{B}$, so is the whole of $\mathfrak{A}$. There must therefore be a finite subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$, a $\mathfrak{D} \in \mathbb{D}$, and a', $a'' \in \mathfrak{C}_0 \otimes \mathfrak{D}$, for each $a \in A$, such that $\bar\mu(a \triangle a') \le \eta$ and $\bar\mu(\pi a \triangle a'') \le \eta$ for every $a \in A$. Note that this implies that

$$|\bar\mu a' - \bar\mu a''| \le |\bar\mu a' - \bar\mu a| + |\bar\mu a - \bar\mu a''| \le 2\eta$$

for every $a \in A$.

(iii) Let $E \in \mathcal{E}$ be the set of atoms of $\mathfrak{C}_0$, D the set of atoms of $\mathfrak{D}$, and γ the common measure of the members of D. For $e \in E$ and $a \in A$, set

$$D'_{ea} = \{d : d \in D,\ \bar\mu((e \otimes d) \cap a) > \tfrac{1}{2}\bar\mu(e \otimes d)\}, \quad b'_{ea} = \sup D'_{ea},$$

$$D''_{ea} = \{d : d \in D,\ \bar\mu((e \otimes d) \cap \pi a) > \tfrac{1}{2}\bar\mu(e \otimes d)\}, \quad b''_{ea} = \sup D''_{ea}.$$

Note that as A is disjoint, $\langle D'_{ea}\rangle_{a\in A}$ is disjoint, for each e; and similarly $\langle D''_{ea}\rangle_{a\in A}$ is disjoint for each e, because $\langle\pi a\rangle_{a\in A}$ is disjoint. Next, for $a \in A$ and $e \in E$, set $D_{ea} = \{d : d \in D,\ e \otimes d \subseteq a'\}$. Then, for each $a \in A$,

$$\begin{aligned}\bar\mu(a'\,\triangle\,\sup_{e\in E}e\otimes b'_{ea}) &= \sum_{\substack{e\in E\\ d\in D'_{ea}\setminus D_{ea}}}\bar\mu(e\otimes d)+\sum_{\substack{e\in E\\ d\in D_{ea}\setminus D'_{ea}}}\bar\mu(e\otimes d)\\ &\le \sum_{\substack{e\in E\\ d\in D'_{ea}\setminus D_{ea}}}2\bar\mu((e\otimes d)\cap a)+\sum_{\substack{e\in E\\ d\in D_{ea}\setminus D'_{ea}}}2\bar\mu((e\otimes d)\setminus a)\\ &\le \sum_{\substack{e\in E\\ d\in D\setminus D_{ea}}}2\bar\mu((e\otimes d)\cap a)+\sum_{\substack{e\in E\\ d\in D_{ea}}}2\bar\mu((e\otimes d)\setminus a)\\ &=2\bar\mu(a\setminus a')+2\bar\mu(a'\setminus a)=2\bar\mu(a\,\triangle\,a')\le 2\eta,\end{aligned}$$

and $\bar\mu(a\,\triangle\,\sup_{e\in E}e\otimes b'_{ea})\le 3\eta$. Similarly, passing through a'' in place of a', we see that $\bar\mu(\pi a\,\triangle\,\sup_{e\in E}e\otimes b''_{ea})\le 3\eta$.

Consequently, for any $a\in A$,

$$\begin{aligned}\sum_{e\in E}\bar\lambda e\cdot\gamma|\#(D'_{ea})-\#(D''_{ea})| &= \sum_{e\in E}|\bar\mu(e\otimes b'_{ea})-\bar\mu(e\otimes b''_{ea})|\\ &\le \sum_{e\in E}\bar\mu((e\otimes b'_{ea})\,\triangle\,((e\otimes 1)\cap a))\\ &\qquad +|\bar\mu((e\otimes 1)\cap a)-\bar\mu((e\otimes 1)\cap\pi a)|\\ &\qquad +\bar\mu(((e\otimes 1)\cap\pi a)\,\triangle\,(e\otimes b''_{ea}))\\ &= \sum_{e\in E}\bar\mu((e\otimes b'_{ea})\,\triangle\,((e\otimes 1)\cap a))\\ &\qquad +\bar\mu(((e\otimes 1)\cap\pi a)\,\triangle\,(e\otimes b''_{ea}))\end{aligned}$$

(because $\pi\in G$, so $(e\otimes 1)\cap\pi a=\pi((e\otimes 1)\cap a)$ for every e)

$$=\bar\mu(a\,\triangle\,\sup_{e\in E}e\otimes b'_{ea})+\bar\mu(\pi a\,\triangle\,\sup_{e\in E}e\otimes b''_{ea})\le 6\eta.$$

(iv) Fix $e\in E$ for the moment. For each $a\in A$, take $\tilde D'_{ea}\subseteq D'_{ea}$, $\tilde D''_{ea}\subseteq D''_{ea}$ such that $\#(\tilde D'_{ea})=\#(\tilde D''_{ea})=\min(\#(D'_{ea}),\#(D''_{ea}))$. As $\langle\tilde D'_{ea}\rangle_{a\in A}$ and $\langle\tilde D''_{ea}\rangle_{a\in A}$ are always disjoint families, there is a permutation $\psi_e:D\to D$ such that $\psi_e[\tilde D'_{ea}]=\tilde D''_{ea}$ for every $a\in A$. Because $(\mathfrak B,\bar\nu)$ is homogeneous, there is a $\phi_e\in\operatorname{Aut}_{\bar\nu}\mathfrak B$ such that $\phi_e d=\psi_e d$ for every $d\in D$.

(v) This gives us a family $\boldsymbol\phi=\langle\phi_e\rangle_{e\in E}$. Consider $\theta_E(\boldsymbol\phi)$. For each $a\in A$,

$$\begin{aligned}\bar\mu(\pi a\,\triangle\,\theta_E(\boldsymbol\phi)(a)) &\le \bar\mu(\pi a\,\triangle\,\sup_{e\in E}e\otimes b''_{ea})+\bar\mu((\sup_{e\in E}e\otimes b''_{ea})\,\triangle\,\theta_E(\boldsymbol\phi)(\sup_{e\in E}e\otimes b'_{ea}))\\ &\qquad +\bar\mu(\theta_E(\boldsymbol\phi)(\sup_{e\in E}e\otimes b'_{ea})\,\triangle\,\theta_E(\boldsymbol\phi)(a))\\ &\le 3\eta+\sum_{e\in E}\bar\mu((e\otimes b''_{ea})\,\triangle\,(e\otimes\phi_e b'_{ea}))+\bar\mu((\sup_{e\in E}e\otimes b'_{ea})\,\triangle\,a)\\ &\le 3\eta+\sum_{e\in E}\bar\lambda e\cdot\gamma\#(D''_{ea}\triangle\psi_e[D'_{ea}])+3\eta\\ &\le 6\eta+\sum_{e\in E}\bar\lambda e\cdot\gamma(\#(D''_{ea}\setminus\tilde D''_{ea})+\#(D'_{ea}\setminus\tilde D'_{ea}))\\ &= 6\eta+\sum_{e\in E}\bar\lambda e\cdot\gamma|\#(D''_{ea})-\#(D'_{ea})|\le 12\eta.\end{aligned}$$

(vi) Now, for each $i\le n$, set $A_i=\{a:a\in A,\ a\subseteq a_i\}$; then

$$\bar\mu(\pi a_i\,\triangle\,\theta_E(\boldsymbol\phi)(a_i))\le\textstyle\sum_{a\in A_i}\bar\mu(\pi a\,\triangle\,\theta_E(\boldsymbol\phi)(a))\le 12\eta\#(A)\le\epsilon,$$

while $\theta_E\boldsymbol\phi\in G_E$. As ϕ, $a_0,\dots,a_n$ and ϵ are arbitrary, $\bigcup_{E\in\mathcal E}G_E$ is dense in G.

(vii) Note also that if $E,\,E'\in\mathcal{E}$, there is an $F\in\mathcal{E}$ such that $G_F\supseteq G_E\cup G_{E'}$. **P** Set $F=\{e\cap e':e\in E,\ e'\in E'\}\setminus\{0\}$. If $\boldsymbol{\phi}=\langle\phi_e\rangle_{e\in E}\in(\mathrm{Aut}_{\bar\nu}\mathfrak{B})^E$, define $\langle\psi_f\rangle_{f\in F}\in(\mathrm{Aut}_{\bar\nu}\mathfrak{B})^F$ by saying that $\psi_f=\phi_e$ whenever $f\in F$, $e\in E$ and $f\subseteq e$. Then it is easy to check that $\theta_F(\langle\psi_f\rangle_{f\in F})=\theta_E(\boldsymbol{\phi})$. So $G_F\supseteq G_E$; similarly, $G_F\supseteq G_{E'}$. **Q**

So $\{G_E:E\in\mathcal{E}\}$ is an upwards-directed family of subgroups of G with dense union in G.

(d) At this point, we start looking at the rest of the hypotheses.

(i) Suppose that $\mathfrak{B}$ is atomless. Then 494I tells us that $\mathrm{Aut}_{\bar\nu}\mathfrak{B}$ is extremely amenable. So all the products $(\mathrm{Aut}_{\bar\nu}\mathfrak{B})^E$ are extremely amenable (493Bd), all the G_E are extremely amenable, and G is extremely amenable by (c) and 493Bb.

(ii) Suppose that $\mathfrak{B}$ is finite. Then $\mathrm{Aut}_{\bar\nu}\mathfrak{B}$ is finite, therefore amenable (449Cg); all the products $(\mathrm{Aut}_{\bar\nu}\mathfrak{B})^E$ are amenable (449Ce), and G is amenable (449Cb).

(e) I have still to finish the case in which $\mathfrak{C}$ is atomless and $\mathfrak{B}$ is finite. If $\mathfrak{B}=\{0\}$ then of course $G=\{\iota\}$ is extremely amenable, so we may take it that $\bar\lambda 1>0$.

(i) Take $\epsilon>0$, a neighbourhood V of the identity in G, a finite set $I\subseteq G$ and a finite family $\mathcal{A}$ of zero sets in G. Let V_1 be a neighbourhood of the identity in G such that $V_1^2\subseteq V^{-1}$. By (c), there is an $E'\in\mathcal{E}$ such that $I\subseteq G_{E'}V_1$. Set $k=\#(E')$. V_1 is a neighbourhood of the identity for the uniform topology on G (494Cd), so there is a $\delta>0$ such that $\pi\in V_1$ whenever $\pi\in G$ and the support of π has measure at most δ (494Cb). Let m be so large that $m\delta\ge k\bar\lambda 1$ and $(\frac{m\delta}{\bar\lambda 1}-1)^2\ge m\ln(\frac{2}{\epsilon})$; set $r=\lfloor\frac{m\delta}{\bar\lambda 1}\rfloor$, so that $2\exp(-\frac{r^2}{m})\le\epsilon$.

(ii) For each $e\in E'$ let D_e be a maximal disjoint set of elements of measure $\frac{1}{m}\bar\lambda 1$ in $\mathfrak{C}_e$; let $E\supseteq\bigcup_{e\in E'}D_e$ be a maximal disjoint set of elements of measure $\frac{1}{m}\bar\lambda 1$ in $\mathfrak{C}$. Note that $c=1\setminus\sup_{e\in E'}\sup D_e$ has measure at most $\frac{k}{m}\bar\lambda 1\le\delta$. Consequently $G_{E'}\subseteq G_EV_1$. **P** If $\pi'\in G_{E'}$, express $\theta_{E'}^{-1}(\pi')$ as $\langle\phi'_e\rangle_{e\in E'}$. Let $\langle\phi_e\rangle_{e\in E}\in(\mathrm{Aut}_{\bar\nu}\mathfrak{B})^E$ be such that $\phi_e=\phi'_{e'}$ whenever $e'\in E'$ and $e\in D_{e'}$, and set $\pi=\theta_E(\langle\phi_e\rangle_{e\in E})$. Then $\pi a=\pi' a$ for every $a\subseteq(1\setminus c)\otimes 1$, so $\pi^{-1}\pi'$ is supported by $c\otimes 1$ and belongs to V_1. Thus $\pi'\in\pi V_1$; as π' is arbitrary, $G_{E'}\subseteq G_EV_1$. **Q** It follows that $I\subseteq G_EV_1^2\subseteq G_EV^{-1}$.

(iii) Set $H=\mathrm{Aut}_{\bar\nu}\mathfrak{B}$, and let λ_0 be the Haar probability measure on H, that is, the uniform probability measure. Let λ be the product measure on H^E, so that λ is the Haar probability measure on H^E. Let ν be the image measure $\lambda\theta_E^{-1}$ on G. If $\pi\in I$, then G_E meets πV, so there is a $\boldsymbol{\phi}\in H^E$ such that $\theta_E(\boldsymbol{\phi})\in\pi V$; now

$$\nu(\theta_E(\boldsymbol{\phi})F)=\lambda\theta_E^{-1}[\theta_E(\boldsymbol{\phi})F]=\lambda(\boldsymbol{\phi}\theta_E^{-1}[F])=\lambda\theta_E^{-1}[F]=\nu F$$

for every $F\subseteq G$, and in particular for every $F\in\mathcal{A}$. Thus ν satisfies condition (ii) of 493C.

(iv) Set

$$U=\{\langle\phi_e\rangle_{e\in E}:\phi_e\in\mathrm{Aut}_{\bar\nu}\mathfrak{B}\text{ for every }e\in E,\ \#(\{e:\phi_e\text{ is not the identity}\})\le r\}.$$

Then $\theta_E[U]\subseteq V$. **P** If $\boldsymbol{\phi}=\langle\phi_e\rangle_{e\in E}$ belongs to U, then $b=\sup\{e:e\in E,\ \phi_e\text{ is not the identity}\}$ has measure at most $\frac{r}{m}\bar\lambda 1\le\delta$, while b supports $\theta_E(\boldsymbol{\phi})$. So $\bar\lambda(a\,\triangle\,\theta_E(\boldsymbol{\phi})(a))\le\delta$ for every $a\in\mathfrak{A}$, and $\theta_E(\boldsymbol{\phi})\in V$. **Q**

Let ρ be the normalized Hamming metric on H^E (492D). If $\boldsymbol{\phi}=\langle\phi_e\rangle_{e\in E}$ and $\boldsymbol{\psi}=\langle\psi_e\rangle_{e\in E}$ belong to H^E and $\rho(\boldsymbol{\phi},\boldsymbol{\psi})\le\frac{r}{m}$, then $\{e:\phi_e\psi_e^{-1}\text{ is not the identity}\}$ has at most r members, and $\boldsymbol{\phi}\boldsymbol{\psi}^{-1}\in U$, that is, $\boldsymbol{\phi}\in U\boldsymbol{\psi}$. So if $W\subseteq H^E$ is such that $\lambda W\ge\frac{1}{2}$,

$$\lambda(UW)\ge\lambda\{\boldsymbol{\phi}:\rho(\boldsymbol{\phi},W)\le\frac{r}{m}\}\ge 1-2\exp(-m(\frac{r}{m})^2)$$

(492Ea)

$$\ge 1-\epsilon$$

by the choice of m and r. Transferring this to G, remembering that $\theta_E:H^E\to G$ is an injective homomorphism, we get

$$\nu(VF)=\lambda\theta_E^{-1}[VF]\ge\lambda(U\theta_E^{-1}[F])\ge 1-\epsilon$$

whenever $F\subseteq G$ and $\nu F\ge\frac{1}{2}$. So ν satisfies the first condition of 493C.

(v) As ϵ, V, I and $\mathcal{A}$ were arbitrary, 493C tells us that G is extremely amenable. This completes the proof.

494K Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and give $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology. Let G be a subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$, and suppose that $G=\{\pi:\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A},\,\pi c=c$ for every $c\in\mathfrak{C}\}$. Then G is amenable, and if every atom of $\mathfrak{A}$ belongs to $\mathfrak{C}$, then G is extremely amenable.

proof (a) We need the structure theorems of §333; the final one 333R is the best adapted to our purposes here. I repeat some of the special notation used in that theorem. For $n\in\mathbb{N}$, set $\mathfrak{B}_n=\mathcal{P}(n+1)$ and let $\bar\nu_n$ be the uniform probability measure on $n+1$, so that $\mathfrak{B}_n$ has $n+1$ atoms of the same measure; for an infinite cardinal κ, let $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ be the measure algebra of the usual measure on $\{0,1\}^\kappa$. Then 333R tells us that there are a partition of unity $\langle c_i\rangle_{i\in I}$ in $\mathfrak{C}$, where I is a countable set of cardinals, and a measure-preserving isomorphism $\theta:\mathfrak{A}\to\mathfrak{A}'=\prod_{i\in I}\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_i$ such that $\theta c=\langle(c\cap c_i)\otimes 1\rangle_{i\in I}$ for every $c\in\mathfrak{C}$. In particular, for any $i\in I$,

$$\begin{aligned}(\theta c_i)(j)&=c_i\otimes 1 \text{ if } j=i,\\ &=0 \text{ otherwise,}\end{aligned}$$

that is, $\theta[\mathfrak{A}_{c_i}]$ is just the principal ideal of $\mathfrak{A}'$ corresponding to the factor $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_i$. Thus we have an isomorphism $\theta_i:\mathfrak{A}_{c_i}\to\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_i$ such that $\theta_i c=c\otimes 1$ for every $c\in\mathfrak{C}_{c_i}$.

(b) For each $i\in I$, set $H_i=\{\pi\restriction\mathfrak{A}_{c_i}:\pi\in G\}$. Because $\pi c_i=c_i$ for every $\pi\in G$, H_i is a subgroup of $\mathrm{Aut}\,\mathfrak{A}_{c_i}$, and $\pi\mapsto\pi\restriction\mathfrak{A}_{c_i}$ is a group homomorphism from G to H_i. Set $\Theta(\pi)=\langle\pi\restriction\mathfrak{A}_{c_i}\rangle_{i\in I}$ for $\pi\in G$. Then $\Theta:G\to\prod_{i\in I}H_i$ is a group homomorphism. Because $\langle c_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, Θ is injective. In the other direction, suppose that $\boldsymbol{\phi}=\langle\phi_i\rangle_{i\in I}$ is such that every ϕ_i is a measure-preserving automorphism of $\mathfrak{A}_{c_i}$ and $\phi_i c=c$ for every $c\in\mathfrak{C}_{c_i}$. Then we have a $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\pi a=\phi_i a$ whenever $i\in I$ and $a\subseteq c_i$; it is easy to check that $\pi\in G$ and now $\Theta(\pi)=\boldsymbol{\phi}$. Thus

$$H_i=\{\phi:\phi\in\mathrm{Aut}\,\mathfrak{A}_{c_i}\text{ is measure-preserving, }\phi c=c\text{ for every }c\in\mathfrak{C}_{c_i}\}$$

for each i, and Θ is a group isomorphism between G and $\prod_{i\in I}H_i$.

(c) As in part (b) of the proof of 494J, the next step is to confirm that Θ is a homeomorphism for the weak topologies. The argument is very similar.

(i) If U is a neighbourhood of the identity in G, then there are a finite set $K\subseteq A$ and an $\epsilon>0$ such that U includes $\{\pi:\pi\in G,\,\bar\mu(a\,\triangle\,\pi a)\le 2\epsilon$ for every $a\in K\}$. Let $J\subseteq I$ be a finite set such that $\sum_{i\in I\setminus J}\bar\mu c_i\le\epsilon$, and set

$$\begin{aligned}V=\{\langle\phi_i\rangle_{i\in I}:\phi_i\in H_i&\text{ for every }i\in I,\\ &\bar\mu((a\cap c_i)\,\triangle\,\phi_i(a\cap c_i))\le\tfrac{\epsilon}{1+\#(J)}\text{ for every }i\in J\text{ and }a\in K\}.\end{aligned}$$

Then V is a neighbourhood of the identity in $\prod_{i\in I}H_i$. If $\boldsymbol{\phi}=\langle\phi_i\rangle_{i\in I}$ belongs to V, and $\pi=\Theta^{-1}(\boldsymbol{\phi})$, then, for $a\in K$,

$$\begin{aligned}\bar\mu(a\,\triangle\,\pi a)&=\sum_{i\in I}\bar\mu((a\cap c_i)\,\triangle\,\pi(a\cap c_i))=\sum_{i\in I}\bar\mu((a\cap c_i)\,\triangle\,\phi_i(a\cap c_i))\\ &\le\sum_{i\in J}\bar\mu((a\cap c_i)\,\triangle\,\phi_i(a\cap c_i))+\sum_{i\in I\setminus J}\bar\mu c_i\le\tfrac{\epsilon\#(J)}{\#(J)+1}+\epsilon\le 2\epsilon,\end{aligned}$$

and $\pi\in U$. As U is arbitrary, Θ^{-1} is continuous.

(ii) If V is a neighbourhood of the identity in $\prod_{i\in I}H_i$, then there are a finite $J\subseteq I$, finite sets $K_j\subseteq\mathfrak{A}_{c_j}$ for $j\in J$, and an $\epsilon>0$ such that $\boldsymbol{\phi}=\langle\phi_i\rangle_{i\in I}$ belongs to V if $\boldsymbol{\phi}\in\prod_{i\in I}H_i$ and $\bar\mu(a\,\triangle\,\phi_j a)\le\epsilon$ whenever $j\in J$ and $a\in K_j$. In this case.

$$U=\{\pi:\pi\in G,\,\bar\mu(a\,\triangle\,\pi a)\le\epsilon\text{ whenever }a\in\textstyle\bigcup_{j\in J}K_j\}$$

is a neighbourhood of the identity in G, and $\Theta(\pi)\in V$ whenever $\pi\in U$. As V is arbitrary, Θ is continuous.

(d) Now observe that under the isomorphism θ_i the group H_i corresponds to the group of measure-preserving automorphisms of $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_i$ fixing $c\otimes 1$ for every $c\in\mathfrak{C}_{c_i}$. By 494J, H_i is amenable. By (b)-(c) and 449Ce, G is amenable.

(e) Finally, suppose that every atom of $\mathfrak{A}$ belongs to $\mathfrak{C}$, and look more closely at the algebras $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_i$ and the groups H_i. If $i\in I$ is an infinite cardinal, then $\mathfrak{B}_i$ is homogeneous and 494J tells us that H_i is extremely amenable.

If $0 \in I$, then $\mathfrak{B}_0 = \{0,1\}$ and $\mathfrak{C}_{c_0}\widehat{\otimes}\mathfrak{B}_0$ is isomorphic to $\mathfrak{C}_{c_0}$; in this case, H_0 consists of the identity alone, and is surely extremely amenable. If $i \in I$ is finite and not 0, then $\mathfrak{B}_i$ is finite; and also $\mathfrak{C}_{c_i}$ is atomless. **P?** If $c \in \mathfrak{C}_{c_i}$ is an atom, take an atom b of $\mathfrak{B}_i$; then $c \otimes b$ is an atom of $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_i$, and $\theta^{-1}(c \otimes b)$ is an atom of $\mathfrak{A}$ not belonging to $\mathfrak{C}$. **XQ** So in this case again, 494J tells us that H_i is extremely amenable. Thus G is isomorphic to a product of extremely amenable groups and is extremely amenable (493Bd).

494L Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and G a full subgroup of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, with the topology induced by the weak topology of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. Then G is amenable. If every atom of $\mathfrak{A}$ with finite measure belongs to the fixed-point subalgebra of G, then G is extremely amenable.

proof (a) To begin with, suppose that $(\mathfrak{A},\bar\mu)$ is totally finite. Let $\mathfrak{C}$ be the fixed-point subalgebra of G, and $G' \supseteq G$ the subgroup $\{\pi : \pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A},\ \pi c = c$ for every $c \in \mathfrak{C}\}$. Then G is dense in G', by 494Ge. $\mathfrak{C}$ is of course the fixed-point subalgebra of G', so G' is amenable (494K) and G is amenable (449F(a-ii)). If every atom of $\mathfrak{A}$ belongs to $\mathfrak{C}$, then G' and G are extremely amenable, by 494K and 493Bf.

(b) Now for the general case.

(i) For each $a \in \mathfrak{A}^f$, set

$$G_a = \{\pi : \pi \in G,\ \pi \text{ is supported by } a\}, \quad H_a = \{\pi\restriction\mathfrak{A}_a : \pi \in G_a\}.$$

Then H_a is a full subgroup of $\operatorname{Aut}_{\bar\mu\restriction\mathfrak{A}_a}\mathfrak{A}_a$ (494Ha), and is isomorphic to G_a; moreover, the isomorphism is a homeomorphism for the weak topologies. **P** Set $\theta(\pi) = \pi\restriction\mathfrak{A}_a$ for $\pi \in G_a$. (α) If V is a neighbourhood of the identity in H_a, let $\delta > 0$ and $K \in [\mathfrak{A}_a]^{<\omega}$ be such that

$$V \supseteq \{\phi : \phi \in H_a,\ \bar\mu(b \,\triangle\, \phi b) \le \delta \text{ for every } b \in K\};$$

then

$$U = \{\pi : \pi \in G_a,\ \bar\mu(b \,\triangle\, \pi b) \le \delta \text{ for every } b \in K\}$$

is a neighbourhood of the identity in G_a, and $\theta(\pi) \in V$ for every $\pi \in U$. So θ is continuous. (β) If U is a neighbourhood of the identity in G_a, let $\delta > 0$ and $K \in [\mathfrak{A}]^{<\omega}$ be such that

$$U \supseteq \{\pi : \pi \in G_a,\ \bar\mu(b \,\triangle\, \pi b) \le \delta \text{ for every } b \in K\};$$

then

$$V = \{\phi : \phi \in G_a,\ \bar\mu((b \cap a) \,\triangle\, \phi(b \cap a)) \le \delta \text{ for every } b \in K\}$$

is a neighbourhood of the identity in G_a, and $\theta^{-1}(\phi) \in U$ for every $\phi \in V$, because

$$\theta^{-1}(\phi)(b) = \phi(b \cap a) \cup (b \setminus a), \quad b \,\triangle\, \theta^{-1}(\phi)(b) = (b \cap a) \,\triangle\, \phi(b \cap a)$$

for every $\phi \in G_a$ and $b \in \mathfrak{A}$. Thus θ^{-1} is continuous. **Q**

By (a), H_a, and therefore G_a, is amenable.

(ii) $H = \bigcup_{a\in\mathfrak{A}^f} G_a$ is dense in G. **P** If $\pi \in G$, $a_0, \dots, a_n \in \mathfrak{A}^f$ and $\epsilon > 0$, set $a = \sup_{i\le n} a_i$ and $b = a \cup \pi a$. Then there is a $\phi \in G$ such that ϕ agrees with π on $\mathfrak{A}_a$ and ϕ is supported by b (494Ga). In this case, $\phi \in G_b$ and $\bar\mu(\pi a_i \,\triangle\, \phi a_i) = 0 \le \epsilon$ for every $i \le n$. **Q**

Since $\langle G_a\rangle_{a\in\mathfrak{A}^f}$ is upwards-directed, and every G_a is amenable, G is amenable (449Cb).

(iii) If every atom of $\mathfrak{A}$ of finite measure is fixed under the action of G, then every atom of $\mathfrak{A}_a$ is fixed under the action of H_a, for every $a \in \mathfrak{A}^f$. So every H_a and every G_a is extremely amenable, and G is extremely amenable, by 493Bb.

494M Lemma Let $\mathfrak{A}$ be a Boolean algebra, G a full subgroup of $\operatorname{Aut}\mathfrak{A}$, and $V \subseteq G$ a symmetric set. Let $\sim_G$ be the orbit equivalence relation on $\mathfrak{A}$ induced by the action of G, so that $a \sim_G b$ iff there is a $\phi \in G$ such that $\phi a = b$. Suppose that $a \in \mathfrak{A}$ and π, $\pi' \in G$ are such that

$\pi = (\overleftarrow{b\ {}_\pi\ c})$ and $\pi' = (\overleftarrow{b'\ {}_{\pi'}\ c'})$ are exchanging involutions supported by a,
$b \sim_G b'$ and $a \setminus (b \cup c) \sim_G a \setminus (b' \cup c')$,
$\pi \in V$,
whenever $\phi \in G$ is supported by a there is a $\psi \in V$ agreeing with ϕ on $\mathfrak{A}_a$.

Then $\pi' \in V^3$.

proof (a) There is a $\phi\in G$, supported by a, such that $\phi\pi'=\pi\phi$. **P** We know that there are ϕ_0, $\phi_1\in G$ such that $\phi_0(a\setminus(b'\cup c'))=a\setminus(b\cup c)$ and $\phi_1 b'=b$. Set $\phi_2=\pi\phi_1\pi'$; then $\phi_2\in G$, $\pi\phi_2=\phi_1\pi'$, $\phi_2\pi'=\pi\phi_1$ and $\phi_2 c'=c$. Because $(a\setminus(b'\cup c'),b',c',1\setminus a)$ and $(a\setminus(b\cup c),b,c,1\setminus a)$ are partitions of unity in $\mathfrak{A}$, there is a $\phi\in\operatorname{Aut}\mathfrak{A}$ such that

$$\begin{aligned}\phi d&=\phi_0 d\text{ if }d\subseteq a\setminus(b'\cup c'),\\&=\phi_1 d\text{ if }d\subseteq b',\\&=\phi_2 d\text{ if }d\subseteq c',\\&=d\text{ if }d\subseteq 1\setminus a\end{aligned}$$

(381C once more); because G is full, $\phi\in G$. Of course ϕ is supported by a. Now

$$\phi\pi'd=\phi d=\pi\phi d\text{ if }d\subseteq a\setminus(b'\cup c')$$

(because $\phi d=\phi_0 d$ is disjoint from $b\cup c$),

$$\begin{aligned}&=\phi_2\pi'd=\pi\phi_1 d=\pi\phi d\text{ if }d\subseteq b',\\&=\phi_1\pi'd=\pi\phi_2 d=\pi\phi d\text{ if }d\subseteq c',\\&=\phi d=d=\pi\phi d\text{ if }d\subseteq 1\setminus a.\end{aligned}$$

So $\phi\pi'=\pi\phi$. **Q**

(b) By our hypothesis, there is a $\psi\in V$ agreeing with ϕ on $\mathfrak{A}_a$. In this case,

$$\begin{aligned}\psi\pi'd=\phi\pi'd&=\pi\phi d=\pi\psi d\text{ if }d\subseteq a,\\&=\psi d=\pi\psi d\text{ if }d\subseteq 1\setminus a.\end{aligned}$$

So $\psi\pi'=\pi\psi$ and

$$\pi'=\psi^{-1}\pi\psi\in V^3$$

because V is symmetric and π, $\psi\in V$.

494N Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $G\subseteq\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ a full subgroup with fixed-point subalgebra $\mathfrak{C}$. Suppose that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$. For $a\in\mathfrak{A}$, let $u_a\in L^\infty(\mathfrak{C})$ be the conditional expectation of χa on $\mathfrak{C}$, and let G_a be $\{\pi:\pi\in G$ is supported by $a\}$. Suppose that $a\subseteq e$ in $\mathfrak{A}$ and $V\subseteq G$ are such that

V is symmetric, that is, $V=V^{-1}$,

for every $\phi\in G_e$ there is a $\psi\in V$ such that ϕ and ψ agree on $\mathfrak{A}_e$,

there is an involution in V with support a,

$u_a\le\frac{2}{3}u_e$.

Then $G_a\subseteq V^{18}=\{\pi_1\ldots\pi_{18}:\pi_1,\ldots,\pi_{18}\in V\}$.

proof (a)(i) Note that 494Gb, in the language of conditional expectations, tells us that if b, $c\in\mathfrak{A}$ then $b\sim_G c$ in the notation of 494M iff $u_b=u_c$. Similarly, 494Gd tells us that if $\langle b_i\rangle_{i\in I}$ and $\langle c_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$ and $u_{b_i}=u_{c_i}$ for every $i\in I$, there is a $\pi\in G$ such that $\pi b_i=c_i$ for every i.

(ii) It follows that every non-zero $b\in\mathfrak{A}$ is the support of an involution in G. **P** Because $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$, there is a $c\subseteq b$ such that $u_c=\frac{1}{2}u_b$ (494Ad); now there is a $\phi\in G$ such that $\phi c=b\setminus c$, and $\pi=(\overleftarrow{c\,{}_\phi\,b\setminus c})$ is an involution, belonging to G (381Sd once more), with support b. **Q**

(b) If $\pi'\in G$ is an involution with support $a'\subseteq e$ and $u_{a'}=u_a$, then $\pi'\in V^3$. **P** Let $\pi_0\in V$ be an involution with support a. Because $\mathfrak{A}$ is Dedekind complete, π_0 is an exchanging involution (382Fa); express it as $(\overleftarrow{b_0\,{}_{\pi_0}\,c_0})$ and π' as $(\overleftarrow{b'\,{}_{\pi'}\,c'})$. Because $\pi b_0=c_0$, $u_{b_0}=u_{c_0}$, while $u_{b_0}+u_{c_0}=u_{b_0\cup c_0}=u_a$; so $u_{b_0}=\frac{1}{2}u_a$. Similarly, $u_{b'}=\frac{1}{2}u_{a'}=u_{b_0}$. On the other hand,

$$u_{e\setminus a}=u_e-u_a=u_e-u_{a'}=u_{e\setminus a'}$$

and $e\setminus a\sim_G e\setminus a'$. So the conditions of 494M are satisfied and $\pi'\in V^3$. **Q**

(c) Now suppose that π is any involution in G with support included in a. Then $\pi\in V^6$. **P** Let b, c be such that $\pi=(\overleftarrow{b\,{}_\pi\,c})$. Once again,

$$u_b = u_c \le \frac{1}{2}u_a, \quad u_{e \setminus a} = u_e - u_a \ge \frac{1}{2}u_a,$$

so we can find $d \subseteq e \setminus a$ such that $u_d = u_c$, while there is also a $b_1 \subseteq b$ such that $u_{b_1} = \frac{1}{2}u_b$. Set

$$c_1 = \pi b_1, \quad b_2 = b \setminus b_1, \quad c_2 = \pi b_2, \quad \pi_1 = (\overleftarrow{b_1 \,{}_{\pi}\, c_1}), \quad \pi_2 = (\overleftarrow{b_2 \,{}_{\pi}\, c_2});$$

then π_1 and π_2 are involutions, with supports $b_1 \cup c_1$ and $b_2 \cup c_2$ respectively, belonging to G and such that $\pi = \pi_1\pi_2$. Next, (a-iii) tells us that there are involutions π_3, $\pi_4 \in G$ with supports d and $a \setminus (b \cup c)$ (if $a = b \cup c$, set $\pi_4 = \iota$). Since π_1, π_2, π_3 and π_4 have disjoint supports, they commute (381Ef). Consequently $\pi_1\pi_3\pi_4$, $\pi_2\pi_3\pi_4$ are involutions, belonging to G, with supports $a_1 = b_1 \cup c_1 \cup (a \setminus (b \cup c)) \cup d$, $a_2 = b_2 \cup c_2 \cup (a \setminus (b \cup c)) \cup d$ respectively. But now observe that

$$u_{a_1} = u_{b_1} + u_{c_1} + u_a - u_b - u_c + u_d = u_b + u_a - u_b = u_a,$$

and similarly $u_{a_2} = u_a$. By (b), both $\pi_1\pi_3\pi_4$ and $\pi_2\pi_3\pi_4$ belong to V^3. But this means that

$$\pi = \pi_1\pi_2 = \pi_1\pi_2\pi_3^2\pi_4^2 = \pi_1\pi_3\pi_4\pi_2\pi_3\pi_4$$

belongs to V^6, as claimed. **Q**

(d) By 382N, every member of G_a is expressible as the product of at most three involutions belonging to G_a, so belongs to V^{18}.

494O Theorem (Kittrell & Tsankov 09) Suppose that $(\mathfrak{A}, \bar\mu)$ is an atomless probability algebra and $G \subseteq \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is a full ergodic subgroup (definition: 395Ge), with the topology induced by the uniform topology of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

(a) If $V \subseteq G$ is symmetric and G can be covered by countably many left translates of V in G, then $V^{38} = \{\pi_1\pi_2\ldots\pi_{38} : \pi_1, \ldots, \pi_{38} \in V\}$ is a neighbourhood of the identity in G.

(b) If H is a topological group such that for every neighbourhood W of the identity in H there is a countable set $D \subseteq H$ such that $H = DW$, and $\theta : G \to H$ is a group homomorphism, then θ is continuous.

proof (a)(i) Let $\langle\psi_n\rangle_{n\in\mathbb{N}}$ be a sequence in G such that $G = \bigcup_{n\in\mathbb{N}} \psi_n V$. It may help if I note straight away that $\iota \in V^2$. **P** There is an $n \in \mathbb{N}$ such that $\iota \in \psi_n V$, that is, $\psi_n^{-1} \in V$; as V is symmetric, $\psi_n \in V$ and $\iota = \psi_n\psi_n^{-1}$ belongs to V^2. **Q**

(ii) As before, set $G_a = \{\pi : \pi \in G,\ \pi \text{ is supported by } a\}$ for $a \in \mathfrak{A}$. Now there is a non-zero $e \in \mathfrak{A}$ such that for every $\pi \in G_e$ there is a $\phi \in V^2$ agreeing with π on $\mathfrak{A}_e$. **P** Because $\mathfrak{A}$ is atomless, there is a disjoint sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A} \setminus \{0\}$. **?** Suppose, if possible, that for every $n \in \mathbb{N}$ there is a $\pi_n \in G_{b_n}$ such that there is no $\phi \in V^2$ agreeing with π_n on $\mathfrak{A}_{b_n}$. If $n \in \mathbb{N}$, then $V^2 = (\psi_n V)^{-1}\psi_n V$ and $\pi_n = \iota^{-1}\pi_n$, so there must be a $\pi'_n \in G_{b_n}$, either ι or π_n, not agreeing with ϕ on $\mathfrak{A}_{b_n}$ for any $\phi \in \psi_n V$. Define $\psi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ by the formula

$$\begin{aligned} \psi d &= \pi'_n d \text{ if } n \in \mathbb{N} \text{ and } d \subseteq b_n, \\ &= d \text{ if } d \cap \sup_{n\in\mathbb{N}} b_n = 0. \end{aligned}$$

Because G is full, $\psi \in G$ and there is an $m \in \mathbb{N}$ such that $\psi \in \psi_m V$. But now π'_m agrees with ψ on $\mathfrak{A}_{b_m}$, contrary to the choice of π'_m. **X** So one of the b_n will serve for e. **Q**

(iii) There is an involution $\pi \in V^2$, supported by e, such that $\bar\mu(\operatorname{supp}\pi) \le \frac{2}{3}\bar\mu e$. **P** Take disjoint b, $b' \subseteq e$ such that $\bar\mu b = \bar\mu b' = \frac{1}{2}\bar\mu e$. Because G is full and ergodic, there is a $\phi \in G$ such that $\phi b = b'$. (By 395Gf, the fixed-point subalgebra of G is $\{0, 1\}$, so we can apply 494Gb.) For every $d \in \mathfrak{A}_b$, set $\phi_d = (\overleftarrow{d \,{}_{\phi}\, \phi d})$. Because G is full, $\phi_d \in G$. Observe that

$$\phi_c\phi_d = \phi_{c\setminus d}\phi_{c\cap d}\phi_{c\cap d}\phi_{d\setminus c} = \phi_{c\setminus d}\phi_{d\setminus c} = \phi_{c\triangle d}$$

for all c, $d \subseteq b$. Set $A_n = \{d : d \in \mathfrak{A}_b,\ \phi_d \in \psi_n V\}$ for each $n \in \mathbb{N}$. Since $\mathfrak{A}_b$ is complete under its measure metric, there is an $n \in \mathbb{N}$ such that A_n is non-meager; because $\mathfrak{A}$ is atomless, $\mathfrak{A}_b$ has no isolated points; so there are d_0, $d_1 \in A_n$ such that $0 < \bar\mu(d_0 \triangle d_1) \le \frac{1}{3}\bar\mu e$. Set $d = d_0 \triangle d_1$. Then

$$\phi_d = \phi_{d_0}\phi_{d_1} = \phi_{d_0}^{-1}\phi_{d_1} \in V^{-1}\psi_n^{-1}\psi_n V = V^2,$$

and we can take ϕ_d for π. **Q**

(iv) Taking $a = \operatorname{supp}\pi$ in (iv), a and e satisfy the conditions of 494N with respect to V^2 and $\mathfrak{C} = \{0, 1\}$, so $G_a \subseteq (V^2)^{18} = V^{36}$.

(v) Finally, there is a $\delta > 0$ such that, in the language of 494C, $G \cap U(1,\delta) \subseteq V^{38}$. **P?** Otherwise, we can find for each $n \in \mathbb{N}$ a $\pi_n \in G \cap U(1, 2^{-n-1}\bar{\mu}a) \setminus V^{38}$. Set $\pi'_n = \psi_n \pi_n \psi_n^{-1}$, $b_n = \operatorname{supp} \pi'_n$; then $\bar{\mu}b_n = \bar{\mu}(\operatorname{supp} \pi_n) \le 2^{-n-1}\bar{\mu}a$ for each n (381Gd). So $b = \sup_{n\in\mathbb{N}} b_n$ has measure at most $\bar{\mu}a$, and there is a $\phi \in G$ such that $\phi b \subseteq a$. In this case, there is an $n \in \mathbb{N}$ such that $\phi^{-1} \in \psi_n V$, that is, $\phi\psi_n \in V^{-1} = V$. Now $\pi = \phi\psi_n\pi_n\psi_n^{-1}\phi^{-1}$ has support $\phi b_n \subseteq a$, so belongs to V^{36}. But this means that $\pi_n = \psi_n^{-1}\phi^{-1}\pi\phi\psi_n$ belongs to V^{38}, contrary to the choice of π_n. **XQ**

So V^{38} is a neighbourhood of ι in G, as claimed.

(b) Let W be a neighbourhood of the identity in H. Then there is a symmetric neighbourhood W_1 of the identity in H such that $W_1^{38} \subseteq W$. Set $V = \theta^{-1}[W_1]$. Let W_2 be a neighbourhood of the identity in H such that $W_2^{-1}W_2 \subseteq W_1$, and $\langle y_n\rangle_{n\in\mathbb{N}}$ a sequence in H such that $H = \bigcup_{n\in\mathbb{N}} y_n W_2$. For each $n \in \mathbb{N}$, choose $\psi_n \in G$ such that $\theta(\psi_n) \in y_n W_2$ whenever $\theta[G]$ meets $y_n W_2$. If $\pi \in G$, there is an $n \in \mathbb{N}$ such that $\theta(\pi) \in y_n W_2$; in this case, $\theta(\psi_n) \in y_n W_2$, so

$$\theta(\psi_n^{-1}\pi) \in W_2^{-1} y_n^{-1} y_n W_2 \subseteq W_1$$

and $\psi_n^{-1}\pi \in V$. Thus $\pi \in \psi_n V$; as π is arbitrary, $G = \bigcup_{n\in\mathbb{N}} \psi_n V$. By (a), V^{38} is a neighbourhood of ι; but $V^{38} \subseteq \theta^{-1}[W_1^{38}] \subseteq \theta^{-1}[W]$, so $\theta^{-1}[W]$ is a neighbourhood of ι. As W is arbitrary, θ is continuous (4A5Fa).

494P Remark Note that if a topological group H is either Lindelöf or ccc, it satisfies the condition of (b) above. **P** Let W be an open neighbourhood of the identity in H. (α) If H is Lindelöf, the result follows immediately from the fact that $\{yW : y \in H\}$ is an open cover of H. (β) If H is ccc, let W_1 be an open neighbourhood of the identity such that $W_1 W_1^{-1} \subseteq W$, and $D \subseteq H$ a maximal set such that $\langle yW_1\rangle_{y\in D}$ is disjoint. Then D is countable. If $x \in H$, there is a $y \in D$ such that $xW_1 \cap yW_1 \ne \emptyset$, that is, $x \in yW_1W_1^{-1} \subseteq yW$; thus $H = DW$. **Q** See also 494Yh.

494Q Some of the same ideas lead to an interesting group-theoretic property of the automorphism groups here.

Theorem (see Droste Holland & Ulbrich 08) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and G a full subgroup of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over the fixed-point subalgebra $\mathfrak{C}$ of G. Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of subsets of G such that $V_n^2 \subseteq V_{n+1}$ for every n and $G = \bigcup_{n\in\mathbb{N}} V_n$. Then there is an $n \in \mathbb{N}$ such that $G = V_n$.

proof (a) For the time being (down to the end of (e) below), suppose that every V_n is symmetric. As in 494N, for each $a \in \mathfrak{A}$ write u_a for the conditional expectation of χa on $\mathfrak{C}$, and set $G_a = \{\pi : \pi \in G,\ \pi \text{ is supported by } a\}$.

(b) There are an $\alpha_1 > 0$, an $a_1 \in \mathfrak{A}$ and an $n_0 \in \mathbb{N}$ such that $u_{a_1} = \alpha_1\chi 1$ and for every $\pi \in G_{a_1}$ there is a $\phi \in V_{n_0}$ agreeing with π on $\mathfrak{A}_{a_1}$. **P?** Otherwise, because $\mathfrak{A}$ is relatively atomless, we can choose inductively a disjoint sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $u_{b_n} = 2^{-n-1}\chi 1$ for each n (use 494Ad). For each $n \in \mathbb{N}$ there must be a $\pi_n \in G_{b_n}$ such that there is no $\phi \in V_n$ agreeing with π_n on $\mathfrak{A}_{b_n}$. Because G is full, there is a $\phi \in G$ agreeing with π_n on $\mathfrak{A}_{b_n}$ for every n. But now $\phi \notin \bigcup_{n\in\mathbb{N}} V_n = G$. **XQ**

(c) There is an $a_0 \subseteq a_1$ such that $u_{a_0} = \frac{2}{3}\alpha_1\chi 1$ and there is an involution $\pi \in G$ with support a_0. **P** Take disjoint a, $a' \subseteq a_1$ such that $u_a = u_{a'} = \frac{1}{3}\alpha_1\chi 1$ (494Ad again). Set $a_0 = a \cup a'$, so that $u_{a_0} = \frac{2}{3}\alpha_1\chi 1$. There is a $\phi \in G$ such that $\phi a = a'$, and $\pi = (\overleftarrow{a_{\,\phi}\,a'})$ is an involution in G with support a_0. **Q** Let $n_1 \ge n_0$ be such that $\pi \in V_{n_1}$.

(d) By 494N, $G_{a_0} \subseteq V_{n_1}^{18} \subseteq V_{n_1+5}$. Taking $k \ge \dfrac{3}{\alpha_1}$, 494Ad once more gives us a disjoint family $\langle d_i\rangle_{i<k}$ in $\mathfrak{A}$ such that $u_{d_i} = \frac{1}{k}\chi 1$ for every $i < k$; since $\sum_{i=0}^{k-1} \bar{\mu}d_i = 1$, $\langle d_i\rangle_{i<k}$ is a partition of unity, while $u_{d_i} \le \frac{1}{3}\alpha_1\chi 1$ for every i. For i, $j < k$, let $\phi_{ij} \in G$ be such that $\phi_{ij}(d_i \cup d_j) \subseteq a_0$ (494Gc). Let $n_2 \ge n_1 + 5$ be such that $\phi_{ij} \in V_{n_2}$ for all i, $j < k$. Then any involution in G belongs to $V_{n_2}^{3k^2}$. **P** Let $\pi \in G$ be an involution; by 382Fa again, we can express it as $(\overleftarrow{e_{\,\pi}\,e'})$. For i, $j < k$, set $e_{ij} = e \cap d_i \cap \pi d_j$, $e'_{ij} = \pi e_{ij} = e' \cap \pi d_i \cap d_j$; set $\pi_{ij} = (\overleftarrow{e_{ij\,\pi}\,e'_{ij}})$. In this case, because all the e_{ij} and e'_{ij} are disjoint, $\langle \pi_{ij}\rangle_{i,j<k}$ is a commuting family, and we can talk of $\prod_{i,j<k} \pi_{ij}$, which of course is equal to π. Now, for each i, $j < k$,

$$\phi_{ij}\pi_{ij}\phi_{ij}^{-1} = (\overleftarrow{\phi_{ij}e_{ij}\ {}_{\phi_{ij}\pi\phi_{ij}^{-1}}\ \phi_{ij}e'_{ij}})$$

(381Sb) belongs to $G_{a_0} \subseteq V_{n_1+5} \subseteq V_{n_2}$. So

$$\pi_{ij} = \phi_{ij}^{-1}\phi_{ij}\pi_{ij}\phi_{ij}^{-1}\phi_{ij}$$

belongs to $V_{n_2}^3$ and $\pi = \prod_{i,j<k} \pi_{ij}$ belongs to $V_{n_2}^{3k^2}$. **Q**

(e) Since, by 382N again, every member of G is expressible as a product of at most three involutions belonging to G, $G \subseteq V_{n_2}^{9k^2} \subseteq V_n$, where $n = n_2 + \lceil \log_2(9k^2) \rceil$.

(f) This completes the proof on the assumption that every V_n is symmetric. For the general case, set $W_n = V_n \cap V_n^{-1}$ for every n. Then $\langle W_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of symmetric sets with union G, and

$$W_n^2 \subseteq V_n^2 \cap V_n^{-2} = V_n^2 \cap (V_n^2)^{-1} \subseteq V_{n+1} \cap V_{n+1}^{-1} = W_{n+1}$$

for every n, so there is an $n \in \mathbb{N}$ such that $G = W_n = V_n$.

494R There are many alternative versions of 494Q; see, for instance, 494Xm. Rather than attempt a portmanteau result to cover them all, I give one which can be applied to the measure algebra of Lebesgue measure on $\mathbb{R}$ and indicates some of the new techniques required.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be an atomless localizable measure algebra, and G a full ergodic subgroup of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. Let $\langle V_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of subsets of G, covering G, with $V_n^2 \subseteq V_{n+1}$ for every n. Then there is an $n \in \mathbb{N}$ such that $G = V_n$.

proof (a) My aim is to mimic the proof of 494Q. We have a simplification because G is ergodic, but 494M will be applied in a different way. As before, it will be enough to consider the case in which every V_n is symmetric; as before, I will write G_a for $\{\pi : \pi \in G, \pi \text{ is supported by } a\}$.

Because G is ergodic, $\mathfrak{A}$ must be quasi-homogeneous (374G); as it is also atomless, there is an infinite cardinal κ such that $\mathfrak{A}_a$ is homogeneous, with Maharam type κ, for every $a \in \mathfrak{A}^f \setminus \{0\}$ (374H). If $(\mathfrak{A}, \bar{\mu})$ is totally finite, then the result is immediate from 494Q, normalizing the measure if necessary. So I will assume that $(\mathfrak{A}, \bar{\mu})$ is not totally finite. In this case, the orbits of G can be described in terms of 'magnitude' (332Ga). If $a \in \mathfrak{A}^f$, $\operatorname{mag} a = \bar{\mu}a$; otherwise, $\operatorname{mag} a$ is the cellularity of $\mathfrak{A}_a$, and there will be a disjoint family in $\mathfrak{A}_a$ of this cardinality (332F). Set $\lambda = \operatorname{mag} 1 \geq \omega$; then whenever $a \in \mathfrak{A}$ and $\operatorname{mag} a = \lambda$, there is a partition of unity $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_a$ such that $\operatorname{mag} a_n = \lambda$ for every n.

(b) The key fact, corresponding to 494Gd, is as follows: if $\langle a_i \rangle_{i\in I}$ and $\langle b_i \rangle_{i\in I}$ are partitions of unity in $\mathfrak{A}$ such that $\operatorname{mag} a_i = \operatorname{mag} b_i$ for every $i \in I$, then there is a $\phi \in G$ such that $\phi a_i = b_i$ for every $i \in I$.

P(i) Consider first the case in which all the a_i, b_i have finite measure. In this case, let $\langle (c_j, \pi_j, d_j) \rangle_{j\in J}$ be a maximal family such that

— $\langle c_j \rangle_{j\in J}$ is a disjoint family in $\mathfrak{A}^f \setminus \{0\}$,

— $\langle d_j \rangle_{j\in J}$ is a disjoint family in $\mathfrak{A}$,

— for every $j \in J$, $\pi_j \in G$, $\pi_j c_j = d_j$ and there is an $i \in I$ such that $c_j \subseteq a_i$ and $d_j \subseteq b_i$.

Set $a = 1 \setminus \sup_{j\in J} c_j$ and $b = 1 \setminus \sup_{j\in J} d_j$. **?** If $a \neq 0$, there is an $i \in I$ such that $a \cap a_i \neq 0$. In this case,

$$\sum_{j\in J} \bar{\mu}(b_i \cap d_j) = \sum_{j\in J, d_j \subseteq b_i} \bar{\mu}d_j = \sum_{j\in J, d_j \subseteq b_i} \bar{\mu}\pi_j^{-1}d_j$$
$$= \sum_{j\in J, c_j \subseteq a_i} \bar{\mu}c_j < \bar{\mu}a_i = \bar{\mu}b_i,$$

and $b \cap b_i \neq 0$. Because G is ergodic, there is a $\pi \in G$ such that $\pi(a \cap a_i) \cap (b \cap b_i) \neq 0$. Setting $d = a \cap a_i \cap \pi^{-1}(b \cap b_i)$, we ought to have added $(d, \pi, \pi d)$ to $\langle (c_j, \pi_j, d_j) \rangle_{j\in J}$. **X**

Thus $a = 0$; similarly, $b = 0$ and $\langle c_j \rangle_{j\in J}$, $\langle d_j \rangle_{j\in J}$ are partitions of unity in $\mathfrak{A}$. Because $\mathfrak{A}$ is Dedekind complete, there is a $\phi \in \mathrm{Aut}\,\mathfrak{A}$ such that $\phi d = \pi_j d$ whenever $j \in J$ and $d \subseteq c_j$, and now $\phi \in G$ and $\phi a_i = b_i$ for every $i \in I$.

(ii) For the general case, refine the partitions $\langle a_i \rangle_{i\in I}$ and $\langle b_i \rangle_{i\in I}$ as follows. For each $i \in I$, if $\bar{\mu}a_i = \bar{\mu}b_i$ is finite, take $\lambda_i = 1$, $c_{i0} = a_i$ and $d_{i0} = b_i$; otherwise, take $\lambda_i = \operatorname{mag} a_i = \operatorname{mag} b_i$, and let $\langle c_{i\xi} \rangle_{\xi<\lambda_i}$, $\langle d_{i\xi} \rangle_{\xi<\lambda_i}$ be partitions of unity in $\mathfrak{A}_{a_i}$, $\mathfrak{A}_{b_i}$ respectively with $\bar{\mu}c_{i\xi} = \bar{\mu}d_{i\xi} = 1$ for every $\xi < \lambda_i$ (332I). Now (i) tells us that there is a $\phi \in G$ such that $\phi c_{i\xi} = d_{i\xi}$ whenever $i \in I$ and $\xi < \lambda_i$, in which case ϕa_i will be equal to b_i for every $i \in I$. **Q**

(c) There are an $a_1 \in \mathfrak{A}$ and an $n_0 \in \mathbb{N}$ such that $\operatorname{mag} a_1 = \operatorname{mag}(1 \setminus a_1) = \lambda$ and whenever $\pi \in G_{a_1}$ there is a $\phi \in V_{n_0}$ agreeing with π on $\mathfrak{A}_{a_1}$. **P?** Otherwise, let $\langle b_n \rangle_{n\in\mathbb{N}}$ be a partition of unity in $\mathfrak{A}$ such that $\operatorname{mag} b_n = \lambda$ for every n. For each $n \in \mathbb{N}$ there must be a $\pi_n \in G_{b_n}$ such that there is no $\phi \in V_n$ agreeing with π_n on $\mathfrak{A}_{b_n}$. Now there is a $\phi \in G$ agreeing with π_n on $\mathfrak{A}_{b_n}$ for every n. But in this case $\phi \notin \bigcup_{n\in\mathbb{N}} V_n = G$. **XQ**

(d) There is an $a_0 \subseteq a_1$ such that $\operatorname{mag} a_0 = \operatorname{mag}(a_1 \setminus a_0) = \lambda$ and there is an involution $\pi_0 \in G$ with support a_0. **P** Take disjoint a, a', $a'' \subseteq a_1$ all of magnitude λ; by (b), there is a $\phi \in G$ such that $\phi a = a'$, and $\pi_0 = (\overleftarrow{a \,{}_{\phi}\, a'})$ is an involution with support $a_0 = a \cup a'$ of magnitude λ, while $a_1 \setminus a_0 \supseteq a''$ also has magnitude λ. **Q** Let $n_1 \ge n_0$ be such that $\pi_0 \in V_{n_1}$.

(e) If $\pi \in G$ is an involution with support $b_1 \subseteq a_1$ and $\operatorname{mag} b_1 = \operatorname{mag}(a_1 \setminus b_1) = \lambda$, then $\pi \in V_{n_1}^3$. **P** Express π_0 and π as $(\overleftarrow{a \,{}_{\pi_0}\, a'})$ and $(\overleftarrow{b \,{}_{\pi}\, b'})$ respectively, and set $\tilde a = a_1 \setminus (a \cup a')$, $\tilde b = a_1 \setminus b_1$; then a, a', b, b', $\tilde a$ and $\tilde b$ must all have magnitude λ. By (b), there is a $\phi_0 \in G$ such that

$$\phi_0 b = a, \quad \phi_0 b' = a', \quad \phi_0 \tilde b = \tilde a, \quad \phi_0(1 \setminus a_1) = 1 \setminus a_1.$$

In particular, $a \sim_G b$ and $\tilde a \sim_G \tilde b$, in the language of 494M, and (c) tells us that the final hypothesis of 494M is satisfied; so $\pi \in V_{n_1}^3$. **Q**

(f) Now suppose that π is any involution in G_{a_0}. Then $\pi \in V_{n_1}^6$. **P** Let b, b' be such that $\pi = (\overleftarrow{b \,{}_{\pi}\, b'})$. Next take disjoint c, $c' \subseteq a_1 \setminus a_0$ such that $\operatorname{mag} c = \operatorname{mag} c' = \operatorname{mag}(a_1 \setminus (a_0 \cup c \cup c')) = \lambda$. Then there there is an involution $\pi' \in G$ exchanging c and c', and π', $\pi\pi'$ are both involutions in G_{a_1} satisfying the conditions of (e). So both belong to $V_{n_1}^3$ and $\pi = \pi\pi'\pi' \in V_{n_1}^6$. **Q**

(g) Set $d_0 = a_0$, $d_1 = a_1 \setminus a_0$ and $d_2 = 1 \setminus a_1$. Then d_0, d_1 and d_2 all have magnitude λ, so for all i, $j < 3$ there is a $\phi_{ij} \in G$ such that $\phi_{ij}(d_i \cup d_j) = d_0$. Let $n_2 \ge n_1 + 3$ be such that $\phi_{ij} \in V_{n_2}$ for all i, $j < 3$. Then any involution in G belongs to $V_{n_2}^{27}$. **P** Let $\pi \in G$ be an involution; express it as $(\overleftarrow{e \,{}_{\pi}\, e'})$. For i, $j < 3$, set $e_{ij} = e \cap d_i \cap \pi d_j$, $e'_{ij} = \pi e_{ij} = e' \cap \pi d_i \cap d_j$; set $\pi_{ij} = (\overleftarrow{e_{ij} \,{}_{\pi}\, e'_{ij}})$. In this case, because all the e_{ij} and e'_{ij} are disjoint, $\langle \pi_{ij} \rangle_{i,j<3}$ is a commuting family, and we can talk of $\prod_{i,j<3} \pi_{ij}$, which of course is equal to π. Now, for each pair i, $j < 3$,

$$\phi_{ij}\pi_{ij}\phi_{ij}^{-1} = (\overleftarrow{\phi_{ij} e_{ij} \,{}_{\phi_{ij}\pi\phi_{ij}^{-1}}\, \phi_{ij} e'_{ij}})$$

is an involution in G_{a_0}, so belongs to $V_{n_1}^6 \subseteq V_{n_1+3} \subseteq V_{n_2}$. So

$$\pi_{ij} = \phi_{ij}^{-1}\phi_{ij}\pi_{ij}\phi_{ij}^{-1}\phi_{ij}$$

belongs to $V_{n_2}^3$ and $\pi = \prod_{i,j<3} \pi_{ij}$ belongs to $V_{n_2}^{27}$. **Q**

(h) Since every member of G is expressible as a product of at most three involutions in G (382N once more), $G = V_{n_2}^{81} = V_{n_2+7}$.

494X Basic exercises (a) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra. Show that the natural action of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ on $\mathfrak{A}^f$ identifies $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ with a subgroup of the isometry group G of $\mathfrak{A}^f$ when $\mathfrak{A}^f$ is given its measure metric, and that the weak topology on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ corresponds to the topology of pointwise convergence on G as described in 441G.

(b) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is purely atomic and has at most finitely many atoms of any fixed measure; (ii) $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is locally compact in its weak topology; (iii) $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is compact in its uniform topology; (iv) $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ has a Haar measure for its weak topology.

(c) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Show that the weak topology on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is that induced by the product topology on $\mathfrak{A}^{\mathfrak{A}}$ if $\mathfrak{A}$ is given its measure-algebra topology.

(d) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, G a subgroup of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $\overline{G}$ its closure for the weak topology on $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. Show that G is ergodic iff $\overline{G}$ is ergodic.

(e) Let I be a set, ν_I the usual measure on $\{0,1\}^I$, and $(\mathfrak{B}_I, \bar\nu_I)$ its measure algebra. Let Ψ be the group of measure space automorphisms g of $\{0,1\}^I$ for which there is a countable set $J \subseteq I$ such that for every $x \in \{0,1\}^I$ there is a finite set $K \subseteq J$ such that $g(x)(i) = x(i)$ for every $i \in I \setminus K$. For $g \in \Psi$, let $\pi_g \in \operatorname{Aut}_{\bar\nu_I}\mathfrak{B}_I$ be the corresponding automorphism defined by saying that $\pi_g(E^\bullet) = g^{-1}[E]^\bullet$ whenever ν_I measures E. (i) Show that $G = \{\pi_g : g \in \Psi\}$ is a full subgroup of $\operatorname{Aut}_{\bar\nu_I}\mathfrak{B}_I$. (ii) Show that G is ergodic and dense in $\operatorname{Aut}_{\bar\nu_I}\mathfrak{B}_I$ for the weak topology on $\operatorname{Aut}_{\bar\nu_I}\mathfrak{B}_I$.

(f) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$, and give $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ its weak topology. (i) Show that the entropy function h of 385M is Borel measurable. (*Hint*: 385Xr.) (ii) Show that the set of ergodic measure-preserving automorphisms is a dense G_δ set. (*Hint*: let $D \subseteq \mathfrak{A}$ be a countable dense set. Show that $\pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is ergodic iff $\inf_{n\in\mathbb{N}} \|\frac{1}{n+1}\sum_{i=0}^{n} \chi(\pi^i d) - \bar\mu d \cdot \chi 1\|_1 = 0$ for every $d \in D$.)

(g) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless semi-finite measure algebra. (i) Show that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is metrizable under its weak topology iff $(\mathfrak{A}, \bar{\mu})$ is σ-finite and has countable Maharam type. (ii) Show that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is metrizable under its uniform topology iff $(\mathfrak{A}, \bar{\mu})$ is σ-finite.

(h) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Show that $\operatorname{supp} : \operatorname{Aut}_{\bar{\mu}}\mathfrak{A} \to \mathfrak{A}$ is continuous for the uniform topology on $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and the measure-algebra topology on $\mathfrak{A}$.

(i) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless homogeneous probability algebra. Show that there is a weakly mixing measure-preserving automorphism of $\mathfrak{A}$ which is not mixing. (*Hint*: 372Yj).

(j) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\pi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and T the corresponding operator on $L^2_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$. (i) Show that π is not ergodic iff there is a non-zero $v \in L^2_{\mathbb{C}}$ such that $\int v = 0$ and $Tv = v$. (ii) Show that π is not weakly mixing iff there is a non-zero $v \in L^2_{\mathbb{C}}$ such that $\int v = 0$ and Tv is a multiple of v. (iii) Let $(\mathfrak{A}\widehat{\otimes}\mathfrak{A}, \bar{\lambda})$ be the probability algebra free product of $(\mathfrak{A}, \bar{\mu})$ with itself (definition: 325K), and $\tilde{\pi} \in \operatorname{Aut}_{\bar{\lambda}}(\mathfrak{A}\widehat{\otimes}\mathfrak{A})$ the automorphism such that $\tilde{\pi}(a \otimes b) = \pi a \otimes \pi b$ for all a, $b \in \mathfrak{A}$. Show that π is weakly mixing iff $\tilde{\pi}$ is ergodic iff $\tilde{\pi}$ is weakly mixing. (*Hint*: consider $T_{\tilde{\pi}}(v \otimes \bar{v})$.)

(k) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. For $\pi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ let T_π be the corresponding operator on $L^2_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$. Show that $(\pi, v) \mapsto T_\pi v : \operatorname{Aut}_{\bar{\mu}}\mathfrak{A} \times L^2_{\mathbb{C}} \to L^2_{\mathbb{C}}$ is continuous if $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is given its weak topology and $L^2_{\mathbb{C}}$ its norm topology.

(l) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra; give $\mathfrak{A}$ its measure metric. Show that the isometry group of $\mathfrak{A}$, with its topology of pointwise convergence, is extremely amenable. (*Hint*: every isometry of $\mathfrak{A}$ is of the form $a \mapsto c \mathbin{\triangle} \pi a$, where $c \in \mathfrak{A}$ and $\pi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$; now use 493Bc.)

(m) Let $\mathfrak{A}$ be a homogeneous Dedekind complete Boolean algebra, and $\langle V_n \rangle_{n\in\mathbb{N}}$ a non-decreasing sequence of subsets of $\operatorname{Aut}\mathfrak{A}$, covering $\operatorname{Aut}\mathfrak{A}$, with $V_n^2 \subseteq V_{n+1}$ for every n. Show that there is an $n \in \mathbb{N}$ such that $\operatorname{Aut}\mathfrak{A} = V_n$.

494Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra. For $\pi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, let $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ be the Riesz space automorphism such that $T_\pi(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$ (364P). Take any $p \in [1, \infty[$ and write $L^p_{\bar{\mu}}$ for $L^p(\mathfrak{A}, \bar{\mu})$ as defined in 366A. Set $G_p = \{T_\pi \restriction L^p_{\bar{\mu}} : \pi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}\}$. (i) Show that $\pi \mapsto T_\pi \restriction L^p_{\bar{\mu}}$ is a topological group isomorphism between $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, with its weak topology, and G_p, with the strong operator topology from $\mathsf{B}(L^p_{\bar{\mu}}; L^p_{\bar{\mu}})$ (3A5I). (ii) Show that G_p is closed in $\mathsf{B}(L^p_{\bar{\mu}}; L^p_{\bar{\mu}})$. (iiI) Show that if $(\mathfrak{A}, \bar{\mu})$ is totally finite, then $\pi \mapsto T_\pi \restriction L^1_{\bar{\mu}}$ is a topological group isomorphism between $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, with its uniform topology, and G_1 with the topology of uniform convergence on weakly compact subsets of $L^1_{\bar{\mu}}$.

(b) Let $\mathfrak{A}$ be any Boolean algebra. For $I \subseteq \mathfrak{A}$, set $U_I = \{\pi : \pi \in \operatorname{Aut}\mathfrak{A},\ \pi a = a \text{ for every } a \in I\}$. (i) Show that $\{U_I : I \in [\mathfrak{A}]^{<\omega}\}$ is a base of neighbourhoods of the identity for a Hausdorff topology on $\operatorname{Aut}\mathfrak{A}$ under which $\operatorname{Aut}\mathfrak{A}$ is a topological group. (ii) Show that if $\mathfrak{A}$ is countable then $\operatorname{Aut}\mathfrak{A}$, with this topology, is a Polish group.

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra. Show that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, with its weak topology, is weakly α-favourable.

(d) Let $\bar{\mu}$ be counting measure on $\mathbb{N}$. (i) Show that if we identify $\operatorname{Aut}_{\bar{\mu}}\mathcal{P}\mathbb{N}$ with the set of permutations on $\mathbb{N}$, the weak topology of $\operatorname{Aut}_{\bar{\mu}}\mathcal{P}\mathbb{N}$ is the topology induced by the usual topology of $\mathbb{N}^{\mathbb{N}}$. (ii) Show that there is a comeager conjugacy class in $\operatorname{Aut}_{\bar{\mu}}\mathcal{P}\mathbb{N}$.

(e) (Rosendal 09) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $[0, 1]$, and give $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ its weak topology. Let $\mathcal{V}$ be a countable base of open neighbourhoods of ι in $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$. (i) Show that if $I \subseteq \mathbb{N}$ is infinite and $V \in \mathcal{V}$, then $\{\pi : \pi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A},\ \pi^n = \iota \text{ for some } n \in I\}$ is dense in $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, and that $B(I, V) = \{\pi : \pi^n \in V \text{ for some } n \in I\}$ is dense and open. (ii) Show that if $I \subseteq \mathbb{N}$ is infinite then $C(I) = \bigcap_{V\in\mathcal{V}} B(I, V)$ is comeager, and is a union of conjugacy classes. (iii) Show that $\bigcap\{C(I) : I \in [\mathbb{N}]^\omega\} = \{\iota\}$. (iv) Show that every conjugacy class in $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is meager.

(f) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Suppose that $\pi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is aperiodic. Show that the set of conjugates of π in $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is dense for the weak topology on $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$.

(g) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. Show that the set of weakly mixing automorphisms, with the subspace topology inherited from the weak topology of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, is weakly α-favourable.

(h) Let G be a Hausdorff topological group. Show that the following are equiveridical: (i) for every neighbourhood V of the identity in G there is a countable set $D \subseteq G$ such that $G = DV$; (ii) there is a family $\langle H_i \rangle_{i \in I}$ of Polish groups such that G is isomorphic, as topological group, to a subgroup of $\prod_{i \in I} H_i$.

(i) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless σ-finite measure algebra, and G a full ergodic subgroup of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. Let $V \subseteq \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ be a symmetric set such that countably many left translates of V cover $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. Show that V^{228} is a neighbourhood of ι for the uniform topology on $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$.

(j) Let $(\mathfrak{A}, \bar{\mu})$ be a purely atomic probability algebra with two atoms of measure 2^{-n-2} for each $n \in \mathbb{N}$; give $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ its uniform topology. (i) Show that $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A} \cong \mathbb{Z}_2^{\mathbb{N}}$ is compact, therefore not extremely amenable, and can be regarded as a linear space over the field $\mathbb{Z}_2$. (ii) Show that there is a strictly increasing sequence of subgroups of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ with union $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. (iii) Show that there is a subgroup V of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$, not open, such that $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is covered by countably many translates of V. (iv) Show that there is a discontinuous homomorphism from $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ to a Polish group.

(k) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that it is not the union of a strictly increasing sequence of subalgebras.

494Z Problems For $k \in \mathbb{N}$, say that a topological group G is ***k*-Steinhaus** if whenever $V \subseteq G$ is a symmetric set, containing the identity, such that countably many left translates of V cover G, then V^k is a neighbourhood of the identity. For your favourite groups, determine the smallest k, if any, for which they are k-Steinhaus. (See ROSENDAL & SOLECKI 07.)

494 Notes and comments In 494B-494C I run through properties of the weak and uniform topologies of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ in parallel. The effect is to emphasize their similarities, but they are of course very different – for instance, consider 494Xg, or the contrast between 494Cg and 494Ge. Both have expressions in terms of standard topologies on spaces of linear operators (494Ya), and the weak topology corresponds to the pointwise topology of an isometry group (494Xa). There are other more or less natural topologies which can be considered (e.g., that of 494Yb), but at present the two examined in this section seem to be the most important. I spell out 494Be and 494Ci to show that the groups here provide interesting examples of Polish groups with striking properties.

The formulation of 494D is specifically designed for the application in the proof of 494E(b-ii); the version in 494Xj(ii) is much closer to the real strength of the idea, and takes us directly to one of the important reasons for being interested in weakly mixing automorphisms in 494Xj(iii). The proof of 494D through Bochner's theorem saves space here, but fails to signal the concept of 'spectral resolution' of a unitary operator on a Hilbert space (RIESZ & SZ.-NAGY 55, §109), which is an important tool in understanding operators T_π and hence automorphisms π.

While 494H and 494G are of some interest in themselves, their function here is to prepare the way to 494L, 494O and 494Q. The first belongs to the series in §493; like the results in that section, it depends on concentration-of-measure theorems, quoted in part (e) of the proof of 494I and again in part (e) of the proof of 494J. In addition, for the generalization from ergodic full groups to arbitrary full groups, we need the structure theory for closed subalgebras developed in §333.

494O and 494Q-494R break new ground. The former, following KITTRELL & TSANKOV 09, examines a curious phenomenon identified by ROSENDAL & SOLECKI 07 in the course of a search for automatic-continuity results. We cannot dispense entirely with the hypotheses that $\mathfrak{A}$ should be atomless and G ergodic (494Yj), though perhaps they can be relaxed. Many examples are now known of k-Steinhaus groups (494O, 494Yi), but as far as I am aware there are no non-trivial cases in which the critical value of k has been determined (494Z). The automatic-continuity corollary in 494Ob is really a result about homomorphisms into Polish groups (see 494Yh), but applies in many other cases (494P).

The phenomenon of 494Q, which we might call a (negative) 'algebraic cofinality' result, has attracted attention with regard to many algebraic structures, starting with BERGMAN 06. Apart from the variations of 494Q in 494R and 494Xm, there is a simple example in 494Yk. 494Yj again indicates one of the limits of the result.

495 Poisson point processes

A classical challenge in probability theory is to formulate a consistent notion of 'random set'. Simple geometric considerations lead us to a variety of measures which are both interesting and important. All these are manifestly special constructions. Even in the most concrete structures, we have to make choices which come to seem arbitrary as soon as we are conscious of the many alternatives. There is however one construction which has a claim to pre-eminence because it is both robust under the transformations of abstract measure theory and has striking properties when applied to familiar measures (to the point, indeed, that it is relevant to questions in physics and chemistry). This gives the 'Poisson point processes' of 495D-495E. In this section I give a brief introduction to the measure-theoretic aspects of this construction.

495A Poisson distributions We need a little of the elementary theory of Poisson distributions.

(a) The **Poisson distribution** with parameter $\gamma > 0$ is the point-supported Radon probability measure ν_γ on $\mathbb{R}$ such that $\nu_\gamma\{n\} = \frac{\gamma^n}{n!}e^{-\gamma}$ for every $n \in \mathbb{N}$. (See 285Q and 285Xo.) Its expectation is $\sum_{n=1}^{\infty} \frac{\gamma^n}{(n-1)!}e^{-\gamma} = \gamma$. Since $\nu_\gamma\mathbb{N} = 1$, ν_γ can be identified with the corresponding subspace measure on $\mathbb{N}$. It will be convenient to allow $\gamma = 0$, so that the Dirac measure on $\mathbb{R}$ or $\mathbb{N}$ concentrated at 0 becomes a 'Poisson distribution with expectation 0'.

(b) The convolution of two Poisson distributions is a Poisson distribution. **P** If α, $\beta > 0$ then

$$(\nu_\alpha * \nu_\beta)(\{n\}) = \int \nu_\beta(\{n\} - t)\nu_\alpha(dt)$$

(444A)

$$= \sum_{i=0}^{n} \frac{\beta^{n-i}}{(n-i)!}e^{-\beta} \cdot \frac{\alpha^i}{i!}e^{-\alpha}$$

$$= e^{-\alpha-\beta}\frac{1}{n!}\sum_{i=0}^{n} \frac{n!}{i!(n-i)!}\alpha^i\beta^{n-i} = \frac{(\alpha+\beta)^n}{n!}e^{-\alpha-\beta}$$

for every $n \in \mathbb{N}$, so $\nu_\alpha * \nu_\beta = \nu_{\alpha+\beta}$. **Q** So if f and g are independent random variables with Poisson distributions then $f + g$ has a Poisson distribution (272T[5]).

(c) If $\langle f_i\rangle_{i\in I}$ is a countable independent family of random variables with Poisson distributions, and $\alpha = \sum_{i\in I} \mathbb{E}(f_i)$ is finite, then $f = \sum_{i\in I} f_i$ is defined a.e. and has a Poisson distribution with expectation α. **P** For finite I we can induce on $\#(I)$, using (b) (and 272L) for the inductive step. For the infinite case we can suppose that $I = \mathbb{N}$. In this case $f_i \ge 0$ a.e. for each i so $f = \sum_{i=0}^{\infty} f_i$ is defined a.e. and has expectation α, by B.Levi's theorem. Setting $g_n = \sum_{i=0}^{n} f_i$ for each n, so that g_n has a Poisson distribution with expectation $\beta_n = \sum_{i=0}^{n} \alpha_i$, we have

$$\Pr(f \le \gamma) = \lim_{n\to\infty} \Pr(g_n \le \gamma) = \lim_{n\to\infty} \sum_{i=0}^{\lfloor\gamma\rfloor} \frac{\beta_n^i}{i!}e^{-\beta_n} = \sum_{i=0}^{\lfloor\gamma\rfloor} \frac{\alpha^i}{i!}e^{-\alpha}$$

for every $\gamma \ge 0$, so f has a Poisson distribution with expectation α. **Q**

(d) I find myself repeatedly calling on the simple fact that $1 - e^{-\gamma}(1+\gamma) = \nu_\gamma(\mathbb{N} \setminus \{0,1\})$ is at most $\frac{1}{2}\gamma^2$ for every $\gamma \ge 0$; this is because $\frac{d}{dt}(\frac{1}{2}t^2 + e^{-t}(1+t)) = t(1-e^{-t}) \ge 0$ for $t \ge 0$.

495B Theorem Let (X, Σ, μ) be a measure space. Set $\Sigma^f = \{E : E \in \Sigma,\ \mu E < \infty\}$. Then for any $\gamma > 0$ there are a probability space $(\Omega, \Lambda, \lambda)$ and a family $\langle g_E\rangle_{E\in\Sigma^f}$ of random variables on Ω such that

(i) for every $E \in \Sigma^f$, g_E has a Poisson distribution with expectation $\gamma\mu E$;

(ii) whenever $\langle E_i\rangle_{i\in I}$ is a disjoint family in Σ^f, then $\langle g_{E_i}\rangle_{i\in I}$ is stochastically independent;

(iii) whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ^f with union $E \in \Sigma^f$, then $g_E =_{\text{a.e.}} \sum_{i=0}^{\infty} g_{E_i}$.

proof (a) Let $\mathcal{H} \subseteq \{H : H \in \Sigma,\ 0 < \mu H < \infty\}$ be a maximal family such that $H \cap H'$ is negligible for all distinct H, $H' \in \mathcal{H}$. For $H \in \mathcal{H}$, let μ'_H be the normalized subspace measure defined by setting $\mu'_H E = \mu E/\mu H$

[5] Formerly 272S.

for $E \in \Sigma \cap \mathcal{P}H$, and λ_H the corresponding product probability measure on $H^{\mathbb{N}}$. Next, for $H \in \mathcal{H}$, let ν_H be the Poisson distribution with expectation $\gamma\mu H$, regarded as a probability measure on $\mathbb{N}$. Let λ be the product measure on $\Omega = \prod_{H\in\mathcal{H}}(\mathbb{N} \times H^{\mathbb{N}})$, giving each $\mathbb{N} \times H^{\mathbb{N}}$ the product measure $\nu_H \times \lambda_H$. For $\omega \in \Omega$, write $m_H(\omega)$, $x_{Hj}(\omega)$ for its coordinates, so that $\omega = \langle(m_H(\omega), \langle x_{Hj}(\omega)\rangle_{j\in\mathbb{N}})\rangle_{H\in\mathcal{H}}$.

(b) For $H \in \mathcal{H}$ and $E \in \Sigma$, set $g_{HE}(\omega) = \#(\{j : j < m_H(\omega),\, x_{Hj}(\omega) \in E\})$ when this is finite. Then g_{HE} is measurable and has a Poisson distribution with expectation $\gamma\mu(H \cap E)$; moreover, if $E_0, \dots , E_r \in \Sigma$ are disjoint, then $g_{HE_0}, \dots , g_{HE_r}$ are independent. **P** It is enough to examine the case in which the E_i cover X. Then for any $n_0, \dots , n_r \in \mathbb{N}$ with sum n,

$$\begin{aligned}
\lambda\{\omega &: g_{HE_i}(\omega) = n_i \text{ for every } i \le r\}\\
&= \lambda\{\omega : \#(\{j : j < m_H(\omega),\, x_{Hj} \in E_i\}) = n_i \text{ for every } i \le r\}\\
&= \lambda\{\omega : m_H(\omega) = n,\, \#(\{j : j < n,\, x_{Hj} \in E_i\}) = n_i \text{ for every } i \le r\}\\
&= \sum_{\substack{J_0,\dots,J_r \text{ partition } n\\ \#(J_i)=n_i \text{ for each } i\le r}} \lambda\{\omega : m_H(\omega) = n,\, x_{Hj} \in E_i \text{ whenever } i \le r,\, j \in J_i\}\\
&= \sum_{\substack{J_0,\dots,J_r \text{ partition } n\\ \#(J_i)=n_i \text{ for each } i\le r}} \frac{(\gamma\mu H)^n}{n!} e^{-\gamma\mu H} \prod_{i=0}^{r} \Bigl(\frac{\mu(H\cap E_i)}{\mu H}\Bigr)^{n_i}\\
&= \frac{n!}{n_0!\dots n_r!}\frac{1}{n!} e^{-\gamma\mu H} \prod_{i=0}^{r} (\gamma\mu(H \cap E_i))^{n_i}\\
&= \prod_{i=0}^{r} \frac{(\gamma\mu(H\cap E_i))^{n_i}}{n_i!} e^{-\gamma\mu(H\cap E_i)},
\end{aligned}$$

which is just what we wanted to know. **Q**

Obviously $g_{HE} = \sum_{i=0}^{\infty} g_{HE_i}$ whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ with union E, and $g_{HE} = 0$ a.e. if $\mu(H \cap E) = 0$.

(c) Suppose that $H_0, \dots , H_m \in \mathcal{H}$ are distinct and $E_0, \dots , E_r \in \Sigma$ are disjoint. Then the random variables $g_{H_jE_i}$ are independent. **P** For each $j \le m$, $g_{H_jE_i}$ is Λ_{H_j}-measurable, where Λ_{H_j} is the σ-algebra of subsets of Ω which are measured by λ and determined by the single coordinate H_j in the product $\prod_{H\in\mathcal{H}}(\mathbb{N} \times H^{\mathbb{N}})$. Now the σ-algebras Λ_{H_j} are independent (272Ma). So if we have any family $\langle n_{ij}\rangle_{i\le r,j\le m}$ in $\mathbb{N}$,

$$\begin{aligned}
\lambda\{\omega : g_{H_jE_i}(\omega) = n_{ij} &\text{ for every } i \le r, j \le m\}\\
&= \prod_{j=0}^{m} \lambda\{\omega : g_{H_jE_i}(\omega) = n_{ij} \text{ for every } i \le r\}\\
&= \prod_{j=0}^{m}\prod_{i=0}^{r} \lambda\{\omega : g_{H_jE_i}(\omega) = n_{ij}\}
\end{aligned}$$

by (b); and this is what we need to know. **Q**

(d) For $E \in \Sigma^f$, set $\mathcal{H}_E = \{H : H \in \mathcal{H},\, \mu(E \cap H) > 0\}$; then $\mathcal{H}_E$ is countable, because $\mathcal{H}$ is almost disjoint, and $\mu E = \sum_{H\in\mathcal{H}_E} \mu(H\cap E)$, because $\mathcal{H}$ is maximal. Set $g_E(\omega) = \sum_{H\in\mathcal{H}_E} g_{HE}(\omega)$ when this is finite. Then g_E is defined a.e. and has a Poisson distribution with expectation $\gamma\mu E$ (495Ac). Also $\langle g_{E_i}\rangle_{i\in I}$ are independent whenever $\langle E_i\rangle_{i\in I}$ is a disjoint family in Σ^f. **P** It is enough to deal with the case of finite I (272Bb). Set $\mathcal{H}^* = \bigcup_{i\in I} \mathcal{H}_{E_i}$, so that $\mathcal{H}^*$ is countable, and for $i \in I$ set $g'_i = \sum_{H\in\mathcal{H}^*} g_{HE_i}$. Then each g'_i is equal almost everywhere to the corresponding g_{E_i}, and $\langle g'_i\rangle_{i\in I}$ is independent, by 272K. (The point is that each g'_i is Λ^*_i-measurable, where Λ^*_i is the σ-algebra generated by $\{g_{HE_i} : H \in \mathcal{H}\}$, and 272K, with (c) above, assures us that the Λ^*_i are independent.) It follows at once that $\langle g_{E_i}\rangle_{i\in I}$ is independent (272H). **Q** This proves (i) and (ii).

(e) Similarly, if $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ^f with union $E \in \Sigma^f$, set $\mathcal{H}^* = \mathcal{H}_E \cup \bigcup_{i\in\mathbb{N}} \mathcal{H}_{E_i}$. For each $i \in \mathbb{N}$, set $g'_i = \sum_{H\in\mathcal{H}^*} g_{HE_i}$; then $g'_i =_{\text{a.e.}} g_{E_i}$. Now

$$\sum_{i=0}^{\infty} g_{E_i} =_{\text{a.e.}} \sum_{i=0}^{\infty} g'_i = \sum_{H\in\mathcal{H}^*} \sum_{i=0}^{\infty} g_{HE_i} = \sum_{H\in\mathcal{H}^*} g_{HE} =_{\text{a.e.}} g_E,$$

as required by (iii).

495C Lemma Let X be a set and $\mathcal{E}$ a subring of the Boolean algebra $\mathcal{P}X$. Let $\mathcal{H}$ be the family of sets of the form

$$\{S : S \subseteq X,\ \#(S \cap E_i) = n_i \text{ for every } i \in I\}$$

where $\langle E_i\rangle_{i\in I}$ is a finite disjoint family in $\mathcal{E}$ and $n_i \in I$ for every $i \in I$. Then the Dynkin class $\mathrm{T} \subseteq \mathcal{P}(\mathcal{P}X)$ generated by $\mathcal{H}$ is the σ-algebra of subsets of $\mathcal{P}X$ generated by $\mathcal{H}$.

proof Let Q be the set of functions q from finite subsets of $\mathcal{E}$ to $\mathbb{N}$, and for $q \in Q$ set

$$H_q = \{S : S \subseteq X,\ \#(S \cap E) = q(E) \text{ for every } E \in \operatorname{dom} q\}.$$

Our family $\mathcal{H}$ is just $\{H_q : q \in Q,\ \operatorname{dom} q \text{ is disjoint}\}$.

If $q \in Q$ and $\operatorname{dom} q$ is a subring of $\mathcal{E}$, then $H_q \in \mathrm{T}$. **P** Being a finite Boolean ring, $\operatorname{dom} q$ is a Boolean algebra; let $\mathcal{A}$ be the set of its atoms. Then H_q is either empty or equal to $H_{q\restriction\mathcal{A}}$; in either case it belongs to T. **Q**

If q is any member of Q, then $H_q \in \mathrm{T}$. **P** Let $\mathcal{E}$ be the subring of $\mathcal{P}X$ generated by $\operatorname{dom} q$. Then $H_q = \bigcup_{q\subseteq q'\in Q,\operatorname{dom} q'=\mathcal{E}} H_{q'}$ is the union of a finite disjoint family in T, so belongs to T. **Q**

Now observe that $\mathcal{H}_1 = \{H_q : q \in Q\} \cup \{\emptyset\}$ is a subset of T closed under finite intersections, so by the Monotone Class Theorem (136B) T includes the σ-algebra generated by $\mathcal{H}_1$, and must be precisely the σ-algebra generated by $\mathcal{H}$.

495D Theorem Let (X, Σ, μ) be an atomless measure space. Set $\Sigma^f = \{E : E \in \Sigma,\ \mu E < \infty\}$; for $E \in \Sigma^f$, set $f_E(S) = \#(S \cap E)$ when $S \subseteq X$ meets E in a finite set. Let T be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S : f_E(S) = n\}$ where $E \in \Sigma^f$ and $n \in \mathbb{N}$. Then for any $\gamma > 0$ there is a unique probability measure ν with domain T such that

(i) for every $E \in \Sigma^f$, f_E is measurable and has a Poisson distribution with expectation $\gamma\mu E$;

(ii) whenever $\langle E_i\rangle_{i\in I}$ is a disjoint family in Σ^f, then $\langle f_{E_i}\rangle_{i\in I}$ is stochastically independent.

proof (a) Let $\mathcal{H}$, $\langle \nu_H\rangle_{H\in\mathcal{H}}$, $\langle \mu_H\rangle_{H\in\mathcal{H}}$, $\langle \mu'_H\rangle_{H\in\mathcal{H}}$, $\langle \lambda_H\rangle_{H\in\mathcal{H}}$, Ω, λ, $\langle \mathcal{H}_E\rangle_{E\in\Sigma^f}$ and $\langle g_E\rangle_{E\in\Sigma^f}$ be as in the proof of 495B. Note that all the μ'_H are atomless (234Nf[6]). Define $\phi : \Omega \to \mathcal{P}X$ by setting

$$\phi(\omega) = \{x_{Hj}(\omega) : H \in \mathcal{H},\ j < m_H(\omega)\}$$

for $\omega \in \Omega$.

(b) For $E \in \Sigma^f$, let A_E be the set of those $\omega \in \Omega$ such that

either there are $H \in \mathcal{H} \setminus \mathcal{H}_E$, $j \in \mathbb{N}$ such that $x_{Hj}(\omega) \in E$

or there are distinct H, $H' \in \mathcal{H}_E$ and $j \in \mathbb{N}$ such that $x_{Hj}(\omega) \in H'$

or there is an $H \in \mathcal{H}$ such that the $x_{Hj}(\omega)$, for $j \in \mathbb{N}$, are not all distinct.

Then for any sequence $\langle E_i\rangle_{i\in\mathbb{N}}$ in Σ^f, $\lambda_*(\bigcup_{i\in\mathbb{N}} A_{E_i}) = 0$. **P** Set $\mathcal{H}^* = \bigcup_{i\in\mathbb{N}} \mathcal{H}_{E_i}$, so that $\mathcal{H}^*$ is a countable subset of $\mathcal{H}$. For $H \in \mathcal{H}$, set

$$F_H = H \setminus (\bigcup\{E_i : i \in \mathbb{N},\ H \cap E_i \text{ is negligible}\} \cup \bigcup\{H' : H' \in \mathcal{H}^*,\ H' \neq H\}),$$

$$W_H = \{\boldsymbol{x} : \boldsymbol{x} \in H^{\mathbb{N}} \text{ is injective}\},$$

so that F_H is μ'_H-conegligible and W_H is λ_H-conegligible (because μ'_H is atomless, see 254V). Now

$$\Omega \setminus \bigcup_{k\in\mathbb{N}} A_{E_k} \supseteq \prod_{H\in\mathcal{H}} (\mathbb{N} \times (W_H \cap F_H^{\mathbb{N}}))$$

has full outer measure in Ω, by 254Lb, and its complement has zero inner measure (413Ec). **Q**

It follows that there is a probability measure $\tilde\lambda$ on Ω, extending λ, such that $\tilde\lambda A_E = 0$ for every $E \in \Sigma^f$ (417A). Let ν_0 be the image measure $\tilde\lambda\phi^{-1}$.

[6] Formerly 234F.

(c) If $E \in \Sigma^f$ and $\omega \in \Omega \setminus A_E$, then $f_E(\phi(\omega)) = g_E(\omega)$ if either is defined. **P** If $H \in \mathcal{H}$, then all the $x_{Hj}(\omega)$ are distinct; if $H \in \mathcal{H} \setminus \mathcal{H}_E$, no $x_{Hj}(\omega)$ can belong to E; if H, $H' \in \mathcal{H}_E$ are distinct, then no $x_{Hj}(\omega)$ can belong to H'. So all the $x_{Hj}(\omega)$, $x_{H'k}(\omega)$ for H, $H' \in \mathcal{H}_E$ and j, $k \in \mathbb{N}$ must be distinct, and

$$\begin{aligned} f_E(\phi(\omega)) &= \#(\{x_{Hj}(\omega) : H \in \mathcal{H},\, j < m_H(\omega),\, x_{Hj}(\omega) \in E\}) \\ &= \#(\{(H,j) : H \in \mathcal{H}_E,\, j < m_H(\omega),\, x_{Hj}(\omega) \in E\}) \\ &= \sum_{H \in \mathcal{H}_E} g_{HE}(\omega) = g_E(\omega) \end{aligned}$$

if any of these is finite. **Q** It follows at once that if $E_0, \dots, E_r \in \Sigma^f$ are disjoint, then $\{\omega : f_{E_i}(\phi(\omega)) = g_{E_i}(\omega)$ for every $i \le r\}$ is $\tilde{\lambda}$-conegligible, so that if $n_0, \dots, n_r \in \mathbb{N}$ then

$$\begin{aligned} \nu_0\{S : f_{E_i}(S) = n_i \text{ for every } i \le r\} &= \tilde{\lambda}\{\omega : f_{E_i}(\phi(\omega)) = n_i \text{ for every } i \le r\} \\ &= \tilde{\lambda}\{\omega : g_{E_i}(\omega) = n_i \text{ for every } i \le r\} \\ &= \lambda\{\omega : g_{E_i}(\omega) = n_i \text{ for every } i \le r\} \\ &= \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}. \end{aligned}$$

Thus every f_{E_i} is finite ν_0-a.e., belongs to $\mathcal{L}^0(\nu_0)$ and has a Poisson distribution with the appropriate expectation, and they are independent.

(d) As T is defined to be the σ-algebra generated by the family $\{f_E : E \in \Sigma^f\}$, it is included in the domain of ν_0. Set $\nu = \nu_0 \restriction \mathrm{T}$; then ν has the properties (i) and (ii). To see that it is unique, observe that if ν' also has these properties, then $\{A : \nu A = \nu' A\}$ is a Dynkin class containing every set of the form

$$\{S : f_{E_i}(S) = n_i \text{ for } i \le r\}$$

where $E_0, \dots, E_r \in \Sigma^f$ are disjoint and $n_0, \dots, n_r \in \mathbb{N}$. By 495C it contains the σ-algebra generated by this family, which is T. So ν and ν' agree on T, and are equal.

495E Definition In the context of 495D, I will call the completion of ν the **Poisson point process** on X with density γ.

Note that the Poisson point process on (X, μ) with density $\gamma > 0$ is identical with the Poisson point process on $(X, \gamma\mu)$ with density 1. There would therefore be no real loss of generality in the main theorems of this section if I spoke only of point processes with density 1. I retain the extra parameter because applications frequently demand it, and the formulae will be more useful with the γs in their proper places; moreover, there are important ideas associated with variations in γ, as in 495Xe.

495F Proposition Let (X, Σ, μ) be a perfect atomless measure space, and $\gamma > 0$. Then the Poisson point process on X with density γ is a perfect probability measure.

proof I refer to the construction in 495B-495D. In (b) of the proof of 495D, use the construction set out in the proof of 417A, so that the domain $\tilde{\Lambda}$ of $\tilde{\lambda}$ is precisely the family of sets of the form $W \triangle A$ where W belongs to the domain Λ of the product measure λ and A belongs to the σ-ideal $\mathcal{A}^*$ generated by $\{A_E : E \in \Sigma^f\}$. Then $\tilde{\lambda}$ is perfect. **P** Let $h : \Omega \to \mathbb{R}$ be a $\tilde{\Lambda}$-measurable function and $W \in \tilde{\Lambda}$ a set of non-zero measure. Then there are a $W' \in \Lambda$ and an $A \in \mathcal{A}^*$ such that $W \triangle W' \subseteq A$ and $h \restriction \Omega \setminus A$ is Λ-measurable; let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence in Σ^f such that $A \subseteq \bigcup_{n \in \mathbb{N}} A_{E_n}$, and $h_1 : \Omega \to \mathbb{R}$ a Λ-measurable function agreeing with h on $\Omega \setminus A$. Set $\mathcal{H}^* = \bigcup_{n \in \mathbb{N}} \mathcal{H}_{E_n}$, so that $\mathcal{H}^*$ is countable. As in the proof of 495D, set

$$F_H = H \setminus (\bigcup\{E_n : n \in \mathbb{N},\, H \cap E_n \text{ is negligible}\} \cup \bigcup\{H' : H' \in \mathcal{H}^*,\, H' \ne H\}),$$

$$W_H = \{\boldsymbol{x} : \boldsymbol{x} \in H^{\mathbb{N}} \text{ is injective}\},$$

for $H \in \mathcal{H}$, so that $W'_H = W_H \cap F_H^{\mathbb{N}}$ is λ_H-conegligible. Set $\Omega' = \prod_{H \in \mathcal{H}} (\mathbb{N} \times W'_H)$. This is disjoint from every A_{E_n} (as in 495D) and therefore from A. The subspace measure $\lambda_{\Omega'}$ on Ω' induced by λ is just the product of the measures on $\mathbb{N} \times W'_H$ (254La). All of these are perfect (451Jc, 451Dc), so $\lambda_{\Omega'}$ also is perfect (451Jc again). Now

$$\lambda_{\Omega'}(W \cap \Omega') = \lambda_{\Omega'}(W' \cap \Omega') = \lambda W' = \tilde{\lambda} W > 0.$$

It follows that there is a compact set $K \subseteq h_1[W \cap \Omega']$ such that $\lambda_{\Omega'}(h_1^{-1}[K] \cap \Omega') > 0$. As h and h_1 agree on Ω', $K \subseteq h[W]$, while

$$\tilde{\lambda} h^{-1}[K] = \tilde{\lambda} h_1^{-1}[K]$$

(because $\tilde{\lambda} A = 0$)

$$= \lambda h_1^{-1}[K] = \lambda_{\Omega'}(h^{-1}[K] \cap \Omega') > 0.$$

As W and h are arbitrary, $\tilde{\lambda}$ is perfect. **Q**

It follows at once that the image measure $\tilde{\lambda}\phi^{-1}$ and its restriction to T are perfect (451Ea); finally, the completion is perfect, by 451Gc.

495G Proposition Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be atomless measure spaces, and $f : X_1 \to X_2$ an inverse-measure-preserving function. Let $\gamma > 0$, and let ν_1, ν_2 be the Poisson point processes on X_1, X_2 respectively with density γ. Then $S \mapsto f[S] : \mathcal{P}X_1 \to \mathcal{P}X_2$ is inverse-measure-preserving for ν_1 and ν_2; in particular, $\mathcal{P}A$ has full outer measure for ν_2 whenever $A \subseteq X_2$ has full outer measure for μ_2.

proof Set $\psi(S) = f[S]$ for $S \subseteq X_1$.

(a) If $F \in \Sigma_2^f$, then $\{S : S \subseteq X_1,\ f{\restriction}f^{-1}[F] \cap S$ is not injective$\}$ is ν_1-negligible. **P** Let $n \in \mathbb{N}$. Set $\alpha = \frac{1}{n+1}\mu_2 F$. Because μ_2 is atomless, we can find a partition of F into sets $F_0, \dots, F_n$ of measure α. Now

$$\{S : f{\restriction}f^{-1}[F] \cap S \text{ is not injective}\} \subseteq \bigcup_{i \le n}\{S : \#(S \cap f^{-1}[F_i]) > 1\}$$

has ν_1-outer measure at most

$$(n+1)(1 - e^{-\gamma\alpha}(1 + \gamma\alpha)) \le \frac{1}{2}(n+1)\alpha^2\gamma^2 = \frac{1}{2(n+1)}(\gamma\mu_2 F)^2.$$

As n is arbitrary, $\{S : f{\restriction}f^{-1}[F] \cap S$ is not injective$\}$ is negligible. **Q**

(b) It follows that, for any $F \in \Sigma_2^f$ and $n \in \mathbb{N}$,

$$\{S : \#(f[S] \cap F) = n\} \triangle \{S : \#(S \cap f^{-1}[F]) = n\}$$

is ν_1-negligible and $\{S : \#(f[S] \cap F) = n\}$ is measured by ν_1. So if T_2 is the σ-algebra of subsets of $\mathcal{P}X_2$ generated by sets of the form $\{T : \#(F \cap T) = n\}$ for $F \in \Sigma_2^f$ and $n \in \mathbb{N}$, then ν_1 measures $\psi^{-1}[H]$ for every $H \in \mathrm{T}_2$. Next, if $\langle F_i \rangle_{i \in I}$ is a finite disjoint family in Σ_2^f and $n_i \in \mathbb{N}$ for $i \in I$,

$$\begin{aligned}\nu_1\{S : \#(f[S] \cap F_i) = n_i \text{ for every } i \in I\}& \\ &= \nu_1\{S : \#(S \cap f^{-1}[F_i]) = n_i \text{ for every } i \in I\} \\ &= \prod_{i \in I} \frac{(\gamma\mu_1 f^{-1}[F_i])^n}{n!} e^{-\gamma\mu_1 f^{-1}[F_i]}\end{aligned}$$

(because $\langle f^{-1}[F_i] \rangle_{i \in I}$ is a disjoint family in Σ_1^f)

$$= \prod_{i \in I} \frac{(\gamma\mu_2 F_i)^n}{n!} e^{-\gamma\mu_2 F_i}.$$

So the image measure $\nu_1\psi^{-1}$ satisfies (i) and (ii) of 495D, and must agree with ν_2 on T_2; that is, ψ is inverse-measure-preserving for ν_1 and $\nu_2{\restriction}\mathrm{T}_2$. As ν_1 is complete, ψ is inverse-measure-preserving for ν_1 and ν_2 (234Ba[7]).

(c) If $A \subseteq X_2$ has full outer measure, then we can take μ_1 to be the subspace measure on $X_1 = A$ and $f(x) = x$ for $x \in A$. In this case, $\mathcal{P}A = \psi[\mathcal{P}A]$ must have full outer measure for ν_2.

495H Lemma Let $(\tilde{X}, \tilde{\Sigma}, \tilde{\mu})$ be an atomless σ-finite measure space, and $\gamma > 0$. Write μ_L for Lebesgue measure on $[0,1]$, μ' for the product measure on $X' = \tilde{X} \times [0,1]$, and λ' for the product measure on $\Omega' = [0,1]^{\tilde{X}}$. Let $\tilde{\nu}$, ν' be the Poisson point processes on $\tilde{X}$, X' respectively with density γ. For $T \subseteq \tilde{X}$ define $\psi_T : \Omega' \to \mathcal{P}X'$ by setting

[7] Formerly 235Hc.

$\psi_T(z) = \{(t, z(t)) : t \in T\}$ for $z \in \Omega'$; let ν'_T be the image measure $\lambda'\psi_T^{-1}$ on $\mathcal{P}X'$. Then $\langle \nu'_T \rangle_{T \subseteq \tilde{X}}$ is a disintegration of ν' over $\tilde{\nu}$ (definition: 452E).

proof (a) Let $E \subseteq X'$ be a measurable set with finite measure, and write $H_E = \{S : S \cap E \neq \emptyset\}$. Then $\nu' H_E = 1 - e^{-\gamma\mu' E} \le \gamma\mu' E$; but also $\overline{\int}\nu'_T(H_E)\tilde{\nu}(dT) \le 2\gamma\mu' E$. **P** We know that $\int \mu_L E[\{t\}]\tilde{\mu}(dt) = \mu' E$ (252D). Let $Y \subseteq \tilde{X}$ be a conegligible set such that $E[\{t\}]$ is measurable for every $t \in Y$ and $t \mapsto \mu_L E[\{t\}] : Y \to [0,1]$ is measurable. Set $F_i = \{t : t \in Y,\ 2^{-i-1} < \mu_L E[\{t\}] \le 2^{-i}\}$ for each $i \in \mathbb{N}$; let $\langle F'_i \rangle_{i \in \mathbb{N}}$ be a sequence of sets of finite measure with union $\tilde{X} \setminus \bigcup_{i \in \mathbb{N}} F_i$. Let W be the set of those $T \subseteq \tilde{X}$ such that $T \cap (\tilde{X} \setminus Y)$ is empty and $T \cap F_i$, $T \cap F'_i$ are finite for every $i \in \mathbb{N}$; then W is $\tilde{\nu}$-conegligible.

For any $T \in W$,

$$\begin{aligned}\psi_T^{-1}[H_E] &= \{z : \psi_T(z) \cap E \neq \emptyset\} \\ &= \bigcup_{i \in \mathbb{N}} \bigcup_{t \in T \cap F_i} \{z : z(t) \in E[\{t\}]\} \cup \bigcup_{i \in \mathbb{N}} \bigcup_{t \in T \cap F'_i} \{z : z(t) \in E[\{t\}]\}\end{aligned}$$

is measured by λ' and has measure at most $\sum_{i=0}^{\infty} 2^{-i}\#(T \cap F_i)$, because $\mu_L E[\{t\}]$ has measure at most 2^{-i} if $t \in T \cap F_i$, and is zero if $t \in T \cap F'_i$. So

$$\begin{aligned}\overline{\int}\nu'_T(H_E)\tilde{\nu}(dT) &= \overline{\int}\lambda'\psi_T^{-1}[H_E]\tilde{\nu}(dT) \le \int \sum_{i=0}^{\infty} 2^{-i}\#(T \cap F_i)\tilde{\nu}(dT) \\ &= \sum_{i=0}^{\infty} 2^{-i} \int \#(T \cap F_i)\tilde{\nu}(dT) = \sum_{i=0}^{\infty} 2^{-i}\gamma\tilde{\mu}F_i\end{aligned}$$

(because $T \mapsto \#(T \cap F_i)$ has expectation $\gamma\tilde{\mu}F_i$)

$$\le 2\gamma \int \mu_L E[\{t\}]\tilde{\nu}(dt) = 2\gamma\mu' E. \text{ **Q**}$$

(b) Suppose that $\langle F_j \rangle_{j<s}$, $\langle C_{ij} \rangle_{i<r, j<s}$ and $\langle n_{ij} \rangle_{i<r, j<s}$ are such that

$r,\ s \in \mathbb{N}$,

$n_{ij} \in \mathbb{N}$ for $i < r$, $j < s$,

$\langle F_j \rangle_{j<s}$ is a disjoint family of subsets of $\tilde{X}$ with finite measure,

for each $j < s$, $\langle C_{ij} \rangle_{i<r}$ is a disjoint family of measurable subsets of $[0,1]$.

Set $E_{ij} = F_j \times C_{ij}$ for $i < r$ and $j < s$, and $H = \{S : S \subseteq X',\ \#(S \cap E_{ij}) = n_{ij}$ for every $i < r,\ j < s\}$. Then $\int \nu'_T(H)\tilde{\nu}(dT) = \nu' H$.

P **(i)** To begin with, suppose that $\bigcup_{i<r} C_{ij} = [0,1]$ for every j. Set $n_j = \sum_{i=0}^{r-1} n_{ij}$ for each j, and let W be the set of those $T \subseteq \tilde{X}$ such that $\#(T \cap F_j) = n_j$ for every j. Then $\tilde{\nu}W = \prod_{j=0}^{s-1} \frac{(\gamma\tilde{\mu}F_j)^{n_j}}{n_j!} e^{-\gamma\tilde{\mu}F_j}$. If $T \subseteq \tilde{X}$ and $z \in \psi_T^{-1}[H]$, then for each $j < s$ we must have

$$\begin{aligned}\#(T \cap F_j) &= \#(\{t : t \in T \cap F_j,\ z(t) \in \bigcup_{i<r} C_{ij}\}) \\ &= \sum_{i=0}^{r-1} \#(\{t : t \in T \cap F_j,\ z(t) \in C_{ij}\}) \\ &= \sum_{i=0}^{r-1} \#(\psi_T(z) \cap E_{ij}) = \sum_{i=0}^{r-1} n_{ij} = n_j.\end{aligned}$$

Turning this round, we see that if $T \notin W$ then $\psi_T^{-1}[H] = \emptyset$ and $\nu'_T H = 0$.

If $T \in W$, let Q be the set of all $q = \langle q(i,j) \rangle_{i<r, j<s}$ such that $\langle q(i,j) \rangle_{i<r}$ is a disjoint family of subsets of $T \cap F_j$ for each j and $\#(q(i,j)) = n_{ij}$ for all i and j. Then $\#(Q) = \prod_{j=0}^{s-1} \frac{n_j!}{\prod_{i=0}^{r-1} n_{ij}!}$. Accordingly

$$\begin{aligned}\nu'_T H &= \lambda'\{z : \psi_T(z) \in H\}\\ &= \lambda'\{z : \#(\{t : t \in T \cap F_j,\, z(t) \in C_{ij}\}) = n_{ij} \text{ for all } i,\, j\}\\ &= \sum_{q\in Q} \lambda\{z : z(t) \in C_{ij} \text{ whenever } i < r,\, j < s \text{ and } t \in q(i,j)\}\\ &= \sum_{q\in Q} \prod_{\substack{i<r,j<s\\ t\in q(i,j)}} \mu_L C_{ij} = \sum_{q\in Q} \prod_{i<r,j<s} (\mu_L C_{ij})^{n_{ij}} = \prod_{j=0}^{s-1}\Bigl(n_j! \prod_{i=0}^{r-1} \frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!}\Bigr).\end{aligned}$$

It follows that

$$\begin{aligned}\int \nu'_T(H)\tilde\nu(dT) &= \tilde\nu W \cdot \prod_{j=0}^{s-1}\Bigl(n_j! \prod_{i=0}^{r-1} \frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!}\Bigr)\\ &= \prod_{j=0}^{s-1} \frac{(\gamma\tilde\mu F_j)^{n_j}}{n_j!} e^{-\gamma\tilde\mu F_j} \cdot \prod_{j=0}^{s-1}\Bigl(n_j! \prod_{i=0}^{r-1} \frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!}\Bigr)\\ &= \prod_{j=0}^{s-1} (\gamma\tilde\mu F_j)^{n_j} e^{-\gamma\tilde\mu F_j} \prod_{i=0}^{r-1} \frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!}\\ &= \prod_{j=0}^{s-1}\prod_{i=0}^{r-1} e^{-\gamma\mu' E_{ij}} \frac{(\gamma\mu' E_{ij})^{n_{ij}}}{n_{ij}!} = \nu' H,\end{aligned}$$

as required.

(ii) For the general case, set $C_{rj} = [0,1] \setminus \bigcup_{i<r} C_{ij}$, $E_{rj} = F_j \times C_{rj}$ for each $j < s$. For $\sigma \in \mathbb{N}^{(r+1)\times s}$, set

$$H_\sigma = \{S : S \subseteq X',\, \#(S \cap E_{ij}) = \sigma(i,j) \text{ for every } i \le r \text{ and } j < s\}.$$

By (i), we have $\nu' H_\sigma = \int \nu'_T(H_\sigma)\tilde\nu(dT)$ for every $\sigma \in \mathbb{N}^{(r+1)\times s}$.

Set

$$J = \{\sigma : \sigma \in \mathbb{N}^{(r+1)\times s},\, \sigma(i,j) = n_{ij} \text{ for } i < r,\, j < s\}, \quad K = \mathbb{N}^{(r+1)\times s} \setminus J,$$

$$H'_1 = \textstyle\bigcup_{\sigma\in J} H_\sigma, \quad H'_2 = \bigcup_{\sigma\in K} H_\sigma.$$

Then $H'_1 \subseteq H$, $H'_2 \cap H = \emptyset$ and

$$H'_1 \cup H'_2 = \{S : S \cap E_{ij} \text{ is finite for all } i \le r,\, j < s\}$$

is ν'-conegligible. Accordingly we have

$$\begin{aligned}\underline{\int} (\nu'_T)_*(H)\tilde\nu(dT) &\ge \underline{\int} (\nu'_T)_*(H'_1)\tilde\nu(dT) \ge \sum_{\sigma\in J} \int \nu'_T(H_\sigma)\tilde\nu(dT)\\ &= \sum_{\sigma\in J} \nu' H_\sigma = 1 - \sum_{\sigma\in K} \nu' H_\sigma\end{aligned}$$

(because $\bigcup_{\sigma\in J\cup K} H_\sigma$ is ν'-conegligible)

$$\begin{aligned}&= 1 - \sum_{\sigma\in K} \int \nu'_T(H_\sigma)\tilde\nu(dT) = \int 1 - \sum_{\sigma\in K} \nu'_T(H_\sigma)\,\tilde\nu(dT)\\ &= \int \nu'_T(\mathcal{P}X' \setminus H'_2)\tilde\nu(dT) \ge \overline{\int} (\nu'_T)^*(H)\tilde\nu(dT).\end{aligned}$$

But this means, first, that $(\nu'_T)_*(H) = (\nu'_T)^*(H)$ for $\tilde\nu$-almost every T; since ν'_T, being an image of the complete measure λ', is always complete, $\nu'_T(H)$ is defined for $\tilde\nu$-almost every T. Finally,

$$\nu' H = \nu' H_1' = \sum_{\sigma\in J} \nu' H_\sigma = \int \nu_T'(H)\tilde\nu(dT),$$

as required. **Q**

(c) Suppose that $\langle E_i\rangle_{i<r}$ is a disjoint family in $\tilde\Sigma\otimes\Sigma_L$ such that all the projections of the E_i onto $\tilde X$ have finite measure, and $n_i\in\mathbb{N}$ for each $i<r$. Set $H=\{S : S\subseteq X', \#(S\cap E_i)=n_i$ for every $i<r\}$. Then $\int \nu_T'(H)\tilde\nu(dT)=\nu' H$.

P Let $\mathcal{E}$ be a finite subalgebra of $\tilde\Sigma$ such that every E_i belongs to $\mathcal{E}\otimes\Sigma_L$, and let $\langle F_j\rangle_{j<s}$ enumerate the atoms of $\mathcal{E}$ of finite measure; extend this to an enumeration $\langle F_j\rangle_{j<s'}$ of all the atoms of $\mathcal{E}$. Then we can express each E_i as $\bigcup_{j<s'} F_j\times C_{ij}$ where each $C_{ij}\in\Sigma_L$; but as the projection of E_i has finite measure, C_{ij} must be empty for every $j\ge s$, so $E_i=\bigcup_{j<s} F_j\times C_{ij}$. Let Q be the set of all $q\in\mathbb{N}^{r\times s}$ such that $\sum_{j=0}^{s-1} q(i,j)=n_i$ for every $i<r$. For $q\in Q$ set

$$H_q=\{S : S\subseteq X',\ \#(S\cap(F_j\times C_{ij}))=q(i,j) \text{ for every } i<r,\ j<s\}.$$

Then $\langle H_q\rangle_{q\in Q}$ is disjoint and has union H, so

$$\int \nu_T'(H)\tilde\nu(dT) = \sum_{q\in Q}\int \nu_T'(H_q)\tilde\nu(dT) = \sum_{q\in Q}\nu' H_q = \nu' H,$$

using (b) for the middle equality. **Q**

(d) Now let $\langle E_i\rangle_{i<r}$ be a finite disjoint family of subsets of X' of finite measure, and $\langle n_i\rangle_{i<r}$ a family in $\mathbb{N}$. Set $H=\{S : S\subseteq X', \#(S\cap E_i)=n_i$ for every $i<r\}$. Then $\int\nu_T'(H)\tilde\nu(dT)$ is defined and equal to $\nu' H$.

P Let $\epsilon>0$. For each $i<r$ we can find an $E_i'\in\tilde\Sigma\otimes\Sigma_L$ such that $\mu'(E_i\triangle E_i')\le\epsilon$ (251Ie). Discarding a negligible set from E_i' if necessary, we may suppose that the projection of E_i' on $\tilde X$ has finite measure. Set $\hat E_i=E_i'\setminus\bigcup_{k<i}E_k'$ for each i, so that $\langle\hat E_i\rangle_{i<r}$ is a disjoint family in $\tilde\Sigma\otimes\Sigma_L$, and the projections of the $\hat E_i$ are still of finite measure. Set $\hat H=\{S : S\subseteq X', \#(S\cap\hat E_i)=n_i$ for every $i<r\}$. Then (c) tells us that $\int\nu_T'(\hat H)\tilde\nu(dT)=\nu'\hat H$.

Set $E=\bigcup_{i<r}(E_i\triangle E_i')$. Then $\mu' E\le r\epsilon$, while E includes $E_i\triangle\hat E_i$ for every i, so

$$\hat H\setminus H_E\subseteq H\subseteq \hat H\cup H_E,$$

where $H_E=\{S : S\cap E\ne\emptyset\}$ as in (a). Accordingly

$$\nu' H - 3r\gamma\epsilon \le \nu'\hat H - 2r\gamma\epsilon$$

(by (a))

$$= \int \nu_T'(\hat H)\tilde\nu(dT) - 2r\gamma\epsilon$$

(by (c))

$$\le \int \nu_T'(\hat H) - \nu_T'(H_E)\tilde\nu(dT)$$

(by the other part of (a))

$$\le \underline{\int}(\nu_T')_*(H)\tilde\nu(dT) \le \overline{\int}(\nu_T')^*(H)\tilde\nu(dT)$$

$$\le \int \nu_T'(\hat H)+\nu_T'(H_E)\tilde\nu(dT) \le \nu'\hat H + 2r\gamma\epsilon \le \nu' H + 3r\gamma\epsilon.$$

As ϵ is arbitrary,

$$\nu' H = \underline{\int}(\nu_T')_*(H)\tilde\nu(dT) = \overline{\int}(\nu_T')^*(H)\tilde\nu(dT).$$

As in (c-ii) above, it follows that $\int\nu_T'(H)\tilde\nu(dT)$ is defined and equal to $\nu' H$. **Q**

(e) So if we write $\mathcal{H}$ for the family of subsets H of $\mathcal{P}X'$ such that $\int\nu_T'(H)\tilde\nu(dT)$ is defined and equal to $\nu' H$, and $\mathcal{H}_0$ for the family of sets of the form $H=\{S : S\subseteq X', \#(S\cap E_i)=n_i$ for every $i<r\}$ where $\langle E_i\rangle_{i<r}$ is a disjoint family of sets of finite measure and $n_i\in\mathbb{N}$ for $i<r$, we have $\mathcal{H}\supseteq\mathcal{H}_0$. But $\mathcal{H}$ is a Dynkin class, so includes the σ-algebra T' generated by $\mathcal{H}_0$, by 495C. Since every ν'-negligible set is included in a ν'-negligible member of T', $\mathcal{H}$ contains every ν'-negligible set, and therefore every set measured by ν'; which is what we need to know.

495I Theorem Let (X,Σ,μ) and $(\tilde{X},\tilde{\Sigma},\tilde{\mu})$ be atomless σ-finite measure spaces and $\gamma>0$. Let ν, $\tilde{\nu}$ be the Poisson point processes on X, $\tilde{X}$ respectively with density γ. Suppose that $f:X\to\tilde{X}$ is inverse-measure-preserving and that $\langle\mu_t\rangle_{t\in\tilde{X}}$ is a disintegration of μ over $\tilde{\mu}$ consistent with f (definition: 452E) such that every μ_t is a probability measure. Write λ for the product measure $\prod_{t\in\tilde{X}}\mu_t$ on $\Omega=X^{\tilde{X}}$, and for $T\subseteq\tilde{X}$ define $\phi_T:\Omega\to\mathcal{P}X$ by setting $\phi_T(z)=z[T]$ for $z\in\Omega$; let ν_T be the image measure $\lambda\phi_T^{-1}$ on $\mathcal{P}X$. Then $\langle\nu_T\rangle_{T\subseteq\tilde{X}}$ is a disintegration of ν over $\tilde{\nu}$. Moreover

(i) setting $\tilde{f}(S)=f[S]$ for $S\subseteq X$, $\langle\nu_T\rangle_{T\subseteq\tilde{X}}$ is consistent with $\tilde{f}:\mathcal{P}X\to\mathcal{P}\tilde{X}$;

(ii) if $\langle\mu_t\rangle_{t\in\tilde{X}}$ is strongly consistent with f, then $\langle\nu_T\rangle_{T\subseteq\tilde{X}}$ is strongly consistent with $\tilde{f}$.

proof (a) For $T\subseteq\tilde{X}$ let V_T be the set of those $z\in\Omega$ such that $fz\restriction T$ is injective. We need to know that

$$W=\{T:T\subseteq\tilde{X},\,T\text{ is countable, }V_T\text{ is }\lambda\text{-conegligible}\}$$

is $\tilde{\nu}$-conegligible. **P** Write $\tilde{\Sigma}^f=\{F:F\in\tilde{\Sigma},\,\tilde{\mu}F<\infty\}$. Because $\tilde{\mu}$ is atomless and σ-finite, there is a countable subalgebra $\mathcal{E}$ of $\tilde{\Sigma}$ such that for every $\epsilon>0$ there is a cover of $\tilde{X}$ by members of $\mathcal{E}$ of measure at most ϵ. Set

$$Y=\{t:t\in\tilde{X},\,\mu_tf^{-1}[F]=(\chi F)(t)\text{ for every }F\in\mathcal{E}\},$$

so that Y is $\tilde{\mu}$-conegligible and $\mathcal{P}Y$ is $\tilde{\nu}$-conegligible. For $F\in\mathcal{E}$, let W_F be the set of those $T\subseteq Y$ such that for every $t\in T\cap F$ there is an $F'\in\mathcal{E}$ such that $T\cap F'=\{t\}$. Now, given $F\in\mathcal{E}\cap\tilde{\Sigma}^f$ and $\epsilon>0$, there is a partition $\langle F_i\rangle_{i\in I}$ of F into members of $\mathcal{E}$ of measure at most ϵ. Then

$$\begin{aligned}\tilde{\nu}^*(\mathcal{P}Y\setminus W_F)&\le\tilde{\nu}\{T:\#(T\cap F_i)>1\text{ for some }i\in I\}\\&\le\sum_{i\in I}1-e^{-\gamma\tilde{\mu}F_i}(1+\gamma\tilde{\mu}F_i)\le\sum_{i\in I}\frac{1}{2}(\gamma\tilde{\mu}F_i)^2\le\frac{1}{2}\epsilon\gamma^2\sum_{i\in I}\tilde{\mu}F_i=\frac{1}{2}\epsilon\gamma^2\tilde{\mu}F.\end{aligned}$$

As ϵ is arbitary, W_F is $\tilde{\nu}$-conegligible; accordingly $W'=\bigcap\{W_F:F\in\mathcal{E}\cap\tilde{\Sigma}^f\}$ is $\tilde{\nu}$-conegligible.

Now suppose that $T\in W'$. Because $\tilde{X}$ is covered by $\mathcal{E}\cap\tilde{\Sigma}^f$, we see that for every $t\in T$ there is an $F\in\mathcal{E}\cap\tilde{\Sigma}^f$ containing t, and now there is an $F'\in\mathcal{E}$ such that $T\cap F'=\{t\}$. In particular, T is countable, so

$$U_T=\{z:z(t)\in f^{-1}[F]\text{ whenever }F\in\mathcal{E}\text{ and }t\in T\cap F\}$$

is λ-conegligible. Take $z\in U_T$. If t, t' are distinct points of T, there is an $F\in\mathcal{E}$ containing t but not t', and now F contains $f(z(t))$ but not $f(z(t'))$. So $fz\restriction T$ is injective. Thus $U_T\subseteq V_T$ and V_T is λ-conegligible. This is true for every $T\in W'$, so $W\supseteq W'$ is $\tilde{\nu}$-conegligible. **Q**

(b) Suppose that $\langle E_i\rangle_{i<r}$ is a disjoint family of subsets of X with finite measure, and $n_i\in\mathbb{N}$ for $i<r$. Set $H=\{S:S\subseteq X,\,\#(S\cap E_i)=n_i\text{ for every }i<r\}$. Then $\int\nu_T(H)\tilde{\nu}(dT)$ is defined and equal to νH. **P** As in 495H, set $X'=\tilde{X}\times[0,1]$ with the product measure μ', and write λ' for the product measure on $\Omega'=[0,1]^{\tilde{X}}$. Let ν' be the Poisson point process on X' with density γ. For $T\subseteq\tilde{X}$ define $\psi_T:\Omega'\to\mathcal{P}X'$ by setting $\psi_T(z)=\{(t,z(t)):t\in T\}$ for $z\in\Omega'$ and let ν'_T be the image measure $\lambda'\psi_T^{-1}$ on $\mathcal{P}X'$. By 495H, $\langle\nu'_T\rangle_{T\subseteq\tilde{X}}$ is a disintegration of ν' over $\tilde{\nu}$.

For each $i<r$, $\int\mu_t(E_i)\tilde{\mu}(dt)=\mu E_i$; set $Y_1=\{t:\mu_tE_i\text{ is defined for every }i<r\}$, so that $Y_1\subseteq\tilde{X}$ is $\tilde{\nu}$-conegligible. Set $g_i(t)=\sum_{j<i}\mu_tE_j$ for $t\in Y_1$ and $i\le r$, and

$$E'_i=\{(t,\alpha):t\in Y_1,\,g_i(t)\le\alpha<g_{i+1}(t)\}$$

for $i<r$. Then $\mu'E'_i=\int g_{i+1}-g_i\,d\tilde{\nu}=\mu E_i$ for each i, by 252N.

Set

$$H'=\{S:S\subseteq X',\,\#(S'\cap E'_i)=n_i\text{ for every }i<r\},$$

$$W_1=\{T:T\in W,\,T\subseteq Y_1,\,\nu'_TH'\text{ is defined}\},$$

so H' is measured by ν' and W_1 is $\tilde{\nu}$-conegligible. Let T be any member of W_1. Let Q be the set of partitions $q=\langle q(i)\rangle_{i\le r}$ of T such that $\#(q(i))=n_i$ for every $i<r$; because T is countable, so is Q. Set $E_r=X\setminus\bigcup_{i<r}E_i$ and $E'_r=X'\setminus\bigcup_{i<r}E'_i$. Then

$$\mu_tE_r=1-\textstyle\sum_{i=0}^{r-1}\mu_tE_i=1-g_r(t)=\mu_LE'_r[\{t\}]$$

for every $t\in Y_1$. Now

$$\begin{aligned}
\nu'_T H' &= \lambda'\{z : z \in \Omega',\, \psi_T(z) \in H'\} \\
&= \lambda'\{z : z \in \Omega',\, \#(\{t : t \in T,\, z(t) \in E'_i[\{t\}]\}) = n_i \text{ for every } i < r\} \\
&= \sum_{q \in Q} \lambda'\{z : z \in \Omega',\, z(t) \in E'_i[\{t\}] \text{ whenever } i \le r \text{ and } t \in q(i)\} \\
&= \sum_{q \in Q} \prod_{i=0}^{r} \prod_{t \in q(i)} \mu_L E'_i[\{t\}] \\
&= \sum_{q \in Q} \prod_{i=0}^{r} \prod_{t \in q(i)} \mu_t E_i \\
&= \sum_{q \in Q} \lambda\{z : z \in \Omega,\, z(t) \in E_i \text{ whenever } i \le r \text{ and } t \in q(i)\} \\
&= \lambda\{z : z \in \Omega,\, \#(\{t : t \in T,\, z(t) \in E_i\}) = n_i \text{ for every } i < r\} \\
&= \lambda\{z : z \in V_T,\, \#(\{t : t \in T,\, z(t) \in E_i\}) = n_i \text{ for every } i < r\} \\
&= \lambda\{z : z \in V_T,\, \#(z[T] \cap E_i) = n_i \text{ for every } i < r\} \\
&= \lambda\{z : z \in V_T,\, \phi_T(z) \in H\} = \nu_T H.
\end{aligned}$$

Since this is true for $\tilde{\nu}$-almost every T,

$$\begin{aligned}
\int \nu_T(H)\tilde{\nu}(dT) = \int \nu'_T(H')\tilde{\nu}(dT) &= \nu' H' \\
&= \prod_{i<r} \frac{(\gamma\mu' E'_i)^{n_i}}{n_i!} e^{-\gamma\mu' E'_i} = \prod_{i<r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i} = \nu H. \quad \mathbf{Q}
\end{aligned}$$

Now, just as in part (e) of the proof of 495H, 495C tells us that $\int \nu_T(H)\tilde{\nu}(dT) = \nu H$ whenever ν measures H, so that $\langle \nu_T \rangle_{T \subseteq \tilde{X}}$ is a disintegration of ν over $\tilde{\nu}$.

(c) Let $\langle F_i \rangle_{i<r}$ be a disjoint family in $\tilde{\Sigma}^f$, and take $n_i \in \mathbb{N}$ for $i < r$. Set

$$F_r = \tilde{X} \setminus \bigcup_{i<r} F_i,$$

$$Y_2 = \{t : t \in \tilde{X},\, \mu_t f^{-1}[F_i] = (\chi F_i)(t) \text{ for every } i \le r\},$$

$$W_2 = \{T : T \in W,\, T \subseteq Y_2\},$$

so that Y_2 is $\tilde{\mu}$-conegligible and W_2 is $\tilde{\nu}$-conegligible. Set

$$\tilde{H} = \{T : T \in W_2,\, \#(T \cap F_i) = n_i \text{ for every } i < r\},$$

$$H = \tilde{f}^{-1}[\tilde{H}] = \{S : S \subseteq X,\, f[S] \in \tilde{H}\}.$$

Then $\nu_T H = \chi\tilde{H}(T)$ for every $T \in \tilde{H}$. **P** For $i \le r$ and $t \in T \cap F_i$, we have

$$\lambda\{z : z(t) \in f^{-1}[F_i]\} = \mu_t f^{-1}[F_i] = 1$$

because $T \subseteq Y_2$. So

$$V = \{z : z \in V_T,\, f(z(t)) \in F_i \text{ whenever } i \le r \text{ and } t \in T \cap F_i\}$$

is λ-conegligible. But if $z \in V$ then $fz{\restriction}T$ is injective, so

$$\#(f[z[T]] \cap F_i) = \#(T \cap (fz)^{-1}[F_i]) = \#(T \cap F_i)$$

for every $i < r$, and $z[T] \in H$ iff $T \in \tilde{H}$. Thus

$$\begin{aligned}
\nu_T H = \lambda\{z : z[T] \in H\} &= \lambda V = 1 \text{ if } T \in \tilde{H}, \\
&= \lambda(\Omega \setminus V) = 0 \text{ otherwise. } \mathbf{Q}
\end{aligned}$$

Setting

$$\mathcal{H} = \{\tilde{H} : \tilde{H} \subseteq \tilde{X},\, \nu_T \tilde{f}^{-1}[\tilde{H}] = \chi\tilde{H}(T) \text{ for } \tilde{\nu}\text{-almost every } T \in \tilde{H}\},$$

it is easy to check that $\mathcal{H}$ is a Dynkin class containing all sets of the form $\{T : T \subseteq \tilde{X},\, \#(T \cap F_i) = n_i$ for every $i < r\}$ where $\langle F_i\rangle_{i<r}$ is a disjoint family in $\tilde{\Sigma}^f$, and therefore including the σ-algebra generated by such sets, by 495C. But as $\mathcal{H}$ also contains any subset of a negligible set belonging to $\mathcal{H}$ (remember that all the ν_T are complete probability measures, like λ), it includes the domain of $\tilde{\nu}$, and $\langle\nu_T\rangle_{T\subseteq\tilde{X}}$ is consistent with $\tilde{f}$.

(d) Now suppose that $\langle\mu_t\rangle_{t\in\tilde{X}}$ is strongly consistent with f. Set $Y_3 = \{t : \mu_t f^{-1}[\{t\}] = 1\}$ and $W_3 = \{T : T \in W,\, T \subseteq Y_3\}$. Then $\nu_T \tilde{f}^{-1}[\{T\}] = 1$ for every $T \in W_3$. **P** Set $V'_T = \{z : z \in \Omega,\, f(z(t)) = t$ for every $t \in T\}$. For each $t \in T$,

$$\lambda\{z : f(z(t)) = t\} = \mu_t f^{-1}[\{t\}] = 1,$$

because $T \subseteq W_3$. As T is countable, V'_T is λ-conegligible. But now

$$\nu_T \tilde{f}^{-1}[\{T\}] = \lambda\{z : f[z[T]] = T\} \ge \lambda V'_T = 1. \text{ **Q**}$$

As W_3 is $\tilde{\nu}$-conegligible, $\langle\nu_T\rangle_{T\subseteq\tilde{X}}$ is strongly consistent with $\tilde{f}$.

495J Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $\gamma > 0$. Then there are a probability algebra $(\mathfrak{B}, \bar{\lambda})$ and a function $\theta : \mathfrak{A} \to \mathfrak{B}$ such that

(i) $\theta(\sup A) = \sup \theta[A]$ for every non-empty $A \subseteq \mathfrak{A}$ such that $\sup A$ is defined in $\mathfrak{A}$;

(ii) $\bar{\lambda}\theta(a) = 1 - e^{-\gamma\bar{\mu}a}$ for every $a \in \mathfrak{A}$, interpreting $e^{-\infty}$ as 0;

(iii) whenever $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}$ generated by $\{\theta(a) : a \subseteq a_i\}$ for each i, then $\langle\mathfrak{C}_i\rangle_{i\in I}$ is stochastically independent.

proof (a) We may suppose that $(\mathfrak{A}, \bar{\mu})$ is the measure algebra of a measure space (X, Σ, μ) (321J). Set $\Sigma^f = \{E : E \in \Sigma,\, \mu E < \infty\}$ and $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\, \bar{\mu}a < \infty\}$. Let $(\Omega, \Lambda, \lambda)$ and $\langle g_E\rangle_{E\in\Sigma^f}$ be as in 495B, and take $(\mathfrak{B}, \bar{\lambda})$ to be the measure algebra of $(\Omega, \Lambda, \lambda)$. Note that if E, $F \in \Sigma^f$ and $\mu(E\triangle F) = 0$, then $g_{E\setminus F}$ and $g_{F\setminus E}$ have Poisson distributions with expectation 0, so are zero almost everywhere, while $g_E =_{\text{a.e.}} g_{E\cap F} + g_{E\setminus F}$ and $g_F =_{\text{a.e.}} g_{E\cap F} + g_{F\setminus E}$; so that $g_E =_{\text{a.e.}} g_F$. This means that we can define $\theta : \mathfrak{A}^f \to \mathfrak{B}$ by setting $\theta(E^\bullet) = \{\omega : g_E(\omega) \ne 0\}^\bullet$ whenever $E \in \Sigma^f$, and we shall have $\bar{\lambda}(\theta a) = 1 - e^{-\gamma\bar{\mu}a}$ because g_E has a Poisson distribution with expectation $\bar{\mu}a$ whenever $E \in \Sigma^f$ and $E^\bullet = a$. For $a \in \mathfrak{A} \setminus \mathfrak{A}^f$ set $\theta(a) = 1_{\mathfrak{B}}$.

(b) If a, $b \in \mathfrak{A}^f$ are disjoint, they can be represented as $E^\bullet$, $F^\bullet$ where E, $F \in \Sigma^f$ are disjoint. In this case, $g_{E\cup F} =_{\text{a.e.}} g_E + g_F$, so $\theta(a \cup b) = \theta(a) \cup \theta(b)$. Of course the same is true if a, $b \in \mathfrak{A}$ are disjoint and either has infinite measure. It follows at once that for any a, $b \in \mathfrak{A}$,

$$\theta(a \cup b) = \theta(a \setminus b) \cup \theta(a \cap b) \cup \theta(b \setminus a) = \theta(a) \cup \theta(b).$$

Consequently $\theta(\sup A) = \sup\theta[A]$ for any finite set $A \subseteq \mathfrak{A}$. If $A \subseteq \mathfrak{A}$ is an infinite set with supremum a^*, then $A' = \{\sup B : B \in [A]^{<\omega}\}$ is an upwards-directed set with supremum a^*, so there is a non-decreasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A' such that $\lim_{n\to\infty} \bar{\mu}a_n = \bar{\mu}a^*$ (321D). In this case, $b^* = \sup_{n\in\mathbb{N}} \theta(a_n)$ is defined in $\mathfrak{B}$ and

$$\bar{\lambda}b^* = \lim_{n\to\infty} \bar{\lambda}\theta(a_n) = \lim_{n\to\infty} 1 - e^{-\gamma\bar{\mu}a_n} = 1 - e^{-\gamma\bar{\mu}a^*} = \bar{\lambda}\theta(a^*).$$

So $b^* = \theta(a^*)$; since $\theta(a^*)$ is certainly an upper bound of $\theta[A']$, it must actually be the supremum of $\theta[A']$ and therefore (because θ preserves finite suprema) of $\theta[A]$.

(c) Thus θ satisfies (i) and (ii). As for (iii), note first that if $\langle a_i\rangle_{i\in I}$ is a finite disjoint family in $\mathfrak{A}$, then $\bar{\lambda}(\inf_{i\in I} \theta(a_i)) = \prod_{i\in I} \bar{\lambda}\theta(a_i)$. **P** Set $J = \{i : i \in I,\, \bar{\mu}a_i < \infty\}$. For $i \in J$, represent a_i as $E_i^\bullet$ where $\langle E_i\rangle_{i\in J}$ is a disjoint family in Σ^f. Then $\langle g_{E_i}\rangle_{i\in J}$ is independent, so

$$\begin{aligned}\bar{\lambda}(\inf_{i\in I} \theta(a_i)) &= \bar{\lambda}(\inf_{i\in J} \theta(a_i)) = \lambda(\Omega \cap \bigcap_{i\in J} \{\omega : g_{E_i}(\omega) = 0\}) \\ &= \prod_{i\in J} \lambda\{\omega : g_{E_i}(\omega) = 0\} = \prod_{i\in J} \bar{\lambda}\theta(a_i) = \prod_{i\in I} \bar{\lambda}\theta(a_i). \text{ **Q**}\end{aligned}$$

Now suppose that $\langle a_i\rangle_{i\in I}$ is a finite disjoint family in $\mathfrak{A}$ and that $\mathfrak{D}_i$ is the subalgebra of $\mathfrak{B}$ generated by $D_i = \{\theta(a) : a \subseteq a_i\}$ for each i. We know that each D_i is closed under $\cup$ (by (i)) and that $\bar{\lambda}(\inf_{i\in J} d_i) = \prod_{i\in J} \bar{\lambda}d_i$ whenever $J \subseteq I$ and $d_i \in D_i$ for each $i \in J$, that is, that $\langle d_i\rangle_{i\in I}$ is stochastically independent whenever $d_i \in D_i$

for each i. Setting $D_i' = \{1 \setminus d : d \in D_i\} \cup \{0\}$, we see that D_i' is closed under $\cap$ and that $\langle d_i \rangle_{i\in I}$ is stochastically independent whenever $d_i \in D_i'$ for each i (as in 272F). An induction on $\#(J)$, using 313Ga for the inductive step, shows that if $J \subseteq I$, $d_i \in \mathfrak{D}_i$ for $i \in J$, and $d_i \in D_i'$ for $i \in I \setminus J$, then $\bar\lambda(\inf_{i\in I} d_i) = \prod_{i\in I} \bar\lambda d_i$. At the end of the induction, we see that $\bar\lambda(\inf_{i\in I} d_i) = \prod_{i\in I} \bar\lambda d_i$ whenever $d_i \in \mathfrak{D}_i$ for each i, and therefore whenever d_i belongs to the topological closure of $\mathfrak{D}_i$ for each i, where $\mathfrak{B}$ is given its measure-algebra topology (§323).

Finally, suppose that $\langle a_i \rangle_{i\in I}$ is any disjoint family in $\mathfrak{A}$, and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}$ generated by $D_i = \{\theta(a) : a \subseteq a_i\}$ for each i. Take a finite set $J \subseteq I$ and $c_i \in \mathfrak{C}_i$ for each $i \in J$. By 323J, $\mathfrak{C}_i$ is the topological closure of the subalgebra $\mathfrak{D}_i$ of $\mathfrak{B}$ generated by $\{\theta(a) : a \subseteq a_i\}$; so $\bar\lambda(\inf_{i\in J} c_i) = \prod_{i\in J} \bar\lambda c_i$. As $\langle c_i \rangle_{i\in J}$ is arbitrary, $\langle \mathfrak{C}_i \rangle_{i\in I}$ is independent.

495K Proposition Let U be any L-space. Then there are a probability space $(\Omega, \Lambda, \lambda)$ and a positive linear operator $T : U \to L^1(\lambda)$ such that $\|Tu\|_1 = \|u\|_1$ whenever $u \in L^1(\mu)^+$ and $\langle Tu_i \rangle_{i\in I}$ is stochastically independent in $L^0(\lambda)$ whenever $\langle u_i \rangle_{i\in I}$ is a disjoint family in $L^1(\mu)$.

Remarks Recall that a family $\langle u_i \rangle_{i\in I}$ in a Riesz space is 'disjoint' if $|u_i| \wedge |u_j| = 0$ for all distinct i, $j \in I$ (352C). A family $\langle v_i \rangle_{i\in I}$ in $L^0(\lambda)$ is 'independent' if $\langle g_i \rangle_{i\in I}$ is an independent family of random variables whenever $g_i \in \mathcal{L}^0(\lambda)$ represents v_i for each i; compare 367W.

proof (a) By Kakutani's theorem, there is a measure algebra $(\mathfrak{A}, \bar\mu)$ such that U is isomorphic, as Banach lattice, to $L^1(\mathfrak{A}, \bar\mu)$; now $(\mathfrak{A}, \bar\mu)$ can be represented as the measure algebra of a measure space (X, Σ, μ), and we can identify U and $L^1(\mathfrak{A}, \bar\mu)$ with $L^1(\mu)$ (365B). Set $\Sigma^f = \{E : E \in \Sigma, \mu E < \infty\}$ and $\mathfrak{A}^f = \{a : a \in \mathfrak{A}, \bar\mu a < \infty\}$ as usual. Take $(\Omega, \Lambda, \lambda)$ and $\langle g_E \rangle_{E\in\Sigma^f}$ from 495B, with $\gamma = 1$. As in the proof of 495J, we have $g_E =_{\text{a.e.}} g_F$ whenever E, $F \in \Sigma^f$ and $\mu(E \triangle F) = 0$; consequently we can define $\psi : \mathfrak{A}^f \to L^1(\lambda)$ by setting $\psi a = g_E^\bullet$ whenever $E \in \Sigma^f$ and $E^\bullet = a$. Again as in 495J, $g_{E\cup F} =_{\text{a.e.}} g_E + g_F$ whenever E, $F \in \Sigma^f$ are disjoint, so ψ is additive. Also

$$\|\psi a\|_1 = \int g_E d\lambda = \mu E = \bar\mu a$$

whenever $E \in \Sigma^f$ represents $a \in \mathfrak{A}^f$. By 365I, there is a unique bounded linear operator $T : L^1(\mathfrak{A}, \bar\mu) \to L^1(\lambda)$ such that $T(\chi a) = \psi a$ for every $a \in \mathfrak{A}^f$. By 365Ka, T is a positive operator. The set $\{u : u \in L^1(\mathfrak{A}, \bar\mu)^+, \|Tu\|_1 = \|u\|_1\}$ is closed under addition, norm-closed and contains $\alpha\chi a$ for every $a \in \mathfrak{A}^f$ and $\alpha \ge 0$, so is the whole of $L^1(\mathfrak{A}, \bar\mu)^+$, by 365F.

Note that if $\langle a_i \rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}^f$, then $\langle \psi a_i \rangle_{i\in I}$ is stochastically independent, by 495B(ii).

(b) Now let $\langle u_i \rangle_{i\in I}$ be a disjoint family in $L^1(\mathfrak{A}, \bar\mu)$. Then $\langle Tu_i \rangle_{i\in I}$ is independent. **P?** Otherwise, there are a finite set $J \subseteq I$ and a family $\langle V_i \rangle_{i\in J}$ such that V_i is a neighbourhood of Tu_i in the topology of convergence in measure on $L^0(\mu)$ for each $i \in J$, and $\langle v_i \rangle_{i\in J}$ is not independent whenever $v_i \in V_i$ for each i (367W). Because the embedding $L^1(\lambda) \subsetneq L^0(\lambda)$ is continuous for the norm topology on $L^1(\lambda)$ and the topology of convergence in measure (245G), there is a $\delta > 0$ such that $Tu_i' \in V_i$ whenever $i \in J$, $u_i' \in L^1(\mathfrak{A}, \bar\mu)$ and $\|u_i' - u_i\|_1 \le \delta$. Now we can find such $u_i' \in S(\mathfrak{A}^f)$ with $|u_i'| \le |u_i|$ (365F).

Express each u_i' as $\sum_{k=0}^{n_i} \alpha_{ik}\chi a_{ik}$ where $\langle a_{ik} \rangle_{k\le n_i}$ is a disjoint family in $\mathfrak{A}^f$ and no α_{ik} is zero (361Eb). In this case, all the a_{ik}, for $i \in J$ and $k \le n_i$, are disjoint, so all the $\psi(a_{ik})$ are independent. But this means that $\langle Tu_i' \rangle_{i\in J} = \langle \sum_{k=0}^{n_i} \alpha_{ik}\psi(a_{ik}) \rangle_{i\in J}$ is independent (272K); which is impossible, because $Tu_i' \in V_i$ for every $i \in J$. **XQ**

So T, regarded as a function from U to $L^1(\lambda)$, has the required properties.

495L The following is a more concrete expression of the same ideas.

Proposition Let (X, Σ, μ) be an atomless measure space, and ν the Poisson point process on X with density $\gamma > 0$.

(a) If $h \in \mathcal{L}^1(\mu)$, $Q_h(S) = \sum_{x\in S\cap\operatorname{dom} h} h(x)$ is defined and finite for ν-almost every $S \subseteq X$, and $\int Q_h d\nu = \gamma \int h\,d\mu$.

(b) We have a positive linear operator $T : L^1(\mu) \to L^1(\nu)$ defined by setting $T(h^\bullet) = Q_h^\bullet$ for every $h \in \mathcal{L}^1(\mu)$.

(c) $\|Tu\|_1 = \gamma\|u\|_1$ whenever $u \in L^1(\mu)^+$ and $\langle Tu_i \rangle_{i\in I}$ is stochastically independent in $L^0(\lambda)$ whenever $\langle u_i \rangle_{i\in I}$ is a disjoint family in $L^1(\mu)$.

proof (a) In the language of 495D, $Q_{\chi E} = f_E$ for every $E \in \Sigma^f$. So $Q_{\chi E} \in \mathcal{L}^1(\nu)$ and $\int Q_{\chi E} d\nu = \gamma\mu E$ for every $E \in \Sigma^f$. If $h = \sum_{i=0}^r \alpha_i \chi E_i$ is a simple function on X, then $Q_h =_{\text{a.e.}} \sum_{i=0}^r \alpha_i Q_{\chi E_i} \in \mathcal{L}^1(\nu)$ and $\int Q_h d\nu = \int h\,d\mu$. If $h \in \mathcal{L}^1(\mu)$ is zero a.e., then $\{S : S \subseteq h^{-1}[\{0\}]\}$ is ν-conegligible, so $Q_h = 0$ a.e. It follows that if $h \in \mathcal{L}^1(\mu)$ is non-negative, and $\langle h_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of simple functions converging to h almost everywhere, then $Q_h =_{\text{a.e.}} \lim_{n\to\infty} Q_{h_n}$, while $Q_{h_n} \le_{\text{a.e.}} Q_{h_{n+1}}$ for every n; so Q_h is ν-integrable and

$$\int Q_h d\nu = \lim_{n\to\infty} \int Q_{h_n} d\nu = \lim_{n\to\infty} \gamma \int h_n d\mu = \gamma \int h d\mu.$$

(b) Since $Q_h =_{\text{a.e.}} Q_{h'}$ whenever $h =_{\text{a.e.}} h'$ in $\mathcal{L}^1(\mu)$, we can define $T : L^1(\mu) \to L^1(\nu)$ by setting $T(h^\bullet) = (Q_h)^\bullet$ for every $h \in \mathcal{L}^1(\mu)$; because $Q_{\alpha h} =_{\text{a.e.}} \alpha Q_h$ and $Q_{h+h'} =_{\text{a.e.}} Q_h + Q_{h'}$ whenever h, $h' \in \mathcal{L}^1(\mu)$ and $\alpha \in \mathbb{R}$, T is linear; because $Q_h \ge 0$ a.e. whenever $h \ge 0$ a.e., T is positive.

(c) Because $\int Q_h d\nu = \gamma \int h\,d\mu$ for every $h \in \mathcal{L}^1(\mu)$, and T is positive, $\|Tu\|_1 = \gamma\|u\|_1$ for every $u \in L^1(\mu)^+$. Finally, if $\langle u_i\rangle_{i\in I}$ is a finite disjoint family in $L^1(\mu)$, we can find a family $\langle h_i\rangle_{i\in I}$ of measurable functions from X to $\mathbb{R}$ such that $h_i^\bullet = u_i$ for each i and the sets $E_i = \{x : h_i(x) \ne 0\}$ are disjoint. For each $i \in I$, let T_i be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S : \#(S \cap E) = n\}$ where $E \subseteq E_i$ has finite measure and $n \in \mathbb{N}$. Then $\langle \mathrm{T}_i\rangle_{i\in I}$ is independent (as in part (c) of the proof of 495J), and each Q_{h_i} is T_i-measurable, so $\langle Q_{h_i}\rangle_{i\in I}$ is independent and $\langle Tu_i\rangle_{i\in I}$ is independent.

495M We can identify the characteristic functions of the random variables Q_f as defined above.

Proposition Let (X, Σ, μ) be an atomless measure space, and ν the Poisson point process on X with density $\gamma > 0$. For $f \in \mathcal{L}^1(\mu)$ set $Q_f(S) = \sum_{x\in S\cap\operatorname{dom} f} f(x)$ when $S \subseteq X$ and the sum is defined in $\mathbb{R}$. Then

$$\int_{\mathcal{P}X} e^{iyQ_f} d\nu = \exp\bigl(\gamma\int_X (e^{iyf} - 1)d\mu\bigr)$$

for any $y \in \mathbb{R}$.

proof Note that Q_f is defined ν-almost everywhere, by 495La.

(a) Consider first the case in which f is a simple function, expressed as $\sum_{j=0}^n \alpha_j\chi F_j$ where $\langle F_j\rangle_{j\le n}$ is a disjoint family of sets of finite measure and $\alpha_j \in \mathbb{R}$ for each j. Then $Q_f(S) = \sum_{j=0}^n \alpha_j\#(S \cap F_j)$ for ν-almost every S, so

$$\int e^{iyQ_f} d\nu = \int \prod_{j=0}^n e^{iy\alpha_j\#(S\cap F_j)}\nu(dS) = \prod_{j=0}^n \int e^{iy\alpha_j\#(S\cap F_j)}\nu(dS)$$

(because the functions $S \mapsto \#(S \cap F_j)$ are independent)

$$= \prod_{j=0}^n \sum_{k=0}^\infty \frac{(\gamma\mu F_j)^k}{k!} e^{-\gamma\mu F_j} e^{iy\alpha_j k} = \prod_{j=0}^n e^{-\gamma\mu F_j} \sum_{k=0}^\infty \frac{(e^{iy\alpha_j}\gamma\mu F_j)^k}{k!}$$

$$= \prod_{j=0}^n \exp\bigl((e^{iy\alpha_j} - 1)\gamma\mu F_j\bigr) = \exp\bigl(\gamma\sum_{j=0}^n (e^{iy\alpha_j} - 1)\mu F_j\bigr)$$

$$= \exp\bigl(\gamma\int (e^{iyf} - 1)d\mu\bigr).$$

(b) Now suppose that f is any integrable function. Then there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of simple functions such that $|f_n| \le_{\text{a.e.}} |f|$ for every n and $\lim_{n\to\infty} f_n =_{\text{a.e.}} f$. Write q_n, q for Q_{f_n}, Q_f. Set

$$D = \{x : x \in \operatorname{dom} f,\ |f_n(x)| \le |f(x)| \text{ for every } n \text{ and } \lim_{n\to\infty} f_n(x) = f(x)\},$$

so that D is μ-conegligible. If $S \subseteq D$ and $Q_{|f|}(S)$ is defined, then $q(S) = \lim_{n\to\infty} q_n(S)$, and this is true for ν-almost every S; so $\int e^{iyq} d\nu = \lim_{n\to\infty} e^{iyq_n} d\nu$, by Lebesgue's Dominated Convergence Theorem. On the other hand,

$$|e^{i\alpha} - 1| = |\int_0^\alpha \frac{1}{i}e^{it}dt| \le \alpha, \quad |e^{-i\alpha} - 1| = |\int_0^\alpha \frac{1}{i}e^{-it}dt| \le \alpha$$

for every $\alpha \ge 0$. So if we set $g(x) = e^{iyf(x)} - 1$, $g_n(x) = e^{iyf_n(x)} - 1$ when these are defined, we have $|g_n| \le_{\text{a.e.}} |yf_n| \le_{\text{a.e.}} |yf|$ for every n. Accordingly

$$\int (e^{iyf} - 1)d\mu = \int \lim_{n\to\infty} (e^{iyf_n} - 1)d\mu = \lim_{n\to\infty} \int (e^{iyf_n} - 1)d\mu$$

by Lebesgue's theorem again. It follows that

$$\int e^{iyQ_f} d\nu = \lim_{n\to\infty} \int e^{iyq_n} d\nu = \lim_{n\to\infty} \exp\bigl(\gamma\int (e^{iyf_n} - 1)d\mu\bigr)$$

(by (a))

$$= \exp\bigl(\gamma \lim_{n\to\infty} \int (e^{iyf_n} - 1)d\mu\bigr) = \exp\bigl(\gamma \int (e^{iyf} - 1)d\mu\bigr),$$

as claimed.

Remark Recall that a Poisson random variable with expectation γ has characteristic function $y \mapsto \exp(\gamma(e^{iy} - 1))$ (part (a) of the proof of 285Q), corresponding to the case $f = \chi F$ where $\mu F = 1$. The random variables Q_f have **compound Poisson** distributions.

495N If our underlying measure is a Radon measure, we can look for Radon measures on $\mathcal{P}X$ to represent the Poisson point processes on X. There seem to be difficulties in general, but I can offer the following. See also 495Yd.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space such that μ is outer regular with respect to the open sets, and $\gamma > 0$. Give the space $\mathcal{C}$ of closed subsets of X its Fell topology (4A2T).

(a) There is a unique quasi-Radon probability measure $\tilde\nu$ on $\mathcal{C}$ such that

$$\tilde\nu\{C : \#(C \cap E) = \emptyset\} = e^{-\gamma\mu E}$$

whenever $E \subseteq X$ is a measurable set of finite measure.

(b) If $E_0, \dots , E_r$ are disjoint sets of finite measure, none including any singleton set of non-zero measure, and $n_i \in \mathbb{N}$ for $i \le r$, then

$$\tilde\nu\{C : \#(C \cap E_i) = n_i \text{ for every } i \le r\} = \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}.$$

(c) Suppose that μ is atomless and ν is the Poisson point process on X with density γ.

(i) $\mathcal{C}$ has full outer measure for ν, and $\tilde\nu$ extends the subspace measure $\nu_{\mathcal{C}}$.

(ii) If moreover μ is σ-finite, then $\mathcal{C}$ is ν-conegligible.

(d) If X is locally compact then $\tilde\nu$ is a Radon measure.

(e) If X is second-countable and μ is atomless then $\tilde\nu = \nu_{\mathcal{C}}$.

proof (a)(i) Set $\Sigma^f = \{E : \mu E < \infty\}$. There is a disjoint family $\mathcal{H}$ of non-empty self-supporting measurable subsets of X of finite measure such that $\mu E = \sum_{H\in\mathcal{H}} \mu(E \cap H)$ for every $E \in \Sigma$ (412I); so if $G \subseteq X$ is an open set of finite measure, $\{H : H \in \mathcal{H},\, G \cap H \ne \emptyset\}$ is countable. If E is any set of finite measure, it is included in an open set of finite measure, because μ is outer regular with respect to the open sets; so once again $\{H : H \in \mathcal{H},\, E \cap H \ne \emptyset\}$ is countable.

Build $\Omega = \prod_{H\in\mathcal{H}} \mathbb{N} \times H^{\mathbb{N}}$, $\langle g_{HE}\rangle_{H\in\mathcal{H},E\in\Sigma}$ and the product measure λ on Ω as in the proof of 495B; as in the proof of 495D, set

$$\phi(\omega) = \{x_{Hj}(\omega) : H \in \mathcal{H},\, j < m_H(\omega)\}$$

for $\omega \in \Omega$.

(ii) If $E \in \Sigma^f$, $\lambda\{\omega : E \cap \phi(\omega) = \emptyset\} = e^{-\gamma\mu E}$. **P** $\mathcal{H}' = \{H : H \in \mathcal{H},\, E \cap H \ne \emptyset\}$ is countable. Now

$$\{\omega : E \cap \phi(\omega) = \emptyset\} = \bigcap_{H\in\mathcal{H}'} \{\omega : x_{Hj} \notin E \text{ for every } j < m_H(\omega)\}$$

has measure

$$\prod_{H\in\mathcal{H}'} \lambda\{\omega : g_{HE}(\omega) = 0\} = \prod_{H\in\mathcal{H}'} e^{-\gamma\mu(H\cap E)} = e^{-\gamma\mu E}$$

because $\mu E = \sum_{H\in\mathcal{H}'} \mu(H \cap E)$. **Q**

Let T_0 be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S : S \cap E = \emptyset\}$ where $E \in \Sigma^f$. By the Monotone Class Theorem (136B), λ measures $\phi^{-1}[W]$ for every $W \in \mathrm{T}_0$; set $\nu_0 W = \lambda\phi^{-1}[W]$ for $W \in \mathrm{T}_0$, so that if $E \in \Sigma$ then $\nu_0\{S : S \cap E = \emptyset\} = e^{-\gamma\mu E}$.

(iii) Give $\mathcal{P}X$ the topology $\mathfrak{S}$ generated by sets of the form

$$\{S : S \cap G \ne \emptyset\}, \quad \{S : S \cap K = \emptyset\}$$

for open sets $G \subseteq X$ and compact sets $K \subseteq X$. (Thus the Fell topology on $\mathcal{C}$ is the subspace topology induced by $\mathfrak{S}$.) Then $\mathcal{P}X$ is compact. **P** Follow the proof of 4A2T(b-iii) word for word, but replacing every $\mathcal{C}$ with $\mathcal{P}X$. **Q**

(iv) ν_0 is inner regular with respect to the $\mathfrak{S}$-closed sets. **P** Write $\mathcal{L}$ for the family of $\mathfrak{S}$-closed sets belonging to T_0. Of course $\mathcal{L}$ is closed under finite unions and countable intersections.

(α) Suppose that $E \in \Sigma^f$ and $W = \{S : S \cap E \ne \emptyset\}$. Let $\epsilon > 0$. Then there is a compact set $K \subseteq E$ such that $\mu(E \setminus K) \le \epsilon$. Set $V = \{S : S \cap K \ne \emptyset\}$; then $V \in \mathcal{L}$, $V \subseteq W$ and

$$\nu_0(W \setminus V) \le \nu_0\{S : S \cap E \setminus K \ne \emptyset\} \le 1 - e^{-\gamma\epsilon}.$$

As ϵ is arbitrary, $\nu_0 W = \sup\{\nu_0 V : V \in \mathcal{L},\, V \subseteq W\}$.

(β) Suppose that $E \in \Sigma^f$ and $W = \{S : S \cap E = \emptyset\}$. Let $\epsilon > 0$. Then there is an open set $G \supseteq E$ such that $\mu(G \setminus E) \le \epsilon$. Set $V = \{S : S \cap G = \emptyset\}$; then $V \in \mathcal{L}$, $V \subseteq W$ and

$$\nu_0(W \setminus V) \le \nu_0\{S : S \cap G \setminus E \ne \emptyset\} \le 1 - e^{-\gamma\epsilon}.$$

As ϵ is arbitrary, $\nu_0 W = \sup\{\nu_0 V : V \in \mathcal{L},\, V \subseteq W\}$.

(γ) By 412C, ν_0 is inner regular with respect to the $\mathfrak{S}$-closed sets. **Q**

(v) Since $\mathfrak{S}$ is a compact topology, the family of $\mathfrak{S}$-closed sets is a compact class, so 413O tells us that ν_0 has an extension to a complete topological measure $\tilde\nu_0$ on $\mathcal{P}X$, inner regular with respect to the closed sets. Of course $\tilde\nu_0$, being a probability measure, is effectively locally finite and locally determined, so it is a quasi-Radon measure with respect to the topology $\mathfrak{S}$. Consequently the subspace measure $\tilde\nu$ on $\mathcal{C}$ is a quasi-Radon measure for the Fell topology on $\mathcal{C}$ (415B).

(vi) $\mathcal{C}$ has full outer measure for $\tilde\nu_0$. **P?** Otherwise, there is a non-empty closed set $V \subseteq \mathcal{P}X \setminus \mathcal{C}$. Consider the family $\mathcal{U}$ of subsets of $\mathcal{P}X$ of the form

$$\{S : S \cap K = \emptyset,\, S \cap G_i \ne \emptyset \text{ for } i < r\}$$

where $K \subseteq X$ is compact and $G_i \subseteq X$ is an open set of finite measure for every $i < r$. Because μ is locally finite, this is a base for $\mathfrak{S}$. So $\mathcal{U}' = \{U : U \in \mathcal{U},\, U \cap W = \emptyset\}$ is a cover of $\mathcal{P}X \setminus W \supseteq \mathcal{C}$. Of course $U \cap \mathcal{C}$ is open in the Fell topology for every $U \in \mathcal{U}$; because $\mathcal{C}$ is compact, there are $U_0, \dots, U_m \in \mathcal{U}'$ covering $\mathcal{C}$.

Express each U_j as $\{S : S \cap K_j = \emptyset,\, S \cap G_{ji} \ne \emptyset \text{ for } i < r_j\}$, where the K_j are all compact and the G_{ji} are all open. Because $\bigcup_{j \le m} U_j$ is disjoint from V, there is an $S \subseteq \mathcal{P}X$ which does not belong to any U_j. Let $\mathcal{E}$ be the finite algebra of subsets of X generated by $\{K_j : j \le m\} \cup \{G_{ji} : j \le m,\, i < r_j\}$; then there is a finite set $C \subseteq S$ such that $C \cap E \ne \emptyset$ whenever $E \in \mathcal{E}$ and $S \cap E \ne \emptyset$. In this case, $C \in \mathcal{C} \setminus \bigcup_{j \le m} U_j$; which is supposed to be impossible. **XQ**

(vii) Consequently $\tilde\nu$ is a probability measure. If $E \in \Sigma^f$, then

$$\begin{aligned}\tilde\nu\{C : C \in \mathcal{C},\, C \cap E = \emptyset\} &= \tilde\nu(\mathcal{C} \cap \{S : S \subseteq X,\, S \cap E = \emptyset\})\\ &= \tilde\nu_0\{S : S \cap E = \emptyset\} = \nu_0\{S : S \cap E = \emptyset\} = e^{-\gamma\mu E}.\end{aligned}$$

(viii) To see that $\tilde\nu$ is uniquely defined, let $\tilde\nu'$ be another quasi-Radon probability measure on $\mathcal{C}$ with the same property.

(α) Suppose that $E_0, \dots, E_r \subseteq X$ are disjoint measurable sets of finite measure, and

$$W = \{C : C \in \mathcal{C},\, C \cap E_0 = \emptyset,\, C \cap E_i \ne \emptyset \text{ for } 1 \le i \le r\}.$$

Then $\tilde\nu W = \tilde\nu' W$. **P** Induce on r. If $r = 0$ the result is immediate. For the inductive step to $r \ge 1$, consider $\{C : C \cap E_0 = \emptyset,\, C \cap E_i \ne \emptyset \text{ for } 1 \le i < r\}$ and $\{C : C \cap (E_0 \cup E_r) = \emptyset,\, C \cap E_i \ne \emptyset \text{ for } 1 \le i < r\}$. By the inductive hypothesis, $\tilde\nu$ and $\tilde\nu'$ agree on these two sets, and therefore on their difference $\{C : C \in \mathcal{C},\, C \cap E_0 = \emptyset,\, C \cap E_i \ne \emptyset$ for $1 \le i \le r\}$. **Q**

(β) Suppose that we have a compact set $K \subseteq X$ and open sets $G_i \subseteq X$ of finite measure, for $i < r$, and set

$$V = \{C : C \in \mathcal{C},\, C \cap K = \emptyset,\, C \cap G_i \ne \emptyset \text{ for every } i < r\}.$$

Then $\tilde\nu V = \tilde\nu' V$. **P** Let $\mathcal{E}$ be the finite subalgebra of $\mathcal{P}X$ generated by $\{G_i : i < r\} \cup \{K\}$, and $\mathcal{A}$ the set of atoms of $\mathcal{E}$ included in $K \cup \bigcup_{i<r} G_i$. For $\mathcal{I} \subseteq \mathcal{A}$ set

$$V_{\mathcal{I}} = \{C : C \in \mathcal{C},\, C \cap E \ne \emptyset \text{ for } E \in \mathcal{I},\, C \cap E = \emptyset \text{ for } E \in \mathcal{A} \setminus \mathcal{I}\}.$$

Then $V = \bigcup_{\mathcal{I} \in \mathfrak{I}} V_{\mathcal{I}}$, where

$$\begin{aligned}\mathfrak{I} = \{\mathcal{I} : \mathcal{I} \subseteq \mathcal{A},\, &A \cap K = \emptyset \text{ for every } A \in \mathcal{I},\\ &\text{for every } i < r \text{ there is an } A \in \mathcal{I} \text{ such that } A \subseteq G_i\}.\end{aligned}$$

Now (α) shows that $\tilde\nu V_{\mathcal{I}} = \tilde\nu' V_{\mathcal{I}}$ for every $\mathcal{I} \subseteq \mathcal{A}$, so that $\tilde\nu V = \tilde\nu' V$. Since sets V of the type described form a base for the Fell topology closed under finite intersections, $\tilde\nu = \tilde\nu'$ (415H(v)). **Q**

This completes the proof of (a).

(b)(i) In the construction of 495B and (a-i) above, all the normalized subspace measures μ'_H are Radon measures (416Rb), while of course all the Poisson distributions ν_H are Radon measures, so the product measure λ on $\Omega = \prod_{H\in\mathcal{H}} \mathbb{N} \times H^{\mathbb{N}}$ has an extension to a Radon measure $\tilde\lambda$ (417Q). Let $\mathcal{W}$ be the family of those sets $W \subseteq \mathcal{P}X$ such that $\tilde\nu_0 W$ and $\tilde\lambda\phi^{-1}[E]$ are defined and equal. Then $\mathcal{W}$ is a Dynkin class. So if $\mathcal{W}_0 \subseteq \mathcal{W}$ is closed under finite intersections, the σ-algebra of subsets of $\mathcal{P}X$ generated by $\mathcal{W}_0$ is included in $\mathcal{W}$. By (a-ii), $\mathrm{T}_0 \subseteq \mathcal{W}$.

(ii) Let $\mathfrak{S}_0$ be the topology on $\mathcal{P}X$ generated by sets of the form $\{S : S \cap G \ne \emptyset\}$ where $G \subseteq X$ is open. (So $\mathfrak{S}_0$ is coarser than the topology $\mathfrak{S}$ of (a-iii) above.) Then $\phi : \Omega \to \mathcal{P}X$ is continuous for the product topology $\mathfrak{U}$ on Ω and $\mathfrak{S}_0$ on $\mathcal{P}X$. **P** If $G \subseteq X$ is open, then

$$\phi^{-1}[\{S : S \cap G \ne \emptyset\}] = \Omega \cap \bigcap_{i<r} \bigcup_{H\in\mathcal{H}, j\in\mathbb{N}} \{\omega : j < m_H(\omega),\, x_{Hj}(\omega) \in G_i\}$$

is open; by 4A2B(a-ii), this is enough. **Q**

(iii) $\mathfrak{S}_0 \subseteq \mathcal{W}$. **P** Because μ is locally finite, the family $\mathcal{U}$ of sets of the form

$$\{S : S \cap G_i \ne \emptyset \text{ for } i < r\},$$

where $G_i \subseteq X$ is an open set of finite measure for each $i < r$, is a base for $\mathfrak{S}_0$; and $\mathcal{U} \subseteq \mathrm{T}_0 \subseteq \mathcal{W}$. So if $W \in \mathfrak{S}_0$, $\mathcal{V} = \{V : V \in \mathfrak{S}_0 \cap \mathrm{T}_0,\, V \subseteq W\}$ is an upwards-directed family of sets with union W. Since $\tilde\nu_0$ and $\tilde\lambda$ are both τ-additive, and $\phi^{-1}[V]$ is open for every $V \in \mathcal{V}$,

$$\tilde\lambda\phi^{-1}[W] = \sup_{V\in\mathcal{V}} \tilde\lambda\phi^{-1}[V] = \sup_{V\in\mathcal{V}} \tilde\nu_0 V = \tilde\nu_0 W,$$

and $W \in \mathcal{W}$. **Q**

(iv) If $G \subseteq X$ is open and $n \in \mathbb{N}$, $W = \{S : \#(S \cap G) \ge n\}$ belongs to $\mathfrak{S}_0$. **P**

$$W = \bigcup\{\{S : S \cap G_i \ne \emptyset \text{ for every } i < n\} : \langle G_i\rangle_{i<n} \text{ is a disjoint family of open subsets of } G \text{ of finite measure}\}.$$ **Q**

(v) If $E_0, \dots, E_r \subseteq X$ are sets of finite measure, and $n_0, \dots, n_r \in \mathbb{N}$, then

$$V = \{S : \#(S \cap E_i) \ge n_i \text{ for } i \le r\}$$

belongs to $\mathcal{W}$. **P** Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $1 - e^{-\gamma\delta} \le \epsilon$. Let $G_0, \dots, G_r$ be open sets such that $E_i \subseteq G_i$ for $i \le r$ and $\sum_{i=0}^r \mu(G_i \setminus E_i) \le \delta$. Set

$$W = \{S : \#(S \cap G_i) \ge n_i \text{ for } i \le r\}, \quad W_0 = \{S : S \cap \bigcup_{i\le r} G_i \setminus E_i \ne \emptyset\};$$

then $W \in \mathfrak{S}_0$ and $W_0 \in \mathrm{T}_0$, so both belong to $\mathcal{W}$, while

$$\tilde\nu_0 W_0 = \tilde\lambda\phi^{-1}[W_0] = 1 - \exp(-\gamma\mu(\bigcup_{i\le r} G_i \setminus E_i)) \le \epsilon.$$

Now

$$W \setminus W_0 \subseteq V \subseteq W, \quad \phi^{-1}[W] \setminus \phi^{-1}[W_0] \subseteq \phi^{-1}[V] \subseteq \phi^{-1}[W].$$

So

$$\tilde\nu_0^* V - (\tilde\nu_0)_* V \le \epsilon, \quad \tilde\lambda^*(\phi^{-1}[V]) - \tilde\lambda_*(\phi^{-1}[V]) \le \epsilon, \quad |\tilde\nu_0^* V - \tilde\lambda^*(\phi^{-1}[V])| \le \epsilon.$$

As ϵ is arbitrary (and $\tilde\nu_0$, $\tilde\lambda$ are complete), V is measured by $\tilde\nu_0$, $\phi^{-1}[V]$ is measured by $\tilde\lambda$, and

$$|\tilde\nu_0 V - \tilde\lambda\phi^{-1}[V]| = |\tilde\nu_0^* V - \tilde\lambda^*(\phi^{-1}[V])| = 0.$$ **Q**

(vi) If $E_0, \dots, E_r \subseteq X$ are sets of finite measure, $n_0, \dots, n_r \in \mathbb{N}$ and $j \le r$, then

$$\{S : \#(S \cap E_i) = n_i \text{ for } i < j,\, \#(S \cap E_i) \ge n_i \text{ for } j \le i \le r\}$$

belongs to $\mathcal{W}$. **P** Induce on j. For $j = 0$ we just have the case of (v). For the inductive step to $j + 1$, we have

$$\begin{aligned}
&\{S : \#(S \cap E_i) = n_i \text{ for } i \le j,\, \#(S \cap E_i) \ge n_i \text{ for } j < i \le r\} \\
&\quad= \{S : \#(S \cap E_i) = n_i \text{ for } i < j,\, \#(S \cap E_i) \ge n_i \text{ for } j \le i \le r\} \\
&\qquad\setminus \{S : \#(S \cap E_i) = n_i \text{ for } i < j,\, \#(S \cap E_j) \ge n_j + 1, \\
&\qquad\qquad \#(S \cap E_i) \ge n_i \text{ for } j < i \le r\} \\
&\quad\in \mathcal{W}
\end{aligned}$$

because $\mathcal{W}$ is a Dynkin class. **Q**

(vii) If $E \in \Sigma$ has finite measure and does not include any non-negligible singleton, then $\#(E\cap\phi(\omega)) = g_E(\omega)$, as defined in 495B, for λ-almost every $\omega \in \Omega$. **P** Let A_E be the set of those $\omega \in \Omega$ such that

either there are an $H \in \mathcal{H}$ and $j \in \mathbb{N}$ such that $\mu(H \cap E) = 0$ and $x_{Hj}(\omega) \in E$

or there are an $H \in \mathcal{H}$ and distinct i, $j \in \mathbb{N}$ such that $x_{Hi}(\omega) = x_{Hj}(\omega) \in E$.

As observed in (a-i) above, $\{H : H \in \mathcal{H},\ H \cap E \ne \emptyset\}$ is countable; while for any $H \in \mathcal{H}$ and distinct i, $j \in \mathbb{N}$ the set $\{\omega : x_{Hi}(\omega) = x_{Hj}(\omega) \in E\}$ is negligible because the subspace measure on E is atomless (414G/416Xa), so the diagonal $\{(x, x) : x \in E\}$ is negligible in X^2. Consequently $\lambda A_E = 0$. But $\#(E \cap \phi(\omega)) = g_E(\omega)$ for every $\omega \in \Omega \setminus A_E$. **Q**

(viii) Now suppose that $E_0, \dots , E_r \subseteq X$ are disjoint sets of finite measure, none including any non-negligible singleton, and $n_0, \dots , n_r \in \mathbb{N}$. Then

$$V = \{S : S \subseteq X,\ \#(S \cap E_i) = n_i \text{ for every } i \le r\}$$

belongs to $\mathcal{W}$, by (vi). Next,

$$\phi^{-1}[V] = \{\omega : \#(E_i \cap \phi(\omega)) = n_i \text{ for every } i \le r\},$$

so

$$\tilde\nu_0 V = \tilde\lambda\phi^{-1}[V] = \tilde\lambda\{\omega : g_{E_i}(\omega) = n_i \text{ for every } i \le r\}$$

(by (vii))

$$= \prod_{i=0}^{r} \tilde\lambda\{\omega : g_{E_i}(\omega) = n_i\} = \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}.$$

Finally, because $\tilde\nu_0^*\mathcal{C} = 1$ and $\tilde\nu$ is the subspace measure on $\mathcal{C}$,

$$\tilde\nu\{C : C \in \mathcal{C},\ \#(C \cap E_i) = n_i \text{ for every } i \le r\} = \tilde\nu(V \cap \mathcal{C}) = \tilde\nu_0 V$$

$$= \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}.$$

This completes the proof of (b).

(c)(i) Taking $\mathrm{T} \supseteq \mathrm{T}_0$ to be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S : \#(S \cap E) = n\}$ where $E \in \Sigma^f$ and $n \in \mathbb{N}$, (b-viii) tells us that $\tilde\nu_0\restriction\mathrm{T}$ satisfies the conditions of 495D, so its completion ν is the Poisson point process as defined in 495E. Because $\tilde\nu_0$ is complete, it extends ν. (The identity map from $\mathcal{P}X$ to itself is inverse-measure-preserving for $\tilde\nu_0$ and $\tilde\nu_0\restriction\mathrm{T}$, therefore also for their completions $\tilde\nu_0$ and ν.) Since $\mathcal{C}$ has full outer measure for $\tilde\nu_0$, by (a-v), it has full outer measure for ν, and

$$\nu_{\mathcal{C}}(V \cap \mathcal{C}) = \nu V = \tilde\nu_0 V = \tilde\nu(V \cap \mathcal{C})$$

whenever ν measures V, so $\tilde\nu$ extends $\nu_{\mathcal{C}}$.

(ii) If μ is σ-finite, then there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of open sets of finite measure covering X. For each $n \in \mathbb{N}$, $\{S : S \subseteq X,\ S \cap H_n \text{ is finite}\}$ is ν-conegligible. So $W = \{S : S \cap H_n \text{ is finite for every } n\}$ is ν-conegligible. But $W \subseteq \mathcal{C}$, so $\mathcal{C}$ is ν-conegligible.

(d) If X is locally compact then $\mathcal{C}$ is Hausdorff (4A2T(e-ii)); so $\tilde\nu$, being a quasi-Radon probability measure on a compact Hausdorff space, is a Radon measure (416G).

(e) Now suppose that X is second-countable.

(i) $\mathcal{C}$ has a countable network consisting of sets in $\mathrm{T}_{\mathcal{C}}$, the subspace σ-algebra induced by the σ-algebra T of (c-i). **P** Let $\mathcal{U}$ be a countable base for $\mathfrak{T}$, closed under finite unions, consisting of sets of finite measure. For $U_0 \in \mathcal{U}$ and finite $\mathcal{U}_0 \subseteq \mathcal{U}$, set

$$V(U_0, \mathcal{U}_0) = \{C : C \in \mathcal{C},\ C \cap U_0 = \emptyset,\ C \cap U \ne \emptyset \text{ for every } U \in \mathcal{U}\} \in \mathrm{T}_{\mathcal{C}}.$$

If $W \subseteq \mathcal{C}$ is open for the Fell topology and $C_0 \in W$, there are a compact set $K \subseteq X$ and a finite family $\mathcal{G} \subseteq \mathfrak{T}$ such that

$$C_0 \in \{C : C \in \mathcal{C},\, C \cap K = \emptyset,\, C \cap G \neq \emptyset \text{ for every } G \in \mathcal{G}\}.$$

For $G \in \mathcal{G}$ let y_G be a point of $C_0 \cap G$. Now there are a $U_0 \in \mathcal{U}$ such that $K \subseteq U_0 \subseteq X \setminus C$ and a family $\langle U_G \rangle_{G \in \mathcal{G}}$ in $\mathcal{U}$ such that $x_G \in U_G \subseteq G$ for every $G \in \mathcal{G}$. In this case,

$$C_0 \in V(U_0, \{U_G : G \in \mathcal{G}\}) \subseteq W.$$

As C_0 and W are arbitrary, the countable set $\{V(U_0, \mathcal{U}_0) : U_0 \in \mathcal{U},\, \mathcal{U}_0 \in [\mathcal{U}]^{<\omega}\}$ is a network for the topology of $\mathcal{C}$. **Q**

(ii) Since $\nu_{\mathcal{C}}$ measures every set in this countable network, it is a topological measure. Since it is also complete, and $\tilde{\nu}$, being a quasi-Radon probability measure, is the completion of its restriction to the Borel σ-algebra of $\mathcal{C}$, $\nu_{\mathcal{C}}$ extends $\tilde{\nu}$, and the two must be equal.

495O Proposition Let $(X, \mathfrak{T})$ be a σ-compact locally compact Hausdorff space and $M_{\mathrm{R}}^{\infty+}(X)$ the set of Radon measures on X. Give $M_{\mathrm{R}}^{\infty+}(X)$ the topology generated by sets of the form $\{\mu : \mu G > \alpha\}$ and $\{\mu : \mu K < \alpha\}$ where $G \subseteq X$ is open, $K \subseteq X$ is compact and $\alpha \in \mathbb{R}$. Let $\mathcal{C}$ be the space of closed subsets of X with its Fell topology, and $P_{\mathrm{R}}(\mathcal{C})$ the set of Radon probability measures on $\mathcal{C}$ with its narrow topology (definition: 437Jd). For $\mu \in M_{\mathrm{R}}^{\infty+}(X)$ and $\gamma > 0$ let $\tilde{\nu}_{\mu,\gamma}$ be the Radon measure on $\mathcal{C}$ defined from μ and γ as in 495N. Then the function $(\mu, \gamma) \mapsto \tilde{\nu}_{\mu,\gamma} : M_{\mathrm{R}}^{\infty+}(X) \times \,]0, \infty[\, \to P_{\mathrm{R}}(\mathcal{C})$ is continuous.

proof (a) Note that because X is σ-compact, every Radon measure on X is σ-finite, therefore outer regular with respect to the open sets (412Wb), and we can apply 495N to build the measures $\tilde{\nu}_{\mu,\gamma}$. Just as in 495E for ordinary Poisson point processes, the uniqueness assertion in 495Na assures us that $\tilde{\nu}_{\mu,\gamma} = \tilde{\nu}_{\gamma\mu,1}$ for all γ and μ. Of course the sets

$$\{(\mu, \gamma) : \gamma\mu G > \alpha\}, \quad \{(\mu, \gamma) : \gamma\mu K < \alpha\}$$

where $G \subseteq X$ is open, $K \subseteq X$ is compact and $\alpha \in \mathbb{R}$, are all open in $M_{\mathrm{R}}^{\infty+}(X) \times \,]0, \infty[$; so the map $(\mu, \gamma) \mapsto \gamma\mu$ is continuous. It will therefore be enough to show that the map $\mu \mapsto \tilde{\nu}_{\mu,1} : M_{\mathrm{R}}^{\infty+}(X) \to P_{\mathrm{R}}(\mathcal{C})$ is continuous. Write $\tilde{\nu}_\mu$ for $\tilde{\nu}_{\mu,1}$.

(b) Fix an open set $W_0 \subseteq \mathcal{C}$, $\alpha_0 > 0$ and $\mu_0 \in M_{\mathrm{R}}^{\infty+}(X)$ such that $\tilde{\nu}_{\mu_0} W_0 > \alpha_0$. Let $\mathcal{E}$ be the family of relatively compact Borel subsets E of X such that $\mu_0(\partial E) = 0$. Then $\mathcal{E}$ is a subring of $\mathcal{P}X$ (4A2Bi). Also $\mu \mapsto \mu E : M_{\mathrm{R}}^{\infty+}(X) \to [0, \infty[$ is continuous at μ_0 for every $E \in \mathcal{E}$. **P** If $E \in \mathcal{E}$ and $\epsilon > 0$, then

$$\begin{aligned}\{\mu : \mu_0 E - \epsilon < \mu E < \mu_0 E + \epsilon\}& \\ &\supseteq \{\mu : \mu(\operatorname{int} E) > \mu_0(\operatorname{int} E) - \epsilon,\, \mu\overline{E} < \mu_0\overline{E} + \epsilon\}\end{aligned}$$

is a neighbourhood of μ_0. **Q**

(c) Next, $\mathcal{U} = \mathcal{E} \cap \mathfrak{T}$ is a base for $\mathfrak{T}$ (411Gi). It follows that the family $\mathcal{V}$ of sets of the form

$$\{C : C \in \mathcal{C},\, C \cap U_i \neq \emptyset \text{ for } i < r,\, C \cap \overline{U} = \emptyset\},$$

where $U, U_0, \ldots \in \mathcal{U}$, is a base for the Fell topology on $\mathcal{C}$. **P** If $W \subseteq \mathcal{C}$ is open for the Fell topology and $C_0 \in W$, there are $r \in \mathbb{N}$, open sets $G_i \subseteq X$ for $i < r$ and a compact set $K \subseteq X$ such that

$$C_0 \in \{C : C \cap G_i \neq \emptyset \text{ for each } i < r,\, C \cap K = \emptyset\} \subseteq W.$$

For each $i < r$ choose $x_i \in C_0 \cap G_i$ and $U_i \in \mathcal{U}$ such that $x_i \in U_i \subseteq G_i$. Because X is locally compact and Hausdorff, it is regular, so every point of K belongs to a member of $\mathcal{U}$ with closure disjoint from C_0; because $\mathcal{U}$ is closed under finite unions, there is a $U \in \mathcal{U}$ such that $K \subseteq U$ and $C_0 \cap \overline{U} = \emptyset$. Now

$$\{C : C \in \mathcal{C},\, C \cap U_i \neq \emptyset \text{ for } i < r,\, C \cap \overline{U} = \emptyset\}$$

belongs to $\mathcal{V}$, contains C_0 and is included in W. As C_0 and W are arbitrary, $\mathcal{V}$ is a base for the Fell topology on $\mathcal{C}$. **Q**

(d) If $V \in \mathcal{V}$, then $\mu \mapsto \tilde{\nu}_\mu V : M_{\mathrm{R}}^{\infty+}(X) \to [0, 1]$ is continuous at μ_0. **P** Express V as $\{C : C \in \mathcal{C},\, C \cap U_i \neq \emptyset$ for $i < r$, $C \cap \overline{U} = \emptyset\}$, where U_i, $U \in \mathcal{U}$. Let $\mathcal{A}$ be the set of atoms of the finite subring of $\mathcal{E}$ generated by $\{U_i : i < r\} \cup \{\overline{U}\}$. For $\mathcal{I} \subseteq \mathcal{A}$ set

$$V_{\mathcal{I}} = \{C : C \in \mathcal{C},\, \mathcal{I} = \{A : A \in \mathcal{A},\, C \cap A \neq \emptyset\}\}.$$

Let $\mathfrak{I}$ be the set of those $\mathcal{I} \subseteq \mathcal{A}$ such that $A \cap \overline{U} = \emptyset$ for every $A \in \mathcal{I}$ and for every $i < r$ there is an $A \in \mathcal{I}$ such that $A \subseteq U_i$. Then $\langle V_{\mathcal{I}} \rangle_{\mathcal{I} \in \mathfrak{I}}$ is a partition of V. Moreover, for any $\mu \in M_{\mathrm{R}}^{\infty +}(X)$ and $\mathcal{I} \subseteq \mathcal{A}$,

$$\tilde{\nu}_\mu V_{\mathcal{I}} = \prod_{A \in \mathcal{A} \setminus \mathcal{I}} e^{-\mu A} \cdot \prod_{A \in \mathcal{I}} (1 - e^{-\mu A}).$$

Since each $\mu \mapsto \mu A$ is continuous at μ_0, by (a), so are the functionals $\mu \mapsto \tilde{\nu}_\mu V_{\mathcal{I}}$, for $\mathcal{I} \subseteq \mathcal{A}$, and $\mu \mapsto \tilde{\nu}_\mu V = \sum_{\mathcal{I} \in \mathfrak{I}} \tilde{\nu}_\mu V_{\mathcal{I}}$. **Q**

(e) Let $\mathcal{V}^*$ be the family of Borel subsets V of $\mathcal{C}$ such that $\mu \mapsto \tilde{\nu}_\mu V : M_{\mathrm{R}}^{\infty +}(X) \to [0, \infty[$ is continuous at μ_0. Then $\mathcal{V} \subseteq \mathcal{V}^*$ (by (c)), $\mathcal{C} \in \mathcal{V}^*$ and $V \setminus V' \in \mathcal{V}^*$ whenever $V, V' \in \mathcal{V}^*$ and $V' \subseteq V$. Because $\mathcal{V}$ is closed under finite intersections, it follows that $\mathcal{V}^*$ includes the algebra of subsets of $\mathcal{C}$ generated by $\mathcal{V}$ (313Ga); in particular, any finite union of members of $\mathcal{V}$ belongs to $\mathcal{V}^*$.

(f) Let us return to the open set $W_0 \subseteq \mathcal{C}$ and the $\alpha_0 \in \mathbb{R}$ of part (a). Because $\tilde{\nu}_{\mu_0}$ is τ-additive and $\mathcal{V}$ is a base for the topology of $\mathcal{C}$ ((b) above), there is a finite family $\mathcal{V}_0 \subseteq \mathcal{V}$ such that $V_0 = \bigcup \mathcal{V}_0$ is included in W_0 and $\tilde{\nu}_{\mu_0} V_0 > \alpha_0$. But this means that

$$\{\mu : \mu \in M_{\mathrm{R}}^{\infty +}(X),\, \tilde{\nu}_\mu W_0 > \alpha_0\} \supseteq \{\mu : \mu \in M_{\mathrm{R}}^{\infty +}(X),\, \tilde{\nu}_\mu V_0 > \alpha_0\}$$

is a neighbourhood of μ_0. As μ_0 is arbitrary, $\{\mu : \tilde{\nu}_\mu W_0 > \alpha_0\}$ is open; as W_0 and α_0 are arbitrary, $\mu \mapsto \tilde{\nu}_\mu$ is continuous.

495P There are many constructions which, in particular cases, can be used as an alternative to the method of 495B-495D in setting up Poisson point processes. I give one which applies to the half-line $[0, \infty[$ with Lebesgue measure.

Theorem Let $\gamma > 0$, and let ν be the Poisson point process on $[0, \infty[$, with Lebesgue measure, with density γ. Let λ_0 be the exponential distribution with expectation $1/\gamma$, regarded as a Radon probability measure on $]0, \infty[$, and λ the corresponding product measure on $]0, \infty[^{\mathbb{N}}$. Define $\phi :]0, \infty[^{\mathbb{N}} \to \mathcal{P}([0, \infty[)$ by setting $\phi(x) = \{\sum_{i=0}^{n} x(i) : n \in \mathbb{N}\}$ for $x \in]0, \infty[^{\mathbb{N}}$. Then ϕ is a measure space isomorphism between $]0, \infty[^{\mathbb{N}}$ and a ν-conegligible subset of $\mathcal{P}([0, \infty[)$.

Remark As I seem not to have mentioned exponential distributions earlier in this treatise, I remark now that the **exponential distribution** with parameter γ has distribution function

$$F(t) = 0 \text{ if } t < 0,\ 1 - e^{-\gamma t} \text{ if } t \geq 0,$$

and probability density function

$$f(t) = 0 \text{ if } t \leq 0,\ \gamma e^{-\gamma t} \text{ if } t > 0;$$

its expectation is

$$\int_0^\infty \gamma t e^{-\gamma t} dt = -\int_0^\infty \frac{d}{dt}\Bigl(\frac{\gamma t + 1}{\gamma} e^{-\gamma t}\Bigr) dt = \frac{1}{\gamma}.$$

Because (when regarded as a Radon probability measure on $\mathbb{R}$, following my ordinary rule set out in §271) it gives measure zero to $]-\infty, 0]$, it can be identified with the subspace measure on $]0, \infty[$, as here.

proof (a) For each $n \in \mathbb{N}$, $\#(S \cap [0, n])$ is finite for ν-almost every S; so the set

$$Q_0 = \{S : S \subseteq [0, \infty[,\ \#(S \cap [0, n]) \text{ is finite for every } n\}$$

is ν-conegligible. Next, the sets $\{S : S \cap [n, n+1[\neq \emptyset\}$ are ν-independent and have measure $1 - e^{-\gamma} > 0$, so

$$\{S : S \cap [n, n+1[\neq \emptyset \text{ for infinitely many } n\}$$

is ν-conegligible (273K). Finally, $\nu\{S : 0 \in S\} = 0$, so $Q = \{S : S \in Q_0,\ 0 \notin S,\ S \text{ is infinite}\}$ is ν-conegligible. For $S \in Q$, let $\langle g_n(S) \rangle_{n \in \mathbb{N}}$ be the increasing enumeration of S. Let T be the σ-algebra of subsets of $\mathcal{P}([0, \infty[)$ generated by sets of the form $\{S : \#(S \cap E) = n\}$ where $E \subseteq [0, \infty[$ has finite measure and $n \in \mathbb{N}$. Then, for $n \in \mathbb{N}$ and $\alpha \geq 0$, $\{S : g_n(S) \leq \alpha\} = \{S : \#(S \cap [0, \alpha]) \geq n + 1\}$ belongs to T, so g_n is T-measurable. Set $h_0(S) = g_0(S)$, $h_n(S) = g_n(S) - g_{n-1}(S)$ for $n \geq 1$, and $h(S) = \langle h_n(S) \rangle_{n \in \mathbb{N}}$; then $h : Q \to]0, \infty[^{\mathbb{N}}$ is a bijection, and its inverse is ϕ.

(b) For each $k \in \mathbb{N}$, $I_S = \{i : i \in \mathbb{N},\ S \cap \bigl[2^{-k} i, 2^{-k}(i+1)\bigr[\neq \emptyset\}$ is infinite for every $S \in Q$. So we can define $g_{kn} : Q \to]0, \infty[$, for each n, by taking $g_{kn}(S) = 2^{-k}(j + 1)$ if $j \in I_S$ and $\#(I_S \cap j) = n$. Because all the sets $\{S : j \in I_S\}$ belong to T, each g_{kn} is T-measurable, and $\langle g_{kn} \rangle_{k \in \mathbb{N}}$ is a non-increasing sequence with limit g_n. Set $h_{k0}(S) = g_{k0}(S)$ and $h_{kn}(S) = g_{kn}(S) - g_{k,n-1}(S)$ for $n \geq 1$. Then $h_n = \lim_{k \to \infty} h_{kn}$.

(c) For any $n \in \mathbb{N}$, $j_0, \dots, j_n \in \mathbb{N}$, $k \in \mathbb{N}$, set $j'_r = \sum_{i=0}^{r} j_i$ for $r \le n$. Then

$$\begin{aligned}
\nu\{S : S \in Q, h_{ki}(S) = 2^{-k}(j_i + 1) \text{ for every } i \le n\}
&= \nu\{S : S \in Q,\ g_{kr}(S) = 2^{-k}(r + 1 + j'_r) \text{ for every } r \le n\}\\
&= \nu\{S : S \cap \left[2^{-k}(r + j'_r), 2^{-k}(r + 1 + j'_r)\right[\ne \emptyset \text{ for every } r \le n,\\
&\qquad S \cap \left[0, 2^{-k} j_0\right[= \emptyset,\\
&\qquad S \cap \left[2^{-k}(r + 1 + j'_r), 2^{-k}(r + 1 + j'_{r+1})\right[= \emptyset \text{ for every } r < n\}\\
&= (1 - \exp(-2^{-k}\gamma))^{n+1} \exp(-2^{-k}\gamma j_0) \prod_{r<n} \exp(-2^{-k}\gamma(j'_{r+1} - j'_r))\\
&= \prod_{i=0}^{n} (1 - \exp(-2^{-k}\gamma)) \exp(-2^{-k}\gamma j_i).
\end{aligned}$$

This means that the h_{ki}, for $i \in \mathbb{N}$ are independent, with

$$\Pr(h_{ki} = 2^{-k}(j + 1)) = (1 - e^{-2^{-k}\gamma}) e^{-2^{-k}\gamma j}$$

for each j. Since $h_{ki} \to h_i$ ν-a.e. for each i, $\langle h_i \rangle_{i\in\mathbb{N}}$ is also independent (367W). Now, for any $\alpha > 0$,

$$\begin{aligned}
\Pr(h_{ki} \le \alpha) &= \sum_{2^{-k}(j+1) \le \alpha} (1 - \exp(-2^{-k}\gamma)) \exp(-2^{-k}\gamma j)\\
&= 1 - \exp(-2^{-k}\gamma \lfloor 2^k \alpha \rfloor) \to 1 - e^{-\gamma\alpha}
\end{aligned}$$

as $k \to \infty$. So

$$\begin{aligned}
\Pr(h_i \le \alpha) &= \inf_{\beta > \alpha} \liminf_{k\to\infty} \Pr(h_{ki} \le \beta)\\
\text{(271L)}\qquad &= \inf_{\beta > \alpha} 1 - e^{-\gamma\beta} = 1 - e^{-\gamma\alpha}
\end{aligned}$$

for every $\alpha \ge 0$ and every $i \in \mathbb{N}$.

(d) Accordingly $\langle h_i \rangle_{i\in\mathbb{N}}$ is an independent sequence of random variables, each exponentially distributed with expectation $1/\gamma$. It follows that $h : Q \to \left]0, \infty\right[^{\mathbb{N}}$ is inverse-measure-preserving for the subspace measure ν_Q and λ (254G).

Observe next that if $E \subseteq [0, \infty[$ is Lebesgue measurable and $n \in \mathbb{N}$, then

$$\{x : x \in \left]0, \infty\right[^{\mathbb{N}}, \#(\phi(x) \cap E) = n\} = \bigcup_{I \in [\mathbb{N}]^n} \{x : \textstyle\sum_{i=0}^{j} x(i) \in E \iff j \in I\} \in \Lambda,$$

writing Λ for the domain of λ. So ϕ is (Λ, T)-measurable. Now, for any $W \in \mathrm{T}$,

$$\lambda\phi^{-1}[W] = \nu(h^{-1}[\phi^{-1}[W]]) = \nu(W \cap Q) = \nu W.$$

So ϕ is inverse-measure-preserving for λ and for $\nu \restriction \mathrm{T}$. Since λ is complete and ν is defined as the completion of its restriction to T, ϕ is inverse-measure-preserving for λ and ν. Thus ϕ and h are the two halves of an isomorphism between $(\left]0, \infty\right[^{\mathbb{N}}, \lambda)$ and the subspace (Q, ν_Q), as claimed.

495X Basic exercises >(a) Let (X, Σ, μ) be an atomless countably separated measure space (definition: 343D) and $\gamma > 0$. Let ν be a complete probability measure on $\mathcal{P}X$ such that $\nu\{S : S \subseteq X, S \cap E = \emptyset\}$ is defined and equal to $e^{-\gamma\mu E}$ whenever $E \in \Sigma$ has finite measure. Show that ν extends the Poisson process with density γ defined in 495D.

(b) Let (X, Σ, μ) be an atomless measure space, and ν a Poisson point process on X. (i) Show that $[X]^{\le\omega}$ has full outer measure for ν. (ii) Show that if μ is semi-finite then $[X]^{\le\omega}$ is conegligible iff μ is σ-finite. (iii) Show that if μ is semi-finite, then $[X]^{<\omega}$ is non-negligible iff $[X]^{<\omega}$ is conegligible iff μ is totally finite.

(c)(i) Let (X,Σ,μ) be an atomless measure space and $\mathcal{E}$ a countable partition of X into measurable sets. Let ν be the Poisson point process on X with density 1, and for each $E\in\mathcal{E}$ let ν_E be the Poisson point process on E with density 1 corresponding to the subspace measure μ_E on E. Let λ be the product of the family $\langle\nu_E\rangle_{E\in\mathcal{E}}$. Show that the map $S\mapsto\langle S\cap E\rangle_{E\in\mathcal{E}}:\mathcal{P}X\to\prod_{E\in\mathcal{E}}\mathcal{P}E$ is a measure space isomorphism for ν and λ. (ii) Let (X,Σ,μ) be a strictly localizable measure space and $\langle X_i\rangle_{i\in I}$ a decomposition of X. Let ν be the Poisson point process on X with density 1, and for each $i\in I$ let ν_i be the Poisson point process on X_i with density 1 corresponding to the subspace measure μ_{X_i} on X_i. Let λ be the product of the family $\langle\nu_i\rangle_{i\in I}$. Show that the map $S\mapsto\langle S\cap X_i\rangle_{i\in I}:\mathcal{P}X\to\prod_{i\in I}\mathcal{P}X_i$ is inverse-measure-preserving for ν and λ.

>(d) Let (X,Σ,μ) be an atomless measure space and $\mathfrak{T}$ a topology on X such that X is covered by a sequence of open sets of finite outer measure. Let ν be a Poisson point process on X. Show that ν-almost every set $S\subseteq X$ is locally finite in the sense that X is covered by the open sets meeting S in finite sets; in particular, if X is T_1, then ν-almost every subset of X is closed.

>(e)(i) Let (X,Σ,μ) be an atomless measure space, and for $\gamma>0$ let ν_γ be the Poisson point process on X with density γ. Show that for any γ, $\delta>0$ the map $(S,T)\mapsto S\cup T:\mathcal{P}X\times\mathcal{P}X\to\mathcal{P}X$ is inverse-measure-preserving for the product measure $\nu_\gamma\times\nu_\delta$ and $\nu_{\gamma+\delta}$. (ii) Let X be a set, Σ a σ-algebra of subsets of X, and $\langle\mu_i\rangle_{i\in I}$ a countable family of measures with domain Σ such that $\mu=\sum_{i\in I}\mu_i$ is atomless. Let ν, ν_i be the Poisson point processes with density 1 corresponding to the measures μ, μ_i. Show that the map $\langle S_i\rangle_{i\in I}\mapsto\bigcup_{i\in I}S_i:(\mathcal{P}X)^I\to\mathcal{P}X$ is inverse-measure-preserving for the product measure $\prod_{i\in I}\nu_i$ and ν. (iii) Compare with 495Xc(i).

(f) Let (X,Σ,μ) be an atomless semi-finite measure space, and ν the Poisson point process on X with density 1. Show that ν is perfect iff μ is.

(g) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Show that there is a probability measure λ on $\mathbb{R}^{\mathfrak{A}^f}$ such that (i) for every $a\in\mathfrak{A}^f$ the corresponding marginal measure on $\mathbb{R}$ is the Poisson distribution with expectation $\bar\mu a$ (ii) whenever $a_0,\dots,a_n\in\mathfrak{A}^f$ are disjoint, the functions $z\mapsto z(a_i):\mathbb{R}^{\mathfrak{A}^f}\to\mathbb{R}$ are stochastically independent with respect to λ. (*Hint*: prove the result for finite $\mathfrak{A}$ and use 454D.) Use this to prove 495J.

(h) Let U be a Hilbert space. Show that there are a probability algebra $(\mathfrak{B},\bar\lambda)$ and a linear operator $T:U\to L^2(\mathfrak{B})$ such that (i) for every $u\in U$, Tu has a normal distribution with expectation 0 and variance $\|u\|_2^2$ (ii) if $\langle u_i\rangle_{i\in I}$ is an orthogonal family in U then $\langle Tu_i\rangle_{i\in I}$ is $\bar\lambda$-independent. (*Hint*: see the proof of 456K.)

(i) Let ν be the Poisson point process with density 1 on $[0,\infty[$ with Lebesgue measure. Set $Q_0=\{S:S\subseteq[0,\infty[,$ $S\cap[0,n]$ is finite for every $n\}$ and for $S\in Q_0$ set $\psi(S)(t)=\#(S\cap[0,t])$ for $t\in[0,\infty[$. Show that ψ is inverse-measure-preserving for the subspace measure ν_{Q_0} and the distribution on $\mathbb{R}^{[0,\infty[}$ corresponding to the Poisson process of 455Xh.

(j) Let (Y,T,ν) be a probability space, and λ_0 the exponential distribution with expectation 1, regarded as a Radon measure on $]0,\infty[$. Let λ be the product measure $\lambda_0^{\mathbb{N}}\times\nu^{\mathbb{N}}$ on $]0,\infty[^{\mathbb{N}}\times Y^{\mathbb{N}}$. Set $\phi(x,y)=\{(\sum_{i=0}^n x(i),y(n)):n\in\mathbb{N}\}$ for $x\in\,]0,\infty[^{\mathbb{N}}$ and $y\in Y^{\mathbb{N}}$. Show that $\phi:\,]0,\infty[^{\mathbb{N}}\times Y^{\mathbb{N}}\to\mathcal{P}(]0,\infty[\times Y)$ is a measure space isomorphism between $(]0,\infty[^{\mathbb{N}}\times Y^{\mathbb{N}},\lambda)$ and a conegligible set for the Poisson point process on $]0,\infty[\times Y$ with density 1 for the c.l.d. product measure $\mu_L\times\nu$, where μ_L is Lebesgue measure.

(k) Let $\mathcal{C}$ be the family of closed subsets of $[0,\infty[$. Let ρ be the usual metric on $[0,\infty[$ and $\tilde\rho$ the corresponding Hausdorff metric on $\mathcal{C}\setminus\{\emptyset\}$ (4A2T). Let ν be the Poisson point process on $[0,\infty[$ with density 1 over Lebesgue measure. Show that every member of $\mathcal{C}\setminus\{\emptyset\}$ has a ν-negligible $\tilde\rho$-neighbourhood.

(l) Show that the topology on $M_{\mathrm{R}}^+(X)$ described in 495O is just the topology induced by the natural embedding of $M_{\mathrm{R}}(X)$ into $C_k(X)^\sim$ (436J) and the weak topology $\mathfrak{T}_s(C_k(X)^\sim,C_k(X))$, where $C_k(X)$ is the Riesz space of continuous real-valued functions on X with compact support.

(m) Let $\mathcal{C}$ be the set of closed subsets of $[0,\infty[$ with its Fell topology. For $\delta\in\,]0,1]$ let λ_δ be the measure on $\{0,1\}^{\mathbb{N}}$ which is the product of copies of the measure on $\{0,1\}$ in which $\{1\}$ is given measure δ. Define $\phi_\delta:\{0,1\}^{\mathbb{N}}\to\mathcal{C}$ by setting $\phi_\delta(x)=\{n\delta:n\in\mathbb{N},\,x(n)=1\}$, and let $\tilde\nu_\delta$ be the Radon measure $\lambda_\delta\phi_\delta^{-1}$ on $\mathcal{C}$. Show that the Radon measure on $\mathcal{C}$ representing the Poisson point process on $[0,\infty[$ with density 1 over Lebesgue measure is the limit $\lim_{\delta\downarrow0}\tilde\nu_\delta$ for the narrow topology on the space of Radon probability measures on $\mathcal{C}$.

(n) Show that the standard gamma distribution with expectation 1 is the exponential distribution with expectation 1.

(o) Let $r \ge 1$ be an integer; let μ be Lebesgue measure on $\mathbb{R}^r$ and β_r the volume of the unit ball in $\mathbb{R}^r$. Set $\psi(t) = (t/\beta_r)^{1/r}$ for $t \ge 0$, so that the volume of a ball of radius $\psi(t)$ is t. Let S_{r-1} be the unit sphere in $\mathbb{R}^r$ and θ the invariant Radon probability measure on S_{r-1}, so that θ is a multiple of $(r-1)$-dimensional Hausdorff measure (see 476I). Let λ_0 be the exponential distribution with expectation 1, regarded as a Radon probability measure on $]0,\infty[$, and λ the product measure $\lambda_0^{\mathbb{N}} \times \theta^{\mathbb{N}}$ on $]0,\infty[^{\mathbb{N}} \times S_{r-1}^{\mathbb{N}}$. Set

$$\phi(x,z) = \{\psi(\sum_{i=0}^n x(i))z(n) : n \in \mathbb{N}\}$$

for $x \in {]0,\infty[}^{\mathbb{N}}$ and $z \in S_{r-1}^{\mathbb{N}}$. Show that $\phi : {]0,\infty[}^{\mathbb{N}} \times S_{r-1}^{\mathbb{N}} \to \mathcal{P}(\mathbb{R}^r)$ is a measure space isomorphism between $]0,\infty[^{\mathbb{N}} \times S_{r-1}^{\mathbb{N}}$ and a conegligible set for the Poisson point process on $\mathbb{R}^r$ with density 1.

495Y Further exercises (a) Let U be an L-space. Show that there are a probability algebra $(\mathfrak{B}, \bar\lambda)$ and a linear operator $T : U \to L^0(\mathfrak{B})$ such that (i) for every $u \in U$, Tu has a Cauchy distribution with centre 0 and scale parameter $\|u\|$ (ii) if $\langle u_i\rangle_{i\in I}$ is a disjoint family in U then $\langle Tu_i\rangle_{i\in I}$ is $\bar\lambda$-independent.

(b) Let U be an L-space. Show that there are a probability algebra $(\mathfrak{B}, \bar\lambda)$ and a linear operator $T : U \to L^1(\mathfrak{B}, \bar\lambda)$ such that (i) for every $u \in U^+$, Tu has a standard gamma distribution (definition: 455Xj) with expectation $\|u\|$ (ii) if $\langle u_i\rangle_{i\in I}$ is a disjoint family in U then $\langle Tu_i\rangle_{i\in I}$ is $\bar\lambda$-independent.

(c) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. For α, $y \in \mathbb{R}$ set $h_y(\alpha) = e^{iy\alpha}$, and let $\bar h_y : L^0(\mathfrak{A}) \to L^0_{\mathbb{C}}(\mathfrak{A})$ (definition: 366M[8]) be the corresponding operator (to be defined, following the ideas of 364H[9] or otherwise). Show that there are a probability algebra $(\mathfrak{B}, \bar\lambda)$ and a positive linear operator $T : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{B}, \bar\lambda)$ such that (i) $\|Tu\|_1 = \|u\|_1$ whenever $u \in L^1(\mathfrak{A}, \bar\mu)^+$ (ii) $\langle Tu_i\rangle_{i\in I}$ is $\bar\lambda$-independent in $L^0(\mathfrak{B})$ whenever $\langle u_i\rangle_{i\in I}$ is a disjoint family in $L^1(\mathfrak{A}, \bar\mu)$ (iii) $\int \bar h_y(Tu)d\bar\lambda = \exp(\int(\bar h_y(u) - \chi 1)d\bar\mu)$ for every $u \in L^1(\mathfrak{A}, \bar\mu)$ and $y \in \mathbb{R}$.

(d) Let (X, ρ) be a totally bounded metric space, μ a Radon measure on X and $\gamma > 0$. Let $\mathcal{C}$ be the set of closed subsets of X, and $\tilde\nu$ the quasi-Radon measure of 495N; let $\tilde\rho$ be the Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$. Show that the subspace measure on $\mathcal{C} \setminus \{\emptyset\}$ induced by $\tilde\nu$ is a Radon measure for the topology induced by $\tilde\rho$.

495 Notes and comments The underlying fact on which this section relies is that the Poisson distributions form a one-parameter semigroup of infinitely divisible distributions, with $\nu_\alpha * \nu_\beta = \nu_{\alpha+\beta}$ for all α, $\beta > 0$. Other well-known families with this property are normal distributions, Cauchy distributions and gamma distributions; for each of these we have results corresponding to 495B and 495K (495Xh, 495Ya, 495Yb). The same distributions appeared, for the same reason, in the Lévy processes of §455. Observe that the version for the normal distribution is related to the Gaussian processes of §456. The 'compound Poisson' distributions of 495M provide further examples, which approach the general form of infinitely divisible distributions (Loève 77, §23, or Fristedt & Gray 97, §16.3).

The special feature of the Poisson point process, in this context, is the fact that (for atomless measure spaces (X, μ)) it can be represented by a measure on $\mathcal{P}X$ rather than on some abstract auxiliary space (495D); so that we have a notion of 'random subset', and can discuss the expected topological properties of subsets of X (495Xb, 495Xd). In Euclidean spaces the geometric properties of these random subsets are also of great interest; see Meester & Roy 96. Here I look at the relations between this construction and others which have been prominent in this book, such as inverse-measure-preserving functions (495G) and disintegrations (495H-495I). In the latter we find ourselves in an interesting difficulty. If, as in 495H, we have a measure space $X = \tilde X \times [0,1]$, where $\tilde X$ is an atomless measure space, then it is natural to suppose that our Poisson process on X can be represented by picking a random subset T of $\tilde X$ and then, for each $t \in T$, a random $(t, \alpha) \in X$. The obvious model for this idea is the map $(T, z) \mapsto \{(t, z(t)) : t \in T\} : \mathcal{P}\tilde X \times [0,1]^{\tilde X} \to \mathcal{P}X$. The problem with this model is that the map is simply not measurable for the standard σ-algebras on $\mathcal{P}\tilde X$, $\mathcal{P}\tilde X \times [0,1]^{\tilde X}$ and $\mathcal{P}X$. When we have a canonical ordering in order type ω of almost every subset of $\tilde X$ ('almost every' with respect to the Poisson point process on $\tilde X$, of course), as in 495Xo, there can be a way around this, cutting $[0,1]^{\tilde X}$ down to a countable product and re-inventing the representation of pairs (T, z) as subsets of X. But in the general case it seems that we have to set up a disintegration

[8] Formerly 364Yn.

[9] Formerly 364I.

of the Poisson point process on X over the Poisson point process on $\tilde{X}$ which does not correspond to any measure on a product $\mathcal{P}\tilde{X} \times \Omega$.

Following my usual custom, I have expressed the theorems of this section in terms of arbitrary (atomless) measure spaces. The results are not quite without interest when applied to totally finite measures, but their natural domain is the class of non-totally-finite σ-finite measures, as in 495N-495P. There is an unavoidable obstacle if we wish to extend the ideas to measure spaces which are not atomless. The functions $S \mapsto \#(S \cap E)$ may no longer have Poisson distributions, since if E is a singleton of positive measure then we shall have a non-trivial two-valued random variable. In 495N-495O I take one of the possible resolutions of this, with measures $\tilde{\nu}$ on spaces of subsets for which at least the sets $\{S : S \cap E = \emptyset\}$, for disjoint E, are independent. An alternative which is sometimes appropriate is to work with functions $h : X \to \mathbb{N}$ and $\sum_{x \in E} h(x)$ in place of subsets S of X and $\#(S \cap E)$; see Fristedt & Gray 97, §29.

In 495J-495L we have a little cluster of results which are relevant to rather different questions, to which I will return in Chapter 52 of Volume 5. The objective here is to connect the structure of a measure algebra or Banach lattice of arbitrarily large cellularity with something which can be realized in a probability space. In each case, disjointness is transformed into stochastic independence. Once again, the special feature of the Poisson point process is that we have a concrete representation of a linear operator which can also be described in a more abstract way (495L).

The construction of 495B-495D seems to be the most straightforward way to generate Poisson point processes. It fails however to give a direct interpretation of one of the most important approaches to these processes, as limits of purely atomic processes in which sets are chosen by including or excluding individual points independently (495Xm). In order to make sense of the limit here it seems that we need to put some further structure onto the underlying measure space, and 'σ-finite locally compact Radon measure space' is sufficient to give a positive result (495O).

496 Maharam submeasures

The old problem of characterizing measurable algebras led, among other things, to the concepts of 'Maharam submeasure' and 'Maharam algebra' (§393). It is known that these can be very different from measures (§394), but the differences are not well understood. In this section I will continue the work of §393 by showing that some, at least, of the ways in which topologies and measures interact apply equally to Maharam submeasures. The most important of these interactions are associated with the concept of 'Radon measure', so the first step is to find a corresponding notion of 'Radon submeasure' (496C, 496Y). In 496D-496K I run through a handful of theorems which parallel results in §§416 and 431-433. Products of submeasures remain problematic, but something can be done (496L-496M).

496A Definitions As we have hardly had 'submeasures' before in this volume, I repeat the essential definitions from Chapter 39. If $\mathfrak{A}$ is a Boolean algebra, a **submeasure** on $\mathfrak{A}$ is a functional $\mu : \mathfrak{A} \to [0, \infty]$ such that $\mu 0 = 0$ and $\mu a \le \mu(a \cup b) \le \mu a + \mu b$ for all a, $b \in \mathfrak{A}$ (392A). μ is **strictly positive** if $\mu a > 0$ for every $a \in \mathfrak{A} \setminus \{0\}$ (392Ba), **exhaustive** if $\lim_{n\to\infty} \mu a_n = 0$ for every disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ (392Bb), **totally finite** if $\mu 1 < \infty$ (392Bd), a **Maharam submeasure** if it is totally finite and $\lim_{n\to\infty} \mu a_n = 0$ for every non-increasing sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with zero infimum (393A). A Maharam submeasure is sequentially order-continuous (393Ba). If μ and ν are two submeasures on a Boolean algebra $\mathfrak{A}$, then μ is **absolutely continuous** with respect to ν if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\mu a \le \epsilon$ whenever $\nu a \le \delta$ (392Bg). A **Maharam algebra** is a Dedekind σ-complete Boolean algebra which carries a strictly positive Maharam submeasure (393E).

496B Basic facts I list some elementary ideas for future reference.

(a) Let μ be a submeasure on a Boolean algebra $\mathfrak{A}$.

(i) Set $I = \{a : a \in \mathfrak{A},\ \mu a = 0\}$. Clearly I is an ideal of $\mathfrak{A}$; write $\mathfrak{C}$ for the quotient Boolean algebra $\mathfrak{A}/I$. Then we have a strictly positive submeasure $\bar{\mu}$ on $\mathfrak{C}$ defined by setting $\bar{\mu} a^\bullet = \mu a$ for every $a \in \mathfrak{A}$. **P** If $a^\bullet = b^\bullet$ then

$$\mu(a \setminus b) = \mu(b \setminus a) = \mu(a \triangle b) = 0, \qquad \mu a = \mu(a \cap b) = \mu b;$$

so $\bar{\mu}$ is well-defined. The formulae defining 'submeasure' transfer directly from μ to $\bar{\mu}$. If $\bar{\mu} a^\bullet = 0$ then $\mu a = 0$, $a \in I$ and $a^\bullet = 0$, so $\bar{\mu}$ is strictly positive. **Q**

(ii) If μ is exhaustive, so is $\bar{\mu}$. **P** If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\langle a_n^\bullet \rangle_{n\in\mathbb{N}}$ is disjoint in $\mathfrak{A}/I$, set $b_n = a_n \setminus \sup_{i<n} a_i$ for each n; then $\langle b_n \rangle_{n\in\mathbb{N}}$ is disjoint so

$$\lim_{n\to\infty} \bar\mu a_n^\bullet = \lim_{n\to\infty} \bar\mu b_n^\bullet = \lim_{n\to\infty} \mu b_n = 0;$$

thus $\bar\mu$ is exhaustive. **Q**

(iii) If $\mathfrak{A}$ is Dedekind σ-complete and μ is a Maharam submeasure, then $\mathfrak{C}$ is a Maharam algebra. **P** As μ is sequentially order-continuous, I is a σ-ideal and $\mathfrak{C}$ is Dedekind σ-complete (314C). Now suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\langle a_n^\bullet\rangle_{n\in\mathbb{N}}$ is non-increasing and has zero infimum in $\mathfrak{C}$. Set $b_n = \inf_{i\le n} a_i$ for each n, and $a = \inf_{n\in\mathbb{N}} a_n$; then $a^\bullet = 0$ so $\mu a = 0$ and (again because μ is sequentially order-continuous)

$$\lim_{n\to\infty} \bar\mu a_n^\bullet = \lim_{n\to\infty} \bar\mu b_n^\bullet = \lim_{n\to\infty} \mu b_n = \mu a = 0.$$

Since we already know that $\bar\mu$ is a strictly positive submeasure, it is a strictly positive Maharam submeasure and $\mathfrak{C}$ is a Maharam algebra. **Q**

In this context I will say that $\mathfrak{C}$ is **the Maharam algebra of** μ.

(b) If μ is a strictly positive totally finite submeasure on a Boolean algebra $\mathfrak{A}$, there is an associated metric $(a,b)\mapsto \mu(a \mathbin{\triangle} b)$ (392H); the corresponding metric completion $\widehat{\mathfrak{A}}$ admits a continuous extension of μ to a strictly positive submeasure $\hat\mu$ on $\widehat{\mathfrak{A}}$. If μ is exhaustive, then $\hat\mu$ is a Maharam submeasure and $\widehat{\mathfrak{A}}$ is a Maharam algebra (393H). A Maharam algebra is ccc, therefore Dedekind complete, and weakly (σ,∞)-distributive (393Eb).

(c) If μ is a submeasure defined on an algebra Σ of subsets of a set X, I will say that the **null ideal** $\mathcal{N}(\mu)$ of μ is the ideal of subsets of X generated by $\{E : E\in\Sigma,\ \mu E = 0\}$. If $\mathcal{N}(\mu)\subseteq\Sigma$ I will say that μ is **complete**. Generally, the **completion** of μ is the functional $\hat\mu$ defined by saying that $\hat\mu(E\triangle A) = \mu E$ whenever $E\in\Sigma$ and $A\in\mathcal{N}(\mu)$; it is elementary to check that $\hat\mu$ is a complete submeasure.

(d) If $\mathfrak{A}$ is a Maharam algebra, and μ, ν are two strictly positive Maharam submeasures on $\mathfrak{A}$, then each is absolutely continuous with respect to the other (393F). Consequently the metrics associated with them are uniformly equivalent, and induce the same topology, the **Maharam-algebra topology** of $\mathfrak{A}$ (393G).

496C Radon submeasures Let X be a Hausdorff space. A **totally finite Radon submeasure** on X is a complete totally finite submeasure μ defined on a σ-algebra Σ of subsets of X such that (i) Σ contains every open set (ii) $\inf\{\mu(E\setminus K) : K\subseteq E \text{ is compact}\} = 0$ for every $E\in\Sigma$.

In this context I will say that a set $E\in\Sigma$ is **self-supporting** if $\mu(E\cap G) > 0$ whenever $G\subseteq X$ is open and $G\cap E\ne\emptyset$.

496D Proposition Let μ be a totally finite Radon submeasure on a Hausdorff space X with domain Σ.

(a) μ is a Maharam submeasure.

(b) $\inf\{\mu(G\setminus E) : G\supseteq E \text{ is open}\} = 0$ for every $E\in\Sigma$.

(c) If $E\in\Sigma$ there is a relatively closed $F\subseteq E$ such that F is self-supporting and $\mu(E\setminus F) = 0$.

(d) If $E\in\Sigma$ and $\epsilon > 0$ there is a compact self-supporting $K\subseteq E$ such that $\mu(E\setminus K)\le\epsilon$.

proof (a) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence in Σ with empty intersection. **?** If $\inf_{n\in\mathbb{N}}\mu E_n = \epsilon > 0$, then for each $n\in\mathbb{N}$ choose a compact set $K_n\subseteq E_n$ such that $\mu(E_n\setminus K_n)\le 2^{-n-2}\epsilon$. For each $n\in\mathbb{N}$,

$$\mu(E_n\setminus \bigcap_{i\le n} K_i) \le \sum_{i=0}^{n} \mu(E_i\setminus K_i) < \epsilon \le \mu E_n,$$

so $\bigcap_{i\le n} K_i\ne\emptyset$. There is therefore a point in $\bigcap_{n\in\mathbb{N}} K_n\subseteq\bigcap_{n\in\mathbb{N}} E_n$. **X** As $\langle E_n\rangle_{n\in\mathbb{N}}$ is arbitrary, μ is a Maharam submeasure.

(b) We have only to observe that

$$\begin{aligned}\inf\{\mu(G\setminus E) &: G\supseteq E \text{ and } G \text{ is open}\}\\ &\le \inf\{\mu((X\setminus E)\setminus K) : K\subseteq X\setminus E \text{ is compact}\} = 0.\end{aligned}$$

(c) Let $\mathcal{G}$ be the family of open subsets G of X such that $\mu(E\cap G) = 0$, and $H = \bigcup\mathcal{G}$. Then $\mathcal{G}$ is upwards-directed. If $\epsilon > 0$ there is a compact set $K\subseteq E\cap H$ such that $\mu(E\cap H\setminus K)\le\epsilon$; now there is a $G\in\mathcal{G}$ such that $K\subseteq G$ and $\mu(E\cap H)\le\epsilon+\mu K = \epsilon$. As ϵ is arbitrary, $H\in\mathcal{G}$; set $F = E\setminus H$.

(d) There is a compact $K_0\subseteq E$ such that $\mu(E\setminus K_0)\le\epsilon$; by (c), there is a closed self-supporting $K\subseteq K_0$ such that $\mu(K_0\setminus K) = 0$.

496E Theorem Let X be a Hausdorff space and $\mathcal{K}$ the family of compact subsets of X. Let $\phi : \mathcal{K} \to [0, \infty[$ be a bounded functional such that

(α) $\phi\emptyset = 0$ and $\phi K \le \phi(K \cup L) \le \phi K + \phi L$ for all K, $L \in \mathcal{K}$;

(β) whenever $K \in \mathcal{K}$ and $\epsilon > 0$ there is an $L \in \mathcal{K}$ such that $L \subseteq X \setminus K$ and $\phi K' \le \epsilon$ whenever $K' \in \mathcal{K}$ is disjoint from $K \cup L$;

(γ) whenever K, $L \in \mathcal{K}$ and $K \subseteq L$ then $\phi L \le \phi K + \sup\{\phi K' : K' \in \mathcal{K}, K' \subseteq L \setminus K\}$.

Then there is a unique totally finite Radon submeasure on X extending ϕ.

proof (a) For $A \subseteq X$ write $\phi_* A = \sup\{\phi K : K \subseteq A$ is compact$\}$. Then ϕ_* extends ϕ, by (α). Also $\phi_*(\bigcup_{n\in\mathbb{N}} G_n) \le \sum_{n=0}^{\infty} \phi_* G_n$ for every sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open subsets of X. **P** If $K \subseteq \bigcup_{n\in\mathbb{N}} G_n$ is compact, it is expressible as $\bigcup_{i\le n} K_i$ where $n \in \mathbb{N}$ and $K_i \subseteq G_i$ is compact for every $i \le n$ (4A2Fj). Now

$$\phi K \le \sum_{i=0}^{n} \phi K_i \le \sum_{i=0}^{\infty} \phi_* G_i.$$

As K is arbitrary, $\phi_*(\bigcup_{n\in\mathbb{N}} G_n) \le \sum_{n=0}^{\infty} \phi_* G_n$. **Q** In particular, because $\phi\emptyset = 0$, $\phi_*(G \cup H) \le \phi_* G + \phi_* H$ for all open G, $H \subseteq X$.

(b) Let Σ be the family of subsets E of X such that for every $\epsilon > 0$ there is a $K \subseteq X$ such that $K \cap E$ and $K \setminus E$ are both compact and $\phi_*(X \setminus K) \le \epsilon$. Then Σ is an algebra of subsets of X including $\mathcal{K}$. **P** (i) Of course $X \setminus E \in \Sigma$ whenever $E \in \Sigma$. (ii) If E, $F \in \Sigma$ and $\epsilon > 0$, let K, $L \subseteq X$ be such that $K \cap E$, $K \setminus E$, $L \cap F$ and $L \setminus F$ are all compact and $\phi_*(X \setminus K)$, $\phi_*(X \setminus L)$ are both at most $\frac{1}{2}\epsilon$. Then $(K \cap L) \cap (E \cup F)$ and $(K \cap L) \setminus (E \cup F)$ are both compact, and

$$\phi_*(X \setminus (K \cap L)) \le \phi_*(X \setminus K) + \phi_*(X \setminus L) \le \epsilon.$$

As ϵ is arbitrary, $E \cup F \in \Sigma$. (iii) By hypothesis (β), $\mathcal{K} \subseteq \Sigma$. **Q**

(c) Σ is a σ-algebra of subsets of X. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ with intersection E, and $\epsilon > 0$. For each $n \in \mathbb{N}$ let $K_n \subseteq X$ be such that $K_n \cap E_n$ and $K_n \setminus E_n$ are compact and $\phi_*(X \setminus K_n) \le 2^{-n}\epsilon$; set $K = \bigcap_{n\in\mathbb{N}} K_n$. Set $L = \bigcap_{n\in\mathbb{N}} K_n \cap E_n$, so that $L \subseteq E$ is compact, and let $L' \subseteq X \setminus L$ be a compact set such that $\phi_*(X \setminus (L \cup L')) \le \epsilon$; set $K' = K \cap (L \cup L')$. Then $\phi_*(X \setminus K') \le 3\epsilon$. As $L' \cap L = \emptyset$ there is an $n \in \mathbb{N}$ such that $L' \cap \bigcap_{i\le n} K_i \cap E_i$ is empty. Now

$$K \cap L' \subseteq \bigcup_{i\le n}(X \setminus (K_i \cap E_i)) \cap \bigcap_{i\le n} K_i \subseteq \bigcup_{i\le n} X \setminus E_i \subseteq X \setminus E,$$

so $K' \cap E = K \cap L$ and $K' \setminus E = K \cap L'$ are compact. As ϵ is arbitrary, $E \in \Sigma$. **Q**

(d) Set $\mu = \phi_* \restriction \Sigma$. Then μ is subadditive. **P** Suppose that E, $F \in \Sigma$ and $K \subseteq E \cup F$ is compact. Let $\epsilon > 0$. Then there are L_1, $L_2 \in \mathcal{K}$ such that $L_1 \cap E$, $L_1 \setminus E$, $L_2 \cap F$ and $L_2 \setminus F$ are all compact, while $\phi_*(X \setminus L_1)$ and $\phi_*(X \setminus L_2)$ are both at most ϵ. Set $K_1 = L_1 \cap E$ and $K_2 = L_2 \cap F$, so that

$$\phi K \le \phi(K \cup K_1 \cup K_2) \le \phi(K_1 \cup K_2) + \phi_*(K \setminus (K_1 \cup K_2))$$

(by hypothesis (γ))

$$\le \phi K_1 + \phi K_2 + \phi_*(X \setminus (L_1 \cap L_2)) \le \phi_* E + \phi_* F + 2\epsilon.$$

As ϵ and K are arbitrary, $\phi_*(E \cup F) \le \phi_* E + \phi_* F$. **Q**

(e) Every open set belongs to Σ. **P** Let $G \subseteq X$ be open, and $\epsilon > 0$. Applying (β) with $K = \emptyset$ we have an $L \in \mathcal{K}$ such that $\phi_*(X \setminus L) \le \epsilon$. Next, there is an $L' \in \mathcal{K}$, disjoint from $L \setminus G$, such that $\phi_*(X \setminus ((L \setminus G) \cup L')) \le \epsilon$. Set $L'' = L \cap ((L \setminus G) \cup L')$. Then $L'' \cap G = L \cap L'$ and $L'' \setminus G = L \setminus G$ are compact and $\phi_*(X \setminus L'') \le 2\epsilon$. **Q**

(f) If $E \subseteq F \in \Sigma$ and $\mu F = 0$ then $E \in \Sigma$. **P** Let $\epsilon > 0$. Let $K \subseteq X$ be such that $K \cap F$ and $K \setminus F$ are both compact and $\phi_*(X \setminus K) \le \epsilon$. Then $(K \setminus F) \cap E$ and $(K \setminus F) \setminus E$ are both compact, and

$$\phi_*(X \setminus (K \setminus F)) = \mu(X \setminus (K \setminus F)) \le \mu(X \setminus K) + \mu F = \phi_*(X \setminus K) \le \epsilon.$$

As ϵ is arbitrary, $E \in \Sigma$. **Q**

(g) If $E \in \Sigma$ and $\epsilon > 0$, there is a compact $K \subseteq E$ such that $\mu(E \setminus K) \le \epsilon$. **P** Let $K_0 \subseteq X$ be such that $K_0 \cap E$ and $K_0 \setminus E$ are both compact and $\phi_*(X \setminus K_0) \le \epsilon$. Set $K = E \cap K_0$. If $L \in \mathcal{K}$ and $L \subseteq E \setminus K$ then $\phi L \le \phi_*(X \setminus K_0) \le \epsilon$; so $\mu(E \setminus K) \le \epsilon$. **Q**

(h) So μ is a totally finite Radon submeasure. To see that it is unique, let μ' be another totally finite Radon submeasure with the same properties, and Σ' its domain. By condition (ii) of 496C, $\mu' = \phi_*\restriction\Sigma'$. If $E \in \Sigma$ there are sequences $\langle K_n\rangle_{n\in\mathbb{N}}$, $\langle L_n\rangle_{n\in\mathbb{N}}$ of compact sets such that $K_n \subseteq E$, $L_n \subseteq X\setminus E$ and $\mu(E\setminus K_n)+\mu((X\setminus E)\setminus L_n) \le 2^{-n}$ for every n. Set $F = \bigcup_{n\in\mathbb{N}} K_n$ and $F' = \bigcup_{n\in\mathbb{N}} L_n$; then $F\cup F'$ belongs to $\Sigma\cap\Sigma'$ and

$$\begin{aligned}\mu'(X\setminus(F\cup F')) &= \phi_*(X\setminus(F\cup F')) = \mu(X\setminus(F\cup F'))\\ &\le \inf_{n\in\mathbb{N}} \mu(X\setminus(K_n\cup L_n)) = 0.\end{aligned}$$

Consequently $E\setminus F \in \Sigma'$ and $E \in \Sigma'$.

The same works with μ and μ' interchanged, so $\Sigma = \Sigma'$ and $\mu' = \phi_*\restriction\Sigma = \mu$.

496F Theorem Let X be a zero-dimensional compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. Let $\nu : \mathcal{E} \to [0,\infty[$ be an exhaustive submeasure. Then there is a unique totally finite Radon submeasure on X extending ν.

proof (a) Let $\mathcal{K}$ be the family of compact subsets of X and for $K \in \mathcal{K}$ set $\phi K = \inf\{\nu E : K \subseteq E \in \mathcal{E}\}$. Then ϕ satisfies the conditions of 496E.

P (α) Of course $\phi\emptyset = 0$ and $\phi K \le \phi L$ whenever $K \subseteq L$ in $\mathcal{K}$. If $K \subseteq E \in \mathcal{E}$ and $L \subseteq F \in \mathcal{E}$, then $K\cup L \subseteq E\cup F \in \mathcal{E}$ and $\nu(E\cup F) \le \nu E + \nu F$, so ϕ is subadditive.

(β) The point is that for every $K \in \mathcal{K}$ and $\epsilon > 0$ there is an $E \in \mathcal{E}$ such that $K \subseteq E$ and $\nu F \le \epsilon$ whenever $F \in \mathcal{E}$ and $F \subseteq E\setminus K$; since otherwise we could find a disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ with $\nu F_n \ge \epsilon$ for every n. But now $L = X\setminus E$ is compact and disjoint from K, and every compact subset of $X\setminus(K\cup L) = E\setminus K$ is included in a member of $\mathcal{E}$ included in $E\setminus K$; so $\sup\{\phi K' : K' \subseteq X\setminus(K\cup L)$ is compact$\} \le \epsilon$.

(γ) If K and L are compact and $K \subseteq L$ and $\epsilon > 0$, take $E \in \mathcal{E}$ such that $K \subseteq E$ and $\nu E \le \phi K + \epsilon$. Set $K' = L\setminus E$. If $F \in \mathcal{E}$ and $F \supseteq K'$, then $E\cup F \supseteq L$, so

$$\phi L \le \nu(E\cup F) \le \nu E + \nu F \le \phi K + \epsilon + \nu F.$$

As F is arbitrary, $\phi L \le \phi K + \phi K' + \epsilon$. **Q**

There is therefore a totally finite Radon submeasure μ extending ϕ and ν.

(b) If μ' is another totally finite Radon submeasure extending ν, then $\mu'\restriction\mathcal{K} = \phi$. **P** Of course $\mu' K \le \phi K$ for every $K \in \mathcal{K}$. **?** If $K \in \mathcal{K}$ and $\epsilon > 0$ and $\mu' K + \epsilon < \phi K$, let $E \in \mathcal{E}$ be such that $K \subseteq E$ and $\phi L \le \epsilon$ whenever $L \subseteq E\setminus K$ is compact, as in (a-β) above. Then

$$\begin{aligned}\mu'(E\setminus K) &= \sup\{\mu' L : L \subseteq E\setminus K \text{ is compact}\}\\ &\le \sup\{\phi L : L \subseteq E\setminus K \text{ is compact}\} \le \epsilon\end{aligned}$$

and

$$\nu E = \mu' E \le \epsilon + \mu' K < \mu K \le \mu E = \nu E. \textbf{ XQ}$$

By the guarantee of uniqueness in 496E, $\mu' = \mu$.

496G Theorem Let $\mathfrak{A}$ be a Maharam algebra, and μ a strictly positive Maharam submeasure on $\mathfrak{A}$. Let Z be the Stone space of $\mathfrak{A}$, and write $\widehat{a}$ for the open-and-closed subset of Z corresponding to each $a \in \mathfrak{A}$. Then there is a unique totally finite Radon submeasure ν on Z such that $\nu\widehat{a} = \mu a$ for every $a \in \mathfrak{A}$. The domain of ν is the Baire-property algebra $\widehat{\mathcal{B}}$ of Z, and the null ideal of ν is the nowhere dense ideal of Z.

proof Let $\mathcal{E}$ be the algebra of open-and-closed subsets of Z, and $\mathcal{M}$ the ideal of meager subsets of Z. Because $\mathfrak{A}$ is Dedekind complete (393Eb/496Bb), $\mathcal{E}$ is the regular open algebra of Z (314S). By 496R(b-ii), $\widehat{\mathcal{B}} = \{E\triangle F : E \in \mathcal{E}, F \in \mathcal{M}\}$.

For $a \in \mathfrak{A}$, let $\widehat{a}$ be the corresponding member of $\mathcal{E}$. By 314M, we have an isomorphism $\theta : \mathfrak{A} \to \widehat{\mathcal{B}}/\mathcal{M}$ defined by setting $\theta(a) = \widehat{a}^\bullet$ for every $a \in \mathfrak{A}$. For $E \in \widehat{\mathcal{B}}$, set $\nu E = \mu(\theta^{-1}E^\bullet)$. Because $E \mapsto E^\bullet$ is a sequentially order-continuous Boolean homomorphism (313P(b-ii)), ν is a Maharam submeasure on $\widehat{\mathcal{B}}$. Because μ is strictly positive, the null ideal of ν is $\mathcal{M}$.

Because $\mathfrak{A}$ is weakly (σ,∞)-distributive (393Eb/496Bb), $\mathcal{M}$ is the ideal of nowhere dense subsets of Z (316I). If $E\in\widehat{\mathcal{B}}$, consider $B=\{b:b\in\mathfrak{A},\,\widehat{b}\subseteq E\}$; set $a=\sup B$ in $\mathfrak{A}$. Now $E\setminus\widehat{a}$ is nowhere dense. **P?** Otherwise, there is a non-zero $c\in\mathfrak{A}$ such that $F=\widehat{c}\setminus(E\setminus\widehat{a})$ is nowhere dense. In this case, the non-empty open set $\widehat{c}\setminus\overline{F}$ is included in $E\setminus\widehat{a}$ and there is a non-zero $b\in\mathfrak{A}$ such that $\widehat{b}\subseteq E\setminus\widehat{a}$. But in this case $b\in B$ and $\widehat{b}\subseteq\widehat{a}$, which is absurd. **XQ**

Set $D=\{a\setminus b:b\in B\}$. Then D is downwards-directed and has infimum 0. Because μ is sequentially order-continuous and $\mathfrak{A}$ is ccc, μ is order-continuous (316Fc), and $\inf_{d\in D}\mu d=0$. Accordingly

$$\begin{aligned}\inf\{\nu(E\setminus K):K\subseteq E\text{ is compact}\}&\le\inf_{b\in B}\nu(E\setminus\widehat{b})=\inf_{b\in B}\nu(\widehat{a}\setminus\widehat{b})\\&=\inf_{b\in B}\mu(a\setminus b)=0.\end{aligned}$$

Thus condition (ii) of 496C is satisfied and ν is a totally finite Radon measure.

By 496F, ν is unique.

496H Theorem Let X be a Hausdorff space, Σ_0 an algebra of subsets of X, and $\mu_0:\Sigma_0\to[0,\infty[$ an exhaustive submeasure such that $\inf\{\mu_0(E\setminus K):K\in\Sigma_0\text{ is compact},\,K\subseteq E\}=0$ for every $E\in\Sigma_0$. Then μ_0 has an extension to a totally finite Radon submeasure μ_1 on X.

proof (a) Let P be the set of all submeasures μ, defined on algebras of subsets of X, which extend μ_0, and have the properties

(α) $\inf\{\mu(E\setminus K):K\in\operatorname{dom}\mu\text{ is compact},\,K\subseteq E\}=0$ for every $E\in\operatorname{dom}\mu$,

(*) for every $E\in\operatorname{dom}\mu$ and $\epsilon>0$ there is an $F\in\Sigma_0$ such that $\mu(E\triangle F)\le\epsilon$.

Order P by extension of functions, so that P is a partially ordered set.

(b) If $\mu\in P$, then μ is exhaustive. **P?** Otherwise, let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\operatorname{dom}\mu$ such that $\epsilon=\inf_{n\in\mathbb{N}}\mu E_n$ is greater than 0. For each $n\in\mathbb{N}$, let $F_n\in\Sigma_0$ be such that $\mu(E_n\triangle F_n)\le 2^{-n-2}\epsilon$; set $G_n=F_n\setminus\bigcup_{i<n}F_i$ for each n. Then

$$E_n\subseteq G_n\cup\bigcup_{i\le n}(E_i\triangle F_i),\quad \epsilon\le\mu G_n+\sum_{i=0}^{n}2^{-i-2}\epsilon\le\mu_0G_n+\frac{1}{2}\epsilon$$

and $\mu_0G_n\ge\frac{1}{2}\epsilon$ for every n. But $\langle G_n\rangle_{n\in\mathbb{N}}$ is disjoint and μ_0 is supposed to be exhaustive. **XQ**

(c) Suppose that $\mu\in P$ has domain Σ, and that $V\subseteq X$ is such that

$\ddagger(V,\mu)$: for every $\epsilon>0$ there is a $K\in\Sigma$ such that $K\cap V$ is compact and $\mu(X\setminus K)\le\epsilon$.

(i) Set $\mathcal{H}=\{H:V\subseteq H\in\Sigma\}$. Then $\mathcal{H}$ is downwards-directed. If $\epsilon>0$ there is an $H\in\mathcal{H}$ such that $\mu(H\setminus H')\le\epsilon$ for every $H'\in\mathcal{H}$. **P?** Otherwise, there would be a non-increasing sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $\mu(H_n\setminus H_{n+1})\ge\epsilon$ for every n; but μ is exhaustive, by (b). **XQ**

(ii) Let $\mathcal{F}$ be the filter on $\mathcal{H}$ generated by sets of the form $\{H':H'\in\mathcal{H},\,H'\subseteq H\}$ for $H\in\mathcal{H}$. Then $\lim_{H\to\mathcal{F}}\mu((E\cap H)\cup(F\setminus H))$ is defined for all E, $F\in\Sigma$. **P** Given $\epsilon>0$, then (i) tells us that there is an $H_0\in\mathcal{H}$ such that $\mu(H\triangle H')\le\mu(H_0\setminus(H\cap H'))\le\epsilon$ whenever H, $H'\in\mathcal{H}$ are included in H_0. Now, for such H and H',

$$((E\cap H)\cup(F\setminus H))\triangle((E\cap H')\cup(F\setminus H'))\subseteq H\triangle H',$$

so

$$|\mu((E\cap H)\cup(F\setminus H))-\mu((E\cap H')\cup(F\setminus H'))|\le\epsilon.\ \mathbf{Q}$$

(iii) If E, F, E', $F'\in\Sigma$ and $(E\cap V)\cup(F\setminus V)=(E'\cap V)\cup(F'\setminus V)$, then

$$\lim_{H\to\mathcal{F}}\mu((E\cap H)\cup(F\setminus H))=\lim_{H\to\mathcal{F}}\mu((E'\cap H)\cup(F'\setminus H)).$$

P Given $\epsilon>0$, there is an $H_0\in\mathcal{H}$ such that $\mu G\le\epsilon$ whenever $G\in\Sigma$ and $G\subseteq H_0\setminus V$, by (i). Now if $H\in\mathcal{H}$ and $H\subseteq H_0$,

$$G=((E\cap H)\cup(F\setminus H))\triangle((E'\cap H)\cup(F'\setminus H))\subseteq H\setminus V,$$

so

$$|\mu((E\cap H)\cup(F\setminus H))-\mu((E'\cap H)\cup(F'\setminus H))|\le\mu G\le\epsilon.$$

As ϵ is arbitrary, the limits are equal. **Q**

(iv) Consequently, taking Σ' to be the algebra $\{(E\cap V)\cup(F\setminus V): E,\, F\in\Sigma\}$ of subsets of X generated by $\Sigma\cup\{V\}$, we have a functional $\mu':\Sigma'\to[0,\infty[$ defined by saying that

$$\mu'((E\cap V)\cup(F\setminus V))=\lim_{H\to\mathcal{F}}\mu((E\cap H)\cup(F\setminus H))$$

whenever E, $F\in\Sigma$.

(v) μ' is a submeasure extending μ. **P** If $E\in\Sigma$, then

$$\mu'E=\mu'((E\cap V)\cup(E\setminus V))=\lim_{H\to\mathcal{F}}\mu((E\cap H)\cup(E\setminus H))=\mu E,$$

so μ' extends μ. If E_1, E_2, F_1, $F_2\in\Sigma$, set $E=E_1\cup E_2$, $F=F_1\cup F_2$; then

$$((E_1\cap A)\cup(F_1\setminus A))\cup((E_2\cap A)\cup(F_2\setminus A))=((E\cap A)\cup(F\setminus A))$$

for every set A, so

$$\begin{aligned}\mu'\bigl(((E_1\cap V)&\cup(F_1\setminus V))\cup((E_2\cap V)\cup(F_2\setminus V))\bigr)\\&=\mu'(((E\cap V)\cup(F\setminus V))\\&=\lim_{H\to\mathcal{F}}\mu((E\cap H)\cup(F\setminus H))\\&=\lim_{H\to\mathcal{F}}\mu\bigl(((E_1\cap H)\cup(F_1\setminus H))\cup((E_2\cap H)\cup(F_2\setminus H))\bigr)\\&\le\lim_{H\to\mathcal{F}}\mu((E_1\cap H)\cup(F_1\setminus H))+\mu((E_2\cap H)\cup(F_2\setminus H))\\&=\mu'((E_1\cap V)\cup(F_1\setminus V))+\mu'((E_2\cap V)\cup(F_2\setminus V)).\end{aligned}$$

Thus μ' is subadditive; monotonicity is easier. **Q**

(vi) μ' has the property (α). **P** Suppose that E, $F\in\Sigma$ and that $\epsilon>0$. Let $H_0\in\mathcal{H}$ be such that $\mu(H_0\setminus H)\le\epsilon$ whenever $H\in\mathcal{H}$ and $H\subseteq H_0$. Let $K_0\in\Sigma$ be such that $\mu(X\setminus K_0)\le\epsilon$ and $K_0\cap V$ is compact. Let $K_1\subseteq E$ and $K_2\subseteq F\setminus H_0$ be compact sets, belonging to Σ, such that $\mu(E\setminus K_1)\le\epsilon$ and $\mu((F\setminus H_0)\setminus K_2)\le\epsilon$. Set $K=(K_1\cap K_0\cap V)\cup K_2$, so that K is a compact set belonging to Σ' and $K\subseteq(E\cap V)\cup(F\setminus V)$. Now if $H\in\mathcal{H}$ and $H\subseteq H_0$,

$$\begin{aligned}\mu\bigl(((E\setminus(K_1\cap K_0))&\cap H)\cup((F\setminus K_2)\setminus H)\bigr)\\&\le\mu(E\setminus K_1)+\mu(X\setminus K_0)+\mu((F\setminus H_0)\setminus K_2)+\mu(H_0\setminus H)\le4\epsilon.\end{aligned}$$

Taking the limit along $\mathcal{F}$,

$$\mu'\bigl(((E\cap V)\cup(F\setminus V))\setminus K\bigr)=\mu'\bigl(((E\setminus(K_1\cap K_0))\cap V)\cup((F\setminus K_2)\setminus V)\bigr)\le4\epsilon.$$

As E, F and ϵ are arbitrary, we have the result. **Q**

(vii) μ' has the property (*). **P** Suppose that E, $F\in\Sigma$ and that $\epsilon>0$. Let $H_0\in\mathcal{H}$ be such that $\mu(H_0\setminus H)\le\epsilon$ whenever $H\in\mathcal{H}$ and $H\subseteq H_0$. Set $G=(E\cap H_0)\cup(F\setminus H_0)\in\Sigma$. Then

$$((E\cap V)\cup(F\setminus V))\triangle G\subseteq H_0\setminus V,$$

so

$$\mu'\bigl(((E\cap V)\cup(F\setminus V))\triangle G\bigr)\le\mu'(H_0\setminus V)=\lim_{H\to\mathcal{F}}\mu(H_0\setminus H)\le\epsilon.\ \mathbf{Q}$$

(d)(i) If $\mu\in P$ and $V\in\mathcal{N}(\mu)$, then $\ddagger(V,\mu)$ is true. **P** Let $\epsilon>0$. There is an $E\in\operatorname{dom}\mu$, including V, such that $\mu E=0$; now there is a compact $K\in\operatorname{dom}\mu$, included in $X\setminus E$, such that

$$\epsilon\ge\mu((X\setminus E)\setminus K)=\mu(X\setminus K),$$

while $K\cap V=\emptyset$ is compact. **Q**

(ii) If $\mu\in P$ and $V\subseteq X$ is closed, then $\ddagger(V,\mu)$ is true. **P** For every $\epsilon>0$, there is a compact $K\in\operatorname{dom}\mu$ such that $\mu(X\setminus K)\le\epsilon$, and now $K\cap V$ is compact. **Q**

(iii) Now suppose that $\mu\in P$ is such that every compact subset of X belongs to the domain Σ of μ, and that $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ with intersection V. Then $\ddagger(V,\mu)$ is true. **P** Let $\epsilon>0$. For each $n\in\mathbb{N}$, there are compact sets $K_n\subseteq E_n$, $K'_n\subseteq X\setminus E_n$ such that

$$\mu(E_n\setminus K_n)+\mu((X\setminus E_n)\setminus K'_n)\le2^{-n-1}\epsilon.$$

Set $K=\bigcap_{n\in\mathbb{N}}K_n\cup K'_n$; then K is compact, so belongs to Σ. If $L\subseteq X\setminus K$ is compact, then there is an $n\in\mathbb{N}$ such that $L\cap\bigcap_{i\le n}K_i\cup K'_i$ is empty, so that

$$\mu L\le\sum_{i=0}^{n}\mu(X\setminus(K_i\cup K'_i))\le\sum_{i=0}^{n}2^{-n-1}\epsilon\le\epsilon.$$

As L is arbitrary, $\mu(X\setminus K)\le\epsilon$. Finally,

$$K\cap V=\bigcap_{n\in\mathbb{N}}(K_n\cup K'_n)\cap E_n=\bigcap_{n\in\mathbb{N}}K_n$$

is compact. **Q**

(e) If $Q\subseteq P$ is a non-empty totally ordered subset of P, $\bigcup Q\in P$. So P has a maximal element μ_1, which is a submeasure, satisfying (α), and extending μ_0. Setting $\Sigma_1=\operatorname{dom}\mu_1$, (c) tells us that $V\in\Sigma_1$ whenever $V\subseteq X$ and $\ddagger(V,\mu_1)$ is true. By (d-i), $\mathcal{N}(\mu_1)\subseteq\Sigma_1$ and μ_1 is complete. By (d-ii), every closed set, and therefore every open set, belongs to Σ_1. So (d-iii) tells us that $\bigcap_{n\in\mathbb{N}}E_n\in\Sigma_1$ for every sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ_1, and Σ_1 is a σ-algebra. Putting these together, all the conditions of 496C are satisfied, and μ_1 is a totally finite Radon submeasure.

496I Theorem Let X be a set, Σ a σ-algebra of subsets of X, and μ a complete Maharam submeasure on Σ.

(a) Σ is closed under Souslin's operation.

(b) If A is the kernel of a Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S}$ in Σ, and $\epsilon>0$, there is a $\psi\in\mathbb{N}^{\mathbb{N}}$ such that

$$\mu(A\setminus\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}},\phi\le\psi}\bigcap_{n\ge1}E_{\phi\restriction n})\le\epsilon.$$

proof (a) Let $\mathcal{N}(\mu)$ be the null ideal of μ. Because μ is exhaustive, every disjoint sequence in $\Sigma\setminus\mathcal{N}(\mu)$ is countable, so 431G tells us that Σ is closed under Souslin's operation.

(b) The argument of 431D applies, with trifling modifications in its expression. For $\sigma\in S^*=\bigcup_{k\in\mathbb{N}}\mathbb{N}^k$, set $A_\sigma=\bigcup_{\sigma\subseteq\phi\in\mathbb{N}^{\mathbb{N}}}\bigcap_{n\ge1}E_{\phi\restriction n}$; then $A_\sigma\in\Sigma$, by (a). Given $\epsilon>0$, let $\langle\epsilon_\sigma\rangle_{\sigma\in S^*}$ be a family of strictly positive real numbers such that $\sum_{\sigma\in S^*}\epsilon_\sigma\le\epsilon$. For each $\sigma\in S^*$, let m_σ be such that $\mu(A_\sigma\setminus\bigcup_{i\le m_\sigma}A_{\sigma^\frown<i>})\le\epsilon_\sigma$. Set

$$\psi(k)=\max\{m_\sigma:\sigma\in\mathbb{N}^k,\ \sigma(i)\le\psi(i)\text{ for every }i<k\}$$

for $k\in\mathbb{N}$; then

$$A\setminus\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}},\phi\le\psi}\bigcap_{n\ge1}E_{\phi\restriction n}\subseteq\bigcup_{\sigma\in S^*}\bigl(A_\sigma\setminus\bigcup_{i\le m_\sigma}A_{\sigma^\frown<i>}\bigr)$$

has submeasure at most ϵ.

496J Theorem Let X be a K-analytic Hausdorff space and μ a Maharam submeasure defined on the Borel σ-algebra of X. Then

$$\inf\{\mu(X\setminus K):K\subseteq X\text{ is compact}\}=0.$$

proof Again, we have only to re-use the ideas of 432B. Let $\hat\mu$ be the completion of μ (496A) and Σ the domain of $\hat\mu$. Let $R\subseteq\mathbb{N}^{\mathbb{N}}\times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}]=X$. For $\sigma\in S^*=\bigcup_{n\ge1}\mathbb{N}^n$ set $I_\sigma=\{\phi:\sigma\subseteq\phi\in\mathbb{N}^{\mathbb{N}}\}$, $F_\sigma=\overline{R[I_\sigma]}$; then X is the kernel of the Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S^*}$. Now, given $\epsilon>0$, 496Ib tells us that there is a $\psi\in\mathbb{N}^{\mathbb{N}}$ such that $\mu(X\setminus F)\le\epsilon$, where $F=\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}},\phi\le\psi}\bigcap_{n\ge1}F_{\phi\restriction n}$; but $F=R[K]$ where K is the compact set $\{\phi:\phi\in\mathbb{N}^{\mathbb{N}},\ \phi\le\psi\}$, so F is compact.

496K Proposition Let μ be a Maharam submeasure on the Borel σ-algebra of an analytic Hausdorff space X. Then the completion of μ is a totally finite Radon submeasure on X.

proof If $E\subseteq X$ is Borel, then it is K-analytic (423Eb); applying 496J to $\mu\restriction\mathcal{P}E$, we see that $\inf\{\mu(E\setminus K):K\subseteq E$ is compact$\}=0$. Consequently, writing Σ for the domain of the completion $\hat\mu$ of μ, $\inf\{\hat\mu(E\setminus K):K\subseteq E$ is compact$\}=0$ for every $E\in\Sigma$. Condition (i) of the definition 496C is surely satisfied by $\hat\mu$, so $\hat\mu$ is a totally finite Radon submeasure.

496L Free products of Maharam algebras If $\mathfrak{A}$, $\mathfrak{B}$ are Boolean algebras with submeasures μ, ν respectively, we have a submeasure $\mu\ltimes\nu$ on the free product $\mathfrak{A}\otimes\mathfrak{B}$ (392K). It is easy to see, in 392K, that if μ and ν are strictly positive so is $\mu\ltimes\nu$; moreover, if μ and ν are exhaustive so is $\mu\ltimes\nu$ (392Ke).

Now suppose that $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a family of Maharam algebras, where I is a finite totally ordered set. Then we can take a strictly positive Maharam submeasure μ_i on each $\mathfrak{A}_i$, form an exhaustive submeasure λ on $\mathfrak{C}_I=\bigotimes_{i\in I}\mathfrak{A}_i$, and use λ to construct a metric completion $\widehat{\mathfrak{C}}_I$ which is a Maharam algebra, as in 393H/496Bb. (If $I = \{i_0, \ldots, i_n\}$ where $i_0 < \ldots < i_n$, then $\lambda = (.(\mu_{i_0} \ltimes \mu_{i_1}) \ltimes \ldots) \ltimes \mu_{i_n}$ (392Kf). By 392Kc, the product is associative, so the arrangement of the brackets is immaterial.) If we change each μ_i to μ_i', where μ_i' is another strictly positive Maharam submeasure on $\mathfrak{A}_i$, then every μ_i' is absolutely continuous with respect to μ_i (393F/496Bd), so the corresponding λ' will be absolutely continuous with respect to λ, and vice versa (392Kd); in which case the metrics on $\mathfrak{C}_I$ are uniformly equivalent and we get the same metric completion $\widehat{\mathfrak{C}}_I$ up to Boolean algebra isomorphism. We can therefore think of $\widehat{\mathfrak{C}}_I$ as 'the' **Maharam algebra free product** of the family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras; as in 392Kf, we shall have an isomorphism between $\widehat{\mathfrak{C}}_{J\cup K}$ and the Maharam algebra free product of $\widehat{\mathfrak{C}}_J$ and $\widehat{\mathfrak{C}}_K$ whenever J, $K \subseteq I$ and $j < k$ for every $j \in J$ and $k \in K$.

From 392Kg we see that if $(\mathfrak{A}, \mu)$ and $(\mathfrak{B}, \nu)$ are probability algebras, then their Maharam algebra free product, regarded as a Boolean algebra, is isomorphic to their probability algebra free product as defined in §325.

496M Representing products of Maharam algebras: Theorem Let X and Y be sets, with σ-algebras Σ and T and Maharam submeasures μ and ν defined on Σ, T respectively. Let $\mathfrak{A}$, $\mathfrak{B}$ be their Maharam algebras and write $\bar{\mu}$, $\bar{\nu}$ for the strictly positive Maharam submeasures on $\mathfrak{A}$ and $\mathfrak{B}$ induced by μ and ν as in 496Ba above. Let $\Sigma\widehat{\otimes}\mathrm{T}$ be the σ-algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma,\ F \in \mathrm{T}\}$.

(a) (Compare 418T.) Give $\mathfrak{B}$ its Maharam-algebra topology (393G/496Bd). If $W \in \Sigma\widehat{\otimes}\mathrm{T}$ then $W[\{x\}] \in \mathrm{T}$ for every $x \in X$ and the function $x \mapsto W[\{x\}]^\bullet : X \to \mathfrak{B}$ is Σ-measurable and has separable range. Consequently $x \mapsto \nu W[\{x\}] : X \to [0, \infty[$ is Σ-measurable.

(b) For $W \in \Sigma\widehat{\otimes}\mathrm{T}$ set

$$\lambda W = \inf\{\epsilon : \epsilon > 0,\ \mu\{x : \nu W[\{x\}] > \epsilon\} \leq \epsilon\}.$$

Then λ is a Maharam submeasure on $\Sigma\widehat{\otimes}\mathrm{T}$, and

$$\lambda^{-1}[\{0\}] = \{W : W \in \Sigma\widehat{\otimes}\mathrm{T},\ \{x : W[\{x\}] \notin \mathcal{N}(\nu)\} \in \mathcal{N}(\mu)\}.$$

(c) Let $\mathfrak{C}$ be the Maharam algebra of λ. Then $\mathfrak{A} \otimes \mathfrak{B}$ can be embedded in $\mathfrak{C}$ by mapping $E^\bullet \otimes F^\bullet$ to $(E \times F)^\bullet$ for all $E \in \Sigma$ and $F \in \mathrm{T}$.

(d) This embedding identifies $(\mathfrak{C}, \bar{\lambda})$ with the metric completion of $(\mathfrak{A} \otimes \mathfrak{B}, \bar{\mu} \ltimes \bar{\nu})$.

proof (a) Write $\mathcal{W}$ for the set of those $W \subseteq X \times Y$ such that $W[\{x\}] \in \mathrm{T}$ for every $x \in X$ and $x \mapsto W[\{x\}]^\bullet : X \to \mathfrak{B}$ is Σ-measurable and has separable range. Then $\Sigma \otimes \mathrm{T}$ (identified with the algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma,\ F \in \mathrm{T}\}$) is included in $\mathcal{W}$.

If $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}$ with union W, then $W \in \mathcal{W}$. **P** Of course $W[\{x\}] = \bigcup_{n\in\mathbb{N}} W_n[\{x\}]$ belongs to T for every $x \in X$. Set $f_n(x) = W_n[\{x\}]^\bullet$ for $n \in \mathbb{N}$ and $x \in X$. For each $x \in X$, $W[\{x\}] \setminus W_n[\{x\}]$ is a non-increasing sequence with empty intersection, so $\lim_{n\to\infty} \nu(W[\{x\}] \setminus W_n[\{x\}]) = 0$ and $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ converges to $f(x) = W[\{x\}]^\bullet$ in $\mathfrak{B}$. By 418Ba, f is measurable. Also $D = \overline{\{f_n(x) : x \in X,\ n \in \mathbb{N}\}}$ is a separable subspace of $\mathfrak{B}$ including $f[X]$. So $W \in \mathcal{W}$. **Q**

Similarly, $\bigcap_{n\in\mathbb{N}} W_n \in \mathcal{W}$ for any non-increasing sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}$. $\mathcal{W}$ therefore includes the σ-algebra generated by $\Sigma \otimes \mathrm{T}$ (136G), which is $\Sigma\widehat{\otimes}\mathrm{T}$.

Now $x \mapsto \nu W[\{x\}] = \bar{\nu} W[\{x\}]^\bullet$ is measurable because $\bar{\nu} : \mathfrak{B} \to \mathbb{R}$ is continuous.

(b) Of course $\lambda\emptyset = 0$ and $\lambda W \leq \lambda W'$ if W, $W' \in \Sigma\widehat{\otimes}\mathrm{T}$ and $W \subseteq W'$. If W_1, $W_2 \in \Sigma\widehat{\otimes}\mathrm{T}$ have union W, $\lambda W_1 = \alpha_1$ and $\lambda W_2 = \alpha_2$, then

$$\{x : \nu W[\{x\}] > \alpha_1 + \alpha_2\} \subseteq \{x : \nu W_1[\{x\}] > \alpha_1\} \cup \{x : \nu W_2[\{x\}] > \alpha_2\},$$

so, setting $\alpha = \alpha_1 + \alpha_2$,

$$\mu\{x : \nu W[\{x\}] > \alpha\} \leq \mu\{x : \nu W_1[\{x\}] > \alpha_1\} + \mu\{x : \nu W_2[\{x\}] > \alpha_2\} \leq \alpha_1 + \alpha_2 = \alpha,$$

and $\lambda W \leq \alpha$. Thus λ is monotonic and subadditive.

If now $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\Sigma\widehat{\otimes}\mathrm{T}$ with empty intersection, and $\epsilon > 0$, set $E_n = \{x : \nu W_n[\{x\}] \geq \epsilon\}$ for each n. Then $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-increasing; moreover, for any $x \in X$, $\langle W_n[\{x\}]\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T with empty intersection, so $\lim_{n\to\infty} \nu W_n[\{x\}] = 0$ and $x \notin \bigcap_{n\in\mathbb{N}} E_n$. There is therefore an n such that $\mu E_n \leq \epsilon$ and $\lambda W_n \leq \epsilon$. As $\langle W_n\rangle_{n\in\mathbb{N}}$ and ϵ are arbitrary, λ is a Maharam submeasure.

Finally, for $W \in \Sigma\widehat{\otimes}\mathrm{T}$,

$$\begin{aligned}\lambda W = 0 &\iff \mu\{x : \nu W[\{x\}] \ge 2^{-n}\} \le 2^{-n} \text{ for every } n \in \mathbb{N}\\ &\iff \mu\{x : \nu W[\{x\}] \ge 2^{-m}\} \le 2^{-n} \text{ for every } m, n \in \mathbb{N}\\ &\iff \mu\{x : \nu W[\{x\}] > 0\} \le 2^{-n} \text{ for every } n \in \mathbb{N}\\ &\iff \mu\{x : \nu W[\{x\}] > 0\} = 0 \iff \{x : W[\{x\}] \notin \mathcal{N}(\nu)\} \in \mathcal{N}(\mu).\end{aligned}$$

(c) If $E \in \Sigma$, then $\lambda(E \times F) = \min(\mu E, \nu F)$ for all $E \in \Sigma$ and $F \in \mathrm{T}$. So $\lambda(E \times F) = 0$ iff $E^\bullet \otimes F^\bullet = 0$ in $\mathfrak{A} \otimes \mathfrak{B}$. Consequently we have injective Boolean homomorphisms from $\mathfrak{A}$ to $\mathfrak{C}$ and from $\mathfrak{B}$ to $\mathfrak{C}$ defined by the formulae

$$E^\bullet \mapsto (E \times Y)^\bullet \text{ for } E \in \Sigma, \quad F^\bullet \mapsto (X \times F)^\bullet \text{ for } F \in \mathrm{T};$$

by 315J and 315Kb[10], we have an injective Boolean homomorphism from $\mathfrak{A} \otimes \mathfrak{B}$ to $\mathfrak{C}$ which maps $E^\bullet \otimes F^\bullet$ to $(E \times F)^\bullet$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$.

(d) $\bar\lambda(\phi e) = (\mu \ltimes \nu)(e)$ for every $e \in \mathfrak{A} \otimes \mathfrak{B}$. **P** Express e as $\sup_{i \in I} a_i \otimes b_i$ where $\langle a_i \rangle_{i \in I}$ is a finite partition of unity in $\mathfrak{A}$ and $b_i \in \mathfrak{B}$ for each i. For each i, we can express a_i, b_i as $E_i^\bullet$, $F_i^\bullet$ where $E_i \in \Sigma$ and $F_i \in \mathrm{T}$; moreover, we can do this in such a way that $\langle E_i \rangle_{i \in I}$ is a partition of X. In this case, $\phi e = W^\bullet$ where $W = \bigcup_{i \in I} E_i \times F_i$, so that, for $\epsilon > 0$,

$$\mu\{x : \nu W[\{x\}] > \epsilon\} = \mu(\bigcup\{E_i : i \in I, \nu F_i > \epsilon\}) = \bar\mu(\sup\{a_i : i \in I, \bar\nu b_i > \epsilon\}).$$

Accordingly

$$\begin{aligned}(\mu \ltimes \nu)(e) &= \inf\{\epsilon : \bar\mu(\sup\{a_i : i \in I, \bar\nu b_i > \epsilon\}) \le \epsilon\}\\ &= \inf\{\epsilon : \mu\{x : \nu W[\{x\}] > \epsilon\} \le \epsilon\} = \lambda W = \bar\lambda W^\bullet = \bar\lambda(\phi e). \quad \mathbf{Q}\end{aligned}$$

Next, $\phi[\mathfrak{A} \otimes \mathfrak{B}]$ is dense in $\mathfrak{C}$ for the metric induced by $\bar\lambda$. **P** Let $\mathfrak{D}$ be the metric closure of $\phi[\mathfrak{A} \otimes \mathfrak{B}]$ and set $\mathcal{V} = \{V : V \in \Sigma \otimes \mathrm{T}, V^\bullet \in \mathfrak{D}\}$. Then $\mathcal{V}$ includes $\Sigma \otimes \mathrm{T}$ and is closed under unions and intersections of monotonic sequences, so is the whole of $\Sigma \widehat{\otimes} \mathrm{T}$, and $\mathfrak{D} = \mathfrak{C}$, as required. **Q** But this means that we can identify $\mathfrak{C}$ with the metric completions of $\phi[\mathfrak{A} \otimes \mathfrak{B}]$ and $\mathfrak{A} \otimes \mathfrak{B}$.

496X Basic exercises (a) Let X and Y be Hausdorff spaces, μ a totally finite Radon submeasure on X, and $f : X \to Y$ a function which is almost continuous in the sense that for every $\epsilon > 0$ there is a compact $K \subseteq X$ such that $f{\restriction}K$ is continuous and $\mu(X \setminus K) \le \epsilon$. Show that the image submeasure μf^{-1}, defined on $\{F : F \subseteq Y, f^{-1}[F] \in \operatorname{dom}\mu\}$, is a totally finite Radon submeasure on Y.

(b) Let X be a Hausdorff space and μ a totally finite Radon submeasure on X. For $A \subseteq X$, set $\mu^* A = \inf\{\mu E : A \subseteq E \in \operatorname{dom}\mu\}$. Show that μ^* is an outer regular Choquet capacity on X.

(c) Let X and Y be compact Hausdorff spaces, $f : X \to Y$ a continuous surjection, and ν a totally finite Radon submeasure on Y. Show that there is a totally finite Radon submeasure μ on X such that ν is the image submeasure μf^{-1}.

(d) Let X be a regular K-analytic Hausdorff space, and μ a Maharam submeasure on the Borel σ-algebra of X which is τ-additive in the sense that whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets in X with union H, then $\inf_{G \in \mathcal{G}} \mu(H \setminus G) = 0$. Show that the completion of μ is a totally finite Radon submeasure on X. (*Hint*: let Σ_0 be the algebra of subsets of X generated by the compact sets; show that there is a totally finite Radon submeasure extending $\mu{\restriction}\Sigma_0$.)

496Y Further exercises In the following exercises, I will say that a **Radon submeasure** is a complete submeasure μ on a Hausdorff space X such that (i) the domain Σ of μ is a σ-algebra of subsets of X containing every open set (ii) every point of X belongs to an open set G such that $\mu G < \infty$ (iii)(α) $\mu E = \sup\{\mu K : K \subseteq E$ is compact$\}$ for every $E \in \Sigma$ (β) $\inf\{\mu(E \setminus K) : K \subseteq E$ is compact$\} = 0$ whenever $E \in \Sigma$ and $\mu E < \infty$ (iv) if $E \subseteq X$ is such that $E \cap K \in \Sigma$ for every compact $K \subseteq X$, then $E \in \Sigma$.

(a) Let μ be a Radon submeasure with domain Σ and null ideal $\mathcal{N}(\mu)$. Show that $\Sigma/\mathcal{N}(\mu)$ is Dedekind complete.

[10] Formerly 315I-315J.

(b) Let X be a Hausdorff space, Y a metrizable space, μ a Radon submeasure on X with domain Σ, and $f : X \to Y$ a Σ-measurable function. Let $\mathcal{H}$ be the family of those $H \in \Sigma$ such that $f{\restriction}H$ is continuous. Show that (α) $\mu E = \sup\{\mu H : H \in \mathcal{H}, H \subseteq E\}$ for every $E \in \Sigma$ (β) $\inf\{\mu(E \setminus H) : H \in \mathcal{H}, H \subseteq E\} = 0$ whenever $E \in \Sigma$ and $\mu E < \infty$.

(c) Let X and Y be Hausdorff spaces, μ a Radon submeasure on X with domain Σ, and $f : X \to Y$ a function. Let $\mathcal{F}$ be the family of those $F \in \Sigma$ such that $f{\restriction}F$ is continuous, and suppose that (α) $\mu E = \sup\{\mu F : F \in \mathcal{F}, F \subseteq E\}$ for every $E \in \Sigma$ (β) $\inf\{\mu(E \setminus F) : F \in \mathcal{F}, F \subseteq E\} = 0$ whenever $E \in \Sigma$ and $\mu E < \infty$. (i) Show that the image submeasure $\nu = \mu f^{-1}$, defined on $\{F : F \subseteq Y, f^{-1}[F] \in \Sigma\}$, is a submeasure on Y defined on a σ-algebra of sets containing every open subset of Y. (ii) Show that if ν is locally finite in the sense that $Y = \bigcup\{H : H \subseteq Y$ is open, $\nu H < \infty\}$, then ν is a Radon submeasure.

(d) Let X be a Hausdorff space and μ a Radon submeasure on X which is either submodular or supermodular. Show that there is a Radon measure on X with the same domain and null ideal as μ. (*Hint*: 413Yf.)

(e) Let X be a topological space, $\mathcal{G}$ the family of cozero subsets of X, $\mathcal{B}\mathfrak{a}$ the Baire σ-algebra of X and $\psi : \mathcal{G} \to [0, \infty[$ a functional. Show that ψ can be extended to a Maharam submeasure with domain $\mathcal{B}\mathfrak{a}$ iff

(α) $\psi G \le \psi H$ whenever $G, H \in \mathcal{G}$ and $G \subseteq H$,

(β) $\psi(\bigcup_{n\in\mathbb{N}} G_n) \le \sum_{n=0}^{\infty} \psi G_n$ for every sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$,

(γ) $\lim_{n\to\infty} \psi G_n = 0$ for every non-increasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ with empty intersection,

and that in this case the extension is unique. (*Hint*: consider the family of sets $E \subseteq X$ such that for every $\epsilon > 0$ there are a cozero set $G \supseteq E$ and a zero set $F \subseteq E$ such that $\psi(G \setminus F) \le \epsilon$.)

(f) Let X be a Hausdorff space and $\mathcal{K}$ the family of compact subsets of X. Let $\phi : \mathcal{K} \to [0, \infty[$ be a functional such that

(α) $\phi\emptyset = 0$ and $\phi K \le \phi(K \cup L) \le \phi K + \phi L$ for all $K, L \in \mathcal{K}$;

(β) for every disjoint sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$, either $\lim_{n\to\infty} \phi K_n = 0$ or $\lim_{n\to\infty} \phi(\bigcup_{i\le n} K_i) = \infty$;

(γ) whenever $K, L \in \mathcal{K}$ and $K \subseteq L$ then $\phi L \le \phi K + \sup\{\phi K' : K' \in \mathcal{K}, K' \subseteq L \setminus K\}$;

(δ) for every $x \in X$ there is an open set G containing x such that $\sup\{\phi K : K \in \mathcal{K}, K \subseteq G\}$ is finite.

Show that there is a unique Radon submeasure on X extending ϕ.

496 Notes and comments 'Submeasures' turn up in all sorts of places, if you are looking out for them; so, as always, I have tried to draw my definitions as wide as practicable. When we come to 'Maharam' and 'Radon' submeasures, however, we certainly want to begin with results corresponding to the familiar properties of totally finite measures, and the new language is complex enough without troubling with infinite submeasures. For the main part of this section, therefore, I look only at totally finite submeasures.

I have tried here to give a sample of the ideas from the present volume which can be applied to submeasures as well as to measures. I think they go farther than most of us would take for granted. One key point concerns the definition of inner regularity: to the familiar '$\mu E = \sup\{\mu K : K \in \mathcal{K}, K \subseteq E\}$' we need to add 'if μE is finite, then $\inf\{\mu(E \setminus K) : K \in \mathcal{K}, K \subseteq E\} = 0$' (496C, 496Y; see also condition (β) of 496Yf). Using this refinement, we can repeat a good proportion of the arguments of measure theory which are based on topology and orderings rather than on arithmetic identities.

497 Tao's proof of Szemerédi's theorem

Szemerédi's celebrated theorem on arithmetic progressions (497L) is not obviously part of measure theory. Remarkably, however, it has stimulated significant developments in the subject. The first was Furstenberg's multiple recurrence theorem (Furstenberg 77, Furstenberg 81, Furstenberg & Katznelson 85). In this section I will give an account of an approach due to T.Tao (Tao 07) which introduces another phenomenon of great interest from a measure-theoretic point of view.

497A Definitions (a) Let (X, Σ, μ) be a probability space, T a subalgebra of Σ (*not* necessarily a σ-subalgebra) and $\langle\Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ. I will say that $\langle\Sigma_i\rangle_{i\in I}$ has **T-removable intersections** if whenever $J \subseteq I$ is finite and not empty, $E_i \in \Sigma_i$ for $i \in J$, $\mu(\bigcap_{i\in J} E_i) = 0$ and $\epsilon > 0$, there is a family $\langle F_i\rangle_{i\in J}$ such that $F_i \in \mathrm{T} \cap \Sigma_i$ and $\mu(E_i \setminus F_i) \le \epsilon$ for each $i \in J$, and $\bigcap_{i\in J} F_i = \emptyset$. (This is a stronger version of what Tao 07 calls the 'uniform intersection property'.)

(b) If X is a set and Σ, Σ' are two σ-algebras of subsets of X, $\Sigma\vee\Sigma'$ will be the σ-algebra generated by $\Sigma\cup\Sigma'$. If $\langle\Sigma_i\rangle_{i\in I}$ is a family of σ-algebras of subsets of X, I will write $\bigvee_{i\in I}\Sigma_i$ for the σ-algebra generated by $\bigcup_{i\in I}\Sigma_i$.

(c) If (X,Σ,μ) is a probability space and $\mathcal{A}\subseteq\mathcal{E}\subseteq\Sigma$, I will say that $\mathcal{A}$ is **metrically dense** in $\mathcal{E}$ if for every $E\in\mathcal{E}$ and $\epsilon>0$ there is an $F\in\mathcal{A}$ such that $\mu(E\triangle F)\le\epsilon$; that is, if $\{F^\bullet:F\in A\}$ is dense in $\{E^\bullet:E\in\mathcal{E}\}$ for the measure-algebra topology on the measure algebra of μ (323A). Note that a subalgebra of Σ is metrically dense in the σ-algebra it generates (compare 323J).

497B Lemma Let (X,Σ,μ) be a probability space and T a subalgebra of Σ. Let $\langle\Sigma_i\rangle_{i\in I}$ be a family of σ-subalgebras of Σ.

(a) $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections iff $\langle\Sigma_i\rangle_{i\in J}$ has T-removable intersections for every finite $J\subseteq I$.

(b) Suppose that $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections and that $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for every i. Let J be any set and $f:J\to I$ a function. Then $\langle\Sigma_{f(j)}\rangle_{j\in J}$ has T-removable intersections.

(c) Suppose that, for each $i\in I$, we are given a σ-subalgebra Σ'_i of Σ_i such that for every $E\in\Sigma_i$ there is an $E'\in\Sigma'_i$ such that $E\triangle E'$ is negligible. If $\langle\Sigma'_i\rangle_{i\in I}$ has T-removable intersections, so has $\langle\Sigma_i\rangle_{i\in I}$.

proof (a) is trivial.

(b) Suppose that $K\subseteq J$ is finite and not empty, that $\langle E_j\rangle_{j\in K}\in\prod_{j\in K}\Sigma_{f(j)}$ is such that $\mu(\bigcap_{j\in K}E_j)=0$, and $\epsilon>0$. Set $n=\#(K)$ and $\eta=\frac{\epsilon}{n+2}>0$. Set $E'_i=\bigcap_{j\in K,f(j)=i}E_j\in\Sigma_i$ for $i\in f[K]$; then $\bigcap_{i\in f[K]}E'_i=\bigcap_{j\in K}E_j$ is negligible, so we have $F'_i\in\mathrm{T}\cap\Sigma_i$, for $i\in f[K]$, such that $\bigcap_{i\in f[K]}F'_i=\emptyset$ and $\mu(E'_i\setminus F'_i)\le\eta$ for every $i\in f[K]$. As $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for each i, we can find $G_j\in\mathrm{T}\cap\Sigma_{f(j)}$ such that $\mu(E_j\triangle G_j)\le\eta$ for each $j\in K$. Set $G'_i=\bigcap_{j\in K,f(j)=i}G_j$ for $i\in f[K]$. Then

$$\mu(G'_i\setminus F'_i)\le\mu(G'_i\setminus E'_i)+\mu(E'_i\setminus F'_i)\le\sum_{j\in K,f(j)=i}\mu(G_j\setminus E_j)+\eta\le(n+1)\eta.$$

Note that $G'_i\in\mathrm{T}\cap\Sigma_i$ for each i. Now set $F_j=G_j\setminus(G'_{f(j)}\setminus F'_{f(j)})$ for $j\in K$. Then $F_j\in\mathrm{T}\cap\Sigma_{f(j)}$ and

$$\mu(E_j\setminus F_j)\le\mu(E_j\setminus G_j)+\mu(G'_{f(j)}\setminus F'_{f(j)})\le(n+2)\eta=\epsilon.$$

Also

$$\begin{aligned}\bigcap_{j\in K}F_j&=\bigcap_{i\in f[K]}\bigcap_{\substack{j\in K\\ f(j)=i}}G_j\setminus(G'_i\setminus F'_i)\\&=\bigcap_{i\in f[K]}G'_i\setminus(G'_i\setminus F'_i)\subseteq\bigcap_{i\in f[K]}F'_i=\emptyset.\end{aligned}$$

As $\langle E_j\rangle_{j\in K}$ and ϵ are arbitrary, $\langle\Sigma_{f(j)}\rangle_{j\in J}$ has T-removable intersections.

(c) If $J\subseteq I$ is finite and not empty, $\langle E_j\rangle_{j\in J}\in\prod_{j\in J}\Sigma_j$, $\bigcap_{j\in J}E_j$ is negligible and $\epsilon>0$, then for each $j\in J$ let $E'_j\in\Sigma'_j$ be such that $E'_j\triangle E_j$ is negligible. In this case, $\bigcap_{j\in J}E'_j$ is negligible, so there are $F_j\in\mathrm{T}\cap\Sigma'_j$, for $j\in J$, such that $\mu(E'_j\setminus F_j)\le\epsilon$ for every $j\in J$ and $\bigcap_{j\in J}F_j$ is empty. Now $\mu(E_j\setminus F_j)\le\epsilon$ for every j. As $\langle E_j\rangle_{j\in J}$ and ϵ are arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

497C Lemma Let (X,Σ,μ) be a probability space and T a subalgebra of Σ. Let I be a set, A an upwards-directed set, and $\langle\Sigma_{\alpha i}\rangle_{\alpha\in A,i\in I}$ a family of σ-subalgebras of Σ such that, setting $\Sigma_i=\bigvee_{\alpha\in A}\Sigma_{\alpha i}$ for each i,

(i) $\Sigma_{\alpha i}\subseteq\Sigma_{\beta i}$ whenever $i\in I$ and $\alpha\le\beta$ in A,

(ii) $\langle\Sigma_{\alpha i}\rangle_{i\in I}$ has T-removable intersections for every $\alpha\in A$,

(iii) Σ_i and $\bigvee_{j\in I}\Sigma_{\alpha j}$ are relatively independent over $\Sigma_{\alpha i}$ for every $i\in I$ and $\alpha\in A$.

Then $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

proof Take a non-empty finite set $J\subseteq I$, a family $\langle E_i\rangle_{i\in J}$ such that $E_i\in\Sigma_i$ for every $i\in J$ and $\bigcap_{i\in J}E_i$ is negligible, and $\epsilon>0$. Set $\delta=\frac{1}{\#(J)+1}$, $\eta=\delta\sqrt{\frac{\epsilon}{2}}>0$. For each $i\in J$ there are an $\alpha\in A$ and an $E'_i\in\Sigma_{\alpha i}$ such that $\mu(E_i\triangle E'_i)\le\eta^2$ (because A is upwards-directed, so $\bigcup_{\alpha\in A}\Sigma_{\alpha i}$ is a subalgebra of Σ and is metrically dense in Σ_i); we can suppose that it is the same α for each i. Let $g_i:X\to[0,1]$ be a $\Sigma_{\alpha i}$-measurable function which is a conditional expectation of χE_i on $\Sigma_{\alpha i}$; then

$$\|\chi E_i-g_i\|_2\le\|\chi E_i-\chi E'_i\|_2\le\eta$$

(cf. 244Nb). Set $E_i'' = \{x : g_i(x) \ge 1-\delta\} \in \Sigma_{\alpha i}$; then

$$\mu(E_i \setminus E_i'') = \mu\{x : \chi E_i(x) - g_i(x) > \delta\} \le \frac{\eta^2}{\delta^2} = \frac{\epsilon}{2}.$$

Set $E = \bigcap_{i\in J} E_i''$. Then $\mu E = 0$. **P** Since $\mu(\bigcap_{i\in J} E_i) = 0$, $\mu E \le \sum_{i\in J}\mu(E\setminus E_i)$. For $i \in J$, set $H_i = X \cap \bigcap_{j\in J\setminus\{i\}} E_j''$ and let h_i be a conditional expectation of χH_i on $\Sigma_{\alpha i}$. Then $X\setminus E_i \in \Sigma_i$ and $H_i \in \bigvee_{j\in I}\Sigma_{\alpha j}$ are relatively independent over $\Sigma_{\alpha i}$, while $\chi X - g_i$ is a conditional expectation of $\chi(X\setminus E_i)$ on $\Sigma_{\alpha i}$, so

$$\mu(E\setminus E_i) = \mu((E_i''\setminus E_i)\cap H_i) = \int_{E_i''} \chi(X\setminus E_i)\times\chi H_i d\mu = \int_{E_i''}(\chi X - g_i)\times h_i d\mu$$

(by the definition of 'relative independence', 458Aa)

$$= \int (\chi X - g_i)\times \chi E_i''\times h_i d\mu \le \int \delta\chi E_i''\times h_i d\mu$$

(by the definition of E_i'')

$$= \delta\int_{E_i''} h_i d\mu = \delta\mu(E_i''\cap H_i) = \delta\mu E.$$

Summing, we have

$$\mu E \le \delta\#(J)\mu E;$$

but $\delta\#(J) < 1$, so $\mu E = 0$. **Q**

Because $\langle\Sigma_{\alpha i}\rangle_{i\in I}$ has T-removable intersections, there are $F_i \in \mathrm{T}\cap\Sigma_{\alpha i} \subseteq \mathrm{T}\cap\Sigma_i$, for $i \in J$, such that $\bigcap_{i\in I} F_i = \emptyset$ and $\mu(E_i''\setminus F_i) \le \frac{\epsilon}{2}$ for each i; in which case $\mu(E_i \setminus F_i) \le \epsilon$ for each i. As $\langle E_i\rangle_{i\in J}$ and ϵ are arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections.

497D Lemma Let (X,Σ,μ) be a probability space, T a subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a finite family of σ-subalgebras of Σ which has T-removable intersections; suppose that $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for each i. Set $\Sigma^* = \bigvee_{i\in I}\Sigma_i$. Suppose that we have a finite set Γ, a function $g : \Gamma \to I$ and a family $\langle\Lambda_\gamma\rangle_{\gamma\in\Gamma}$ of σ-subalgebras of Σ such that

$\langle\Lambda_\gamma\rangle_{\gamma\in\Gamma}$ is relatively independent over Σ^*,

for each $\gamma\in\Gamma$, Λ_γ and Σ^* are relatively independent over $\Sigma_{g(\gamma)}$,

for each $\gamma\in\Gamma$, $\mathrm{T}\cap\Lambda_\gamma$ is metrically dense in Λ_γ.

Let A be a finite set and $f : A \to I$, $\phi : A \to \mathcal{P}\Gamma$ functions such that $\Sigma_{g(\gamma)} \subseteq \Sigma_{f(\alpha)}$ whenever $\alpha\in A$ and $\gamma\in\phi(\alpha)$. Suppose that

for each $\alpha\in A$, $\bigvee_{\gamma\in\phi(\alpha)}\Lambda_\gamma$ and $\Sigma^*\vee\bigvee_{\gamma\in\Gamma\setminus\phi(\alpha)}\Lambda_\gamma$ are relatively independent over $\Sigma_{f(\alpha)}$.

Set $\tilde\Sigma_\alpha = \Sigma_{f(\alpha)}\vee\bigvee_{\gamma\in\phi(\alpha)}\Lambda_\gamma$ for $\alpha\in A$. Then $\langle\tilde\Sigma_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

proof Of course we can suppose that A is non-empty, and that $\Gamma = \bigcup_{\alpha\in A}\phi(\alpha)$.

(a) To begin with, suppose that every Λ_γ is actually a finite subalgebra of T.

(i) Take a non-empty set $B\subseteq A$, a family $\langle E_\alpha\rangle_{\alpha\in B} \in \prod_{\alpha\in B}\tilde\Sigma_\alpha$ such that $\bigcap_{\alpha\in B} E_\alpha$ is negligible, and $\epsilon > 0$. Set $\Delta = \bigcup_{\alpha\in B}\phi(\alpha)$. Let $\mathcal{A}$ be the set of atoms of $\bigvee_{\gamma\in\Delta}\Lambda_\gamma$ and set $\eta = \frac{\epsilon}{\#(\mathcal{A})} > 0$.

(ii) For each $H\in\mathcal{A}$ and $\alpha\in B$, let $C(H,\alpha)$ be the atom of $\bigvee_{\gamma\in\phi(\alpha)}\Lambda_\gamma$ including H. Then there is a family $\langle F_{H\alpha}\rangle_{\alpha\in B}$, with empty intersection, such that $F_{H\alpha}\in\mathrm{T}\cap\Sigma_\alpha$ and $\mu(E_\alpha\cap C(H,\alpha)\setminus F_{H\alpha}) \le \eta$ for each $\alpha\in B$. **P** For each $\gamma\in\Delta$, let H_γ be the atom of Λ_γ including H, $h_\gamma : X \to [0,1]$ a $\Sigma_{g(\gamma)}$-measurable function which is a conditional expectation of χH_γ on $\Sigma_{g(\gamma)}$, and $G_\gamma = \{x : h_\gamma(x) > 0\}$. Note that

$$C(H,\alpha) = X\cap\bigcap\nolimits_{\gamma\in\phi(\alpha)} H_\gamma \text{ for every } \alpha\in B,$$

$$H = X\cap\bigcap\nolimits_{\gamma\in\Delta} H_\gamma = \bigcap\nolimits_{\alpha\in B} C(H,\alpha).$$

Because Λ_γ and Σ^* are relatively independent over $\Sigma_{g(\gamma)}$, and $\Sigma_{g(\gamma)}\subseteq\Sigma^*$, h_γ is a conditional expectation of χH_γ on Σ^* for each γ (458Fb). Because $\langle\Lambda_\gamma\rangle_{\gamma\in\Delta}$ is relatively independent over Σ^*, $h = \prod_{\gamma\in\Delta} h_\gamma$ is a conditional expectation of $\chi H = \chi(X\cap\bigcap_{\gamma\in\Delta} H_\gamma)$ on Σ^*. (For the trivial case in which $\Delta=\emptyset$, take $h = \chi X$.) For each $\alpha\in B$

we have $E_\alpha \in \Sigma_{f(\alpha)} \vee \bigvee_{\gamma\in\phi(\alpha)} \Lambda_\gamma$, so there is an an $E'_\alpha \in \Sigma_{f(\alpha)}$ such that $E_\alpha \cap C(H,\alpha) = E'_\alpha \cap C(H,\alpha)$. Now $\bigcap_{\alpha\in B} E'_\alpha \in \Sigma^*$, so

$$\int_{\bigcap_{\alpha\in B} E'_\alpha} h = \mu(\bigcap_{\alpha\in B} E'_\alpha \cap \bigcap_{\gamma\in\Delta} H_\gamma) \le \mu(\bigcap_{\alpha\in B}(E_\alpha \cap C(H,\alpha))) = 0;$$

accordingly $\bigcap_{\alpha\in B} E'_\alpha \cap \bigcap_{\gamma\in\Delta} G_\gamma$ is negligible. Set $E''_\alpha = E'_\alpha \cap \bigcap_{\gamma\in\phi(\alpha)} G_\gamma$ for each $\alpha \in B$; then $E''_\alpha \in \Sigma_{f(\alpha)}$, because $G_\gamma \in \Sigma_{g(\gamma)} \subseteq \Sigma_{f(\alpha)}$ whenever $\gamma \in \phi(\alpha)$. Also $\bigcap_{\alpha\in B} E''_\alpha$ is negligible.

Because $\langle\Sigma_i\rangle_{i\in I}$ has T-removable intersections and $\mathrm{T}\cap\Sigma_i$ is metrically dense in Σ_i for each i, $\langle\Sigma_{f(\alpha)}\rangle_{\alpha\in B}$ has T-removable intersections (497Bb). So we have $F_{H\alpha} \in \mathrm{T}\cap\Sigma_{f(\alpha)}$, for $\alpha \in B$, such that $\bigcap_{\alpha\in A} F_{H\alpha} = \emptyset$ and $\mu(E''_\alpha \setminus F_{H\alpha}) \le \eta$ for every α.

If $\alpha \in B$ and $\gamma \in \phi(\alpha)$,

$$0 = \int_{E'_\alpha\setminus G_\gamma} h_\gamma = \mu(H_\gamma \cap E'_\alpha \setminus G_\gamma)$$

because h_γ is a conditional expectation of χH_γ on Σ^*. So if $\alpha \in B$,

$$\begin{aligned}\mu(E_\alpha \cap C(H,\alpha) \setminus F_{H\alpha}) &= \mu(E'_\alpha \cap C(H,\alpha) \setminus F_{H\alpha})\\ &\le \mu(E''_\alpha \setminus F_{H\alpha}) + \sum_{\gamma\in\phi(\alpha)} \mu(E'_\alpha \cap H_\gamma \setminus G_\gamma)\end{aligned}$$

(because $C(H,\alpha) = X \cap \bigcap_{\gamma\in\phi(\alpha)} H_\gamma$)

$$\le \eta,$$

as required. **Q**

(iii) For $\alpha \in B$ let $\mathcal{A}_\alpha$ be the set of atoms of $\bigvee_{\gamma\in\phi(\alpha)} \Lambda_\gamma$ and set

$$F_\alpha = \bigcup_{G\in\mathcal{A}_\alpha} \bigcap_{H\in\mathcal{A}, H\subseteq G} F_{H\alpha}.$$

Then $F_\alpha \in \mathrm{T}\cap\tilde\Sigma_\alpha$ and

$$\begin{aligned}\mu(E_\alpha \setminus F_\alpha) &= \sum_{G\in\mathcal{A}_\alpha} \mu(E_\alpha \cap G \setminus F_\alpha)\\ &\le \sum_{G\in\mathcal{A}_\alpha} \sum_{\substack{H\in\mathcal{A}\\ H\subseteq G}} \mu(E_\alpha \cap G \setminus F_{H\alpha})\\ &= \sum_{H\in\mathcal{A}} \mu(E_\alpha \cap C(H,\alpha) \setminus F_{H\alpha}) \le \eta\#(\mathcal{A}) = \epsilon.\end{aligned}$$

If $H \in \mathcal{A}$ then $H \subseteq C(H,\alpha) \in \mathcal{A}_\alpha$ and

$$H \cap F_\alpha \subseteq F_\alpha \cap C(H,\alpha) \subseteq F_{H\alpha},$$

for each α. So $H \cap \bigcap_{\alpha\in A} F_\alpha$ is empty. But $X = \bigcup\mathcal{A}$ so $\bigcap_{\alpha\in A} F_\alpha = \emptyset$. As $\langle E_\alpha\rangle_{\alpha\in B}$ and ϵ are arbitrary, $\langle\tilde\Sigma_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

(b) Next, suppose that each Λ_γ is the σ-algebra generated by $\mathrm{T}\cap\Lambda_\gamma$.

(i) For $L \in [\mathrm{T}]^{<\omega}$, $\gamma \in \Gamma$ and $\alpha \in A$ write $\Lambda_{L\gamma}$ for the algebra σ-generated by $\Lambda_\gamma \cap L$ and $\tilde\Sigma_{L\alpha} = \Sigma_{f(\alpha)} \vee \bigvee_{\gamma\in\phi(\alpha)} \Lambda_{L\gamma}$. Then

$$\tilde\Sigma_{\Delta\alpha} \subseteq \tilde\Sigma_{L\alpha} \text{ whenever } \alpha \in A \text{ and } \Delta \subseteq L \in [\mathrm{T}]^{<\omega},$$

$$\bigvee_{L\in[\mathrm{T}]^{<\omega}} \tilde\Sigma_{L\alpha} = \Sigma_{f(\alpha)} \vee \bigvee_{\gamma\in\phi(\alpha)} \bigvee_{L\in[\mathrm{T}]^{<\omega}} \Lambda_{L\gamma} = \Sigma_{f(\alpha)} \vee \bigvee_{\gamma\in\phi(\alpha)} \Lambda_\gamma = \tilde\Sigma_\alpha$$

because each Λ_γ is the σ-algebra generated by $\mathrm{T}\cap\Lambda_\gamma = \bigcup_{L\in[\mathrm{T}]^{<\omega}} \Lambda_{L\gamma}$. By (a), $\langle\tilde\Sigma_{L\alpha}\rangle_{\alpha\in A}$ has T-removable intersections for every $L \in [\mathrm{T}]^{<\omega}$.

(ii) Suppose that $\alpha \in A$ and $L \in [\mathrm{T}]^{<\omega}$. Then $\bigvee_{\gamma\in\phi(\alpha)} \Lambda_\gamma$ and $\Sigma^* \vee \bigvee_{\gamma\in\Gamma\setminus\phi(\alpha)} \Lambda_\gamma$ are relatively independent over $\Sigma_{f(\alpha)}$, by hypothesis. So $\tilde\Sigma_\alpha = \Sigma_{f(\alpha)} \vee \bigvee_{\gamma\in\phi(\alpha)} \Lambda_\gamma$ and $\Sigma^* \vee \bigvee_{\gamma\in\Gamma\setminus\phi(\alpha)} \Lambda_\gamma$ are relatively independent over $\Sigma_{f(\alpha)}$

(458Db). Because $\Sigma_{f(\alpha)} \subseteq \tilde{\Sigma}_{L\alpha} \subseteq \tilde{\Sigma}_\alpha$, $\tilde{\Sigma}_\alpha$ and $\Sigma^* \vee \bigvee_{\gamma\in\Gamma\setminus\phi(\alpha)} \Lambda_\gamma$ are relatively independent over $\tilde{\Sigma}_{L\alpha}$ (458Dc). Because $\bigvee_{\gamma\in\phi(\alpha)} \Lambda_{L\gamma} \subseteq \tilde{\Sigma}_{L\alpha}$, $\tilde{\Sigma}_\alpha$ and $\Sigma^* \vee \bigvee_{\gamma\in\Gamma} \Lambda_{L\gamma}$ are relatively independent over $\tilde{\Sigma}_{L\alpha}$ (458Db again). So $\tilde{\Sigma}_\alpha$ and

$$\bigvee_{\beta\in A} \tilde{\Sigma}_{L\beta} = \bigvee_{\beta\in A}(\Sigma_{f(\beta)} \vee \bigvee_{\gamma\in\phi(\beta)} \Lambda_{L\gamma}) \subseteq \Sigma^* \vee \bigvee_{\gamma\in\Gamma} \Lambda_{L\gamma}$$

are relatively independent over $\tilde{\Sigma}_{L\alpha}$.

(iii) With (i), this shows that the family $\langle\tilde{\Sigma}_{L\alpha}\rangle_{L\in[\mathrm{T}]^{<\omega},\alpha\in A}$ satisfies the conditions of 497C, and $\langle\tilde{\Sigma}_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

(c) Finally, for the general case, let Λ'_γ be the σ-algebra generated by $\Lambda_\gamma \cap \mathrm{T}$ for $\gamma \in \Gamma$, and $\Sigma'_\alpha = \Sigma_{f(\alpha)} \vee \bigvee_{\gamma\in\phi(\alpha)} \Lambda'_\gamma$ for $\alpha \in A$. If $\gamma \in \Gamma$ and $F \in \Lambda_\gamma$, there is an $F' \in \Lambda'_\gamma$ such that $F\triangle F'$ is negligible; so if $\alpha \in A$ and $E \in \tilde{\Sigma}_\alpha$, there is an $E' \in \Sigma'_\alpha$ such that $E\triangle E'$ is negligible. By (b), $\langle\Sigma'_\alpha\rangle_{\alpha\in A}$ has T-removable intersections; by 497Bc, $\langle\tilde{\Sigma}_\alpha\rangle_{\alpha\in A}$ has T-removable intersections.

497E Theorem (Tao 07) Let (X, Σ, μ) be a probability space, and T a subalgebra of Σ. Let Γ be a partially ordered set such that $\gamma \wedge \delta = \inf\{\gamma, \delta\}$ is defined in Γ for all γ, $\delta \in \Gamma$, and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ a family of σ-subalgebras of Σ such that

(i) $\mathrm{T} \cap \Sigma_\gamma$ is metrically dense in Σ_γ for every $\gamma \in \Gamma$,

(ii) if γ, $\delta \in \Gamma$ and $\gamma \le \delta$ then $\Sigma_\gamma \subseteq \Sigma_\delta$,

(iii) if $\gamma \in \Gamma$ and Δ, Δ' are finite subsets of Γ such that $\delta \wedge \gamma \in \Delta'$ for every $\delta \in \Delta$, then Σ_γ and $\bigvee_{\delta\in\Delta} \Sigma_\delta$ are relatively independent over $\bigvee_{\delta\in\Delta'} \Sigma_\delta$.

Then $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections.

proof (a) To begin with (down to the end of (d) below) suppose that Γ is finite. In this case, we have a rank function $r : \Gamma \to \mathbb{N}$ such that $r(\gamma) = \min\{n : n \in \mathbb{N}, r(\delta) < n$ for every $\delta < \gamma\}$ for each $\gamma \in \Gamma$. For $a \subseteq \Gamma$ set $\tilde{\Sigma}_a = \bigvee_{\gamma\in a} \Sigma_\gamma$; note that $\mathrm{T} \cap \tilde{\Sigma}_a$ is always metrically dense in $\tilde{\Sigma}_a$.

Let A be the family of those sets $a \subseteq \Gamma$ such that $\gamma \in a$ whenever $\gamma \le \delta \in a$. For $n \in \mathbb{N}$ set $\Gamma_n = \{\gamma : r(\gamma) = n\}$ and $A_n = \{a : a \in A, r(\gamma) < n$ for every $\gamma \in a\}$

(b) Suppose that a, b, c are subsets of Γ and that $\gamma \wedge \delta \in c$ whenever $\gamma \in a$ and $\delta \in b \cup (a \setminus \{\gamma\})$. Then

(i) $\langle\Sigma_\gamma\rangle_{\gamma\in a}$ is relatively independent over $\tilde{\Sigma}_c$,

(ii) $\tilde{\Sigma}_a$ and $\tilde{\Sigma}_b$ are relatively independent over $\tilde{\Sigma}_c$.

P Induce on $\#(a)$. If $a = \emptyset$ then $\tilde{\Sigma}_a = \{\emptyset, X\}$ and the result is trivial. For the inductive step, take $\gamma_0 \in a$ and set $a' = a\setminus\{\gamma_0\}$. Then the inductive hypothesis tells us that $\langle\Sigma_\gamma\rangle_{\gamma\in a'}$ is relatively independent over $\tilde{\Sigma}_c$ and that $\tilde{\Sigma}_{a'}$ and $\tilde{\Sigma}_b$ are relatively independent over $\tilde{\Sigma}_c$. We also see that $\gamma_0 \wedge \delta \in c$ whenever $\delta \in a'$, so that Σ_{γ_0} and $\tilde{\Sigma}_{a'}$ are relatively independent over $\tilde{\Sigma}_c$, by condition (iii) of this theorem. But this means that $\langle\Sigma_\gamma\rangle_{\gamma\in a}$ is relatively independent over $\tilde{\Sigma}_c$ (458Hb). Similarly, because in fact $\gamma_0 \wedge \delta \in c$ for every $\delta \in a' \cup b$, Σ_{γ_0} and $\tilde{\Sigma}_{a'} \vee \tilde{\Sigma}_b$ are relatively independent over $\tilde{\Sigma}_c$; so the triple Σ_{γ_0}, $\tilde{\Sigma}_{a'}$ and $\tilde{\Sigma}_b$ are relatively independent over $\tilde{\Sigma}_c$ (458Hb again), and $\tilde{\Sigma}_a = \Sigma_{\gamma_0} \vee \tilde{\Sigma}_{a'}$ and $\tilde{\Sigma}_b$ are relatively independent over $\tilde{\Sigma}_c$ (458Ha). Thus the induction continues. **Q**

(c) For each $n \in \mathbb{N}$, $\langle\tilde{\Sigma}_a\rangle_{a\in A_n}$ has T-removable intersections. **P** Induce on n. If $n = 0$ then $A_n = \{\emptyset\}$ and the result is trivial. For the inductive step to $n + 1 \ge 1$, apply 497D, as follows. The inductive hypothesis tells us that $\langle\tilde{\Sigma}_a\rangle_{a\in A_n}$ has T-removable intersections, and we know that $\mathrm{T} \cap \tilde{\Sigma}_a$ is always metrically dense in $\tilde{\Sigma}_a$. Set

$$\Sigma^* = \bigvee_{a\in A_n} \tilde{\Sigma}_a = \tilde{\Sigma}_d$$

where $d = \bigcup_{m<n} \Gamma_m$ is the largest member of A_n. Define $g : \Gamma_n \to A_n$ by setting $g(\gamma) = \{\delta : \delta < \gamma\}$. Then $\gamma \wedge \delta \in d$ for all distinct γ, $\delta \in \Gamma_n$, so $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma_n}$ is relatively independent over Σ^*, by (b-i) just above. If $\gamma \in \Gamma_n$ and $\delta \in d$, $\gamma \wedge \delta \in g(\gamma)$, so $\Sigma_\gamma = \tilde{\Sigma}_{\{\gamma\}}$ and $\Sigma^* = \tilde{\Sigma}_d$ are relatively independent over $\tilde{\Sigma}_{g(\gamma)}$, by (b-ii). Of course $\mathrm{T} \cap \Sigma_\gamma$ is metrically dense in Σ_γ for every $\gamma \in \Gamma_n$.

For $a \in A_{n+1}$, set $\phi(a) = a \cap \Gamma_n$ and

$$f(a) = a \setminus \phi(a) = a \cap \bigcup_{m<n} \Gamma_m \in A_n.$$

If $\gamma \in \phi(a)$ then $g(\gamma) \subseteq a$, by the definition of A, so $g(\gamma) \subseteq f(a)$ and $\tilde{\Sigma}_{g(\gamma)} \subseteq \tilde{\Sigma}_{f(a)}$. Finally, by (b-ii), $\bigvee_{\gamma\in\phi(a)} \Sigma_\gamma$ and $\Sigma^* \vee \bigvee_{\gamma\in\Gamma_n\setminus\phi(a)} \Sigma_\gamma$ are relatively independent over $\tilde{\Sigma}_{f(a)}$, because if $\gamma \in \phi(a)$ and $\delta \in d \cup (\Gamma_n \setminus \phi(a))$ then $\gamma \wedge \delta \in g(\gamma) \subseteq f(a)$.

So all the hypotheses of 497D are satisfied, and

$$\langle\tilde{\Sigma}_{f(a)} \vee \bigvee_{\gamma\in\phi(a)} \Sigma_\gamma\rangle_{a\in A_{n+1}} = \langle\tilde{\Sigma}_a\rangle_{a\in A_{n+1}}$$

has T-removable intersections. Thus the induction proceeds. **Q**

(d) Because Γ is finite, there is some n such that $A = A_n$. Now, for each $\gamma \in \Gamma$, set $e_\gamma = \{\delta : \delta \le \gamma\}$; then $e_\gamma \in A$ and $\Sigma_\gamma = \tilde{\Sigma}_{e_\gamma}$. By 497Bb, or otherwise, $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections, as required.

(e) Thus the theorem is true when Γ is finite. For the general case, take any finite $\Gamma_0 \subseteq \Gamma$ and set $\Gamma' = \{\inf a : a \subseteq \Gamma_0$ is non-empty$\}$. Then Γ' is finite and closed under $\wedge$, and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma'}$ satisfies the conditions of the theorem. So $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma'}$ and $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma_0}$ have T-removable intersections. As Γ_0 is arbitrary, $\langle\Sigma_\gamma\rangle_{\gamma\in\Gamma}$ has T-removable intersections (497Ba), and the proof is complete.

497F Invariant measures on $\mathcal{P}([I]^{<\omega})$ (a) Let I be a set. Then $\mathcal{P}([I]^{<\omega})$ is a compact Hausdorff space, if we give it its usual topology, generated by sets of the form $\{R : a \in R \subseteq [I]^{<\omega},\ b \notin R\}$ for finite sets a, $b \subseteq I$. (You should perhaps fix on the case $I = \mathbb{N}$ for the first reading of this paragraph, so that $[I]^{<\omega}$ will be a relatively familiar countable set, and you can remember that $\mathcal{P}([I]^{<\omega})$ is homeomorphic to the Cantor set.) Let G_I be the set of permutations of I, and for $\phi \in G_I$, $R \subseteq [I]^{<\omega}$ set

$$\phi\bullet R = \{\phi[a] : a \in R\} = \{a : a \in [I]^{<\omega},\ \phi^{-1}[a] \in R\},$$

so that $\bullet$ is an action of G_I on $\mathcal{P}([I]^{<\omega})$, and $R \mapsto \phi\bullet R$ is a homeomorphism for every $\phi \in G_I$. Let P_I be the set of Radon probability measures on $\mathcal{P}([I]^{<\omega})$. Then we have an action of G_I on P_I defined by saying that

$$\phi\bullet E = \{\phi\bullet R : R \in E\}$$

for $\phi \in G_I$ and $E \subseteq \mathcal{P}([I]^{<\omega})$, and

$$(\phi\bullet\mu)(E) = \mu(\phi^{-1}\bullet E)$$

for $\phi \in G_I$, $\mu \in P_I$ and Borel sets $E \subseteq \mathcal{P}([I]^{<\omega})$. Because $R \mapsto \phi\bullet R$ is a homeomorphism, the map $\mu \mapsto \phi\bullet\mu$ is a homeomorphism when P_I is given its narrow topology, corresponding to the weak* topology on $C(\mathcal{P}([I]^{<\omega}))^*$ (437J, 437Kc).

(b) If $\mu \in P_I$, I will say that μ is **permutation-invariant** if $\mu = \phi\bullet\mu$ for every $\phi \in G_I$.

(c) For $R \subseteq [I]^{<\omega}$ and $J \subseteq I$ I write $R\lceil J$ for the trace $R \cap \mathcal{P}J \subseteq [J]^{<\omega}$ of R on J. Let $\mathcal{V}$ be the family of sets of the form $V_{JS} =\{R : R \subseteq [I]^{<\omega},\ R\lceil J = S\}$ where $J \subseteq I$ is finite and $S \subseteq \mathcal{P}J$. If μ, $\nu \in P_I$ agree on $\mathcal{V}$, they are equal. **P** If $E \subseteq \mathcal{P}([I])^{<\omega}$ is open-and-closed, it is determined by coordinates in some finite subset $\mathcal{K}$ of $[I]^{<\omega}$, in the sense that if $R \in E$, $R' \subseteq [I]^{<\omega}$ and $R \cap \mathcal{K} = R' \cap \mathcal{K}$, then $R' \in E$. Let $J \subseteq I$ be a finite set such that $\mathcal{K} \subseteq [J]^{<\omega}$, and set $\mathcal{S} = \{R\lceil J : R \in E\}$. Now $\langle V_{JS}\rangle_{S\in\mathcal{S}}$ is a disjoint family in $\mathcal{V}$ with union E, so

$$\mu E = \sum_{S\in\mathcal{S}} \mu V_{JS} = \nu E.$$

As E is arbitrary, $\mu = \nu$ (416Qa). **Q**

(d) If I, J are sets and $f : I \to J$ is a function, I define $\tilde{f} : \mathcal{P}([J]^{<\omega}) \to \mathcal{P}([I]^{<\omega})$ by setting $\tilde{f}(R) = \{a : a \in [I]^{<\omega},\ f[a] \in R\}$ for $R \subseteq [J]^{<\omega}$. Note that $\tilde{f}$ is continuous, since $\{R : a \in \tilde{f}(R)\} = \{R : f[a] \in R\}$ is a basic open-and-closed set in $\mathcal{P}([J]^{<\omega})$ for every $a \in [I]^{<\omega}$. If $I \subseteq J$ and f is the identity function, then $\tilde{f}(R) = R\lceil I$ for every $R \subseteq [J]^{<\omega}$. Observe that when $\phi \in G_I$ and $R \subseteq [I]^{<\omega}$ then $\tilde{\phi}(R) = \phi^{-1}\bullet R$.

497G Theorem (Tao 07) Let I be an infinite set and $\mathcal{J}$ a filter on I not containing any finite set. Let T be the algebra of open-and-closed subsets of $\mathcal{P}([I]^{<\omega})$, and $\mu \in P_I$ a permutation-invariant measure. For $J \subseteq I$, write Σ_J for the σ-algebra of subsets of $\mathcal{P}([I]^{<\omega})$ generated by sets of the form $E_a = \{R : a \in R \subseteq [I]^{<\omega}\}$ where $a \in [J]^{<\omega}$. Then $\langle\Sigma_J\rangle_{J\in\mathcal{J}}$ has T-removable intersections with respect to μ.

proof I seek to apply 497E with $\Gamma = \mathcal{J}$, ordered by $\subseteq$. If $J \in \mathcal{J}$ and $a \in [J]^{<\omega}$ then $\{R : a \in R \subseteq [I]^{<\omega}\}$ belongs to $\mathrm{T} \cap \Sigma_J$; accordingly Σ_J is the σ-algebra generated by $\mathrm{T} \cap \Sigma_J$ and $\mathrm{T} \cap \Sigma_J$ is metrically dense in Σ_J. Condition (ii) of 497E is obviously satisfied. As for condition (iii), we can use 459I, as follows. Taking $X = \mathcal{P}([I]^{<\omega})$, we have the action $\bullet$ of G_I on X described in 497Fa, and $R \mapsto \phi\bullet R$ is inverse-measure-preserving for each ϕ because μ is permutation-invariant. Now we see easily that

— for every $J \subseteq I$, $\bigcup_{K\in[J]^{<\omega}} \Sigma_K$ contains E_a for every $a \in [J]^{<\omega}$, so σ-generates Σ_J;

— if $a \in [I]^{<\omega}$ and $\phi \in G_I$,

$$E_{\phi[a]} = \{R : \phi[a] \in R\} = \{\phi\bullet R : \phi[a] \in \phi\bullet R\} = \{\phi\bullet R : a \in R\} = \phi\bullet E_a;$$

— if $J \subseteq I$, then $\{E : \phi\bullet E \in \Sigma_{\phi[J]}\}$ is a σ-algebra of sets containing $\phi^{-1}\bullet E_{\phi[a]} = E_a$ whenever $a \in [J]^{<\omega}$, so it includes Σ_J, and $\phi\bullet E \in \Sigma_{\phi[J]}$ for every $E \in \Sigma_J$;

— if $J \subseteq I$ and $\phi \in G_I$ is such that $\phi(i) = i$ for every $i \in J$, then $\{E : \phi\bullet E = E\}$ is a σ-algebra of sets containing E_a for every $a \in [J]^{<\omega}$, so it includes Σ_J, and $\phi\bullet E = E$ for every $E \in \Sigma_J$.

Thus the conditions of 459I are satisfied. So if $J \in \mathcal{J}$ and $\mathcal{K}$, $\mathcal{K}'$ are (finite) subsets of $\mathcal{J}$ such that $J \cap K \in \mathcal{K}'$ for every $K \in \mathcal{K}$, 459I tells us that Σ_J and $\bigvee_{K\in\mathcal{K}} \Sigma_K$ are relatively independent over $\bigvee_{K\in\mathcal{K}'} \Sigma_K$, as required by (iii) of 497E.

So 497E gives the result we seek.

497H I come now to the next essential ingredient of the proof.

Construction Suppose we are given a sequence $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ and a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that

(α) $\langle m_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}\setminus\{0\}$ and $\lim_{n\to\mathcal{F}} m_n = \infty$,

(β) $T_n \subseteq \mathcal{P}m_n$ for each n.

Then for any set I there is a permutation-invariant $\mu \in P_I$ such that

$$\mu\{R : R\lceil K = S\} = \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(K)}}\#(\{z : z \in m_n^K,\, \tilde{z}(T_n) = S\})$$

whenever $K \subseteq I$ is finite and $S \subseteq \mathcal{P}K$.

proof (a) For each $n \in \mathbb{N}$ let ν_n be the usual measure on m_n^I, the product of I copies of the uniform probability measure on the finite set m_n. The function $w \mapsto \tilde{w}(T_n) : m_n^I \to \mathcal{P}([I]^{<\omega})$ is continuous, since for any $a \in [I]^{<\omega}$ the set $\{w : a \in \tilde{w}(T_n)\} = \{w : w[a] \in T_n\}$ is determined by coordinates in the finite set a. So we have a corresponding Radon probability measure μ_n on $\mathcal{P}([I]^{<\omega})$ defined by saying that $\mu_n E = \nu_n\{w : \tilde{w}(T_n) \in E\}$ for every set $E \subseteq \mathcal{P}([I]^{<\omega})$ such that ν_n measures $\{w : \tilde{w}(T_n) \in E\}$ (418I). If $K \subseteq I$ is finite and $w \in m_n^I$, then

$$\tilde{w}(T_n)\lceil K = \{a : a \subseteq K,\, w[a] \in T_n\} = \{a : a \subseteq K,\, (w\restriction K)[a] \in T_n\} = (w\restriction K)^{\sim}(T_n).$$

So if $S \subseteq \mathcal{P}K$, then

$$\begin{aligned}\mu_n\{R : R\lceil K = S\} &= \nu_n\{w : w \in m_n^I,\, \tilde{w}(T_n)\lceil K = S\}\\ &= \nu_n\{w : w \in m_n^I,\, (w\restriction K)^{\sim}(T_n) = S\}\\ &= \frac{1}{m_n^{\#(K)}}\#(\{z : z \in m_n^K,\, \tilde{z}(T_n) = S\}).\end{aligned}$$

Let μ be the limit $\lim_{n\to\mathcal{F}} \mu_n$ in the narrow topology on P_I; then

$$\mu\{R : R\lceil K = S\} = \lim_{n\to\mathcal{F}} \mu_n\{R : R\lceil K = S\}$$

(because $\{R : R\lceil K = S\}$ is open-and-closed; see 437Jf)

$$= \lim_{n\to\mathcal{F}} \frac{1}{m_n^{\#(K)}}\#(\{z : z \in m_n^K,\, \tilde{z}(T_n) = S\})$$

whenever $K \subseteq I$ is finite and $S \subseteq \mathcal{P}K$.

(b) Now let $\phi : I \to I$ be any permutation and $\tilde{\phi} : \mathcal{P}([I]^{<\omega}) \to \mathcal{P}([I]^{<\omega})$ the corresponding permutation. Then for any finite $K \subseteq I$,

$$\begin{aligned}\tilde{\phi}(R)\lceil K &= \{a : a \subseteq K,\, a \in \tilde{\phi}(R)\}\\ &= \{a : a \subseteq K,\, \phi[a] \in R\} = \{a : a \in [I]^{<\omega},\, \phi[a] \in R\lceil\phi[K]\}.\end{aligned}$$

Fix $n \in \mathbb{N}$ for the moment. If $K \subseteq I$ is finite, and $S \subseteq \mathcal{P}K$, then

$$\begin{aligned}\mu_n\tilde{\phi}^{-1}\{R : R\lceil K = S\} &= \mu_n\{R : \tilde{\phi}(R)\lceil K = S\}\\ &= \mu_n\{R : S = \{a : a \in [I]^{<\omega},\ \phi[a] \in R\lceil\phi[K]\}\}\\ &= \mu_n\{R : R\lceil\phi[K] = \{\phi[a] : a \in S\}\}\\ &= \nu_n\{w : \tilde{w}(T_n)\lceil\phi[K] = \{\phi[a] : a \in S\}\}\\ &= \nu_n\{w : \{a : a \subseteq \phi[K],\ w[a] \in T_n\} = \{\phi[a] : a \in S\}\}\\ &= \nu_n\{w : \{\phi[a] : a \subseteq K,\ w[\phi[a]] \in T_n\} = \{\phi[a] : a \in S\}\}\\ &= \nu_n\{w : \{a : a \subseteq K,\ (w\phi)[a] \in T_n\} = S\}\\ &= \nu_n\{w : \{a : a \subseteq K,\ w[a] \in T_n\} = S\}\end{aligned}$$

(because $w \mapsto w\phi : m_n^I \to m_n^I$ is an automorphism for the measure ν_n)

$$= \nu_n\{w : \tilde{w}(T_n)\lceil K = S\} = \mu_n\{R : R\lceil K = S\}.$$

So μ_n and $\mu_n\tilde{\phi}^{-1}$ agree on the family $\mathcal{V}$ of basic open-and-closed sets described in 497F. As this is true for every n, we also have $\mu V = \mu\tilde{\phi}^{-1}[V]$ for every $V \in \mathcal{V}$, and $\mu = \mu\tilde{\phi}^{-1}$. As ϕ is arbitrary, μ is permutation-invariant.

497I Definition If I, J are sets, $R \subseteq \mathcal{P}I$ and $S \subseteq \mathcal{P}J$, I will say for the purposes of the next two results that an **embedding** of (I, R) in (J, S) is an injective function $f : I \to J$ such that $f[a] \in S$ for every $a \in R$, that is (when $S \subseteq [J]^{<\omega}$), $R \subseteq \tilde{f}(S)$.

497J Theorem (Nagle Rödl & Schacht 06) Let L be a finite set with r members, and $T \subseteq \mathcal{P}L$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever I is a non-empty finite set, $R \subseteq \mathcal{P}I$ and the number of embeddings of (L, T) in (I, R) is at most $\delta\#(I)^r$, there is an $S \subseteq \mathcal{P}I$ such that $\#(S \cap [I]^k) \le \epsilon\#(I)^k$ for every k and there is no embedding of (L, T) in $(I, R \setminus S)$.

proof (Tao 07) **?** Suppose, if possible, otherwise.

(a) We have a sequence $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ such that

$m_n \in \mathbb{N}\setminus\{0\}$, $T_n \subseteq \mathcal{P}m_n$; the number of embeddings of (L, T) in (m_n, T_n) is at most $2^{-n}m_n^r$; if $S \subseteq \mathcal{P}m_n$ and $\#(S \cap [m_n]^k) \le \epsilon m_n^k$ for every k then there is an embedding of (L, T) in $(m_n, T_n \setminus S)$

for every $n \in \mathbb{N}$. Of course (L, T) always has at least one embedding in (m_n, T_n) so $\lim_{n\to\infty} m_n = \infty$. Let I be an infinite set including L and $\mathcal{F}$ a non-principal ultrafilter on $\mathbb{N}$. Let $\mu \in P_I$ be the permutation-invariant measure defined from $\langle(m_n, T_n)\rangle_{n\in\mathbb{N}}$ and $\mathcal{F}$ by the process of 497H.

(b) For $c \subseteq L$ set $J_c = c \cup (I \setminus L)$, so that Σ_{J_c}, in the notation of 497G, is the σ-algebra of subsets of $\mathcal{P}([I]^{<\omega})$ generated by sets of the form $E_a = \{R : a \in R \subseteq [I]^{<\omega}\}$ where $a \in [c \cup (I \setminus L)]^{<\omega}$. Note that every member of Σ_{J_c} is determined by coordinates in $\mathcal{P}J_c$, in the sense that if $R \in E \in \Sigma_{J_c}$, $R' \subseteq \mathcal{P}([I]^{<\omega})$ and $R \cap \mathcal{P}J_c) = R' \cap \mathcal{P}J_c$, then $R' \in E$.

By 497G, applied to the filter $\mathcal{J}$ on I generated by $\{I \setminus L\}$, $\langle\Sigma_{J_c}\rangle_{c\subseteq L}$ has T-removable intersections with respect to μ, where T is the algebra of open-and-closed subsets of $\mathcal{P}([I]^{<\omega})$. *A fortiori*, $\langle\Sigma_{J_c}\rangle_{c\in T}$ has T-removable intersections with respect to μ.

(c) $E_c \in \Sigma_{J_c}$ for every $c \in T$, and

$$\begin{aligned}\mu(\bigcap_{c\in T} E_c) = \mu\{R : T \subseteq R\} &= \sum_{T\subseteq T'\subseteq\mathcal{P}L} \mu\{R : R\lceil L = T'\}\\ &= \sum_{T\subseteq T'\subseteq\mathcal{P}L} \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z \in m_n^L,\ \tilde{z}(T_n) = T'\})\\ &= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r} \sum_{T\subseteq T'\subseteq\mathcal{P}L} \#(\{z : z \in m_n^L,\ \tilde{z}(T_n) = T'\})\\ &= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z \in m_n^L,\ T \subseteq \tilde{z}(T_n)\})\\ &= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z \in m_n^L \text{ is injective},\ T \subseteq \tilde{z}(T_n)\})\end{aligned}$$

(because $\lim_{n\to\infty} \frac{k_n}{m_n^r} = 0$, where $k_n = m_n^r - \frac{m_n!}{(m_n-r)!}$ is the number of non-injective functions from L to m_n)

$$= \lim_{n\to\mathcal{F}} \frac{1}{m_n^r}\#(\{z : z \text{ is an embedding of } (L,T) \text{ in } (m_n,T_n)\}) = 0.$$

(d) Take $\eta > 0$ such that $2\eta\#(T) < \epsilon$, and $\langle F_c\rangle_{c\in T}$ such that $\bigcap_{c\in T} F_c = \emptyset$ and $F_c \in \mathrm{T}\cap\Sigma_{J_c}$ and $\mu(E_c\setminus F_c) \le \eta$ for every $c \in T$. Every F_c is open-and-closed, so there is an $M \in [I]^{<\omega}$ such that $L \subseteq M$ and every F_c is determined by coordinates in $\mathcal{P}M$. In this case, each F_c is determined by coordinates in $\mathcal{P}M\cap\mathcal{P}J_c = \mathcal{P}(c\cup(M\setminus L))$. Setting

$$F'_c = \{R\lceil M : R \in F_c\},\quad E'_c = \{R\lceil M : R \in E_c\} = \{R : c \in R \subseteq \mathcal{P}M\},$$

we have

$$F_c = \{R : R \subseteq [I]^{<\omega},\ R\lceil M \in F'_c\},\quad E_c = \{R : R \subseteq [I]^{<\omega},\ R\lceil M \in E'_c\},$$

while both E'_c and F'_c, and therefore $E'_c\setminus F'_c$, regarded as subsets of $\mathcal{P}(\mathcal{P}M)$, are determined by coordinates in $\mathcal{P}(c\cup(M\setminus L))$. Because $\bigcap_{c\in T} F_c$ is empty, so is $\bigcap_{c\in T} F'_c$.

(e) Let $n \ge r$ be such that

$$\begin{aligned}\frac{1}{m_n^{\#(M)}}\#(\{z : z \in m_n^M,\ \tilde z(T_n) \in E'_c\setminus F'_c\}) &\le \eta + \mu\{R : R\lceil M \in E'_c\setminus F'_c\}\\ &= \eta + \mu\{R : R \in E_c\setminus F_c\} \le 2\eta\end{aligned}$$

for every $c \in T$. For $c \in T$ set

$$Q_c = \{z : z \in m_n^M,\ \tilde z(T_n) \in E'_c\setminus F'_c\},$$

so that $\#(Q_c) \le 2\eta m_n^{\#(M)}$. Since

$$\begin{aligned}\sum_{\substack{c\in T\\ w\in m_n^{M\setminus L}}} \#(\{z : w \subseteq z \in Q_c\}) &= \sum_{c\in T}\#(Q_c) \le 2\eta\#(T)m_n^{\#(M)}\\ &\le \epsilon m_n^{\#(M)} = \epsilon\#(m_n^{M\setminus L})m_n^r,\end{aligned}$$

there must be a $w \in m_n^{M\setminus L}$ such that

$$\textstyle\sum_{c\in T, w\in m_n^{M\setminus L}} \#(\{z : w \subseteq z \in Q_c\}) \le \epsilon m_n^r;$$

set

$$Q'_c = \{z : w \subseteq z \in Q_c,\ z\restriction c \text{ is injective}\}$$

for $c \in T$, so that $\sum_{c\in T}\#(Q'_c) \le \epsilon m_n^r$.

If $c \in T$ and $\#(c) = k$, then

$$\#(\{z[c] : z \in Q'_c\}) = \frac{1}{m_n^{r-k}}\#(Q'_c).$$

P If $z,\ z' \in m_n^M$ and $z\restriction(c\cup(M\setminus L)) = z'\restriction(c\cup(M\setminus L))$, then $\tilde z(T_n)\lceil(c\cup(M\setminus L)) = \tilde z'(T_n)\lceil(c\cup(M\setminus L))$, so $\tilde z(T_n) \in E'_c\setminus F'_c$ iff $\tilde z'(T_n) \in E'_c\setminus F'_c$, that is, $z \in Q_c$ iff $z' \in Q_c$. So if $a = z[c]$ for some $z \in Q'_c$, then

$$\{z' : z' \in Q'_c,\ z'[c] = a\} = \{z' : z' \in m_n^M,\ z'\restriction(c\cup(M\setminus L)) = z\restriction(c\cup(M\setminus L))\}$$

has just $\#(m_n^{L\setminus c}) = m_n^{r-k}$ members. **Q**

(f) Consider

$$S = \{z[c] : c \in T,\ z \in Q'_c\}.$$

Then

$$\begin{aligned}\#(S\cap[m_n]^k) &= \#([m_n]^k \cap \{z[c] : c \in T,\ z \in Q'_c\})\\ &= \#(\{z[c] : c \in T\cap[L]^k,\ z \in Q'_c\})\end{aligned}$$

(because every member of Q'_c is injective on c)

$$\leq \sum_{\substack{c\in T\\ \#(c)=k}} \frac{1}{m_n^{r-k}}\#(Q'_c)$$

(by the last remark in (e))

$$\leq \frac{1}{m_n^{r-k}}\epsilon m_n^r = \epsilon m_n^k$$

for every k. So by the choice of (m_n, T_n) there is an embedding v of (L, T) in $(m_n, T_n \setminus S)$; take $z = v \cup w$, so that $w \subseteq z \in m_n^M$ and $z\restriction L = v$ is injective and $z[c] \notin S$ for every $c \in T$. However, there is some $c \in T$ such that $\tilde{z}(T_n) \notin F'_c$. As $c \in \tilde{z}(T_n)$, $\tilde{z}(T_n) \in E'_c$. But now $z \in Q'_c$ and $z[c] \in S$. **X**

This contradiction proves the theorem.

497K Corollary: the Hypergraph Removal Lemma For every $\epsilon > 0$ and $r \geq 1$ there is a $\delta > 0$ such that whenever I is a finite set, $R \subseteq [I]^r$ and $\#(\{J : J \in [I]^{r+1},\ [J]^r \subseteq R\}) \leq \delta\#(I)^{r+1}$, there is an $S \subseteq [I]^r$ such that $\#(S) \leq \epsilon\#(I)^r$ and there is no $J \in [I]^{r+1}$ such that $[J]^r \subseteq R \setminus S$.

proof In 497J, take L to be a set of size $r+1$, and set $T = [L]^r$ in 497J. Then there is a $\delta_0 > 0$ such that whenever I is a finite set, $R \subseteq [I]^r$ and the number of embeddings of $(L, [L]^r)$ in (I, R) is at most $\delta_0\#(I)^{r+1}$, there is an $S \subseteq [I]^r$ such that $\#(S) \leq \epsilon\#(I)^r$ and there is no embedding of $(L, [L^r])$ in $(I, R \setminus S)$. Try $\delta = \frac{1}{(r+1)!}\delta_0$. If I is finite, $R \subseteq [I]^r$ and $\mathcal{J} = \{J : J \in [I]^{r+1},\ [J]^r \subseteq R\}$ has at most $\delta\#(I)^{r+1}$ members, then an embedding of $(L, [L]^r)$ in (I, R) is an injective function $f : L \to I$ such that $f[J] \in R$ for every $J \in [L]^r$, that is, $f[L] \in \mathcal{J}$. So the number of such embeddings is $(r+1)!\#(\mathcal{J}) \leq \delta_0\#(I)^{r+1}$. There is therefore an $S \subseteq [I]^r$ such that $\#(S) \leq \epsilon\#(I)^r$ and there is no embedding of $(L, [L]^r)$ in $(I, R \setminus S)$, that is, there is no $J \in [I]^{r+1}$ such that $[J]^r \subseteq R \setminus S$.

497L Corollary: Szemerédi's Theorem (Szemerédi 75) For every $\epsilon > 0$ and $r \geq 2$ there is an $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, $A \subseteq n$ and $\#(A) \geq \epsilon n$ there is an arithmetic progression of length $r+1$ in A.

proof (Frankl & Rödl 02) Set $\eta = \frac{1}{r!}(\frac{\epsilon}{2r!})^r$. Take $\delta > 0$ such that whenever I is a finite set, $R \subseteq [I]^r$ and $\#(\{J : J \in [I]^{r+1},\ [J]^r \subseteq R\}) \leq \delta\#(I)^{r+1}$, there is an $S \subseteq [I]^r$ such that $\#(S) \leq \frac{\eta}{2(r+1)^r}\#(I)^r$ and there is no $J \in [I]^{r+1}$ such that $[J]^r \subseteq R \setminus S$. Let n_0 be such that $\epsilon n \geq 2r \cdot r!$ and $n(r+1)^{r+1}\delta \geq 1$ whenever $n \geq n_0$. Take $n \geq n_0$ and $A \subseteq n$ such that $\#(A) \geq \epsilon n$.

Let $C \subseteq n^r$ be the set

$$\{(i_0, i_1, \dots, i_{r-1}) : \textstyle\sum_{j=0}^{r-1}(j+1)i_j \in A\}.$$

Then $\#(C) \geq \eta n^r$.[11] **P** For $m < r!$ set $A_m = \{i : i \in A,\ i \equiv m \mod r!\}$. Then there is an m such that $\#(A_m) \geq \frac{\epsilon n}{r!}$. Now we have an injection $\phi : [A_m]^r \to C$ given by saying that if $l_0 < \dots < l_{r-1}$ in A_m then

$$\begin{aligned}\phi(\{l_0, \dots, l_{r-1}\})(j) &= l_0 \text{ if } j = 0\\ &= \tfrac{1}{j+1}(l_j - l_{j-1}) \text{ if } 0 < j < r.\end{aligned}$$

So

$$\#(C) \geq \#([A_m]^r) \geq \frac{1}{r!}(\frac{\epsilon n}{r!} - r)^r \geq \frac{1}{r!}(\frac{\epsilon n}{2r!})^r = \eta n^r. \textbf{ Q}$$

Let I be $n \times (r+1)$ and for $c = (i_0, \dots, i_{r-1}) \in C$ set

$$J_c = \{(i_j, j) : j < r\} \cup \{(\textstyle\sum_{j=0}^{r-1} i_j, r)\} \in [I]^{r+1}.$$

Observe that if c, $c' \in C$ are distinct, then $[J_c]^r \cap [J_{c'}]^r = \emptyset$, since given any face of the r-simplex J_c we can read off all but at most one of the coordinates of c and calculate the last. Set $R = \bigcup_{c\in C}[J_c]^r \subseteq [I]^r$.

? Suppose, if possible, that the only r-simplices $J \in [I]^{r+1}$ such that $[J]^r \subseteq R$ are of the form J_c for some $c \in C$. Then there are at most

[11] For the rest of this proof, and also in 497M and 497N below, I will use the formula n^r both for the set of functions from $r = \{0, \dots, r-1\}$ to $n = \{0, \dots, n-1\}$ and for its cardinal interpreted as a real number; I trust that this will not lead to any confusion.

$$\#(C) \le n^r \le n^r \cdot n(r+1)^{r+1}\delta = \delta\#(I)^{r+1}$$

such simplices; by the choice of δ, there is an $S \subseteq [I]^r$ such that $R \setminus S$ covers no r-simplices and

$$\#(S) \le \frac{\eta}{2(r+1)^r}\#(I)^r = \frac{\eta}{2}n^r < \#(C).$$

But every J_c must have a face in S, and no two J_c share a face, so this is impossible. **X**

So we have an r-simplex $J \in [I]^{r+1}$, which is not of the form J_c where $c \in C$, such that $[J]^r \subseteq R$. Now since the only faces put into R come from the J_c, and therefore meet each of the $r+1$ levels $n \times \{k\}$ in at most one point, J must be of the form $\{(i_j, j) : j < r\} \cup \{(l, r)\}$. Since $\{(i_j, j) : j < r\}$ is a face of some J_c, $c = (i_0, \dots, i_{r-1}) \in C$. Set $l' = i_0 + \ldots + i_{r-1}$; then $l' \ne l$ because $J \ne J_c$. For each $k < r$, $J \setminus \{(i_k, k)\}$ is a face of J and therefore of $J_{c'}$ for some $c' \in C$; now $J_{c'}$ must be

$$(J \setminus \{(i_k, k)\}) \cup \{(l - \textstyle\sum_{j<r, j\ne k} i_j, k)\} = (J \setminus \{(i_k, k)\}) \cup \{(i_k + l - l', k)\}$$

and

$$\textstyle\sum_{j=0}^{r}(j+1)i_j + (k+1)(l - l')$$

belongs to A. Since this is true for every $k < r$, and we also have $\sum_{j=0}^{r}(j+1)i_j \in A$ because $c \in C$, we have an arithmetic progression in A of length $r+1$, as required.

497M For a full-strength version of the multiple recurrence theorem it seems that the ideas described above are inadequate; for an adaptation which goes farther, see Austin 10a and Austin 10b. However the methods here can reach the following.

Lemma (cf. Solymosi 03) Suppose that $r \ge 1$ and $n \in \mathbb{N}$. For $0 \le j, k < r$ set $e_j(k) = 1$ if $k = j$, 0 otherwise. For $z \in n^r$ and $C \subseteq n^r$ write

$$\Delta(z, C) = \{k : k \in \mathbb{Z},\ z + ke_i \in C \text{ for every } i < r\}, \quad q(z, C) = \#(\Delta(z, C)).$$

Then for every $\epsilon > 0$ there is a $\delta > 0$ such that $\#(\{z : z \in n^r,\ q(z, C) \ge \delta n\}) \ge \delta n^r$ whenever $n \in \mathbb{N}$, $C \subseteq n^r$ and $\#(C) \ge \epsilon n^r$.

proof We can use some of the same ideas as in 497L. Let $\epsilon' > 0$ be such that $2^r r^r \epsilon' < \epsilon$. Let $\delta > 0$ be such that whenever I is finite, $R \subseteq [I]^r$ and $\#(\{J : J \in [I]^{r+1},\ [J]^r \subseteq R\}) \le \delta\#(I)^{r+1}$ there is an $S \subseteq R$ such that $\#(S) \le \epsilon'\#(I)^r$ and there is no $J \in [I]^{r+1}$ such that $[J]^r \subseteq R \setminus S$ (497K).

Take $n \in \mathbb{N}$ and $C \subseteq n^r$ such that $\#(C) \ge \epsilon n^r$. Set $I = (n \times r) \cup (nr \times \{r\})$, so that $\#(I) = 2nr$. For $c \in C$ set

$$J_c = \{(c(i), i) : i < r\} \cup \{(\textstyle\sum_{i=0}^{r-1} c(i), r)\} \in [I]^{r+1};$$

set $R = \bigcup_{c\in C}[J_c]^r$. Observe that if c, $c' \in C$ are distinct then $[J_c]^r$ and $[J_{c'}]^r$ are disjoint. If $S \subseteq R$ and $\#(S) \le \epsilon'\#(I)^r$, then $\#(S) < \epsilon n^r$ and there must be a $c \in C$ such that $[J_c]^r \cap S = \emptyset$ and $[J_c]^r \subseteq R \setminus S$. Consequently

$$\mathcal{K} = \{K : K \in [I]^{r+1},\ [K]^r \subseteq R\}$$

must have more than $\delta\#(I)^{r+1} \ge 2\delta n^{r+1}$ members, by the choice of δ.

Next, $\#(\mathcal{K}) = \sum_{z\in n^r} q(z, C)$. **P** Set $B = \{(z, k) : z \in n^r,\ k \in \mathbb{Z},\ z + ke_i \in C$ for every $i < r\}$; then $\#(B) = \sum_{z\in n^r} q(z, C)$. For any $K \in \mathcal{K}$, there must be a $c_K \in C$ such that $(c_K(i), i) \in K$ for every $i < r$ while $(k_K + \sum_{i=0}^{r-1} c_K(i), r) \in K$ for some k_K; in this case, $c_K + k_K e_i \in C$ for every $i < r$ and $(c_K, k_K) \in B$. Conversely, starting from $(z, k) \in B$, $\{(z(i), i) : i < r\} \cup \{(k + \sum_{i=0}^{r-1} z(i), r)\}$ belongs to K. So $K \mapsto (c_K, k_K)$ is a bijection from $\mathcal{K}$ to B and $\#(\mathcal{K}) = \#(B)$. **Q**

Thus $\sum_{z\in n^r} q(z, C) \ge 2\delta n^{r+1}$. Of course

$$q(z, C) \le \#(\{k : z + ke_0 \in n^r\} \le n$$

for every $z \in n^r$. So setting $D = \{z : z \in n^r,\ q(z, C) \ge \delta n\}$, we have

$$2\delta n^{r+1} \le n\#(D) + \delta n \cdot n^r \le n\#(D) + \delta n^{r+1}$$

and $\#(D) \ge \delta n^r$, as claimed.

497N Theorem (Furstenberg 81) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\langle\pi_i\rangle_{i<r}$ a non-empty finite commuting family of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself. If $a \in \mathfrak{A} \setminus \{0\}$, there is an $\eta > 0$ such that

$$\sum_{k=0}^{n-1} \bar{\mu}(\inf_{i<r} \pi_i^k a) \geq \eta n$$

for every $n \in \mathbb{N}$.

proof (a) To being with, suppose that every π_i is an automorphism. Doubling r if necessary, we can suppose that for every $i < r$ there is a $j < r$ such that $\pi_j = \pi_i^{-1}$. Let (Z, Σ, μ) be the Stone space of $(\mathfrak{A}, \bar{\mu})$ (321K), and set $E = \widehat{a}$, the open-and-closed subset of Z corresponding to $a \in \mathfrak{A}$. For each $i < r$ let $T_i : Z \to Z$ be the homeomorphism corresponding to $\pi_i : \mathfrak{A} \to \mathfrak{A}$, so that $T_i^{-1}[\widehat{b}] = \widehat{\pi_i b}$ for every $b \in \mathfrak{A}$ (312Q[12]); note that $T_i T_j$ corresponds to $\pi_j \pi_i$ (312R[13]) and $T_i T_j = T_j T_i$ (because the representations in 312Q are unique), for all $i, j < r$.

In 497M, set $\epsilon = \frac{1}{2}\mu E = \frac{1}{2}\bar{\mu}a$ and take a corresponding $\delta > 0$; set $\eta = \frac{1}{2}\delta^2\epsilon$. Now, given $n \geq 1$, then for $z \in n^r$ set $\tilde{T}_z = \prod_{i<r} T_i^{z(i)}$. (We can speak of the product without inhibitions because the T_i commute.) Consider the set $W = \{(x, z) : z \in n^r,\ \tilde{T}_z(x) \in E\}$. Then $W^{-1}[\{z\}]$ has measure μE for every z, so if we set $F = \{x : x \in E, \#(W[\{x\}]) \geq \epsilon n^r\}$ we have

$$n^r \mu E \leq n^r \mu F + \epsilon n^r, \quad \mu F \geq \epsilon.$$

In the notation of 497M, set

$$V = \{(x, z) : (x, z) \in W,\ q(z, W[\{x\}]) \geq \delta n\};$$

then for any $x \in F$ we have $\#(V[\{x\}]) \geq \delta n^r$, by the choice of δ. There must therefore be a $z \in n^r$ such that $\mu V^{-1}[\{z\}] \geq \delta \mu F \geq \delta\epsilon$. Take any $x \in V^{-1}[\{z\}]$ and $k \in \Delta(z, W[\{x\}])$. Setting $e_i(i) = 1$ and $e_i(j) = 0$ for $i < r$ and $j \in r \setminus \{i\}$, $z + ke_i \in W[\{x\}]$, that is, $T_i^k \tilde{T}_z(x) \in E$, for every $i < r$. Also $|k| < n$. Set $G = \tilde{T}_z[V^{-1}[\{z\}]]$, so that $\mu G \geq \delta\epsilon$ and for every $y \in G$ we have $\#(\{k : |k| < n,\ T_i^k(y) \in E \text{ for every } i < r\}) \geq \delta n$. But this means that

$$\sum_{k=0}^{n-1} \bar{\mu}(\inf_{i<r} \pi_i^k a) = \sum_{k=0}^{n-1} \mu\{x : T_i^k(x) \in E \text{ for every } i < r\}$$

$$\geq \frac{1}{2} \sum_{|k|<n} \mu\{x : T_i^k(x) \in E \text{ for every } i < r\}$$

(because if $T_i^k(x) \in E$ for every $i < r$ then $T_i^{|k|}(x) \in E$ for every $i < r$)

$$= \frac{1}{2} \int \#(\{k : |k| < n,\ T_i^k(x) \in E \text{ for every } i < r\})\mu(dx)$$

$$\geq \frac{1}{2}\delta n \mu G \geq \frac{1}{2}\delta^2 \epsilon n = \eta n,$$

as required.

(b) For the general case, 328J tells us that there are a probability algebra $(\mathfrak{C}, \bar{\lambda})$, a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ and a commuting family $\langle \tilde{\pi}_i \rangle_{i<r}$ of measure-preserving automorphisms of $\mathfrak{C}$ such that $\tilde{\pi}_i \pi = \pi \pi_i$ for every $i < r$. Now $\pi a \in \mathfrak{C} \setminus \{0\}$, so there is an $\eta > 0$ such that

$$\eta n \leq \sum_{k=0}^{n-1} \bar{\lambda}(\inf_{i<r} \tilde{\pi}_i^k \pi a) = \sum_{k=0}^{n-1} \bar{\lambda}(\inf_{i<r} \pi \pi_i^k a) = \sum_{k=0}^{n-1} \bar{\mu}(\inf_{i<r} \pi_i^k a)$$

for every $n \in \mathbb{N}$.

497X Basic exercises (a) Let (X, Σ, μ) be a probability space, $\langle \Sigma_i \rangle_{i \in I}$ an independent family of σ-subalgebras of Σ, and T a subalgebra of Σ such that $\mathrm{T} \cap \Sigma_i$ is metrically dense in Σ_i for every $i \in I$. For $J \subseteq I$ set $\tilde{\Sigma}_J = \bigvee_{i \in J} \Sigma_i$. Show that $\langle \tilde{\Sigma}_J \rangle_{J \subseteq I}$ has T-removable intersections.

(b) Let I be a set, G_I the group of permutations of I with its topology of pointwise convergence (441G, 449Xh), and $\bullet$ the action of G_I on $\mathcal{P}([I]^{<\omega})$ described in 497F. Show that $\bullet$ is continuous.

(c) In 497F, show that $\{\mu : \mu \in P_I \text{ is permutation-invariant}\}$ is a closed subset of P_I.

[12] Formerly 312P.
[13] Formerly 312Q.

497Y Further exercises (a) (i) Show that if $A \subseteq \mathbb{N}$ has non-zero upper asymptotic density then there is a translation-invariant additive functional $\nu : \mathcal{P}\mathbb{Z} \to [0,1]$ such that $\nu A > 0$. (ii) Consider the statement

(†) If $\epsilon > 0$ and $A \subseteq \mathbb{N}$ are such that $\#(A \cap n) \ge \epsilon n$ for every n then A includes arithmetic progressions of all finite lengths.

Use Theorem 497N to prove (†). (iii) Find a direct proof that (†) implies Szemerédi's theorem.

497 Notes and comments I am grateful to T.D.Austin for introducing me to a preprint of Tao 07, on which this section is based.

Regarded as a proof of Szemerédi's theorem, the argument above has the virtues of reasonable brevity and (I hope) of completeness and correctness. It depends, of course, on non-trivial ideas from measure theory, which for anyone except a measure theorist will compromise the claim of 'brevity'; and even measure theorists may find that the proofs here demand close attention. There are further, more significant, defects. The outstanding problem associated with Szemerédi's theorem is the estimation of n_0 as a function of r and ϵ; and while in a theoretical sense it must be possible to trace through the arguments above to establish rigorous bounds, the methods are not well adapted to such an exercise, and one would not expect the bounds obtained to be good. There is also the point that I have made uninhibited use of the axiom of choice. The ultrafilter in 497J can easily be replaced by an appropriate sequence, but all standard treatments of measure theory assume at least the countable axiom of choice, and Szemerédi's theorem is clearly true in significantly weaker theories than ordinary ZF.

The first 'measure-theoretic' proof of Szemerédi's theorem was due to Furstenburg 77, and relied on a deep analysis of the structure of measure-preserving transformations. While the methods described here do not seem to give us any information on this structure, it is apparently a folklore result that the hypergraph removal lemma provides a quick proof of the basic theorem used in Furstenburg's approach (497N, 497Ya).

The value of the work here, therefore, lies less in its applications to the hypergraph removal lemma and Szemerédi's theorem, than in the idea of 'removable intersections', where Theorems 497E and 497G give us two remarkable results, and useful exercises in the theory of relative independence from §458. We also have an instructive example of a more general phenomenon. Given a sequence of finite objects with quantititative aspects, it is often profitable to seek a measure μ reflecting the asymptotic behaviour of this sequence; this is the idea of the construction in 497H. The 'quantitative aspects' here, as developed in 497J, are the proportion of functions from L to m_n which are embeddings of (L,T) in (m_n,T_n), and the proportion of simplices in $[m_n]^k$ which must be removed from T_n in order to destroy all these embeddings. The measure μ is set up to describe the limits of these proportions as measures of appropriate sets.

Returning to the definition 497Aa, most of its clauses can be expressed in terms of the measure algebra of the measure μ; but the final '$\bigcap_{i\in J} F_i = \emptyset$' has to be taken literally, and makes sense only in terms of the measure space itself. In the key application (part (d) of the proof of 497J), the original sets E_c, with negligible intersection, already belong to the algebra T, but the adjustment to sets F_c with empty intersection is still non-trivial, because of the requirement that each F_c must belong to the prescribed σ-algebra Σ_{J_c}.

I said in 497F that you could note that $\mathcal{P}([\mathbb{N}]^{<\omega})$ is homeomorphic to the Cantor set, so that $P_{\mathbb{N}}$ is isomorphic to the space of Radon probability measures on $\{0,1\}^{\mathbb{N}}$. However the point of the construction there is that we are looking at a particular action of the symmetric group $G_{\mathbb{N}}$ on $\mathcal{P}([\mathbb{N}]^{<\omega})$; and this has very little to do with the natural actions of $G_{\mathbb{N}}$ on $\mathcal{P}\mathbb{N}$ or $\{0,1\}^{\mathbb{N}}$, as studied in 459E and 459H, for instance. In particular, permutation-invariant measures, in the sense of 497Fb, will not normally be invariant under the much larger group derived from all permutations of $[\mathbb{N}]^{<\omega}$ rather than just those corresponding to permutations of $\mathbb{N}$.

I express 497N in terms of measure-preserving automorphisms of probability algebras in order to connect it with the treatment of ergodic theory in Chapter 38, but you will observe that the proof presented immediately shifts to a more traditional formulation in terms of probability spaces. This is only one of many multiple recurrence theorems, some of them much stronger (and, it seems, deeper) than 497N or, indeed, 497J.

498 Cubes in product spaces

I offer a brief note on a special property of (Radon) product measures.

498A Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra with its measure-algebra topology (323A). Suppose that $A \subseteq \mathfrak{A}$ is an uncountable analytic set. Then there is a compact set $L \subseteq A$, homeomorphic to $\{0,1\}^{\mathbb{N}}$, such that $\inf L \ne 0$ in $\mathfrak{A}$.

proof $A\setminus\{0\}$ is still an uncountable analytic subset of $\mathfrak{A}$. By 423J, it has a subset homeomorphic to $\{0,1\}^{\mathbb{N}}\cong\mathcal{P}\mathbb{N}$; let $f:\mathcal{P}\mathbb{N}\to A\setminus\{0\}$ be an injective continuous function. Because $(\mathfrak{A},\bar\mu)$ is semi-finite, there is an $a\subseteq f(\emptyset)$ such that $0<\bar\mu a<\infty$; set $\delta=\bar\mu a$. Note that $(I,J)\mapsto\bar\mu(a\cap f(I)\setminus f(J)):(\mathcal{P}\mathbb{N})^2\to\mathbb{R}$ is continuous. Choose a sequence $\langle k_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ inductively, as follows. Given $\langle k_i\rangle_{i<n}$, set $K_n=\{k_i:i<n\}$. For each $J\subseteq K_n$ we have $\lim_{r\to\infty}\bar\mu(a\cap f(J)\setminus f(J\cup\{r\}))=0$, so there is a k_n, greater than k_i for every $i<n$, such that $\bar\mu(a\cap f(J)\setminus f(J\cup\{k_n\})\le 2^{-2n-2}\delta$ for every $J\subseteq K_n$; continue.

Now

$$\bar\mu(a\cap\inf_{J\subseteq K_n}f(J))\ge\delta(\frac{1}{2}+2^{-n-1})$$

for every $n\in\mathbb{N}$. **P** Induce on n. If $n=0$, then $a\cap\inf_{J\subseteq K_n}f(J)=a\cap f(\emptyset)$ has measure $\delta=\delta(\frac{1}{2}+2^{-1})$. For the inductive step to $n+1\ge 1$, observe that

$$\begin{aligned}\bar\mu(a\cap\inf_{J\subseteq K_{n+1}}f(J))&=\bar\mu(a\cap\inf_{J\subseteq K_n}f(J)\cap f(J\cup\{k_n\}))\\&\ge\bar\mu(a\cap\inf_{J\subseteq K_n}f(J))-\sum_{J\subseteq K_n}\bar\mu(a\cap f(J)\setminus f(J\cup\{k_n\}))\\&\ge\delta(\frac{1}{2}+2^{-n-1})-\sum_{J\subseteq K_n}2^{-2n-2}\delta\end{aligned}$$

(by the inductive hypothesis and the choice of k_n)

$$=\delta(\frac{1}{2}+2^{-n-2}).$$

So the induction proceeds. **Q**

Set $K=\{k_i:i\in\mathbb{N}\}$, $c=\inf\{f(J):J\subseteq K\text{ is finite}\}$. Then

$$\bar\nu(a\cap c)=\inf_{n\in\mathbb{N}}\bar\mu(a\cap\inf_{J\subseteq K_n}f(J))\ge\frac{1}{2}\delta,$$

and $c\ne 0$. But now observe that $L=f[\mathcal{P}K]$ is a subset of A homeomorphic to $\mathcal{P}K$ and therefore to $\{0,1\}^{\mathbb{N}}$. Also $\{b:b\supseteq c\}$ is closed (323D(d-i)), so $C=\{J:f(J)\supseteq c\}$ is closed in $\mathcal{P}K$; as it includes the dense set $[K]^{<\omega}$, $C=\mathcal{P}K$ and $\inf L\supseteq c$ is non-zero.

498B Proposition (see Brodskiĭ 49, Eggleston 54) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an atomless Radon measure space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ an effectively locally finite τ-additive topological measure space, and $\tilde\lambda$ the τ-additive product measure on $X\times Y$ (417G). Then if $W\subseteq X\times Y$ and $\tilde\lambda W>0$ there are a non-scattered compact set $K\subseteq X$ and a set $F\subseteq Y$ of positive measure such that $K\times F\subseteq W$.

proof (a) To begin with (down to the end of (c)), let us suppose that both μ and ν are totally finite. Let $(\mathfrak{B},\bar\nu)$ be the measure algebra of ν. Because $\tilde\lambda$ is inner regular with respect to the closed sets, we may suppose that W is closed. In this case, writing λ for the c.l.d. product measure on $X\times Y$, there is a $W'\supseteq W$ such that $\lambda W'$ is defined and equal to $\tilde\lambda W$ (417C(iv)). By 418Tb, there is a μ-conegligible set X_0 such that $W'[\{x\}]\in\mathrm{T}$ for every $x\in X_0$, $B=\{W'[\{x\}]^\bullet:x\in X_0\}$ is separable for the measure-algebra topology of $\mathfrak{B}$, and $x\mapsto W'[\{x\}]^\bullet:X_0\to\mathfrak{B}$ is measurable. Now Fubini's theorem, applied in the form of 252D to λ and in the form of 417Ha to $\tilde\lambda$, tells us that

$$\int\nu W'[\{x\}]\mu(dx)=\lambda W'=\tilde\lambda W=\int\nu W[\{x\}]\mu(dx).$$

So $X_1=\{x:x\in X_0,\ W'[\{x\}]^\bullet=W[\{x\}]^\bullet\}$ is μ-conegligible. Since the topology of $\mathfrak{B}$ is metrizable (323Ad or 323Gb), B is separable and metrizable, and $x\mapsto W[\{x\}]^\bullet:X_1\to B$ is almost continuous (418J, applied to the subspace measure on X_1). Let $K^*\subseteq X_1$ be a compact set of non-zero measure such that $x\mapsto W[\{x\}]^\bullet:K^*\to\mathfrak{B}$ is continuous.

(b) There is a non-zero $c\in\mathfrak{B}$ such that $K_c=\{x:x\in K^*,\ c\subseteq W[\{x\}]^\bullet\}$ is compact and not scattered. **P** Because $x\mapsto W[\{x\}]^\bullet$ is continuous on K^*, $B^*=\{W[\{x\}]^\bullet:x\in K^*\}$ is compact and every K_c is compact. (i) If B^* is countable, then $K^*=\bigcup_{b\in B^*}K_b$, so there is some $c\in B^*$ such that $\mu K_c>0$. Let E be a non-negligible self-supporting subset of K_c; then (because μ is atomless, therefore zero on singletons) E has no isolated points. So K_c is not scattered. (ii) If B^* is uncountable, then by 498A there is a set $D\subseteq B^*$, homeomorphic to $\{0,1\}^{\mathbb{N}}$, with a non-zero lower bound c in $\mathfrak{B}$. Now $\{W[\{x\}]^\bullet : x \in K_c\}$ includes D, so $\{0,1\}^{\mathbb{N}}$ and therefore $[0,1]$ are continuous images of closed subsets of K_c and K_c is not scattered (4A2G(j-iv)). **Q**

(c) Set $K = K_c$. Then 414Ac tells us that

$$(\bigcap_{x\in K} W[\{x\}])^\bullet = \inf_{x\in K} W[\{x\}]^\bullet \supseteq c$$

is non-zero, so $F = \bigcap_{x\in K} W[\{x\}]$ is non-negligible; while $K \times F \subseteq W$. So we have found appropriate sets K and F, at least when μ and ν are totally finite.

(d) For the general case, we need observe only that by 417C(iii) there are $X' \in \Sigma$ and $Y' \in \mathrm{T}$, both of finite measure, such that $\tilde\lambda(W \cap (X' \times Y')) > 0$. Now the subspace measure $\mu_{X'}$ on X' is atomless and Radon (214Ka, 416Rb), the subspace measure $\nu_{Y'}$ on Y' is τ-additive (414K), and the τ-additive product of $\mu_{X'}$ and $\nu_{Y'}$ is the subspace measure on $X' \times Y'$ induced by $\tilde\lambda$ (417I). So we can apply (a)-(c) to $\mu_{X'}$ and $\nu_{Y'}$ to see that there are a non-scattered compact set $K \subseteq X'$ and a non-negligible measurable set $F \subseteq Y'$ such that $K \times F \subseteq W$.

498C Proposition (see Ciesielski & Pawlikowski 03) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of atomless Radon probability spaces, and $\tilde\lambda$ the product Radon probability measure on $X = \prod_{i\in I} X_i$. If $W \subseteq X$ and $\tilde\lambda W > 0$, there is a family $\langle K_i\rangle_{i\in I}$ such that $K_i \subseteq X_i$ is a non-scattered compact set for each $i \in I$ and $\prod_{i\in I} K_i \subseteq W$.

proof (a) To begin with, let us suppose that $I = \mathbb{N}$. For each $n \in \mathbb{N}$, set $Y_n = \prod_{i\ge n} X_i$ and let $\tilde\lambda_n$ be the product Radon probability measure on Y_n, so that $\tilde\lambda_0 = \tilde\lambda$ and $\tilde\lambda_n$ can be identified with the product of μ_n and $\tilde\lambda_{n+1}$ (417J). Using 498B repeatedly, we can find non-scattered compact sets $K_n \subseteq X_n$ and closed non-negligible sets $W_n \subseteq Y_n$ such that $W_0 \subseteq W$ and $K_n \times W_{n+1} \subseteq W_n$ for every n. In this case, $\prod_{i<n} K_i \times W_n \subseteq W_0$ for every n. If $x \in \prod_{i\in\mathbb{N}} K_i$, then there is for each $n \in \mathbb{N}$ an $x_n \in \prod_{i<n} K_i \times W_n$ such that $x_n\restriction n = x\restriction n$, just because W_n is not empty. But now every x_n belongs to W_0 and

$$x = \lim_{n\to\infty} x_n \in W_0 \subseteq W.$$

As x is arbitrary, $\prod_{i\in\mathbb{N}} K_i \subseteq W$.

(b) For the general case, we may suppose that $I \subseteq \mathbb{N}$. For $i \in \mathbb{N}\setminus I$, take $(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)$ to be $[0,1]$ with Lebesgue measure. Set $\tilde W = \{x : x \in \prod_{i\in\mathbb{N}} X_i,\ x\restriction I \in W\}$. By (a), there are non-scattered compact sets $K_i \subseteq X_i$ such that $\prod_{i\in\mathbb{N}} K_i \subseteq \tilde W$, in which case $\prod_{i\in I} K_i \subseteq W$, as required.

498X Basic exercises (a) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ an effectively locally finite τ-additive topological measure space, and $\tilde\lambda$ the τ-additive product measure on $X \times Y$. Show that if $W \subseteq X \times Y$ and $\tilde\lambda W > 0$ there are a compact set $K \subseteq X$ and a set $F \subseteq Y$ of positive measure such that $K \times F \subseteq W$ and K is either non-scattered or non-negligible.

(b) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an atomless Radon measure space, (Y,T,ν) any measure space, and λ the c.l.d. product measure on $X \times Y$. Show that if $W \subseteq X \times Y$ and $\lambda W > 0$ there are a non-scattered compact set $K \subseteq X$ and a set $F \subseteq Y$ of positive measure such that $K \times F \subseteq W$. (*Hint*: reduce to the case in which ν is totally finite and T is countably generated, so that the completion of ν is a quasi-Radon measure for an appropriate second-countable topology.)

(c) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any family of atomless Radon probability spaces, and λ the ordinary product measure on $X = \prod_{i\in I} X_i$. Show that if $W \subseteq X$ and $\lambda W > 0$ then there are non-scattered compact sets $K_i \subseteq X_i$ for $i \in I$ such that $\prod_{i\in I} K_i \subseteq W$.

(d) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of atomless Radon probability spaces, and $W \subseteq \prod_{i\in I} X_i$ a set with positive measure for the Radon product of $\langle\mu_i\rangle_{i\in I}$. Show that there are atomless Radon probability measures ν_i on X_i such that W is conegligible for the Radon product of $\langle\nu_i\rangle_{i\in I}$. (*Hint*: 439Xh(vii).)

498Y Further exercises (a) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of atomless perfect probability spaces, and λ the ordinary product measure on $X = \prod_{i\in I} X_i$. Show that if $W \subseteq X$ and $\lambda W > 0$ then there are sets $K_i \subseteq X_i$ for $i \in I$, all of cardinal $\mathfrak{c}$, such that $\prod_{i\in I} K_i \subseteq W$.

498 Notes and comments I have already noted (325Yd) that a set W of positive measure in a product space need not include the product of two sets of positive measure; this fact is also the basis of 419E. Here, however, we see that if one of the factors is a Radon measure space then W does include the product of a non-trivial compact set and a set of positive measure. There are many possible variations on the result, corresponding to different product measures (498B, 498Xb) and different notions of 'non-trivial' (498Xa, 498Ya). The most important of the latter seems to be the idea of a 'non-scattered' compact set K; this is a quick way of saying that $[0,1]$ is a continuous image of K, which is a little stronger than saying that $\#(K) \geq \mathfrak{c}$, and arises naturally from the proof of 498B.

Appendix to Volume 4

Useful facts

As is to be expected, we are coming in this volume to depend on a wide variety of more or less recondite information, and only an exceptionally broad mathematical education will have covered it all. While all the principal ideas are fully expressed in standard textbooks, there are many minor points where I need to develop variations on the familiar formulations. A little under half the material, by word-count, is in general topology (§4A2), where I begin with some pages of definitions. I follow this with a section on Borel and Baire σ-algebras, Baire-property algebras and cylindrical algebras (§4A3), worked out a little more thoroughly than the rest of the material. The other sections are on set theory (§4A1), linear analysis (§4A4), topological groups (§4A5) and Banach algebras (§4A6).

4A1 Set theory

For this volume, we need fragments from four topics in set theory and one in Boolean algebra. The most important are the theory of closed cofinal sets and stationary sets (4A1B-4A1C) and infinitary combinatorics (4A1D-4A1H). Rather more specialized, we have the theory of normal (ultra)filters (4A1J-4A1L) and a mention of Ostaszewski's ♣ (4A1M-4A1N), used for an example in §439. I conclude with a simple result on the cardinality of σ-algebras (4A1O).

4A1A Cardinals again (a) An infinite cardinal κ is **regular** if it is not the supremum of fewer than κ smaller cardinals, that is, if $\operatorname{cf}\kappa = \kappa$. Any infinite successor cardinal is regular. (Kunen 80, I.10.37; Just & Weese 96, 11.18; Jech 78, p. 40; Levy 79, IV.3.11.) In particular, $\omega_1 = \omega^+$ is regular.

(b) If ζ is an ordinal and X is a set then I say that a family $\langle x_\xi \rangle_{\xi<\zeta}$ in X **runs over X with cofinal repetitions** if $\{\xi : \xi < \zeta,\, x_\xi = x\}$ is cofinal with ζ for every $x \in X$. Now if X is any non-empty set and κ is a cardinal greater than or equal to $\max(\omega, \#(X))$, there is a family $\langle x_\xi \rangle_{\xi<\kappa}$ running over X with cofinal repetitions. **P** By 3A1Ca, there is a surjection $\xi \mapsto (x_\xi, \alpha_\xi) : \kappa \to X \times \kappa$. **Q**

(c) The cardinal $\mathfrak{c}$ (i) Every non-trivial interval in $\mathbb{R}$ has cardinal $\mathfrak{c}$. (Enderton 77, p. 131.)

(ii) If $\#(A) \le \mathfrak{c}$ and D is countable, then $\#(A^D) \le \mathfrak{c}$. ($\#(A^D) \le \#(\mathcal{P}\mathbb{N})^{\mathbb{N}}) = \#(\mathcal{P}(\mathbb{N}\times\mathbb{N})) = \#(\mathcal{P}\mathbb{N})$.)

(iii) $\operatorname{cf}(2^\kappa) > \kappa$ for every infinite cardinal κ; in particular, $\operatorname{cf}\mathfrak{c} > \omega$. (Kunen 80, I.10.40; Just & Weese 96, 11.24; Jech 78, p. 46; Jech 03, 5.11; Levy 79, V.5.2.)

(d) The Continuum Hypothesis This is the statement '$\mathfrak{c} = \omega_1$'; it is neither provable nor disprovable from the ordinary axioms of mathematics, including the Axiom of Choice. As such, it belongs to Volume 5 rather than to the present volume. But I do at one point refer to one of its immediate consequences. If the continuum hypothesis is true, then there is a well-ordering $\preccurlyeq$ of $[0,1]$ such that $([0,1], \preccurlyeq)$ has order type ω_1 (because there is a bijection $f : [0,1] \to \omega_1$, and we can set $s \preccurlyeq t$ if $f(s) \le f(t)$).

4A1B Closed cofinal sets Let α be an ordinal.

(a) Note that a subset F of α is closed in the order topology iff $\sup A \in F$ whenever $A \subseteq F$ is non-empty and $\sup A < \alpha$. (4A2S(a-ii).)

(b) If α has uncountable cofinality, and $A \subseteq \alpha$ has supremum α, then $A' = \{\xi : 0 < \xi < \alpha,\, \xi = \sup(A \cap \xi)\}$ is a closed cofinal set in α. **P** (α) For any $\eta < \alpha$ we can choose inductively a strictly increasing sequence $\langle \xi_n \rangle_{n\in\mathbb{N}}$ in A starting from $\xi_0 \ge \eta$; now $\xi = \sup_{n\in\mathbb{N}} \xi_n \in A'$ and $\xi \ge \eta$; this shows that A' is cofinal with α. (β) If $\emptyset \ne B \subseteq A'$ and $\sup B = \xi < \alpha$, then $A \cap \xi \supseteq A \cap \eta$ for every $\eta \in B$, so

$$\xi = \sup B = \sup_{\eta\in B} \sup(A\cap\eta) = \sup(\bigcup_{\eta\in B} A\cap\eta) \le \sup A\cap\xi \le \xi$$

and $\xi \in A'$. **Q**

In particular, taking $A = \alpha = \omega_1$, the set of non-zero countable limit ordinals is a closed cofinal set in ω_1.

(c)(i) If $\langle F_\xi \rangle_{\xi<\alpha}$ is a family of subsets of α, the **diagonal intersection** of $\langle F_\xi \rangle_{\xi<\alpha}$ is $\{\xi : \xi < \alpha,\, \xi \in F_\eta$ for every $\eta < \xi\}$.

(ii) If κ is a regular uncountable cardinal and $\langle F_\xi\rangle_{\xi<\kappa}$ is any family of closed cofinal sets in κ, its diagonal intersection F is again a closed cofinal set in κ. **P** $F=\bigcap_{\xi<\kappa}(F_\xi\cup[0,\xi])$ is certainly closed. To see that it is cofinal, argue as follows. Start from any $\zeta_0<\kappa$. Given $\zeta_n<\kappa$, set

$$\zeta_{n+1}=\sup_{\xi<\zeta_n}(\min(F_\xi\setminus\zeta_n)+1);$$

this is defined because every F_ξ is cofinal with κ, and is less than κ because $\operatorname{cf}\kappa=\kappa$. At the end of the induction, set $\zeta^*=\sup_{n\in\mathbb{N}}\zeta_n$; then $\zeta_0\le\zeta^*$ and $\zeta^*<\kappa$ because $\operatorname{cf}\kappa>\omega$. If ξ, $\eta<\zeta^*$, there is an $n\in\mathbb{N}$ such that $\max(\xi,\eta)<\zeta_n$, in which case $F_\xi\cap(\zeta^*\setminus\eta)\supseteq F_\xi\cap\zeta_{n+1}\setminus\zeta_n$ is non-empty. As η is arbitrary and F_ξ is closed, $\zeta^*\in F_\xi$; as ξ is arbitrary, $\zeta^*\in F$; as ζ_0 is arbitrary, F is cofinal. **Q**

(iii) In particular, if $f:\kappa\to\kappa$ is any function, then $\{\xi:\xi<\kappa,\ f(\eta)<\xi$ for every $\eta<\xi\}$ is a closed cofinal set in κ, being the diagonal intersection of $\langle\kappa\setminus(f(\xi)+1)\rangle_{\xi<\kappa}$.

(d) If α has uncountable cofinality, $\mathcal{F}$ is a non-empty family of closed cofinal sets in α and $\#(\mathcal{F})<\operatorname{cf}\alpha$, then $\bigcap\mathcal{F}$ is a closed cofinal set in α. **P** Being the intersection of closed sets it is surely closed. Set $\lambda=\max(\omega,\#(\mathcal{F}))$ and let $\langle F_\xi\rangle_{\xi<\lambda}$ run over $\mathcal{F}$ with cofinal repetitions. Starting from any $\zeta_0<\alpha$, we can choose $\langle\zeta_\xi\rangle_{1\le\xi\le\lambda}$ such that

– if $\xi<\lambda$ then $\zeta_\xi\le\zeta_{\xi+1}\in F_\xi$;

– if $\xi\le\lambda$ is a non-zero limit ordinal, $\zeta_\xi=\sup_{\eta<\xi}\zeta_\eta$.

(Because $\lambda<\operatorname{cf}\alpha$, $\zeta_\xi<\alpha$ for every ξ.) Now $\zeta_0\le\zeta_\lambda<\alpha$, and if $F\in\mathcal{F}$, $\zeta<\zeta_\lambda$ there is a $\xi<\lambda$ such that $F=F_\xi$ and $\zeta\le\zeta_\xi$, in which case $\zeta\le\zeta_{\xi+1}\le\zeta_\lambda$ and $\zeta_{\xi+1}\in F$. This shows that either $\zeta_\lambda\in F$ or $\zeta_\lambda=\sup(F\cap\zeta_\lambda)$, in which case again $\zeta_\lambda\in F$. As F is arbitrary, $\zeta_\lambda\in\bigcap\mathcal{F}$; as ζ_0 is arbitrary, $\bigcap\mathcal{F}$ is cofinal. **Q**

In particular, the intersection of any sequence of closed cofinal sets in ω_1 is again a closed cofinal set in ω_1.

4A1C Stationary sets (a) Let κ be a cardinal. A subset of κ is **stationary** in κ if it meets every closed cofinal set in κ; otherwise it is **non-stationary**.

(b) If κ is a cardinal of uncountable cofinality, the intersection of any stationary subset of κ with a closed cofinal set in κ is again a stationary set (because the intersection of two closed cofinal sets is a closed cofinal set); the family of non-stationary subsets of κ is a σ-ideal, the **non-stationary ideal** of κ. (Kunen 80, II.6.9; Just & Weese 97, Lemma 21.11; Jech 03, p. 93; Levy 79, IV.4.35.)

(c) Pressing-Down Lemma (Fodor's theorem) If κ is a regular uncountable cardinal, $A\subseteq\kappa$ is stationary and $f:A\to\kappa$ is such that $f(\xi)<\xi$ for every $\xi\in A$, then there is a stationary set $B\subseteq A$ such that f is constant on B. (Kunen 80, II.6.15; Just & Weese 97, Theorem 21.2; Jech 78, Theorem 22; Jech 03, 8.7; Levy 79, IV.4.40.)

(d) There are disjoint stationary sets A, $B\subseteq\omega_1$. (This is easily deduced from 419G or 438Cd, and is also a special case of very much stronger results. See 541Ya in Volume 5, or Kunen 80, II.6.12; Just & Weese 97, Corollary 23.4; Jech 78, p. 59; Jech 03, 8.8; Levy 79, IV.4.48.)

4A1D Δ-systems (a) A family $\langle I_\xi\rangle_{\xi\in A}$ of sets is a **Δ-system** with **root** I if $I_\xi\cap I_\eta=I$ for all distinct ξ, $\eta\in A$.

(b) Δ-system Lemma If $\#(A)$ is a regular uncountable cardinal and $\langle I_\xi\rangle_{\xi\in A}$ is any family of finite sets, there is a set $D\subseteq A$ such that $\#(D)=\#(A)$ and $\langle I_\xi\rangle_{\xi\in D}$ is a Δ-system. (Kunen 80, II.1.6; Just & Weese 97, Theorem 16.3. For the present volume we need only the case $\#(A)=\omega_1$, which is treated in Jech 78, p. 225 and Jech 03, 9.18.)

4A1E Free sets (a) Let A be a set of cardinal at least ω_2, and $\langle J_\xi\rangle_{\xi\in A}$ a family of countable sets. Then there are distinct ξ, $\eta\in A$ such that $\xi\notin J_\eta$ and $\eta\notin J_\xi$. **P** Let $K\subseteq A$ be a set of cardinal ω_1, and set $L=K\cup\bigcup_{i\in K}J_i$. Then L has cardinal ω_1, so there is a $\xi\in A\setminus L$. Now there is an $\eta\in K\setminus J_\xi$, and this pair (ξ,η) serves. **Q**

(b) If $\langle K_\xi\rangle_{\xi\in A}$ is a disjoint family of sets indexed by an uncountable subset A of ω_1, and $\langle J_\eta\rangle_{\eta<\omega_1}$ is a family of countable sets, there is an uncountable $B\subseteq A$ such that $K_\xi\cap J_\eta=\emptyset$ whenever η, $\xi\in B$ and $\eta<\xi$. **P** Choose $\langle\zeta_\xi\rangle_{\xi<\omega_1}$ inductively in such a way that $\zeta_\xi\in A$ and $K_{\zeta_\xi}\cap J_{\zeta_\eta}=\emptyset$, $\zeta_\xi>\zeta_\eta$ for every $\eta<\xi$. Set $B=\{\zeta_\xi:\xi<\omega_1\}$. **Q**

4A1F Selecting subsequences (a) Let $\langle K_i\rangle_{i\in I}$ be a countable family of sets such that $\bigcap_{i\in J}K_i$ is infinite for every finite subset J of I. Then there is an infinite set K such that $K\setminus K_i$ is finite and $K_i\setminus K$ is infinite for every

$i \in I$. **P** We can suppose that $I \subseteq \mathbb{N}$. Choose $\langle k_n \rangle_{n\in\mathbb{N}}$ inductively such that $k_n \in \bigcap_{i\in I, i\le n} K_i \setminus \{k_i : i < n\}$ for every $n \in \mathbb{N}$, and set $K = \{k_{2n} : n \in \mathbb{N}\}$. **Q**

Consequently there is a family $\langle K_\xi \rangle_{\xi<\omega_1}$ of infinite subsets of $\mathbb{N}$ such that $K_\xi \setminus K_\eta$ is finite if $\eta \le \xi$, infinite if $\xi < \eta$. (Choose the K_ξ inductively.)

(b) Let $\langle \mathcal{J}_i \rangle_{i\in I}$ be a countable family of subsets of $[\mathbb{N}]^\omega$ such that $\mathcal{J}_i \cap \mathcal{P}K \ne \emptyset$ for every $K \in [\mathbb{N}]^\omega$ and $i \in I$. Then there is an infinite $K \subseteq \mathbb{N}$ such that for every $i \in I$ there is a $J \in \mathcal{J}_i$ such that $K \setminus J$ is finite. **P** The case $I = \emptyset$ is trivial; suppose that $\langle i_n \rangle_{n\in\mathbb{N}}$ runs over I. Choose K_n, k_n inductively, for $n \in \mathbb{N}$, by taking

$$K_0 = \mathbb{N}, \quad k_n \in K_n, \quad K_{n+1} \subseteq K_n \setminus \{k_n\}, \quad K_{n+1} \in \mathcal{J}_{i_n}$$

for every n; set $K = \{k_n : n \in \mathbb{N}\}$. **Q**

4A1G Ramsey's theorem If $n \in \mathbb{N}$, K is finite and $h : [\mathbb{N}]^n \to K$ is any function, there is an infinite $I \subseteq \mathbb{N}$ such that h is constant on $[I]^n$. (Bollobás 79, p. 105, Theorem 3; Just & Weese 97, 15.3; Jech 78, 29.1; Jech 03, 9.1; Levy 79, IX.3.7. For the present volume we need only the case $n = \#(K) = 2$.)

4A1H The Marriage Lemma again In 449L it will be useful to have an infinitary version of the Marriage Lemma available.

Proposition Let X and Y be sets, and $R \subseteq X\times Y$ a set such that $R[\{x\}]$ is finite for every $x \in X$ and $\#(R[I]) \ge \#(I)$ for every finite set $I \subseteq X$. Then there is an injective function $f : X \to Y$ such that $(x, f(x)) \in R$ for every $x \in X$.

proof For each finite $J \subseteq X$ there is an injective function $f_J : J \to Y$ such that $f_J \subseteq R$ (identifying f_J with its graph), by the ordinary Marriage Lemma (3A1K) applied to $R \cap (J \times R[J])$. Let $\mathcal{F}$ be any ultrafilter on $[X]^{<\omega}$ containing $\{J : I \subseteq J \in [X]^{<\omega}\}$ for every $I \in [X]^{<\omega}$; then for each $x \in X$ there must be an $f(x) \in R[\{x\}]$ such that $\{J : f_J(x) = f(x)\} \in \mathcal{F}$, because $R[\{x\}]$ is finite. Now $f \subseteq R$ is a function from X to Y and must be injective, because for any x, $x' \in X$ there is a $J \in [X]^{<\omega}$ such that f and f_J agree on $\{x, x'\}$.

4A1I Filters (a) Let X be a non-empty set. If $\mathcal{E} \subseteq \mathcal{P}X$ is non-empty and has the finite intersection property,

$$\mathcal{F} = \{A : A \subseteq X,\ A \supseteq \bigcap \mathcal{E}' \text{ for some non-empty finite } \mathcal{E}' \subseteq \mathcal{E}\}$$

is the smallest filter on X including $\mathcal{E}$, the filter **generated** by $\mathcal{E}$.

If $\mathcal{E} \subseteq \mathcal{P}X$ is non-empty and downwards-directed, then it has the finite intersection property iff it does not contain $\emptyset$; in this case we say that $\mathcal{E}$ is a **filter base**; $\mathcal{F} = \{A : A \subseteq X,\ A \supseteq E \text{ for some } E \in \mathcal{E}\}$, and $\mathcal{E}$ is a base for the filter $\mathcal{F}$.

In general, if $\mathcal{E}$ is a family of subsets of X, then there is a filter on X including $\mathcal{E}$ iff $\mathcal{E}$ has the finite intersection property; in this case, there is an ultrafilter on X including $\mathcal{E}$ (2A1O).

(b) If κ is a cardinal and $\mathcal{F}$ is a filter then $\mathcal{F}$ is **κ-complete** if $\bigcap \mathcal{E} \in \mathcal{F}$ whenever $\mathcal{E} \subseteq \mathcal{F}$ and $0 < \#(\mathcal{E}) < \kappa$. Every filter is ω-complete.

(c) A filter $\mathcal{F}$ on a regular uncountable cardinal κ is **normal** if (α) $\kappa\setminus\xi \in \mathcal{F}$ for every $\xi < \kappa$ (β) whenever $\langle F_\xi \rangle_{\xi<\kappa}$ is a family in $\mathcal{F}$, its diagonal intersection belongs to $\mathcal{F}$.

4A1J Lemma A normal filter $\mathcal{F}$ on a regular uncountable cardinal κ is κ-complete.

proof If $\lambda < \kappa$ and $\langle F_\xi \rangle_{\xi<\lambda}$ is a family in $\mathcal{F}$, set $F_\xi = \kappa$ for $\lambda \le \xi < \kappa$, and let F be the diagonal intersection of $\langle F_\xi \rangle_{\xi<\kappa}$; then $\bigcap_{\xi<\lambda} F_\xi \supseteq F \setminus \lambda$ belongs to $\mathcal{F}$.

4A1K Theorem Let X be a set and $\mathcal{F}$ a non-principal ω_1-complete ultrafilter on X. Let κ be the least cardinal of any non-empty set $\mathcal{E} \subseteq \mathcal{F}$ such that $\bigcap \mathcal{E} \notin \mathcal{F}$. Then κ is a regular uncountable cardinal, $\mathcal{F}$ is κ-complete, and there is a function $g : X \to \kappa$ such that $g[[\mathcal{F}]]$ is a normal ultrafilter on κ.

proof (a) By the definition of κ, $\mathcal{F}$ is κ-complete. Because $\mathcal{F}$ is ω_1-complete, $\kappa > \omega$. Let H be the set of all functions $h : X \to \kappa$ such that $h^{-1}[\kappa \setminus \xi] \in \mathcal{F}$ for every $\xi < \kappa$. Then H is not empty. **P** Let $\langle E_\xi \rangle_{\xi<\kappa}$ be a family in $\mathcal{F}$ such that $E = \bigcap_{\xi<\kappa} E_\xi \notin \mathcal{F}$. Because $\mathcal{F}$ is an ultrafilter, $X \setminus E \in \mathcal{F}$. Set $h(x) = 0$ if $x \in E$, $h(x) = \min\{\xi : x \notin E_\xi\}$ if $x \in X \setminus E$; then

$$h^{-1}[\kappa \setminus \xi] \supseteq (X \setminus E) \cap \bigcap_{\eta<\xi} E_\eta \in \mathcal{F}$$

for every $\xi < \kappa$, because $\mathcal{F}$ is κ-complete, so $h \in H$. **Q**

(b) For h, $h' \in H$, say that $h \prec h'$ if $\{x : h(x) < h'(x)\} \in \mathcal{F}$. Then there is a $g \in H$ such that $h \not\prec g$ for any $h \in H$. **P?** Otherwise, there is a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in H such that $h_{n+1} \prec h_n$ for every $n \in \mathbb{N}$. In this case $E_n = \{x : h_{n+1}(x) < h_n(x)\} \in \mathcal{F}$ for every n. Because $\mathcal{F}$ is ω_1-complete, there is an $x \in \bigcap_{n\in\mathbb{N}} E_n$; but now $\langle h_n(x)\rangle_{n\in\mathbb{N}}$ is a strictly decreasing sequence of ordinals, which is impossible. **XQ**

(c) I should check that κ is regular. **P** If $\langle\alpha_\xi\rangle_{\xi<\lambda}$ is any family in κ with $\lambda < \kappa$, then $g^{-1}[\kappa \setminus \alpha_\xi] \in \mathcal{F}$ for every ξ, so (because $\mathcal{F}$ is κ-complete)

$$g^{-1}[\kappa \setminus \sup_{\xi<\lambda} \alpha_\xi] = \bigcap_{\xi<\lambda} g^{-1}[\kappa \setminus \alpha_\xi] \in \mathcal{F},$$

and $\sup_{\xi<\lambda} \alpha_\xi \ne \kappa$. **Q**

(d) The image filter $g[[\mathcal{F}]]$ is an ultrafilter on κ, by 2A1N. Because $g \in H$, $g^{-1}[\kappa \setminus \xi] \in \mathcal{F}$ and $\kappa \setminus \xi \in g[[\mathcal{F}]]$ for any $\xi < \kappa$. **?** Suppose, if possible, that $g[[\mathcal{F}]]$ is not normal. Then there is a family $\langle A_\xi\rangle_{\xi<\kappa}$ in $g[[\mathcal{F}]]$ such that its diagonal intersection A does not belong to $g[[\mathcal{F}]]$, that is, $g^{-1}[A] \notin \mathcal{F}$ and $X \setminus g^{-1}[A] \in \mathcal{F}$. Define $h : X \to \kappa$ by setting

$$\begin{aligned} h(x) &= 0 \text{ if } g(x) \in A,\\ &= \min\{\eta : \eta < g(x),\, g(x) \notin A_\eta\} \text{ if } g(x) \notin A. \end{aligned}$$

Then

$$h^{-1}[\kappa \setminus \xi] \supseteq (X \setminus g^{-1}[A]) \cap \bigcap_{\eta<\xi} g^{-1}[A_\eta] \in \mathcal{F}$$

for every $\xi < \kappa$. Thus $h \in H$. But also $h(x) < g(x)$ for every $x \in X \setminus g^{-1}[A]$, so $h \prec g$, contrary to the choice of g. **X**

Thus $g[[\mathcal{F}]]$ is a normal filter, and the theorem is proved.

4A1L Theorem Let κ be a regular uncountable cardinal, and $\mathcal{F}$ a normal ultrafilter on κ. If $S \subseteq [\kappa]^{<\omega}$, there is a set $F \in \mathcal{F}$ such that, for each $n \in \mathbb{N}$, $[F]^n$ is either a subset of S or disjoint from S.

proof (a) For each $n \in \mathbb{N}$ there is an $F_n \in \mathcal{F}$ such that either $[F_n]^n \subseteq S$ or $[F_n]^n \cap S = \emptyset$. **P** Induce on n. If $n = 0$ we can take $F_n = \kappa$, because $[\kappa]^0 = \{\emptyset\}$. For the inductive step to $n+1$, set $S_\xi = \{I : I \in [\kappa]^{<\omega},\, I \cup \{\xi\} \in S\}$ for each $\xi < \kappa$. By the inductive hypothesis, there is for each $\xi < \kappa$ a set $E_\xi \in \mathcal{F}$ such that either $[E_\xi]^n \subseteq S_\xi$ or $[E_\xi]^n \cap S_\xi = \emptyset$. Let E be the diagonal intersection of $\langle E_\xi\rangle_{\xi<\kappa}$, so that $E \in \mathcal{F}$.

Suppose that $A = \{\xi : [E_\xi]^n \subseteq S_\xi\}$ belongs to $\mathcal{F}$. Then $E \cap A \in \mathcal{F}$. If $I \in [E \cap A]^{n+1}$, set $\xi = \min I$. Then $I \setminus \{\xi\} \subseteq E_\xi$, so that $I \setminus \{\xi\} \in S_\xi$ and $I \in S$. Thus $[E \cap A]^{n+1} \subseteq S$. Similarly, if $A \notin \mathcal{F}$, then $E \setminus A \in \mathcal{F}$ and $[E \setminus A]^{n+1} \cap S = \emptyset$. Thus we can take one of $E \cap A$, $E \setminus A$ for F_{n+1}, and the induction continues. **Q**

(b) At the end of the induction, take $F = \bigcap_{n\in\mathbb{N}} F_n$; this serves.

4A1M Ostaszewski's ♣ This is the statement

> Let Ω be the family of non-zero countable limit ordinals. Then there is a family $\langle\theta_\xi(n)\rangle_{\xi\in\Omega,n\in\mathbb{N}}$ such that (α) for each $\xi \in \Omega$, $\langle\theta_\xi(n)\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence with supremum ξ (β) for any uncountable $A \subseteq \omega_1$ there is a $\xi \in \Omega$ such that $\theta_\xi(n) \in A$ for every $n \in \mathbb{N}$.

This is an immediate consequence of Jensen's $\diamondsuit$ (Just & Weese 97, Exercise 22.9), which is itself a consequence of Gödel's Axiom of Constructibility (Kunen 80, §II.7; Just & Weese 97, §22; Jech 78, §22; Jech 03, 13.21).

4A1N Lemma Assume ♣. Then there is a family $\langle C_\xi\rangle_{\xi<\omega_1}$ of sets such that (i) $C_\xi \subseteq \xi$ for every $\xi < \omega_1$ (ii) $C_\xi \cap \eta$ is finite whenever $\eta < \xi < \omega_1$ (iii) for any uncountable sets A, $B \subseteq \omega_1$ there is a $\xi < \omega_1$ such that $A \cap C_\xi$ and $B \cap C_\xi$ are both infinite.

proof (a) Let $\langle\theta_\xi(n)\rangle_{\xi\in\Omega,n\in\mathbb{N}}$ be a family as in 4A1M. Let $f : \omega_1 \to [\omega_1]^2$ be a surjection (3A1Cd). For $\xi \in \Omega$, set

$$C_\xi = \bigcup_{i\in\mathbb{N}} f(\theta_\xi(i+1)) \cap \xi \setminus \theta_\xi(i).$$

Then $C_\xi \subseteq \xi$, and if $\eta < \xi$ there is some $n \in \mathbb{N}$ such that $\theta_\xi(n) \ge \eta$, so that

$$C_\xi \cap \eta \subseteq \bigcup_{i\le n} f(\theta_\xi(i))$$

is finite. For $\xi\in\omega_1\setminus\Omega$ set $C_\xi=\emptyset$. Then $\langle C_\xi\rangle_{\xi<\omega_1}$ satisfies (i) and (ii) above.

(b) Now suppose that A, $B\subseteq\omega_1$ are uncountable. Choose $\langle\alpha_\xi\rangle_{\xi<\omega_1}$, $\langle\beta_\xi\rangle_{\xi<\omega_1}$, $\langle I_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. β_ξ is to be the smallest ordinal such that $\{\alpha_\eta:\eta<\xi\}\cup\bigcup_{\eta<\xi}I_\eta\subseteq\beta_\xi$; I_ξ is to be a doubleton subset of $\omega_1\setminus(\beta_\xi\cup\bigcup_{\eta\le\beta_\xi}f(\eta))$ meeting both A and B; and $\alpha_\xi<\omega_1$ is to be such that $f(\alpha_\xi)=I_\xi$. Set $D=\{\alpha_\xi:\xi<\omega_1\}$. This construction ensures that $\langle\alpha_\xi\rangle_{\xi<\omega_1}$ and $\langle\beta_\xi\rangle_{\xi<\omega_1}$ are strictly increasing, with $\beta_\xi<\alpha_\xi<\beta_{\xi+1}$ for every ξ, so that $f(\delta)$ meets both A and B for every $\delta\in D$, while $f(\delta)\subseteq\delta'$ and $f(\delta')\cap(\delta\cup f(\delta))=\emptyset$ whenever $\delta<\delta'$ in D.

By the choice of $\langle\theta_\xi(n)\rangle_{\xi\in\Omega,n\in\mathbb{N}}$, there is a $\xi\in\Omega$ such that $\theta_\xi(n)\in D$ for every $n\in\mathbb{N}$. But this means that

$$f(\theta_\xi(i))\subseteq\theta_\xi(i+1)\subseteq\xi,\quad f(\theta_\xi(i+1))\cap(\theta_\xi(i)\cup f(\theta_\xi(i)))=\emptyset$$

for every $i\in\mathbb{N}$, so $C_\xi=\bigcup_{i\ge1}f(\theta_\xi(i))$ meets both A and B in infinite sets.

4A1O The size of σ-algebras: Proposition Let $\mathfrak{A}$ be a Boolean algebra, B a subset of $\mathfrak{A}$, and $\mathfrak{B}$ the σ-subalgebra of $\mathfrak{A}$ generated by B (331E). Then $\#(\mathfrak{B})\le\max(4,\#(B^{\mathbb{N}}))$. In particular, if $\#(B)\le\mathfrak{c}$ then $\#(\mathfrak{B})\le\mathfrak{c}$.

proof (a) If $\#(B)\le1$, this is trivial, since then $\#(\mathfrak{B})\le4$. So we need consider only the case $\#(B)\ge2$.

(b) Set $\kappa=\#(B^{\mathbb{N}})$; then whenever $\#(A)\le\kappa$, that is, there is an injection from A to $B^{\mathbb{N}}$, then

$$\#(A^{\mathbb{N}})\le\#((B^{\mathbb{N}})^{\mathbb{N}})=\#(B^{\mathbb{N}\times\mathbb{N}})=\#(B^{\mathbb{N}})=\kappa.$$

As we are supposing that B has more than one element, $\kappa\ge\#(\{0,1\}^{\mathbb{N}})=\mathfrak{c}\ge\omega_1$.

(c) Define $\langle B_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. $B_0=B\cup\{0\}$. Given $\langle B_\eta\rangle_{\eta<\xi}$, where $0<\xi<\omega_1$, set $B'_\xi=\bigcup_{\eta<\xi}B_\eta$ and

$$\begin{aligned}B_\xi=\{1\setminus b:b\in B'_\xi\}&\\ \cup\{\sup_{n\in\mathbb{N}}b_n:\langle b_n\rangle_{n\in\mathbb{N}}&\text{ is a sequence in }B'_\xi\text{ with a supremum in }\mathfrak{A}\};\end{aligned}$$

continue.

An easy induction on ξ (relying on 3A1Cc and (b) above) shows that every B_ξ has cardinal at most κ. So $C=\bigcup_{\xi<\omega_1}B_\xi$ has cardinal at most κ.

(d) Now $\langle B_\xi\rangle_{\xi<\omega_1}$ is a non-decreasing family, so if $\langle c_n\rangle_{n\in\mathbb{N}}$ is any sequence in C there is some $\xi<\omega_1$ such that every c_n belongs to $B_\xi\subseteq B'_{\xi+1}$. But this means that if $\sup_{n\in\mathbb{N}}c_n$ is defined in $\mathfrak{A}$, it belongs to $B_{\xi+1}\subseteq C$. At the same time,

$$1\setminus c_0\in B_{\xi+1}\subseteq C.$$

This shows that C is closed under complementation and countable suprema; since it contains 0, it is a σ-subalgebra of $\mathfrak{A}$; since it includes B, it includes $\mathfrak{B}$, and $\#(\mathfrak{B})\le\#(C)\le\kappa$, as claimed.

(d) Finally, if $\#(B)\le\mathfrak{c}$, $\#(\mathfrak{B})\le\max(4,\#(B^{\mathbb{N}}))\le\mathfrak{c}$ by 4A1A(c-ii).

4A1P An incidental fact If I is a countable set and $\epsilon>0$, there is a family $\langle\epsilon_i\rangle_{i\in I}$ of strictly positive real numbers such that $\sum_{i\in I}\epsilon_i\le\epsilon$. **P** Let $f:I\to\mathbb{N}$ be an injection and set $\epsilon_i=2^{-f(i)-1}\epsilon$. **Q**

4A2 General topology

Even more than in previous volumes, naturally enough, the work of this volume depends on results from general topology. We have now reached the point where some of the facts I rely on are becoming hard to find as explicitly stated theorems in standard textbooks. I find myself therefore writing out rather a lot of proofs. You should not suppose that the results to which I attach proofs, rather than references, are particularly deep; on the contrary, in many cases I am merely spelling out solutions to classic exercises.

The style of 'general' topology, as it has evolved over the last hundred years, is to develop a language capable of squeezing the utmost from every step of argument. While this does sometimes lead to absurdly obscure formulations, it remains a natural, and often profitable, response to the remarkably dense network of related ideas in this area. I therefore follow the spirit of the subject in giving the results I need in the full generality achievable within the terminology I use. For the convenience of anyone coming to the theory for the first time, I repeat some of them in the forms in which they are actually applied. I should remark, however, that in some cases materially stronger results can be proved with little extra effort; as always, this appendix is to be thought of not as a substitute for a thorough study of the subject, but as a guide connecting standard approaches to the general theory with the special needs of this volume.

4A2A Definitions I begin the section with a glossary of terms not defined elsewhere.

Baire space A topological space X is a **Baire space** if $\bigcap_{n\in\mathbb{N}} G_n$ is dense in X whenever $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of dense open subsets of X.

Base of neighbourhoods If X is a topological space and $x \in X$, a **base of neighbourhoods** of x is a family $\mathcal{V}$ of neighbourhoods of x such that every neighbourhood of x includes some member of $\mathcal{V}$.

boundary If X is a topological space and $A \subseteq X$, the **boundary** of A is $\partial A = \overline{A} \setminus \operatorname{int} A = \overline{A} \cap \overline{X \setminus A}$.

càdlàg If X is a Hausdorff space, a function $f : [0,\infty[\, \to X$ is **càdlàg** ('continue à droit, limite à gauche') (or **RCLL** ('right continuous, left limits'), an ***R*-function**,) if $\lim_{s\downarrow t} f(s) = f(t)$ for every $t \ge 0$ and $\lim_{s\uparrow t} f(s)$ is defined in X for every $t > 0$.

càllàl If X is a Hausdorff space, a function $f : [0,\infty[\, \to X$ is **càllàl** ('continue à l'une, limite à l'autre') if $f(0) = \lim_{s\downarrow 0} f(s)$ and, for every $t > 0$, $\lim_{s\downarrow t} f(s)$ and $\lim_{s\uparrow t} f(s)$ are defined in X, and at least one of them is equal to $f(t)$.

Čech-complete A completely regular Hausdorff topological space X is **Čech-complete** if it is homeomorphic to a G_δ subset of a compact Hausdorff space.

closed interval Let X be a totally ordered set. A **closed interval** in X is an interval of one of the forms $\emptyset$, $[x,y]$, $]-\infty,y]$, $[x,\infty[$ or $X = \,]-\infty,\infty[$ where x, $y \in X$ (see the definition of 'interval' below).

coarser topology If $\mathfrak{S}$ and $\mathfrak{T}$ are two topologies on a set X, we say that $\mathfrak{S}$ is **coarser** than $\mathfrak{T}$ if $\mathfrak{S} \subseteq \mathfrak{T}$. (Equality allowed.)

compact support Let X be a topological space and $f : X \to \mathbb{R}$ a function. I say that f has **compact support** if $\overline{\{x : x \in X,\, f(x) \ne 0\}}$ is compact in X.

countably compact A topological space X is **countably compact** if every countable open cover of X has a finite subcover. (**Warning!** some authors reserve the term for Hausdorff spaces.) A subset of a topological space is countably compact if it is countably compact in its subspace topology.

countably paracompact A topological space X is **countably paracompact** if given any countable open cover $\mathcal{G}$ of X there is a locally finite family $\mathcal{H}$ of open sets which refines $\mathcal{G}$ and covers X. (**Warning!** some authors reserve the term for Hausdorff spaces.)

countably tight A topological space X is **countably tight** (or has **countable tightness**) if whenever $A \subseteq X$ and $x \in \overline{A}$ there is a countable set $B \subseteq A$ such that $x \in \overline{B}$.

direct sum, disjoint union Let $\langle (X_i,\mathfrak{T}_i)\rangle_{i\in I}$ be a family of topological spaces, and set $X = \{(x,i) : i \in I,\, x \in X_i\}$. The **disjoint union topology** on X is $\mathfrak{T} = \{G : G \subseteq X,\, \{x : (x,i) \in G\} \in \mathfrak{T}_i$ for every $i \in I\}$; $(X,\mathfrak{T})$ is the **(direct) sum** of $\langle (X_i,\mathfrak{T}_i)\rangle_{i\in I}$.

If X is a set, $\langle X_i\rangle_{i\in I}$ a partition of X, and $\mathfrak{T}_i$ a topology on X_i for every $i \in I$, then the **disjoint union topology** on X is $\{G : G \subseteq X,\, G \cap X_i \in \mathfrak{T}_i$ for every $i \in I\}$.

dyadic A Hausdorff space is **dyadic** if it is a continuous image of $\{0,1\}^I$ for some set I.

equicontinuous If X is a topological space, $(Y,\mathcal{W})$ a uniform space, and F a set of functions from X to Y, then F is **equicontinuous** if for every $x \in X$ and $W \in \mathcal{W}$ the set $\{y : (f(x),f(y)) \in W$ for every $f \in F\}$ is a neighbourhood of x.

finer topology If $\mathfrak{S}$ and $\mathfrak{T}$ are two topologies on a set X, we say that $\mathfrak{S}$ is **finer** than $\mathfrak{T}$ if $\mathfrak{S} \supseteq \mathfrak{T}$. (Equality allowed.)

first-countable A topological space X is **first-countable** if every point has a countable base of neighbourhoods.

half-open Let X be a totally ordered set. A **half-open interval** in X is a set of one of the forms $[x,y[$, $]x,y]$ where x, $y \in X$ and $x < y$ (see the definition of 'interval' below).

hereditarily Lindelöf A topological space is **hereditarily Lindelöf** if every subspace is Lindelöf.

hereditarily metacompact A topological space is **hereditarily metacompact** if every subspace is metacompact.

indiscrete If X is any set, the **indiscrete** topology on X is the topology $\{\emptyset, X\}$.

interval Let $(P,\le)$ be a partially ordered set. An **interval** in P is a set of one of the forms $[p,q] = \{r : p \le r \le q\}$, $[p,q[\, = \{r : p \le r < q\}$, $]p,q] = \{r : p < r \le q\}$, $]p,q[\, = \{r : p < r < q\}$, $[p,\infty[\, = \{r : p \le r\}$, $]-\infty,q] = \{r : r \le q\}$, $]p,\infty[\, = \{r : p < r\}$, $]-\infty,q[\, = \{r : r < q\}$, $]-\infty,\infty[\, = P$, where p, $q \in P$. Note that every interval is order-convex, but even in a totally ordered set not every order-convex set need be an interval in this sense; an interval always has end-points, if we allow $\pm\infty$.

irreducible If X and Y are topological spaces, a continuous surjection $f : X \to Y$ is **irreducible** if $f[F] \ne Y$ for any closed proper subset F of X.

isolated If X is a topological space, a family $\mathcal{A}$ of subsets of X is **isolated** if $A \cap \overline{\bigcup(\mathcal{A} \setminus \{A\})}$ is empty for every $A \in \mathcal{A}$; that is, if $\mathcal{A}$ is disjoint and every member of $\mathcal{A}$ is a relatively open set in $\bigcup \mathcal{A}$.

Lindelöf A topological space is **Lindelöf** if every open cover has a countable subcover. (**Warning!** some authors reserve the term for regular spaces.)

Lipschitz If (X, ρ) and (Y, σ) are metric spaces, a function $f : X \to Y$ is γ**-Lipschitz**, where $\gamma \ge 0$, if $\sigma(f(x), f(y)) \le \gamma\rho(x, y)$ for all x, $y \in X$. $f : X \to Y$ is **Lipschitz** if it is γ-Lipschitz for some $\gamma \ge 0$.

locally finite If X is a topological space, a family $\mathcal{A}$ of subsets of X is **locally finite** if for every $x \in X$ there is an open set which contains x and meets only finitely many members of $\mathcal{A}$.

lower semi-continuous If X is a topological space and T a totally ordered set, a function $f : X \to T$ is **lower semi-continuous** if $\{x : f(x) > t\}$ is open for every $t \in T$. (Cf. 225H, 3A3Cf.)

metacompact A topological space is **metacompact** if every open cover has a point-finite refinement which is an open cover. (**Warning!** some authors reserve the term for Hausdorff spaces.)

neighbourhood If X is a topological space and $x \in X$, a **neighbourhood** of x is any subset of X including an open set which contains x.

network Let $(X, \mathfrak{T})$ be a topological space. A **network** for $\mathfrak{T}$ is a family $\mathcal{E} \subseteq \mathcal{P}X$ such that whenever $x \in G \in \mathfrak{T}$ there is an $E \in \mathcal{E}$ such that $x \in E \subseteq G$.

normal A topological space X is **normal** if for any disjoint closed sets E, $F \subseteq X$ there are disjoint open sets G, H such that $E \subseteq G$ and $F \subseteq H$. (**Warning!** some authors reserve the term for Hausdorff spaces.)

open interval Let X be a totally ordered set. An **open interval** in X is a set of one of the the forms $]x, y[$, $]x, \infty[$, $]-\infty, x[$ or $]-\infty, \infty[\; = X$ where x, $y \in X$ (see the definition of 'interval' above).

open map If $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ are topological spaces, a function $f : X \to Y$ is **open** if $f[G] \in \mathfrak{S}$ for every $G \in \mathfrak{T}$.

order-convex Let $(P, \le)$ be a partially ordered set. A subset C of P is **order-convex** if $[p, q] = \{r : p \le r \le q\}$ is included in C whenever p, $q \in C$.

order topology Let $(X, \le)$ be a totally ordered set. Its **order topology** is that generated by intervals of the form $]x, \infty[\; = \{y : y > x\}$, $]-\infty, x[\; = \{y : y < x\}$ as x runs over X.

paracompact A topological space is **paracompact** if every open cover has a locally finite refinement which is an open cover. (**Warning!** some authors reserve the term for Hausdorff spaces.)

perfect A topological space is **perfect** if it is compact and has no isolated points.

perfectly normal A topological space is **perfectly normal** if it is normal and every closed set is a G_δ set. (**Warning!** remember that some authors reserve the term 'normal' for Hausdorff spaces.)

point-countable, point-finite A family $\mathcal{A}$ of sets is **point-countable** if no point belongs to more than countably many members of $\mathcal{A}$. Similarly, an indexed family $\langle A_i \rangle_{i \in I}$ of sets is **point-finite** if $\{i : x \in A_i\}$ is finite for every x.

Polish A topological space X is **Polish** if it is separable and its topology can be defined from a metric under which X is complete.

pseudometrizable A topological space $(X, \mathfrak{T})$ is **pseudometrizable** if $\mathfrak{T}$ is defined by a single pseudometric (2A3F).

refine(ment) If $\mathcal{A}$ is a family of sets, a **refinement** of $\mathcal{A}$ is a family $\mathcal{B}$ of sets such that every member of $\mathcal{B}$ is included in some member of $\mathcal{A}$; in this case I say that $\mathcal{B}$ **refines** $\mathcal{A}$. (**Warning!** I do not suppose that $\bigcup \mathcal{B} = \bigcup \mathcal{A}$.)

relatively countably compact If X is a topological space, a subset A of X is **relatively countably compact** if every sequence in A has a cluster point in X. (**Warning!** This is *not* the same as supposing that A is included in a countably compact subset of X.)

scattered A topological space X is **scattered** if every non-empty subset of X has an isolated point (in its subspace topology).

second-countable A topological space is **second-countable** if the topology has a countable base, that is, if its weight is at most ω.

semi-continuous *see* lower semi-continuous, upper semi-continuous.

sequential A topological space is **sequential** if every sequentially closed set in X is closed.

sequentially closed If X is a topological space, a subset A of X is **sequentially closed** if $x \in A$ whenever $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence in A converging to $x \in X$.

sequentially compact A topological space is **sequentially compact** if every sequence has a convergent sequence. A subset of a topological space is sequentially compact if it is sequentially compact in its subspace topology. (**Warning!** some authors reserve the term for Hausdorff spaces.)

sequentially continuous If X and Y are topological spaces, a function $f : X \to Y$ is **sequentially continuous** if $\langle f(x_n)\rangle_{n\in\mathbb{N}} \to f(x)$ in Y whenever $\langle x_n\rangle_{n\in\mathbb{N}} \to x$ in X.

subbase If $(X, \mathfrak{T})$ is a topological space, a **subbase** for $\mathfrak{T}$ is a family $\mathcal{U} \subseteq \mathfrak{T}$ which generates $\mathfrak{T}$, in the sense that $\mathfrak{T}$ is the coarsest topology on X including $\mathcal{U}$. (**Warning!** most authors reserve the term for families $\mathcal{U}$ with union X.)

totally bounded If $(X, \mathcal{W})$ is a uniform space, a subset A of X is **totally bounded** if for every $W \in \mathcal{W}$ there is a finite set $I \subseteq X$ such that $A \subseteq W[I]$. If (X, ρ) is a metric space, a subset of X is totally bounded if it is totally bounded for the associated uniformity (3A4B).

uniform convergence If X is a set, (Y, σ) is a metric space and $\mathcal{A}$ is a family of subsets of X then the **topology of uniform convergence** on members of $\mathcal{A}$ is the topology on Y^X generated by the pseudometrics $(f, g) \mapsto \min(1, \sup_{x\in A} \sigma(f(x), g(x)))$ as A runs over $\mathcal{A} \setminus \{\emptyset\}$. (It is elementary to verify that the formula here defines a pseudometric.)

upper semi-continuous If X is a topological space and T is a totally ordered set, a function $f : X \to T$ is **upper semi-continuous** if $\{x : f(x) < t\}$ is open for every $t \in T$.

weakly α-favourable A topological space $(X, \mathfrak{T})$ is **weakly α-favourable** if there is a function $\sigma : \bigcup_{n\in\mathbb{N}}(\mathfrak{T} \setminus \{\emptyset\})^{n+1} \to \mathfrak{T} \setminus \{\emptyset\}$ such that (i) $\sigma(G_0, \dots, G_n) \subseteq G_n$ whenever $G_0, \dots, G_n$ are non-empty open sets (ii) whenever $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{T} \setminus \{\emptyset\}$ such that $G_{n+1} \subseteq \sigma(G_0, \dots, G_n)$ for every n, then $\bigcap_{n\in\mathbb{N}} G_n$ is non-empty.

weight If X is a topological space, its **weight** $w(X)$ is the smallest cardinal of any base for the topology.

C_b If X is a topological space, $C_b(X)$ is the space of bounded continuous real-valued functions defined on X.

F_σ If X is a topological space, an $\mathbf{F}_\sigma$ set in X is one expressible as the union of a sequence of closed sets.

G_δ If X is a topological space, a $\mathbf{G}_\delta$ set in X is one expressible as the intersection of a sequence of open sets.

K_σ If X is a topological space, a $\mathbf{K}_\sigma$ set in X is one expressible as the union of a sequence of compact sets.

$\mathcal{P}X$ If X is any set, the **usual topology** on $\mathcal{P}X$ is that generated by the sets $\{a : a \subseteq X,\ a \cap J = K\}$ where $J \subseteq X$ is finite and $K \subseteq J$.

T_0 If $(X, \mathfrak{T})$ is a topological space, we say that it is $\mathbf{T}_0$ if for any two distinct points of X there is an open set containing one but not the other.

T_1 If $(X, \mathfrak{T})$ is a topological space, we say that it is $\mathbf{T}_1$ if singleton sets are closed.

π-base If $(X, \mathfrak{T})$ is a topological space, a **π-base** for $\mathfrak{T}$ is a set $\mathcal{U} \subseteq \mathfrak{T}$ such that every non-empty open set includes a non-empty member of $\mathcal{U}$.

σ-compact A topological space X is **σ-compact** if there is a sequence of compact subsets of X covering X.

σ-disjoint A family of sets is **σ-disjoint** if it is expressible as $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ where every $\mathcal{A}_n$ is disjoint.

σ-isolated If X is a topological space, a family of subsets of X is **σ-isolated** if it is expressible as $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ where every $\mathcal{A}_n$ is an isolated family.

σ-metrically-discrete If (X, ρ) is a metric space, a family of subsets of X is **σ-metrically-discrete** if it is expressible as $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ where $\rho(x, y) \ge 2^{-n}$ whenever $n \in \mathbb{N}$, A and B are distinct members of $\mathcal{A}_n$, $x \in A$ and $y \in B$.

4A2B Elementary facts about general topological spaces (a) Bases and networks (i) Let $(X, \mathfrak{T})$ be a topological space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then $\{X\} \cup \{U_0 \cap U_1 \cap \ldots \cap U_n : U_0, \dots, U_n \in \mathcal{U}\}$ is a base for $\mathfrak{T}$. (For this is a base for a topology, by 3A3Mc.)

(ii) Let X and Y be topological spaces, and $\mathcal{V}$ a subbase for the topology of Y. Then a function $f : X \to Y$ is continuous iff $f^{-1}[V]$ is open for every $V \in \mathcal{V}$. (Engelking 89, 1.4.1(ii)).

(iii) If X and Y are topological spaces, $\mathcal{E}$ is a network for the topology of Y, and $f : X \to Y$ is a function such that $f^{-1}[E]$ is open for every $E \in \mathcal{E}$, then f is continuous. (The topology generated by $\mathcal{E}$ includes the given topology on Y.)

(iv) If X is a topological space and $\mathcal{U}$ is a subbase for the topology of X, then a filter $\mathcal{F}$ on X converges to $x \in X$ iff $\{U : x \in U \in \mathcal{U}\} \subseteq \mathcal{F}$. (If the condition is satisfied, $\mathcal{F} \cup \{A : A \subseteq X,\ x \notin A\}$ is a topology on X including $\mathcal{U}$.)

(v) If X and Y are topological spaces with subbases $\mathcal{U}$, $\mathcal{V}$ respectively, then $\{U \times Y : U \in \mathcal{U}\} \cup \{X \times V : V \in \mathcal{V}\}$ is a subbase for the product topology of $X \times Y$. (Kuratowski 66, §15.I.)

(vi) If $\mathcal{U}$ is a (sub-)base for a topology on X, and $Y \subseteq X$, then $\{Y \cap U : U \in \mathcal{U}\}$ is a (sub-)base for the subspace topology of Y. (Császár 78, 2.3.13(e)-(f).)

(vii) If X is a topological space, $\mathcal{E}$ is a network for the topology of X, and Y is a subset of X, then $\{E \cap Y : E \in \mathcal{E}\}$ is a network for the topology of Y.

(viii) If X is a topological space and $\mathcal{A}$ is a (σ-)isolated family of subsets of X, then $\{A\cap Y : A\in\mathcal{A}'\}$ is (σ-)isolated whenever $Y\subseteq X$ and $\mathcal{A}'\subseteq\mathcal{A}$.

(ix) If a topological space X has a σ-isolated network, so has every subspace of X.

(b) If $\langle H_i\rangle_{i\in I}$ is a partition of a topological space X into open sets and $F_i\subseteq H_i$ is closed (either in X or in H_i) for each $i\in I$, then $F=\bigcup_{i\in I}F_i$ is closed in X. ($X\setminus F=\bigcup_{i\in I}(H_i\setminus F_i)$.)

(c) If X is a topological space, $A\subseteq X$ and $x\in X$, then $x\in\overline{A}$ iff there is an ultrafilter on X, containing A, which converges to x. ($\{A\}\cup\{G : x\in G\subseteq X,\ G$ is open$\}$ has the finite intersection property; use 4A1Ia.)

(d) Semi-continuity Let X be a topological space.

(i) A function $f:X\to\mathbb{R}$ is lower semi-continuous iff $-f$ is upper semi-continuous. (Čech 66, 18D.8.) A function $f:X\to\mathbb{R}$ is lower semi-continuous iff $\Omega=\{(x,\alpha): x\in X,\ \alpha\ge f(x)\}$ is closed in $X\times\mathbb{R}$. (If f is lower semi-continuous and $\alpha<\beta<f(x)$ then $\{y : f(y)>\beta\}\times\,]-\infty,\beta[$ is a neighbourhood of (x,α); so Ω is closed. If Ω is closed then for any $\gamma\in\mathbb{R}$ the set $\{x : f(x)>\gamma\}=\{x:(x,\gamma)\notin\Omega\}$ is open; so f is lower semi-continuous.)

(ii) If T is a totally ordered set, $f:X\to T$ is lower semi-continuous, Y is another topological space, and $g:Y\to X$ is continuous, then $fg:Y\to T$ is lower semi-continuous. ($\{y:(fg)(y)>t\}=g^{-1}[\{x:f(x)>t\}]$.) In particular, if $f:X\to T$ is lower semi-continuous and $Y\subseteq X$, then $f{\restriction}Y$ is lower semi-continuous. Similarly, if $f:X\to T$ is upper semi-continuous and $g:Y\to X$ is continuous, then $fg:Y\to T$ is upper semi-continuous.

(iii) If f, $g:X\to\,]-\infty,\infty]$ are lower semi-continuous so is $f+g:X\to\,]-\infty,\infty]$. (Čech 66, 18D.8.)

(iv) If f, $g:X\to[0,\infty]$ are lower semi-continuous so is $f\times g:X\to[0,\infty]$. (Čech 66, 18D.8.)

(v) If Φ is any non-empty set of lower semi-continuous functions from X to $[-\infty,\infty]$, then $x\mapsto\sup_{f\in\Phi}f(x): X\to[-\infty,\infty]$ is lower semi-continuous.

(vi) $f:X\to\mathbb{R}$ is continuous iff f is both upper semi-continuous and lower semi-continuous iff f and $-f$ are both lower semi-continuous.

(vii) If $f:X\to[-\infty,\infty]$ is lower semi-continuous, and $\mathcal{F}$ is a filter on X converging to $y\in X$, then $f(y)\le\liminf_{x\to\mathcal{F}}f(x)$.

(viii) If X is compact and not empty, and $f:X\to[-\infty,\infty]$ is lower semi-continuous then $K=\{x:f(x)=\inf_{y\in X}f(y)\}$ is non-empty and compact. **P** Setting $\gamma=\inf_{y\in X}f(y)\in[-\infty,\infty]$, $\{\{x:f(x)\le\alpha\}:\alpha>\gamma\}$ is a downwards-directed family of non-empty closed sets, so its intersection K is a non-empty closed set. **Q**

(ix) If f, $g:X\to[0,\infty]$ are lower semi-continuous and $f+g$ is continuous at $x\in X$ and finite there, then f and g are continuous at x. **P** If $\epsilon>0$ there is a neighbourhood G of x such that $(f+g)(y)\le(f+g)(x)+\epsilon$ for every $y\in G$ and $g(y)\ge g(x)-\epsilon$ for every $y\in G$, so that $f(y)\le f(x)+2\epsilon$ for every $y\in G$. **Q**

(e) Separable spaces (i) If $\langle A_i\rangle_{i\in I}$ is a countable family of separable subsets of a topological space X then $\bigcup_{i\in I}A_i$ and $\overline{\bigcup_{i\in I}A_i}$ are separable. (If $D_i\subseteq A_i$ is countable and dense for each i, $\bigcup_{i\in I}D_i$ is countable and dense in both $\bigcup_{i\in I}A_i$ and its closure.)

(ii) If $\langle X_i\rangle_{i\in I}$ is a family of separable topological spaces and $\#(I)\le\mathfrak{c}$, then $\prod_{i\in I}X_i$ is separable. (Engelking 89, 2.3.16.)

(iii) A continuous image of a separable topological space is separable. (Engelking 89, 1.4.11.)

(f) Open maps (i) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces, with product X. If $J\subseteq I$ is any set, and we write X_J for $\prod_{i\in J}X_i$, then the canonical map $x\mapsto x{\restriction}J:X\to X_J$ is open. (Engelking 89, p. 79.)

(ii) Let X and Y be topological spaces and $f:X\to Y$ a continuous open map. Then $\operatorname{int}f^{-1}[B]=f^{-1}[\operatorname{int}B]$ and $\overline{f^{-1}[B]}=f^{-1}[\overline{B}]$ for every $B\subseteq Y$. **P** Because f is continuous, $f^{-1}[\operatorname{int}B]$ is an open set included in $f^{-1}[B]$, so is included in $\operatorname{int}f^{-1}[B]$. Because f is open, $f[\operatorname{int}f^{-1}[B]]$ is an open set included in $f[f^{-1}[B]]\subseteq B$, so $f[\operatorname{int}f^{-1}[B]]\subseteq\operatorname{int}B$, that is, $\operatorname{int}f^{-1}[B]\subseteq f^{-1}[\operatorname{int}B]$. Now apply this to $Y\setminus B$ and take complements. **Q**

It follows that $f^{-1}[B]$ is nowhere dense in X whenever $B\subseteq Y$ is nowhere dense in Y. ($\operatorname{int}\overline{f^{-1}[B]}=\operatorname{int}f^{-1}[\overline{B}]=f^{-1}[\operatorname{int}\overline{B}]=\emptyset$.) If f is surjective and $B\subseteq Y$, then B is nowhere dense in Y iff $f^{-1}[B]$ is nowhere dense in X. (For $\operatorname{int}\overline{f^{-1}[B]}=f^{-1}[\operatorname{int}\overline{B}]$ is empty iff $\operatorname{int}\overline{B}$ is empty.)

(iii) Let X and Y be topological spaces and $f:X\to Y$ a continuous open map. Then $H\mapsto f^{-1}[H]$ is an order-continuous Boolean homomorphism from the regular open algebra of Y to the regular open algebra of X. **P** If $H\subseteq Y$ is a regular open set,

$$\operatorname{int}\overline{f^{-1}[H]}=\operatorname{int}f^{-1}[\overline{H}]=f^{-1}[\operatorname{int}\overline{H}]=f^{-1}[H]$$

by (ii), so $f^{-1}[H]$ is a regular open set in X. If $F \subseteq Y$ is nowhere dense, then $f^{-1}[F]$ is nowhere dense in X, as noted in (ii) above. By 314Ra, $H \mapsto f^{-1}[H] = \operatorname{int} \overline{f^{-1}[H]}$ is an order-continuous Boolean homomorphism from $\mathrm{RO}(Y)$ to $\mathrm{RO}(X)$. **Q** If f is surjective, then the homomorphism is injective (because $f^{-1}[H] \neq \emptyset$ whenever $H \neq \emptyset$), and for $H \subseteq Y$, H is a regular open set in Y iff $f^{-1}[H]$ is a regular open set in X (because in this case $f^{-1}[H] = f^{-1}[\operatorname{int} \overline{H}]$).

(iv) If X_0, Y_0, X_1, Y_1 are topological spaces, and $f_i : X_i \to Y_i$ is an open map for each i, then $(x_0, x_1) \mapsto (f_0(x_0), f_1(x_1)) : X_0 \times X_1 \to Y_0 \times Y_1$ is open. (Engelking 89, 2.3.29.)

(g) Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X.

(i) If $A \subseteq X$ is determined by coordinates in $J \subseteq I$ in the sense of 254M, then $\overline{A}$ and $\operatorname{int} A$ are also determined by coordinates in J. **P** Let $\pi : X \to \prod_{i \in J} X_i$ be the canonical map. Then $A = \pi^{-1}[\pi[A]]$, so (f) tells us that $\operatorname{int} A = \pi^{-1}[\operatorname{int} \pi[A]]$ and $\overline{A} = \pi^{-1}[\overline{\pi[A]}]$; but these are both determined by coordinates in J. **Q**

(ii) If $F \subseteq X$ is closed, there is a smallest set $J^* \subseteq I$ such that F is determined by coordinates in J^*.

P Let $\mathcal{J}$ be the family of all those sets $J \subseteq I$ such that F is determined by coordinates in $\mathcal{J}$. If J_1, $J_2 \in \mathcal{J}$, then $J_1 \cap J_2 \in \mathcal{J}$ (254Ta). Set $J^* = \bigcap \mathcal{J}$. **?** Suppose, if possible, that F is not determined by coordinates in J^*. Then there are $x \in F$, $y \in X \setminus F$ such that $x \restriction J^* = y \restriction J^*$. Because $X \setminus F$ is open, there is a finite set $K \subseteq I$ such that $z \notin F$ whenever $z \in X$ and $z \restriction K = y \restriction K$. Because $\mathcal{J}$ is closed under finite intersections, there is a $J \in \mathcal{J}$ such that $K \cap J = K \cap J^*$. Define $z \in X$ by setting $z(i) = x(i)$ for $i \in J$, $z(i) = y(i)$ for $i \in I \setminus J$. Then $z \restriction J = x \restriction J$, so $z \in F$, but $z \restriction K = y \restriction K$, so $z \notin F$. **X**

Thus $J^* \in \mathcal{J}$ and is the required smallest member of $\mathcal{J}$. **Q**

(h) Let X be a topological space.

(i) If $\mathcal{E}$ is a locally finite family of closed subsets of X, then $\bigcup \mathcal{E}'$ is closed for every $\mathcal{E}' \subseteq \mathcal{E}$. (Engelking 89, 1.1.11.)

(ii) If $\langle f_i \rangle_{i \in I}$ is a family in $C(X)$ such that $\langle \{x : f_i(x) \neq 0\} \rangle_{i \in I}$ is locally finite, then we have a continuous function $f : X \to \mathbb{R}$ defined by setting $f(x) = \sum_{i \in I} f_i(x)$ for every $x \in X$. **P** For any x, $\{i : f_i(x) \neq 0\}$ is finite, so f is well-defined. If $x_0 \in X$ and $\epsilon > 0$, there is a neighbourhood V of x_0 such that $J = \{i : i \in I, \ f_i(x) \neq 0$ for some $x \in V\}$ is finite; now there is a neighbourhood W of x_0, included in V, such that $\sum_{i \in J} |f_i(x) - \sum_{i \in J} f_i(x_0)| < \epsilon$ for every $x \in W$, so that $|f(x) - f(x_0)| < \epsilon$ for every $x \in W$. As x_0 and ϵ are arbitrary, f is continuous. **Q**

(i) Let X be a topological space and A, B two subsets of X. Then the boundary $\partial(A * B)$ is included in $\partial A \cup \partial B$, where $*$ is any of $\cup$, $\cap$, $\setminus$, $\triangle$. (Generally, if $F \subseteq X$, $\{A : \partial A \subseteq F\} = \{A : \overline{A} \setminus F \subseteq \operatorname{int} A\}$ is a subalgebra of $\mathcal{P}X$.)

(j) Let X be a topological space and D a dense subset of X, endowed with its subspace topology.

(i) A set $A \subseteq D$ is nowhere dense in D iff it is nowhere dense in X. **P**

$$A \text{ is nowhere dense in } X \iff X \setminus \overline{A} \text{ is dense in } X$$

(writing $\overline{A}$ for the closure of A in X)

$$\iff D \setminus \overline{A} \text{ is dense in } X$$

(3A3Ea)

$$\iff D \setminus \overline{A} \text{ is dense in } D$$
$$\iff D \setminus (D \cap \overline{A}) \text{ is dense in } D$$
$$\iff A \text{ is nowhere dense in } D$$

because $D \cap \overline{A} = \overline{A}^{(D)}$ is the closure of A in D. **Q**

(ii) A set $G \subseteq D$ is a regular open set in D iff it is expressible as $D \cap H$ for some regular open set $H \subseteq X$. **P** (α) If G is a regular open subset of D, set $H = \operatorname{int} \overline{G}$, taking both the closure and the interior in X. Then H is a regular open set in X. Now $D \cap H$ is a relatively open subset of D included in $D \cap \overline{G} = \overline{G}^{(D)}$, so $D \cap H \subseteq \operatorname{int}_D \overline{G}^{(D)} = G$. In the other direction, $\overline{G} \cup \overline{D \setminus G} = \overline{D} = X$, so $\overline{G} \supseteq X \setminus \overline{D \setminus G}$ and $H \supseteq X \setminus \overline{D \setminus G} \supseteq G$. So $G = H \cap D$ is of the required form. (β) If $H \subseteq X$ is a regular open set such that $G = D \cap H$, set $V = X \setminus \overline{H}$; then $H = X \setminus \overline{V}$. Now

$$\overline{V \cap D}^{(D)} = D \cap \overline{V \cap D} = D \cap \overline{V} = D \setminus H = D \setminus G,$$

so $G = D \setminus \overline{V \cap D}^{(D)}$ is the complement of the closure of an open set in D, and is a regular open set in D. **Q**

4A2C G_δ, F_σ, zero and cozero sets Let X be a topological space.

(a)(i) The union of two G_δ sets in X is a G_δ set. (Engelking 89, p. 26; Kuratowski 66, §5.V.)

(ii) The intersection of countably many G_δ sets is a G_δ set. (Engelking 89, p. 26; Kuratowski 66, §5.V.)

(iii) If Y is another topological space, $f : X \to Y$ is continuous and $E \subseteq Y$ is G_δ in Y, then $f^{-1}[E]$ is G_δ in X. ($f^{-1}[\bigcap_{n\in\mathbb{N}} H_n] = \bigcap_{n\in\mathbb{N}} f^{-1}[H_n]$.)

(iv) If Y is a G_δ set in X and $Z \subseteq Y$ is a G_δ set for the subspace topology of Y, then Z is a G_δ set in X. (Kuratowski 66, §5.V.)

(v) A set $E \subseteq X$ is an F_σ set iff $X \setminus E$ is a G_δ set. (Kuratowski 66, §5.V.)

(b)(i) A zero set is closed. A cozero set is open.

(ii) The union of two zero sets is a zero set. (Császár 78, 4.2.36.) The intersection of two cozero sets is a cozero set.

(iii) The intersection of a sequence of zero sets is a zero set. (If $f_n : X \to \mathbb{R}$ is continuous for each n, $x \mapsto \sum_{n=0}^{\infty} \min(2^{-n}, |f_n(x)|)$ is continuous.) The union of a sequence of cozero sets is a cozero set.

(iv) If Y is another topological space, $f : X \to Y$ is continuous and $L \subseteq Y$ is a zero set, then $f^{-1}[L]$ is a zero set. If $f : X \to Y$ is continuous and $H \subseteq Y$ is a cozero set, then $f^{-1}[H]$ is a cozero set. (Čech 66, 28B.3.) If $K \subseteq X$ and $L \subseteq Y$ are zero sets then $K \times L$ is a zero set in $X \times Y$. ($K \times L = \pi_1^{-1}[K] \cap \pi_2^{-1}[L]$.)

(v) If $H \subseteq X$ is a (co-)zero set and $Y \subseteq X$, then $H \cap Y$ is a (co-)zero set in Y. (Use (iv).)

(vi) A cozero set is the union of a non-decreasing sequence of zero sets. (If $f : X \to \mathbb{R}$ is continuous, $X \setminus f^{-1}[\{0\}]) = \bigcup_{n\in\mathbb{N}} g_n^{-1}[\{0\}]$, where $g_n(x) = \max(0, 2^{-n} - |f(x)|)$.) In particular, a cozero set is an F_σ set; taking complements, a zero set is a G_δ set.

(vii) If $\mathcal{G}$ is a partition of X into open sets, and $H \subseteq X$ is such that $H \cap G$ is a cozero set in G for every $G \in \mathcal{G}$, then H is a cozero set in X. (If $f_G : G \to \mathbb{R}$ is continuous for every $G \in \mathcal{G}$, then $f : X \to \mathbb{R}$ is continuous, where $f(x) = f_G(x)$ for $x \in G \in \mathcal{G}$.) Similarly, if $F \subseteq X$ is such that $F \cap G$ is a zero set in G for every $G \in \mathcal{G}$, then F is a zero set in X.

4A2D Weight Let X be a topological space.

(a)(i) $w(Y) \le w(X)$ for every subspace Y of X (4A2B(a-vi)).

(ii) If $X = \prod_{i\in I} X_i$ then $w(X) \le \max(\omega, \#(I), \sup_{i\in I} w(X_i))$. (Engelking 89, 2.3.13.)

(b) A disjoint family $\mathcal{G}$ of non-empty open sets in X has cardinal at most $w(X)$. (If $\mathcal{U}$ is a base for the topology of X, then every non-empty member of $\mathcal{G}$ includes a non-empty member of $\mathcal{U}$, so we have an injective function from $\mathcal{G}$ to $\mathcal{U}$.)

(c) A point-countable family $\mathcal{G}$ of open sets in X has cardinal at most $\max(\omega, w(X))$. **P** If $X = \emptyset$, this is trivial. Otherwise, let $\mathcal{U}$ be a base for the topology of X with $\#(\mathcal{U}) = w(X) > 0$. Choose a function $f : \mathcal{G} \to \mathcal{U}$ such that $\emptyset \ne f(G) \subseteq G$ whenever $G \in \mathcal{G} \setminus \{\emptyset\}$. Then $\mathcal{G}_U = \{G : f(G) = U\}$ is countable for every $U \in \mathcal{U}$, so there is an injection $h_U : \mathcal{G}_U \to \mathbb{N}$; now $G \mapsto (f(G), h_{f(G)}(G)) : \mathcal{G} \to \mathcal{U} \times \mathbb{N}$ is injective, so $\#(\mathcal{G}) \le \#(\mathcal{U} \times \mathbb{N}) = \max(\omega, w(X))$. **Q**

(d) If X is a dyadic Hausdorff space then X is a continuous image of $\{0,1\}^{w(X)}$. **P** There are a set I and a continuous surjection $f : \{0,1\}^I \to X$; because any power of $\{0,1\}$ is compact, so is X. If $w(X)$ is finite, $\#(X) = w(X) \le \#(\{0,1\}^{w(X)})$ and the result is trivial; so we may suppose that $w(X)$ is infinite. Let $\mathcal{U}$ be a base for the topology of X with cardinality $w(X)$. Set $Z = \{0,1\}^I$ and let $\mathcal{E}$ be the algebra of subsets of Z determined by coordinates in finite sets, so that $\mathcal{E}$ is an algebra of subsets of Z and is a base for the topology of Z. For each pair U, V of members of $\mathcal{U}$ such that $\overline{U} \subseteq V$, $f^{-1}[V] \subseteq Z$ is open; the set $\{E : E \in \mathcal{E},\, E \subseteq f^{-1}[V]\}$ is upwards-directed and covers the compact set $f^{-1}[\overline{U}]$, so there is an $E_{UV} \in \mathcal{E}$ such that $f^{-1}[\overline{U}] \subseteq E_{UV} \subseteq f^{-1}[V]$. Let $J \subseteq I$ be a set of cardinal at most $\max(\omega, w(X))$ such that every E_{UV} is determined by coordinates in J. Fix any $w \in \{0,1\}^{I\setminus J}$ and define $g : \{0,1\}^J \to X$ by setting $g(z) = f(z, w)$ for every $z \in \{0,1\}^J$, identifying Z with $\{0,1\}^J \times \{0,1\}^{I\setminus J}$. Then g is continuous. **?** If g is not surjective, set $H = X \setminus g[\{0,1\}^J]$. Take $x \in H$; take $V \in \mathcal{U}$ such that $x \in V \subseteq H$; take an open set G such that $x \in G \subseteq \overline{G} \subseteq V$ (this must be possible because X, being compact and Hausdorff, is regular); take $U \in \mathcal{U}$ such that $x \in U \subseteq G$, so that $x \in U \subseteq \overline{U} \subseteq V$. Because f is surjective, there is a $(u, v) \in \{0,1\}^J \times \{0,1\}^{I\setminus J}$ such that $f(u, v) = x$. Now $(u, v) \in f^{-1}[U] \subseteq E_{UV}$; as E_{UV} is determined by coordinates in J, $(u, w) \in E_{UV} \subseteq f^{-1}[V]$ and $g(u) = f(u, w) \in V$; but V is supposed to be disjoint from $g[\{0,1\}^J]$. **X** So g is surjective, and X is a continuous image of $\{0,1\}^J$. Since $\#(J) \le \max(\omega, w(X)) = w(X)$, $\{0,1\}^J$ and X are continuous images of $\{0,1\}^{w(X)}$. **Q**

(e) If X is a dyadic Hausdorff space then X is separable iff it is a continuous image of $\{0,1\}^{\mathfrak{c}}$. **P** $\{0,1\}^{\mathfrak{c}}$ is separable (4A2B(e-ii)), so any continuous image of it is separable. If X is a separable dyadic Hausdorff space, let $A \subseteq X$ be a countable dense set. If G, $G' \subseteq X$ are distinct regular open sets, then $G \cap A \neq G \cap A'$. Thus X has at most $\mathfrak{c}$ regular open sets; since X is compact and Hausdorff, therefore regular, its regular open sets form a base (4A2F(b-ii)), and $w(X) \le \mathfrak{c}$. By (d), X is a continuous image of $\{0,1\}^{\max(\omega,w(X))}$ which is in turn a continuous image of $\{0,1\}^{\mathfrak{c}}$. **Q**

4A2E The countable chain condition (a)(i) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces. If $\prod_{i\in J} X_i$ is ccc for every finite $J \subseteq I$, then $\prod_{i\in I} X_i$ is ccc. (Kunen 80, II.1.9; Fremlin 84, 12I.)

(ii) A separable topological space is ccc. (If D is a countable dense set and $\mathcal{G}$ is a disjoint family of non-empty open sets, we have a surjection from a subset of D onto $\mathcal{G}$.)

(iii) The product of any family of separable topological spaces is ccc. **P** By 4A2B(e-ii) and (ii) here, the product of finitely many separable spaces is separable, therefore ccc; so we can apply (i). **Q**

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, and suppose that $X = \prod_{i\in I} X_i$ is ccc. For $J \subseteq I$ and $x \in X$ set $X_J = \prod_{i\in J} X_i$, $\pi_J(x) = x\restriction J$.

(i) If $G \subseteq X$ is open, there is an open set $W \subseteq G$ determined by coordinates in a countable subset of I such that $G \subseteq \overline{W}$. **P** Let $\mathcal{W}$ be the family of subsets of X determined by coordinates in countable sets. Then $\mathcal{W}$ is a σ-algebra (254Mb) including the standard base $\mathcal{U}$ for the topology of X. Let $\mathcal{U}_0$ be a maximal disjoint family in $\{U : U \in \mathcal{U}, U \subseteq G\}$. Then $\mathcal{U}_0$ is countable, so $W = \bigcup\mathcal{U}_0$ belongs to $\mathcal{W}$. No member of $\mathcal{U}$ can be included in $G \setminus W$, so $G \setminus \overline{W}$ must be empty, and we have a suitable set. **Q** So $\overline{G} = \overline{W}$ and $\operatorname{int}\overline{G}$ are determined by coordinates in a countable set (4A2B(g-i)); in particular, if G is a regular open set, then it is determined by coordinates in a countable set.

(ii) If $f : X \to \mathbb{R}$ is continuous, there are a countable set $J \subseteq I$ and a continuous function $g : X_J \to \mathbb{R}$ such that $f = g\pi_J$. **P** For each $q \in \mathbb{Q}$, set $F_q = \overline{\{x : f(x) < q\}}$. By (i), F_q is determined by coordinates in a countable set. Because $\mathbb{Q}$ is countable, there is a countable $J \subseteq I$ such that every F_q is determined by coordinates in J. Also $\{x : f(x) < \alpha\} = \bigcup_{q\in\mathbb{Q},q<\alpha} F_q$ is determined by coordinates in J for every $\alpha \in \mathbb{R}$, so $f(x) = f(y)$ whenever $x\restriction J = y\restriction J$, and there is a $g : X_J \to \mathbb{R}$ such that $f = g\pi_J$. Now if $H \subseteq \mathbb{R}$ is open, $g^{-1}[H] = \pi_J[f^{-1}[H]]$ is open (4A2B(f-i)), so g is continuous. **Q**

(iii) If $A \subseteq X$ is nowhere dense there is a countable set $J \subseteq I$ such that $\pi_J^{-1}[\pi_J[A]]$ is nowhere dense. **P** By (ii), there are a countable set J and an open set $W \subseteq X \setminus \overline{A}$ such that W is determined by coordinates in J and $X \setminus \overline{A} \subseteq \overline{W}$; now W is dense in X and $\pi_J^{-1}[\pi_J[A]] \subseteq X \setminus W$ is nowhere dense. **Q**

4A2F Separation axioms (a) Hausdorff spaces (i) A Hausdorff space is T_1. (Čech 66, 27A.1.) Any subspace of a Hausdorff space is Hausdorff. (Engelking 89, 2.1.6; Čech 66, 27A.3; Kuratowski 66, §5.VIII; Császár 78, 2.5.21.)

(ii) If X is a Hausdorff space and $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X, then a point x of X is a cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$ iff there is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $x = \lim_{n\to\mathcal{F}} x_n$. (If x is a cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$, apply 4A1Ia to $\{\{n : n \ge n_0, x_n \in G\} : n_0 \in \mathbb{N}, G \subseteq X \text{ is open}, x \in G\}$.)

(iii) A topological space X is Hausdorff iff $\{(x,x) : x \in X\}$ is closed in $X \times X$. (Čech 66 27A.7; Kuratowski 66, I.15.IV.)

(b) Regular spaces (i) A regular T_1 space is Hausdorff. (Čech 66, 27B.7; Gaal 64, p. 81.) Any subspace of a regular space is regular. (Engelking 89, 2.1.6; Kuratowski 66, §14.I.)

(ii) If X is a regular topological space, the regular open subsets of X form a base for the topology. **P** If G is open and $x \in G$, there is an open set H such that $x \in H \subseteq \overline{H} \subseteq G$; now $\operatorname{int}\overline{H}$ is a regular open set containing x and included in G. **Q**

(c) Completely regular spaces In a completely regular space, the cozero sets form a base for the topology. (Čech 66, 28B.5.)

(d) Normal spaces (i) **Urysohn's Lemma** If X is normal and E, F are disjoint closed subsets of X, then there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in E$ and $f(x) = 1$ for $x \in F$. (Engelking 89, 1.5.11; Kuratowski 66, §14.IV.)

(ii) A regular normal space is completely regular.

(iii) A normal T_1 space is Hausdorff (Gaal 64, p. 86) and completely regular (Császár 78, 4.2.5; Gaal 64, p. 110).

(iv) If X is normal and E, F are disjoint closed sets in X there is a zero set including E and disjoint from F. (Take a continuous function f which is zero on E and 1 on F, and set $Z = \{x : f(x) = 0\}$.)

(v) In a normal space a closed G_δ set is a zero set. (Engelking 89, 1.5.12.)

(vi) If X is a normal space and $\langle G_i \rangle_{i \in I}$ is a point-finite cover of X by open sets, there is a family $\langle H_i \rangle_{i \in I}$ of open sets, still covering X, such that $\overline{H}_i \subseteq G_i$ for every i. (Engelking 89, 1.5.18; Čech 66, 29C.1; Gaal 64, p. 89.)

(vii) If X is a normal space and $\langle G_i \rangle_{i \in I}$ is a point-finite cover of X by open sets, there is a family $\langle H'_i \rangle_{i \in I}$ of cozero sets, still covering X, such that $H'_i \subseteq G_i$ for every i. (Take $\langle H_i \rangle_{i \in I}$ from (vi), and apply (iv) to the disjoint closed sets $X \setminus G_i$, $\overline{H}_i$ to find a suitable cozero set H'_i for each i.)

(viii) If X is a normal space and $\langle G_i \rangle_{i \in I}$ is a locally finite cover of X by open sets, there is a family $\langle g_i \rangle_{i \in I}$ of continuous functions from X to $[0,1]$ such that $g_i \le \chi G_i$ for every $i \in I$ and $\sum_{i \in I} g_i(x) = 1$ for every $x \in X$. (Engelking 89, proof of 5.1.9.)

(ix) **Tietze's theorem** Let X be a normal space, F a closed subset of X and $f : F \to \mathbb{R}$ a continuous function. Then there is a continuous function $g : X \to \mathbb{R}$ extending f. (Engelking 89, 2.1.8; Kuratowski 66, §14.IV; Gaal 64, p. 203.) It follows that if $F \subseteq X$ is closed and $f : F \to [0,1]^I$ is a continuous function from F to any power of the unit interval, there is a continuous function from X to $[0,1]^I$ extending f. (Extend each of the functionals $x \mapsto f(x)(i)$ for $i \in I$.)

(e) Paracompact spaces A Hausdorff paracompact space is regular. (Engelking 89, 5.1.5.) A regular paracompact space is normal. (Engelking 89, 5.1.5; Gaal 64, p. 160.)

(f) Countably paracompact spaces A normal space X is countably paracompact iff whenever $\langle F_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of closed subsets of X with empty intersection, there is a sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ of open sets, also with empty intersection, such that $F_n \subseteq G_n$ for every $n \in \mathbb{N}$. (Engelking 89, 5.2.2; Császár 78, 8.3.56(f).)

(g) Metacompact spaces (i) A paracompact space is metacompact.

(ii) A closed subspace of a metacompact space is metacompact.

(iii) A normal metacompact space is countably paracompact. (Engelking 89, 5.2.6; Császár 78, 8.3.56(c).)

(h) Separating compact sets (i) If X is a Hausdorff space and K and L are disjoint compact subsets of X, there are disjoint open sets G, $H \subseteq X$ such that $K \subseteq G$ and $L \subseteq H$. (Császár 78, 5.3.18.) If T is an algebra of subsets of X including a subbase for the topology of X, there is an open $V \in \mathrm{T}$ such that $K \subseteq V \subseteq X \setminus L$. **P** By 4A2B(a-i), T includes a base for the topology of X. So $\mathcal{E} = \{U : U \in \mathrm{T} \text{ is open}, U \subseteq G\}$ has union G and there must be a finite $\mathcal{E}_0 \subseteq \mathcal{E}$ covering K; set $V = \bigcup \mathcal{E}_0$. **Q**

(ii) If X is a regular space, $F \subseteq X$ is closed, and $K \subseteq X \setminus F$ is compact, there are disjoint open sets G, $H \subseteq X$ such that $K \subseteq G$ and $F \subseteq H$. (Engelking 89, 3.1.6.)

(iii) If X is a completely regular space, $G \subseteq X$ is open and $K \subseteq G$ is compact, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ for $x \in K$ and $f(x) = 0$ for $x \in X \setminus G$. **P** For each $x \in K$ there is a continuous function $f_x : X \to [0,1]$ such that $f_x(x) = 1$ and $f_x(y) = 0$ for $y \in X \setminus G$. Set $H_x = \{y : f_x(y) > \frac{1}{2}\}$. Then $\bigcup_{x \in K} H_x \supseteq K$, so there is a finite set $I \subseteq K$ such that $K \subseteq \bigcup_{x \in I} H_x$. Set $f(y) = \min(1, 2\sum_{x \in I} f_x(y))$ for $y \in X$. **Q**

(iv) If X is a completely regular Hausdorff space and K and L are disjoint compact subsets of X, there are disjoint cozero sets G, $H \subseteq X$ such that $K \subseteq G$ and $L \subseteq H$. **P** By (i), there are disjoint open sets G', H' such that $K \subseteq G'$ and $L \subseteq H'$. By (iii), there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ for $x \in K$ and $f(x) = 0$ for $x \in X \setminus G'$; set $G = \{x : f(x) \ne 0\}$, so that G is a cozero set and $K \subseteq G \subseteq G'$. Similarly there is a cozero set H including L and included in H'. **Q**

(v) If X is a completely regular space and $K \subseteq X$ is a compact G_δ set, then K is a zero set. **P** Let $\langle G_n \rangle_{n \in \mathbb{N}}$ be a sequence of open sets with intersection K. For each $n \in \mathbb{N}$ there is a continuous function $f_n : X \to [0,1]$ such that $f_n(x) = 1$ for $x \in K$ and $f_n(x) = 0$ for $x \in X \setminus G_n$, by (iii). Now $K = \bigcap_{n \in \mathbb{N}} \{x : 1 - f_n(x) = 0\}$ is a zero set, by 4A2C(b-iii). **Q**

(vi) If $\langle X_n \rangle_{n \in \mathbb{N}}$ is a sequence of topological spaces with product X, $K \subseteq X$ is compact, $F \subseteq X$ is closed and $K \cap F = \emptyset$, there is some $n \in \mathbb{N}$ such that $x \restriction n \ne y \restriction n$ for any $x \in F$ and $y \in K$.

P For $n \in \mathbb{N}$ and $x \in X$ set $\pi_n(x) = x \restriction n$; set $F_n = \overline{\pi_n^{-1}[\pi_n[F]]}$. Since $\langle \pi_n^{-1}[\pi_n[F]] \rangle_{n \in \mathbb{N}}$ is non-increasing, so is $\langle F_n \rangle_{n \in \mathbb{N}}$. If $x \in K$, there is an open set $G \subseteq X$, determined by coordinates in a finite set, such that $x \in G \subseteq X \setminus F$;

in this case there is an $n \in \mathbb{N}$ such that $\pi_n^{-1}[\pi_n[G]] = G$ is disjoint from F, so that $\pi_n[G] \cap \pi_n[F] = \emptyset$, G does not meet $\pi_n^{-1}[\pi_n[F]]$ and $x \notin F_n$. As x is arbitrary, $\langle K \cap F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of relatively closed subsets of K with empty intersection; as K is compact, there is an n such that $K \cap F_n = \emptyset$, so that $K \cap \pi_n^{-1}[\pi_n[F]] = \emptyset$ and $x\restriction n \ne y\restriction n$ whenever $x \in F$ and $y \in K$. **Q**

(vii) If X is a compact Hausdorff space, $f : X \to \mathbb{R}$ is continuous, and $\mathcal{U}$ is a subbase for $\mathfrak{T}$, then there is a countable set $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $f(x) = f(y)$ whenever $\{U : x \in U \in \mathcal{U}_0\} = \{U : y \in U \in \mathcal{U}_0\}$. (Apply (i) to sets of the form $K = \{x : f(x) \le \alpha\}$, $L = \{x : f(x) \ge \beta\}$.)

(i) Perfectly normal spaces A topological space X is perfectly normal iff every closed set is a zero set. (Engelking 89, 1.4.9.)

Consequently, every open set in a perfectly normal space is a cozero set (and, of course, an F_σ set).

(j) Covers of compact sets Let X be a Hausdorff space, K a compact subset of X, and $\langle G_i\rangle_{i\in I}$ a family of open subsets of X covering K. Then there are a finite set $J \subseteq I$ and a family $\langle K_i\rangle_{i\in J}$ of compact sets such that $K = \bigcup_{i\in J} K_i$ and $K_i \subseteq G_i$ for every $i \in J$. **P** (i) Suppose first that $I = \{i, j\}$ has just two members. Then $K \setminus G_j$ and $K \setminus G_i$ are disjoint compact sets. By (h-i), there are disjoint open sets H_i, H_j such that $K \setminus G_j \subseteq H_i$ and $K \setminus G_i \subseteq H_j$; setting $K_i = K \setminus H_j$ and $K_j = K \setminus H_i$ we have a suitable pair K_i, K_j. (ii) Inducing on $\#(I)$ we get the result for finite I. (iii) In general, there is certainly a finite $J \subseteq I$ such that $K \subseteq \bigcup_{i\in J} G_i$, and we can apply the result to $\langle G_i\rangle_{i\in J}$. **Q**

4A2G Compact and locally compact spaces (a) In any topological space, the union of two compact subsets is compact.

(b) A compact Hausdorff space is normal. (Engelking 89, 3.1.9; Császár 78, 5.3.23; Gaal 64, p. 139.)

(c)(i) If X is a compact Hausdorff space, $Y \subseteq X$ is a zero set and $Z \subseteq Y$ is a zero set in Y, then Z is a zero set in X. (By 4A2C(b-vi) and 4A2C(a-iv), Z is a G_δ set in X; now use 4A2F(d-v).)

(ii) Let X and Y be compact Hausdorff spaces, $f : X \to Y$ a continuous open map and $Z \subseteq X$ a zero set in X. Then $f[Z]$ is a zero set in Y. **P** Let $g : X \to \mathbb{R}$ be a continuous function such that $Z = g^{-1}[\{0\}]$. Set $G_n = \{x : x \in X, |g(x)| < 2^{-n}\}$ for each $n \in \mathbb{N}$. If $y \in \bigcap_{n\in\mathbb{N}} f[G_n]$, then $f^{-1}[\{y\}]$ is a compact set meeting all the closed sets $\overline{G}_n$, so meets their intersection, which is Z. Thus $f[Z] = \bigcap_{n\in\mathbb{N}} f[G_n]$ is a G_δ set. By 4A2F(d-v), it is a zero set. **Q**

(d) If X is a Hausdorff space, $\mathcal{V}$ is a downwards-directed family of compact neighbourhoods of a point x of X and $\bigcap\mathcal{V} = \{x\}$, then $\mathcal{V}$ is a base of neighbourhoods of x. **P** Let G be any open set containing x. Fix any $V_0 \in \mathcal{V}$. Note that because X is Hausdorff, every member of $\mathcal{V}$ is closed (3A3Dc). So $\{V_0 \cap V \setminus G : V \in \mathcal{V}\}$ is a family of (relatively) closed subsets of V_0 with empty intersection, cannot have the finite intersection property (3A3Da), and there is a $V \in \mathcal{V}$ such that $V_0 \cap V \setminus G = \emptyset$. Now there is a $V' \in \mathcal{V}$ such that $V' \subseteq V_0 \cap V$ and $V' \subseteq G$. As G is arbitrary, $\mathcal{V}$ is a base of neighbourhoods of x. **Q**

(e) Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space.

(i) If $K \subseteq X$ is a compact set and $G \supseteq K$ is open, then there is a continuous $f : X \to [0, 1]$ with compact support such that $\chi K \le f \le \chi G$. (Let $\mathcal{V}$ be the family of relatively compact open subsets of X. Then $\mathcal{V}$ is upwards-directed and covers X, so there is a $V \in \mathcal{V}$ including K. By 3A3Bb, $\mathfrak{T}$ is completely regular; now 4A2F(h-iii) tells us that there is a continuous $f : X \to [0, 1]$ such that $\chi K \le f \le \chi(G \cap V)$, so that f has compact support.)

(ii) $\mathfrak{T}$ is the coarsest topology on X such that every $\mathfrak{T}$-continuous real-valued function with compact support is continuous. **P** Let Φ be the set of continuous functions of compact support for $\mathfrak{T}$. If $\mathfrak{S}$ is a topology on X such that every member of Φ is continuous, and $x \in G \in \mathfrak{T}$, then there is an $f \in \Phi$ such that $f(x) = 1$ and $f(y) = 0$ for $y \in X \setminus G$, by (i). Now f is $\mathfrak{S}$-continuous, by hypothesis, so $H = \{y : f(y) > \frac{1}{2}\}$ belongs to $\mathfrak{S}$ and $x \in H \subseteq G$. As x is arbitrary, $G = \mathrm{int}_{\mathfrak{S}}\, G$ belongs to $\mathfrak{S}$; as G is arbitrary, $\mathfrak{T} \subseteq \mathfrak{S}$. **Q**

(f)(i) A topological space X is countably compact iff every sequence in X has a cluster point in X, that is, X is relatively countably compact in itself. (Engelking 89, 3.10.3; Császár 78, 5.3.31(e); Gaal 64, p. 129.)

(ii) If X is a countably compact topological space and $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence of closed sets such that $\bigcap_{i\le n} F_i \ne \emptyset$ for every $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} F_n \ne \emptyset$. (Engelking 89, 3.10.3; Császár 78, 5.3.31(c).)

(iii) In any topological space, a relatively compact set is relatively countably compact (2A3Ob).

(iv) Let X and Y be topological spaces and $f : X \to Y$ a continuous function. If $A \subseteq X$ is relatively countably compact in X, then $f[A]$ is relatively countably compact in Y. **P** Let $\langle y_n \rangle_{n\in\mathbb{N}}$ be a sequence in $f[A]$. Then there is a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in A such that $f(x_n) = y_n$ for every $n \in \mathbb{N}$. Because A is relatively countably compact, $\langle x_n \rangle_{n\in\mathbb{N}}$ has a cluster point $x \in X$. If $n_0 \in \mathbb{N}$ and H is an open set containing $f(x)$, there is an $n \geq n_0$ such that $x_n \in f^{-1}[H]$, so that $y_n \in H$. Thus $f(x)$ is a cluster point of $\langle y_n \rangle_{n\in\mathbb{N}}$; as $\langle y_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $f[A]$ is relatively countably compact. **Q**

(v) A relatively countably compact set in $\mathbb{R}$ must be bounded. (If $A \subseteq \mathbb{R}$ is unbounded there is a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in A such that $|x_n| \geq n$ for every n.) So if X is a topological space, $A \subseteq X$ is relatively countably compact and $f : X \to \mathbb{R}$ is continuous, then $f[A]$ is bounded.

(vi) If X and Y are topological spaces and $f : X \to Y$ is continuous, then $f[A]$ is countably compact whenever $A \subseteq X$ is countably compact. (Engelking 89, 3.10.5.)

(g)(i) Let X and Y be topological spaces and $\phi : X \times Y \to \mathbb{R}$ a continuous function. Define $\theta : X \to C(Y)$ by setting $\theta(x)(y) = \phi(x, y)$ for $x \in X$, $y \in Y$. Then θ is continuous if we give $C(Y)$ the topology of uniform convergence on compact subsets of Y. (As noted in Engelking 89, pp. 157-158, the topology of uniform convergence on compact sets is the 'compact-open' topology of $C(Y)$, as defined in 441Yh, so the result here is covered by Engelking 89, 3.4.1.)

(ii) In particular, if Y is compact then θ is continuous if we give $C(Y)$ its usual norm topology.

(h)(i) Suppose that X is a compact space such that there are no non-trivial convergent sequences in X, that is, no convergent sequences which are not eventually constant. If $\langle F_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of infinite closed subsets of X, then $F = \bigcap_{n\in\mathbb{N}} F_n$ is infinite. **P** Because X is compact, F cannot be empty (3A3Da). Choose a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ such that $x_n \in F_n \setminus \{x_i : i < n\}$ for each $n \in \mathbb{N}$. If $G \supseteq F$ is an open set, then $\bigcap_{n\in\mathbb{N}} F_n \setminus G = \emptyset$, so there must be some $n \in \mathbb{N}$ such that $F_n \subseteq G$, and $x_i \in G$ for $i \geq n$. **?** If $F = \{y_0, \dots, y_k\}$, let $l \leq k$ be the first point such that whenever $G \supseteq \{y_0, \dots, y_l\}$ is open, then $\{i : x_i \notin G\}$ is finite. Then there is an open set $G' \supseteq \{y_j : j < l\}$ such that $I = \{i : x_i \notin G'\}$ is infinite. But if H is any open set containing y_l, then $\{i : x_i \notin G' \cup H\}$ is finite, so $\{i : i \in I,\, x_i \notin H\}$ is finite. Thus if we re-enumerate $\langle x_i \rangle_{i\in I}$ as $\langle x'_n \rangle_{n\in\mathbb{N}}$, $\langle x'_n \rangle_{n\in\mathbb{N}}$ converges to y_l and is a non-trivial convergent sequence. **X** Thus F is infinite, as claimed. **Q**

(ii) If X is an infinite scattered compact Hausdorff space it has a non-trivial convergent sequence. **P** Let $\langle x_n \rangle_{n\in\mathbb{N}}$ be any sequence of distinct points in X. Set $F_n = \overline{\{x_i : i \geq n\}}$ for each n, so that $F = \bigcap_{n\in\mathbb{N}} F_n$ is a non-empty set. Because X is scattered, F has an isolated point z say; let G be an open set such that $F \cap G = \{z\}$, and H an open set such that $z \in H \subseteq \overline{H} \subseteq G$ (3A3Bb). In this case, $I = \{i : x_i \in H\}$ must be infinite; re-enumerate $\langle x_i \rangle_{i\in I}$ as $\langle x'_n \rangle_{n\in\mathbb{N}}$. **?** If $\langle x'_n \rangle_{n\in\mathbb{N}}$ does not converge to z, there is an open set H' containing z such that $\{n : x'_n \notin H'\}$ is infinite, that is, $\{i : i \in I,\, x_i \notin H'\}$ is infinite. In this case, $F_n \cap \overline{H} \setminus H'$ is non-empty for every $n \in \mathbb{N}$, but $F \cap \overline{H} \setminus H' = \emptyset$, which is impossible. **X** Thus $\langle x'_n \rangle_{n\in\mathbb{N}}$ is a non-trivial convergent sequence in X. **Q**

(iii) If X is an extremally disconnected Hausdorff space (definition: 3A3Af), it has no non-trivial convergent sequence. **P?** Suppose, if possible, that there is a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ converging to $x \in X$ such that $\{n : x_n \neq x\}$ is infinite. Choose $\langle n_i \rangle_{i\in\mathbb{N}}$ and $\langle G_i \rangle_{i\in\mathbb{N}}$ inductively, as follows. Given that $x \notin \overline{G}_j$ for $j < i$, there is an n_i such that $x_{n_i} \neq x$ and $x_{n_i} \notin \overline{G}_j$ for every $j < i$; now let G_i be an open set such that $x_{n_i} \in G_i$ and $x \notin \overline{G}_i$, and continue.

Since all the n_i must be distinct, $\langle x_{n_i} \rangle_{i\in\mathbb{N}} \to x$. But consider

$$G = \bigcup_{i\in\mathbb{N}} G_{2i} \setminus \bigcup_{j<2i} \overline{G}_j, \quad H = \bigcup_{i\in\mathbb{N}} G_{2i+1} \setminus \bigcup_{j\leq 2i} \overline{G}_j.$$

Then G and H are disjoint open sets and $x_{n_{2i}} \in G$, $x_{n_{2i+1}} \in H$ for every i. So $x \in \overline{G} \cap \overline{H}$. But $\overline{G}$ is open (because X is extremally disconnected), and is disjoint from H, and now $\overline{H}$ is disjoint from $\overline{G}$; so they cannot both contain x. **XQ**

(i)(i) If X and Y are compact Hausdorff spaces and $f : X \to Y$ is a continuous surjection then there is a closed set $K \subseteq X$ such that $f[K] = Y$ and $f{\restriction}K$ is irreducible. **P** Let $\mathcal{E}$ be the family of closed sets $F \subseteq X$ such that $f[F] = Y$. If $\mathcal{F} \subseteq \mathcal{E}$ is non-empty and downwards-directed, then for any $y \in Y$ the family $\{F \cap f^{-1}[\{y\}] : F \in \mathcal{F}\}$ is a downwards-directed family of non-empty closed sets, so (because X is compact) has non-empty intersection; this shows that $\bigcap \mathcal{F} \in \mathcal{E}$. By Zorn's Lemma, $\mathcal{E}$ has a minimal element K say. Now $f[K] = Y$ but $f[F] \neq Y$ for any closed proper subset of K, so $f{\restriction}K$ is irreducible. **Q**

(ii) If X and Y are compact Hausdorff spaces and $f : X \to Y$ is an irreducible continuous surjection, then (α) if $\mathcal{U}$ is a π-base for the topology of Y then $\{f^{-1}[U] : U \in \mathcal{U}\}$ is a π-base for the topology of X (β) if Y has a countable π-base so does X (γ) if x is an isolated point in X then $f(x)$ is an isolated point in Y (δ) if Y has no isolated points, nor does X. **P** (α) If $G \subseteq X$ is a non-empty open set then $f[X \setminus G] \neq Y$. As $f[X \setminus G]$ is closed, there is a non-empty

$U \in \mathcal{U}$ disjoint from $f[X \setminus G]$. Now $f^{-1}[U]$ is a non-empty subset of G. (β) Follows at once from (α). (γ) By (α), with $\mathcal{U}$ the family of all open subsets of Y, there is a non-empty open set $U \subseteq Y$ such that $f^{-1}[U] \subseteq \{x\}$, that is, $U = \{f(x)\}$. (δ) Follows at once from (γ). **Q**

(j)(i) Let X be a non-empty compact Hausdorff space without isolated points. Then there are a closed set $F \subseteq X$ and a continuous surjection $f : F \to \{0,1\}^{\mathbb{N}}$. **P** For $\sigma \in \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ choose closed sets $V_\sigma \subseteq X$ inductively, as follows. $V_\emptyset = X$. Given that V_σ is a closed set with non-empty interior, there are distinct points x, $y \in \operatorname{int} V_\sigma$ (because X has no isolated points); let G, H be disjoint open subsets of X such that $x \in G$ and $y \in H$; and let $V_{\sigma^\frown<0>}$ and $V_{\sigma^\frown<1>}$ be closed sets such that

$$x \in \operatorname{int} V_{\sigma^\frown<0>} \subseteq G \cap \operatorname{int} V_\sigma, \quad y \in \operatorname{int} V_{\sigma^\frown<1>} \subseteq H \cap \operatorname{int} V_\sigma.$$

(This is possible because X is regular.) The construction ensures that $V_\tau \subseteq V_\sigma$ whenever $\tau \in \{0,1\}^n$ extends $\sigma \in \{0,1\}^m$, and that $V_\tau \cap V_\sigma = \emptyset$ whenever τ, $\sigma \in \{0,1\}^n$ are different. Set $F = \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in\{0,1\}^n} V_\sigma$; then F is a closed subset of X and we have a continuous function $f : F \to \{0,1\}^{\mathbb{N}}$ defined by saying that $f(x)(i) = \sigma(i)$ whenever $n \in \mathbb{N}$, $\sigma \in \{0,1\}^n$, $i < n$ and $x \in V_\sigma$. Finally, f is surjective, because if $z \in \{0,1\}^{\mathbb{N}}$ then $\langle V_{z\restriction n}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed sets in the compact space X, so has non-empty intersection V say, and $f(x) = z$ for any $x \in V$. **Q**

(ii) If X is a non-empty compact Hausdorff space without isolated points, then $\#(X) \ge \mathfrak{c}$. (Use (i).)

(iii) If X is a compact Hausdorff space which is not scattered, it has an infinite closed subset with a countable π-base and no isolated points. **P** Because X is not scattered, it has a non-empty subset A without isolated points. Then $\overline{A}$ is compact and has no isolated points; by (i), there are a closed set $F_0 \subseteq \overline{A}$ and a continuous surjection $f : F_0 \to \{0,1\}^{\mathbb{N}}$. By (i-i) above, there is a closed $F \subseteq F_0$ such that $f[F] = \{0,1\}^{\mathbb{N}}$ and $f\restriction F$ is surjective. Of course F is infinite; by (i-ii), it has a countable π-base and no isolated points. **Q**

(iv) Let X be a compact Hausdorff space. Then there is a continuous surjection from X onto $[0,1]$ iff X is not scattered. **P** (α) Suppose that $f : X \to [0,1]$ is a continuous surjection. By (i-i) again, there is a closed set $F \subseteq X$ such that $f[F] = [0,1]$ and $f\restriction F$ is irreducible; by (i-ii) F has no isolated points. So X is not scattered. (β) If X is not scattered, let $A \subseteq X$ be a non-empty set with no isolated points. Then $\overline{A}$ is a non-empty compact subset of X with no isolated points, so there is a continuous surjection $g : \overline{A} \to \{0,1\}^{\mathbb{N}}$ ((i) of this subparagraph). Now there is a continuous surjection $h : \{0,1\}^{\mathbb{N}} \to [0,1]$ (e.g., set $h(y) = \sum_{n=0}^{\infty} 2^{-n-1}y(n)$ for $y \in \{0,1\}^{\mathbb{N}}$), so we have a continuous surjection $hg : \overline{A} \to [0,1]$. By Tietze's theorem (4A2F(d-ix)), there is a continuous function $f_0 : X \to \mathbb{R}$ extending hg; setting $f(x) = \operatorname{med}(0, f_0(x), 1)$ for $x \in X$, we have a continuous surjection $f : X \to [0,1]$. **Q**

(v) A Hausdorff continuous image of a scattered compact Hausdorff space is scattered. (Immediate from (iv).)

(vi) If X is an uncountable first-countable compact Hausdorff space, it is not scattered. **P** Let $\mathcal{G}$ be the family of countable open subsets of X, and G^* its union. No finite subset of $\mathcal{G}$ can cover X, so $X \setminus G^*$ is non-empty. **?** If x is an isolated point of $X \setminus G^*$, then $\{x\} \cup G^*$ is a neighbourhood of x; let $\langle U_n\rangle_{n\in\mathbb{N}}$ run over a base of open neighbourhoods of x with $\overline{U}_0 \subseteq \{x\} \cup G^*$. For each $n \in \mathbb{N}$, $F_n = \overline{U}_0 \setminus U_n$ is a compact set included in G^*, so is covered by finitely many members of $\mathcal{G}$, and is countable. But this means that $U_0 = \{x\} \cup \bigcup_{n\in\mathbb{N}} U_0 \setminus U_n$ is countable, and $x \in G^*$. **X** Thus $X \setminus G^*$ is a non-empty set with no isolated points, and X is not scattered. **Q**

It follows that there is a continuous surjection from X onto $[0,1]$, by (iv).

(k) A locally compact Hausdorff space is Čech-complete. (Engelking 89, p. 196.)

(l) If X is a topological space, $f : X \to \mathbb{R}$ is lower semi-continuous, and $K \subseteq X$ is compact and not empty, then there is an $x_0 \in K$ such that $f(x_0) = \inf_{x\in K} f(x)$. (Gaal 64, p. 209 Theorem 3.) Similarly, if $g : X \to \mathbb{R}$ is upper semi-continuous, there is an $x_1 \in K$ such that $g(x_1) = \sup_{x\in K} g(x)$.

(m) If X is a Hausdorff space, Y is a compact space and $F \subseteq X \times Y$ is closed, then its projection $\{x : (x, y) \in F\}$ is a closed subset of X. (Engelking 89, 3.1.16.)

(n) If X is a locally compact topological space, Y is a topological space and $f : X \to Y$ is a continuous open surjection, then Y is locally compact. (Engelking 89, 3.3.15.)

4A2H Lindelöf spaces (a) If X is a topological space, then a subset Y of X is Lindelöf (in its subspace topology) iff for every family $\mathcal{G}$ of open subsets of X covering Y there is a countable subfamily of $\mathcal{G}$ still covering Y.

(b)(i) A regular Lindelöf space X is paracompact, therefore normal and completely regular. (Engelking 89, 3.8.11 & 5.1.2.)

(ii) If X is a Lindelöf space and $\mathcal{A}$ is a locally finite family of subsets of X then $\mathcal{A}$ is countable. **P** The family $\mathcal{G}$ of open sets meeting only finitely many members of $\mathcal{A}$ is an open cover of X. If $\mathcal{G}_0 \subseteq \mathcal{G}$ is a countable cover of X then $\{A : A \in \mathcal{A},\ A$ meets some member of $\mathcal{G}_0\} = \mathcal{A} \setminus \{\emptyset\}$ is countable. **Q**

(c)(i) A topological space X is hereditarily Lindelöf iff for any family $\mathcal{G}$ of open subsets of X there is a countable family $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\bigcup \mathcal{G}_0 = \bigcup \mathcal{G}$. **P** (α) If X is hereditarily Lindelöf and $\mathcal{G}$ is a family of open subsets of X, then $\mathcal{G}$ is an open cover of $\bigcup \mathcal{G}$, so has a countable subcover. (β) If X is not hereditarily Lindelöf, let $Y \subseteq X$ be a non-Lindelöf subspace, and $\mathcal{H}$ a cover of Y by relatively open sets which has no countable subcover; setting $\mathcal{G} = \{G : G \subseteq X$ is open, $G \cap Y \in \mathcal{H}\}$, there can be no countable $\mathcal{G}_0 \subseteq \mathcal{G}$ with union $\bigcup \mathcal{G}$. **Q**

(ii) Let X be a regular hereditarily Lindelöf space. Then X is perfectly normal. **P** Let $F \subseteq X$ be closed. Let $\mathcal{G}$ be the family of open sets $G \subseteq X$ such that $\overline{G} \cap F = \emptyset$; because X is regular, $\bigcup \mathcal{G} = X \setminus F$; because X is hereditarily Lindelöf, there is a sequence $\langle G_n \rangle_{n\in\mathbb{N}}$ in G such that $X \setminus F = \bigcup_{n\in\mathbb{N}} G_n$. This means that $F = \bigcap_{n\in\mathbb{N}} X \setminus \overline{G_n}$ is a G_δ set. But X is normal ((b) above), so is perfectly normal. **Q**

(d) Any σ-compact topological space is Lindelöf. (Engelking 89, 3.8.5.)

4A2I Stone-Čech compactifications (a) Let X be a completely regular Hausdorff space. Then there is a compact Hausdorff space βX, the **Stone-Čech compactification** of X, in which X can be embedded as a dense subspace. If Y is another compact Hausdorff space, then every continuous function from X to Y has a unique continuous extension to a continuous function from βX to Y. (Engelking 89, 3.6.1; Császár 78, 6.4d; Čech 66, 41D.5.)

(b) Let I be any set, and write βI for its Stone-Čech compactification when I is given its discrete topology. Let Z be the Stone space of the Boolean algebra $\mathcal{P}I$.

(i) There is a canonical homeomorphism $\phi : \beta I \to Z$ defined by saying that $\phi(i)(a) = \chi a(i)$ for every $i \in I$ and $a \subseteq I$. **P** Recall that Z is the set of ring homomorphisms from $\mathcal{P}I$ onto $\mathbb{Z}_2$ (311E). If $i \in I$, let $\hat{i}$ be the corresponding member of Z defined by setting $\hat{i}(a) = \chi a(i)$ for every $a \subseteq I$. Then Z is compact and Hausdorff (311I), and $i \mapsto \hat{i} : I \to Z$ is continuous, so has a unique extension to a continuous function $\phi : \beta I \to Z$.

If $G \subseteq Z$ is open and not empty, it includes a set of the form $\widehat{a} = \{\theta : \theta \in Z,\ \theta(a) = 1\}$ where $a \subseteq I$ is not empty; if i is any member of a, $\hat{i} \in \widehat{a} \subseteq G$ so $G \cap \phi[\beta I] \neq \emptyset$. This shows that $\phi[\beta I]$ is dense in Z; as βI is compact, $\phi[\beta I]$ is compact, therefore closed, and is equal to Z. Thus ϕ is surjective.

If t, u are distinct points of βI, there is an open subset H of βI such that $t \in H$ and $u \notin \overline{H}$. Set $a = H \cap I$. Then $t \in \overline{a}$, the closure of a regarded as a subset of βI, so $\phi(t) \in \overline{\phi[a]}$ (3A3Cd). But $\phi[a] = \{\hat{i} : i \in a\} \subseteq \widehat{a}$, which is open-and-closed, so $\phi(t) \in \widehat{a}$. Similarly, setting $b = I \setminus \overline{H}$, $\phi(u) \in \widehat{b}$; since $\widehat{a} \cap \widehat{b} = \widehat{a \cap b}$ is empty, $\phi(t) \neq \phi(u)$. This shows that ϕ is injective, therefore a homeomorphism between βI and Z (3A3Dd). **Q**

Note that if $z : \mathcal{P}I \to \mathbb{Z}_2$ is a Boolean homomorphism, then $\{J : z(J) = 1\}$ is an ultrafilter on I; and conversely, if $\mathcal{F}$ is an ultrafilter on I, we have a Boolean homomorphism $z : \mathcal{P}I \to \mathbb{Z}_2$ such that $\mathcal{F} = z^{-1}[\{1\}]$. So we can identify βI with the set of ultrafilters on I. Under this identification, the canonical embedding of I in βI corresponds to matching each member of I with the corresponding principal ultrafilter on I.

(ii) $C(\beta I)$ is isomorphic, as Banach lattice, to $\ell^\infty(I)$. **P** By 363Ha, we can identify $\ell^\infty(I)$, as Banach lattice, with $L^\infty(\mathcal{P}I) = C(Z)$. But (i) tells us that we have a canonical identification between $C(Z)$ and $C(\beta I)$. **Q**

(iii) We have a one-to-one correspondence between filters $\mathcal{F}$ on I and non-empty closed sets $F \subseteq \beta I$, got by matching $\mathcal{F}$ with $\bigcap\{\widehat{a} : a \in \mathcal{F}\}$, or F with $\{a : a \subseteq I,\ F \subseteq \widehat{a}\}$, where $\widehat{a} \subseteq \beta I$ is the open-and-closed set corresponding to $a \subseteq I$. **P** The identification of βI with Z means that we can regard the map $a \mapsto \widehat{a}$ as a Boolean isomorphism between $\mathcal{P}I$ and the algebra of open-and-closed subsets of βI (311I). For any filter $\mathcal{F}$ on I, set $H(\mathcal{F}) = \bigcap\{\widehat{a} : a \in \mathcal{F}\}$; because $\{\widehat{a} : a \in \mathcal{F}\}$ is a downwards-directed family of non-empty closed sets in the compact Hausdorff space βI, $H(\mathcal{F})$ is a non-empty closed set. If $F \subseteq \beta I$ is a non-empty closed set, then it is elementary to check that $\mathcal{H}(F) = \{a : F \subseteq \widehat{a}\}$ is a filter on I, and evidently $H(\mathcal{H}(F)) \supseteq F$. But if $t \in \beta I \setminus F$, then (because $\{\widehat{a} : a \subseteq I\}$ is a base for the topology of βI, see 311I again) there is an $a \subseteq I$ such that $t \in \widehat{a}$ and $F \cap \widehat{a} = \emptyset$, that is, $F \subseteq \widehat{I \setminus a}$; so $I \setminus a \in \mathcal{H}(F)$ and $H(\mathcal{H}(F)) \subseteq \widehat{I \setminus a}$ and $t \notin H(\mathcal{H}(F))$. Thus $H(\mathcal{H}(F)) = F$ for every non-empty closed set $F \subseteq \beta I$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on I and $a \in \mathcal{F}_1 \setminus \mathcal{F}_2$, then $\{\widehat{b \setminus a} : b \in \mathcal{F}_2\}$ is a downwards-directed family of non-empty closed sets in βI, so has non-empty intersection; if $t \in \widehat{b \setminus a} = \widehat{b} \setminus \widehat{a}$ for every $b \in \mathcal{F}_2$, then $t \in H(\mathcal{F}_2) \setminus H(\mathcal{F}_1)$. This shows that $\mathcal{F} \mapsto H(\mathcal{F})$ is injective. It follows that $\mathcal{F} \mapsto H(\mathcal{F})$, $F \mapsto \mathcal{H}(F)$ are the two halves of a bijection, as claimed. **Q**

(iv) βI is extremally disconnected. (Because $\mathcal{P}I$ is Dedekind complete, Z is extremally disconnected (314S).)

(v) There are no non-trivial convergent sequences in βI. (4A2G(h-iii). Compare ENGELKING 89, 3.6.15.)

4A2J Uniform spaces (See §3A4.) Let $(X, \mathcal{W})$ be a uniform space; give X the associated topology $\mathfrak{T}$ (3A4Ab).

(a) $\mathcal{W}$ is generated by a family of pseudometrics. (ENGELKING 89, 8.1.10; BOURBAKI 66, IX.1.4; CSÁSZÁR 78, 4.2.32.) More precisely: if $\langle W_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{W}$, there is a pseudometric ρ on X such that (α) $\{(x,y) : \rho(x,y) \le \epsilon\} \in \mathcal{W}$ for every $\epsilon > 0$ (β) whenever $n \in \mathbb{N}$ and $\rho(x,y) < 2^{-n}$ then $(x,y) \in W_n$ (ENGELKING 89, 8.1.10).

It follows that $\mathfrak{T}$ is completely regular, therefore regular (3A3Be). $\mathfrak{T}$ is defined by the bounded uniformly continuous functions, in the sense that it is the coarsest topology $\mathfrak{S}$ on X such that these are all continuous. **P** Let P be the family of pseudometrics compatible with $\mathcal{W}$ in the sense of (α) just above. If $x \in G \in \mathfrak{T}$, there is a $\rho \in$ P such that $\{y : \rho(x,y) < 1\} \subseteq G$; setting $f(y) = \rho(x,y)$, we see that f is uniformly continuous, therefore $\mathfrak{S}$-continuous, and that $x \in \operatorname{int}_{\mathfrak{S}} G$. As x is arbitrary, $G \in \mathfrak{S}$; as G is arbitrary, $\mathfrak{T} \subseteq \mathfrak{S}$; but of course $\mathfrak{S} \subseteq \mathfrak{T}$ just because uniformly continuous functions are continuous. **Q**

(b) If $\mathcal{W}$ is countably generated and $\mathfrak{T}$ is Hausdorff, there is a metric ρ on X defining $\mathcal{W}$ and $\mathfrak{T}$. (ENGELKING 89, 8.1.21.)

(c) If $W \in \mathcal{W}$ and $x \in X$ then $x \in \operatorname{int} W[\{x\}]$. (ENGELKING 89, 8.1.3.) If $A \subseteq X$ then $\overline{A} = \bigcap_{W\in\mathcal{W}} W[A]$. (ENGELKING 89, 8.1.4.)

(d) Any subset of a totally bounded set in X is totally bounded. (ENGELKING 89, 8.3.2; CSÁSZÁR 78, 3.2.70.) The closure of a totally bounded set is totally bounded. **P** If A is totally bounded and $W \in \mathcal{W}$, take $W' \in \mathcal{W}$ such that $W' \circ W' \subseteq W$. Then there is a finite set $I \subseteq X$ such that $A \subseteq W'[I]$. In this case

$$\overline{A} \subseteq W'[A] \subseteq W'[W'[I]] = (W' \circ W')[I] \subseteq W[I]$$

by (b). As W is arbitrary, $\overline{A}$ is totally bounded. **Q**

(e) A subset of X is compact iff it is complete (definition: 3A4F) (for its subspace uniformity) and totally bounded. (ENGELKING 89, 8.3.16; ČECH 66, 41A.8; CSÁSZÁR 78, 5.2.22; GAAL 64, pp. 278-279.) So if X is complete, every closed totally bounded subset of X is compact, and the totally bounded sets are just the relatively compact sets. (A closed subspace of a complete space is complete.)

(f) If $f : X \to \mathbb{R}$ is a continuous function with compact support, it is uniformly continuous. **P** Set $K = \overline{\{x : f(x) \ne 0\}}$. Let $\epsilon > 0$. For each $x \in X$, there is a $W_x \in \mathcal{W}$ such that $|f(y) - f(x)| \le \frac{1}{2}\epsilon$ whenever $y \in W_x[\{x\}]$. Let $W'_x \in \mathcal{W}$ be such that $W'_x \circ W'_x \subseteq W_x$. Set $G_x = \operatorname{int} W'_x[\{x\}]$; then $x \in G_x$, by (b). Because K is compact, there is a finite set $I \subseteq K$ such that $K \subseteq \bigcup_{x\in I} G_x$. Set $W = (X \times X) \cap \bigcap_{x\in I} W'_x \in \mathcal{W}$. Take any $(y,z) \in W \cap W^{-1}$. If neither y nor z belongs to K, then of course $|f(y) - f(z)| \le \epsilon$. If $y \in K$, let $x \in I$ be such that $y \in G_x$. Then

$$y \in W'_x[\{x\}] \subseteq W_x[\{x\}], \quad z \in W[W'_x[\{x\}]] \subseteq W'_x[W'_x[\{x\}]] \subseteq W_x[\{x\}],$$

so

$$|f(y) - f(z)| \le |f(y) - f(x)| + |f(z) - f(x)| \le \epsilon.$$

The same idea works if $z \in K$. So $|f(y) - f(z)| \le \epsilon$ for all y, $z \in W \cap W^{-1}$; as ϵ is arbitrary, f is uniformly continuous. **Q**

(g)(i) If $(Y, \mathfrak{S})$ is a completely regular space, there is a uniformity on Y compatible with $\mathfrak{S}$. (ENGELKING 89, 8.1.20.)

(ii) If $(Y, \mathfrak{S})$ is a compact Hausdorff space, there is exactly one uniformity on Y compatible with $\mathfrak{S}$; it is induced by the set of all those pseudometrics on Y which are continuous as functions from $Y \times Y$ to $\mathbb{R}$. (ENGELKING 89, 8.3.13; GAAL 64, p. 304.)

(iii) If $(Y, \mathfrak{S})$ is a compact Hausdorff space and $\mathcal{V}$ is the uniformity on Y compatible with $\mathfrak{S}$, then any continuous function from Y to X is uniformly continuous. (GAAL 64, p. 305 Theorem 8.)

(h) The set U of uniformly continuous real-valued functions on X is a Riesz subspace of $\mathbb{R}^X$ containing the constant functions. If a sequence in U converges uniformly, the limit function again belongs to U. (CSÁSZÁR 78, 3.2.64; GAAL 64, p. 237 Lemma 4.)

(i) Let $(Y,\mathcal{V})$ be another uniform space. If $\mathcal{F}$ is a Cauchy filter on X and $f:X\to Y$ is a uniformly continuous function, then $f[[\mathcal{F}]]$ is a Cauchy filter on Y. (Császár 78, 5.1.2.)

4A2K First-countable, sequential and countably tight spaces (a) Let X be a countably tight topological space. If $\langle F_\xi\rangle_{\xi<\zeta}$ is a non-decreasing family of closed subsets of X indexed by an ordinal ζ, then $E=\bigcup_{\xi<\zeta}F_\xi$ is an F_σ set, and is closed unless $\operatorname{cf}\zeta=\omega$. **P** If $\operatorname{cf}\zeta=0$, that is, $\zeta=0$, then $E=\emptyset$ is closed. If $\operatorname{cf}\zeta=1$, that is, $\zeta=\xi+1$ for some ordinal ξ, then $E=F_\xi$ is closed. If $\operatorname{cf}\zeta=\omega$, there is a sequence $\langle\xi_n\rangle_{n\in\mathbb{N}}$ in ζ with supremum ζ, so that $E=\bigcup_{n\in\mathbb{N}}F_{\xi_n}$ is F_σ. If $\operatorname{cf}\zeta>\omega$, take $x\in\overline{E}$. Then there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in E such that $x\in\overline{\{x_n:n\in\mathbb{N}\}}$. For each n there is a $\xi_n<\zeta$ such that $x_n\in F_{\xi_n}$, and now $\xi=\sup_{n\in\mathbb{N}}\xi_n<\zeta$ and $x\in\overline{F}_\xi=F_\xi\subseteq E$. As x is arbitrary, E is closed. **Q**

(b) If X is countably tight, any subspace of X is countably tight, just because if $A\subseteq Y\subseteq X$ then the closure of A in Y is the intersection of Y with the closure of A in X. If X is compact and countably tight, then any Hausdorff continuous image of X is countably tight. **P** Let $f:X\to Y$ be a continuous surjection, where Y is Hausdorff, B a subset of Y and $y\in\overline{B}$. Set $A=f^{-1}[B]$. Then $\overline{A}$ is compact, so $f[\overline{A}]$ is compact, therefore closed; because f is surjective, $y\in\overline{f[A]}\subseteq f[\overline{A}]$, and there is an $x\in\overline{A}$ such that $f(x)=y$. Now there is a countable set $A_0\subseteq A$ such that $x\in\overline{A}_0$, in which case

$$y=f(x)\in f[\overline{A}_0]\subseteq\overline{f[A_0]},$$

while $f[A_0]$ is a countable subset of B. **Q**

(c) If X is a sequential space, it is countably tight. **P** Suppose that $A\subseteq X$ and $x\in\overline{A}$. Set $B=\bigcup\{\overline{C}:C\subseteq A$ is countable$\}$. If $\langle y_n\rangle_{n\in\mathbb{N}}$ is a sequence in B converging to $y\in X$, then for each $n\in\mathbb{N}$ we can find a countable set $C_n\subseteq A$ such that $y_n\in\overline{C}_n$, and now $C=\bigcup_{n\in\mathbb{N}}C_n$ is a countable subset of A such that $y\in\overline{C}\subseteq B$. So B is sequentially closed, therefore closed, and $x\in B$. As A and x are arbitrary, X is countably tight. **Q**

(d) If X is a sequential space, Y is a topological space and $f:X\to Y$ is sequentially continuous, then f is continuous. (Engelking 89, 1.6.15.)

(e) First-countable spaces are sequential. (Engelking 89, 1.6.14.)

(f) Let X be a locally compact Hausdorff space in which every singleton set is G_δ. Then X is first-countable. **P** If $\{x\}=\bigcap_{n\in\mathbb{N}}G_n$ where each G_n is open, then for each $n\in\mathbb{N}$ we can find a compact set F_n such that $x\in\operatorname{int}F_n\subseteq G_n$. By 4A2Gd, $\{\bigcap_{i\le n}F_i:n\in\mathbb{N}\}$ is a base of neighbourhoods of x. **Q**

4A2L (Pseudo-)metrizable spaces 'Pseudometrizable' spaces, as such, hardly appear in this volume, for the usual reasons; they surface briefly in §463. It is perhaps worth noting, however, that all the ideas, and very nearly all the results, in this paragraph apply equally well to pseudometrics and pseudometrizable topologies. If X is a set and ρ is a pseudometric on X, set $U(x,\delta)=\{y:\rho(x,y)<\delta\}$ for $x\in X$ and $\delta>0$.

(a) Any subspace of a (pseudo-)metrizable space is (pseudo-)metrizable (2A3J). A topological space is metrizable iff it is pseudometrizable and Hausdorff (2A3L).

(b) Metrizable spaces are paracompact (Engelking 89, 5.1.3; Császár 78, 8.3.16; Čech 66, 30C.2; Gaal 64, p. 155), therefore hereditarily metacompact ((a) above and 4A2F(g-i)).

(c) A metrizable space is perfectly normal (Engelking 89, 4.1.13; Császár 78, 8.4.5.), so every closed set is a zero set and every open set is a cozero set (in particular, is F_σ).

(d) If X is a pseudometrizable space, it is first-countable. (If ρ is a pseudometric defining the topology of X, and x is any point of X, then $\{\{y:\rho(y,x)<2^{-n}\}:n\in\mathbb{N}\}$ is a base of neighbourhoods of x.) So X is sequential and countably tight (4A2Ke, 4A2Kc), and if Y is another topological space and $f:X\to Y$ is sequentially continuous, then f is continuous (4A2Kd).

(e) Relative compactness Let X be a pseudometrizable space and A a subset of X. Then the following are equiveridical: (α) A is relatively compact; (β) A is relatively countably compact; (γ) every sequence in A has a subsequence with a limit in X. **P** Fix a pseudometric ρ defining the topology of X. (α)$\Rightarrow$(β) by 4A2G(f-iii). If

$\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in A with a cluster point $x\in X$, then we can choose $\langle n_i\rangle_{i\in\mathbb{N}}$ inductively such that $\rho(x_{n_i},x)\le 2^{-i}$ and $n_{i+1}>n_i$ for every i; now $\langle x_{n_i}\rangle_{i\in\mathbb{N}}\to x$; it follows that $(\beta)\Rightarrow(\gamma)$. Now assume that (α) is false. Then there is an ultrafilter $\mathcal{F}$ on X containing A which has no limit in X (3A3Be, 3A3De). If $\mathcal{F}$ is a Cauchy filter, choose $F_n\in\mathcal{F}$ such that $\rho(x,y)\le 2^{-n}$ whenever x, $y\in F_n$, and $x_n\in A\cap\bigcap_{i\le n}F_i$ for each n; then it is easy to see that $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in A with no convergent subsequence. If $\mathcal{F}$ is not a Cauchy filter, let $\epsilon>0$ be such that there is no $F\in\mathcal{F}$ such that $\rho(x,y)\le\epsilon$ for every x, $y\in F$. Then $X\setminus U(x,\frac{1}{2}\epsilon)\in\mathcal{F}$ for every $x\in X$, so we can choose $\langle x_n\rangle_{n\in\mathbb{N}}$ inductively such that $x_n\in A\setminus\bigcup_{i<n}U(x_i,\frac{1}{2}\epsilon)$ for every $n\in\mathbb{N}$, and again we have a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in A with no convergent subsequence in X. Thus not-$(\alpha)\Rightarrow$ not-(γ) and the proof is complete. **Q**

(f) Compactness If X is a pseudometrizable space, it is compact iff it is countably compact iff it is sequentially compact. ((e) above, using 4A2G(f-i). Compare Engelking 89, 4.1.17, and Császár 78, 5.3.33 & 5.3.47.)

(g)(i) If (X,ρ) is a metric space, its topology has a base which is σ-metrically-discrete. **P** Enumerate X as $\langle x_\xi\rangle_{\xi<\kappa}$ where κ is a cardinal. Let $\langle(q_n,q_n')\rangle_{n\in\mathbb{N}}$ be a sequence running over $\{(q,q'):q,\,q'\in\mathbb{Q},\,0<q<q'\}$ in such a way that $q_n'-q_n\ge 2^{-n}$ for every $n\in\mathbb{N}$. For $n\in\mathbb{N}$, $\xi<\kappa$ set $G_{n\xi}=\{x:\rho(x,x_\xi)<q_n,\,\inf_{\eta<\xi}\rho(x,x_\eta)>q_n'\}$ (interpreting $\inf\emptyset$ as ∞). Then $\mathcal{U}=\langle G_{n\xi}\rangle_{\xi<\kappa,n\in\mathbb{N}}$ is a σ-metrically-discrete family of open sets. If $G\subseteq X$ is open and $x\in G$, let $\epsilon>0$ be such that $U(x,2\epsilon)\subseteq G$. Let $\xi<\kappa$ be minimal such that $\rho(x,x_\xi)<\epsilon$, and let $n\in\mathbb{N}$ be such that $\rho(x,x_\xi)<q_n<q_n'<\epsilon$; then $x\in G_{n\xi}\subseteq G$. As x and G are arbitrary, $\mathcal{U}$ is a base for the topology of X. **Q**

(ii) Consequently, any metrizable space has a σ-disjoint base. (Compare Engelking 89, 4.4.3; Császár 78, 8.4.5; Kuratowski 66, §21.XVII.)

(h) The product of a countable family of metrizable spaces is metrizable. (Engelking 89, 4.2.2; Császár 78, 7.3.27.)

(i) Let X be a metrizable space and $\kappa\ge\omega$ a cardinal. Then $w(X)\le\kappa$ iff X has a dense subset of cardinal at most κ. (Engelking 89, 4.1.15.)

(j) If (X,ρ) is any metric space, then the balls $B(x,\delta)=\{y:\rho(y,x)\le\delta\}$ are all closed sets (cf. 1A2G). In particular, in a normed space $(X,\|\,\|)$, the balls $B(x,\delta)=\{y:\|y-x\|\le\delta\}$ are closed.

4A2M Complete metric spaces (a) Baire's theorem for complete metric spaces Every complete metric space is a Baire space. (Engelking 89, 4.3.36 & 3.9.4; Kechris 95, 8.4; Császár 78, 9.2.1 & 9.2.8; Gaal 64, p. 287.) So a non-empty complete metric space is not meager (cf. 3A3Ha).

(b) Let $\langle(X_i,\rho_i)\rangle_{i\in I}$ be a countable family of complete metric spaces. Then there is a complete metric on $X=\prod_{i\in I}X_i$ which defines the product topology on X. (Engelking 89, 4.3.12; Kuratowski 66, §33.III.)

(c) Let (X,ρ) be a complete metric space, and $E\subseteq X$ a G_δ set. Then there is a complete metric on E which defines the subspace topology of E. (Engelking 89, 4.3.23; Kuratowski 66, §33.VI; Kechris 95, 3.11.)

(d) Let (X,ρ) be a complete metric space. Then it is Čech-complete. (Engelking 89, 4.3.26.)

(e) A non-empty complete metric space without isolated points is uncountable. (If x is not isolated, $\{x\}$ is nowhere dense.)

4A2N Countable networks: Proposition (a) If X is a topological space with a countable network, any subspace of X has a countable network.

(b) Let X be a space with a countable network. Then X is hereditarily Lindelöf. If it is regular, it is perfectly normal.

(c) If X is a topological space, and $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X each of which has a countable network (for its subspace topology), then $A=\bigcup_{n\in\mathbb{N}}A_n$ has a countable network.

(d) A continuous image of a space with a countable network has a countable network.

(e) Let $\langle X_i\rangle_{i\in I}$ be a countable family of topological spaces with countable networks, with product X. Then X has a countable network.

(f) If X is a Hausdorff space with a countable network, there is a countable family $\mathcal{G}$ of open sets such that whenever x, y are distinct points in X there are disjoint G, $H\in\mathcal{G}$ such that $x\in G$ and $y\in H$.

(g) If X is a regular topological space with a countable network, it has a countable network consisting of closed sets.

(h) A compact Hausdorff space with a countable network is second-countable.

(i) If a topological space X has a countable network, then any dense set in X includes a countable dense set; in particular, X is separable.

(j) If a topological space X has a countable network, then $C(X)$, with the topology of pointwise convergence inherited from the product topology of $\mathbb{R}^X$, has a countable network.

proof (a) If $\mathcal{E}$ is a countable network for the topology of X, and $Y \subseteq X$, then $\{Y \cap E : E \in \mathcal{E}\}$ is a countable network for the topology of Y.

(b) By Engelking 89, 3.8.12 X is Lindelöf. Since any subspace of X has a countable network ((a) above), it also is Lindelöf, and X is hereditarily Lindelöf. By 4A2H(c-ii), if X is regular, it is perfectly normal.

(c) If $\mathcal{E}_n$ is a countable network for the topology of A_n for each n, then $\bigcup_{n\in\mathbb{N}} \mathcal{E}_n$ is a countable network for the topology of A.

(d) Let X be a topological space with a countable network $\mathcal{E}$, and Y a continuous image of X. Let $f : X \to Y$ be a continuous surjection. Then $\{f[E] : E \in \mathcal{E}\}$ is a network for the topology of Y. **P** If $H \subseteq Y$ is open and $y \in H$, then $f^{-1}[H]$ is an open subset of X and there is an $x \in X$ such that $f(x) = y$. Now there must be an $E \in \mathcal{E}$ such that $x \in E \subseteq f^{-1}[H]$, so that $y \in f[E] \subseteq H$. **Q** But $\{f[E] : E \in \mathcal{E}\}$ is countable, so Y has a countable network.

(e) For each $i \in I$ let $\mathcal{E}_i$ be a countable network for the topology of X_i. For each finite $J \subseteq I$, let $\mathcal{C}_J$ be the family of sets expressible as $\prod_{i\in I} E_i$ where $E_i \in \mathcal{E}_i$ for each $i \in J$ and $E_i = X_i$ for $i \in I\setminus J$; then $\mathcal{C}_J$ is countable because $\mathcal{E}_i$ is countable for each $i \in J$. Because the family $[I]^{<\omega}$ of finite subsets of I is countable (3A1Cd), $\mathcal{E} = \bigcup\{\mathcal{C}_J : J \in [I]^{<\omega}\}$ is countable. But $\mathcal{E}$ is a network for the topology of X. **P** If $G \subseteq X$ is open and $x \in G$, then there is a family $\langle G_i\rangle_{i\in I}$ such that every $G_i \subseteq X_i$ is open, $J = \{i : G_i \neq X_i\}$ is finite, and $x \in \prod_{i\in I} G_i$. For $i \in J$, there is an $E_i \in \mathcal{E}_i$ such that $x(i) \in E_i \subseteq G_i$; set $E_i = X_i$ for $i \in I \setminus J$. Then

$$E = \prod_{i\in I} E_i \in \mathcal{C}_J \subseteq \mathcal{E}$$

and $x \in E \subseteq G$. **Q**

So $\mathcal{E}$ is a countable network for the topology of X.

(f) By (b) and (e), $X \times X$ is hereditarily Lindelöf. In particular, $W = \{(x, y) : x \neq y\}$ is Lindelöf. Set

$$\mathcal{V} = \{G \times H : G,\ H \subseteq X \text{ are open}, G \cap H = \emptyset\}.$$

Because X is Hausdorff, $\mathcal{V}$ is a cover of W. So there is a countable $\mathcal{V}_0 \subseteq \mathcal{V}$ covering W. Set

$$\mathcal{G} = \{G : G \times H \in \mathcal{V}_0\} \cup \{H : G \times H \in \mathcal{V}_0\}.$$

Then $\mathcal{G}$ is a countable family of open sets separating the points of X.

(g) Let $\mathcal{E}$ be a countable network for the topology of X. Set $\mathcal{E}' = \{\overline{E} : E \in \mathcal{E}\}$. If $G \subseteq X$ is open and $x \in G$, then (because the topology is regular) there is an open set H such that $x \in H \subseteq \overline{H} \subseteq G$. Now there is an $E \in \mathcal{E}$ such that $x \in E \subseteq H$, in which case $\overline{E} \in \mathcal{E}'$ and $x \in \overline{E} \subseteq G$. So $\mathcal{E}'$ is a countable network for X consisting of closed sets.

(h) Engelking 89, 3.1.19.

(i) Let $D \subseteq X$ be dense, and $\mathcal{E}$ a countable network for the topology of X. Let $D' \subseteq D$ be a countable set such that $D' \cap E \neq \emptyset$ whenever $E \in \mathcal{E}$ and $D \cap E \neq \emptyset$. If $G \subseteq X$ is open and not empty, there is an $x \in D \cap G$; now there is an $E \in \mathcal{E}$ such that $x \in E \subseteq G$, and as $x \in D \cap E$ there must be an $x' \in D' \cap E$, so that $x' \in D' \cap G$. As G is arbitrary, D' is dense in X.

Taking $D = X$, we see that X has a countable dense subset.

(j) Let $\mathcal{E}$ be a countable network for the topology of X and $\mathcal{U}$ a countable base for the topology of $\mathbb{R}$ (4A2Ua). For $E \in \mathcal{E}$ and $U \in \mathcal{U}$ set $H(E, U) = \{f : f \in C(X),\ E \subseteq f^{-1}[U]\}$. Then the set of finite intersections of sets of the form $H(E, U)$ is a countable network for the topology of pointwise convergence on $C(X)$. (Compare 4A2Oe.)

4A2O Second-countable spaces (a) Let $(X, \mathfrak{T})$ be a topological space and $\mathcal{U}$ a countable subbase for $\mathfrak{T}$. Then $\mathfrak{T}$ is second-countable. $(\{X\} \cup \{U_0 \cap U_1 \cap \ldots \cap U_n : U_0, \ldots , U_n \in \mathcal{U}\}$ is countable and is a base for $\mathfrak{T}$, by 4A2B(a-i).)

(b) Any base of a second-countable space includes a countable base. (Császár 78, 2.4.17.)

(c) A second-countable space has a countable network (because a base is also a network), so is separable and hereditarily Lindelöf (ENGELKING 89, 1.3.8 & 3.8.1, 4A2Nb, 4A2Ni).

(d) The product of a countable family of second-countable spaces is second-countable. (ENGELKING 89, 2.3.14.)

(e) If X is a second-countable space then $C(X)$, with the topology of uniform convergence on compact sets, has a countable network. **P** (See ENGELKING 89, Ex. 3.4H.) Let $\mathcal{U}$ be a countable base for the topology of X and $\mathcal{V}$ a countable base for the topology of $\mathbb{R}$ (4A2Ua). For $U \in \mathcal{U}$, $V \in \mathcal{V}$ set $H(U, V) = \{f : f \in C(X), U \subseteq f^{-1}[V]\}$. Then the set of finite intersections of sets of the form $H(U, V)$ is a countable network for the topology of uniform convergence on compact subsets of X. **Q**

4A2P Separable metrizable spaces (a)(i) A metrizable space is second-countable iff it is separable. (ENGELKING 89, 4.1.16; CSÁSZÁR 78, 2.4.16; GAAL 64 p. 120.)

(ii) A compact metrizable space is separable (ENGELKING 89, 4.1.18; CSÁSZÁR 78, 5.3.35; KURATOWSKI 66, §21.IX), so is second-countable and has a countable network.

(iii) Any base of a separable metrizable space includes a countable base (4A2Ob), which is also a countable network, so the space is hereditarily Lindelöf (4A2Nb).

(iv) Any subspace of a separable metrizable space is separable and metrizable (4A2La, 4A2Na, 4A2Ni).

(v) A countable product of separable metrizable spaces is separable and metrizable (4A2B(e-ii), 4A2Lh).

(b) A topological space is separable and metrizable iff it is second-countable, regular and Hausdorff. (ENGELKING 89, 4.2.9; CSÁSZÁR 78, 7.1.57; KURATOWSKI 66, §22.II.)

(c) A Hausdorff continuous image of a compact metrizable space is metrizable. (It is a compact Hausdorff space, by 2A3N(b-ii), with a countable network, by 4A2Nd, so is metrizable, by 4A2Nh.)

(d) A metrizable space is separable iff it is ccc iff it is Lindelöf. (ENGELKING 89, 4.1.16.)

(e) If X is a compact metrizable space, then $C(X)$ is separable under its usual norm topology defined from the norm $\| \|_\infty$. (4A2Oe, or ENGELKING 89, 3.4.16.)

4A2Q Polish spaces: Proposition (a) A countable discrete space is Polish.

(b) A compact metrizable space is Polish.

(c) The product of a countable family of Polish spaces is Polish.

(d) A G_δ subset of a Polish space is Polish in its subspace topology; in particular, a set which is either open or closed is Polish.

(e) The disjoint union of countably many Polish spaces is Polish.

(f) If X is any set and $\langle \mathfrak{T}_n \rangle_{n \in \mathbb{N}}$ is a sequence of Polish topologies on X such that $\mathfrak{T}_m \cap \mathfrak{T}_n$ is Hausdorff for all m, $n \in \mathbb{N}$, then the topology $\mathfrak{T}_\infty$ generated by $\bigcup_{n \in \mathbb{N}} \mathfrak{T}_n$ is Polish.

(g) If X is a Polish space, it is homeomorphic to a G_δ set in a compact metrizable space.

(h) If X is a locally compact Hausdorff space, it is Polish iff it has a countable network iff it is metrizable and σ-compact.

proof (a) Any set X is complete under the discrete metric ρ defined by setting $\rho(x, y) = 1$ whenever x, $y \in X$ are distinct. This defines the discrete topology, and if X is countable it is separable, therefore Polish.

(b) By 4A2P(a-ii), it is separable; by 4A2Je, any metric defining the topology is complete.

(c) If $\langle X_i \rangle_{i \in I}$ is a countable family of Polish spaces with product X, then surely X is separable (4A2B(e-ii)); and 4A2Mb tells us that its topology is defined by a complete metric.

(d) If X is Polish and E is a G_δ set in X, then E is separable, by 4A2P(a-iv), and its topology is defined by a complete metric, by 4A2Mc. So E is Polish. Any open set is of course a G_δ set, and any closed set is a G_δ set by 4A2Lc.

(e) Let $\langle X_i \rangle_{i \in I}$ be a countable disjoint family of Polish spaces, and $X = \bigcup_{i \in I} X_i$. For each $i \in I$ let ρ_i be a complete metric on X_i defining the topology of X_i. Define $\rho : X \times X \to [0, \infty[$ by setting $\rho(x, y) = \min(1, \rho_i(x, y))$ if $i \in I$ and x, $y \in X_i$, $\rho(x, y) = 1$ otherwise. It is easy to check that ρ is a complete metric on X defining the disjoint union topology on X. X is separable, by 4A2B(e-i), therefore Polish.

(f) This result is in KECHRIS 95, 13.3; but I spell out the proof because it is an essential element of some measure-theoretic arguments. On $X^{\mathbb{N}}$ take the product topology $\mathfrak{T}$ of the topologies $\mathfrak{T}_n$. This is Polish, by (c). Consider the diagonal $\Delta = \{x : x \in X^{\mathbb{N}}, x(m) = x(n) \text{ for all } m, n \in \mathbb{N}\}$. This is closed in $X^{\mathbb{N}}$. **P** If $x \in X^{\mathbb{N}} \setminus \Delta$, let $m, n \in \mathbb{N}$ be such that $x(m) \neq x(n)$. Because $\mathfrak{T}_m \cap \mathfrak{T}_n$ is Hausdorff, there are disjoint G, $H \in \mathfrak{T}_m \cap \mathfrak{T}_n$ such that $x(m) \in G$ and $x(n) \in H$. Now $\{y : y \in X^{\mathbb{N}}, y(m) \in G, y(n) \in H\}$ is an open set in $X^{\mathbb{N}}$ containing x and disjoint from Δ. As x is arbitrary, Δ is closed. **Q**

By (d), Δ, with its subspace topology, is a Polish space. Let $f : X \to \Delta$ be the natural bijection, setting $f(t) = x$ if $x(n) = t$ for every n, and let $\mathfrak{S}$ be the topology on X which makes f a homeomorphism. The topology on Δ is generated by $\{\{x : x \in \Delta, x(n) \in G\} : n \in \mathbb{N}, G \in \mathfrak{T}_n\}$, so $\mathfrak{S}$ is generated by $\{\{t : t \in X, t \in G\} : n \in \mathbb{N}, G \in \mathfrak{T}_n\} = \bigcup_{n\in\mathbb{N}} \mathfrak{T}_n$. Thus $\mathfrak{S} = \mathfrak{T}_\infty$ and $\mathfrak{T}_\infty$ is Polish.

(g) KECHRIS 95, 4.14.

(h) If X is Polish, then it is separable, therefore Lindelöf (4A2P(a-iii)). Since the family $\mathcal{G}$ of relatively compact open subsets of X covers X, there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ covering X, and $\{\overline{G} : G \in \mathcal{G}_0\}$ witnesses that X is σ-compact. Also, of course, X is metrizable.

If X is metrizable and σ-compact, let $\langle K_n \rangle_{n\in\mathbb{N}}$ be a sequence of compact sets covering X; each K_n has a countable network (4A2P(a-ii)), so $X = \bigcup_{n\in\mathbb{N}} K_n$ has a countable network (4A2Nc).

If X has a countable network, let $Z = X \cup \{\infty\}$ be its one-point compactification (3A3O). This is compact and Hausdorff and has a countable network, by 4A2Nc again, so is second-countable (4A2Nh) and metrizable (4A2Pb) and Polish ((b) above). So X also, being an open set in Z, is Polish ((d) above).

4A2R Order topologies Let $(X, \leq)$ be a totally ordered set and $\mathfrak{T}$ its order topology.

(a) The set $\mathcal{U}$ of open intervals in X (definition: 4A2A) is a base for $\mathfrak{T}$.

(b) $[x, y]$, $[x, \infty[$ and $]-\infty, x]$ are closed sets for all x, $y \in X$.

(c) $\mathfrak{T}$ is Hausdorff, normal and countably paracompact.

(d) If $A \subseteq X$ then $\overline{A}$ is the set of elements of X expressible as either suprema or infima of non-empty subsets of A.

(e) A subset of X is closed iff it is order-closed.

(f) If $\langle x_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in X with supremum x, then $x = \lim_{n\to\infty} x_n$.

(g) A set $K \subseteq X$ is compact iff $\sup A$ and $\inf A$ are defined in X and belong to K for every non-empty $A \subseteq K$.

(h) X is Dedekind complete iff $[x, y]$ is compact for all x, $y \in X$.

(i) X is compact iff it is either empty or Dedekind complete with greatest and least elements.

(j) Any open set $G \subseteq X$ is expressible as a union of disjoint open order-convex sets; if X is Dedekind complete, these will be open intervals.

(k) If X is well-ordered it is locally compact.

(l) In $X \times X$, $\{(x, y) : x < y\}$ is open and $\{(x, y) : x \leq y\}$ is closed.

(m) If $F \subseteq X$ and *either* F is order-convex *or* F is compact *or* X is Dedekind complete and F is closed, then the subspace topology on F is induced by the inherited order of F.

(n) If X is ccc it is hereditarily Lindelöf, therefore perfectly normal.

(o) If Y is another totally ordered set with its order topology, an order-preserving function from X to Y is continuous iff it is order-continuous.

proof (a) Put the definition of 'order topology' (4A2A) together with 4A2B(a-i).

(b) Their complements are either X, or members of $\mathcal{U}$, or unions of two members of $\mathcal{U}$.

(c) Fix a well-ordering $\preccurlyeq$ of X.

(i) If $x < y \in X$, define U_{xy}, U_{yx} as follows: if $]x, y[$ is empty, $U_{xy} =]-\infty, x] =]-\infty, y[$ and $U_{yx} = [y, \infty[=]x, \infty[$; otherwise, let z be the $\preccurlyeq$-least member of $]x, y[$ and set $U_{xy} =]-\infty, z[$, $U_{yx} =]z, \infty[$.

This construction ensures that if x, y are any distinct points of X, U_{xy} and U_{yx} are disjoint open sets containing x, y respectively, so $\mathfrak{T}$ is Hausdorff.

(ii) Now suppose that $F \subseteq X$ is closed and that $x \in X \setminus F$. Then $V_{xF} = \operatorname{int}(X \cap \bigcap_{y\in F} U_{xy})$ contains x. **P** There are u, $v \in X \cup \{-\infty, \infty\}$ such that $x \in]u, v[\subseteq X \setminus F$. If $]u, x[= \emptyset$, set $u' = u$; otherwise, let u' be the $\preccurlyeq$-least member of $]u, x[$. Similarly, if $]x, v[= \emptyset$, set $v' = v$; otherwise, let v' be the $\preccurlyeq$-least member of $]x, v[$. Then $u' < x < v'$. Now suppose that $y \in F$ and $y > x$. If $]x, v[= \emptyset$, then $U_{xy} \supseteq]-\infty, x] =]-\infty, v'[$. Otherwise,

$v' \in \,]x, v[\, \subseteq \,]x, y[$, so $U_{xy} = \,]-\infty, z[$ where z is the $\preccurlyeq$-least member of $]x, y[$. But this means that $z \preccurlyeq v'$ and either $z = v'$ or $z \notin \,]x, v[$; in either case, $v' \le z$ and $]-\infty, v'[\, \subseteq U_{xy}$.

Similarly, $]u', \infty[\, \subseteq U_{xy}$ whenever $y \in F$ and $y < x$. So $x \in \,]u', v'[\, \subseteq V_{xF}$. **Q**

(iii) Let E and F be disjoint closed sets. Set $G = \bigcup_{x\in E} V_{xF}$, $H = \bigcup_{y\in F} V_{yE}$. Then G and H are open sets including E, F respectively. If $x \in E$ and $y \in F$, then $V_{xF} \cap V_{yE} \subseteq U_{xy} \cap U_{yx} = \emptyset$, so $G \cap H = \emptyset$. As E and F are arbitrary, $\mathfrak{T}$ is normal.

(iv) Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of closed sets with empty intersection. Let $\mathcal{I}$ be the family of open intervals $I \subseteq X$ such that $I \cap F_n = \emptyset$ for some n. Because the F_n are closed and have empty intersection, $\mathcal{I}$ covers X. If I, $I' \in \mathcal{I}$ are not disjoint, $I \cup I' \in \mathcal{I}$; so we have an equivalence relation $\sim$ on X defined by saying that $x \sim y$ if there is some $I \in \mathcal{I}$ containing both x and y. The corresponding equivalence classes are open and therefore closed, and are order-convex. Let $\mathcal{G}$ be the set of equivalence classes for $\sim$.

For each $G \in \mathcal{G}$, fix $x_G \in G$. Set $G^+ = G \cap [x_G, \infty[$. Then we have a non-decreasing sequence $\langle G_n^+\rangle_{n\in\mathbb{N}}$ of closed sets, with union G^+, such that $G_n^+ \cap F_n = \emptyset$ for each n. **P** If there is some $m \in \mathbb{N}$ such that $G^+ \cap F_m = \emptyset$, set $G_n^+ = \emptyset$ if $G^+ \cap F_n \ne \emptyset$, G^+ if $G^+ \cap F_n = \emptyset$. Otherwise, given $x \in G^+$ and $n \in \mathbb{N}$, there is some m such that $[x_G, x]$ does not meet F_m, and an $x' \in G^+ \cap F_{\max(m,n)}$, so that $x' \in F_n$ and $x' > x$. We can therefore choose a strictly increasing sequence $\langle x_k\rangle_{k\in\mathbb{N}}$ such that $x_0 = x_G$ and $x_{k+1} \in G^+ \cap F_k$ for each k. If x is any upper bound of $\{x_k : k \in \mathbb{N}\}$ then $x \not\sim x_G$, so $G^+ = \bigcup_{k\in\mathbb{N}} [x_G, x_k]$. Now, for each n, there is a least $k(n)$ such that $[x_G, x_{k(n)}] \cap F_n \ne \emptyset$; set $G_n^+ = \emptyset$ if $k(n) = 0$, $[x_G, x_{k(n-1)}]$ otherwise. As $F_{n+1} \subseteq F_n$, $k(n+1) \ge k(n)$ for each n. Since each $[x_G, x_k]$ is disjoint from some F_n, and therefore from all but finitely many F_n, $\lim_{n\to\infty} k(n) = \infty$ and $G^+ = \bigcup_{n\in\mathbb{N}} G_n^+$. **Q**

Similarly, $G^- = G \cap \,]-\infty, x_G]$ can be expressed as the union of a non-decreasing sequence $\langle G_n^-\rangle_{n\in\mathbb{N}}$ of closed sets such that $G_n^- \cap F_n = \emptyset$ for every n. Now set $F'_n = \bigcup_{G\in\mathcal{G}} G_n^+ \cup G_n^-$ for each n. Because every G_n^+ and G_n^- is closed, and every G is open-and-closed, F'_n is closed. So $\langle F'_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of closed sets with union X, and F'_n is disjoint from F_n for each n. Accordingly $\langle X \setminus F'_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of open sets with empty intersection enveloping the F_n. As $\langle F_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{T}$ is countably paracompact (4A2Ff).

(d) Let B be the set of such suprema and infima. For $x \in X$ set $A_x = A \cap \,]-\infty, x]$, $A'_x = A \cap [x, \infty[$. Then $x \in B$ iff either $x = \sup A_x$ or $x = \inf A'_x$, so

$$\begin{aligned} x \notin B &\iff x \ne \sup A_x \text{ and } x \ne \inf A'_x \\ &\iff \text{ there are } u < x,\ v > x \text{ such that } A_x \subseteq \,]-\infty, u] \text{ and } A'_x \subseteq [v, \infty[\\ &\iff \text{ there are } u,\ v \text{ such that } x \in \,]u, v[\, \subseteq X \setminus A \\ &\iff x \notin \overline{A}. \end{aligned}$$

Thus $B = \overline{A}$, as claimed.

(e) Because X is totally ordered, all its subsets are both upwards-directed and downwards-directed; so we have only to join the definition in 313Da to (d) above.

(f) If $x \in \,]u, v[$ then there is some $n \in \mathbb{N}$ such that $x_n \ge u$, and now $x_i \in \,]u, v[$ for every $i \ge n$.

(g)(i) If K is compact and $A \subseteq K$ is non-empty, let B be the set of upper bounds for A in $X \cup \{-\infty, \infty\}$, and set $\mathcal{G} = \{]-\infty, a[\, : a \in A\} \cup \{]b, \infty[\, : b \in B\}$. Then no finite subfamily of $\mathcal{G}$ can cover K; and if $c \in K \setminus \bigcup \mathcal{G}$ then $c = \sup A$. Similarly, any non-empty subset of K has an infimum in X which belongs to K.

(ii) Now suppose that K satisfies the condition. By (d) above, it is closed. If it is empty it is certainly compact. Otherwise, $a_0 = \inf K$ and $b_0 = \sup K$ are defined in X and belong to K. Let $\mathcal{G}$ be an open cover of K. Set

$$A = \{x : x \in X,\ K \cap [a_0, x] \text{ is not covered by any finite } \mathcal{G}_0 \subseteq \mathcal{G}\}.$$

Note that A is bounded below by a_0. **?** If $b_0 \in A$, then $c = \inf A$ is defined and belongs to $[a_0, b_0]$, because X is Dedekind complete. If $c \notin K$ then there are u, v such that $c \in \,]u, v[\, \subseteq X \setminus K$; if $c \in K$ then there are u, v such that $c \in \,]u, v[\, \subseteq G$ for some $G \in \mathcal{G}$. In either case, $u \notin A$, so that $K \cap [a_0, v[\, \subseteq (K \cap [a_0, u]) \cup \,]u, v[$ is covered by a finite subset of $\mathcal{G}$, and A does not meet $[a_0, v[$, that is, $A \subseteq [v, \infty[$ and v is a lower bound of A. **X** Thus $b_0 \notin A$, and $K = K \cap [a_0, b_0]$ is covered by a finite subset of $\mathcal{G}$. As $\mathcal{G}$ is arbitrary, K is compact.

(h) Use (g).

(i) Use (h).

(j) (Compare 2A2I.) For x, $y \in G$ write $x \sim y$ if either $x \le y$ and $[x, y] \subseteq G$ or $y \le x$ and $[y, x] \subseteq G$. It is easy to check that $\sim$ is an equivalence relation on G. Let $\mathcal{C}$ be the set of equivalence classes under $\sim$. Then $\mathcal{C}$ is a partition of G into order-convex sets. Now every $C \in \mathcal{C}$ is open. **P** If $x \in C \in \mathcal{C}$ then there are u, $v \in X \cup \{-\infty, \infty\}$ such that $x \in \,]u, v[\, \subseteq G$; now $]u, v[\, \subseteq C$. **Q** So we have our partition of G into disjoint open order-convex sets.

If X is Dedekind complete, then every member of $\mathcal{C}$ is an open interval. **P** Take $C \in \mathcal{C}$. Set

$$A = \{u : u \in X \cup \{-\infty\},\ u < x \text{ for every } x \in C\},$$

$$B = \{v : v \in X \cup \{\infty\},\ x < v \text{ for every } x \in C\},$$

$$a = \sup A, \quad b = \inf B;$$

these are defined because X is Dedekind complete. If $a < x < b$, there are y, $z \in C$ such that $y \le x \le z$, so that $[y, x] \subseteq [y, z] \subseteq G$ and $y \sim x$ and $x \in C$; thus $]a, b[\, \subseteq C$. If $x \in C$, there is an open interval $]u, v[$ containing x and included in G; now $]u, v[\, \subseteq C$, so $a \le u < x < v \le b$ and $x \in \,]a, b[$. Thus $C = \,]a, b[$ is an open interval. **Q**

(k) Use (h).

(l) Write W for $\{(x, y) : x < y\}$. If $x < y$, then either there is a z such that $x < z < y$, in which case $]-\infty, z[\, \times \,]z, \infty[$ is an open set containing (x, y) and included in W, or $]x, y[\, = \emptyset$, so $]-\infty, y[\, \times \,]x, \infty[$ is an open set containing (x, y) and included in W. Thus W is open.

Now $\{(x, y) : x \le y\} = (X \times X) \setminus \{(x, y) : y < x\}$ is closed.

(m) The subspace topology $\mathfrak{T}_F$ on F is generated by sets of the form $F \cap \,]-\infty, x[$, $F \cap \,]x, \infty[$ where $x \in X$ (4A2B(a-vi)), while the order topology $\mathfrak{S}$ on F is generated by sets of the form $F \cap \,]-\infty, x[$, $F \cap \,]x, \infty[$ where $x \in F$. So $\mathfrak{S} \subseteq \mathfrak{T}_F$.

Now suppose that one of the three conditions is satisfied, and that $x \in X$. If $x \in F$, or $F \cap \,]-\infty, x[$ is either F or $\emptyset$, then of course $F \cap \,]-\infty, x[\, \in \mathfrak{S}$. Otherwise, F meets both $]-\infty, x[$ and $[x, \infty[$ and does not contain x, so is not order-convex. In this case $x' = \inf(F \cap [x, \infty[)$ is defined and belongs to F. **P** If F is compact, this is covered by (g). If X is Dedekind complete and F is closed, then x' is defined, and belongs to F by (e). **Q** Now $F \cap \,]-\infty, x[\, = F \cap \,]-\infty, x'[\, \in \mathfrak{S}$.

Similarly, $F \cap \,]x, \infty[\, \in \mathfrak{S}$ for every $x \in X$. But this means that $\mathfrak{T}_F \subseteq \mathfrak{S}$, so the two topologies are equal, as stated.

(n)(i) Let $\mathcal{G}$ be a family of open subsets of X with union H. Set

$$\mathcal{A} = \{A : A \subseteq \bigcup \mathcal{G}_0 \text{ for some countable } \mathcal{G}_0 \subseteq \mathcal{G}\}.$$

(I seek to show that $H \in \mathcal{A}$.) Of course $\bigcup \mathcal{A}_0 \in \mathcal{A}$ for every countable subset $\mathcal{A}_0$ of $\mathcal{A}$.

(ii) Let $\mathcal{C}$ be the family of order-convex members of $\mathcal{A}$, and $\mathcal{C}^*$ the family of maximal members of $\mathcal{C}$. If $C \in \mathcal{C}$ is not included in any member of $\mathcal{C}^*$, then there is a $C' \in \mathcal{C}$ such that $C \subseteq C'$ and $\operatorname{int}(C' \setminus C) \ne \emptyset$. **P** Since no member of $\mathcal{C}$ including C can be maximal, we can find $C' \in \mathcal{C}$ such that $C \subseteq C'$ and $\#(C' \setminus C) \ge 5$. Because C is order-convex, every point of $X \setminus C$ is either a lower bound or an upper bound of C, and there must be three points $x < y < z$ of $C' \setminus C$ on the same side of C. In this case,

$$y \in \,]x, z[\, \subseteq \operatorname{int}(C' \setminus C),$$

so we have an appropriate C'. **Q**

(iii) In fact, every member of $\mathcal{C}$ is included in a member of $\mathcal{C}^*$. **P?** Suppose, if possible, otherwise. Then we can choose a strictly increasing family $\langle C_\xi \rangle_{\xi < \omega_1}$ in $\mathcal{C}$ inductively, as follows. Start from any non-empty $C_0 \in \mathcal{C}$ not included in any member of $\mathcal{C}^*$. Given that $C_0 \subseteq C_\xi \in \mathcal{C}$, then C_ξ cannot be included in any member of $\mathcal{C}^*$, so by (β) above there is a $C_{\xi+1} \in \mathcal{C}$ such that $\operatorname{int}(C_{\xi+1} \setminus C_\xi)$ is non-empty. Given $\langle C_\eta \rangle_{\eta < \xi}$ where $\xi < \omega_1$ is a non-zero countable limit ordinal, set $C_\xi = \bigcup_{\eta < \xi} C_\eta$; then C_ξ is order-convex, because $\{C_\eta : \eta < \xi\}$ is upwards-directed, and belongs to $\mathcal{A}$, because $\mathcal{A}$ is closed under countable unions, so $C_\xi \in \mathcal{C}$ and the induction proceeds.

Now, however, $\langle \operatorname{int}(C_{\xi+1} \setminus C_\xi) \rangle_{\xi < \omega_1}$ is an uncountable disjoint family of non-empty open sets, and X is not ccc. **XQ**

(iv) Since $C \cup C'$ is order-convex whenever C, $C' \in \mathcal{C}$ and $C \cap C' \ne \emptyset$, $\mathcal{C}^*$ is a disjoint family. Moreover, if $x \in H$, there is some open interval containing x and belonging to $\mathcal{C}$, so $x \in \operatorname{int} C$ for some $C \in \mathcal{C}^*$; this shows that $\mathcal{C}^*$ is an open cover of H. Because X is ccc, $\mathcal{C}^*$ is countable, so $H = \bigcup \mathcal{C}^* \in \mathcal{A}$. Thus there is some countable $\mathcal{G}_0 \subseteq \mathcal{G}$ with union H; as $\mathcal{G}$ is arbitrary, X is hereditarily Lindelöf, by 4A2H(c-i).

(v) By 4A2H(c-ii), X is perfectly normal.

(o)(i) Suppose that f is continuous. If $A \subseteq X$ is a non-empty set with supremum x in X, then $x \in \overline{A}$, by (d), so $f(x) \in \overline{f[A]}$ (3A3Cc) and $f(x)$ is less than or equal to any upper bound of $f[A]$; but $f(x)$ is an upper bound of $f[A]$, because f is order-preserving, so $f(x) = \sup f[A]$. Similarly, $f(\inf A) = \inf f[A]$ whenever A is non-empty and has an infimum, so f is order-continuous.

(ii) Now suppose that f is order-continuous. Take any $y \in Y$ and consider $A = f^{-1}[\,]-\infty, y[\,]$, $B = X \setminus A$. If $x \in A$ then $f(x)$ cannot be $\inf f[B]$ so x cannot be $\inf B$ and there is an $x' \in X$ such that $x < x' \le z$ for every $z \in B$; in which case $x \in \,]-\infty, x'[\, \subseteq A$. So A is open. Similarly, $f^{-1}[\,]y, \infty[\,]$ is open. By 4A2B(a-ii), f is continuous.

4A2S Order topologies on ordinals (a) Let ζ be an ordinal with its order topology.

(i) ζ is locally compact (4A2Rk); all the sets $[0, \eta] = \,]-\infty, \eta+1[$, for $\eta < \zeta$, are open and compact (4A2Rh). If ζ is a successor ordinal, it is compact, being of the form $[0, \eta]$ where $\zeta = \eta + 1$.

(ii) For any $A \subseteq \zeta$, $\overline{A} = \{\sup B : \emptyset \ne B \subseteq A, \sup B < \zeta\}$. (4A2Rd, because $\inf B \in B \subseteq A$ for every non-empty $B \subseteq A$.)

(iii) If $\xi \le \zeta$, then the subspace topology on ξ induced by the order topology of ζ is the order topology of ξ. (4A2Rm.)

(b) Give ω_1 its order topology.

(i) ω_1 is first-countable. **P** If $\xi < \omega_1$ is either zero or a successor ordinal, then $\{\xi\}$ is open so $\{\{\xi\}\}$ is a base of neighbourhoods of ξ. If ξ is a non-zero limit ordinal, there is a sequence $\langle \xi_n \rangle_{n \in \mathbb{N}}$ in ξ with supremum ξ, and $\{]\xi_n, \xi] : n \in \mathbb{N}\}$ is a base of neighbourhoods of ξ. **Q**

(ii) Singleton subsets of ω_1 are zero sets. (Assemble 4A2F(d-v), 4A2Rc and (i) above.)

(iii) If $f : \omega_1 \to \mathbb{R}$ is continuous, there is a $\xi < \omega_1$ such that $f(\eta) = f(\xi)$ for every $\eta \ge \xi$. **P?** Otherwise, we may define a strictly increasing family $\langle \zeta_\xi \rangle_{\xi < \omega_1}$ in ω_1 by saying that $\zeta_0 = 0$,

$$\zeta_{\xi+1} = \min\{\eta : \eta > \zeta_\xi,\ f(\eta) \ne f(\zeta_\xi)\}$$

for every $\xi < \omega_1$,

$$\zeta_\xi = \sup\{\zeta_\eta : \eta < \xi\}$$

for non-zero countable limit ordinals ξ. Now

$$\omega_1 = \bigcup_{k \in \mathbb{N}} \{\xi : |f(\zeta_{\xi+1}) - f(\zeta_\xi)| \ge 2^{-k}\},$$

so there is a $k \in \mathbb{N}$ such that $A = \{\xi : |f(\zeta_{\xi+1}) - f(\zeta_\xi)| \ge 2^{-k}\}$ is infinite. Take a strictly increasing sequence $\langle \xi_n \rangle_{n \in \mathbb{N}}$ in A and set $\xi = \sup_{n \in \mathbb{N}} \xi_n = \sup_{n \in \mathbb{N}} (\xi_n + 1)$. Then $\langle \zeta_{\xi_n} \rangle_{n \in \mathbb{N}}$ and $\langle \zeta_{\xi_n+1} \rangle_{n \in \mathbb{N}}$ are strictly increasing sequences with supremum ζ_ξ, so both converge to ζ_ξ in the order topology of ω_1 (4A2Rf), and

$$f(\zeta_\xi) = \lim_{n \to \infty} f(\zeta_{\xi_n}) = \lim_{n \to \infty} f(\zeta_{\xi_n+1}).$$

But this means that

$$\lim_{n \to \infty} f(\zeta_{\xi_n}) - f(\zeta_{\xi_n+1}) = 0,$$

which is impossible, because $|f(\zeta_{\xi_n}) - f(\zeta_{\xi_n+1})| \ge 2^{-k}$ for every n. **XQ**

4A2T Topologies on spaces of subsets In §§446, 476 and 479 it will be useful to be able to discuss topologies on spaces of closed sets. In fact everything we really need can be expressed in terms of Fell topologies ((a-ii) here), but it may help if I put these in the context of two other constructions, Vietoris topologies and Hausdorff metrics (see (a) and (g) below), which may be more familiar to some readers. Let X be a topological space, and $\mathcal{C} = \mathcal{C}_X$ the family of closed subsets of X.

(a)(i) The **Vietoris topology** on $\mathcal{C}$ is the topology generated by sets of the forms

$$\{F : F \in \mathcal{C},\ F \cap G \ne \emptyset\}, \quad \{F : F \in \mathcal{C},\ F \subseteq G\}$$

where $G \subseteq X$ is open.

(ii) The **Fell topology** on $\mathcal{C}$ is the topology generated by sets of the forms

$$\{F : F \in \mathcal{C},\ F \cap G \ne \emptyset\}, \quad \{F : F \in \mathcal{C},\ F \cap K = \emptyset\}$$

where $G \subseteq X$ is open and $K \subseteq X$ is compact. If X is Hausdorff then the Fell topology is coarser than the Vietoris topology. If X is compact and Hausdorff the two topologies agree.

(iii) Suppose X is metrizable, and that ρ is a metric on X inducing its topology. For a non-empty subset A of X, write $\rho(x, A) = \inf_{y \in A} \rho(x, y)$ for every $x \in X$. Note that $\rho(x, A) \le \rho(x, y) + \rho(y, A)$ for all x, $y \in X$, so that $x \mapsto \rho(x, A) : X \to \mathbb{R}$ is 1-Lipschitz.

For E, $F \in \mathcal{C} \setminus \{\emptyset\}$, set

$$\tilde{\rho}(E, F) = \min(1, \max(\sup_{x \in E} \rho(x, F), \sup_{y \in F} \rho(y, E))).$$

If E, $F \in \mathcal{C} \setminus \{\emptyset\}$ and $z \in X$, then $\rho(z, F) \le \rho(z, E) + \sup_{x \in E} \rho(x, F)$; from this it is easy to see that $\tilde{\rho}$ is a metric on $\mathcal{C} \setminus \{\emptyset\}$, the **Hausdorff metric**. Observe that $\tilde{\rho}(\{x\}, \{y\}) = \min(1, \rho(x, y))$ for all x, $y \in X$.

Remarks The formula I give for $\tilde{\rho}$ has a somewhat arbitrary feature 'min$(1, \dots)$'. Any number strictly greater than 0 can be used in place of '1' here. Many authors prefer to limit themselves to the family of non-empty closed sets of finite diameter, rather than the whole of $\mathcal{C} \setminus \{\emptyset\}$; this makes it more more natural to omit the truncation, and work with $(E, F) \mapsto \max(\sup_{x \in E} \rho(x, F), \sup_{y \in F} \rho(y, E))$. All such variations produce uniformly equivalent metrics. A more radical approach redefines 'metric' to allow functions which take the value ∞; but this seems a step too far.

Given that I am truncating my Hausdorff metrics by the value 1, there would be no extra problems if I defined $\tilde{\rho}(\emptyset, E) = 1$ for every non-empty closed set E; but I think I am following the majority in regarding Hausdorff distance as defined only for non-empty sets.

(b)(i) The Fell topology is T_1. **P** If $F \subseteq X$ is closed and $x \in X$, then $\{E : E \in \mathcal{C},\ E \cap (X \setminus F) = \emptyset\}$ and $\{E : E \in \mathcal{C},\ E \cap \{x\} \ne \emptyset\}$ are complements of open sets, so are closed. Now if $F \in \mathcal{C}$ then

$$\{F\} = \{E : E \subseteq F\} \cap \bigcap_{x \in F} \{E : x \in E\}$$

is closed. **Q**

(ii) The map $(E, F) \mapsto E \cup F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is continuous for the Fell topology. **P** If $G \subseteq X$ is open and $K \subseteq X$ is compact, then

$$\{(E, F) : (E \cup F) \cap G \ne \emptyset\} = \{(E, F) : E \cap G \ne \emptyset \text{ or } F \cap G \ne \emptyset\},$$

$$\{(E, F) : (E \cup F) \cap K = \emptyset\} = \{E : E \cap K = \emptyset\} \times \{F : F \cap K = \emptyset\}$$

are open in the product topology. So the result follows by 4A2B(a-ii). **Q**

(iii) $\mathcal{C}$ is compact in the Fell topology. **P** Let $\mathfrak{F}$ be an ultrafilter on $\mathcal{C}$. Set $F_0 = \bigcap_{\mathcal{E} \in \mathfrak{F}} \overline{\bigcup \mathcal{E}}$. (α) If $G \subseteq X$ is open and $G \cap F_0 \ne \emptyset$, set $\mathcal{E} = \{F : F \in \mathcal{C},\ F \cap G = \emptyset\}$. Then $\overline{\bigcup \mathcal{E}}$ does not meet G, so does not include F_0, and $\mathcal{E} \notin \mathfrak{F}$. Accordingly $\{F : F \cap G \ne \emptyset\} = \mathcal{C} \setminus \mathcal{E}$ belongs to $\mathfrak{F}$. (β) If $K \subseteq X$ is compact and $F_0 \cap K = \emptyset$, then $\{K \cap \overline{\bigcup \mathcal{E}} : \mathcal{E} \in \mathfrak{F}\}$ is a downwards-directed family of relatively closed subsets of K with empty intersection so must contain the empty set, and there is an $\mathcal{E} \in \mathfrak{F}$ such that $K \cap F = \emptyset$ for every $F \in \mathcal{E}$, that is, $\{F : F \cap K = \emptyset\}$ belongs to $\mathfrak{F}$. (γ) By 4A2B(a-iv), $\mathfrak{F} \to F_0$. As $\mathfrak{F}$ is arbitrary, $\mathcal{C}$ is compact. **Q**

(c) If X is Hausdorff, $x \mapsto \{x\}$ is continuous for the Fell topology on $\mathcal{C}$. **P** If $G \subseteq X$ is open, then $\{x : \{x\} \cap G \ne \emptyset\} = G$ is open. If $K \subseteq X$ is compact, then $\{x : \{x\} \cap K = \emptyset\} = X \setminus K$ is open (3A3Dc). **Q**

(d) If X and another topological space Y are regular, and $\mathcal{C}_Y$, $\mathcal{C}_{X \times Y}$ are the families of closed subsets of Y and $X \times Y$ respectively, then $(E, F) \mapsto E \times F : \mathcal{C}_X \times \mathcal{C}_Y \to \mathcal{C}_{X \times Y}$ is continuous when each space is given its Fell topology. **P** (i) Suppose that $W \subseteq X \times Y$ is open, and consider $\mathcal{V}_W = \{(E, F) : E \in \mathcal{C}_X,\ F \in \mathcal{C}_Y,\ (E \times F) \cap W \ne \emptyset\}$. If $(E_0, F_0) \in \mathcal{V}_W$, take $(x_0, y_0) \in (E_0 \times F_0) \cap W$. Let $G \subseteq X$ and $H \subseteq Y$ be open sets such that $(x_0, y_0) \in G \times H \subseteq W$. Then $\{(E, F) : E \in \mathcal{C}_X,\ F \in \mathcal{C}_Y,\ E \cap G \ne \emptyset,\ F \cap H \ne \emptyset\}$ is an open set in $\mathcal{C}_X \times \mathcal{C}_Y$ containing (E_0, F_0) and included in $\mathcal{V}_W$. As (E_0, F_0) is arbitrary, $\mathcal{V}_W$ is open in $\mathcal{C}_X \times \mathcal{C}_Y$. (ii) Suppose that $K \subseteq X \times Y$ is compact, and consider $\mathcal{W}_K = \{(E, F) : E \in \mathcal{C}_X,\ F \in \mathcal{C}_Y,\ (E \times F) \cap K = \emptyset\}$. If $(E_0, F_0) \in \mathcal{W}_K$, then set $K_1 = K \cap (E_0 \times Y)$ and $K_2 = K \cap (X \times F_0)$. These are disjoint closed subsets of K, so there are disjoint open subsets U_1, U_2 of $X \times Y$ including K_1, K_2 respectively (4A2F(h-ii)). Now $K_1' = K \setminus U_1$ and $K_2' = K \setminus U_2$ are compact subsets of K with union K.

Let π_1, π_2 be the projections from $X \times Y$ onto X, Y respectively; then $\{(E, F) : E \in \mathcal{C}_X,\ E \cap \pi_1[K_1'] = F \cap \pi_2[K_2'] = \emptyset\}$ is an open set in $\mathcal{C}_X \times \mathcal{C}_Y$ containing (E_0, F_0) and included in $\mathcal{W}_K$. As (E_0, F_0) is arbitrary, $\mathcal{W}_K$ is open. (iii) Putting these together with 4A2B(a-ii), we see that $(E, F) \mapsto E \times F$ is continuous. **Q**

(e) Suppose that X is locally compact and Hausdorff.

(i) The set $\{(E,F) : E,\ F \in \mathcal{C},\ E \subseteq F\}$ is closed in $\mathcal{C}\times\mathcal{C}$ for the product topology defined from the Fell topology on $\mathcal{C}$. **P** Suppose that $E_0,\ F_0 \in \mathcal{C}$ and $E_0 \not\subseteq F_0$. Take $x \in E_0 \setminus F_0$. Because X is locally compact and Hausdorff, there is a relatively compact open set G such that $x \in G$ and $\overline{G}\cap F_0 = \emptyset$. Now $\mathcal{V} = \{E : E\cap G \neq \emptyset\}$ and $\mathcal{W} = \{F : F\cap\overline{G} = \emptyset\}$ are open sets in $\mathcal{C}$ containing E_0, F_0 respectively, and $E \not\subseteq F$ for every $E \in \mathcal{V}$ and $F \in \mathcal{W}$. This shows that $\{(E,F) : E \not\subseteq F\}$ is open, so that its complement is closed. **Q**

It follows that $\{(x,F) : x \in F\}$= $\{(x,F) : \{x\} \subseteq F\}$ is closed in $X\times\mathcal{C}$ when $\mathcal{C}$ is given its Fell topology, since $x \mapsto \{x\}$ is continuous, by (c) above.

(ii) The Fell topology on $\mathcal{C}$ is Hausdorff. **P** The set

$$\{(E,E) : E \in \mathcal{C}\} = \{(E,F) : E \subseteq F \text{ and } F \subseteq E\}$$

is closed in $\mathcal{C}\times\mathcal{C}$, by (i). So 4A2F(a-iii) applies. **Q**

It follows that if $\langle F_i\rangle_{i\in I}$ is a family in $\mathcal{C}$, and $\mathcal{F}$ is an ultrafilter on I, then we have a well-defined limit $\lim_{i\to\mathcal{F}} F_i$ defined in $\mathcal{C}$ for the Fell topology, because $\mathcal{C}$ is compact ((b-iii) above).

(iii) If $\mathcal{L} \subseteq \mathcal{C}$ is compact, then $\bigcup\mathcal{L}$ is a closed subset of X. **P** Take $x \in X\setminus\bigcup\mathcal{L}$. For every $C \in \mathcal{L}$, there is a relatively compact open set G containing x such that $C\cap\overline{G}$ is empty; now finitely many such open sets G must suffice for every $C \in \mathcal{L}$, and the intersection of these G is a neighbourhood of x not meeting $\bigcup\mathcal{L}$. **Q** (Compare 4A2Gm.)

(f) Suppose that X is metrizable, locally compact and separable. Then the Fell topology on $\mathcal{C}$ is metrizable. **P** X is second-countable (4A2P(a-i)); let $\mathcal{U}$ be a countable base for the topology of X consisting of relatively compact open sets (4A2Ob) and closed under finite unions. Let $\mathbb{V}$ be the set of open sets in $\mathcal{C}$ expressible in the form

$$\{F : F\cap U \neq \emptyset \text{ for every } U \in \mathcal{U}_0,\ F\cap\overline{V} = \emptyset\}$$

where $\mathcal{U}_0 \subseteq \mathcal{U}$ is finite and $V \in \mathcal{U}$. Then $\mathbb{V}$ is countable. If $\mathcal{V} \subseteq \mathcal{C}$ is open and $F_0 \in \mathcal{V}$, then there are a finite family $\mathcal{G}$ of open sets in X and a compact $K \subseteq X$ such that

$$F_0 \in \{F : F\cap G \neq \emptyset \text{ for every } G \in \mathcal{G},\ F\cap K = \emptyset\} \subseteq \mathcal{V}.$$

For each $G \in \mathcal{G}$ there is a $U_G \in \mathcal{U}$ such that $U_G \subseteq G$ and $F\cap U_G \neq \emptyset$. Next, each point of K belongs to a member of $\mathcal{U}$ with closure disjoint from F_0, so (because K is compact and $\mathcal{U}$ is closed under finite unions) there is a $V \in \mathcal{U}$ such that $K \subseteq V$ and $F_0\cap\overline{V} = \emptyset$. Now

$$\mathcal{V}' = \{F : F\cap U_G \neq \emptyset \text{ for every } G \in \mathcal{G},\ F\cap\overline{V} = \emptyset\}$$

belongs to $\mathbb{V}$, contains F_0 and is included in $\mathcal{V}$. This shows that $\mathbb{V}$ is a base for the Fell topology, and the Fell topology is second-countable. Since we already know that it is compact and Hausdorff, therefore regular, it is metrizable (4A2Pb). **Q**

(g) Suppose that X is metrizable, and that ρ is a metric inducing the topology of X; let $\tilde\rho$ be the corresponding Hausdorff metric on $\mathcal{C}\setminus\{\emptyset\}$.

(i) The topology $\mathfrak{S}_{\tilde\rho}$ defined by $\tilde\rho$ is finer than the Fell topology $\mathfrak{S}_F$ on $\mathcal{C}\setminus\{\emptyset\}$. **P** Let $G \subseteq X$ be open, and consider the set $\mathcal{V}_G = \{F : F \in \mathcal{C},\ F\cap G \neq \emptyset\}$. If $E \in \mathcal{V}_G$, take $x \in E\cap G$ and $\epsilon > 0$ such that $U(x,\epsilon) \subseteq G$; then $\{F : \tilde\rho(F,E) < \epsilon\} \subseteq \mathcal{V}_G$. As E is arbitrary, $\mathcal{V}_G$ is $\mathfrak{S}_{\tilde\rho}$-open. Next, suppose that $K \subseteq X$ is compact, and consider the set $\mathcal{W}_K = \{F : F \in \mathcal{C}\setminus\{\emptyset\},\ F\cap K = \emptyset\}$. If $E \in \mathcal{W}_K$, the function $x \mapsto \rho(x,E) : K \to]0,\infty[$ is continuous, so has a non-zero lower bound ϵ say; now $\{F : \tilde\rho(F,E) < \epsilon\} \subseteq \mathcal{W}_K$. As E is arbitrary, $\mathcal{W}_K$ is $\mathfrak{S}_{\tilde\rho}$-open. So $\mathfrak{S}_{\tilde\rho}$ is finer than the topology $\mathfrak{S}_F$ generated by the sets $\mathcal{V}_G$ and $\mathcal{W}_K$. **Q**

(ii) If X is compact, then $\mathfrak{S}_{\tilde\rho}$ and $\mathfrak{S}_F$ are the same, and both are compact. **P** Suppose that $E \in \mathcal{V} \in \mathfrak{S}_{\tilde\rho}$. Let $\epsilon \in]0,1[$ be such that $F \in \mathcal{V}$ whenever $F \in \mathcal{C}\setminus\{\emptyset\}$ and $\tilde\rho(E,F) < 2\epsilon$. Because X is compact, E is ρ-totally bounded (4A2Je) and there is a finite set $I \subseteq E$ such that $E \subseteq \bigcup_{x\in I} U(x,\epsilon)$. Because $x \mapsto \rho(x,E)$ is continuous, $K = \{x : \rho(x,E) \geq \epsilon\}$ is closed, therefore compact; now

$$\mathcal{W} = \{F : F \in \mathcal{C},\ F\cap K = \emptyset,\ F\cap U(x,\epsilon) \neq \emptyset \text{ for every } x \in I\}$$

is a neighbourhood of E for $\mathfrak{S}_F$ included in $\mathcal{V}$. Thus $\mathcal{V}$ is a neighbourhood of E for $\mathfrak{S}_F$; as E and $\mathcal{V}$ are arbitrary, $\mathfrak{S}_F$ is finer than $\mathfrak{S}_{\tilde\rho}$. So the two topologies are equal.

Observe finally that $\{\emptyset\} = \{F : F \in \mathcal{C},\ F\cap X = \emptyset\}$ is open for the Fell topology on $\mathcal{C}$, so $\mathcal{C}\setminus\{\emptyset\}$ is closed, therefore compact, by (b-iii). So $\mathfrak{S}_{\tilde\rho} = \mathfrak{S}_F$ is compact. **Q**

4A2U Old friends (a) $\mathbb{R}$, with its usual topology, is metrizable (2A3Ff) and separable (the countable set $\mathbb{Q}$ is dense), so is second-countable (4A2P(a-i)). Every subset of $\mathbb{R}$ is separable (4A2P(a-iv)); in particular, every dense subset of $\mathbb{R}$ has a countable subset which is still dense.

(b) $\mathbb{N}^{\mathbb{N}}$ is Polish in its usual topology (4A2Qc), so has a countable network (4A2P(a-iii) or 4A2Ne), and is hereditarily Lindelöf (4A2Nb or 4A2Pd). Moreover, it is homeomorphic to $[0,1]\setminus\mathbb{Q}$ (Kuratowski 66, §36.II; Kechris 95, 7.7).

(c) The map $x\mapsto\frac{2}{3}\sum_{j=0}^{\infty}3^{-j}x(j)$ (cf. 134Gb) is a homeomorphism between $\{0,1\}^{\mathbb{N}}$ and the Cantor set $C\subseteq[0,1]$. (It is a continuous bijection.)

(d) If I is any set, then the map $A\mapsto\chi A:\mathcal{P}I\to\{0,1\}^I$ is a homeomorphism (for the usual topologies on $\mathcal{P}I$ and $\{0,1\}^I$, as described in 4A2A and 3A3K). So $\mathcal{P}I$ is zero-dimensional, compact (3A3K) and Hausdorff. If I is countable, then $\mathcal{P}I$ is metrizable, therefore Polish (4A2Qb).

(e) Give the space $C([0,\infty[)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact sets.

(i) $C([0,\infty[)$ is a Polish locally convex linear topological space. **P** $\mathfrak{T}_c$ is determined by the seminorms $f\mapsto\sup_{t\le n}|f(t)|$ for $n\in\mathbb{N}$, so it is a metrizable linear space topology. By 4A2Oe, it has a countable network, so is separable. Any function which is continuous on every set $[0,n]$ is continuous on $[0,\infty[$, so $C([0,\infty[)$ is complete under the metric $(f,g)\mapsto\sum_{n=0}^{\infty}\min(2^{-n},\sup_{t\le n}|f(t)-g(t)|)$; as this metric defines $\mathfrak{T}_c$, $\mathfrak{T}_c$ is Polish. **Q**

(ii) Suppose that $A\subseteq C([0,\infty[)$ is such that $\{f(0):f\in A\}$ is bounded and for every $a\ge0$ and $\epsilon>0$ there is a $\delta>0$ such that $|f(s)-f(t)|\le\epsilon$ whenever $f\in A$, s, $t\in[0,a]$ and $|s-t|\le\delta$. Then A is relatively compact for $\mathfrak{T}_c$. **P** Note first that $\{f(a):f\in A\}$ is bounded for every $a\ge0$, since if $\delta>0$ is such that $|f(s)-f(t)|\le1$ whenever $f\in A$, s, $t\in[0,a]$ and $|s-t|\le\delta$, then $|f(a)|\le|f(0)|+\lceil\frac{a}{\delta}\rceil$ for every $f\in A$. So if $\mathcal{F}$ is an ultrafilter on $C([0,\infty[)$ containing A, $g(a)=\lim_{f\to\mathcal{F}}f(a)$ is defined for every $a\ge0$. If $a\ge0$ and $\epsilon>0$, let $\delta\in\,]0,1]$ be such that $|f(s)-f(t)|\le\epsilon$ whenever $f\in A$, s, $t\in[0,a+1]$ and $|s-t|\le\delta$; then $|g(s)-g(a)|\le\epsilon$ whenever $|s-a|\le\delta$; as a and ϵ are arbitrary, $g\in C([0,\infty[)$. If $a\ge0$ and $\epsilon>0$, let $\delta>0$ be such that $|f(s)-f(t)|\le\frac{1}{3}\epsilon$ whenever $f\in A$, s, $t\in[0,a]$ and $|s-t|\le\delta$. Then

$$A'=\{f:f\in A,\ |f(i\delta)-g(i\delta)|\le\frac{1}{3}\epsilon\text{ for every }i\le\lceil\frac{a}{\delta}\rceil\}$$

belongs to $\mathcal{F}$. Now $|f(t)-g(t)|\le\epsilon$ for every $f\in A'$ and $t\in[0,a]$. As a and ϵ are arbitrary, $\mathcal{F}\to g$ for $\mathfrak{T}_c$. As $\mathcal{F}$ is arbitrary, A is relatively compact (3A3De). **Q**

4A3 Topological σ-algebras

I devote a section to some σ-algebras which can be defined on topological spaces. While 'measures' will not be mentioned here, the manipulation of these σ-algebras is an essential part of the technique of measure theory, and I will give proofs and exercises as if this were part of the main work. I look at Borel σ-algebras (4A3A-4A3J), Baire σ-algebras (4A3K-4A3P), Baire-property algebras (4A3Q, 4A3R), cylindrical σ-algebras on linear spaces (4A3T-4A3V) and spaces of càdlàg functions (4A3W).

4A3A Borel sets If $(X,\mathfrak{T})$ is a topological space, the **Borel σ-algebra** of X is the σ-algebra $\mathcal{B}(X)$ of subsets of X generated by $\mathfrak{T}$. Its elements are the **Borel sets** of X. If $(Y,\mathfrak{S})$ is another topological space with Borel σ-algebra $\mathcal{B}(Y)$, a function $f:X\to Y$ is **Borel measurable** if $f^{-1}[H]\in\mathcal{B}(X)$ for every $H\in\mathfrak{S}$, and is a **Borel isomorphism** if it is a bijection and $\mathcal{B}(Y)=\{F:F\subseteq Y,\ f^{-1}[F]\in\mathcal{B}(X)\}$, that is, f is an isomorphism between the structures $(X,\mathcal{B}(X))$ and $(Y,\mathcal{B}(Y))$.

4A3B (Σ,T)-measurable functions It is time I put the following idea into bold type.

(a) Let X and Y be sets, with σ-algebras $\Sigma\subseteq\mathcal{P}X$ and $\mathrm{T}\subseteq\mathcal{P}Y$. A function $f:X\to Y$ is **(Σ,T)-measurable** if $f^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$.

(b) If Σ, T and Υ are σ-algebras of subsets of X, Y and Z respectively, and $f:X\to Y$ is (Σ,T)-measurable while $g:Y\to Z$ is (T,Υ)-measurable, then $gf:X\to Z$ is (Σ,Υ)-measurable. (If $H\in\Upsilon$, $g^{-1}[H]\in\mathrm{T}$ so $(gf)^{-1}[H]=f^{-1}[g^{-1}[H]]\in\Sigma$.)

(c) Let $\langle X_i\rangle_{i\in I}$ be a family of sets with product X, Y another set, and $f:X\to Y$ a function. If $\mathrm{T}\subseteq\mathcal{P}Y$, $\Sigma_i\subseteq\mathcal{P}X_i$ are σ-algebras for each i, then f is $(\mathrm{T},\widehat{\bigotimes}_{i\in I}\Sigma_i)$-measurable iff $\pi_i f:Y\to X_i$ is (T,Σ_i)-measurable for every i, where $\pi_i:X\to X_i$ is the coordinate map. **P** π_i is $(\widehat{\bigotimes}_{j\in I}\Sigma_j,\Sigma_i)$-measurable, so if f is $(\mathrm{T},\widehat{\bigotimes}_{j\in I}\Sigma_j)$-measurable then $\pi_i f$ must be (T,Σ_i)-measurable. In the other direction, if every $\pi_i f$ is measurable, then $\{H:H\subseteq X,\ f^{-1}[E]\in\mathrm{T}\}$ is a σ-algebra of subsets of X containing $\pi_i^{-1}[E]$ whenever $i\in I$ and $E\in\Sigma_i$, so includes $\widehat{\bigotimes}_{i\in I}\Sigma_i$, and f is measurable. **Q**

4A3C Elementary facts (a) If X is a topological space and Y is a subspace of X, then $\mathcal{B}(Y)$ is just the subspace σ-algebra $\{E\cap Y:E\in\mathcal{B}(X)\}$. **P** $\{E:E\subseteq X,\ E\cap Y\in\mathcal{B}(Y)\}$ and $\{E\cap Y:E\in\mathcal{B}(X)\}$ are σ-algebras containing all open sets, so include $\mathcal{B}(X)$, $\mathcal{B}(Y)$ respectively. **Q**

(b) If X is a set, Σ is a σ-algebra of subsets of X, $(Y,\mathfrak{S})$ is a topological space and $f:X\to Y$ is a function, then f is $(\Sigma,\mathcal{B}(Y))$-measurable iff $f^{-1}[H]\in\Sigma$ for every $H\in\mathfrak{S}$. **P** If f is $(\Sigma,\mathcal{B}(Y))$-measurable then $f^{-1}[H]\in\Sigma$ for every $H\in\mathfrak{S}$ just because $\mathfrak{S}\subseteq\mathcal{B}(Y)$. If $f^{-1}[H]\in\Sigma$ for every $H\in\mathfrak{S}$, then $\{F:F\subseteq Y,\ f^{-1}[F]\in\Sigma\}$ is a σ-algebra of subsets of Y (111Xc; cf. 234C[1]) including $\mathfrak{S}$, so includes $\mathcal{B}(Y)$, and f is $(\Sigma,\mathcal{B}(Y))$-measurable. **Q**

(c) If X and Y are topological spaces, and $f:X\to Y$ is a function, then f is Borel measurable iff it is $(\mathcal{B}(X),\mathcal{B}(Y))$-measurable. (Apply (b) with $\Sigma=\mathcal{B}(X)$.) So if X, Y and Z are topological spaces and $f:X\to Y$, $g:Y\to Z$ are Borel measurable functions, then $gf:X\to Z$ is Borel measurable. (Use 4A3Bb.)

(d) If X and Y are topological spaces and $f:X\to Y$ is continuous, it is Borel measurable. (Immediate from the definitions in 4A3A.)

(e) If X is a topological space and $f:X\to[-\infty,\infty]$ is lower semi-continuous, then it is Borel measurable. (The inverse image of a half-open interval $]\alpha,\beta]$ is a difference of open sets, so is a Borel set, and every open subset of $[-\infty,\infty]$ is a countable union of such half-open intervals.)

(f) If $\langle X_i\rangle_{i\in I}$ is a family of topological spaces with product X, then $\mathcal{B}(X)\supseteq\widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i)$. (Put (d) and 4A3Bc together.)

(g) Let X be a topological space.

(i) The algebra $\mathfrak{A}$ of subsets generated by the open sets is precisely the family of sets expressible as a disjoint union $\bigcup_{i\le n}G_i\cap F_i$ where every G_i is open and every F_i is closed. **P** Write $\mathcal{A}$ for the family of sets expressible in this form. Of course $\mathcal{A}\subseteq\mathfrak{A}$. In the other direction, observe that

$X\in\mathcal{A}$,
if E, $E'\in\mathcal{A}$ then $E\cap E'\in\mathcal{A}$,
if $E\in\mathcal{A}$ then $X\setminus E\in\mathcal{A}$

because if G_i is open and F_i is closed for $i\le n$, then (identifying $\{0,\dots,n\}$ with $n+1$)

$$\begin{aligned}X\setminus\bigcup_{i\le n}(G_i\cap F_i)&=\bigcap_{i\le n}(X\setminus G_i)\cup(G_i\setminus F_i)\\&=\bigcup_{I\subseteq n+1}\Bigl(\bigcap_{i\in I}(G_i\setminus F_i)\cap\bigcap_{i\in(n+1)\setminus I}(X\setminus G_i)\Bigr)\end{aligned}$$

belongs to $\mathcal{A}$. So $\mathcal{A}$ is an algebra of sets and must be equal to $\mathfrak{A}$. **Q**

(ii) $\mathcal{B}(X)$ is the smallest family $\mathcal{E}\supseteq\mathfrak{A}$ such that $\bigcup_{n\in\mathbb{N}}E_n\in\mathcal{E}$ for every non-decreasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ and $\bigcap_{n\in\mathbb{N}}E_n\in\mathcal{E}$ for every non-increasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$. (136G.)

4A3D Hereditarily Lindelöf spaces (a) Suppose that X is a hereditarily Lindelöf space and $\mathcal{U}$ is a subbase for the topology of X. Then $\mathcal{B}(X)$ is the σ-algebra of subsets of X generated by $\mathcal{U}$. **P** Write Σ for the σ-algebra generated by $\mathcal{U}$. Of course $\Sigma\subseteq\mathcal{B}(X)$ just because every member of $\mathcal{U}$ is open. In the other direction, set

$$\mathcal{V}=\{X\}\cup\{U_0\cap U_1\cap\ldots\cap U_n:U_0,\dots,U_n\in\mathcal{U}\};$$

then $\mathcal{V}\subseteq\Sigma$ and $\mathcal{V}$ is a base for the topology of X (4A2B(a-i)). If $G\subseteq X$ is open, set $\mathcal{V}_1=\{V:V\in\mathcal{V},\ V\subseteq G\}$; then $G=\bigcup\mathcal{V}_1$. Because X is hereditarily Lindelöf, there is a countable set $\mathcal{V}_0\subseteq\mathcal{V}_1$ such that $G=\bigcup\mathcal{V}_0$ (4A2H(c-i)), so that $G\in\Sigma$. Thus every open set belongs to Σ and $\mathcal{B}(X)\subseteq\Sigma$. **Q**

[1] Formerly 112E.

(b) Let X be a set, Σ a σ-algebra of subsets of X, Y a hereditarily Lindelöf space, $\mathcal{U}$ a subbase for the topology of Y, and $f : X \to Y$ a function. If $f^{-1}[U] \in \Sigma$ for every $U \in \mathcal{U}$, then f is $(\Sigma, \mathcal{B}(Y))$-measurable. **P** $\{F : F \subseteq Y, f^{-1}[F] \in \Sigma\}$ is a σ-algebra of subsets of Y including $\mathcal{U}$, so contains every open set, by (a). **Q**

(c) Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X. Suppose that X is hereditarily Lindelöf.

(i) $\mathcal{B}(X) = \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$. **P** By 4A3Cf, $\mathcal{B}(X) \supseteq \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$. On the other hand, $\widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$ is a σ-algebra including

$$\mathcal{U} = \{\pi_i^{-1}[G] : i \in I, \, G \subseteq X_i \text{ is open}\},$$

where $\pi_i(x) = x(i)$ for $i \in I$ and $x \in X$; since $\mathcal{U}$ is a subbase for the topology of X, (a) tells us that $\widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$ includes $\mathcal{B}(X)$. **Q**

(ii) If Y is another topological space, then a function $f : Y \to X$ is Borel measurable iff $\pi_i f : Y \to X_i$ is Borel measurable for every $i \in I$, where $\pi_i : X \to X_i$ is the canonical map. (Use 4A3Bc.)

4A3E Applications Recall that any topological space with a countable network (in particular, any second-countable space, any separable metrizable space) is hereditarily Lindelöf (4A2Nb, 4A2Oc, 4A2P(a-iii)); and so is any ccc totally ordered space (4A2Rn). So 4A3Da-4A3Db will be applicable to these. As for product spaces, the product of a countable family of spaces with countable networks again has a countable network (4A2Ne), so for such spaces we shall be able to use 4A3Dc. For instance, $\mathcal{B}(\{0,1\}^{\mathbb{N}})$ is the σ-algebra of subsets of $\{0,1\}^{\mathbb{N}}$ generated by the sets $\{x : x(n) = 1\}$ for $n \in \mathbb{N}$.

4A3F Spaces with countable networks (a) Let X be a topological space with a countable network. Then $\#(\mathcal{B}(X)) \le \mathfrak{c}$. **P** Let $\mathcal{E}$ be a countable network for the topology of X and Σ the σ-algebra of subsets of X generated by $\mathcal{E}$. Then $\#(\Sigma) \le \mathfrak{c}$ (4A1O). If $G \subseteq X$ is open, there is a subset $\mathcal{E}'$ of $\mathcal{E}$ such that $\bigcup \mathcal{E}' = G$; but $\mathcal{E}'$ is necessarily countable, so $G \in \Sigma$. It follows that $\mathcal{B}(X) \subseteq \Sigma$ and $\#(\mathcal{B}(X)) \le \mathfrak{c}$. **Q**

(b) $\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) = \mathfrak{c}$. **P** $\mathbb{N}^{\mathbb{N}}$ has a countable network (4A2Ub), so $\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) \le \mathfrak{c}$. On the other hand, $\mathcal{B}(\mathbb{N}^{\mathbb{N}})$ contains all singletons, so

$$\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) \ge \#(\mathbb{N}^{\mathbb{N}}) \ge \#(\{0,1\}^{\mathbb{N}}) = \mathfrak{c}. \text{ } \mathbf{Q}$$

4A3G Second-countable spaces (a) Suppose that X is a second-countable space and Y is any topological space. Then $\mathcal{B}(X \times Y) = \mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$. **P** By 4A3Cf, $\mathcal{B}(X \times Y) \supseteq \mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$. On the other hand, let $\mathcal{U}$ be a countable base for the topology of X. If $W \subseteq X \times Y$ is open, set

$$V_U = \bigcup \{H : H \subseteq Y \text{ is open}, \, U \times H \subseteq W\}$$

for $U \in \mathcal{U}$. Then $W = \bigcup_{U \in \mathcal{U}} U \times V_U$ belongs to $\mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$. As W is arbitrary, $\mathcal{B}(X \times Y) \subseteq \mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$. **Q**

(b) If X is any topological space, Y is a T_0 second-countable space, and $f : X \to Y$ is Borel measurable, then (the graph of) f is a Borel set in $X \times Y$. **P** Let $\mathcal{U}$ be a countable base for the topology of Y. Because Y is T_0,

$$f = \bigcap_{U \in \mathcal{U}} (\{(x, y) : x \in f^{-1}[U], \, y \in U\} \cup \{(x, y) : x \in X \setminus f^{-1}[U], \, y \in Y \setminus U\})$$

which is a Borel subset of $X \times Y$ by 4A3Cc and 4A3Cf. **Q**

4A3H Borel sets in Polish spaces: Proposition Let $(X, \mathfrak{T})$ be a Polish space and $E \subseteq X$ a Borel set. Then there is a Polish topology $\mathfrak{S}$ on X, including $\mathfrak{T}$, for which E is open.

proof Let $\mathcal{E}$ be the union of all the Polish topologies on X including $\mathfrak{T}$. Of course $X \in \mathcal{E}$. If $E \in \mathcal{E}$ then $X \setminus E \in \mathcal{E}$. **P** There is a Polish topology $\mathfrak{S} \supseteq \mathfrak{T}$ such that $E \in \mathfrak{S}$. As both E and $X \setminus E$ are Polish in the subspace topologies $\mathfrak{S}_E$, $\mathfrak{S}_{X \setminus E}$ induced by $\mathfrak{S}$ (4A2Qd), the disjoint union topology $\mathfrak{S}'$ of $\mathfrak{S}_E$ and $\mathfrak{S}_{X \setminus E}$ is also Polish (4A2Qe). Now $\mathfrak{S}' \supseteq \mathfrak{S} \supseteq \mathfrak{T}$ and $X \setminus E \in \mathfrak{S}'$, so $X \setminus E \in \mathcal{E}$. **Q** Moreover, the union of any sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$ belongs to $\mathcal{E}$. **P** For each $n \in \mathbb{N}$ let $\mathfrak{S}_n \supseteq \mathfrak{T}$ be a Polish topology containing E_n. If $m, n \in \mathbb{N}$ then $\mathfrak{S}_m \cap \mathfrak{S}_n$ includes $\mathfrak{T}$, so is Hausdorff. By 4A2Qf, the topology $\mathfrak{S}$ generated by $\bigcup_{n \in \mathbb{N}} \mathfrak{S}_n$ is Polish. Of course $\mathfrak{S} \supseteq \mathfrak{T}$, and $\bigcup_{n \in \mathbb{N}} E_n \in \mathfrak{S}$, so $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{E}$. **Q**

Thus $\mathcal{E}$ is a σ-algebra. Since it surely includes $\mathfrak{T}$, it includes $\mathcal{B}(X, \mathfrak{T})$, as claimed.

4A3I Corollary If $(X,\mathfrak{T})$ is a Polish space and $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence of Borel subsets of X, then there is a zero-dimensional Polish topology $\mathfrak{S}$ on X, including $\mathfrak{T}$, for which every E_n is open-and-closed.

proof (a) For each $n\in\mathbb{N}$ we can find a Polish topology $\mathfrak{T}_n\supseteq\mathfrak{T}$ containing E_k (if $n=2k$ is even) or $X\setminus E_k$ (if $n=2k+1$ is odd); now the topology $\mathfrak{T}'$ generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{T}_n$ is Polish, by 4A2Qf, and every E_n is open-and-closed for $\mathfrak{T}'$.

(b) Now choose $\langle\mathcal{V}_n\rangle_{n\in\mathbb{N}}$ and $\langle\mathfrak{T}'_n\rangle_{n\in\mathbb{N}}$ inductively such that

$\mathfrak{T}'_0=\mathfrak{T}'$,

given that $\mathfrak{T}'_n$ is a Polish topology on X, then $\mathcal{V}_n$ is a countable base for $\mathfrak{T}'_n$ and $\mathfrak{T}'_{n+1}$ is a Polish topology on X including $\mathfrak{T}'_n\cup\{X\setminus V:V\in\mathcal{V}_n\}$

(using (a) for the inductive step). Now the topology $\mathfrak{S}$ generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{T}'_n$ is Polish, includes $\mathfrak{T}$, contains every E_n and has a base $\bigcup_{n\in\mathbb{N}}\mathcal{V}_n$ consisting of open-and-closed sets.

4A3J Borel sets in ω_1: Proposition A set $E\subseteq\omega_1$ is a Borel set iff either E or its complement includes a closed cofinal set.

proof (a) Let Σ be the family of all those sets $E\subseteq\omega_1$ such that either E or $\omega_1\setminus E$ includes a closed cofinal set. Of course Σ is closed under complements. Because the intersection of a sequence of closed cofinal sets is a closed cofinal set (4A1Bd), the union of any sequence in Σ belongs to Σ; so Σ is a σ-algebra. If E is closed, then either it is cofinal with ω_1, and is a closed cofinal set, or there is a $\xi<\omega_1$ such that $E\subseteq\xi$, in which case $\omega_1\setminus E$ includes the closed cofinal set $\omega_1\setminus\xi$; in either case, $E\in\Sigma$. Thus every open set belongs to Σ, and Σ includes $\mathcal{B}(\omega_1)$.

(b) Now suppose that $E\subseteq\omega_1$ is such that there is a closed cofinal set $F\subseteq\omega_1\setminus E$. For each $\xi<\omega_1$ let $f_\xi:\xi\to\mathbb{N}$ be an injective function. Define $g:E\to\mathbb{N}$ by setting $g(\eta)=f_{\alpha(\eta)}(\eta)$, where $\alpha(\eta)=\min(F\setminus\eta)$ for $\eta\in E$. Set $A_n=g^{-1}[\{n\}]$ for $n\in\mathbb{N}$, so that $E=\bigcup_{n\in\mathbb{N}}A_n$. If $\xi\in\overline{A}_n\setminus A_n$ and $\xi'<\xi$, there must be η, $\eta'\in A_n$ such that $\xi'\le\eta<\eta'<\xi$. But now, because $f_{\alpha(\eta')}$ is injective, while $g(\eta)=g(\eta')=n$, $\alpha(\eta)\ne\alpha(\eta')$, so $\alpha(\eta)\in F\cap\,]\xi',\xi[$. As ξ' is arbitrary, $\xi\in\overline{F}=F$. This shows that $\overline{A}_n\subseteq A_n\cup F$ and $A_n=\overline{A}_n\setminus F$ is a Borel set. This is true for every $n\in\mathbb{N}$, so $E=\bigcup_{n\in\mathbb{N}}A_n$ is a Borel set.

(c) If $E\subseteq\omega_1$ includes a closed cofinal set, then (b) tells us that $\omega_1\setminus E$ and E are Borel sets. Thus $\Sigma\subseteq\mathcal{B}(\omega_1)$ and $\Sigma=\mathcal{B}(\omega_1)$, as claimed.

4A3K Baire sets When we come to study measures in terms of the integrals of continuous functions (§436), we find that it is sometimes inconvenient or even impossible to apply them to arbitrary Borel sets, and we need to use a smaller σ-algebra, as follows.

(a) Definition Let X be a topological space. The **Baire σ-algebra** $\mathcal{B}\mathfrak{a}(X)$ of X is the σ-algebra generated by the zero sets. Members of $\mathcal{B}\mathfrak{a}(X)$ are called **Baire** sets. (**Warning!** Do not confuse 'Baire sets' in this sense with 'sets with the Baire property' in the sense of 4A3Q, nor with 'sets which are Baire spaces in their subspace topologies'.)

(b) For any topological space X, $\mathcal{B}\mathfrak{a}(X)\subseteq\mathcal{B}(X)$ (because every zero set is closed, therefore Borel). If $\mathfrak{T}$ is perfectly normal – for instance, if it is metrizable (4A2Lc), or is regular and hereditarily Lindelöf (4A2H(c-ii)) – then $\mathcal{B}\mathfrak{a}(X)=\mathcal{B}(X)$ (because every closed set is a zero set, by 4A2Fi, so every open set belongs to $\mathcal{B}\mathfrak{a}(X)$).

(c) Let X and Y be topological spaces, with Baire σ-algebras $\mathcal{B}\mathfrak{a}(X)$, $\mathcal{B}\mathfrak{a}(Y)$ respectively. If $f:X\to Y$ is continuous, it is $(\mathcal{B}\mathfrak{a}(X),\mathcal{B}\mathfrak{a}(Y))$-measurable. **P** Let T be the σ-algebra $\{F:F\subseteq Y,\ f^{-1}[F]\in\mathcal{B}\mathfrak{a}(X)\}$. If $g:Y\to\mathbb{R}$ is continuous, then $gf:X\to\mathbb{R}$ is continuous, so

$$f^{-1}[\{y:g(y)=0\}]=\{x:gf(x)=0\}\in\mathcal{B}\mathfrak{a}(X),$$

and $\{y:g(y)=0\}\in\mathrm{T}$. Thus every zero set belongs to T, and $\mathrm{T}\supseteq\mathcal{B}\mathfrak{a}(Y)$. **Q**

(d) In particular, if X is a subspace of Y, then $E\cap X\in\mathcal{B}\mathfrak{a}(X)$ whenever $E\in\mathcal{B}\mathfrak{a}(Y)$. More fundamentally, $F\cap X$ is a zero set in X for every zero set $F\subseteq Y$, just because $g\restriction X$ is continuous for any continuous $g:Y\to\mathbb{R}$.

(e) If X is a topological space and Y is a separable metrizable space, a function $f:X\to Y$ is **Baire measurable** if $f^{-1}[H]\in\mathcal{B}\mathfrak{a}(X)$ for every open $H\subseteq Y$. Observe that f is Baire measurable in this sense iff it is $(\mathcal{B}\mathfrak{a}(X),\mathcal{B}(Y))$-measurable iff it is $(\mathcal{B}\mathfrak{a}(X),\mathcal{B}\mathfrak{a}(Y))$-measurable.

4A3L Lemma Let $(X, \mathfrak{T})$ be a topological space. Then $\mathcal{B}\mathfrak{a}(X)$ is just the smallest σ-algebra of subsets of X with respect to which every continuous real-valued function on X is measurable.

proof (a) Let $f : X \to \mathbb{R}$ be a continuous function and $\alpha \in \mathbb{R}$. Set $g(x) = \max(0, f(x) - \alpha)$ for $x \in X$; then g is continuous, so

$$\{x : f(x) \le \alpha\} = \{x : g(x) = 0\}$$

is a zero set and belongs to $\mathcal{B}\mathfrak{a}(X)$. As α is arbitrary, f is $\mathcal{B}\mathfrak{a}(X)$-measurable.

(b) On the other hand, if Σ is any σ-algebra of subsets of X such that every continuous real-valued function on X is Σ-measurable, and $F \subseteq X$ is a zero set, then there is a continuous g such that $F = g^{-1}[\{0\}]$, so that $F \in \Sigma$; as F is arbitrary, $\Sigma \supseteq \mathcal{B}\mathfrak{a}(X)$.

4A3M Product spaces Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X.

(a) $\mathcal{B}\mathfrak{a}(X) \supseteq \widehat{\bigotimes}_{i \in I} \mathcal{B}\mathfrak{a}(X_i)$. (Apply 4A3Bc to the identity map from X to itself; compare 4A3Cf.)

(b) Suppose that X is ccc. Then every Baire subset of X is determined by coordinates in a countable set. **P** By 254Mb, the family $\mathcal{W}$ of sets determined by coordinates in countable sets is a σ-algebra of subsets of X. By 4A2E(b-ii), every continuous real-valued function is $\mathcal{W}$-measurable, so $\mathcal{W}$ contains every zero set and every Baire set. **Q**

4A3N Products of separable metrizable spaces: Proposition Let $\langle X_i \rangle_{i \in I}$ be a family of separable metrizable spaces, with product X.

(a) $\mathcal{B}\mathfrak{a}(X) = \widehat{\bigotimes}_{i \in I} \mathcal{B}\mathfrak{a}(X_i) = \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$.

(b) $\mathcal{B}\mathfrak{a}(X)$ is the family of Borel subsets of X which are determined by coordinates in countable sets.

(c) A set $Z \subseteq X$ is a zero set iff it is closed and determined by coordinates in a countable set.

(d) If Y is a dense subset of X, then the Baire σ-algebra $\mathcal{B}\mathfrak{a}(Y)$ of Y is just the subspace σ-algebra $\mathcal{B}\mathfrak{a}(X)_Y$ induced by $\mathcal{B}\mathfrak{a}(X)$.

(e) If Y is a set, T is a σ-algebra of subsets of Y, and $f : Y \to X$ is a function, then f is $(\mathrm{T}, \mathcal{B}\mathfrak{a}(X))$-measurable iff $\pi_i f : Y \to X_i$ is $(\mathrm{T}, \mathcal{B}(X_i))$-measurable for every $i \in I$, where $\pi_i(x) = x(i)$ for $x \in X$ and $i \in I$.

proof (a) X is ccc (4A2E(a-iii)), so if $f : X \to \mathbb{R}$ is continuous, there are a countable set $J \subseteq I$ and a continuous function $g : X_J \to \mathbb{R}$ such that $f = g\tilde{\pi}_J$, where $X_J = \prod_{i \in J} X_i$ and $\tilde{\pi}_J : X \to X_J$ is the canonical map (4A2E(b-ii)). Now X_J is separable and metrizable (4A2P(a-v)), therefore hereditarily Lindelöf (4A2P(a-iii)), so $\mathcal{B}(X_J) = \widehat{\bigotimes}_{i \in J} \mathcal{B}(X_i)$, by 4A3D(c-i). By 4A3Bc, $\tilde{\pi}_J$ is $(\widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i), \widehat{\bigotimes}_{j \in J} \mathcal{B}(X_j))$-measurable, so f is $\widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$-measurable. As f is arbitrary, $\mathcal{B}\mathfrak{a}(X) \subseteq \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$ (4A3L). Also $\mathcal{B}(X_i) = \mathcal{B}\mathfrak{a}(X_i)$ for every i (4A3Kb), so $\widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i) = \widehat{\bigotimes}_{i \in I} \mathcal{B}\mathfrak{a}(X_i)$. With 4A3Ma, this proves the result.

(b)(i) If $W \in \mathcal{B}\mathfrak{a}(X)$ it is certainly a Borel set (4A3Kb), and by 4A3Mb it is determined by coordinates in a countable set.

(ii) If W is a Borel subset of X determined by coordinates in a countable subset J of X, then write $X_J = \prod_{i \in J} X_i$ and $X_{I \setminus J} = \prod_{i \in I \setminus J} X_i$; let $\tilde{\pi}_J : X \to X_J$ be the canonical map. We can identify W with $W' \times X_{I \setminus J}$, where W' is some subset of X_J. Now if $z \in X_{I \setminus J}$, $W' = \{w : w \in X_J, (w, z) \in W\}$ is a Borel subset of X_J, because $w \mapsto (w, z) : X_J \to X$ is continuous. (I am passing over the trivial case $X = \emptyset$.) Since X_J is metrizable (4A2P(a-v)), $W' \in \mathcal{B}\mathfrak{a}(X_J)$ (4A3Kb) and $W = \tilde{\pi}_J^{-1}[W']$ is a Baire set (4A3Kc).

(c)(i) If Z is a zero set, it is surely closed; and it is determined by coordinates in a countable set by (b) above, or directly from 4A2E(b-ii).

(ii) If Z is closed and determined by coordinates in a countable set J, then (in the language of (b) above) it can be identified with $Z' \times X_{I \setminus J}$ for some $Z' \subseteq X_J$. As in the proof of (b), Z' is closed (at least, if $X_{I \setminus J} \ne \emptyset$), so is a zero set (4A2Lc), and $Z = \tilde{\pi}_J^{-1}[Z']$ is a zero set (4A2C(b-iv)).

(d)(i) $\mathcal{B}\mathfrak{a}(Y) \supseteq \mathcal{B}\mathfrak{a}(X)_Y$ by 4A3Kd.

(ii) Let $f : Y \to \mathbb{R}$ be any continuous function. For each $n \in \mathbb{N}$, there is an open set $G_n \subseteq X$ such that $G_n \cap Y = \{y : f(y) > 2^{-n}\}$. Now $\overline{G}_n$ is determined by coordinates in a countable set (4A2E(b-i)), so is a zero set, by

(c) here. Because Y is dense in X, $\overline{G}_n = \overline{G_n \cap Y}$ does not meet $\{y : f(y) = 0\}$, and $\{y : f(y) > 0\} = Y \cap \bigcup_{n\in\mathbb{N}} \overline{G}_n$ belongs to $\mathcal{B}\mathfrak{a}(X)_Y$. Thus $\mathcal{B}\mathfrak{a}(X)_Y$ contains every cozero subset of Y and includes $\mathcal{B}\mathfrak{a}(Y)$.

(e) Put (a) and 4A3Bc together.

4A3O Compact spaces (a) Let $(X, \mathfrak{T})$ be a topological space, $\mathcal{U}$ a subbase for $\mathfrak{T}$, and $\mathfrak{A}$ the algebra of subsets of X generated by $\mathcal{U}$. If $H \subseteq X$ is open and $K \subseteq H$ is compact, there is an $E \in \mathfrak{A}$ such that $K \subseteq E \subseteq H$. **P** Set $\mathcal{V} = \{X\} \cup \{U_0 \cap U_1 \cap \ldots \cap U_n : U_0, \ldots, U_n \in \mathcal{U}\}$, so that $\mathcal{V}$ is a base for $\mathfrak{T}$ and $\mathcal{V} \subseteq \mathfrak{A}$. $\{U : U \in \mathcal{V}, U \subseteq H\}$ is an open cover of the compact set K, so there is a finite set $\mathcal{U}_0 \subseteq \mathcal{V}$ such that $E = \bigcup \mathcal{U}_0$ includes K and is included in H; now $E \in \mathfrak{A}$. **Q**

(b) Let $(X, \mathfrak{T})$ be a compact space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then every open-and-closed subset of X belongs to the algebra of subsets of X generated by $\mathcal{U}$. (If $F \subseteq X$ is open-and-closed, it is also compact; apply (a) here with $K = H = F$.)

(c) Let $(X, \mathfrak{T})$ be a compact space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then $\mathcal{B}\mathfrak{a}(X)$ is included in the σ-algebra of subsets of X generated by $\mathcal{U}$. **P** Let Σ be the σ-algebra generated by $\mathcal{U}$. If $Z \subseteq X$ is a zero set, there is a sequence $\langle H_n \rangle_{n\in\mathbb{N}}$ of open sets with intersection Z (4A2C(b-vi)); now we can find a sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ in Σ such that $Z \subseteq E_n \subseteq H_n$ for every n, by (a), so that $Z = \bigcap_{n\in\mathbb{N}} E_n \in \Sigma$. This shows that every zero set belongs to Σ, so Σ must include $\mathcal{B}\mathfrak{a}(X)$. **Q**

(d) Let $\langle X_i \rangle_{i\in I}$ be a family of compact Hausdorff spaces with product X. Then $\mathcal{B}\mathfrak{a}(X) = \widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$. **P** By 4A3Ma, $\mathcal{B}\mathfrak{a}(X) \supseteq \widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$. On the other hand, let $\mathcal{U}_i$ be the family of cozero sets in X_i for each i. Because X_i is completely regular (3A3Bb), $\mathcal{U}_i$ is a base for its topology (4A2Fc). Set

$$\mathcal{W} = \{\prod_{i\in I} U_i : U_i \in \mathcal{U}_i \text{ for every } i \in I, \ \{i : U_i \neq X_i\} \text{ is finite}\},$$

so that $\mathcal{W} \subseteq \widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$ is a base for the topology of X. By (c) above, $\mathcal{B}\mathfrak{a}(X)$ is the σ-algebra generated by $\mathcal{W}$, and $\mathcal{B}\mathfrak{a}(X) \subseteq \widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$. **Q**

(e) In a compact Hausdorff zero-dimensional space the Baire σ-algebra is the σ-algebra generated by the open-and-closed sets. (Apply (c) with $\mathcal{U}$ the family of open-and-closed sets.)

(f) In particular, for any set I, $\mathcal{B}\mathfrak{a}(\{0,1\}^I)$ is the σ-algebra generated by sets of the form $\{x : x(i) = 1\}$ as i runs over I.

4A3P Proposition The Baire σ-algebra $\mathcal{B}\mathfrak{a}(\omega_1)$ of ω_1 is just the countable-cocountable algebra (211R).

proof We see from 4A2S(b-iii) that every continuous function is measurable with respect to the countable-cocountable algebra, so $\mathcal{B}\mathfrak{a}(\omega_1)$ is included in the countable-cocountable algebra. On the other hand,

$$[0,\xi] = \{\eta : \eta \le \xi\} = [0, \xi+1[\, = \omega_1 \setminus \,]\xi, \omega_1[$$

is an open-and-closed set (4A2S(a-i)), therefore a zero set, therefore belongs to $\mathcal{B}\mathfrak{a}(\omega_1)$, for every $\xi < \omega_1$. Now if $\xi < \omega_1$, it is itself a countable set, so

$$[0,\xi[\, = \bigcup_{\eta<\xi}[0,\eta] \in \mathcal{B}\mathfrak{a}(\omega_1), \qquad \{\xi\} = [0,\xi] \setminus [0,\xi[\, \in \mathcal{B}\mathfrak{a}(\omega_1).$$

It follows that every countable set belongs to $\mathcal{B}\mathfrak{a}(\omega_1)$ and the countable-cocountable algebra is included in $\mathcal{B}\mathfrak{a}(\omega_1)$.

4A3Q Baire property Let X be a topological space, and $\mathcal{M}$ the ideal of meager subsets of X. A subset X has the **Baire property** if it is expressible in the form $G \triangle M$ where $G \subseteq X$ is open and $M \in \mathcal{M}$; that is, $A \subseteq X$ has the Baire property if there is an open set $G \subseteq X$ such that $G \triangle A$ is meager. (For $A = G \triangle M$ iff $M = G \triangle A$.) The family $\widehat{\mathcal{B}}(X)$ of all such sets is the **Baire-property algebra** of X. (See 4A3R.) (**Warning!** do not confuse the 'Baire-property algebra' $\widehat{\mathcal{B}}$ with the 'Baire σ-algebra' $\mathcal{B}\mathfrak{a}$ as defined in 4A3K.) The quotient algebra $\widehat{\mathcal{B}}(X)/\mathcal{M}$ is the **category algebra** of X.

4A3R Proposition Let X be a topological space.

(a) Let $A \subseteq X$ be any set.

(i) There is a smallest regular open set $H \subseteq X$ such that $A \setminus H$ is meager.

(ii) $H \cap G$ is empty whenever $G \subseteq X$ is open and $A \cap G$ is meager; in particular, $H \subseteq \overline{A}$.

(iii) H is in itself a Baire space.

(iv) If $A \in \widehat{\mathcal{B}}(X)$, $H \triangle A$ is meager.

(v) If X is a Baire space and $A \in \widehat{\mathcal{B}}(X)$, then H is the largest open subset of X such that $H \setminus A$ is meager.

(b)(i) $\widehat{\mathcal{B}}(X)$ is a σ-algebra of subsets of X including $\mathcal{B}(X)$.

(ii) $\widehat{\mathcal{B}}(X) = \{G \triangle M : G \subseteq X \text{ is a regular open set}, M \in \mathcal{M}\}$.

(c) If X has a countable network, its category algebra has a countable order-dense set (definition: 313J).

proof (a) (See Kechris 95, 8.29.)

(i) Set $\mathcal{G} = \{G : G \subseteq X \text{ is open}, A \cap G \text{ is meager}\}$. Let $\mathcal{G}_0 \subseteq \bigcup \mathcal{G}$ be a maximal disjoint set, and $G_0 = \bigcup \mathcal{G}_0$. Then $A \cap G_0$ is meager. **P** For each $G \in \mathcal{G}_0$, let $\langle F_{Gn} \rangle_{n \in \mathbb{N}}$ be a sequence of nowhere dense closed sets covering $A \cap G$. Set $A_n = \bigcup_{G \in \mathcal{G}_0} G \cap F_{Gn}$. If $U \subseteq X$ is any non-empty open set, either $U \cap A_n$ is empty or there is a $G \in \mathcal{G}_0$ such that $U \cap G \neq \emptyset$, in which case $U \cap G \setminus F_{Gn}$ is a non-empty open subset of U not meeting A_n. Thus A_n is nowhere dense. This is true for every n, so $A \cap G_0 \subseteq \bigcup_{n \in \mathbb{N}} A_n$ is meager. **Q**

Set $H = \operatorname{int}(X \setminus G_0)$, so that H is a regular open set (definition: 314O). $A \setminus H \subseteq (\overline{G}_0 \setminus G_0) \cup (A \cap G_0)$ is meager. If H' is another regular open set such that $A \setminus H'$ is meager, then $H \setminus \overline{H'} \in \mathcal{G}$; as $H \setminus \overline{H'}$ does not meet G_0, it must be empty, by the maximality of $\mathcal{G}_0$. So $H \subseteq \operatorname{int} \overline{H'} = H'$. Thus H is the smallest regular open set such that $A \setminus H$ is meager.

(ii) If $G \in \mathcal{G}$ then $G \cap H = G \setminus \overline{G}_0$ belongs to $\mathcal{G}$ and is disjoint from every member of $\mathcal{G}_0$; by the maximality of $\mathcal{G}_0$, it is empty.

In particular, $X \setminus \overline{A}$ does not meet H, and $H \subseteq \overline{A}$.

(iii) If now $\langle H_n \rangle_{n \in \mathbb{N}}$ is a sequence of open subsets of H which are dense in H, and $H' \subseteq H$ is any non-empty open set, $H' \setminus H_n$ is nowhere dense for every n, so $H' \setminus \bigcap_{n \in \mathbb{N}} H_n$ is meager. On the other hand, $A \cap H'$ is non-meager so H' also is, and H' must meet $\bigcap_{n \in \mathbb{N}} H_n$. As H' is arbitrary, $\bigcap_{n \in \mathbb{N}} H_n$ is dense in H; as $\langle H_n \rangle_{n \in \mathbb{N}}$ is arbitrary, H is a Baire space in its subspace topology.

(iv) If A has the Baire property, there is an open set G such that $A \triangle G$ is meager. In this case, $A \cap H \setminus \overline{G}$ must be meager, so $H \subseteq \overline{G}$ and $H \setminus A \subseteq (G \setminus A) \cup (\overline{G} \setminus G)$ is meager and $H \triangle A = (A \setminus H) \cup (H \setminus A)$ is meager.

(v) By (iv), $H \setminus A$ is meager. If $G \subseteq X$ is open and $G \setminus A$ is meager, set $G' = G \setminus \overline{H}$. Then $G' \setminus A$ and $A \setminus G'$ are both meager, so G' is meager, and must be empty, since X is a Baire space. Thus $G \subseteq \overline{H}$ and $G \subseteq H$, because H is a regular open set.

(b)(i) (See Čech 66, §22C; Kuratowski 66, §11.III; Kechris 95, 8.22.) Of course $X = X \triangle \emptyset$ belongs to $\widehat{\mathcal{B}}(X)$. If $E \in \widehat{\mathcal{B}}(X)$, let $G \subseteq X$ be an open set such that $E \triangle G$ is meager. Then

$$(X \setminus \overline{G}) \triangle (X \setminus E) \subseteq (\overline{G} \setminus G) \cup (G \triangle E)$$

is meager, so $X \setminus E \in \widehat{\mathcal{B}}(X)$. If $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\widehat{\mathcal{B}}(X)$, then for each $n \in \mathbb{N}$ we can find an open set G_n such that $G_n \triangle E_n$ is meager, and now

$$(\bigcup_{n \in \mathbb{N}} G_n) \triangle (\bigcup_{n \in \mathbb{N}} E_n) \subseteq \bigcup_{n \in \mathbb{N}} G_n \triangle E_n$$

is meager, so $\bigcup_{n \in \mathbb{N}} E_n \in \widehat{\mathcal{B}}(X)$.

This shows that $\widehat{\mathcal{B}}(X)$ is a σ-algebra of subsets of X. Since it contains every open set, it must include the Borel σ-algebra.

(ii) All we have to observe is that $\overline{G} \setminus G$ and $F \setminus \operatorname{int} F$ are meager for all open sets G and closed sets F, so that if G is open then there is a regular open set $H = \operatorname{int} \overline{G}$ such that $G \triangle H \in \mathcal{M}$.

(c) Suppose that X has a countable network $\mathcal{A}$. For $A \in \mathcal{A}$, set $d_A = \overline{A}^\bullet$, the equivalence class of $\overline{A} \in \widehat{\mathcal{B}}$ in $\widehat{\mathcal{B}}/\mathcal{M}$. If $b \in \widehat{\mathcal{B}}/\mathcal{M}$ is non-zero, take $E \in \widehat{\mathcal{B}}$ such that $E^\bullet = b$, and an open set $G \subseteq X$ such that $E \triangle G \in \mathcal{M}$. Then G is not meager. Set $\mathcal{A}_1 = \{A : A \in \mathcal{A}, A \subseteq G\}$; then $\mathcal{A}_1$ is countable and $G = \bigcup \mathcal{A}_1$, so there is a non-meager $A \in \mathcal{A}_1$. In this case $\overline{A}$ is not meager, so $d_A \neq 0$, while $\overline{A} \setminus G \subseteq \overline{G} \setminus G$ is nowhere dense, so $d_A \subseteq b$. As b is arbitrary, $\{d_A : A \in \mathcal{A}\}$ is order-dense in $\widehat{\mathcal{B}}/\mathcal{M}$, and is countable because $\mathcal{A}$ is.

***4A3S** The following result will be useful in §424 and in Volume 5.

Lemma Let X and Y be sets, Σ a σ-algebra of subsets of X, T a σ-algebra of subsets of Y and $\mathcal{J}$ a σ-ideal of T. Suppose that the quotient Boolean algebra $\mathrm{T}/\mathcal{J}$ has a countable order-dense set.

(a) $\{x : x \in X, W[\{x\}] \cap A \in \mathcal{J}\}$ belongs to Σ for any $W \in \Sigma\widehat{\otimes}\mathrm{T}$ and $A \subseteq Y$.

(b) For every $W \in \Sigma\widehat{\otimes}\mathrm{T}$ there are sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ, $\langle V_n\rangle_{n\in\mathbb{N}}$ in T such that $(W\triangle W_1)[\{x\}] \in \mathcal{J}$ for every $x \in X$, where $W_1 = \bigcup_{n\in\mathbb{N}} E_n \times V_n$.

proof (Compare Kechris 95, 16.1.)

(a) Let $D \subseteq \mathrm{T}/\mathcal{J}$ be a countable order-dense set. Let $\mathcal{V} \subseteq \mathrm{T}$ be a countable set such that $D = \{V^\bullet : V \in \mathcal{V}\}$. Let Λ be the family of subsets W of $X \times Y$ such that

$$W[\{x\}] \in \mathrm{T} \text{ for every } x \in X,$$

$$\{x : x \in X, W[\{x\}] \cap A \in \mathcal{J}\} \in \Sigma \text{ for every } A \subseteq Y.$$

(i) Of course $\emptyset \in \Lambda$, because if $W = \emptyset$ then $W[\{x\}] = \emptyset$ for every $x \in X$, and $\{x : W[\{x\}] \cap A \in \mathcal{J}\} = X$ for every $A \subseteq Y$.

(ii) Suppose that $W \in \Lambda$, and set $W' = (X \times Y) \setminus W$. Then

$$W'[\{x\}] = Y \setminus W[\{x\}] \in \mathrm{T}$$

for every $x \in X$. Now suppose that $A \subseteq Y$. Set $\mathcal{V}^* = \{V : V \in \mathcal{V}, A \cap V \notin \mathcal{J}\}$,

$$E = \{x : W'[\{x\}] \cap A \in \mathcal{J}\},$$

$$E' = \{x : V \cap W[\{x\}] \notin \mathcal{J} \text{ for every } V \in \mathcal{V}^*\}.$$

Then $E = E'$. **P** (α) If $x \in E$ and $V \in \mathcal{V}^*$, then $A \cap W'[\{x\}] = A \setminus W[\{x\}]$ belongs to $\mathcal{J}$, so $V \cap A \setminus W[\{x\}] \in \mathcal{J}$ and $V \cap W[\{x\}] \supseteq V \cap A \cap W[\{x\}]$ is not in $\mathcal{J}$. As V is arbitrary, $x \in E'$; as x is arbitrary, $E \subseteq E'$. (β) If $x \notin E$, then $W'[\{x\}] \cap A \notin \mathcal{J}$. Set $\mathcal{V}_1 = \{V : V \in \mathcal{V}, V \setminus W'[\{x\}] \in \mathcal{J}\}$, $D_1 = \{V^\bullet : V \in \mathcal{V}_1\}$. Then $D_1 = \{d : d \in D, d \subseteq W'[\{x\}]^\bullet\}$, so $W'[\{x\}]^\bullet = \sup D_1$ in $\mathrm{T}/\mathcal{J}$ (313K). Because $\mathcal{J}$ is a σ-ideal, the map $F \mapsto F^\bullet : \mathrm{T} \to \mathrm{T}/\mathcal{J}$ is sequentially order-continuous (313Qb), and $(\bigcup\mathcal{V}_1)^\bullet = \sup D_1$ (313Lc), that is, $W'[\{x\}]\triangle\bigcup\mathcal{V}_1 \in \mathcal{J}$. There must therefore be a $V \in \mathcal{V}_1$ such that $V \cap A \notin \mathcal{J}$, that is, $V \in \mathcal{V}^*$. At the same time, $V \cap W[\{x\}] = V \setminus W'[\{x\}]$ belongs to $\mathcal{J}$, so V witnesses that $x \notin E'$. As x is arbitrary, $E' \subseteq E$. **Q**

Since $W \in \Lambda$,

$$E = E' = X \setminus \bigcup_{V\in\mathcal{V}^*}\{x : V \cap W[\{x\}] \in \mathcal{J}\}$$

belongs to T. As A is arbitrary, $W' \in \Lambda$. Thus the complement of any member of Λ belongs to Λ.

(iii) If $\langle W_n\rangle_{n\in\mathbb{N}}$ is any sequence in Λ with union W, then

$$W[\{x\}] = \bigcup_{n\in\mathbb{N}} W_n[\{x\}] \in \mathrm{T}$$

for every x, while

$$\{x : W[\{x\}] \cap A \in \mathcal{J}\} = \bigcap_{n\in\mathbb{N}}\{x : W_n[\{x\}] \cap A \in \mathcal{J}\} \in \Sigma$$

for every $A \subseteq Y$. So $W \in \Lambda$.

(iv) What this shows is that Λ is a σ-algebra of subsets of $X \times Y$. But if $W = E \times F$, where $E \in \Sigma$ and $F \in \mathrm{T}$, then

$$W[\{x\}] \in \{\emptyset, F\} \subseteq \mathrm{T} \text{ for every } x \in X,$$

$$\{x : W[\{x\}] \cap A \in \mathcal{J}\} \in \{X \setminus E, X\} \subseteq \Sigma$$

for every $A \subseteq Y$; so $W \in \Lambda$. Accordingly Λ must include $\Sigma\widehat{\otimes}\mathrm{T}$, as claimed.

(b) Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{V} \cup \{\emptyset\}$. For each $n \in \mathbb{N}$, set

$$E_n = \{x : V_n \setminus W[\{x\}] \in \mathcal{J}\};$$

by (a), $E_n \in \Sigma$. Set $W_1 = \bigcup_{n\in\mathbb{N}} E_n \times V_n$. Take any $x \in X$. Then $W_1[\{x\}] = \bigcup_{n\in I} V_n$ where $I = \{n : x \in E_n\}$. Since $V_n \setminus W[\{x\}] \in \mathcal{J}$ for every $n \in I$, $W_1[\{x\}] \setminus W[\{x\}] \in \mathcal{J}$. **?** If $W[\{x\}] \setminus W_1[\{x\}] \notin \mathcal{J}$, there is a $d \in D$ such that

$0 \neq d \subseteq W[\{x\}]^\bullet \setminus W_1[\{x\}]^\bullet$. Let $n \in \mathbb{N}$ be such that $V_n^\bullet = d$; then $V_n \setminus W[\{x\}] \in \mathcal{J}$, so $n \in I$ and $V_n \subseteq W_1[\{x\}]$ and $d \subseteq W_1[\{x\}]^\bullet$, which is impossible. **X**

As x is arbitrary, W_1 has the required properties.

4A3T Cylindrical σ-algebras I offer a note on a particular type of Baire σ-algebra.

Definition Let X be a linear topological space. Then the **cylindrical σ-algebra** of X is the smallest σ-algebra Σ of subsets of X such that every continuous linear functional on X is Σ-measurable.

4A3U Proposition Let X be a linear topological space and $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$ its weak topology. Then the cylindrical σ-algebra of X is just the Baire σ-algebra of $(X, \mathfrak{T}_s)$.

proof (a) Let $\langle f_i \rangle_{i \in I}$ be a Hamel basis for X^* (4A4Ab). For $x \in X$, set $Tx = \langle f_i(x) \rangle_{i \in I}$; then $T : X \to \mathbb{R}^I$ is a linear operator. Now $Y = T[X]$ is dense in $\mathbb{R}^I$. **P** Y is a linear subspace of $\mathbb{R}^I$, so its closure $\overline{Y}$ also is (2A5Ec). If $\phi \in (\mathbb{R}^I)^*$ is such that $\phi(Tx) = 0$ for every $x \in X$, there are a finite set $J \subseteq I$ and a family $\langle \alpha_i \rangle_{i \in J} \in \mathbb{R}^J$ such that $\phi(y) = \sum_{i \in J} \alpha_i y(i)$ for every $y \in \mathbb{R}^I$ (4A4Be). In this case, $\sum_{i \in J} \alpha_i f_i(x) = \phi(Tx) = 0$ for every $x \in X$; but $\langle f_i \rangle_{i \in I}$ is linearly independent, so $\alpha_i = 0$ for every $i \in J$ and $\phi = 0$. By 4A4Eb, $\overline{Y}$ must be the whole of $\mathbb{R}^I$. **Q**

(b)(i) Set $\mathfrak{T}_s' = \{T^{-1}[H] : H \subseteq Y \text{ is open}\}$. Then $\mathfrak{T}_s' = \mathfrak{T}_s$. **P** Because every f_i is continuous, T is continuous, so $\mathfrak{T}_s' \subseteq \mathfrak{T}_s$. On the other hand, any $f \in X^*$ is a linear combination of the f_i, so is $\mathfrak{T}_s'$-continuous, and $\mathfrak{T}_s \subseteq \mathfrak{T}_s'$. **Q**

(ii) If $g : X \to \mathbb{R}$ is $\mathfrak{T}_s$-continuous, there is a continuous $g_1 : Y \to \mathbb{R}$ such that $g = g_1 T$. **P** If x, $x' \in X$ are such that $Tx = Tx'$, then every $\mathfrak{T}_s$-open set containing one must contain the other, so $g(x) = g(x')$. This means that there is a function $g_1 : Y \to \mathbb{R}$ such that $g = g_1 T$. Next, if $U \subseteq \mathbb{R}$ is open, $T^{-1}[g_1^{-1}[U]] = g^{-1}[U]$ belongs to $\mathfrak{T}_s = \mathfrak{T}_s'$, so $g_1^{-1}[U]$ must be open in Y; as U is arbitrary, g_1 is continuous. **Q**

(c) Now let Σ be the cylindrical σ-algebra of X. Then every $f_i : X \to \mathbb{R}$ is Σ-measurable, so $T : X \to \mathbb{R}^I$ is $(\Sigma, \mathcal{B}\mathfrak{a}(\mathbb{R}^I))$-measurable, by 4A3Ne. Because Y is dense in $\mathbb{R}^I$, $\mathcal{B}\mathfrak{a}(Y)$ is the subspace σ-algebra induced by $\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$ (4A3Nd). So $T : X \to Y$ is $(\Sigma, \mathcal{B}\mathfrak{a}(Y))$-measurable. Now if $g : X \to \mathbb{R}$ is a continuous function, there is a continuous function $g_1 : Y \to \mathbb{R}$ such that $g = g_1 T$; g_1 is $\mathcal{B}\mathfrak{a}(Y)$-measurable, so g is Σ-measurable. As this is true for every $\mathfrak{T}_s$-continuous $g : X \to \mathbb{R}$, $\mathcal{B}\mathfrak{a}(X) \subseteq \Sigma$.

(d) On the other hand, $\Sigma \subseteq \mathcal{B}\mathfrak{a}(X)$ just because every member of X^* is $\mathfrak{T}_s$-continuous. So $\Sigma = \mathcal{B}\mathfrak{a}(X)$, as claimed.

4A3V Proposition Let $(X, \mathfrak{T})$ be a separable metrizable locally convex linear topological space, and $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$ its weak topology. Then the cylindrical σ-algebra of X is also both the Baire σ-algebra and the Borel σ-algebra for both $\mathfrak{T}$ and $\mathfrak{T}_s$.

proof (a) For any linear topological space, we have

$$\Sigma \subseteq \mathcal{B}\mathfrak{a}(X, \mathfrak{T}_s) \subseteq \mathcal{B}\mathfrak{a}(X, \mathfrak{T}) \subseteq \mathcal{B}(X, \mathfrak{T}), \quad \mathcal{B}\mathfrak{a}(X, \mathfrak{T}_s) \subseteq \mathcal{B}(X, \mathfrak{T}_s) \subseteq \mathcal{B}(X, \mathfrak{T}),$$

writing Σ for the cylindrical σ-algebra and $\mathcal{B}\mathfrak{a}(X, \mathfrak{T}_s)$, $\mathcal{B}\mathfrak{a}(X, \mathfrak{T})$, $\mathcal{B}(X, \mathfrak{T}_s)$ and $\mathcal{B}(X, \mathfrak{T})$ for the Baire and Borel σ-algebras of the two topologies. So all I have to do is to show that $\mathcal{B}(X, \mathfrak{T}) \subseteq \Sigma$.

(b) Let $\langle \tau_n \rangle_{n \in \mathbb{N}}$ be a sequence of seminorms defining the topology of X (4A4Cf), and $D \subseteq X$ a countable dense set; for $n \in \mathbb{N}$, $\delta > 0$ and $x \in X$, set $U_n(x, \delta) = \{y : \tau_i(y - x) < \delta \text{ for every } i \leq n\}$. Then every $U_n(x, \delta)$ is a convex open set. Set

$$\mathcal{U} = \{U_n(z, 2^{-m}) : z \in D,\ m,\ n \in \mathbb{N}\}.$$

(c) If $G \subseteq X$ is any convex open set, $\overline{G} \in \Sigma$. **P** Set $\mathcal{U}_G = \{U : U \in \mathcal{U},\ U \cap G = \emptyset\}$. For each $U \in \mathcal{U}_G$, there are $f_U \in X^*$, $\alpha_U \in \mathbb{R}$ such that $f_U(x) < \alpha_U$ for every $x \in G$ and $f_U(x) > \alpha_U$ for every $x \in U$ (4A4Db). So $F = \{x : f_U(x) \leq \alpha_U \text{ for every } U \in \mathcal{U}_G\}$ belongs to Σ. Of course $F \supseteq \overline{G}$. On the other hand, if $x \notin \overline{G}$, there are m, $n \in \mathbb{N}$ such that $G \cap U_n(x, 2^{-m}) = \emptyset$; if we take $z \in D \cap U_n(x, 2^{-m-1})$, $U = U_n(z, 2^{-m-1})$ then $x \in U \in \mathcal{U}_G$, so $f_U(x) > \alpha_U$ and $x \notin F$. Thus $\overline{G} = F$ belongs to Σ. **Q**

(d) In particular, $\mathcal{V} = \{\overline{U} : U \in \mathcal{U}\}$ is included in Σ. But $\mathcal{V}$ is a countable network for $\mathfrak{T}$. So every member of $\mathfrak{T}$ is a union of countably many members of Σ and belongs to Σ. It follows at once that $\mathcal{B}(X, \mathfrak{T}) \subseteq \Sigma$, as required.

4A3W Càdlàg functions Let X be a Polish space, and C_{dlg} the set of càdlàg functions (definition: 4A2A) from $[0,\infty[$ to X, with its topology of pointwise convergence inherited from $X^{[0,\infty[}$.

(a) $\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}})$ is the subspace σ-algebra induced by $\mathcal{B}\mathfrak{a}(X^{[0,\infty[})$.

(b) $(C_{\mathrm{dlg}},\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}}))$ is a standard Borel space.

(c)(i) For any $t\ge 0$, let $\mathcal{B}\mathfrak{a}_t(C_{\mathrm{dlg}})$ be the σ-algebra of subsets of C_{dlg} generated by the functions $\omega\mapsto\omega(s)$ for $s\le t$. Then $(s,\omega)\mapsto\omega(s):C_{\mathrm{dlg}}\times[0,t]\to X$ is $\mathcal{B}([0,t])\widehat{\otimes}\mathcal{B}\mathfrak{a}_t(C_{\mathrm{dlg}})$-measurable.

(ii) $(\omega,t)\mapsto\omega(t):C_{\mathrm{dlg}}\times[0,\infty[\,\to X$ is $\mathcal{B}([0,\infty[)\widehat{\otimes}\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}})$-measurable.

(d) The set $C([0,\infty[\,;X)$ of continuous functions from $[0,\infty[$ to X belongs to $\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}})$.

proof (a) Use 4A3Nd.

(b)(i) Fix a complete metric ρ on X defining its topology. For $A\subseteq B\subseteq\mathbb{R}$, $f\in X^B$ and $\epsilon>0$, set

$$\begin{aligned}\mathrm{jump}_A(f,\epsilon)=\sup\{n: &\text{ there is an } I\in[A]^n \text{ such that } \rho(f(s),f(t))>\epsilon\\ &\text{whenever } s<t \text{ are successive elements of } I\}\end{aligned}$$

(see 438P). Set $D=\mathbb{Q}\cap[0,\infty[$. Then any set of the form $\{f:\mathrm{jump}_A(f,\epsilon)>m\}$ is open, so

$$\begin{aligned}E=\bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}&\{f:f\in X^D,\ \mathrm{jump}_{D\cap[0,n]}(f,2^{-n})\le m\}\\ &\cap\bigcap_{\substack{q'\in D\\ n\in\mathbb{N}}}\bigcup_{m\ge n}\{f:f\in X^D,\ \rho(f(q'+2^{-m}),f(q'))\le 2^{-n}\}\end{aligned}$$

is Borel in the Polish space X^D.

(ii) If $\omega\in C_{\mathrm{dlg}}$, then $\mathrm{jump}_{[0,n]}(\omega,2^{-n})$ is finite for every $n\in\mathbb{N}$. (Apply 438Pa to an extension of ω to a member of $X^{\mathbb{R}}$ which is constant on $]-\infty,0[$; or make the trifling required changes to the argument of 438Pa.) Since $\lim_{n\to\infty}\omega(q'+2^{-n})=\omega(q')$ for every $q'\in D$, $\omega\restriction D\in E$.

(iii) Conversely, given $f\in E$, $\mathrm{jump}_{[0,n]}(f,\epsilon)$ is finite for every $n\in\mathbb{N}$ and $\epsilon>0$, so $\lim_{q\in D,q\downarrow t}f(q)$ is defined in X for every $t\ge0$, and $\lim_{q\in D,q\uparrow t}f(q)$ is defined in X for every $t>0$. (Apply the argument of (a-ii) of the proof of 438P.) Set $\omega_f(t)=\lim_{q\in D,q\downarrow t}f(q)$ for $t\ge0$. Because $f(q)$ is a cluster point of $\langle f(q+2^{-n})\rangle_{n\in\mathbb{N}}$ for every $q\in D$, ω_f extends f. It is easy to see that $\mathrm{jump}_{[0,n[}(\omega_f,\epsilon)=\mathrm{jump}_{[0,n[}(f,\epsilon)$ is finite for every $n\in\mathbb{N}$ and $\epsilon>0$, so $\lim_{s\downarrow t}\omega_f(s)$ is defined for every $t\ge0$ and $\lim_{s\uparrow t}\omega_f(s)$ is defined for every $t>0$. Also, for $t\ge0$,

$$\lim\nolimits_{s\downarrow t}\omega_f(s)=\lim\nolimits_{q\in D,q\downarrow t}\omega_f(q)=\lim\nolimits_{q\in D,q\downarrow t}f(q)=\omega_f(t).$$

Thus $\omega_f\in C_{\mathrm{dlg}}$. Clearly a member of C_{dlg} is uniquely determined by its values on D, so $f\mapsto\omega_f$ and $\omega\mapsto\omega\restriction D$ are the two halves of a bijection between E and C_{dlg}.

(iv) By 424G, E, with its Borel (or Baire) σ-algebra $\mathcal{B}(E)$, is a standard Borel space. Of course the map $\omega\mapsto\omega\restriction D$ is $(\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}}),\mathcal{B}(E))$-measurable. But also the map $f\mapsto\omega_f(t):E\to X$ is $\mathcal{B}(E)$-measurable for every $t\ge0$. **P** If $\langle q_n\rangle_{n\in\mathbb{N}}$ is a sequence in D decreasing to t, $\omega_f(t)=\lim_{n\to\infty}f(q_n)$ for every $f\in E$, and we can use 418Ba. **Q** Since $\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}})$ is the σ-algebra induced by $\mathcal{B}\mathfrak{a}(X^{[0,\infty[})$ (4A3Nd), and $\mathcal{B}\mathfrak{a}(X^{[0,\infty[})=\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(X)$ (4A3Na), this is enough to show that $f\mapsto\omega_f$ is $(\mathcal{B}(E),\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}}))$-measurable.

Thus $(C_{\mathrm{dlg}},\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}}))\cong(E,\mathcal{B}(E))$ is a standard Borel space.

(c)(i) For $n\in\mathbb{N}$, $\omega\in C_{\mathrm{dlg}}$ and $s\in[0,t]$, set $h_n(s,\omega)=\omega(\min(t,2^{-n}i))$ if $i\in\mathbb{N}$ and $2^{-n}(i-1)<s\le2^{-n}i$. If $G\subseteq X$ is open then

$$\begin{aligned}\{(s,\omega):s\le t,\ h_n(s,\omega)\in G\}=\bigcup_{i\in\mathbb{N}}&([0,t]\cap\,]2^{-n}(i-1),2^{-n}i])\\ &\times\{\omega:\omega(\min(t,2^{-n}i))\in G\}\\ &\in\mathcal{B}([0,t])\widehat{\otimes}\mathcal{B}\mathfrak{a}_t(C_{\mathrm{dlg}}),\end{aligned}$$

so h_n is $\mathcal{B}([0,t])\widehat{\otimes}\mathcal{B}\mathfrak{a}_t(C_{\mathrm{dlg}})$-measurable. Now $\omega(s)=\lim_{n\to\infty}h_n(s,\omega)$ for every $\omega\in C_{\mathrm{dlg}}$ and $s\in[0,t]$, so $(s,\omega)\mapsto\omega(s)$ is measurable in the same sense, by 418Ba again.

(ii) If $G\subseteq X$ is open then

$$\{(t,\omega):t\ge 0,\,\omega(t)\in G\}=\bigcup_{n\in\mathbb{N}}\{(t,\omega):t\in[0,n],\,\omega(t)\in G\}\in\mathcal{B}([0,\infty[)\widehat{\otimes}\mathcal{B}\mathfrak{a}(C_{\mathrm{dlg}}).$$

(d)

$$C([0,\infty[\,;X)=\bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\{\omega:\omega\in C_{\mathrm{dlg}},\,\rho(\omega(q),\omega(q'))\le 2^{-n}$$
$$\text{whenever } q,\,q'\in\mathbb{Q}\cap[0,n] \text{ and } |q-q'|\le 2^{-m}\}.$$

4A3X Basic exercises (a) (i) Let X be a regular space with a countable network and Y any topological space. Show that $\mathcal{B}(X\times Y)=\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. (*Hint*: 4A2Ng.) (ii) Set $X=[0,1]$ with the topology generated by $\{X\setminus\{x\}:x\in X\}$ and $Y=[0,1]$ with its discrete topology. Show that X has a countable network and that $\{(x,x):x\in[0,1]\}$ is a closed subset of $X\times Y$ not belonging to $\mathcal{B}(X)\times\mathcal{B}(Y)$.

(b) Let X be a topological space, and $E\subseteq X$. Show that the following are equiveridical: (i) $E\in\mathcal{B}\mathfrak{a}(X)$; (ii) there are a continuous function $f:X\to[0,1]^{\mathbb{N}}$ and $F\in\mathcal{B}([0,1]^{\mathbb{N}})$ such that $E=f^{-1}[F]$.

>(c) Let X be a topological space, and $K\subseteq X$ a compact set such that $K\in\mathcal{B}\mathfrak{a}(X)$. Show that K is a zero set. (*Hint*: 4A3Xb.)

(d) Let X be a compact Hausdorff space such that $\mathcal{B}\mathfrak{a}(X)=\mathcal{B}(X)$. Show that X is perfectly normal.

(e) Let $\mathfrak{S}$ be the topology on $\mathbb{R}$ generated by the usual topology and $\{\{x\}:x\in\mathbb{R}\setminus\mathbb{Q}\}$. Show that $\mathfrak{S}$ is completely regular and Hausdorff and that $\mathbb{Q}$ is a closed Baire set which is not a zero set.

>(f) Let X be a ccc completely regular topological space. Show that any nowhere dense set is included in a nowhere dense zero set.

(g) Let $\langle X_i\rangle_{i\in I}$ be a family of spaces with countable networks, and X their product. Show that $\mathcal{B}\mathfrak{a}(X)=\widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(X_i)$.

>(h)(i) Let I be an uncountable set with its discrete topology, and X the one-point compactification of I. Show that $\mathcal{B}\mathfrak{a}(I)$ is not the subspace σ-algebra generated by $\mathcal{B}\mathfrak{a}(X)$. (ii) Show that ω_1, with its order topology, has a subset I such that $\mathcal{B}\mathfrak{a}(I)$ is not the subspace σ-algebra induced by $\mathcal{B}\mathfrak{a}(\omega_1)$.

4A3Y Further exercises (a) Let X be a Čech-complete space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence of Borel sets in X. Show that there is a Čech-complete topology on X, finer than the given topology, with the same weight, and containing every E_n.

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of regular spaces with countable networks, with product X. Show that (i) $\mathcal{B}\mathfrak{a}(X)$ is the family of Borel subsets of X which are determined by coordinates in countable sets; (ii) a set $Z\subseteq X$ is a zero set iff it is closed and determined by coordinates in a countable set; (iii) if $Y\subseteq X$ is dense, then $\mathcal{B}\mathfrak{a}(Y)$ is just the subspace σ-algebra $\mathcal{B}\mathfrak{a}(X)_Y$ induced by $\mathcal{B}\mathfrak{a}(X)$.

(c) Let X be a normal topological space and Y a closed subset of X. Show that every Baire subset of Y is the intersection of Y with a Baire subset of X. (*Hint*: use Tietze's theorem.)

4A3 Notes and comments Much of this section consists of easy technicalities. It is however not always easy to guess at the exact results obtainable by these methods. It is important to notice that Baire σ-algebras on subspaces can give difficulties which do not arise with Borel σ-algebras (4A3Xh, 4A3Ca). I have expressed 4A3D in terms of 'hereditarily Lindelöf' spaces. Of course the separable metrizable spaces form by far the most important class of these, but there are others (the split interval, for instance) which are of great interest in measure theory. Similarly, there are important products of non-metrizable spaces which are ccc (e.g., 417Xt(vii)), so that 4A3Mb has something to say.

4A3H-4A3I are a most useful tool in studying Borel subsets of Polish spaces, especially in conjunction with the First Separation Theorem (422I); see 424G and 424H. I include 4A3Ya and 4A3Yb to show that some more of the arguments here can be adapted to non-separable or non-metrizable spaces.

You will note my caution in the definition of 'Baire measurable' function (4A3Ke). This is supposed to cover the case of functions taking values in $[-\infty,\infty]$ without taking a position on functions between general topological spaces.

It is relatively easy to show that spaces of càdlàg functions have standard Borel structures (4A3Wb). To exhibit usable complete metrics generating these is another matter; see chap. 3 of Billingsley 99.

4A4 Locally convex spaces

As in §3A5, all the ideas, and nearly all the results as stated below, are applicable to complex linear spaces; but for the purposes of this volume the real case will almost always be sufficient, and for definiteness you may take it that the scalar field is $\mathbb{R}$, except in 4A4J-4A4K. (Complex Hilbert spaces arise naturally in §445.)

4A4A Linear spaces (a) If U is a linear space, a **Hamel basis** for U is a maximal linearly independent family $\langle u_i\rangle_{i\in I}$ in U, so that every member of U is uniquely expressible as $\sum_{i\in J}\alpha_i u_i$ for some finite $J\subseteq I$ and $\langle\alpha_i\rangle_{i\in J}\in(\mathbb{R}\setminus\{0\})^J$.

(b) Every linear space has a Hamel basis. (Schaefer 71, p. 10; Köthe 69, §7.3.)

(c) If U is a linear space, I write U' for the **algebraic dual** of U, the linear space of all linear functionals from U to $\mathbb{R}$.

4A4B Linear topological spaces (see §2A5) **(a)** If U is a linear topological space, and V is a linear subspace of U, then V, with the linear structure and topology induced by those of U, is again a linear topological space. (Bourbaki 87, I.1.3; Schaefer 71, §I.2; Köthe 69, §15.2.)

(b) If $\langle U_i\rangle_{i\in I}$ is any family of linear topological spaces, then $U=\prod_{i\in I}U_i$, with the product linear space structure and topology, is again a linear topological space. (Bourbaki 87, I.1.3; Schaefer 71, §I.2; Köthe 69, §15.4.) In particular, $\mathbb{R}^X$, with its usual linear and topological structures, is a linear topological space, for any set X.

(c) If U and V are linear topological spaces, the set of continuous linear operators from U to V is a linear subspace of the space $\mathsf{L}(U;V)$ of all linear operators from U to V. If U, V and W are linear topological spaces, and $T:U\to V$ and $S:V\to W$ are continuous linear operators, then $ST:U\to W$ is a continuous linear operator.

(d) If U is a linear topological space, I will write U^* for the **dual** of U, the space of all continuous linear functionals from U to $\mathbb{R}$ (compare 2A4H). U^* is a linear subspace of U' as defined in 4A4Ac. The **weak topology** on U, $\mathfrak{T}_s(U,U^*)$, is that defined by the method of 2A5B from the seminorms $u\mapsto|f(u)|$ as f runs over U^* (compare 2A5Ia). The **weak* topology** on U^*, $\mathfrak{T}_s(U^*,U)$, is that defined from the seminorms $f\mapsto|f(u)|$ as u runs over U (compare 2A5Ig). By 2A5B, both are linear space topologies. If U and V are linear topological spaces, $T:U\to V$ is a continuous linear operator, and $g\in V^*$, then $gT\in U^*$ ((c) above); consequently T is $(\mathfrak{T}_s(U,U^*),\mathfrak{T}_s(V,V^*))$-continuous.

(e) If $U=\prod_{i\in I}U_i$ is a product of linear topological spaces, then every element of U^* is of the form $u\mapsto\sum_{i\in J}f_i(u(i))$ where $J\subseteq I$ is finite and $f_i\in U_i^*$ for every $i\in J$. (Bourbaki 87, II.6.6; Schaefer 71, IV.4.3; Köthe 69, §22.5.) Consequently the weak topology on U is the product of the weak topologies on the U_i.

(f) Let U be a linear topological space. For $A\subseteq U$ write A° for its **polar** set $\{f:f\in U^*,\ f(x)\le 1$ for every $x\in A\}$ in U^*. If G is a neighbourhood of 0 in U, then G° is a $\mathfrak{T}_s(U^*,U)$-compact subset of U^* (compare 3A5F). (Schaefer 71, III.4.3; Köthe 69, §20.9; Rudin 91, 3.15.)

(g) Let U be a linear topological space. If $D\subseteq U$ is non-empty and closed under addition and multiplication by rationals, $\overline{D}$ is a linear subspace of U. **P** The linear span

$$V=\{\textstyle\sum_{i=0}^n\alpha_i u_i:u_0,\dots,u_n\in D,\ \alpha_0,\dots,\alpha_n\in\mathbb{R}\}$$

of

$$D=\{\textstyle\sum_{i=0}^n\alpha_i u_i:u_0,\dots,u_n\in D,\ \alpha_0,\dots,\alpha_n\in\mathbb{Q}\}$$

is included in $\overline{D}$, because addition and scalar multiplication are continuous; so $\overline{D}=\overline{V}$ is a linear subspace. **Q** If $A\subseteq U$ is separable, then the closed linear subspace generated by A is separable. **P** Let $D_0\subseteq A$ be a countable dense subset; then

$$D=\{\sum_{i=0}^{n}\alpha_i u_i : u_0,\dots,u_n\in D_0,\ \alpha_0,\dots,\alpha_n\in\mathbb{Q}\}$$

is countable, and $\overline{D}$ is separable; but $\overline{D}$ is the closed linear subspace generated by A. **Q**

(h) If $\langle u_i\rangle_{i\in I}$ is an indexed family in a Hausdorff linear topological space U and $u\in U$, we say that $u=\sum_{i\in I}u_i$ if for every neighbourhood G of u there is a finite set $J\subseteq I$ such that $\sum_{i\in K}u_i\in G$ whenever $K\subseteq I$ is finite and $J\subseteq K$ (compare 226Ad).

If $\langle v_i\rangle_{i\in I}$ is another family with the same index set, and $v=\sum_{i\in I}v_i$ is defined, then $\sum_{i\in I}(u_i+v_i)$ is defined and equal to $u+v$. **P** If G is a neighbourhood of $u+v$, there are neighbourhoods H, H' of u, v respectively such that $H+H'\subseteq G$; there are finite sets J, $J'\subseteq I$ such that $\sum_{i\in K}u_i\in H$ whenever $J\subseteq K\in[I]^{<\omega}$ and $\sum_{i\in K}v_i\in H'$ whenever $J'\subseteq K\in[I]^{\omega}$; now $\sum_{i\in K}u_i+v_i\in G$ whenever $J\cup J'\subseteq K\in[I]^{<\omega}$. **Q**

If now V is another Hausdorff linear topological space and $T:U\to V$ is a continuous linear operator, $\sum_{i\in I}Tu_i=T(\sum_{i\in I}u_i)$ if the right-hand-side is defined. **P** Set $u=\sum_{i\in I}u_i$. If H is an open set containing Tu, then $T^{-1}[H]$ is an open set containing u, so there is a $J\in[I]^{<\omega}$ (notation: 3A1J) such that $\sum_{i\in K}u_i\in T^{-1}[H]$ and $\sum_{i\in K}Tu_i\in H$ whenever $J\subseteq K\in[I]^{<\omega}$. **Q**

(i) If U is a Hausdorff linear topological space, then any finite-dimensional linear subspace of U is closed. (Schaefer 71I.3.3; Taylor 64, 3.12-C; Rudin 91, 1.2.1.)

(j) If U is a first-countable Hausdorff linear topological space which (regarded as a linear topological space) is complete, then there is a metric ρ on U, defining its topology, under which U is complete. **P** Let $\mathcal{W}$ be the uniformity of U (3A4Ad). We know there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ running over a base of neighbourhoods of 0 in U; setting $W_n=\{(u,v):u-v\in G_n\}$ for each n, $\{W_n:n\in\mathbb{N}\}$ generates $\mathcal{W}$. So there is a metric ρ on U defining $\mathcal{W}$ (4A2Jb). Because X is $\mathcal{W}$-complete, it is ρ-complete (Engelking 89, 8.3.5). **Q**

4A4C Locally convex spaces (a) A linear topological space is **locally convex** if the convex open sets form a base for the topology.

(b) A linear topological space is locally convex iff its topology can be defined by a family of seminorms (2A5B, 2A5D). (Bourbaki 87, II.4.1; Schaefer 71, §II.4; Köthe 69, §18.1.)

(c) Let U be a linear space and τ a seminorm on U. Then $N_\tau=\{u:\tau(u)=0\}$ is a linear subspace of X. On the quotient space U/N_τ we have a norm defined by setting $\|u^\bullet\|=\tau(u)$ for every $u\in U$. (Bourbaki 87, II.1.3; Schaefer 71, II.5.4; Rudin 91, 1.43.)

(d) Let U be a locally convex linear topological space, and T the family of continuous seminorms on U. For each $\tau\in\mathrm{T}$, write $N_\tau=\{u:\tau(u)=0\}$, as in (c) above, and π_τ for the canonical map from U to $U_\tau=U/N_\tau$. Give each U_τ its norm, and set $\mathcal{G}_\tau=\{\pi_\tau^{-1}[H]:H\subseteq U_\tau \text{ is open}\}$. Then $\bigcup_{\tau\in\mathrm{T}}\mathcal{G}_\tau$ is a base for the topology of X closed under finite unions. (Schaefer 71, II.5.4.)

(e) A linear subspace of a locally convex linear topological space is locally convex. (Bourbaki 87, II.4.3; Köthe 69, §18.3.) The product of any family of locally convex linear topological spaces (4A4Bb) is locally convex. (Bourbaki 87, II.4.3; Köthe 69, §18.3.)

(f) If U is a metrizable locally convex linear topological space, its topology can be defined by a sequence of seminorms. (Bourbaki 87, II.4.1; Köthe 69, §18.2.)

(g) Let U be a linear space and V a linear subspace of the space U' of all linear functionals on U. Let $\mathfrak{T}_s(V,U)$ be the topology on V generated by the seminorms $f\mapsto|f(u)|$ as u runs over U (compare 4A4Bd), and let $\phi:V\to\mathbb{R}$ be a $\mathfrak{T}_s(V,U)$-continuous linear functional. Then there is a $u\in U$ such that $\phi(f)=f(u)$ for every $f\in V$. (Bourbaki 87, IV.1.1; Schaefer 71, IV.1.2; Köthe 69, §20.2; Rudin 91, 3.10; Dunford & Schwartz 57, II.3.9; Taylor 64, 3.81-A.)

(h) Grothendieck's theorem If U is a complete locally convex Hausdorff linear topological space, and ϕ is a linear functional on the dual U^* such that $\phi\restriction G^\circ$ is $\mathfrak{T}_s(U^*,U)$-continuous for every neighbourhood G of 0 in U, then ϕ is of the form $f\mapsto f(u)$ for some $u\in U$. (Bourbaki 87, III.3.6; Schaefer 71, IV.6.2; Köthe 69, §21.9.)

4A4D Hahn-Banach theorem (a) Let U be a linear space and $\theta : U \to [0,\infty[$ a seminorm.

(i) If $V \subseteq U$ is a linear subspace and $g : V \to \mathbb{R}$ is a linear functional such that $|g(v)| \le \theta(v)$ for every $v \in V$, then there is a linear functional $f : U \to \mathbb{R}$, extending g, such that $|f(u)| \le \theta(u)$ for every $u \in U$.

(ii) If $u_0 \in U$ then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u_0) = \theta(u_0)$ and $|f(u)| \le \theta(u)$ for every $u \in U$. (Bourbaki 87, II.3.2; Rudin 91, 3.3; Dunford & Schwartz 57, II.3.11; Taylor 64, 3.7-C; or use 3A5Aa.)

(b) Let U be a linear topological space and G, H two disjoint convex sets in U, of which one has non-empty interior. Then there are a non-zero $f \in U^*$ and an $\alpha \in \mathbb{R}$ such that $f(u) \le \alpha \le f(v)$ for every $u \in G$, $v \in H$, so that $f(u) < \alpha$ for every $u \in \operatorname{int} G$ and $\alpha < f(v)$ for every $u \in \operatorname{int} H$. (Bourbaki 87, II.5.2; Schaefer 71, II.9.1; Köthe 69, §17.1.)

4A4E The Hahn-Banach theorem in locally convex spaces Let U be a locally convex linear topological space.

(a) If $V \subseteq U$ is a linear subspace, then every member of V^* extends to a member of U^* (compare 3A5Ab). (Bourbaki 87, II.4.1; Schaefer 71, II.4.2; Köthe 69, §20.1; Rudin 91, 3.6; Taylor 64, 3.8-D.)

Consequently $\mathfrak{T}_s(V,V^*)$ is just the subspace topology on V induced by $\mathfrak{T}_s(U,U^*)$.

(b) Let $C \subseteq U$ be a non-empty closed convex set. If $u \in U$ then $u \in C$ iff $f(u) \le \sup_{v\in C} f(v)$ for every $f \in U^*$ iff $f(u) \ge \inf_{v\in C} f(v)$ for every $f \in U^*$. (Bourbaki 87, II.5.3; Schaefer 71, II.9.2; Köthe 69, §20.7; Dunford & Schwartz 57, V.2.12.)

If $V \subseteq U$ is a closed linear subspace and $u \in U \setminus V$ there is an $f \in U^*$ such that $f(u) \ne 0$ and $f(v) = 0$ for every $v \in V$. (Bourbaki 87, II.5.3; Köthe 69, §20.1; Rudin 91, 3.5; Taylor 64, 3.8-E.)

(c) If U is Hausdorff, U^* separates its points (compare 3A5Ae). (Bourbaki 87, II.4.1; Rudin 91, 3.4.)

(d) If $u \in U$ belongs to the $\mathfrak{T}_s(U,U^*)$-closure of a convex set $C \subseteq U$, it belongs to the closure of C (compare 3A5Ee). (Schaefer 71, II.9.2; Köthe 69, §20.7; Rudin 91, 3.12.) In particular, if C is closed, it is $\mathfrak{T}_s(U,U^*)$-closed. (Dunford & Schwartz 57, V.2.13.)

(e) If C, $C' \subseteq U$ are disjoint non-empty closed convex sets, of which one is compact, there is an $f \in U^*$ such that $\sup_{u\in C} f(u) < \inf_{u\in C'} f(u)$. (Apply (b) to $C - C'$. See Bourbaki 87, II.5.3; Schaefer 71, II.9.2; Köthe 69, §20.7; Rudin 91, 3.4; Dunford & Schwartz 57, V.3.13.)

(f) Let V be a linear subspace of U'. Let $K \subseteq U$ be a non-empty $\mathfrak{T}_s(U,V)$-compact convex set, and $\phi_0 : V \to \mathbb{R}$ a linear functional such that $\phi_0(f) \le \sup_{u\in K} f(u)$ for every $f \in V$. Then there is a $u_0 \in K$ such that $\phi_0(f) = f(u_0)$ for every $f \in V$. **P** Give U the topology $\mathfrak{T}_s(U,V)$ and V' the topology $\mathfrak{T}_s(V',V)$ (4A4Cg). For $u \in U$, $f \in V$ set $\hat{u}(f) = f(u)$; then $u \mapsto \hat{u}$ is a continuous linear operator from U to V' (use 2A3H), so $\hat{K} = \{\hat{u} : u \in K\}$ is a compact convex subset of V'.

? Suppose, if possible, that $\phi_0 \notin \hat{K}$. By (b), there is a continuous linear functional $\boldsymbol{\theta} : V' \to \mathbb{R}$ such that $\boldsymbol{\theta}(\phi_0) > \sup_{u\in K} \boldsymbol{\theta}(\hat{u})$. But there is an $f \in V$ such that $\boldsymbol{\theta}(\phi) = \phi(f)$ for every $\phi \in V'$ (4A4Cg), so that $\phi_0(f) > \sup_{u\in K} f(u)$, contrary to hypothesis. **X** So there is a $u_0 \in K$ such that $\phi_0 = \hat{u}_0$, as claimed. **Q**

(g) The Bipolar Theorem Let $A \subseteq U'$ be a non-empty set. Set $A^\circ = \{u : u \in U,\ f(u) \le 1 \text{ for every } f \in A\}$ (compare 4A4Bf). If $g \in U'$ is such that $g(u) \le 1$ for every $u \in A^\circ$, then g belongs to the $\mathfrak{T}_s(U',U)$-closed convex hull of $A \cup \{0\}$. **P** Put 4A4Cg and (b) above together, as in (f). See Bourbaki 87, II.6.3; Schaefer 71, IV.1.5; Köthe 69, 20.8. **Q**

(h) Let W be a linear subspace of U' separating the points of U. Then W is $\mathfrak{T}_s(U',U)$-dense in U'. (For $W^0 = \{0\}$.)

4A4F The Mackey topology Let U be a linear space and V a linear subspace of U'. The **Mackey topology** $\mathfrak{T}_k(V,U)$ on V is the topology of uniform convergence on convex $\mathfrak{T}_s(U,V)$-compact subsets of U. Every $\mathfrak{T}_k(V,U)$-continuous linear functional on V is of the form $f \mapsto f(u)$ for some $u \in U$ (use 4A4Ef). So every $\mathfrak{T}_k(V,U)$-closed convex set is $\mathfrak{T}_s(V,U)$-closed, by 4A4Ed. (See Bourbaki 87, IV.1.1; Schaefer 71, IV.3.2; Köthe 69, 21.4.)

4A4G Extreme points (a) Let X be a real linear space, and $C \subseteq X$ a convex set. An element of C is an **extreme** point of C if it is not expressible as a convex combination of two other members of C; equivalently, if it is not expressible as $\frac{1}{2}(x+y)$ where x, $y \in C$ are distinct.

(b) The Kreĭn-Mil'man theorem Let U be a Hausdorff locally convex linear topological space and $K \subseteq U$ a compact convex set. Then K is the closed convex hull of the set of its extreme points. (Bourbaki 87, II.7.1; Schaefer 71, II.10.4; Köthe 69, §25.1; Rudin 91, 3.22.)

(c) Let U and V be Hausdorff locally convex linear topological spaces, $T : U \to V$ a continuous linear operator, $K \subseteq X$ a compact convex set and v any extreme point of $T[K] \subseteq V$. Then there is an extreme point u of K such that $Tu = v$. **P** Set $K_1 = \{u' : u' \in K, Tu' = v\}$. Then K_1 is a compact convex set so has an extreme point u. **?** If u is not an extreme point of K, it is expressible as $\alpha u_1 + (1-\alpha)u_2$ where u_1, u_2 are distinct points of K and $\alpha \in]0,1[$. So $v = \alpha Tu_1 + (1-\alpha)Tu_2$, and we must have $Tu_1 = Tv = Tu_2$, because v is an extreme point of $T[K]$; but this means that u_1, $u_2 \in K_1$ and u is not an extreme point of K_1. **X** So u has the required properties. **Q**

4A4H Proposition Let I be a set, W a closed linear subspace of $\mathbb{R}^I$, U a linear topological space and V a Hausdorff linear topological space. Let $K \subseteq U$ be a compact set and $T : U \times \mathbb{R}^I \to V$ a continuous linear operator. Then $T[K \times W]$ is closed.

proof Take $v_0 \in \overline{T[K \times W]}$. Let $J \subseteq I$ be a maximal set such that

whenever $L \subseteq J$ is finite and $H \subseteq V$ is an open set containing v_0, there are a $u \in K$ and an $x \in W$ such that $x(i) = 0$ for every $i \in L$ and $T(u,x) \in H$.

Let $\mathcal{F}$ be the filter on $U \times \mathbb{R}^I$ generated by the closed set $K \times W$, the sets $\{(u,x) : x(i) = 0\}$ for $i \in J$, and the sets $T^{-1}[H]$ for open sets H containing v_0. Then for any $j \in I$ there is an $F \in \mathcal{F}$ such that $\{x(j) : (u,x) \in F\}$ is bounded. **P?** Suppose, if possible, otherwise. Then, in particular, $j \notin J$. So there must be a finite set $L \subseteq J$ and an open set H containing v_0 such that $x(j) \neq 0$ whenever $u \in K$, $x \in W$, $x(i) = 0$ for every $i \in L$ and $T(u,x) \in H$. By 2A5C, or otherwise, there is a neighbourhood G_0 of 0 in V such that $v_0 + G_0 - G_0 \subseteq H$ and $\alpha v \in G_0$ whenever $v \in G_0$ and $|\alpha| \le 1$. Fix $u^* \in K$, $x^* \in W$ such that $x^*(i) = 0$ for every $i \in L$ and $T(u^*,x^*) \in v_0 + G_0$. If $x \in W$ and $x(i) = 0$ for every $i \in L$ and $x(j) = x^*(j)$, then $T(u^*, x^* - x) \notin v_0 + G_0 - G_0$ so $T(0,x) \notin G_0$. It follows that $T(0,x) \notin G_0$ whenever $x \in W$ and $x(i) = 0$ for every $i \in L$ and $|x(j)| \ge |x^*(j)|$.

Let G be a neighbourhood of 0 in V such that $G + G - G - G \subseteq G_0$. We are supposing that $\{x(j) : (u,x) \in F\}$ is unbounded for every $F \in \mathcal{F}$. So for every $n \in \mathbb{N}$ there are $u_n \in K$ and $x_n \in W$ such that $x_n(i) = 0$ for $i \in L$, $T(u_n,x_n) \in v_0 + G$ and $|x_n(j)| \ge n$. Let $u \in K$ be a cluster point of $\langle u_n \rangle_{n\in\mathbb{N}}$. Then $T(u,0)$ is a cluster point of $\langle T(u_n,0) \rangle_{n\in\mathbb{N}}$, so $M = \{n : T(u_n,0) \in T(u,0) + G\}$ is infinite. For $n \in M$, $T(0,x_n) = T(u_n,x_n) - T(u_n,0) \in v_0 - T(u,0) + G - G$; so if m, $n \in M$, $T(0, x_m - x_n) \in G - G - (G - G) \subseteq G_0$. But note now that $(x_m - x_n)(i) = 0$ for all m, $n \in \mathbb{N}$ and $i \in L$, and that because M is infinite there are certainly m, $n \in M$ such that $|x_m(j) - x_n(j)| \ge |x^*(j)|$; which contradicts the last paragraph. **XQ**

Now let $\mathcal{G}$ be any ultrafilter on $U \times \mathbb{R}^I$ including $\mathcal{F}$. Then for every $i \in I$ there is a $\gamma_i < \infty$ such that $\mathcal{G}$ contains $\{(u,x) : |x(i)| \le \gamma_i\}$. It follows that $\mathcal{G}$ has a limit $(\hat{u},\hat{x})$ in $K \times W$. Now the image filter $T[[\mathcal{G}]]$ (2A1Ib) converges to $T(\hat{u},\hat{x})$; since $T[[\mathcal{F}]] \to v_0$, and the topology of V is Hausdorff, $v_0 = T(\hat{u},\hat{x}) \in T[K \times W]$. As v_0 is arbitrary, $T[K \times W]$ is closed.

4A4I Normed spaces (a) Two norms $\| \;\|$, $\| \;\|'$ on a linear space U give rise to the same topology iff they are **equivalent** in the sense that, for some $M \ge 0$,

$$\|x\| \le M\|x\|', \quad \|x\|' \le M\|x\|$$

for every $x \in U$. (Köthe 69, §14.2; Taylor 64, 3.1-D; Jameson 74, 2.8.)

(b) If U and V are normed spaces, $T : U \to V$ is a linear operator and $gT : U \to \mathbb{R}$ is continuous for every $g \in V^*$, then T is a bounded operator. (Jameson 74, 27.6.)

(c) If U is any normed space, its dual U^*, under its usual norm (2A4H), is a Banach space. (Rudin 91, 4.1; Dunford & Schwartz 57, II.3.9; Köthe 69, §14.5.)

(d) If U is a separable normed space, its dual U^* (regarded as a normed space) is isometrically isomorphic to a closed linear subspace of ℓ^∞. **P** Let $\langle x_n \rangle_{n\in\mathbb{N}}$ run over a dense subset of the unit ball of U (4A2P(a-iv)); define $T : U^* \to \ell^\infty$ by setting $(Tf)(n) = f(x_n)$ for every n. T is a linear isometry between U^* and $T[U^*]$, which is closed because U^* is complete (4A4Ic, 3A4Fd). **Q**

(e) Let U be a Banach space. Suppose that $\langle u_i\rangle_{i\in I}$ is a family in U such that $\gamma=\sum_{i\in I}\|u_i\|<\infty$.

(i) $\sum_{i\in I}u_i$ is defined in the sense of 4A4Bh. **P** For $J\in[I]^{<\omega}$, set $v_J=\sum_{i\in J}u_i$. For each $n\in\mathbb{N}$, there is a $J_n\in[I]^{<\omega}$ such that $\gamma-\sum_{i\in K}\|u_i\|\le 2^{-n}$ whenever $J_n\subseteq K\in[I]^{<\omega}$. Now

$$\|v_{J_m}-v_{J_n}\|\le\sum_{i\in J_m\triangle J_n}\|u_i\|\le 2^{-m}+2^{-n}$$

for all m, $n\in\mathbb{N}$, so $\langle v_{J_n}\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence and has a limit v say. If now $n\in\mathbb{N}$ and $J_n\subseteq K\in[I]^{<\omega}$,

$$\begin{aligned}\|v-\sum_{i\in K}u_i\|&=\lim_{m\to\infty}\|v_{J_m}-\sum_{i\in K}u_i\|\le\limsup_{m\to\infty}\sum_{i\in J_m\triangle K}\|u_i\|\\&\le\lim_{m\to\infty}2^{-m}+2^{-n}=2^{-n},\end{aligned}$$

so $v=\sum_{i\in I}u_i$. **Q**

(ii) Now if $\langle I_j\rangle_{j\in J}$ is any partition of I, $w_j=\sum_{i\in I_j}u_i$ is defined for every j, and $\sum_{j\in J}w_j$ is defined and equal to $\sum_{i\in I}u_i$.

(f) Let U be a normed space. For $u\in U$, define $\hat{u}\in U^{**}=(U^*)^*$ by setting $\hat{u}(f)=f(u)$ for every $f\in U^*$. Then $\{\hat{u}:u\in U,\ \|u\|\le 1\}$ is weak*-dense in $\{\phi:\phi\in U^{**},\ \|\phi\|=1\}$. (Apply 4A4Eg with $A=\{\hat{u}:\|u\|\le 1\}$.)

4A4J Inner product spaces (a) Let U be an inner product space over $\begin{smallmatrix}\mathbb{R}\\\mathbb{C}\end{smallmatrix}$ (3A5M). An **orthonormal family** in U is a family $\langle e_i\rangle_{i\in I}$ in U such that $(e_i|e_j)=0$ if $i\ne j$, 1 if $i=j$. An **orthonormal basis** in U is an orthonormal family $\langle e_i\rangle_{i\in I}$ in U such that the closed linear subspace of U generated by $\{e_i:i\in I\}$ is U itself.

(b) If U, V are inner product spaces over $\mathbb{R}$ and $T:U\to V$ is an isometry such that $T(0)=0$, then $(Tu|Tv)=(u|v)$ for all u, $v\in U$ and T is linear. **P** (α)

$$(Tu|Tv)=\frac{1}{2}(\|Tu\|^2+\|Tv\|^2-\|Tu-Tv\|^2)=\frac{1}{2}(\|u\|^2+\|v\|^2-\|u-v\|^2)=(u|v).$$

(β) For any u, $v\in U$,

$$\begin{aligned}\|T(u+v)-Tu-Tv\|^2&=\|T(u+v)\|^2+\|Tu\|^2+\|Tv\|^2\\&\qquad-2(T(u+v)|Tu)-2(T(u+v)|Tv)+2(Tu|Tv)\\&=\|u+v\|^2+\|u\|^2+\|v\|^2\\&\qquad-2(u+v|u)-2(u+v|v)+2(u|v)\\&=0.\end{aligned}$$

So T is additive. (γ) Consequently $T(qu)=qTu$ for every $u\in U$ and $q\in\mathbb{Q}$; as T is continuous, it is linear. **Q**

(c) If U, V are inner product spaces over $\mathbb{C}$ and $T:U\to V$ is a linear operator such that $\|Tu\|=\|u\|$ for every $u\in U$, then $(Tu|Tv)=(u|v)$ for all u, $v\in U$. **P**

$$\mathcal{R}\mathrm{e}(Tu|Tv)=\frac{1}{2}(\|Tu\|^2+\|Tv\|^2-\|Tu-Tv\|^2)=\mathcal{R}\mathrm{e}(u|v),$$

$$\mathcal{I}\mathrm{m}(Tu|Tv)=-\mathcal{R}\mathrm{e}(i(Tu|Tv))=-\mathcal{R}\mathrm{e}(T(iu)|Tv)=-\mathcal{R}\mathrm{e}(iu|v)=\mathcal{I}\mathrm{m}(u|v).\ \mathbf{Q}$$

(d) If U is an inner product space over $\begin{smallmatrix}\mathbb{R}\\\mathbb{C}\end{smallmatrix}$, a linear operator $T:U\to U$ is **self-adjoint** if $(Tu|v)=(u|Tv)$ for all u, $v\in U$.

(e) If U is a finite-dimensional inner product space over $\mathbb{R}$, it is isomorphic to Euclidean space $\mathbb{R}^r$, where $r=\dim U$. (Taylor 64, 3.21-A.) In particular, any finite-dimensional inner product space is a Hilbert space.

(f) If U is an inner product space over $\begin{smallmatrix}\mathbb{R}\\\mathbb{C}\end{smallmatrix}$ and $V\subseteq U$ is a linear subspace of U, then $V^\perp=\{x:x\in U,\ (x|y)=0$ for every $y\in V\}$ is a linear subspace of U, and $\|x+y\|^2=\|x\|^2+\|y\|^2$ for $x\in V$, $y\in V^\perp$. If V is complete (in particular, if V is finite-dimensional), then $U=V\oplus V^\perp$. (Rudin 91, 12.4; Bourbaki 87, V.1.6; Taylor 64, 4.82-A.)

(g) If U is an inner product space over $\mathbb{R}$ and v_1, $v_2 \in U$ are such that $\|v_1\| = \|v_2\| = 1$, there is a linear operator $T : U \to U$ such that $Tv_1 = v_2$ and $\|Tu\| = \|u\|$ and $\|Tu - u\| \le \|v_1 - v_2\|\|u\|$ for every $u \in U$. **P** If v_2 is a multiple of v_1, say $v_2 = \alpha v_1$, take $Tu = \alpha u$ for every u. Otherwise, set $w = v_2 - (v_2|v_1)v_1$ and $w_1 = \frac{1}{\|w\|}w$, so that $v_2 = \cos\theta v_1 + \sin\theta w_1$, where $\theta = \arccos(v_2|v_1)$. Let V be the two-dimensional linear subspace of U generated by v_1 and w_1, so that $U = V \oplus V^\perp$. Define a linear operator $T : U \to U$ by saying that $Tv_1 = v_2$, $Tw_1 = \cos\theta w_1 - \sin\theta v_1$ and $Tu = u$ for $u \in V^\perp$. Then T acts on V as a simple rotation through an angle θ, so $\|Tv\| = 1$ and $\|Tv - v\| = \|v_2 - v_1\|$ whenever $v \in V$ and $\|v\| = 1$; generally, if $u \in U$, then $\|Tu\| = \|u\|$ and

$$\|Tu - u\| = \|T(Pu) - Pu\| = \|v_2 - v_1\|\|Pu\| \le \|v_2 - v_1\|\|u\|,$$

where P is the orthogonal projection of U onto V. **Q**

(h) Let U be an inner product space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $\langle u_i\rangle_{i\in I}$ a countable family in U. Then there is a countable orthonormal family $\langle v_j\rangle_{j\in J}$ in U such that $\{v_j : j \in J\}$ and $\{u_i : i \in I\}$ span the same linear subspace of U. **P** We can suppose that $I \subseteq \mathbb{N}$; set $u_i = 0$ for $i \in \mathbb{N} \setminus I$. Define $\langle v_n\rangle_{n\in\mathbb{N}}$ inductively by setting $v'_n = u_n - \sum_{i<n}(u_n|v_i)v_i$, $v_n = 0$ if $v'_n = 0$, $\frac{1}{\|v'_n\|}v'_n$ otherwise. Set $J = \{n : v_n \ne 0\}$. **Q**

(i) Let U be an inner product space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $\langle e_i\rangle_{i\in I}$ an orthonormal family in U. Then $\sum_{i\in I}|(u|e_i)|^2 \le \|u\|^2$ for every $u \in U$. (Dunford & Schwartz 57, p. 252; Taylor 64, 3.2-D.)

(j) Let U be an inner product space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$, and $C \subseteq U$ a convex set. Then there is at most one point $u \in C$ such that $\|u\| \le \|v\|$ for every $v \in C$. **P** If u, u' both have this property, then $v = \frac{1}{2}(u + u') \in C$, and $\|u\| = \|u'\| \le \|v\|$; but $4\|v\|^2 + \|u - u'\|^2 = 2(\|u\|^2 + \|u'\|^2)$, so $\|u - u'\| = 0$ and $u = u'$. **Q**

For such a u, $\|u\|^2 \le \mathcal{R}\mathrm{e}(u|v)$ for every $v \in C$. **P** For $\alpha \in \,]0, 1]$,

$$\|u\|^2 \le \|\alpha v + (1-\alpha)u\|^2 = \|u\|^2 + 2\alpha(\mathcal{R}\mathrm{e}(u|v) - \|u\|^2) + \alpha^2\|v - u\|^2,$$

so $\mathcal{R}\mathrm{e}(u|v) - \|u\|^2 \ge -\lim_{\alpha\downarrow 0}\frac{1}{2}\alpha\|v - u\|^2$. **Q**

4A4K Hilbert spaces (a) If U is a real or complex Hilbert space, its unit ball is compact in the weak topology $\mathfrak{T}_s(U, U^*)$; any bounded set is relatively compact for $\mathfrak{T}_s(U, U^*)$. (Bourbaki 87, V.1.7; Dunford & Schwartz 57, IV.4.6.)

(b) If U is a real or complex Hilbert space, any norm-bounded sequence in U has a weakly convergent subsequence. (462D; Dunford & Schwartz 57, IV.4.7.)

(c) If U is a real or complex Hilbert space and $\langle u_i\rangle_{i\in I}$ is any orthonormal family in U, then it can be extended to an orthonormal basis. (Dunford & Schwartz 57, IV.4.10; Taylor 64, 3.2-I.) In particular, U has an orthonormal basis.

4A4L Compact operators (see 3A5La) **(a)** Let U, V and W be Banach spaces. If $T \in \mathsf{B}(U;V)$ and $S \in \mathsf{B}(V;W)$ and either S or T is a compact operator, then ST is compact. (Dunford & Schwartz 57, VI.5.4; Jameson 74, 34.2.)

(b) If U is a Banach space, $T \in \mathsf{B}(U;U)$ is a compact linear operator and $\gamma \ne 0$ then $\{u : Tu = \gamma u\}$ is finite-dimensional. (Rudin 91, 4.18; Taylor 64, 5.5-C; Dunford & Schwartz 57, VII.4.5; Jameson 74, 34.8.)

4A4M Self-adjoint compact operators If U is a Hilbert space and $T : U \to U$ is a self-adjoint compact linear operator, then $T[U]$ is included in the closed linear span of $\{Tv : v$ is an eigenvector of $T\}$. (Taylor 64, 6.4-B.)

4A4N Max-flow Min-cut Theorem (Ford & Fulkerson 56) Let (V, E, γ) be a (finite) transportation network, that is,

V is a finite set of 'vertices',
$E \subseteq \{(v, v') : v, v' \in V, v \ne v'\}$ is a set of (directed) 'edges',
$\gamma : E \to [0, \infty[$ is a function;

we regard a member $e=(v,v')$ of E as 'starting' at v and 'ending' at v', and $\gamma(e)$ is the 'capacity' of the edge e. Suppose that v_0, $v_1\in V$ are distinct vertices such that no edge ends at v_0 and no edge starts at v_1. Then we have a 'flow' $\phi:E\to[0,\infty[$ and a 'cut' $X\subseteq E$ such that

(i) for every $v\in V\setminus\{v_0,v_1\}$,

$$\sum_{e\in E,e\text{ starts at }v}\phi(e)=\sum_{e\in E,e\text{ ends at }v}\phi(e),$$

(ii) $\phi(e)\le\gamma(e)$ for every $e\in E$,

(iii) there is no path from v_0 to v_1 using only edges in $E\setminus X$,

(iv) $\sum_{e\in E,e\text{ starts at }v_0}\phi(e)=\sum_{e\in E,e\text{ ends at }v_1}\phi(e)=\sum_{e\in X}\gamma(e)$.

proof Bollobás 79, §III.1; Anderson 87, 12.3.1.

4A5 Topological groups

For Chapter 44 we need a variety of facts about topological groups. Most are essentially elementary, and all the non-trivial ideas are covered by at least one of Császár 78 and Hewitt & Ross 63. In 4A5A-4A5C and 4A5I I give some simple definitions concerning groups and group actions. Topological groups, properly speaking, appear in 4A5D. Their simplest properties are in 4A5E-4A5G. I introduce 'right' and 'bilateral' uniformities in 4A5H; the latter are the more interesting (4A5M-4A5O), but the former are also important (see the proof of 4A5P). 4A5J-4A5L deal with quotient spaces, including spaces of cosets of non-normal subgroups. I conclude with notes on metrizable groups (4A5Q-4A5S).

4A5A Notation If X is a group, $x_0\in X$, and A, $B\subseteq X$ I write

$$x_0A=\{x_0x:x\in A\},\quad Ax_0=\{xx_0:x\in A\},$$

$$AB=\{xy:x\in A,\,y\in B\},\quad A^{-1}=\{x^{-1}:x\in A\}.$$

A is **symmetric** if $A=A^{-1}$. Observe that $(AB)C=A(BC)$, $(AB)^{-1}=B^{-1}A^{-1}$ for any A, B, $C\subseteq X$.

4A5B Group actions (a) If X is a group and Z is a set, an **action** of X on Z is a function $(x,z)\mapsto x{\bullet}z:X\times Z\to Z$ such that

$$(xy){\bullet}z=x{\bullet}(y{\bullet}z)\text{ for all }x,\,y\in X\text{ and }z\in Z,$$

$$e{\bullet}z=z\text{ for every }z\in Z$$

where e is the identity of X.

In this context I may say that 'X acts on Z', taking the operation $\bullet$ for granted.

(b) An action $\bullet$ of a group X on a set Z is **transitive** if for every w, $z\in Z$ there is an $x\in X$ such that $x{\bullet}w=z$.

(c) If $\bullet$ is an action of a group X on a set Z, I write $x{\bullet}A=\{x{\bullet}z:z\in A\}$ whenever $x\in X$ and $A\subseteq Z$.

(d) If $\bullet$ is an action of a group X on a set Z, then $z\mapsto x{\bullet}z:Z\to Z$ is a permutation for every $x\in X$. (For it has an inverse $z\mapsto x^{-1}{\bullet}z$.) So if Z is a topological space and $z\mapsto x{\bullet}z$ is continuous for every x, it is a homeomorphism for every x.

(e) An action $\bullet$ of a group X on a set Z is **faithful** if whenever x, $y\in X$ are distinct there is a $z\in Z$ such that $x{\bullet}z\ne y{\bullet}z$; that is, the natural homomorphism from X to the group of permutations of Z is injective. An action of X on Z is faithful iff for any $x\in X$ which is not the identity there is a $z\in Z$ such that $x{\bullet}z\ne z$.

(f) If $\bullet$ is an action of a group X on a set Z, then $Y_z=\{x:x\in X,\,x{\bullet}z=z\}$ is a subgroup of X (the **stabilizer** of z) for every $z\in Z$. If $\bullet$ is transitive, then Y_w and Y_z are conjugate subgroups for all w, $z\in Z$. (If $x{\bullet}w=z$, then $Y_z=xY_wx^{-1}$.)

(g) If $\bullet$ is an action of a group X on a set Z, then sets of the form $\{a{\bullet}z:a\in X\}$ are called **orbits** of the action; they are the equivalence classes under the equivalence relation $\sim$, where $z\sim z'$ if there is an $a\in X$ such that $z'=a{\bullet}z$.

4A5C Examples Let X be any group.

(a) Write

$$x\bullet_l y = xy, \quad x\bullet_r y = yx^{-1}, \quad x\bullet_c y = xyx^{-1}$$

for x, $y \in X$. These are all actions of X on itself, the **left**, **right** and **conjugacy** actions.

(b) If $A \subseteq X$, we have an action of X on the set $\{yA : y \in X\}$ of left cosets of A defined by setting $x\bullet(yA) = xyA$ for x, $y \in X$.

(c)(i) Let $\bullet$ be an action of a group X on a set Z. If f is any function defined on a subset of Z, and $x \in X$, write $x\bullet f$ for the function defined by saying that $(x\bullet f)(z) = f(x^{-1}\bullet z)$ whenever $z \in Z$ and $x^{-1}\bullet z \in \operatorname{dom} f$. It is easy to check that this defines an action of X on the class of all functions with domains included in Z. Observe that

$$x\bullet(f+g) = (x\bullet f) + (x\bullet g), \quad x\bullet(f \times g) = (x\bullet f) \times (x\bullet g), \quad x\bullet(f/g) = (x\bullet f)/(x\bullet g)$$

whenever $x \in X$ and f, g are real-valued functions with domains included in Z.

(ii) In (i), if $X = Z$, we have corresponding actions $\bullet_l$, $\bullet_r$ and $\bullet_c$ of X on the class of functions with domains included in X:

$$(x\bullet_l f)(y) = f(x^{-1}y), \quad (x\bullet_r f)(y) = f(yx), \quad (x\bullet_c f)(y) = f(x^{-1}yx)$$

whenever these are defined. These are the left, right and conjugacy **shift actions**.

Note that

$$x\bullet_l \chi A = \chi(xA), \quad x\bullet_r \chi A = \chi(Ax^{-1}), \quad x\bullet_c \chi A = \chi(xAx^{-1})$$

whenever $A \subseteq X$ and $x \in X$. In this context, the following idea is sometimes useful. If f is a function with domain included in X, set $\overset{\leftrightarrow}{f}(y) = f(y^{-1})$ when $y \in X$ and $y^{-1} \in \operatorname{dom} f$. Then

$$(\overset{\leftrightarrow}{f})^{\leftrightarrow} = f, \quad x\bullet_l \overset{\leftrightarrow}{f} = (x\bullet_r f)^{\leftrightarrow}, \quad x\bullet_r \overset{\leftrightarrow}{f} = (x\bullet_l f)^{\leftrightarrow}, \quad x\bullet_c \overset{\leftrightarrow}{f} = (x\bullet_c f)^{\leftrightarrow}$$

for any such f and any $x \in X$.

(d) If $\bullet$ is an action of a group X on a set Z, $Y \subseteq X$ is a subgroup of X, and $W \subseteq Z$ is Y-invariant in the sense that $y\bullet w \in W$ whenever $y \in Y$ and $w \in W$, then $\bullet\restriction Y \times W$ is an action of Y on W. In the context of (c-i) above, this means that if V is any set of functions with domains included in W such that $y\bullet f \in V$ whenever $y \in Y$ and $f \in V$, then we have an action of Y on V.

4A5D Definitions (a) A **topological group** is a group X endowed with a topology such that the operations $(x,y) \mapsto xy : X \times X \to X$ and $x \mapsto x^{-1} : X \to X$ are continuous.

(b) A **Polish group** is a topological group in which the topology is Polish.

4A5E Elementary facts Let X be any topological group.

(a) For any $x \in X$, the functions $y \mapsto xy$, $y \mapsto yx$ and $y \mapsto y^{-1}$ are all homeomorphisms from X to itself. (Hewitt & Ross 63, 4.2; Folland 95, 2.1.)

(b) The maps $(x,y) \mapsto x^{-1}y$, $(x,y) \mapsto xy^{-1}$ and $(x,y) \mapsto xyx^{-1}$ from $X \times X$ to X are continuous.

(c) $\{G : G \text{ is open}, e \in G, G^{-1} = G\}$ is a base of neighbourhoods of the identity e of X. (Hewitt & Ross 63, 4.6; Folland 95, 2.1.)

(d) If $G \subseteq X$ is an open set, then AG and GA are open for any set $A \subseteq X$. (Hewitt & Ross 63, 4.4.)

(e) If $F \subseteq X$ is closed and $x \in X \setminus F$, there is a neighbourhood U of e such that $UxUU \cap FUU = \emptyset$. **P** Set $U_1 = X \setminus x^{-1}F$. Let U_2 be a neighbourhood of e such that $U_2U_2U_2U_2^{-1}U_2^{-1} \subseteq U_1$. Let U be a neighbourhood of e such that $U \subseteq U_2 \cap xU_2x^{-1}$; this works. **Q**

(f) If $K \subseteq X$ is compact and $F \subseteq X$ is closed then KF and FK are closed. If K, $L \subseteq X$ are compact so is KL. (Hewitt & Ross 63, 4.4.)

(g) If there is any compact set $K \subseteq X$ such that $\operatorname{int} K$ is non-empty, then X is locally compact.

(h) If $K \subseteq X$ is compact and $\mathcal{F}$ is a downwards-directed family of closed subsets of X with intersection F_0, then $KF_0 = \bigcap_{F\in\mathcal{F}} KF$ and $F_0K = \bigcap_{F\in\mathcal{F}} FK$. **P** Of course $KF_0 \subseteq \bigcap_{F\in\mathcal{F}} KF$. If $x \in X \setminus KF_0$, then $K^{-1}x \cap F_0$ is empty; because $K^{-1}x$ is compact, there is some $F \in \mathcal{F}$ such that $K^{-1}x \cap F = \emptyset$ (3A3Db), so that $x \notin KF$. Accordingly $KF_0 = \bigcap_{F\in\mathcal{F}} KF$. Similarly, $F_0K = \bigcap_{F\in\mathcal{F}} FK$. **Q**

(i) If $K \subseteq X$ is compact and $G \subseteq X$ is open, then $W = \{(x, y) : xKy \subseteq G\}$ is open in $X \times X$. **P** It is enough to deal with the case $K \neq \emptyset$. Take $(x_0, y_0) \in W$. For each $z \in K$, there is an open neighbourhood U_z of e such that $U_z zU_zU_z \subseteq x_0^{-1}Gy_0^{-1}$ (apply (e) with $F = X \setminus x_0^{-1}Gy_0^{-1}$). Now $\{zU_z : z \in K\}$ is an open cover of K so there are $z_0, \dots , z_n \in K$ such that $K \subseteq \bigcup_{i\le n} z_iU_{z_i}$. Set $U = \bigcap_{i\le n} U_{z_i}$; then $UKU \subseteq x_0^{-1}Gy_0^{-1}$ and $(x, y) \in W$ whenever $x \in x_0U$ and $y \in Uy_0$. Accordingly $(x_0, y_0) \in \operatorname{int} W$; as (x_0, y_0) is arbitrary, W is open. **Q**

It follows that $\{x : xK \subseteq G\}$, $\{x : Kx \subseteq G\}$ and $\{x : xKx^{-1} \subseteq G\}$ are open in X.

(j) If X is Hausdorff, $K \subseteq X$ is compact and U is a neighbourhood of e, there is a neighbourhood V of e such that $xy \in U$ whenever x, $y \in K$ and $yx \in V$; that is, $y^{-1}zy \in U$ whenever $z \in V$ and $y \in K$. **P** If U is open, then $\{yx : x, y \in K, xy \notin U\}$ is a closed set not containing e. Compare 4A5Oc below. **Q**

(k) Any open subgroup of X is also closed. (Császár 78, 11.2.12; Hewitt & Ross 63, 5.5; Folland 95, 2.1.)

(l) If X is locally compact, it has an open subgroup which is σ-compact. (Hewitt & Ross 63, 5.14; Folland 95, 2.3.)

(m) If Y is a subgroup of X, its closure $\overline{Y}$ is a subgroup of X. (Hewitt & Ross 63, 5.3; Folland 95, 2.1.)

4A5F Proposition (a) Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological groups. If $\phi : X \to Y$ is a group homomorphism which is continuous at the identity of X, it is continuous. (Császár 78, 11.2.17; Hewitt & Ross 63, 5.40.)

(b) Let X be a group and $\mathfrak{S}$, $\mathfrak{T}$ two topologies on X both making X a topological group. If every $\mathfrak{S}$-neighbourhood of the identity is a $\mathfrak{T}$-neighbourhood of the identity, then $\mathfrak{S} \subseteq \mathfrak{T}$. (Apply (a) to the identity map from $(X, \mathfrak{T})$ to $(X, \mathfrak{S})$.)

4A5G Proposition If $\langle X_i\rangle_{i\in I}$ is any family of topological groups, then $\prod_{i\in I} X_i$, with the product topology and the product group structure, is again a topological group. (Hewitt & Ross 63, 6.2.)

4A5H The uniformities of a topological group Let $(X, \mathfrak{T})$ be a topological group. Write $\mathcal{U}$ for the set of open neighbourhoods of the identity e of X.

(a) For $U \in \mathcal{U}$, set $W_U = \{(x, y) : xy^{-1} \in U\} \subseteq X \times X$. The family $\{W_U : U \in \mathcal{U}\}$ is a filter base, and the filter on $X \times X$ which it generates is a uniformity on X, the **right uniformity** of X. (**Warning!!** Some authors call this the 'left uniformity'.) This uniformity induces the topology $\mathfrak{T}$ (Császár 78, 11.2.7). It follows that $\mathfrak{T}$ is completely regular, therefore regular (4A2Ja, or Hewitt & Ross 63, 8.4).

(b) For $U \in \mathcal{U}$, set $\tilde{W}_U = \{(x, y) : xy^{-1} \in U, x^{-1}y \in U\} \subseteq X \times X$. The family $\{\tilde{W}_U : U \in \mathcal{U}\}$ is a filter base, and the filter on $X \times X$ which it generates is a uniformity on X, the **bilateral uniformity** of X. This uniformity induces the topology $\mathfrak{T}$. (Császár 78, 11.3.c.)

(c) $x \mapsto x^{-1}$ is uniformly continuous for the bilateral uniformity. (The check is elementary.)

(d) If X and Y are topological groups and $\phi : X \to Y$ is a continuous homomorphism, then ϕ is uniformly continuous for the bilateral uniformities. **P** If V is a neighbourhood of the identity in Y and $W_V = \{(y, z) : yz^{-1}$, $y^{-1}z$ both belong to $V\}$ is the corresponding member of the bilateral uniformity on Y, then $U = \phi^{-1}[V]$ is a neighbourhood of the identity in X and $(\phi(x), \phi(w)) \in W_V$ whenever $(x, w) \in W_U$. **Q**

(e) If X is an abelian topological group, then the right and bilateral uniformities on X coincide, and may be called 'the' topological group uniformity of X; cf. 3A4Ad.

4A5I Definitions If X is a topological group and Z a topological space, an action of X on Z is 'continuous' or 'Borel measurable' if it is continuous, or Borel measurable, when regarded as a function from $X \times Z$ to Z.

Of course the left, right and conjugacy actions of a topological group on itself are all continuous.

4A5J Quotients under group actions, and quotient groups: Theorem (a) Let X be a topological space, Y a topological group, and $\bullet$ a continuous action of Y on X. Let Z be the set of orbits of the action, and for $x \in X$ write $\pi(x) \in Z$ for the orbit containing x.

(i) We have a topology on Z defined by saying that $V \subseteq Z$ is open iff $\pi^{-1}[V]$ is open in X. The canonical map $\pi : X \to Z$ is continuous and open.

(ii)(α) If Y is compact and X is Hausdorff, then Z is Hausdorff.

(β) If X is locally compact then Z is locally compact.

(b) Let X be a topological group, Y a subgroup of X, and Z the set of left cosets of Y in X. Set $\pi(x) = xY$ for $x \in X$.

(i) We have a topology on Z defined by saying that $V \subseteq Z$ is open iff $\pi^{-1}[V]$ is open in X. The canonical map $\pi : X \to Z$ is continuous and open.

(ii)(α) Z is Hausdorff iff Y is closed.

(β) If X is locally compact, so is Z.

(γ) If X is locally compact and Polish and Y is closed, then Z is Polish.

(δ) If X is locally compact and σ-compact and Y is closed and Z is metrizable, then Z is Polish.

(iii) We have a continuous action of X on Z defined by saying that $x\bullet\pi(x') = \pi(xx')$ for any x, $x' \in X$.

(iv) If Y is a normal subgroup of X, then the group operation on Z renders it a topological group.

proof (a)(i) It is elementary to check that $\{V : \pi^{-1}[V] \text{ is open}\}$ is a topology such that $\pi : X \to Z$ is continuous. To see that π is open, take an open set $U \subseteq X$ and consider

$$\pi^{-1}[\pi[U]] = \bigcup_{x' \in U} \{x : \pi(x) = \pi(x')\} = \bigcup_{x' \in U, y \in Y} \{x : x = y\bullet x'\} = \bigcup_{y \in Y} y\bullet U.$$

But as $x \mapsto y\bullet x$ is a homeomorphism for every $y \in Y$ (4A5Ea), every $y\bullet U$ is open, and the union $\pi^{-1}[\pi[U]]$ is open. So $\pi[U]$ is open in Z; as U is arbitrary, π is an open map.

(ii)(α) Set $F = \{((x,x'),y) : x \in X,\ y \in Y,\ y\bullet x = x'\}$. Because the function $((x,x'),y) \mapsto (y\bullet x, x') : (X \times X) \times Y \to X \times X$ is continuous and $\{(x,x) : x \in X\}$ is closed in $X \times X$ (4A2F(a-iii)), F is closed. By 4A2Gm, the projection $\{(x,x') : \exists\, y \in Y,\ y\bullet x = x'\} = \{(x,x') : \pi(x) = \pi(x')\}$ is closed in $X \times X'$ and $\{(x,x') : \pi(x) \ne \pi(x')\}$ is open. Since $(x,x') \mapsto (\pi(x),\pi(x')) : X \times X \to Z \times Z$ is an open mapping (4A2B(f-iv)), $\{(z,z') : z \ne z'\}$ is open in $Z \times Z$, and Z is Hausdorff by 4A2F(a-iii) in the other direction.

(β) Use 4A2Gn.

(b)(i) Apply (a-i) to the right action $(y,x) \mapsto xy^{-1}$ of Y on X, or see Hewitt & Ross 63, 5.15-5.16.

(ii)(α) By Hewitt & Ross 63, 5.21, Z is Hausdorff iff Y is closed.

(β) Use (a-ii-β), or see Hewitt & Ross 63, 5.22 or Folland 95, 2.2.

(γ) X has a countable network (4A2P(a-ii)), so Z also has (4A2Nd); since we have just seen that Z is locally compact and Hausdorff, it must be Polish (4A2Qh).

(δ) Because X is σ-compact, so is its continuous image Z; we know from (α)-(β) that Z is locally compact and Hausdorff; we are supposing that it is metrizable; so it is Polish, by the other half of 4A2Qh.

(iii) I have noted in 4A5Cb that the formula given defines an action. If $V \subseteq Z$ is open and $x_0 \in X$, $z_0 \in Z$ are such that $x_0\bullet z_0 \in V$, take $x_0' \in X$ such that $\pi(x_0') = z_0$, and observe that $x_0x_0' \in \pi^{-1}[V]$, which is open. So there are open neighbourhoods V_0, V_0' of x_0, x_0' respectively such that $V_0V_0' \subseteq \pi^{-1}[V]$, and $x\bullet z \in V$ whenever $x \in V_0$ and $z \in \pi[V_0']$. Since $\pi[V_0']$ is an open neighbourhood of z_0, this is enough to show that $\bullet$ is continuous at (x_0,z_0).

(iv) Császár 78, 11.2.15; Hewitt & Ross 63, 5.26; Folland 95, 2.2.

4A5K Proposition Let X be a topological group with identity e.

(a) $Y = \overline{\{e\}}$ is a closed normal subgroup of X.

(b) Writing $\pi : X \to X/Y$ for the canonical map,

(i) a subset of X is open iff it is the inverse image of an open subset of X/Y,

(ii) a subset of X is closed iff it is the inverse image of a closed subset of X/Y,
(iii) $\pi[G]$ is a regular open set in X/Y for every regular open set $G\subseteq X$,
(iv) $\pi[F]$ is nowhere dense in X/Y for every nowhere dense set $F\subseteq X$,
(v) $\pi^{-1}[V]$ is nowhere dense in X for every nowhere dense $V\subseteq X/Y$.

proof (a) Császár 78, 11.2.13; Hewitt & Ross 63, 5.4; Folland 95, 2.3.

(b)(i)-(ii) Because π is continuous, the inverse image of an open or closed set is open or closed. In the other direction, if $G\subseteq X$ is open and $x\in G$, then $x\overline{\{e\}}=\overline{\{x\}}\subseteq G$, because X is regular (4A5Ha). So $G=GY=\pi^{-1}[\pi[G]]$. Since π is an open map (4A5J(a-i)), $\pi[G]$ is open and G is the inverse image of an open set. If $F\subseteq X$ is closed, $\pi[F]=(X/Y)\setminus\pi[X\setminus F]$ is closed and

$$F=X\setminus\pi^{-1}[\pi[X\setminus F]]=\pi^{-1}[(X/Y)\setminus\pi[X\setminus F]]$$

is the inverse image of a closed set.

(iii) If $A\subseteq X$, then $\pi[\overline{A}]$ is a closed set included in $\overline{\pi[A]}$ (because π is continuous), so $\overline{\pi[A]}=\pi[\overline{A}]$. If $G\subseteq X$ is a regular open set, then $\pi^{-1}[\operatorname{int}\pi[\overline{G}]]$ is an open subset of $\pi^{-1}[\pi[\overline{G}]]=\overline{G}$, so is included in $\operatorname{int}\overline{G}=G$. But this means that the open set $\pi[G]$ includes $\operatorname{int}\pi[\overline{G}]=\operatorname{int}\overline{\pi[G]}$, and $\pi[G]=\operatorname{int}\overline{\pi[G]}$ is a regular open set.

(iv) If $F\subseteq X$ is nowhere dense, then its closure is of the form $\pi^{-1}[V]$ for some closed set $V\subseteq X/Y$. Now if $H\subseteq X/Y$ is a non-empty open set, $\pi^{-1}[H]$ is a non-empty open subset of X, so is not included in F, and H cannot be included in V. Thus V is nowhere dense; but $V\supseteq\pi[F]$, so $\pi[F]$ is nowhere dense.

(v) If $V\subseteq X/Y$ is nowhere dense, and $G\subseteq X$ is open and not empty, then $G=\pi^{-1}[H]$ for some non-empty open $H\subseteq X/Y$. In this case, $H\setminus\overline{V}$ is non-empty, so $\pi^{-1}[H\setminus\overline{V}]$ is a non-empty open subset of G disjoint from $\pi^{-1}[V]$. As G is arbitrary, $\pi^{-1}[V]$ is nowhere dense.

4A5L Theorem Let X be a topological group and Y a normal subgroup of X. Let $\pi:X\to X/Y$ be the canonical homomorphism.
(a) If X' is another topological group and $\phi:X\to X'$ a continuous homomorphism with kernel including Y, then we have a continuous homomorphism $\psi:X/Y\to X'$ defined by the formula $\psi\pi=\phi$; ψ is injective iff Y is the kernel of ϕ.
(b) Suppose that K_1, K_2 are two subgroups of X/Y such that $K_2\lhd K_1$. Set $Y_1=\pi^{-1}[K_1]$ and $Y_2=\pi^{-1}[K_2]$. Then $Y_2\lhd Y_1$ and Y_1/Y_2 and K_1/K_2 are isomorphic as topological groups.

proof (a) This is elementary group theory, except for the claim that ψ is continuous. But if $H\subseteq X'$ is open, then $\psi^{-1}[H]=\pi[\phi^{-1}[H]]$ is open because ϕ is continuous and π is open (4A5J(a-i)); so ψ is continuous.

(b) See Hewitt & Ross 63, 5.35.

4A5M Proposition Let X be a topological group.
(a) Let Y be any subgroup of X. If X is given its bilateral uniformity, then the subspace uniformity on Y is the bilateral uniformity of Y. (Császár 78, 11.3.13.)
(b) If X is locally compact it is complete under its right uniformity. (Császár 78, 11.3.21.) If X is complete under its right uniformity it is complete under its bilateral uniformity. (Császár 78, 11.3.10.)
(c) Suppose that X is Hausdorff and that Y is a subgroup of X which is locally compact in its subspace topology. Then Y is closed in X. **P** Putting (a) and (b) together, we see that Y is complete in its subspace uniformity, therefore closed (3A4Fd). **Q**

4A5N Theorem Let X be a Hausdorff topological group. Then its completion $\widehat{X}$ under its bilateral uniformity can be endowed (in exactly one way) with a group structure rendering it a Hausdorff topological group in which the natural embedding of X in $\widehat{X}$ represents X as a dense subgroup of $\widehat{X}$. (Császár 78, 11.3.15.) If X has a neighbourhood of the identity which is totally bounded for the bilateral uniformity, then $\widehat{X}$ is locally compact. (Császár 78, 11.3.24.)

4A5O Proposition Let X be a topological group.
(a) If $A\subseteq X$, then the following are equiveridical: (i) A is totally bounded for the bilateral uniformity of X; (ii) for every neighbourhood U of the identity there is a finite set $I\subseteq X$ such that $A\subseteq IU\cap UI$.

(b) If A, $B \subseteq X$ are totally bounded for the bilateral uniformity of X, so are $A \cup B$, A^{-1} and AB. In particular, $\bigcup_{i \le n} x_i B$ is totally bounded for any $x_0, \ldots , x_n \in X$.

(c) If $A \subseteq X$ is totally bounded for the bilateral uniformity, and U is any neighbourhood of the identity, then $\{y : xyx^{-1} \in U$ for every $x \in A\}$ is a neighbourhood of the identity.

(d) If X is the product of a family $\langle X_i \rangle_{i \in I}$ of topological groups, a subset A of X is totally bounded for the bilateral uniformity of X iff it is included in a product $\prod_{i \in I} A_i$ where $A_i \subseteq X_i$ is totally bounded for the bilateral uniformity of X_i for every $i \in I$.

(e) If X is locally compact, a subset of X is totally bounded for the bilateral uniformity iff it is relatively compact.

proof (a)(i)⇒(ii) Suppose that A is totally bounded, and that U is a neighbourhood of the identity e of X. Set

$$W = \{(x, y) : xy^{-1} \in U^{-1}, \, x^{-1}y \in U\} = \{(x, y) : y \in Ux \cap xU\};$$

then W belongs to the uniformity, so there is a finite set $I \subseteq X$ such that $A \subseteq W[I]$. But $W[I] \subseteq UI \cap IU$, so $A \subseteq UI \cap IU$.

(ii)⇒(i) Now suppose that A satisfies the condition, and that W belongs to the uniformity. Then there is a neighbourhood U of e such that $\{(x, y) : xy^{-1} \in U, \, x^{-1}y \in U\} \subseteq W$. Let V be a neighbourhood of e such that $VV^{-1} \subseteq U$ and $V^{-1}V \subseteq U$. Let $I \subseteq X$ be a finite set such that $A \subseteq VI \cap IV$. For w, $z \in I$ set $A_{wz} = A \cap Vw \cap zV$. If x, $y \in A_{wz}$, $xy^{-1} \in Vww^{-1}V^{-1} \subseteq U$ and $x^{-1}y \in V^{-1}z^{-1}zV \subseteq U$. But this means that $A_{wz} \times A_{wz} \subseteq W$. So if we take a finite set J which meets every non-empty A_{wz}, $A \subseteq W[J]$. As W is arbitrary, A is totally bounded.

(b) Of course $A \cup B$ is totally bounded; this is immediate from the definition of 'totally bounded'. If U is a neighbourhood of e, so is U^{-1}, so there is a finite set $I \subseteq X$ such that $A \subseteq IU^{-1} \cap U^{-1}I$ and $A^{-1} \subseteq UI^{-1} \cap I^{-1}U$; as U is arbitrary, A^{-1} is totally bounded.

To see that AB also is totally bounded, let U be a neighbourhood of e, and take a neighbourhood V of e such that $VV \subseteq U$. Then there is a finite set $I \subseteq X$ such that $A \subseteq VI$ and $B \subseteq IV$. Let W be a neighbourhood of e such that $zWz^{-1} \cup z^{-1}Wz \subseteq V$ for every $z \in I$, and J a finite set such that $B \subseteq WJ$ and $A \subseteq JW$. Then

$$zW \subseteq Vz, \quad Wz \subseteq zV$$

for every $z \in I$, so

$$IW \subseteq VI, \quad WI \subseteq IV$$

and

$$AB \subseteq VIWJ \subseteq VVIJ \subseteq UK, \quad AB \subseteq JWIV \subseteq JIVV \subseteq KU$$

where $K = IJ \cup JI$ is finite. As U is arbitrary, AB is totally bounded.

(c) Let V be a neighbourhood of e such that $VVV^{-1} \subseteq U$. Let I be a finite set such that $A \subseteq VI$. Let W be a neighbourhood of e such that $zWz^{-1} \subseteq V$ for every $z \in I$. If now $y \in W$ and $x \in A$, there is a $z \in I$ such that $x \in Vz$, so that

$$xyx^{-1} \in VzWz^{-1}V^{-1} \subseteq VVV^{-1} \subseteq U.$$

Turning this round, $\{y : xyx^{-1} \in U$ for every $x \in A\}$ includes W and is a neighbourhood of e.

(d)(i) Suppose that A is totally bounded. Set $A_i = \pi_i[A]$ for each $i \in I$, where $\pi_i(x) = x(i)$ for $x \in X$. If U is a neighbourhood of the identity in X_i, then $V = \pi_i^{-1}[U]$ is a neighbourhood of the identity in X, so there is a finite set $J \subseteq X$ such that $A \subseteq JV \cap VJ$; now $A_i \subseteq KU \cap UK$, where $K = \pi_i[J]$ is finite. As U is arbitrary, A_i is totally bounded. This is true for every i, while $A \subseteq \prod_{i \in I} A_i$.

(ii) Suppose that $A \subseteq \prod_{i \in I} A_i$ where $A_i \subseteq X_i$ is totally bounded for each $i \in I$. If A is empty, of course it is totally bounded; assume that $A \neq \emptyset$. If $I = \emptyset$, then $X = \{\emptyset\}$ is the trivial group, and again A is totally bounded; so assume that I is non-empty. Let V be a neighbourhood of the identity in X. Then there are a non-empty finite set $L \subseteq I$ and a family $\langle U_i \rangle_{i \in L}$ such that U_i is a neighbourhood of the identity in X_i for each $i \in L$, and $V \supseteq \bigcap_{i \in L} \pi_i^{-1} U_i$. For each $i \in L$, let J_i be a finite subset of X_i such that $A_i \subseteq J_i U_i \cap U_i J_i$. Set

$$J = \{x : x \in X, \, x(i) \text{ is the identity for } i \in I \setminus L, \, x(i) \in J_i \text{ for } i \in L\}.$$

Then J is finite and $A \subseteq JV \cap VJ$. As V is arbitrary, A is totally bounded.

(e) Use (a).

4A5P Lemma Let X be a locally compact Hausdorff topological group. Take $f \in C_k(X)$, the space of continuous real-valued functions on X with compact supports.

(a) Let $K \subseteq X$ be a compact set. Then for any $\epsilon > 0$ there is a neighbourhood W of the identity e of X such that $|f(xay) - f(xby)| \le \epsilon$ whenever $x \in K$, $y \in X$ and $ab^{-1} \in W$.

(b) For any $x_0 \in X$, there is a non-negative $f^* \in C_k(X)$ such that for every $\epsilon > 0$ there is an open set G containing x_0 such that $|f(xy) - f(x_0 y)| \le \epsilon f^*(y)$ for every $x \in G$ and $y \in X$.

proof (a) By 4A2Jf and 4A5Ha, f is uniformly continuous for the right uniformity of X. There is therefore a symmetric neighbourhood U of e such that $|f(y_1) - f(y_2)| \le \epsilon$ whenever y_1, $y_2 \in X$ and $y_1 y_2^{-1} \in U$. By 4A5Oc, there is a symmetric neighbourhood W of e such that $xzx^{-1} \in U$ whenever $x \in K$ and $z \in W$.

Now suppose that $x \in K$, $y \in X$ and $ab^{-1} \in W$. Then $(xay)(xby)^{-1} = xab^{-1}x^{-1} \in U$, so $|f(xay) - f(xby)| \le \epsilon$, as required.

(b) We need a trifling refinement of the ideas above.

(i) Suppose for the moment that $x_0 = e$. Set $L = \overline{\{x : f(x) \ne 0\}}$ and let V be a compact symmetric neighbourhood of the identity e, so that L and VL are compact. Let $f^* \in C_k(X)$ be such that $f^* \ge \chi(VL)$ (4A2G(e-i)). Given $\epsilon > 0$, take U as in (a), so that U is a symmetric neighbourhood of e and $|f(y_1) - f(y_2)| \le \epsilon$ whenever $y_1 y_2^{-1} \in U$; this time arrange further that $U \subseteq V$. Then if $x \in U$ and $y \in X$,

either y and xy belong to VL, while $(xy)y^{-1} \in U$, so $|f(xy) - f(y)| \le \epsilon \le \epsilon f^*(y)$

or neither y nor xy belongs to L, so $|f(xy) - f(y)| = 0 \le \epsilon f^*(y)$.

(ii) For the general case, set $f_0(x) = f(x_0 x)$ for $x \in X$. Because $x \mapsto x_0 x$ is a homeomorphism, $f_0 \in C_k(X)$. By (i), we have a non-negative $f^* \in C_k(X)$ such that for every $\epsilon > 0$ there is a neighbourhood G_ϵ of e such that $|f_0(xy) - f_0(y)| \le \epsilon f^*(y)$ whenever $x \in G_\epsilon$ and $y \in X$. Now, given $\epsilon > 0$, $G' = x_0 G_\epsilon$ is a neighbourhood of x_0 and $|f(xy) - f(x_0 y)| \le \epsilon f^*(y)$ whenever $x \in G'$ and $y \in X$. So f^* witnesses that the result is true.

4A5Q Metrizable groups: Proposition Let $(X, \mathfrak{T})$ be a topological group. Then the following are equiveridical:

(i) X is metrizable;

(ii) the identity e of X has a countable neighbourhood base;

(iii) there is a metric ρ on X, inducing the topology $\mathfrak{T}$, which is **right-translation-invariant**, that is, $\rho(x_1, x_2) = \rho(x_1 y, x_2 y)$ for all x_1, x_2, $y \in X$;

(iv) there is a right-translation-invariant metric on X which induces the right uniformity of X;

(v) the bilateral uniformity of X is metrizable.

proof Császár 78, 11.2.10 and 11.3.2.

Warning! A Polish group (4A5Db) is of course metrizable, so has a right-translation-invariant metric inducing its topology. At the same time, it has a complete metric inducing its topology. But there is no suggestion that these two metrics should be the same, or even induce the same uniformity (441Xq).

4A5R Corollary If X is a locally compact topological group and $\{e\}$ is a G_δ set in X, then X is metrizable. (Put 4A5Q and 4A2Kf together, or see Hewitt & Ross 63, 8.5.)

4A5S Lemma Let X be a σ-compact locally compact Hausdorff topological group and $\langle U_n \rangle_{n \in \mathbb{N}}$ any sequence of neighbourhoods of the identity in X. Then X has a compact normal subgroup $Y \subseteq \bigcap_{n \in \mathbb{N}} U_n$ such that $Z = X/Y$ is Polish.

proof Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be a sequence of compact sets covering X. Choose inductively a sequence $\langle V_n \rangle_{n \in \mathbb{N}}$ of compact neighbourhoods of e such that, for each $n \in \mathbb{N}$,

(α) $V_{n+1} \subseteq V_n^{-1}$, $\quad V_{n+1} V_{n+1} \subseteq V_n$, $\quad V_n \subseteq U_n$,

(β) $xyx^{-1} \in V_n$ whenever $y \in V_{n+1}$ and $x \in \bigcup_{i \le n} K_i$.

(When we come to choose V_{n+1}, we can achieve (α) because inversion and multiplication are continuous, and (β) by 4A5Oc; and we can then shrink V_{n+1} to a compact neighbourhood of e because X is locally compact.) Set $Y = \bigcap_{n \in \mathbb{N}} V_n$. Then ($\alpha$) is enough to ensure that Y is a compact subgroup of X included in $\bigcap_{n \in \mathbb{N}} U_n$, while ($\beta$) ensures that Y is normal, because for any $x \in X$ there is an $n \in \mathbb{N}$ such that $xV_{m+1}x^{-1} \subseteq V_m$ for every $m \ge n$.

Let $\pi : X \to Z$ be the canonical map. Then $C = \bigcap_{n \in \mathbb{N}} \pi[\operatorname{int} V_n]$ is a G_δ subset of Z, because π is open (4A5J(a-i) again). But

$$\pi^{-1}[C] \subseteq \bigcap_{n\in\mathbb{N}} \pi^{-1}[\pi[V_{n+1}]] = \bigcap_{n\in\mathbb{N}} V_{n+1}Y \subseteq \bigcap_{n\in\mathbb{N}} V_n = Y,$$

so $C = \{e_Z\}$, writing e_Z for the identity of Z. Thus $\{e_Z\}$ is a G_δ set; as Z is locally compact and Hausdorff (4A5J(b-ii)), it is metrizable (4A5R). By 4A5J(b-ii-δ), Z is Polish.

***4A5T** I shall not rely on the following fact, but it will help you to make sense of some of the results of this volume.

Theorem A compact Hausdorff topological group is dyadic.

proof Uspenskiĭ 88.

4A6 Banach algebras

I give results which are needed for Chapter 44. Those down to 4A6K should be in any introductory text on normed algebras; 4A6L-4A6O, as expressed here, are a little more specialized. As with normed spaces or linear topological spaces, Banach algebras may be defined over either $\mathbb{R}$ or $\mathbb{C}$. In §445 we need complex Banach algebras, but in §446 I think the ideas are clearer in the context of real Banach algebras. Accordingly, as in §2A4, I express as much as possible of the theory in terms applicable equally to either, speaking of 'normed algebras' or 'Banach algebras' without qualification, and using the symbol $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ to represent the field of scalars. Since (at least, if you keep to the path indicated here) the ideas are independent of which field we work with, you will have no difficulty in applying the arguments given in Folland 95 or Hewitt & Ross 63 for the complex case to the real case. In 4A6B and 4A6I-4A6K, however, we have results which apply only to 'complex' Banach algebras, in which the underlying field is taken to be $\mathbb{C}$.

4A6A Definition (a) I repeat a definition from §2A4. A **normed algebra** is a normed space U together with a multiplication, a binary operator $\times$ on U, such that

$$u \times (v \times w) = (u \times v) \times w,$$

$$u \times (v + w) = (u \times v) + (u \times w), \quad (u + v) \times w = (u \times w) + (v \times w),$$

$$(\alpha u) \times v = u \times (\alpha v) = \alpha(u \times v),$$

$$\|u \times v\| \le \|u\|\|v\|$$

for all u, v, $w \in U$ and $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$. A normed algebra is **commutative** if its multiplication is commutative.

(b) A **Banach algebra** is a normed algebra which is a Banach space. A **unital Banach algebra** is a Banach algebra with a multiplicative identity e such that $\|e\| = 1$. (**Warning:** some authors reserve the term 'Banach algebra' for what I call a 'unital Banach algebra'.)

In a unital Banach algebra I will always use the letter e for the identity.

4A6B Stone-Weierstrass Theorem: fourth form Let X be a locally compact Hausdorff space, and $C_0 = C_0(X;\mathbb{C})$ the complex Banach algebra of continuous functions $f : X \to \mathbb{C}$ such that $\{x : |f(x)| \ge \epsilon\}$ is compact for every $\epsilon > 0$. Let $A \subseteq C_0$ be such that

A is a linear subspace of C_0,

$f \times g \in A$ for every f, $g \in A$,

the complex conjugate $\bar{f}$ of f belongs to A for every $f \in A$,

for every $x \in X$ there is an $f \in A$ such that $f(x) \ne 0$,

whenever x, y are distinct points of X there is an $f \in A$ such that $f(x) \ne f(y)$.

Then A is $\| \|_\infty$-dense in C_0.

proof Let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X (3A3O). For $f \in C_0$ write $f^\#$ for the extension of f to $X \cup \{\infty\}$ with $f^\#(\infty) = 0$, so that $f^\# \in C_b(X_\infty;\mathbb{C})$. Let $B \subseteq C_b(X_\infty;\mathbb{C})$ be the set of all functions of the form

$f^\# + \alpha\chi X_\infty$ where $f \in A$ and $\alpha \in \mathbb{C}$. Then B is a subalgebra of $C_b(X \cup \{\infty\})$ which contains complex conjugates of its members and constant functions and separates the points of X_∞.

Take any $h \in C_0$ and $\epsilon > 0$. By the 'third form' of the Stone-Weierstrass theorem (281G), there is a $g \in B$ such that $\|g - h^\#\|_\infty \le \frac{1}{2}\epsilon$. Express g as $f^\# + \alpha\chi X_\infty$ where $f \in A$ and $\alpha \in \mathbb{C}$. Then

$$|\alpha| = |g(\infty)| = |g(\infty) - h^\#(\infty)| \le \tfrac{1}{2}\epsilon,$$

so

$$\|h - f\|_\infty = \|h^\# - f^\#\|_\infty \le \|h^\# - g\|_\infty + \|g - f^\#\|_\infty \le \tfrac{1}{2}\epsilon + |\alpha| \le \epsilon.$$

As h and ϵ are arbitrary, A is dense in C_0.

4A6C Proposition If U is any Banach space other than $\{0\}$, then the space $\mathsf{B}(U;U)$ of bounded linear operators from U to itself is a unital Banach algebra. (Köthe 69, 14.6.)

4A6D Proposition Any normed algebra U can be embedded as a subalgebra of a unital Banach algebra V, in such a way that if U is commutative so is V. (Folland 95, §1.3; Hewitt & Ross 63, C.3.)

4A6E Proposition Let U be a unital Banach algebra and $W \subseteq U$ a closed proper ideal. Then U/W, with the quotient linear structure, ring structure and norm, is a unital Banach algebra. (Hewitt & Ross 63, C.2.)

4A6F Proposition If U is a Banach algebra and $\phi : U \to {}^{\mathbb{R}}_{\mathbb{C}}$ is a multiplicative linear functional, then $|\phi(u)| \le \|u\|$ for every $u \in U$.

proof ? Otherwise, there is a u such that $|\phi(u)| > \|u\|$; set $v = \frac{1}{\phi(u)}u$, so that $\phi(v) = 1$ and $\|v\| < 1$. Since $\|v^n\| \le \|v\|^n$ for every $n \ge 1$, $w = \sum_{n\in\mathbb{N}\setminus\{0\}} v^n$ is defined in U (4A4Ie), and $w = vw + v$ (because $u \mapsto vu$ is a continuous linear operator, so we can use 4A4Bh to see that $vw = \sum_{n\in\mathbb{N}\setminus\{0\}} v^{n+1}$). But this means that $\phi(w) = \phi(v)\phi(w) + \phi(v) = \phi(w) + 1$, which is impossible. **X**

4A6G Definition Let U be a normed algebra and $u \in U$.

(a) For any $u \in U$, $\lim_{n\to\infty} \|u^n\|^{1/n}$ is defined and equal to $\inf_{n\ge1} \|u^n\|^{1/n}$. (Hewitt & Ross 63, C.4.)

(b) This common value is called the **spectral radius** of u.

4A6H Theorem If U is a unital Banach algebra, then the set R of invertible elements is open, and $u \mapsto u^{-1}$ is a continuous function from R to itself. If $v \in U$ and $\|v - e\| < 1$, then $v \in R$ and $\|v^{-1} - e\| \le \frac{\|v-e\|}{1-\|v-e\|}$. (Folland 95, 1.4; Hewitt & Ross 63, C.8 & C.10; Rudin 91, 10.7 & 10.12.)

4A6I Theorem Let U be a complex unital Banach algebra and $u \in U$. Write r for the spectral radius of u.

(a) If $\zeta \in \mathbb{C}$ and $|\zeta| > r$ then $\zeta e - u$ is invertible.

(b) There is a ζ such that $|\zeta| = r$ and $\zeta e - u$ is not invertible.

proof Folland 95, 1.8; Hewitt & Ross 63, C.24; Rudin 91, 1.13.

4A6J Theorem Let U be a commutative complex unital Banach algebra, and $u \in U$. Then for any $\zeta \in \mathbb{C}$ the following are equiveridical:

(i) there is a non-zero multiplicative linear functional $\phi : U \to \mathbb{C}$ such that $\phi(u) = \zeta$;

(ii) $\zeta e - u$ is not invertible.

proof Folland 95, 1.13; Hewitt & Ross 63, C.20; Rudin 91, 11.5.

4A6K Corollary Let U be a commutative complex Banach algebra and $u \in U$. Then its spectral radius $r(u)$ is $\max\{|\phi(u)| : \phi$ is a multiplicative linear functional on $U\}$. (Folland 95, 1.13; Hewitt & Ross 63, C.20; Rudin 91, 11.9.)

4A6L Exponentiation Let U be a unital Banach algebra. For any $u \in U$, $\sum_{k=0}^{\infty} \|\frac{1}{k!}u^k\| \le \sum_{k=0}^{\infty} \frac{1}{k!}\|u\|^k$ is finite, so

$$\exp(u) = \sum_{k\in\mathbb{N}} \frac{1}{k!}u^k$$

is defined in U (4A4Ie). (In this formula, interpret u^0 as e for every u.)

4A6M Lemma Let U be a unital Banach algebra.

(a) If u, $v \in U$ and $\max(\|u\|, \|v\|) \le \gamma$ then $\|\exp(u)-\exp(v)-u+v\| \le \|u-v\|(\exp\gamma-1)$. So if $\max(\|u\|, \|v\|) \le \frac{2}{3}$ and $\exp(u) = \exp(v)$ then $u = v$.

(b) If $\|u - e\| \le \frac{1}{6}$ then there is a v such that $\exp(v) = u$ and $\|v\| \le 2\|u - e\|$.

(c) If u, $v \in U$ and $uv = vu$ then $\exp(u + v) = \exp(u)\exp(v)$.

proof (a) Note first that if $k \ge 1$ then

$$\begin{aligned}\|u^k - v^k\| &= \|\sum_{i=0}^{k-1} u^{k-i}v^i - u^{k-i-1}v^{i+1}\| = \|\sum_{i=0}^{k-1} u^{k-i-1}(u-v)v^i\| \\ &\le \sum_{i=0}^{k-1}\|u\|^{k-i-1}\|u-v\|\|v\|^i \le \sum_{i=0}^{k-1}\gamma^{k-1}\|u-v\| = k\gamma^{k-1}\|u-v\|.\end{aligned}$$

So

$$\begin{aligned}\|\exp(u)-\exp(v)-u+v\| &= \|\sum_{k\in\mathbb{N}\setminus\{0,1\}}\frac{1}{k!}(u^k - v^k)\| \le \sum_{k=2}^{\infty}\frac{1}{k!}\|u^k - v^k\| \\ &\le \sum_{k=2}^{\infty}\frac{k}{k!}\gamma^{k-1}\|u-v\| = \|u-v\|(\exp\gamma - 1).\end{aligned}$$

Now if $\exp(u) = \exp(v)$ and $\gamma \le \frac{2}{3}$, $0 \le \exp\gamma - 1 < 1$ so $\|u - v\| = 0$ and $u = v$.

(b) Set $\gamma = \|u - e\|$. Define $\langle v_n\rangle_{n\in\mathbb{N}}$ in U by setting $v_0 = 0$, $v_{n+1} = v_n + u - \exp(v_n)$ for $n \in \mathbb{N}$. Then

$$\|v_{n+1} - v_n\| = \|u - \exp(v_n)\| \le 2^{-n}\gamma, \quad \|v_n\| \le 2(1 - 2^{-n})\gamma$$

for every $n \in \mathbb{N}$. **P** Induce on n. The induction starts with $\|v_0\| = 0$ and $\|u - \exp(v_0)\| = \|u - e\| = \gamma$. Given that $\|v_n\| \le 2(1 - 2^{-n})\gamma$ and $\|u - \exp(v_n)\| \le 2^{-n}\gamma$, then

$$\|v_{n+1}\| \le \|v_n\| + \|u - \exp(v_n)\| \le 2(1 - 2^{-n})\gamma + 2^{-n}\gamma = 2(1 - 2^{-n-1})\gamma.$$

Now $\max(\|v_{n+1}\|, \|v_n\|) \le 2\gamma \le \frac{1}{3}$, so

$$\begin{aligned}\|u - \exp(v_{n+1})\| &= \|v_{n+1} - v_n + \exp(v_n) - \exp(v_{n+1})\| \le \|v_{n+1} - v_n\|(\exp\frac{1}{3} - 1) \\ &\le \frac{1}{2}\|v_{n+1} - v_n\| = \frac{1}{2}\|u - \exp(v_n)\| \le 2^{-n-1}\gamma,\end{aligned}$$

and the induction continues. **Q**

Since $\sum_{n=0}^{\infty}\|v_{n+1} - v_n\|$ is finite, $v = \lim_{n\to\infty} v_n$ is defined in U, and $\|v\| = \lim_{n\to\infty}\|v_n\| \le 2\gamma$. Accordingly

$$\begin{aligned}\|\exp(v) - \exp(v_n)\| &\le \|v - v_n\| + \|\exp(v) - \exp(v_n) - v + v_n\| \\ &\le \|v - v_n\|(1 + \exp\frac{1}{3} - 1) \to 0\end{aligned}$$

as $n \to \infty$, and $\exp(v) = \lim_{n\to\infty}\exp(v_n) = u$.

(c) Because $uv = vu$, $(u+v)^m = \sum_{j=0}^{m}\frac{m!}{j!(m-j)!}u^jv^{m-j}$ for every $m \in \mathbb{N}$ (induce on m; the point is that $uv^j = v^ju$ for every $j \in \mathbb{N}$). Next, $\sum_{j,k\in\mathbb{N}}\frac{1}{j!k!}\|u\|^j\|v\|^k$ is finite. So

$$\exp(u+v)=\sum_{m\in\mathbb{N}}\frac{1}{m!}(u+v)^m$$
$$=\sum_{m\in\mathbb{N}}\bigl(\sum_{j+k=m}\frac{1}{j!k!}u^jv^k\bigr)=\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}}\frac{1}{j!k!}u^jv^k$$

(using 4A4I(e-ii))

$$=\sum_{j\in\mathbb{N}}\bigl(\sum_{k\in\mathbb{N}}\frac{1}{j!k!}u^jv^k\bigr)$$

(4A4I(e-ii) again)

$$=\sum_{j\in\mathbb{N}}\bigl(\frac{1}{j!}u^j\sum_{k\in\mathbb{N}}\frac{1}{k!}v^k\bigr)$$

(by 4A4Bh, because $w\mapsto\frac{1}{j!}u^jw$ is a continuous linear operator for each j)

$$=\sum_{j\in\mathbb{N}}\frac{1}{j!}u^j\exp(v)=\bigl(\sum_{j\in\mathbb{N}}\frac{1}{j!}u^j\bigr)\exp(v)$$

(4A4Bh again)

$$=\exp(u)\exp(v),$$

as claimed.

4A6N Lemma If U is a unital Banach algebra, $u\in U$ and $\|u^n-e\|\le\frac{1}{6}$ for every $n\in\mathbb{N}$, then $u=e$.

proof For every $n\in\mathbb{N}$ there is a $v_n\in U$ such that $\exp(v_n)=u^{2^n}$ and $\|v_n\|\le\frac{1}{3}$ (4A6Mb). Then $\exp(v_{n+1})=\exp(v_n)^2=\exp(2v_n)$ (4A6Mc), $\|v_{n+1}\|\le\frac{1}{3}$ and $\|2v_n\|\le\frac{1}{3}$ so $v_{n+1}=2v_n$ for every n (4A6Ma). Inducing on n, $v_n=2^nv_0$ for every n, so that $\|v_0\|\le 2^{-n}\|v_n\|\to 0$ as $n\to\infty$, and $u=\exp(v_0)=e$.

4A6O Proposition Let U be a normed algebra, and U^*, U^{**} its dual and bidual as a normed space. For a bounded linear operator $T:U\to U$ let $T':U^*\to U^*$ be the adjoint of T and $T'':U^{**}\to U^{**}$ the adjoint of T'.

(a) We have bilinear maps, all of norm at most 1,

$$(f,x)\mapsto f\circ x:U^*\times U\to U^*,$$
$$(\phi,f)\mapsto\phi\circ f:U^{**}\times U^*\to U^*,$$
$$(\phi,\psi)\mapsto\phi\circ\psi:U^{**}\times U^{**}\to U^{**}$$

defined by the formulae

$$(f\circ x)(y)=f(xy),$$
$$(\phi\circ f)(x)=\phi(f\circ x),$$
$$(\phi\circ\psi)(f)=\phi(\psi\circ f)$$

for all x, $y\in U$, $f\in U^*$ and ϕ, $\psi\in U^{**}$.

(b)(i) Suppose that $S:U\to U$ is a bounded linear operator such that $S(xy)=(Sx)y$ for all x, $y\in U$. Then $S''(\phi\circ\psi)=(S''\phi)\circ\psi$ for all ϕ, $\psi\in U^{**}$.

(ii) Suppose that $T:U\to U$ is a bounded linear operator such that $T(xy)=x(Ty)$ for all x, $y\in U$. Then $T''(\phi\circ\psi)=\phi\circ(T''\psi)$ for all ϕ, $\psi\in U^{**}$.

proof (a) The calculations are elementary if we take them one at a time.

(b)(i)($\boldsymbol{\alpha}$)$(S'f)\circ x=f\circ(Sx)$ for every $f\in U^*$ and $x\in U$. **P**

$$((S'f)\circ x)(y)=(S'f)(xy)=f(S(xy))=f((Sx)y)=(f\circ(Sx))(y)$$

for every $y\in U$. **Q**

(β) $\psi\circ(S'f)=S'(\psi\circ f)$ for every $f\in U^*$. **P**

$$(\psi\circ(S'f))(x)=\psi((S'f)\circ x)=\psi(f\circ(Sx))=(\psi\circ f)(Sx)=(S'(\psi\circ f))(x)$$

for every $x\in U$. **Q**

(γ) So

$$\begin{aligned}(S''(\phi\circ\psi))(f)&=(\phi\circ\psi)(S'f)=\phi(\psi\circ(S'f))\\&=\phi(S'(\psi\circ f))=(S''\phi)(\psi\circ f)=((S''\phi)\circ\psi)(f)\end{aligned}$$

for every $f\in U^*$, and $S''(\phi\circ\psi)=(S''\phi)\circ\psi$.

(ii)(α) $(T'f)\circ x=T'(f\circ x)$ for every $f\in U^*$ and $x\in U$. **P**

$$((T'f)\circ x)(y)=(T'f)(xy)=f(T(xy))=f(x(Ty))=(f\circ x)(Ty)=(T'(f\circ x))(y)$$

for every $y\in U$. **Q**

(β) $\psi\circ(T'f)=(T''\psi)\circ f$ for every $f\in U^*$. **P**

$$\begin{aligned}(\psi\circ(T'f))(x)&=\psi((T'f)\circ x)=\psi(T'(f\circ x))\\&=(T''\psi)(f\circ x)=((T''\psi)\circ f)(x)\end{aligned}$$

for every $x\in U$. **Q**

(γ) So

$$(T''(\phi\circ\psi))(f)=(\phi\circ\psi)(T'f)=\phi(\psi\circ(T'f))=\phi((T''\psi)\circ f)=(\phi\circ(T''\psi))(f)$$

for every $f\in U'$, and $T''(\phi\circ\psi)=\phi\circ(T''\psi)$.

Remark I must not abandon you at this point without telling you that $\circ:U^{**}\times U^{**}\to U^{**}$ is an **Arens multiplication**, and that it is associative, so that that U^{**} is a Banach algebra.

4A7 'Later editions only'

In this edition of Volume 4 I refer to a handful of fragments which I have interpolated into earlier volumes and which have not yet appeared in a printed version. For the time being they may be found, in context, in the online drafts listed in `http://www.essex.ac.uk/maths/people/fremlin/mtcont.htm`.

235Xn Exercise Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $\phi:X\to Y$ an inverse-measure-preserving function. Show that $\overline{\int}h\phi\,d\mu\le\overline{\int}h\,d\nu$ for every real-valued function h defined almost everywhere in Y. (Compare 234Bf.)

254Yh Exercise Let $f:[0,1]\to[0,1]^2$ be a function which is inverse-measure-preserving for Lebesgue planar measure on $[0,1]^2$ and Lebesgue linear measure on $[0,1]$, as in 134Yl; let f_1, f_2 be the coordinates of f. Define $g:[0,1]\to[0,1]^{\mathbb{N}}$ by setting $g(t)=\langle f_1f_2^n(t)\rangle_{n\in\mathbb{N}}$ for $0\le t\le 1$. Show that g is inverse-measure-preserving. (*Hint*: show that $g_n:[0,1]\to[0,1]^{n+1}$ is inverse-measure-preserving for every $n\ge 1$, where $g_n(t)=(f_1(t),f_1f_2(t),\dots,f_1f_2^{n-1}(t),f_2^n(t))$ for $t\in[0,1]$.)

2A5B Definition If U is a linear space over $\substack{\mathbb{R}\\ \mathbb{C}}$, a functional $\tau:U\to[0,\infty[$ is an **F-seminorm** if

(i) $\tau(u+v)\le\tau(u)+\tau(v)$ for all u, $v\in U$, $\tau\in\mathrm{T}$;
(ii) $\tau(\alpha u)\le\tau(u)$ if $u\in U$, $|\alpha|\le 1$, $\tau\in\mathrm{T}$;
(iii) $\lim_{\alpha\to 0}\tau(\alpha u)=0$ for every $u\in U$, $\tau\in\mathrm{T}$.

535Yd Exercise Let (X,Σ,μ) be a countably separated perfect complete strictly localizable measure space, $\mathfrak{A}$ its measure algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ of cardinal at most $\min(\operatorname{add}\mathcal{N},\mathfrak{p})$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$. Show that there is an action $\bullet$ of G on X such that $\pi\bullet E=\{\pi\bullet x:x\in E\}$ belongs to Σ and $(\pi\bullet E)^\bullet=\pi(E^\bullet)$ whenever $\pi\in G$ and $E\in\Sigma$. (*Hint*: 344C, 425Ya.)

536C Proposition Let (X, Σ, μ) be a probability space such that the π-weight $\pi(\mu)$ of μ is at most $\mathfrak{p}$. If $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-compact then it is $\mathfrak{T}_m$-compact.

Concordance to chapters 46-49

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to these chapters, and which have since been changed.

463H Fremlin's Alternative The dichotomy for sequences of measurable functions on a perfect measures space, referred to in BOGACHEV 07, is now 463K.

479Xe Exercise 479Xe on Choquet-Newton capacity, referred to in the 2008 edition of Volume 5, is now 479Xi.

4A2Jf Uniformities on completely regular spaces 4A2Jf, referred to in the 2009 edition of Volume 5, has been moved to 4A2Jg.

4A4B Bounded sets in linear topological spaces 4A4Bg, referred to in the 2008 edition of Volume 5, has been moved to 3A5Nb.

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Index to volumes 1-4

Principal topics and results

The general index below is intended to be comprehensive. Inevitably the entries are voluminous to the point that they are often unhelpful. I have therefore prepared a shorter, better-annotated, index which will, I hope, help readers to focus on particular areas. It does not mention definitions, as the bold-type entries in the main index are supposed to lead efficiently to these; and if you draw blank here you should always, of course, try again in the main index. Entries in the form of mathematical assertions frequently omit essential hypotheses and should be checked against the formal statements in the body of the work.

General index

References in **bold** type refer to definitions; references in *italics* are passing references. Definitions marked with **>** are those in which my usage is dangerously at variance with that of some other author or authors.

[1] A.M.Lyapunov, 1857-1918
[2] A.A.Lyapunov, 1911-1973