MEASURE THEORY

Volume 4

Part I

D.H.Fremlin

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MEASURE THEORY

Volume 4

Topological Measure Spaces

Part I

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to the Publisher

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General introduction In this treatise I aim to give a comprehensive description of modern abstract measure theory, with some indication of its principal applications. The first two volumes are set at an introductory level; they are intended for students with a solid grounding in the concepts of real analysis, but possibly with rather limited detailed knowledge. As the book proceeds, the level of sophistication and expertise demanded will increase; thus for the volume on topological measure spaces, familiarity with general topology will be assumed. The emphasis throughout is on the mathematical ideas involved, which in this subject are mostly to be found in the details of the proofs.

My intention is that the book should be usable both as a first introduction to the subject and as a reference work. For the sake of the first aim, I try to limit the ideas of the early volumes to those which are really essential to the development of the basic theorems. For the sake of the second aim, I try to express these ideas in their full natural generality, and in particular I take care to avoid suggesting any unnecessary restrictions in their applicability. Of course these principles are to to some extent contradictory. Nevertheless, I find that most of the time they are very nearly reconcilable, *provided* that I indulge in a certain degree of repetition. For instance, right at the beginning, the puzzle arises: should one develop Lebesgue measure first on the real line, and then in spaces of higher dimension, or should one go straight to the multidimensional case? I believe that there is no single correct answer to this question. Most students will find the one-dimensional case easier, and it therefore seems more appropriate for a first introduction, since even in that case the technical problems can be daunting. But certainly every student of measure theory must at a fairly early stage come to terms with Lebesgue area and volume as well as length; and with the correct formulations, the multidimensional case differs from the one-dimensional case only in a definition and a (substantial) lemma. So what I have done is to write them both out (§§114-115). In the same spirit, I have been uninhibited, when setting out exercises, by the fact that many of the results I invite students to look for will appear in later chapters; I believe that throughout mathematics one has a better chance of understanding a theorem if one has previously attempted something similar alone.

The plan of the work is as follows:

Volume 1: The Irreducible Minimum
Volume 2: Broad Foundations
Volume 3: Measure Algebras
Volume 4: Topological Measure Spaces
Volume 5: Set-theoretic Measure Theory.

Volume 1 is intended for those with no prior knowledge of measure theory, but competent in the elementary techniques of real analysis. I hope that it will be found useful by undergraduates meeting Lebesgue measure for the first time. Volume 2 aims to lay out some of the fundamental results of pure measure theory (the Radon-Nikodým theorem, Fubini's theorem), but also gives short introductions to some of the most important applications of measure theory (probability theory, Fourier analysis). While I should like to believe that most of it is written at a level accessible to anyone who has mastered the contents of Volume 1, I should not myself have the courage to try to cover it in an undergraduate course, though I would certainly attempt to include some parts of it. Volumes 3 and 4 are set at a rather higher level, suitable to postgraduate courses; while Volume 5 will assume a wide-ranging competence over large parts of analysis and set theory.

There is a disclaimer which I ought to make in a place where you might see it in time to avoid paying for this book. I make no attempt to describe the history of the subject. This is not because I think the history uninteresting or unimportant; rather, it is because I have no confidence of saying anything which would not be seriously misleading. Indeed I have very little confidence in anything I have ever read concerning the history of ideas. So while I am happy to honour the names of Lebesgue and Kolmogorov and Maharam in more or less appropriate places, and I try to include in the bibliographies the works which I have myself consulted, I leave any consideration of the details to those bolder and better qualified than myself.

For the time being, at least, printing will be in short runs. I hope that readers will be energetic in commenting on errors and omissions, since it should be possible to correct these relatively promptly. An inevitable consequence of this is that paragraph references may go out of date rather quickly. I shall be most flattered if anyone chooses to rely on this book as a source for basic material; and I am willing to attempt to maintain a concordance to such references, indicating where migratory results have come to rest for the moment, if authors will supply me with copies of papers which use them.

I mention some minor points concerning the layout of the material. Most sections conclude with lists of 'basic exercises' and 'further exercises', which I hope will be generally instructive and occasionally entertaining. How many of these you should attempt must be for you and your teacher, if any, to decide, as no two students will have quite the same needs. I mark with a **>** those which seem to me to be particularly important. But while you may not need to write out solutions to all the 'basic exercises', if you are in any doubt as to your capacity to do so you should take this as a warning to slow down a bit. The 'further exercises' are unbounded in difficulty, and are unified only by a presumption that each has at least one solution based on ideas already introduced. Occasionally I add a final 'problem', a question to which I do not know the answer and which seems to arise naturally in the course of the work.

The impulse to write this book is in large part a desire to present a unified account of the subject. Cross-references are correspondingly abundant and wide-ranging. In order to be able to refer freely across the whole text, I have chosen a reference system which gives the same code name to a paragraph wherever it is being called from. Thus 132E is the fifth paragraph in the second section of the third chapter of Volume 1, and is referred to by that name throughout. Let me emphasize that cross-references are supposed to help the reader, not distract her. Do not take the interpolation '(121A)' as an instruction, or even a recommendation, to lift Volume 1 off the shelf and hunt for §121. If you are happy with an argument as it stands, independently of the reference, then carry on. If, however, I seem to have made rather a large jump, or the notation has suddenly become opaque, local cross-references may help you to fill in the gaps.

Each volume will have an appendix of 'useful facts', in which I set out material which is called on somewhere in that volume, and which I do not feel I can take for granted. Typically the arrangement of material in these appendices is directed very narrowly at the particular applications I have in mind, and is unlikely to be a satisfactory substitute for conventional treatments of the topics touched on. Moreover, the ideas may well be needed only on rare and isolated occasions. So as a rule I recommend you to ignore the appendices until you have some direct reason to suppose that a fragment may be useful to you.

During the extended gestation of this project I have been helped by many people, and I hope that my friends and colleagues will be pleased when they recognise their ideas scattered through the pages below. But I am especially grateful to those who have taken the trouble to read through earlier versions and comment on obscurities and errors.

Introduction to Volume 4

I return in this volume to the study of measure *spaces* rather than measure *algebras*. For fifty years now measure theory has been intimately connected with general topology. Not only do a very large proportion of the measure spaces arising in applications carry topologies related in interesting ways to their measures, but many questions in abstract measure theory can be effectively studied by introducing suitable topologies. Consequently any course in measure theory at this level must be frankly dependent on a substantial knowledge of topology. With this proviso, I hope that the present volume will be accessible to graduate students, and will lead them to the most important ideas of modern abstract measure theory.

The first and third chapters of the volume seek to provide a thorough introduction into the ways in which topologies and measures can interact. They are divided by a short chapter on descriptive set theory, on the borderline between set theory, logic, real analysis and general topology, which I single out for detailed exposition because I believe that it forms an indispensable part of the background of any measure theorist. Chapter 41 is dominated by the concepts of inner regularity and τ-additivity, coming together in Radon measures (§416). Chapter 43 concentrates rather on questions concerning properties of a topological space which force particular relationships with measures on that space. But plenty of side-issues are treated in both, such as Lusin measurability (§418), the definition of measures from linear functionals (§436) and measure-free cardinals (§438). Chapters 45 and 46 continue some of the same themes, with particular investigations into 'disintegrations' or regular conditional probabilities (§§452-453), stochastic processes (§§454-456), Talagrand's theory of stable sets (§465) and the theory of measures on normed spaces (§§466-467).

In contrast with the relatively amorphous structure of Chapters 41, 43, 45 and 46, four chapters of this volume have definite topics. I have already said that Chapter 42 is an introduction to descriptive set theory; like Chapters 31 and 35 in the preceding volume, it is a kind of appendix brought into the main stream of the argument. Chapter 44 deals with topological groups. Most of it is of course devoted to Haar measure, giving the Pontryagin-van Kampen duality theorem (§445) and the Ionescu Tulcea theorem on the existence of translation-invariant liftings (§447). But there are also sections on Polish groups (§448) and amenable groups (§449), and some of the general theory of measures on measurable groups (§444). Chapter 47 is a second excursion, after Chapter 26, into geometric measure theory. It starts with Hausdorff measures (§471), gives a proof of the Di Giorgio-Federer Divergence Theorem (§475), and then examines a number of examples of 'concentration of measure' (§476). In the second half of the chapter, §§477-479, I decribe Brownian motion and use it as a basis of the theory of Newtonian capacity. In Chapter 48, I set out the elementary theory of gauge integrals, with sections on the Henstock and Pfeffer integrals (§§483-484). Finally, in Chapter 49, I give notes on seven special topics: equidistributed sequences (§491), combinatorial forms of concentration of measure (§492), extremely amenable groups and groups of measure-preserving automorphisms (§§493-494), Poisson point processes (§495), submeasures (§496), Szemerédi's theorem (§497) and subproducts in product spaces (§498).

I had better mention prerequisites, as usual. To embark on this material you will certainly need a solid foundation in measure theory. Since I do of course use my own exposition as my principal source of references to the elementary ideas, I advise readers to ensure that they have easy access to all three previous volumes before starting serious work on this one. But you may not need to read very much of them. It might be prudent to glance through the detailed contents of Volume 1 and the first five chapters of Volume 2 to check that most of the material there is more or less familiar. I think §417 might be difficult to read without at least the results-only version of Chapter 25 to hand. But Volume 3, and the last three chapters of Volume 2, can probably be left on one side for the moment. Of course you will need the Lifting Theorem (Chapter 34) for §§447, 452 and 453, and Chapter 26 is essential background for Chapter 47, while Chapter 28 (on Fourier analysis) may help to make sense of Chapter 44, and parts of Chapter 27 (on probability theory) are necessary for §§455-456 and 458-459. You will certainly need some Fourier analysis for §479. And measure algebras are mentioned in every chapter except (I think) Chapter 48; but I hope that the cross-references are precise enough to lead you to what you need to know at any particular point. Even Maharam's theorem is hardly used in this volume.

What you will need, apart from any knowledge of measure theory, is a sound background in general topology. This volume calls on a great many miscellaneous facts from general topology, and the list in §4A2 is not a good place to start if continuity and compactness and the separation axioms are unfamiliar. My primary reference for topology is Engelking 89. I do not insist that you should have read this book (though of course I hope you will do so sometime); but I do think you should make sure that you can use it.

In the general introduction to this treatise, I wrote 'I make no attempt to describe the history of the subject', and I have generally been casual – some would say negligent – in my attributions of results to their discoverers. Through much of the first three volumes I did at least have the excuse that the history exists in print in far more detail than I am qualified to describe. In the present volume I find my position more uncomfortable, in that I have been watching the evolution of the subject relatively closely over the last forty years, and ought to be able to say something about it. Nevertheless I remain reluctant to make definite statements crediting one person rather than another with originating an idea. My more intimate knowledge of the topic makes me even more conscious than elsewhere of the danger of error and of the breadth of reading that would be necessary to produce a balanced account. In some cases I do attach a result to a specific published paper, but these attributions should never be regarded as an assertion that any particular author has priority; at most, they declare that a historian should examine the source cited before coming to any decision. I assure my friends and colleagues that my omissions are not intended to slight either them or those we all honour. What I have tried to do is to include in the bibliography to this volume all the published work which (as far as I am consciously aware) has influenced me while writing it, so that those who wish to go into the matter will have somewhere to start their investigations.

Note on second printing

I fear that there were even more errors, not all of them trivial, in the first printing of this volume than there were in previous volumes. I have tried to correct those which I have noticed; many surely remain. Apart from these, there are many minor expansions and elaborations, and a couple of new results, but few new ideas and no dramatic rearrangements. Details may be found in `http://www.essex.ac.uk/maths/people/fremlin/mterr4.03.pdf`.

Both printers and readers found that the 945-page format of the first printing was hard to handle. I have therefore divided the volume into two parts for the second printing. I hope you will find that the additional convenience is worth the the increase in cost.

Note on second ('Lulu') edition

I was right about many errors remaining (particularly in §458, on relative independence), and I hope I have cleared some of them out of the way. There are substantial additions in the new edition, the most important being a vastly expanded §455 on Lévy processes, an account of Brownian motion and Newtonian potential in §§477-479, and Tao's proof of Szemerédi's theorem in §497. I have included theorems of A.Törnquist and G.W.Mackey on the realization of group actions on measure algebras, some material on a version of the Kantorovich-Rubinstein distance between two measures, and a section on Maharam submeasures (§496).

Chapter 41

Topologies and Measures I

I begin this volume with an introduction to some of the most important ways in which topologies and measures can interact, and with a description of the forms which such constructions as subspaces and product spaces take in such contexts. By far the most important concept is that of Radon measure (411Hb, §416). In Radon measure spaces we find both the richest combinations of ideas and the most important applications. But, as usual, we are led both by analysis of these ideas and by other interesting examples to consider wider classes of topological measure space, and the greater part of the chapter, by volume, is taken up by a description of the many properties of Radon measures individually and in partial combinations.

I begin the chapter with a short section of definitions (§411), including a handful of more or less elementary examples. The two central properties of a Radon measure are 'inner regularity' (411B) and 'τ-additivity' (411C). The former is an idea of great versatility which I look at in an abstract setting in §412. I take a section (§413) to describe some methods of constructing measure spaces, extending the rather limited range of constructions offered in earlier volumes. There are two sections on τ-additive measures, §§414 and 417; the former covers the elementary ideas, and the latter looks at product measures, where it turns out that we need a new technique to supplement the purely measure-theoretic constructions of Chapter 25. On the way to Radon measures in §416, I pause over 'quasi-Radon' measures (411Ha, §415), where inner regularity and τ-additivity first come effectively together.

The possible interactions of a topology and a measure on the same space are so varied that even a brief account makes a long chapter; and this is with hardly any mention of results associated with particular types of topological space, most of which must wait for later chapters. But I include one section on the two most important classes of functions acting between topological measure spaces (§418), and another describing some examples to demonstrate special phenomena (§419).

411 Definitions

In something of the spirit of §211, but this time without apologising, I start this volume with a list of definitions. The rest of Chapter 41 will be devoted to discussing these definitions and relationships between them, and integrating the new ideas into the concepts and constructions of earlier volumes; I hope that by presenting the terminology now I can give you a sense of the directions the following sections will take. I ought to remark immediately that there are many cases in which the exact phrasing of a definition is important in ways which may not be immediately apparent.

411A I begin with a phrase which will be a useful shorthand for the context in which most, but not all, of the theory here will be developed.

Definition A **topological measure space** is a quadruple $(X,\mathfrak{T},\Sigma,\mu)$ where (X,Σ,μ) is a measure space and $\mathfrak{T}$ is a topology on X such that $\mathfrak{T}\subseteq\Sigma$, that is, every open set (and therefore every Borel set) is measurable.

411B Now I come to what are in my view the two most important concepts to master; jointly they will dominate the chapter.

Definition Let (X,Σ,μ) be a measure space and $\mathcal{K}$ a family of sets. I say that μ is **inner regular with respect to** $\mathcal{K}$ if

$$\mu E=\sup\{\mu K : K\in\Sigma\cap\mathcal{K},\ K\subseteq E\}$$

for every $E\in\Sigma$. (Cf. 256Ac, 342Aa.)

Remark Note that in this definition I do not assume that $\mathcal{K}\subseteq\Sigma$, nor even that $\mathcal{K}\subseteq\mathcal{P}X$. But of course μ will be inner regular with respect to $\mathcal{K}$ iff it is inner regular with respect to $\mathcal{K}\cap\Sigma$.

It is convenient in this context to interpret $\sup\emptyset$ as 0, so that we have to check the definition only when $\mu E>0$, and need not insist that $\emptyset\in\mathcal{K}$.

411C Definition Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X. I say that μ is **τ-additive** (the phrase **τ-regular** has also been used) if whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets such that $\mathcal{G}\subseteq\Sigma$ and $\bigcup\mathcal{G}\in\Sigma$ then $\mu(\bigcup\mathcal{G})=\sup_{G\in\mathcal{G}}\mu G$.

Remark Note that in this definition I do not assume that every open set is measurable. Consequently we cannot take it for granted that an extension of a τ-additive measure will be τ-additive; on the other hand, the restriction of a τ-additive measure to any σ-subalgebra will be τ-additive.

411D Complementary to 411B we have the following.

Definition Let (X, Σ, μ) be a measure space and $\mathcal{H}$ a family of subsets of X. Then μ is **outer regular with respect to** $\mathcal{H}$ if

$$\mu E = \inf\{\mu H : H \in \Sigma \cap \mathcal{H},\ H \supseteq E\}$$

for every $E \in \Sigma$.

Note that a totally finite measure on a topological space is inner regular with respect to the family of closed sets iff it is outer regular with respect to the family of open sets.

411E I delay discussion of most of the relationships between the concepts here to later in the chapter. But it will be useful to have a basic fact set out immediately.

Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X. If μ is inner regular with respect to the compact sets, it is τ-additive.

proof Let $\mathcal{G}$ be a non-empty upwards-directed family of measurable open sets such that $H = \bigcup \mathcal{G} \in \Sigma$. If $\gamma < \mu H$, there is a compact set $K \subseteq H$ such that $\mu K \ge \gamma$; now there must be a $G \in \mathcal{G}$ which includes K, so that $\mu G \ge \gamma$. As γ is arbitrary, $\sup_{G \in \mathcal{G}} \mu G = \mu H$.

411F In order to deal efficiently with measures which are not totally finite, I think we need the following ideas.

Definitions Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) I say that μ is **locally finite** if every point of X has a neighbourhood of finite measure, that is, if the open sets of finite outer measure cover X.

(b) I say that μ is **effectively locally finite** if for every non-negligible measurable set $E \subseteq X$ there is a measurable open set $G \subseteq X$ such that $\mu G < \infty$ and $E \cap G$ is not negligible.

Note that an effectively locally finite measure must measure many open sets, while a locally finite measure need not.

(c) This seems a convenient moment at which to introduce the following term. A real-valued function f defined on a subset of X is **locally integrable** if for every $x \in X$ there is an open set G containing x such that $\int_G f$ is defined (in the sense of 214D) and finite.

411G Elementary facts (a) If μ is a locally finite measure on a topological space X, then $\mu^* K < \infty$ for every compact set $K \subseteq X$. **P** The family $\mathcal{G}$ of open sets of finite outer measure is upwards-directed and covers X, so there must be some $G \in \mathcal{G}$ including K, in which case $\mu^* K \le \mu^* G$ is finite. **Q**

(b) A measure μ on $\mathbb{R}^r$ is locally finite iff every bounded set has finite outer measure (cf. 256Ab). **P** (i) If every bounded set has finite outer measure then, in particular, every open ball has finite outer measure, so that μ is locally finite. (ii) If μ is locally finite and $A \subseteq \mathbb{R}^r$ is bounded, then its closure $\overline{A}$ is compact (2A2F), so that $\mu^* A \le \mu^* \overline{A}$ is finite, by (a) above. **Q**

(c) I should perhaps remark immediately that a locally finite topological measure need not be effectively locally finite (419A), and an effectively locally finite measure need not be locally finite (411P).

(d) An effectively locally finite measure must be semi-finite.

(e) A locally finite measure on a Lindelöf space X is σ-finite. **P** Let $\mathcal{G}$ be the family of open sets of finite outer measure. Because μ is locally finite, $\mathcal{G}$ is a cover of X. Because X is Lindelöf, there is a sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{G}$ covering X. For each $n \in \mathbb{N}$, there is a measurable set $E_n \supseteq G_n$ of finite measure, and now $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence of sets of finite measure covering X. **Q**

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space such that μ is locally finite and inner regular with respect to the compact sets. Then μ is effectively locally finite. **P** Suppose that $\mu E > 0$. Then there is a measurable compact set $K \subseteq E$ such that $\mu K > 0$. As in the argument for (a) above, there is an open set G of finite measure including K, so that $\mu(E \cap G) > 0$. **Q**

(g) Corresponding to (a) above, we have the following fact. If μ is a measure on a topological space and $f \in \mathcal{L}^0(\mu)$ is locally integrable, then $\int_K f d\mu$ is finite for every compact $K \subseteq X$, because K can be covered by a finite family of open sets G such that $\int_G |f| d\mu < \infty$.

(h) If μ is a locally finite measure on a topological space X, and $f \in \mathcal{L}^p(\mu)$ for some $p \in [1, \infty]$, then f is locally integrable; this is because $\int_G |f| \le \int_E |f| \le \|f\|_p \|\chi E\|_q$ is finite whenever $G \subseteq E$ and $\mu E < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$, by Hölder's inequality (244Eb).

(i) If $(X, \mathfrak{T})$ is a completely regular space and μ is a locally finite topological measure on X, then the collection of open sets with negligible boundaries is a base for $\mathfrak{T}$. **P** If $x \in G \in \mathfrak{T}$, let $H \subseteq G$ be an open set of finite measure containing x, and $f : X \to [0, 1]$ a continuous function such that $f(x) = 1$ and $f(y) = 0$ for $y \in X \setminus H$. Then $\{f^{-1}[\{\alpha\}] : 0 < \alpha < 1\}$ is an uncountable disjoint family of measurable subsets of H, so there must be some $\alpha \in \,]0, 1[$ such that $f^{-1}[\{\alpha\}]$ is negligible. Set $U = \{y : f(y) > \alpha\}$; then U is an open neighbourhood of x included in G and its boundary $\partial U \subseteq f^{-1}[\{\alpha\}]$ is negligible. **Q**

411H Two particularly important combinations of the properties above are the following.

Definitions (a) A **quasi-Radon measure space** is a topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that (i) (X, Σ, μ) is complete and locally determined (ii) μ is τ-additive, inner regular with respect to the closed sets and effectively locally finite.

(b) A **Radon measure space** is a topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that (i) (X, Σ, μ) is complete and locally determined (ii) $\mathfrak{T}$ is Hausdorff (iii) μ is locally finite and inner regular with respect to the compact sets.

411I Remarks (a) You may like to seek your own proof that a Radon measure space is always quasi-Radon, before looking it up in §416 below.

(b) Note that a measure on Euclidean space $\mathbb{R}^r$ is a Radon measure on the definition above iff it is a Radon measure as described in 256Ad. **P** In 256Ad, I said that a measure μ on $\mathbb{R}^r$ is 'Radon' if it is a locally finite complete topological measure, inner regular with respect to the compact sets. (The definition of 'locally finite' in 256A was not the same as the one above, but I have already covered this point in 411Gb.) So the only thing to add is that μ is necessarily locally determined, because it is σ-finite (256Ba). **Q**

411J The following special types of inner regularity are of sufficient importance to have earned separate names.

Definitions (a) If $(X, \mathfrak{T})$ is a topological space, I will say that a measure μ on X is **tight** if it is inner regular with respect to the closed compact sets.

(b) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a topological measure space, I will say that μ is **completion regular** if it is inner regular with respect to the zero sets (definition: 3A3Qa).

411K Borel and Baire measures If $(X, \mathfrak{T})$ is a topological space, I will call a measure with domain (exactly) the Borel σ-algebra of X (4A3A) a **Borel measure** on X, and a measure with domain (exactly) the Baire σ-algebra of X (4A3K) a **Baire measure** on X.

Of course a Borel measure is a topological measure in the sense of 411A. On a metrizable space, the Borel and Baire measures coincide (4A3Kb). The most important measures in this chapter will be c.l.d. versions of Borel measures.

411L When we come to look at functions defined on a topological measure space, we shall have to relate ideas of continuity and measurability. Two basic concepts are the following.

Definition Let X be a set, Σ a σ-algebra of subsets of X and $(Y, \mathfrak{S})$ a topological space. I will say that a function $f : X \to Y$ is **measurable** if $f^{-1}[G] \in \Sigma$ for every open set $G \subseteq Y$.

Remarks (a) Note that a function $f : X \to \mathbb{R}$ is measurable on this definition (when $\mathbb{R}$ is given its usual topology) iff it is measurable according to the familiar definition in 121C, which asks only that sets of the form $\{x : f(x) < \alpha\}$ should be measurable (121Ef).

(b) For any topological space $(Y, \mathfrak{S})$, a function $f : X \to Y$ is measurable iff f is $(\Sigma, \mathcal{B}(Y))$-measurable, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y (4A3Cb).

411M Definition Let (X, Σ, μ) be a measure space, $\mathfrak{T}$ a topology on X, and $(Y, \mathfrak{S})$ another topological space. I will say that a function $f : X \to Y$ is **almost continuous** or **Lusin measurable** if μ is inner regular with respect to the family of subsets A of X such that $f \restriction A$ is continuous.

411N Finally, I introduce some terminology to describe ways in which (sometimes) measures can be located in one part of a topological space rather than another.

Definitions Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) I will call a set $A \subseteq X$ **self-supporting** if $\mu^*(A \cap G) > 0$ for every open set G such that $A \cap G$ is non-empty. (Such sets are sometimes called **of positive measure everywhere**.)

(b) A **support** of μ is a closed self-supporting set F such that $X \setminus F$ is negligible.

(c) Note that μ can have at most one support. **P** If F_1, F_2 are supports then $\mu^*(F_1 \setminus F_2) \le \mu^*(X \setminus F_2) = 0$ so $F_1 \setminus F_2$ must be empty. Similarly, $F_2 \setminus F_1 = \emptyset$, so $F_1 = F_2$. **Q**

(d) If μ is a τ-additive topological measure it has a support. **P** Let $\mathcal{G}$ be the family of negligible open sets, and F the closed set $X \setminus \bigcup \mathcal{G}$. Then $\mathcal{G}$ is an upwards-directed family in $\mathfrak{T} \cap \Sigma$ and $\bigcup \mathcal{G} \in \mathfrak{T} \cap \Sigma$, so

$$\mu(X \setminus F) = \mu(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}} \mu G = 0.$$

If G is open and $G \cap F \neq \emptyset$ then $G \notin \mathcal{G}$ so $\mu^*(G \cap F) = \mu(G \cap F) = \mu G > 0$; thus F is self-supporting and is the support of μ. **Q**

(e) Let X and Y be topological spaces with topological measures μ, ν respectively and a continuous inverse-measure-preserving function $f : X \to Y$. Suppose that μ has a support E. Then $\overline{f[E]}$ is the support of ν. **P** We have only to observe that for an open set $H \subseteq Y$

$$\begin{aligned} \nu H > 0 &\iff \mu f^{-1}[H] > 0 \iff f^{-1}[H] \cap E \neq \emptyset \\ &\iff H \cap f[E] \neq \emptyset \iff H \cap \overline{f[E]} \neq \emptyset. \ \mathbf{Q} \end{aligned}$$

(f) μ is **strictly positive** (with respect to $\mathfrak{T}$) if $\mu^* G > 0$ for every non-empty open set $G \subseteq X$, that is, X itself is the support of μ.

***(g)** If $(X, \mathfrak{T})$ is a topological space, and μ is a strictly positive σ-finite measure on X such that the domain Σ of μ includes a π-base $\mathcal{U}$ for $\mathfrak{T}$, then X is ccc. **P** Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence of sets of finite measure covering X. Let $\mathcal{G}$ be a disjoint family of non-empty open sets. For each $G \in G$, take $U_G \in \mathcal{U} \setminus \{\emptyset\}$ such that $U_G \subseteq G$; then $\mu U_G > 0$, so there is an $n(G)$ such that $\mu(E_{n(G)} \cap U_G) > 0$. Now $\sum_{G \in G, n(G) = k} \mu(E_k \cap U_G) \le \mu E_k$ is finite for every k, so $\{G : n(G) = k\}$ must be countable and $\mathcal{G}$ is countable. **Q**

411O Example Lebesgue measure on $\mathbb{R}^r$ is a Radon measure (256Ha); in particular, it is locally finite and tight. It is therefore τ-additive and effectively locally finite (411E, 411Gf). It is completion regular (because every compact set is a zero set, see 4A2Lc), outer regular with respect to the open sets (134Fa) and strictly positive.

411P Example: Stone spaces (a) Let $(Z, \mathfrak{T}, \Sigma, \mu)$ be the Stone space of a semi-finite measure algebra $(\mathfrak{A}, \bar{\mu})$, so that $(Z, \mathfrak{T})$ is a zero-dimensional compact Hausdorff space, (Z, Σ, μ) is complete and semi-finite, the open-and-closed sets are measurable, the negligible sets are the nowhere dense sets, and every measurable set differs by a nowhere dense set from an open-and-closed set (311I, 321K, 322Bd, 322Ra[1]).

(b) μ is inner regular with respect to the open-and-closed sets (322Ra); in particular, it is completion regular and tight. Consequently it is τ-additive (411E).

(c) μ is strictly positive, because the open-and-closed sets form a base for $\mathfrak{T}$ (311I) and a non-empty open-and-closed set has non-zero measure. μ is effectively locally finite. **P** Suppose that $E \in \Sigma$ is not negligible. There is a measurable set $F \subseteq E$ such that $0 < \mu F < \infty$; now there is a non-empty open-and-closed set G included in F, in which case $\mu G < \infty$ and $\mu(E \cap G) > 0$. **Q**

[1] Formerly 322Qa.

(d) The following are equiveridical, that is, if one is true so are the others:

(i) $(\mathfrak{A},\bar{\mu})$ is localizable;

(ii) μ is strictly localizable;

(iii) μ is locally determined;

(iv) μ is a quasi-Radon measure.

P The equivalence of (i)-(iii) is Theorem 322O[2]. (iv)$\Rightarrow$(iii) is trivial. If one, therefore all, of (i)-(iii) are true, then μ is a topological measure, because if $G \subseteq Z$ is open, then $\overline{G}$ is open-and-closed, by 314S, therefore measurable, and $\overline{G} \setminus G$ is nowhere dense, therefore also measurable. We know already that μ is complete, effectively locally finite and τ-additive, so that if it is also locally determined it is a quasi-Radon measure. **Q**

(e) The following are equiveridical:

(i) μ is a Radon measure;

(ii) μ is totally finite;

(iii) μ is locally finite;

(iv) μ is outer regular with respect to the open sets.

P (ii)$\Rightarrow$(iv) If μ is totally finite and $E \in \Sigma$, then for any $\epsilon > 0$ there is a closed set $F \subseteq Z \setminus E$ such that $\mu F \geq \mu(Z \setminus E) - \epsilon$, and now $G = Z \setminus F$ is an open set including E with $\mu G \leq \mu E + \epsilon$. (iv)$\Rightarrow$(iii) Suppose that μ is outer regular with respect to the open sets, and $z \in Z$. Because Z is Hausdorff, $\{z\}$ is closed. If it is open it is measurable, and because μ is semi-finite it must have finite measure. Otherwise it is nowhere dense, therefore negligible, and must be included in open sets of arbitrarily small measure. Thus in both cases z belongs to an open set of finite measure; as z is arbitrary, μ is locally finite. (iii)$\Rightarrow$(ii) Because Z is compact, this is a consequence of 411Ga. (i)$\Rightarrow$(iii) is part of the definition of 'Radon measure'. Finally, (ii)$\Rightarrow$(i), again directly from the definition and the facts set out in (a)-(b) above. **Q**

(f) Let $W \subseteq Z$ be the union of all the open subsets of Z with finite measure. Because μ is effectively locally finite, W has full outer measure, so $(\mathfrak{A},\bar{\mu})$ can be identified with the measure algebra of the subspace measure μ_W (322Jb). By the definition of W, μ_W is locally finite. If $(\mathfrak{A},\bar{\mu})$ is localizable, then μ_W is a Radon measure. **P** Every open subset of W belongs to Σ, by (d), and therefore to the domain of μ_W, and μ_W is a topological measure. By 214Ka, μ_W is complete and locally determined. Because μ is inner regular with respect to the compact sets, so is μ_W. **Q**

411Q Example: Dieudonné's measure Recall that a set $E \subseteq \omega_1$ is a Borel set iff either E or its complement includes a cofinal closed set (4A3J). So we may define a Borel measure μ on ω_1 by saying that $\mu E = 1$ if E includes a cofinal closed set and $\mu E = 0$ if E is disjoint from a cofinal closed set. If E is disjoint from some cofinal closed set, so is any subset of E, so μ is complete. Since μ takes only the values 0 and 1, it is a purely atomic probability measure.

μ is a topological measure; being totally finite, it is surely locally finite and effectively locally finite. It is inner regular with respect to the closed sets (because if $\mu E > 0$, there is a cofinal closed set $F \subseteq E$, and now F is a closed set with $\mu F = \mu E$), therefore outer regular with respect to the open sets. It is not τ-additive (because $\xi = [0,\xi[$ is an open set of zero measure for every $\xi < \omega_1$, and the union of these sets is a measurable open set of measure 1).

μ is not completion regular, because the set of countable limit ordinals is a closed set (4A1Bb) which does not include any uncountable zero set (see 411R below).

The only self-supporting subset of ω_1 is the empty set (because there is a cover of ω_1 by negligible open sets). In particular, μ does not have a support.

Remark There is a measure of this type on any ordinal of uncountable cofinality; see 411Xj.

411R Example: The Baire σ-algebra of ω_1 The Baire σ-algebra $\mathcal{B}\mathfrak{a}(\omega_1)$ of ω_1 is the countable-cocountable algebra (4A3P). The countable-cocountable measure μ on ω_1 is therefore a Baire measure on the definition of 411K. Since all sets of the form $]\xi,\omega_1[$ are zero sets, μ is inner regular with respect to the zero sets and outer regular with respect to the cozero sets. Since sets of the form $[0,\xi[$ $(=\xi)$ form a cover of ω_1 by measurable open sets of zero measure, μ is not τ-additive.

411X Basic exercises >(a) Let (X,Σ,μ) be a totally finite measure space and $\mathfrak{T}$ a topology on X. Show that μ is inner regular with respect to the closed sets iff it is outer regular with respect to the open sets, and is inner regular with respect to the zero sets iff it is outer regular with respect to the cozero sets.

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[2]Formerly 322N.

(b) Let μ be a Radon measure on $\mathbb{R}^r$, where $r \ge 1$, and $f \in \mathcal{L}^0(\mu)$. Show that f is locally integrable in the sense of 411Fc iff it is locally integrable in the sense of 256E, that is, $\int_E f d\nu < \infty$ for every bounded set $E \subseteq \mathbb{R}^r$.

(c) Let μ be a measure on a topological space, $\hat{\mu}$ its completion and $\tilde{\mu}$ its c.l.d. version. Show that μ is locally finite iff $\hat{\mu}$ is locally finite, and in this case $\tilde{\mu}$ is locally finite.

>(d) Let μ be an effectively locally finite measure on a topological space X. (i) Show that the completion and c.l.d. version of μ are effectively locally finite. (ii) Show that if μ is complete and locally determined, then the union of the measurable open sets of finite measure is conegligible. (iii) Show that if X is hereditarily Lindelöf then μ must be σ-finite.

(e) Let X be a topological space and μ a measure on X. Let $U \subseteq L^0(\mu)$ be the set of equivalence classes of locally integrable functions in $\mathcal{L}^0(\mu)$. Show that U is a solid linear subspace of $L^0(\mu)$. Show that if μ is locally finite then U includes $L^p(\mu)$ for every $p \in [1, \infty]$.

(f) Let X be a topological space. (i) Let μ, ν be two totally finite Borel measures which agree on the closed sets. Show that they are equal. (*Hint*: 136C.) (ii) Let μ, ν be two totally finite Baire measures which agree on the zero sets. Show that they are equal.

(g) Let $(X, \mathfrak{T})$ be a topological space, μ a measure on X, and Y a subset of X; let $\mathfrak{T}_Y$, μ_Y be the subspace topology and measure. Show that if μ is a topological measure, or locally finite, or a Borel measure, so is μ_Y.

(h) Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of measure spaces, with direct sum (X, Σ, μ); suppose that we are given a topology $\mathfrak{T}_i$ on each X_i, and let $\mathfrak{T}$ be the disjoint union topology on X (definition: 4A2A). Show that μ is a topological measure, or locally finite, or effectively locally finite, or a Borel measure, or a Baire measure, or strictly positive, iff every μ_i is.

(i) Let (X, Σ, μ) and (Y, T, ν) be two measure spaces, with c.l.d. product measure λ on $X \times Y$. Suppose we are given topologies $\mathfrak{T}$, $\mathfrak{S}$ on X, Y respectively, and give $X \times Y$ the product topology. Show that λ is locally finite, or effectively locally finite, if μ and ν are.

(j) Let α be any ordinal of uncountable cofinality with its order topology (definitions: 3A1Fb, 4A2A). Show that there is a complete topological probability measure μ on α defined by saying that $\mu E = 1$ if E includes a cofinal closed set in α, 0 if E is disjoint from some cofinal closed set. Show that μ is inner regular with respect to the closed sets but is not completion regular.

(k) Let $\langle (X_i, \mathfrak{T}_i) \rangle_{i \in I}$ be a family of topological spaces, and μ_i a strictly positive probability measure on X_i for each i. Show that the product measure on $\prod_{i \in I} X_i$ is strictly positive.

411Y Further exercises (a) Let r, $s \ge 1$ be integers. Show that a function $f : \mathbb{R}^r \to \mathbb{R}^s$ is measurable iff it is almost continuous (where $\mathbb{R}^r$ is endowed with Lebesgue measure and its usual topology, of course). (*Hint*: 256F.)

(b) Let (X, ρ) be a metric space, $r \ge 0$, and write μ_{Hr} for r-dimensional Hausdorff measure on X (264K, 471A). (i) Show that μ_{Hr} is a topological measure, outer regular with respect to the Borel sets. (ii) Show that the c.l.d. version $\tilde{\mu}_{Hr}$ of μ_{Hr} is inner regular with respect to the closed totally bounded sets. (iii) Show that $\tilde{\mu}_{Hr}$ is completion regular. (iv) Show that if X is complete then $\tilde{\mu}_{Hr}$ is tight.

(c) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space. Set $\mathcal{E} = \{E : E \subseteq X,\ \mu(\partial E) = 0\}$, where ∂E is the boundary of A. (i) Show that $\mathcal{E}$ is a subalgebra of $\mathcal{P}X$, and that every member of $\mathcal{E}$ is measured by the completion of μ. ($\mathcal{E}$ is sometimes called the **Jordan algebra** of $(X, \mathfrak{T}, \Sigma, \mu)$. Do not confuse with the 'Jordan algebras' of abstract algebra.) (ii) Suppose that μ is totally finite and inner regular with respect to the closed sets, and that $\mathfrak{T}$ is normal. Show that $\{E^\bullet : E \in \mathcal{E} \cap \Sigma\}$ is dense in the measure algebra of μ endowed with its usual topology. (iii) Suppose that μ is a quasi-Radon measure and $\mathfrak{T}$ is completely regular. Show that $\{E^\bullet : E \in \mathcal{E}\}$ is dense in the measure algebra of μ. (*Hint*: 414Aa.)

(d) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a second-countable atomless topological probability space with a strictly positive measure, $\mathcal{E}$ the Jordan algebra of μ as defined in 411Yc, $(\mathfrak{A}, \bar{\mu})$ the measure algebra of μ and $\mathfrak{C}$ the image $\{E^\bullet : E \in \mathcal{E}\} \subseteq \mathfrak{A}$. Let $\mathfrak{B}$ be a Boolean algebra and $\nu : \mathfrak{B} \to [0, 1]$ a finitely additive functional. Show that $(\mathfrak{B}, \nu) \cong (\mathfrak{C}, \bar{\mu} \restriction \mathfrak{C})$ iff (α) ν

is strictly positive and properly atomless in the sense of 326F[3], and $\nu 1 = 1$ (β) there is a countable subalgebra $\mathfrak{B}_0$ of $\mathfrak{B}$ such that $\nu b = \sup\{\nu c : c \in \mathfrak{B}_0,\ c \subseteq b\}$ for every $b \in \mathfrak{B}$ (γ) whenever $A,\ B \subseteq \mathfrak{B}$ are upwards-directed sets such that $a \cap b = 0$ for every $a \in A$ and $b \in B$ and $\sup\{\nu(a \cup b) : a \in A,\ b \in B\} = 1$, then $\sup A$ is defined in $\mathfrak{B}$.

411 Notes and comments Of course the list above can give only a rough idea of the ways in which topologies and measures can interact. In particular I have rather arbitrarily given a sort of priority to three particular relationships between the domain Σ of a measure and the topology: 'topological measure space' (in which Σ includes the Borel σ-algebra), 'Borel measure' (in which Σ is precisely the Borel σ-algebra) and 'Baire measure' (in which Σ is the Baire σ-algebra).

Abstract topological measure theory is a relatively new subject, and there are many technical questions on which different authors take different views. For instance, the phrase 'Radon measure' is commonly used to mean what I would call a 'tight locally finite Borel measure' (cf. 416F); and some writers enlarge the definition of 'topological measure' to include Baire measures as defined above.

I give very few examples at this stage, two drawn from the constructions of Volumes 1-3 (Lebesgue measure and Stone spaces, 411O-411P) and one new one ('Dieudonné's measure', 411Q), with a glance at the countable-cocountable measure of ω_1 (411R). The most glaring omission is that of the product measures on $\{0,1\}^I$ and $[0,1]^I$. I pass these by at the moment because a proper study of them requires rather more preparation than can be slipped into a parenthesis. (I return to them in 416U.) I have also omitted any discussion of 'measurable' and 'almost continuous' functions, except for a reference to a theorem in Volume 2 (411Ya), which will have to be repeated and amplified later on (418K). There is an obvious complementarity between the notions of 'inner' and 'outer' regularity (411B, 411D), but it works well only for totally finite spaces (411Xa); in other cases it may not be obvious what will happen (411O, 411Pe, 412W).

412 Inner regularity

As will become apparent as the chapter progresses, the concepts introduced in §411 are synergic; their most interesting manifestations are in combinations of various kinds. Any linear account of their properties will be more than usually like a space-filling curve. But I have to start somewhere, and enough results can be expressed in terms of inner regularity, more or less by itself, to be a useful beginning.

After a handful of elementary basic facts (412A) and a list of standard applications (412B), I give some useful sufficient conditions for inner regularity of topological and Baire measures (412D, 412E, 412G), based on an important general construction (412C). The rest of the section amounts to a review of ideas from Volume 2 and Chapter 32 in the light of the new concept here. I touch on completions (412H), c.l.d. versions and complete locally determined spaces (412H, 412J, 412L), strictly localizable spaces (412I), inverse-measure-preserving functions (412K, 412M), measure algebras (412N), subspaces (412O, 412P), indefinite-integral measures (412Q) and product measures (412R-412V), with a brief mention of outer regularity (412W); most of the hard work has already been done in Chapters 21 and 25.

412A I begin by repeating a lemma from Chapter 34, with some further straightforward facts.

Lemma (a) Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of sets such that

whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$.

Then whenever $E \in \Sigma$ there is a countable disjoint family $\langle K_i \rangle_{i \in I}$ in $\mathcal{K} \cap \Sigma$ such that $K_i \subseteq E$ for every i and $\sum_{i \in I} \mu K_i = \mu E$. If moreover

(†) $K \cup K' \in \mathcal{K}$ whenever $K,\ K'$ are disjoint members of $\mathcal{K}$,

then μ is inner regular with respect to $\mathcal{K}$. If $\bigcup_{i \in I} K_i \in \mathcal{K}$ for every countable disjoint family $\langle K_i \rangle_{i \in I}$ in $\mathcal{K}$, then for every $E \in \Sigma$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K = \mu E$.

(b) Let (X, Σ, μ) be a measure space, T a σ-subalgebra of Σ, and $\mathcal{K}$ a family of sets. If μ is inner regular with respect to T and $\mu \restriction \mathrm{T}$ is inner regular with respect to $\mathcal{K}$, then μ is inner regular with respect to $\mathcal{K}$.

(c) Let (X, Σ, μ) be a semi-finite measure space and $\langle \mathcal{K}_n \rangle_{n \in \mathbb{N}}$ a sequence of families of sets such that μ is inner regular with respect to $\mathcal{K}_n$ and

(‡) if $\langle K_i \rangle_{i \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}_n$, then $\bigcap_{i \in \mathbb{N}} K_i \in \mathcal{K}_n$

for every $n \in \mathbb{N}$. Then μ is inner regular with respect to $\bigcap_{n \in \mathbb{N}} \mathcal{K}_n$.

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[3]Formerly 326Ya.

proof (a) This is 342B-342C.

(b) If $E \in \Sigma$ and $\gamma < \mu E$, there are an $F \in \mathrm{T}$ such that $F \subseteq E$ and $\mu F > \gamma$, and a $K \in \mathcal{K} \cap \mathrm{T}$ such that $K \subseteq F$ and $\mu K \geq \gamma$.

(c) Suppose that $E \in \Sigma$ and that $0 \leq \gamma < \mu E$. Because μ is semi-finite, there is an $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$ (213A). Choose $\langle K_i \rangle_{i \in \mathbb{N}}$ inductively, as follows. Start with $K_0 = F$. Given that $K_i \in \Sigma$ and $\gamma < \mu K_i$, then let $n_i \in \mathbb{N}$ be such that $2^{-n_i}(i+1)$ is an odd integer, and choose $K_{i+1} \in \mathcal{K}_{n_i}$ such that $K_{i+1} \subseteq K_i$ and $\mu K_{i+1} > \gamma$; this will be possible because μ is inner regular with respect to $\mathcal{K}_{n_i}$. Consider $K = \bigcap_{i \in \mathbb{N}} K_i$. Then $K \subseteq E$ and $\mu K = \lim_{i \to \infty} \mu K_i \geq \gamma$. But also

$$K = \bigcap_{j \in \mathbb{N}} K_{2^n(2j+1)} \in \mathcal{K}_n$$

because $\langle K_{2^n(2j+1)} \rangle_{j \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}_n$, for each n. So $K \in \bigcap_{n \in \mathbb{N}} \mathcal{K}_n$. As E and γ are arbitrary, μ is inner regular with respect to $\bigcap_{n \in \mathbb{N}} \mathcal{K}_n$.

412B Corollary Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X. Suppose that $\mathcal{K}$ is

either the family of Borel subsets of X

or the family of closed subsets of X

or the family of compact subsets of X

or the family of zero sets in X,

and suppose that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$. Then μ is inner regular with respect to $\mathcal{K}$.

proof In every case, $\mathcal{K}$ satisfies the condition (†) of 412Aa.

412C The next lemma provides a particularly useful method of proving that measures are inner regular with respect to 'well-behaved' families of sets.

Lemma Let (X, Σ, μ) be a semi-finite measure space, and suppose that $\mathcal{A} \subseteq \Sigma$ is such that

$\emptyset \in \mathcal{A} \subseteq \Sigma$,

$X \setminus A \in \mathcal{A}$ for every $A \in \mathcal{A}$.

Let T be the σ-subalgebra of Σ generated by $\mathcal{A}$. Let $\mathcal{K}$ be a family of subsets of X such that

(†) $K \cup K' \in \mathcal{K}$ whenever $K, K' \in \mathcal{K}$,

(‡) $\bigcap_{n \in \mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}$,

whenever $A \in \mathcal{A}$, $F \in \Sigma$ and $\mu(A \cap F) > 0$, there is a $K \in \mathcal{K} \cap \mathrm{T}$ such that $K \subseteq A$ and $\mu(K \cap F) > 0$.

Then $\mu \restriction \mathrm{T}$ is inner regular with respect to $\mathcal{K}$.

proof (a) Write $\mathfrak{A}$ for the measure algebra of (X, Σ, μ), and $\mathcal{L} = \mathcal{K} \cap \mathrm{T}$, so that $\mathcal{L}$ also is closed under finite unions and countable intersections. Set

$$\mathcal{H} = \{E : E \in \Sigma,\ \sup_{L \in \mathcal{L}, L \subseteq E} L^\bullet = E^\bullet\} \text{ in } \mathfrak{A},$$

$$\mathrm{T}' = \{E : E \in \mathcal{H},\ X \setminus E \in \mathcal{H}\},$$

so that the last two conditions tell us that $\mathcal{A} \subseteq \mathrm{T}'$.

(b) The intersection of any sequence in $\mathcal{H}$ belongs to $\mathcal{H}$. **P** Let $\langle H_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{H}$ with intersection H. Write A_n for $\{L^\bullet : L \in \mathcal{L},\ L \subseteq H_n\} \subseteq \mathfrak{A}$ for each $n \in \mathbb{N}$. Since μ is semi-finite, $\mathfrak{A}$ is weakly (σ, ∞)-distributive (322F). As A_n is upwards-directed and $\sup A_n = H_n^\bullet$ for each $n \in \mathbb{N}$,

$$H^\bullet = \inf_{n \in \mathbb{N}} H_n^\bullet$$

(because $F \mapsto F^\bullet : \Sigma \to \mathfrak{A}$ is sequentially order-continuous, by 321H)

$$= \inf_{n \in \mathbb{N}} \sup A_n = \sup\{\inf_{n \in \mathbb{N}} a_n : a_n \in A_n \text{ for every } n \in \mathbb{N}\}$$

(316H(iv))

$$= \sup\{(\bigcap_{n\in\mathbb{N}} L_n)^\bullet : L_n \in \mathcal{L},\ L_n \subseteq H_n \text{ for every } n \in \mathbb{N}\}$$

$$\subseteq \{L^\bullet : L \in \mathcal{L},\ L \subseteq H\}$$

(by (‡))

$$\subseteq H^\bullet,$$

and $H \in \mathcal{H}$. **Q**

(c) The union of any sequence in $\mathcal{H}$ belongs to $\mathcal{H}$. **P** If $\langle H_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ with union H then

$$\sup_{L\in\mathcal{L},L\subseteq H} L^\bullet \supseteq \sup_{n\in\mathbb{N}} \sup_{L\in\mathcal{L},L\subseteq E_n} L^\bullet = \sup_{n\in\mathbb{N}} H_n^\bullet = H^\bullet,$$

so $H \in \mathcal{H}$. **Q**

(d) T' is a σ-subalgebra of Σ. **P** (i) $\emptyset$ and X belong to $\mathcal{A} \subseteq \mathcal{H}$, so $\emptyset \in \mathrm{T}'$. (ii) Obviously $X \setminus E \in \mathrm{T}'$ whenever $E \in \mathrm{T}'$. (iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in T' with union E then $E \in \mathcal{H}$, by (c); but also $X \setminus E = \bigcap_{n\in\mathbb{N}}(X \setminus E_n)$ belongs to $\mathcal{H}$, by (b). So $E \in \mathrm{T}'$. **Q**

(e) Accordingly $\mathrm{T} \subseteq \mathrm{T}'$, and $E^\bullet = \sup_{L\in\mathcal{L},L\subseteq E} L^\bullet$ for every $E \in \mathrm{T}$. It follows at once that if $E \in \mathrm{T}$ and $\mu E > 0$, there must be an $L \in \mathcal{L}$ such that $L \subseteq E$ and $\mu L > 0$; since (†) is true, and $\mathcal{L} \subseteq \mathrm{T}$, we can apply 412Aa to see that $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{L}$, therefore with respect to $\mathcal{K}$.

412D As corollaries of the last lemma I give two-and-a-half basic theorems.

Theorem Let $(X,\mathfrak{T})$ be a topological space and μ a semi-finite Baire measure on X. Then μ is inner regular with respect to the zero sets.

proof Write Σ for the Baire σ-algebra of X, the domain of μ, $\mathcal{K}$ for the family of zero sets, and $\mathcal{A}$ for $\mathcal{K} \cup \{X \setminus K : K \in \mathcal{K}\}$. Since the union of two zero sets is a zero set (4A2C(b-ii)), the intersection of a sequence of zero sets is a zero set (4A2C(b-iii)), and the complement of a zero set is the union of a sequence of zero sets (4A2C(b-vi)), the conditions of 412C are satisfied; and as the σ-algebra generated by $\mathcal{A}$ is just Σ, μ is inner regular with respect to $\mathcal{K}$.

412E Theorem Let $(X,\mathfrak{T})$ be a perfectly normal topological space (e.g., any metrizable space). Then any semi-finite Borel measure on X is inner regular with respect to the closed sets.

proof Because the Baire and Borel σ-algebras are the same (4A3Kb), this is a special case of 412D.

412F Lemma Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X such that μ is effectively locally finite with respect to $\mathfrak{T}$. Then

$$\mu E = \sup\{\mu(E \cap G) : G \text{ is a measurable open set of finite measure}\}$$

for every $E \in \Sigma$.

proof Apply 412Aa with $\mathcal{K}$ the family of subsets of measurable open sets of finite measure.

412G Theorem Let (X,Σ,μ) be a measure space with a topology $\mathfrak{T}$ such that μ is effectively locally finite with respect to $\mathfrak{T}$ and Σ is the σ-algebra generated by $\mathfrak{T} \cap \Sigma$. If

$$\mu G = \sup\{\mu F : F \in \Sigma \text{ is closed},\ F \subseteq G\}$$

for every measurable open set G of finite measure, then μ is inner regular with respect to the closed sets.

proof In 412C, take $\mathcal{K}$ to be the family of measurable closed subsets of X, and $\mathcal{A}$ to be the family of measurable sets which are *either* open *or* closed. If $G \in \Sigma\cap\mathfrak{T}$, $F \in \Sigma$ and $\mu(G\cap F) > 0$, then there is an open set H of finite measure such that $\mu(H \cap G \cap F) > 0$, because μ is effectively locally finite; now there is a $K \in \mathcal{K}$ such that $K \subseteq H \cap G$ and $\mu K > \mu(H \cap G) - \mu(H \cap G \cap F)$, so that $\mu(K \cap F) > 0$. This is the only non-trivial item in the list of hypotheses in 412C, so we can conclude that $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{K}$, where T is the σ-algebra generated by $\mathcal{A}$; but of course this is just Σ.

Remark There is a similar result in 416F(iii) below.

412H Proposition Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of sets.

(a) If μ is inner regular with respect to $\mathcal{K}$, so are its completion $\hat{\mu}$ (212C) and c.l.d. version $\tilde{\mu}$ (213E).

(b) Now suppose that

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$.

If *either* $\hat{\mu}$ is inner regular with respect to $\mathcal{K}$ *or* μ is semi-finite and $\tilde{\mu}$ is inner regular with respect to $\mathcal{K}$, then μ is inner regular with respect to $\mathcal{K}$.

proof (a) If F belongs to the domain of $\hat{\mu}$, then there is an $E \in \Sigma$ such that $E \subseteq F$ and $\hat{\mu}(F \setminus E) = 0$. So if $0 \le \gamma < \hat{\mu}F = \mu E$, there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E \subseteq F$ and $\hat{\mu}K = \mu K \ge \gamma$.

If H belongs to the domain of $\tilde{\mu}$ and $0 \le \gamma < \tilde{\mu}H$, there is an $E \in \Sigma$ such that $\mu E < \infty$ and $\hat{\mu}(E \cap H) > \gamma$ (213D). Now there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E \cap H$ and $\mu K \ge \gamma$. As $\mu K < \infty$, $\tilde{\mu}K = \mu K \ge \gamma$.

(b) Write $\check{\mu}$ for whichever of $\hat{\mu}$, $\tilde{\mu}$ is supposed to be inner regular with respect to $\mathcal{K}$. Then $\check{\mu}$ is inner regular with respect to Σ (212Ca, 213Fc), so is inner regular with respect to $\mathcal{K} \cap \Sigma$ (412Ab). Also $\check{\mu}$ extends μ (212D, 213Hc). Take $E \in \Sigma$ and $\gamma < \mu E = \check{\mu}E$. Then there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\gamma < \check{\mu}K = \mu K$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$.

412I Lemma Let (X, Σ, μ) be a strictly localizable measure space and $\mathcal{K}$ a family of sets such that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$.

(a) There is a decomposition $\langle X_i\rangle_{i\in I}$ of X such that at most one X_i does not belong to $\mathcal{K}$, and that exceptional one, if any, is negligible.

(b) There is a disjoint family $\mathcal{L} \subseteq \mathcal{K} \cap \Sigma$ such that $\mu^*A = \sum_{L\in\mathcal{L}} \mu^*(A \cap L)$ for every $A \subseteq X$.

(c) If μ is σ-finite then the family $\langle X_i\rangle_{i\in I}$ of (a) and the set $\mathcal{L}$ of (b) can be taken to be countable.

proof (a) Let $\langle E_j\rangle_{j\in J}$ be any decomposition of X. For each $j \in J$, let $\mathcal{K}_j$ be a maximal disjoint subset of

$$\{K : K \in \mathcal{K} \cap \Sigma,\ K \subseteq E_j,\ \mu K > 0\}.$$

Because $\mu E_j < \infty$, $\mathcal{K}_j$ must be countable. Set $E'_j = E_j \setminus \bigcup \mathcal{K}_j$. By the maximality of $\mathcal{K}_j$, E'_j cannot include any non-negligible set in $\mathcal{K} \cap \Sigma$; but this means that $\mu E'_j = 0$. Set $X' = \bigcup_{j\in J} E'_j$. Then

$$\mu X' = \sum_{j\in J} \mu(X' \cap E_j) = \sum_{j\in J} \mu E'_j = 0.$$

Note that if j, $j' \in J$ are distinct, $K \in \mathcal{K}_j$ and $K' \in \mathcal{K}_{j'}$, then $K \cap K' = \emptyset$; thus $\mathcal{L} = \bigcup_{j\in J} \mathcal{K}_j$ is disjoint. Let $\langle X_i\rangle_{i\in I}$ be any indexing of $\{X'\} \cup \mathcal{L}$. This is a partition (that is, disjoint cover) of X into sets of finite measure. If $E \subseteq X$ and $E \cap X_i \in \Sigma$ for every $i \in I$, then for every $j \in J$

$$E \cap E_j = (E \cap X' \cap E_j) \cup \bigcup_{K\in\mathcal{K}_j} E \cap K$$

belongs to Σ, so that $E \in \Sigma$ and

$$\mu E = \sum_{j\in J} \mu(E \cap E_j) = \sum_{j\in J} \sum_{K\in\mathcal{K}_j} \mu(E \cap K) = \sum_{i\in I} \mu(E \cap X_i).$$

Thus $\langle X_i\rangle_{i\in I}$ is a decomposition of X, and it is of the right type because every X_i but one belongs to $\mathcal{L} \subseteq \mathcal{K}$.

(b) If now $A \subseteq X$ is any set,

$$\mu^*A = \mu_A A = \sum_{i\in I} \mu_A(A \cap X_i) = \sum_{i\in I} \mu^*(A \cap X_i)$$

by 214Ia, writing μ_A for the subspace measure on A. So we have

$$\mu^*A = \mu^*(A \cap X') + \sum_{L\in\mathcal{L}} \mu^*(A \cap L) = \sum_{L\in\mathcal{L}} \mu^*(A \cap L),$$

while $\mathcal{L} \subseteq \mathcal{K}$ is disjoint.

(c) If μ is σ-finite we can take J to be countable, so that I and $\mathcal{L}$ will also be countable.

412J Proposition Let (X, Σ, μ) be a complete locally determined measure space, and $\mathcal{K}$ a family of sets such that μ is inner regular with respect to $\mathcal{K}$.

(a) If $E \subseteq X$ is such that $E \cap K \in \Sigma$ for every $K \in \mathcal{K} \cap \Sigma$, then $E \in \Sigma$.

(b) If $E \subseteq X$ is such that $E \cap K$ is negligible for every $K \in \mathcal{K} \cap \Sigma$, then E is negligible.

(c) For any $A \subseteq X$, $\mu^*A = \sup_{K\in\mathcal{K}\cap\Sigma} \mu^*(A \cap K)$.

(d) Let f be a non-negative $[0, \infty]$-valued function defined on a subset of X. If $\int_K f$ is defined in $[0, \infty]$ for every $K \in \mathcal{K}$, then $\int f$ is defined and equal to $\sup_{K\in\mathcal{K}} \int_K f$.

(e) If f is a μ-integrable function and $\epsilon > 0$, there is a $K \in \mathcal{K}$ such that $\int_{X\setminus K} |f| \le \epsilon$.

Remark In (c), we must interpret $\sup \emptyset$ as 0 if $\mathcal{K} \cap \Sigma = \emptyset$.

proof (a) If $F \in \Sigma$ and $\mu F < \infty$, then $E \cap F \in \Sigma$. **P** If $\mu F = 0$, this is trivial, because μ is complete and $E \cap F$ is negligible. Otherwise, there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K} \cap \Sigma$ such that $K_n \subseteq F$ for each n and $\sup_{n\in\mathbb{N}} \mu K_n = \mu F$. Now $E \cap F \setminus \bigcup_{n\in\mathbb{N}} K_n$ is negligible, therefore measurable, while $E \cap K_n$ is measurable for every $n \in \mathbb{N}$, by hypothesis; so $E \cap F$ is measurable. **Q** As μ is locally determined, $E \in \Sigma$, as claimed.

(b) By (a), $E \in \Sigma$; and because μ is inner regular with respect to $\mathcal{K}$, μE must be 0.

(c) Let μ_A be the subspace measure on A. Because μ is complete and locally determined, μ_A is semi-finite (214Id). So if $0 \le \gamma < \mu^* A = \mu_A A$, there is an $H \subseteq A$ such that $\mu_A H$ is defined, finite and greater than γ. Let $E \in \Sigma$ be a measurable envelope of H (132Ee), so that $\mu E = \mu^* H > \gamma$. Then there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K \ge \gamma$. In this case

$$\mu^*(A \cap K) \ge \mu^*(H \cap K) = \mu(E \cap K) = \mu K \ge \gamma.$$

As γ is arbitrary,

$$\mu^* A \le \sup_{K\in\mathcal{K}\cap\Sigma} \mu^*(A \cap K);$$

but the reverse inequality is trivial, so we have the result.

(d) Applying (b) with $E = X \setminus \operatorname{dom} f$, we see that f is defined almost everywhere in X. Applying (a) with $E = \{x : x \in \operatorname{dom} f,\ f(x) \ge \alpha\}$ for each $\alpha \in \mathbb{R}$, we see that f is measurable. So $\int f$ is defined in $[0, \infty]$, and of course $\int f \ge \sup_{K\in\mathcal{K}} \int_K f$. If $\gamma < \int f$, there is a non-negative simple function g such that $g \le_{\text{a.e.}} f$ and $\int g > \gamma$; taking $E = \{x : g(x) > 0\}$, there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu(E \setminus K)\|g\|_\infty \le \int g - \gamma$, so that $\int_K f \ge \int_K g \ge \gamma$. As γ is arbitrary, $\int f = \sup_{K\in\mathcal{K}} \int_K f$.

(e) By (d), there is a $K \in \mathcal{K}$ such that $\int_K |f| \ge \int |f| - \epsilon$.

Remark See also 413F below.

412K Proposition Let (X, Σ, μ) be a complete locally determined measure space, (Y, T, ν) a measure space and $f : X \to Y$ a function. Suppose that $\mathcal{K} \subseteq \mathrm{T}$ is such that

(i) ν is inner regular with respect to $\mathcal{K}$;

(ii) $f^{-1}[K] \in \Sigma$ and $\mu f^{-1}[K] = \nu K$ for every $K \in \mathcal{K}$;

(iii) whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $\nu K < \infty$ and $\mu(E \cap f^{-1}[K]) > 0$.

Then f is inverse-measure-preserving for μ and ν.

proof (a) If $F \in \mathrm{T}$, $E \in \Sigma$ and $\mu E < \infty$, then $E \cap f^{-1}[F] \in \Sigma$. **P** Let H_1, $H_2 \in \Sigma$ be measurable envelopes for $E \cap f^{-1}[F]$ and $E \setminus f^{-1}[F]$ respectively. **?** If $\mu(H_1 \cap H_2) > 0$, there is a $K \in \mathcal{K}$ such that νK is finite and $\mu(H_1 \cap H_2 \cap f^{-1}[K]) > 0$. Because ν is inner regular with respect to $\mathcal{K}$, there are K_1, $K_2 \in \mathcal{K}$ such that $K_1 \subseteq K \cap F$, $K_2 \subseteq K \setminus F$ and

$$\begin{aligned}\nu K_1 + \nu K_2 &> \nu(K \cap F) + \nu(K \setminus F) - \mu(H_1 \cap H_2 \cap f^{-1}[K])\\ &= \nu K - \mu(H_1 \cap H_2 \cap f^{-1}[K]).\end{aligned}$$

Now

$$\mu(H_1 \cap f^{-1}[K_2]) = \mu^*(E \cap f^{-1}[F] \cap f^{-1}[K_2]) = 0,$$

$$\mu(H_2 \cap f^{-1}[K_1]) = \mu^*(E \cap f^{-1}[K_1] \setminus f^{-1}[F]) = 0,$$

so $\mu(H_1 \cap H_2 \cap f^{-1}[K_1 \cup K_2]) = 0$ and

$$\begin{aligned}\mu(H_1 \cap H_2 \cap f^{-1}[K]) &\le \mu(f^{-1}[K] \setminus f^{-1}[K_1 \cup K_2])\\ &= \mu f^{-1}[K] - \mu f^{-1}[K_1] - \mu f^{-1}[K_2]\\ &= \nu K - \nu K_1 - \nu K_2 < \mu(H_1 \cap H_2 \cap f^{-1}[K]),\end{aligned}$$

which is absurd. **X**

Now $(E\cap H_1)\setminus(E\cap f^{-1}[F])\subseteq H_1\cap H_2$ is negligible, therefore measurable (because μ is complete), and $E\cap f^{-1}[F]\in\Sigma$, as claimed. **Q**

(b) It follows (because μ is locally determined) that $f^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$.

(c) If $F\in\mathrm{T}$ and $\nu F=0$ then $\mu f^{-1}[F]=0$. **P?** Otherwise, there is a $K\in\mathcal{K}$ such that $\nu K<\infty$ and

$$0<\mu(f^{-1}[F]\cap f^{-1}[K])=\mu f^{-1}[F\cap K].$$

Let $K'\in\mathcal{K}$ be such that $K'\subseteq K\setminus F$ and $\nu K'>\nu K-\mu f^{-1}[F\cap K]$. Then $f^{-1}[K']\cap f^{-1}[F\cap K]=\emptyset$, so

$$\nu K=\mu f^{-1}[K]\ge\mu f^{-1}[K']+\mu f^{-1}[F\cap K]>\nu K'+\nu K-\nu K'=\nu K,$$

which is absurd. **XQ**

(d) Finally, $\mu f^{-1}[F]=\nu F$ for every $F\in\mathrm{T}$. **P** Let $\langle K_i\rangle_{i\in I}$ be a countable disjoint family in $\mathcal{K}$ such that $K_i\subseteq F$ for every i and $\sum_{i\in I}\nu K_i=\nu F$ (412Aa). Set $F'=F\setminus\bigcup_{i\in I}K_i$. Then

$$\mu f^{-1}[F]=\mu f^{-1}[F']+\textstyle\sum_{i\in I}\mu f^{-1}[K_i]=\mu f^{-1}[F']+\sum_{i\in I}\nu K_i=\mu f^{-1}[F']+\nu F.$$

If $\nu F=\infty$ then surely $\mu f^{-1}[F]=\infty=\nu F$. Otherwise, $\nu F'=0$ so $\mu f^{-1}[F']=0$ (by (c)) and again $\mu f^{-1}[F]=\nu F$. **Q**

Thus f is inverse-measure-preserving.

412L Corollary Let X be a set and $\mathcal{K}$ a family of subsets of X. Suppose that μ and ν are two complete locally determined measures on X, with domains including $\mathcal{K}$, agreeing on $\mathcal{K}$, and both inner regular with respect to $\mathcal{K}$. Then they are identical (and, in particular, have the same domain).

proof Apply 412K with $X=Y$ and f the identity function to see that μ extends ν; similarly, ν extends μ and the two measures are the same.

412M Corollary Let (X,Σ,μ) be a complete probability space, (Y,T,ν) a probability space and $f:X\to Y$ a function. Suppose that whenever $F\in\mathrm{T}$ and $\nu F>0$ there is a $K\in\mathrm{T}$ such that $K\subseteq F$, $\nu K>0$, $f^{-1}[K]\in\Sigma$ and $\mu f^{-1}[K]\ge\nu K$. Then f is inverse-measure-preserving.

proof Set $\mathcal{K}^*=\{K:K\in\mathrm{T},\,f^{-1}[K]\in\Sigma,\,\mu f^{-1}[K]\ge\nu K\}$. Then $\mathcal{K}^*$ is closed under countable disjoint unions and includes $\mathcal{K}$, so for every $F\in\mathrm{T}$ there is a $K\in\mathcal{K}^*$ such that $K\subseteq F$ and $\nu K=\nu F$, by 412Aa. But this means that $\mu f^{-1}[K]=\nu K$ for every $K\in\mathcal{K}^*$. **P** There is a $K'\in\mathcal{K}^*$ such that $K'\subseteq Y\setminus K$ and $\nu K'=1-\nu K$; but in this case

$$\mu f^{-1}[K']+\mu f^{-1}[K]\le 1=\nu K'+\nu K,$$

so $\mu f^{-1}[K]$ must be equal to νK. **Q** Moreover, there is a $K^*\in\mathcal{K}^*$ such that $\nu K^*=\nu Y=1$, so $\mu f^{-1}[K^*]=\mu X=1$ and $\mu(E\cap f^{-1}[K^*])>0$ whenever $\mu E>0$. Applying 412K to $\mathcal{K}^*$ we have the result.

412N Lemma Let (X,Σ,μ) be a measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$. Then

$$E^\bullet=\sup\{K^\bullet:K\in\mathcal{K}\cap\Sigma,\,K\subseteq E\}$$

in the measure algebra $\mathfrak{A}$ of μ, for every $E\in\Sigma$. In particular, $\{K^\bullet:K\in\mathcal{K}\cap\Sigma\}$ is order-dense in $\mathfrak{A}$; and if $\mathcal{K}$ is closed under finite unions, then $\{K^\bullet:K\in\mathcal{K}\cap\Sigma\}$ is topologically dense in $\mathfrak{A}$ for the measure-algebra topology.

proof ? If $E^\bullet\ne\sup\{K^\bullet:K\in\mathcal{K}\cap\Sigma,\,K\subseteq E\}$, there is a non-zero $a\in\mathfrak{A}$ such that $a\subseteq E^\bullet\setminus K^\bullet$ whenever $K\in\mathcal{K}\cap\Sigma$ and $K\subseteq E$. Express a as $F^\bullet$ where $F\subseteq E$. Then $\mu F>0$, so there is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq F$ and $\mu K>0$. But in this case $0\ne K^\bullet\subseteq a$, while $K\subseteq E$. **X**

It follows at once that $D=\{K^\bullet:K\in\mathcal{K}\cap\Sigma\}$ is order-dense. If $\mathcal{K}$ is closed under finite unions, and $a\in\mathfrak{A}$, then $D_a=\{d:d\in D,\,d\subseteq a\}$ is upwards-directed and has supremum a, so $a\in\overline{D}_a\subseteq\overline{D}$ (323D(a-ii)).

412O Lemma Let (X,Σ,μ) be a measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$.

(a) If $E\in\Sigma$, then the subspace measure μ_E (131B) is inner regular with respect to $\mathcal{K}$.

(b) Let $Y\subseteq X$ be any set such that the subspace measure μ_Y (214B) is semi-finite. Then μ_Y is inner regular with respect to $\mathcal{K}_Y=\{K\cap Y:K\in\mathcal{K}\}$.

proof (a) This is elementary.

(b) Suppose that F belongs to the domain Σ_Y of μ_Y and $0 \le \gamma < \mu_Y F$. Because μ_Y is semi-finite there is an $F' \in \Sigma_Y$ such that $F' \subseteq F$ and $\gamma < \mu_Y F' < \infty$. Let $G \in \Sigma$ be such that $F' = G \cap Y$, and let $E \subseteq G$ be a measurable envelope for F' with respect to μ, so that

$$\mu E = \mu^* F' = \mu_Y F' > \gamma.$$

There is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K \ge \gamma$, in which case $K \cap Y \in \mathcal{K}_Y \cap \Sigma_Y$ and

$$\mu_Y(K \cap Y) = \mu^*(K \cap Y) = \mu^*(K \cap F') = \mu(K \cap E) = \mu K \ge \gamma.$$

As F and γ are arbitrary, μ_Y is inner regular with respect to $\mathcal{K}_Y$.

Remark Recall from 214Ic that if (X, Σ, μ) has locally determined negligible sets (in particular, is either strictly localizable or complete and locally determined), then all its subspaces are semi-finite.

412P Proposition Let (X, Σ, μ) be a measure space, $\mathfrak{T}$ a topology on X and Y a subset of X; write $\mathfrak{T}_Y$ for the subspace topology of Y and μ_Y for the subspace measure on Y. Suppose that *either* $Y \in \Sigma$ *or* μ_Y is semi-finite.

(a) If μ is a topological measure, so is μ_Y.
(b) If μ is inner regular with respect to the Borel sets, so is μ_Y.
(c) If μ is inner regular with respect to the closed sets, so is μ_Y.
(d) If μ is inner regular with respect to the zero sets, so is μ_Y.
(e) If μ is effectively locally finite, so is μ_Y.

proof (a) is an immediate consequence of the definitions of 'subspace measure', 'subspace topology' and 'topological measure'. The other parts follow directly from 412O if we recall that

(i) a subset of Y is Borel for $\mathfrak{T}_Y$ whenever it is expressible as $Y \cap E$ for some Borel set $E \subseteq X$ (4A3Ca);
(ii) a subset of Y is closed in Y whenever it is expressible as $Y \cap F$ for some closed set $F \subseteq X$;
(iii) a subset of Y is a zero set in Y whenever it is expressible as $Y \cap F$ for some zero set $F \subseteq X$ (4A2C(b-v));
(iv) μ is effectively locally finite iff it is inner regular with respect to subsets of open sets of finite measure.

412Q Proposition Let (X, Σ, μ) be a measure space, and ν an indefinite-integral measure over μ (definition: 234J[4]). If μ is inner regular with respect to a family $\mathcal{K}$ of sets, so is ν.

proof Because μ and its completion $\hat{\mu}$ give the same integrals, ν is an indefinite-integral measure over $\hat{\mu}$ (234Ke); and as $\hat{\mu}$ is still inner regular with respect to $\mathcal{K}$ (412Ha), we may suppose that μ itself is complete. Let f be a Radon-Nikodým derivative of ν with respect to μ; by 234Ka[5], we may suppose that $f : X \to [0, \infty[$ is Σ-measurable.

Suppose that $F \in \operatorname{dom} \nu$ and that $\gamma < \nu F$. Set $G = \{x : f(x) > 0\}$, so that $F \cap G \in \Sigma$ (234La[6]). For $n \in \mathbb{N}$, set $H_n = \{x : x \in F,\ 2^{-n} \le f(x) \le 2^n\}$, so that $H_n \in \Sigma$ and

$$\nu F = \int f \times \chi F d\mu = \lim_{n\to\infty} \int f \times \chi H_n d\mu.$$

Let $n \in \mathbb{N}$ be such that $\int f \times \chi H_n d\mu > \gamma$.

If $\mu H_n = \infty$, there is a $K \in \mathcal{K}$ such that $K \subseteq H_n$ and $\mu K \ge 2^n \gamma$, so that $\nu K \ge \gamma$. If μH_n is finite, there is a $K \in \mathcal{K}$ such that $2^n(\mu H_n - \mu K) \le \int f \times \chi H_n d\mu - \gamma$, so that $\int f \times \chi(H_n \setminus K) d\mu + \gamma \le \int f \times \chi H_n d\mu$ and $\nu K = \int f \times \chi K\, d\mu \ge \gamma$. Thus in either case we have a $K \in \mathcal{K}$ such that $K \subseteq F$ and $\nu K \ge \gamma$; as F and γ are arbitrary, ν is inner regular with respect to $\mathcal{K}$.

412R Lemma Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$ (251F). Suppose that $\mathcal{K} \subseteq \mathcal{P}X$, $\mathcal{L} \subseteq \mathcal{P}Y$ and $\mathcal{M} \subseteq \mathcal{P}(X \times Y)$ are such that

(i) μ is inner regular with respect to $\mathcal{K}$;
(ii) ν is inner regular with respect to $\mathcal{L}$;
(iii) $K \times L \in \mathcal{M}$ for all $K \in \mathcal{K}$, $L \in \mathcal{L}$;
(iv) $M \cup M' \in \mathcal{M}$ whenever M, $M' \in \mathcal{M}$;
(v) $\bigcap_{n\in\mathbb{N}} M_n \in \mathcal{M}$ for every sequence $\langle M_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{M}$.

Then λ is inner regular with respect to $\mathcal{M}$.

proof Write $\mathcal{A} = \{E \times F : E \in \Sigma\} \cup \{X \times F : F \in \mathrm{T}\}$. Then the σ-algebra of subsets of $X \times Y$ generated by $\mathcal{A}$ is $\Sigma \widehat{\otimes} \mathrm{T}$. If $V \in \mathcal{A}$, $W \in \Lambda$ and $\lambda(W \cap V) > 0$, there is an $M \in \mathcal{M} \cap (\Sigma \widehat{\otimes} \mathrm{T})$ such that $M \subseteq W$ and $\lambda(M \cap V) > 0$.

[4] Formerly 234B.
[5] Formerly 234C.
[6] Formerly 234D.

P Suppose that $V = E \times Y$ where $E \in \Sigma$. There must be $E_0 \in \Sigma$ and $F_0 \in \mathrm{T}$, both of finite measure, such that $\lambda(W \cap V \cap (E_0 \times F_0)) > 0$ (251F). Now there are $K \in \mathcal{K} \cap \Sigma$ and $L \in \mathcal{L} \cap \mathrm{T}$ such that $K \subseteq E \cap E_0$, $L \subseteq F \cap F_0$ and

$$\mu((E \cap E_0) \setminus K) \cdot \nu F_0 + \mu E_0 \cdot \nu((F \cap F_0) \setminus L) < \lambda(W \cap V \cap (E_0 \times F_0));$$

but this means that $M = K \times L$ is included in V and $\mu(W \cap M) > 0$, while $M \in \mathcal{M} \cap (\Sigma \widehat{\otimes} \mathrm{T})$. Reversing the roles of the coordinates, the same argument deals with the case in which $V = X \times F$ for some $F \in \mathrm{T}$. **Q**

By 412C, $\lambda \restriction \Sigma \widehat{\otimes} \mathrm{T}$ is inner regular with respect to $\mathcal{M}$. But λ is inner regular with respect to $\Sigma \widehat{\otimes} \mathrm{T}$ (251Ib) so is also inner regular with respect to $\mathcal{M}$ (412Ab).

412S Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Let $\mathfrak{T}$, $\mathfrak{S}$ be topologies on X and Y respectively, and give $X \times Y$ the product topology.

(a) If μ and ν are inner regular with respect to the closed sets, so is λ.

(b) If μ and ν are tight (that is, inner regular with respect to the closed compact sets), so is λ.

(c) If μ and ν are inner regular with respect to the zero sets, so is λ.

(d) If μ and ν are inner regular with respect to the Borel sets, so is λ.

(e) If μ and ν are effectively locally finite, so is λ.

proof We have only to read the conditions (i)-(v) of 412R carefully and check that they apply in each case. (In part (e), recall that 'effectively locally finite' is the same thing as 'inner regular with respect to the subsets of open sets of finite measure'.)

412T Lemma Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of probability spaces, with product probability space (X, Λ, λ) (§254). Suppose that $\mathcal{K}_i \subseteq \mathcal{P}X_i$, $\mathcal{M} \subseteq \mathcal{P}X$ are such that

(i) μ_i is inner regular with respect to $\mathcal{K}_i$ for each $i \in I$;

(ii) $\pi_i^{-1}[K] \in \mathcal{M}$ for every $i \in I$ and $K \in \mathcal{K}_i$, writing $\pi_i(x) = x(i)$ for $x \in X$;

(iii) $M \cup M' \in \mathcal{M}$ whenever M, $M' \in \mathcal{M}$;

(iv) $\bigcap_{n \in \mathbb{N}} M_n \in \mathcal{M}$ for every sequence $\langle M_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{M}$.

Then λ is inner regular with respect to $\mathcal{M}$.

proof (Compare 412R.) Write $\mathcal{A} = \{\pi_i^{-1}[E] : i \in I,\, E \in \Sigma_i\}$. If $V \in \mathcal{A}$, $W \in \Lambda$ and $\lambda(W \cap V) > 0$, express V as $\pi_i^{-1}[E]$, where $i \in I$ and $E \in \Sigma_i$, and take $K \in \mathcal{K}_i$ such that $K \subseteq E$ and $\mu_i(E \setminus K) < \lambda(W \cap V)$; then $M = \pi_i^{-1}[K]$ belongs to $\mathcal{M} \cap \mathcal{A}$, is included in V, and meets W in a non-negligible set. So the conditions of 412C are met.

It follows that $\lambda_0 = \lambda \restriction \widehat{\bigotimes}_{i \in I} \Sigma_i$ is inner regular with respect to $\mathcal{M}$. But λ is the completion of λ_0 (254Fd, 254Ff), so is also inner regular with respect to $\mathcal{M}$ (412Ha).

412U Proposition Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of probability spaces, with product probability space (X, Λ, λ). Suppose that we are given a topology $\mathfrak{T}_i$ on each X_i, and let $\mathfrak{T}$ be the product topology on X.

(a) If every μ_i is inner regular with respect to the closed sets, so is λ.

(b) If every μ_i is inner regular with respect to the zero sets, so is λ.

(c) If every μ_i is inner regular with respect to the Borel sets, so is λ.

proof This follows from 412T just as 412S follows from 412R.

412V Corollary Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of probability spaces, with product probability space (X, Λ, λ). Suppose that we are given a Hausdorff topology $\mathfrak{T}_i$ on each X_i, and let $\mathfrak{T}$ be the product topology on X. Suppose that every μ_i is tight, and that X_i is compact for all but countably many $i \in I$. Then λ is tight.

proof By 412Ua, λ is inner regular with respect to the closed sets. If $W \in \Lambda$ and $\gamma < \lambda W$, let $V \subseteq W$ be a measurable closed set such that $\lambda V > \gamma$. Let J be the set of those $i \in I$ such that X_i is not compact; we are supposing that J is countable. Let $\langle \epsilon_i \rangle_{i \in J}$ be a family of strictly positive real numbers such that $\sum_{i \in J} \epsilon_j \le \lambda V - \gamma$ (4A1P). For each $i \in J$, let $K_i \subseteq X_i$ be a compact measurable set such that $\mu_i(X_i \setminus K_i) \le \epsilon_i$; and for $i \in I \setminus J$, set $K_i = X_i$. Then $K = \prod_{i \in I} K_i$ is a compact measurable subset of X, and

$$\lambda(X \setminus K) \le \textstyle\sum_{i \in J} \mu(X_i \setminus K_i) \le \lambda V - \gamma,$$

so $\lambda(K \cap V) \ge \gamma$; while $K \cap V$ is a compact measurable subset of W. As W and γ are arbitrary, λ is tight.

***412W Outer regularity** I have already mentioned the complementary notion of 'outer regularity' (411D). In this book it will not be given much prominence. It is however a useful tool when dealing with Lebesgue measure (see, for instance, the proof of 225K), for reasons which the next proposition will make clear.

Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) Suppose that μ is outer regular with respect to the open sets. Then for any integrable function $f : X \to [0, \infty]$ and $\epsilon > 0$, there is a lower semi-continuous measurable function $g : X \to [0, \infty]$ such that $f \le g$ and $\int g \le \epsilon + \int f$.

(b) Now suppose that there is a sequence of measurable open sets of finite measure covering X. Then the following are equiveridical:

(i) μ is inner regular with respect to the closed sets;

(ii) μ is outer regular with respect to the open sets;

(iii) for any measurable set $E \subseteq X$ and $\epsilon > 0$, there are a measurable closed set $F \subseteq E$ and a measurable open set $H \supseteq E$ such that $\mu(H \setminus F) \le \epsilon$;

(iv) for every measurable function $f : X \to [0, \infty[$ and $\epsilon > 0$, there is a lower semi-continuous measurable function $g : X \to [0, \infty]$ such that $f \le g$ and $\int g - f \le \epsilon$;

(v) for every measurable function $f : X \to \mathbb{R}$ and $\epsilon > 0$, there is a lower semi-continuous measurable function $g : X \to]-\infty, \infty]$ such that $f \le g$ and $\mu\{x : g(x) \ge f(x) + \epsilon\} \le \epsilon$.

proof (a) Let $\eta \in]0, 1]$ be such that $\eta(7 + \int f d\mu) \le \epsilon$. For $n \in \mathbb{Z}$, set $E_n = \{x : (1 + \eta)^n \le f(x) < (1 + \eta)^{n+1}\}$, and let $E'_n \in \Sigma$ be a measurable envelope of E_n; let $G_n \supseteq E'_n$ be a measurable open set such that $\mu G_n \le 3^{-|n|}\eta + \mu E'_n$. Set $g = \sum_{n=-\infty}^{\infty}(1 + \eta)^{n+1}\chi G_n$. Then g is lower semi-continuous (4A2B(d-iii), 4A2B(d-v)), $f \le g$ and

$$\begin{aligned}\int g d\mu &= \sum_{n=-\infty}^{\infty}(1 + \eta)^{n+1}\mu G_n \\ &\le (1 + \eta)\sum_{n=-\infty}^{\infty}(1 + \eta)^n \mu E'_n + \sum_{n=-\infty}^{\infty}(1 + \eta)^{n+1}3^{-|n|}\eta \\ &\le (1 + \eta)\int f d\mu + 7\eta \le \int f d\mu + \epsilon,\end{aligned}$$

as required.

(b) Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a sequence of open sets of finite measure covering X; replacing it by $\langle\bigcup_{i<n} G_i\rangle_{n\in\mathbb{N}}$ if necessary, we may suppose that $\langle G_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and that $G_0 = \emptyset$.

(i)⇒(iii) Suppose that μ is inner regular with respect to the closed sets, and that $E \in \Sigma$, $\epsilon > 0$. For each $n \in \mathbb{N}$ let $F_n \subseteq G_n \setminus E$ be a measurable closed set such that $\mu F_n \ge \mu(G_n \setminus E) - 2^{-n-2}\epsilon$. Then $H = \bigcup_{n\in\mathbb{N}}(G_n \setminus F_n)$ is a measurable open set including E and $\mu(H \setminus E) \le \frac{1}{2}\epsilon$. Applying the same argument to $X \setminus E$, we get a closed set $F \subseteq E$ such that $\mu(E \setminus F) \le \frac{1}{2}\epsilon$, so that $\mu(H \setminus F) \le \epsilon$.

(ii)⇒(iii) The same idea works. Suppose that μ is outer regular with respect to the open sets, and that $E \in \Sigma$, $\epsilon > 0$. For each $n \in \mathbb{N}$, let $H_n \supseteq G_n \cap E$ be an open set such that $\mu(H_n \setminus E) \le 2^{-n-2}\epsilon$; then $H = \bigcup_{n\in\mathbb{N}} H_n$ is a measurable open set including E, and $\mu(H \setminus E) \le \frac{1}{2}\epsilon$. Now repeat the argument on $X \setminus E$ to find a measurable closed set $F \subseteq E$ such that $\mu(E \setminus F) \le \frac{1}{2}\epsilon$.

(iii)⇒(iv) Assume (iii), and let $f : X \to [0, \infty[$ be a measurable function, $\epsilon > 0$. Set $\eta_n = 2^{-n}\epsilon/(16 + 4\mu G_n)$ for each $n \in \mathbb{N}$. For $k \in \mathbb{N}$ set $E_k = \bigcup_{n\in\mathbb{N}}\{x : x \in G_n,\ k\eta_n \le f(x) < (k + 1)\eta_n\}$, and choose an open set $H_k \supseteq E_k$ such that $\mu(H_k \setminus E_k) \le 2^{-k}$. Set

$$g = \sup_{k,n\in\mathbb{N}}(k + 1)\eta_n\chi(G_n \cap H_k).$$

Then $g : X \to [0, \infty]$ is lower semi-continuous (4A2B(d-v) again). Since

$$\sup_{k,n\in\mathbb{N}} k\eta_n\chi(G_n \cap E_k) \le f \le \sup_{k,n\in\mathbb{N}}(k + 1)\eta_n\chi(G_n \cap E_k),$$

$f \le g$ and

$$g - f \le \sup_{k,n\in\mathbb{N}}(k + 1)\eta_n\chi(G_n \cap H_k \setminus E_k) + \sup_{k,n\in\mathbb{N}} \eta_n\chi(G_n \cap E_k)$$

has integral at most

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(k + 1)\eta_n 2^{-k} + \sum_{n=0}^{\infty} \eta_n\mu G_n \le \epsilon.$$

(i)⇒(v) Assume (i), and suppose that $f : X \to \mathbb{R}$ is measurable and $\epsilon > 0$. For each $n \in \mathbb{N}$, let $\alpha_n \ge 0$ be such that $\mu E_n < 2^{-n-1}\epsilon$, where $E_n = \{x : x \in G_{n+1} \setminus G_n,\ f(x) \le -\alpha_n\}$. Let $F_n \subseteq (G_{n+1} \setminus G_n) \setminus E_n$ be a measurable closed set such that $\mu((G_{n+1} \setminus G_n) \setminus F_n) \le 2^{-n-2}\epsilon$. Because $\langle F_n\rangle_{n\in\mathbb{N}}$ is disjoint, $h = \sum_{n=0}^{\infty} \alpha_n \chi F_n$ is defined as a function from X to $\mathbb{R}$. $\{F_n : n \in \mathbb{N}\}$ is locally finite, so $\{x : h(x) \ge \alpha\} = \bigcup_{n\in\mathbb{N},\alpha_n\ge\alpha} F_n$ is closed for every $\alpha > 0$ (4A2B(h-i)), and h is upper semi-continuous. Now $f_1 = f + h$ is a measurable function. Since (i)⇒(iii)⇒(iv), there is a measurable lower semi-continuous function $g_1 : X \to [0,\infty]$ such that $f_1^+ \le g_1$ and $\int g_1 - f_1^+ \le \frac{1}{2}\epsilon^2$, where $f_1^+ = \max(0, f_1)$. But if we now set $g = g_1 - h$, g is lower semi-continuous, $f \le g$ and

$$\begin{aligned}\{x : f(x) + \epsilon \le g(x)\} &\subseteq \{x : f_1^+(x) + \epsilon \le g_1(x)\} \cup \{x : f_1(x) < 0\}\\ &\subseteq \{x : f_1^+(x) + \epsilon \le g_1(x)\} \cup \bigcup_{n\in\mathbb{N}} (G_{n+1} \setminus G_n) \setminus F_n\end{aligned}$$

has measure at most ϵ, as required.

(iv)⇒(ii) and **(v)⇒(ii)** Suppose that either (iv) or (v) is true, and that $E \in \Sigma$, $\epsilon > 0$. Then there is a measurable lower semi-continuous function $g : X \to]0,\infty]$ such that $\chi E \le g$ and $\mu\{x : \chi E(x) + \frac{1}{2} \le g(x)\} \le \epsilon$, since this is certainly true if $\int g - \chi E \le \frac{1}{2}\epsilon$. Set $G = \{x : g(x) > \frac{1}{2}\}$; then $E \subseteq G$ and $\mu(G \setminus E) \le \epsilon$.

(iii)⇒(i) is trivial. Assembling these fragments, the proof is complete.

412X Basic exercises (a) Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets *and* with respect to the compact sets. Show that μ is tight.

(b) Explain how 213A is a special case of 412Aa.

(c) Let X be a set and $\mathcal{K}$ a family of sets. Suppose that μ and ν are two semi-finite measures on X with the same domain and the same null ideal. Show that if one is inner regular with respect to $\mathcal{K}$, so is the other. (*Hint*: show that if $\nu F < \infty$ then $\nu F = \sup\{\nu E : E \subseteq F,\ \mu E < \infty\}$.)

>(d) Let (X, Σ, μ) be a measure space, and Σ_0 a σ-subalgebra of Σ such that μ is inner regular with respect to Σ_0. Show that if $1 \le p < \infty$ then every member of $L^p(\mu)$ is of the form $f^\bullet$ for some Σ_0-measurable $f : X \to \mathbb{R}$.

>(e) Let (X, Σ, μ) be a semi-finite measure space and $\mathcal{A} \subseteq \Sigma$ an algebra of sets such that the σ-algebra generated by $\mathcal{A}$ is Σ. Write $\mathcal{K}$ for $\{\bigcap_{n\in\mathbb{N}} E_n : E_n \in \mathcal{A}$ for every $n \in \mathbb{N}\}$. Show that μ is inner regular with respect to $\mathcal{K}$.

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite Hausdorff topological measure space such that μ is inner regular with respect to the Borel sets. Suppose that $\mu G = \sup\{\mu K : K \subseteq G$ is compact$\}$ for every open set $G \subseteq X$. Show that μ is tight.

(g) Let $(X, \mathfrak{T})$ be a topological space such that every open set is an F_σ set. Show that any effectively locally finite Borel measure on X is inner regular with respect to the closed sets.

(h) Let $(X, \mathfrak{T})$ be a normal topological space and μ a topological measure on X which is inner regular with respect to the closed sets. Show that $\mu G = \max\{\mu H : H \subseteq G$ is a cozero set$\}$ for every open set $G \subseteq X$. Show that if μ is totally finite, then $\mu F = \min\{\mu H : H \supseteq F$ is a zero set$\}$ for every closed set $F \subseteq X$.

(i) Let (X, Σ, μ) be a complete locally determined measure space, and suppose that μ is inner regular with respect to a family $\mathcal{K}$ of sets. Let Σ_0 be the σ-algebra of subsets of X generated by $\mathcal{K} \cap \Sigma$. (i) Show that μ is the c.l.d. version of $\mu\restriction\Sigma_0$. (*Hint*: 412J-412L.) (ii) Show that if μ is σ-finite, it is the completion of $\mu\restriction\Sigma_0$.

>(j)(i) Let (X, Σ, μ) be a σ-finite measure space and T a σ-subalgebra of Σ. Show that if μ is inner regular with respect to T then the completion of $\mu\restriction\mathrm{T}$ extends μ, so that μ and $\mu\restriction\mathrm{T}$ have the same negligible sets. (ii) Show that if μ is a σ-finite topological measure which is inner regular with respect to the Borel sets, then every μ-negligible set is included in a μ-negligible Borel set.

(k) Devise a direct proof of 412L, not using 412K, by (i) showing that $\mu^*(A \cap K) = \nu^*(A \cap K)$ whenever $A \subseteq X$ and $K \in \mathcal{K}$ (ii) showing that $\mu^* = \nu^*$ (iii) quoting 213C.

(l) Let (X, Σ, μ) be a complete locally determined measure space, Y a set and $f : X \to Y$ a function. Show that the following are equiveridical: (i) μ is inner regular with respect to $\{f^{-1}[B] : B \subseteq Y\}$ (ii) $f^{-1}[f[E]] \setminus E$ is negligible for every $E \in \Sigma$.

(m) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X, Σ, μ). Suppose that for each $i \in I$ we are given a topology $\mathfrak{T}_i$ on X_i, and let $\mathfrak{T}$ be the corresponding disjoint union topology on X. Show that (i) μ is inner regular with respect to the closed sets iff every μ_i is (ii) μ is inner regular with respect to the compact sets iff every μ_i is (iii) μ is inner regular with respect to the zero sets iff every μ_i is (iv) μ is inner regular with respect to the Borel sets iff every μ_i is.

(n) Use 412L and 412Q to shorten the proof of 253I.

(o) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and suppose that we are given, for each $i \in I$, a σ-algebra Σ_i of subsets of X_i and a topology $\mathfrak{T}_i$ on X_i. Let $\mathfrak{T}$ be the product topology on $X = \prod_{i\in I} X_i$, and $\Sigma = \widehat{\bigotimes}_{i\in I}\Sigma_i$. Let μ be a totally finite measure with domain Σ, and set $\mu_i = \mu\pi_i^{-1}$ for each $i \in I$, where $\pi_i(x) = x(i)$ for $i \in I$, $x \in X$. (i) Show that μ is inner regular with respect to the family $\mathcal{K}$ of sets expressible as $X \setminus \bigcup_{n\in\mathbb{N}} \prod_{i\in I} E_{ni}$ where $E_{ni} \in \Sigma_i$ for every n, i and $\{i : E_{ni} \ne X_i\}$ is finite for each n. (ii) Show that if every μ_i is inner regular with respect to the closed sets, so is μ. (iii) Show that if every μ_i is inner regular with respect to the zero sets, so is μ. (iv) Show that if every μ_i is inner regular with respect to the Borel sets, so is μ. (v) Show that if every μ_i is tight, and all but countably many of the X_i are compact, then μ is tight.

(p) Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a Lindelöf topology on X such that μ is locally finite. (i) Show that μ is σ-finite. (ii) Show that μ is inner regular with respect to the closed sets iff it is outer regular with respect to the open sets.

(q) Let X be a topological space and μ a measure on X which is outer regular with respect to the open sets. Show that for any $Y \subseteq X$ the subspace measure on Y is outer regular with respect to the open sets.

(r) Let X be a topological space and μ a measure on X which is outer regular with respect to the open sets. Show that if $f : X \to \mathbb{R}$ is integrable and $\epsilon > 0$ then there is a lower semi-continuous $g : X \to \left]-\infty, \infty\right]$ such that $f \le g$ and $\int g - f \le \epsilon$.

>(s) Let X be a topological space and μ a measure on X which is effectively locally finite and inner regular with respect to the closed sets. (i) Show that if $\mu E < \infty$ and $\epsilon > 0$ there is a measurable open set G such that $\mu(E \triangle G) \le \epsilon$. (ii) Show that if f is a non-negative integrable function and $\epsilon > 0$ there is a measurable lower semi-continuous function $g : X \to [0, \infty[$ such that $\int |f - g| \le \epsilon$. (iii) Show that if f is an integrable real-valued function there are measurable lower semi-continuous functions g_1, $g_2 : X \to [0, \infty]$ such that $f =_{\text{a.e.}} g_1 - g_2$ and $\int g_1 + g_2 \le \int |f| + \epsilon$. (iv) Now suppose that μ is σ-finite. Show that for every measurable $f : X \to \mathbb{R}$ there are measurable lower semi-continuous functions g_1, $g_2 : X \to [0, \infty]$ such that $f =_{\text{a.e.}} g_1 - g_2$.

(t) Let (X, Σ, μ) be a semi-finite measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}^0(\mu)$ which converges almost everywhere to $f \in \mathcal{L}^0(\mu)$. Show that μ is inner regular with respect to $\{E : \langle f_n \restriction E\rangle_{n\in\mathbb{N}}$ is uniformly convergent$\}$. (Cf. 215Yb.)

(u) In 216E, give $\{0, 1\}^I$ its usual compact Hausdorff topology. Show that the measure μ described there is inner regular with respect to the zero sets.

(v) Let $\langle\mu_i\rangle_{i\in I}$ be a family of measures on a set X, with sum μ (234G[7]). Suppose that $\mathcal{K}$ is a family of sets such that every μ_i is inner regular with respect to $\mathcal{K}$. Show that if *either* $\mathcal{K} \subseteq \operatorname{dom}\mu$ *or* $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ for every non-increasing sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$, then μ is inner regular with respect to $\mathcal{K}$.

(w) Let μ, ν be c.l.d. measures on a set X, both inner regular with respect to a family $\mathcal{K} \subseteq \operatorname{dom}\mu \cap \operatorname{dom}\nu$. Suppose that $\mu K \le \nu K < \infty$ for every $K \in \mathcal{K}$. Show that $\mu \le \nu$ in the sense of 234P.

(x) Let X be a topological space, and μ a tight topological measure on X. Suppose that $\mathcal{F}$ is a non-empty downwards-directed family of closed compact subsets of X with intersection F_0, and that $\gamma = \inf_{F\in\mathcal{F}} \mu F$ is finite. Show that $\mu F_0 = \gamma$.

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[7] Formerly 112Ya.

412Y Further exercises (a) Let $\mathcal{K}$ be the family of subsets of $\mathbb{R}$ which are homeomorphic to the Cantor set. Show that Lebesgue measure is inner regular with respect to $\mathcal{K}$. (*Hint*: show that if $F \subseteq \mathbb{R} \setminus \mathbb{Q}$ is an uncountable compact set, then $\{x : [x-\delta, x+\delta] \cap F$ is uncountable for every $\delta > 0\}$ belongs to $\mathcal{K}$.)

(b)(i) Show that if X is a perfectly normal space then any semi-finite topological measure on X which is inner regular with respect to the Borel sets is inner regular with respect to the closed sets. (ii) Show that any subspace of a perfectly normal space is perfectly normal. (iii) Show that ω_1, with its order topology, is completely regular, normal and Hausdorff, but not perfectly normal. (iv) Show that $[0,1]^I$ is perfectly normal iff I is countable.

(c) Let (X, Σ, μ) be a measure space, and suppose that μ is inner regular with respect to $\mathcal{K}$. Write Σ^f for $\{E : E \in \Sigma,\ \mu E < \infty\}$. Show that $\{E^\bullet : E \in \mathcal{K} \cap \Sigma^f\}$ is dense in $\{E^\bullet : E \in \Sigma^f\}$ for the strong measure-algebra topology.

(d) Let (X, Σ, μ) be $[0,1]$ with Lebesgue measure, and $Y = [0,1]$ with counting measure ν; give X its usual topology and Y its discrete topology, and let λ be the c.l.d. product measure on $X \times Y$. (i) Show that μ, ν and λ are all tight (for the appropriate topologies) and therefore completion regular. (ii) Let λ_0 be the primitive product measure on $X \times Y$ (definition: 251C). Show that λ_0 is not tight. (*Hint*: 252Yk.) *Remark*: it is undecidable in ZFC whether λ_0 is inner regular with respect to the closed sets.

(e) Give an example of a Hausdorff topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that μ is complete, strictly localizable and outer regular with respect to the open sets, but not inner regular with respect to the closed sets.

412 Notes and comments In this volume we are returning to considerations which have been left on one side for almost the whole of Volume 3 – the exceptions being in Chapter 34, where I looked at realization of homomorphisms of measure algebras by functions between measure spaces, and was necessarily dragged into an investigation of measure spaces which had enough points to be adequate codomains (343B). The idea of 'inner regularity' is to distinguish families $\mathcal{K}$ of sets which will be large enough to describe the measure entirely, but whose members will be of recognisable types. For an example of this principle see 412Ya. Of course we cannot always find a single type of set adequate to fill a suitable family $\mathcal{K}$, though this happens oftener than one might expect, but it is surely easier to think about an arbitrary zero set (for instance) than an arbitrary measurable set, and whenever a measure is inner regular with respect to a recognisable class it is worth knowing about it.

I have tried to use the symbols † and ‡ (412A, 412C) consistently enough for them to act as a guide to some of the ideas which will be used repeatedly in this chapter. Note the emphasis on disjoint unions and countable intersections; I mentioned similar conditions in 136Xi-136Xj. You will recognise 412Aa as an exhaustion principle; note that it is enough to use disjoint unions, as in 313K. In the examples of this section this disjointness is not important. Of course inner regularity has implications for the measure algebra (412N), but it is important to recognise that 'μ is inner regular with respect to $\mathcal{K}$' is saying much more than '$\{K^\bullet : K \in \mathcal{K}\}$ is order-dense in the measure algebra'; the latter formulation tells us only that whenever $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $K \setminus E$ is negligible and $\mu K > 0$, while the former tells us that we can take K to be actually a subset of E.

412D, 412E and 412G are all of great importance. 412D looks striking, but of course the reason it works is just that the Baire σ-algebra is very small. In 412E the Baire and Borel σ-algebras coincide, so it is nothing but a special case of 412D; but as metric spaces are particularly important it is worth having it spelt out explicitly. In 412D and 412E the hypothesis 'semi-finite' is sufficient, while in 412G we need 'effectively locally finite'; this is because in both 412D and 412E the open sets we are looking at are countable unions of measurable closed sets. There are interesting non-metrizable spaces in which the same thing happens (412Yb). As you know, I am strongly biased in favour of complete and locally determined measures, and the Baire and Borel measures dealt with in these three results are rarely complete; but they can still be applied to completions and c.l.d. versions of these measures, using 412Ab or 412H.

412O-412V are essentially routine. For subspace measures, the only problem we need to come to terms with is the fact that subspaces of semi-finite measure spaces need not be semi-finite (216Xa). For product measures the point is that the c.l.d. product of two measure spaces, and the product of any family of probability spaces, as I defined them in Chapter 25, are inner regular with respect to the σ-algebra of sets generated by the cylinder sets. This is not in general true of the 'primitive' product measure (412Yd), which is one of my reasons for being prejudiced against it. I should perhaps warn you of a trap in the language I use here. I say that if the factor measures are inner regular with respect to the closed sets, so is the c.l.d. product measure. But I do not say that all closed sets in the product are measured by the product measure, even if closed sets in the factors are measured by the factor measures. So the path is open for a different product measure to exist, still inner regular with respect to the closed sets; and indeed I shall be going down that path in §417. The uniqueness result in 412L specifically refers to complete locally determined measures defined on all sets of the family $\mathcal{K}$.

There is one special difficulty in 412V: in order to ensure that there are enough compact measurable sets in $X = \prod_{i\in I} X_i$, we need to know that all but countably many of the X_i are actually compact. When we come to look more closely at products of Radon probability spaces we shall need to consider this point again (417Q, 417Xq).

In fact some of the ideas of 412U-412V are not restricted to the product measures considered there. Other measures on the product space will have inner regularity properties if their images on the factors, their 'marginals' in the language of probability theory, are inner regular; see 412Xo. I will return to this in §454.

This section is almost exclusively concerned with *inner* regularity. The complementary notion of *outer* regularity is not much use except in σ-finite spaces (415Xh), and not always then (416Yd). In totally finite spaces, of course, and some others, any version of inner regularity corresponds to a version of outer regularity, as in 412Wb(i)-(ii); and when we have something as strong as 412Wb(iii) available it is worth knowing about it.

413 Inner measure constructions

I now turn in a different direction, giving some basic results on the construction of inner regular measures. The first step is to describe 'inner measures' (413A) and a construction corresponding to the Carathéodory construction of measures from outer measures (413C). Just as every measure gives rise to an outer measure, it gives rise to an inner measure (413D). Inner measures form an effective tool for studying complete locally determined measures (413F).

The most substantial results of the section concern the construction of measures as extensions of functionals defined on various classes $\mathcal{K}$ of sets. Typically, $\mathcal{K}$ is closed under finite unions and countable intersections, though it we can sometimes relax the hypotheses a bit. The methods here make it possible to distinguish arguments which produce finitely additive functionals (413H, 413N, 413P, 413Q) from the succeeding steps to countably additive measures (413I, 413O, 413S). 413H-413M investigate conditions on a functional $\phi : \mathcal{K} \to [0, \infty[$ sufficient to produce a measure extending ϕ, necessarily unique, which is inner regular with respect to $\mathcal{K}$ or $\mathcal{K}_\delta$, the set of intersections of sequences in $\mathcal{K}$. 413N-413O look instead at functionals defined on sublattices of the class $\mathcal{K}$ of interest, and at sufficient conditions to ensure the existence of a measure, not normally unique, defined on the whole of $\mathcal{K}$, inner regular with respect to $\mathcal{K}$ and extending the given functional. Finally, 413P-413S are concerned with majorizations rather than extensions; we seek a measure μ such that $\mu K \geq \lambda K$ for $K \in \mathcal{K}$, while μX is as small as possible.

413A I begin with some material from the exercises of earlier volumes.

Definition Let X be a set. An **inner measure** on X is a functional $\phi : \mathcal{P}X \to [0, \infty]$ such that

$\phi\emptyset = 0$;

(α) $\phi(A \cup B) \geq \phi A + \phi B$ for all disjoint A, $B \subseteq X$;

(β) if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of X and $\phi A_0 < \infty$ then $\phi(\bigcap_{n\in\mathbb{N}} A_n) = \inf_{n\in\mathbb{N}} \phi A_n$;

$(*)$ $\phi A = \sup\{\phi B : B \subseteq A,\, \phi B < \infty\}$ for every $A \subseteq X$.

413B The following fact will be recognised as an element of Carathéodory's method. There will be an application later in which it will be useful to know that it is not confined to proving countable additivity.

Lemma Let X be a set and $\phi : X \to [0, \infty]$ any functional such that $\phi\emptyset = 0$. Then

$$\Sigma = \{E : E \subseteq X,\, \phi A = \phi(A \cap E) + \phi(A \setminus E) \text{ for every } A \subseteq X\}$$

is an algebra of subsets of X, and $\phi(E \cup F) = \phi E + \phi F$ for all disjoint E, $F \in \Sigma$.

proof The symmetry of the definition of Σ ensures that $X \setminus E \in \Sigma$ whenever $E \in \Sigma$. If E, $F \in \Sigma$ and $A \subseteq X$, then

$$\begin{aligned}
\phi(A \cap (E \cup F)) &+ \phi(A \setminus (E \cup F)) \\
&= \phi(A \cap (E \cup F) \cap E) + \phi(A \cap (E \cup F) \setminus E) + \phi(A \setminus (E \cup F)) \\
&= \phi(A \cap E) + \phi((A \setminus E) \cap F) + \phi((A \setminus E) \setminus F) \\
&= \phi(A \cap E) + \phi(A \setminus E) = \phi A.
\end{aligned}$$

As A is arbitrary, $E \cup F \in \Sigma$. Finally, if $A \subseteq X$,

$$\phi(A \cap \emptyset) + \phi(A \setminus \emptyset) = \phi\emptyset + \phi A = \phi A$$

because $\phi\emptyset = 0$; so $\emptyset \in \Sigma$.

Thus Σ is an algebra of sets. If E, $F \in \Sigma$ and $E \cap F = \emptyset$, then

$$\phi(E \cup F) = \phi((E \cup F) \cap E) + \phi((E \cup F) \setminus E) = \phi E + \phi F.$$

413C Measures from inner measures I come now to a construction corresponding to Carathéodory's method of defining measures from outer measures.

Theorem Let X be a set and $\phi : X \to [0, \infty]$ an inner measure. Set

$$\Sigma = \{E : E \subseteq X,\ \phi(A \cap E) + \phi(A \setminus E) = \phi A \text{ for every } A \subseteq X\}.$$

Then $(X, \Sigma, \phi{\restriction}\Sigma)$ is a complete measure space.

proof (Compare 113C.)

(a) The first step is to note that if $A \subseteq B \subseteq X$ then

$$\phi B \geq \phi A + \phi(B \setminus A) \geq \phi A.$$

Next, a subset E of X belongs to Σ iff $\phi A \leq \phi(A \cap E) + \phi(A \setminus E)$ whenever $A \subseteq X$ and $\mu A < \infty$. **P** Of course any element of Σ satisfies the condition. If E satisfies the condition and $A \subseteq X$, then

$$\begin{aligned}\phi A &= \sup\{\phi B : B \subseteq A,\ \phi B < \infty\}\\ &\leq \sup\{\phi(B \cap E) + \phi(B \setminus E) : B \subseteq A\}\\ &= \phi(A \cap E) + \phi(A \setminus E) \leq \phi A,\end{aligned}$$

so $E \in \Sigma$. **Q**

(b) By 413B, Σ is an algebra of subsets of X. Now suppose that $\langle E_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in Σ, with union E. If $A \subseteq X$ and $\phi A < \infty$, then

$$\phi(A \setminus E) = \inf_{n \in \mathbb{N}} \phi(A \setminus E_n) = \lim_{n \to \infty} \phi(A \setminus E_n)$$

because $\langle A \setminus E_n \rangle_{n \in \mathbb{N}}$ is non-increasing and $\phi(A \setminus E_0)$ is finite; so

$$\phi(A \cap E) + \phi(A \setminus E) \geq \lim_{n \to \infty} \phi(A \cap E_n) + \phi(A \setminus E_n) = \phi A.$$

By (a), $E \in \Sigma$. So Σ is a σ-algebra.

(c) If E, $F \in \Sigma$ and $E \cap F = \emptyset$ then $\phi(E \cup F) = \phi E + \phi F$, by 413B. If $\langle E_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in Σ with union E, then

$$\mu E \geq \mu(\bigcup_{i \leq n} E_i) = \sum_{i=0}^{n} \mu E_i$$

for every n, so $\mu E \geq \sum_{i=0}^{\infty} \mu E_i$. **?** If $\mu E > \sum_{i=0}^{\infty} \mu E_i$, there is an $A \subseteq E$ such that $\sum_{i=0}^{\infty} \mu E_i < \phi A < \infty$. But now, setting $F_n = \bigcup_{i \leq n} E_i$ for each n, we have $\lim_{n \to \infty} \phi(A \setminus F_n) = 0$, so that

$$\phi A = \lim_{n \to \infty} \phi(A \cap F_n) + \phi(A \setminus F_n) = \sum_{i=0}^{\infty} \phi(A \cap E_i) < \phi A,$$

which is absurd. **X** Thus $\mu E = \sum_{i=0}^{\infty} \mu E_i$. As $\langle E_n \rangle_{n \in \mathbb{N}}$ is arbitrary, μ is a measure.

(d) Finally, suppose that $B \subseteq E \in \Sigma$ and $\mu E = 0$. Then for any $A \subseteq X$ we must have

$$\phi(A \cap B) + \phi(A \setminus B) \geq \phi(A \setminus E) = \phi(A \cap E) + \phi(A \setminus E) = \phi A,$$

so $B \in \Sigma$. Thus μ is complete.

Remark For a simple example see 213Yd[8].

413D The inner measure defined by a measure Let (X, Σ, μ) be any measure space. Just as μ has an associated outer measure μ^* defined by the formula

[8] Formerly 213Yc.

$$\mu^* A = \inf\{\mu E : A \subseteq E \in \Sigma\}$$

(132A-132B), it gives rise to an inner measure μ_* defined by the formula

$$\mu_* A = \sup\{\mu E : E \in \Sigma^f,\ E \subseteq A\},$$

where I write Σ^f for $\{E : E \in \Sigma,\ \mu E < \infty\}$. **P** $\mu_*\emptyset = \mu\emptyset = 0$. ($\alpha$) If $A \cap B = \emptyset$, and $E \subseteq A$, $F \subseteq B$ belong to Σ^f, then $E \cup F \subseteq A \cup B$ also has finite measure, so

$$\mu_*(A \cup B) \ge \mu(E \cup F) = \mu E + \mu F;$$

taking the supremum over E and F, $\mu_*(A \cup B) \ge \mu_* A + \mu_* B$. ($\beta$) If $\langle A_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of sets with intersection A and $\mu_* A_0 < \infty$, then for each $n \in \mathbb{N}$ we can find an $E_n \subseteq A_n$ such that $\mu E_n \ge \mu_* A_n - 2^{-n}$. In this case,

$$\mu(\bigcup_{m \in \mathbb{N}} E_m) = \sup_{n \in \mathbb{N}} \mu(\bigcup_{m \le n} E_m) \le \mu_* A_0 < \infty.$$

Set

$$E = \bigcap_{n \in \mathbb{N}} \bigcup_{m \ge n} E_m \subseteq A.$$

Then $E \in \Sigma^f$, so

$$\mu_* A \ge \mu E \ge \limsup_{n \to \infty} \mu E_n = \lim_{n \to \infty} \mu_* A_n \ge \mu_* A.$$

($*$) If $A \subseteq X$ and $\mu_* A = \infty$ then

$$\sup\{\mu_* B : B \subseteq A,\ \mu_* B < \infty\} \ge \sup\{\mu E : E \in \Sigma^f,\ E \subseteq A\} = \infty. \text{ **Q**}$$

Warning Many authors use the formula

$$\mu_* A = \sup\{\mu E : A \supseteq E \in \Sigma\}.$$

In 'ordinary' cases, when (X, Σ, μ) is semi-finite, this agrees with my usage (413Ed); but for non-semi-finite spaces there is a difference. See 413Yg.

413E I note the following simple facts concerning inner measures defined from measures.

Proposition Let (X, Σ, μ) be a measure space. Write Σ^f for $\{E : E \in \Sigma,\ \mu E < \infty\}$.

(a) For every $A \subseteq X$ there is an $E \in \Sigma$ such that $E \subseteq A$ and $\mu E = \mu_* A$.

(b) $\mu_* A \le \mu^* A$ for every $A \subseteq X$.

(c) If $E \in \Sigma$ and $A \subseteq X$, then $\mu_*(E \cap A) + \mu^*(E \setminus A) \le \mu E$, with equality if either (i) $\mu E < \infty$ or (ii) μ is semi-finite.

(d) In particular, $\mu_* E \le \mu E$ for every $E \in \Sigma$, with equality if either $\mu E < \infty$ or μ is semi-finite.

(e) If μ is inner regular with respect to $\mathcal{K}$, then $\mu_* A = \sup\{\mu K : K \in \mathcal{K} \cap \Sigma^f,\ K \subseteq A\}$ for every $A \subseteq X$.

(f) If $A \subseteq X$ is such that $\mu_* A = \mu^* A < \infty$, then A is measured by the completion of μ.

(g) If $\hat{\mu}$, $\tilde{\mu}$ are the completion and c.l.d. version of μ, then $\hat{\mu}_* = \tilde{\mu}_* = \mu_*$.

(h) If (Y, T, ν) is another measure space, and $f : X \to Y$ is an inverse-measure-preserving function, then

$$\mu^*(f^{-1}[B]) \le \nu^* B, \quad \mu_*(f^{-1}[B]) \ge \nu_* B$$

for every $B \subseteq Y$, and

$$\nu^*(f[A]) \ge \mu^* A$$

for every $A \subseteq X$.

(i) Suppose that μ is semi-finite. If $A \subseteq E \in \Sigma$, then E is a measurable envelope of A iff $\mu_*(E \setminus A) = 0$.

proof (a) There is a sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in Σ^f such that $E_n \subseteq A$ for each n and $\lim_{n \to \infty} \mu E_n = \mu_* A$; now set $E = \bigcup_{n \in \mathbb{N}} E_n$.

(b) If $E \subseteq A \subseteq F$ we must have $\mu E \le \mu F$.

(c) If $F \subseteq E \cap A$ and $F \in \Sigma^f$, then

$$\mu F + \mu^*(E \setminus A) \le \mu F + \mu(E \setminus F) = \mu E;$$

taking the supremum over F, $\mu_*(E \cap A) + \mu^*(E \setminus A) \le \mu E$. If $\mu E < \infty$, then

$$\begin{aligned}\mu_*(E\cap A)&=\sup\{\mu F:F\in\Sigma,\,F\subseteq E\cap A\}\\&=\mu E-\inf\{\mu(E\setminus F):F\in\Sigma,\,F\subseteq E\cap A\}\\&=\mu E-\inf\{\mu F:F\in\Sigma,\,E\setminus A\subseteq F\subseteq E\}=\mu E-\mu^*(E\setminus A).\end{aligned}$$

If μ is semi-finite, then

$$\begin{aligned}\mu_*(E\cap A)+\mu^*(E\setminus A)&\ge\sup\{\mu_*(F\cap A)+\mu^*(F\setminus A):F\in\Sigma^f,\,F\subseteq E\}\\&=\sup\{\mu F:F\in\Sigma^f,\,F\subseteq E\}=\mu E.\end{aligned}$$

(d) Take $A=E$ in (c).

(e)

$$\begin{aligned}\mu_*A&=\sup\{\mu E:E\in\Sigma^f,\,E\subseteq A\}\\&=\sup\{\mu K:K\in\mathcal{K}\cap\Sigma,\,\exists\,E\in\Sigma^f,\,K\subseteq E\subseteq A\}\\&=\sup\{\mu K:K\in\mathcal{K}\cap\Sigma^f,\,K\subseteq A\}.\end{aligned}$$

(f) By (a) above and 132Aa, there are E, $F\in\Sigma$ such that $E\subseteq A\subseteq F$ and

$$\mu E=\mu_*A=\mu^*A=\mu F<\infty;$$

now $\mu(F\setminus E)=0$, so $F\setminus A$ and A are measured by the completion of μ.

(g) Write $\check\mu$ for either $\hat\mu$ or $\tilde\mu$, and $\check\Sigma$ for its domain, and let $A\subseteq X$. (i) If $\gamma<\mu_*A$, there is an $E\in\Sigma$ such that $E\subseteq A$ and $\gamma\le\mu E<\infty$; now $\check\mu E=\mu E$ (212D, 213Fa), so $\check\mu_*A\ge\gamma$. As γ is arbitrary, $\mu_*A\le\check\mu_*A$. (ii) If $\gamma<\check\mu_*A$, there is an $E\in\check\Sigma$ such that $E\subseteq A$ and $\gamma\le\check\mu E<\infty$. Now there is an $F\in\Sigma$ such that $F\subseteq E$ and $\mu F=\check\mu E$ (212C, 213Fc), so that $\mu_*A\ge\gamma$. As γ is arbitrary, $\mu_*A\ge\check\mu_*A$.

(h) This is elementary; all we have to note is that if F, $F'\in\mathrm{T}$ and $F\subseteq B\subseteq F'$, then $f^{-1}[F]\subseteq f^{-1}[B]\subseteq f^{-1}[F']$, so that

$$\nu F=\mu f^{-1}[F]\le\mu_*f^{-1}[B]\le\mu^*f^{-1}[B]\le\mu f^{-1}[F']=\nu F'.$$

Now, for $A\subseteq X$,

$$\mu^*A\le\mu^*(f^{-1}[f[A]])\le\nu^*(f[A]).$$

(i)(i) If E is a measurable envelope of A and $F\in\Sigma$ is included in $E\setminus A$, then

$$\mu F=\mu(F\cap E)=\mu^*(F\cap A)=0;$$

as F is arbitrary, $\mu_*(E\setminus A)=0$. (ii) If E is not a measurable envelope of A, there is an $F\in\Sigma$ such that $\mu^*(F\cap A)<\mu(F\cap E)$. Let $G\in\Sigma$ be such that $F\cap A\subseteq G$ and $\mu G=\mu^*(F\cap A)$. Then $\mu(F\cap E\setminus G)>0$; because μ is semi-finite, $\mu_*(E\setminus A)\ge\mu_*(F\cap E\setminus G)>0$.

413F The language of 413D makes it easy to express some useful facts about complete locally determined measure spaces, complementing 412J.

Lemma Let (X,Σ,μ) be a complete locally determined measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$. Then for $E\subseteq X$ the following are equiveridical:

(i) $E\in\Sigma$;
(ii) $E\cap K\in\Sigma$ whenever $K\in\Sigma\cap\mathcal{K}$;
(iii) $\mu^*(K\cap E)+\mu^*(K\setminus E)=\mu^*K$ for every $K\in\mathcal{K}$;
(iv) $\mu_*(K\cap E)+\mu_*(K\setminus E)=\mu_*K$ for every $K\in\mathcal{K}$;
(v) $\mu^*(E\cap K)=\mu_*(E\cap K)$ for every $K\in\mathcal{K}\cap\Sigma$;
(vi) $\min(\mu^*(K\cap E),\mu^*(K\setminus E))<\mu K$ whenever $K\in\mathcal{K}\cap\Sigma$ and $0<\mu K<\infty$;
(vii) $\max(\mu_*(K\cap E),\mu_*(K\setminus E))>0$ whenever $K\in\mathcal{K}\cap\Sigma$ and $\mu K>0$.

proof (a) Assume (i). Then of course $E\cap K\in\Sigma$ for every $K\in\Sigma\cap\mathcal{K}$, and (ii) is true. For any $K\in\mathcal{K}$ there is an $F\in\Sigma$ such that $F\supseteq K$ and $\mu F=\mu^*K$ (132Aa); now

$$\mu^*K \le \mu^*(K \cap E) + \mu^*(K \setminus E) \le \mu(F \cap E) + \mu(F \setminus E) = \mu F = \mu^* K,$$

so (iii) is true. Next, for any $K \in \mathcal{K}$,

$$\mu_*(K \cap E) + \mu_*(K \setminus E) \le \mu_* K = \sup\{\mu F : F \in \Sigma^f,\, F \subseteq K\}$$

(writing Σ^f for $\{F : F \in \Sigma,\, \mu F < \infty\}$)

$$= \sup\{\mu(F \cap E) + \mu(F \setminus E) : F \in \Sigma^f,\, F \subseteq K\}$$
$$\le \mu_*(K \cap E) + \mu_*(K \setminus E).$$

So (iv) is true. If $K \in \mathcal{K} \cap \Sigma$, then

$$\mu_*(E \cap K) = \sup\{\mu F : F \in \Sigma^f,\, F \subseteq E \cap K\} = \mu(E \cap K) = \mu^*(E \cap K)$$

because μ is semi-finite. So (v) is true. Since (iii)$\Rightarrow$(vi) and (iv)$\Rightarrow$(vii), we see that all the conditions are satisfied.

(b) Now suppose that $E \notin \Sigma$; I have to show that (ii)-(vii) are all false. Because μ is locally determined, there is an $F \in \Sigma^f$ such that $E \cap F \notin \Sigma$. Take measurable envelopes H, H' of $F \cap E$ and $F \setminus E$ respectively (132Ee). Then $F \setminus H' \subseteq F \cap E \subseteq F \cap H$, so

$$G = (F \cap H) \setminus (F \setminus H') = F \cap H \cap H'$$

cannot be negligible. Take $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq G$ and $\mu K > 0$. As $G \subseteq F$, $\mu K < \infty$. Now

$$\mu^*(K \cap E) = \mu^*(K \cap F \cap E) = \mu(K \cap H) = \mu K,$$

$$\mu^*(K \setminus E) = \mu^*(K \cap F \setminus E) = \mu(K \cap H') = \mu K.$$

But this means that

$$\mu_*(K \cap E) = \mu K - \mu^*(K \setminus E) = 0, \quad \mu_*(K \setminus E) = \mu K - \mu^*(K \cap E) = 0$$

by 413Ec. Now we see that this K witnesses that (ii)-(vii) are all false.

413G The ideas of 413F can be used to give criteria for measurability of real-valued functions. I spell out one which is particularly useful.

Lemma Let (X, Σ, μ) be a complete locally determined measure space and suppose that μ is inner regular with respect to $\mathcal{K} \subseteq \Sigma$. Suppose that $f : X \to \mathbb{R}$ is a function, and for $\alpha \in \mathbb{R}$ set $E_\alpha = \{x : f(x) \le \alpha\}$, $F_\alpha = \{x : f(x) \ge \beta\}$. Then f is Σ-measurable iff

$$\min(\mu^*(E_\alpha \cap K), \mu^*(F_\beta \cap K)) < \mu K$$

whenever $K \in \mathcal{K}$, $0 < \mu K < \infty$ and $\alpha < \beta$.

proof (a) If f is measurable, then

$$\mu^*(E_\alpha \cap K) + \mu^*(F_\beta \cap K) = \mu(E_\alpha \cap K) + \mu(F_\beta \cap K) \le \mu K$$

whenever $K \in \Sigma$ and $\alpha < \beta$, so if $0 < \mu K < \infty$ then we must have $\min(\mu^*(E_\alpha \cap K), \mu^*(F_\beta \cap K)) < \mu K$.

(b) If f is not measurable, then there is some $\alpha \in \mathbb{R}$ such that E_α is not measurable. 413F(vi) tells us that there is a $K \in \mathcal{K}$ such that $0 < \mu K < \infty$ and $\mu^*(E_\alpha \cap K) = \mu^*(K \setminus E_\alpha) = \mu K$. Note that K is a measurable envelope of $K \cap E_\alpha$ (132Eb). Now $\langle K \cap F_{\alpha+2^{-n}} \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence with union $K \setminus E_\alpha$, so there is some $\beta > \alpha$ such that $K \cap F_\beta$ is not negligible. Let $H \subseteq K$ be a measurable envelope of $K \cap F_\beta$, and $K' \in \mathcal{K}$ such that $K' \subseteq H$ and $\mu K' > 0$; then

$$\mu^*(K' \cap E_\alpha) = \mu^*(K' \cap K \cap E_\alpha) = \mu(K' \cap K) = \mu K',$$

$$\mu^*(K' \cap F_\beta) = \mu^*(K' \cap H \cap F_\beta) = \mu(K' \cap H) = \mu K',$$

so K', α and β witness that the condition is not satisfied.

413H Inner measure constructions based on 413C are important because they offer an efficient way of setting up measures which are inner regular with respect to given families of sets. Two of the fundamental results are 413I and 413J. I proceed by means of a lemma on finitely additive functionals.

Lemma Let X be a set and $\mathcal{K}$ a family of subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$.

Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$.

Set

$$\phi A = \sup\{\phi_0 K : K \in \mathcal{K},\ K \subseteq A\} \text{ for } A \subseteq X,$$

$$\Sigma = \{E : E \subseteq X,\ \phi A = \phi(A \cap E) + \phi(A \setminus E) \text{ for every } A \subseteq X\}.$$

Then Σ is an algebra of subsets of X, including $\mathcal{K}$, and $\phi\restriction\Sigma : \Sigma \to [0, \infty]$ is an additive functional extending ϕ_0.

proof (a) To see that Σ is an algebra of subsets and $\phi\restriction\Sigma$ is additive, all we need to know is that $\phi\emptyset = 0$ (413B); and this is because, applying hypothesis (α) with $K = L = \emptyset$, $\phi_0\emptyset = \phi_0\emptyset + \phi_0\emptyset$, so $\phi_0\emptyset = 0$. (α) also assures us that $\phi_0 L \le \phi_0 K$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$, so $\phi K = \phi_0 K$ for every $K \in \mathcal{K}$.

(b) To check that $\mathcal{K} \subseteq \Sigma$, we have a little more work to do. First, observe that (†) and (α) together tell us that $\phi_0(K \cup K') = \phi_0 K + \phi_0 K'$ for all disjoint K, $K' \in \mathcal{K}$. So if A, $B \subseteq X$ and $A \cap B = \emptyset$ then

$$\begin{aligned} \phi A + \phi B &= \sup_{K \in \mathcal{K}, K \subseteq A} \phi_0 K + \sup_{L \in \mathcal{K}, L \subseteq A} \phi_0 L \\ &= \sup_{K, L \in \mathcal{K}, K \subseteq A, L \subseteq B} \phi_0(K \cup L) \le \phi(A \cup B). \end{aligned}$$

(c) $\mathcal{K} \subseteq \Sigma$. **P** Take $K \in \mathcal{K}$ and $A \subseteq X$. If $L \in \mathcal{K}$ and $L \subseteq A$, then

$$\phi_0 L = \phi_0(K \cap L) + \sup\{\phi_0 L' : L' \in \mathcal{K},\ L' \subseteq L \setminus K\} \le \phi(A \cap K) + \phi(A \setminus K).$$

(Note the use of the hypothesis (‡).) As L is arbitrary, $\phi A \le \phi(A \cap K) + \phi(A \setminus K)$. We already know that $\phi(A \cap K) + \phi(A \setminus K) \le \phi A$; as A is arbitrary, $K \in \Sigma$. **Q**

This completes the proof.

413I Theorem (Topsøe 70a) Let X be a set and $\mathcal{K}$ a family of subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $\bigcap_{n \in \mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathcal{K}$.

Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{n \in \mathbb{N}} \phi_0 K_n = 0$ whenever $\langle K_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection.

Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}$.

proof (a) Set

$$\phi A = \sup\{\phi_0 K : K \in \mathcal{K},\ K \subseteq A\} \text{ for } A \subseteq X,$$

$$\Sigma = \{E : E \subseteq X,\ \phi A = \phi(A \cap E) + \phi(A \setminus E) \text{ for every } A \subseteq X\}.$$

Then 413H tells us that Σ is an algebra of subsets of X, including $\mathcal{K}$, and $\mu = \phi\restriction\Sigma$ is an additive functional extending ϕ_0.

(b) Now $\mu(\bigcap_{n \in \mathbb{N}} K_n) = \inf_{n \in \mathbb{N}} \mu K_n$ whenever $\langle K_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$. **P** Set $L = \bigcap_{n \in \mathbb{N}} K_n$. Of course $\mu L \le \inf_{n \in \mathbb{N}} \mu K_n$. For the reverse inequality, take $\epsilon > 0$. Then (α) tells us that there is a $K' \in \mathcal{K}$ such that $K' \subseteq K_0 \setminus L$ and $\mu K_0 \le \mu L + \mu K' + \epsilon$. Since $\langle K_n \cap K' \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, (β) tells us that there is an $n \in \mathbb{N}$ such that $\mu(K_n \cap K') \le \epsilon$. Now

$$\begin{aligned} \mu K_0 - \mu L = \mu(K_0 \setminus L) &= \mu(K_0 \setminus (K' \cup L)) + \mu K' \\ &\le \epsilon + \mu(K_n \cap K') + \mu(K' \setminus K_n) \le 2\epsilon + \mu(K_0 \setminus K_n) = 2\epsilon + \mu K_0 - \mu K_n. \end{aligned}$$

(These calculations depend, of course, on the additivity of μ and the finiteness of μK_0.) So $\mu L \geq \mu K_n - 2\epsilon$. As ϵ is arbitrary, $\mu L = \inf_{n\in\mathbb{N}} \mu K_n$. **Q**

(c) If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of X, with intersection A, and $\phi A_0 < \infty$, then $\phi A = \inf_{n\in\mathbb{N}} \phi A_n$. **P** Of course $\phi A \leq \phi A_n$ for every n. Given $\epsilon > 0$, then for each $n \in \mathbb{N}$ choose $K_n \in \mathcal{K}$ such that $K_n \subseteq A_n$ and $\phi_0 K_n \geq \phi A_n - 2^{-n}\epsilon$ (this is where I use the hypothesis that ϕA_0 is finite); set $L_n = \bigcap_{i\leq n} K_i$ for each n, and $L = \bigcap_{n\in\mathbb{N}} L_n$. Then we have

$$\begin{aligned}\phi A_{n+1} - \mu L_{n+1} &= \phi A_{n+1} - \mu(K_{n+1} \cap L_n)\\ &= \phi A_{n+1} - \mu K_{n+1} - \mu L_n + \mu(K_{n+1} \cup L_n)\\ &\leq 2^{-n-1}\epsilon - \mu L_n + \phi A_n\end{aligned}$$

because $K_{n+1} \subseteq A_{n+1} \subseteq A_n$ and $L_n \subseteq K_n \subseteq A_n$. Inducing on n, we see that $\mu L_n \geq \phi A_n - 2\epsilon + 2^{-n}\epsilon$ for every n. So

$$\phi A \geq \mu L = \inf_{n\in\mathbb{N}} \mu L_n \geq \inf_{n\in\mathbb{N}} \phi A_n - 2\epsilon,$$

using (b) above for the middle equality. As ϵ is arbitrary, $\phi A = \inf_{n\in\mathbb{N}} \phi A_n$. **Q**

(d) It follows that ϕ is an inner measure. **P** The arguments of parts (a) and (b) of the proof of 413H tell us that $\phi\emptyset = 0$ and $\phi(A\cup B) \leq \phi A + \phi B$ whenever $A, B \subseteq X$ are disjoint. We have just seen that $\phi(\bigcap_{n\in\mathbb{N}} A_n) = \inf_{n\in\mathbb{N}} \phi A_n$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of sets and $\phi A_0 < \infty$. Finally, $\phi K = \phi_0 K$ is finite for every $K \in \mathcal{K}$, so $\phi A = \sup\{\phi B : B \subseteq A, \phi B < \infty\}$ for every $A \subseteq X$. Putting these together, ϕ is an inner measure. **Q**

(e) So 413C tells us that μ is a complete measure, and of course it is inner regular with respect to $\mathcal{K}$, by the definition of ϕ. It is semi-finite because $\mu K = \phi_0 K$ is finite for every $K \in \mathcal{K}$. Now suppose that $E \subseteq X$ and that $E \cap F \in \Sigma$ whenever $\mu F < \infty$. Take any $A \subseteq X$. If $L \in \mathcal{K}$ and $L \subseteq A$, we have $L \in \Sigma$ and $\mu L < \infty$, so

$$\phi_0 L = \mu L = \mu(L \cap E) + \mu(L \setminus E) = \phi(L \cap E) + \phi(L \setminus E) \leq \phi(A \cap E) + \phi(A \setminus E);$$

taking the supremum over L, $\phi A \leq \phi(A \cap E) + \phi(A \setminus E)$. As A is arbitrary, $E \in \Sigma$; as E is arbitrary, μ is locally determined.

(f) Finally, μ is unique by 412L.

413J Theorem Let X be a set and $\mathcal{K}$ a family of subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever $K, K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ whenever $K, K' \in \mathcal{K}$.

Let $\phi_0 : \mathcal{K} \to [0,\infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K}, K' \subseteq K \setminus L\}$ whenever $K, L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection.

Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}_\delta$, the family of sets expressible as intersections of sequences in $\mathcal{K}$.

proof (a) Set

$$\psi A = \sup\{\phi_0 K : K \in \mathcal{K}, K \subseteq A\} \text{ for } A \subseteq X,$$

$$\mathrm{T} = \{E : E \subseteq X, \psi A = \psi(A \cap E) + \psi(A \setminus E) \text{ for every } A \subseteq X\}.$$

Then 413H tells us that T is an algebra of subsets of X, including $\mathcal{K}$, and $\nu = \psi\restriction\mathrm{T}$ is an additive functional extending ϕ_0.

(b) Write T^f for $\{E : E \in \mathrm{T}, \nu E < \infty\}$. If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f with empty intersection, $\lim_{n\to\infty} \nu E_n = 0$. **P** Given $\epsilon > 0$, we can choose a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $K_n \subseteq E_n$ and

$$\nu K_n = \phi_0 K_n \geq \nu E_n - 2^{-n}\epsilon$$

for each n. Set $L_n = \bigcap_{i\leq n} K_i$ for each n; then

$$\lim_{n\to\infty} \nu L_n = \lim_{n\to\infty} \phi_0 L_n = 0$$

by hypothesis (β). But also, for each n,

$$\nu E_n \le \nu L_n + \sum_{i=0}^{n} \nu(E_i \setminus K_i) \le \nu L_n + 2\epsilon,$$

because ν is additive and non-negative and $E_n \subseteq L_n \cup \bigcup_{i \le n}(E_i \setminus K_i)$. So $\limsup_{n\to\infty} \nu E_n \le 2\epsilon$; as ϵ is arbitrary, $\lim_{n\to\infty} \nu E_n = 0$. **Q**

(c) Write T_δ^f for the family of sets expressible as intersections of sequences in T^f, and for $H \in \mathrm{T}_\delta^f$ set $\phi_1 H = \inf\{\nu E : H \subseteq E \in \mathrm{T}\}$. Note that because $E \cap F \in \mathrm{T}^f$ whenever E, $F \in \mathrm{T}^f$, every member of T_δ^f can be expressed as the intersection of a non-increasing sequence in T^f.

(i) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f with intersection $H \in \mathrm{T}_\delta^f$, $\phi_1 H = \lim_{n\to\infty} \nu E_n$. **P** Of course

$$\phi_1 H \le \inf_{n\in\mathbb{N}} \nu E_n = \lim_{n\to\infty} \nu E_n.$$

On the other hand, if $H \subseteq E \in \mathrm{T}$, then $\langle E_n \setminus E\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f with empty intersection, and

$$\nu E \ge \lim_{n\to\infty} \nu(E_n \cap E) = \lim_{n\to\infty} \nu E_n - \lim_{n\to\infty} \nu(E_n \setminus E) = \lim_{n\to\infty} \nu E_n$$

by (b) above. As E is arbitrary, $\phi_1(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \nu E_n$. **Q**

(ii) Because $\mathcal{K} \subseteq \mathrm{T}^f$, $\mathcal{K}_\delta \subseteq \mathrm{T}_\delta^f$. Now for any $H \in \mathrm{T}_\delta^f$, $\phi_1 H = \sup\{\phi_1 L : L \in \mathcal{K}_\delta,\ L \subseteq H\}$. **P** Express H as $\bigcap_{n\in\mathbb{N}} E_n$ where $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f. Given $\epsilon > 0$, we can choose a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $K_n \subseteq E_n$ and $\nu K_n \ge \nu E_n - 2^{-n}\epsilon$ for each n. Setting $L_n = \bigcap_{i\le n} K_i$ for each n and $L = \bigcap_{n\in\mathbb{N}} L_n$, we have $L \in \mathcal{K}_\delta$, $L \subseteq H$ and

$$\phi_1 H = \lim_{n\to\infty} \nu E_n \le \lim_{n\to\infty}(\nu L_n + \sum_{i=0}^{n} \nu(E_i \setminus K_i)) \le \phi_1 L + 2\epsilon.$$

As ϵ is arbitrary, this gives the result. **Q**

(d) We find that T_δ^f and ϕ_1 satisfy the conditions of 413I. **P** Of course $\emptyset \in \mathrm{T}_\delta^f$. If G, $H \in \mathrm{T}_\delta^f$ and $G \cap H = \emptyset$, express them as $\bigcap_{n\in\mathbb{N}} E_n$, $\bigcap_{n\in\mathbb{N}} F_n$ where $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ are non-increasing sequences in T^f. Then

$$G \cup H = \bigcap_{n\in\mathbb{N}} E_n \cup F_n$$

belongs to T_δ^f, and

$$\begin{aligned}\phi_1(G \cup H) &= \lim_{n\to\infty} \nu(E_n \cup F_n) = \lim_{n\to\infty} \nu E_n + \nu F_n - \nu(E_n \cap F_n)\\ &= \lim_{n\to\infty} \nu E_n + \nu F_n\end{aligned}$$

(by (b))

$$= \phi_1 G + \phi_1 H.$$

The definition of T_δ^f as the set of intersections of sequences in T^f ensures that the intersection of any sequence in T_δ^f will belong to T_δ^f.

Now suppose that G, $H \in \mathrm{T}_\delta^f$ and that $G \subseteq H$. Express them as intersections $\bigcap_{n\in\mathbb{N}} E_n$, $\bigcap_{n\in\mathbb{N}} F_n$ of non-increasing sequences in T^f, so that $\phi_1 G = \lim_{n\to\infty} \nu E_n$ and $\phi_1 H = \lim_{n\to\infty} \nu F_n$. For each n, set $H_n = \bigcap_{m\in\mathbb{N}} F_m \setminus E_n$, so that $H_n \in \mathrm{T}_\delta^f$, $H_n \subseteq H \setminus G$, and

$$\begin{aligned}\phi_1 H_n &= \lim_{m\to\infty} \nu(F_m \setminus E_n) = \lim_{m\to\infty} \nu F_m - \nu(F_m \cap E_n)\\ &\ge \lim_{m\to\infty} \nu F_m - \nu E_n = \phi_1 H - \nu E_n.\end{aligned}$$

Accordingly

$$\sup\{\phi_1 G' : G' \in \mathrm{T}_\delta^f,\ G' \subseteq H \setminus G\} \ge \sup_{n\in\mathbb{N}} \phi_1 H - \nu E_n = \phi_1 H - \phi_1 G.$$

On the other hand, if $G' \in \mathrm{T}_\delta^f$ and $G' \subseteq H \setminus G$, then

$$\phi_1 G + \phi_1 G' = \phi_1(G \cup G') \le \phi_1 H$$

because of course ϕ_1 is non-decreasing, as well as being additive on disjoint sets. So

$$\sup\{\phi_1 G' : G' \in \mathrm{T}_\delta^f,\ G' \subseteq H \setminus G\} = \phi_1 H - \phi_1 G$$

as required by condition (α) of 413I. Finally, suppose that $\langle H_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T_δ^f with empty intersection. For each $n \in \mathbb{N}$, let $\langle E_{ni}\rangle_{i\in\mathbb{N}}$ be a non-increasing sequence in T^f with intersection H_n, and set

$F_m = \bigcap_{n\le m} E_{nn}$ for each m. Then $\langle F_m\rangle_{m\in\mathbb{N}}$ is a non-increasing sequence in T^f with empty intersection, while $H_m \subseteq F_m$ for each m, so

$$\lim_{m\to\infty}\phi_1 H_m \le \lim_{m\to\infty}\nu F_m = 0.$$

Thus condition 413I(β) is satisfied, and we have the full list. **Q**

(e) By 413I, we have a complete locally determined measure μ, extending ϕ_1, and inner regular with respect to T^f_δ. Since $\phi_1 K = \nu K = \phi_0 K$ for $K \in \mathcal{K}$, μ extends ϕ_0. If G belongs to the domain of μ, and $\gamma < \mu G$, there is an $H \in \mathrm{T}^f_\delta$ such that $H \subseteq G$ and $\gamma < \mu H = \phi_1 H$; by (c-ii), there is an $L \in \mathcal{K}_\delta$ such that $L \subseteq H$ and $\gamma \le \phi_1 L = \mu L$. Thus μ is inner regular with respect to $\mathcal{K}_\delta$. To see that μ is unique, observe that if μ' is any other measure with these properties, and $L \in \mathcal{K}_\delta$, then L is expressible as $\bigcap_{n\in\mathbb{N}} K_n$ where $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$. Now

$$\mu L = \lim_{n\to\infty}\mu(\bigcap_{i\le n} K_i) = \lim_{n\to\infty}\phi_0(\bigcap_{i\le n} K_i) = \mu' L.$$

So μ and μ' must agree on $\mathcal{K}_\delta$, and by 412L they are identical.

413K Corollary (a) Let X be a set, Σ a subring of $\mathcal{P}X$, and $\nu : \Sigma \to [0,\infty[$ a non-negative finitely additive functional such that $\lim_{n\to\infty}\nu E_n = 0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection. Then ν has a unique extension to a complete locally determined measure on X which is inner regular with respect to the family Σ_δ of intersections of sequences in Σ.

(b) Let X be a set, Σ a subalgebra of $\mathcal{P}X$, and $\nu : \Sigma \to [0,\infty[$ a non-negative finitely additive functional such that $\lim_{n\to\infty}\nu E_n = 0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection. Then ν has a unique extension to a measure defined on the σ-algebra of subsets of X generated by Σ.

proof (a) Take Σ, ν in place of $\mathcal{K}$, ϕ_0 in 413J.

(b) Let ν_1 be the complete extension as in (a), and let ν_1' be the restriction of ν_1 to the σ-algebra Σ' generated by Σ; this is the extension required here. To see that ν_1' is unique, use the Monotone Class Theorem (136C).

Remark These are versions of the **Hahn extension theorem**. You will sometimes see (b) above stated as 'an additive functional on an algebra of sets extends to a measure iff it is countably additive'. But this formulation depends on a different interpretation of the phrase 'countably additive' from the one used in this book; see the note after the definition in 326I[9].

413L It will be useful to have a definition extending an idea in §342.

Definition A **countably compact class** (or **semicompact paving**) is a family $\mathcal{K}$ of sets such that $\bigcap_{n\in\mathbb{N}} K_n \ne \emptyset$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $\bigcap_{i\le n} K_i \ne \emptyset$ for every $n \in \mathbb{N}$.

413M Corollary Let X be a set and $\mathcal{K}$ a countably compact class of subsets of X such that

$\emptyset \in \mathcal{K}$,

($\dagger$) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

($\ddagger$) $K \cap K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$.

Let $\phi_0 : \mathcal{K} \to [0,\infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$.

Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}_\delta$, the family of sets expressible as intersections of sequences in $\mathcal{K}$.

proof The point is that the hypothesis (β) of 413J is necessarily satisfied: if $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, then, because $\mathcal{K}$ is countably compact, there must be some n such that $K_n = \emptyset$. Since hypothesis (α) here is already enough to ensure that $\phi_0\emptyset = 0$ and $\phi_0 K \ge 0$ for every $K \in \mathcal{K}$, we must have $\inf_{n\in\mathbb{N}}\phi_0 K_n = 0$. So we apply 413J to get the result.

413N I now turn to constructions of a different kind, being extension theorems in which the extension is not uniquely defined. Again I start with a theorem on finitely additive functionals.

[9] Formerly 326E.

Theorem Let X be a set, T_0 a subring of $\mathcal{P}X$, and $\nu_0 : \mathrm{T}_0 \to [0, \infty[$ a finitely additive functional. Suppose that $\mathcal{K} \subseteq \mathcal{P}X$ is a family of sets such that

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$,

every member of $\mathcal{K}$ is included in some member of T_0,

and ν_0 is inner regular with respect to $\mathcal{K}$ in the sense that

(α) $\nu_0 E = \sup\{\nu_0 K : K \in \mathcal{K} \cap \mathrm{T}_0,\ K \subseteq E\}$ for every $E \in \mathrm{T}_0$.

Then ν_0 has an extension to a non-negative finitely additive functional ν_1, defined on a subring T_1 of $\mathcal{P}X$ including $\mathrm{T}_0 \cup \mathcal{K}$, inner regular with respect to $\mathcal{K}$, and such that whenever $E \in \mathrm{T}_1$ and $\epsilon > 0$ there is an $E_0 \in \mathrm{T}_0$ such that $\nu_1(E \triangle E_0) \le \epsilon$.

proof (a) Let P be the set of all non-negative additive real-valued functionals ν, defined on subrings of $\mathcal{P}X$, inner regular with respect to $\mathcal{K}$, and such that

($*$) whenever $E \in \operatorname{dom}\nu$ and $\epsilon > 0$ there is an $E_0 \in \mathrm{T}_0$ such that $\nu(E \triangle E_0) \le \epsilon$.

Order P by extension of functions, so that P is a partially ordered set.

(b) It will be convenient to borrow some notation from the theory of countably additive functionals. If T is a subring of $\mathcal{P}X$ and $\nu : \mathrm{T} \to [0, \infty[$ is a non-negative additive functional, set

$$\nu^* A = \inf\{\nu E : A \subseteq E \in \mathrm{T}\}, \quad \nu_* A = \sup\{\nu E : A \supseteq E \in \mathrm{T}\}$$

for every $A \subseteq X$ (interpreting $\inf \emptyset$ as ∞ if necessary). Now if $A \subseteq X$ and E, $F \in \mathrm{T}$ are disjoint,

$$\nu^*(A \cap (E \cup F)) = \nu^*(A \cap E) + \nu^*(A \cap F),$$

$$\nu_*(A \cap (E \cup F)) = \nu_*(A \cap E) + \nu_*(A \cap F).$$

$$
\begin{aligned}
\mathbf{P}\ \nu^*(A \cap (E \cup F)) &= \inf\{\nu G : G \in \mathrm{T},\ A \cap (E \cup F) \subseteq G\} \\
&= \inf\{\nu G : G \in \mathrm{T},\ A \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\
&= \inf\{\nu(G \cap E) + \nu(G \cap F) : G \in \mathrm{T},\ A \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\
&= \inf\{\nu G_1 + \nu G_2 : G_1,\ G_2 \in \mathrm{T},\ A \cap E \subseteq G_1 \subseteq E,\ A \cap F \subseteq G_2 \subseteq F\} \\
&= \inf\{\nu G_1 : G_1 \in \mathrm{T},\ A \cap E \subseteq G_1 \subseteq E\} \\
&\qquad\qquad + \inf\{\nu G_2 : G_2 \in \mathrm{T},\ A \cap F \subseteq G_2 \subseteq F\} \\
&= \nu^*(E \cap A) + \nu^*(F \cap A),
\end{aligned}
$$

$$
\begin{aligned}
\nu_*(A \cap (E \cup F)) &= \sup\{\nu G : G \in \mathrm{T},\ A \cap (E \cup F) \supseteq G\} \\
&= \sup\{\nu(G \cap E) + \nu(G \cap F) : G \in \mathrm{T},\ A \cap (E \cup F) \supseteq G\} \\
&= \sup\{\nu G_1 + \nu G_2 : G_1,\ G_2 \in \mathrm{T},\ A \cap E \supseteq G_1,\ A \cap F \supseteq G_2\} \\
&= \sup\{\nu G_1 : G_1 \in \mathrm{T},\ A \cap E \supseteq G_1\} \\
&\qquad\qquad + \sup\{\nu G_2 : G_2 \in \mathrm{T},\ A \cap F \supseteq G_2\} \\
&= \nu_*(E \cap A) + \nu_*(F \cap A).\ \mathbf{Q}
\end{aligned}
$$

(c) The key to the proof is the following fact: if $\nu \in P$ and $M \in \mathcal{K}$, there is a $\nu' \in P$ such that ν' extends ν and $M \in \operatorname{dom}\nu'$. $\mathbf{P}$ Set $\mathrm{T} = \operatorname{dom}\nu$, $\mathrm{T}' = \{(E \cap M) \cup (F \setminus M) : E,\ F \in \mathrm{T}\}$. For $H \in \mathrm{T}'$, set

$$\nu' H = \nu^*(H \cap M) + \nu_*(H \setminus M).$$

Now we have to check the following.

(i) T' is a subring of $\mathcal{P}X$, because if E, F, E', $F' \in \mathrm{T}$ then

$$((E \cap M) \cup (F \setminus M)) * ((E' \cap M) \cup (F' \setminus M)) = ((E * E') \cap M) \cup ((F * F') \setminus M)$$

for both the Boolean operations $* = \triangle$ and $* = \cap$. $\mathrm{T}' \supseteq \mathrm{T}$ because $E = (E \cap M) \cup (E \setminus M)$ for every $E \in \mathrm{T}$. (Cf. 312N.) $M \in \mathrm{T}'$ because there is some $E \in \mathrm{T}_0$ such that $M \subseteq E$, so that $M = (E \cap M) \cup (\emptyset \setminus M) \in \mathrm{T}'$.

(ii) ν' is finite-valued because if $H = (E \cap M) \cup (F \setminus M)$, where E, $F \in \mathrm{T}$, then $\nu' H \le \nu E + \nu F$. If H, $H' \in \mathrm{T}$ are disjoint, they can be expressed as $(E \cap M) \cup (F \setminus M)$, $(E' \cap M) \cup (F' \setminus M)$ where E, F, E', F' belong to T; replacing E', F' by $E' \setminus E$ and $F' \setminus F$ if necessary, we may suppose that $E \cap E' = F \cap F' = \emptyset$. Now

$$\begin{aligned}\nu'(H \cup H') &= \nu^*((E \cup E') \cap M) + \nu_*((F \cup F') \cap (X \setminus M))\\ &= \nu^*(E \cap M) + \nu^*(E' \cap M) + \nu_*(F \cap (X \setminus M)) + \nu_*(F' \cap (X \setminus M))\end{aligned}$$

(by (b) above)

$$= \nu' H + \nu' H'.$$

Thus ν' is additive.

(iii) If $E \in \mathrm{T}$, then

$$\begin{aligned}\nu_*(E \setminus M) &= \sup\{\nu F : F \in \mathrm{T},\, F \subseteq E \setminus M\}\\ &= \sup\{\nu E - \nu(E \setminus F) : F \in \mathrm{T},\, F \subseteq E \setminus M\}\\ &= \sup\{\nu E - \nu F : F \in \mathrm{T},\, E \cap M \subseteq F \subseteq E\}\\ &= \nu E - \inf\{\nu F : F \in \mathrm{T},\, E \cap M \subseteq F \subseteq E\} = \nu E - \nu^*(E \cap M).\end{aligned}$$

So

$$\nu' E = \nu^*(E \cap M) + \nu_*(E \setminus M) = \nu E.$$

Thus ν' extends ν.

(iv) If $H \in \mathrm{T}'$ and $\epsilon > 0$, express H as $(E \cap M) \cup (F \setminus M)$, where E, $F \in \mathrm{T}$. Then we can find (α) a $K \in \mathcal{K} \cap \mathrm{T}$ such that $K \subseteq E$ and $\nu(E \setminus K) \le \epsilon$ (β) an $F' \in \mathrm{T}$ such that $F' \subseteq F \setminus M$ and $\nu F' \ge \nu_*(F \setminus M) - \epsilon$ (γ) a $K' \in \mathcal{K} \cap \mathrm{T}$ such that $K' \subseteq F'$ and $\nu K' \ge \nu F' - \epsilon$. Set $L = (K \cap M) \cup K' \in \mathrm{T}'$; by the hypotheses (†) and (‡), $L \in \mathcal{K}$. Now $L \subseteq H$ and

$$\begin{aligned}\nu' L &= \nu'(K \cap M) + \nu' K' = \nu'(E \cap M) - \nu'((E \setminus K) \cap M) + \nu K'\\ &= \nu^*(H \cap M) - \nu^*((E \setminus K) \cap M) + \nu K' \ge \nu^*(H \cap M) - \nu(E \setminus K) + \nu F' - \epsilon\\ &\ge \nu^*(H \cap M) + \nu_*(F \setminus M) - 3\epsilon = \nu' H - 3\epsilon.\end{aligned}$$

As H and ϵ are arbitrary, ν is inner regular with respect to $\mathcal{K}$.

(v) Finally, given $H \in \mathrm{T}'$ and $\epsilon > 0$, take E, $F \in \mathrm{T}$ such that $H \cap M \subseteq E$, $F \subseteq H \setminus M$, $\nu E \le \nu^*(H \cap M) + \epsilon$ and $\nu F \ge \nu_*(H \setminus M) - \epsilon$. In this case,

$$\nu'(E \setminus (H \cap M)) = \nu' E - \nu'(H \cap M) = \nu E - \nu^*(H \cap M) \le \epsilon,$$

$$\nu'((H \setminus M) \setminus F) = \nu'(H \setminus M) - \nu' F = \nu_*(H \setminus M) - \nu F \le \epsilon.$$

But as

$$H \triangle (E \cup F) \subseteq (E \setminus (H \cap M)) \cup ((H \setminus M) \setminus F),$$

$\nu'(H \triangle (E \cup F)) \le 2\epsilon$. Now ν satisfies the condition $(*)$, so there is an $E_0 \in \mathrm{T}_0$ such that $\nu((E \cup F) \triangle E_0) \le \epsilon$, and $\nu'(H \triangle E_0) \le 3\epsilon$. As H and ϵ are arbitrary, ν' satisfies $(*)$.

This completes the proof that ν' is a member of P extending ν. **Q**

(d) It is easy to check that if $Q \subseteq P$ is a non-empty totally ordered subset, the smallest common extension ν' of the functions in Q belongs to P. (To see that ν' is inner regular with respect to $\mathcal{K}$, observe that if $E \in \operatorname{dom} \nu'$ and $\gamma < \nu' E$, there is some $\nu \in Q$ such that $E \in \operatorname{dom} \nu$; now there is a $K \in \mathcal{K} \cap \operatorname{dom} \nu$ such that $K \subseteq E$ and $\nu K \ge \gamma$, so that $K \in \mathcal{K} \cap \operatorname{dom} \nu'$ and $\nu' K \ge \gamma$.) And of course P is not empty, because $\nu_0 \in P$. So by Zorn's Lemma P has a maximal element ν_1 say; write T_1 for the domain of ν_1. If $M \in \mathcal{K}$ there is an element of P, with a domain containing M, extending ν_1; as ν_1 is maximal, this must be ν_1 itself, so $M \in \mathrm{T}_1$. Thus $\mathcal{K} \subseteq \mathrm{T}_1$, and ν_1 has all the required properties.

413O Corollary Let (X, Σ_0, μ_0) be a measure space and $\mathcal{K}$ a countably compact class of subsets of X such that

(†) $K \cup K' \in \mathcal{K}$ whenever $K, K' \in \mathcal{K}$ are disjoint,

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

$\mu_0^* K < \infty$ for every $K \in \mathcal{K}$,

μ_0 is inner regular with respect to $\mathcal{K}$.

Then μ_0 has an extension to a complete locally determined measure μ, defined on every member of $\mathcal{K}$, inner regular with respect to $\mathcal{K}$, and such that whenever $E \in \operatorname{dom}\mu$ and $\mu E < \infty$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$.

proof (a) Set $\mathrm{T}_0 = \{E : E \in \Sigma_0,\ \mu_0 E < \infty\}$, $\nu_0 = \mu_0 \restriction \mathrm{T}_0$. Then ν_0, T_0 satisfy the conditions of 413N; take ν_1, T_1 as in 413N. If $K, L \in \mathcal{K}$ and $L \subseteq K$, then

$$\nu_1 L + \sup\{\nu_1 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\} = \nu_1 L + \nu_1(K \setminus L) = \nu_1 K.$$

So $\nu_1 \restriction \mathcal{K}$ satisfies the conditions of 413M and there is a complete locally determined measure μ, extending $\nu_1 \restriction \mathcal{K}$, and inner regular with respect to $\mathcal{K}$.

(b) Write Σ for the domain of μ. Then $\mathrm{T}_1 \subseteq \Sigma$. **P** If $E \in \mathrm{T}_1$ and $K \in \mathcal{K}$,

$$\begin{aligned}
&\mu_*(K \cap E) + \mu_*(K \setminus E)\\
&\qquad \ge \sup\{\mu K' : K' \in \mathcal{K},\ K' \subseteq K \cap E\} + \sup\{\mu K' : K' \in \mathcal{K},\ K' \subseteq K \setminus E\}\\
&\qquad = \sup\{\nu_1 K' : K' \in \mathcal{K},\ K' \subseteq K \cap E\} + \sup\{\nu_1 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus E\}\\
&\qquad = \nu_1(K \cap E) + \nu_1(K \setminus E) = \nu_1 K = \mu K.
\end{aligned}$$

By 413F(iv), $E \in \Sigma$. **Q** It follows at once that μ extends ν_1, since if $E \in \mathrm{T}_1$

$$\nu_1 E = \sup\{\nu_1 K : K \in \mathcal{K},\ K \subseteq E\} = \sup\{\mu K : K \in \mathcal{K},\ K \subseteq E\} = \mu E.$$

(c) In particular, μ agrees with μ_0 on T_0. Now in fact μ extends μ_0. **P** Take $E \in \Sigma_0$. If $K \in \mathcal{K}$, there is an $E_0 \in \Sigma_0$ such that $K \subseteq E_0$ and $\mu_0 E_0 < \infty$. Since $E \cap E_0 \in \mathrm{T}_0 \subseteq \Sigma$, $E \cap K = E \cap E_0 \cap K \in \Sigma$. As K is arbitrary, $E \in \Sigma$, by 413F(ii). Next, because every member of $\mathcal{K}$ is included in a member of T_0,

$$\begin{aligned}
\mu_0 E &= \sup\{\mu_0 K : K \in \mathcal{K} \cap \Sigma_0,\ K \subseteq E\} = \sup\{\mu_0(E \cap E_0) : E_0 \in \mathrm{T}_0\}\\
&= \sup\{\mu(E \cap E_0) : E_0 \in \mathrm{T}_0\} = \sup\{\mu K : K \in \mathcal{K},\ K \subseteq E\} = \mu E. \text{ **Q**}
\end{aligned}$$

(d) Finally, suppose that $E \in \Sigma$ and $\mu E < \infty$. For each $n \in \mathbb{N}$ we can find $K_n \in \mathcal{K}$ and $E_n \in \Sigma_0$ such that $K_n \subseteq E$, $\mu(E \setminus K_n) \le 2^{-n}$ and $\nu_1(K_n \triangle E_n) \le 2^{-n}$. In this case $\sum_{n=0}^{\infty} \mu(E_n \triangle E) < \infty$, so $\mu(E \triangle E') = 0$, where $E' = \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} E_m \in \Sigma_0$.

Thus μ has all the required properties.

413P I now describe an alternative route to some of the applications of 413N. As before, I do as much as possible in the context of finitely additive functionals.

Lemma Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let $\lambda : \mathcal{K} \to [0, \infty[$ be a bounded functional such that

$$\lambda\emptyset = 0, \quad \lambda K \le \lambda K' \text{ whenever } K, K' \in \mathcal{K} \text{ and } K \subseteq K',$$

$$\lambda(K \cup K') + \lambda(K \cap K') \ge \lambda K + \lambda K' \text{ for all } K, K' \in \mathcal{K}.$$

Then there is a finitely additive functional $\nu : \mathcal{P}X \to [0, \infty[$ such that

$$\nu X = \sup_{K\in\mathcal{K}} \lambda K, \quad \nu K \ge \lambda K \text{ for every } K \in \mathcal{K}.$$

proof (a) Let us consider first the case in which $\mathcal{K}$ is finite. I induce on $n = \#(\mathcal{K})$. If $n = 1$ then $\mathcal{K} = \{\emptyset\}$ and ν must be the zero functional. For the inductive step to $n > 1$, let K_0 be a minimal member of $\mathcal{K} \setminus \{\emptyset\}$. If $K \in \mathcal{K}$ then $K \cap K_0$ is a member of $\mathcal{K}$ included in K_0, so is either empty or K_0, that is, either $K \cap K_0 = \emptyset$ or $K \supseteq K_0$. Set $Y = X \setminus K_0$ and $\mathcal{L} = \{K \setminus K_0 : K \in \mathcal{K}\}$. Then $\mathcal{L}$ is a sublattice of $\mathcal{P}Y$ containing $\emptyset$, and $K \mapsto K \setminus K_0 : \mathcal{K} \to \mathcal{L}$ is surjective but not injective, so $\#(\mathcal{L}) < n$.

For $L \in \mathcal{L}$, observe that $L \cup K_0 \in \mathcal{K}$. **P** There is a $K \in \mathcal{K}$ such that $L = K \setminus K_0$. If K is disjoint from K_0, then $L \cup K_0 = K \cup K_0$ belongs to $\mathcal{K}$; if K includes K_0 then $L \cup K_0 = K$ belongs to $\mathcal{K}$. **Q**

We can therefore define $\lambda' : \mathcal{L} \to [0, \infty[$ by setting

$$\lambda' L = \lambda(L \cup K_0) - \lambda K_0$$

for every $L \in \mathcal{L}$. Of course $\lambda'\emptyset = 0$ and $\lambda' L \le \lambda' L'$ whenevery $L \subseteq L'$. If L, $L' \in \mathcal{L}$ then

$$\begin{aligned}\lambda'(L \cup L') + \lambda'(L \cap L') &= \lambda(L \cup L' \cup K_0) + \lambda((L \cap L') \cup K_0) - 2\lambda K_0 \\ &= \lambda((L \cup K_0) \cup (L' \cup K_0)) + \lambda((L \cup K_0) \cap (L' \cup K_0)) - 2\lambda K_0 \\ &\ge \lambda(L \cup K_0) + \lambda(L' \cup K_0) - 2\lambda K_0 = \lambda' L + \lambda' L'.\end{aligned}$$

So the hypotheses of this lemma are satisfied by Y, $\mathcal{L}$ and λ', and by the inductive hypothesis there is a finitely additive functional $\nu' : \mathcal{P}Y \to [0, \infty[$ such that

$$\nu' Y = \sup_{L \in \mathcal{L}} \lambda' L, \quad \nu' L \ge \lambda' L \text{ for every } L \in \mathcal{L}.$$

Fix any $x_0 \in K_0$ and define $\nu : \mathcal{P}X \to [0, \infty[$ by setting

$$\begin{aligned}\nu A &= \lambda K_0 + \nu'(A \cap Y) \text{ if } x_0 \in A \subseteq X, \\ &= \nu'(A \cap Y) \text{ for other } A \subseteq X.\end{aligned}$$

Then ν is additive. If $K \in \mathcal{K}$ is disjoint from K_0 then

$$\nu K = \nu' K \ge \lambda' K = \lambda(K \cup K_0) - \lambda K_0 \ge \lambda K - \lambda(K \cap K_0) = \lambda K.$$

If $K \in \mathcal{K}$ includes K_0 then

$$\nu K = \lambda K_0 + \nu'(K \setminus K_0) \ge \lambda K_0 + \lambda'(K \setminus K_0) = \lambda K_0 + \lambda K - \lambda K_0 = \lambda K.$$

Finally,

$$\begin{aligned}\nu X = \lambda K_0 + \nu' Y &= \lambda K_0 + \sup_{L \in \mathcal{L}} \lambda' L \\ &= \lambda K_0 + \sup_{K \in \mathcal{K}} (\lambda(K \cup K_0) - \lambda K_0) = \sup_{K \in \mathcal{K}} \lambda(K \cup K_0) = \sup_{K \in \mathcal{K}} \lambda K.\end{aligned}$$

So ν has the required properties and the induction continues.

(b) For the general case, set $\gamma = \sup_{K \in \mathcal{K}} \lambda K$. We need to know that every finite subset of $\mathcal{K}$ is included in a finite sublattice of $\mathcal{K}$; this is because it is included in a finite subalgebra $\mathcal{E}$ of $\mathcal{P}X$ and $\mathcal{K} \cap \mathcal{E}$ is a sublattice. Let N be the set of all finitely additive functionals $\nu : \mathcal{P}X \to [0, \gamma]$. Then N is a closed subset of $[0, \gamma]^{\mathcal{P}X}$, so is compact. For each $K \in \mathcal{K}$ set $\mathrm{N}_K = \{\nu : \nu \in \mathrm{N}, \nu K \ge \lambda K\}$. Then N_K is a closed subset of N. If $\mathcal{K}_0 \subseteq \mathcal{K}$ is finite, there is a finite sublattice $\mathcal{L}$ of $\mathcal{K}$ including $\mathcal{K}_0 \cup \{\emptyset\}$, and now (a) tells us that there is a $\nu \in \bigcap_{K \in \mathcal{L}} \mathrm{N}_K$. Thus $\{\mathrm{N}_K : K \in \mathcal{K}\}$ has the finite intersection property and there is a $\nu \in \bigcap_{K \in \mathcal{K}} \mathrm{N}_K$. In this case, $\nu : \mathcal{P}X \to [0, \gamma]$ is a finitely additive functional dominating λ; it follows that $\nu X = \gamma$ and the proof is complete.

Remark If P is a lattice, a function $f : P \to]-\infty, \infty]$ such that $f(p \vee q) + f(p \wedge q) \ge f(p) + f(q)$ for all p, $q \in P$ is called **supermodular**.

413Q Theorem Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let Σ be the algebra of subsets of X generated by $\mathcal{K}$, and $\nu_0 : \Sigma \to [0, \infty[$ a finitely additive functional. Then there is a finitely additive functional $\nu : \Sigma \to [0, \infty[$ such that

(i) $\nu X = \sup_{K \in \mathcal{K}} \nu_0 K$,

(ii) $\nu K \ge \nu_0 K$ for every $K \in \mathcal{K}$,

(iii) ν is inner regular with respect to $\mathcal{K}$ in the sense that $\nu E = \sup\{\nu K : K \in \mathcal{K}, K \subseteq E\}$ for every $E \in \Sigma$.

proof (a) Set $\gamma = \sup_{K \in \mathcal{K}} \nu_0 K$. Let P be the set of all functionals $\lambda : \mathcal{K} \to [0, \gamma]$ such that

$$\lambda K + \lambda K' \le \lambda(K \cup K') + \lambda(K \cap K')$$

for every K, $K' \in \mathcal{K}$. Give P the natural partial order inherited from $\mathbb{R}^{\mathcal{K}}$. Note that $\nu_0 \restriction \mathcal{K}$ belongs to P. If $Q \subseteq P$ is non-empty and upwards-directed, then $\sup Q$, taken in $\mathbb{R}^{\mathcal{K}}$, belongs to P; so there is a maximal $\lambda \in P$ such that $\nu_0 \restriction \mathcal{K} \le \lambda$. By 413P, there is a non-negative additive functional ν on $\mathcal{P}X$ such that $\nu K \ge \lambda K$ for every $K \in \mathcal{K}$ and $\nu X = \gamma$. Since $\nu \restriction \mathcal{K}$ also belongs to P, we must have $\nu K = \lambda K$ for every $K \in \mathcal{K}$.

(b) Now for any $K_0 \in \mathcal{K}$,

$$\nu K_0 + \sup\{\nu L : L \in \mathcal{K},\, L \subseteq X \setminus K_0\} = \gamma.$$

P (i) Set $\mathcal{L} = \{L : L \in \mathcal{K},\, L \subseteq X \setminus K_0\}$. For $A \subseteq X$, set $\theta_0 A = \sup_{L \in \mathcal{L}} \nu(A \cap L)$. Because $\mathcal{L}$ is upwards-directed, $\theta_0 : \mathcal{P}X \to \mathbb{R}$ is additive, and of course $0 \le \theta_0 \le \nu$. Set $\theta_1 = \nu - \theta_0$, so that θ_1 is another additive functional, and write

$$\lambda' K = \theta_0 K + \sup\{\theta_1 M : M \in \mathcal{K},\, M \cap K_0 \subseteq K\}$$

for $K \in \mathcal{K}$.

(ii) If K, $K' \in \mathcal{K}$ and $\epsilon > 0$, there are M, $M' \in \mathcal{K}$ such that $M \cap K_0 \subseteq K$, $M' \cap K_0 \subseteq K'$ and

$$\theta_0 K + \theta_1 M \ge \lambda' K - \epsilon, \quad \theta_0 K' + \theta_1 M' \ge \lambda' K' - \epsilon.$$

Now

$$M \cup M' \in \mathcal{K}, \quad M \cap M' \in \mathcal{K},$$

$$(M \cup M') \cap K_0 \subseteq K \cup K', \quad (M \cap M') \cap K_0 \subseteq K \cap K',$$

so

$$\begin{aligned}\lambda'(K \cup K') + \lambda'(K \cap K') &\ge \theta_0(K \cup K') + \theta_1(M \cup M') + \theta_0(K \cap K') + \theta_1(M \cap M')\\ &= \theta_0 K + \theta_1 M + \theta_0 K' + \theta_1 M' \ge \lambda' K + \lambda' K' - 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\lambda'(K \cup K') + \lambda'(K \cap K') \ge \lambda K + \lambda K'$.

(iii) Suppose that K, $M \in \mathcal{K}$ are such that $M \cap K_0 \subseteq K$. If $L \in \mathcal{L}$, then

$$\begin{aligned}\nu(K \cap L) + \theta_1 M &= \nu(K \cap L) + \nu M - \theta_0 M\\ &= \nu(M \cap K \cap L) + \nu(M \cup (K \cap L)) - \theta_0 M \le \gamma\end{aligned}$$

because $K \cap L \in \mathcal{L}$; taking the supremum over L and M, $\lambda' K \le \gamma$. As K is arbitrary, $\lambda' \in P$.

(iv) If $K \in \mathcal{K}$, then of course $K \cap K_0 \subseteq K$, so

$$\lambda' K \ge \theta_0 K + \theta_1 K = \nu K = \lambda K.$$

Thus $\lambda' \ge \lambda$. Because λ is maximal, $\lambda' = \lambda$. But this means that

$$\lambda K_0 = \lambda' K_0 = \theta_0 K_0 + \sup\{\theta_1 M : M \in \mathcal{K},\, M \cap K_0 \subseteq K_0\} = \sup_{M \in \mathcal{K}} \theta_1 M.$$

Now given $\epsilon > 0$ there is an $M \in \mathcal{K}$ such that

$$\gamma - \epsilon \le \nu_0 M \le \lambda M = \nu M,$$

so that

$$\nu K_0 = \lambda K_0 \ge \theta_1 M = \nu M - \theta_0 M \ge \gamma - \epsilon - \theta_0 M \ge \gamma - \epsilon - \sup_{L \in \mathcal{L}} \nu L,$$

and $\nu K_0 + \sup_{L \in \mathcal{L}} \nu L \ge \gamma - \epsilon$. As ϵ is arbitrary, $\nu K_0 + \sup_{L \in \mathcal{L}} \nu L \ge \gamma$. But of course $\nu K_0 + \nu L \le \nu X = \gamma$ for every $L \in \mathcal{L}$, so $\nu K_0 + \sup_{L \in \mathcal{L}} \nu L = \gamma$, as claimed. **Q**

(c) It follows that if K, $L \in \mathcal{K}$ and $L \subseteq K$,

$$\nu K = \nu L + \sup\{\nu K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\}.$$

P Because ν is additive and non-negative, we surely have

$$\nu K \ge \nu L + \sup\{\nu K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\}.$$

On the other hand, given $\epsilon > 0$, there is an $M \in \mathcal{K}$ such that $M \subseteq X \setminus L$ and $\nu L + \nu M \ge \gamma - \epsilon$, so that $M \cap K \in \mathcal{K}$, $M \cap K \subseteq K \setminus L$ and

$$\nu L + \nu(M \cap K) = \nu L + \nu K + \nu M - \nu(M \cup K) \ge \nu K + \gamma - \epsilon - \gamma = \nu K - \epsilon.$$

As ϵ is arbitrary,

$$\nu K \le \nu L + \sup\{\nu K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\}$$

and we have equality. **Q**

(d) By 413H, we have an additive functional $\nu' : \Sigma \to [0,\infty[$ such that $\nu' E = \sup\{\nu K : K \in \mathcal{K}, K \subseteq E\}$ for every $E \in \Sigma$. It is easy to show that ν' and ν must agree on Σ, but even without doing so we can see that ν' has the properties (i)-(iii) required in the theorem.

413R The following lemma on countably compact classes, corresponding to 342Db, will be useful.

Lemma (MARCZEWSKI 53) Let $\mathcal{K}$ be a countably compact class of sets. Then there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ such that $K \cup L \in \mathcal{K}^*$ and $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}^*$ whenever K, $L \in \mathcal{K}^*$ and $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}^*$.

proof (a) Write $\mathcal{K}_s$ for $\{K_0 \cup \ldots \cup K_n : K_0, \ldots, K_n \in \mathcal{K}\}$. Then $\mathcal{K}_s$ is countably compact. **P** Let $\langle L_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}_s$ such that $\bigcap_{i\le n} L_i \ne \emptyset$ for each $n \in \mathbb{N}$. Then there is an ultrafilter $\mathcal{F}$ on $X = \bigcup \mathcal{K}$ containing every L_n. For each n, L_n is a finite union of members of $\mathcal{K}$, so there must be a $K_n \in \mathcal{K}$ such that $K_n \subseteq L_n$ and $K_n \in \mathcal{F}$. Now $\bigcap_{i\le n} K_i \ne \emptyset$ for every n, so $\bigcap_{n\in\mathbb{N}} K_n \ne \emptyset$ and $\bigcap_{n\in\mathbb{N}} L_n \ne \emptyset$. As $\langle L_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{K}_s$ is countably compact. **Q**

Note that $L \cup L' \in \mathcal{K}_s$ for all L, $L' \in \mathcal{K}_s$.

(b) Write $\mathcal{K}^*$ for

$$\{\bigcap \mathcal{L}_0 : \mathcal{L}_0 \subseteq \mathcal{K}_s \text{ is non-empty and countable}\}.$$

Then $\mathcal{K}^*$ is countably compact. **P** If $\langle M_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{K}^*$ such that $\bigcap_{i\le n} M_i \ne \emptyset$ for every $n \in \mathbb{N}$, then for each $n \in \mathbb{N}$ let $\mathcal{L}_n \subseteq \mathcal{K}_s$ be a countable non-empty set such that $M_n = \bigcap \mathcal{L}_n$. Let $\langle L_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\bigcup_{n\in\mathbb{N}} \mathcal{L}_n$; then $\bigcap_{i\le n} L_i \ne \emptyset$ for every n, so $\bigcap_{n\in\mathbb{N}} L_n = \bigcap_{n\in\mathbb{N}} M_n$ is non-empty. As $\langle M_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{K}^*$ is countably compact. **Q**

(c) Of course $\mathcal{K} \subseteq \mathcal{K}_s \subseteq \mathcal{K}^*$. It is immediate from the definition of $\mathcal{K}^*$ that it is closed under countable intersections. Finally, if M_1, $M_2 \in \mathcal{K}^*$, let $\mathcal{L}_1$, $\mathcal{L}_2 \subseteq \mathcal{K}_s$ be countable sets such that $M_1 = \bigcap \mathcal{L}_1$ and $M_2 = \bigcap \mathcal{L}_2$; then $\mathcal{L} = \{L_1 \cup L_2 : L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2\}$ is a countable subset of $\mathcal{K}_s$, so $M_1 \cup M_2 = \bigcap \mathcal{L}$ belongs to $\mathcal{K}^*$.

413S Corollary Let X be a set and $\mathcal{K}$ a countably compact class of subsets of X. Let T be a subalgebra of $\mathcal{P}X$ and $\nu : \mathrm{T} \to \mathbb{R}$ a non-negative finitely additive functional.

(a) There is a complete measure μ on X such that $\mu X \le \nu X$, $\mathcal{K} \subseteq \operatorname{dom}\mu$ and $\mu K \ge \nu K$ for every $K \in \mathcal{K} \cap \mathrm{T}$.

(b) If

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$,

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

we may arrange that μ is inner regular with respect to $\mathcal{K}$.

proof By 413R, there is always a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ satisfying (†) and (‡); for case (b), take $\mathcal{K}^* = \mathcal{K}$. By 391G, there is an extension of ν to a finitely additive functional $\nu' : \mathcal{P}X \to \mathbb{R}$. Let T_1 be the subalgebra of $\mathcal{P}X$ generated by $\mathcal{K}^*$. By 413Q, there is a non-negative additive functional $\nu_1 : \mathrm{T}_1 \to \mathbb{R}$ such that $\nu_1 X \le \nu' X = \nu X$, $\nu_1 K \ge \nu' K = \nu K$ for every $K \in \mathcal{K}^* \cap \mathrm{T}$ and $\nu_1 E = \sup\{\nu_1 K : K \in \mathcal{K}^*, K \subseteq E\}$ for every $E \in \mathrm{T}_1$. In particular, if K, $L \in \mathcal{K}^*$,

$$\nu_1 L + \sup\{\nu_1 K' : K' \in \mathcal{K}^*, K' \subseteq K \setminus L\} = \nu_1 L + \nu_1(K \setminus L) = \nu_1 K.$$

So $\mathcal{K}^*$ and $\nu_1 \restriction \mathcal{K}^*$ satisfy the hypotheses of 413M. Accordingly we have a complete measure μ extending $\nu_1 \restriction \mathcal{K}^*$ and inner regular with respect to $\mathcal{K}^* = \mathcal{K}^*_\delta$; in which case

$$\mu K = \nu_1 K \ge \nu K$$

for every $K \in \mathcal{K} \cap \mathrm{T}$, and

$$\mu X = \sup_{K\in\mathcal{K}^*} \mu K = \sup_{K\in\mathcal{K}^*} \nu_1 K \le \nu_1 X \le \nu X,$$

as required.

413T The following fact is interesting and not quite obvious.

Proposition Let (X, Σ, μ) be a complete totally finite measure space, (Y, T, ν) a measure space, and $\mathfrak{S}$ a Hausdorff topology on Y such that ν is inner regular with respect to the closed sets. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of inverse-measure-preserving functions from X to Y. If $f(x) = \lim_{n\to\infty} f_n(x)$ is defined in Y for every $x \in X$, then f is inverse-measure-preserving.

proof Let μ_* be the inner measure associated with μ (413D). If $F \in \mathrm{T}$ is closed then $\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n} f_m^{-1}[F] \subseteq f^{-1}[F]$, so

$$\mu_* f^{-1}[F] \ge \mu(\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n} f_m^{-1}[F]) \ge \liminf_{n\to\infty} \mu f_n^{-1}[F] = \nu F.$$

So if H is any member of T,

$$\begin{aligned}\mu_* f^{-1}[H] &\ge \sup\{\mu_* f^{-1}[F] : F \in \mathrm{T},\ F \subseteq H \text{ and } F \text{ is closed}\}\\ &\ge \sup\{\nu F : F \in \mathrm{T},\ F \subseteq H \text{ and } F \text{ is closed}\} = \nu H.\end{aligned}$$

Taking complements,

$$\begin{aligned}\mu^* f^{-1}[H] &= \mu X - \mu_* f^{-1}[Y \setminus H]\\ &\le \nu Y - \nu(Y \setminus H) = \nu H \qquad (413\text{Ec})\end{aligned}$$

(of course $\nu Y = \mu f_0^{-1}[Y] = \mu X$). So $\mu^* f^{-1}[H] = \mu_* f^{-1}[H] = \nu H$. Because μ is complete, $\mu f^{-1}[H]$ is defined and equal to νH (413Ef). As H is arbitrary, f is inverse-measure-preserving.

413X Basic exercises (a) Define $\phi : \mathcal{P}\mathbb{N} \to [0,\infty[$ by setting $\phi A = 0$ if A is finite, ∞ otherwise. Check that ϕ satisfies conditions (α) and (β) of 413A, but that if we attempt to reproduce the construction of 413C then we obtain $\Sigma = \mathcal{P}\mathbb{N}$ and $\mu = \phi$, so that μ is not countably additive.

(b) Let ϕ_1, ϕ_2 be two inner measures on a set X, inducing measures μ_1 and μ_2 by the method of 413C. (i) Show that $\phi = \phi_1 + \phi_2$ is an inner measure. (ii) Show that the measure μ induced by ϕ extends the measure $\mu_1 + \mu_2$ defined on $\operatorname{dom}\mu_1 \cap \operatorname{dom}\mu_2$.

>(c) Let X be a set, ϕ an inner measure on X, and μ the measure constructed from it by the method of 413C. (i) Let Y be a subset of X. Show that $\phi\restriction\mathcal{P}Y$ is an inner measure on Y, and that the measure on Y defined from it extends the subspace measure μ_Y induced on Y by μ. (ii) Now suppose that ϕX is finite. Let Y be a set and $f : X \to Y$ a function. Show that $B \mapsto \phi f^{-1}[B]$ is an inner measure on Y, and that it defines a measure on Y which extends the image measure μf^{-1}.

(d) Let (X, Σ, μ) be a measure space. Set $\theta A = \frac{1}{2}(\mu^* A + \mu_* A)$ for every $A \subseteq X$. Show that θ is an outer measure on X, and that if μ is semi-finite then the measure defined from θ by Carathéodory's method extends μ.

>(e) Show that there is a partition $\langle A_n\rangle_{n\in\mathbb{N}}$ of $[0,1]$ such that $\mu_*(\bigcup_{i\le n} A_i) = 0$ for every n, where μ_* is Lebesgue inner measure. (*Hint*: set $A_n = (A + q_n) \cap [0,1]$ where $\langle q_n\rangle_{n\in\mathbb{N}}$ is an enumeration of $\mathbb{Q}$ and A is a suitable set; cf. 134B.)

(f) Let (X, Σ, μ) be a measure space. (i) Show that $\mu_*\restriction\Sigma$ is the semi-finite version μ_{sf} of μ as constructed in 213Xc. (ii) Show that if A is any subset of X, and Σ_A the subspace σ-algebra, then $\mu_*\restriction\Sigma_A$ is a semi-finite measure on A.

>(g) Let (X, Σ, μ) and (Y, T, ν) be two measure spaces, and λ the c.l.d. product measure on $X \times Y$. Show that $\lambda_*(A \times B) = \mu_* A \cdot \nu_* B$ for all $A \subseteq X$ and $B \subseteq Y$. (*Hint*: use Fubini's theorem to show that $\lambda_*(A \times B) \le \mu_* A \cdot \nu_* B$.)

(h)(i) Let (X, Σ, μ) be a σ-finite measure space and $f : X \to \mathbb{R}$ a function such that $\overline{\int} f d\mu$ is finite. Show that for every $\epsilon > 0$ there is a measure ν on X extending μ such that $\int f d\nu \ge \overline{\int} f d\mu - \epsilon$. (*Hint*: 133Ja, 215B(viii), 417Xa.) (ii) Let (X, Σ, μ) be a totally finite measure space and $f : X \to \mathbb{R}$ a bounded function. Show that there is a finitely additive functional $\nu : \mathcal{P}X \to [0,\infty[$, extending μ, such that $\oint f d\nu$, defined as in 363L, is equal to $\overline{\int} f d\mu$.

(i) Let X be a set and μ, ν two complete locally determined measures on X with domains Σ, T respectively, both inner regular with respect to $\mathcal{K} \subseteq \Sigma \cap \mathrm{T}$. Suppose that, for $K \in \mathcal{K}$, $\mu K = 0$ iff $\nu K = 0$. Show that $\Sigma = \mathrm{T}$ and that μ and ν have the same null ideals.

>(j) Let (X,T,ν) be a measure space. (i) Show that the measure constructed by the method of 413C from the inner measure ν_* is the c.l.d. version of ν. (ii) Set $\mathcal{K}=\{E:E\in\mathrm{T},\,\nu E<\infty\}$, $\phi_0=\nu\restriction\mathcal{K}$. Show that $\mathcal{K}$ and ϕ_0 satisfy the conditions of 413I, and that the measure constructed by the method there is again the c.l.d. version of ν.

(k) Let (X,Σ,μ) be a complete locally determined measure space and $\mathcal{L}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{L}$. Set $\mathcal{K}=\{K:K\in\mathcal{L}\cap\Sigma,\,\mu K<\infty\}$ and $\phi_0=\mu\restriction\mathcal{K}$. Show that $\mathcal{K}$ and ϕ_0 satisfy the conditions of 413J and that the measure constructed from them by the method there is just μ.

>(l) Let $\mathcal{K}$ be the family of subsets of $\mathbb{R}$ expressible as disjoint finite unions of bounded closed intervals. (i) Show from first principles that there is a unique functional $\phi_0:\mathcal{K}\to[0,\infty[$ such that $\phi_0[\alpha,\beta]=\beta-\alpha$ whenever $\alpha\le\beta$ and ϕ_0 satisfies the conditions of 413J. (ii) Show that the measure on $\mathbb{R}$ constructed from ϕ_0 by the method of 413J is Lebesgue measure.

(m) Let X be a set, Σ a subring of $\mathcal{P}X$, and $\nu:\Sigma\to[0,\infty[$ a non-negative additive functional such that $\lim_{n\to\infty}\nu E_n=0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection, as in 413K. Define $\theta:\mathcal{P}X\to[0,\infty]$ by setting

$$\theta A=\inf\{\sum_{n=0}^{\infty}\nu E_n:\langle E_n\rangle_{n\in\mathbb{N}}\text{ is a sequence in }\Sigma\text{ covering }A\}$$

for $A\subseteq X$, interpreting $\inf\emptyset$ as ∞ if necessary. Show that θ is an outer measure. Let μ_θ be the measure defined from θ by Carathéodory's method. Show that the measure defined from ν by the process of 413K is the c.l.d. version of μ_θ. (*Hint*: the c.l.d. version of μ_θ is inner regular with respect to Σ_δ.)

>(n) Let X be a set, Σ a subring of $\mathcal{P}X$, and $\nu:\Sigma\to[0,\infty[$ a non-negative additive functional. Show that the following are equiveridical: (i) ν has an extension to a measure on X; (ii) $\lim_{n\to\infty}\nu E_n=0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection; (iii) $\nu(\bigcup_{n\in\mathbb{N}}E_n)=\sum_{n=0}^{\infty}\nu E_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ such that $\bigcup_{n\in\mathbb{N}}E_n\in\Sigma$.

>(o) Let $\langle(X_n,\Sigma_n,\mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of probability spaces, and $\mathcal{F}$ a non-principal ultrafilter on $\mathbb{N}$. For x, $y\in\prod_{n\in\mathbb{N}}X_n$, write $x\sim y$ if $\{n:x(n)=y(n)\}\in\mathcal{F}$. (i) Show that $\sim$ is an equivalence relation; write X for the set of equivalence classes, and $x^\bullet\in X$ for the equivalence class of $x\in\prod_{n\in\mathbb{N}}X_n$. (Compare 351M.) (ii) Let Σ be the set of subsets of X expressible in the form $Q(\langle E_n\rangle_{n\in\mathbb{N}})=\{x^\bullet:x\in\prod_{n\in\mathbb{N}}E_n\}$, where $E_n\in\Sigma_n$ for each $n\in\mathbb{N}$. Show that Σ is an algebra of subsets of X, and that there is a well-defined additive functional $\nu:\Sigma\to[0,1]$ defined by setting $\nu(Q(\langle E_n\rangle_{n\in\mathbb{N}}))=\lim_{n\to\mathcal{F}}\mu_nE_n$. (iii) Show that for any non-decreasing sequence $\langle H_i\rangle_{i\in\mathbb{N}}$ in Σ there is an $H\in\Sigma$ such that $H\subseteq\bigcap_{i\in\mathbb{N}}H_i$ and $\nu H=\lim_{n\to\infty}\nu H_n$. (*Hint*: express each H_i as $Q(\langle E_{in}\rangle_{n\in\mathbb{N}})$. Do this in such a way that $E_{i+1,n}\subseteq E_{in}$ for all i, n. Take a decreasing sequence $\langle J_i\rangle_{i\in\mathbb{N}}$ in $\mathcal{F}$, with empty intersection, such that $\nu H_i\le\mu E_{in}+2^{-i}$ for $n\in J_i$. Set $E_n=E_{in}$ for $n\in J_i\setminus J_{i+1}$.) (iv) Show that there is a unique extension of ν to a complete probability measure μ on X which is inner regular with respect to Σ. (This is a kind of **Loeb measure**.)

(p) Let $\mathfrak{A}$ be a Boolean algebra and $K\subseteq\mathfrak{A}$ a sublattice containing 0. Suppose that $\lambda:K\to[0,\infty[$ is a bounded functional such that $\lambda0=0$, $\lambda a\le\lambda a'$ whenever a, $a'\in K$ and $a\subseteq a'$, and $\lambda(a\cup a')+\lambda(a\cap a')\ge\lambda a+\lambda a'$ for all a, $a'\in K$. Show that there is a non-negative additive functional $\nu:\mathfrak{A}\to\mathbb{R}$ such that $\nu a\ge\lambda a$ for every $a\in K$ and $\nu1=\sup_{a\in K}\lambda a$.

(q) Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let $\lambda:\mathcal{K}\to\mathbb{R}$ be an order-preserving function such that $\lambda\emptyset=0$ and $\lambda(K\cup K')+\lambda(K\cap K')=\lambda K+\lambda K'$ for all K, $K'\in\mathcal{K}$. Show that there is a non-negative additive functional $\nu:\mathcal{P}X\to[0,\infty]$ extending λ. (*Hint*: start with the case $X\in\mathcal{K}$.) (If P is a lattice, a functional $f:P\to\mathbb{R}$ such that $f(p\vee q)+f(p\wedge q)=f(p)+f(q)$ for all p, $q\in P$ is called **modular**.)

(r) Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$. Let $\lambda:\mathcal{K}\to[0,1]$ be a functional such that

$$\lambda K\le\lambda K'\text{ whenever }K,\,K'\in\mathcal{K}\text{ and }K\subseteq K',\quad\inf_{K\in\mathcal{K}}\lambda K=0,$$

$$\lambda(K\cup K')+\lambda(K\cap K')\le\lambda K+\lambda K'\text{ for all }K,\,K'\in\mathcal{K}$$

Show that there is a finitely additive functional $\nu:\mathcal{P}X\to[0,1]$ such that

$$\nu X=\sup_{K\in\mathcal{K}}\lambda K,\quad\nu K\le\lambda K\text{ for every }K\in\mathcal{K}.$$

(s) Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let $\lambda : \mathcal{K} \to [0, \infty[$ be such that $\lambda K \le \sum_{n=0}^{\infty}(\lambda K_n - \lambda L_n)$ whenever $K \in \mathcal{K}$ and $\langle K_n \rangle_{n\in\mathbb{N}}$, $\langle L_n \rangle_{n\in\mathbb{N}}$ are sequences in $\mathcal{K}$ such that $L_n \subseteq K_n$ for every n and $\langle K_n \setminus L_n \rangle_{n\in\mathbb{N}}$ is a disjoint cover of K. Show that there is a measure on X extending λ. (*Hint*: Show that we can apply 413Xq. Show that if T is the ring of subsets of X generated by $\mathcal{K}$, every member of T is a finite union of differences of members of $\mathcal{K}$. Now apply 413Ka. See Kelley & Srinivasan 71.)

413Y Further exercises (a) Give an example of two inner measures ϕ_1, ϕ_2 on a set X such that the measure defined by $\phi_1 + \phi_2$ strictly extends the sum of the measures defined by ϕ_1 and ϕ_2.

(b) Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i\in I}$ be any family of probability spaces, and λ the product measure on $X = \prod_{i\in I} X_i$. Show that $\lambda_*(\prod_{i\in I} A_i) \le \prod_{i\in I} (\mu_i)_* A_i$ whenever $A_i \subseteq X_i$ for every i, with equality if I is countable.

(c) Let (X, Σ, μ) be a totally finite measure space, and Z the Stone space of the Boolean algebra Σ. For $E \in \Sigma$ write $\widehat{E}$ for the corresponding open-and-closed subset of Z. Show that there is a unique function $f : X \to Z$ such that $f^{-1}[\widehat{E}] = E$ for every $E \in \Sigma$. Show that there is a measure ν on Z, inner regular with respect to the open-and-closed sets, such that f is inverse-measure-preserving with respect to μ and ν, and that f represents an isomorphism between the measure algebras of μ and ν. Use this construction to prove (vi)$\Rightarrow$(i) in Theorem 343B without appealing to the Lifting Theorem.

(d) Let X be a set, T a subalgebra of $\mathcal{P}X$, and $\nu : \mathrm{T} \to [0, \infty[$ a finitely additive functional. Suppose that there is a set $\mathcal{K} \subseteq \mathrm{T}$, containing $\emptyset$, such that (i) $\mu F = \sup\{\mu K : K \in \mathcal{K}, K \subseteq F\}$ for every $F \in \mathrm{T}$ (ii) $\mathcal{K}$ is **monocompact**, that is, $\bigcap_{n\in\mathbb{N}} K_n \ne \emptyset$ for every non-increasing sequence in $\mathcal{K}$. Show that ν extends to a measure on X.

(e)(i) Let X be a topological space. Show that the family of closed countably compact subsets of X is a countably compact class. (ii) Let X be a Hausdorff space. Show that the family of sequentially compact subsets of X is a countably compact class.

(f) Let $\mathfrak{A}$ be a Boolean algebra and ν a totally finite submeasure on $\mathfrak{A}$ which is *either* supermodular *or* exhaustive and submodular. Show that ν is uniformly exhaustive.

(g)(i) Find a measure space (X, Σ, μ), with $\mu X > 0$, and a sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ of subsets of X, covering X, such that whenever $E \in \Sigma$, $n \in \mathbb{N}$ and $\mu E > 0$, there is an $F \in \Sigma$ such that $F \subseteq E \setminus X_n$ and $\mu F = \mu E$. (ii) For $A \subseteq X$ set $\phi A = \sup\{\mu E : E \in \Sigma, E \subseteq A\}$. Set $\mathrm{T} = \{G : G \subseteq X, \phi A = \phi(A \cap G) + \phi(A \setminus G)$ for every $A \subseteq X\}$. Show that $\phi\restriction\mathrm{T}$ is not a measure.

(h) Let X be a set, $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$, and $f : \mathcal{K} \to \mathbb{R}$ a modular functional such that $f(\emptyset) = 0$. Show that there is an additive functional $\nu : \mathcal{K} \to \mathbb{R}$ extending f.

413 Notes and comments I gave rather few methods of constructing measures in the first three volumes of this treatise; in the present volume I shall have to make up for lost time. In particular I used Carathéodory's construction for Lebesgue measure (Chapter 11), product measures (Chapter 25) and Hausdorff measures (Chapter 26). The first two, at least, can be tackled in quite different ways if we choose. The first alternative approach I offer is the 'inner measure' method of 413C. Note the exact definition in 413A; I do not think it is an obvious one. In particular, while (α) seems to have something to do with subadditivity, and (β) is a kind of sequential order-continuity, there is no straightforward way in which to associate an outer measure with an inner measure, unless they both happen to be derived from measures (132B, 413D), even when they are finite-valued; and for an inner measure which is allowed to take the value ∞ we have to add the semi-finiteness condition $(*)$ of 413A (see 413Xa).

Once we have got these points right, however, we have a method which rivals Carathéodory's in scope, and in particular is especially well adapted to the construction of inner regular measures. As an almost trivial example, we have a route to the c.l.d. version of a measure μ (413Xj(i)), which can be derived from the inner measure μ_* defined from μ (413D). Henceforth μ_* will be a companion to the familiar outer measure μ^*, and many calculations will be a little easier with both available, as in 413E-413F.

The intention behind 413I-413J is to find a minimal set of properties of a functional ϕ_0 which will ensure that it has an extension to a measure. Indeed it is easy to see that, in the context of 413I, given a family $\mathcal{K}$ with the properties (†) and (‡) there, a functional ϕ_0 on $\mathcal{K}$ can have an extension to an inner regular measure iff it satisfies the conditions (α) and (β), so in this sense 413I is the best possible result. Note that while Carathéodory's construction is liable to produce wildly infinite measures (like Hausdorff measures, or primitive product measures), the construction here always gives us locally determined measures, provided only that ϕ_0 is finite-valued.

We have to work rather hard for the step from 413I to 413J. Of course 413I is a special case of 413J, and I could have saved a little space by giving a direct proof of the latter result. But I do not think that this would have made it easier; 413J really does require an extra step, because somehow we have to extend the functional ϕ_0 from $\mathcal{K}$ to $\mathcal{K}_\delta$. The method I have chosen uses 413B and 413H to cast as much of the argument as possible into the context of algebras of sets with additive functionals, where I hope the required manipulations will seem natural. (But perhaps I should insist that you must not take them too much for granted, as some of the time we have a finitely additive functional taking infinite values, and must take care not to subtract illegally, as well as not to take limits in the wrong places.) Note that the progression $\phi_0 \to \phi_1 \to \mu$ in the proof of 413J involves first an approximation from outside (if $K \in \mathcal{K}_\delta$, then $\phi_1 K$ will be $\inf\{\phi_0 K' : K \subseteq K' \in \mathcal{K}\}$) and then an approximation from inside (if $E \in \Sigma$, then $\mu E = \sup\{\phi_1 K : K \in \mathcal{K}_\delta,\, K \subseteq E\}$). The essential difficulty in the proof is just that we have to take successive non-exchangeable limits. I have slipped 413K in as a corollary of 413J; but it can be regarded as one of the fundamental results of measure theory. A non-negative finitely additive functional ν on an algebra Σ of sets can be extended to a countably additive measure iff it is 'relatively countably additive' in the sense that $\nu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ such that $\bigcup_{n\in\mathbb{N}} E_n \in \Sigma$ (413Xn). Of course the same result can easily be got from an outer measure construction (413Xm). Note that the outer measure construction also has repeated limits, albeit simpler ones: in the formula

$$\theta A = \inf\{\sum_{n=0}^{\infty} \nu E_n : \langle E_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in } \Sigma \text{ covering } A\}$$

the sum $\sum_{n=0}^{\infty} \nu E_n = \sup_{n\in\mathbb{N}} \sum_{i=0}^{n} \nu E_i$ can be regarded as a crude approximation from inside, while the infimum is an approximation from outside. To get the result as stated in 413K, of course, the outer measure construction needs a third limiting process, to obtain the c.l.d. version automatically provided by the inner measure method, and the inner regularity with respect to Σ_δ, while easily checked, also demands a few words of argument.

Many applications of the method of 413I-413J pass through 413M; if the family $\mathcal{K}$ is a countably compact class then the sequential order-continuity hypothesis (β) of 413I or 413J becomes a consequence of the other hypotheses. The essence of the method is the inner regularity hypothesis (α). I have tried to use the labels †, ‡, α and β consistently enough to suggest the currents which I think are flowing in this material.

In 413N we strike out in a new direction. The object here is to build an extension which is not going to be unique, and for which choices will have to be made. As with any such argument, the trick is to specify the allowable intermediate stages, that is, the partially ordered set P to which we shall apply Zorn's Lemma. But here the form of the theorem makes it easy to guess what P should be: it is the set of functionals satisfying the hypotheses of the theorem which have not wandered outside the boundary set by the conclusion, that is, which satisfy the condition (∗) of part (a) of the proof of 413N. The finitistic nature of the hypotheses makes it easy to check that totally ordered subsets of P have upper bounds (that is to say, if we did this by transfinite induction there would be no problem at limit stages), and all we have to prove is that maximal elements of P are defined on adequately large domains; which amounts to showing that a member of P not defined on every element of $\mathcal{K}$ has a proper extension, that is, setting up a construction for the step to a successor ordinal in the parallel transfinite induction (part (c) of the proof).

Of course the principal applications of 413N in this book will be in the context of countably additive functionals, as in 413O.

It is clear that 413N and 413Q overlap to some extent. I include both because they have different virtues. 413N provides actual extensions of functionals in a way that 413Q, as given, does not; but its chief advantage, from the point of view of the work to come, is the approximation of members of T_1, in measure, by members of T_0. This will eventually enable us to retain control of the Maharam types of measures constructed by the method of 413O. In 413S we have a different kind of control; we can specify a lower bound for the measure of each member of our basic class $\mathcal{K}$, provided only that our specifications are consistent with some *finitely* additive functional.

414 τ-additivity

The second topic I wish to treat is that of 'τ-additivity'. Here I collect results which do not depend on any strong kind of inner regularity. I begin with what I think of as the most characteristic feature of τ-additivity, its effect on the properties of semi-continuous functions (414A), with a variety of corollaries, up to the behaviour of subspace measures (414K). A very important property of τ-additive topological measures is that they are often strictly localizable (414J).

The theory of inner regular τ-additive measures belongs to the next section, but here I give two introductory results: conditions under which a τ-additive measure will be inner regular with respect to closed sets (414M) and conditions under which a measure which is inner regular with respect to closed sets will be τ-additive (414N). I end the section with notes on 'density' and 'lifting' topologies (414P-414R).

414A Theorem Let $(X,\mathfrak{T})$ be a topological space and μ an effectively locally finite τ-additive measure on X with domain Σ and measure algebra $\mathfrak{A}$.

(a) Suppose that $\mathcal{G}$ is a non-empty family in $\Sigma\cap\mathfrak{T}$ such that $H=\bigcup\mathcal{G}$ also belongs to Σ. Then $\sup_{G\in\mathcal{G}}G^\bullet=H^\bullet$ in $\mathfrak{A}$.

(b) Write $\mathcal{L}$ for the family of Σ-measurable lower semi-continuous functions from X to $\mathbb{R}$. Suppose that $\emptyset\ne A\subseteq\mathcal{L}$ and set $g(x)=\sup_{f\in A}f(x)$ for every $x\in X$. If g is Σ-measurable and finite almost everywhere, then $\tilde g^\bullet=\sup_{f\in A}f^\bullet$ in $L^0(\mu)$, where $\tilde g(x)=g(x)$ whenever $g(x)$ is finite.

(c) Suppose that $\mathcal{F}$ is a non-empty family of measurable closed sets such that $\bigcap\mathcal{F}\in\Sigma$. Then $\inf_{F\in\mathcal{F}}F^\bullet=(\bigcap\mathcal{F})^\bullet$ in $\mathfrak{A}$.

(d) Write $\mathcal{U}$ for the family of Σ-measurable upper semi-continuous functions from X to $\mathbb{R}$. Suppose that $A\subseteq\mathcal{U}$ is non-empty and set $g(x)=\inf_{f\in A}f(x)$ for every $x\in X$. If g is Σ-measurable and finite almost everywhere, then $\tilde g^\bullet=\inf_{f\in A}f^\bullet$ in $L^0(\mu)$, where $\tilde g(x)=g(x)$ whenever $g(x)$ is finite.

proof (a) ? If $H^\bullet\ne\sup_{G\in\mathcal{G}}G^\bullet$, there is a non-zero $a\in\mathfrak{A}$ such that $a\subseteq H^\bullet$ but $a\cap G^\bullet=0$ for every $G\in\mathcal{G}$. Express a as $E^\bullet$ where $E\in\Sigma$ and $E\subseteq H$. Because μ is effectively locally finite, there is a measurable open set H_0 of finite measure such that $\mu(H_0\cap E)>0$. Now $\{H_0\cap G:G\in\mathcal{G}\}$ is an upwards-directed family of measurable open sets with union $H_0\cap H\supseteq H_0\cap E$; as μ is τ-additive, there is a $G\in\mathcal{G}$ such that $\mu(H_0\cap G)>\mu H_0-\mu(H_0\cap E)$. But in this case $\mu(G\cap E)>0$, which is impossible, because $G^\bullet\cap E^\bullet=0$. **X**

(b) For any $\alpha\in\mathbb{R}$,

$$\{x:g(x)>\alpha\}=\bigcup_{f\in A}\{x:f(x)>\alpha\},$$

and these are all measurable open sets. Identifying $\{x:g(x)>\alpha\}^\bullet\in\mathfrak{A}$ with $[\![\tilde g^\bullet>\alpha]\!]$ (364Ib[10]), we see from (a) that $[\![\tilde g^\bullet>\alpha]\!]=\sup_{f\in A}[\![f^\bullet>\alpha]\!]$ for every α. But this means that $\tilde g^\bullet=\sup_{f\in A}f^\bullet$, by 364L(a-ii)[11].

(c) Apply (a) to $\mathcal{G}=\{X\setminus F:F\in\mathcal{F}\}$.

(d) Apply (b) to $\{-f:f\in A\}$.

414B Corollary Let X be a topological space and μ an effectively locally finite τ-additive topological measure on X.

(a) Suppose that A is a non-empty upwards-directed family of lower semi-continuous functions from X to $[0,\infty]$. Set $g(x)=\sup_{f\in A}f(x)$ in $[0,\infty]$ for every $x\in X$. Then $\int g=\sup_{f\in A}\int f$ in $[0,\infty]$.

(b) Suppose that A is a non-empty downwards-directed family of non-negative continuous real-valued functions on X, and that $g(x)=\inf_{x\in A}f(x)$ for every $x\in X$. If any member of A is integrable, then $\int g=\inf_{f\in A}\int f$.

proof (a) Of course all the $f\in A$, and also g, are measurable functions. Set $g_n=g\wedge n\chi X$ for every $n\in\mathbb{N}$. Then

$$g_n(x)=\sup_{f\in A}(f\wedge n\chi X)(x)$$

for every $x\in X$, so $g_n^\bullet=\sup_{f\in A}(f\wedge n\chi X)^\bullet$, by 414Ab, and

$$\int g_n=\int g_n^\bullet=\sup_{f\in A}\int(f\wedge n\chi X)^\bullet=\sup_{f\in A}\int f\wedge n\chi X$$

by 365Dh. But now, of course,

$$\int g=\sup_{n\in\mathbb{N}}\int g_n=\sup_{n\in\mathbb{N},f\in A}\int f\wedge n\chi X=\sup_{f\in A}\int f,$$

as claimed.

(b) Take an integrable $f_0\in A$, and apply (a) to $\{(f_0-f)^+:f\in A\}$.

414C Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ be an effectively locally finite τ-additive topological measure space and $\mathcal{F}$ a non-empty downwards-directed family of closed sets. If $\inf_{F\in\mathcal{F}}\mu F$ is finite, this is the measure of $\bigcap\mathcal{F}$.

[10] Formerly 364Jb.

[11] Formerly 364Mb.

proof Setting $F_0=\bigcap\mathcal{F}$, then $F_0^\bullet=\inf_{F\in\mathcal{F}}F^\bullet$, by 414Ac; now

$$\mu F_0=\bar\mu F_0^\bullet=\inf_{F\in\mathcal{F}}\bar\mu F^\bullet=\inf_{F\in\mathcal{F}}\mu F$$

by 321F.

414D Corollary Let μ be an effectively locally finite τ-additive measure on a topological space X. If ν is a totally finite measure with the same domain as μ, truly continuous with respect to μ, then ν is τ-additive. In particular, if μ is σ-finite and ν is absolutely continuous with respect to μ, then ν is τ-additive.

proof We have a functional $\bar\nu:\mathfrak{A}\to[0,\infty[$, where $\mathfrak{A}$ is the measure algebra of μ, such that $\bar\nu E^\bullet=\nu E$ for every E in the common domain Σ of μ and ν. Now $\bar\nu$ is continuous for the measure-algebra topology of $\mathfrak{A}$ (327Cd), therefore completely additive (327Ba), therefore order-continuous (326Oc[12]). So if $\mathcal{G}$ is an upwards-directed family of open sets belonging to Σ with union $G_0\in\Sigma$,

$$\sup_{G\in\mathcal{G}}\nu G=\sup_{G\in\mathcal{G}}\bar\nu G^\bullet=\bar\nu G_0^\bullet=\nu G_0$$

because $G_0^\bullet=\sup_{G\in\mathcal{G}}G^\bullet$.

The last sentence follows at once, because on a σ-finite space an absolutely continuous countably additive functional is truly continuous (232Bc).

414E Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ be an effectively locally finite τ-additive topological measure space. Suppose that $\mathcal{G}\subseteq\mathfrak{T}$ is non-empty and upwards-directed, and $H=\bigcup\mathcal{G}$. Then

(a) $\mu(E\cap H)=\sup_{G\in\mathcal{G}}\mu(E\cap G)$ for every $E\in\Sigma$;

(b) if f is a non-negative virtually measurable real-valued function defined almost everywhere in X, then $\int_H f=\sup_{G\in\mathcal{G}}\int_G f$ in $[0,\infty]$.

proof (a) In the measure algebra $(\mathfrak{A},\bar\mu)$ of μ,

$$\begin{aligned}(E\cap H)^\bullet=E^\bullet\cap H^\bullet&=E^\bullet\cap\sup_{G\in\mathcal{G}}G^\bullet\\&=\sup_{G\in\mathcal{G}}E^\bullet\cap G^\bullet=\sup_{G\in\mathcal{G}}(E\cap G)^\bullet,\end{aligned}$$

using 414Aa and the distributive law 313Ba. So

$$\mu(E\cap H)=\bar\mu(E\cap H)^\bullet=\sup_{G\in\mathcal{G}}\bar\mu(E\cap G)^\bullet=\sup_{G\in\mathcal{G}}\mu(E\cap G)$$

by 321D, because $\mathcal{G}$ and $\{(E\cap G)^\bullet:G\in\mathcal{G}\}$ are upwards-directed.

(b) For each $G\in\mathcal{G}$,

$$\int_G f=\int f\times\chi G=\int(f\times\chi G)^\bullet=\int f^\bullet\times\chi G^\bullet,$$

where $\chi G^\bullet$ can be interpreted either as $(\chi G)^\bullet$ (in $L^0(\mu)$) or as $\chi(G^\bullet)$ (in $L^0(\mathfrak{A})$, where $\mathfrak{A}$ is the measure algebra of μ); see 364J[13]. Now $H^\bullet=\sup_{G\in\mathcal{G}}G^\bullet$ (414Aa); since χ and $\times$ are order-continuous (364Jc, 364N[14]), $f^\bullet\times\chi H^\bullet=\sup_{G\in\mathcal{G}}f^\bullet\times\chi G^\bullet$; so

$$\int_H f=\int f^\bullet\times\chi H^\bullet=\sup_{G\in\mathcal{G}}\int f^\bullet\times\chi G^\bullet=\sup_{G\in\mathcal{G}}\int_G f$$

by 365Dh again.

414F Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ be an effectively locally finite τ-additive topological measure space. Then for every $E\in\Sigma$ there is a unique relatively closed self-supporting set $F\subseteq E$ such that $\mu(E\setminus F)=0$.

proof Let $\mathcal{G}$ be the set $\{G:G\in\mathfrak{T},\ \mu(G\cap E)=0\}$. Then $\mathcal{G}$ is upwards-directed, so $\mu(E\cap G^*)=\sup_{G\in\mathcal{G}}\mu(E\cap G)=0$, where $G^*=\bigcup\mathcal{G}$. Set $F=E\setminus G^*$. Then $F\subseteq E$ is relatively closed, and $\mu(E\setminus F)=0$. If $H\in\mathfrak{T}$ and $H\cap F\ne\emptyset$, then $H\notin\mathcal{G}$ so $\mu(F\cap H)=\mu(E\cap H)>0$; thus F is self-supporting. If $F'\subseteq E$ is another self-supporting relatively closed set such that $\mu(E\setminus F')=0$, then $\mu(F\setminus F')=\mu(F'\setminus F)=0$; but as $F\setminus F'$ is relatively open in F, and $F'\setminus F$ is relatively open in F', these must both be empty, and $F=F'$.

[12] Formerly 326Kc.

[13] Formerly 364K.

[14] Formerly 364P.

414G Corollary If $(X,\mathfrak{T},\Sigma,\mu)$ is a Hausdorff effectively locally finite τ-additive topological measure space and $E\in\Sigma$ is an atom for μ (definition: 211I), then there is an $x\in E$ such that $E\setminus\{x\}$ is negligible.

proof Let $F\subseteq E$ be a self-supporting set such that $\mu(E\setminus F)=0$. Since $\mu F=\mu E>0$, F is not empty; take $x\in F$. **?** If $F\ne\{x\}$, let $y\in F\setminus\{x\}$. Because $\mathfrak{T}$ is Hausdorff, there are disjoint open sets G, H containing x, y respectively; and in this case $\mu(E\cap G)=\mu(F\cap G)$ and $\mu(E\cap H)=\mu(F\cap H)$ are both non-zero, which is impossible, since E is an atom. **X**

So $F=\{x\}$ and $E\setminus\{x\}$ is negligible.

414H Corollary If $(X,\mathfrak{T},\Sigma,\mu)$ is an effectively locally finite τ-additive topological measure space and ν is an indefinite-integral measure over μ (definition: 234J[15]), then ν is a τ-additive topological measure.

proof Because ν measures every set in Σ (234La[16]), it is a topological measure. To see that it is τ-additive, apply 414Eb to a Radon-Nikodým derivative of ν.

414I Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space. If $E\subseteq X$ and $\mathcal{G}\subseteq\mathfrak{T}$ are such that $E\subseteq\bigcup\mathcal{G}$ and $E\cap G\in\Sigma$ for every $G\in\mathcal{G}$, then $E\in\Sigma$.

proof Set $\mathcal{K}=\{K:K\in\Sigma,\ E\cap K\in\Sigma\}$. Then whenever $F\in\Sigma$ and $\mu F>0$ there is a $K\in\mathcal{K}$ included in F with $\mu K>0$. **P** Set $K_1=F\setminus\bigcup\mathcal{G}$. Then K_1 is a member of $\mathcal{K}$ included in F. If $\mu K_1>0$ then we can stop. Otherwise, $\mathcal{G}^*=\{G_0\cup\ldots\cup G_n:G_0,\ldots,G_n\in\mathcal{G}\}$ is an upwards-directed family of open sets, and

$$\sup_{G\in\mathcal{G}^*}\mu(F\cap G)=\mu(F\cap\bigcup\mathcal{G}^*)=\mu F>0,$$

by 414Ea. So there is a $G\in\mathcal{G}^*$ such that $\mu(F\cap G)>0$; but now $E\cap G\in\Sigma$ so $F\cap G\in\mathcal{K}$. **Q**

By 412Aa, μ is inner regular with respect to $\mathcal{K}$; by 412Ja, $E\in\Sigma$.

414J Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space. Then μ is strictly localizable.

proof Let $\mathcal{F}$ be a maximal disjoint family of self-supporting measurable sets of finite measure. Then whenever $E\in\Sigma$ and $\mu E>0$, there is an $F\in\mathcal{F}$ such that $\mu(E\cap F)>0$. **P?** Otherwise, let G be an open set of finite measure such that $\mu(G\cap E)>0$, and set $\mathcal{F}_0=\{F:F\in\mathcal{F},\ F\cap G\ne\emptyset\}$. Then $\mu(F\cap G)>0$ for every $F\in\mathcal{F}_0$, while $\mu G<\infty$ and $\mathcal{F}_0$ is disjoint, so $\mathcal{F}_0$ is countable and $\bigcup\mathcal{F}_0\in\Sigma$. Set $E'=E\setminus\bigcup\mathcal{F}_0$; then $E\setminus E'=E\cap\bigcup\mathcal{F}_0$ is negligible, so $\mu(G\cap E')>0$. By 414F, there is a self-supporting set $F'\subseteq G\cap E'$ such that $\mu F'>0$. But in this case $F'\cap F=\emptyset$ for every $F\in\mathcal{F}$, so we ought to have added F' to $\mathcal{F}$. **XQ**

This means that $\mathcal{F}$ satisfies the criterion of 213Oa. Because (X,Σ,μ) is complete and locally determined, it is strictly localizable.

414K Proposition Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X, and $Y\subseteq X$ a subset such that the subspace measure μ_Y is semi-finite (see the remark following 412O). If μ is an effectively locally finite τ-additive topological measure, so is μ_Y.

proof By 412Pe, μ_Y is an effectively locally finite topological measure. Now suppose that $\mathcal{H}$ is a non-empty upwards-directed family in $\mathfrak{T}_Y$ with union H^*. Set

$$\mathcal{G}=\{G:G\in\mathfrak{T},\ G\cap Y\in\mathcal{H}\},\quad G^*=\bigcup\mathcal{G},$$

so that $\mathcal{G}$ is upwards-directed and $H^*=Y\cap G^*$. Let $\mathcal{K}$ be the family of sets $K\subseteq X$ such that $K\cap G^*\setminus G=\emptyset$ for some $G\in\mathcal{G}$. If $E\in\Sigma$,

$$\mu E=\mu(E\setminus G^*)+\mu(E\cap G^*)=\mu(E\setminus G^*)+\sup_{G\in\mathcal{G}}\mu(E\cap G)$$

(414Ea)

$$=\sup_{G\in\mathcal{G}}\mu(E\setminus(G^*\setminus G)),$$

[15] Formerly 234B.

[16] Formerly 234D.

so μ is inner regular with respect to $\mathcal{K}$. By 412Ob, μ_Y is inner regular with respect to $\{K \cap Y : K \in \mathcal{K}\}$. So if $\gamma < \mu_Y H^*$, there is a $K \in \mathcal{K}$ such that $K \cap Y \subseteq H^*$ and $\mu_Y(K \cap Y) \geq \gamma$. But now there is a $G \in \mathcal{G}$ such that $K \cap G^* \setminus G = \emptyset$, so that $K \cap Y \subseteq G \cap Y \in \mathcal{H}$ and $\sup_{H \in \mathcal{H}} \mu H \geq \gamma$. As $\mathcal{H}$ and γ are arbitrary, μ is τ-additive.

Remarks Recall from 214Ic that if (X, Σ, μ) has locally determined negligible sets (in particular, is either strictly localizable or complete and locally determined), then all its subspaces are semi-finite. In 419C below I describe a tight locally finite Borel measure with a subset on which the subspace measure is not semi-finite, therefore not effectively locally finite or τ-additive. In 419A I describe a σ-finite locally finite τ-additive topological measure, inner regular with respect to the closed sets, with a closed subset on which the subspace measure is totally finite but not τ-additive.

414L Lemma Let $(X, \mathfrak{T})$ be a topological space, and μ, ν two effectively locally finite Borel measures on X which agree on the open sets. Then they are equal.

proof Write $\mathfrak{T}^f$ for the family of open sets of finite measure. (I do not need to specify which measure I am using here.) For $G \in \mathfrak{T}^f$, set $\mu_G E = \mu(G \cap E)$, $\nu_G E = \nu(G \cap E)$ for every Borel set E. Then μ_G and ν_G are totally finite Borel measures which agree on $\mathfrak{T}$. By the Monotone Class Theorem (136C), μ_G and ν_G agree on the σ-algebra generated by $\mathfrak{T}$, that is, the Borel σ-algebra $\mathcal{B}$. Now, for any $E \in \mathcal{B}$,

$$\mu E = \sup_{G \in \mathfrak{T}^f} \mu_G E = \sup_{G \in \mathfrak{T}^f} \nu_G E = \nu E,$$

by 412F. So $\mu = \nu$.

414M Proposition Let (X, Σ, μ) be a measure space with a regular topology $\mathfrak{T}$ such that μ is effectively locally finite and τ-additive and Σ includes a base for $\mathfrak{T}$.

(a) $\mu G = \sup\{\mu F : F \in \Sigma \text{ is closed}, F \subseteq G\}$ for every open set $G \in \Sigma$.

(b) If μ is inner regular with respect to the σ-algebra generated by $\mathfrak{T} \cap \Sigma$, it is inner regular with respect to the closed sets.

proof (a) For $U \in \Sigma \cap \mathfrak{T}$, the set

$$\mathcal{H}_U = \{H : H \in \Sigma \cap \mathfrak{T}, \overline{H} \subseteq U\}$$

is an upwards-directed family of open sets, and $\bigcup \mathcal{H}_U = U$ because $\mathfrak{T}$ is regular and Σ includes a base for $\mathfrak{T}$. Because μ is τ-additive, $\mu U = \sup\{\mu H : H \in \mathcal{H}_U\}$. Now, given $\gamma < \mu G$, we can choose $\langle U_n \rangle_{n \in \mathbb{N}}$ in $\Sigma \cap \mathfrak{T}$ inductively, as follows. Start by taking $U_0 \subseteq G$ such that $\gamma < \mu U_0 < \infty$ (using the hypothesis that μ is effectively locally finite). Given $U_n \in \Sigma \cap \mathfrak{T}$ and $\mu U_n > \gamma$, take $U_{n+1} \in \Sigma \cap \mathfrak{T}$ such that $\overline{U}_{n+1} \subseteq U_n$ and $\mu U_{n+1} > \gamma$. On completing the induction, set

$$F = \bigcap_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} \overline{U}_n;$$

then F is a closed set belonging to Σ, $F \subseteq G$ and $\mu F \geq \gamma$. As γ is arbitrary, we have the result.

(b) Let Σ_0 be the σ-algebra generated by $\Sigma \cap \mathfrak{T}$ and set $\mu_0 = \mu \restriction \Sigma_0$. Then $\Sigma_0 \cap \mathfrak{T} = \Sigma \cap \mathfrak{T}$ is still a base for $\mathfrak{T}$ and μ_0 is still τ-additive and effectively locally finite, so by (a) and 412G it is inner regular with respect to the closed sets. Now we are supposing that μ is inner regular with respect to Σ_0, so μ is inner regular with respect to the closed sets, by 412Ab.

414N Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X. Suppose that (i) μ is semi-finite and inner regular with respect to the closed sets (ii) whenever $\mathcal{F}$ is a non-empty downwards-directed family of measurable closed sets with empty intersection and $\inf_{F \in \mathcal{F}} \mu F < \infty$, then $\inf_{F \in \mathcal{F}} \mu F = 0$. Then μ is τ-additive.

proof Let $\mathcal{G}$ be a non-empty upwards-directed family of measurable open sets with measurable union H. Take any $\gamma < \mu H$. Because μ is semi-finite, there is a measurable set $E \subseteq H$ such that $\gamma < \mu E < \infty$. Now there is a measurable closed set $F \subseteq E$ such that $\mu F \geq \gamma$. Consider $\mathcal{F} = \{F \setminus G : G \in \mathcal{G}\}$. This is a downwards-directed family of closed sets of finite measure with empty intersection. So $\inf_{G \in \mathcal{G}} \mu(F \setminus G) = 0$, that is,

$$\gamma \leq \mu F = \sup_{G \in \mathcal{G}} \mu(F \cap G) \leq \sup_{G \in \mathcal{G}} \mu G.$$

As γ is arbitrary, $\mu H = \sup_{G \in \mathcal{G}} \mu G$; as $\mathcal{G}$ is arbitrary, μ is τ-additive.

414O The following elementary result is worth noting.

Proposition If X is a hereditarily Lindelöf space (e.g., if it is separable and metrizable) then every measure on X is τ-additive.

proof If μ is a measure on X, with domain Σ, and $\mathcal{G} \subseteq \Sigma$ is a non-empty upwards-directed family of measurable open sets, then there is a sequence $\langle G_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ such that $\bigcup \mathcal{G} = \bigcup_{n\in\mathbb{N}} G_n$. Now

$$\mu(\bigcup \mathcal{G}) = \lim_{n\to\infty} \mu(\bigcup_{i\le n} G_i) \le \sup_{G\in\mathcal{G}} \mu G.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive.

414P Density topologies Recall that a lower density for a measure space (X,Σ,μ) is a function $\underline{\phi} : \Sigma \to \Sigma$ such that $\underline{\phi}E = \underline{\phi}F$ whenever E, $F \in \Sigma$ and $\mu(E\triangle F) = 0$, $\mu(E\triangle\underline{\phi}E) = 0$ for every $E \in \Sigma$, $\underline{\phi}\emptyset = \emptyset$ and $\underline{\phi}(E\cap F) = \underline{\phi}E \cap \underline{\phi}F$ for all E, $F \in \Sigma$ (341C).

Proposition Let (X,Σ,μ) be a complete locally determined measure space and $\underline{\phi} : \Sigma \to \Sigma$ a lower density such that $\underline{\phi}X = X$. Set

$$\mathfrak{T} = \{E : E \in \Sigma,\ E \subseteq \underline{\phi}E\}.$$

Then $\mathfrak{T}$ is a topology on X, the **density topology** associated with $\underline{\phi}$, and $(X,\mathfrak{T},\Sigma,\mu)$ is an effectively locally finite τ-additive topological measure space; μ is strictly positive and inner regular with respect to the open sets.

proof (a)(i) For any $E \in \Sigma$, $\underline{\phi}(E\cap\underline{\phi}E) = \underline{\phi}E$ because $E\setminus\underline{\phi}E$ is negligible; consequently $E\cap\underline{\phi}E \in \mathfrak{T}$. In particular, $\emptyset = \emptyset\cap\underline{\phi}\emptyset$ and $X = X\cap\underline{\phi}X$ belong to $\mathfrak{T}$. If E, $F \in \mathfrak{T}$ then

$$\underline{\phi}(E\cap F) = \underline{\phi}E \cap \underline{\phi}F \supseteq E\cap F,$$

so $E\cap F \in \mathfrak{T}$.

(ii) Suppose that $\mathcal{G} \subseteq \mathfrak{T}$ and $H = \bigcup \mathcal{G}$. By 341M, μ is (strictly) localizable, so $\mathcal{G}$ has an essential supremum $F \in \Sigma$ such that $F^\bullet = \sup_{G\in\mathcal{G}} G^\bullet$ in the measure algebra $\mathfrak{A}$ of μ; that is, for $E \in \Sigma$, $\mu(G\setminus E) = 0$ for every $G \in \mathcal{G}$ iff $\mu(F\setminus E) = 0$. Now $F\setminus H$ is negligible, by 213K. On the other hand,

$$G \subseteq \underline{\phi}G = \underline{\phi}(G\cap F) \subseteq \underline{\phi}F$$

for every $G \in \mathcal{G}$, so $H \subseteq \underline{\phi}F$, and $H\setminus F \subseteq \underline{\phi}F\setminus F$ is negligible. But as μ is complete, this means that $H \in \Sigma$. Also $\underline{\phi}H = \underline{\phi}F \supseteq H$, so $H \in \mathfrak{T}$. Thus $\mathfrak{T}$ is closed under arbitrary unions and is a topology.

(b) By its definition, $\mathfrak{T}$ is included in Σ, so μ is a topological measure. If $E \in \Sigma$ then $E\cap\underline{\phi}E$ belongs to $\mathfrak{T}$, is included in E and has the same measure as E; so μ is inner regular with respect to the open sets. As μ is semi-finite, it is inner regular with respect to the open sets of finite measure, and is effectively locally finite. If $E \in \mathfrak{T}$ is non-empty, then $\underline{\phi}E \supseteq E$ is non-empty, so $\mu E > 0$; thus μ is strictly positive. Finally, if $\mathcal{G}$ is a non-empty upwards-directed family in $\mathfrak{T}$, then the argument of (a-ii) shows that $(\bigcup\mathcal{G})^\bullet = \sup_{G\in\mathcal{G}} G^\bullet$ in $\mathfrak{A}$, so that $\mu(\bigcup\mathcal{G}) = \sup_{G\in\mathcal{G}} \mu G$. Thus μ is τ-additive.

414Q Lifting topologies Let (X,Σ,μ) be a measure space and $\phi : \Sigma \to \Sigma$ a lifting, that is, a Boolean homomorphism such that $\phi E = \emptyset$ whenever $\mu E = 0$ and $\mu(E\triangle\phi E) = 0$ for every $E \in \Sigma$ (341A). The **lifting topology** associated with ϕ is the topology generated by $\{\phi E : E \in \Sigma\}$. Note that $\{\phi E : E \in \Sigma\}$ is a topology base, so is a base for the lifting topology.

414R Proposition Let (X,Σ,μ) be a complete locally determined measure space and $\phi : \Sigma \to \Sigma$ a lifting with lifting topology $\mathfrak{S}$ and density topology $\mathfrak{T}$. Then $\mathfrak{S} \subseteq \mathfrak{T} \subseteq \Sigma$, and μ is τ-additive, effectively locally finite and strictly positive with respect to $\mathfrak{S}$. Moreover, $\mathfrak{S}$ is zero-dimensional.

proof Of course ϕ is a lower density, so we can talk of its density topology, and since $\phi^2E = \phi E$, $\phi E \in \mathfrak{T}$ for every $E \in \Sigma$, so $\mathfrak{S} \subseteq \mathfrak{T}$. Because μ is τ-additive and strictly positive with respect to $\mathfrak{T}$, it must also be τ-additive and strictly positive with respect to $\mathfrak{S}$. If $E \in \Sigma$ and $\mu E > 0$ there is an $F \subseteq E$ such that $0 < \mu F < \infty$, and now ϕF is an $\mathfrak{S}$-open set of finite measure meeting E in a non-negligible set; so μ is effectively locally finite with respect to $\mathfrak{S}$. Of course $\mathfrak{S}$ is zero-dimensional because $\phi[\Sigma]$ is a base for $\mathfrak{S}$ consisting of open-and-closed sets.

414X Basic exercises >(a) Let (X,Σ,μ) and (Y,T,ν) be measure spaces with topologies $\mathfrak{T}$ and $\mathfrak{S}$, and $f : X \to Y$ a continuous inverse-measure-preserving function. Show that if μ is τ-additive with respect to $\mathfrak{T}$ then ν is τ-additive with respect to $\mathfrak{S}$. Show that if ν is locally finite, so is μ.

(b) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X,Σ,μ); suppose that we are given a topology $\mathfrak{T}_i$ on X_i for each i, and let $\mathfrak{T}$ be the disjoint union topology on X. Show that μ is τ-additive iff every μ_i is.

>(c) Let $(X,\mathfrak{T})$ be a topological space and μ a totally finite measure on X which is inner regular with respect to the closed sets. Suppose that $\mu X=\sup_{G\in\mathcal{G}}\mu G$ whenever $\mathcal{G}$ is an upwards-directed family of measurable open sets covering X. Show that μ is τ-additive.

(d) Let μ be an effectively locally finite τ-additive σ-finite measure on a topological space X, and $\nu:\operatorname{dom}\mu\to[0,\infty[$ a countably additive functional which is absolutely continuous with respect to μ. Show from first principles that ν is τ-additive.

(e) Give an example of an indefinite-integral measure over Lebesgue measure on $\mathbb{R}$ which is not effectively locally finite. (*Hint*: arrange for every non-trivial interval to have infinite measure.)

(f) Let $(X,\mathfrak{T})$ be a topological space and μ a complete locally determined effectively locally finite τ-additive topological measure on X. Show that if f is a real-valued function, defined on a subset of X, which is locally integrable in the sense of 411Fc, then f is measurable.

(g) Let $(X,\mathfrak{T})$ be a topological space and μ an effectively locally finite τ-additive measure on X. Let $\mathcal{G}$ be a cover of X consisting of measurable open sets, and $\mathcal{K}$ the ideal of subsets of X generated by $\mathcal{G}$. Show that μ is inner regular with respect to $\mathcal{K}$.

(h) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space, and A a subset of X. Suppose that for every $x\in A$ there is an open set G containing x such that $A\cap G$ is negligible. Show that A is negligible.

(i) Give an alternative proof of 414K based on the fact that the canonical map from the measure algebra of μ to the measure algebra of μ_Y is order-continuous (322Yd).

>(j)(i) If μ is an effectively locally finite τ-additive Borel measure on a regular topological space, show that the c.l.d. version of μ is a quasi-Radon measure. (ii) If μ is a locally finite, effectively locally finite τ-additive Borel measure on a locally compact Hausdorff space, show that μ is tight, so that the c.l.d. version of μ is a Radon measure.

>(k) Let (X,Σ,μ) be a complete locally determined measure space and $\underline{\phi}$ a lower density for μ such that $\underline{\phi}X=X$; let $\mathfrak{T}$ be the corresponding density topology. (i) Show that a dense open subset of X must be conegligible. (ii) Show that a subset of X is nowhere dense for $\mathfrak{T}$ iff it is negligible iff it is meager for $\mathfrak{T}$. (iii) Show that a function $f:X\to\mathbb{R}$ is Σ-measurable iff it is $\mathfrak{T}$-continuous at almost every point of X. (*Hint*: if f is measurable, set $E_q=\{x:f(x)>q\}$, $F_q=\{x:f(x)<q\}$; show that f is continuous at every point of $X\setminus\bigcup_{q\in\mathbb{Q}}((E_q\setminus\underline{\phi}E_q)\cup(F_q\setminus\underline{\phi}F_q))$.) (iv) Show that Σ is both the Borel σ-algebra of $(X,\mathfrak{T})$ and the Baire-property algebra of $(X,\mathfrak{T})$. (v) Show that $(X,\mathfrak{T})$ is a Baire space.

(l) Let (X,Σ,μ) be a complete locally determined measure space and $\underline{\phi}:\Sigma\to\Sigma$ a lower density such that $\underline{\phi}X=X$, with density topology $\mathfrak{T}$. Show that if $A\subseteq X$ and E is a measurable envelope of A then the $\mathfrak{T}$-closure of A is just $A\cup(X\setminus\underline{\phi}(X\setminus E))$.

(m) Let μ be Lebesgue measure on $\mathbb{R}^r$, Σ its domain, $\mathrm{int}^*:\Sigma\to\Sigma$ lower Lebesgue density (341E) and $\mathfrak{T}$ the corresponding density topology. (i) Show that $\mathfrak{T}$ is finer than the usual Euclidean topology of $\mathbb{R}^r$. (ii) Show that for any $A\subseteq\mathbb{R}$, the closure of A for $\mathfrak{T}$ is just $A\cup\{x:\limsup_{\delta\downarrow 0}\frac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)}>0\}$, and the interior is $A\cap\{x:\lim_{\delta\downarrow 0}\frac{\mu_*(A\cap B(x,\delta))}{\mu B(x,\delta)}=1\}$.

(n) Let (X,Σ,μ) be a complete locally determined measure space and $\underline{\phi}:\Sigma\to\Sigma$ a lower density such that $\underline{\phi}X=X$; let $\mathfrak{T}$ be the associated density topology. Let A be a subset of X and E a measurable envelope of A; let Σ_A be the subspace σ-algebra and μ_A the subspace measure on A. (i) Show that we have a lower density $\underline{\phi}_A:\Sigma_A\to\Sigma_A$ defined by setting $\underline{\phi}_A(F\cap A)=A\cap\underline{\phi}(E\cap F)$ for every $F\in\Sigma$. (ii) Show that $\underline{\phi}_A A=A$ iff $A\subseteq\underline{\phi}E$, and that in this case the density topology on A derived from $\underline{\phi}_A$ is just the subspace topology.

(o) Let (X,Σ,μ) be a complete locally determined measure space and $\phi:\Sigma\to\Sigma$ a lifting, with density topology $\mathfrak{T}$ and lifting topology $\mathfrak{S}$. (i) Show that

$$\mathfrak{T}=\{H\cap G:G\in\mathfrak{S},\ H\text{ is conegligible}\}=\{H\cap\phi E:E\in\Sigma,\ H\text{ is conegligible}\}.$$

(ii) Show that if $A\subseteq X$ and E is a measurable envelope of A then the $\mathfrak{T}$-closure of A is $A\cup\phi E$.

(p) Let (X,Σ,μ) be a complete locally determined measure space and $\phi:\Sigma\to\Sigma$ a lifting; let $\mathfrak{S}$ be its lifting topology. Let A be a subset of X such that $A\subseteq\phi E$ for some (therefore any) measurable envelope E of A. Let Σ_A be the subspace σ-algebra and μ_A the subspace measure on A. (i) Show that we have a lifting $\phi_A:\Sigma_A\to\Sigma_A$ defined by setting $\phi_A(F\cap A)=A\cap\phi F$ for every $F\in\Sigma$. (ii) Show that the lifting topology on A derived from ϕ_A is just the subspace topology.

(q) Let (X,Σ,μ) and (Y,T,ν) be complete locally determined measure spaces and $f:X\to Y$ an inverse-measure-preserving function. Suppose that we have lower densities $\underline{\phi}:\Sigma\to\Sigma$ and $\underline{\psi}:\mathrm{T}\to\mathrm{T}$ such that $\underline{\phi}X=X$, $\underline{\psi}Y=Y$ and $\underline{\phi}f^{-1}[F]=f^{-1}[\underline{\psi}F]$ for every $F\in\mathrm{T}$. (i) Show that f is continuous for the density topologies of $\underline{\phi}$ and $\underline{\psi}$. (ii) Show that if $\underline{\phi}$ and $\underline{\psi}$ are liftings then f is continuous for the lifting topologies.

(r) Let (X,Σ,μ) be a complete locally determined measure space and $\phi:\Sigma\to\Sigma$ a lifting, with associated lifting topology $\mathfrak{S}$. Show that a function $f:X\to\mathbb{R}$ is Σ-measurable iff there is a conegligible set H such that $f\restriction H$ is $\mathfrak{S}$-continuous. (Compare 414Xk, 414Xt.)

(s) Let (X,Σ,μ) be a complete locally determined measure space and $\phi:\Sigma\to\Sigma$ a lifting. Let (Z,T,ν) be the Stone space of the measure algebra of μ, and $f:X\to Z$ the inverse-measure-preserving function associated with ϕ (341P). Show that the lifting topology on X is just $\{f^{-1}[G]:G\subseteq Z\text{ is open}\}$.

(t) Let (X,Σ,μ) be a strictly localizable measure space and $\phi:\Sigma\to\Sigma$ a lifting. Write $\mathcal{L}^\infty$ for the Banach lattice of bounded Σ-measurable real-valued functions on X, identified with $\mathcal{L}^\infty(\Sigma)$ (363H); let $T:\mathcal{L}^\infty\to\mathcal{L}^\infty$ be the Riesz homomorphism associated with ϕ (363F). (i) Show that $T^2=T$. (ii) Show that if X is given the lifting topology $\mathfrak{S}$ defined by ϕ, then $T[\mathcal{L}^\infty]$ is precisely the space of bounded continuous real-valued functions on X. (iii) Show that if $f\in\mathcal{L}^\infty$, $x\in X$ and $\epsilon>0$ there is an $\mathfrak{S}$-open set U containing x such that $|(Tf)(x)-\frac{1}{\mu V}\int_V fd\mu|\le\epsilon$ for every non-negligible measurable set V included in U.

414Y Further exercises (a) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a totally finite topological measure space. For $E\in\Sigma$ set

$$\mu_\tau E=\inf\{\sup_{G\in\mathcal{G}}\mu(E\cap G):\mathcal{G}\subseteq\mathfrak{T}\text{ is an upwards-directed set with union }X\}.$$

Suppose *either* that μ is inner regular with respect to the closed sets *or* that $\mathfrak{T}$ is regular. Show that μ_τ is a τ-additive measure, the largest τ-additive measure with domain Σ which is dominated by μ.

(b) Let X be a set, Σ an algebra of subsets of X, and $\mathfrak{T}$ a topology on X. Let M be the L-space of bounded finitely additive real-valued functionals on Σ (362B). Let $N\subseteq M$ be the set of those functionals ν such that $\inf_{G\in\mathcal{G}}|\nu|(H\setminus G)=0$ whenever $\mathcal{G}\subseteq\mathfrak{T}\cap\Sigma$ is a non-empty upwards-directed family with union $H\in\Sigma$. Show that N is a band in M. (Cf. 362Xi.)

(c) Find a probability space (X,Σ,μ) and a topology $\mathfrak{T}$ on X such that Σ includes a base for $\mathfrak{T}$ and μ is τ-additive, but there is a set $E\in\Sigma$ such that the subspace measure μ_E is not τ-additive.

(d) Let int* be lower Lebesgue density on $\mathbb{R}^r$, and $\mathfrak{T}$ the associated density topology. Show that every $\mathfrak{T}$-Borel set is an F_σ set for $\mathfrak{T}$.

(e) Let (X,ρ) be a metric space and μ a strictly positive locally finite quasi-Radon measure on X; write $\mathfrak{T}$ for the topology of X and Σ for the domain of μ. For $E\in\Sigma$ set $\underline{\phi}(E)=\{x:x\in X,\ \lim_{\delta\downarrow 0}\frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)}=1\}$. Suppose that $E\setminus\underline{\phi}(E)$ is negligible for every $E\in\Sigma$ (cf. 261D, 472D). (i) Show that $\underline{\phi}$ is a lower density for μ, with $\underline{\phi}(X)=X$. Let $\mathfrak{T}_d$ be the associated density topology. (ii) Suppose that $H\in\mathfrak{T}_d$ and that $K\subseteq H$ is $\mathfrak{T}$-closed and ρ-totally bounded. Show that there is a $\mathfrak{T}$-closed, ρ-totally bounded $K'\subseteq H$ such that K is included in the $\mathfrak{T}_d$-interior of K'. (iii) Show that $\mathfrak{T}_d$ is completely regular. (*Hint*: Lukeš Malý & Zajíček 86.)

(f) Show that the density topology on $\mathbb{R}$ associated with lower Lebesgue density is not normal.

(g) Let μ be Lebesgue measure on $\mathbb{R}^r$, Σ its domain, $\mathrm{int}^* : \Sigma \to \Sigma$ lower Lebesgue density and $\mathfrak{T}$ the corresponding density topology. (i) Show that if $f : \mathbb{R}^r \to \mathbb{R}^r$ is a permutation such that f and f^{-1} are both differentiable everywhere, with continuous derivatives, then f is a homeomorphism for $\mathfrak{T}$. (*Hint*: 263D.) (ii) Show that if $\phi : \Sigma \to \Sigma$ is a lifting and $\mathfrak{S}$ the corresponding lifting topology, then $x \mapsto -x$ is not a homeomorphism for $\mathfrak{S}$. (*Hint*: 345Xc.)

414 Notes and comments I have remarked before that it is one of the abiding frustrations of measure theory, at least for anyone ambitious to apply the power of modern general topology to measure-theoretic problems, that the basic convergence theorems are irredeemably confined to sequences. In Volume 3 I showed that if we move to measure algebras and function spaces, we can hope that the countable chain condition or the countable sup property will enable us to replace arbitrary directed sets with monotonic sequences, thereby giving theorems which apply to apparently more general types of convergence. In 414A and its corollaries we come to a quite different context in which a measure, or integral, behaves like an order-continuous functional. Of course the theorems here depend directly on the hypothesis of τ-additivity, which rather begs the question; but we shall see in the rest of the chapter that this property does indeed often appear. For the moment, I remark only that as Lebesgue measure is τ-additive we certainly have a non-trivial example to work with.

The hypotheses of the results above move a touch awkwardly between those with the magic phrase 'topological measure' and those without. The point is that (as in 412G, for instance) it is sometimes useful to be able to apply these ideas to Baire measures on completely regular spaces, which are defined on a base for the topology but may not be defined on every open set. The device I have used in the definition of τ-additivity (411C) makes this possible, at the cost of occasional paradoxical phenomena like 414Yc.

I hope that no confusion will arise between the two topologies associated with a lifting on a complete locally determined space. I have called them the 'density topology' and the 'lifting topology' because the former can be defined directly from a lower density; but it would be equally reasonable to call them the 'fine' and 'coarse' lifting topologies. The density topology has the apparent advantage of giving us a measure which is inner regular with respect to the Borel sets, but at the cost of being rather odd regarded as a topological space (414P, 414Xk, 414Ye, 414Yf). It has the important advantage that there are densities (like the Lebesgue lower density) which have some claim to be called canonical, and others with useful special properties, as in §346, while liftings are always arbitrary and invariance properties for them sometimes unachievable. So, for instance, the Lebesgue density topology on $\mathbb{R}^r$ is invariant under diffeomorphisms, which no lifting topology can be (414Yg). The lifting topology is well-behaved as a topology, but only in special circumstances (as in 453Xd) is the measure inner regular with respect to its Borel sets, and even the closure of a set can be difficult to determine.

As with inner regularity, τ-additivity can be associated with the band structure of the space of bounded additive functionals on an algebra (414Yb); there will therefore be corresponding decompositions of measures into τ-additive and 'purely non-τ-additive' parts (cf. 414Ya).

415 Quasi-Radon measure spaces

We are now I think ready to draw together the properties of inner regularity and τ-additivity. Indeed this section will unite several of the themes which have been running through the treatise so far: (strict) localizability, subspaces and products as well as the new concepts of this chapter. In these terms, the principal results are that a quasi-Radon space is strictly localizable (415A), any subspace of a quasi-Radon space is quasi-Radon (415B), and the product of a family of strictly positive quasi-Radon probability measures on separable metrizable spaces is quasi-Radon (415E). I describe a basic method of constructing quasi-Radon measures (415K), with details of one of the standard ways of applying it (415L, 415N) and some notes on how to specify a quasi-Radon measure uniquely (415H-415I). I spell out useful results on indefinite-integral measures (415O) and L^p spaces (415P), and end the section with a discussion of the Stone space Z of a localizable measure algebra $\mathfrak{A}$ and an important relation in $Z \times X$ when $\mathfrak{A}$ is the measure algebra of a quasi-Radon measure space X (415Q-415R).

It would be fair to say that the study of quasi-Radon spaces for their own sake is a minority interest. If you are not already well acquainted with Radon measure spaces, it would make good sense to read this section in parallel with the next. In particular, the constructions of 415K and 415L derive much of their importance from the corresponding constructions in §416.

415A Theorem A quasi-Radon measure space is strictly localizable.

proof This is a special case of 414J.

415B Theorem Any subspace of a quasi-Radon measure space is quasi-Radon.

proof Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $(Y, \mathfrak{T}_Y, \Sigma_Y, \mu_Y)$ a subspace with the induced topology and measure. Because μ is complete, locally determined and localizable (by 415A), so is μ_Y (214Ie). Because μ_Y is semi-finite and μ is an effectively locally finite τ-additive topological measure, so is μ_Y (414K). Because μ is inner regular with respect to the closed sets and μ_Y is semi-finite, μ_Y is inner regular with respect to the relatively closed subsets of Y (412Pc). So μ_Y is a quasi-Radon measure.

415C In regular topological spaces, the condition 'inner regular with respect to the closed sets' in the definition of 'quasi-Radon measure' can be weakened or omitted.

Proposition Let $(X, \mathfrak{T})$ be a regular topological space.

(a) If μ is a complete locally determined effectively locally finite τ-additive topological measure on X, inner regular with respect to the Borel sets, then it is a quasi-Radon measure.

(b) If μ is an effectively locally finite τ-additive Borel measure on X, its c.l.d. version is a quasi-Radon measure.

proof (a) By 414Mb, μ is inner regular with respect to the closed sets, which is the only feature missing from the given hypotheses.

(b) The c.l.d. version of μ satisfies the hypotheses of (a).

415D In separable metrizable spaces, among others, we can even omit τ-additivity.

Proposition Let $(X, \mathfrak{T})$ be a hereditarily Lindelöf topological space; e.g., a separable metrizable space (4A2P(a-iii)), indeed any space with a countable network (4A2Nb).

(i) If μ is a complete effectively locally finite measure on X, inner regular with respect to the Borel sets, and its domain includes a base for $\mathfrak{T}$, then it is a quasi-Radon measure.

(ii) If μ is an effectively locally finite Borel measure on X, then its completion is a quasi-Radon measure.

(iii) Any quasi-Radon measure on X is σ-finite.

(iv) If X is regular, any quasi-Radon measure on X is completion regular.

proof (a) The basic fact we need is that if $\mathcal{G}$ is any family of open sets in X, then there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\bigcup \mathcal{G}_0 = \bigcup \mathcal{G}$ (4A2H(c-i)). Consequently any effectively locally finite measure μ on X is σ-finite. **P** Let $\mathcal{G}$ be the family of measurable open sets of finite measure. Let $\mathcal{G}_0 \subseteq \mathcal{G}$ be a countable set with the same union as $\mathcal{G}$. Then $E = X \setminus \bigcup \mathcal{G}_0$ is measurable, and $E \cap G = \emptyset$ for every $G \in \mathcal{G}$, so $\mu E = 0$; accordingly $\mathcal{G}_0 \cup \{E\}$ is a countable cover of X by sets of finite measure, and μ is σ-finite. **Q**

Moreover, any measure on X is τ-additive. **P** If $\mathcal{G}$ is a non-empty upwards-directed family of open measurable sets, there is a sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{G}$ with union $\bigcup \mathcal{G}$. If $n \in \mathbb{N}$ there is a $G \in \mathcal{G}$ such that $\bigcup_{i \le n} G_i \subseteq G$, so

$$\mu(\bigcup \mathcal{G}) = \mu(\bigcup_{n \in \mathbb{N}} G_n) = \sup_{n \in \mathbb{N}} \mu(\bigcup_{i \le n} G_i) \le \sup_{G \in \mathcal{G}} \mu G.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive. **Q**

(b)(i) Now let μ be a complete effectively locally finite measure on X, inner regular with respect to the Borel sets, and with domain Σ including a base for the topology of X. If $H \in \mathfrak{T}$, then $\mathcal{G} = \{G : G \in \Sigma \cap \mathfrak{T}, G \subseteq H\}$ has union H, because $\Sigma \cap \mathfrak{T}$ is a base for $\mathfrak{T}$; but in this case there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $H = \bigcup \mathcal{G}_0$, so that $H \in \Sigma$. Thus μ is a topological measure. We know also from (a) that it is τ-additive and σ-finite, therefore locally determined. By 415Ca, it is a quasi-Radon measure.

(ii) If μ is an effectively locally finite Borel measure on X, then its completion $\hat{\mu}$ satisfies the conditions of (i), so is a quasi-Radon measure.

(iii) If μ is a quasi-Radon measure on X, it is surely effectively locally finite, therefore σ-finite.

(iv) If X is regular, then every closed set is a zero set (4A2H(c-ii)), so any measure which is inner regular with respect to the closed sets is completion regular.

415E I am delaying most of the theory of products of (quasi-)Radon measures to §417. However, there is one result which is so important that I should like to present it here, even though some of the ideas will have to be repeated later.

Theorem Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of separable metrizable quasi-Radon probability spaces such that every μ_i is strictly positive, and λ the product measure on $X=\prod_{i\in I}X_i$. Then

(i) λ is a completion regular quasi-Radon measure;

(ii) if $F\subseteq X$ is a closed self-supporting set, there is a countable set $J\subseteq I$ such that F is determined by coordinates in J, so F is a zero set.

proof (a) Write Λ for the domain of λ, and $\mathcal{U}$ for the family of subsets of X of the form $\prod_{i\in I}G_i$ where $G_i\in\mathfrak{T}_i$ for every $i\in I$ and $\{i:G_i\ne X_i\}$ is finite. Then $\mathcal{U}$ is a base for the topology of X, included in Λ. For $J\subseteq I$ let λ_J be the product measure on $X_J=\prod_{i\in J}X_i$ and Λ_J its domain. Write $\mathcal{U}^{(J)}$ for the family of subsets of X_J of the form $\prod_{i\in J}G_i$ where $G_i\in\mathfrak{T}_i$ for each $i\in J$ and $\{i:G_i\ne X_i\}$ is finite.

(b) Consider first the case in which I is countable. In this case X also is separable and metrizable (4A2P(a-v)), while Λ includes a base for its topology. Also λ is a complete probability measure and inner regular with respect to the closed sets (412Ua), so must be a quasi-Radon measure, by 415D(i).

(c) Now consider uncountable I. The key to the proof is the following fact: if $\mathcal{V}\subseteq\mathcal{U}$ has union W, then $W\in\Lambda$ and $\lambda W=\sup_{V\in\mathcal{V}^*}\lambda V$, where $\mathcal{V}^*$ is the set of unions of finite subsets of $\mathcal{V}$.

P (i) By 215B(iv), there is a countable set $\mathcal{V}_1\subseteq\mathcal{V}$ such that $\lambda(U\setminus W_1)=0$ for every $U\in\mathcal{V}$, where $W_1=\bigcup\mathcal{V}_1$. Every member of $\mathcal{U}$ is determined by coordinates in some finite set (see 254M for this concept), so there is a countable set $J\subseteq I$ such that every member of $\mathcal{V}_1$ is determined by coordinates in J, and W_1 also is determined by coordinates in J. Let $\pi_J:X\to X_J$ be the canonical map. Because it is an open map (4A2B(f-i)), $\pi_J[W]$ and $\pi_J[W_1]$ are open in X_J, and belong to Λ_J, by (b).

(ii) ? Suppose, if possible, that $\lambda_J\pi_J[W]>\lambda_J\pi_J[W_1]$. Since $\pi_J[W]=\bigcup\{\pi_J[U]:U\in\mathcal{V}\}$, while λ_J is quasi-Radon and all the sets $\pi_J[U]$ are open, there must be some $U\in\mathcal{V}$ such that $\lambda_J(\pi_J[U]\setminus\pi_J[W_1])>0$ (414Ea). Now π_J is inverse-measure-preserving (254Oa), so

$$0<\lambda\pi_J^{-1}[\pi_J[U]\setminus\pi_J[W_1]]=\lambda(\pi_J^{-1}[\pi_J[U]]\setminus\pi_J^{-1}[\pi_J[W_1]])=\lambda(\pi_J^{-1}[\pi_J[U]]\setminus W_1),$$

because W_1 is determined by coordinates in J.

At this point note that U is of the form $\prod_{i\in I}G_i$, where $G_i\in\mathfrak{T}_i$ for each I, so we can express U as $U'\cap U''$, where $U'=\pi_J^{-1}[\pi_J[U]]$ and $U''=\pi_{I\setminus J}^{-1}[\pi_{I\setminus J}[U]]$. U' is determined by coordinates in J and U'' is determined by coordinates in $I\setminus J$. In this case

$$\lambda(U\setminus W_1)=\lambda(U''\cap U'\setminus W_1)=\lambda U''\cdot\lambda(U'\setminus W_1),$$

because U'' is determined by coordinates in $I\setminus J$ and $U'\setminus W_1$ is determined by coordinates in J, and we can identify λ with the product $\lambda_{I\setminus J}\times\lambda_J$ (254N). But now recall that every μ_i is strictly positive. Since U is surely not empty, no G_i can be empty and no μ_iG_i can be 0. Consequently $\prod_{i\in I}\mu_iG_i>0$ (because only finitely many terms in the product are less than 1) and $\lambda U>0$; more to the point, $\lambda U''>0$. Since we chose U so that $\lambda(U'\setminus W_1)>0$, we have $\lambda(U\setminus W_1)>0$. But this contradicts the first sentence of (i) just above. **X**

(iii) Thus $\lambda_J\pi_J[W]=\lambda_J\pi_J[W_1]$. But this means that $\lambda\pi_J^{-1}[\pi_J[W]]=\lambda W_1$. Since λ is complete and $W_1\subseteq W\subseteq\pi_J^{-1}[\pi_J[W]]$, λW is defined and equal to λW_1.

Taking $\langle V_n\rangle_{n\in\mathbb{N}}$ to be a sequence running over $\mathcal{V}_1\cup\{\emptyset\}$, we have

$$\lambda W=\lambda W_1=\lambda(\bigcup_{n\in\mathbb{N}}V_n)=\sup_{n\in\mathbb{N}}\lambda(\bigcup_{i\le n}V_i)\le\sup_{V\in\mathcal{V}^*}\lambda V\le\lambda W,$$

so $\lambda W=\sup_{V\in\mathcal{V}^*}\lambda V$, as required. **Q**

(d) Thus we see that λ is a topological measure. But it is also τ-additive. **P** If $\mathcal{W}$ is an upwards-directed family of open sets in X with union W^*, set

$$\mathcal{V}=\{U:U\in\mathcal{U},\ \exists\,W\in\mathcal{W},\ U\subseteq W\}.$$

Then $W^*=\bigcup\mathcal{V}$, so $\lambda W^*=\sup_{V\in\mathcal{V}^*}\lambda V$, where $\mathcal{V}^*$ is the set of finite unions of members of $\mathcal{V}$. But because $\mathcal{W}$ is upwards-directed, every member of $\mathcal{V}^*$ is included in some member of $\mathcal{W}$, so

$$\lambda W^*=\sup_{V\in\mathcal{V}^*}\lambda V\le\sup_{W\in\mathcal{W}}\lambda W\le\lambda W^*.$$

As $\mathcal{W}$ is arbitrary, λ is τ-additive. **Q**

(e) As in (b) above, we know that λ is a complete probability measure and is inner regular with respect to the closed sets, so it is a quasi-Radon measure. Because λ is inner regular with respect to the zero sets (412Ub), it is completion regular.

(f) Now suppose that $F \subseteq X$ is a closed self-supporting set. By 254Oc, there is a set $W \subseteq X$, determined by coordinates in some countable set $J \subseteq I$, such that $W \triangle F$ is negligible. **?** Suppose, if possible, that $x \in F$ and $y \in X \setminus F$ are such that $x \restriction J = y \restriction J$. Then there is a $U \in \mathcal{U}$ such that $y \in U \subseteq X \setminus F$. As in (b-ii) above, we can express U as $U' \cap U''$ where U', $U'' \in \mathcal{U}$ are determined by coordinates in J and $I \setminus J$ respectively. In this case,

$$\begin{aligned}\lambda(F \cap U) &= \lambda(W \cap U) = \lambda(W \cap U') \cdot \lambda U'' \\ &= \lambda(F \cap U') \cdot \lambda U'' > 0,\end{aligned}$$

because $x \in F \cap U'$ and F is self-supporting, while $U'' \neq \emptyset$ and λ is strictly positive. But $F \cap U = \emptyset$, so this is impossible. **X**

Thus F is determined by coordinates in the countable set J. Consequently it is of the form $\pi_J^{-1}[\pi_J[F]]$. But $\pi_J[X \setminus F]$ is open (4A2B(f-i) again), so its complement $\pi_J[F]$ is closed. Now X_J is metrizable (4A2P(a-v)), so $\pi_J[F]$ is a zero set (4A2Lc) and F is a zero set (4A2C(b-iv)).

415F Corollary (a) If Y is either $[0,1[$ or $]0,1[$, endowed with Lebesgue measure, and I is any set, then Y^I, with the product topology and measure, is a quasi-Radon measure space.

(b) If $\langle \nu_i \rangle_{i \in I}$ is a family of probability distributions on $\mathbb{R}$, in the sense of §271 (that is, Radon probability measures), and every ν_i is strictly positive, then the product measure on $\mathbb{R}^I$ is a quasi-Radon measure.

Remark See also 416U below, and 453I, where there is an alternative proof of the main step in 415E, applicable to some further cases. Yet another approach, most immediately applicable to $[0,1[^I$, is in 443Xp. For further facts about these product measures, see §417, particularly 417M.

415G Comparing quasi-Radon measures: Proposition Let X be a topological space, and μ, ν two quasi-Radon measures on X. Then the following are equiveridical:

(i) $\mu F \le \nu F$ for every closed set $F \subseteq X$;

(ii) $\operatorname{dom} \nu \subseteq \operatorname{dom} \mu$ and $\mu E \le \nu E$ for every $E \in \operatorname{dom} \nu$.

If ν is locally finite, we can add

(iii) $\mu G \le \nu G$ for every open set $G \subseteq X$;

(iv) there is a base $\mathcal{U}$ for the topology of X such that $G \cup H \in \mathcal{U}$ for all G, $H \in \mathcal{U}$ and $\mu G \le \nu G$ for $G \in \mathcal{U}$.

proof (a) Of course (ii)$\Rightarrow$(i). Suppose that (i) is true. Observe that if $E \in \operatorname{dom} \mu \cap \operatorname{dom} \nu$ (for instance, if $E \subseteq X$ is Borel), then

$$\mu E = \sup\{\mu F : F \subseteq E \text{ is closed}\} \le \sup\{\nu F : F \subseteq E \text{ is closed}\} = \nu E.$$

Set $\mathcal{H} = \{H : H \subseteq X \text{ is open}, \nu H < \infty\}$, and $W = \bigcup \mathcal{H}$. Then $\nu(X \setminus W) = 0$, because ν is effectively locally finite, so $\mu(X \setminus W) = 0$. Set $\mathcal{F} = \{F : F \subseteq X \text{ is closed}, \mu F < \infty\}$.

Take any $E \in \operatorname{dom} \nu$. If $F \in \mathcal{F}$ and $\epsilon > 0$, then $\mu(F \cap W) = \mu F$, so there is an $H \in \mathcal{H}$ such that $\mu(F \cap H) \ge \mu F - \epsilon$. Now there are closed sets $F_1 \subseteq F \cap H \cap E$, $F_2 \subseteq F \cap H \setminus E$ such that $\nu F_1 + \nu F_2 \ge \nu(F \cap H) - \epsilon$, that is, $\nu((F \cap H) \setminus (F_1 \cup F_2)) \le \epsilon$, so that $\mu((F \cap H) \setminus (F_1 \cup F_2)) \le \epsilon$ and $\mu F_1 + \mu F_2 \ge \mu(F \cap H) - \epsilon$. This means that

$$\mu_*(F \cap E) + \mu_*(F \setminus E) \ge \mu(F \cap H) - \epsilon \ge \mu F - \epsilon.$$

As ϵ is arbitrary, $\mu_*(F \cap E) + \mu_*(F \setminus E) \ge \mu F$; as μ is inner regular with respect to $\mathcal{F}$, μ measures E, by 413F(vii).

Thus $\operatorname{dom} \nu \subseteq \operatorname{dom} \mu$; and we have already observed that $\mu E \le \nu E$ whenever E is measured by both.

(b) The first sentence in the proof of (a) shows that (i)$\Rightarrow$(iii), and (iii)$\Rightarrow$(iv) is trivial. If (iv) is true and $G \subseteq X$ is open, then $\mathcal{V} = \{V : V \in \mathcal{U}, V \subseteq G\}$ is upwards-directed and has union G, so

$$\mu G = \sup_{V \in \mathcal{V}} \mu V \le \sup_{V \in \mathcal{V}} \nu V = \nu G.$$

Thus (iv)$\Rightarrow$(iii).

Now assume that ν is locally finite and that (iii) is true. **?** Suppose, if possible, that $F \subseteq X$ is a closed set such that $\nu F < \mu F$. Then $\mathcal{H}$, as defined in part (a) of the proof, is upwards-directed and has union X, so there is an $H \in \mathcal{H}$ such that $\nu F < \mu(F \cap H)$. Now there is a closed set $F' \subseteq H \setminus F$ such that

$$\nu F' > \nu(H \setminus F) - \mu(F \cap H) + \nu F \ge \nu H - \mu(F \cap H).$$

Set $G = H \setminus F'$, so that $F \cap H \subseteq G$ and

$$\nu G = \nu H - \nu F' < \mu(F \cap H) \le \mu G,$$

which is impossible. **X**

This shows that (provided that ν is locally finite) (iii)⇒(i).

415H Uniqueness of quasi-Radon measures: Proposition Let $(X,\mathfrak{T})$ be a topological space and μ, ν two quasi-Radon measures on X. Then the following are equiveridical:

(i) $\mu=\nu$;

(ii) $\mu F=\nu F$ for every closed set $F\subseteq X$;

(iii) $\mu G=\nu G$ for every open set $G\subseteq X$;

(iv) there is a base $\mathcal{U}$ for the topology of X such that $G\cup H\in\mathcal{U}$ for every G, $H\in\mathcal{U}$ and $\mu\restriction\mathcal{U}=\nu\restriction\mathcal{U}$;

(v) there is a base $\mathcal{U}$ for the topology of X such that $G\cap H\in\mathcal{U}$ for every G, $H\in\mathcal{U}$ and $\mu\restriction\mathcal{U}=\nu\restriction\mathcal{U}$.

proof Of course (i) implies all the others. (ii)⇒(i) is immediate from 415G (see also 412L). If (iii) is true, then, for any closed set $F\subseteq X$,

$$\begin{aligned}\mu F&=\sup\{\mu(G\cap F):G\in\mathfrak{T},\,\mu G<\infty\}\\&=\sup\{\mu G-\mu(G\setminus F):G\in\mathfrak{T},\,\mu G<\infty\}\\&=\sup\{\nu G-\nu(G\setminus F):G\in\mathfrak{T},\,\nu G<\infty\}=\nu F;\end{aligned}$$

so (iii)⇒(ii). (iv)⇒(iii) by the argument of (iv)⇒(iii) in the proof of 415G.

Finally, suppose that (v) is true. Then $\mu(G_0\cup\ldots\cup G_n)=\nu(G_0\cup\ldots\cup G_n)$ for all $G_0,\ldots,G_n\in\mathcal{U}$. **P** Induce on n. For the inductive step to $n\ge1$, if any G_i has infinite measure (for either measure) the result is trivial. Otherwise,

$$\begin{aligned}\mu(G_0\cup\ldots\cup G_n)&=\mu(\bigcup_{i<n}G_i)+\mu G_n-\mu(\bigcup_{i<n}G_n\cap G_i)\\&=\nu(\bigcup_{i<n}G_i)+\nu G_n-\nu(\bigcup_{i<n}G_n\cap G_i)=\nu(G_0\cup\ldots\cup G_n).\ \mathbf{Q}\end{aligned}$$

So μ and ν agree on the base $\{G_0\cup\ldots\cup G_n:G_0,\ldots,G_n\in\mathcal{U}\}$, and (iv) is true.

415I Proposition Let X be a completely regular topological space and μ, ν two quasi-Radon measures on X such that $\int f d\mu=\int f d\nu$ whenever $f:X\to\mathbb{R}$ is a bounded continuous function integrable with respect to both measures. Then $\mu=\nu$.

proof ? Otherwise, there is an open set $G\subseteq X$ such that $\mu G\ne\nu G$; suppose $\mu G<\nu G$. Because ν is effectively locally finite, there is an open set $G'\subseteq G$ such that $\mu G<\nu G'<\infty$. Now the cozero sets form a base for the topology of X, so $\mathcal{H}=\{H:H\subseteq G'$ is a cozero set$\}$ has union G'; as ν is τ-additive, there is an $H\in\mathcal{H}$ such that $\nu H>\mu G$. Express H as $\{x:g(x)>0\}$ where $g:X\to[0,\infty[$ is continuous. For each $n\in\mathbb{N}$, set $f_n=ng\wedge\chi X$; then $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with limit χH, so there is an $n\in\mathbb{N}$ such that $\int f_n d\nu>\mu G\ge\int f_n d\mu$. But f_n is both μ-integrable and ν-integrable because μG and νH are both finite. **X**

415J Proposition Let X be a regular topological space, Y a subspace of X, and ν a quasi-Radon measure on Y. Then there is a quasi-Radon measure μ on X such that $\mu E=\nu(E\cap Y)$ whenever μ measures E, that is, Y has full outer measure in X and ν is the subspace measure on Y.

proof Write $\mathcal{B}$ for the Borel σ-algebra of X, and set $\mu_0E=\nu(E\cap Y)$ for every $E\in\mathcal{B}$. Then it is easy to see that μ_0 is a τ-additive Borel measure on X. Moreover, μ_0 is effectively locally finite. **P** If $E\in\mathcal{B}$ and $\mu_0E>0$, there is a relatively open set $H\subseteq Y$ such that $\nu H<\infty$ and $\nu(H\cap E\cap Y)>0$. Now H is of the form $G\cap Y$ where $G\subseteq X$ is open, and we have $\mu_0G=\nu H<\infty$, $\mu_0(E\cap G)=\nu(H\cap E\cap Y)>0$. **Q**

By 415Cb, the c.l.d. version μ of μ_0 is a quasi-Radon measure on X. If $E\in\operatorname{dom}\mu$, then $E\cap Y\in\operatorname{dom}\nu$. **P** Let $\mathcal{F}_Y$ be the set of relatively closed subsets of Y of finite measure for ν. If $F\in\mathcal{F}_Y$, it is expressible as $F'\cap Y$ where F' is a closed subset of X, and $\mu F'=\mu_0F'=\nu F$ is finite. So there are E_1, $E_2\in\mathcal{B}$ such that $E_1\subseteq E\cap F'\subseteq E_2$ and $\mu E_1=\mu(E\cap F')=\mu E_2$. Accordingly

$$E_1\cap Y\subseteq E\cap Y\cap F\subseteq E_2\cap Y$$

and

$$\nu(E_1\cap Y)=\nu(E_2\cap Y)=\mu(E\cap F')$$

is finite. This means that $E \cap Y \cap F \in \operatorname{dom} \nu$; because ν is complete and locally determined and inner regular with respect to $\mathcal{F}_Y$, $E \cap Y \in \operatorname{dom} \nu$, by 412Ja. **Q**

If $E \in \operatorname{dom} \mu$, then

$$\begin{aligned}\mu E &= \sup\{\mu F : F \subseteq E \text{ is closed}\}\\ &= \sup\{\nu(F \cap Y) : F \subseteq E \text{ is closed}\} \le \nu(E \cap Y).\end{aligned}$$

On the other hand, if $\gamma < \nu(E \cap Y)$, there is a relatively open set $H \subseteq Y$ such that $\nu H < \infty$ and $\nu(E \cap Y \cap H) \ge \gamma$ (412F). Let $G \subseteq X$ be an open set such that $G \cap Y = H$. Then

$$\mu E \ge \mu G - \mu(G \setminus E) = \nu H - \nu(H \setminus E) = \nu(E \cap Y \cap H) \ge \gamma.$$

As γ is arbitrary, $\mu E = \nu(E \cap Y)$.

Thus $\mu E = \nu(E \cap Y)$ whenever μ measures E. So if E, $F \in \operatorname{dom} \mu$ and $E \cap Y \subseteq F$,

$$\mu E = \nu(E \cap Y) \le \nu(F \cap Y) = \mu F;$$

as F is arbitrary, $\mu^*(E \cap Y) = \mu E$; as E is arbitrary, Y has full outer measure in X. Moreover, if μ_Y is the subspace measure on Y, $\mu_Y H = \mu^* H = \nu H$ whenever $H \in \operatorname{dom} \mu_Y$, that is, $H = E \cap Y$ for some $E \in \operatorname{dom} \mu$. Now μ_Y, like ν, is a quasi-Radon measure on Y (415B), and they agree on the (relatively) closed subsets of Y, so are equal, by 415H.

415K I come now to a couple of basic results on the construction of quasi-Radon measures. The first follows 413J.

Theorem Let X be a topological space and $\mathcal{K}$ a family of closed subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $F \in \mathcal{K}$ whenever $K \in \mathcal{K}$ and $F \subseteq K$ is closed.

Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{K \in \mathcal{K}'} \phi_0 K = 0$ whenever $\mathcal{K}'$ is a non-empty downwards-directed subset of $\mathcal{K}$ with empty intersection,

(γ) whenever $K \in \mathcal{K}$ and $\phi_0 K > 0$, there is an open set G such that the supremum $\sup_{K' \in \mathcal{K}, K' \subseteq G} \phi_0 K'$ is finite, while $\phi_0 K' > 0$ for some $K' \in \mathcal{K}$ such that $K' \subseteq K \cap G$.

Then there is a unique quasi-Radon measure on X extending ϕ_0 and inner regular with respect to $\mathcal{K}$.

proof By 413J, there is a complete locally determined measure μ on X, inner regular with respect to $\mathcal{K}$, and extending ϕ_0; write Σ for the domain of μ. If $F \subseteq X$ is closed, then $K \cap F \in \mathcal{K} \subseteq \Sigma$ for every $K \in \mathcal{K}$, so $F \in \Sigma$, by 413F(ii); accordingly every open set is measurable. Because μ is inner regular with respect to $\mathcal{K}$ it is surely inner regular with respect to the closed sets. If $E \in \Sigma$ and $\mu E > 0$, there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$; now (γ) tells us that there is an open set G such that $\mu G < \infty$ and $\mu(G \cap K) > 0$, so that $\mu(G \cap E) > 0$. As E is arbitrary, μ is effectively locally finite. Now suppose that $\mathcal{G}$ is a non-empty upwards-directed family of open sets with union H, and that $\gamma < \mu H$. Then there is a $K \in \mathcal{K}$ such that $K \subseteq H$ and $\mu K > \gamma$. Applying the hypothesis (β) to $\mathcal{K}' = \{K \setminus G : G \in \mathcal{G}\}$, we see that $\inf_{G \in \mathcal{G}} \mu(K \setminus G) = 0$, so that

$$\sup_{G \in \mathcal{G}} \mu G \ge \sup_{G \in \mathcal{G}} \mu(K \cap G) = \mu K \ge \gamma.$$

As $\mathcal{G}$ and γ are arbitrary, μ is τ-additive. So μ is a quasi-Radon measure.

415L Proposition Let (X, Σ_0, μ_0) be a measure space and $\mathfrak{T}$ a topology on X such that μ_0 is τ-additive, effectively locally finite and inner regular with respect to the closed sets, and Σ_0 includes a base for $\mathfrak{T}$. Then μ_0 has a unique extension to a quasi-Radon measure μ on X such that

(i) $\mu F = \mu_0^* F$ whenever $F \subseteq X$ is closed and $\mu_0^* F < \infty$,

(ii) $\mu G = (\mu_0)_* G$ whenever $G \subseteq X$ is open,

(iii) the embedding $\Sigma_0 \subsetneqq \Sigma$ identifies the measure algebra $(\mathfrak{A}_0, \bar\mu_0)$ of μ_0 with an order-dense subalgebra of the measure algebra $(\mathfrak{A}, \bar\mu)$ of μ, so that the subrings $\mathfrak{A}_0^f$, $\mathfrak{A}^f$ of elements of finite measure coincide, and $L^p(\mu_0)$ may be identified with $L^p(\mu)$ for $1 \le p < \infty$,

(iv) whenever $E \in \Sigma$ and $\mu E < \infty$, there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$,

(v) for every μ-integrable real-valued function f there is a μ_0-integrable function g such that $f = g$ μ-a.e.

If μ_0 is complete and locally determined, then we have

(i)′ $\mu F = \mu_0^* F$ for every closed $F \subseteq X$.

If μ_0 is localizable, then we have

(iii)′ $\mathfrak{A}_0 = \mathfrak{A}$, so that $L^0(\mu) \cong L^0(\mu_0)$ and $L^\infty(\mu) \cong L^\infty(\mu_0)$,

(iv)′ for every $E \in \Sigma$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$,

(v)′ for every Σ-measurable real-valued function f there is a Σ_0-measurable real-valued function g such that $f = g$ μ-a.e.

proof (a) Let $\mathcal{K}$ be the set of closed subsets of X of finite outer measure for μ_0. Note that μ_0 is inner regular with respect to $\mathcal{K}$, because it is inner regular with respect to the closed sets and also with respect to the sets of finite measure.

It is obvious from its definition that $\mathcal{K}$ satisfies (†) and (‡) of 415K. For $K \in \mathcal{K}$, set $\phi_0 K = \mu_0^* K$. Then ϕ_0 satisfies (α)-(γ) of 415K.

P $(\boldsymbol{\alpha})$ If K, $L \in \mathcal{K}$ and $L \subseteq K$, take measurable envelopes E_0, $E_1 \in \Sigma_0$ of K, L respectively. (i) Let $\epsilon > 0$. Because μ_0 is inner regular with respect to the closed sets, there is a closed set $F \in \Sigma_0$ such that $F \subseteq E_0 \setminus E_1$ and $\mu F \geq \mu_0(E_0 \setminus E_1) - \epsilon$. Set $K' = F \cap K$. Then $K' \in \mathcal{K}$ and

$$\phi_0 K' = \mu_0^*(F \cap K) = \mu_0(F \cap E_0) = \mu_0 F \geq \mu_0 E_0 - \mu_0 E_1 - \epsilon = \phi_0 K - \phi_0 L - \epsilon.$$

As ϵ is arbitrary, we have

$$\phi_0 K \leq \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}.$$

(ii) On the other hand, **?** suppose, if possible, that there is a closed set $K' \subseteq K \setminus L$ such that $\mu_0^* L + \mu_0^* K' > \mu^* K$. Let E_2 be a measurable envelope of K', so that $\mu_0 E_1 + \mu_0 E_2 > \mu_0 E_0$; since

$$\mu_0(E_1 \setminus E_0) = \mu_0^*(L \setminus E_0) = \mu_0^* \emptyset = 0, \quad \mu_0(E_2 \setminus E_0) = \mu_0^*(K' \setminus E_0) = 0,$$

$\mu_0(E_1 \cap E_2) > 0$. Because μ_0 is effectively locally finite, there is a measurable open set G_0, of finite measure, such that $\mu_0(G_0 \cap E_1 \cap E_2) > 0$. Set

$$\mathcal{G} = \{G \cup G' : G,\ G' \in \Sigma_0 \cap \mathfrak{T},\ G \subseteq G_0 \setminus L,\ G' \subseteq G_0 \setminus K'\}.$$

Then $\mathcal{G}$ is an upwards-directed family of measurable open sets, and because Σ_0 includes a base for the topology of X, its union is $(G_0 \setminus L) \cup (G_0 \setminus K') = G_0$. So there is an $H \in \mathcal{G}$ such that $\mu_0 H > \mu_0 G_0 - \mu_0(E_1 \cap E_2)$, that is, there are open sets G, $G' \in \Sigma_0$ such that $G \subseteq G_0 \setminus L$, $G' \subseteq G_0 \setminus K'$ and $\mu_0((G \cup G') \cap E_1 \cap E_2)) > 0$. But we must have

$$\mu_0(G \cap E_1) = \mu_0^*(G \cap L) = 0, \quad \mu_0(G' \cap E_2) = \mu_0^*(G' \cap K') = 0,$$

so this is impossible. **X**

Accordingly

$$\phi_0 K \geq \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\},$$

so that ϕ_0 satisfies condition (α) of 415K.

$(\boldsymbol{\beta})$ Let $\mathcal{K}' \subseteq \mathcal{K}$ be a non-empty downwards-directed family with empty intersection. Fix $K_0 \in \mathcal{K}'$ and $\epsilon > 0$. Let E_0 be a measurable envelope of K_0 and G_0 a measurable open set of finite measure such that $\mu_0(G_0 \cap E_0) \geq \mu_0 E_0 - \epsilon$. Then

$$\mathcal{G} = \{G : G \in \Sigma_0 \cap \mathfrak{T},\ G \subseteq G_0 \setminus K \text{ for some } K \in \mathcal{K}' \text{ such that } K \subseteq K_0\}$$

is an upwards-directed family of measurable open sets, and its union is $G_0 \setminus \bigcap \mathcal{K}' = G_0$, again because Σ_0 includes a base for the topology $\mathfrak{T}$. So there is a $G \in \mathcal{G}$ such that $\mu_0 G \geq \mu_0 G_0 - \epsilon$. Let $K \in \mathcal{K}'$ be such that $K \subseteq K_0$ and $G \cap K = \emptyset$; then

$$\phi_0 K = \mu_0^* K \leq \mu_0(E_0 \setminus G) \leq \mu_0(E_0 \setminus G_0) + \mu_0(G_0 \setminus G) \leq 2\epsilon.$$

As ϵ is arbitrary, $\inf_{K \in \mathcal{K}'} \phi_0 K = 0$.

$(\boldsymbol{\gamma})$ If $K \in \mathcal{K}$ and $\phi_0 K > 0$, let E_0 be a measurable envelope of K. Then there is a measurable open set G of finite measure such that $\mu_0(G \cap E_0) > 0$. Of course $\sup_{K' \in \mathcal{K}, K' \subseteq G} \phi_0 K' \leq \mu_0 G < \infty$; but also there is a measurable closed set $K' \subseteq G \cap E_0$ such that $\mu_0 K' > 0$, in which case $\phi_0(K \cap K') = \mu_0(E_0 \cap K') > 0$. So ϕ_0 satisfies condition (γ). **Q**

(b) By 415K, ϕ_0 has an extension to a quasi-Radon measure μ on X which is inner regular with respect to $\mathcal{K}$. Write Σ for the domain of μ. Note that, for $K \in \mathcal{K}$,

$$\mu K = \phi_0 K = \mu_0^* K,$$

so we can already be sure that the conclusion (i) of this theorem is satisfied. Now μ extends μ_0.

P(i) Take any $K \in \mathcal{K}$. Let $E_0 \in \Sigma_0$ be a measurable envelope of K for the measure μ_0. If $E \in \Sigma_0$, then surely

$$\begin{aligned}\mu_*(K \cap E) &= \sup\{\mu K' : K' \in \mathcal{K},\ K' \subseteq K \cap E\}\\ &= \sup\{\mu_0^* K' : K' \in \mathcal{K},\ K' \subseteq K \cap E\} \le \mu_0^*(K \cap E).\end{aligned}$$

On the other hand, given $\gamma < \mu_0^*(K \cap E) = \mu_0(E_0 \cap E)$, there is a closed set $F \in \Sigma_0$ such that $F \subseteq E_0 \cap E$ and $\mu_0 F \ge \gamma$, so that

$$\mu_*(K \cap E) \ge \mu(K \cap F) = \mu_0^*(K \cap F) = \mu_0(E_0 \cap F) \ge \gamma.$$

Thus $\mu_*(K \cap E) = \mu_0^*(K \cap E)$ for every $K \in \mathcal{K}$ and $E \in \Sigma_0$.

(ii) If $K \in \mathcal{K}$ and $E \in \Sigma_0$ then

$$\mu_*(K \cap E) + \mu_*(K \setminus E) = \mu_0^*(K \cap E) + \mu_0^*(K \setminus E) = \mu_0^* K = \mu K.$$

Because μ is complete and locally determined and inner regular with respect to $\mathcal{K}$, $E \in \Sigma$ (413F(iv)). Thus $\Sigma_0 \subseteq \Sigma$.

(iii) For any $E \in \Sigma_0$, we now have

$$\begin{aligned}\mu E &= \sup\{\mu K : K \in \mathcal{K},\ K \subseteq E\} = \sup\{\mu_0^* K : K \in \mathcal{K},\ K \subseteq E\}\\ &\le \mu_0 E = \sup\{\mu_0 K : K \in \mathcal{K} \cap \Sigma_0,\ K \subseteq E\} \le \mu E.\end{aligned}$$

As E is arbitrary, μ extends μ_0. **Q**

(c) Because $\Sigma_0 \cap \mathfrak{T}$ is a base for $\mathfrak{T}$, closed under finite unions, μ is unique, by 415H(iv).

(d) Now for the conditions (i)-(v). I have already noted that (i) is guaranteed by the construction. Concerning (ii), if $G \subseteq X$ is open, we surely have $(\mu_0)_* G \le \mu_* G = \mu G$ because μ extends μ_0. On the other hand, writing $\mathcal{G} = \{G' : G' \in \Sigma_0 \cap \mathfrak{T},\ G' \subseteq G\}$, $\mathcal{G}$ is upwards-directed and has union G, so

$$\mu G = \sup_{G' \in \mathcal{G}} \mu G' = \sup_{G' \in \mathcal{G}} \mu_0 G' \le (\mu_0)_* G.$$

So (ii) is true.

Because μ extends μ_0, the embedding $\Sigma_0 \subsetneq \Sigma$ corresponds to a measure-preserving embedding of $\mathfrak{A}_0$ as a σ-subalgebra of $\mathfrak{A}$. To see that $\mathfrak{A}_0$ is order-dense in $\mathfrak{A}$, take any non-zero $a \in \mathfrak{A}$. This is expressible as $E^\bullet$ for some $E \in \Sigma$ with $\mu E > 0$. Now there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$. There is an $E_0 \in \Sigma_0$ which is a measurable envelope for K with respect to μ_0, so that

$$\mu E_0 = \mu_0 E_0 = \mu_0^* K = \mu K.$$

But this means that

$$0 \ne E_0^\bullet = K^\bullet \subseteq E^\bullet = a$$

in $\mathfrak{A}$, while $E_0^\bullet \in \mathfrak{A}_0$. As a is arbitrary, $\mathfrak{A}_0$ is order-dense in $\mathfrak{A}$.

If $a \in \mathfrak{A}^f$, then $B = \{b : b \in \mathfrak{A}_0,\ b \subseteq a\}$ is upwards-directed and $\sup_{b \in B} \bar{\mu}_0 b \le \bar{\mu} a$ is finite; accordingly B has a supremum in $\mathfrak{A}_0$ (321C), which must also be its supremum in $\mathfrak{A}$, which is a (313O, 313K). So $a \in \mathfrak{A}_0$. Thus $\mathfrak{A}^f$ can be identified with $\mathfrak{A}_0^f$. But this means that, for any $p \in [1, \infty[$, $L^p(\mu) \cong L^p(\mathfrak{A}, \bar{\mu})$ is identified with $L^p(\mathfrak{A}_0, \bar{\mu}_0) \cong L^p(\mu)$ (366H). This proves (iii).

Of course (iv) and (v) are just translations of this. If $E \in \Sigma$ and $\mu E < \infty$, then $E^\bullet \in \mathfrak{A}^f \subseteq \mathfrak{A}_0$, that is, there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$. If f is μ-integrable, then $f^\bullet \in L^1(\mu) = L^1(\mu_0)$, that is, there is a μ_0-integrable function f_0 such that $f = f_0$ μ-a.e.

(e) If μ_0 is complete and locally determined and $F \subseteq X$ is an arbitrary closed set, then

$$\mu_0^*F = \sup_{K\in\mathcal{K}} \mu_0^*(F\cap K) = \sup_{K\in\mathcal{K}} \mu(F\cap K) = \sup_{K\in\mathcal{K},K\subseteq F} \mu K = \mu F$$

by 412Jc, because μ and μ_0 are both inner regular with respect to $\mathcal{K}$.

(f) If μ_0 is localizable, $\mathfrak{A}_0$ is Dedekind complete; as it is order-dense in $\mathfrak{A}$, the two must coincide (314Ib). Consequently

$$L^0(\mu) \cong L^0(\mathfrak{A}) = L^0(\mathfrak{A}_0) \cong L^0(\mu_0), \quad L^\infty(\mu) \cong L^\infty(\mathfrak{A}) = L^\infty(\mathfrak{A}_0) \cong L^\infty(\mu_0).$$

So (iii)$'$ is true; now (iv)$'$ and (v)$'$ follow at once.

415M Corollary Let $(X,\mathfrak{T})$ be a regular topological space and μ_0 an effectively locally finite τ-additive measure on X, defined on the σ-algebra generated by a base for $\mathfrak{T}$. Then μ_0 has a unique extension to a quasi-Radon measure on X.

proof By 414Mb, μ_0 is inner regular with respect to the closed sets. So 415L gives the result.

415N Corollary Let $(X,\mathfrak{T})$ be a completely regular topological space, and μ_0 a τ-additive effectively locally finite Baire measure on X. Then μ_0 has a unique extension to a quasi-Radon measure on X.

proof This is a special case of 415M, because the domain $\mathcal{B}\mathfrak{a}(X)$ of μ_0 is generated by the family of cozero sets, which form a base for $\mathfrak{T}$ (4A2Fc).

415O Proposition (a) Let $(X,\mathfrak{T})$ be a topological space, and μ, ν two quasi-Radon measures on X. Then ν is an indefinite-integral measure over μ (definition: 234J[17]) iff $\nu F = 0$ whenever $F \subseteq X$ is closed and $\mu F = 0$.

(b) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space, and ν an indefinite-integral measure over μ. If ν is effectively locally finite it is a quasi-Radon measure.

proof (a) If ν is an indefinite-integral measure over μ, then of course it is zero on all μ-negligible closed sets. So let us suppose that the condition is satisfied. Write $\Sigma = \operatorname{dom}\mu$ and $\mathrm{T} = \operatorname{dom}\nu$.

(i) If $E \subseteq X$ is a μ-negligible Borel set it is ν-negligible, because every closed subset of E must be μ-negligible, therefore ν-negligible, and ν is inner regular with respect to the closed sets. In particular, taking U^* to be the union of the family $\mathcal{U} = \{U : U \in \mathfrak{T},\ \mu U < \infty\}$, $\nu(X\setminus U^*) = \mu(X\setminus U^*) = 0$ because μ is effectively locally finite. Also, of course, taking V^* to be the union of the family $\mathcal{V} = \{V : V \in \mathfrak{T},\ \nu V < \infty\}$, $\nu(X\setminus V^*) = 0$ because ν is effectively locally finite. Setting $\mathcal{G} = \mathcal{U}\cap\mathcal{V}$ and $G^* = \bigcup\mathcal{G}$, we have $G^* = U^*\cap V^*$, so G^* is ν-conegligible.

(ii) In fact, every μ-negligible set E is ν-negligible. **P?** Otherwise, $\nu^*(E\cap G^*) > 0$. Because the subspace measure ν_E is quasi-Radon (415B), there is a $G \in \mathcal{G}$ such that $\nu^*(E\cap G) > 0$. But there is an F_σ set $H \subseteq G\setminus E$ such that $\mu H = \mu(G\setminus E)$, and now $E\cap G$ is included in the μ-negligible Borel set $G\setminus H$, so that $\nu(E\cap G) = \nu(G\setminus H) = 0$. **XQ**

(iii) Let $\mathcal{K}$ be the family of closed subsets F of X such that either F is included in some member of $\mathcal{G}$ or $F\cap G^* = \emptyset$. If $E \in \operatorname{dom}\mu$ and $\mu E > 0$, then there is an $F \in \mathcal{K}$ such that $F \subseteq E$ and $\mu F > 0$. **P** If $\mu(E\setminus G^*) > 0$ take any closed set $F \subseteq E\setminus G^*$ with $\mu F > 0$. Otherwise, $\mu(E\cap G^*) > 0$. Because the subspace measure μ_E is quasi-Radon, there is a $G \in \mathcal{G}$ such that $\mu(E\cap G) > 0$; and now we can find a closed set $F \subseteq E\cap G$ with $\mu F > 0$, and $F \in \mathcal{K}$. **Q**

(iv) By 412Ia, there is a decomposition $\langle X_i\rangle_{i\in I}$ for μ such that every X_i except perhaps one belongs to $\mathcal{K}$ and that exceptional one, if any, is μ-negligible. Now $\langle X_i\rangle_{i\in I}$ is a decomposition for ν. **P** Every X_i is measured by ν because it is either closed or μ-negligible, and of finite measure for ν because it is either ν-negligible or included in a member of $\mathcal{G}$. If $E \subseteq X$ and $\nu E > 0$, then $\nu(E\cap G^*) > 0$, so there must be some $G \in \mathcal{G}$ such that $\nu(E\cap G) > 0$. Now $J = \{i : i \in I,\ \mu(X_i\cap G) > 0\}$ is countable, and $\nu(G\setminus\bigcup_{i\in J} X_i) = \mu(G\setminus\bigcup_{i\in J} X_i) = 0$, so there is an $i \in J$ such that $\nu(X_i\cap E) > 0$. By 213Ob, $\langle X_i\rangle_{i\in I}$ is a decomposition for ν. **Q**

(v) It follows that $\Sigma \subseteq \mathrm{T}$. **P** If $E \in \Sigma$, then for every $i \in I$ there is an F_σ set $H \subseteq E\cap X_i$ such that $E\cap X_i\setminus H$ is μ-negligible, therefore ν-negligible, and $E\cap X_i \in \mathrm{T}$. As i is arbitrary, $E \in \mathrm{T}$. **Q** In fact, ν is the completion of $\nu\restriction\Sigma$. **P** If $F \in \mathrm{T}$, then for every $i \in I$ there is an F_σ set $H_i \subseteq F\cap X_i$ such that $F\cap X_i\setminus H_i$ is ν-negligible. Set $H = \bigcup_{i\in I} H_i$; because $H\cap X_i = H_i$ belongs to Σ for every i, $H \in \Sigma$; and $\nu(F\setminus H) = \sum_{i\in I}\nu(F\cap X_i\setminus H) = 0$. Similarly, there is

[17] Formerly 234B.

an $H'\in\Sigma$ such that $H'\subseteq X\setminus F$ and $\nu((X\setminus F)\setminus H')=0$, so that $H\subseteq F\subseteq X\setminus H'$ and $\nu((X\setminus H')\setminus H)=0$. So F is measured by the completion of $\nu\restriction\Sigma$. Since ν itself is complete, it must be the completion of $\nu\restriction\Sigma$. **Q**

(vi) By (iv), ν is inner regular with respect to $\{E:E\in\Sigma,\ \mu E<\infty\}$. By 234O[18], ν is an indefinite-integral measure over μ.

(b) Let $f\in\mathcal{L}^0(\mu)$ be a non-negative function such that $\nu F=\int f\times\chi F\,d\mu$ whenever this is defined. Because μ is complete and locally determined, so is ν (234Nb[19]). Because μ is an effectively locally finite τ-additive topological measure, ν is a τ-additive topological measure (414H). Because μ is inner regular with respect to the closed sets, so is ν (412Q). Since we are assuming in the hypotheses that ν is effectively locally finite, it is a quasi-Radon measure.

415P Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space.

(a) Suppose that $(X,\mathfrak{T})$ is completely regular. If $1\le p<\infty$ and $f\in\mathcal{L}^p(\mu)$, then for any $\epsilon>0$ there is a bounded continuous function $g:X\to\mathbb{R}$ such that $\mu\{x:g(x)\neq 0\}<\infty$ and $\|f-g\|_p\le\epsilon$.

(b) Suppose that $(X,\mathfrak{T})$ is regular and Lindelöf. Let $f\in\mathcal{L}^0(\mu)$ be locally integrable. Then for any $\epsilon>0$ there is a continuous function $g:X\to\mathbb{R}$ such that $\|f-g\|_1\le\epsilon$.

proof (a) Write $\mathfrak{C}$ for the set of bounded continuous functions $g:X\to\mathbb{R}$ such that $\{x:g(x)\neq 0\}$ has finite measure. Then $\mathfrak{C}$ is a linear subspace of $\mathbb{R}^X$ included in $\mathcal{L}^p=\mathcal{L}^p(\mu)$. Let $\mathfrak{U}$ be the closure of $\mathfrak{C}$ in $\mathcal{L}^p$, that is, the set of $h\in\mathcal{L}^p$ such that for every $\epsilon>0$ there is a $g\in\mathfrak{C}$ such that $\|h-g\|_p\le\epsilon$. Then $\mathfrak{U}$ is closed under addition and scalar multiplication. Also $\chi E\in\mathfrak{U}$ whenever $\mu E<\infty$. **P** Let $\epsilon>0$. Set $\delta=\frac{1}{4}\epsilon^{1/p}$. Write $\mathcal{G}$ for the family of open sets of finite measure. Because μ is effectively locally finite, there is a $G\in\mathcal{G}$ such that $\mu(E\setminus G)\le\delta$. Let $F\subseteq G\setminus E$ be a closed set such that $\mu F\ge\mu(G\setminus E)-\delta$; then $\mu(E\triangle(G\setminus F))\le 2\delta$. Write $\mathcal{H}$ for the family of cozero sets. Because $\mathfrak{T}$ is completely regular, $\mathcal{H}$ is a base for $\mathfrak{T}$; because $\mathcal{H}$ is closed under finite unions (4A2C(b-iii)) and μ is τ-additive, there is an $H\in\mathcal{H}$ such that $H\subseteq G\setminus F$ and $\mu H\ge\mu(G\setminus F)-\delta$, so that $\mu(E\triangle H)\le 3\delta$. Express H as $\{x:g(x)>0\}$ where $g:X\to\mathbb{R}$ is a continuous function. For each $n\in\mathbb{N}$, set $g_n=ng\wedge\chi X\in\mathfrak{C}$; then

$$|\chi E-g_n|^p\le\chi(E\triangle H)+(\chi H-g_n)^p$$

for every n, so

$$\int|\chi E-g_n|^p\le\mu(E\triangle H)+\int(\chi H-g_n)^p\to\mu(E\triangle H)$$

as $n\to\infty$, because $g_n\to\chi H$. So there is an $n\in\mathbb{N}$ such that $\int|\chi E-g_n|^p\le 4\delta$, that is, $\|\chi E-g_n\|_p\le\epsilon$. As ϵ is arbitrary, $\chi E\in\mathfrak{U}$. **Q**

Accordingly every simple function belongs to $\mathfrak{U}$. But if $f\in\mathcal{L}^p$ and $\epsilon>0$, there is a simple function h such that $\|f-h\|_p\le\frac{1}{2}\epsilon$ (244Ha); now there is a $g\in\mathfrak{C}$ such that $\|h-g\|_p\le\frac{1}{2}\epsilon$ and $\|f-g\|_p\le\epsilon$, as claimed.

(b) This time, write $\mathcal{G}$ for the family of open subsets of X such that $\int_G f$ is finite, so that $\mathcal{G}$ is an open cover of X. As X is paracompact (4A2H(b-i)), there is a locally finite family $\mathcal{G}_0\subseteq\mathcal{G}$ covering X, which must be countable (4A2H(b-ii)).

Let $\langle\epsilon_G\rangle_{G\in\mathcal{G}_0}$ be a family of strictly positive real numbers such that $\sum_{G\in\mathcal{G}_0}\epsilon_G\le\epsilon$ (4A1P). Since X is completely regular (4A2H(b-i)), we can apply (a) to see that, for each $G\in\mathcal{G}_0$, there is a continuous function $g_G:X\to\mathbb{R}$ such that $\int|g_G-f\times\chi G|\le\epsilon_G$. Next, because X is normal (4A2H(b-i)), there is a family $\langle h_G\rangle_{G\in\mathcal{G}_0}$ of continuous functions from X to $[0,1]$ such that $h_G\le\chi G$ for every $G\in\mathcal{G}_0$ and $\sum_{G\in\mathcal{G}_0}h_G(x)=1$ for every $x\in X$ (4A2F(d-viii)).

Set $g(x)=\sum_{G\in\mathcal{G}_0}g_G(x)h_G(x)$ for every $x\in X$. Because $\mathcal{G}_0$ is locally finite, $g:X\to\mathbb{R}$ is continuous (4A2Bh). Now

$$\begin{aligned}\int|f-g|&=\int|\sum_{G\in\mathcal{G}_0}(f-g_G)\times h_G|\le\sum_{G\in\mathcal{G}_0}\int|(f-g_G)\times h_G|\\&\le\sum_{G\in\mathcal{G}_0}\int_G|f-g_G|\le\sum_{G\in\mathcal{G}_0}\epsilon_G\le\epsilon,\end{aligned}$$

as required.

415Q Recall (411P) that if $(\mathfrak{A},\bar{\mu})$ is a localizable measure algebra, with Stone space $(Z,\mathfrak{S},\mathrm{T},\nu)$, then ν is a strictly positive completion regular quasi-Radon measure, inner regular with respect to the open-and-closed sets

[18] Formerly 234G.

[19] Formerly 234F.

(which are all compact). The following construction is primarily important for Radon measure spaces (see 416V), but is also of interest for general quasi-Radon spaces.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Let $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ be the Stone space of $(\mathfrak{A}, \bar{\mu})$. For $E \in \Sigma$ let $E^* \subseteq Z$ be the open-and-closed set corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Define $R \subseteq Z \times X$ by saying that $(z, x) \in R$ iff $x \in F$ whenever $F \subseteq X$ is closed and $z \in F^*$. Set $Q = R^{-1}[X]$.

(a) R is a closed subset of $Z \times X$.

(b) For any $E \in \Sigma$, $R[E^*]$ is the smallest closed set such that $\mu(E \setminus R[E^*]) = 0$. In particular, if $F \subseteq X$ is closed then $R[F^*]$ is the self-supporting closed set included in F such that $\mu(F \setminus R[F^*]) = 0$; and $R[Z]$ is the support of μ.

(c) Q is of full outer measure for ν.

(d) For any $E \in \Sigma$, $R^{-1}[E] \triangle (Q \cap E^*)$ is negligible; consequently $\nu^* R^{-1}[E] = \mu E$ and $R^{-1}[E] \cap R^{-1}[X \setminus E]$ is negligible.

(e) For any $A \subseteq X$, $\nu^* R^{-1}[A] = \mu^* A$.

(f) If $(X, \mathfrak{T})$ is regular, then $R^{-1}[G]$ is relatively open in Q for every open set $G \subseteq X$, $R^{-1}[F]$ is relatively closed in Q for every closed set $F \subseteq X$ and $R^{-1}[X \setminus E] = Q \setminus R^{-1}[E]$ for every Borel set $E \subseteq X$.

proof (a)

$$R = \bigcap_{F \subseteq X \text{ is closed}} ((Z \setminus F^*) \times X) \cup (Z \times F)$$

is an intersection of closed sets, therefore closed.

(b) Let $\mathcal{G}$ be the family of open sets $G \subseteq X$ such that $\mu(E \cap G) = 0$, and $G_0 = \bigcup \mathcal{G}$; then $G_0 \in \mathcal{G}$ (414Ea). Set $F_0 = X \setminus G_0$, so that F_0 is the smallest closed set such that $E \setminus F_0$ is negligible, and $F_0^* \supseteq E^*$. If $(z, x) \in R$ and $z \in E^*$ we must have $x \in F_0$. Thus $R[E^*] \subseteq F_0$. On the other hand, if $x \in F_0$, and G is an open set containing x, then $G \notin \mathcal{G}$ so $\mu(G \cap E) > 0$ and $(E \cap G)^* \neq \emptyset$. Accordingly $\{(G \cap E)^* : x \in G \in \mathfrak{T}\}$ is a downwards-directed family of non-empty open-and-closed sets in the compact space Z and has non-empty intersection, containing a point z say. If $H \subseteq X$ is closed and $z \in H^*$, then $X \setminus H$ is open and $z \notin (X \setminus H)^*$, so x cannot belong to $X \setminus H$, that is, $x \in H$; as H is arbitrary, $(z, x) \in R$ and $x \in R[E^*]$; as x is arbitrary, $R[E^*] = F_0$, as claimed.

Of course, when E is actually closed, $R[E^*] = F_0 \subseteq E$. Taking $E = X$ we see that $R[Z] = R[X^*]$ is the support of μ.

(c) If $W \in \mathrm{T}$ and $\nu W > 0$, there is a non-empty open-and-closed set $U \subseteq W$, by 322Ra, which must be of the form E^* for some $E \in \Sigma$. By (b), $R[E^*]$ cannot be empty; but $E^* \subseteq W$, so $R[W] \neq \emptyset$, that is, $W \cap Q \neq \emptyset$. As W is arbitrary, Q has full outer measure in Z.

(d)(i) Let $\mathcal{F}$ be the set of closed subsets of X included in E. Then $\sup_{F \in \mathcal{F}} F^\bullet = E^\bullet$ in $\mathfrak{A}$ (412N), so $E^* \setminus \bigcup_{F \in \mathcal{F}} F^*$ is nowhere dense and negligible. Now for each $F \in \mathcal{F}$, $R[F^*] \subseteq F$, so $Q \cap F^* \subseteq R^{-1}[F] \subseteq R^{-1}[E]$. Accordingly

$$Q \cap E^* \setminus R^{-1}[E] \subseteq E^* \setminus \textstyle\bigcup_{F \in \mathcal{F}} F^*$$

is nowhere dense and negligible.

(ii) ? Suppose, if possible, that $\nu^*(R^{-1}[E] \setminus E^*) > 0$. Then there is an open-and-closed set U of finite measure such that $\nu^*(R^{-1}[E] \cap U \setminus E^*) > 0$ (use 412Jc). Express U as H^*, where $\mu H < \infty$, and let $F \subseteq H \setminus E$ be a closed set such that $\mu((H \setminus E) \setminus F) < \nu^*(R^{-1}[E] \cap H^* \setminus E^*)$. Then we must have $\nu^*(R^{-1}[E] \cap F^*) > 0$. But $R[F^*] \subseteq F \subseteq X \setminus E$ so $F^* \cap R^{-1}[E] = \emptyset$, which is impossible. **X**

(iii) Putting these together, $(Q \cap E^*) \triangle R^{-1}[E]$ is negligible.

(iv) It follows at once that (because Z is a measurable envelope for Q)

$$\nu^* R^{-1}[E] = \nu^*(Q \cap E^*) = \nu E^* = \mu E.$$

Moreover, applying the result to $X \setminus E$,

$$R^{-1}[X \setminus E] \cap R^{-1}[E] \subseteq (R^{-1}[X \setminus E] \triangle (Q \cap (X \setminus E)^*)) \cup (R^{-1}[E] \triangle (Q \cap E^*))$$

is negligible.

(e)(i) Take $E \in \Sigma$ such that $A \subseteq E$ and $\mu E = \mu^* A$; then $R^{-1}[A] \subseteq R^{-1}[E]$, so

$$\nu^* R^{-1}[A] \leq \nu^* R^{-1}[E] = \mu E = \mu^* A.$$

(ii) ? Suppose, if possible, that $\nu^*R^{-1}[A] < \mu^*A$. Let $W \in \mathrm{T}$ be a measurable envelope of $F \in \Sigma$ such that $\nu(W \triangle F^*) = 0$. Since

$$\mu F = \nu F^* = \nu W < \mu^* A,$$

$\mu^*(A \setminus F) > 0$; let G be a measurable envelope of $A \setminus F$ disjoint from F. Then $G^* \cap F^* = \emptyset$ so

$$\nu(G^* \setminus W) = \nu G^* = \mu G > 0$$

and there is a non-empty open-and-closed $V \subseteq G^* \setminus W$; let $H \in \Sigma$ be such that $H \subseteq G$ and $V = H^*$. In this case, $R[V]$ is closed and $\mu(H \setminus R[V]) = 0$, by (b), so that $H \cap R[V]$ is measurable, not negligible, and included in G. But $H \cap R[V] \cap A$ is empty, because $V \cap R^{-1}[A]$ is empty, so $\mu^*(H \cap R[V] \cap A) < \mu(H \cap R[V])$, and G cannot be a measurable envelope of $A \setminus F$. **X**

Thus $\nu^*R^{-1}[A] = \mu^*A$, as claimed.

(f) Suppose now that $(X, \mathfrak{T})$ is regular.

(i) If $G \subseteq X$ is open, $R^{-1}[G] \cap R^{-1}[X \setminus G] = \emptyset$. **P** If $z \in R^{-1}[G]$, then there is an $x \in G$ such that $(z, x) \in R$. Let H be an open set containing x such that $\overline{H} \subseteq G$. Then $x \notin X \setminus H$ so $z \notin (X \setminus H)^*$, that is, $z \in H^*$. But

$$R[H^*] \subseteq R[\overline{H}^*] \subseteq \overline{H} \subseteq G,$$

so $H^* \cap R^{-1}[X \setminus G] = \emptyset$ and $z \notin R^{-1}[X \setminus G]$. **Q**

(ii) It is easy to check that

$$\begin{aligned}\mathcal{E} &= \{E : E \subseteq X,\ R^{-1}[E] \cap R^{-1}[X \setminus E] = \emptyset\} \\ &= \{E : E \subseteq X,\ R^{-1}[X \setminus E] = Q \setminus R^{-1}[E]\}\end{aligned}$$

is a σ-algebra of subsets of X (indeed, an algebra closed under arbitrary unions), just because $R \subseteq Z \times X$ and $R^{-1}[X] = Q$. Because it contains all open sets, $\mathcal{E}$ must contain all Borel sets.

(iii) Now suppose once again that $G \subseteq X$ is open and that $z \in R^{-1}[G]$. As in (i) above, there is an open set $H \subseteq G$ such that $z \in H^* \subseteq Z \setminus R^{-1}[X \setminus G]$, so that $z \in H^* \cap Q \subseteq R^{-1}[G]$. As z is arbitrary, $R^{-1}[G]$ is relatively open in Q.

(iv) Finally, if $F \subseteq X$ is closed, $R^{-1}[F] = Q \setminus R^{-1}[X \setminus F]$ is relatively closed in Q.

415R Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Hausdorff quasi-Radon measure space and $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ the Stone space of its measure algebra. Let $R \subseteq Z \times X$ be the relation described in 415Q. Then

(a) R is (the graph of) a function f;

(b) f is inverse-measure-preserving for the subspace measure ν_Q on $Q = \operatorname{dom} f$, and in fact μ is the image measure $\nu_Q f^{-1}$;

(c) if $(X, \mathfrak{T})$ is regular, then f is continuous.

proof (a) If $z \in Z$ and x, $y \in X$ are distinct, let G, H be disjoint open sets containing x, y respectively. Then

$$(X \setminus G)^* \cup (X \setminus H)^* = ((X \setminus G) \cup (X \setminus H))^* = Z,$$

defining * as in 415Q, so z must belong to at least one of $(X \setminus G)^*$, $(X \setminus H)^*$. In the former case $(z, x) \notin R$ and in the latter case $(z, y) \notin R$. This shows that R is a function; to remind us of its new status I will henceforth call it f. The domain of f is just $Q = R^{-1}[X]$.

(b) By 415Qd, f is inverse-measure-preserving for ν_Q and μ. Suppose that $A \subseteq X$ and $f^{-1}[A]$ is in the domain T_Q of ν_Q, that is, is of the form $Q \cap U$ for some $U \in \mathrm{T}$. Take any $E \in \Sigma$ such that $\mu E > 0$; then either $\nu(E^* \cap U) > 0$ or $\nu(E^* \setminus U) > 0$. (α) Suppose that $\nu(E^* \cap U) > 0$. Because ν is inner regular with respect to the open-and-closed sets, there is an $H \in \Sigma$ such that $H^* \subseteq E^* \cap U$ and $\mu H = \nu H^* > 0$. Now there is a closed set $F \subseteq E \cap H$ with $\mu F > 0$. In this case, $f[F^*] \subseteq F \subseteq E$, by 415Qb, while $F^* \cap Q \subseteq U \cap Q = f^{-1}[A]$, so $f[F^*] \subseteq E \cap A$. But this means that

$$\mu_*(E \cap A) \ge \mu f[F^*] = \mu F > 0.$$

(β) If $\nu(E^* \setminus U) > 0$, then the same arguments show that $\mu_*(E \setminus A) > 0$. (γ) Thus $\mu_*(E \cap A) + \mu_*(E \setminus A) > 0$ whenever $\mu E > 0$. Because μ is complete and locally determined, $A \in \Sigma$ (413F(vii)).

Thus we see that $\{A : A \subseteq X,\ f^{-1}[A] \in \mathrm{T}_Q\}$ is included in Σ, and μ is the image measure $\nu_Q f^{-1}$.

(c) If $\mathfrak{T}$ is regular, then 415Qf tells us that f is continuous.

415X Basic exercises >(a) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space and $E\in\Sigma$ an atom for the measure. Show that there is a closed set $F\subseteq E$ such that $\mu F>0$ and F is an atom of Σ, in the sense that the only measurable subsets of F are $\emptyset$ and F. (*Hint*: 414G.) Show that μ is atomless iff all countable subsets of X are negligible.

(b) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any family of quasi-Radon measure spaces. Show that the direct sum measure on $X=\{(x,i):i\in I,\,x\in X_i\}$ is a quasi-Radon measure when X is given its disjoint union topology.

(c) Let $\mathfrak{S}$ be the **right-facing Sorgenfrey topology** or **lower limit topology** on $\mathbb{R}$, that is, the topology generated by the half-open intervals of the form $[\alpha,\beta[$. Show that Lebesgue measure is completion regular and quasi-Radon for $\mathfrak{S}$. (*Hint*: 114Yj or 221Yb, or 419L.)

(d) Let X be a topological space and μ a complete measure on X, and suppose that there is a conegligible closed measurable set $Y\subseteq X$ such that the subspace measure on Y is quasi-Radon. Show that μ is quasi-Radon.

(e) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space. Show that μ is inner regular with respect to the family of self-supporting closed sets included in open sets of finite measure.

(f) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space. Show that whenever $E\in\Sigma$ and $\epsilon>0$ there is an open set G such that $\mu G\le\mu E+\epsilon$ and $E\setminus G$ is negligible.

(g) Find a compact Hausdorff quasi-Radon measure space which is not σ-finite.

(h) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an atomless quasi-Radon measure space which is outer regular with respect to the open sets. Show that it is σ-finite. (*Hint*: if not, take a decomposition $\langle X_i\rangle_{i\in I}$ in which every X_i except one is self-supporting, and a set A meeting every X_i in just one point.)

(i) Let (X,Σ,μ) be a σ-finite measure space in which Σ is countably generated as a σ-algebra. Show that, for a suitable topology on X, the completion of μ is a quasi-Radon measure. (*Hint*: take the topology generated by a countable subalgebra of Σ, and use the arguments of 415D.)

(j) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces such that every μ_i is strictly positive, and λ the product measure on $X=\prod_{i\in I}X_i$. Show that if every $\mathfrak{T}_i$ has a countable network, λ is a quasi-Radon measure.

(k) Let $\langle X_i\rangle_{i\in I}$ be a family of separable metrizable spaces, and μ a quasi-Radon measure on $X=\prod_{i\in I}X_i$. Show that μ is completion regular iff every self-supporting closed set in X is determined by coordinates in a countable set. (*Hint*: 4A2Eb.)

(l) Find two quasi-Radon measures μ, ν on the unit interval such that $\mu G\le\nu G$ for every open set G but there is a closed set F such that $\nu F<\mu F$.

(m) Let X be a topological space and μ, ν two quasi-Radon measures on X. (i) Suppose that $\mu F=\nu F$ whenever $F\subseteq X$ is closed and both μF and νF are finite. Show that $\mu=\nu$. (ii) Suppose that $\mu G=\nu G$ whenever $G\subseteq X$ is open and both μG and νG are finite. Show that $\mu=\nu$.

(n) In 415L, write $\tilde\mu_0$ for the c.l.d. version of μ_0 (213E). Show that μ extends $\tilde\mu_0$. Show that $\tilde\mu_0$ is τ-additive and inner regular with respect to the closed sets.

(o) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite paracompact Hausdorff quasi-Radon measure space, and $f\in\mathcal{L}^0(\mu)$ a locally integrable function. Show that for any $\epsilon>0$ there is a continuous function $g:X\to\mathbb{R}$ such that $\int|f-g|\le\epsilon$.

>(p) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite completely regular quasi-Radon measure space. (i) Show that for every $E\in\Sigma$ there is an F in the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$ of X such that $\mu(E\triangle F)=0$. (*Hint*: start with an open set E of finite measure.) (ii) Show that for every Σ-measurable function $f:X\to\mathbb{R}$ there is a $\mathcal{B}\mathfrak{a}(X)$-measurable function equal almost everywhere to f.

(q) Let (X,Σ,μ) be a measure space and f a μ-integrable real-valued function. Show that there is a unique quasi-Radon measure λ on $\mathbb{R}$ such that $\lambda\{0\}=0$ and $\lambda\,[\alpha,\infty[\,=\mu^*\{x:x\in\operatorname{dom}f,\,f(x)\ge\alpha\}$, $\lambda\,]-\infty,-\alpha]=\mu^*\{x:x\in\operatorname{dom}f,\,f(x)\le-\alpha\}$ whenever $\alpha>0$; and that $\int h\,d\lambda=\int hf\,d\mu$ whenever $h\in\mathcal{L}^0(\lambda)$ and $h(0)=0$ and either integral is defined in $[-\infty,\infty]$. (*Hint*: set $\lambda E=\mu^*f^{-1}[E\setminus\{0\}]$ for Borel sets $E\subseteq\mathbb{R}$, and use 414Mb, 414O and 235Gb[20].)

[20] Formerly 235I.

(r) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite quasi-Radon measure space with $\mu X > 0$. Show that there is a quasi-Radon probability measure on X with the same measurable sets and the same negligible sets as μ.

(s) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, $(\mathfrak{A}, \bar{\mu})$ its measure algebra, and $\mathfrak{A}^f$ the ideal $\{a : a \in \mathfrak{A}, \bar{\mu}a < \infty\}$. Show that $\{G^\bullet : G \in \mathfrak{T}, \mu G < \infty\}$ is dense in $\mathfrak{A}^f$ for the strong measure-algebra topology (323Ad).

(t) Find a second-countable Hausdorff topological space X with a τ-additive Borel probability measure which is not inner regular with respect to the closed sets.

415Y Further exercises (a) Give an example of two quasi-Radon measures μ, ν on $\mathbb{R}$ such that their sum, as defined in 234G[21], is not effectively locally finite, therefore not a quasi-Radon measure.

(b) Show that any quasi-Radon measure space is isomorphic, as topological measure space, to a subspace of a compact quasi-Radon measure space. (*Hint*: if X is a T_1 quasi-Radon measure space, let $\hat{X}$ be its Wallman compactification (Engelking 89, 3.6.21).)

(c) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space. Show that the following are equiveridical: (i) μ is outer regular with respect to the open sets; (ii) every negligible subset of X is included in an open set of finite measure; (iii) $\{x : \mu\{x\} = 0\}$ can be covered by a sequence of open sets of finite measure.

(d) Show that $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous for the right-facing Sorgenfrey topology.

(e) Let $r \geq 1$. On $\mathbb{R}^r$ let $\mathfrak{S}$ be the topology generated by the half-open intervals $[a, b[$ where a, $b \in \mathbb{R}^r$ (definition: 115Ab). (i) Show that $\mathfrak{S}$ is the product topology if each factor is given the right-facing Sorgenfrey topology (415Xc). (ii) Show that Lebesgue measure is quasi-Radon for $\mathfrak{S}$. (*Hint*: induce on r. See also 417Yi.)

(f) Let $Y \subseteq [0, 1]$ be a set of full outer measure and zero inner measure for Lebesgue measure μ. Give $[0, 1]$ the topology $\mathfrak{T}$ generated by $\mathfrak{S} \cup \{Y\}$ where $\mathfrak{S}$ is the usual topology. Show that the subspace measure $\nu = \mu_Y$ is quasi-Radon for the subspace topology $\mathfrak{T}_Y$, but that there is no measure λ on X which is quasi-Radon for $\mathfrak{T}$ and such that the subspace measure λ_Y is equal to ν.

(g) Find a base $\mathcal{U}$ for the topology of $X = \{0, 1\}^{\mathbb{N}}$ and two totally finite (quasi-)Radon measures μ, ν on X such that $G \cap H \in \mathcal{U}$ for all G, $H \in \mathcal{U}$, $\mu G \leq \nu G$ for every $G \in \mathcal{U}$, but $\nu X < \mu X$.

(h) Let X be a topological space and $\mathcal{G}$ an open cover of X. Suppose that for each $G \in \mathcal{G}$ we are given a quasi-Radon measure μ_G on G such that $\mu_G(U) = \mu_H(U)$ whenever G, $H \in \mathcal{G}$ and $U \subseteq G \cap H$ is open. Show that there is a unique quasi-Radon measure on X such that each μ_G is the subspace measure on G. (*Hint*: if $\langle \mu_G \rangle_{G \in \mathcal{G}}$ is a maximal family with the given properties, then $\mathcal{G}$ is upwards-directed.)

(i) Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X, and suppose that there is a family $\mathcal{U} \subseteq \Sigma \cap \mathfrak{T}$ such that

$\mu U < \infty$ for every $U \in \mathcal{U}$,
for every $U \in \mathcal{U}$, $\mathfrak{T} \cap \Sigma \cap \mathcal{P}U$ is a base for the subspace topology of U,
if $\mathcal{G}$ is an upwards-directed family in $\mathfrak{T} \cap \Sigma$ and $\bigcup \mathcal{G} \in \mathcal{U}$, then $\mu(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}} \mu G$,
μ is inner regular with respect to the closed sets,
if $E \in \Sigma$ and $\mu E > 0$ then there is a $U \in \mathcal{U}$ such that $\mu(E \cap U) > 0$.

Show that μ has an extension to a quasi-Radon measure on X.

(j) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space such that $\mathfrak{T}$ is normal (but not necessarily Hausdorff or regular). Show that if $1 \leq p < \infty$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$, there is a bounded continuous function $g : X \to \mathbb{R}$ such that $\|f - g\|_p \leq \epsilon$ and $\{x : g(x) \neq 0\}$ has finite measure.

(k) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a completely regular quasi-Radon measure space and suppose that we are given a uniformity defining the topology $\mathfrak{T}$. Show that if $1 \leq p < \infty$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$, there is a bounded uniformly continuous function $g : X \to \mathbb{R}$ such that $\|f - g\|_p \leq \epsilon$ and $\{x : g(x) \neq 0\}$ has finite measure.

[21] Formerly 112Xe.

(l) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a completely regular quasi-Radon measure space and τ an extended Fatou norm on $L^0(\mu)$ such that (i) $\tau \restriction L^\tau$ is an order-continuous norm (ii) whenever $E \in \Sigma$ and $\mu E > 0$ there is an open set G such that $\mu(E \cap G) > 0$ and $\tau(\chi G^\bullet) < \infty$. Show that $L^\tau \cap \{f^\bullet : f : X \to \mathbb{R}$ is continuous$\}$ is norm-dense in L^τ.

(m) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space. Show that μ is a compact measure (definition: 342Ac or 451Ab) iff there is a locally compact topology $\mathfrak{S}$ on X such that $(X, \mathfrak{S}, \Sigma, \mu)$ is quasi-Radon.

415 Notes and comments 415B is particularly important because a very high proportion of the quasi-Radon measure spaces we study are actually subspaces of Radon measure spaces. I would in fact go so far as to say that when you have occasion to wonder whether all quasi-Radon measure spaces have a property, you should as a matter of habit look first at subspaces of Radon measure spaces; if the answer is affirmative for them, you will have most of what you want, even if the generalization to arbitrary quasi-Radon spaces gives difficulties. Of course the reverse phenomenon can also occur. Stone spaces (411P) can be thought of as quasi-Radon compactifications of Radon measure spaces (416V). But this is relatively rare. Indeed the reason why I give so few examples of quasi-Radon spaces at this point is just that the natural ones arise from Radon measure spaces. Note however that the quasi-Radon product of an uncountable family of Radon probability spaces need not be Radon (see 417Xq), so that 415E here and 417O below are sources of non-Radon quasi-Radon measure spaces. Density and lifting topologies can also provide us with quasi-Radon measure spaces (453Xd, 453Xg).

415K is the second in a series of inner-regular-extension theorems; there will be a third in 416J.

I have been saying since Volume 1 that the business of measure theory, since Lebesgue's time, has been to measure as many sets and integrate as many functions as possible. I therefore take seriously any theorem offering a canonical extension of a measure. 415L and its corollaries can all be regarded as improvement theorems, showing that a good measure can be made even better. We have already had such improvement theorems in Chapter 21: the completion and c.l.d. version of a measure (212C, 213E). In all such theorems we need to know exactly what effect our improvement is having on the other constructions we are interested in; primarily, the measure algebra and the function spaces. The machinery of Chapter 36 shows that if we understand the measure algebra(s) involved then the function spaces will give us no further surprises. Completion of a measure does not affect the measure algebra at all (322Da). Taking the c.l.d. version does not change $\mathfrak{A}^f = \{a : \bar{\mu}a < \infty\}$ or L^1 (213Fc, 213G, 322Db, 366H), but can affect the rest of the measure algebra and therefore L^0 and L^∞. In this respect, what we might call the 'quasi-Radon version' behaves like the c.l.d. version (as could be expected, since the quasi-Radon version must itself be complete and locally determined; cf. 415Xn). The archetypal application of 415L is 415N. We shall see later how Baire measures arise naturally when studying Banach spaces of continuous functions (436E). 415N will be one of the keys to applying the general theory of topological measure spaces in such contexts. A virtue of Baire measures is that inner regularity with respect to closed sets comes almost free (412D); but there can be unsurmountable difficulties if we wish to extend them to Borel measures (439M), and it is important to know that τ-additivity, even in the relatively weak form allowed by the definition I use here (411C), is often enough to give a canonical extension to a well-behaved measure defined on every Borel set. In 415C we have inner regularity for a different reason, and the measure is already known to be defined on every Borel set, so in fact the quasi-Radon version of the measure is just the c.l.d. version.

One interpretation of the Lifting Theorem is that for a complete strictly localizable measure space (X, Σ, μ) there is a function $g : X \to Z$, where Z is the Stone space of the measure algebra of μ, such that $E \triangle g^{-1}[E^*]$ is negligible for every $E \in \Sigma$, where $E^* \subseteq Z$ is the open-and-closed set corresponding to the image of E in the measure algebra (341Q). For a Hausdorff quasi-Radon measure space we have a function $f : Q \to X$, where Q is a dense subset of Z, such that $(Q \cap E^*) \triangle f^{-1}[E]$ is negligible for every $E \in \Sigma$ (415Qd, 415R); moreover, there is a canonical construction for this function. For completeness' sake, I have given the result for general, not necessarily Hausdorff, spaces X (415Q); but evidently it will be of greatest interest for regular Hausdorff spaces (415Rc). Perhaps I should remark that in the most important applications, Q is the whole of Z (416Xx). Of course the question arises, whether fg can be the identity. (Z typically has larger cardinal than X, so asking for gf to be the identity is a bit optimistic.) This is in fact an important question; I will return to it in 453M.

416 Radon measure spaces

We come now to the results for which the chapter so far has been preparing. The centre of topological measure theory is the theory of 'Radon' measures (411Hb), measures inner regular with respect to compact sets. Most of the section is devoted to pulling the earlier work together, and in particular to re-stating theorems on quasi-Radon measures in the new context. Of course this has to begin with a check that Radon measures are quasi-Radon (416A). It follows immediately that Radon measures are (strictly) localizable (416B). After presenting a miscellany of elementary facts, I turn to the constructions of §413, which take on simpler and more dramatic forms in this context (416J-416P). I proceed to investigate subspace measures (416R-416T) and some special product measures (416U). I end the section with further notes on the forms which earlier theorems on Stone spaces (416V) and compact measure spaces (416W) take when applied to Radon measure spaces.

416A Proposition A Radon measure space is quasi-Radon.

proof Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space. Because $\mathfrak{T}$ is Hausdorff, every compact set is closed, so μ is inner regular with respect to the closed sets. By 411E, μ is τ-additive; by 411Gf, it is effectively locally finite. Thus all parts of condition (ii) of 411Ha are satisfied, and μ is a quasi-Radon measure.

416B Corollary A Radon measure space is strictly localizable.

proof Put 416A and 415A together.

416C In order to use the results of §415 effectively, it will be helpful to spell out elementary conditions ensuring that a quasi-Radon measure is Radon.

Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a locally finite Hausdorff quasi-Radon measure space. Then the following are equiveridical:

(i) μ is a Radon measure;

(ii) whenever $E\in\Sigma$ and $\mu E>0$ there is a compact set K such that $\mu(E\cap K)>0$;

(iii) $\sup\{K^\bullet : K\subseteq X \text{ is compact}\}=1$ in the measure algebra of μ.

If μ is totally finite we can add

(iv) $\sup\{\mu K : K\subseteq X \text{ is compact}\}=\mu X$.

proof (i)$\Rightarrow$(ii) and (ii)$\Leftrightarrow$(iii) are trivial. For (ii)$\Rightarrow$(i), observe that if $E\in\Sigma$ and $\mu E>0$ there is a compact set $K\subseteq E$ such that $\mu K>0$. **P** There is a compact set K' such that $\mu(E\cap K')>0$, by hypothesis; now there is a closed set $K\subseteq E\cap K'$ such that $\mu K>0$, because μ is inner regular with respect to the closed sets, and K is compact. **Q** By 412B, μ is tight. Being a complete, locally determined, locally finite topological measure, it is a Radon measure.

When $\mu X<\infty$, of course, we also have (ii)$\Leftrightarrow$(iv).

416D Some further elementary facts are worth writing out plainly.

Lemma (a) In a Radon measure space, every compact set has finite measure.

(b) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, and $E\subseteq X$ a set such that $E\cap K\in\Sigma$ for every compact $K\subseteq X$. Then $E\in\Sigma$.

(c) A Radon measure is inner regular with respect to the self-supporting compact sets.

(d) Let X be a Hausdorff space and μ a tight locally finite complete locally determined measure on X. If μ measures every compact set, μ is a Radon measure.

(e) Let X be a Hausdorff space and $\langle\mu_i\rangle_{i\in I}$ a family of Radon measures on X. Let $\mu=\sum_{i\in I}\mu_i$ be their sum (definition: 234G[22]). Suppose that μ is locally finite. Then it is a Radon measure.

proof (a) 411Ga.

(b) We have only to remember that μ is complete, locally determined and tight, and apply 413F(ii).

(c) If $(X,\mathfrak{T},\Sigma,\mu)$ is a Radon measure space, $E\in\Sigma$ and $\gamma<\mu E$, there is a compact set $K\subseteq E$ such that $\mu K\ge\gamma$. By 414F, there is a self-supporting relatively closed set $L\subseteq K$ such that $\mu L=\mu K$; but now of course L is compact, while $L\subseteq E$ and $\mu L\ge\gamma$.

[22] Formerly 112Ya.

(d) Let $\mathcal{K}$ be the family of compact subsets of X; write Σ for the domain of μ. If $F \subseteq X$ is closed, then $F \cap K \in \mathcal{K} \subseteq \Sigma$ for every $K \in \mathcal{K}$; accordingly $F \in \Sigma$, by 412Ja. But this means that every closed set, therefore every open set, belongs to Σ, and μ is a Radon measure.

(e) Because every μ_i is a topological measure, so is μ; because every μ_i is complete, so is μ (234Ha). By hypothesis, μ is locally finite. If $\mu E > 0$, then there is some $i \in I$ such that $\mu_i E > 0$; now there is a compact $K \subseteq E$ such that $0 < \mu_i K \le \mu K$. So μ is inner regular with respect to the compact sets.

Now suppose that $E \subseteq X$ is such that μ measures $E \cap F$ whenever $\mu F < \infty$. Then, in particular, $E \cap K$ is measured by μ, therefore measured by every μ_i, whenever $K \subseteq X$ is compact. By 413F(ii), μ_i measures E for every i, so μ measures E. As E is arbitrary, μ is locally determined and is a Radon measure.

Remark In (e) above, note that if I is finite then μ is necessarily locally finite.

416E Specification of Radon measures In 415H I described some conditions which enable us to be sure that two quasi-Radon measures on a given topological space are the same. In the case of Radon measures we have a similar list. This time I include a note on the natural ordering of Radon measures.

Proposition Let X be a Hausdorff space and μ, ν two Radon measures on X.

(a) The following are equiveridical:

(i) $\nu \le \mu$ in the sense of 234P, that is, νE is defined and $\nu E \le \mu E$ whenever μ measures E;
(ii) $\mu K \le \nu K$ for every compact set $K \subseteq X$;
(iii) $\mu G \le \nu G$ for every open set $G \subseteq X$;
(iv) $\mu F \le \nu F$ for every closed set $F \subseteq X$.

If X is locally compact, we can add

(v) $\int f d\mu \le \int f d\nu$ for every non-negative continuous function $f : X \to \mathbb{R}$ with compact support.

(b) The following are equiveridical:

(i) $\mu = \nu$;
(ii) $\mu K = \nu K$ for every compact set $K \subseteq X$;
(iii) $\mu G = \nu G$ for every open set $G \subseteq X$;
(iv) $\mu F = \nu F$ for every closed set $F \subseteq X$.

If X is locally compact, we can add

(v) $\int f d\mu = \int f d\nu$ for every continuous function $f : X \to \mathbb{R}$ with compact support.

proof (a)(i)⇒(iv)⇒(ii) and **(i)⇒(iii)** are trivial, if we recall that $\nu \le \mu$ when $\operatorname{dom}\nu \supseteq \operatorname{dom}\mu$ and $\nu E \le \mu E$ for every $E \in \operatorname{dom}\mu$.

(ii)⇒(i) If (ii) is true, then

$$\mu E = \sup_{K \subseteq E \text{ is compact}} \mu K \le \sup_{K \subseteq E \text{ is compact}} \nu K = \nu E$$

for every set E measured by both μ and ν. Also $\operatorname{dom}\nu \subseteq \operatorname{dom}\mu$. **P** Suppose that $E \in \operatorname{dom}\nu$ and that $K \subseteq X$ is a compact set such that $\mu K > 0$. Then there are compact sets $K_1 \subseteq K \cap E$, $K_2 \subseteq K \setminus E$ such that

$$\nu K_1 + \nu K_2 > \nu(K \cap E) + \nu(K \setminus E) - \mu K = \nu K - \mu K.$$

So

$$\mu(K \setminus (K_1 \cup K_2)) \le \nu(K \setminus (K_1 \cup K_2)) < \mu K$$

and $\mu K_1 + \mu K_2 > 0$. This shows that $\mu_*(K \cap E) + \mu_*(K \setminus E) > 0$. As K is arbitrary, $E \in \operatorname{dom}\mu$ (413F(vii)). **Q**

So (i) is true.

(iii)⇒(ii) The point is that if $K \subseteq X$ is compact, then $\mu K = \inf\{\mu G : G \subseteq X \text{ is open}, K \subseteq G\}$. **P** Because $X = \bigcup\{\mu G : G \subseteq X \text{ is open}, \mu G < \infty\}$, there is an open set G_0 of finite measure including K. Now, for any $\gamma > \mu K$, there is a compact set $L \subseteq G_0 \setminus K$ such that $\mu L \ge \mu G_0 - \gamma$, so that $\mu G \le \gamma$, where $G = G_0 \setminus L$ is an open set including K. **Q**

The same is true for ν. So, if (iii) is true,

$$\mu K = \inf_{G \supseteq K \text{ is open}} \mu G \le \inf_{G \supseteq K \text{ is open}} \nu G = \nu K$$

for every compact $K \subseteq X$, and (ii) is true.

(iii)⇒(v) If (iii) is true and $f : X \to [0, \infty[$ is a non-negative continuous function, then

$$\int f d\mu = \int_0^\infty \mu\{x : f(x) > t\} dt$$

(252O)

$$\le \int_0^\infty \nu\{x : f(x) > t\} dt = \int f d\nu.$$

(v)⇒(iii) If X is locally compact and (v) is true, take any open set $G \subseteq X$, and consider the set A of continuous functions $f : X \to [0,1]$ with compact support such that $f \le \chi G$. Then A is upwards-directed and $\sup_{f\in A} f(x) = \chi G(x)$ for every $x \in X$, by 4A2G(e-i). So

$$\mu G = \sup_{f\in A} \int f d\mu \le \sup_{f\in A} \int f d\nu = \nu G$$

by 414Ba. As G is arbitrary, (iii) is true.

(b) now follows at once, or from 415H.

416F Proposition Let X be a Hausdorff space and μ a Borel measure on X. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) μ is locally finite and tight;

(iii) μ is locally finite and effectively locally finite, and $\mu G = \sup\{\mu K : K \subseteq G \text{ is compact}\}$ for every open set $G \subseteq X$;

(iv) μ is locally finite, effectively locally finite and τ-additive, and $\mu G = \sup\{\mu(G \cap K) : K \subseteq X \text{ is compact}\}$ for every open set $G \subseteq X$.

In this case the extension is unique; it is the c.l.d. version of μ.

proof (a)(i)⇒(iv) If $\mu = \tilde{\mu}\restriction\mathcal{B}(X)$ where $\tilde{\mu}$ is a Radon measure and $\mathcal{B}(X)$ is the Borel σ-algebra of X, then of course μ is locally finite and effectively locally finite and τ-additive because $\tilde{\mu}$ is (see 416A) and every open set belongs to $\mathcal{B}(X)$. Also

$$\mu G = \sup\{\mu K : K \subseteq G \text{ is compact}\} \le \sup\{\mu(G \cap K) : K \subseteq X \text{ is compact}\} \le \mu G$$

for every open set $G \subseteq X$, because $\tilde{\mu}$ is tight and compact sets belong to $\mathcal{B}(X)$.

(b)(iv)⇒(iii) Suppose that (iv) is true. Of course μ is locally finite and effectively locally finite. Suppose that $G \subseteq X$ is open and that $\gamma < \mu G$. Then there is a compact $K \subseteq X$ such that $\mu(G \cap K) > \gamma$. By 414K, the subspace measure μ_K is τ-additive. Now K is a compact Hausdorff space, therefore regular. By 414Ma there is a closed set $F \subseteq G \cap K$ such that $\mu_K F \ge \gamma$. Now F is compact, $F \subseteq G$ and $\mu F \ge \gamma$. As G and γ are arbitrary, (iii) is true.

(c)(iii)⇒(ii) I have to show that if μ satisfies the conditions of (iii) it is tight. Let $\mathcal{K}$ be the family of compact subsets of X and $\mathcal{A}$ the family of subsets of X which are either open or closed. Then whenever $A \in \mathcal{A}$, $F \in \Sigma$ and $\mu(A \cap F) > 0$, there is a $K \in \mathcal{K}$ such that $K \subseteq A$ and $\mu(K \cap F) > 0$. **P** Because μ is effectively locally finite, there is an open set G of finite measure such that $\mu(G \cap A \cap F) > 0$. (α) If A is open, then there will be a compact set $K \subseteq G \cap A$ such that $\mu K > \mu(G \cap A) - \mu(G \cap A \cap F)$, so that $\mu(K \cap F) > 0$. (β) If A is closed, then let $L \subseteq G$ be a compact set such that $\mu L > \mu G - \mu(G \cap A \cap F)$; then $K = L \cap A$ is compact and $\mu(K \cap F) > 0$. **Q**

By 412C, μ is inner regular with respect to $\mathcal{K}$, as required.

(d)(ii)⇒(i) If μ is locally finite and tight, let $\tilde{\mu}$ be the c.l.d. version of μ. Then $\tilde{\mu}$ is complete, locally determined, locally finite (because μ is), a topological measure (because μ is) and tight (because μ is, using 412Ha); so is a Radon measure. Every compact set has finite measure for μ, so μ is semi-finite and $\tilde{\mu}$ extends μ (213Hc).

(e) By 416Eb there can be at most one Radon measure extending μ, and we have observed in (d) above that in the present case it is the c.l.d. version of μ.

416G One of the themes of §434 will be the question: on which Hausdorff spaces is every locally finite quasi-Radon measure a Radon measure? I do not think we are ready for a general investigation of this, but I can give one easy special result.

Proposition Let $(X,\mathfrak{T})$ be a locally compact Hausdorff space and μ a locally finite quasi-Radon measure on X. Then μ is a Radon measure.

proof μ satisfies condition (ii) of 416C. **P** Take $E \in \operatorname{dom}\mu$ such that $\mu E > 0$. Let $\mathcal{G}$ be the family of relatively compact open subsets of X; then $\mathcal{G}$ is upwards-directed and has union X. By 414Ea, there is a $G \in \mathcal{G}$ such that $\mu(E \cap G) > 0$. But now $\overline{G}$ is compact and $\mu(E \cap \overline{G}) > 0$. **Q** By 416C, μ is a Radon measure.

416H Corollary Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space, and μ a locally finite, effectively locally finite, τ-additive Borel measure on X. Then μ is tight and its c.l.d. version is a Radon measure, the unique Radon measure on X extending μ.

proof By 415Cb, the c.l.d. version $\tilde{\mu}$ of μ is a quasi-Radon measure extending μ. Because μ is locally finite, so is $\tilde{\mu}$; by 416G, $\tilde{\mu}$ is a Radon measure. By 416Eb, the extension is unique. Now

$$\mu E = \tilde{\mu} E = \sup_{K \subseteq E \text{ is compact}} \tilde{\mu} K = \sup_{K \subseteq E \text{ is compact}} \mu K$$

for every Borel set $E \subseteq X$, so μ itself is tight.

416I While on the subject of locally compact spaces, I mention an important generalization of a result from Chapter 24.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally compact Radon measure space. Write C_k for the space of continuous real-valued functions on X with compact supports. If $1 \le p < \infty$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$, there is a $g \in C_k$ such that $\|f - g\|_p \le \epsilon$.

proof By 415Pa, there is a bounded continuous function $h_1 : X \to \mathbb{R}$ such that $G = \{x : h_1(x) \ne 0\}$ has finite measure and $\|f - h_1\|_p \le \frac{1}{2}\epsilon$. Let $K \subseteq G$ be a compact set such that $\|h_1\|_\infty(\mu(G \setminus K))^{1/p} \le \frac{1}{2}\epsilon$, and let $h_2 \in C_k$ be such that $\chi K \le h_2 \le \chi G$ (4A2G(e-i)). Set $g = h_1 \times h_2$. Then $g \in C_k$ and

$$\int |h_1 - g|^p \le \int_{G \setminus K} |h_1|^p \le \mu(G \setminus K)\|h_1\|_\infty^p,$$

so $\|h_1 - g\|_p \le \frac{1}{2}\epsilon$ and $\|f - g\|_p \le \epsilon$, as required.

416J I turn now to constructions of Radon measures based on ideas in §413.

Theorem Let X be a Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X and $\phi_0 : \mathcal{K} \to [0, \infty[$ a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(γ) for every $x \in X$ there is an open set G containing x such that $\sup\{\phi_0 K : K \in \mathcal{K},\ K \subseteq G\}$ is finite.

Then there is a unique Radon measure on X extending ϕ_0.

proof By 413M, there is a unique complete locally determined measure μ on X, extending ϕ_0, which is inner regular with respect to $\mathcal{K}$. By (γ), μ is locally finite; by 416Dd, it is a Radon measure.

416K Proposition (see Topsøe 70a) Let X be a Hausdorff space, T a subring of $\mathcal{P}X$ such that $\mathcal{H} = \{G : G \in \mathrm{T}$ is open$\}$ covers X, and $\nu : \mathrm{T} \to [0, \infty[$ a finitely additive functional (definition: 361B). Then there is a Radon measure μ on X such that $\mu K \ge \nu K$ for every compact $K \in \mathrm{T}$ and $\mu G \le \nu G$ for every open $G \in \mathrm{T}$.

proof (a) For $H \in \mathcal{H}$ set $\mathrm{T}_H = \mathrm{T} \cap \mathcal{P}H$; then T_H is an algebra of subsets of H, and $\nu_H = \nu{\restriction}\mathrm{T}_H$ is additive. By 391G, there is an additive functional $\tilde{\nu}_H : \mathcal{P}H \to [0, \infty[$ extending ν_H. Let $\mathfrak{F}$ be an ultrafilter on $\mathcal{H}$ containing $\{H : H_0 \subseteq H \in \mathcal{H}\}$ for every $H_0 \in \mathcal{H}$, and $\tilde{\mathrm{T}}$ the ideal of subsets of X generated by $\mathcal{H}$. If $A \in \tilde{\mathrm{T}}$ then there is an $H_0 \in \mathcal{H}$ including A, and now

$$\tilde{\nu}_H(A \cap H) = \tilde{\nu}_H(A \cap H_0) \le \nu_H(H \cap H_0) = \nu(H \cap H_0) \le \nu H_0$$

for every $H \in \mathcal{H}$, so $\tilde{\nu} A = \lim_{H \to \mathfrak{F}} \nu_H(A \cap H)$ is defined in $[0, \infty[$. Note that if $A \in \mathrm{T} \cap \tilde{\mathrm{T}}$ then there is an $H_0 \in \mathcal{H}$ including A, so that $\nu_H A = \nu A$ whenever $H \in \mathcal{H}$ and $H \supseteq H_0$, and $\tilde{\nu} A = \nu A$. Also $\tilde{\nu} : \tilde{\mathrm{T}} \to [0, \infty[$ is additive because all the functionals $A \mapsto \tilde{\nu}_H(A \cap H)$ are.

(b) Let $\mathcal{K}$ be the family of compact subsets of X. Because $X = \bigcup \mathcal{H}$, $\mathcal{K} \subseteq \tilde{\mathrm{T}}$. For $K \in \mathcal{K}$, set

$$\phi_0 K = \inf\{\tilde{\nu} G : G \in \tilde{\mathrm{T}} \text{ is open}, K \subseteq G\}.$$

Then ϕ_0 satisfies the conditions of 416J. **P** (α) Take K, $L \in \mathcal{K}$ such that $L \subseteq K$, and $\epsilon > 0$. Then there are open sets G_0, $H_0 \in \tilde{\mathrm{T}}$ such that

$$K \subseteq G_0, \quad \tilde{\nu} G_0 \leq \phi_0 K + \epsilon, \quad L \subseteq H_0, \quad \tilde{\nu} H_0 \leq \phi_0 L + \epsilon.$$

(i) If $K' \in \mathcal{K}$ is such that $K' \subseteq K \setminus L$, there are disjoint open sets H, $H' \subseteq X$ such that $L \subseteq H$ and $K' \subseteq H'$ (4A2F(h-i)). So

$$\phi_0 L + \phi_0 K' \leq \tilde{\nu}(G_0 \cap H) + \tilde{\nu}(G_0 \cap H') \leq \tilde{\nu} G_0 \leq \phi_0 K + \epsilon.$$

(ii) In the other direction, consider $K_1 = K \setminus H_0$. Then there is an open set $H_1 \in \tilde{\mathfrak{T}}$ such that $K_1 \subseteq H_1$ and $\tilde{\nu} H_1 \leq \phi_0 K_1 + \epsilon$, so that

$$\phi_0 K \leq \tilde{\nu}(H_0 \cup H_1) \leq \tilde{\nu} H_0 + \tilde{\nu} H_1 \leq \phi_0 L + \phi_0 K' + 2\epsilon.$$

As ϵ is arbitrary,

$$\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\}$$

as required by 416J(α). (γ) If $x \in X$ there is an $H_0 \in \mathcal{H}$ containing x, and now

$$\sup\{\phi_0 K : K \in \mathcal{K},\, K \subseteq H_0\} \leq \tilde{\nu} H_0$$

is finite. So the second hypothesis also is satisfied. **Q**

(c) By 416J, we have a Radon measure μ on X extending ϕ_0. If $K \in \mathfrak{T}$ is compact, then $\mu K \geq \nu K$. **P** Since $K \in \mathfrak{T} \cap \tilde{\mathfrak{T}}$, $\tilde{\nu} K = \nu K$. Now

$$\mu K = \phi_0 K = \inf\{\tilde{\nu} G : G \in \tilde{\mathfrak{T}} \text{ is open, } K \subseteq G\} \geq \tilde{\nu} K = \nu K. \textbf{ Q}$$

If $G \in \mathfrak{T}$ is open, then $\mu G = \sup\{\mu K : K \subseteq G \text{ is compact}\}$. But if $K \subseteq G$ is compact then

$$\mu K = \phi_0 K \leq \tilde{\nu} G = \nu G,$$

so $\mu G \leq \nu G$.

Thus μ has the required properties.

416L Proposition Let X be a regular Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X, and $\phi_0 : \mathcal{K} \to [0, \infty[$ a functional such that

(α_1) $\phi_0 K \leq \phi_0(K \cup L) \leq \phi_0 K + \phi_0 L$ for all K, $L \in \mathcal{K}$,

(α_2) $\phi_0(K \cup L) = \phi_0 K + \phi_0 L$ whenever K, $L \in \mathcal{K}$ and $K \cap L = \emptyset$,

(γ) for every $x \in X$ there is an open set G containing X such that $\sup\{\phi_0 K : K \in \mathcal{K},\, K \subseteq G\}$ is finite.

Then there is a unique Radon measure μ on X such that

$$\mu K = \inf_{G \subseteq X \text{ is open}, K \subseteq G} \sup_{L \subseteq G \text{ is compact}} \phi_0 L$$

for every $K \in \mathcal{K}$.

proof (a) For open sets $G \subseteq X$ set

$$\psi G = \sup_{L \in \mathcal{K}, L \subseteq G} \phi_0 L,$$

and for compact sets $K \subseteq X$ set

$$\phi_1 K = \inf\{\psi G : G \subseteq X \text{ is open, } K \subseteq G\}.$$

Evidently $\psi G \leq \psi H$ whenever $G \subseteq H$. We need to know that $\psi(G \cup H) \leq \psi G + \psi H$ for all open sets G, $H \subseteq X$. **P** If $L \subseteq G \cup H$ is compact, then the disjoint compact sets $L \setminus G$, $L \setminus H$ can be separated by disjoint open sets H', G' (4A2F(h-i) again); now $L \setminus G' \subseteq H$, $L \setminus H' \subseteq G$ are compact and cover L, so

$$\phi_0 L \leq \phi_0(L \setminus G') + \phi_0(L \setminus H') \leq \psi H + \psi G.$$

As L is arbitrary, $\psi(G \cup H) \leq \psi G + \psi H$. **Q**

Moreover, $\psi(G \cup H) = \psi G + \psi H$ if $G \cap H = \emptyset$. **P** If $K \subseteq G$, $L \subseteq H$ are compact, then

$$\phi_0 K + \phi_0 L = \phi_0(K \cup L) \leq \psi(G \cup H).$$

As K and L are arbitrary, $\psi G + \psi H \leq \psi(G \cup H)$. **Q**

(b) It follows that $\phi_1 K$ is finite for every compact $K \subseteq X$. **P** Set $\mathcal{G} = \{G : G \subseteq X \text{ is open, } \psi G < \infty\}$. Then (a) tells us that $\mathcal{G}$ is upwards-directed. But also we are supposing that $\mathcal{G}$ covers X, by (γ). So if $K \subseteq X$ is compact there is a member of $\mathcal{G}$ including K and $\phi_1 K < \infty$. **Q**

(c) Now ϕ_1 satisfies the conditions of 416J.

$\mathbf{P}$**($\boldsymbol{\alpha}$)** Suppose that K, $L \in \mathcal{K}$ and $L \subseteq K$. Set $\gamma = \sup\{\phi_1 M : M \in \mathcal{K},\, M \subseteq K \setminus L\}$. Take any $\epsilon > 0$.

Let G be an open set such that $K \subseteq G$ and $\psi G \le \phi_1 K + \epsilon$. If $M \in \mathcal{K}$ and $M \subseteq K \setminus L$, there are disjoint open sets U, V such that $L \subseteq U$ and $M \subseteq V$ (4A2F(h-i) once more); we may suppose that $U \cup V \subseteq G$. In this case,

$$\phi_1 L + \phi_1 M \le \psi U + \psi V = \psi(U \cup V)$$

(by the second part of (a) above)

$$\le \psi G \le \phi_1 K + \epsilon.$$

As M is arbitrary, $\gamma \le \phi_1 K - \phi_1 L + \epsilon$.

On the other hand, there is an open set H such that $L \subseteq H$ and $\psi H \le \phi_1 L + \epsilon$. Set $F = K \setminus H$, so that F is a compact subset of $K \setminus L$. Then there is an open set V such that $F \subseteq V$ and $\psi V \le \phi_1 F + \epsilon$. In this case $K \subseteq H \cup V$, so

$$\begin{aligned}\phi_1 K &\le \psi(H \cup V) \le \psi H + \psi V\\ &\le \phi_1 L + \phi_1 F + 2\epsilon \le \phi_1 L + \gamma + 2\epsilon,\end{aligned}$$

so $\gamma \ge \phi_1 K - \phi_1 L - 2\epsilon$.

As ϵ is arbitrary, $\gamma = \phi_1 K - \phi_1 L$; as K and L are arbitrary, ϕ_1 satisfies condition (α) of 416J.

($\boldsymbol{\gamma}$) Any $x \in X$ is contained in an open set G such that $\psi G < \infty$; but now $\sup\{\phi_1 K : K \in \mathcal{K},\, K \subseteq G\} \le \psi G$ is finite. So ϕ_1 satisfies condition (γ) of 416J. $\mathbf{Q}$

(d) By 416J, there is a unique Radon measure on X extending ϕ_1, as claimed.

416M Corollary Let X be a locally compact Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X, and $\phi_0 : \mathcal{K} \to [0, \infty[$ a functional such that

$\phi_0 K \le \phi_0(K \cup L) \le \phi_0 K + \phi_0 L$ for all K, $L \in \mathcal{K}$,

$\phi_0(K \cup L) = \phi_0 K + \phi_0 L$ whenever K, $L \in \mathcal{K}$ and $K \cap L = \emptyset$.

Then there is a unique Radon measure μ on X such that

$$\mu K = \inf\{\phi_0 K' : K' \in \mathcal{K},\, K \subseteq \operatorname{int} K'\}$$

for every $K \in \mathcal{K}$.

proof Observe that ϕ_0 satisfies the conditions of 416L; 416L(γ) is true because X is locally compact. Define ψ, ϕ_1 as in the proof of 416L, and set

$$\phi_1' K = \inf\{\phi_0 K' : K' \in \mathcal{K},\, K \subseteq \operatorname{int} K'\}$$

for every $K \in \mathcal{K}$. Then $\phi_1' = \phi_1$. $\mathbf{P}$ Let $K \in \mathcal{K}$, $\epsilon > 0$. (i) There is an open set $G \subseteq X$ such that $K \subseteq G$ and $\psi G \le \phi_1 K + \epsilon$. Now the relatively compact open subsets with closures included in G form an upwards-directed cover of K, so there is a $K' \in \mathcal{K}$ such that $K \subseteq \operatorname{int} K'$ and $K' \subseteq G$. Accordingly

$$\phi_1' K \le \phi_0 K' \le \psi G \le \phi K + \epsilon.$$

(ii) There is an $L \in \mathcal{K}$ such that $K \subseteq \operatorname{int} L$ and $\phi_0 L \le \phi_1' K + \epsilon$, so that

$$\phi_1 K \le \psi(\operatorname{int} L) \le \phi_0 L \le \phi_1' K + \epsilon.$$

(iii) As K and ϵ are arbitrary, $\phi_1' = \phi_1$. $\mathbf{Q}$

Now 416L tells us that there is a unique Radon measure extending ϕ_1, and this is the measure we seek.

416N The extension theorems in the second half of §413 also have important applications to Radon measures.

Henry's theorem (Henry 69) Let X be a Hausdorff space and μ_0 a measure on X which is locally finite and tight. Then μ_0 has an extension to a Radon measure μ on X; and the extension may be made in such a way that whenever $\mu E < \infty$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$.

proof All the work has been done in §413; we need to check here only that the family $\mathcal{K}$ of compact subsets of X and the measure μ_0 satisfy the hypotheses of 413O. But (†) and (‡) there are elementary, and $\mu_0^* K < \infty$ for every $K \in \mathcal{K}$ by 411Ga.

Now take the measure μ from 413O. It is complete, locally determined and inner regular with respect to $\mathcal{K}$; also $\mathcal{K} \subseteq \operatorname{dom}\mu$. Because μ_0 is locally finite and μ extends μ_0, μ is locally finite. By 416Dd, μ is a Radon measure. And the construction of 413O ensures that every set of finite measure for μ differs from a member of Σ_0 by a μ-negligible set.

416O Theorem Let X be a Hausdorff space and T a subalgebra of $\mathcal{P}X$. Let $\nu : \mathrm{T} \to [0, \infty[$ be a finitely additive functional such that

$$\nu E = \sup\{\nu F : F \in \mathrm{T},\, F \subseteq E,\, F \text{ is closed}\} \text{ for every } E \in \mathrm{T},$$

$$\nu X = \sup_{K \subseteq X \text{ is compact}} \inf_{F \in \mathrm{T}, F \supseteq K} \nu F.$$

Then there is a Radon measure μ on X extending ν.

proof (a) For $A \subseteq X$, write

$$\nu^* A = \inf_{F \in \mathrm{T}, F \supseteq A} \nu F.$$

Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be a sequence of compact subsets of X such that $\lim_{n\to\infty} \nu^* K_n = \nu X$; replacing K_n by $\bigcup_{i<n} K_i$ if necessary, we may suppose that $\langle K_n \rangle_{n \in \mathbb{N}}$ is non-decreasing and that $K_0 = \emptyset$. For each $n \in \mathbb{N}$, set

$$\nu'_n E = \nu^*(E \cap K_n)$$

for every $E \in \mathrm{T}$. Then ν'_n is additive. **P** (I copy from the proof of 413N.) If $E, F \in \mathrm{T}$ and $E \cap F = \emptyset$,

$$\begin{aligned}
\nu'_n(E \cup F) &= \inf\{\nu G : G \in \mathrm{T},\, K_n \cap (E \cup F) \subseteq G\} \\
&= \inf\{\nu G : G \in \mathrm{T},\, K_n \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\
&= \inf\{\nu(G \cap E) + \nu(G \cap F) : G \in \mathrm{T},\, K_n \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\
&= \inf\{\nu G_1 + \nu G_2 : G_1, G_2 \in \mathrm{T},\, K_n \cap E \subseteq G_1 \subseteq E,\, K_n \cap F \subseteq G_2 \subseteq F\} \\
&= \inf\{\nu G_1 : G_1 \in \mathrm{T},\, K_n \cap E \subseteq G_1 \subseteq E\} \\
&\qquad\qquad + \inf\{\nu G_2 : G_2 \in \mathrm{T},\, K_n \cap F \subseteq G_2 \subseteq F\} \\
&= \nu'_n E + \nu'_n F.
\end{aligned}$$

As E and F are arbitrary, ν'_n is additive. **Q**

(b) For each $n \in \mathbb{N}$, set $\nu_n E = \nu'_{n+1} E - \nu'_n E$ for every $E \in \mathrm{T}$; then ν_n is additive. Because $K_{n+1} \supseteq K_n$, ν_n is non-negative.

If $E \in \mathrm{T}$ and $E \cap K_{n+1} = \emptyset$, then $\nu_n E = \nu'_{n+1} E = 0$. So if we set $\mathrm{T}_n = \{E \cap K_{n+1} : E \in \mathrm{T}\}$, we have an additive functional $\tilde\nu_n : \mathrm{T}_n \to [0, \infty[$ defined by setting $\tilde\nu_n(E \cap K_{n+1}) = \nu_n E$ for every $E \in \mathrm{T}$. Also $\tilde\nu_n H = \sup\{\tilde\nu_n K : K \in \mathrm{T}_n,\, K \subseteq H,\, K \text{ is compact}\}$ for every $H \in \mathrm{T}_n$. **P** Express H as $E \cap K_{n+1}$ where $E \in \mathrm{T}$. Given $\epsilon > 0$, there is a closed set $F \in \mathrm{T}$ such that $F \subseteq E$ and $\nu F \ge \nu E - \epsilon$; but now $K = F \cap K_{n+1} \in \mathrm{T}_n$ is a compact subset of H, and

$$\tilde\nu_n(H \setminus K) = \nu_n(E \setminus F) \le \nu'_{n+1}(E \setminus F) \le \nu(E \setminus F) \le \epsilon,$$

so $\tilde\nu_n K \ge \tilde\nu_n H - \epsilon$. **Q**

(c) For each $n \in \mathbb{N}$, we have a Radon measure μ_n on K_{n+1}, with domain Σ_n say, such that $\mu_n K_{n+1} \le \tilde\nu_n K_{n+1}$ and $\mu_n K \ge \tilde\nu_n K$ for every compact set $K \subseteq K_{n+1}$ (416K). Since K_{n+1} is itself compact, we must have $\mu_n K_{n+1} = \tilde\nu_n K_{n+1}$. But this means that μ_n extends $\tilde\nu_n$. **P** If $H \in \mathrm{T}_n$ and $\epsilon > 0$ there is a compact set $K \in \mathrm{T}_n$ such that $K \subseteq H$ and $\tilde\nu_n K \ge \tilde\nu_n H - \epsilon$, so that $(\mu_n)_* H \ge \mu_n K \ge \tilde\nu_n H - \epsilon$; as ϵ is arbitrary, $(\mu_n)_* H \ge \tilde\nu_n H$. So there is an $F_1 \in \Sigma_n$ such that $F_1 \subseteq H$ and $\mu_n F_1 \ge \tilde\nu_n H$. Similarly, there is an $F_2 \in \Sigma_n$ such that $F_2 \subseteq K_{n+1} \setminus H$ and $\mu_n F_2 \ge \tilde\nu_n(K_{n+1} \setminus H)$. But in this case $H \setminus F_1 \subseteq K_{n+1} \setminus (F_1 \cup F_2)$ is μ_n-negligible, because

$$\mu_n F_1 + \mu_n F_2 \ge \tilde\nu_n H + \tilde\nu_n(K_{n+1} \setminus H) = \tilde\nu_n K_{n+1} = \mu_n K_{n+1}.$$

So $H \setminus F_1$ and H belong to Σ_n and $\mu_n H = \mu_n F_1 = \tilde\nu_n H$. **Q**

(d) Set

$$\Sigma = \{E : E \subseteq X,\, E \cap K_{n+1} \in \Sigma_n \text{ for every } n \in \mathbb{N}\},$$

$$\mu E = \sum_{n=0}^{\infty} \mu_n(E \cap K_{n+1}) \text{ for every } E \in \Sigma.$$

Then μ is a Radon measure on X extending ν.

P (i) It is easy to check that Σ is a σ-algebra of subsets of X including T, just because each Σ_n is a σ-algebra of subsets of K_{n+1} including T_n; and that μ is a complete measure because every μ_n is.

(ii) If $E \in \mathrm{T}$, then

$$\begin{aligned}\mu E &= \sum_{n=0}^{\infty} \mu_n(E \cap K_{n+1}) = \sum_{n=0}^{\infty} \tilde{\nu}_n(E \cap K_{n+1}) = \sum_{n=0}^{\infty} \nu_n E = \lim_{n\to\infty} \sum_{i=0}^{n} \nu_i E \\ &= \lim_{n\to\infty} \sum_{i=0}^{n} \nu^*(E \cap K_{i+1}) - \nu^*(E \cap K_i) = \lim_{n\to\infty} \nu^*(E \cap K_n) \le \nu E.\end{aligned}$$

On the other hand,

$$\mu X = \lim_{n\to\infty} \nu^* K_{n+1} = \nu X,$$

so in fact $\mu E = \nu E$ for every $E \in \mathrm{T}$, that is, μ extends ν. In particular, μ is totally finite, therefore locally determined and locally finite.

(iii) If $G \subseteq X$ is open, then $G \cap K_{n+1} \in \Sigma_n$ for every n, so $G \in \Sigma$; thus μ is a topological measure. If $\mu E > 0$, there is some $n \in \mathbb{N}$ such that $\mu_n(E \cap K_{n+1}) > 0$; now there is a compact set $K \subseteq E \cap K_{n+1}$ such that $\mu_n K > 0$, so that $\mu K > 0$. This shows that μ is tight, so is a Radon measure, as required. **Q**

Remark Observe that in this construction

$$\begin{aligned}\mu K_{n+1} &= \sum_{i=0}^{\infty} \mu_i(K_{n+1} \cap K_{i+1}) = \sum_{i=0}^{\infty} \tilde{\nu}_i(K_{n+1} \cap K_{i+1}) = \sum_{i=0}^{\infty} \nu_i(K_{n+1} \cap K_{i+1}) \\ &= \sum_{i=0}^{\infty} \nu'_{i+1}(K_{n+1} \cap K_{i+1}) - \nu'_i(K_{n+1} \cap K_{i+1}) \\ &= \sum_{i=0}^{\infty} \nu^*(K_{n+1} \cap K_{i+1}) - \nu^*(K_{n+1} \cap K_i) \\ &= \sum_{i=0}^{n} \nu^*(K_{n+1} \cap K_{i+1}) - \nu^*(K_{n+1} \cap K_i) = \nu^* K_{n+1}\end{aligned}$$

for every $n \in \mathbb{N}$. What this means is that if instead of the hypothesis

$$\nu X = \sup_{K \subseteq X \text{ is compact}} \inf_{F \in \mathrm{T}, F \supseteq K} \nu F$$

we are presented with a specified non-decreasing sequence $\langle L_n \rangle_{n\in\mathbb{N}}$ of compact subsets of X such that $\nu X = \sup_{n\in\mathbb{N}} \nu^* L_n$, then we can take $K_{n+1} = L_n$ in the argument above and we shall have $\mu L_n = \nu^* L_n$ for every n.

416P Theorem Let X be a Hausdorff space and μ a locally finite measure on X which is inner regular with respect to the closed sets. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) for every non-negligible measurable set $E \subseteq X$ there is a compact set $K \subseteq E$ such that $\mu^* K > 0$.

If μ is totally finite, we can add

(iii) $\sup\{\mu^* K : K \subseteq X \text{ is compact}\} = \mu X$.

proof Write Σ for the domain of μ.

(a)(i)⇒(ii) If λ is a Radon measure extending μ, and $\mu E > 0$, then $\lambda E > 0$, so there is a compact set $K \subseteq E$ such that $\lambda K > 0$; but now, because λ is an extension of μ,

$$\mu^* K \ge \lambda^* K = \lambda K > 0.$$

(b)(ii)⇒(i) & (iii)($\boldsymbol{\alpha}$) Let $\mathcal{E}$ be the family of measurable envelopes of compact sets. Then $\mu E < \infty$ for every $E \in \mathcal{E}$. **P** If $E \in \mathcal{E}$, there is a compact set K such that E is a measurable envelope of K. Now $\mu E = \mu^* K$ is finite by 411Ga, as usual. **Q**

Next, $\mathcal{E}$ is closed under finite unions, by 132Ed. The hypothesis (ii) tells us that if $\mu E > 0$ then there is some $F \in \mathcal{E}$ such that $F \subseteq E$ and $\mu F > 0$; for there is a compact set $K \subseteq E$ such that $\mu^* K > 0$, K has a measurable envelope F_0, and $F = E \cap F_0$ is still a measurable envelope of K. So in fact μ is inner regular with respect to $\mathcal{E}$ (412Aa). In particular, μ is semi-finite.

If $\gamma < \mu X$ there is an $F \in \mathcal{E}$ such that $\mu F \geq \gamma$, and now there is a compact set K such that F is a measurable envelope of K, so that $\mu^* K = \mu F \geq \gamma$. As γ is arbitrary, (iii) is true.

(β) Because μ is inner regular with respect to $\mathcal{E}$, $D = \{E^\bullet : E \in \mathcal{E}\}$ is order-dense in the measure algebra $(\mathfrak{A}, \bar{\mu})$ of μ (412N), so there is a family $\langle d_i \rangle_{i \in I}$ in D which is a partition of unity in $\mathfrak{A}$ (313K). For each $i \in I$, take $E_i \in \mathcal{E}$ such that $E_i^\bullet = d_i$. Then

$$\sum_{i \in I} \mu(E \cap E_i) = \sum_{i \in I} \bar{\mu}(E^\bullet \cap d_i) = \bar{\mu} E^\bullet$$

(321E)

$$= \mu E$$

for every $E \in \Sigma$.

(γ) For each $i \in I$, let μ_i be the subspace measure on E_i. Then there is a Radon measure λ_i on E_i extending μ_i. **P** Because μ is inner regular with respect to the closed sets, μ_i is inner regular with respect to the relatively closed subsets of E_i (412Oa). Also there is a compact subset $K \subseteq E_i$ such that

$$\mu_i E_i = \mu E_i = \mu^* K = \mu_i^* K,$$

so μ_i satisfies the conditions of 416O and has an extension to a Radon measure. **Q**

(δ) Define

$$\lambda E = \sum_{i \in I} \lambda_i (E \cap E_i)$$

whenever $E \subseteq X$ is such that λ_i measures $E \cap E_i$ for every $i \in I$. Then λ is a Radon measure on X extending μ. **P** It is easy to check that it is a measure, just because every λ_i is a measure, and it extends μ by (β) above. If $G \subseteq X$ is open, then $G \cap E_i$ is relatively open for every $i \in I$, so λ measures G; thus λ is a topological measure. If $\lambda E = 0$ and $A \subseteq E$, then $\lambda_i(A \cap E_i) \leq \lambda(E \cap E_i) = 0$ for every i, so $\lambda A = 0$; thus λ is complete. For all distinct i, $j \in I$,

$$\lambda_i(E_i \cap E_j) = \mu_i(E_i \cap E_j) = \mu(E_i \cap E_j) = \bar{\mu}(d_i \cap d_j) = 0,$$

so $\lambda E_i = \lambda_i E_i = \mu_i E_i$ is finite. This means that if $E \subseteq X$ is such that λ measures $E \cap F$ whenever $\lambda F < \infty$, then λ must measure $E \cap E_i$ for every i, and λ measures E; thus λ is locally determined. If $\lambda E > 0$ there are an $i \in I$ such that $\lambda_i(E \cap E_i) > 0$ and a compact $K \subseteq E \cap E_i$ such that $0 < \lambda_i K = \lambda K$; consequently λ is tight. Finally, if $x \in X$, there is an $E \in \Sigma$ such that $x \in \operatorname{int} E$ and $\lambda E = \mu E < \infty$, so λ is locally finite. Thus λ is a Radon measure. **Q**

So (i) is true.

(c) Finally, suppose that μ is totally finite and (iii) is true. Then we can appeal directly to 416O to see that (i) is true.

416Q Proposition (a) Let X be a compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. Then any non-negative finitely additive functional from $\mathcal{E}$ to $\mathbb{R}$ has an extension to a Radon measure on X. If X is zero-dimensional then the extension is unique.

(b) Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space. Then there is a one-to-one correspondence between non-negative additive functionals ν on $\mathfrak{A}$ and Radon measures μ on Z given by the formula

$$\nu a = \mu \widehat{a} \text{ for every } a \in \mathfrak{A},$$

where for $a \in \mathfrak{A}$ I write $\widehat{a}$ for the corresponding open-and-closed subset of Z.

proof (a) Let $\nu : \mathcal{E} \to [0, \infty[$ be a non-negative additive functional. Then ν satisfies the conditions of 416O (because every member of $\mathcal{E}$ is closed, while X is compact), so has an extension to a Radon measure μ. If X is zero-dimensional, $\mathcal{E}$ is a base for the topology of X closed under finite unions and intersections, so μ is unique, by 415H(iv) or 415H(v).

(b) The map $a \mapsto \widehat{a}$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra $\mathcal{E}$ of open-and-closed subsets of Z, so we have a one-to-one correspondence between non-negative additive functionals ν on $\mathfrak{A}$ and non-negative additive functionals ν' on $\mathcal{E}$ defined by the formula $\nu'\widehat{a} = \nu a$. Now Z is compact, Hausdorff and zero-dimensional, so ν' has a unique extension to a Radon measure on Z, by part (a). And of course every Radon measure μ on Z gives us a non-negative additive functional $\mu{\restriction}\mathcal{E}$ on $\mathcal{E}$, corresponding to a non-negative additive functional on $\mathfrak{A}$.

416R Theorem (a) Any subspace of a Radon measure space is a quasi-Radon measure space.

(b) A measurable subspace of a Radon measure space is a Radon measure space.

(c) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Hausdorff complete locally determined topological measure space, and $Y \subseteq X$ is such that the subspace measure μ_Y on Y is a Radon measure, then $Y \in \Sigma$.

proof (a) Put 416A and 415B together.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $(E, \mathfrak{T}_E, \Sigma_E, \mu_E)$ a member of Σ with the induced topology and measure. Because μ is complete and locally determined, so is μ_E (214Ka). Because $\mathfrak{T}$ is Hausdorff, so is $\mathfrak{T}_E$ (4A2F(a-i)). Because μ is locally finite, so is μ_E. Because μ is tight (and a subset of E is compact for $\mathfrak{T}_E$ whenever it is compact for $\mathfrak{T}$), μ_E is tight (412Oa).

(c) ? If $Y \notin \Sigma$, then there is a set $F \in \Sigma$ such that $\mu_*(Y \cap F) < \mu^*(Y \cap F)$ (413F(v)). But now $\mu^*(Y \cap F) = \mu_Y(Y \cap F)$, so there is a compact set $K \subseteq Y \cap F$ such that $\mu_Y K > \mu_*(Y \cap F)$. When regarded as a subset of X, K is still compact; because $\mathfrak{T}$ is Hausdorff, K is closed, so belongs to Σ, and

$$\mu_*(Y \cap F) \ge \mu K = \mu_Y K > \mu_*(Y \cap F),$$

which is absurd. **X**

416S Corresponding to 415O, we have the following.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space.

(a) If ν is a locally finite indefinite-integral measure over μ, it is a Radon measure.

(b) If ν is a Radon measure on X and $\nu K = 0$ whenever $K \subseteq X$ is compact and $\mu K = 0$, then ν is an indefinite-integral measure over μ.

proof (a) Because μ is complete and locally determined, so is ν (234Nb[23]). Because μ is tight, so is ν (412Q). So if ν is also locally finite, it is a Radon measure.

(b) Write T for the domain of ν.

(i) If $E \in \Sigma \cap \mathrm{T}$ and $\mu E = 0$, then $\nu K = \mu K = 0$ for every compact $K \subseteq E$, so $\nu E = 0$.

(ii) $\mathrm{T} \supseteq \Sigma$. **P** If $E \in \Sigma$ and $K \subseteq X$ is compact, there are Borel sets F, F' such that $F \subseteq E \cap K \subseteq F' \subseteq K$ and $\mu(F' \setminus F) = 0$. Consequently $\nu(F' \setminus F) = 0$ and $E \cap K \in \mathrm{T}$, because ν is complete. By 416Db, $E \in \mathrm{T}$; as E is arbitrary, $\Sigma \subseteq \mathrm{T}$. **Q**

(iii) If $E \in \mathrm{T}$, there is an $F \in \Sigma$ such that $E \subseteq F$ and $\nu(F \setminus E) = 0$. **P** By 416Dc and 412Ia, there is a decomposition $\langle X_i \rangle_{i \in I}$ of X for the measure μ such that at most one of the X_i is not a compact μ-self-supporting set, and that exceptional one, if any, is μ-negligible. For each i, let F_i be such that

— if X_i is a compact μ-self-supporting set, then F_i is a Borel subset of X_i, $F_i \supseteq E \cap X_i$ and $\nu(F_i \setminus E) = 0$,

— if X_i is not a compact μ-self-supporting set, $F_i = X_i$.

Then $F_i \in \Sigma$ for every i so $F = \bigcup_{i \in I} F_i$ belongs to Σ. We also have $\nu(F_i \setminus E) = 0$ for every i, because if X_i is not compact and μ-self-supporting then $\nu X_i = \mu X_i = 0$. Of course $E \subseteq F$. If $K \subseteq X$ is compact, there is an open set $G \supseteq X$ such that $\mu G < \infty$; consequently $\{i : \mu(X_i \cap G) \ge \epsilon\}$ is finite for every $\epsilon > 0$, so $\{i : X_i \cap K \ne \emptyset\}$ is countable and $\nu(K \cap F \setminus E) = 0$. By 412Jb, $\nu(F \setminus E) = 0$. **Q**

Applying the same argument to $X \setminus E$, we can get an $F' \in \Sigma$ such that $F' \subseteq E$ and $\nu(E \setminus F') = 0$. As E is arbitrary, ν is the completion of its restriction to Σ.

(iv) Now look at the conditions of 234O. We know that μ is localizable and ν is semi-finite. We saw in (ii) above that $\mathrm{T} \supseteq \Sigma$ and in (i) that ν is zero on μ-negligible sets. In (iii) we saw that ν is the completion of $\nu{\restriction}\Sigma$. And if $\nu E > 0$ there is a compact $K \subseteq E$ such that $\nu K > 0$, while $\mu K < \infty$. So 234O tells us that ν is an indefinite-integral measure over μ.

[23] Formerly 234F.

416T I said in the notes to §415 that the most important quasi-Radon measure spaces are subspaces of Radon measure spaces. I do not know of a useful necessary and sufficient condition, but the following deals with completely regular spaces.

Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a locally finite completely regular Hausdorff quasi-Radon measure space. Then it is isomorphic, as topological measure space, to a subspace of a locally compact Radon measure space.

proof (a) Write βX for the Stone-Čech compactification of X (4A2I); I will take it that X is actually a subspace of βX. Let $\mathcal{U}$ be the set of those open subsets U of βX such that $\mu(U\cap X)<\infty$; then $\mathcal{U}$ is upwards-directed and covers X, because μ is locally finite. Set $W=\bigcup\mathcal{U}\supseteq X$. Then W is an open subset of βX, so is locally compact.

(b) Let $\mathcal{B}(W)$ be the Borel σ-algebra of W. Then $V\cap X$ is a Borel subset of X for every $V\in\mathcal{B}(W)$ (4A3Ca), so we have a measure $\nu:\mathcal{B}(W)\to[0,\infty]$ defined by setting $\nu V=\mu(X\cap V)$ for every $V\in\mathcal{B}(W)$. Now ν satisfies the conditions of 415Cb. **P** (α) If $\nu V>0$, then, because μ is effectively locally finite, there is an open set $G\subseteq X$ such that $\mu(G\cap V)>0$ and $\mu G<\infty$. There is an open set $U\subseteq\beta X$ such that $U\cap X=G$, in which case $U\subseteq W$, $\nu U<\infty$ and $\nu(U\cap V)>0$. Thus ν is effectively locally finite. (β) If $\mathcal{U}$ is an upwards-directed family of open subsets of W, then $\{U\cap X:U\in\mathcal{U}\}$ is an upwards-directed family of open subsets of X, so

$$\begin{aligned}\nu(\bigcup\mathcal{U})&=\mu(X\cap\bigcup\mathcal{U})=\mu(\bigcup\{U\cap X:U\in\mathcal{U}\})\\&=\sup_{U\in\mathcal{U}}\mu(X\cap U)=\sup_{U\in\mathcal{U}}\nu U.\ \mathbf{Q}\end{aligned}$$

So the c.l.d. version $\tilde\nu$ of ν is a quasi-Radon measure on W (415Cb).

(c) The construction of W ensures that ν and $\tilde\nu$ are locally finite. By 416G, $\tilde\nu$ is a Radon measure. So the subspace measure $\tilde\nu_X$ is a quasi-Radon measure on X (416Ra). But $\tilde\nu_X G=\mu G$ for every open set $G\subseteq X$. **P** Note first that as ν effectively locally finite, therefore semi-finite, $\tilde\nu$ extends ν (213Hc). If $K\subseteq W$ is a compact set not meeting X, then

$$\tilde\nu K=\nu K=\mu(K\cap X)=0;$$

accordingly $\tilde\nu_*(W\setminus X)=0$, by 413Ee. Now there is an open set $U\subseteq W$ such that $G=X\cap U$, and

$$\begin{aligned}\tilde\nu_X G=\tilde\nu^*G\le\tilde\nu U&=\nu U=\mu(U\cap X)=\mu G\\&=\tilde\nu^*(U\cap X)+\tilde\nu_*(U\setminus X)\end{aligned}$$

(by 413E(c-ii), because $\tilde\nu$ is semi-finite)

$$\le\tilde\nu^*(U\cap X)+\tilde\nu_*(W\setminus X)=\tilde\nu^*G.\ \mathbf{Q}$$

So 415H(iii) tells us that $\mu=\tilde\nu_X$ is the subspace measure induced by ν.

416U Theorem (a) If $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ is a family of compact metrizable Radon probability spaces such that every μ_i is strictly positive, the product measure on $X=\prod_{i\in I}X_i$ is a completion regular Radon measure.

(b) In particular, the usual measures on $\{0,1\}^I$ and $[0,1]^I$ and $\mathcal{P}I$ are completion regular Radon measures, for any set I.

proof (a) By 415E, it is a completion regular quasi-Radon probability measure; but X is a compact Hausdorff space, so it is a Radon measure, by 416G or otherwise.

(b) follows at once. (I suppose it is obvious that by the 'usual measure on $[0,1]^I$' I mean the product measure when each copy of $[0,1]$ is given Lebesgue measure. Recall also that the 'usual measure on $\mathcal{P}I$' is just the copy of the usual measure on $\{0,1\}^I$ induced by the standard bijection $A\leftrightarrow\chi A$ (254Jb), which is a homeomorphism (4A2Ud).

416V Stone spaces The results of 415Q-415R become simpler and more striking in the present context.

Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, and $(Z,\mathfrak{S},\mathrm{T},\nu)$ the Stone space of its measure algebra $(\mathfrak{A},\bar\mu)$. For $E\in\Sigma$ let E^* be the open-and-closed set in Z corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Define $R\subseteq Z\times X$ by saying that $(z,x)\in R$ iff $x\in F$ whenever $F\subseteq X$ is closed and $z\in F^*$.

(a) R is the graph of a function $f:Q\to X$, where $Q=R^{-1}[X]$. If we set $W=\bigcup\{K^*:K\subseteq X$ is compact$\}$, then $W\subseteq Q$ is a ν-conegligible open set, and the subspace measure ν_W on W is a Radon measure.

(b) Setting $g = f\restriction W$, g is continuous and μ is the image measure $\nu_W g^{-1}$.

(c) If X is compact, $W = Q = Z$ and $\mu = \nu g^{-1}$.

proof (a) By 415Ra, R is the graph of a function. If $K \subseteq X$ is compact and $z \in K^*$, then $\mathcal{F} = \{F : F \subseteq X$ is closed, $z \in F^*\}$ is a family of non-empty closed subsets of X, closed under finite intersections, and containing the compact set K; so it has non-empty intersection, and there is an $x \in K$ such that $(z, x) \in R$, that is, $z \in Q$ and $f(z) \in K$. Thus $W \subseteq Q$. Of course W is an open set, being the union of a family of open-and-closed sets; but it is also conegligible, because $\sup\{K^\bullet : K \subseteq X$ is compact$\} = 1$ in $\mathfrak{A}$ (412N), so $Z \setminus W$ must be nowhere dense, therefore negligible. Now the subspace measure ν_W is quasi-Radon because ν is (411P(d-iv), 415B); but W is a union of compact open sets of finite measure, so ν_W is locally finite and W is locally compact; by 416G, ν_W is a Radon measure.

(b) g is continuous. **P** Let $G \subseteq X$ be an open set and $z \in g^{-1}[G]$. Let $K \subseteq X$ be a compact set such that $z \in K^*$. As remarked above, $g(z) = f(z)$ belongs to K. K, being a compact Hausdorff space, is regular (3A3Bb), so there is an open set H containing $g(z)$ such that $L = \overline{H \cap K} \subseteq G$. Note that L is compact, so $L^* \subseteq W$. Now $g(z)$ does not belong to the closed set $X \setminus H$, so $z \notin (X \setminus H)^*$ and $z \in H^*$; accordingly $z \in (H \cap K)^* \subseteq L^*$. If $w \in L^*$, $g(w) \in L \subseteq G$; so $L^* \subseteq g^{-1}[G]$, and $z \in \operatorname{int} g^{-1}[G]$. As z is arbitrary, $g^{-1}[G]$ is open; as G is arbitrary, g is continuous. **Q**

By 415Rb, we know that $\mu = \nu_Q f^{-1}$, where ν_Q is the subspace measure on Q. But as ν is complete and both Q and W are conegligible, we have

$$\nu_Q f^{-1}[A] = \nu f^{-1}[A] = \nu g^{-1}[A] = \nu_W g^{-1}[A]$$

whenever $A \subseteq X$ and any of the four terms is defined, so that $\mu = \nu_Q f^{-1} = \nu_W g^{-1}$.

(c) If X is compact, then $Z = X^* \subseteq W$, so $W = Q = Z$ and $\nu g^{-1} = \nu_W g^{-1} = \mu$.

416W Compact measure spaces Recall that a semi-finite measure space (X, Σ, μ) is 'compact' (as a measure space) if there is a family $\mathcal{K} \subseteq \Sigma$ such that μ is inner regular with respect to $\mathcal{K}$ and $\bigcap \mathcal{K}' \neq \emptyset$ whenever $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property (342A); while (X, Σ, μ) is 'perfect' if whenever $f : X \to \mathbb{R}$ is measurable and $\mu E > 0$, there is a compact set $K \subseteq f[E]$ such that $\mu f^{-1}[K] > 0$ (342K). In §342 I introduced these concepts in order to study the realization of homomorphisms between measure algebras. The following result is now very easy.

Proposition (a) Any Radon measure space is a compact measure space, therefore perfect.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$, and (Y, T, ν) a complete strictly localizable measure space, with measure algebra $(\mathfrak{B}, \bar{\nu})$. If $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, there is a function $f : Y \to X$ such that $f^{-1}[E] \in \mathrm{T}$ and $f^{-1}[E]^\bullet = \pi E^\bullet$ for every $E \in \Sigma$. If π is measure-preserving, f is inverse-measure-preserving.

proof (a) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Radon measure space, μ is inner regular with respect to the compact class consisting of the compact subsets of X, so (X, Σ, μ) is a compact measure space. By 342L, it is perfect.

(b) Use (i)$\Rightarrow$(v) of Theorem 343B. (Of course f is inverse-measure-preserving iff π is measure-preserving.)

416X Basic exercises >(a) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $E \in \Sigma$ an atom for the measure. Show that there is a point $x \in E$ such that $\mu\{x\} = \mu E$.

(b) Let X be a topological space and μ a point-supported measure on X, as described in 112Bd. (i) Show that μ is tight, so is a Radon measure iff it is locally finite. In particular, show that if X has its discrete topology then counting measure on X is a Radon measure. (ii) Show that every purely atomic Radon measure is of this type.

>(c) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of Radon measure spaces, with direct sum (X, Σ, μ) (214L). Give X its disjoint union topology. Show that μ is a Radon measure.

(d) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Show that $\alpha\mu$, defined on Σ, is a Radon measure for any $\alpha > 0$.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite Radon measure space with $\mu X > 0$. Show that there is a Radon probability measure on X with the same measurable sets and the same negligible sets as μ.

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. (i) Show that μ has a decomposition $\langle X_i \rangle_{i \in I}$ in which every X_i except at most one is a self-supporting compact set, and the exceptional one, if any, is negligible. (ii) Show that μ has a decomposition $\langle X_i \rangle_{i \in I}$ in which every X_i is expressible as the intersection of a closed set with an open set. (*Hint*: enumerate the open sets of finite measure as $\langle G_\xi \rangle_{\xi < \kappa}$, and set $X_\xi = G_\xi \setminus \bigcup_{\eta < \xi} G_\eta$.)

(g) Let X be a Hausdorff space and μ, ν two Radon measures on X such that $\nu G = \mu G$ whenever $G \subseteq X$ is open and $\min(\mu G, \nu G) < \infty$. Show that $\mu = \nu$.

(h) Explain how to prove 416H from 416F, without appealing to §415.

(i) Give a direct proof of 416I not relying on 415O.

(j) Let $(X, \mathfrak{T})$ be a completely regular Hausdorff space and μ a locally finite topological measure on X which is inner regular with respect to the closed sets. Show that

$$\begin{aligned}\mu K &= \inf\{\mu G : G \supseteq K \text{ is a cozero set}\}\\ &= \inf\{\mu F : F \supseteq K \text{ is a zero set}\} = \inf\{\int f d\mu : \chi K \le f \in C(X)\}\end{aligned}$$

for every compact set $K \subseteq X$.

(k) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally compact Radon measure space, and C_k the space of continuous real-valued functions on X with compact supports. Show that $\{f^\bullet : f \in C_k\}$ is dense in $L^0(\mu)$ for the topology of convergence in measure.

(l) Let X be a completely regular Hausdorff space and ν a locally finite Baire measure on X. (i) Show that $\nu^* K = \inf\{\nu G : G \subseteq X$ is a cozero set, $K \subseteq G\}$ for every compact set $K \subseteq X$. (ii) Show that there is a Radon measure μ on X such that $\mu K = \nu^* K$ for every compact set $K \subseteq X$. (*Hint*: in the language of the proof of 416L, $\phi_1 = \nu^* \restriction \mathcal{K}$.)

(m) Let X be a Hausdorff space and ν a non-negative finitely additive functional defined on some algebra of subsets of X. Show that there is a Radon measure μ on X such that $\mu X \le \nu X$ and $\mu K \ge \nu E$ whenever $E \in \mathrm{T}$, $K \subseteq X$ is compact and $E \subseteq K$. (*Hint*: start by extending ν to $\mathcal{P}X$.)

(n) Let $(X, \mathfrak{T})$ be a Hausdorff space, $\Sigma \supseteq \mathfrak{T}$ a σ-algebra of subsets of X, and $\nu : \Sigma \to [0, \infty[$ a finitely additive functional such that $\nu E = \sup\{\nu K : K \subseteq E$ is compact$\}$ for every $E \in \Sigma$. Show that ν is countably additive and that its completion is a Radon measure on X.

(o) Explain how to prove 416Rb from 416C and 415B.

(p) Let X be a Hausdorff space, μ a complete locally finite measure on X, and Y a conegligible subset of X. Show that μ is a Radon measure iff the subspace measure on Y is a Radon measure.

(q) Let X be a Hausdorff space, Y a subset of X, and ν a Radon measure on Y. Define a measure μ on X by setting $\mu E = \nu(E \cap Y)$ whenever ν measures $E \cap Y$. Show that if *either* Y is closed *or* ν is totally finite, μ is a Radon measure on X. (Cf. 418I.)

(r) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space and $\mathcal{E} \subseteq \Sigma$ a non-empty upwards-directed family. Set $\nu F = \sup_{E \in \mathcal{E}} \mu(E \cap F)$ whenever $F \subseteq X$ is such that μ measures $E \cap F$ for every $E \in \mathcal{E}$. Show that ν is a Radon measure on X.

(s) Let $\langle X_n \rangle_{n \in \mathbb{N}}$ be a sequence of Hausdorff spaces with product X; write $\mathcal{B}(X_n)$ for the Borel σ-algebra of X_n. Let T be the σ-algebra $\widehat{\bigotimes}_{n \in \mathbb{N}} \mathcal{B}(X_n)$ (definition: 254E). Let $\nu : \mathrm{T} \to [0, \infty[$ be a finitely additive functional such that $E \mapsto \nu \pi_n^{-1}[E] : \mathcal{B}(X_n) \to [0, \infty[$ is countably additive and tight for each $n \in \mathbb{N}$, writing $\pi_n(x) = x(n)$ for $x \in X$, $n \in \mathbb{N}$. Show that there is a unique Radon measure on X extending ν.

(t) Set $S_2 = \bigcup_{n \in \mathbb{N}} \{0,1\}^n$, and let $\phi : S_2^* \to [0, \infty[$ be a functional such that $\phi(\sigma) = \phi(\sigma^\frown{<}0{>}) + \phi(\sigma^\frown{<}1{>})$ for every $\sigma \in S_2$, writing $\sigma^\frown{<}0{>}$ and $\sigma^\frown{<}1{>}$ for the two members of $\{0,1\}^{n+1}$ extending any $\sigma \in \{0,1\}^n$. Show that there is a unique Radon measure μ on $\{0,1\}^{\mathbb{N}}$ such that $\mu\{x : x \restriction \{0, \dots, n-1\} = \sigma\} = \phi(\sigma)$ whenever $n \in \mathbb{N}$, $\sigma \in \{0,1\}^{\mathbb{N}}$. (*Hint*: use 416Xs or 416Q.)

(u) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Show that a measure ν on X is an indefinite-integral measure over μ iff (α) ν is a complete, locally determined topological measure (β) ν is tight (γ) $\nu K = 0$ whenever $K \subseteq X$ is compact and $\mu K = 0$.

(v) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a compact Hausdorff quasi-Radon measure space. Let $W\subseteq X$ be the union of the open subsets of X of finite measure. Show that the subspace measure on W is a Radon measure.

(w) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a completely regular Radon measure space. Show that it is isomorphic, as topological measure space, to a measurable subspace of a locally compact Radon measure space.

(x) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a compact Radon measure space and $(Z,\mathfrak{S},\mathrm{T},\nu)$ the Stone space of its measure algebra. For $E\in\Sigma$ let E^* be the corresponding open-and-closed subset of Z, as in 416V. Show that the function described in 416V is the unique continuous function $h:Z\to X$ such that $\nu(E^*\triangle h^{-1}[E])=0$ for every $E\in\Sigma$.

(y) Show that the right-facing Sorgenfrey line (415Xc), with Lebesgue measure, is a quasi-Radon measure space which, regarded as a measure space, is compact, but, regarded as a topological measure space, is not a Radon measure space.

(z) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $(Z,\mathfrak{T},\Sigma,\mu)$ its Stone space. Show that if ν is a strictly positive Radon measure on Z then μ is an indefinite-integral measure over ν.

416Y Further exercises (a) Let X be a Hausdorff space and ν a countably additive real-valued functional defined on a σ-algebra Σ of subsets of X. Show that the following are equiveridical: (i) $|\nu|:\Sigma\to[0,\infty[$, defined as in 362B, is a Radon measure on X; (ii) ν is expressible as $\mu_1-\mu_2$, where μ_1, μ_2 are Radon measures on X and $\Sigma=\operatorname{dom}\mu_1\cap\operatorname{dom}\mu_2$.

(b) Let X be a topological space and μ_0 a semi-finite measure on X which is inner regular with respect to the family $\mathcal{K}_{\mathrm{ccc}}$ of closed countably compact sets. Show that μ_0 has an extension to a complete locally determined topological measure μ on X, still inner regular with respect to $\mathcal{K}_{\mathrm{ccc}}$; and that the extension may be done in such a way that whenever $\mu E<\infty$ there is an $E_0\in\Sigma_0$ such that $\mu(E\triangle E_0)=0$. (*Hint*: use the argument of 416N, but with $\mathcal{K}=\{K:K\in\mathcal{K}_{\mathrm{ccc}},\,\mu_0^*K<\infty\}$.)

(c) Let X be a topological space and μ_0 a semi-finite measure on X which is inner regular with respect to the family $\mathcal{K}_{\mathrm{sc}}$ of sequentially compact sets. Show that μ_0 has an extension to a complete locally determined topological measure μ on X, still inner regular with respect to $\mathcal{K}_{\mathrm{sc}}$; and that the extension may be done in such a way that whenever $\mu E<\infty$ there is an $E_0\in\Sigma_0$ such that $\mu(E\triangle E_0)=0$.

(d) Let $X\subseteq\beta\mathbb{N}$ be the union of all those open sets $G\subseteq\beta\mathbb{N}$ such that $\sum_{n\in G\cap\mathbb{N}}\frac{1}{n+1}$ is finite. For $E\subseteq X$ set $\mu E=\sum_{n\in E\cap\mathbb{N}}\frac{1}{n+1}$. Show that μ is a σ-finite Radon measure on the locally compact Hausdorff space X. Show that μ is not outer regular with respect to the open sets.

(e) Set $S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$, and let $\phi:S\to[0,\infty[$ be a functional such that $\phi(\sigma)=\sum_{i=0}^{\infty}\phi(\sigma^\frown{<}i{>})$ for every $\sigma\in S$, writing $\sigma^\frown{<}i{>}$ for the members of $\mathbb{N}^{n+1}$ extending any $\sigma\in\mathbb{N}^n$. Show that there is a unique Radon measure μ on $\mathbb{N}^{\mathbb{N}}$ such that $\mu I_\sigma=\phi(\sigma)$ for every $\sigma\in S$, where $I_\sigma=\{x:x\restriction\{0,\dots,n-1\}=\sigma\}$ for any $n\in\mathbb{N}$, $\sigma\in\mathbb{N}^{\mathbb{N}}$. (*Hint*: set $\theta A=\inf\{\sum_{\sigma\in R}\phi(\sigma):R\subseteq S,\,A\subseteq\bigcup_{\sigma\in R}I_\sigma\}$ for every $A\subseteq\mathbb{N}^{\mathbb{N}}$, and use Carathéodory's method.)

(f) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space. Show that a measure ν on X is an indefinite-integral measure over μ iff (i) there is a topology $\mathfrak{S}$ on X, including $\mathfrak{T}$, such that ν is a Radon measure with respect to $\mathfrak{S}$ (ii) $\nu K=0$ whenever K is a $\mathfrak{T}$-compact set and $\mu K=0$.

(g) Let $\langle x_n\rangle_{n\in\mathbb{N}}$ enumerate a dense subset of $X=\{0,1\}^{\mathfrak{c}}$ (4A2B(e-ii)). Let $\nu_{\mathfrak{c}}$ be the usual measure on X, and set $\mu E=\frac{1}{2}\nu_{\mathfrak{c}}E+\sum_{n=0}^{\infty}2^{-n-2}\chi E(x_n)$ for $E\in\operatorname{dom}\nu_{\mathfrak{c}}$. (i) Show that μ is a strictly positive Radon probability measure on X with Maharam type $\mathfrak{c}$. (ii) Let $I\in[\mathfrak{c}]^{\le\omega}$ be such that $x_m\restriction I\ne x_n\restriction I$ whenever $m\ne n$. Set $Z=\{0,1\}^I$ and let $\pi:X\to Z$ be the canonical map. Show that if $f\in C(X)$ is such that $\int f\times g\pi\,d\mu=0$ for every $g\in C(Z)$, then $f=0$. (*Hint*: otherwise, take $n\in\mathbb{N}$ such that $|f(x_n)|\ge\frac{1}{2}\|f\|_\infty$, and let $g\ge0$ be such that $g(\pi x_n)=1$ and $\int g\,d(\mu\pi^{-1})<3\cdot2^{-n-3}$; show that $\int f\times g\pi\,d\mu>0$.) (iii) Show that there is no orthonormal basis for $L^2(\mu)$ in $\{f^\bullet:f\in C(X)\}$. (See Hart & Kunen 99.)

(h) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space and $\mathcal{A}\subseteq\Sigma$ a countable set. Let $\mathfrak{S}$ be the topology generated by $\mathfrak{T}\cup\mathcal{A}$. Show that μ is $\mathfrak{S}$-Radon.

(i) In 254Yh[24], show that if we start from a continuous inverse-measure-preserving $f:[0,1]\to[0,1]^2$, as in 134Yl, we get a continuous inverse-measure-preserving surjection $g:[0,1]\to[0,1]^{\mathbb{N}}$.

416 Notes and comments The original measures studied by Radon (Radon 1913) were, in effect, what I call differences of Radon measures on $\mathbb{R}^r$, as introduced in §256. Successive generalizations moved first to Radon measures on general compact Hausdorff spaces, then to locally compact Hausdorff spaces, and finally to arbitrary Hausdorff spaces, as presented in this section. I ought perhaps to remark that, following Bourbaki 65, many authors use the term 'Radon measure' to describe a linear functional on a space of continuous functions; I will discuss the relationship between such functionals and the measures of this chapter in §436. For the moment, observe that by 415I a Radon measure on a completely regular space can be determined from the integrals it assigns to continuous functions. It is also common for the phrase 'Radon measure' to be used for what I would call a tight Borel measure; you have to check each author to see whether local finiteness is also assumed. In my usage, a Radon measure is necessarily the c.l.d. version of a Borel measure. The Borel measures which correspond to Radon measures are described in 416F.

In §256, I discussed Radon measures on $\mathbb{R}^r$ as a preparation for a discussion of convolutions of measures. It should now be becoming clear why I felt that it was impossible, in that context, to give you a proper idea of what a Radon measure, in the modern form, 'really' is. In Euclidean space, too many concepts coincide. As a trivial example, the simplest definition of 'local finiteness' (256Ab) is not the right formulation in other spaces (411Fa). Next, because every closed set is a countable union of compact sets, there is no distinction between 'inner regular with respect to closed sets' and 'inner regular with respect to compact sets', so one cannot get any intuition for which is important in which arguments. (When we come to subspace measures on non-measurable subsets, of course, this changes; quasi-Radon measures on subsets of Euclidean space are important and interesting.) Third, the fact that the c.l.d. product of two Radon measures on Euclidean space is already a Radon measure (256K) leaves us with no idea of what to do with a general product of Radon measures. (There are real difficulties at this point, which I will attack in the next section. For the moment I offer just 416U.) And finally, we simply cannot represent a product of uncountably many Radon probability measures on Euclidean spaces as a measure on Euclidean space.

As you would expect, a very large proportion of the results of this chapter, and many theorems from earlier volumes, were originally proved for compact Radon measure spaces. The theory of general totally finite Radon measures is, in effect, the theory of measurable subspaces of compact Radon measure spaces, while the theory of quasi-Radon measures is pretty much the theory of non-measurable subspaces of Radon measure spaces. Thus the theorem that every quasi-Radon measure space is strictly localizable is almost a consequence of the facts that every Radon measure space is strictly localizable and any subspace of a strictly localizable space is strictly localizable.

The cluster of results between 416J and 416Q form only a sample, I hope a reasonably representative sample, of the many theorems on construction of Radon measures from functionals on algebras or lattices of sets. (See also 416Ye.) The essential simplification, compared with the theorems in §413 and §415, is that we do not need to mention any σ- or τ-additivity condition of the type 413I(β) or 415K(β), because we are dealing with a 'compact class', the family of compact subsets of a Hausdorff space. We can use this even at some distance, as in 416O (where the hypotheses do not require any non-empty compact set to belong to the domain of the original functional). The particular feature of 416O which makes it difficult to prove from such results as 413J and 413O above is that we have to retain control of the outer measures of a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of non-measurable sets. In general this is hard to do, and is possible here principally because the sequence is non-decreasing, so that we can make sense of the functionals $\nu_nE=\nu^*(E\cap K_{n+1})-\nu^*(E\cap K_n)$; compare 214P.

417 τ-additive product measures

The 'ordinary' product measures introduced in Chapter 25 have served us well for a volume and a half. But we come now to a fundamental obstacle. If we start with two Radon measure spaces, their product measure, as defined in §251, need not be a Radon measure (419E). Furthermore, the counterexample is one of the basic compact measure spaces of the theory; and while it is dramatically non-metrizable, there is no other reason to set it aside. Consequently, if we wish (as we surely do) to create Radon measure spaces as products of Radon measure spaces, we need a new construction. This is the object of the present section. It turns out that the construction can be adapted to work well beyond the special context of Radon measure spaces; the methods here apply to general effectively locally finite τ-additive topological measures (for the product of finitely many factors) and to τ-additive topological probability measures (for the product of infinitely many factors).

[24] Later editions only.

The fundamental theorems are 417C and 417E, listing the essential properties of what I call 'τ-additive product measures', which are extensions of the c.l.d. product measures and product probability measures of Chapter 25. They depend on a straightforward lemma on the extension of a measure to make every element of a given class of sets negligible (417A). It is relatively easy to prove that the extensions are more or less canonical (417D, 417F). We still have Fubini's theorem for the new product measures (417H), and the basic operations from §254 still apply (417J, 417K, 417M).

It is easy to check that if we start with quasi-Radon measures, then the τ-additive product measure is again quasi-Radon (417N, 417O). The τ-additive product of two Radon measures is Radon (417P), and the τ-additive product of Radon probability measures with compact supports is Radon (417Q).

In the last part of the section I look at continuous real-valued functions and Baire σ-algebras; it turns out that for these the ordinary product measures are adequate (417U, 417V).

417A Lemma Let (X,Σ,μ) be a semi-finite measure space, and $\mathcal{A}\subseteq\mathcal{P}X$ a family of sets such that $\mu_*(\bigcup_{n\in\mathbb{N}}A_n)=0$ for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$. Then there is a measure μ' on X, extending μ, such that

(i) $\mu'A$ is defined and zero for every $A\in\mathcal{A}$;

(ii) μ' is complete if μ is;

(iii) for every F in the domain Σ' of μ' there is an $E\in\Sigma$ such that $\mu'(F\triangle E)=0$;

(iv) whenever $\mathcal{K}$, $\mathcal{G}$ are families of sets such that

(α) μ is inner regular with respect to $\mathcal{K}$,

(β) $K\cup K'\in\mathcal{K}$ for all K, $K'\in\mathcal{K}$,

(γ) $\bigcap_{n\in\mathbb{N}}K_n\in\mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

(δ) for every $A\in\mathcal{A}$ there is a $G\in\mathcal{G}$, including A, such that $G\setminus A\in\Sigma$,

(ϵ) $K\setminus G\in\mathcal{K}$ whenever $K\in\mathcal{K}$ and $G\in\mathcal{G}$,

then μ' is inner regular with respect to $\mathcal{K}$.

In particular, μ and μ' have isomorphic measure algebras, so that μ' is localizable if μ is.

proof (a) Let $\mathcal{A}^*$ be the collection of subsets of X which can be covered by a countable subfamily of $\mathcal{A}$. Then $\mathcal{A}^*$ is a σ-ideal of subsets of X and $\mu_*A=0$ for every $A\in\mathcal{A}^*$. Set

$$\Sigma'=\{E\triangle A:E\in\Sigma,\,A\in\mathcal{A}^*\}.$$

Then Σ' is a σ-algebra of subsets of X. **P** (i) $\emptyset=\emptyset\triangle\emptyset\in\Sigma'$. (ii) If $E\in\Sigma$, $A\in\mathcal{A}^*$ then $X\setminus(E\triangle A)=(X\setminus E)\triangle A\in\Sigma'$. (iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle A_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ, $\mathcal{A}^*$ respectively, then

$$E=\bigcup_{n\in\mathbb{N}}E_n\in\Sigma,\quad A=E\triangle\bigcup_{n\in\mathbb{N}}(E_n\triangle A_n)\subseteq\bigcup_{n\in\mathbb{N}}A_n\in\mathcal{A}^*,$$

so $\bigcup_{n\in\mathbb{N}}(E_n\triangle A_n)=E\triangle A\in\Sigma'$. **Q**

(b) If E, $E'\in\Sigma$, A, $A'\in\mathcal{A}^*$ and $E\triangle A=E'\triangle A'$, then $E\triangle E'=A\triangle A'\in\mathcal{A}^*$ and $\mu_*(E\triangle E')=0$; because μ is semi-finite, $\mu(E\triangle E')=0$ and $\mu E=\mu E'$. Accordingly we can define $\mu':\Sigma'\to[0,\infty]$ by setting

$$\mu'(E\triangle A)=\mu E\text{ whenever }E\in\Sigma,\,A\in\mathcal{A}^*.$$

Evidently μ' extends μ and $\mu'A=0$ for every $A\in\mathcal{A}$. Also μ' is a measure. **P** (i) $\mu'\emptyset=\mu\emptyset=0$. (ii) If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ', with union F, express each F_n as $E_n\triangle A_n$ where $E_n\in\Sigma$, $A_n\in\mathcal{A}^*$; set $E=\bigcup_{n\in\mathbb{N}}E_n$, so that $F\triangle E\in\mathcal{A}^*$ and $\mu'F=\mu E$. If $m\neq n$, then $E_m\cap E_n\subseteq A_m\cup A_n$, so $\mu(E_m\cap E_n)=0$; accordingly

$$\mu'F=\mu E=\sum_{n=0}^{\infty}\mu E_n=\sum_{n=0}^{\infty}\mu'F_n.\ \mathbf{Q}$$

(c) A subset of X is μ'-negligible iff it can be included in a set of the form $E\triangle A$ where $\mu E=0$ and $A\in\mathcal{A}^*$, so μ' is complete if μ is. The embedding $\Sigma\subsetneq\Sigma'$ induces a measure-preserving homomorphism from the measure algebra of μ to the measure algebra of μ' which is an isomorphism just because every member of Σ' is the symmetric difference of a member of Σ and a μ'-negligible set.

(d) This deals with (i)-(iii) in the statement of the lemma. Now suppose that $\mathcal{K}$ and $\mathcal{G}$ are as in (iv). Take $F\in\Sigma'$ and $\gamma<\mu'F$. Take $E\in\Sigma$ and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $E\triangle F\subseteq\bigcup_{n\in\mathbb{N}}A_n$. Then $\mu E=\mu'F>\gamma$ so (again because μ is semi-finite) there is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E$ and $\gamma<\mu K<\infty$; set $\epsilon=\frac{1}{2}(\mu K-\gamma)>0$. For each $n\in\mathbb{N}$, choose $G_n\in\mathcal{G}$ and $K_n\in\mathcal{K}\cap\Sigma$ such that

$$A_n\subseteq G_n,\quad E_n=G_n\setminus A_n\text{ belongs to }\Sigma,$$

$$K_n\subseteq K\cap E_n,\quad\mu(K\cap E_n\setminus K_n)\le 2^{-n}\epsilon.$$

Set $L = \bigcap_{n\in\mathbb{N}}(K \setminus G_n) \cup K_n$. Putting the hypotheses (iv-ϵ), (iv-β) and (iv-γ) together, we see that $L \in \mathcal{K}$; moreover, since $G_n = E_n \cup A_n$ belongs to Σ' for every n, $L \in \Sigma'$. Next, setting $H = \bigcap_{n\in\mathbb{N}}(K \setminus E_n) \cup K_n$, $L = H \setminus \bigcup_{n\in\mathbb{N}} A_n$, so $\mu' L = \mu H$ and $L \subseteq F$. But $K \setminus H \subseteq \bigcup_{n\in\mathbb{N}}(K \cap E_n \setminus K_n)$ so

$$\mu' L = \mu H \ge \mu K - \sum_{n=0}^{\infty} 2^{-n}\epsilon = \gamma.$$

As F and γ are arbitrary, μ' is inner regular with respect to $\mathcal{K}$.

417B Lemma Let X and Y be topological spaces, and ν a τ-additive topological measure on Y.

(a) If $W \subseteq X \times Y$ is open, then $x \mapsto \nu W[\{x\}] : X \to [0,\infty]$ is lower semi-continuous.

(b) If ν is effectively locally finite and σ-finite and $W \subseteq X \times Y$ is a Borel set, then $x \mapsto \nu W[\{x\}]$ is Borel measurable.

(c) If $f : X \times Y \to [0,\infty]$ is a lower semi-continuous function, then $x \mapsto \int f(x,y)\nu(dy) : X \to [0,\infty]$ is lower semi-continuous.

(d) If ν is totally finite and $f : X \times Y \to \mathbb{R}$ is a bounded continuous function, then $x \mapsto \int f(x,y)\nu(dy)$ is continuous.

(e) If ν is totally finite and $W \subseteq X \times Y$ is a Baire set, then $x \mapsto \nu W[\{x\}]$ is Baire measurable.

proof (a) If $x \in X$ and $\nu W[\{x\}] > \alpha$, then

$$\mathcal{H} = \{H : H \subseteq Y \text{ is open, there is an open set } G \text{ containing } x \text{ such that } G \times H \subseteq W\}$$

is an upwards-directed family of open sets with union $W[\{x\}]$, so there is an $H \in \mathcal{H}$ such that $\nu H \ge \alpha$. Now there is an open set G containing x such that $G \times H \subseteq W$, so that $\nu W[\{x'\}] \ge \alpha$ for every $x' \in G$.

(b)(i) Suppose to begin with that ν is totally finite. In this case, the set

$$\{W : W \subseteq X \times Y,\ x \mapsto \nu W[\{x\}] \text{ is a Borel measurable function defined everywhere on } X\}$$

is a Dynkin class containing every open set, so contains every Borel set, by the Monotone Class Theorem (136B).

(ii) For the general case, let $\langle Y_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of sets of finite measure covering Y, and for $n \in \mathbb{N}$ let ν_n be the subspace measure on Y_n. Then ν_n is effectively locally finite and τ-additive (414K). If $W \subseteq X \times Y$ is a Borel set, then $W_n = W \cap (X \times Y_n)$ is a relatively Borel set for each n, so that $x \mapsto \nu_n W_n[\{x\}]$ is Borel measurable, by (i). Since $\nu W[\{x\}] = \sum_{n=0}^{\infty} \nu_n W_n[\{x\}]$ for every x, $x \mapsto \nu W[\{x\}]$ is Borel measurable.

(c) For i, $n \in \mathbb{N}$ set $W_{ni} = \{(x,y) : f(x,y) > 2^{-n}i\}$, so that $W_{ni} \subseteq X \times Y$ is open. Set $f_n = 2^{-n}\sum_{i=1}^{4^n} \chi W_{ni}$; then $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum f. For $n \in \mathbb{N}$ and $x \in X$, $\int f_n(x,y)\nu(dy) = 2^{-n}\sum_{i=1}^{4^n} \nu W_{ni}[\{x\}]$, so $x \mapsto \int f_n(x,y)\nu(dy)$ is lower semi-continuous, by (a) and 4A2B(d-iii). By 414Ba, $\int f(x,y)\nu(dy)$ is the supremum $\sup_{n\in\mathbb{N}} \int f_n(x,y)\nu(dy)$ for every x, so $x \mapsto \int f(x,y)\nu(dy)$ is lower semi-continuous (4A2B(d-v)).

(d) Applying (c) to $f + \|f\|_\infty \chi(X \times Y)$, we see that $x \mapsto \int f(x,y)\nu(dy)$ is lower semi-continuous. Similarly, $x \mapsto -\int f(x,y)\nu(dy)$ is lower semi-continuous, so $x \mapsto \int f(x,y)\nu(dy)$ is continuous (4A2B(d-vi)).

(e) Suppose first that W is a cozero set; let $f : X \times Y \to [0,1]$ be a continuous function such that $W = \{(x,y) : f(x,y) > 0\}$. For $n \in \mathbb{N}$ set $f_n = nf \wedge \chi(X \times Y)$. Then $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of continuous functions with supremum χW. By (d), all the functions $x \mapsto \int f_n(x,y)\nu(dy)$ are continuous, so their limit $x \mapsto \nu W[\{x\}]$ is Baire measurable.

Now

$$\{W : W \subseteq X \times Y,\ x \mapsto \nu W[\{x\}] \text{ is a Baire measurable function defined everywhere on } X\}$$

is a Dynkin class containing every cozero set, so contains every Baire set, by the Monotone Class Theorem again.

417C Theorem (Ressel 77) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Then λ has an extension to a τ-additive topological measure $\tilde\lambda$ on $X \times Y$. Moreover, we can arrange that:

(i) $\tilde\lambda$ is complete, locally determined and effectively locally finite, therefore strictly localizable;

(ii) if Q belongs to the domain $\tilde{\Lambda}$ of $\tilde{\lambda}$, there is a $Q_1 \in \Lambda$ such that $\tilde{\lambda}(Q \triangle Q_1) = 0$; that is to say, the embedding $\Lambda \subsetneqq \tilde{\Lambda}$ induces an isomorphism between the measure algebras of λ and $\tilde{\lambda}$;

(iii) if $Q \in \tilde{\Lambda}$, then

$$\tilde{\lambda}Q = \sup\{\tilde{\lambda}(Q \cap (G \times H)) : G \in \mathfrak{T},\ \mu G < \infty,\ H \in \mathfrak{S},\ \nu H < \infty\};$$

(iv) if $W \subseteq X \times Y$ is open, then

(α) there is an open set $W_0 \in \Lambda$ such that $W_0 \subseteq W$ and $\lambda W_0 = \tilde{\lambda} W$,

(β) $\tilde{\lambda} W = \lambda_* W = \int \nu W[\{x\}] \mu(dx) = \int \mu W^{-1}[\{y\}] \nu(dy)$;

in particular, $\tilde{\lambda}(G \times H) = \mu G \cdot \nu H$ whenever $G \in \mathfrak{T}$ and $H \in \mathfrak{S}$;

(v) the support of λ is the product of the supports of μ and ν;

(vi) if μ and ν are both inner regular with respect to the Borel sets, so is $\tilde{\lambda}$;

(vii) if μ and ν are both inner regular with respect to the closed sets, so is $\tilde{\lambda}$;

(viii) if μ and ν are both tight (that is, inner regular with respect to the closed compact sets), so is $\tilde{\lambda}$.

proof Write

$$\Sigma^f = \{E : E \in \Sigma,\ \mu E < \infty\}, \quad \mathrm{T}^f = \{F : F \in \mathrm{T},\ \nu F < \infty\},$$

$$\mathfrak{T}^f = \mathfrak{T} \cap \Sigma^f, \quad \mathfrak{S}^f = \mathfrak{S} \cap \mathrm{T}^f.$$

(a) Let $\mathcal{U}$ be $\{G \times H : G \in \mathfrak{T}^f,\ H \in \mathfrak{S}^f\}$. Because $\mathfrak{T} \subseteq \Sigma$ and $\mathfrak{S} \subseteq \mathrm{T}$, $\mathcal{U} \subseteq \Lambda$. $\mathcal{U}$ need not be a base for the topology of $X \times Y$, unless μ and ν are locally finite, but if an open subset of $X \times Y$ is included in a member of $\mathcal{U}$ it is the union of the members of $\mathcal{U}$ it includes. Moreover, if $Q \in \Lambda$, then $\lambda Q = \sup_{U \in \mathcal{U}} \lambda(Q \cap U)$. **P** By 412R, λ is inner regular with respect to $\bigcup_{U \in \mathcal{U}} \mathcal{P}U$. **Q**

Write $\mathcal{U}_s$ for the set of finite unions of members of $\mathcal{U}$, and $\mathfrak{V}$ for the set of non-empty upwards-directed families $\mathcal{V} \subseteq \mathcal{U}_s$ such that $\sup_{V \in \mathcal{V}} \lambda V < \infty$. For each $\mathcal{V} \in \mathfrak{V}$, fix on a countable $\mathcal{V}' \subseteq \mathcal{V}$ such that $\sup_{V \in \mathcal{V}'} \lambda V = \sup_{V \in \mathcal{V}} \lambda V$; because $\mathcal{V}$ is upwards-directed, we may suppose that $\mathcal{V}' = \{V_n : n \in \mathbb{N}\}$ for some non-decreasing sequence $\langle V_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{V}$. Set $A(\mathcal{V}) = \bigcup \mathcal{V} \setminus \bigcup \mathcal{V}'$.

(b)(i) For $V \in \mathcal{U}_s$, set $f_V(x) = \nu V[\{x\}]$ for every $x \in X$. This is always defined because V is open; moreover, f_V is lower semi-continuous, by 417Ba. Because V is a finite union of products of sets of finite measure, $\int f_V d\mu = \lambda V$.

(ii) The key to the proof is the following fact: for any $\mathcal{V} \in \mathfrak{V}$, almost every vertical section of $A(\mathcal{V})$ is negligible. **P** $\langle f_V \rangle_{V \in \mathcal{V}}$ is a non-empty upwards-directed set of lower semi-continuous functions. Set

$$g(x) = \nu(\bigcup_{V \in \mathcal{V}} V[\{x\}]), \quad h(x) = \nu(\bigcup_{V \in \mathcal{V}'} V[\{x\}])$$

for every $x \in X$. Because $\mathcal{V}$ is upwards-directed and ν is τ-additive,

$$g(x) = \sup_{V \in \mathcal{V}} \nu V[\{x\}] = \sup_{V \in \mathcal{V}} f_V(x)$$

in $[0, \infty]$ for each x, so, by 414Ba again,

$$\int g\, d\mu = \sup_{V \in \mathcal{V}} \int f_V d\mu = \sup_{V \in \mathcal{V}} \lambda V = \sup_{V \in \mathcal{V}'} \lambda V = \int h\, d\mu.$$

Since $h \le g$ and $\sup_{V \in \mathcal{V}} \lambda V$ is finite, $g(x) = h(x) < \infty$ for μ-almost every x. But for any such x, we must have

$$\nu(\bigcup \mathcal{V})[\{x\}] = \nu(\bigcup \mathcal{V}')[\{x\}] < \infty,$$

so that

$$A(\mathcal{V})[\{x\}] = (\bigcup \mathcal{V})[\{x\}] \setminus (\bigcup \mathcal{V}')[\{x\}]$$

is negligible. **Q**

(c) ? Suppose, if possible, that there is a sequence $\langle \mathcal{V}_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{V}$ such that $\lambda_*(\bigcup_{n \in \mathbb{N}} A(\mathcal{V}_n)) > 0$. Take $W \in \Lambda$ such that $W \subseteq \bigcup_{n \in \mathbb{N}} A(\mathcal{V}_n)$ and $\lambda W > 0$. Because almost every vertical section of every $A(\mathcal{V}_n)$ is negligible, almost every vertical section of W is negligible. But this contradicts Fubini's theorem (252F). **X**

(d) Let λ' be an extension of λ as described in 417A, $\tilde{\lambda}$ the c.l.d. version of λ' (213E), and Λ', $\tilde{\Lambda}$ the domains of λ', $\tilde{\lambda}$ respectively.

(i) If $W \in \Lambda$, then $W \in \Lambda' \subseteq \tilde{\Lambda}$. Also, because λ is semi-finite,

$$\lambda' W = \lambda W = \sup\{\lambda W' : W' \subseteq W,\, W \in \Lambda,\, \lambda W' < \infty\}$$
$$\le \sup\{\lambda' W' : W' \subseteq W,\, W \in \Lambda',\, \lambda' W' < \infty\} = \tilde\lambda W \le \lambda' W.$$

Thus $\lambda W = \tilde\lambda W$; as W is arbitrary, $\tilde\lambda$ extends λ.

(ii) It will be useful if we go directly to one of the targets: if $\tilde Q \in \tilde\Lambda$ and $\gamma < \tilde\lambda\tilde Q$, there is a $U \in \mathcal{U}$ such that $\tilde\lambda(\tilde Q \cap U) \ge \gamma$. **P** There is a $Q' \in \Lambda'$ such that $Q' \subseteq \tilde Q$ and $\gamma < \lambda' Q' < \infty$. By 417A(iii), there is a $Q \in \Lambda$ such that $\lambda'(Q \triangle Q') = 0$, so that

$$\lambda Q = \lambda' Q = \lambda' Q' > \gamma.$$

There is a $U \in \mathcal{U}$ such that $\lambda(Q \cap U) \ge \gamma$, by (a). Now

$$\tilde\lambda(\tilde Q \cap U) \ge \lambda'(Q' \cap U) = \lambda'(Q \cap U) = \lambda(Q \cap U) \ge \gamma. \text{ } \mathbf{Q}$$

(iii) $\tilde\lambda$ is a topological measure. **P** Let $W \subseteq X \times Y$ be an open set. Suppose that $\tilde Q \in \tilde\Lambda$ and $\tilde\lambda\tilde Q > 0$. By (ii), there is a $U \in \mathcal{U}$ such that $\tilde\lambda(\tilde Q \cap U) > 0$. Let $\mathcal{V}$ be $\{V : V \in \mathcal{U}_s,\, V \subseteq W \cap U\}$. Then $\mathcal{V} \in \mathfrak{V}$, so $\tilde\lambda A(\mathcal{V}) = \lambda' A(\mathcal{V}) = 0$, by 417A(i); since $\bigcup \mathcal{V}' \in \Lambda$,

$$W \cap U = \bigcup \mathcal{V} \in \Lambda' \subseteq \tilde\Lambda.$$

But this means that $\tilde Q \cap U \cap W$ and $\tilde Q \cap U \setminus W = (\tilde Q \cap U) \setminus (W \cap U)$ belong to $\tilde\Lambda$ and

$$\tilde\lambda_*(\tilde Q \cap W) + \tilde\lambda_*(\tilde Q \setminus W) \ge \tilde\lambda(\tilde Q \cap U \cap W) + \tilde\lambda(\tilde Q \cap U \setminus W) = \tilde\lambda(\tilde Q \cap U) > 0.$$

Because $\tilde\lambda$ is complete and locally determined, and $\tilde Q$ is arbitrary, this is enough to ensure that $W \in \tilde\Lambda$ (413F(vii)). **Q**

(iv) $\tilde\lambda$ is τ-additive. **P?** Suppose, if possible, otherwise; that there is a non-empty upwards-directed family $\mathcal{W}$ of open sets in $X \times Y$ such that $\tilde\lambda W^* > \gamma = \sup_{W \in \mathcal{W}} \tilde\lambda W$, where $W^* = \bigcup \mathcal{W}$. In this case, we can find a $Q' \in \Lambda'$ such that $Q' \subseteq W^*$ and $\lambda' Q' > \gamma$, a $Q \in \Lambda$ such that $\lambda'(Q' \triangle Q) = 0$, and a $U \in \mathcal{U}$ such that $\lambda(Q \cap U) > \gamma$ (using (a) again). Let $\mathcal{V} \in \mathfrak{V}$ be the set of those $V \in \mathcal{U}_s$ such that $V \subseteq W \cap U$ for some $W \in \mathcal{W}$. Then $\bigcup \mathcal{V} = W^* \cap U$, so

$$\gamma < \lambda(Q \cap U) = \lambda'(Q \cap U) = \lambda'(Q' \cap U)$$
$$\le \tilde\lambda(W^* \cap U) = \tilde\lambda(\bigcup \mathcal{V}) = \tilde\lambda(\bigcup \mathcal{V}')$$

(because $\tilde\lambda A(\mathcal{V}) = 0$)

$$= \sup_{V \in \mathcal{V}'} \tilde\lambda V$$

(because $\mathcal{V}'$ is countable and upwards-directed)

$$\le \sup_{W \in \mathcal{W}} \tilde\lambda W \le \gamma,$$

which is absurd. **XQ**

(e) Now for the supplementary properties (i)-(viii), in order.

(i) $\tilde\lambda$ was constructed to be complete and locally determined. If $\tilde\lambda\tilde Q > 0$, then by (d-ii) there is a $U \in \mathcal{U}$ such that $\tilde\lambda(\tilde Q \cap U) > 0$; since U is open and $\tilde\lambda U = \lambda U$ is finite, this shows that $\tilde\lambda$ is effectively locally finite. By 414J, it is strictly localizable.

(ii) The point is that λ also is (strictly) localizable. **P** Let $\tilde\mu$, $\tilde\nu$ be the c.l.d. versions of μ and ν. These are τ-additive topological measures (because μ and ν are), complete and locally determined (by construction), and are still effectively locally finite (cf. 412Ha), so are strictly localizable (414J again). Now λ is the c.l.d. product of $\tilde\mu$ and $\tilde\nu$ (251T), therefore strictly localizable (251O). **Q**

As remarked in 417A, it follows that λ' is localizable, so that every member $\tilde Q$ of $\tilde\Lambda$ differs by a $\tilde\lambda$-negligible set from a member Q' of Λ' (213Hb). Now there is a $Q \in \Lambda$ such that $\lambda'(Q \triangle Q') = 0$, in which case $\tilde\lambda(Q \triangle \tilde Q) = 0$.

(iii) This is just (d-ii) above.

(iv) Let $W \subseteq X \times Y$ be an open set. Set $\mathcal{V} = \{V : V \in \mathcal{U}_s,\, V \subseteq W\}$. Then $\mathcal{V}$ is upwards-directed and has union $W \cap (X^* \times Y^*)$, where X^* and Y^* are the unions of the open sets of finite measure in X and Y respectively. Because

μ and ν are effectively locally finite, X^* and Y^* are both conegligible, and $X^* \times Y^*$ is λ-conegligible, therefore $\tilde{\lambda}$-conegligible. Now, as before,

$$\tilde{\lambda}W = \tilde{\lambda}(W \cap (X^* \times Y^*)) = \sup_{V\in\mathcal{V}} \tilde{\lambda}V = \sup_{V\in\mathcal{V}} \lambda V \le \lambda_* W \le \tilde{\lambda}W.$$

If we take a countable set $\mathcal{V}_0 \subseteq \mathcal{V}$ such that $\sup_{V\in\mathcal{V}_0} \lambda V = \sup_{V\in\mathcal{V}} \lambda V$, and set $W_0 = \bigcup \mathcal{V}_0$, then W_0 is an open set, belonging to λ and included in W, and $\lambda W_0 = \tilde{\lambda}W$.

Next, defining the functions f_V as in part (a) of this proof, we have $\int f_V d\mu = \lambda V$ for every $V \in \mathcal{V}$; and setting $g(x) = \nu W[\{x\}]$, we have $g(x) = \sup_{V\in\mathcal{V}} f_V(x)$ for every $x \in X^*$. Once more, 414Ba tells us that

$$\int \nu W[\{x\}]\mu(dx) = \int g\,d\mu = \sup_{V\in\mathcal{V}} \int f_V d\mu = \sup_{V\in\mathcal{V}} \lambda V = \tilde{\lambda}W.$$

Similarly,

$$\int \mu W^{-1}[\{y\}]\nu(dy) = \tilde{\lambda}W.$$

(The point here is that while the arguments of part (b) of this proof give different roles to μ and ν, the asserted properties of the extension in part (d), and the following deductions, are symmetric between the two factors.)

Now, given $G \in \mathfrak{T}$ and $H \in \mathfrak{S}$, set $W = G \times H$; then

$$\tilde{\lambda}(G \times H) = \int \mu W^{-1}[\{y\}]\nu(dy) = \int_H \mu G\,\nu(dy) = \mu G \cdot \nu H.$$

(v) Let Z and Z' be the supports of μ, ν respectively. (By 411Nd these are defined.) Then $Z \times Z'$ is a closed subset of $X \times Y$. Because $X \setminus Z$ and $Y \setminus Z'$ are negligible, $Z \times Z'$ is conegligible for both λ and $\tilde{\lambda}$. If $W \subseteq X \times Y$ is an open set and $(x, y) \in W \cap (Z \times Z')$, there are open sets $G \subseteq X$, $H \subseteq Y$ such that $(x, y) \in G \times H \subseteq W$. Now

$$\tilde{\lambda}(W \cap (Z \times Z')) \ge \lambda((G \times H) \cap (Z \times Z')) = \mu(G \cap Z) \cdot \nu(H \cap Z') > 0.$$

This shows that $Z \times Z'$ is self-supporting, so is the support of λ.

(vi), (vii), (viii) These are all consequences of 417A(iv). In each case, take $\mathcal{G}$ to be the set of open subsets of $X \times Y$. Take $\mathcal{K}$ to be either the family of Borel subsets of $X \times Y$ (for (vi)) or the family of closed subsets of $X \times Y$ (for (vii)) or the family of closed compact subsets of $X \times Y$ (for (viii)). Then (β), (γ) and (ϵ) of 417A(iv) are all satisfied. As for 417A(iv-δ), given any $A \in \mathcal{A}$, let $\mathcal{V} \in \mathfrak{V}$ be such that $A = A(\mathcal{V})$, and set $G = \bigcup \mathcal{V} \in \mathcal{G}$; then $A \subseteq G$ and $G \setminus A = \bigcup \mathcal{V}'$ belongs to Λ, as required. Finally, the hypotheses of each part are just what we need in order to be sure that λ is inner regular with respect to $\mathcal{K}$ (412Sd, 412Sa, 412Sb), as required in 417A(iv-α), so we can conclude that λ' is inner regular with respect to $\mathcal{K}$. By 412Ha, its c.l.d. version $\tilde{\lambda}$ also is inner regular with respect to $\mathcal{K}$.

417D Multiple products Just as with the c.l.d. product measure (see 251W), we can apply the construction of 417C repeatedly to obtain measures on the products of finite families of τ-additive measure spaces.

Proposition (a) Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a finite family of effectively locally finite τ-additive topological measure spaces. Then there is a unique complete locally determined effectively locally finite τ-additive topological measure $\tilde{\lambda}$ on $X = \prod_{i\in I} X_i$, inner regular with respect to the Borel sets, such that $\tilde{\lambda}(\prod_{i\in I} G_i) = \prod_{i\in I} \mu_i G_i$ whenever $G_i \in \mathfrak{T}_i$ for every $i \in I$.

(b) If now $\langle I_k\rangle_{k\in K}$ is a partition of I, and $\tilde{\lambda}_k$ is the product measure defined by the construction of (a) on $Z_k = \prod_{i\in I_k} X_i$ for each $k \in K$, then the natural bijection between X and $\prod_{k\in K} Z_k$ identifies $\tilde{\lambda}$ with the product of the $\tilde{\lambda}_k$ defined by the construction of (a).

proof (a)(i) Suppose first that every μ_i is inner regular with respect to the Borel sets. Then a direct induction on $\#(I)$, using 417C for the inductive step, tells us that there is a measure $\tilde{\lambda}$ with the required properties. Note that 417C(vi) ensures that (in the present context) all our product measures will be inner regular with respect to the Borel sets.

(ii) For the general case, apply (a) to $\mu_i{\restriction}\mathcal{B}(X_i)$, where $\mathcal{B}(X_i)$ is the Borel σ-algebra of X_i for each i.

(iii) To see that $\tilde{\lambda}$ is unique, suppose that λ' is another measure with the same properties. Let $\mathcal{U}$ be the set $\{\prod_{i\in I} G_i : G_i \in \mathfrak{T}_i$ for every $i \in I\}$, and $\mathcal{U}_s$ the set of finite unions of members of $\mathcal{U}$. Then $\tilde{\lambda}$ and λ' agree on $\mathcal{U}_s$. **P** Suppose that $U = \bigcup_{j\le n} \prod_{i\in I} G_{ji}$, where $G_{ji} \in \mathfrak{T}_i$ for $j \le n$, $i \in I$. If there is any j such that $\prod_{i\in I} \mu_i G_{ji} = \infty$, then $\tilde{\lambda}U = \lambda' U = \infty$. Otherwise, set $L = \{j : j \le n, \prod_{i\in I} \mu_i G_{ji} > 0\}$ and $G^*_i = \bigcup_{j\in L} G_{ji}$ for $i \in I$. Then

$$\tilde{\lambda}(U \setminus \prod_{i\in I} G^*_i) = 0 = \lambda'(U \setminus \prod_{i\in I} G^*_i).$$

On the other hand, $\mu_i G_{ji}$ must be finite whenever $j \in L$ and $i \in I$, so $\mu_i G_i^*$ is finite for every i. Consider $\mathcal{I} = \{\prod_{i\in I} G_i : G_i \in \mathfrak{T}_i,\ G_i \subseteq G_i^* \text{ for every } i \in I\}$. Then $V \cap V' \in \mathcal{I}$ for all V, $V' \in \mathcal{I}$, and $\tilde{\lambda}$, λ' agree on $\mathcal{I}$. It follows from the Monotone Class Theorem (136C), or otherwise, that $\tilde{\lambda}$ and λ' agree on the algebra of subsets of $\prod_{i\in I} G_i^*$ generated by $\mathcal{I}$. In particular, $\tilde{\lambda}(U \cap \prod_{i\in I} G_i^*) = \lambda'(U \cap \prod_{i\in I} G_i^*)$, so that $\tilde{\lambda}U = \lambda' U$. **Q**

But now, because $\tilde{\lambda}$ and λ' are τ-additive,

$$\tilde{\lambda}W = \sup\{\tilde{\lambda}U : U \in \mathcal{U}_s,\ U \subseteq W\} = \sup\{\lambda' U : U \in \mathcal{U}_s,\ U \subseteq W\} = \lambda' W$$

for every open set $W \subseteq X$. Writing $\mathcal{B} = \mathcal{B}(X)$ for the Borel σ-algebra of X, $\tilde{\lambda}\restriction\mathcal{B}$ and $\lambda'\restriction\mathcal{B}$ are effectively locally finite Borel measures which agree on the open sets, so must be equal, by 414L. Since both $\tilde{\lambda}$ and λ' are complete locally determined measures defined on $\mathcal{B}$ and inner regular with respect to $\mathcal{B}$, they also are equal, by 412L.

(b) Let λ' be the measure on X corresponding to the τ-additive product of the $\tilde{\lambda}_k$ on $\prod_{k\in K} Z_k$. Then λ' is an effectively locally finite complete locally determined τ-additive topological measure inner regular with respect to the zero sets, and if $G_i \in \mathfrak{T}_i$ for every $i \in I$ then

$$\lambda'(\textstyle\prod_{i\in I} G_i) = \prod_{k\in K} \tilde{\lambda}_k(\prod_{i\in I_k} G_i) = \prod_{k\in K}\prod_{i\in I_k} \mu_i G_i = \prod_{i\in I} \mu_i G_i,$$

so $\lambda' = \tilde{\lambda}$.

417E Theorem Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces, with product probability space (X, Λ, λ). Then λ is τ-additive, and has an extension to a τ-additive topological measure $\tilde{\lambda}$ on X. Moreover, we can arrange that:

(i) $\tilde{\lambda}$ is complete;

(ii) if $\tilde{Q}$ is measured by $\tilde{\lambda}$, there is a $Q \in \Lambda$ such that $\tilde{\lambda}(\tilde{Q} \triangle Q) = 0$; that is to say, the embedding $\Lambda \subsetneqq \tilde{\Lambda}$ induces an isomorphism between the measure algebras of λ and $\tilde{\lambda}$;

(iii) $\tilde{\lambda}W = \lambda_* W$ for every open set $W \subseteq X$, and $\tilde{\lambda}F = \lambda^* F$ for every closed set $F \subseteq X$;

(iv) the support of λ is the product of the supports of the μ_i;

(v) if λ is inner regular with respect to the Borel sets, so is $\tilde{\lambda}$;

(vi) if λ is inner regular with respect to the closed sets, so is $\tilde{\lambda}$;

(vii) if λ is tight, so is $\tilde{\lambda}$.

proof The strategy of the proof is the same as in 417C, subject to some obviously necessary modifications. The key step, showing that every union $\bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)$ has zero inner measure, is harder, but we do save a little work because we no longer have to worry about sets of infinite measure.

(a) I begin by setting up some machinery. Let $\mathcal{C}$ be the family of subsets of X expressible in the form $\prod_{i\in I} E_i$, where $E_i \in \Sigma_i$ for every i and $\{i : E_i \neq X_i\}$ is finite. Let $\mathcal{U} \subseteq \mathcal{C}$ be the standard basis for the topology $\mathfrak{T}$ of X, consisting of sets expressible as $\prod_{i\in I} G_i$ where $G_i \in \mathfrak{T}_i$ for every $i \in I$ and $\{i : G_i \neq X_i\}$ is finite. Write $\mathcal{U}_s$ for the set of finite unions of members of $\mathcal{U}$, and $\mathfrak{V}$ for the set of non-empty upwards-directed families in $\mathcal{U}_s$. Note that every member of $\mathcal{U}_s$ is determined by coordinates in some finite subset of I (definition: 254M).

If $J \subseteq I$, write λ_J for the product measure on $\prod_{i\in J} X_i$; we shall need $\lambda_\emptyset$, which is the unique probability measure on the single-point set $\{\emptyset\} = \prod_{i\in\emptyset} X_i$. For $J \subseteq I$, $v \in \prod_{i\in J} X_i$ and $W \subseteq X$ set

$$f_W(v) = \lambda_{I\setminus J}\{w : (v, w) \in W\}$$

if this is defined, identifying $\prod_{i\in J} X_i \times \prod_{i\in I\setminus J} X_i$ with X.

(b) We need two easy facts.

(i) $f_W(v) = \int f_W(v^\frown\langle t\rangle)\mu_j(dt)$ whenever $W \in \widehat{\bigotimes}_{i\in I}\Sigma_i$, $J \subseteq I$, $v \in \prod_{i\in J} X_i$ and $j \in I \setminus J$, writing $v^\frown\langle t\rangle$ for the member $v \cup \{(j,t)\}$ of $\prod_{i\in J\cup\{j\}} X_i$ extending v and taking the value t at the coordinate j. **P** Let $\mathcal{A}$ be the family of sets W satisfying the property. Then $\mathcal{A}$ is a Dynkin class including $\mathcal{C}$, so includes the σ-algebra generated by $\mathcal{C}$, which is $\widehat{\bigotimes}_{i\in I}\Sigma_i$. **Q**

(ii) If $J \subseteq I$, $v \in \prod_{i\in J} X_i$, $j \in I \setminus J$ and $V \in \mathcal{U}_s$, and we set $g(t) = f_V(v^\frown\langle t\rangle)$ for $t \in X_j$, then g is lower semi-continuous. **P** We can express V as $\bigcup_{n\le m}\prod_{i\in I} G_{ni}$, where $G_{ni} \subseteq X_i$ is open for every $n \le m$, $i \in I$. Now if $t \in X_j$, we shall have

$$\{w : (v^\frown\langle t\rangle, w) \in V\} \subseteq \{w : (v^\frown\langle t'\rangle, w) \in V\}$$

whenever

$$t' \in H = X_j \cap \bigcap\{G_{nj} : n \le m,\, t \in G_{nj}\}.$$

So $g(t') \ge g(t)$ for every $t' \in H$, which is an open neighbourhood of t. As t is arbitrary, g is lower semi-continuous. **Q**

(c) For each $\mathcal{V} \in \mathfrak{V}$, fix, for the remainder of this proof, a countable $\mathcal{V}' \subseteq \mathcal{V}$ such that $\sup_{V\in\mathcal{V}'}\lambda V = \sup_{V\in\mathcal{V}}\lambda V$; because $\mathcal{V}$ is upwards-directed, we may suppose that $\mathcal{V}' = \{V_n : n \in \mathbb{N}\}$ for some non-decreasing sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$. Set $A(\mathcal{V}) = \bigcup\mathcal{V} \setminus \bigcup\mathcal{V}'$.

? Suppose, if possible, that there is a sequence $\langle\mathcal{V}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{V}$ such that $\lambda_*(\bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) > 0$.

(i) We have $\lambda^*(X \setminus \bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) < 1$; let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$ such that

$$X \setminus \textstyle\bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n) \subseteq \bigcup_{n\in\mathbb{N}} C_n, \quad \sum_{n=0}^{\infty}\lambda C_n = \gamma_0 < 1$$

(see 254A-254C). For each $n \in \mathbb{N}$, express $\mathcal{V}'_n$ as $\{V_{nr} : r \in \mathbb{N}\}$ where $\langle V_{nr}\rangle_{r\in\mathbb{N}}$ is non-decreasing, and set $W_n = \bigcup\mathcal{V}'_n = \bigcup_{r\in\mathbb{N}} V_{nr}$. Let $J \subseteq I$ be a countable set such that every C_n and every V_{nr} is dependent on coordinates in J. Express J as $\bigcup_{k\in\mathbb{N}} J_k$ where $J_0 = \emptyset$ and, for each k, J_{k+1} is equal either to J_k or to J_k with one point added. (As in the proof of 254Fa, I am using a formulation which will apply equally to finite and infinite I, though of course the case of finite I is elementary once we have 417C.)

(ii) For each $n \in \mathbb{N}$, set

$$W'_n = \textstyle\bigcup_{k\in\mathbb{N}}\{x : x \in X,\, f_{W_n}(x\restriction J_k) = 1\}.$$

Then $\lambda(W'_n \setminus W_n) = 0$. **P** For any $k \in \mathbb{N}$, if we think of λ as the product of λ_{J_k} and $\lambda_{I\setminus J_k}$ and of f_{W_n} as a measurable function on $\prod_{i\in J_k} X_i$, we see that $\{x : f_{W_n}(x\restriction J_k) = 1\}$ is of the form $F_k \times \prod_{i\in I\setminus J_k} X_i$, where $F_k \subseteq \prod_{i\in J_k} X_i$ is measurable; and

$$\lambda((F_k \times \textstyle\prod_{i\in I\setminus J_k} X_i) \setminus W_n) = \int_{F_k}(1 - f_{W_n}(v))\lambda_{J_k}(dv) = 0.$$

Summing over k, we see that $W'_n \setminus W_n$ is negligible. **Q**

Observe that every W'_n, like W_n, is determined by coordinates in J. So $\bigcup_{n\in\mathbb{N}} W'_n \setminus W_n$ is of the form $E \times \prod_{i\in I\setminus J} X_i$ where $\lambda_J E = 0$ (254Ob). There is therefore a sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of measurable cylinders in $\prod_{i\in J} X_i$ such that $E \subseteq \bigcup_{n\in\mathbb{N}} D_n$ and $\sum_{n=0}^{\infty}\lambda_J D_n < 1 - \gamma_0$. Set $C'_n = \{x : x \in X,\, x\restriction J \in D_n\} \in \mathcal{C}$ for each n. Then $\bigcup_{n\in\mathbb{N}} W'_n \setminus W_n \subseteq \bigcup_{n\in\mathbb{N}} C'_n$, so

$$(X \setminus \textstyle\bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) \cup \bigcup_{n\in\mathbb{N}} W'_n \setminus W_n \subseteq \bigcup_{n\in\mathbb{N}} C_n \cup \bigcup_{n\in\mathbb{N}} C'_n,$$

$$\gamma = \textstyle\sum_{n=0}^{\infty}\lambda C_n + \sum_{n=0}^{\infty}\lambda C'_n < 1,$$

while each C_n and each C'_n is determined by coordinates in a finite subset of J.

(iii) For $k \in \mathbb{N}$, let P_k be the set of those $v \in \prod_{i\in J_k} X_i$ such that

$$\textstyle\sum_{n=0}^{\infty} f_{C_n}(v) + f_{C'_n}(v) \le \gamma, \quad f_V(v) \le f_{W_n}(v) \text{ whenever } n \in \mathbb{N} \text{ and } V \in \mathcal{V}_n.$$

Our hypothesis is that

$$\textstyle\sum_{n=0}^{\infty} f_{C_n}(\emptyset) + f_{C'_n}(\emptyset) = \sum_{n=0}^{\infty}\lambda C_n + \lambda C'_n \le \gamma,$$

and the $\mathcal{V}'_n$ were chosen such that

$$f_V(\emptyset) = \lambda V \le \lambda W_n = f_{W_n}(\emptyset)$$

for every $n \in \mathbb{N}$, $V \in \mathcal{V}_n$; that is, $\emptyset \in P_0$.

(iv) Now if $k \in \mathbb{N}$ and $v \in P_k$ there is a $v' \in P_{k+1}$ extending v. **P** If $J_{k+1} = J_k$ we can take $v' = v$. Otherwise, $J_{k+1} = J_k \cup \{j\}$ for some $j \in I \setminus J_k$. Now

$$\gamma \ge \sum_{n=0}^{\infty} f_{C_n}(v) + f_{C'_n}(v) = \sum_{n=0}^{\infty}\int f_{C_n}(v^\frown{<}t{>}) + f_{C'_n}(v^\frown{<}t{>})\,\mu_j(dt)$$

((b-i) above)

$$= \int\sum_{n=0}^{\infty} f_{C_n}(v^\frown{<}t{>}) + f_{C'_n}(v^\frown{<}t{>})\,\mu_j(dt),$$

so

$$H = \{t : t \in X_j,\, \sum_{n=0}^{\infty} f_{C_n}(v^\frown{<}t{>}) + f_{C'_n}(v^\frown{<}t{>})\mu_j(dt) \le \gamma\}$$

has positive measure.

Next, for $V \in \mathcal{U}_s$, set $g_V(t) = f_V(v^\frown{<}t{>})$ for each $t \in X_j$. Then g_V is lower semi-continuous, by (b-ii) above. For each $n \in \mathbb{N}$, $\{g_V : V \in \mathcal{V}_n\}$ is an upwards-directed family of lower semi-continuous functions, so its supremum g_n^* is lower semi-continuous, and because μ_j is τ-additive,

$$\int g_n^* d\mu_j = \sup_{V\in\mathcal{V}_n} \int g_V d\mu_j = \sup_{V\in\mathcal{V}_n} f_V(v) \le f_{W_n}(v) = \int f_{W_n}(v^\frown{<}t{>})\mu_j(dt)$$

(using 414B and (b-i) again). But also, because $\langle V_{nr}\rangle_{r\in\mathbb{N}}$ is non-decreasing and has union W_n,

$$f_{W_n}(v^\frown{<}t{>}) = \sup_{r\in\mathbb{N}} f_{V_{nr}}(v^\frown{<}t{>}) \le g_n^*(t)$$

for every $t \in X_j$. So we must have

$$f_{W_n}(v^\frown{<}t{>}) = g_n^*(t) \text{ a.e.}(t).$$

And this is true for every $n \in \mathbb{N}$.

There is therefore a $t \in H$ such that

$$f_{W_n}(v^\frown{<}t{>}) = g_n^*(v^\frown{<}t{>}) \text{ for every } n \in \mathbb{N}.$$

Fix on such a t and set $v' = v^\frown{<}t{>} \in \prod_{i\in J_{k+1}} X_i$; then $v' \in P_{k+1}$, as required. **Q**

(v) We can therefore choose a sequence $\langle v_k\rangle_{k\in\mathbb{N}}$ such that $v_k \in P_k$ and v_{k+1} extends v_k for each k. Choose $x \in X$ such that $x(i) = v_k(i)$ whenever $k \in \mathbb{N}$ and $i \in J_k$, and $x(i)$ belongs to the support of μ_i whenever $i \in I \setminus J$. (Once again, 411Nd tells us that every μ_i has a support.)

We need to know that if k, $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ then $f_{V\setminus W_n}(v_k) = 0$. **P** For any $r \in \mathbb{N}$ there is a $V' \in \mathcal{V}_n$ such that $V \cup V_{nr} \subseteq V'$, so

$$f_{V\cup V_{nr}}(v_k) \le f_{V'}(v_k) \le f_{W_n}(v_k),$$

and

$$f_{V\setminus W_n}(v_k) \le f_{V\setminus V_{nr}}(v_k) = f_{V\cup V_{nr}}(v_k) - f_{V_{nr}}(v_k) \le f_{W_n}(v_k) - f_{V_{nr}}(v_k) \to 0$$

as $r \to \infty$. **Q**

(vi) If $n \in \mathbb{N}$, then $x \notin C_n \cup C'_n$. **P** C_n and C'_n are determined by coordinates in a finite subset of J, so must be determined by coordinates in J_k for some $k \in \mathbb{N}$. Now $f_{C_n}(v_k) + f_{C'_n}(v_k) \le \gamma < 1$, so $\{y : y{\restriction}J_k = v_k\}$ cannot be included in $C_n \cup C'_n$, and must be disjoint from it; accordingly $x \notin C_n \cup C'_n$. **Q**

(vii) Because

$$(X \setminus \bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) \cup \bigcup_{n\in\mathbb{N}} W'_n \setminus W_n \subseteq \bigcup_{n\in\mathbb{N}} C_n \cup \bigcup_{n\in\mathbb{N}} C'_n,$$

there is some $n \in \mathbb{N}$ such that

$$x \in A(\mathcal{V}_n) \setminus (W'_n \setminus W_n) \subseteq (\bigcup \mathcal{V}_n) \setminus W'_n,$$

that is, there is some $V \in \mathcal{V}_n$ such that $x \in V \setminus W'_n$. Let $U \in \mathcal{U}$ be such that $x \in U \subseteq V$. Express U as $U' \cap U''$ where $U' \in \mathcal{U}$ is determined by coordinates in a finite subset of J and $U'' \in \mathcal{U}$ is determined by coordinates in a finite subset of $I \setminus J$. Let $k \in \mathbb{N}$ be such that U' is determined by coordinates in J_k. Then

$$f_{U\setminus W_n}(v_k) \le f_{V\setminus W_n}(v_k) = 0$$

by (v) above. Now

$$\{w : w \in \prod_{i\in I\setminus J_k} X_i,\, (v_k, w) \in U \setminus W_n\} = \{w : (v_k, w) \in U'' \setminus W_n\}$$

(because $(v_k, w) = (x{\restriction}J_k, w) \in U'$ for every w), while

$$\{w : (v_k, w) \in U''\}, \quad \{w : (v_k, w) \in W_n\}$$

are stochastically independent because the former is determined by coordinates in $I\setminus J$, while the latter is determined by coordinates in $J \setminus J_k$. So we must have

$$\begin{aligned} 0 = f_{U\setminus W_n}(v_k) &= \lambda_{I\setminus J_k}\{w : (v_k, w) \in U \setminus W_n\} \\ &= \lambda_{I\setminus J_k}\{w : (v_k, w) \in U'' \setminus W_n\} \\ &= \lambda_{I\setminus J_k}\{w : (v_k, w) \in U''\}(1 - \lambda_{I\setminus J_k}\{w : (v_k, w) \in W_n\}). \end{aligned}$$

At this point, recall that $x(i)$ belongs to the support of μ_i for every $i \in I\setminus J$, while $x \in U''$. So if $U'' = \{y : y(i) \in H_i$ for $i \in K\}$, where $K \subseteq I \setminus J$ is finite and $H_i \subseteq X_i$ is open for every i, we must have $\mu_i H_i > 0$ for every i, and

$$\lambda_{I\setminus J_k}\{w : (v_k, w) \in U''\} = \prod_{i\in K} \mu_i H_i > 0.$$

On the other hand, we are also supposing that $x \notin W'_n$, so

$$\lambda_{I\setminus J_k}\{w : (v_k, w) \in W_n\} = f_{W_n}(v_k) = f_{W_n}(x\restriction J_k) < 1.$$

But this means that we have expressed 0 as the product of two non-zero numbers, which is absurd. **X**

(d) Thus $\lambda_*(\bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) = 0$ for every sequence $\langle\mathcal{V}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{V}$. Accordingly there is an extension of λ to a measure $\tilde\lambda$ on X as in 417A.

Now $\tilde\lambda$ is a topological measure. **P** If $W \subseteq X$ is open, then $\mathcal{V} = \{V : V \in \mathcal{U}_s,\ V \subseteq W\}$ belongs to $\mathfrak{V}$, and $\bigcup\mathcal{V} = W$. Since $\bigcup\mathcal{V}' \in \Lambda$ (because $\mathcal{V}'$ is countable),

$$W = \bigcup\mathcal{V}' \cup A(\mathcal{V})$$

is measured by $\tilde\lambda$. **Q**

Also, $\tilde\lambda$ is τ-additive. **P** Let $\mathcal{W}$ be a non-empty upwards-directed family of open subsets of X with union W^*. Set

$$\mathcal{V} = \{V : V \in \mathcal{U}_s,\ \exists\, W \in \mathcal{W},\ V \subseteq W\}.$$

Then $\mathcal{V} \in \mathfrak{V}$ and $\bigcup\mathcal{V} = W^*$, so $\tilde\lambda A(\mathcal{V}) = 0$ and

$$\tilde\lambda W^* = \tilde\lambda(\bigcup\mathcal{V}') = \sup_{V\in\mathcal{V}'} \tilde\lambda V \le \sup_{W\in\mathcal{W}} \tilde\lambda W \le \tilde\lambda W^*$$

(using the fact that $\mathcal{V}'$ is upwards-directed). As $\mathcal{W}$ is arbitrary, $\tilde\lambda$ is τ-additive. **Q**

Of course it follows at once that λ also is τ-additive.

(e) Now for the supplementary properties (i)-(vi) listed in the theorem.

(i) Because λ is complete, so is $\tilde\lambda$, by 417A(ii).

(ii) As always, the construction ensures that every member of $\tilde\Lambda$ differs by a $\tilde\lambda$-negligible set from some member of Λ.

(iii) Let $W \subseteq X$ be an open set. Set $\mathcal{V} = \{V : V \in \mathcal{U}_s,\ V \subseteq W\}$. Then

$$\tilde\lambda W = \sup_{V\in\mathcal{V}} \tilde\lambda V = \sup_{V\in\mathcal{V}} \lambda V \le \lambda_* W \le \tilde\lambda W$$

just because $\tilde\lambda$ is a τ-additive extension of λ. Now if $F \subseteq X$ is closed,

$$\tilde\lambda F = 1 - \tilde\lambda(X \setminus F) = 1 - \lambda_*(X \setminus F) = \lambda^* F.$$

(iv) For each $i \in I$ write Z_i for the support of μ_i, and set $Z = \prod_{i\in I} Z_i$. This is closed because every Z_i is. Its complement is covered by the negligible open sets $\{x : x \in X,\ x(i) \in X_i \setminus Z_i\}$ as i runs over I; as $\tilde\lambda$ is τ-additive, the union of the negligible open sets is negligible, and Z is conegligible. If $W \subseteq X$ is open and $x \in Z \cap W$, let $U \in \mathcal{U}$ be such that $x \in U \subseteq W$. Express U as $\prod_{i\in I} G_i$ where $G_i \in \mathfrak{T}_i$ for every $i \in I$ and $J = \{i : G_i \ne X_i\}$ is finite. Then $x(i) \in G_i \cap Z_i$, so $\mu_i G_i > 0$, for every i. Accordingly

$$\tilde\lambda(W \cap Z) = \tilde\lambda W \ge \lambda U = \prod_{i\in J} \mu_i G_i > 0.$$

Thus Z is self-supporting and is the support of λ.

(v), (vi), (vii) As in the proof of 417C, apply 417A(iv) with $\mathcal{G}$ the family of open subsets of X and $\mathcal{K}$ either the Borel σ-algebra of X, or the family of closed subsets of X, or the family of closed compact subsets of X, this time using 412U to confirm that λ is inner regular with respect to $\mathcal{K}$.

417F Corollary Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that μ_i is inner regular with respect to the Borel sets for each i. Then there is a unique complete τ-additive topological measure $\tilde{\lambda}$ on $X = \prod_{i\in I} X_i$ which extends the ordinary product measure and is inner regular with respect to the Borel sets.

proof By 417E(v) we have a measure $\tilde{\lambda}$ with the right properties. If λ' is any other complete τ-additive topological measure, extending λ and inner regular with respect to the family $\mathcal{B}$ of Borel sets, then $\lambda'W = \tilde{\lambda}W$ for every open set $W \subseteq X$. **P** By the argument of (e-iii) of the proof of 417E, $\lambda'W = \lambda_*W = \tilde{\lambda}W$. **Q** By 414L, applied to the Borel measures $\lambda'\restriction\mathcal{B}$ and $\tilde{\lambda}\restriction\mathcal{B}$, $\lambda'W = \tilde{\lambda}W$ for every Borel set W. Now λ' and $\tilde{\lambda}$ are supposed to be complete topological probability measures inner regular with respect to $\mathcal{B}$, so they must be identical, by 412L or otherwise.

417G Notation In the context of 417D or 417F, I will call $\tilde{\lambda}$ the **τ-additive product measure** on $\prod_{i\in I} X_i$.

Note that the uniqueness assertions in 417D and 417F mean that for the products of finitely many probability spaces we do not need to distinguish between the two constructions. The latter also shows that we can relate 415E to the new method: if every $\mathfrak{T}_i$ is separable and metrizable and every μ_i is strictly positive, then the 'ordinary' product measure λ is a complete topological measure. Since it is also inner regular with respect to the Borel sets (412Uc), and τ-additive (because we now know that it has an extension to a τ-additive measure) it must be exactly the τ-additive product measure as described here.

417H Fubini's theorem for τ-additive product measures Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be two complete locally determined effectively locally finite τ-additive topological measure spaces such that both μ and ν are inner regular with respect to the Borel sets. Let $\tilde{\lambda}$ be the τ-additive product measure on $X\times Y$, and $\tilde{\Lambda}$ its domain.

(a) Let f be a $[-\infty,\infty]$-valued function such that $\int f\,d\tilde{\lambda}$ is defined in $[-\infty,\infty]$ and $(X\times Y)\setminus\{(x,y):(x,y)\in \operatorname{dom} f,\ f(x,y)=0\}$ can be covered by a set of the form $X\times\bigcup_{n\in\mathbb{N}} Y_n$ where $\nu Y_n<\infty$ for every $n\in\mathbb{N}$. Then the repeated integral $\iint f(x,y)\nu(dy)\mu(dx)$ is defined and equal to $\int f d\tilde{\lambda}$.

(b) Let $f : X\times Y\to[0,\infty]$ be lower semi-continuous. Then

$$\iint f(x,y)\nu(dy)\mu(dx) = \iint f(x,y)\mu(dx)\nu(dy) = \int f d\tilde{\lambda}$$

in $[0,\infty]$.

(c) Let f be a $\tilde{\Lambda}$-measurable real-valued function defined $\tilde{\lambda}$-a.e. on $X\times Y$. If either $\iint|f(x,y)|\nu(dy)\mu(dx)$ or $\iint|f(x,y)|\mu(dx)\nu(dy)$ is defined and finite, then f is $\tilde{\lambda}$-integrable.

proof (a) I use 252B.

(i) Write $\mathcal{W}$ for the set of those $W\in\tilde{\Lambda}$ such that $\int\nu W[\{x\}]\mu(dx)$ is defined in $[0,\infty]$ and equal to $\tilde{\lambda}W$. Then open sets belong to $\mathcal{W}$, by 417C(iv). Next, any Borel subset of an open set of finite measure belongs to $\mathcal{W}$. **P** If W_0 is an open set of finite measure, then $\{W : W\subseteq X\times Y,\ W\cap W_0\in\mathcal{W}\}$ is a Dynkin class containing every open set, so contains all Borel subsets of $X\times Y$. **Q**

Now suppose that $W\subseteq X\times Y$ is $\tilde{\lambda}$-negligible and included in $X\times\bigcup_{n\in\mathbb{N}} Y_n$, where $\nu Y_n<\infty$ for every n. Then $W\in\mathcal{W}$. **P** Set $A=\{x : x\in X,\ \nu^*W[\{x\}]>0\}$. For each n, let $H_n\subseteq Y$ be an open set of finite measure such that $\nu(Y_n\setminus H_n)\le 2^{-n}$; we may arrange that $H_{n+1}\supseteq H_n$ for each n. Set $H=\bigcup_{n\in\mathbb{N}} H_n$, so that $W[\{x\}]\setminus H\subseteq\bigcup_{n\in\mathbb{N}} Y_n\setminus H$ is negligible for every $x\in X$.

Fix an open set $G\subseteq X$ of finite measure and $n\in\mathbb{N}$ for the moment. Because $\tilde{\lambda}$ is inner regular with respect to the Borel sets, there is a Borel set $V\subseteq(G\times H_n)\setminus W$ such that $\tilde{\lambda}V=\tilde{\lambda}((G\times H_n)\setminus W)$, that is, $\tilde{\lambda}V'=0$, where $V'=(G\times H_n)\setminus V\supseteq(G\times H_n)\cap W$. We know that $V'\in\mathcal{W}$, so

$$\int\nu V'[\{x\}]dx=\tilde{\lambda}V'=0,$$

and $\nu V'[\{x\}]=0$ for almost every $x\in X$; but this means that $H_n\cap W[\{x\}]$ is negligible for almost every $x\in G$.

At this point, recall that n was arbitrary, so $H\cap W[\{x\}]$ and $W[\{x\}]$ are negligible for almost every $x\in G$, that is, $A\cap G$ is negligible. This is true for every open set $G\subseteq X$ of finite measure. Because μ is inner regular with respect to subsets of open sets of finite measure, and is complete and locally determined, A is negligible (412Jb). But this means that $\int\nu W[\{x\}]\mu(dx)$ is defined and equal to zero, so that $W\in\mathcal{W}$. **Q**

(ii) Now suppose that $\int f d\tilde{\lambda}$ is defined in $[-\infty,\infty]$ and that there is a sequence $\langle Y_n\rangle_{n\in\mathbb{N}}$ of sets of finite measure in Y such that $f(x,y)$ is defined and zero whenever $x\in X$ and $y\in Y\setminus\bigcup_{n\in\mathbb{N}} Y_n$. Set $Z=\bigcup_{n\in\mathbb{N}} Y_n$. Write λ for the c.l.d. product measure on $X\times Y$ and Λ for its domain. Then there is a Λ-measurable function $g : X\times Y\to[-\infty,\infty]$ which is equal $\tilde{\lambda}$-almost everywhere to f. **P** For $q\in\mathbb{Q}$ set $W_q=\{(x,y):(x,y)\in\operatorname{dom} f,\ f(x,y)\ge q\}\in\tilde{\Lambda}$, and choose $V_q\in\Lambda$ such that $\tilde{\lambda}(W_q\triangle V_q)=0$ (417C(ii)); set $g(x,y)=\sup\{q : q\in\mathbb{Q},\ (x,y)\in V_q\}$ for $x\in X$, $y\in Y$, interpreting $\sup\emptyset$ as $-\infty$. **Q** Adjusting g if necessary, we may suppose that it is zero on $X\times(Y\setminus Z)$. Set

$$A = (X \times Y) \setminus \{(x, y) : f(x, y) = g(x, y)\},$$

so that A is $\tilde{\lambda}$-negligible and included in $X \times Z$. By (i), $\nu A[\{x\}] = 0$, that is, $y \mapsto f(x, y)$ and $y \mapsto g(x, y)$ are equal ν-a.e., for μ-almost every x. Write $\lambda_{X\times Z}$ for the subspace measure induced by λ on $X \times Z$; note that this is the c.l.d. product of μ with the subspace measure ν_Z on Z, by 251Q(ii-α).

Now we have

$$\int f d\tilde{\lambda} = \int g\, d\tilde{\lambda} = \int g\, d\lambda$$

(by 235Gb, because the identity map from $(X \times Y, \tilde{\lambda})$ to $(X \times Y, \lambda)$ is inverse-measure-preserving)

$$= \int_{X\times Z} g\, d\lambda = \int_{X\times Z} g\, d\lambda_{X\times Z} = \iint_Z g(x, y)\nu_Z(dy)\mu(dx)$$

(by 252B, because ν_Z is σ-finite)

$$= \iint g(x, y)\nu(dy)\mu(dx)$$

(because $g(x, y) = 0$ if $y \in Y \setminus Z$)

$$= \iint f(x, y)\nu(dy)\mu(dx).$$

(b) If f is non-negative and lower semi-continuous, set

$$W_{ni} = \{(x, y) : f(x, y) > 2^{-n}i\}$$

for n, $i \in \mathbb{N}$, and

$$f_n = 2^{-n}\textstyle\sum_{i=1}^{4^n} \chi W_{ni}$$

for $n \in \mathbb{N}$. Applying 417C(iv) we see that

$$\textstyle\int f_n d\tilde{\lambda} = \iint f_n(x, y)dydx = \iint f_n(x, y)dxdy$$

in $[0, \infty]$ for every n; taking the limit as $n \to \infty$,

$$\textstyle\int f d\tilde{\lambda} = \iint f(x, y)dydx = \iint f(x, y)dxdy,$$

because $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with limit f.

(c) ? Suppose, if possible, that $\gamma = \iint |f(x, y)|dydx$ is finite, but that f is not integrable. Because $\tilde{\lambda}$ is semi-finite, there must be a non-negative $\tilde{\lambda}$-simple function g such that $g \leq_{\text{a.e.}} |f|$ and $\int g\, d\tilde{\lambda} > \gamma$ (213B). For each $n \in \mathbb{N}$, there are open sets $G_n \subseteq X$, $H_n \subseteq Y$ of finite measure such that $\tilde{\lambda}(\{(x, y) : g(x, y) \geq 2^{-n}\} \setminus (G_n \times H_n)) \leq 2^{-n}$, by 417C(iii); now $g \times \chi(G_n \times H_n) \to g$ a.e., so there is some n such that $\int_{G_n\times H_n} g\, d\tilde{\lambda} > \gamma$. In this case, setting $g'(x, y) = \min(g(x, y), |f(x, y)|)$ for $(x, y) \in (G_n \times H_n) \cap \operatorname{dom} f$, 0 otherwise, we have $g = g'$ a.e. on $G_n \times H_n$, so that $\int g' d\tilde{\lambda} > \gamma$. But we can apply (a) to g' to see that

$$\textstyle\gamma < \int g' d\tilde{\lambda} = \iint g'(x, y)dydx \leq \iint |f(x, y)|dydx \leq \gamma,$$

which is absurd. **X**

So if $\iint |f(x, y)|dydx$ is finite, f must be $\tilde{\lambda}$-integrable. Of course the same arguments, reversing the roles of X and Y, show that f is $\tilde{\lambda}$-integrable if $\iint |f(x, y)|dxdy$ is defined and finite.

417I The constructions here have most of the properties one would hope for. I give several in the exercises (417Xd-417Xf, 417Xj). One fact which is particularly useful, and also has a trap in it, is the following.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces in which the measures are inner regular with respect to the Borel sets, and $\tilde{\lambda}$ the τ-additive product measure on $X \times Y$. Suppose that $A \subseteq X$ and $B \subseteq Y$, and write μ_A, ν_B for the corresponding subspace measures; assume that both μ_A and ν_B are semi-finite. Then these are also effectively locally finite, τ-additive and inner regular with respect to the Borel sets, and the subspace measure $\tilde{\lambda}_{A\times B}$ induced by $\tilde{\lambda}$ on $A \times B$ is just the τ-additive product measure of μ_A and ν_B.

proof (a) To check that μ_A and ν_B are effectively locally finite, τ-additive and inner regular with respect to the Borel sets, see 414K and 412P. Of course $\tilde{\lambda}_{A\times B}$ inherits the same properties from $\tilde{\lambda}$, and is in addition complete and strictly localizable, because $\tilde{\lambda}$ is (417C(i), 214Ia).

(b) Now if $C \in \operatorname{dom}\mu_A$ and $D \in \operatorname{dom}\nu_B$, then $\tilde{\lambda}^*(C\times D) = \mu_A C\cdot\nu_B D$. **P** ($\alpha$) There are $E\in\Sigma$, $F\in\mathrm{T}$ such that $C\subseteq E$, $D\subseteq F$, $\mu E=\mu^*C$ and $\nu F=\nu^*D$; in which case

$$\tilde{\lambda}^*(C\times D)\le\tilde{\lambda}(E\times F)=\lambda(E\times F)=\mu E\cdot\nu F$$

(251J)

$$=\mu^*C\cdot\nu^*D=\mu_AC\cdot\nu_BD.$$

(β) If $\gamma<\mu_AC\cdot\nu_BD$ then, because μ_A and ν_B are semi-finite, there are $C'\subseteq C$, $D'\subseteq D$ such that both have finite outer measure and $\mu^*C'\cdot\nu^*D'\ge\gamma$. In this case, take $E'\in\Sigma$, $F'\in\mathrm{T}$ such that $C'\subseteq E'$, $D'\subseteq F'$ and both E' and F' have finite measure. Now if $W\in\operatorname{dom}\tilde{\lambda}$ and $C\times D\subseteq W$, we have $C'\times D'\subseteq W\cap(E\times F)$, so that $\nu(W\cap(E\times F))[\{x\}]\ge\nu^*D'$ for every $x\in C'$, and

$$\tilde{\lambda}W\ge\int_E\nu(W\cap(E\times F))[\{x\}]\mu(dx)\ge\mu^*C'\cdot\nu^*D'\ge\gamma,$$

by 417Ha. As W is arbitrary, $\tilde{\lambda}^*(C\times D)\ge\gamma$; as γ is arbitrary, $\tilde{\lambda}^*(C\times D)\ge\mu_AC\cdot\nu_BD$. **Q**

(c) In particular, if $U\subseteq A$ and $V\subseteq B$ are relatively open,

$$\tilde{\lambda}_{A\times B}(U\times V)=\tilde{\lambda}^*(U\times V)=\mu_AU\cdot\nu_BV.$$

But now 417D tells us that $\tilde{\lambda}_{A\times B}$ must be exactly the τ-additive product measure of μ_A and ν_B.

417J In order to use 417H effectively in the theory of infinite products, we need an 'associative law' corresponding to 254N.

Theorem Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $\langle K_j\rangle_{j\in J}$ a partition of I. For each $j\in J$ let $\tilde{\lambda}_j$ be the τ-additive product measure on $Z_j=\prod_{i\in K_j}X_i$, and write $\tilde{\lambda}$ for the τ-additive product measure on $X=\prod_{i\in I}X_i$. Then the natural bijection

$$x\mapsto\phi(x)=\langle x\restriction K_j\rangle_{j\in J}:X\to\prod_{j\in J}Z_j$$

identifies $\tilde{\lambda}$ with the τ-additive product of the family $\langle\tilde{\lambda}_j\rangle_{j\in J}$.

In particular, if $K\subseteq I$ is any set, then $\tilde{\lambda}$ can be identified with the τ-additive product of the τ-additive product measures on $\prod_{i\in K}X_i$ and $\prod_{i\in I\setminus K}X_i$.

proof We have a lot of measures to keep track of; I hope that the following notation will not be too opaque. Write λ for the ordinary product measure on X, and for $j\in J$ write λ_j, $\tilde{\lambda}_j$ for the ordinary and τ-additive product measures on Z_j. Write θ for the *ordinary* product measure on $Z=\prod_{j\in J}Z_j$ of the *τ-additive* product measures $\tilde{\lambda}_j$, and $\tilde{\theta}$ for the τ-additive product of the $\tilde{\lambda}_j$. Write $\tilde{\lambda}^\#$ for the measure on X corresponding to $\tilde{\theta}$ on Z. (If you like, $\tilde{\lambda}^\#$ is the image measure $\tilde{\theta}(\phi^{-1})^{-1}$ defined from $\tilde{\theta}$ and the function $\phi^{-1}:Z\to X$.) Then $\tilde{\lambda}^\#$, like $\tilde{\theta}$, is a complete τ-additive topological measure, inner regular with respect to the Borel sets, because $\phi:X\to Z$ is a homeomorphism. If $C\subseteq X$ is a measurable cylinder, it is of the form $\prod_{i\in I}E_i$ where $E_i\in\Sigma_i$ for each i and $\{i:i\in I,\,E_i\ne\Sigma_i\}$ is finite. So $\phi[C]$ is of the form $\prod_{j\in J}C_j$, where $C_j=\prod_{i\in K_j}E_i$, and

$$\begin{aligned}\tilde{\lambda}^\#C&=\tilde{\theta}(\prod_{j\in J}C_j)=\theta(\prod_{j\in J}C_j)=\prod_{j\in J}\tilde{\lambda}_jC_j\\&=\prod_{j\in J}\lambda_jC_j=\prod_{j\in J}\prod_{i\in K_j}\mu_iE_i=\prod_{i\in I}\mu_iE_i=\lambda C.\end{aligned}$$

But this means, applying 254G to the identity map from $(X,\tilde{\lambda}^\#)$ to (X,λ), that $\tilde{\lambda}^\#$ extends λ. So it is a complete τ-additive topological measure, inner regular with respect to the Borel sets, extending the ordinary product measure, and by the uniqueness declared in 417F, must be identical to the τ-additive product measure on X, as claimed.

417K Proposition Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $(X,\tilde\Lambda,\tilde\lambda)$ their τ-additive product. For $J\subseteq I$ let $\tilde\lambda_J$ be the τ-additive product measure on $X_J=\prod_{i\in J}X_i$, and $\tilde\Lambda_J$ its domain; let $\pi_J:X\to X_J$ be the canonical map. Then $\tilde\lambda_J$ is the image measure $\tilde\lambda\pi_J^{-1}$. In particular, if $W\in\tilde\Lambda$ is determined by coordinates in $J\subseteq I$, then $\pi_J[W]\in\tilde\Lambda_J$ and $\tilde\lambda_J\pi_J[W]=\tilde\lambda W$.

proof Because $\tilde\lambda$ is an extension of the ordinary product measure λ on X, $\tilde\lambda\pi_J^{-1}$ is an extension of $\lambda\pi_J^{-1}$, which is the ordinary product measure on X_J (254Oa). Because $\tilde\lambda$ is a τ-additive topological measure and π_J is continuous, $\tilde\lambda\pi_J^{-1}$ is a τ-additive topological measure; because $\tilde\lambda$ is a complete probability measure, so is $\tilde\lambda\pi_J^{-1}$. Finally, $\tilde\lambda\pi_J^{-1}$ is inner regular with respect to the Borel sets. **P** Recall that we may identify $\tilde\lambda$ with the τ-additive product of $\tilde\lambda_J$ and $\tilde\lambda_{I\setminus J}$ (417J). If $V\in\operatorname{dom}\tilde\lambda\pi_J^{-1}$, that is, $\pi_J^{-1}[V]\in\tilde\Lambda$, we can think of $\pi_J^{-1}[V]\subseteq X$ as $V\times X_{I\setminus J}\subseteq X_I\times X_J$. In this case, we must have

$$\tilde\lambda\pi_J^{-1}[V]=\int\tilde\lambda_J V\,d\lambda_{I\setminus J},$$

by Fubini's theorem for τ-additive products (417Ha); that is, $\tilde\lambda_J V$ must be defined and equal to $\tilde\lambda\pi_J^{-1}[V]$. Now if $\gamma<\tilde\lambda\pi_J^{-1}[V]$, there must be a Borel set $V'\subseteq V$ such that $\tilde\lambda_J V'\ge\gamma$. In this case, because π_J is continuous, $\pi_J^{-1}[V']$ also is Borel, and $\tilde\lambda\pi_J^{-1}[V']$ is defined. As with V, this measure must be $\tilde\lambda_J V'\ge\gamma$. Since V and γ are arbitrary, $\tilde\lambda\pi_J^{-1}$ is inner regular with respect to the Borel sets, as claimed. **Q**

By the uniqueness assertion in 417F, $\tilde\lambda\pi_J^{-1}$ must be $\tilde\lambda_J$ exactly.

If now $W\in\tilde\Lambda$ is determined by coordinates in J, then

$$\tilde\lambda_J\pi_J[W]=\tilde\lambda\pi_J^{-1}[\pi_J[W]]=\tilde\lambda W.$$

417L Corollary Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $(X,\tilde\Lambda,\tilde\lambda)$ their τ-additive product. Let $\langle K_j\rangle_{j\in J}$ be a disjoint family of subsets of I, and for $j\in J$ write $\tilde\Lambda_j$ for the σ-algebra of members of Λ determined by coordinates in K_j. Then $\langle\tilde\Lambda_j\rangle_{j\in J}$ is a stochastically independent family of σ-algebras (definition: 272Ab).

proof It is enough to consider the case in which J is finite (272Bb), no K_j is empty (since if $K_j=\emptyset$ then $\tilde\Lambda_j=\{\emptyset,X\}$) and $\bigcup_{j\in J}K_j=I$ (adding an extra term if necessary). In this case, if $W_j\in\tilde\Lambda_j$ for each j, then the identification between X and $\prod_{j\in J}\prod_{i\in K_j}X_i$, as described in 417J, matches $\bigcap_{j\in J}W_j$ with $\prod_{j\in J}\pi_{K_j}[W_j]$, writing $\pi_{K_j}(x)$ for $x\restriction K_j$. Now if $\tilde\lambda_j$ is the τ-additive product measure on $Z_j=\prod_{i\in K_j}X_i$, we have $\tilde\lambda_j\pi_{K_j}[W_j]=\tilde\lambda W_j$, by 417K. Since $\tilde\lambda$ can be identified with the τ-additive product of $\langle\tilde\lambda_j\rangle_{j\in J}$ (417J),

$$\tilde\lambda(\textstyle\bigcap_{j\in J}W_j)=\prod_{j\in J}\tilde\lambda_j\pi_{K_j}[W_j]=\prod_{j\in J}\tilde\lambda W_j.$$

As $\langle W_j\rangle_{j\in J}$ is arbitrary, $\langle\tilde\Lambda_j\rangle_{j\in J}$ is independent.

417M Proposition Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets and strictly positive. For $J\subseteq I$ let π_J be the canonical map from X onto $X_J=\prod_{i\in J}X_i$; write λ_J, $\tilde\lambda_J$ for the ordinary and τ-additive product measures on X_J, and Λ_J, $\tilde\Lambda_J$ for their domains. Set $\tilde\lambda=\tilde\lambda_I$, $\tilde\Lambda=\tilde\Lambda_I$, $\lambda=\lambda_I$, $\Lambda=\Lambda_I$.

(a) Let $F\subseteq X$ be a closed self-supporting set, and J the smallest subset of I such that F is determined by coordinates in J (4A2B(g-ii)). Then

(i) if $W\in\tilde\Lambda$ is such that $W\triangle F$ is $\tilde\lambda$-negligible and determined by coordinates in $K\subseteq I$, then $K\supseteq J$;

(ii) J is countable;

(iii) there is a $W\in\Lambda$, determined by coordinates in J, such that $W\triangle F$ is $\tilde\lambda$-negligible.

(b) $\tilde\lambda$ is inner regular with respect to the family of sets of the form $\bigcap_{n\in\mathbb{N}}V_n$ where each $V_n\in\tilde\Lambda$ is determined by finitely many coordinates.

(c) If $W\in\tilde\Lambda$, there are a countable $J\subseteq I$ and sets W', $W''\in\tilde\Lambda$, determined by coordinates in J, such that $W'\subseteq W\subseteq W''$ and $\tilde\lambda(W''\setminus W')=0$. Consequently $\tilde\lambda\pi_J^{-1}[\pi_J[W]]=\tilde\lambda W$.

proof (a)(i) ? Suppose, if possible, otherwise. Then F is not determined by coordinates in K, so there are $x\in F$, $y\in X\setminus F$ such that $x\restriction K=y\restriction K$. Let U be an open set containing y, disjoint from F, and of the form $\prod_{i\in I}G_i$, where $G_i\in\mathfrak{T}_i$ for every i and $L=\{i:G_i\ne X_i\}$ is finite. Set

$$U' = \{z : z \in X,\ z(i) \in G_i \text{ for every } i \in L \cap K\},$$

$$U'' = \{z : z(i) \in G_i \text{ for every } i \in L \setminus K\}.$$

Then $U' \cap W$ is determined by coordinates in K, while U'' is determined by coordinates in $I \setminus K$, so

$$0 = \tilde{\lambda}(F \cap U) = \tilde{\lambda}(W \cap U) = \tilde{\lambda}(W \cap U' \cap U'') = \tilde{\lambda}(W \cap U') \cdot \tilde{\lambda}U''$$

(by 417L)

$$= \tilde{\lambda}(F \cap U') \cdot \tilde{\lambda}U'' = \tilde{\lambda}(F \cap U') \cdot \prod_{i \in L \setminus K} \mu_i G_i.$$

But $y \in U'$, and $x{\restriction}K = y{\restriction}K$, so $x \in F \cap U'$; as F is self-supporting, $\tilde{\lambda}(F \cap U') > 0$. Because every μ_i is strictly positive, and no G_i is empty, $\prod_{i \in L \setminus K} \mu_i G_i > 0$; and this is impossible. **X**

(ii) By 417E(ii), there is a $W_0 \in \Lambda$ such that $\tilde{\lambda}(F \triangle W_0) = 0$. By 254Oc there is a $W_1 \in \Lambda$, determined by coordinates in a countable subset K of I, such that $\lambda(W_0 \triangle W_1) = 0$. Now $\tilde{\lambda}(F \triangle W_1) = 0$, so (i) tells us that $J \subseteq K$ is countable.

(iii) By 417K, $\pi_J[F] \in \tilde{\Lambda}_J$. By 417E(ii), there is a $V \in \Lambda_J$ such that $V \triangle \pi_J[F]$ is $\tilde{\lambda}_J$-negligible. Set $W = \pi_J^{-1}[V]$. Then $W \in \Lambda$, W is determined by coordinates in J, and $W \triangle F = \pi_J^{-1}[V \triangle \pi_J[F]]$ is $\tilde{\lambda}$-negligible.

(b)(i) Write $\mathcal{V}$ for the set of those members of $\tilde{\Lambda}$ which are determined by finitely many coordinates, and $\mathcal{V}_\delta$ for the set of intersections of sequences in $\mathcal{V}$. Because $\mathcal{V}$ is closed under finite unions, so is $\mathcal{V}_\delta$; $\mathcal{V}_\delta$ is surely closed under countable intersections, and $\emptyset$, X belong to $\mathcal{V}_\delta$.

(ii) We need to know that every self-supporting closed set $F \subseteq X$ belongs to $\mathcal{V}_\delta$. **P** By (a), F is determined by a countable set J of coordinates. Express J as the union of a non-decreasing sequence $\langle J_n \rangle_{n \in \mathbb{N}}$ of finite sets. Then $F_n = \pi_{J_n}^{-1}[\overline{\pi_{J_n}[F]}] \in \mathcal{V}$ for each n, and $F = \bigcap_{n \in \mathbb{N}} F_n \in \mathcal{V}_\delta$. **Q**

(iii) Let $\mathcal{A}$ be the family of subsets of X which are either open or closed. Then if $A \in \mathcal{A}$, $V \in \tilde{\Lambda}$ and $\tilde{\lambda}(A \cap V) > 0$, there is a $K \in \mathcal{V}_\delta \cap \mathcal{A}$ such that $K \subseteq A$ and $\tilde{\lambda}(K \cap V) > 0$. **P** (α) If A is open, set

$$\mathcal{U} = \{U : U \in \mathcal{V} \text{ is open}, U \subseteq A\}.$$

Because $\mathcal{V}$ includes a base for the topology of X, $\bigcup \mathcal{U} = A$; because $\tilde{\lambda}$ is τ-additive and $\mathcal{V}$ is closed under finite unions, there is a $U \in \mathcal{U}$ such that $U \subseteq A$ and $\tilde{\lambda}U > \tilde{\lambda}A - \tilde{\lambda}(A \cap V)$, so that $\tilde{\lambda}(U \cap V) > 0$. (β) If A is closed, then it includes a self-supporting closed set V of the same measure (414F), which belongs to $\mathcal{V}_\delta$, by (ii) just above. **Q**

(iv) By 412C, the restriction $\tilde{\lambda}{\restriction}\mathcal{B}$ of $\tilde{\lambda}$ to the Borel σ-algebra of X is inner regular with respect to $\mathcal{V}_\delta$. But $\tilde{\lambda}$ is just the completion of $\tilde{\lambda}{\restriction}\mathcal{B}$, so it also is inner regular with respect to $\mathcal{V}_\delta$ (412Ha).

(c) By (b), we have sequences $\langle V_n \rangle_{n \in \mathbb{N}}$, $\langle V'_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{V}_\delta$ such that $V_n \subseteq W$, $V'_n \subseteq X \setminus W$, $\tilde{\lambda}V_n \ge \tilde{\lambda}W - 2^{-n}$ and $\tilde{\lambda}V'_n \ge \tilde{\lambda}(X \setminus W) - 2^{-n}$ for every $n \in \mathbb{N}$. Each V_n, V'_n is determined by a countable set of coordinates, so there is a single countable set $J \subseteq I$ such that every V_n and every V'_n is determined by coordinates in J. Set $W' = \bigcup_{n \in \mathbb{N}} V_n$, $W'' = X \setminus \bigcup_{n \in \mathbb{N}} V'_n$; then W', W'' are both determined by coordinates in J, $W' \subseteq W \subseteq W''$ and $\tilde{\lambda}(W'' \setminus W') = 0$, as required.

417N Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be two quasi-Radon measure spaces. Then the τ-additive product measure $\tilde{\lambda}$ on $X \times Y$ is a quasi-Radon measure, the unique quasi-Radon measure on $X \times Y$ such that $\tilde{\lambda}(E \times F) = \mu E \cdot \nu F$ for every $E \in \Sigma$ and $F \in \mathrm{T}$.

proof $\tilde{\lambda}$ is a complete, locally determined, effectively locally finite, τ-additive topological measure, inner regular with respect to the closed sets (417C(vii)). But this says just that it is a quasi-Radon measure. By 417D, it is the unique quasi-Radon measure with the right values on measurable rectangles.

417O Theorem Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of quasi-Radon probability spaces. Then the τ-additive product measure $\tilde{\lambda}$ on $X = \prod_{i \in I} X_i$ is a quasi-Radon measure, the unique quasi-Radon measure on X extending the ordinary product measure.

proof By 417E(vi), $\tilde{\lambda}$ is inner regular with respect to the closed sets, so is a quasi-Radon measure, which is unique by 417F.

417P Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces. Then the τ-additive product measure $\tilde\lambda$ on $X\times Y$ is a Radon measure, the unique Radon measure on $X\times Y$ such that $\tilde\lambda(E\times F)=\mu E\cdot\nu F$ whenever $E\in\Sigma$ and $F\in\mathrm{T}$.

proof Of course $X\times Y$ is Hausdorff, and $\tilde\lambda$ is locally finite (because $\tilde\lambda(G\times H)=\mu G\cdot\nu H$ is finite whenever μG and νH are finite). By 417C(viii), $\tilde\lambda$ is tight, so is a Radon measure. As in 417N, it is uniquely defined by its values on measurable rectangles.

417Q Theorem Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of Radon probability spaces, and $\tilde\lambda$ the τ-additive product measure on $X=\prod_{i\in I}X_i$. For each $i\in I$, let $Z_i\subseteq X_i$ be the support of μ_i. Suppose that $J=\{i:i\in I,\,Z_i$ is not compact$\}$ is countable. Then $\tilde\lambda$ is a Radon measure, the unique Radon measure on X extending the ordinary product measure.

proof Of course X, being a product of Hausdorff spaces, is Hausdorff, and $\tilde\lambda$, being totally finite, is locally finite. Now, given $\epsilon\in\,]0,1]$, let $\langle\epsilon_j\rangle_{j\in J}$ be a family of strictly positive numbers such that $\sum_{j\in J}\epsilon_j\le\epsilon$, and for $j\in J$ choose a compact set $K_j\subseteq X_j$ such that $\mu_jK_j\ge1-\epsilon_j$; for $i\in I\setminus J$, set $K_i=Z_i$, so that K_i is compact and $\mu_iK_i=1$. Consider $K=\prod_{i\in I}K_i$. Then, using 417E(iii) and 254Lb for the two equalities,

$$\tilde\lambda K=\lambda^*K=\prod_{i\in I}\mu_iK_i\ge\prod_{j\in J}1-\epsilon_j\ge1-\epsilon,$$

where λ is the ordinary product measure on X. As ϵ is arbitrary, $\tilde\lambda$ satisfies the condition (iv) of 416C, and is a Radon measure. As in 417F, it is the unique Radon measure on X extending λ.

417R Notation I will use the phrase **quasi-Radon product measure** for a τ-additive product measure which is in fact a quasi-Radon measure; similarly, a **Radon product measure** is a τ-additive product measure which is a Radon measure.

417S Later I will give an example in which a τ-additive product measure is different from the corresponding c.l.d. product measure (419E). In 415E-415F, 415Ye and 416U I have described cases in which c.l.d. measures are τ-additive product measures. It remains very unclear when to expect this to happen. I can however give a couple of results which show that sometimes, at least, we can be sure that the two measures coincide.

Proposition (a) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be effectively locally finite τ-additive topological measure spaces such that both μ and ν are inner regular with respect to the Borel sets, and λ the c.l.d. product measure on $X\times Y$. If every open subset of $X\times Y$ is measured by λ, then λ is the τ-additive product measure on $X\times Y$.

(b) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and λ the ordinary product measure on $X=\prod_{i\in I}X_i$. If every open subset of X is measured by λ, then λ is the τ-additive product measure on X.

(c) In (b), let λ_J be the ordinary product measure on $X_J=\prod_{i\in J}X_i$ for each $J\subseteq I$, and $\tilde\lambda_J$ the τ-additive product measure. If $\lambda_J=\tilde\lambda_J$ for every finite $J\subseteq I$, and every μ_i is strictly positive, then $\lambda=\tilde\lambda_I$ is the τ-additive product measure on X.

proof (a), (b) In both cases, λ is a complete locally determined effectively locally finite τ-additive measure which is inner regular with respect to the Borel sets (assembling facts from 251I, 254F, 412S, 412U, 417C and 417E). The extra hypothesis added here is that λ is a topological measure, so itself satisfies the conditions of 417D or 417F, and is the τ-additive product measure.

(c)(i) The first step is to note that $\lambda_J=\tilde\lambda_J$ for every countable $J\subseteq I$. **P** Express J as $\bigcup_{n\in\mathbb{N}}J_n$ where $\langle J_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of finite sets. If $F\subseteq X_J$ is closed, then it is $\bigcap_{n\in\mathbb{N}}\pi_n^{-1}[\overline{\pi_n[F]}]$, where $\pi_n:X_J\to X_{J_n}$ is the canonical map for each n. But every $\overline{\pi_n[F]}$ is a closed subset of X_{J_n}, therefore measured by λ_{J_n}; because π_n is inverse-measure-preserving (417K), $\pi_n^{-1}[\overline{\pi_n[F]}]\in\operatorname{dom}\lambda_J$ for each n, and $F\in\operatorname{dom}\lambda_J$. Thus every closed set, therefore every open set is measured by λ_J, and λ_J is a topological measure; by (b), $\lambda_J=\tilde\lambda_J$. **Q**

(ii) Suppose that $W\subseteq X$ is open. By 417M, there are W', W'' measured by $\tilde\lambda$ such that $W'\subseteq W\subseteq W''$, both W'' and W' are determined by coordinates in a countable set, and $\tilde\lambda_I(W''\setminus W')=0$. Let $J\subseteq I$ be a countable set such that W' and W'' are determined by coordinates in J. Then $\lambda_J=\tilde\lambda_J$ measures $\pi_J[W']$, by 417K, so λ measures $W'=\pi_J^{-1}[\pi_J[W]]$, by 254Oa. Similarly, λ measures W''. Now $\lambda(W''\setminus W')=\tilde\lambda_I(W''\setminus W')=0$, so λ measures W. As W is arbitrary, λ is a topological measure and must be the τ-additive product measure, by (a).

417T Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be effectively locally finite τ-additive topological measure spaces such that both μ and ν are inner regular with respect to the Borel sets, and λ the c.l.d. product measure on $X\times Y$. If X has a conegligible subset with a countable network (e.g., if X is metrizable and μ is σ-finite), then λ is the τ-additive product measure on $X\times Y$.

proof (a) Suppose to begin with that μ and ν are totally finite, and that X itself has a countable network; let $\langle A_n\rangle_{n\in\mathbb{N}}$ run over a network for X. Let $\hat\mu$ be the completion of μ and $\hat\Sigma$ its domain. Let $\tilde\lambda$ be the τ-additive product measure on $X\times Y$. (We are going to need Fubini's theorem both for λ and for $\tilde\lambda$. I will use a sprinkling of references to §§251-252 to indicate which parts of the argument below depend on the properties of λ.)

Let $W\subseteq X\times Y$ be an open set. For each $n\in\mathbb{N}$, set

$$H_n=\bigcup\{H:H\in\mathfrak{S},\,A_n\times H\subseteq W\},$$

so that H_n is open. Then $W=\bigcup_{n\in\mathbb{N}}A_n\times H_n$. **P** Of course $A_n\times H_n\subseteq W$ for every $n\in\mathbb{N}$. If $(x,y)\in W$, there are open sets $G\subseteq X$, $H\subseteq Y$ such that $(x,y)\in G\times H\subseteq W$; now there is an $n\in\mathbb{N}$ such that $x\in A_n\subseteq G$, so that $H\subseteq H_n$ and $(x,y)\in A_n\times H_n$. **Q**

By 417C(iv), there is an open set W_0 in the domain Λ of λ such that $W_0\subseteq W$ and $\tilde\lambda(W\setminus W_0)=0$. By 417Ha, applied to $\chi(W\setminus W_0)$, $A=\{x:\nu(W[\{x\}]\setminus W_0[\{x\}])>0\}$ is μ-negligible. For each $n\in\mathbb{N}$, $x\in X$ set $f_n(x)=\nu(H_n\cap W_0[\{x\}])$; then 252B tells us that $\int f_n d\mu$ is defined and equal to $\lambda(W_0\cap(X\times H_n))$. In particular, f_n is $\hat\Sigma$-measurable. Set $E_n=\{x:f_n(x)=\nu H_n\}\in\hat\Sigma$. If $x\in A_n$, then $H_n\subseteq W[\{x\}]$, so $A_n\setminus E_n\subseteq A$.

Now, by 252B again,

$$\begin{aligned}\lambda((E_n\times H_n)\setminus W_0)&=\int_{E_n}\nu(H_n\setminus W_0[\{x\}])\mu(dx)\\&=\int_{E_n}\nu H_n-\nu(H_n\cap W_0[\{x\}])\mu(dx)=0.\end{aligned}$$

So if we set $W_1=\bigcup_{n\in\mathbb{N}}E_n\times H_n$, $W_1\setminus W\subseteq W_1\setminus W_0$ is λ-negligible. On the other hand,

$$W\setminus W_1\subseteq\textstyle\bigcup_{n\in\mathbb{N}}(A_n\setminus E_n)\times H_n\subseteq A\times Y$$

is also λ-negligible. Because λ is complete, $W\in\Lambda$. As W is arbitrary, λ is a topological measure and is equal to $\tilde\lambda$, by 417Sa.

(b) Now consider the general case. Let Z be a conegligible subset of X with a countable network; since any subset of a space with a countable network again has a countable network (4A2Na), we may suppose that $Z\in\Sigma$. Again let W be an open set in $X\times Y$. This time, take arbitrary $E\in\Sigma$, $F\in\mathrm{T}$ of finite measure, and consider the subspace measures $\mu_{E\cap Z}$ and ν_F. These are still effectively locally finite and τ-additive (414K), and are now totally finite. Also $E\cap Z$ has a countable network. So (a) tells us that the relatively open set $W\cap((E\cap Z)\times F)$ is measured by the c.l.d. product of $\mu_{E\cap Z}$ and ν_F, which is the subspace measure on $(E\cap Z)\times F$ induced by λ (251Q). Since λ surely measures $E\times F$, it measures $W\cap(Z\times Y)\cap(E\times F)$. As E and F are arbitrary, λ measures $W\cap(Z\times Y)$ (251H). But $\lambda((X\setminus Z)\times Y)=\mu(X\times Z)\cdot\nu Y=0$ (251Ia), so λ also measures W. As W is arbitrary, λ is the τ-additive product measure.

417U Proposition Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces. Let λ be the ordinary product probability measure on $X=\prod_{i\in I}X_i$ and Λ its domain. Then every continuous function $f:X\to\mathbb{R}$ is Λ-measurable, so Λ includes the Baire σ-algebra of X.

proof (a) Let $\tilde\lambda$ be a τ-additive topological measure extending λ (417E), and $\tilde\Lambda$ its domain; then f is $\tilde\Lambda$-measurable, just because $\tilde\lambda$ is a topological measure. For $\alpha\in\mathbb{R}$, set

$$G_\alpha=\{x:x\in X,\,f(x)<\alpha\},\quad H_\alpha=\{x:x\in X,\,f(x)>\alpha\},$$

$$F_\alpha=\{x:x\in X,\,f(x)=\alpha\}.$$

Then $\langle F_\alpha\rangle_{\alpha\in\mathbb{R}}$ is disjoint, so $A=\{\alpha:\alpha\in\mathbb{R},\,\tilde\lambda F_\alpha>0\}$ is countable, and $A'=\mathbb{R}\setminus A$ is dense in $\mathbb{R}$; let $Q\subseteq A'$ be a countable dense set.

For each $q\in Q$, let $V_q\subseteq G_q$, $W_q\subseteq H_q$ be such that

$$\lambda V_q=\lambda_*G_q=\tilde\lambda G_q,\quad\lambda W_q=\lambda_*H_q=\tilde\lambda H_q$$

(413Ea, 417E(iii)). Then

$$\lambda^*(G_q \setminus V_q) \le \lambda(X \setminus (V_q \cup W_q)) = 1 - \lambda V_q - \lambda W_q = \tilde{\lambda}(X \setminus (G_q \cup H_q)) = 0.$$

Because λ is complete, $G_q \setminus V_q$ and G_q belong to Λ. But now, if $\alpha \in \mathbb{R}$,

$$\{x : f(x) < \alpha\} = \bigcup_{q \in Q, q < \alpha} G_q \in \Lambda,$$

so f is Λ-measurable.

(b) It follows that every zero set belongs to Λ, so that Λ must include the Baire σ-algebra of X.

417V Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces, and $(X \times Y, \Lambda, \lambda)$ their c.l.d. product. Then every continuous function $f : X \times Y \to \mathbb{R}$ is Λ-measurable, and the Baire σ-algebra of $X \times Y$ is included in Λ.

proof Let $Z \subseteq X \times Y$ be a zero set. If $E \in \Sigma$, $F \in \mathrm{T}$ are sets of finite measure, then $Z \cap (E \times F)$ is a zero set for the relative topology of $E \times F$. Now the subspace measures μ_E and ν_F are τ-additive topological measures (414K), so $Z \cap (E \times F)$ is measured by the c.l.d. product $\mu_E \times \nu_F$ of μ_E and ν_F. **P** If either μE or νF is zero, this is trivial. Otherwise, they have scalar multiples μ'_E, ν'_F which are probability measures, and of course are still τ-additive topological measures. By 417U, $Z \cap (E \times F)$ is measured by $\mu'_E \times \nu'_F$. Since $\mu_E \times \nu_F$ is just a scalar multiple of $\mu'_E \times \nu'_F$, $Z \cap (E \times F)$ is measured by $\mu_E \times \nu_F$. **Q** But $\mu_E \times \nu_F$ is the subspace measure $\lambda_{E \times F}$ (251Q), so $Z \cap (E \times F) \in \Lambda$. As E and F are arbitrary, $Z \in \Lambda$ (251H).

Thus every zero set belongs to Λ; accordingly Λ must include the Baire σ-algebra, and every continuous function must be Λ-measurable.

417X Basic exercises (a) Let (X, Σ, μ) be a semi-finite measure space and $\mathcal{A}$ a family of subsets of X. Show that the following are equiveridical: (i) there is a measure μ' on X, extending μ, such that $\mu' A = 0$ for every $A \in \mathcal{A}$; (ii) $\mu_*(\bigcup_{n \in \mathbb{N}} A_n) = 0$ for every sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{A}$.

(b) Let (X, Σ, μ) and (Y, T, ν) be measure spaces with topologies with respect to which μ and ν are locally finite. Show that the c.l.d. product measure on $X \times Y$ is locally finite.

>(c) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be topological measure spaces such that μ and ν are both effectively locally finite τ-additive Borel measures. Show that there is a unique effectively locally finite τ-additive Borel measure λ' on $X \times Y$ such that $\lambda'(G \times H) = \mu G \cdot \nu H$ for all open sets $G \subseteq X$, $H \subseteq Y$.

>(d) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of topological probability spaces in which every μ_i is a τ-additive Borel measure. Show that there is a unique τ-additive Borel measure λ' on $X = \prod_{i \in I} X_i$ such that $\lambda'(\prod_{i \in I} F_i) = \prod_{i \in I} \mu_i F_i$ whenever $F_i \subseteq X_i$ is closed for every $i \in I$.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces in which the measures are inner regular with respect to the Borel sets, and $\tilde{\lambda}$ the τ-additive product measure on $X \times Y$. Let $\langle X_i \rangle_{i \in I}$, $\langle Y_j \rangle_{j \in J}$ be decompositions for μ, ν respectively (definition: 211E). Show that $\langle X_i \times Y_j \rangle_{i \in I, j \in J}$ is a decomposition for $\tilde{\lambda}$. (Cf. 251O.)

>(f) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of τ-additive topological probability spaces such that μ_i is inner regular with respect to the Borel sets for every i, and $\tilde{\lambda}$ the τ-additive product measure on $X = \prod_{i \in I} X_i$. Take $A_i \subseteq X_i$ for each $i \in I$. (i) Show that if $\mu_i^* A_i = 1$ for every i, then the subspace measure induced by $\tilde{\lambda}$ on $A = \prod_{i \in I} A_i$ is just the τ-additive product $\tilde{\lambda}^{\#}$ of the subspace measures on the A_i. (*Hint*: show that if we set $\lambda' W = \tilde{\lambda}^{\#}(W \cap A)$ for Borel sets $W \subseteq X$, then λ' satisfies the conditions of 417Xd.) (ii) Show that in any case $\tilde{\lambda}^* A = \prod_{i \in I} \mu_i^* A_i$. (Cf. 254L.)

(g) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ and $\langle (Y_i, \mathfrak{S}_i, \mathrm{T}_i, \nu_i) \rangle_{i \in I}$ be two families of τ-additive topological probability spaces in which every μ_i and every ν_i is inner regular with respect to the Borel sets. Let $\tilde{\lambda}$, $\tilde{\lambda}'$ be the τ-additive product measures on $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$ respectively. Suppose that for each $i \in I$ we are given a continuous inverse-measure-preserving function $\phi_i : X_i \to Y_i$. Show that the function $\phi : X \to Y$ defined by setting $\phi(x)(i) = \phi_i(x(i))$ for $x \in X$, $i \in I$ is inverse-measure-preserving.

(h) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be two complete locally determined effectively locally finite τ-additive topological measure spaces such that both μ and ν are inner regular with respect to the Borel sets. Let $\tilde{\lambda}$ be the τ-additive product measure on $X \times Y$, and $\tilde{\Lambda}$ its domain. Suppose that ν is σ-finite. Show that for any $W \in \tilde{\Lambda}$, $W[\{x\}] \in \mathrm{T}$ for almost every $x \in X$, and $x \mapsto \nu W[\{x\}]$ is measurable.

>(i) Let $(X,\mathfrak{T},\Sigma,\mu)$ be $[0,1]$ with its usual topology and Lebesgue measure, and let $(Y,\mathfrak{S},\mathrm{T},\nu)$ be $[0,1]$ with its discrete topology and counting measure. (i) Show that both are Radon measure spaces. (ii) Show that the c.l.d. product measure on $X\times Y$ is a Radon measure. (*Hint*: 252Kc, or use 417T and 417P.) (iii) Show that 417Ha can fail if we omit the hypothesis on $\{(x,y):f(x,y)\ne 0\}$.

(j) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be two effectively locally finite τ-additive topological measure spaces. Let λ be the c.l.d. product measure and $\tilde\lambda$ the τ-additive product measure on $X\times Y$. Show that $\lambda^*(A\times B)=\tilde\lambda^*(A\times B)$ for all sets $A\subseteq X$, $B\subseteq Y$. (*Hint*: start with A, B of finite outer measure, so that 417I applies.)

(k) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces with strictly positive measures, all inner regular with respect to the Borel sets, and $(X,\mathfrak{T},\tilde\Lambda,\tilde\lambda)$ their τ-additive product. For $J\subseteq I$ let $\tilde\lambda_J$ be the τ-additive product measure on $X_J=\prod_{i\in J}X_i$, and $\tilde\Lambda_J$ its domain. (i) Show that if f is a real-valued $\tilde\Lambda$-measurable function defined $\tilde\lambda$-almost everywhere on X, we can find a countable set $J\subseteq I$ and a $\tilde\Lambda_J$-measurable function g, defined $\tilde\lambda_J$-almost everywhere on X_J, such that f extends $g\pi_J$. (ii) In (i), show that $\int f\,d\tilde\lambda=\int g\,d\tilde\lambda_J$ if either is defined in $[-\infty,\infty]$.

(l) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $(X,\mathfrak{T},\tilde\Lambda,\tilde\lambda)$ their τ-additive product. Show that for any $W\in\tilde\Lambda$ there is a smallest set $J\subseteq I$ for which there is a $W'\in\tilde\Lambda$, determined by coordinates in J, with $\tilde\lambda(W\triangle W')=0$. (*Hint*: 254R.)

(m) What needs to be added to 417M and 415Xk to complete a proof of 415E?

(n) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an atomless τ-additive topological probability space such that μ is inner regular with respect to the Borel sets, and I a set of cardinal at most that of the support of μ. Show that the set of injective functions from I to X has full outer measure for the τ-additive product measure on X^I.

>(o) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces. Show that the Radon product measure on $X\times Y$ is the unique Radon measure $\tilde\lambda$ such that $\tilde\lambda(K\times L)=\mu K\cdot\nu L$ for all compact sets $K\subseteq X$, $L\subseteq Y$.

>(p) Let I be an uncountable set, and λ, $\tilde\lambda$ be the ordinary and τ-additive product measures on $X=\{0,1\}^I$ when each factor is given its usual topology and the Dirac measure concentrated at 1. Show that $\tilde\lambda$ properly extends λ, and that the support of $\tilde\lambda$ is not determined by any countable set of coordinates. Find a $\tilde\lambda$-negligible open set $W\subseteq X$ such that its projection onto $\{0,1\}^J$ is conegligible for every proper subset J of I.

(q) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of Radon probability spaces, and $\tilde\lambda$ the quasi-Radon product measure on $X=\prod_{i\in I}X_i$. For each $i\in I$, let $Z_i\subseteq X_i$ be the support of μ_i. Show that $\tilde\lambda$ is a Radon measure iff $\{i:i\in I,\ Z_i$ is not compact$\}$ is countable. In particular, show that the ordinary product measure on $[0,1[^I$, where I is uncountable and each copy of $[0,1[$ is given Lebesgue measure, is a quasi-Radon measure, but not a Radon measure.

(r) Let $\langle(X_n,\mathfrak{T}_n,\Sigma_n,\mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of Radon probability spaces. Show that the Radon product measure on $X=\prod_{n\in\mathbb{N}}X_n$ is the unique Radon measure $\tilde\lambda$ on X such that $\tilde\lambda(\prod_{n\in\mathbb{N}}K_n)=\prod_{n=0}^\infty\mu_nK_n$ whenever $K_n\subseteq X_n$ is compact for every n.

(s) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and λ, $\tilde\lambda$ the ordinary and τ-additive product measures on $X=\prod_{i\in I}X_i$. Show that if $A\subseteq X$ has $\tilde\lambda$-negligible boundary, then A is measured by λ.

(t) Let us say that a topological space X is **chargeable** if there is an additive functional $\nu:\mathcal{P}X\to[0,\infty[$ such that $\nu G>0$ for every non-empty open set $G\subseteq X$. (i) Show that if there is a σ-finite measure μ on X such that $\mu_*G>0$ for every non-empty open set G, then X is chargeable. (*Hint*: 215B(vii), 391G.) (ii) Show that any separable space is chargeable. (iii) Show that X is chargeable iff its regular open algebra is chargeable in the sense of 391Bb. (*Hint*: see the proof of 314P.) (iv) Show that any open subspace of a chargeable space is chargeable. (v) Show that if $Y\subseteq X$ is dense, then X is chargeable iff Y is chargeable. (vi) Show that if X is expressible as the union of countably many chargeable subspaces, then it is chargeable. (vii) Show that any product of chargeable spaces is chargeable. (Cf. 391Xb(iii).) (viii) Show that if $\langle X_i\rangle_{i\in I}$ is a family of chargeable spaces with product X, then all regular open subsets of X and all Baire subsets of X are determined by coordinates in countable sets. (*Hint*: 4A2Eb, 4A3Mb.) (ix) Show that a continuous image of a chargeable space is chargeable. (x) Show that a compact Hausdorff space is chargeable iff it carries a strictly positive Radon measure. (*Hint*: 416K.)

(u) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be quasi-Radon measure spaces such that $\mu X\cdot\nu Y>0$. Show that the quasi-Radon product measure on $X\times Y$ is completion regular iff it is equal to the c.l.d. product measure and μ and ν are both completion regular. (*Hint*: 412Sc; if μE, νF are finite and $Z\subseteq E\times F$ is a zero set of positive measure, use Fubini's theorem to show that Z has sections of positive measure.)

(v) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces. Show that the quasi-Radon product measure on $\prod_{i\in I}X_i$ is completion regular iff it is equal to the ordinary product measure and every μ_i is completion regular.

(w) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and λ the τ-additive product measure on $X=\prod_{i\in I}X_i$; write Λ for its domain. (i) Show that if $W\in\Lambda$, $\lambda W>0$ and $\epsilon>0$ then there are a finite $J\subseteq I$ and a $W'\in\Lambda$ such that $\lambda W'\ge 1-\epsilon$ and for every $x\in W'$ there is a $y\in W$ such that $x\restriction I\setminus J=y\restriction I\setminus J$. (Cf. 254Sb.) (ii) Show that if $A\subseteq X$ is determined by coordinates in $I\setminus\{i\}$ for every $i\in I$ then $\lambda^*A\in\{0,1\}$. (Cf. 254Sa.)

417Y Further exercises (a) (i) Show that if, in 417A, μ is strictly localizable, then it has a strictly localizable extension μ' with the properties (i)-(iv) there. (ii) Give an example to show that the construction offered in 417A may not immediately achieve this result.

(b) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be effectively locally finite τ-additive topological measure spaces such that μ and ν are both inner regular with respect to the Borel sets.

(i) Fix open sets $G\subseteq X$, $H\subseteq Y$ of finite measure. Let $\mathcal{W}_{GH}$ be the set of those $W\subseteq X\times Y$ such that $\theta_{GH}(W)=\int_G\hat\nu(W[\{x\}]\cap H)dx$ is defined, where $\hat\nu$ is the completion of ν. (α) Show that every open set belongs to $\mathcal{W}_{GH}$. (β) Show that θ_{GH} is countably additive in the sense that $\theta_{GH}(\bigcup_{n\in\mathbb{N}}W_n)=\sum_{n=0}^\infty\theta_{GH}(W_n)$ for every disjoint sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}_{GH}$, and τ-additive in the sense that $\theta_{GH}(\bigcup\mathcal{V})=\sup_{V\in\mathcal{V}}\theta_{GH}(V)$ for every non-empty upwards-directed family $\mathcal{V}$ of open sets in $X\times Y$. (γ) Show that every Borel set belongs to $\mathcal{W}_{GH}$. (*Hint*: Monotone Class Theorem.) (δ) Writing $\mathcal{B}$ for the Borel σ-algebra of $X\times Y$, show that $\theta_{GH}\restriction\mathcal{B}$ is a τ-additive Borel measure; let λ_{GH} be its completion. (ϵ) Show that $\lambda_{GH}=\theta_{GH}\restriction\Lambda_{GH}$, where $\Lambda_{GH}=\operatorname{dom}\lambda_{GH}$. (ζ) Show that $\lambda_{GH}(E\times F)$ is defined and equal to $\mu E\cdot\nu F$ whenever $E\in\Sigma$, $F\in\mathrm{T}$, $E\subseteq G$ and $F\subseteq H$. (*Hint*: start with open E and F, move to Borel E and F with the Monotone Class Theorem.) (η) Writing λ for the c.l.d. product measure on $X\times Y$, show that $\lambda_{GH}(W)$ is defined and equal to $\lambda(W\cap(G\times H))$ whenever $W\in\operatorname{dom}\lambda$.

(ii) Now take $\tilde\Lambda$ to be $\bigcap\{\Lambda_{GH}:G\in\mathfrak{T},\,H\in\mathfrak{S},\,\mu G<\infty,\,\nu H<\infty\}$ and $\tilde\lambda W=\sup_{G,H}\lambda_{GH}(W)$ for $W\in\tilde\Lambda$. Show that $\tilde\lambda$ is an extension of λ to a complete locally determined effectively locally finite τ-additive topological measure on $X\times Y$ which is inner regular with respect to the Borel sets, so is the τ-additive product measure as defined in 417G.

(c) Let (X,Σ,μ) and (Y,T,ν) be complete measure spaces with topologies $\mathfrak{T}$, $\mathfrak{S}$. Suppose that μ and ν are effectively locally finite and τ-additive and moreover that their domains include bases for the two topologies. Show that the c.l.d. product measure on $X\times Y$ has the same properties. (*Hint*: start by assuming that μX and νY are both finite. If $\mathcal{V}$ is an upwards-directed family of measurable open sets with measurable open union W, look at $g_V(x)=\nu V[\{x\}]$ for $V\in\mathcal{V}$.)

(d) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $(X,\mathfrak{T},\tilde\Lambda,\tilde\lambda)$ their τ-additive product. (i) Show that the following are equiveridical: (α) μ_i is strictly positive for all but countably many $i\in I$; (β) whenever $W\in\tilde\Lambda$ there are a countable $J\subseteq I$ and W_1, $W_2\in\tilde\Lambda$, determined by coordinates in J, such that $W_1\subseteq W\subseteq W_2$ and $\tilde\lambda(W_2\setminus W_1)=0$. (ii) Show that when these are false, $\tilde\lambda$ cannot be equal to the ordinary product measure on X.

(e) Let (X,Σ,μ) and (Y,T,ν) be measure spaces with Hausdorff topologies $\mathfrak{T}$, $\mathfrak{S}$ such that both μ and ν are inner regular with respect to the families of sequentially compact sets in each space. Show that the c.l.d. product measure λ on $X\times Y$ is also inner regular with respect to the sequentially compact sets, so has an extension to a topological measure which is inner regular with respect to the sequentially compact sets. (*Hint*: 412R, 416Yc.)

(f) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces with topologies $\mathfrak{T}_i$ such that every μ_i is inner regular with respect to the family of closed countably compact sets in X_i and every X_i is compact. Show that the ordinary product measure λ on $X=\prod_{i\in I}X_i$ is also inner regular with respect to the closed countably compact sets, so has an extension to a topological measure $\tilde{\lambda}$ which is inner regular with respect to the closed countably compact sets in X. Show that this can be done in such a way that for every $W \in \operatorname{dom}\tilde{\lambda}$ there is a $V \in \operatorname{dom}\lambda$ such that $\tilde{\lambda}(W \triangle V) = 0$. (*Hint*: 412T, 416Yb.)

(g) Let $\langle(X_n, \Sigma_n, \mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of probability spaces with Hausdorff topologies $\mathfrak{T}_n$ such that every μ_n is inner regular with respect to the family of sequentially compact sets in X_n. Show that the ordinary product measure λ on $X = \prod_{n\in\mathbb{N}} X_n$ is also inner regular with respect to the sequentially compact sets, so has an extension to a topological measure $\tilde{\lambda}$ which is inner regular with respect to the sequentially compact sets in X. Show that this can be done in such a way that for every $W \in \operatorname{dom}\tilde{\lambda}$ there is a $V \in \operatorname{dom}\lambda$ such that $\tilde{\lambda}(W \triangle V) = 0$.

(h) Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces, and λ, $\tilde{\lambda}$ the ordinary and quasi-Radon product measures on $X = \prod_{i\in I} X_i$. Suppose that all but *one* of the $\mathfrak{T}_i$ have countable networks and all but *countably* many of the μ_i are strictly positive. Show that $\lambda = \tilde{\lambda}$.

(i) Let us say that a quasi-Radon measure space $(X, \mathfrak{T}, \Sigma, \mu)$ has the **simple product property** if the c.l.d. product measure on $X \times Y$ is equal to the quasi-Radon product measure for every quasi-Radon measure space $(Y, \mathfrak{S}, \mathrm{T}, \nu)$. (i) Show that if $(X, \mathfrak{T})$ has a countable network then $(X, \mathfrak{T}, \Sigma, \mu)$ has the simple product property. (ii) Show that if a quasi-Radon measure space has the simple product property so do all its subspaces. (iii) Show that the quasi-Radon product of two quasi-Radon measure spaces with the simple product property has the simple product property. (iv) Show that the quasi-Radon product of any family of quasi-Radon probability spaces with the simple product property has the simple product property. (v) Show that the real line with the right-facing Sorgenfrey topology (415Xc) and Lebesgue measure has the simple product property.

417 Notes and comments The general problem of determining just when a measure can be extended to a measure with given properties is one which will recur throughout this volume. I have more than once mentioned the Banach-Ulam problem; if you like, this is the question of whether there can ever be an extension of the countable-cocountable measure on a set X to a measure defined on the whole algebra $\mathcal{P}X$. This particular question appears to be undecidable from the ordinary axioms of set theory; but for many sets (for instance, if $X = \omega_1$) it is known that the answer is 'no'. (See 419G and 438C.) This being so, we have to take each manifestation of the general question on its own merits. In 417C and 417E the challenge is to take a product measure λ defined in terms of the factor measures alone, disregarding their topological properties, and extend it to a topological measure, preferably τ-additive. Of course there are important cases in which λ is itself already a topological measure; for instance, we know that the c.l.d. product of Lebesgue measure on $\mathbb{R}$ with itself is Lebesgue measure on $\mathbb{R}^2$ (251N), and other examples are in 415E, 415Ye, 416U, 417S-417T, 417Yh and 453I. But in general not every open set in the product belongs to the domain of λ, even when we have the product of two Radon measures on compact Hausdorff spaces (419E).

Once we have resolved to grasp the nettle, however, there is a natural strategy for the proof. It is easy to see that if λ, in 417C or 417E, is to have an extension to a τ-additive topological measure $\tilde{\lambda}$, then we must have $\tilde{\lambda}A(\mathcal{V}) = 0$ for every $\mathcal{V}$ belonging to the class $\mathfrak{V}$. Now 417A descibes a sufficient (and obviously necessary) condition for there to be an extension of λ with this property. So all we have to do is check. The check is not perfectly straightforward; in 417E it uses all the resources of the original proof that there is a product measure on an arbitrary product of probability spaces (which I suppose is to be expected), with 414B (of course) to apply the hypothesis that the factor measures are τ-additive, and a couple of extra wrinkles (the W'_n and C'_n of part (c-ii) of the proof of 417E, and the use of supports in part (c-vii)).

It is worth noting that (both for finite and for infinite products) the measure algebras of λ and $\tilde{\lambda}$ are identical (417C(ii), 417E(ii)), so there is no new work to do in identifying the measure algebra of $\tilde{\lambda}$ and the associated function spaces.

An obstacle we face in 417C-417E is the fact that *not* every τ-additive measure μ has an extension to a τ-additive topological measure, even when μ is totally finite and its domain includes a base for the topology. (I give an example in 419J.) Consequently it is not enough, in 417C or 417E, to show that the ordinary product measure λ is τ-additive. But perhaps I should remark that if λ is inner regular with respect to the closed sets, this obstacle evaporates (415L). Accordingly, for the principal applications (to quasi-Radon and Radon product measures, and in particular whenever the topological spaces involved are regular) we have rather easier proofs available, based on the constructions of §415. For completely regular spaces, there is yet another approach, because the product measures can be described in terms of the integrals of continuous functions (415I), which by 417U and 417V can be calculated from the ordinary product measures. Of course the proof that λ itself is τ-additive is by no means trivial, especially in the case of infinite products, corresponding to 417E; but for finite products there are relatively direct arguments, applying indeed to slightly more general situations (417Yc). If we have measures which are inner regular with respect to countably compact classes of sets, then there may be other ways of approaching the extension, using theorems from §413 (see 417Ye-417Yg), and for compact Radon measure spaces, λ becomes tight (412Sb, 412V), so its τ-additivity is elementary.

As always, it is important to recognise which constructions are in some sense canonical. The arguments of 417C and 417E allow for the possibility that the factor measures are defined on σ-algebras going well beyond the Borel sets. For all the principal applications, however, the measures will be c.l.d. versions of Borel measures, and in particular will be inner regular with respect to the Borel sets. In such a context it is natural to ask for product measures with the same property, and in this case we can identify a canonical τ-additive topological product measure, as in 417D and 417F. (If you prefer to restrict your measures to Borel σ-algebras, you again get canonical product Borel measures (417Xc-417Xd).) Having done so, we can reasonably expect 'commutative' and 'associative' and 'distributive' laws, as in 417Db, 417J and 417Xe. Subspaces mostly behave themselves (417I, 417Xf).

Of course extending the product measure means that we get new integrable functions on the product, so that Fubini's theorem has to be renegotiated. Happily, it remains valid, at least in the contexts in which it was effective before (417Ha); we still need, in effect, one of the measures to be σ-finite. The theorem still fails for arbitrary integrable functions on products of Radon measure spaces, and the same example works as before (417Xi). In fact this means that we have an alternative route to the construction of the τ-additive product of two measures (417Yb). But note that on this route 'commutativity', the identification of the product measure on $X \times Y$ with that on $Y \times X$, becomes something which can no longer be taken for granted, because if we *define* $\tilde{\lambda}W$ to be $\int \nu W[\{x\}]dx$ we have to worry about when, and why, this will be equal to $\int \mu W^{-1}[\{y\}]dy$.

A version of Tonelli's theorem follows from Fubini's theorem, as before (417Hc). We also have results corresponding to most of the theorems of §254. But note that there are two traps. In the theorem that a measurable set can be described in terms of a projection onto a countable subproduct (254O, 417M) we need to suppose that the factor measures are strictly positive, and in the theorem that a product of Radon measures is a Radon measure (417Q) we need to suppose that the factor measures have compact supports. The basic examples to note in this context are 417Xp and 417Xq.

It is not well understood when we can expect c.l.d. product measures to be topological measures, even in the case of compact Radon probability spaces. Example 419E remains a rather special case, but of course much more effort has gone into seeking positive results. Note that the ordinary product measures of this section are always effectively locally finite and τ-additive (417C, 417E), so that they will be equal to the τ-additive products iff they measure every open set (417S). Regarding infinite products, the τ-additive product measure can fail to be the ordinary product measure in just two ways: if one of the *finite* product measures is not a topological measure, or if uncountably many of the factor measures are not strictly positive (417Sc, 417Xp, 417Yd). So it is finite products which need to be studied.

Whenever we have a subset F of an infinite product $X = \prod_{i \in I} X_i$, it is important to know when F is determined by coordinates in a proper subset of I; in measure theory, we are particularly interested in sets determined by coordinates in countable subsets of I (254Mb). It may happen that there is a *smallest* set J such that F is determined by coordinates in J; for instance, when we have a topological product and F is closed (4A2Bg). When we have a product of probability spaces, we sometimes wish to identify sets J such that F is 'essentially' determined by coordinates in J, in the sense that there is an F', determined by coordinates in J, such that $F \triangle F'$ is negligible. In this context, again, there is a smallest such set (254Rd), which can be identified in terms of the probability algebra free product of the measure algebras (325Mb). In 417Ma the two ideas come together: under the conditions there, we get the same smallest J by either route.

In 417Ma, we have a product of strictly positive τ-additive topological probability measures. If we keep the 'strictly positive' but abandon everything else, we still have very striking results just because the product topology is ccc, so that we can apply 4A2Eb. An abstract expression of this idea is in 417Xt.

418 Measurable functions and almost continuous functions

In this section I work through the basic properties of measurable and almost continuous functions, as defined in 411L and 411M. I give the results in the full generality allowed by the terminology so far introduced, but most of the ideas are already required even if you are interested only in Radon measure spaces as the domains of the functions involved. Concerning the codomains, however, there is a great difference between metrizable spaces and others, and among metrizable spaces separability is of essential importance.

I start with the elementary properties of measurable functions (418A-418C) and almost continuous functions (418D). Under mild conditions on the domain space, almost continuous functions are measurable (418E); for a separable metrizable codomain, we can expect that measurable functions should be almost continuous (418J). Before coming to this, I spend a couple of paragraphs on image measures: a locally finite image measure under a measurable function is Radon if the measure on the domain is Radon and the function is almost continuous (418I).

418L-418Q are important results on expressing given Radon measures as image measures associated with continuous functions, first dealing with ordinary functions $f : X \to Y$ (418L) and then coming to Prokhorov's theorem on projective limits of probability spaces (418M).

The machinery of the first part of the section can also be used to investigate representations of vector-valued functions in terms of product spaces (418R-418T).

418A Proposition Let X be a set, Σ a σ-algebra of subsets of X, Y a topological space and $f : X \to Y$ a measurable function.

(a) $f^{-1}[F] \in \Sigma$ for every Borel set $F \subseteq Y$.

(b) If $A \subseteq X$ is any set, endowed with the subspace σ-algebra, then $f{\restriction}A : A \to Y$ is measurable.

(c) Let $(Z, \mathfrak{T})$ be another topological space. Then $gf : X \to Z$ is measurable for every Borel measurable function $g : Y \to Z$; in particular, for every continuous function $g : Y \to Z$.

proof (a) The set $\{F : F \subseteq Y, f^{-1}[F] \in \Sigma\}$ is a σ-algebra of subsets of Y containing every open set, so contains every Borel subset of Y.

(b) is obvious from the definition of 'subspace σ-algebra' (121A).

(c) If $H \subseteq Z$ is open, then $g^{-1}[H]$ is a Borel subset of Y so $(gf)^{-1}[H] = f^{-1}[g^{-1}[H]]$ belongs to Σ.

418B Proposition Let X be a set and Σ a σ-algebra of subsets of X.

(a) If Y is a metrizable space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable functions from X to Y such that $f(x) = \lim_{n\to\infty} f_n(x)$ is defined in Y for every $x \in X$, then $f : X \to Y$ is measurable.

(b) If Y is a topological space, Z is a separable metrizable space and $f : X \to Y$, $g : X \to Z$ are functions, then $x \mapsto (f(x), g(x)) : X \to Y \times Z$ is measurable iff f and g are measurable.

(c) If Y is a hereditarily Lindelöf space, $\mathcal{U}$ a family of open sets generating its topology, and $f : X \to Y$ a function such that $f^{-1}[U] \in \Sigma$ for every $U \in \mathcal{U}$, then f is measurable.

(d) If $\langle Y_i\rangle_{i\in I}$ is a countable family of separable metrizable spaces, with product Y, then a function $f : X \to Y$ is measurable iff $\pi_i f : X \to Y_i$ is measurable for every i, writing $\pi_i(y) = y(i)$ for $y \in Y$, $i \in \mathbb{N}$.

proof (a) Let ρ be a metric defining the topology of Y. Let $G \subseteq Y$ be any open set, and for each $n \in \mathbb{N}$ set

$$F_n = \{y : y \in Y, \rho(y, z) \ge 2^{-n} \text{ for every } z \in Y \setminus G\}.$$

Then F_n is closed, so $f_i^{-1}[F_n] \in \Sigma$ for every n, $i \in \mathbb{N}$. But this means that

$$f^{-1}[G] = \bigcup_{n\in\mathbb{N}} \bigcap_{i\ge n} f_i^{-1}[F_i] \in \Sigma.$$

As G is arbitrary, f is measurable.

(b)(i) The functions $(y, z) \mapsto y$, $(y, z) \mapsto z$ are continuous, so if $x \mapsto (f(x), g(x))$ is measurable, so are f and g, by 418Ac.

(ii) Now suppose that f and g are measurable, and that $W \subseteq Y \times Z$ is open. By 4A2P(a-i), the topology of Z has a countable base $\mathcal{H}$; let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{H} \cup \{\emptyset\}$. For each n, set

$$G_n = \bigcup\{G : G \subseteq Y \text{ is open}, G \times H_n \subseteq W\};$$

then G_n is open and $G_n \times H_n \subseteq W$. Accordingly $W \supseteq \bigcup_{n\in\mathbb{N}} G_n \times H_n$. But in fact $W = \bigcup_{n\in\mathbb{N}} G_n \times H_n$. **P** If $(y, z) \in W$, there are open sets $G \subseteq Y$, $H \subseteq Z$ such that $(y, z) \in G \times H \subseteq W$. Now there is an $n \in \mathbb{N}$ such that $z \in H_n \subseteq H$, in which case $G \times H_n \subseteq W$ and $G \subseteq G_n$ and $(y, z) \in G_n \times H_n$. **Q**

Accordingly

$$\{x : (f(x), g(x)) \in W\} = \bigcup_{n\in\mathbb{N}} f^{-1}[G_n] \cap g^{-1}[H_n] \in \Sigma.$$

As W is arbitrary, $x \mapsto (f(x), g(x))$ is measurable.

(c) This is just 4A3Db.

(d) If f is measurable, so is every $\pi_i f$, by 418Ac. If every $\pi_i f$ is measurable, set

$$\mathcal{U} = \{\pi_i^{-1}[H] : i \in I,\ H \subseteq Y_i \text{ is open}\}.$$

Then $\mathcal{U}$ generates the topology of Y, and if $U = \pi_i^{-1}[H]$ then $f^{-1}[U] = (\pi_i f)^{-1}[H]$, so $f^{-1}[U] \in \Sigma$ for every U. Also Y is hereditarily Lindelöf (4A2P(a-iii)), so f is measurable, by (c).

418C Proposition Let (X, Σ, μ) be a measure space and Y a Polish space. Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of measurable functions from X to Y. Then

$$\{x : x \in X,\ \lim_{n\to\infty} f_n(x) \text{ is defined in } Y\}$$

belongs to Σ.

proof (Compare 121H.) Let ρ be a complete metric on Y defining the topology of Y.

(a) For m, $n \in \mathbb{N}$ and $\delta > 0$, the set $\{x : \rho(f_m(x), f_n(x)) \le \delta\}$ belongs to Σ. **P** The function $x \mapsto (f_m(x), f_n(x)) : X \to Y^2$ is measurable, by 418Bb, and the function $\rho : Y^2 \to \mathbb{R}$ is continuous, so $x \mapsto \rho(f_m(x), f_n(x))$ is measurable and $\{x : \rho(f_m(x), f_n(x)) \le \delta\} \in \Sigma$. **Q**

(b) Now $\langle f_n(x) \rangle_{n\in\mathbb{N}}$ is convergent iff it is Cauchy, because Y is complete. But

$$\{x : x \in X, \langle f_n(x) \rangle_{n\in\mathbb{N}} \text{ is Cauchy}\} = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{i\ge m} \{x : \rho(f_i(x), f_m(x)) \le 2^{-n}\}$$

belongs to Σ.

418D Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) Suppose that Y is a topological space. Then any continuous function from X to Y is almost continuous.

(b) Suppose that Y and Z are topological spaces, $f : X \to Y$ is almost continuous and $g : Y \to Z$ is continuous. Then $gf : X \to Z$ is almost continuous.

(c) Suppose that $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ is a σ-finite topological measure space, Z is a topological space, $g : Y \to Z$ is almost continuous and $f : X \to Y$ is inverse-measure-preserving and almost continuous. Then $gf : X \to Z$ is almost continuous.

(d) Suppose that μ is semi-finite, and that $\langle Y_i \rangle_{i\in I}$ is a countable family of topological spaces with product Y. Then a function $f : X \to Y$ is almost continuous iff $f_i = \pi_i f$ is almost continuous for every $i \in I$, writing $\pi_i(y) = y(i)$ for $i \in I$, $y \in Y$.

proof (a) is trivial.

(b) The set $\{A : A \subseteq X,\ gf \restriction A \text{ is continuous}\}$ includes $\{A : A \subseteq X,\ f \restriction A \text{ is continuous}\}$; so if μ is inner regular with respect to the latter, it is inner regular with respect to the former.

(c) Take $E \in \Sigma$ and $\gamma < \mu E$; take $\epsilon > 0$. We have a cover of Y by a non-decreasing sequence $\langle Y_n \rangle_{n\in\mathbb{N}}$ of measurable sets of finite measure; now $\langle f^{-1}[Y_n] \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence covering E, so there is an $n \in \mathbb{N}$ such that $\mu(E \cap f^{-1}[Y_n]) \ge \gamma$. Because f is inverse-measure-preserving, $E \cap f^{-1}[Y_n]$ has finite measure. Now we can find measurable sets $F \subseteq Y_n$, $E_1 \subseteq E \cap f^{-1}[Y_n]$ such that $f \restriction E_1$, $g \restriction F$ are continuous and $\nu F \ge \nu Y_n - \epsilon$, $\mu E_1 \ge \mu(E \cap f^{-1}[Y_n] \setminus E_1) - \epsilon$. In this case $E_0 = E_1 \cap f^{-1}[F]$ has measure at least $\gamma - 2\epsilon$ and $gf \restriction E_0$ is continuous. As E, γ and ϵ are arbitrary, gf is almost continuous.

(d)(i) If f is almost continuous, every f_i must be almost continuous, by (b).

(ii) Now suppose that every f_i is almost continuous. Take $E \in \Sigma$ and $\gamma < \mu E$. There is an $E_0 \subseteq E$ such that $E_0 \in \Sigma$ and $\gamma < \mu E_0 < \infty$. Let $\langle \epsilon_i \rangle_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I} \epsilon_i \le \mu E_0 - \gamma$. For each $i \in I$ choose a measurable set $F_i \subseteq E_0$ such that $\mu F_i \ge \mu E_0 - \epsilon_i$ and $f_i \restriction F_i$ is continuous. Then $F = E_0 \cap \bigcap_{i\in I} F_i$ is a subset of E with measure at least γ, and $f \restriction F$ is continuous because $f_i \restriction F$ is continuous for every i (3A3Ib).

418E Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined topological measure space, Y a topological space, and $f : X \to Y$ an almost continuous function. Then f is measurable.

proof Set $\mathcal{K} = \{K : K \in \Sigma,\ f \restriction K \text{ is continuous}\}$; then μ is inner regular with respect to $\mathcal{K}$. If $H \subseteq Y$ is open and $K \in \mathcal{K}$, then $K \cap f^{-1}[H]$ is relatively open in K, that is, there is an open set $G \subseteq X$ such that $K \cap f^{-1}[H] = K \cap G$. Because μ is a topological measure, $G \in \Sigma$ so $K \cap f^{-1}[H] \in \Sigma$. As K is arbitrary, and μ is complete and locally determined, $f^{-1}[H] \in \Sigma$ (412Ja). As H is arbitrary, f is measurable.

418F Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a semi-finite topological measure space, Y a metrizable space, and $f:X\to Y$ a function. Suppose there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of almost continuous functions from X to Y such that $f(x)=\lim_{n\to\infty}f_n(x)$ for almost every $x\in X$. Then f is almost continuous.

proof Suppose that $E\in\Sigma$ and that $\gamma<\mu E$, $\epsilon>0$. Then there is a measurable set $F\subseteq E$ such that $\gamma\le\mu F<\infty$; discarding a negligible set if necessary, we may arrange that $f(x)=\lim_{n\to\infty}f_n(x)$ for every $x\in F$. Let ρ be a metric on Y defining its topology. For each $n\in\mathbb{N}$, let $F_n\subseteq F$ be a measurable set such that $f_n\restriction F_n$ is continuous and $\mu(F_n\setminus F)\le 2^{-n}\epsilon$; set $G=\bigcap_{n\in\mathbb{N}}F_n$, so that $\mu G\ge\gamma-2\epsilon$ and $f_n\restriction G$ is continuous for every $n\in\mathbb{N}$.

For m, $n\in\mathbb{N}$, the functions $x\mapsto(f_m(x),f_n(x)):G\to Y^2$ and $x\mapsto\rho(f_m(x),f_n(x)):G\to\mathbb{R}$ are continuous, therefore measurable, because μ is a topological measure. Also $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x\in G$. So if we set $G_{kn}=\{x:x\in G,\,\rho(f_i(x),f_j(x))\le 2^{-k}$ for all i, $j\ge n\}$, $\langle G_{kn}\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of measurable sets with union G for each $k\in\mathbb{N}$, and we can find a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ such that $\mu(G\setminus G_{kn_k})\le 2^{-k}\epsilon$ for every k. Setting $H=\bigcap_{k\in\mathbb{N}}G_{kn_k}$, $\mu H\ge\mu G-2\epsilon\ge\gamma-4\epsilon$ and $\rho(f_i(x),f_{n_k}(x))\le 2^{-k}$ whenever $x\in H$ and $i\ge n_k$; consequently $\rho(f(x),f_{n_k}(x))\le 2^{-k}$ whenever $x\in H$ and $k\in\mathbb{N}$. But this means that $\langle f_{n_k}\rangle_{k\in\mathbb{N}}$ converges to f uniformly on H, while every f_{n_k} is continuous on H, so $f\restriction H$ is continuous (3A3Nb). And of course $H\subseteq E$.

As E, γ and ϵ are arbitrary, f is almost continuous.

418G Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite quasi-Radon measure space, Y a metrizable space and $f:X\to Y$ an almost continuous function. Then there is a conegligible set $X_0\subseteq X$ such that $f[X_0]$ is separable.

proof (a) Let $\mathcal{K}$ be the family of self-supporting measurable sets K of finite measure such that $f\restriction K$ is continuous. Then μ is inner regular with respect to $\mathcal{K}$. **P** If $E\in\Sigma$ and $\gamma<\mu E$, there is an $F\in\Sigma$ such that $F\subseteq E$ and $\gamma<\mu F<\infty$; there is an $H\in\Sigma$ such that $H\subseteq F$, $\gamma\le\mu H$ and $f\restriction H$ is continuous; and there is a measurable self-supporting $K\subseteq H$ with the same measure as H (414F), in which case $K\in\mathcal{K}$ and $K\subseteq E$ and $\mu K\ge\gamma$. **Q**

(b) Now $f[K]$ is ccc for every $K\in\mathcal{K}$. **P** If $\mathcal{G}$ is a disjoint family of non-empty relatively open subsets of $f[K]$, then $\langle K\cap f^{-1}[G]\rangle_{G\in\mathcal{G}}$ is a disjoint family of non-empty relatively open subsets of K, because $f\restriction K$ is continuous, and $\sum_{G\in\mathcal{G}}\mu(K\cap f^{-1}[G])\le\mu K$. Because K is self-supporting, $\mu(K\cap f^{-1}[G])>0$ for every $G\in\mathcal{G}$; because μK is finite, $\mathcal{G}$ is countable. As $\mathcal{G}$ is arbitrary, $f[K]$ is ccc. **Q**

Because Y is metrizable, $f[K]$ must be separable (4A2Pd).

(c) Because μ is σ-finite, there is a countable family $\mathcal{L}\subseteq\mathcal{K}$ such that $X_0=\bigcup\mathcal{L}$ is conegligible (412Ic). Now $f[X_0]=\bigcup_{L\in\mathcal{L}}f[L]$ is a countable union of separable spaces, so is separable (4A2B(e-i)).

418H Proposition (a) Let X and Y be topological spaces, μ an effectively locally finite τ-additive measure on X, and $f:X\to Y$ an almost continuous function. Then the image measure μf^{-1} is τ-additive.

(b) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a totally finite quasi-Radon measure space, $(Y,\mathfrak{S})$ a regular topological space, and $f:X\to Y$ an almost continuous function. Then there is a unique quasi-Radon measure ν on Y such that f is inverse-measure-preserving for μ and ν.

proof (a) Let $\mathcal{H}$ be an upwards-directed family of open subsets of Y, all measured by μf^{-1}, and suppose that $H^*=\bigcup\mathcal{H}$ also is measurable. Take any $\gamma<(\mu f^{-1})(H^*)=\mu f^{-1}[H^*]$. Then there is a measurable set $E\subseteq f^{-1}[H^*]$ such that $\mu E\ge\gamma$ and $f\restriction E$ is continuous. Consider $\{E\cap f^{-1}[H]:H\in\mathcal{H}\}$. This is an upwards-directed family of relatively open measurable subsets of E with measurable union E. By 414K, the subspace measure on E is τ-additive, so

$$\gamma\le\mu E\le\sup_{H\in\mathcal{H}}\mu(E\cap f^{-1}[H])\le\sup_{H\in\mathcal{H}}\mu f^{-1}[H].$$

As γ is arbitrary, $\mu f^{-1}[H^*]\le\sup_{H\in\mathcal{H}}\mu f^{-1}[H]$; as $\mathcal{H}$ is arbitrary, μf^{-1} is τ-additive.

(b) By 418E, f is measurable. Let ν_0 be the restriction of μf^{-1} to the Borel σ-algebra of Y; by (a), ν_0 is τ-additive, and f is inverse-measure-preserving with respect to μ and ν_0. Because Y is regular, the completion ν of ν_0 is a quasi-Radon measure (415Cb). Because μ is complete, f is still inverse-measure-preserving with respect to μ and ν (234Ba[25]).

To see that ν is unique, observe that its values on Borel sets are determined by the requirement that f be inverse-measure-preserving, so that 415H gives the result.

[25]Formerly 235Hc.

418I The next theorem is one of the central properties of Radon measures. I have already presented what amounts to a special case in 256G.

Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, Y a Hausdorff space, and $f:X\to Y$ an almost continuous function. If the image measure $\nu=\mu f^{-1}$ is locally finite, it is a Radon measure.

proof (a) By 418E, f is measurable, that is, $f^{-1}[H]\in\Sigma$ for every open set $H\subseteq Y$; but this means that the domain T of ν contains every open set, and ν is a topological measure.

(b) ν is inner regular with respect to the compact sets. **P** If $F\in\mathrm{T}$ and $\nu F>0$, then $\mu f^{-1}[F]>0$, so there is an $E\subseteq f^{-1}[F]$ such that $\mu E>0$ and $f\restriction E$ is continuous. Next, there is a compact set $K\subseteq E$ such that $\mu K>0$. In this case, $L=f[K]$ is a compact subset of F, and

$$\nu L=\mu f^{-1}[L]\ge\mu K>0.$$

By 412B, this is enough to prove that ν is tight. **Q** Note that because ν is locally finite, $\nu L<\infty$ for every compact $L\subseteq Y$ (411Ga).

(c) Because μ is complete, so is ν (234Eb[26]). Next, ν is locally determined. **P** Suppose that $H\subseteq Y$ is such that $H\cap F\in\mathrm{T}$ whenever $\nu F<\infty$. Then, in particular, $H\cap f[K]\in\mathrm{T}$ whenever $K\subseteq X$ is compact and $f\restriction K$ is continuous. But setting

$$\mathcal{K}=\{K:K\subseteq X\text{ is compact, }f\restriction K\text{ is continuous}\},$$

μ is inner regular with respect to $\mathcal{K}$ (412Ac). And if $K\in\mathcal{K}$,

$$K\cap f^{-1}[H]=K\cap f^{-1}[H\cap f[K]]\in\Sigma.$$

Because μ is complete and locally determined, this is enough to show that $f^{-1}[H]\in\Sigma$ (412Ja), that is, $H\in\mathrm{T}$. As H is arbitrary, ν is locally determined. **Q**

(d) Thus ν is a complete locally determined locally finite topological measure which is inner regular with respect to the compact sets; that is, it is a Radon measure.

418J Theorem Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a second-countable space (for instance, Y might be separable and metrizable), and $f:X\to Y$ is measurable. Then f is almost continuous.

proof Let $\mathcal{H}$ be a countable base for the topology of Y, and $\langle H_n\rangle_{n\in\mathbb{N}}$ a sequence running over $\mathcal{H}\cup\{\emptyset\}$. Take $E\in\Sigma$ and $\gamma<\mu E$. Choose $\langle E_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. There is an $E_0\in\Sigma$ such that $E_0\subseteq E$ and $\gamma<\mu E_0<\infty$. Given $E_n\in\Sigma$ with $\gamma<\mu E_n<\infty$, $E_n\setminus f^{-1}[H_n]\in\Sigma$, so there is a closed set $F_n\in\Sigma$ such that

$$F_n\subseteq E_n\setminus f^{-1}[H_n],\quad \mu((E_n\setminus f^{-1}[H_n])\setminus F_n)<\mu E_n-\gamma;$$

set $E_{n+1}=(E_n\cap f^{-1}[H_n])\cup F_n$, so that

$$E_{n+1}\in\Sigma,\quad E_{n+1}\subseteq E_n,\quad \mu E_{n+1}>\gamma,\quad E_{n+1}\setminus f^{-1}[H_n]=F_n.$$

Continue.

At the end of the induction, set $F=\bigcap_{n\in\mathbb{N}}E_n$. Then $F\subseteq E$, $\mu F\ge\gamma$, and for every $n\in\mathbb{N}$

$$F\cap f^{-1}[H_n]=F\cap E_{n+1}\cap f^{-1}[H_n]=F\setminus F_n$$

is relatively open in F. It follows that $f\restriction F$ is continuous (4A2B(a-ii)). As E, γ are arbitrary, f is almost continuous.

Remark For variations on this idea, see 418Yg, 433E and 434Yb; also 418Yh.

418K Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space and Y a separable metrizable space. Then a function $f:X\to Y$ is measurable iff it is almost continuous.

proof Put 418E and 418J together.

Remark This generalizes 256F.

[26] Formerly 212Bd.

418L In all the results above, the measure starts on the left of the diagram $f : X \to Y$; in 418H-418I, it is transferred to an image measure on Y. If X has enough compact sets, a measure can move in the reverse direction, as follows.

Theorem Let $(X,\mathfrak{T})$ be a Hausdorff space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ a Radon measure space and $f : X \to Y$ a continuous function such that whenever $F \in \mathrm{T}$ and $\nu F > 0$ there is a compact set $K \subseteq X$ such that $\nu(F \cap f[K]) > 0$. Then there is a Radon measure μ on X such that ν is the image measure μf^{-1} and the inverse-measure-preserving function f induces an isomorphism between the measure algebras of ν and μ.

proof (a) Note first that ν is inner regular with respect to $\mathcal{L} = \{f[K] : K \in \mathcal{K}\}$, where $\mathcal{K}$ is the family of compact subsets of X. **P** If $\nu F > 0$, there is a $K \in \mathcal{K}$ such that $\nu(F \cap f[K]) > 0$; now there is a closed set $F' \subseteq F \cap f[K]$ such that $\nu F' > 0$, and $K' = K \cap f^{-1}[F']$ is compact, while $f[K'] \subseteq F$ has non-zero measure. As $\mathcal{L}$ is closed under finite unions, this is enough to show that ν is inner regular with respect to $\mathcal{L}$ (412Aa). **Q**

(b) Consequently there is a disjoint set $\mathcal{L}_0 \subseteq \mathcal{L}$ such that every non-negligible $F \in \mathrm{T}$ meets some member of $\mathcal{L}_0$ in a non-negligible set (412Ib). We can express $\mathcal{L}_0$ as $\{f[K] : K \in \mathcal{K}_0\}$ where $\mathcal{K}_0 \subseteq \mathcal{K}$ is disjoint. Set $X_0 = \bigcup \mathcal{K}_0$.

(c) Set

$$\Sigma_0 = \{X_0 \cap f^{-1}[F] : F \in \mathrm{T}\}.$$

Then Σ_0 is a σ-algebra of subsets of X_0. If F, $F' \in \mathrm{T}$ and $\nu F \neq \nu F'$, then there must be some $K \in \mathcal{K}_0$ such that $f[K] \cap (F \triangle F') \neq \emptyset$, so that $X_0 \cap f^{-1}[F] \neq X_0 \cap f^{-1}[F']$; we therefore have a functional $\mu_0 : \Sigma_0 \to [0,\infty]$ defined by setting $\mu_0(X_0 \cap f^{-1}[F]) = \nu F$ whenever $F \in \mathrm{T}$. It is easy to check that μ_0 is a measure on X_0. Now μ_0 is inner regular with respect to $\mathcal{K}$. **P** If $E \in \Sigma_0$ and $\mu E > 0$, there is an $F \in \mathrm{T}$ such that $E = X_0 \cap f^{-1}[F]$ and $\nu F > 0$. There are a $K \in \mathcal{K}_0$ such that $\nu(F \cap f[K]) > 0$, and a closed set $F' \subseteq F \cap f[K]$ such that $\nu F' > 0$; now $K \cap f^{-1}[F'] = X_0 \cap f^{-1}[F']$ belongs to $\Sigma_0 \cap \mathcal{K}$, is included in E and has measure greater than 0. Because $\mathcal{K}$ is closed under finite unions, this is enough to show that μ_0 is inner regular with respect to $\mathcal{K}$. **Q**

(d) Set

$$\Sigma_1 = \{E : E \subseteq X,\ E \cap X_0 \in \Sigma_0\}, \quad \mu_1 E = \mu_0(E \cap X_0) \text{ for every } E \in \Sigma_1.$$

Then μ_1 is a measure on X (being the image measure $\mu_0 \iota^{-1}$, where $\iota : X_0 \to X$ is the identity map), and is inner regular with respect to $\mathcal{K}$. If $F \in \mathrm{T}$, then

$$\mu_1 f^{-1}[F] = \mu_0(X_0 \cap f^{-1}[F]) = \nu F,$$

so f is inverse-measure-preserving for μ_1 and ν. Consequently μ_1 is locally finite. **P** If $x \in X$, there is an open set $H \subseteq Y$ such that $f(x) \in H$ and $\nu H < \infty$; now $f^{-1}[H]$ is an open subset of X of finite measure containing x. **Q** In particular, $\mu_1^* K < \infty$ for every compact $K \subseteq X$ (411Ga).

(e) By 413O, there is an extension of μ_1 to a complete locally determined measure μ on X which is inner regular with respect to $\mathcal{K}$, defined on every member of $\mathcal{K}$, and such that whenever E belongs to the domain Σ of μ and $\mu E < \infty$, there is an $E_1 \in \Sigma_1$ such that $\mu(E \triangle E_1) = 0$. Now μ is locally finite because μ_1 is, so μ is a Radon measure; and f is inverse-measure-preserving for μ and ν because it is inverse-measure-preserving for μ_1 and ν.

(f) The image measure μf^{-1} extends ν, so is locally finite, and is therefore a Radon measure (418I); since it agrees with ν on the compact subsets of Y, it must be identical with ν.

(g) I have still to check that the corresponding measure-preserving homomorphism π from the measure algebra $\mathfrak{B}$ of ν to the measure algebra $\mathfrak{A}$ of μ is actually an isomorphism, that is, is surjective. If $a \in \mathfrak{A}$ and $\bar{\mu} a < \infty$, we can find $E \in \Sigma$ such that $E^\bullet = a$ and $E_1 \in \Sigma_1$ such that $\mu(E \triangle E_1) = 0$. Now $E_1 \cap X_0 = f^{-1}[F] \cap X_0$ for some $F \in \mathrm{T}$; but in this case

$$\mu(E_1 \triangle f^{-1}[F]) = \mu_1(E_1 \triangle f^{-1}[F]) = 0, \quad a = E_1^\bullet = (f^{-1}[F])^\bullet = \pi F^\bullet.$$

Accordingly $\pi[\mathfrak{B}]$ includes $\{a : \bar{\mu} a < \infty\}$, and is order-dense in $\mathfrak{A}$. But as π is injective and $\mathfrak{B}$ is Dedekind complete (being the measure algebra of a Radon measure, which is strictly localizable), it follows that $\pi[\mathfrak{B}] = \mathfrak{A}$ (314Ia). Thus π is an isomorphism, as required.

Remarks Of course this result is most commonly applied when X and Y are both compact and f is a surjection, in which case the condition

(*) whenever $F \in \mathrm{T}$ and $\nu F > 0$ there is a compact set $K \subseteq X$ such that $\nu(F \cap f[K]) > 0$

is trivially satisfied.

Evidently (*) is necessary if there is to be any Radon measure on X for which f is inverse-measure-preserving, so in this sense the result is best possible. In 433D, however, there is a version of the theorem in which f is not required to be continuous.

418M Prokhorov's theorem Suppose that $(I,\le)$, $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$, $\langle f_{ij}\rangle_{i\le j\in I}$, $(X,\mathfrak{T})$ and $\langle g_i\rangle_{i\in I}$ are such that

$(I,\le)$ is a non-empty upwards-directed partially ordered set,

every $(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)$ is a Radon probability space,

$f_{ij}:X_j\to X_i$ is an inverse-measure-preserving function whenever $i\le j$ in I,

$(X,\mathfrak{T})$ is a Hausdorff space,

$g_i:X\to X_i$ is a continuous function for every $i\in I$,

$g_i=f_{ij}g_j$ whenever $i\le j$ in I.

Suppose moreover that

for every $\epsilon>0$ there is a compact set $K\subseteq X$ such that $\mu_i g_i[K]\ge 1-\epsilon$ for every $i\in I$.

Then there is a Radon probability measure μ on X such that every g_i is inverse-measure-preserving for μ. If the family $\langle g_i\rangle_{i\in I}$ separates the points of X, then μ is uniquely defined.

proof (a) Set

$$\mathrm{T}=\{g_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}\subseteq\mathcal{P}X.$$

Then T is a subalgebra of $\mathcal{P}X$. **P** (i) There is an $i\in I$, so $\emptyset=g_i^{-1}[\emptyset]$ belongs to T. (ii) If $H\in\mathrm{T}$ there are $i\in I$, $E\in\Sigma_i$ such that $H=g_i^{-1}[E]$; now $X\setminus H=g_i^{-1}[X_i\setminus E]$ belongs to T. (iii) If G, $H\in\mathrm{T}$, there are i, $j\in I$ and $E\in\Sigma_i$, $F\in\Sigma_j$ such that $G=g_i^{-1}[E]$ and $H=g_j^{-1}[F]$. Now I is upwards-directed, so there is a $k\in I$ such that $i\le k$ and $j\le k$. Because f_{ik} and f_{jk} are inverse-measure-preserving, $f_{ik}^{-1}[E]$ and $f_{jk}^{-1}[F]$ belong to Σ_k, so that

$$\begin{aligned}G\cap H&=g_i^{-1}[E]\cap g_j^{-1}[F]=(f_{ik}g_k)^{-1}[E]\cap(f_{jk}g_k)^{-1}[F]\\&=g_k^{-1}[f_{ik}^{-1}[E]\cap f_{jk}^{-1}[F]]\in\mathrm{T}.\ \mathbf{Q}\end{aligned}$$

(b) There is an additive functional $\nu:\mathrm{T}\to[0,1]$ defined by writing $\nu g_i^{-1}[E]=\mu_iE$ whenever $i\in I$ and $E\in\Sigma_i$.

P (i) Suppose that i, $j\in I$ and $E\in\Sigma_i$, $F\in\Sigma_j$ are such that $g_i^{-1}[E]=g_j^{-1}[F]$. Let $k\in I$ be such that $i\le k$ and $j\le k$. Then

$$g_k^{-1}[f_{ik}^{-1}[E]\triangle f_{jk}^{-1}[F]]=g_i^{-1}[E]\triangle g_j^{-1}[F]=\emptyset,$$

so $g_k[X]\cap(f_{ik}^{-1}[E]\triangle f_{jk}^{-1}[F])=\emptyset$. But now remember that for every $\epsilon>0$ there is a set $K\subseteq X$ such that $\mu_kg_k[K]\ge1-\epsilon$. This means that $\mu_kg_k[X]$ must be 1, so that $f_{ik}^{-1}[E]\triangle f_{jk}^{-1}[F]$ must be negligible, and

$$\mu_iE=\mu_kf_{ik}^{-1}[E]=\mu_kf_{jk}^{-1}[F]=\mu_jF.$$

Thus the proposed formula for ν defines a function on T.

(ii) Now suppose that G, $H\in\mathrm{T}$ are disjoint. Again, take i, $j\in I$ and $E\in\Sigma_i$, $F\in\Sigma_j$ such that $G=g_i^{-1}[E]$ and $H=g_j^{-1}[F]$, and $k\in I$ such that $i\le k$ and $j\le k$. Then

$$\begin{aligned}\nu G+\nu H&=\mu_iE+\mu_jF=\mu_kf_{ik}^{-1}[E]+\mu_kf_{jk}^{-1}[F]\\&=\mu_k(f_{ik}^{-1}[E]\cup f_{jk}^{-1}[F])+\mu_k(f_{ik}^{-1}[E]\cap f_{jk}^{-1}[F])\\&=\nu g_k^{-1}[f_{ik}^{-1}[E]\cup f_{jk}^{-1}[F]]+\nu g_k^{-1}[f_{ik}^{-1}[E]\cap f_{jk}^{-1}[F]]\\&=\nu(G\cup H)+\nu(G\cap H).\end{aligned}$$

But as $\nu\emptyset$ is certainly 0, we get $\nu(G\cup H)=\nu G+\nu H$. As G, H are arbitrary, ν is additive. **Q**

Note that $\nu X=1$.

(c) $\nu G=\sup\{\nu H:H\in\mathrm{T},\,H\subseteq G,\,H\text{ is closed}\}$ for every $G\in\mathrm{T}$. **P** If $\gamma<\nu G$, there are an $i\in I$ and an $E\in\Sigma_i$ such that $G=g_i^{-1}[E]$. In this case $\mu_iE=\nu G>\gamma$; let $L\subseteq E$ be a compact set such that $\mu_iL\ge\gamma$; then $H=g_i^{-1}[L]$ is a closed subset of G and $\nu H=\mu_iL\ge\gamma$. **Q**

$\nu X = \sup_{K\subseteq X \text{ is compact}} \inf_{G\in\mathrm{T},G\supseteq K} \nu G$. **P** If $\epsilon > 0$, there is a compact $K \subseteq X$ such that $\mu_i g_i[K] \ge 1-\epsilon$ for every $i \in I$, by the final hypothesis of the theorem. If $G \in \mathrm{T}$ and $G \supseteq K$, there are an $i \in I$ and an $E \in \Sigma_i$ such that $G = g_i^{-1}[E]$, in which case $g_i[K] \subseteq E$, so that

$$\nu G = \mu_i E \ge \mu_i g_i[K] \ge 1-\epsilon.$$

Thus $\inf_{G\in\mathrm{T},G\supseteq K} \nu G \ge 1-\epsilon$; as ϵ is arbitrary, we have the result. **Q**

This means that the conditions of 416O are satisfied, and there is a Radon measure μ extending ν. Of course this means that every g_i is inverse-measure-preserving.

(d) Now suppose that $\langle g_i\rangle_{i\in I}$ separates the points of X. Then whenever K, $L \subseteq X$ are disjoint there is an $i \in I$ such that $g_i[K] \cap g_i[L] = \emptyset$. **P** Set $V_i = \{(x,y) : x \in K,\ y \in L,\ g_i(x) = g_i(y)\}$ for $i \in I$. Because g_i is continuous and $\mathfrak{T}_i$ is Hausdorff, V_i is closed. If $i \le j$ in I, then $g_i = f_{ij}g_i$ so $V_j \subseteq V_i$; accordingly $\langle V_i\rangle_{i\in I}$ is downwards-directed. Because $\langle g_i\rangle_{i\in I}$ separates the points of X, $\bigcap_{i\in I} V_i$ is empty. As $K \times L$ is compact, there is an $i \in I$ such that $V_i = \emptyset$, that is, $g_i[K]$ and $g_i[L]$ are disjoint. **Q**

Let ν be any Radon probability measure on X such that g_i is inverse-measure-preserving for ν and μ_i for every $i \in I$. Let $K \subseteq X$ be compact. **?** If $\mu K < \nu K$ then there is a compact $L \subseteq X \setminus K$ such that $\mu L + \nu K > 1$. Let $i \in I$ be such that $g_i[K] \cap g_i[L] = \emptyset$; then

$$1 < \mu L + \nu K \le \mu g_i^{-1}[g_i[L]] + \nu g_i^{-1}[g_i[K]] = \mu_i g_i[L] + \mu_i g_i[K] \le 1,$$

which is impossible. **X** So $\nu K \le \mu K$. Similarly, $\mu K \le \nu K$. By 416Eb, $\mu = \nu$. Thus μ is uniquely determined.

418N Remarks (a) Taking I to be a singleton, we get a version of 418L in which Y is a probability space, and omitting the check that the function g induces an isomorphism of the measure algebras. Taking I to be the family of finite subsets of a set T, and every X_i to be a product $\prod_{t\in i} Z_t$ of Radon probability spaces with its product Radon measure, we obtain a method of constructing products of arbitrary families of compact probability spaces from finite products.

(b) In the hypotheses of 418M, I asked only that the f_{ij} should be measurable, and omitted any check on the compositions $f_{ij}f_{jk}$ when $i \le j \le k$. But it is easy to see that the f_{ij} must in fact be almost continuous, and that $f_{ij}f_{jk}$ must be equal almost everywhere to f_{ik} (418Xt), just as in 418P below.

(c) In the theorem as written out above, the space X and the functions $g_i : X \to X_i$ are part of the data. Of course in many applications we start with a structure

$$(\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}, \langle f_{ij}\rangle_{i\le j\in I}),$$

and the first step is to find a suitable X and g_i, as in 418O and 418P.

(d) There are important questions concerning possible relaxations of the hypotheses in 418M, especially in the special case already mentioned, in which $X_i = \prod_{t\in i} Z_t$, $f_{ij}(x) = x\restriction i$ when $i \subseteq j \in [T]^{<\omega}$, $X = \prod_{t\in T} Z_t$, and $g_i(x) = x\restriction i$ for $x \in X$ and $i \in I$, but there is no suggestion that the μ_i are product measures. For a case in which we can dispense with auxiliary topologies on the X_i, see 451Yb.

(e) A typical class of applications of Prokhorov's theorem is in the theory of stochastic processes, in which we have large families $\langle X_t\rangle_{t\in T}$ of random variables; for definiteness, imagine that $T = [0,\infty[$, so that we are looking at a system evolving over time. Not infrequently our intuition leads us to a clear description of the joint distributions ν_J of finite subfamilies $\langle X_t\rangle_{t\in J}$ without providing any suggestion of a measure space on which the whole family $\langle X_t\rangle_{t\in T}$ might be defined. (As I tried to explain in the introduction to Chapter 27, probability spaces themselves are often very shadowy things in true probability theory.) Each ν_J can be thought of as a Radon measure on $\mathbb{R}^J$, and for $I \subseteq J \in [T]^{<\omega}$ we have a natural map $f_{IJ} : \mathbb{R}^J \to \mathbb{R}^I$, setting $f_{IJ}(y) = y\restriction I$ for $y \in \mathbb{R}^J$. If our distributions ν_J mean anything at all, every f_{IJ} will surely be inverse-measure-preserving; this is simply saying that ν_I is the joint distribution of a subfamily of $\langle X_t\rangle_{t\in J}$. If we can find a Hausdorff space Ω and a continuous function $g : \Omega \to \mathbb{R}^T$ such that, for every finite $J \subseteq T$ and $\epsilon > 0$, there is a compact set $K \subseteq \Omega$ such that $\nu_J g_J[K] \ge 1-\epsilon$ (where $g_J(x) = g(x)\restriction J$), then Prokhorov's theorem will give us a measure μ on Ω which will then provide us with a suitable realization of $\langle X_t\rangle_{t\in T}$ as a family of random variables on a genuine probability space, writing $X_t(\omega) = g(\omega)(t)$. That they become continuous functions on a Radon measure space is a valuable shield against irrelevant complications.

Clearly, if this can be done at all it can be done with $\Omega = \mathbb{R}^T$; but some of the central results of probability theory are specifically concerned with the possibility of using other sets Ω (e.g., Ω a set of càllàl functions, as in 455H, or continuous functions, as in 477B).

(f) In (e) above, we do always have the option of regarding each ν_J as a measure on the compact space $[-\infty,\infty]^J$. In this case, by 418O or otherwise, we can be sure of finding a measure on $[-\infty,\infty]^T$ to support functions X_t, at the cost of either allowing the values $\pm\infty$ or (as I should myself ordinarily do) accepting that each X_t would be undefined on a negligible set. The advantage of this is just that it gives us confidence in applying the Kolmogorov-Lebesgue theory to the whole family $\langle X_t\rangle_{t\in T}$ at once, rather than to finite or countable subfamilies. For an example of what can happen if we try to do similar things with non-compact measures, see 419K. For an example of the problems which can arise with uncountable families, see 418Xu.

418O I mention two cases in which we can be sure that the projective limit $(X,\langle g_i\rangle_{i\in I})$ required in Prokhorov's theorem will exist.

Proposition Suppose that $(I,\le)$, $\langle(X_i,\mathfrak{T}_i,\mu_i,\Sigma_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\le j\in I}$ are such that

$(I,\le)$ is a non-empty upwards-directed partially ordered set,

every $(X_i,\mathfrak{T}_i,\mu_i,\Sigma_i)$ is a compact Radon measure space,

$f_{ij}:X_j\to X_i$ is a continuous inverse-measure-preserving function whenever $i\le j$ in I,

$f_{ij}f_{jk}=f_{ik}$ whenever $i\le j\le k$ in I.

Then there are a compact Hausdorff space X and a family $\langle g_i\rangle_{i\in I}$ such that I, $\langle X_i\rangle_{i\in I}$, $\langle f_{ij}\rangle_{i\le j\in I}$, X and $\langle g_i\rangle_{i\in I}$ satisfy all the hypotheses of 418M.

proof For each i, let F_i be the support of μ_i; because X_i is compact, so is F_i. If $i\le j$, then $F_i=\overline{f_{ij}[F_j]}=f_{ij}[F_j]$, by 411Ne. Set

$$X=\{x:x\in\prod_{i\in I}F_i,\,f_{ij}x(j)=x(i)\text{ whenever }i\le j\in I\},$$

$$g_i(x)=x(i)\text{ for }x\in X,\,i\in I.$$

Of course $g_i=f_{ij}g_j$ whenever $i\le j$. Also $g_i[X]\supseteq F_i$ for every $i\in I$. **P** (i) If X is empty, then there is some $j\in I$ such that $F_j=\emptyset$. In this case, taking any k greater than or equal to both i and j,

$$\mu_iX_i=\mu_kX_k=\mu_jX_j=0$$

and $F_i=\emptyset$, so we can stop. (ii) Otherwise, take any $y\in F_i$. For each finite set $J\subseteq I$,

$$H_J=\{x:x\in X,\,x(i)=y,\,f_{jk}x(k)=x(j)\text{ whenever }j\le k\in J\}$$

is a closed set. H_J is always non-empty, because if k is an upper bound of $J\cup\{i\}$ there is a $z\in F_k$ such that $f_{ik}(z)=y$, in which case $x\in H_J$ whenever $x\in X$ and $x(j)=f_{jk}(z)$ for every $j\in J\cup\{i\}$. Now $\{H_J:J\in[I]^{<\omega}\}$ is a downwards-directed family of non-empty closed sets in the compact space $\prod_{j\in I}F_j$, so has non-empty intersection, and if x is any point of the intersection then $x\in X$ and $g_i(x)=y$. **Q**

Accordingly $\mu_ig_i[X]=\mu_iX_i$ for every i; as X itself is compact, the final condition of 418M is satisfied.

418P Proposition Let $(I,\le)$, $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\le j\in I}$ be such that

$(I,\le)$ is a countable non-empty upwards-directed partially ordered set,

every $(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)$ is a Radon probability space,

$f_{ij}:X_j\to X_i$ is an inverse-measure-preserving almost continuous function whenever $i\le j$ in I,

$f_{ij}f_{jk}=f_{ik}$ μ_k-a.e. whenever $i\le j\le k$ in I.

Then there are a Radon probability space $(X,\mathfrak{T},\Sigma,\mu)$ and continuous inverse-measure-preserving functions $g_i:X\to X_i$ such that $g_i=f_{ij}g_j$ whenever $i\le j$ in I.

proof (a) We can use nearly the same formula as in 418O:

$$X=\{x:x\in\prod_{i\in I}X_i,\,f_{ij}x(j)=x(i)\text{ whenever }i\le j\in I\},$$

$$g_i(x)=x(i)\text{ for }x\in X,\,i\in I.$$

As before, the consistency relation $g_i=f_{ij}g_j$ is a trivial consequence of the definition of X. For the rest, we have to check that the final condition of 418M is satisfied. Fix $\epsilon\in\,]0,1[$. Start by taking a family $\langle\epsilon_{ij}\rangle_{i\le j\in I}$ of strictly positive numbers such that $\sum_{i\le j\in I}\epsilon_{ij}\le\frac{1}{2}\epsilon$. (This is where we need to know that I is countable.) Set $\epsilon_j=\sum_{i\le j}\epsilon_{ij}$ for each j, so that $\sum_{j\in I}\epsilon_j\le\frac{1}{2}\epsilon$.

For $i\le j\le k$ in I, set

$$E_{ijk} = \{x : x \in X_k,\ f_{ik}(x) = f_{ij}f_{jk}(x)\},$$

so that E_{ijk} is μ_k-conegligible; set $E_k = \bigcap_{i\le j\le k} E_{ijk}$, so that E_k is μ_k-conegligible. For $i \le j \in I$, choose compact sets $K_{ij} \subseteq E_j$ such that $\mu_j K_{ij} \ge 1-\epsilon_{ij}$ and $f_{ij}\restriction K_{ij}$ is continuous. Now we seem to need a three-stage construction, as follows:

for $j \in I$, set $K_j = \bigcap_{i\le j} K_{ij}$;

for $j \in I$, set $K_j^* = K_j \cap \bigcap_{i\le j} f_{ij}^{-1}[K_i]$;

finally, set $K = X \cap \prod_{j\in I} K_j^*$.

Let us trace the properties of these sets stage by stage.

(b) For each $j \in I$, $K_j \subseteq K_{jj} \subseteq E_j$ is compact and

$$\mu_j(X_j \setminus K_j) \le \sum_{i\le j} \mu_j(X_j \setminus K_{ij}) \le \sum_{i\le j} \epsilon_{ij} = \epsilon_j,$$

so that $\mu_j K_j \ge 1-\epsilon_j$. Note that f_{ik} agrees with $f_{ij}f_{jk}$ on K_k whenever $i \le j \le k$, and that $f_{ij}\restriction K_j$ is continuous whenever $i \le j$.

(c) Every K_j^* is compact, and if $i \le j \le k$ then f_{ik} agrees with $f_{ij}f_{jk}$ on K_k^*, while $f_{ij}\restriction K_j^*$ is always continuous. Also

$$\begin{aligned}\mu_j(X_j \setminus K_j^*) &\le \mu_j(X_j \setminus K_j) + \sum_{i\le j} \mu_j(X_j \setminus f_{ij}^{-1}[K_i])\\ &\le \epsilon_j + \sum_{i\le j} \epsilon_i \le \epsilon,\end{aligned}$$

so $\mu_j K_j^* \ge 1-\epsilon$, for every $j \in I$.

The point of moving from K_j to K_j^* is that $f_{jk}[K_k^*] \subseteq K_j^*$ whenever $j \le k$ in I. **P** $K_k^* \subseteq f_{jk}^{-1}[K_j]$, so $f_{jk}[K_k^*] \subseteq K_j$. If $i \le j$, then

$$K_k^* = K_k^* \cap f_{ik}^{-1}[K_i] = K_k^* \cap f_{jk}^{-1}[f_{ij}^{-1}[K_i]]$$

because $f_{ij}f_{jk}$ agrees with f_{ik} on K_k^*. So $f_{jk}[K_k^*] \subseteq f_{ij}^{-1}[K_i]$. As i is arbitrary, $f_{jk}[K_k^*] \subseteq K_j^*$. **Q**

Again because f_{ik} agrees with $f_{ij}f_{jk}$ on K_k^*, we have $f_{ik}[K_k^*] = f_{ij}[f_{jk}[K_k^*]] \subseteq f_{ij}[K_j^*]$ whenever $i \le j \le k$. And because $f_{ij}\restriction K_j^*$ is always continuous, all the sets $f_{ij}[K_j^*]$ are compact.

(d)(i) K is compact. **P**

$$K = \{x : x \in \prod_{i\in I} K_i^*,\ f_{ij}x(j) = x(i) \text{ whenever } i \le j \in I\}$$

is closed in $\prod_{i\in I} K_i^*$ because $f_{ij}\restriction K_j^*$ is always continuous (and every X_i is Hausdorff). Since $\prod_{i\in I} K_i^*$ is compact, so is K. **Q**

(ii) $\mu_i g_i[K] \ge 1-\epsilon$ for every $i \in I$. **P** By (c), $f_{ik}[K_k^*] \subseteq f_{ij}[K_j^*]$ whenever $i \le j \le k$. So $\{f_{ij}[K_j^*] : j \ge i\}$ is a downwards-directed family of compact sets; write L for their intersection. Since

$$\mu_i f_{ij}[K_j^*] = \mu_j f_{ij}^{-1}[f_{ij}[K_j^*]] \ge \mu_j K_j^* \ge 1-\epsilon$$

for every $j \ge i$, $\mu_i L \ge 1-\epsilon$ (414C). If $z \in L$, then for every $k \ge i$ the set

$$F_k = \{x : x \in \prod_{j\in I} K_j^*,\ x(k) = z,\ f_{jk}x(k) = x(j) \text{ whenever } j \le k\}$$

is a closed set in $\prod_{j\in I} K_j^*$, while $F_k \subseteq F_j$ when $j \le k$. Also F_k is non-empty, because there is a $t \in K_k^*$ such that $f_{ik}(t) = z$, and now if we take any $x \in \prod_{j\in I} K_j^*$ such that $x(j) = f_{jk}(t)$ for every $j \le k$, we shall have $x \in F_k$. So $\{F_k : k \ge i\}$ is a downwards-directed family of non-empty closed sets in a compact space, and has non-empty intersection. But if $x \in \bigcap_{k\ge i} F_k$, then $x \in K$ and $x(k) = z$, so $z \in g_i[K]$. Thus $g_i[K] \supseteq L$ and $\mu_i g_i[K] \ge 1-\epsilon$. **Q**

(e) As ϵ is arbitrary, the final condition of 418M is satisfied. But now 418M gives the result.

418Q Corollary Let $\langle(X_n, \mathfrak{T}_n, \Sigma_n, \mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of Radon probability spaces, and suppose we are given an inverse-measure-preserving almost continuous function $f_n : X_{n+1} \to X_n$ for each n. Set

$$X = \{x : x \in \prod_{n\in\mathbb{N}} X_n,\ f_n(x(n+1)) = x(n) \text{ for every } n \in \mathbb{N}\}.$$

Then there is a unique Radon probability measure μ on X such that all the coordinate maps $x \mapsto x(n) : X \to X_n$ are inverse-measure-preserving.

proof For $i \le j \in \mathbb{N}$, define $f_{ij} : X_j \to X_i$ by writing

$$f_{ii}(x) = x \text{ for every } x \in X_i,$$

$$f_{i,j+1} = f_{ij} f_j \text{ for every } j \ge i.$$

It is easy to check that $f_{ij} f_{jk} = f_{ik}$ whenever $i \le j \le k$, and that every f_{ij} is inverse-measure-preserving and almost continuous (using 418Dc). So we are exactly in the situation of 418P, and we know that there is a Radon probability measure on X for which every g_i is inverse-measure-preserving; moreover, the coordinate functionals g_i separate the points of X, so μ is unique.

418R I turn now to a special kind of measurable function, corresponding to a new view of product spaces.

Theorem Let X be a set, Σ a σ-algebra of subsets of X, and (Y, T, ν) a σ-finite measure space. Give $L^0(\nu)$ the topology of convergence in measure (§245). Write $\mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$ for the space of $\Sigma\widehat{\otimes}\mathrm{T}$-measurable real-valued functions on $X \times Y$, where $\Sigma\widehat{\otimes}\mathrm{T}$ is the σ-algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma,\, F \in \mathrm{T}\}$. Then for a function $f : X \to L^0(\nu)$ the following are equiveridical:

(i) $f[X]$ is separable and f is measurable;

(ii) there is an $h \in \mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$ such that $f(x) = h_x^\bullet$ for every $x \in X$, where $h_x(y) = h(x, y)$ for $x \in X$, $y \in Y$.

proof Let $\langle Y_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence of subsets of Y of finite measure covering Y.

(a)(i)⇒(ii) For each $n \in \mathbb{N}$, let ρ_n be the continuous pseudometric on $L^0(\nu)$ defined by saying that $\rho_n(g_1^\bullet, g_2^\bullet) = \int_{Y_n} \min(1, |g_1 - g_2|) d\nu$ for g_1, $g_2 \in \mathcal{L}^0(\mathrm{T})$, writing $\mathcal{L}^0(\mathrm{T})$ for the space of T-measurable real-valued functions on Y (245A). Then $\{\rho_n : n \in \mathbb{N}\}$ defines the topology of $L^0(\nu)$ (see the proof of 245Eb). Because $f[X]$ is separable, there is a sequence $\langle v_k \rangle_{k \in \mathbb{N}}$ in $L^0(\nu)$ such that $f[X] \subseteq \overline{\{v_k : k \in \mathbb{N}\}}$. For each k, choose $g_k \in \mathcal{L}^0(\mathrm{T})$ such that $g_k^\bullet = v_k$. For n, $k \in \mathbb{N}$ set

$$E_{nk} = \{x : x \in X,\, \rho_n(f(x), v_k) < 2^{-n}\},$$

$$H_{nk} = E_{nk} \setminus \bigcup_{i<k} E_{ni}.$$

Then every E_{nk} belongs to Σ (because f is measurable) and $\bigcup_{k \in \mathbb{N}} E_{nk} = X$ (because $\{v_k : k \in \mathbb{N}\}$ is dense); so $\langle H_{nk} \rangle_{k \in \mathbb{N}}$ is a partition of X into measurable sets. Set $h^{(n)}(x, y) = g_k(y)$ whenever $k \in \mathbb{N}$, $x \in H_{nk}$ and $y \in Y$; then $h^{(n)} \in \mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$.

Fix $x \in X$ for the moment. Then for each $n \in \mathbb{N}$ there is a unique k_n such that $x \in H_{nk_n}$, and $\rho_n(f(x), v_{k_n}) \le 2^{-n}$. So if $n \le m$,

$$\begin{aligned}\int_{Y_n} \min(1, |h_x^{(m+1)} - h_x^{(m)}|) &= \int_{Y_n} \min(1, |g_{k_{m+1}} - g_{k_m}|) = \rho_n(g_{k_{m+1}}^\bullet, g_{k_m}^\bullet) \\ &\le \rho_n(g_{k_{m+1}}^\bullet, f(x)) + \rho_n(f(x), g_{k_m}^\bullet) \\ &\le \rho_{m+1}(g_{k_{m+1}}^\bullet, f(x)) + \rho_m(f(x), g_{k_m}^\bullet) \le 3 \cdot 2^{-m-1}.\end{aligned}$$

But this means that $\sum_{m=0}^\infty \int_{Y_n} \min(1, |h_x^{(m+1)} - h_x^{(m)}|)$ is finite, so that $\langle h_x^{(m)} \rangle_{m \in \mathbb{N}}$ must be convergent almost everywhere in Y_n. As this is true for every n, $\langle h_x^{(m)} \rangle_{m \in \mathbb{N}}$ is convergent a.e. on Y. Moreover,

$$\lim_{m \to \infty} (h_x^{(m)})^\bullet = \lim_{m \to \infty} g_{k_m}^\bullet = f(x)$$

in $L^0(\nu)$.

Since this is true for every x,

$$W = \{(x, y) : \langle h^{(m)}(x, y) \rangle_{m \in \mathbb{N}} \text{ converges in } \mathbb{R}\}$$

has conegligible vertical sections, while of course $W \in \Sigma\widehat{\otimes}\mathrm{T}$ because every $h^{(m)}$ is $\Sigma\widehat{\otimes}\mathrm{T}$-measurable (418C). If we set $h(x, y) = \lim_{m \to \infty} h^{(m)}(x, y)$ for $(x, y) \in W$, 0 for other $(x, y) \in X \times Y$, then $h \in \mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$, while (by 245Ca)

$$h_x^\bullet = \lim_{m \to \infty} (h_x^{(m)})^\bullet = f(x)$$

in $L^0(\nu)$ for every $x \in X$. So we have a suitable h.

(b)(ii)⇒(i) Let Φ be the set of those $h \in \mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$ such that (i) is satisfied; that is, $x \mapsto h_x^\bullet$ is measurable, and $\{h_x^\bullet : x \in X\}$ is separable.

(α) Φ is closed under addition. **P** If h, $\tilde{h}$ belong to Φ, set $A=\{h_x^\bullet : x\in X\}$, $\tilde{A}=\{\tilde{h}_x^\bullet : x\in X\}$. Then both A and $\tilde{A}$ are separable metrizable spaces, so $A\times\tilde{A}$ is separable and metrizable and $x\mapsto(h_x^\bullet,\tilde{h}_x^\bullet):X\to A\times\tilde{A}$ is measurable (418Bb). But addition on $L^0(\nu)$ is continuous (245Da), so

$$x\mapsto h_x^\bullet+\tilde{h}_x^\bullet=(h+\tilde{h})_x^\bullet$$

is measurable (418Ac), and

$$\{(h+\tilde{h})_x^\bullet : x\in X\}\subseteq\{u+\tilde{u} : u\in A,\,\tilde{u}\in\tilde{A}\}$$

is separable (4A2B(e-iii), 4A2P(a-iv)). Thus $h+\tilde{h}\in\Phi$. **Q**

(β) Φ is closed under scalar multiplication, just because $u\mapsto\alpha u:L^0(\nu)\to L^0(\nu)$ is always continuous.

(γ) If $\langle h^{(n)}\rangle_{n\in\mathbb{N}}$ is a sequence in Φ and $h(x,y)=\lim_{n\to\infty}h^{(n)}(x,y)$ for all $x\in X$, $y\in Y$, then $h\in\Phi$. **P** Setting $A_n=\{(h_x^{(n)})^\bullet : x\in X\}$ for each n, then $A=\{h_x^\bullet : x\in X\}$ is included in $\overline{\bigcup_{n\in\mathbb{N}}A_n}$, which is separable (4A2B(e-i) again), so A is separable (4A2P(a-iv) again); moreover, $h_x^\bullet=\lim_{n\to\infty}(h_x^{(n)})^\bullet$ for every $x\in X$, so $x\mapsto h_x^\bullet$ is measurable, by 418Ba. **Q**

(δ) What this means is that if we set $\mathcal{W}=\{W : W\in\Sigma\widehat{\otimes}\mathrm{T},\,\chi W\in\Phi\}$, then $W\setminus W'\in\mathcal{W}$ whenever W, $W'\in\mathcal{W}$ and $W'\subseteq W$, and $\bigcup_{n\in\mathbb{N}}W_n\in\mathcal{W}$ whenever $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}$. Also, it is easy to see that $E\times F\in\mathcal{W}$ whenever $E\in\Sigma$ and $F\in\mathrm{T}$. By the Monotone Class Theorem (136B), $\mathcal{W}$ includes the σ-algebra generated by $\{E\times F : E\in\Sigma,\,F\in\mathrm{T}\}$, that is, is equal to $\Sigma\widehat{\otimes}\mathrm{T}$. It follows at once, from (α) and (β), that $\sum_{i=0}^n\alpha_i\chi W_i\in\Phi$ whenever $W_0,\dots,W_n\in\Sigma\widehat{\otimes}\mathrm{T}$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$, and hence (using ($\gamma$)) that $\mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})\subseteq\Phi$, which is what we had to prove.

418S Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces with c.l.d. product $(X\times Y,\Lambda,\lambda)$. Give $L^0(\nu)$ the topology of convergence in measure. Write $\mathcal{L}^0(\lambda)$ for the space of Λ-measurable real-valued functions defined λ-a.e. on $X\times Y$, as in 241A.

(a) If $h\in\mathcal{L}^0(\lambda)$, set $h_x(y)=h(x,y)$ whenever this is defined. Then

$$\{x : f(x)=h_x^\bullet\text{ is defined in }L^0(\nu)\}$$

is μ-conegligible, and includes a conegligible set X_0 such that $f:X_0\to L^0(\nu)$ is measurable and $f[X_0]$ is separable.

(b) If $f:X\to L^0(\nu)$ is measurable and there is a conegligible set $X_0\subseteq X$ such that $f[X_0]$ is separable, then there is an $h\in\mathcal{L}^0(\lambda)$ such that $f(x)=h_x^\bullet$ for almost every $x\in X$.

proof (a) The point is that λ is just the completion of its restriction to $\Sigma\widehat{\otimes}\mathrm{T}$ (251K). So there is a conegligible set $W\in\Sigma\widehat{\otimes}\mathrm{T}$ such that $h\restriction W$ is $\Sigma\widehat{\otimes}\mathrm{T}$-measurable (212Fa). Setting $\tilde{h}(x,y)=h(x,y)$ for $(x,y)\in W$, 0 otherwise, and setting $\tilde{f}(x)=\tilde{h}_x^\bullet$ for every $x\in X$, we see from 418R that $\tilde{f}$ is measurable and that $\tilde{f}[X]$ is separable. But 252D tells us that

$$X_0=\{x : ((X\times Y)\setminus W)[\{x\}]\text{ is negligible}\}$$

is conegligible; and if $x\in X_0$ then $h_x=\tilde{h}_x$ ν-a.e., so that $f(x)$ is defined and equal to $\tilde{f}(x)$. This proves the result.

(b) $f\restriction X_0$ satisfies 418R(i). So, setting $f_1(x)=f(x)$ for $x\in X_0$, 0 otherwise, there is some $h\in\mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$ such that $f_1(x)=h_x^\bullet$ for every x, so that $f(x)=h_x^\bullet$ for almost every x, and (ii) is true.

418T Corollary (Mauldin & Stone 81) Let (Y,T,ν) be a σ-finite measure space, and $(\mathfrak{B},\bar{\nu})$ its measure algebra, with its measure-algebra topology (§323).

(a) Let X be a set, Σ a σ-algebra of subsets of X, and $f:X\to\mathfrak{B}$ a function. Then the following are equiveridical:

(i) $f[X]$ is separable and f is measurable;

(ii) there is a $W\in\Sigma\widehat{\otimes}\mathrm{T}$ such that $f(x)=W[\{x\}]^\bullet$ for every $x\in X$.

(b) Let (X,Σ,μ) be a σ-finite measure space and Λ the domain of the c.l.d. product measure λ on $X\times Y$.

(i) Suppose that ν is complete. If $W\in\Lambda$, then

$$\{x : f(x)=W[\{x\}]^\bullet\text{ is defined in }\mathfrak{B}\}$$

is μ-conegligible, and includes a conegligible set X_0 such that $f:X_0\to\mathfrak{B}$ is measurable and $f[X_0]$ is separable.

(ii) If $f:X\to\mathfrak{B}$ is measurable and there is a conegligible set $X_0\subseteq X$ such that $f[X_0]$ is separable, then there is a $W\in\Sigma\widehat{\otimes}\mathrm{T}$ such that $f(x)=W[\{x\}]^\bullet$ for almost every $x\in X$.

proof Everything follows directly from 418R and 418S if we observe that $\mathfrak{B}$ is homeomorphically embedded in $L^0(\nu)$ by the function $F^\bullet \mapsto (\chi F)^\bullet$ for $F \in \mathrm{T}$ (323Xa, 367R). We do need to check, for (i)$\Rightarrow$(ii) of part (a), that if $h \in \mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$ and $h_x^\bullet$ is always of the form $(\chi F)^\bullet$, then there is some $W \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $h_x^\bullet = (\chi W[\{x\}])^\bullet$ for every x; but of course this is true if we just take $W = \{(x,y) : h(x,y) = 1\}$. Now (b-ii) follows from (a) just as 418Sb followed from 418R.

***418U Independent families of measurable functions** In §455 we shall have occasion to look at independent families of random variables taking values in spaces other than $\mathbb{R}$. We can use the same principle as in §272: a family $\langle X_i\rangle_{i\in I}$ of random variables is independent if $\langle\Sigma_i\rangle_{i\in I}$ is independent, where Σ_i is the σ-subalgebra defined by X_i for each i (272D). Of course this depends on agreement about the definition of Σ_i. The natural thing to do, in the context of this section, is to follow 272C, as follows. Let (X,Σ,μ) be a probability space, Y a topological space, and f a Y-valued function defined on a conegligible subset $\operatorname{dom} f$ of X, which is μ-virtually measurable, that is, such that f is measurable with respect to the subspace σ-algebra on $\operatorname{dom} f$ induced by $\hat\Sigma = \operatorname{dom}\hat\mu$, where $\hat\mu$ is the completion of μ. (Note that if Y is not second-countable this may not imply that $f\restriction D$ is Σ-measurable for a conegligible subset D of X.) The 'σ-algebra defined by f' will be

$$\{f^{-1}[F] : F \in \mathcal{B}(Y)\} \cup \{(\Omega \setminus \operatorname{dom} f) \cup f^{-1}[F] : F \in \mathcal{B}(Y)\} \subseteq \hat\Sigma,$$

where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y; that is, the σ-algebra of subsets of X generated by $\{f^{-1}[G] : G \subseteq Y$ is open$\}$.

Now, given a family $\langle(f_i,Y_i)\rangle_{i\in I}$ where each Y_i is a topological space and each f_i is a $\hat\Sigma$-measurable Y_i-valued function defined on a conegligible subset of X, I will say that $\langle f_i\rangle_{i\in I}$ is **independent** if $\langle\Sigma_i\rangle_{i\in I}$ is independent (with respect to $\hat\mu$), where Σ_i is the σ-algebra defined by f_i for each i.

Corresponding to 272D, we can use the Monotone Class Theorem to show that $\langle f_i\rangle_{i\in I}$ is independent iff

$$\hat\mu(\bigcap_{j\le n} f_{i_j}^{-1}[G_j]) = \prod_{j\le n}\hat\mu f_{i_j}^{-1}[G_j]$$

whenever $i_0,\dots,i_n \in I$ are distinct and $G_j \subseteq Y_{i_j}$ is open for every $j \le n$.

418X Basic exercises >(a) Let (X,Σ,μ) be a measure space, Y a set and $h : X \to Y$ a function; give Y the image measure μh^{-1}. Show that for any function g from Y to a topological space Z, g is measurable iff $gh : X \to Z$ is measurable.

>(b) Let X be a set, Σ a σ-algebra of subsets of X, $\langle Y_n\rangle_{n\in\mathbb{N}}$ a sequence of topological spaces with product Y, and $f : X \to Y$ a function. Show that f is measurable iff $\psi_n f : X \to \prod_{i\le n} Y_i$ is measurable for every $n \in \mathbb{N}$, where $\psi_n(y) = (y(0),\dots,y(n))$ for $y \in Y$, $n \in \mathbb{N}$.

(c) Let (X,Σ,μ) be a semi-finite measure space, $(Y,\mathfrak{S})$ a metrizable space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of measurable functions from X to Y such that $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ is convergent for almost every $x \in X$. Show that μ is inner regular with respect to $\{E : \langle f_n\restriction E\rangle_{n\in\mathbb{N}}$ is uniformly convergent$\}$. (Cf. 412Xt.)

>(d) Set $Y = [0,1]^{[0,1]}$, with the product topology. For $n \in \mathbb{N}$ and $x \in [0,1]$ define $f_n(x) \in Y$ by saying that $f_n(x)(t) = \max(0, 1 - 2^n|x-t|)$ for $t \in [0,1]$. Check that (i) each f_n is continuous, therefore measurable; (ii) $f(x) = \lim_{n\to\infty} f_n(x)$ is defined in Y for every $x \in [0,1]$; (iii) for each $t \in [0,1]$, the coordinate functional $x \mapsto f(x)(t)$ is continuous except at t, and in particular is almost continuous and measurable; (iv) $f\restriction F$ is not continuous for any infinite closed set $F \subseteq [0,1]$, and in particular f is not almost continuous; (v) every subset of $[0,1]$ is of the form $f^{-1}[H]$ for some open set $H \subseteq Y$; (vi) f is not measurable; (vii) the image measure μf^{-1}, where μ is Lebesgue measure on $[0,1]$, is neither a topological measure nor tight.

(e) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space, Y a topological space, and $f : X \to Y$ a function. Suppose that for every $x \in X$ there is an open set G containing x such that $f\restriction G$ is almost continuous with respect to the subspace measure on G. Show that f is almost continuous.

(f) For $i = 1$, 2 let $(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)$ and $(Y_i,\mathfrak{S}_i,\mathrm{T}_i,\nu_i)$ be quasi-Radon measure spaces, and $f_i : X_i \to Y_i$ an almost continuous inverse-measure-preserving function. Show that $(x_1,x_2) \mapsto (f_1(x_1), f_2(x_2))$ is almost continuous and inverse-measure-preserving for the product topologies and quasi-Radon product measures.

(g) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ and $\langle(Y_i,\mathfrak{T}_i,\Sigma_i,\nu_i)\rangle_{i\in I}$ be two families of topological spaces with τ-additive Borel probability measures, and let μ, ν be the τ-additive product measures on $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$. Suppose that every ν_i is strictly positive. Show that if $f_i : X_i \to Y_i$ is almost continuous and inverse-measure-preserving for each i, then $x \mapsto \langle f_i(x(i))\rangle_{i\in I} : X \to Y$ is inverse-measure-preserving, but need not be almost continuous.

(h) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be quasi-Radon measure spaces, $(Z,\mathfrak{U})$ a topological space and $f:X\times Y\to Z$ a function which is almost continuous with respect to the quasi-Radon product measure on $X\times Y$. Suppose that ν is σ-finite. Show that $y\mapsto f(x,y)$ is almost continuous for almost every $x\in X$.

(i) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an effectively locally finite τ-additive topological measure space, Y a topological space and $f:X\to Y$ an almost continuous function. (i) Show that the image measure μf^{-1} is τ-additive. (ii) Show that if μ is a totally finite quasi-Radon measure and the topology on Y is regular, then μf^{-1} is quasi-Radon.

(j) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a topological measure space and U a linear topological space. Show that if $f:X\to U$ and $g:X\to U$ are almost continuous, then $f+g:X\to U$ is almost continuous.

(k) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be topological measure spaces, and $(Z,\mathfrak{U})$ a topological space; let $f:X\to Y$ be almost continuous and inverse-measure-preserving, and $g:Y\to Z$ almost continuous. Show that if *either* μ is a Radon measure and ν is locally finite *or* μ is τ-additive and effectively locally finite and ν is effectively locally finite, then $gf:X\to Z$ is almost continuous. (*Hint*: show that if $\mu E>0$ there is a set F such that $\nu F<\infty$ and $\mu(E\cap f^{-1}[F])>0$.)

(l) Let (X,Σ,μ) be a complete strictly localizable measure space, $\underline{\phi}:\Sigma\to\Sigma$ a lower density such that $\underline{\phi}X=X$, and $\mathfrak{T}$ the associated density topology on X (414P). Let $f:X\to\mathbb{R}$ be a function. Show that the following are equiveridical: (i) f is measurable; (ii) f is almost continuous; (iii) f is continuous at almost every point; (iv) there is a conegligible set $H\subseteq X$ such that $f{\restriction}H$ is continuous. (Cf. 414Xk.)

(m) Let (X,Σ,μ) be a complete strictly localizable measure space, $\phi:\Sigma\to\Sigma$ a lifting, and $\mathfrak{S}$ the lifting topology on X (414Q). Let $f:X\to\mathbb{R}$ be a function. Show that the following are equiveridical: (i) f is measurable; (ii) f is almost continuous; (iii) there is a conegligible set $H\subseteq X$ such that $f{\restriction}H$ is continuous. (Cf. 414Xr.)

(n) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space, $(Y,\mathfrak{S})$ a regular topological space and $f:X\to Y$ an almost continuous function. Show that there is a quasi-Radon measure ν on Y such that f is inverse-measure-preserving for μ and ν iff $\bigcup\{f^{-1}[H]:H\subseteq Y\text{ is open},\ \mu f^{-1}[H]<\infty\}$ is conegligible in X.

(o) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, $(Y,\mathfrak{S})$ and $(Z,\mathfrak{U})$ Hausdorff spaces, $f:X\to Y$ an almost continuous function such that $\nu=\mu f^{-1}$ is locally finite, and $g:Y\to Z$ a function. Show that g is almost continuous with respect to ν iff gf is almost continuous with respect to μ.

(p) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be topological probability spaces, and $f:X\to Y$ a measurable function such that $\mu f^{-1}[H]\ge\nu H$ for every $H\in\mathfrak{S}$. Show that (i) $\int gf\,d\mu=\int g\,d\nu$ for every $g\in C_b(Y)$ (ii) $\mu f^{-1}[F]=\nu F$ for every Baire set $F\subseteq Y$ (iii) if μ is a Radon measure and f is almost continuous, then $\mu f^{-1}[F]=\nu F$ for every Borel set $F\subseteq Y$, so that if in addition ν is complete and inner regular with respect to the Borel sets then it is a Radon measure.

(q) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite topological measure space in which the topology $\mathfrak{T}$ is normal and μ is outer regular with respect to the open sets. Show that if $f:X\to\mathbb{R}$ is a measurable function and $\epsilon>0$ there is a continuous $g:X\to\mathbb{R}$ such that $\mu\{x:g(x)\ne f(x)\}\le\epsilon$.

(r) Let X and Y be Hausdorff spaces, ν a totally finite Radon measure on Y, and $f:X\to Y$ an injective continuous function. Show that the following are equiveridical: (i) there is a Radon measure μ on X such that f is inverse-measure-preserving; (ii) $f[X]$ is conegligible and $f^{-1}:f[X]\to X$ is almost continuous.

(s) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces and $f:X\to Y$ an almost continuous inverse-measure-preserving function. Show that (i) $\mu_*A\le\nu_*f[A]$ for every $A\subseteq X$ (ii) ν is precisely the image measure μf^{-1}.

(t) In 418M, show that all the f_{ij} must be almost continuous. Show that if $i\le j\le k$ then $f_{ij}f_{jk}=f_{ik}$ almost everywhere in X_k.

>(u) Let $\mathcal{I}$ be the family of finite subsets of $[0,1]$, and for each $I\in\mathcal{I}$ let $(X_I,\mathfrak{T}_I,\Sigma_I,\mu_I)$ be $[0,1]\setminus I$ with its subspace topology and measure induced by Lebesgue measure. For $I\subseteq J\in\mathcal{I}$ and $y\in X_J$ set $f_{IJ}(y)=y$. Show that these X_I, f_{IJ} satisfy nearly all the hypotheses of 418O, but that there are no X, g_I which satisfy the hypotheses of 418M.

(v) Let T be any set, and X the set of total orders on T. (i) Regarding each member of X as a subset of $T\times T$, show that X is a closed subset of $\mathcal{P}(T\times T)$. (ii) Show that there is a unique Radon measure μ on X such that $\Pr(t_1\le t_2\le\ldots\le t_n)=\frac{1}{n!}$ for all distinct $t_1,\ldots,t_n\in T$. (*Hint*: for $I\in[T]^{<\omega}$, let X_I be the set of total orders on I with the uniform probability measure giving the same measure to each singleton; show that the natural map from X_I to X_J is inverse-measure-preserving whenever $J\subseteq I$.)

(w) In 418Sb, suppose that $f_1:X\to L^0(\nu)$ and $f_2:X\to L^0(\nu)$ correspond to h_1, $h_2\in\mathcal{L}^0(\lambda)$. Show that $f_1(x)\le f_2(x)$ μ-a.e.(x) iff $h_1\le h_2$ λ-a.e. Hence show that (if we assign appropriate algebraic operations to the space of functions from X to $L^0(\nu)$) we have an f-algebra isomorphism between $L^0(\lambda)$ and the space of equivalence classes of measurable functions from X to $L^0(\nu)$ with separable ranges.

(x) Let μ be Lebesgue measure on $\mathbb{R}^r$, where $r\ge1$, X a Hausdorff space and $f:\mathbb{R}^r\to X$ an almost continuous function. Show that for almost every $x\in\mathbb{R}^r$ there is a measurable set $E\subseteq\mathbb{R}^r$ such that x is a density point of E and $\lim_{y\in E,y\to x}f(y)=f(x)$.

(y) Let X be a compact Hausdorff space. Show that there is an atomless Radon probability measure on X iff X is non-scattered.

418Y Further exercises (a) Let X be a set, Σ a σ-algebra of subsets of X, Y a topological space and $f:X\to Y$ a function. Set $\mathrm{T}=\{F:F\subseteq Y,\ f^{-1}[F]\in\Sigma\}$. Suppose that Y is hereditarily Lindelöf and its topology is generated by some subset of T. Show that f is measurable.

(b) Let (X,Σ,μ) be a measure space, Y and Z topological spaces and $f:X\to Y$, $g:X\to Z$ measurable functions. Show that if Z has a countable network consisting of Borel sets (e.g., Z is second-countable, or Z is regular and has a countable network), then $x\mapsto(f(x),g(x)):X\to Y\times Z$ is measurable.

(c) Let X be a set, Σ a σ-algebra of subsets of X, and $\langle Y_i\rangle_{i\in I}$ a countable family of topological spaces with product Y. Suppose that every Y_i has a countable network, and that $f:X\to Y$ is a function such that $\pi_i f$ is measurable for every $i\in I$, writing $\pi_i(y)=y(i)$. Show that f is measurable.

(d) Find strictly localizable Hausdorff topological measure spaces $(X,\mathfrak{T},\Sigma,\mu)$, $(Y,\mathfrak{S},\mathrm{T},\nu)$ and $(Z,\mathfrak{U},\Lambda,\lambda)$ and almost continuous inverse-measure-preserving functions $f:X\to Y$, $g:Y\to Z$ such that gf is not almost continuous.

(e) Let (X,Σ,μ) be a σ-finite measure space and $\mathfrak{T}$ a topology on X such that μ is effectively locally finite and τ-additive. Let Y be a topological space and $f:X\to Y$ an almost continuous function. Show that there is a conegligible subset X_0 of X such that $f[X_0]$ is ccc.

(f) Show that if μ is Lebesgue measure on $\mathbb{R}$, $\mathfrak{T}$ is the usual topology on $\mathbb{R}$ and $\mathfrak{S}$ is the right-facing Sorgenfrey topology, then the identity map from $(\mathbb{R},\mathfrak{T},\mu)$ to $(\mathbb{R},\mathfrak{S})$ is measurable, but not almost continuous, and the image measure is not a Radon measure.

(g) Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a topological space with a countable network consisting of Borel sets, and that $f:X\to Y$ is measurable. Show that f is almost continuous.

(h) Find a topological probability space $(X,\mathfrak{T},\Sigma,\mu)$ in which μ is inner regular with respect to the closed sets, a topological space Y with a countable network and a measurable function $f:X\to Y$ which is not almost continuous.

(i) Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Let $A\subseteq L^\infty(\mu)$ be a norm-compact set. Show that there is a set B of bounded real-valued measurable functions on X such that (i) $A=\{f^\bullet:f\in B\}$ (ii) B is norm-compact in $\ell^\infty(X)$ (iii) μ is inner regular with respect to $\{E:f\restriction E$ is continuous for every $f\in B\}$.

(j) Let μ be Lebesgue measure on $[0,1]$. For $t\in[0,1]$ set $u_t=\chi[0,t]^\bullet\in L^0(\mu)$. Show that $A=\{u_t:t\in[0,1]\}$ is norm-compact in $L^p(\mu)$ for every $p\in[1,\infty[$ and also compact for the topology of convergence in measure on $L^0(\mu)$. Show that if B is a set of measurable functions such that $A=\{f^\bullet:f\in B\}$ then μ is not inner regular with respect to $\{E:f\restriction E$ is continuous for every $f\in B\}$.

(k) Suppose that $(I,\le)$, $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\le j\in I}$ are such that (α) $(I,\le)$ is a non-empty upwards-directed partially ordered set (β) every $(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)$ is a completely regular Hausdorff quasi-Radon probability space (γ) $f_{ij}:X_j\to X_i$ is a continuous inverse-measure-preserving function whenever $i\le j$ in I (δ) $f_{ij}f_{jk}=f_{ik}$ whenever $i\le j\le k$ in I. Let X'_i be the support of μ_i for each i; show that $f_{ij}[X'_j]$ is a dense subset of X'_i whenever $i\le j$. Let Z_i be the Stone-Čech compactification of X'_i and let $\tilde f_{ij}:Z_j\to Z_i$ be the continuous extension of $f_{ij}\restriction X'_j$ for $i\le j$; let $\tilde\mu_i$ be the Radon probability measure on Z_i corresponding to $\mu_i\restriction\mathcal{P}X'_i$ (416V). Show that Z_i, $\tilde f_{ij}$ satisfy the conditions of 418O, so that we have a projective limit Z, $\langle g_i\rangle_{i\in I}$, μ as in 418M.

(l) Suppose that $(I,\le)$, $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$, X, $\langle f_i\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\le j\in I}$ are such that (α) $(I,\le)$ is a non-empty upwards-directed partially ordered set (β) every (X_i,Σ_i,μ_i) is a probability space (γ) $f_{ij}:X_j\to X_i$ is inverse-measure-preserving whenever $i\le j$ in I (δ) $f_{ij}f_{jk}=f_{ik}$ whenever $i\le j\le k$ (ϵ) $f_i:X\to X_i$ is a function for every $i\in I$ (ζ) $f_i=f_{ij}f_j$ whenever $i\le j$ (η) whenever $\langle i_n\rangle_{n\in\mathbb{N}}$, $\langle x_n\rangle_{n\in\mathbb{N}}$ are such that $\langle i_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in I, $x_n\in X_{i_n}$ for every $n\in\mathbb{N}$ and $f_{i_ni_{n+1}}(x_{n+1})=x_n$ for every $n\in\mathbb{N}$, then there is an $x\in X$ such that $f_{i_n}(x)=x_n$ for every n. Show that there is a probability measure on X such that every f_i is inverse-measure-preserving.

(m) Let μ be Lebesgue measure on $[0,1]$, and $A\subseteq[0,1]$ a set with inner measure 0 and outer measure 1; let $\mathfrak{T}$ be the usual topology on $[0,1]$. Let $\mathcal{I}$ be the family of sets $I\subseteq A$ such that every point of A has a neighbourhood containing at most one point of I. Show that $\mathfrak{S}=\{G\setminus I:G\in\mathfrak{T},\,I\in\mathcal{I}\}$ is a topology on $[0,1]$ with a countable network. Show that the identity map from $[0,1]$ to itself, regarded as a map from $([0,1],\mathfrak{T},\mu)$ to $([0,1],\mathfrak{S})$, is measurable but not almost continuous.

(n) Let X be a set, Σ a σ-algebra of subsets of X and (Y,T,ν) a σ-finite measure space with countable Maharam type. (i) Let $f:X\to L^1(\nu)$ be a function such that $x\mapsto\int_F f(x)d\nu$ is Σ-measurable for every $F\in\mathrm{T}$. Show that f is Σ-measurable for the norm topology on $L^1(\nu)$. (ii) Let $g:X\times Y\to\mathbb{R}$ be a function such that $\int g(x,y)\nu(dy)$ is defined for every $x\in X$, and $x\mapsto\int_F g(x,y)\nu(dy)$ is Σ-measurable for every $F\in\mathrm{T}$. Show that there is an $h\in\mathcal{L}^0(\Sigma\otimes\mathrm{T})$ such that, for every $x\in X$, $g(x,y)=h(x,y)$ for ν-almost every y.

(o) Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y and Z are separable metrizable spaces,and $f:X\times Y\to Z$ is a function such that $x\mapsto f(x,y)$ is measurable for every $y\in Y$, and $y\mapsto f(x,y)$ is continuous for every $x\in X$. Show that μ is inner regular with respect to $\{F:F\subseteq X,\,f\restriction F\times Y\text{ is continuous}\}$. (*Hint*: Let ρ, σ be metrics defining the topologies of Y and Z. For $y\in Y$ and $n\in\mathbb{N}$ set $g_{yn}(x)=\sup\{s:\sigma(f(x,y'),f(x,y))\le 2^{-n}$ whenever $\rho(y',y)\le s\}$, $f_y(x)=f(x,y)$. Show that if $D\subseteq Y$ is dense and $F\subseteq X$ is such that $g_{yn}\restriction F$ and $f_y\restriction F$ are continuous whenever $y\in D$ and $n\in\mathbb{N}$, then $f\restriction F\times X$ is continuous.)

(p) Use 418M and 418O to prove 328H.

(q) Let X be a set, Σ a σ-algebra of subsets of X, (Y,T,ν) a σ-finite measure space and $W\in\Sigma\widehat{\otimes}\mathrm{T}$. Then there is a $V\subseteq W$ such that $V\in\Sigma\widehat{\otimes}\mathrm{T}$, $W[\{x\}]\setminus V[\{x\}]$ is negligible for every $x\in X$, and $\bigcap_{x\in I}V[\{x\}]$ is either empty or non-negligible for every finite $I\subseteq X$.

(r) Let X be a compact Hausdorff space, Y a Hausdorff space, ν a Radon probability measure on Y and $R\subseteq X\times Y$ a closed set such that $\nu^*R[X]=1$. Show that there is a Radon probability measure μ on X such that $\mu R^{-1}[F]\ge\nu F$ for every closed set $F\subseteq Y$.

418 Notes and comments The message of this section is that measurable functions are dangerous, but that almost continuous functions behave themselves. There are two fundamental problems with measurable functions: a function $x\mapsto(f(x),g(x))$ may not be measurable when the components f and g are measurable (419Xg), and an image measure under a measurable function can lose tightness, even when both domain and codomain are Radon measure spaces and the function is inverse-measure-preserving (419Xh). (This is the 'image measure catastrophe' mentioned in 235H and the notes to §343.) Consequently, as long as we are dealing with measurable functions, we often have to impose strong conditions on the range spaces – commonly, we have to restrict ourselves to separable metrizable spaces (418B, 418C), or something similar, which indeed often means that a measurable function is actually almost continuous (418J, 433E). Indeed, for functions taking values in metrizable spaces, 'almost continuity' is very close to 'measurable with essentially separable range' (418G, 418J). The condition 'separable and metrizable' is a little stronger than is strictly necessary (418Yb, 418Yc, 418Yg), but covers the principal applications other than

433E. If we keep the 'metrizable' we can very substantially relax the 'separable' (438E, 438F), and it is in fact the case that a measurable function from a Radon measure space to any metrizable space is almost continuous (451T). These extensions apply equally to the results in 418R-418T (438Xi-438Xj, 451Xp). But both take us deeper into set theory than seems appropriate at the moment.

For almost continuous functions, the two problems mentioned above do not arise (418Dd, 418I, 418Xs). Indeed we rather expect almost continuous functions to behave as if they were continuous. But we still have to be careful. The limit of a sequence of almost continuous functions need not be almost continuous (418Xd), unless the codomain is metrizable (418F); and if we have a function f from a topological measure space to an uncountable product of topological spaces, it can happen that every coordinate of f is an almost continuous function while f is not (418Xd again). But for many purposes, intuitions gained from the study of measurable functions between Euclidean spaces can be transferred to general almost continuous functions.

Theorems 418L and 418M are of a quite different kind, but seem to belong here as well as anywhere. Even in the simplest application of 418L (when $Y=[0,1]$ with Lebesgue measure, and $X\subseteq[0,1]^2$ is a closed set meeting every vertical line) it is not immediately obvious that there will be a measure with the right projection onto the horizontal axis, though there are at least two proofs which are easier than the general case treated in 413N-413O-418L.

As I explain in 418N, the really interesting question concerning 418M is when, given the projective system $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$, $\langle f_{ij}\rangle_{i\le j\in I}$, we can expect to find X and $\langle g_i\rangle_{i\in I}$ satisfying the rest of the hypotheses, and once past the elementary results 418O-418Q this can be hard to determine. I describe a method in 418Yk which can sometimes be used, but (like the trick in 418Nf) it is too easy and too abstract to be often illuminating. See 454G below for something rather deeper.

The results of 418R-418T stand somewhat aside from anything else considered in this chapter, but they form part of an important technique. A special case has already been mentioned in 253Yg. I do not discuss vector-valued measurable functions in this book, except incidentally, but 418R is one of the fundamental results on their representation; it means, for instance, that if V is any of the Banach function spaces of Chapter 36 we can expect to represent Bochner integrable V-valued functions (253Yf) in terms of functions on product spaces, because V will be continuously embedded in an L^0 space (367O). The measure-algebra version in 418T will be useful in Volume 5 when establishing relationships between properties of measure spaces and corresponding properties of measure algebras.

419 Examples

In §216, I went much of the way to describing examples of spaces with all the possible combinations of the properties considered in Chapter 21. When we come to topological measure spaces, the number of properties involved makes it unreasonable to seek any such comprehensive list. I therefore content myself with seven examples to indicate some of the boundaries of the theory developed here.

The first example (419A) is supposed to show that the hypothesis 'effectively locally finite' which appears in so many of the theorems of this chapter cannot as a rule be replaced by 'locally finite'. The next two (419C-419D) address technical questions concerning the definition of 'Radon measure', and show how small variations in the definition can lead to very different kinds of measure space. The fourth example (419E) shows that the τ-additive product measures of §417 are indeed new constructions. 419J is there to show that extension theorems of the types proved in §415 and §417 cannot be taken for granted. The classic example 419K exhibits one of the obstacles to generalizations of Prokhorov's theorem (418M, 418Q). Finally, I return to the split interval (419L) to describe its standard topology and its relation to the measure introduced in 343J.

419A Example There is a locally compact Hausdorff space X with a complete, σ-finite, locally finite, τ-additive topological measure μ, inner regular with respect to the closed sets, which has a closed subset Y, of measure 1, such that the subspace measure μ_Y on Y is not τ-additive. In particular, μ is not effectively locally finite.

proof (a) Let Q be a countably infinite set, not containing any ordinal. Fix an enumeration $\langle q_n\rangle_{n\in\mathbb{N}}$ of Q, and for $A\subseteq Q$ set $\nu A=\sum\{\frac{1}{n+1}:q_n\in A\}$. Let $\mathcal{I}$ be the ideal $\{A:A\subseteq Q,\ \nu A<\infty\}$. For any sets I, J, say that $I\subseteq^* J$ if $I\setminus J$ is finite; then $\subseteq^*$ is a reflexive transitive relation.

Let κ be the smallest cardinal of any family $\mathcal{K}\subseteq\mathcal{I}$ for which there is no $I\in\mathcal{I}$ such that $K\subseteq^* I$ for every $K\in\mathcal{K}$. Then κ is uncountable. **P** (i) Of course κ is infinite. (ii) If $\langle I_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{I}$, then for each $n\in\mathbb{N}$ we can find a finite $I'_n\subseteq I_n$ such that $\nu(I_n\setminus I'_n)\le 2^{-n}$; setting $I=\bigcup_{n\in\mathbb{N}}I_n\setminus I'_n$, we have $\nu I\le 2<\infty$, while $I_n\subseteq^* I$ for every n. Thus $\kappa>\omega$. **Q**

(b) There is a family $\langle I_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$ such that (i) $I_\eta \subseteq^* I_\xi$ whenever $\eta \le \xi < \kappa$ (ii) there is no $I \in \mathcal{I}$ such that $I_\xi \subseteq^* I$ for every $\xi < \kappa$. **P** Take a family $\langle K_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$ such that there is no $I \in \mathcal{I}$ such that $K_\xi \subseteq^* I$ for every $\xi < \kappa$. Choose $\langle I_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$ inductively in such a way that

$$K_\xi \subseteq^* I_\xi, \quad I_\eta \subseteq^* I_\xi \text{ for every } \eta < \xi.$$

(This can be done because $\{K_\xi\} \cup \{I_\eta : \eta < \xi\}$ will always be a subset of $\mathcal{I}$ of cardinal less than κ.) If now $I_\xi \subseteq^* I$ for every $\xi < \kappa$, then $K_\xi \subseteq^* I$ for every $\xi < \kappa$, so $I \notin \mathcal{I}$. **Q**

Hence, or otherwise, we see that κ is regular. **P** If $A \subseteq \kappa$ and $\#(A) < \kappa$, then there is an $I \in \mathcal{I}$ such that $I_\zeta \subseteq^* I$ for every $\zeta \in A$; now there must be a $\xi < \kappa$ such that $I_\xi \not\subseteq^* I$, in which case $\zeta < \xi$ for every $\zeta \in A$, and A is not cofinal with κ. **Q**

(c) Set $X = Q \cup \kappa$. (This is where it is helpful to have arranged at the start that no ordinal belongs to Q, so that $Q \cap \kappa = \emptyset$.) Let $\mathfrak{T}$ be the family of sets $G \subseteq X$ such that

$G \cap \kappa$ is open for the order topology of κ,

for every $\xi \in G \cap \kappa \setminus \{0\}$ there is an $\eta < \xi$ such that $I_\xi \setminus I_\eta \subseteq^* G$,

if $0 \in G$ then $I_0 \subseteq^* G$.

(i) This is a Hausdorff topology on X. **P** (α) It is easy to check that $X \in \mathfrak{T}$, $\emptyset \in \mathfrak{T}$ and $\bigcup \mathcal{G} \in \mathfrak{T}$ for every $\mathcal{G} \subseteq \mathfrak{T}$. (β) Suppose that G, $H \in \mathfrak{T}$. Then $(G \cap H) \cap \kappa = (G \cap \kappa) \cap (H \cap \kappa)$ is open for the order topology of κ. If $\xi \in G \cap H \cap \kappa \setminus \{0\}$ there are η, $\zeta < \xi$ such that $I_\xi \setminus I_\eta \subseteq^* G$ and $I_\xi \setminus I_\zeta \subseteq^* H$, and now

$$\alpha = \max(\eta, \zeta) < \xi, \quad I_\eta \cup I_\zeta \subseteq^* I_\alpha,$$

so

$$I_\xi \setminus I_\alpha \subseteq^* (I_\xi \setminus I_\eta) \cap (I_\xi \setminus I_\zeta) \subseteq^* G \cap H.$$

Finally, if $0 \in G \cap H$ then $I_0 \subseteq^* G \cap H$. So $G \cap H \in \mathfrak{T}$. Thus $\mathfrak{T}$ is a topology on X. (γ) For any $\xi < \kappa$, the set $E_\xi = (\xi+1) \cup I_\xi$ is open-and-closed for $\mathfrak{T}$; for any $q \in Q$, $\{q\}$ is open-and-closed. Since these sets separate the points of X, $\mathfrak{T}$ is Hausdorff. **Q**

(ii) The sets E_ξ of the last paragraph are all compact for $\mathfrak{T}$. **P** Let $\mathcal{F}$ be an ultrafilter on X containing E_ξ. (α) If a finite set K belongs to $\mathcal{F}$, then $\mathcal{F}$ must contain $\{x\}$ for some $x \in K$, and converges to x. So suppose henceforth that $\mathcal{F}$ contains no finite set. (β) If $E_0 \in \mathcal{F}$, then for any open set G containing 0, $E_0 \setminus G$ is finite, so does not belong to $\mathcal{F}$, and $G \in \mathcal{F}$; as G is arbitrary, $\mathcal{F} \to 0$. (γ) If $E_0 \notin \mathcal{F}$, let $\eta \le \xi$ be the least ordinal such that $E_\eta \in \mathcal{F}$. If G is an open set containing η, there are ζ', $\zeta'' < \eta$ such that $I_\eta \setminus I_{\zeta'} \subseteq^* G$ and $]\zeta'', \eta] \subseteq G$; so that $E_\eta \setminus E_\zeta \subseteq^* G$, where $\zeta = \max(\zeta', \zeta'') < \eta$. Now $E_\eta \in \mathcal{F}$, $E_\zeta \notin \mathcal{F}$ and $(E_\eta \setminus E_\zeta) \setminus G \notin \mathcal{F}$, so that $G \in \mathcal{F}$. As G is arbitrary, $\mathcal{F} \to \eta$. (δ) As $\mathcal{F}$ is arbitrary, E_ξ is compact. **Q**

(iii) It follows that $\mathfrak{T}$ is locally compact. **P** For $q \in Q$, $\{q\}$ is a compact open set containing q; for $\xi < \kappa$, E_ξ is a compact open set containing ξ. **Q**

(iv) The definition of $\mathfrak{T}$ makes it clear that $Q \in \mathfrak{T}$, that is, that κ is a closed subset of X. We need also to check that the subspace topology $\mathfrak{T}_\kappa$ on κ induced by $\mathfrak{T}$ is just the order topology of κ. **P** (α) By the definition of $\mathfrak{T}$, $G \cap \kappa$ is open for the order topology of κ for every $G \in \mathfrak{T}$. (β) For any $\xi < \kappa$, E_ξ is open-and-closed for $\mathfrak{T}$ so $\xi + 1 = E_\xi \cap \kappa$ is open-and-closed for $\mathfrak{T}_\kappa$. But this means that all sets of the forms $[0, \xi[= \bigcup_{\eta<\xi} \eta + 1$ and $]\xi, \kappa[= \kappa \setminus (\xi + 1)$ belong to $\mathfrak{T}_\kappa$; as these generate the order topology, every open set for the order topology belongs to $\mathfrak{T}_\kappa$, and the two topologies are equal. **Q**

(d) Now let $\mathcal{F}$ be the filter on X generated by the cofinal closed sets in κ. Because the intersection of any sequence of closed cofinal sets in κ is another (4A1Bd), the intersection of any sequence in $\mathcal{F}$ belongs to $\mathcal{F}$. So

$$\Sigma = \mathcal{F} \cup \{X \setminus F : F \in \mathcal{F}\}$$

is a σ-algebra of subsets of X, and we have a measure $\mu_1 : \Sigma \to \{0, 1\}$ defined by saying that $\mu_1 F = 1$, $\mu_1(X \setminus F) = 0$ if $F \in \mathcal{F}$.

(e) Set $\mu E = \nu(E \cap Q) + \mu_1 E$ for $E \in \Sigma$. Then μ is a measure. Let us work through the properties called for.

(i) If $\mu E = 0$ and $A \subseteq E$, then $X \setminus A \supseteq X \setminus E \in \mathcal{F}$, so $A \in \Sigma$. Thus μ is complete.

(ii) $\mu\kappa = 1$ and $\mu\{q\}$ is finite for every $q \in Q$, so μ is σ-finite.

(iii) If $G \subseteq X$ is open, then $\kappa \setminus G$ is closed, in the order topology of κ; if it is cofinal with κ, it belongs to $\mathcal{F}$; otherwise, $\kappa \cap G \in \mathcal{F}$. Thus in either case $G \in \Sigma$, and μ is a topological measure.

(iv) The next thing to note is that $\mu G=\nu(G\cap Q)$ for every open set $G\subseteq X$. **P** If $G\notin\mathcal{F}$ this is trivial. If $G\in\mathcal{F}$, then $\kappa\setminus G$ cannot be cofinal with κ, so there is a $\xi<\kappa$ such that $\kappa\setminus\xi\subseteq G$. **?** If $G\cap Q\in\mathcal{I}$, then $(G\cap Q)\cup I_\xi\in\mathcal{I}$. There must be a least $\eta<\kappa$ such that $I_\eta\not\subseteq^*(G\cap Q)\cup I_\xi$; of course $\eta>\xi$, so $\eta\in G$. There is some $\zeta<\eta$ such that $I_\eta\setminus I_\zeta\subseteq^* G$; but as $I_\zeta\subseteq^* G\cup I_\xi$, by the choice of η, we must also have $I_\eta\subseteq^* G\cup I_\xi$, which is impossible. **X** Thus $G\cap Q\notin\mathcal{I}$ and $\mu G=\nu(G\cap Q)=\infty$. **Q**

(v) It follows that μ is τ-additive. **P** Suppose that $\mathcal{G}\subseteq\mathfrak{T}$ is a non-empty upwards-directed set with union H. Then

$$\mu H=\nu(H\cap Q)=\sup_{G\in\mathcal{G}}\nu(G\cap Q)=\sup_{G\in\mathcal{G}}\mu G$$

because ν is τ-additive (indeed, is a Radon measure) with respect to the discrete topology on Q. **Q**

(vi) μ is inner regular with respect to the closed sets. **P** Take $E\in\Sigma$ and $\gamma<\mu E$. Then $\gamma-\mu_1E<\nu(E\cap Q)$, so there is a finite $I\subseteq E\cap Q$ such that $\nu I>\gamma-\mu_1E$. If $\mu_1E=0$, then $I\subseteq E$ is already a closed set with $\mu I>\gamma$. Otherwise, $E\cap\kappa\in\mathcal{F}$, so there is a cofinal closed set $F\subseteq\kappa$ such that $F\subseteq E$; now F is closed in X (because κ is closed in X and the subspace topology on κ is the order topology), so $I\cup F$ is closed, and $\mu(I\cup F)>\gamma$. As E and γ are arbitrary, μ is inner regular with respect to the closed sets. **Q**

(vii) μ is locally finite. **P** For any $\xi<\kappa$, E_ξ is an open set containing ξ, and $\mu E_\xi=\nu I_\xi$ is finite. For any $q\in Q$, $\{q\}$ is an open set containing q, and $\mu\{q\}=\nu\{q\}$ is finite. **Q**

(viii) Now consider $Y=\kappa$. This is surely a closed set, and $\mu\kappa=1$. I noted in (c-iv) above that the subspace topology $\mathfrak{T}_\kappa$ is just the order topology of κ. But this means that $\{\xi:\xi<\kappa\}$ is an upwards-directed family of negligible relatively open sets with union κ, so that the subspace measure $\mu_\kappa=\mu_1$ is not τ-additive.

(ix) It follows from 414K that μ cannot be effectively locally finite; but it is also obvious from the work above that κ is a measurable set, of non-zero measure, such that $\mu(\kappa\cap G)=0$ whenever G is an open set of finite measure.

419B Lemma For any non-empty set I, there is a dense G_δ set in $[0,1]^I$ which is negligible for the usual measure on $[0,1]^I$.

proof Fix on some $i_0\in I$, and set $\pi(x)=x(i_0)$ for each $x\in[0,1]^I$, so that π is continuous and inverse-measure-preserving for the usual topologies and measures on $[0,1]^I$ and $[0,1]$. For each $n\in\mathbb{N}$ let $G_n\supseteq[0,1]\cap\mathbb{Q}$ be an open subset of $[0,1]$ with measure at most 2^{-n}, so that $\pi^{-1}[G_n]$ is an open set of measure at most 2^{-n}, and $E=\bigcap_{n\in\mathbb{N}}\pi^{-1}[G_n]$ is a G_δ set of measure 0. If $H\subseteq[0,1]^I$ is any non-empty open set, its image $\pi[H]$ is open in $[0,1]$, so contains some rational number, and meets $\bigcap_{n\in\mathbb{N}}G_n$; but this means that $H\cap E\ne\emptyset$, so E is dense.

419C Example (Fremlin 75b) There is a completion regular Radon measure space $(X,\mathfrak{T},\Sigma,\mu)$ such that

(i) there is an $E\in\Sigma$ such that $\mu(F\triangle E)>0$ for every Borel set $F\subseteq X$, that is, not every element of the measure algebra of μ can be represented by a Borel set;

(ii) μ is not outer regular with respect to the Borel sets;

(iii) writing ν for the restriction of μ to the Borel σ-algebra of X, ν is a locally finite, effectively locally finite, tight (that is, inner regular with respect to the compact sets) τ-additive completion regular topological measure, and there is a set $Y\subseteq X$ such that the subspace measure ν_Y is not semi-finite.

proof (a) For each $\xi<\omega_1$ set $X_\xi=[0,1]^{\omega_1\setminus\xi}$, and take μ_ξ to be the usual measure on X_ξ; write Σ_ξ for its domain. Note that μ_ξ is a completion regular Radon measure for the usual topology $\mathfrak{T}_\xi$ of X_ξ (416U). Set $X=\bigcup_{\xi<\omega_1}X_\xi$, and let μ be the direct sum measure on X (214L), that is, write

$$\Sigma=\{E:E\subseteq X,\,E\cap X_\xi\in\Sigma_\xi\text{ for every }\xi<\omega_1\},$$

$$\mu E=\sum_{\xi<\omega_1}\mu_\xi(E\cap X_\xi)\text{ for every }E\in\Sigma.$$

Then μ is a complete locally determined (in fact, strictly localizable) measure on X. Write Σ for its domain.

(b) For each $\eta<\omega_1$ let $\langle\beta_{\xi\eta}\rangle_{\xi\le\eta}$ be a summable family of strictly positive real numbers with $\beta_{\eta\eta}=1$ (4A1P). Define $g_\eta:X\to\mathbb{R}$ by setting

$$\begin{aligned}g_\eta(x)&=\frac{1}{\beta_{\xi\eta}}x(\eta)\text{ if }x\in X_\xi\text{ where }\xi\le\eta,\\&=0\text{ if }x\in X_\xi\text{ where }\xi>\eta.\end{aligned}$$

Now define $f : X \to \omega_1 \times \mathbb{R}^{\omega_1}$ by setting

$$f(x) = (\xi, \langle g_\eta(x)\rangle_{\eta<\omega_1})$$

if $x \in X_\xi$. Note that f is injective. Let $\mathfrak{T}$ be the topology on X defined by f, that is, the family $\{f^{-1}[W] : W \subseteq \omega_1 \times \mathbb{R}^{\omega_1}$ is open$\}$, where ω_1 and $\mathbb{R}^{\omega_1}$ are given their usual topologies (4A2S, 3A3K), and their product is given its product topology. Because f is injective, $\mathfrak{T}$ can be identified with the subspace topology on $f[X]$; it is Hausdorff and completely regular.

(c) For ξ, $\eta < \omega_1$, $g_\eta \restriction X_\xi$ is continuous for the compact topology $\mathfrak{T}_\xi$. Consequently $f\restriction X_\xi$ is continuous, and the subspace topology on X_ξ induced by $\mathfrak{T}$ must be $\mathfrak{T}_\xi$ exactly. It follows that μ is a Radon measure for $\mathfrak{T}$. **P** (i) We know already that μ is complete and locally determined. (ii) If $G \in \mathfrak{T}$ then $G \cap X_\xi \in \mathfrak{T}_\xi \subseteq \Sigma_\xi$ for every $\xi < \omega_1$, so $G \in \Sigma$; thus μ is a topological measure. (iii) If $E \in \Sigma$ and $\mu E > 0$, there is a $\xi < \omega_1$ such that $\mu_\xi(E \cap X_\xi) > 0$. Because μ_ξ is a Radon measure, there is a $\mathfrak{T}_\xi$-compact set $F \subseteq E \cap X_\xi$ such that $\mu_\xi F > 0$. Now F is $\mathfrak{T}$-compact and $\mu F > 0$. As E is arbitrary, μ is tight (using 412B). (iv) If $x \in X$, take that $\xi < \omega_1$ such that $x \in X_\xi$, and consider

$$G = f^{-1}[(\xi + 1) \times \{w : w \in \mathbb{R}^{\omega_1},\ w(\xi) < 2\}].$$

Because $\xi + 1$ is open in ω_1, $G \in \mathfrak{T}$. Because $g_\xi(x) = x(\xi) \le 1$, $x \in G$. Now for $\zeta \le \xi$,

$$\begin{aligned}\mu_\zeta(G \cap X_\zeta) &= \mu_\zeta\{x : x \in X_\zeta,\ g_\xi(x) < 2\}\\ &= \mu_\zeta\{x : x \in X_\zeta,\ \beta_{\zeta\xi}^{-1}x(\xi) < 2\}\\ &= \mu_\zeta\{x : x \in X_\zeta,\ x(\xi) < 2\beta_{\zeta\xi}\} \le 2\beta_{\zeta\xi},\end{aligned}$$

so

$$\mu G = \textstyle\sum_{\zeta\le\xi} \mu_\zeta(G \cap X_\zeta) \le 2\sum_{\zeta\le\xi} \beta_{\zeta\xi} < \infty.$$

As x is arbitrary, μ is locally finite, therefore a Radon measure. **Q**

We also find that μ is completion regular. **P** If $E \subseteq X$ and $\mu E > 0$, then there is a $\xi < \omega_1$ such that $\mu(E\cap X_\xi) > 0$. Because μ_ξ is completion regular, there is a set $F \subseteq E \cap X_\xi$, a zero set for $\mathfrak{T}_\xi$, such that $\mu F > 0$. Now X_ξ is a G_δ set in X (being the intersection of the open sets $\bigcup_{\eta<\zeta<\xi+1} X_\zeta$ for $\eta < \xi$, unless $\xi = 0$, in which case X_ξ is actually open), so F is a G_δ set in X (4A2C(a-iv)); being a compact G_δ set in a completely regular space, it is a zero set (4A2F(h-v)).

Thus every set of positive measure includes a zero set of positive measure. So μ is inner regular with respect to the zero sets (412B). **Q**

(d) The key to the example is the following fact: if $G \subseteq X$ is open, then *either* there is a cofinal closed set $V \subseteq \omega_1$ such that $G \cap X_\xi = \emptyset$ for every $\xi \in V$ *or* $\{\xi : \mu(G \cap X_\xi) \ne 1\}$ is countable. **P** Suppose that $A = \{\xi : G \cap X_\xi \ne \emptyset\}$ meets every cofinal closed set, that is, is stationary (4A1C). Then $B = A \cap \Omega$ is stationary, where Ω is the set of non-zero countable limit ordinals (4A1Bb, 4A1Cb). Let $H \subseteq \omega_1 \times \mathbb{R}^{\omega_1}$ be an open set such that $G = f^{-1}[H]$.

For each $\xi \in B$ choose $x_\xi \in G\cap X_\xi$. Then $f(x_\xi) \in H$, so there must be a $\zeta_\xi < \xi$, a finite set $I_\xi \subseteq \omega_1$, and a $\delta_\xi > 0$ such that $z \in H$ whenever $z = (\gamma, \langle t_\eta\rangle_{\eta<\omega_1}) \in \omega_1 \times \mathbb{R}^{\omega_1}$, $\zeta_\xi < \gamma \le \xi$ and $|t_\eta - g_\eta(x)| < \delta_\xi$ for every $\eta \in I_\xi$. Because ξ is a non-zero limit ordinal, $\zeta'_\xi = \sup(\{\zeta_\xi\} \cup (I_\xi \cap \xi)) < \xi$.

By the Pressing-Down Lemma (4A1Cc), there is a $\zeta < \omega_1$ such that $C = \{\xi : \xi \in B,\ \zeta'_\xi = \zeta\}$ is uncountable. **?** Suppose, if possible, that $\zeta < \eta < \omega_1$ and $\mu(G\cap X_\eta) < 1$. Then there is a measurable subset F of $X_\eta \setminus G$, determined by coordinates in a countable set $J \subseteq \omega_1 \setminus \eta$, such that $\mu F = \mu_\eta F > 0$ (254Ff). Let $\xi \in C$ be such that $\eta < \xi$ and $J \subseteq \xi$, and take any $y \in F$. If we define $y' \in X_\eta$ by setting

$$\begin{aligned}y'(\gamma) &= y(\gamma) \text{ for } \gamma \in \xi \setminus \eta\\ &= x_\xi(\gamma) \text{ for } \gamma \in \omega_1 \setminus \xi,\end{aligned}$$

then $y' \in F$. But also $\zeta_\xi \le \zeta'_\xi = \zeta < \eta < \xi$ and $\xi \setminus \eta \subseteq \xi \setminus \zeta'_\xi$ is disjoint from I_ξ, so $g_\gamma(y') = g_\gamma(x_\xi)$ for every $\gamma \in I_\xi$, since both are zero if $\gamma < \eta$ and otherwise $y'(\gamma) = x_\xi(\gamma)$. By the choice of ζ_ξ and I_ξ we must have $f(y') \in H$ and $y' \in F \cap G$; which is impossible. **X**

Thus $\mu(G \cap X_\eta) = 1$ for every $\eta > \zeta$, as required by the second alternative. **Q**

(e) For each $\xi < \omega_1$, let $\mathcal{I}_\xi$ be the family of negligible meager subsets of X_ξ. Then $\mathcal{I}_\xi$ is a σ-ideal; note that it contains every closed negligible set, because μ_ξ is strictly positive. Set

$$\mathrm{T}_\xi = \mathcal{I}_\xi \cup \{X_\xi \setminus F : F \in \mathcal{I}_\xi\},$$

so that T_ξ is a σ-algebra of subsets of X_ξ, containing every conegligible open set, and $\mu_\xi F \in \{0,1\}$ for every $F \in \mathrm{T}_\xi$. Set

$$\mathrm{T} = \{E : E \in \Sigma,\, \{\xi : E \cap X_\xi \notin \mathrm{T}_\xi\} \text{ is non-stationary}\}.$$

Then T is a σ-subalgebra of Σ (because the non-stationary sets form a σ-ideal of subsets of ω_1, 4A1Cb), and contains every open set, by (d); so includes the Borel σ-algebra $\mathcal{B}$ of X.

If we set

$$E_\xi = \{x : x \in X_\xi,\, x(\xi) \le \tfrac{1}{2}\} \text{ for each } \xi < \omega_1, \quad E = \bigcup_{\xi<\omega_1} E_\xi,$$

then $E \in \Sigma$. But if $F \subseteq X$ is a Borel set, $F \in \mathrm{T}$ so $\mu(E \triangle F) = \infty$. This proves the property (i) claimed for the example.

(f) Next, for each $\xi < \omega_1$, take a negligible dense G_δ set $E'_\xi \subseteq X_\xi$ (419B). Set $Y = \bigcup_{\xi<\omega_1} E'_\xi$, so that $\mu Y = 0$. If $F \supseteq Y$ is a Borel set, then $F \cap X_\xi \supseteq E_\xi \notin \mathcal{I}_\xi$ for every $\xi < \omega_1$, while $F \in \mathrm{T}$, so $\{\xi : \mu_\xi(F \cap X_\xi) = 0\}$ is non-stationary and $\mu F = \infty$. Thus μ is not outer regular with respect to the Borel sets. Taking $\nu = \mu\restriction\mathcal{B}$, the subspace measure ν_Y is not semi-finite. **P** We have just seen that $\nu_Y Y = \nu^* Y$ is infinite. If $F \in \mathcal{B}$ and $\nu F < \infty$, then $A = \{\xi : \mu_\xi(F \cap X_\xi) > 0\}$ is countable, so $F_0 = \bigcup_{\xi\in A} E'_\xi$ and $F_1 = F \setminus \bigcup_{\xi\in A} X_\xi$ are negligible Borel sets; since $F \cap Y \subseteq F_0 \cup F_1$, $\nu_Y(F \cap Y) = 0$. But this means that ν_Y takes no values in $]0,\infty[$ and is not semi-finite. **Q**

Remark X here is not locally compact. But as it is Hausdorff and completely regular, it can be embedded as a subspace of a locally compact Radon measure space $(X', \mathfrak{T}', \Sigma', \mu')$ (416T). Now μ' still has the properties (i)-(iii).

419D Example (Fremlin 75b) There is a complete locally determined τ-additive completion regular topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ in which μ is tight and compact sets have finite measure, but μ is not localizable.

proof (a) Let I be a set of cardinal greater than $\mathfrak{c}$. Set $X = [0,1]^I$. For $i \in I$, $t \in [0,1]$ set $X_{it} = \{x : x \in X,\, x(i) = t\}$. Give X_{it} its natural topology $\mathfrak{T}_{it}$ and measure μ_{it}, with domain Σ_{it}, defined from the expression of X_{it} as $[0,1]^{I\setminus\{i\}} \times \{t\}$, each factor $[0,1]$ being given its usual topology and Lebesgue measure, and the singleton factor $\{t\}$ being given its unique (discrete) topology and (atomic) probability measure. By 416U, μ_{it} is a completion regular Radon measure. Set

$$\mathfrak{T} = \{G : G \subseteq X,\, G \cap X_{it} \in \mathfrak{T}_{it} \text{ for all } i \in I,\, t \in [0,1]\},$$

$$\Sigma = \{E : E \subseteq X,\, E \cap X_{it} \in \Sigma_{it} \text{ for all } i \in I,\, t \in [0,1]\},$$

$$\mu E = \sum_{i\in I, t\in[0,1]} \mu_{it}(E \cap X_{it}) \text{ for every } E \in \Sigma.$$

(Compare 216D.) Then it is easy to check that $\mathfrak{T}$ is a topology. $\mathfrak{T}$ is Hausdorff because it is finer(= larger) than the usual topology $\mathfrak{S}$ on X; because each $\mathfrak{T}_{it}$ is the subspace topology induced by $\mathfrak{S}$, it is also the subspace topology induced by $\mathfrak{T}$. Next, the definition of μ makes it a locally determined measure; it is a tight complete topological measure because every μ_{it} is.

(b) If $K \subseteq X$ is compact, $\mu K < \infty$. **P?** Otherwise, $M = \{(i,t) : i \in I,\, t \in [0,1],\, \mu_{it}(K \cap X_{it}) > 0\}$ must be infinite. Take any sequence $\langle (i_n, t_n)\rangle_{n\in\mathbb{N}}$ of distinct elements of M. Choose a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in K inductively, as follows. Given $\langle x_m\rangle_{m<n}$, then set $C_{ni} = \{x_m(i) : m < n\}$ for each $i \in I$, and

$$A_n = \{x : x \in X_{i_n t_n},\, x(i) \notin C_{ni} \text{ for } i \in I \setminus \{i_n\}\};$$

then $\mu^*_{i_n t_n} A_n = 1$ (254Lb). Since $\mu_{i_n t_n}(X_{it} \cap X_{ju}) = 0$ whenever $(i,t) \ne (j,u)$, there must be some $x_n \in K \cap A_n \setminus \bigcup_{m\ne n} X_{i_m,t_m}$. Continue.

This construction ensures that if $i \in I$ and $m < n$, either $i \ne i_n$ so $x_n(i) \notin C_{ni}$ and $x_n(i) \ne x_m(i)$, or $i = i_n \ne i_m$ and $x_m \notin X_{i_n t_n}$ so $x_n(i) = t_n \ne x_m(i)$, or $i = i_m = i_n$ and $x_n(i) = t_n \ne t_m = x_m(i)$. But this means that $\{x_n : n \in \mathbb{N}\}$ is an infinite set meeting each X_{it} in at most one point, and is closed for $\mathfrak{T}$; so $\langle x_n\rangle_{n\in\mathbb{N}}$ has no cluster point for $\mathfrak{T}$, which is impossible. **XQ**

(c) μ is not localizable. **P** Fix on any $k \in I$ and consider $\mathcal{E} = \{X_{kt} : t \in [0,1]\}$. **?** If $E \in \Sigma$ is an essential supremum for $\mathcal{E}$, then $E \cap X_{kt}$ must be μ_{kt}-conegligible for every $t \in [0,1]$. We can therefore find a countable set $J_t \subseteq I$ and a μ_{kt}-conegligible set $F_t \subseteq E \cap X_{kt}$, determined by coordinates in J_t. At this point recall that $\#(I) > \mathfrak{c}$, so there is some $j \in I \setminus (\{k\} \cup \bigcup_{t\in[0,1]} J_t)$. Since $X_{j0} \cap X_{kt}$ is negligible for every $t \in [0,1]$, $X_{j0} \cap E$ must be negligible, and $\int_0^1 \nu H_t dt = 0$, where

$$H_t = \{y : y \in [0,1]^{I\setminus\{j,k\}},\ (y,0,t) \in E\}$$

and ν is the usual measure on $[0,1]^{I\setminus\{j,k\}}$, identifying X with $[0,1]^{I\setminus\{j,k\}} \times [0,1] \times [0,1]$. But because F_t is determined by coordinates in $I \setminus \{j\}$, we can identify it with $F'_t \times [0,1] \times \{t\}$ where F'_t is a ν-conegligible subset of $[0,1]^{I\setminus\{j,k\}}$, and $F'_t \subseteq H_t$, so $\nu H_t = 1$ for every t, which is absurd. **X**

Thus $\mathcal{E}$ has no essential supremum in Σ, and μ cannot be localizable. **Q**

(d) I have still to check that μ is completion regular. **P** If $E \in \Sigma$ and $\mu E > 0$, there are $i \in I$, $t \in [0,1]$ such that $\mu_{it}(E \cap X_{it}) > 0$, and an $F \subseteq E \cap X_{it}$, a zero set for the subspace topology of X_{it}, such that $\mu_{it} F > 0$. But now observe that X_{it} is a zero set in X for the usual topology $\mathfrak{S}$, so that F is a zero set for $\mathfrak{S}$ (4A2G(c-i)) and therefore for the finer topology $\mathfrak{T}$. By 412B, this is enough to show that μ is inner regular with respect to the zero sets. **Q**

Remark It may be worth noting that the topology $\mathfrak{T}$ here is not regular. See Fremlin 75b, p. 106.

419E Example (Fremlin 76) Let $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, so that ν is a strictly positive completion regular Radon probability measure (411P). Then the c.l.d. product measure λ on $Z \times Z$ is not a topological measure, so is not equal to the τ-additive product measure $\tilde{\lambda}$, and $\tilde{\lambda}$ is not completion regular.

proof Consider the sets W, $\tilde{W}$ described in 346K. We have $W \in \Lambda = \operatorname{dom}\lambda$ and $\tilde{W} = \bigcup \mathcal{V}$, where

$$\mathcal{V} = \{G \times H : G,\ H \subseteq Z \text{ are open-and-closed, } (G \times H) \setminus W \text{ is negligible}\}.$$

$\tilde{W}$ is a union of open sets, therefore must be open in Z^2. And $\lambda_* \tilde{W} \le \lambda W$. **P?** Otherwise, there is a $V \in \Lambda$ such that $V \subseteq \tilde{W}$ and $\lambda V > \lambda W$. Now λ is tight, by 412Sb, so there is a compact set $K \subseteq V$ such that $K \in \Lambda$ and $\lambda K > \lambda W$. There must be $U_0, \dots, U_n \in \mathcal{V}$ such that $K \subseteq \bigcup_{i \le n} U_i$. But $\lambda(U_i \setminus W) = 0$ for every i, so $\lambda(K \setminus W) = 0$ and $\lambda K \le \lambda W$. **XQ**

However, the construction of 346K arranged that $\lambda^* \tilde{W}$ should be 1 and λW strictly less than 1. So $\lambda_* \tilde{W} < \lambda^* \tilde{W}$ and $\tilde{W} \notin \Lambda$. Accordingly λ is not a topological measure and cannot be equal to the Radon measure $\tilde{\lambda}$ of 417P.

We know that λ is inner regular with respect to the zero sets (412Sc) and is defined on every zero set (417V), while $\tilde{\lambda}$ properly extends λ. But this means that $\tilde{\lambda}$ cannot be inner regular with respect to the zero sets, by 412L, that is, cannot be completion regular.

419F Theorem (Rao 69) $\mathcal{P}(\omega_1 \times \omega_1) = \mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1$, the σ-algebra of subsets of ω_1 generated by $\{E \times F : E,\ F \subseteq \omega_1\}$.

proof (a) Because $\omega_1 \le \mathfrak{c}$, there is an injection $h : \omega_1 \to \{0,1\}^{\mathbb{N}}$; set $E_i = \{\xi : h(\xi)(i) = 1\}$ for each $i \in \mathbb{N}$.

(b) Suppose that $A \subseteq \omega_1$ has countable vertical sections. Then $A \in \mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1$. **P** Set $B = A^{-1}[\omega_1]$ and for $\xi \in B$ choose a surjection $f_\xi : \mathbb{N} \to A[\{\xi\}]$. Set $g_n(\xi) = f_\xi(n)$ for $\xi \in B$ and $n \in \mathbb{N}$, and $A_n = \{(\xi, f_\xi(n)) : \xi \in B\}$ for $n \in \mathbb{N}$. Then

$$\begin{aligned}
A_n &= \{(\xi,\eta) : \xi \in B,\ \eta = g_n(\xi)\} \\
&= \{(\xi,\eta) : \xi \in B,\ \eta < \omega_1,\ h(g_n(\xi)) = h(\eta)\} \\
&= \bigcap_{i\in\mathbb{N}} \bigl(\{(\xi,\eta) : \xi \in g_n^{-1}[E_i],\ \eta \in E_i\} \cup \{(\xi,\eta) : \xi \in B \setminus g_n^{-1}[E_i],\ \eta \in \omega_1 \setminus E_i\}\bigr) \\
&\in \mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1.
\end{aligned}$$

So

$$A = \textstyle\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1. \ \mathbf{Q}$$

(c) Similarly, if a subset of $\omega_1 \times \omega_1$ has countable horizontal sections, it belongs to $\mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1$. But for any $A \subseteq \omega_1 \times \omega_1$, $A = A' \cup A''$ where

$$\begin{aligned}
A' &= \{(\xi,\eta) : (\xi,\eta) \in A,\ \eta \le \xi\} \text{ has countable vertical sections,} \\
A'' &= \{(\xi,\eta) : (\xi,\eta) \in A,\ \xi \le \eta\} \text{ has countable horizontal sections,}
\end{aligned}$$

so both A' and A'' belong to $\mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1$ and A also does.

419G Corollary (Ulam 30) Let Y be a set of cardinal at most ω_1 and μ a semi-finite measure with domain $\mathcal{P}Y$. Then μ is point-supported; in particular, if μ is σ-finite there is a countable conegligible set $A \subseteq Y$.

proof ? Suppose, if possible, otherwise.

(a) We can suppose that Y is either countable or actually equal to ω_1. Let μ_0 be the point-supported part of μ, that is, $\mu_0 A = \sum_{y \in A} \mu\{y\}$ for every $A \subseteq Y$; then μ_0 is a point-supported measure (112Bd), so is not equal to μ. Let $A \subseteq Y$ be such that $\mu_0 A \neq \mu A$. Then $\mu_0 A < \mu A$; because μ is semi-finite, there is a set $B \subseteq A$ such that $\mu_0 A < \mu B < \infty$. Set $\nu C = \mu(B \cap C) - \mu_0(B \cap C)$ for $C \subseteq Y$; then ν is a non-zero totally finite measure with domain $\mathcal{P}Y$, and is zero on singletons.

(b) As $\nu C = 0$ for every countable $C \subseteq Y$, Y is uncountable and $Y = \omega_1$. Let $\lambda = \nu \times \nu$ be the product measure on $\omega_1 \times \omega_1$. By 419F, the domain of λ is the whole of $\mathcal{P}(\omega_1 \times \omega_1)$; in particular, it contains the set $V = \{(\xi, \eta) : \xi \leq \eta < \omega_1\}$. Now by Fubini's theorem

$$\lambda V = \int \nu V[\{\xi\}] \nu(d\xi) = \int \nu(\omega_1 \setminus \xi) \nu(d\xi) = (\nu\omega_1)^2 > 0,$$

and also

$$\lambda V = \int \nu V^{-1}[\{\eta\}] \nu(d\eta) = \int \nu(\eta + 1) \nu(d\eta) = 0. \text{ ✖}$$

Remark I ought to remark that this result, though not 419F, is valid for many other cardinals besides ω_1; see, in particular, 438C below. There will be more on this topic in Chapter 54 of Volume 5.

419H For the next two examples it will be helpful to know some basic facts about Lebesgue measure which seemed a little advanced for Volume 1 and for which I have not found a suitable place since.

Lemma If $(X, \mathfrak{T}, \Sigma, \mu)$ is an atomless Radon measure space and $E \in \Sigma$ has non-zero measure, then $\#(E) \geq \mathfrak{c}$.

proof The subspace measure on E is a Radon measure (416Rb) therefore compact (416Wa) and perfect (342L), and is not purely atomic; by 344H, there is in fact a negligible subset of E of cardinal $\mathfrak{c}$.

419I The next result is a strengthening of 134D.

Lemma Let μ be Lebesgue measure on $\mathbb{R}$, and H any measurable subset of $\mathbb{R}$. Then there is a disjoint family $\langle A_\alpha \rangle_{\alpha < \mathfrak{c}}$ of subsets of H such that H is a measurable envelope of every A_α; in particular, $\mu_* A_\alpha = 0$ and $\mu^* A_\alpha = \mu H$ for every $\alpha < \mathfrak{c}$.

proof If $\mu H = 0$, we can take every A_α to be empty; so suppose that $\mu H > 0$. Let $\mathcal{E}$ be the family of closed subsets of H of non-zero measure. By 4A3Fa, $\#(\mathcal{E}) \leq \mathfrak{c}$; enumerate $\mathcal{E} \times \mathfrak{c}$ as $\langle (F_\xi, \alpha_\xi) \rangle_{\xi < \mathfrak{c}}$ (3A1Ca). Choose $\langle x_\xi \rangle_{\xi < \mathfrak{c}}$ inductively, as follows. Given $\langle x_\eta \rangle_{\eta < \xi}$, where $\xi < \mathfrak{c}$, F_ξ has cardinal (at least) $\mathfrak{c}$, by 419H, so cannot be included in $\{x_\eta : \eta < \xi\}$; take any $x_\xi \in F_\xi \setminus \{x_\eta : \eta < \xi\}$, and continue.

At the end of the induction, set

$$A_\alpha = \{x_\xi : \xi < \mathfrak{c}, \, \alpha_\xi = \alpha\}.$$

Then the A_α are disjoint just because the x_ξ are distinct.

? Suppose, if possible, that H is not a measurable envelope of A_α for some α. Then $\mu_*(H \setminus A_\alpha) > 0$ (413Ei), so there is a non-negligible measurable set $E \subseteq H \setminus A_\alpha$. Now there is an $F \in \mathcal{E}$ such that $F \subseteq E$. Let $\xi < \mathfrak{c}$ be such that $F = F_\xi$ and $\alpha = \alpha_\xi$; then $x_\xi \in A_\alpha \cap F$, which is impossible. ✖

Thus H is always a measurable envelope of A_α. It follows from the definition of 'measurable envelope' that $\mu^* A_\alpha = \mu H$. But also, if $\alpha < \mathfrak{c}$, $\mu_* A_\alpha \leq \mu_*(H \setminus A_{\alpha+1})$, which is 0, as we have just seen. So we have a suitable family.

419J Example There is a complete probability space (X, Σ, μ) with a Hausdorff topology $\mathfrak{T}$ on X such that μ is τ-additive and inner regular with respect to the Borel sets, $\mathfrak{T}$ is generated by $\mathfrak{T} \cap \Sigma$, but μ has no extension to a topological measure.

proof (a) Set $Y = \omega_1 + 1 = \omega_1 \cup \{\omega_1\}$. Let T be the σ-algebra of subsets of Y generated by $\{\{\xi\} : \xi < \omega_1\}$. Let ν be the probability measure with domain T defined by the formula

$$\nu F = \frac{1}{2} \#(F \cap \{0, \omega_1\}) \text{ for every } F \in \mathrm{T}.$$

Set

$$\mathfrak{S} = \{\emptyset, Y\} \cup \{H : 0 \in H \subseteq \omega_1\}.$$

This is a topology on Y, and every subset of Y is a Borel set for $\mathfrak{S}$; so ν is surely inner regular with respect to the Borel sets.

Note that

$$\{\{0, \alpha\} : \alpha < \omega_1\} \cup \{Y\}$$

is a base for $\mathfrak{S}$ included in T.

(b) Set $Z = Y^{\mathbb{N}} \times [0, 1]$. Let λ be the product probability measure on Z when each copy of Y is given the measure ν and $[0, 1]$ is given Lebesgue measure μ_L; let $\mathfrak{U}$ be the product topology when each copy of Y is given the topology $\mathfrak{S}$ and $[0, 1]$ its usual topology. Let Λ be the domain of λ and Λ_0 the σ-algebra generated by sets of the form $\{(x, t) : x(i) \in F,\, t \in [0, 1],\, t < q\}$ where $i \in \mathbb{N}$, $F \in \mathrm{T}$ and $q \in \mathbb{Q}$; then ν is inner regular with respect to Λ_0 (see 254Ff). Note that $\mathfrak{U} \cap \Lambda_0$ is a base for $\mathfrak{U}$ because $\mathfrak{S} \cap \mathrm{T}$ is a base for $\mathfrak{S}$, and λ is inner regular with respect to the $\mathfrak{U}$-Borel sets (412Uc, or otherwise).

Define $\phi : \{0, 1\}^{\mathbb{N}} \to Y^{\mathbb{N}}$ by setting

$$\begin{aligned}\phi(u, t)(n) &= 0 \text{ if } u(n) = 0,\\ &= \omega_1 \text{ if } u(n) = 1,\end{aligned}$$

and set $\psi(u, t) = (\phi(u), t)$ for $u \in \{0, 1\}^{\mathbb{N}}$, $t \in [0, 1]$. Then ψ is continuous, just because $\mathfrak{U}$ is a product topology. Let $\nu_\omega \times \mu_L$ be the product measure on $\{0, 1\}^{\mathbb{N}} \times [0, 1]$; then ψ is inverse-measure-preserving for $\nu_\omega \times \mu_L$ and λ, by 254H.

If $V \in \Lambda_0$ there is an $\alpha < \omega_1$ such that

if x, $y \in Y^{\mathbb{N}}$, $t \in [0, 1]$ and $y(i) = x(i)$ whenever $\min(x(i), y(i)) < \alpha$, then $(x, t) \in V$ iff $(y, t) \in V$.

P Let $\mathcal{W}$ be the family of sets $V \subseteq Z$ with this property. Then $\mathcal{W}$ is a σ-algebra of subsets of Z containing every measurable cylinder, so includes Λ_0. **Q**

(c) $\#(\Lambda_0) \le \mathfrak{c}$. **P** Set

$$A_{i\xi} = \{(x, t) : x \in Y^{\mathbb{N}},\, t \in [0, 1],\, x(i) = \xi\}, \quad A'_q = \{(x, t) : x \in Y^{\mathbb{N}},\, t \in [0, 1],\, t \le q\};$$

for $i \in \mathbb{N}$, $\xi < \omega_1$ and $q \in \mathbb{Q}$, and

$$\mathcal{A} = \{A_{i\xi} : i \in \mathbb{N},\, \xi < \omega_1\} \cup \{A'_q : q \in \mathbb{Q}\}.$$

Then Λ_0 is the σ-algebra of subsets of Z generated by $\mathcal{A}$, and $\#(\mathcal{A}) = \omega_1 \le \mathfrak{c}$, so $\#(\Lambda_0) \le \mathfrak{c}$ (4A1O). **Q**

(d) There is a family $\langle z_\xi \rangle_{\xi < \mathfrak{c}}$ in Z such that

(α) whenever $W \in \Lambda$ and $\lambda W > 0$ there is a $\xi < \mathfrak{c}$ such that $z_\xi \in W$ and $\lambda(H \cap W) > 0$ whenever H is a measurable open subset of Z containing z_ξ,

(β) setting $z_\xi = (x_\xi, t_\xi)$ for each ξ, there is for every $\xi < \mathfrak{c}$ a $j \in \mathbb{N}$ such that $0 < x_\xi(j) < \omega_1$,

(γ) $t_\xi \ne t_\eta$ if $\eta < \xi < \mathfrak{c}$.

P By 4A3Fa, the set of closed subsets of $\{0, 1\}^{\mathbb{N}} \times [0, 1]$ has cardinal at most $\mathfrak{c}$, so there is a family $\langle (K_\xi, V_\xi) \rangle_{\xi < \mathfrak{c}}$ running over all pairs (K, V) such that $V \in \Lambda_0$ and $K \subseteq \psi^{-1}[V]$ is a non-negligible compact set. Choose $\langle (x_\xi, t_\xi) \rangle_{\xi < \mathfrak{c}}$ inductively, as follows. Given $\langle t_\eta \rangle_{\eta < \xi}$ where $\xi < \mathfrak{c}$, then

$$\{t : t \in [0, 1],\, \nu_\omega K_\xi^{-1}[\{t\}] > 0\}$$

is a non-negligible measurable subset of $[0, 1]$, so has cardinal $\mathfrak{c}$ (419H); let t_ξ be a point of this set distinct from every t_η for $\eta < \xi$. Now Lemma 345E tells us that there are points u, $u' \in K_\xi^{-1}[\{t_\xi\}]$ which differ at exactly one coordinate $j \in \mathbb{N}$; suppose that $u(j) = 1$ and $u'(j) = 0$.

Let $\alpha < \omega_1$ be such that if x, $y \in Y^{\mathbb{N}}$, $t \in [0, 1]$ and $y(i) = x(i)$ whenever $\min(x(i), y(i)) < \alpha$, then $(x, t) \in V_\xi$ iff $(y, t) \in V_\xi$. Define $x_\xi \in Y^{\mathbb{N}}$ by setting $x_\xi(j) = \alpha$ and $x_\xi(i) = \phi(u)(i)$ for $i \ne j$. Then $z_\xi = (x_\xi, t_\xi)$ belongs to V_ξ. If $H \subseteq Z$ is any open set containing z_ξ, we have a sequence $\langle H_i \rangle_{i \in \mathbb{N}}$ in $\mathfrak{S}$ such that $x_\xi \in \prod_{i \in \mathbb{N}} H_i$ and $\prod_{i \in \mathbb{N}} H_i \times \{t_\xi\} \subseteq H$; now $H_j \ne \emptyset$ so $0 \in H_j$ and $\phi(u') \in \prod_{i \in \mathbb{N}} H_i$, so that $(u', t_\xi) \in K_\xi \cap \psi^{-1}[H]$. Continue.

The construction ensures that (β) and (γ) are satisfied. Now, if $\lambda W > 0$, let $V \in \Lambda_0$ be such that $V \subseteq W$ and $\lambda V > 0$. In this case, $(\nu_\omega \times \mu_L)(\psi^{-1}[V]) > 0$; let $K \subseteq \psi^{-1}[V]$ be a self-supporting non-negligible compact set. Let $\xi < \mathfrak{c}$ be such that $(K, V) = (K_\xi, V_\xi)$. Then $z_\xi \in V_\xi = V \subseteq W$. If H is a measurable open subset of Z containing z_ξ, then $K \cap \psi^{-1}[H]$ is not empty; as ψ is continuous and inverse-measure-preserving and K is self-supporting,

$$0 < (\nu_\omega \times \mu_L)(K \cap \psi^{-1}[H]) \le (\nu_\omega \times \mu_L)\psi^{-1}[V \cap H] = \lambda(V \cap H) \le \lambda(W \cap H).$$

So (α) is satisfied. **Q**

(e) Set $X = \{z_\xi : \xi < \mathfrak{c}\}$ and let μ be the subspace measure on X induced by λ; let $\mathfrak{T}$ be the subspace topology on X.

(i) λ is complete, so μ also is. Next, $\mu X = \lambda^* X = 1$. **P?** Otherwise, there is a $W \in \Lambda$ such that $\lambda W > 0$ and $X \cap W = \emptyset$. But we know that there is now some $\xi < \mathfrak{c}$ such that $z_\xi \in W$. **XQ**

(ii) $\mathfrak{T}$ is Hausdorff because the projection from X to $[0,1]$ is injective and continuous. $\mathfrak{T}$ is generated by $\mathfrak{T} \cap \Sigma$ because $\mathfrak{U}$ is generated by $\mathfrak{U} \cap \Lambda$. μ is inner regular with respect to the Borel sets because λ is (412Pb).

(iii) μ is τ-additive. **P?** Suppose, if possible, otherwise. Then there is an upwards-directed family $\mathcal{G}$ of measurable open subsets of X such that $G^* = \bigcup \mathcal{G}$ is measurable and $\mu G^* > \sup_{G \in \mathcal{G}} \mu G$. Let $\mathcal{H}$ be the family of sets $H \in \Lambda \cap \mathfrak{U}$ such that $H \cap X$ is included in some member of $\mathcal{G}$; because $\mathfrak{U}$ is generated by $\mathfrak{U} \cap \Lambda$, $G^* = W \cap X$, where $W = \bigcup \mathcal{H}$. At the same time, there is a $V \in \Lambda$ such that $G^* = X \cap V$.

Because $\mathcal{G}$ is upwards-directed, so is $\mathcal{H}$. Because X has full outer measure,

$$\sup_{H \in \mathcal{H}} \lambda H = \sup_{H \in \mathcal{H}} \mu(X \cap H) \le \sup_{G \in \mathcal{G}} \mu G < \mu G^* = \lambda V.$$

Let $\langle H_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence in $\mathcal{H}$ such that $\sup_{n \in \mathbb{N}} \lambda H_n = \sup_{H \in \mathcal{H}} \lambda H$, and set $W_0 = \bigcup_{n \in \mathbb{N}} H_n$; then $\lambda W_0 < \lambda V$ and $\lambda(H \setminus W_0) = 0$ for every $H \in \mathcal{H}$. However, $\lambda(V \setminus W_0) > 0$, so there is a $z \in X \cap V \setminus W_0$ such that $\lambda(H \cap V \setminus W_0) > 0$ for every measurable open set H containing z. As $z \in X \cap V = X \cap W$, there must be an $H \in \mathcal{H}$ containing z, so this is impossible. **XQ**

(iv) ? Suppose, if possible, that there is a topological measure $\tilde{\mu}$ on X agreeing with μ on every open set in the domain of μ. For each $i \in \mathbb{N}$, set $\pi_i(x) = x(i)$ for $(x,t) \in X$. Every subset of Y is a Borel set for $\mathfrak{S}$; because π_i is continuous for $\mathfrak{T}$ and $\mathfrak{S}$, the image measure $\tilde{\mu}\pi_i^{-1}$ has domain $\mathcal{P}Y$. Now $\#(Y) = \omega_1$, so there must be a countable conegligible set (419G), and there must be some $\alpha_i < \omega_1$ such that $\tilde{\mu}\pi_i^{-1}(\omega_1 \setminus \alpha_i) = 0$. On the other hand,

$$\tilde{\mu}\pi_i^{-1}(\alpha_i \setminus \{0\}) = \mu\pi_i^{-1}(\alpha_i \setminus \{0\}) = \lambda\{(x,t) : 0 < x(i) < \alpha_i\} = \nu(\alpha_i \setminus \{0\}) = 0,$$

so $\tilde{\mu}\pi_i^{-1}(\omega_1 \setminus \{0\}) = 0$.

But (d-β) ensures that

$$X = \bigcup_{i \in \mathbb{N}} \pi_i^{-1}(\omega_1 \setminus \{0\}),$$

so this is impossible. **X**

Thus we have the required example.

Remark I note that the topology of X is not regular. Of course the phenomenon here cannot arise with regular spaces, by 415M.

419K Example (Blackwell 56) There are sequences $\langle X_n \rangle_{n \in \mathbb{N}}$, $\langle \mathfrak{T}_n \rangle_{n \in \mathbb{N}}$ and $\langle \nu_n \rangle_{n \in \mathbb{N}}$ such that (i) for each n, $(X_n, \mathfrak{T}_n)$ is a separable metrizable space and ν_n is a quasi-Radon probability measure on $Z_n = \prod_{i \le n} X_i$ (ii) for $m \le n$ the canonical map $\pi_{mn} : Z_n \to Z_m$ is inverse-measure-preserving (iii) there is no probability measure on $Z = \prod_{i \in \mathbb{N}} X_i$ such that all the canonical maps from Z to Z_n are inverse-measure-preserving.

proof Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a disjoint sequence of subsets of $[0,1]$ such that $\mu_*([0,1] \setminus A_n) = 0$, that is, $\mu^* A_n = 1$ for every n, where μ is Lebesgue measure (using 419I). Set $X_n = \bigcup_{i \ge n} A_i$, so that $\langle X_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of sets of outer measure 1 with empty intersection. For each $n \ge 1$, we have a map $f_n : X_n \to Z_n$ defined by setting $f_n(x)(i) = x$ for every $i \le n$, $x \in X_n$. Let ν_n be the image measure $\mu_{X_n} f_n^{-1}$, where μ_{X_n} is the subspace measure on X_n induced by μ. Note that f_n is a homeomorphism between X_n and the diagonal $\Delta_n = \{z : z \in Z_n, z(i) = z(j)$ for all $i, j \le n\}$, which is a closed subset of Z_n; so that ν_n, like μ_{X_n}, is a quasi-Radon probability measure.

If $m \le n$, then π_{mn} is inverse-measure-preserving, where $\pi_{mn}(z)(i) = z(i)$ for $z \in Z_n$ and $i \le m$. **P** If $W \subseteq Z_m$ is measured by ν_m, then $f_m^{-1}[W]$ is measured by μ_{X_m}, so is of the form $X_m \cap E$ where E is Lebesgue measurable. But in this case $f_n^{-1}[\pi_{mn}^{-1}[W]] = X_n \cap E$, so that

$$\nu_n(\pi_{mn}^{-1}[W]) = \mu_{X_n}(f_n^{-1}[\pi_{mn}^{-1}[W]]) = \mu^*(X_n \cap E) = \mu E = \mu^*(X_m \cap E) = \nu_m W. \text{ **Q**}$$

? But suppose, if possible, that there is a probability measure ν on $Z = \prod_{i \in \mathbb{N}} X_i$ such that $\pi_n : Z \to Z_n$ is inverse-measure-preserving for every n, where $\pi_n(z)(i) = z(i)$ for $z \in Z$ and $i \le n$. Then

$$\nu\pi_n^{-1}[\Delta_n] = \nu_n \Delta_n = \mu_{X_n} f_n^{-1}[\Delta_n] = 1$$

for each n, so

$$1=\nu(\bigcap_{n\in\mathbb{N}}\pi_n^{-1}[\Delta_n])=\nu\{z:z\in Z,\,z(i)=z(j)\text{ for all }i,\,j\in\mathbb{N}\}=\nu\emptyset,$$

because $\bigcap_{n\in\mathbb{N}}X_n=\emptyset$; which is impossible. **X**

419L The split interval again (a) For the sake of an example in §343, I have already introduced the 'split interval' or 'double arrow space'. As this construction gives us a topological measure space of great interest, I repeat it here. Let $I^{\|}$ be the set $\{a^+:a\in[0,1]\}\cup\{a^-:a\in[0,1]\}$. Order it by saying that

$$a^+\le b^+\iff a^-\le b^+\iff a^-\le b^-\iff a\le b,\quad a^+\le b^-\iff a<b.$$

Then it is easy to check that $I^{\|}$ is a totally ordered space, and that it is Dedekind complete. (If $A\subseteq[0,1]$ is a non-empty set, then $\sup_{a\in A}a^-=(\sup A)^-$, while $\sup_{a\in A}a^+$ is either $(\sup A)^+$ or $(\sup A)^-$, depending on whether $\sup A$ belongs to A or not.) Its greatest element is 1^+ and its least element is 0^-. Consequently the order topology on $I^{\|}$ is a compact Hausdorff topology (4A2Ri, Alexandroff & Urysohn 1929). Note that $Q=\{q^+:q\in[0,1]\cap\mathbb{Q}\}\cup\{q^-:q\in[0,1]\cap\mathbb{Q}\}$ is dense, because it meets every non-trivial interval in $I^{\|}$. By 4A2E(a-ii) and 4A2Rn, $I^{\|}$ is ccc and hereditarily Lindelöf.

(b) If we define $h:I^{\|}\to[0,1]$ by writing $h(a^+)=h(a^-)=a$ for every $a\in[0,1]$, then h is continuous, because $\{x:h(x)<a\}=\{x:x<a^-\}$, $\{x:h(x)>a\}=\{x:x>a^+\}$ for every $a\in[0,1]$. Now we can describe the Borel sets of $I^{\|}$, as follows: a set $E\subseteq I^{\|}$ is Borel iff there is a Borel set $F\subseteq[0,1]$ such that $E\triangle h^{-1}[F]$ is countable. **P** Write Σ_0 for the family of subsets E of $I^{\|}$ such that $E\triangle h^{-1}[F]$ is countable for some Borel set $F\subseteq[0,1]$. It is easy to check that Σ_0 is a σ-algebra of subsets of $I^{\|}$. (If $E\triangle h^{-1}[F]$ is countable, so is $(I^{\|}\setminus E)\triangle h^{-1}[[0,1]\setminus F]$; if $E_n\triangle h^{-1}[F_n]$ is countable for every n, so is $(\bigcup_{n\in\mathbb{N}}E_n)\triangle h^{-1}[\bigcup_{n\in\mathbb{N}}F_n]$.) Because the topology of $I^{\|}$ is Hausdorff, every singleton set is closed, so every countable set is Borel. Also $h^{-1}[F]$ is Borel for every Borel set $F\subseteq[0,1]$, because h is continuous (4A3Cd). So if $E\triangle h^{-1}[F]$ is countable for some Borel set $F\subseteq[0,1]$, $E=h^{-1}[F]\triangle(E\triangle h^{-1}[F])$ is a Borel set in $I^{\|}$. Thus Σ_0 is included in the Borel σ-algebra $\mathcal{B}$ of $I^{\|}$. On the other hand, if $J\subseteq I^{\|}$ is an interval, $h[J]$ also is an interval, therefore a Borel set, and $h^{-1}[h[J]]\setminus J$ can contain at most two points, so $J\in\Sigma_0$. If $G\subseteq I^{\|}$ is open, it is expressible as $\bigcup_{i\in I}J_i$, where $\langle J_i\rangle_{i\in I}$ is a disjoint family of non-empty open intervals (4A2Rj). As X is ccc, I must be countable. Thus G is expressed as a countable union of members of Σ_0 and belongs to Σ_0. But this means that the Borel σ-algebra $\mathcal{B}$ must be included in Σ_0, by the definition of 'Borel algebra'. So $\mathcal{B}=\Sigma_0$, as claimed. **Q**

(c) In 343J I described the standard measure μ on $I^{\|}$; its domain is the set $\Sigma=\{h^{-1}[F]\triangle M:F\in\Sigma_L,\,M\subseteq I^{\|},\,\mu_Lh[M]=0\}$, where Σ_L is the set of Lebesgue measurable subsets of $[0,1]$ and μ_L is Lebesgue measure, and $\mu E=\mu_Lh[E]$ for $E\in\Sigma$. h is inverse-measure-preserving for μ and μ_L.

The new fact I wish to mention is: μ is a completion regular Radon measure. **P** I noted in 343Ja that it is a complete probability measure; *a fortiori*, it is locally determined and locally finite. If $G\subseteq I^{\|}$ is open, then we can express it as $h^{-1}[F]\triangle C$ for some Borel set $F\subseteq[0,1]$ and countable $C\subseteq I^{\|}$ ((b) above), so it belongs to Σ; thus μ is a topological measure. If $E\in\Sigma$ and $\mu E>\gamma$, then $F=[0,1]\setminus h[I^{\|}\setminus E]$ is Lebesgue measurable, and $\mu E=\mu_LF$. So there is a compact set $L\subseteq F$ such that $\mu_LL\ge\gamma$. But now $K=h^{-1}[L]\subseteq E$ is closed, therefore compact, and $\mu K\ge\gamma$. Moreover, L is a zero set, being a closed set in a metrizable space (4A2Lc), so K is a zero set (4A2C(b-iv)). As E and γ are arbitrary, μ is inner regular with respect to the compact zero sets, and is a completion regular Radon measure. **Q**

419X Basic exercises (a) Show that the topological space X of 419A is zero-dimensional.

(b) Give an example of a compact Radon probability space in which every dense G_δ set is conegligible. (*Hint*: 411P.)

(c) In 419E, show that we can start from any atomless probability measure in place of Lebesgue measure on $[0,1]$.

>(d)(i) Show that if $E\subseteq\mathbb{R}^2$ is Lebesgue measurable, with non-zero measure, then it cannot be covered by fewer than $\mathfrak{c}$ lines. (*Hint*: if $H=\{t:\mu_1E[\{t\}]>0\}$, where μ_1 is Lebesgue measure on $\mathbb{R}$, then $\mu_1H>0$, so $\#(H)=\mathfrak{c}$. So if we have a family $\mathcal{L}$ of lines, with $\#(\mathcal{L})<\mathfrak{c}$, there must be a $t\in H$ such that $L_t=\{t\}\times\mathbb{R}$ does not belong to $\mathcal{L}$. Now $\#(L_t\cap E)=\mathfrak{c}$ and each member of $\mathcal{L}$ meets $L_t\cap E$ in at most one point.) (ii) Show that there is a subset A of $\mathbb{R}^2$, of full outer measure, which meets every vertical line and every horizontal line in exactly one point. (*Hint*: enumerate $\mathbb{R}$ as $\langle t_\xi\rangle_{\xi<\mathfrak{c}}$ and the closed sets of non-zero measure as $\langle F_\xi\rangle_{\xi<\mathfrak{c}}$. Choose $\langle I_\xi\rangle_{\xi<\mathfrak{c}}$ such that every I_ξ is finite, no two points of $I_\xi \cup \bigcup_{\eta<\xi} I_\eta$ lie on any horizontal or vertical line, the lines $\{t_\xi\} \times \mathbb{R}$ and $\mathbb{R} \times \{t_\xi\}$ both meet $I_\xi \cup \bigcup_{\eta<\xi} I_\eta$, and I_ξ meets F_ξ.) (iii) Show that there is a subset B of $\mathbb{R}^2$, of full outer measure, such that every straight line meets B in exactly two points. (*Hint*: enumerate the straight lines in $\mathbb{R}^2$ as $\langle L_\xi \rangle_{\xi<\mathfrak{c}}$. Choose $\langle J_\xi \rangle_{\xi<\mathfrak{c}}$ such that every J_ξ is finite, no three points of $J_\xi \cup \bigcup_{\eta<\xi} J_\eta$ lie on any line, $L_\xi \cap (J_\xi \cup \bigcup_{\eta<\xi} J_\eta)$ has just two points and $J_\xi \cap F_\xi \neq \emptyset$.)

(e) Show that there is a subset A of the Cantor set C (134G) such that $A+A$ is not Lebesgue measurable. (*Hint*: enumerate the closed non-negligible subsets of $C+C=[0,2]$ as $\langle F_\xi \rangle_{\xi<\mathfrak{c}}$. Choose $x_\xi \in C$, $y_\xi \in C$, $z_\xi \in F_\xi$ so that $z_\xi \notin A_\xi + A_\xi$ and $x_\xi + y_\xi \in F_\xi$ and $A_{\xi+1} + A_{\xi+1}$ does not meet $\{z_\eta : \eta \le \xi\}$, where $A_\xi = \{x_\eta : \eta < \xi\} \cup \{y_\eta : \eta < \xi\}$.)

(f) Let $\mathbb{R}^{\|}$ be the **split line**, that is, the set $\{a^+ : a \in \mathbb{R}\} \cup \{a^- : a \in \mathbb{R}\}$, ordered by the rules in 419L. Show that $\mathbb{R}^{\|}$ is a Dedekind complete totally ordered set, so that its order topology $\mathfrak{T}$ is locally compact. Write μ_L for Lebesgue measure on $\mathbb{R}$ and Σ_L for its domain. Set $h(a^+) = h(a^-) = a$ for $a \in \mathbb{R}$, $\Sigma = \{E : E \subseteq X, h[E] \in \Sigma_L, \mu_L(h[E] \cap h[X \setminus E]) = 0\}$, $\mu E = \mu_L h[E]$ for $E \in \Sigma$. Show that μ is a completion regular Radon measure on $\mathbb{R}^{\|}$ and that h is continuous and inverse-measure-preserving for μ and μ_L. Show that the set $\{a^+ : a \in \mathbb{R}\}$, with the induced topology and measure, is isomorphic, as quasi-Radon measure space, to the right-facing Sorgenfrey line (415Xc) with Lebesgue measure. Show that $\mathbb{R}^{\|}$ and the Sorgenfrey line are hereditarily Lindelöf.

(g) Let μ be Lebesgue measure on $[0,1]$ and Σ its domain. Let $I^{\|}$ be the split interval. (i) Show that the functions $x \mapsto x^+ : [0,1] \to I^{\|}$ and $x \mapsto x^- : [0,1] \to I^{\|}$ are measurable. (*Hint*: 419Lb.) (ii) Show that the function $x \mapsto (x^+, x^-) : [0,1] \to (I^{\|})^2$ is not measurable. (*Hint*: the subspace topology on $\{(x^+, x^-) : x \in [0,1]\}$ is discrete.)

>(h)(i) Again writing $I^{\|}$ for the split interval, show that the function which exchanges x^+ and x^- for every $x \in [0,1]$ is a Borel automorphism and an automorphism for the usual Radon measure ν on $I^{\|}$, but is not almost continuous. (ii) Show that if we set $f(x) = x^+$ for $x \in [0,1]$, then f is inverse-measure-preserving for Lebesgue measure μ_L on $[0,1]$, but the image measure $\mu_L f^{-1}$ is not ν (nor, indeed, a Radon measure).

(i) Show that the split interval $I^{\|}$ is perfectly normal, but that $I^{\|} \times I^{\|}$ is not perfectly normal.

419Y Further exercises (a) In the example of 419E, show that there is a Borel set $V \subseteq Z^2$ such that $\tilde{\lambda} V = 0$ and $\lambda^* V = 1$.

(b) Show that if $\mathcal{A} \subseteq \mathcal{P}\omega_1$ is any family with $\#(\mathcal{A}) \le \omega_1$, there is a countably generated σ-algebra Σ of subsets of ω_1 such that $\mathcal{A} \subseteq \Sigma$.

(c) Show that the split interval with its usual topology and measure has the simple product property (417Yi).

419 Notes and comments The construction of the locally compact space X in 419A from the family $\langle I_\xi \rangle_{\xi<\kappa}$ is a standard device which has been used many times. The relation $\subseteq^*$ also appears in many contexts. In effect, part of the argument is taking place in the quotient algebra $\mathfrak{A} = \mathcal{P}Q/[Q]^{<\omega}$, since $I \subseteq^* J$ iff $I^\bullet \subseteq J^\bullet$ in $\mathfrak{A}$; setting $\mathcal{I}^\# = \{I^\bullet : I \in \mathcal{I}\}$, the cardinal κ is $\min\{\#(A) : A \subseteq \mathcal{I}^\#$ has no upper bound in $\mathcal{I}^\#\}$, the 'additivity' of the partially ordered set $\mathcal{I}^\#$. Additivities of partially ordered sets will be one of the important concerns of Volume 5. I remark that we do not need to know whether (for instance) $\kappa = \omega_1$ or $\kappa = \mathfrak{c}$. This is an early taste of the kind of manoeuvre which has become a staple of set-theoretic analysis. It happens that the cardinal κ here is one of the most important cardinals of set-theoretic measure theory; it is 'the additivity of Lebesgue measure' (529Xe[27]), and under that name will appear repeatedly in Volume 5.

Observe that the measure μ of 419A only just fails to be a quasi-Radon measure; it is locally finite instead of being effectively locally finite. And it would be a Radon measure if it were inner regular with respect to the compact sets, rather than just with respect to the closed sets.

419C and 419D are relevant to the question: have I given the 'right' definition of Radon measure space? 419C is perhaps more important. Here we have a Radon measure space (on my definition) for which the associated Borel measure is not localizable. (If $\mathfrak{A}$ is the measure algebra of the measure μ, and $\mathfrak{B}$ the measure algebra of $\mu \restriction \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra of X, then the embedding $\mathcal{B} \subsetneqq \Sigma$ induces an embedding of $\mathfrak{B}$ in $\mathfrak{A}$ which represents $\mathfrak{B}$ as an order-dense subalgebra of $\mathfrak{A}$, just because μ is inner regular with respect to $\mathcal{B}$. Property (i) of 419C shows that

[27] Formerly 529Xc.

$\mathfrak{B} \neq \mathfrak{A}$, so $\mathfrak{B}$ cannot be Dedekind complete in itself, by 314Ib.) Since (I believe) localizable versions of measure spaces should almost always be preferred, I take this as strong support for my prejudice in favour of insisting that 'Radon' measure spaces should be locally determined as well as complete. Property (ii) of 419C is not I think of real significance, but is further evidence, to be added to 415Xh, that outer regularity is like an exoskeleton: it may inhibit growth above a certain size.

In 419D I explore the consequences of omitting the condition 'locally finite' from the definition of Radon measure. Even if we insist instead that compact sets should have finite measure, we are in danger of getting a non-localizable measure. Of course this particular space is pathological in terms of most of the criteria of this chapter – for instance, every non-empty open set has infinite measure, and the topology is not regular.

Perhaps the most important example in the section is 419E. The analysis of τ-additive product measures in §417 was long and difficult, and if these were actually equal to the familiar product measures in all important cases the structure of the theory would be very different. But we find that for one of the standard compact Radon probability spaces of the theory, the c.l.d. product measure on its square is not a Radon measure, and something has to be done about it.

I present 419J here to indicate one of the obstacles to any simplification of the arguments in 417C and 417E. It is not significant in itself, but it offers a welcome excuse to describe some fundamental facts about ω_1 (419F-419G). Similarly, 419K asks for some elementary facts about Lebesgue measure (419H-419I) which seem to have got left out. This example really is important in itself, as it touches on the general problem of representing stochastic processes, to which I will return in Chapter 45.

Chapter 42

Descriptive set theory

At this point, I interpolate an auxiliary chapter, in the same spirit as Chapters 31 and 35 in the last volume. As with Boolean algebras and Riesz spaces, it is not just that descriptive set theory provides essential tools for modern measure theory; it also offers deep intuitions, and for this reason demands study well beyond an occasional glance at an appendix. Several excellent accounts have been published; the closest to what we need here is probably Rogers 80; at a deeper level we have Moschovakis 80, and an admirable recent treatment is Kechris 95. Once again, however, I indulge myself by extracting those parts of the theory which I shall use directly, giving proofs and exercises adapted to the ideas I am trying to emphasize in this volume and the next.

The first section describes Souslin's operation and its basic set-theoretic properties up to first steps in the theory of 'constituents' (421N-421Q), mostly steering away from topological ideas, but with some remarks on σ-algebras and Souslin-F sets. §422 deals with usco-compact relations and K-analytic spaces, working through the topological properties which will be useful later, and giving a version of the First Separation Theorem (422I-422J). §423 looks at 'analytic' or 'Souslin' spaces, treating them primarily as a special kind of K-analytic space, with the von Neumann-Jankow selection theorem (423N). §424 is devoted to 'standard Borel spaces'; it is largely a series of easy applications of results in §423, but there is a substantial theorem on Borel measurable actions of Polish groups (424H). Finally, I add a note on A.Törnquist's theorem on representation of groups of automorphisms of quotient algebras (425D).

421 Souslin's operation

I introduce Souslin's operation $\mathcal{S}$ (421B) and show that it is idempotent (421D). I describe alternative characterizations of members of $\mathcal{S}(\mathcal{E})$, where $\mathcal{E} \subseteq \mathcal{P}X$, as projections of sets in $\mathbb{N}^{\mathbb{N}} \times X$ (421G-421J). I briefly mention Souslin-F sets (421J-421L) and a special property of 'inner Souslin kernels' (421M). At the end of the section I set up an abstract theory of 'constituents' for kernels of Souslin schemes and their complements (421N-421Q).

421A Notation Throughout this chapter, and frequently in the next, I shall regard a member of $\mathbb{N}$ as the set of its predecessors, so that a finite power X^k can be identified with the set of functions from k to X, and if $\phi \in X^{\mathbb{N}}$ and $k \in \mathbb{N}$, we can speak of the restriction $\phi{\restriction}k \in X^k$. In the same spirit, identifying functions with their graphs, I can write '$\sigma \subseteq \phi$' when $\sigma \in X^k$, $\phi \in X^{\mathbb{N}}$ and ϕ extends σ. On occasion I may write $\#(\sigma)$ for the 'length' of a finite function σ – again identifying σ with its graph – so that $\#(\sigma) = k$ if $\sigma \in X^k$. And if $k = 0$, identified with $\emptyset$, then the only function from k to X is the empty function, so X^0 becomes $\{\emptyset\}$.

I shall sometimes refer to the 'usual topology of $\mathbb{N}^{\mathbb{N}}$'; this is the product topology if each copy of $\mathbb{N}$ is given its discrete topology. S will always be the set $\bigcup_{k\in\mathbb{N}} \mathbb{N}^k$, and $S^* = S \setminus \{\emptyset\}$ the set $\bigcup_{k\ge1} \mathbb{N}^k$; for $\sigma \in S$, I_σ will be $\{\phi : \phi \in \mathbb{N}^{\mathbb{N}}, \phi \supseteq \sigma\}$. Then $I_\emptyset = \mathbb{N}^{\mathbb{N}}$ and $\{I_\sigma : \sigma \in S^*\}$ is a base for the topology of $\mathbb{N}^{\mathbb{N}}$ consisting of open-and-closed sets. If $\sigma \in \mathbb{N}^k$ and $i \in \mathbb{N}$ I write $\sigma^\frown{<}i{>}$ for the member τ of $\mathbb{N}^{k+1}$ such that $\tau(k) = i$ and $\tau(j) = \sigma(j)$ for $j < k$.

421B Definition If $\mathcal{E}$ is a family of sets, I write $\mathcal{S}(\mathcal{E})$ for the family of sets expressible in the form

$$\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\ge1} E_{\phi{\restriction}k}$$

for some family $\langle E_\sigma\rangle_{\sigma\in S^*}$ in $\mathcal{E}$.

A family $\langle E_\sigma\rangle_{\sigma\in S^*}$ is called a **Souslin scheme**; the corresponding set $\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\ge1} E_{\phi{\restriction}k}$ is its **kernel**; the operation

$$\langle E_\sigma\rangle_{\sigma\in S^*} \mapsto \bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\ge1} E_{\phi{\restriction}k}$$

is **Souslin's operation** or **operation $\mathcal{A}$**. Thus $\mathcal{S}(\mathcal{E})$ is the family of sets obtainable from sets in $\mathcal{E}$ by Souslin's operation. If $\mathcal{E} = \mathcal{S}(\mathcal{E})$, we say that $\mathcal{E}$ is **closed under Souslin's operation**.

Remark I should perhaps warn you that some authors use $\bigcup_{k\in\mathbb{N}} \mathbb{N}^k$ here in place of S^*; so that their Souslin kernels are of the form $\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\ge0} E_{\phi{\restriction}k} \subseteq E_\emptyset$. Consequently, for such authors, any member of $\mathcal{S}(\mathcal{E})$ is included in some member of $\mathcal{E}$. If $\mathcal{E}$ has a greatest member (or, fractionally more generally, if any sequence in $\mathcal{E}$ is bounded above in $\mathcal{E}$) this makes no difference; but if, for instance, $\mathcal{E}$ is the family of compact subsets of a topological space, the two definitions of $\mathcal{S}$ may not quite coincide. I believe that on this point, for once, I am following the majority.

421C Elementary facts (a) It is worth noting straight away that if $\mathcal{E}$ is any family of sets, then $\bigcup_{n\in\mathbb{N}} E_n$ and $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{S}(\mathcal{E})$ for any sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$. **P** Set

$$F_\sigma = E_{\sigma(0)} \text{ for every } \sigma \in S^*,$$

$$G_\sigma = E_k \text{ whenever } k \in \mathbb{N},\ \sigma \in \mathbb{N}^{k+1};$$

then

$$\textstyle\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{k\ge 1} F_{\phi\restriction k} \in \mathcal{S}(\mathcal{E}),$$

$$\textstyle\bigcap_{n\in\mathbb{N}} E_n = \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{k\ge 1} G_{\phi\restriction k} \in \mathcal{S}(\mathcal{E}). \text{ **Q**}$$

In particular, $\mathcal{E} \subseteq \mathcal{S}(\mathcal{E})$. But note that there is no reason why $E \setminus F$ should belong to $\mathcal{S}(\mathcal{E})$ for E, $F \in \mathcal{E}$.

(b) Let X and Y be sets, and $f : X \to Y$ a function. Let $\langle F_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{P}Y$, with kernel B. Then $f^{-1}[B]$ is the kernel of the Souslin scheme $\langle f^{-1}[F_\sigma]\rangle_{\sigma\in S^*}$. **P**

$$f^{-1}[B] = f^{-1}[\textstyle\bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{n\ge 1} F_{\phi\restriction n}] = \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{n\ge 1} f^{-1}[F_{\phi\restriction n}]. \text{ **Q**}$$

(c) Let X and Y be sets, and $f : X \to Y$ a function. Let $\mathcal{F}$ be a family of subsets of Y. Then

$$\{f^{-1}[B] : B \in \mathcal{S}(\mathcal{F})\} = \mathcal{S}(\{f^{-1}[F] : F \in \mathcal{F}\}).$$

P For a set $A \subseteq X$, $A \in \mathcal{S}(\{f^{-1}[F] : F \in \mathcal{F}\})$ iff there is some Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S^*}$ in $\{f^{-1}[F] : F \in \mathcal{F}\}$ such that A is the kernel of $\langle E_\sigma\rangle_{\sigma\in S^*}$, that is, iff there is some Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S^*}$ in $\mathcal{F}$ such that A is the kernel of $\langle f^{-1}[F_\sigma]\rangle_{\sigma\in S^*}$, that is, iff $A = f^{-1}[B]$ where B is the kernel of some Souslin scheme in $\mathcal{F}$. **Q**

(d) Let X and Y be sets, and $f : X \to Y$ a surjective function. Let $\mathcal{F}$ be a family of subsets of Y. Then

$$\mathcal{S}(\mathcal{F}) = \{B : B \subseteq Y,\ f^{-1}[B] \in \mathcal{S}(\{f^{-1}[F] : F \in \mathcal{F}\})\}.$$

P If $B \in \mathcal{S}(\mathcal{F})$, then $f^{-1}[B] \in \mathcal{S}(\{f^{-1}[F] : F \in \mathcal{F}\})$, by (c) above. If $B \subseteq Y$ and $f^{-1}[B] \in \mathcal{S}(\{f^{-1}[F] : F \in \mathcal{F}\})$, then there is a Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S^*}$ in $\mathcal{F}$ such that $f^{-1}[B]$ is the kernel of $\langle f^{-1}[F_\sigma]\rangle_{\sigma\in S^*}$, that is, $f^{-1}[B] = f^{-1}[C]$ where C is the kernel of $\langle F_\sigma\rangle_{\sigma\in S^*}$. Because f is surjective, $B = C \in \mathcal{S}(\mathcal{F})$. **Q**

(e) Souslin's operation can be thought of as a projection operator, as follows. Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme with kernel A. Set

$$R = \textstyle\bigcap_{n\ge 1} \bigcup_{\sigma\in\mathbb{N}^n} I_\sigma \times E_\sigma.$$

Then $R[\mathbb{N}^\mathbb{N}] = A$. **P** For any x, and any $\phi \in \mathbb{N}^\mathbb{N}$,

$$\begin{aligned}(\phi, x) \in R &\iff \text{for every } n \ge 1 \text{ there is a } \sigma \in \mathbb{N}^n \text{ such that } x \in E_\sigma,\ \phi \in I_\sigma\\ &\iff x \in E_{\phi\restriction n} \text{ for every } n \ge 1.\end{aligned}$$

But this means that

$$\begin{aligned}x \in R[\mathbb{N}^\mathbb{N}] &\iff \text{there is a } \phi \in \mathbb{N}^\mathbb{N} \text{ such that } (\phi, x) \in R\\ &\iff \text{there is a } \phi \in \mathbb{N}^\mathbb{N} \text{ such that } x \in \bigcap_{n\ge 1} E_{\phi\restriction n} \iff x \in A. \text{ **Q**}\end{aligned}$$

421D The first fundamental theorem is that the operation $\mathcal{S}$ is idempotent.

Theorem (SOUSLIN 1917) For any family $\mathcal{E}$ of sets, $\mathcal{S}(\mathcal{E})$ is closed under Souslin's operation.

proof (a) Let $\langle A_\sigma\rangle_{\sigma\in S^*}$ be a family in $\mathcal{S}(\mathcal{E})$, and set $A = \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{k\ge 1} A_{\phi\restriction k}$; I have to show that $A \in \mathcal{S}(\mathcal{E})$. For each $\sigma \in S$, let $\langle E_{\sigma\tau}\rangle_{\tau\in S^*}$ be a family in $\mathcal{E}$ such that $A_\sigma = \bigcup_{\psi\in\mathbb{N}^\mathbb{N}} \bigcap_{m\ge 1} E_{\sigma,\psi\restriction m}$. Then

$$A = \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{k\ge 1} \bigcup_{\psi\in\mathbb{N}^\mathbb{N}} \bigcap_{m\ge 1} E_{\phi\restriction k,\psi\restriction m} = \bigcup_{\substack{\phi\in\mathbb{N}^\mathbb{N}\\ \boldsymbol{\psi}\in(\mathbb{N}^\mathbb{N})^{\mathbb{N}\setminus\{0\}}}} \bigcap_{k,m\ge 1} E_{\phi\restriction k,\psi_k\restriction m},$$

writing $\boldsymbol{\psi} = \langle \psi_k \rangle_{k \geq 1}$ for $\boldsymbol{\psi} \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}\setminus\{0\}}$. The idea of the proof is simply that $\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}\setminus\{0\}}$ is essentially identical to $\mathbb{N}^{\mathbb{N}}$, so that all we have to do is to organize new names for the $E_{\sigma\tau}$. But as it is by no means a trivial matter to devise a coding scheme which really works, I give the details at length.

(b) The first step is to note that S^* and $(S^*)^2$ are countable, so there is a sequence $\langle H_n \rangle_{n \in \mathbb{N}}$ running over $\{E_{\sigma\tau} : \sigma, \tau \in S^*\}$. Next, choose any injective function $q : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \setminus \{0\}$ such that $q(0,0) = 1$ and $q(0,1) = 2$. For k, $m \geq 1$ set $J_{km} = \{(i,0) : i < k\} \cup \{(i,k) : i < m\}$, so that $J_{11} = \{(0,0),(0,1)\}$, and choose a family $\langle (k_n, m_n) \rangle_{n \geq 3}$ running over $(\mathbb{N} \setminus \{0\})^2$ such that $q[J_{k_n,m_n}] \subseteq n$ for every $n \geq 3$. (The pairs (k_n, m_n) need not all be distinct, so this is easy to achieve.)

Now, for $\upsilon \in \mathbb{N}^n$, where $n \geq 3$, set $F_\upsilon = E_{\sigma\tau}$ where

$$\sigma \in \mathbb{N}^{k_n},\ \sigma(i) = \upsilon(q(i,0)) \text{ for } i < k_n,$$

$$\tau \in \mathbb{N}^{m_n},\ \tau(i) = \upsilon(q(i,k_n)) \text{ for } i < m_n;$$

these are well-defined because $q[J_{k_n,m_n}] \subseteq n$. For $\upsilon \in \mathbb{N}^1 \cup \mathbb{N}^2$, set $F_\upsilon = H_{\upsilon(0)}$.

(c) This defines a Souslin scheme $\langle F_\upsilon \rangle_{\upsilon \in S^*}$ in $\mathcal{E}$. Let A' be its kernel, so that $A' \in \mathcal{S}(\mathcal{E})$. The point is that $A' = A$.

P **(i)** If $x \in A$, there must be $\phi \in \mathbb{N}^{\mathbb{N}}$, $\boldsymbol{\psi} \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}\setminus\{0\}}$ such that $x \in \bigcap_{k,m \geq 1} E_{\phi\restriction k, \psi_k\restriction m}$. Choose $\theta \in \mathbb{N}^{\mathbb{N}}$ such that

$$H_{\theta(0)} = E_{\phi\restriction 1, \psi_1\restriction 1},$$
$$\theta(q(i,0)) = \phi(i) \text{ for every } i \in \mathbb{N},$$
$$\theta(q(i,k)) = \psi_k(i) \text{ for every } k \geq 1,\ i \in \mathbb{N}.$$

(This is possible because $q : \mathbb{N}^2 \to \mathbb{N} \setminus \{0\}$ is injective.) Now

$$F_{\theta\restriction 1} = F_{\theta\restriction 2} = H_{\theta(0)} = E_{\phi\restriction 1, \psi_1\restriction 1}$$

certainly contains x. And for $n \geq 3$, $F_{\theta\restriction n} = E_{\sigma\tau}$ where $\sigma(i) = \theta(q(i,0))$ for $i < k_n$, $\tau(i) = \theta(q(i,k_n))$ for $i < m_n$, that is, $\sigma = \phi\restriction k_n$ and $\tau = \psi_{k_n}\restriction m_n$, so again $x \in F_{\theta\restriction n}$. Thus

$$x \in \textstyle\bigcap_{n \geq 1} F_{\theta\restriction n} \subseteq A'.$$

As x is arbitrary, $A \subseteq A'$.

(ii) Now take any $x \in A'$. Let $\theta \in \mathbb{N}^{\mathbb{N}}$ be such that $x \in \bigcap_{n \geq 1} F_{\theta\restriction n}$. Define $\phi \in \mathbb{N}^{\mathbb{N}}$, $\boldsymbol{\psi} \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}\setminus\{0\}}$ by setting

$$\phi(i) = \theta(q(i,0)) \text{ for } i \in \mathbb{N},$$
$$\psi_k(i) = \theta(q(i,k)) \text{ for } k \geq 1,\ i \in \mathbb{N}.$$

If k, $m \geq 1$, let $n \geq 3$ be such that $k = k_n$, $m = m_n$. Then $x \in F_{\theta\restriction n} = E_{\sigma\tau}$, where

$$\sigma(i) = \theta(q(i,0)) \text{ for } i < k_n, \quad \tau(i) = \theta(q(i,k_n)) \text{ for } i < m_n,$$

that is, $\sigma = \phi\restriction k_n = \phi\restriction k$ and $\tau = \psi_{k_n}\restriction m_n = \psi_k\restriction m$. As m and n are arbitrary,

$$x \in \textstyle\bigcap_{m,n \geq 1} E_{\phi\restriction k, \psi_k\restriction m} \subseteq A.$$

As x is arbitrary, $A' \subseteq A$. **Q**

Accordingly we must have $A \in \mathcal{S}(\mathcal{E})$, and the proof is complete.

421E Corollary For any family $\mathcal{E}$ of sets, $\mathcal{S}(\mathcal{E})$ is closed under countable unions and intersections.

proof For 421Ca tells us that the union and intersection of any sequence in $\mathcal{S}(\mathcal{E})$ will belong to $\mathcal{S}\mathcal{S}(\mathcal{E}) = \mathcal{S}(\mathcal{E})$.

421F Corollary Let X be a set and $\mathcal{E}$ a family of subsets of X. Suppose that X and $\emptyset$ belong to $\mathcal{S}(\mathcal{E})$ and that $X \setminus E \in \mathcal{S}(\mathcal{E})$ for every $E \in \mathcal{E}$. Then $\mathcal{S}(\mathcal{E})$ includes the σ-algebra of subsets of X generated by $\mathcal{E}$.

proof The set

$$\Sigma = \{F : F \in \mathcal{S}(\mathcal{E}),\ X \setminus F \in \mathcal{S}(\mathcal{E})\}$$

is closed under complements (necessarily), contains $\emptyset$ (because $\emptyset$ and X belong to $\mathcal{S}(\mathcal{E})$), and is also closed under countable unions, by 421E. So it is a σ-algebra; but the hypotheses also ensure that $\mathcal{E} \subseteq \Sigma$, so that the σ-algebra generated by $\mathcal{E}$ is included in Σ and in $\mathcal{S}(\mathcal{E})$.

421G Proposition Let $\mathcal{E}$ be a family of sets such that $\emptyset \in \mathcal{E}$. Then

$$\begin{aligned}\mathcal{S}(\mathcal{E}) &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\{I_\sigma \times E : \sigma \in S^*,\, E \in \mathcal{E}\})\} \\ &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\{I_\sigma \times E : \sigma \in S^*,\, E \in \mathcal{E}\}),\, R^{-1}[\{x\}] \text{ is closed for every } x\}.\end{aligned}$$

proof Set $\mathcal{F} = \{I_\sigma \times E : \sigma \in S^*,\, E \in \mathcal{E}\}$.

(a) Suppose first that $A \in \mathcal{S}(\mathcal{E})$. Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{E}$ with kernel A. Set

$$R = \bigcap_{k\ge1} \bigcup_{\sigma\in\mathbb{N}^k} I_\sigma \times E_\sigma.$$

Then $R \in \mathcal{S}(\mathcal{F})$, by 421E, and $R[\mathbb{N}^{\mathbb{N}}] = A$, by 421Ce. Also

$$R^{-1}[\{x\}] = \bigcap_{k\ge1} \bigcup\{I_\sigma : \sigma \in \mathbb{N}^k,\, x \in E_\sigma\}$$

is closed, for every x.

(b) Now suppose that $A = R[\mathbb{N}^{\mathbb{N}}]$ for some $R \in \mathcal{S}(\mathcal{F})$. Let $\langle I_{\tau(\sigma)} \times E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{F}$ with kernel R. For $k \ge 1$, $\sigma \in \mathbb{N}^k$ set

$$\begin{aligned}F_\sigma &= E_\sigma \text{ if } \bigcap_{1\le n\le k} I_{\tau(\sigma\restriction n)} \ne \emptyset, \\ &= \emptyset \text{ otherwise.}\end{aligned}$$

Then $\langle F_\sigma\rangle_{\sigma\in S^*}$ is a Souslin scheme in $\mathcal{E}$, so its kernel A' belongs to $\mathcal{S}(\mathcal{E})$.

The point is that $A' = A$. **P** (i) If $x \in A$, there are a $\phi \in \mathbb{N}^{\mathbb{N}}$ such that $(\phi, x) \in R$ and a $\psi \in \mathbb{N}^{\mathbb{N}}$ such that $(\phi, x) \in \bigcap_{n\ge1} I_{\tau(\psi\restriction n)} \times E_{\psi\restriction n}$. Now, for any $k \ge 1$, we have

$$\phi \in \bigcap_{1\le n\le k} I_{\tau(\psi\restriction n)} = \bigcap_{1\le n\le k} I_{\tau((\psi\restriction k)\restriction n)},$$

so that $F_{\psi\restriction k} = E_{\psi\restriction k}$ contains x; thus $x \in \bigcap_{k\ge1} F_{\psi\restriction k} \subseteq A'$. As x is arbitrary, $A \subseteq A'$. (ii) If $x \in A'$, take $\psi \in \mathbb{N}^{\mathbb{N}}$ such that $x \in \bigcap_{n\ge1} F_{\psi\restriction n}$. In this case we must have $F_{\psi\restriction k} \ne \emptyset$, so $\bigcap_{1\le n\le k} I_{\tau(\psi\restriction n)} \ne \emptyset$, for every $k \ge 1$. But what this means is that, setting $\tau_n = \tau(\psi\restriction n)$ for each $n \ge 1$, $\tau_n(i) = \tau_m(i)$ whenever $i \in \mathbb{N}$ is such that both are defined. So $\{\tau_n : n \ge 1\}$ must have a common extension $\phi \in \mathbb{N}^{\mathbb{N}}$, and $\phi \in \bigcap_{n\ge1} I_{\tau(\psi\restriction n)}$. Now

$$(\phi, x) \in \bigcap_{n\ge1} I_{\tau(\psi\restriction n)} \times E_{\psi\restriction n} \subseteq R,$$

so $x \in A$. Thus $A' \subseteq A$ and the two are equal. **Q**

This shows that

$$\{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{F})\} \subseteq \mathcal{S}(\mathcal{E}),$$

and the proof is complete.

421H When the class $\mathcal{E}$ is a σ-algebra, the last proposition can be extended.

Proposition Let X be a set, and Σ a σ-algebra of subsets of X. Let $\mathcal{B}$ be the algebra of Borel subsets of $\mathbb{N}^{\mathbb{N}}$. Then

$$\begin{aligned}\mathcal{S}(\Sigma) &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{B}\widehat{\otimes}\Sigma\} \\ &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\{I_\sigma \times E : \sigma \in S^*,\, E \in \Sigma\})\} \\ &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma)\}.\end{aligned}$$

Notation Recall that $\mathcal{B}\widehat{\otimes}\Sigma$ is the σ-algebra of subsets of $\mathbb{N}^{\mathbb{N}} \times X$ generated by $\{H \times E : H \in \mathcal{B},\, E \in \Sigma\}$.

proof (a) Suppose first that $A \in \mathcal{S}(\Sigma)$. As in 421G, let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in Σ with kernel A, and set

$$R = \bigcap_{k\ge1} \bigcup_{\sigma\in\mathbb{N}^k} I_\sigma \times E_\sigma,$$

so that $A = R[\mathbb{N}^{\mathbb{N}}]$ (421Ce again). Because every I_σ is an open-and-closed set in $\mathbb{N}^{\mathbb{N}}$, $R \in \mathcal{B}\widehat{\otimes}\Sigma$. Thus

$$\mathcal{S}(\Sigma) \subseteq \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{B}\widehat{\otimes}\Sigma\}.$$

(b) Set $\mathcal{F} = \{I_\sigma \times E : \sigma \in S^*,\, E \in \Sigma\}$. Then $\mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma) = \mathcal{S}(\mathcal{F})$. **P** If $E \in \Sigma$ and $\sigma \in \mathbb{N}^k$ then

$$(\mathbb{N}^{\mathbb{N}} \times X) \setminus (I_\sigma \times E) = (I_\sigma \times (X \setminus E)) \cup \bigcup_{\tau \in \mathbb{N}^k, \tau \neq \sigma} I_\tau \times X \in \mathcal{S}(\mathcal{F}).$$

Also

$$\mathbb{N}^{\mathbb{N}} \times X = \bigcup_{\sigma \in \mathbb{N}^1} I_\sigma \times X, \quad \emptyset = I_\tau \times \emptyset$$

(where τ is any member of S^*) belong to $\mathcal{S}(\mathcal{F})$. By 421F, $\mathcal{S}(\mathcal{F})$ includes the σ-algebra Λ of sets generated by $\mathcal{F}$. Now if $E \in \Sigma$ and $H \subseteq \mathbb{N}^{\mathbb{N}}$ is open, $H = \bigcup_{\sigma \in T} I_\sigma$ for some $T \subseteq S^*$; as T is necessarily countable,

$$H \times E = \bigcup_{\sigma \in T} I_\sigma \times E \in \Lambda.$$

Since $\{F : F \subseteq \mathbb{N}^{\mathbb{N}}, F \times E \in \Lambda\}$ is a σ-algebra of subsets of $\mathbb{N}^{\mathbb{N}}$, and we have just seen that it contains all the open sets, it must include $\mathcal{B}$; thus $F \times E \in \Lambda$ for every $F \in \mathcal{B}$, $E \in \Sigma$. So $\mathcal{B}\widehat{\otimes}\Sigma \subseteq \Lambda \subseteq \mathcal{S}(\mathcal{F})$, and

$$\mathcal{S}(\mathcal{F}) \subseteq \mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma) \subseteq \mathcal{S}\mathcal{S}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$$

(421D). **Q**

(c) Now we have

$$\mathcal{S}(\Sigma) \subseteq \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{B}\widehat{\otimes}\Sigma\}$$

(by (a))

$$\subseteq \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma)\} = \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{F})\}$$

(by (b))

$$= \mathcal{S}(\Sigma)$$

by 421G.

421I There is a particularly simple description of sets obtainable by Souslin's operation from closed sets in a topological space.

Lemma Let X be a topological space and $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ a closed set. Then

$$R[A] = \bigcup_{\phi \in A} \bigcap_{n \geq 1} \overline{R[I_{\phi \restriction n}]}.$$

for any $A \subseteq \mathbb{N}^{\mathbb{N}}$. In particular, $R[\mathbb{N}^{\mathbb{N}}]$ is the kernel of the Souslin scheme $\langle \overline{R[I_\sigma]} \rangle_{\sigma \in S^*}$.

proof Set

$$B = \bigcup_{\phi \in A} \bigcap_{n \geq 1} \overline{R[I_{\phi \restriction n}]}.$$

(i) If $x \in R[A]$, there is a $\phi \in A$ such that $(\phi, x) \in R$. In this case, $\phi \in I_{\phi \restriction n}$ so

$$x \in R[I_{\phi \restriction n}] \subseteq \overline{R[I_{\phi \restriction n}]}$$

for every n, and $x \in B$. Thus $R[A] \subseteq B$. (ii) If $x \in B$, let $\phi \in A$ be such that $x \in \overline{R[I_{\phi \restriction n}]}$ for every $n \in \mathbb{N}$. **?** If $(\phi, x) \notin R$, then (because R is closed) there are a $\sigma \in S^*$ and an open $G \subseteq X$ such that $\phi \in I_\sigma$, $x \in G$ and $(I_\sigma \times G) \cap R = \emptyset$. But this means that $G \cap R[I_\sigma] = \emptyset$ so $G \cap \overline{R[I_\sigma]} = \emptyset$ and $x \notin \overline{R[I_\sigma]}$; which is absurd, because $\sigma = \phi \restriction n$ for some $n \geq 1$. **X** Thus $(\phi, x) \in R$ and $x \in R[A]$. As x is arbitrary, $B \subseteq R[A]$ and $B = R[A]$, as required.

421J Proposition Let X be a topological space, and $\mathcal{F}$ the family of closed subsets of X. Then a set $A \subseteq X$ belongs to $\mathcal{S}(\mathcal{F})$ iff there is a closed set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that A is the projection of R on X.

proof (a) Suppose that $A \in \mathcal{S}(\mathcal{F})$. Let $\langle F_\sigma \rangle_{\sigma \in S^*}$ be a Souslin scheme in $\mathcal{F}$ with kernel A. Set

$$R = \bigcap_{n \geq 1} \bigcup_{\sigma \in \mathbb{N}^n} I_\sigma \times F_\sigma.$$

For each $n \geq 1$,

$$\bigcup_{\sigma \in \mathbb{N}^n} I_\sigma \times F_\sigma = (\mathbb{N}^{\mathbb{N}} \times X) \setminus \bigcup_{\sigma \in \mathbb{N}^n} I_\sigma \times (X \setminus F_\sigma)$$

is closed in $\mathbb{N}^{\mathbb{N}} \times X$, so R is closed; and the projection $R[\mathbb{N}^{\mathbb{N}}]$ is A, by 421Ce.

(b) Suppose that $R \subseteq \mathbb{N}^{\mathbb{N}}$ is a closed set with projection A. Then A is the kernel of the Souslin scheme $\langle \overline{R[I_\sigma]} \rangle_{\sigma \in S^*}$, by 421I, so belongs to $\mathcal{S}(\mathcal{F})$.

421K Definition Let X be a topological space. A subset of X is a **Souslin-F** set in X if it is obtainable from closed subsets of X by Souslin's operation; that is, is the projection of a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$.

For a subset of $\mathbb{R}^r$, or, more generally, of any Polish space, it is common to say 'Souslin set' for 'Souslin-F set'; see 421Xl.

421L Proposition Let X be any topological space. Then every Baire subset of X is Souslin-F.

proof Let $\mathcal{Z}$ be the family of zero sets in X. If $F \in \mathcal{Z}$ then $X \setminus F$ is a countable union of zero sets (4A2C(b-vi)), so belongs to $\mathcal{S}(\mathcal{Z})$. By 421F, the σ-algebra generated by $\mathcal{Z}$ is included in $\mathcal{S}(\mathcal{Z}) \subseteq \mathcal{S}(\mathcal{F})$, where $\mathcal{F}$ is the family of closed subsets of X; that is, every Baire set is Souslin-F.

421M Proposition Let $\mathcal{E}$ be any family of sets such that $\emptyset \in \mathcal{E}$ and $E \cup E'$, $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{E}$ for every E, $E' \in \mathcal{E}$ and all sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$. (For instance, $\mathcal{E}$ could be the family of closed subsets of a topological space, or a σ-algebra of sets.) Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{E}$, and $K \subseteq \mathbb{N}^{\mathbb{N}}$ a set which is compact for the usual topology on $\mathbb{N}^{\mathbb{N}}$. Then $\bigcup_{\phi\in K}\bigcap_{n\ge1} E_{\phi\restriction n} \in \mathcal{E}$.

proof Set $A = \bigcup_{\phi\in K}\bigcap_{n\ge1} E_{\phi\restriction n}$. For $k \in \mathbb{N}$, set $K_k = \{\phi\restriction k : \phi \in K\}$; note that $K_k \subseteq \mathbb{N}^k$ is compact, because $\phi \mapsto \phi\restriction k$ is continuous, therefore finite, because the topology of $\mathbb{N}^k$ is discrete. Set

$$H = \bigcap_{k\ge1}\bigcup_{\phi\in K_k}\bigcap_{1\le n\le k} E_{\phi\restriction n}.$$

Because $\mathcal{E}$ is closed under finite unions and countable intersections, $H \in \mathcal{E}$. Now $A = H$. **P** (i) If $x \in A$, take $\phi \in K$ such that $x \in E_{\phi\restriction n}$ for every $n \ge 1$; then $\phi\restriction k \in K_k$ and $x \in \bigcap_{1\le n\le k} E_{(\phi\restriction k)\restriction n}$ for every $k \ge 1$, so $x \in H$. Thus $A \subseteq H$. (ii) If $x \in H$, then for each $k \in \mathbb{N}$ we have a $\sigma_k \in K_k$ such that $x \in \bigcap_{1\le n\le k} E_{\sigma_k\restriction n}$. Choose $\phi_k \in K$ such that $\phi_k\restriction k = \sigma_k$ for each k. Now K is supposed to be compact, so the sequence $\langle\phi_k\rangle_{k\in\mathbb{N}}$ has a cluster point ϕ in K.

If $n \ge 1$, then $I_{\phi\restriction n}$ is a neighbourhood of ϕ in $\mathbb{N}^{\mathbb{N}}$, so must contain ϕ_k for infinitely many k; let $k \ge n$ be such that $\phi_k\restriction n = \phi\restriction n$. In this case

$$x \in E_{\sigma_k\restriction n} = E_{\phi_k\restriction n} = E_{\phi\restriction n}.$$

As n is arbitrary,

$$x \in \bigcap_{n\ge1} E_{\phi\restriction n} \subseteq A.$$

As x is arbitrary, $H \subseteq A$ and $H = A$, as claimed. **Q**

So $A \in \mathcal{E}$.

***421N** I now embark on preparations for the theory of 'constituents' of analytic and coanalytic sets. It turns out that much of the work can be done in the abstract context of this section.

Trees and derived trees (a) Let $\mathcal{T}$ be the family of subsets T of $S^* = \bigcup_{n\ge1}\mathbb{N}^n$ such that $\sigma\restriction k \in T$ whenever $\sigma \in T$ and $1 \le k \le \#(\sigma)$. Note that the intersection and union of any non-empty family of members of $\mathcal{T}$ again belong to $\mathcal{T}$. Members of $\mathcal{T}$ are often called **trees**.

(b) For $T \in \mathcal{T}$, set

$$\partial T = \{\sigma : \sigma \in S^*, \exists\, i \in \mathbb{N}, \sigma^\frown\!<\!i\!> \in T\},$$

so that $\partial T \in \mathcal{T}$ and $\partial T \subseteq T$. Of course $\partial T_0 \subseteq \partial T_1$ whenever T_0, $T_1 \in \mathcal{T}$ and $T_0 \subseteq T_1$.

(c) For $T \in \mathcal{T}$, define $\langle\partial^\xi T\rangle_{\xi<\omega_1}$ inductively by setting $\partial^0 T = T$ and, for $\xi > 0$, $\partial^\xi T = \bigcap_{\eta<\xi}\partial(\partial^\eta T)$. An easy induction shows that $\partial^\xi T \in \mathcal{T}$, $\partial^\xi T \subseteq \partial^\eta T$ and $\partial^{\xi+1}T = \partial(\partial^\xi T)$ whenever $\eta \le \xi < \omega_1$.

(d) For any $T \in \mathcal{T}$, there is a $\xi < \omega_1$ such that $\partial^\xi T = \partial^\eta T$ whenever $\xi \le \eta < \omega_1$. **P** Set $T_1 = \bigcap_{\xi<\omega_1}\partial^\xi T$. For each $\sigma \in S^* \setminus T_1$, there is a $\xi_\sigma < \omega_1$ such that $\sigma \notin \partial^{\xi_\sigma}T$. Set $\xi = \sup\{\xi_\sigma : \sigma \in S^* \setminus T_1\}$; because S^* is countable, $\xi < \omega_1$, and we must now have $\partial^\xi T = T_1$, so that $\partial^\xi T = \partial^\eta T$ whenever $\xi \le \eta < \omega_1$. **Q**

(e) For $T \in \mathcal{T}$, its **rank** is the first ordinal $r(T) < \omega_1$ such that $\partial^{r(T)}T = \partial^{r(T)+1}T$; of course $\partial^{r(T)}T = \partial^\eta T$ whenever $r(T) \le \eta < \omega_1$, and $\partial(\partial^{r(T)}T) = \partial^{r(T)}T$.

(f) For $T \in \mathcal{T}$, the following are equiveridical: (α) $\partial^{r(T)}T \ne \emptyset$; (β) there is a $\phi \in \mathbb{N}^{\mathbb{N}}$ such that $\phi\restriction n \in T$ for every $n \ge 1$. **P** (i) If $\sigma \in \partial^{r(T)}T$ then $\sigma \in \partial(\partial^{r(T)}T)$ so there is an $i \in \mathbb{N}$ such that $\sigma^\frown\!<\!i\!> \in \partial^{r(T)}T$. We can therefore choose $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ inductively so that $\sigma_n \in \partial^{r(T)}T$ and σ_{n+1} properly extends σ_n for every n. At the end of the induction, $\phi = \bigcup_{n\in\mathbb{N}}\sigma_n$ belongs to $\mathbb{N}^{\mathbb{N}}$ and

$$\phi\restriction n = \sigma_n\restriction n \in \partial^{r(T)}T \subseteq T$$

for every $n \ge 1$. (ii) If $\phi \in \mathbb{N}^{\mathbb{N}}$ is such that $\phi\restriction n \in T$ for every $n \ge 1$, then an easy induction shows that $\phi\restriction n \in \partial^\xi T$ for every $\xi < \omega_1$ and every $n \ge 1$, so that $\partial^{r(T)}T$ is non-empty. **Q**

(g) Now suppose that $\langle A_\sigma\rangle_{\sigma\in S^*}$ is a Souslin scheme. For any x we have a tree $T_x \in \mathcal{T}$ defined by saying that

$$T_x = \{\sigma : \sigma \in S^*,\, x \in \bigcap_{1\le i\le\#(\sigma)} A_{\sigma\restriction i}\}.$$

Now the kernel of $\langle A_\sigma\rangle_{\sigma\in S^*}$ is just

$$\begin{aligned} A &= \{x : \exists\, \phi \in \mathbb{N}^{\mathbb{N}},\, x \in \bigcap_{n\ge1} A_{\phi\restriction n}\} \\ &= \{x : \exists\, \phi \in \mathbb{N}^{\mathbb{N}},\, \phi\restriction n \in T_x\ \forall\, n \ge 1\} = \{x : \partial^{r(T)}T \ne \emptyset\} \end{aligned}$$

by (f).

The sets

$$\{x : x \in X \setminus A,\, r(T_x) = \xi\} = \{x : x \in X,\, r(T_x) = \xi,\, \partial^\xi T_x = \emptyset\},$$

for $\xi < \omega_1$, are called **constituents** of $X \setminus A$. (Of course they should properly be called 'the constituents of the Souslin scheme $\langle A_\sigma\rangle_{\sigma\in S^*}$'.)

***421O Theorem** Let X be a set and Σ a σ-algebra of subsets of X. Let $\langle A_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in Σ with kernel A, and for $x \in X$ set

$$T_x = \{\sigma : \sigma \in S^*,\, x \in \bigcap_{1\le i\le\#(\sigma)} A_{\sigma\restriction i}\} \in \mathcal{T}$$

as in 421Ng.

(a) For every $\xi < \omega_1$ and $\sigma \in S^*$, $\{x : x \in X,\, \sigma \in \partial^\xi T_x\} \in \Sigma$.

(b) For every $\xi < \omega_1$, $\{x : x \in A,\, r(T_x) \le \xi\}$ and $\{x : x \in X \setminus A,\, r(T_x) \le \xi\}$ belong to Σ. In particular, all the constituents of $X \setminus A$ belong to Σ.

proof (a) Induce on ξ. For $\xi = 0$, we have

$$\{x : x \in X,\, \sigma \in \partial^0 T_x\} = \{x : x \in X,\, \sigma \in T_x\} = \bigcap_{1\le i\le\#(\sigma)} A_{\sigma\restriction i} \in \Sigma.$$

For the inductive step to $\xi > 0$, we have

$$\begin{aligned} \{x : \sigma \in \partial^\xi T_x\} &= \{x : \sigma \in \bigcap_{\eta<\xi} \partial(\partial^\eta T_x)\} \\ &= \bigcap_{\eta<\xi}\bigcup_{i\in\mathbb{N}}\{x : \sigma^\frown{<}i{>} \in \partial^\eta T_x\} \in \Sigma \end{aligned}$$

because ξ is countable and all the sets $\{x : \sigma^\frown{<}i{>} \in \partial^\eta T_x\}$ belong to Σ by the inductive hypothesis.

(b) Now, given $\xi < \omega_1$, we see that $r(T_x) \le \xi$ iff $\partial^{\xi+1}T_x \supseteq \partial^\xi T_x$, so that if we set $E_\xi = \{x : x \in X,\, r(T_x) \le \xi\}$ then

$$E_\xi = \bigcap_{\sigma\in S^*}\{x : x \in X,\, \sigma \in \partial^{\xi+1}T_x \text{ or } \sigma \notin \partial^\xi T_x\}$$

belongs to Σ. If $x \in E_\xi$, so that $\partial^{r(T_x)}T_x = \partial^\xi T_x$, 421Ng tells us that $x \in A$ iff $\partial^\xi T_x \ne \emptyset$; so that

$$E_\xi \cap A = E_\xi \cap \bigcup_{\sigma\in S^*}\{x : \sigma \in \partial^\xi T_x\}$$

and $E_\xi \setminus A$ both belong to Σ.

Now the constituents of $X \setminus A$ are the sets $(E_\xi \setminus A) \setminus \bigcup_{\eta<\xi} E_\eta$ for $\xi < \omega_1$, which all belong to Σ.

***421P Corollary** Let X be a set and Σ a σ-algebra of subsets of X. If $A \in \mathcal{S}(\Sigma)$ then both A and $X \setminus A$ can be expressed as the union of at most ω_1 members of Σ.

proof In the language of 421O, we have

$$A = \bigcup_{\xi<\omega_1} E_\xi \cap A, \quad X \setminus A = \bigcup_{\xi<\omega_1} E_\xi \setminus A.$$

***421Q Lemma** Let X be a set and $\langle A_\sigma\rangle_{\sigma\in S^*}$ and $\langle B_\sigma\rangle_{\sigma\in S^*}$ two Souslin schemes of subsets of X. Suppose that whenever ϕ, $\psi\in\mathbb{N}^\mathbb{N}$ there is an $n\ge 1$ such that $\bigcap_{1\le i\le n}A_{\phi\restriction i}\cap B_{\psi\restriction i}=\emptyset$. For $x\in X$ set

$$T_x=\bigcup_{n\ge 1}\{\sigma:\sigma\in\mathbb{N}^n,\,x\in\bigcap_{1\le i\le n}A_{\sigma\restriction i}\}$$

as in 421Ng, and let B be the kernel of $\langle B_\sigma\rangle_{\sigma\in S^*}$. Then $\sup_{x\in B}r(T_x)<\omega_1$.

proof For $\sigma\in S^*$ set $A'_\sigma=\bigcap_{1\le i\le\#(\sigma)}A_{\sigma\restriction i}$, $B'_\sigma=\bigcap_{1\le i\le\#(\sigma)}B_{\sigma\restriction i}$. Then $T_x=\{\sigma:\sigma\in S^*,\,x\in A'_\sigma\}$ for each $x\in X$, B is the kernel of $\langle B'_\sigma\rangle_{\sigma\in S^*}$, and for every ϕ, $\psi\in\mathbb{N}^\mathbb{N}$ there is an $n\in\mathbb{N}$ such that $A'_{\phi\restriction n}\cap B'_{\psi\restriction n}=\emptyset$.

Define $\langle Q_\xi\rangle_{\xi<\omega_1}$ inductively by setting

$$Q_0=\{(\sigma,\tau):\sigma,\,\tau\in S^*,\,A'_\sigma\cap B'_\tau\ne\emptyset\},$$

and, for $0<\xi<\omega_1$,

$$Q_\xi=\bigcap_{\eta<\xi}\{(\sigma,\tau):\sigma,\,\tau\in S^*,\,\exists\,i,\,j\in\mathbb{N},\,(\sigma^\frown{<}i{>},\tau^\frown{<}j{>})\in Q_\eta\}.$$

Then the same arguments as in 421Na-421Nd show that there is a $\zeta<\omega_1$ such that $Q_{\zeta+1}=Q_\zeta$. **?** If $Q_\zeta\ne\emptyset$, then, just as in 421Nf, there must be ϕ, $\psi\in\mathbb{N}^\mathbb{N}$ such that $(\phi\restriction m,\psi\restriction n)\in Q_\zeta\subseteq Q_0$ for every m, $n\ge 1$; but this means that $A'_{\phi\restriction n}\cap B'_{\psi\restriction n}\ne\emptyset$ for every $n\ge 1$, which is supposed to be impossible. **X**

Now suppose that $x\in B$. Then there is a $\psi\in\mathbb{N}^\mathbb{N}$ such that $x\in B'_{\psi\restriction n}$ for every $n\ge 1$. But this means that $(\sigma,\psi\restriction n)\in Q_0$ for every $\sigma\in T_x$ and every $n\ge 1$. An easy induction shows that $(\sigma,\psi\restriction n)\in Q_\xi$ whenever $\xi<\omega_1$, $\sigma\in\partial^\xi T_x$ and $n\ge 1$. But as $Q_\zeta=\emptyset$ we must have $\partial^\zeta T_x=\emptyset$ and $r(T_x)\le\zeta$. Thus $\sup_{x\in B}r(T_x)\le\zeta<\omega_1$, and the proof is complete.

421X Basic exercises (a) Let X be a set and $\mathcal{E}$ a family of subsets of X. (i) Show that $\emptyset\in\mathcal{S}(\mathcal{E})$ iff there is a sequence in $\mathcal{E}$ with empty intersection. (ii) Show that $X\in\mathcal{S}(\mathcal{E})$ iff there is a sequence in $\mathcal{E}$ with union X.

(b) Let $\mathcal{E}$ be a family of sets and F any set. Show that

$$\mathcal{S}(\{E\cap F:E\in\mathcal{E}\})=\{A\cap F:A\in\mathcal{S}(\mathcal{E})\},$$

$$\mathcal{S}(\{E\cup F:E\in\mathcal{E}\})=\{A\cup F:A\in\mathcal{S}(\mathcal{E})\}.$$

(c) Suppose that $\mathcal{E}$ is a family of sets with $\#(\mathcal{E})\le\mathfrak{c}$. Show that $\#(\mathcal{S}(\mathcal{E}))\le\mathfrak{c}$. (*Hint*: $\#(\mathcal{E}^{S^*})\le\#((\mathcal{P}\mathbb{N})^{S^*})=\#(\mathcal{P}(\mathbb{N}\times S^*))$.)

(d) Let $\mathcal{E}$ be the family of half-open intervals $[2^{-n}k,2^{-n}(k+1)[$, where $n\in\mathbb{N}$, $k\in\mathbb{Z}$; let $\mathcal{G}$ be the set of open subsets of $\mathbb{R}$; let $\mathcal{F}$ be the set of closed subsets of $\mathbb{R}$; let $\mathcal{K}$ be the set of compact subsets of $\mathbb{R}$; let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$. Show that $\mathcal{S}(\mathcal{E})=\mathcal{S}(\mathcal{F})=\mathcal{S}(\mathcal{G})=\mathcal{S}(\mathcal{K})=\mathcal{S}(\mathcal{B})$. (*Hint*: 421F.)

(e) Let $\mathcal{I}$ be the family $\{I_\sigma:\sigma\in\bigcup_{k\in\mathbb{N}}\mathbb{N}^k\}$ (421A); let $\mathcal{G}$ be the set of open subsets of $\mathbb{N}^\mathbb{N}$; let $\mathcal{F}$ be the set of closed subsets of $\mathbb{N}^\mathbb{N}$; let $\mathcal{K}$ be the set of compact subsets of $\mathbb{N}^\mathbb{N}$; let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{N}^\mathbb{N}$. Show that $\mathcal{S}(\mathcal{I})=\mathcal{S}(\mathcal{F})=\mathcal{S}(\mathcal{G})=\mathcal{S}(\mathcal{B})$, but that $\mathcal{S}(\mathcal{K})$ is strictly smaller than these. (*Hint*: if $\langle K_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{K}$, set $\phi(i)=1+\sup_{\psi\in K_i}\psi(i)$ for each $i\in\mathbb{N}$, so that $\phi\notin\bigcup_{n\in\mathbb{N}}K_n$; hence show that $\mathbb{N}^\mathbb{N}\notin\mathcal{S}(\mathcal{K})$.)

(f) Let X be a separable metrizable space with at least two points; let $\mathcal{U}$ be any base for its topology, and $\mathcal{B}$ its Borel σ-algebra. Show that $\mathcal{S}(\mathcal{U})=\mathcal{S}(\mathcal{B})$. What can happen if $\#(X)\le 1$? What about hereditarily Lindelöf spaces?

(g) Let X be a topological space; let $\mathcal{Z}$ be the set of zero sets in X, $\mathcal{G}$ the set of cozero sets, and $\mathcal{B}\mathfrak{a}$ the Baire σ-algebra. Show that $\mathcal{S}(\mathcal{Z})=\mathcal{S}(\mathcal{G})=\mathcal{S}(\mathcal{B}\mathfrak{a})$.

(h) Let X be a set, $\mathcal{E}$ a family of subsets of X, and Σ the σ-algebra of subsets of X generated by $\mathcal{E}$. Show that if $\#(\mathcal{E})\le\mathfrak{c}$ then $\#(\Sigma)\le\mathfrak{c}$. (*Hint*: $\#(\mathfrak{c}^{S^*})=\#(\mathcal{P}(\mathbb{N}\times\mathbb{N}))=\mathfrak{c}$ and $\Sigma\subseteq\mathcal{S}(\mathcal{E}\cup\{X\setminus E:E\in\mathcal{E}\})$.)

(i) Let X be a topological space such that every open set is Souslin-F. Show that every Borel set is Souslin-F.

(j) Let X be a topological space and $\mathcal{B}(X)$ its Borel σ-algebra. Show that $\mathcal{S}(\mathcal{B}(X))$ is just the set of projections on X of Borel subsets of $\mathbb{N}^\mathbb{N}\times X$. (*Hint*: 4A3G.)

(k) Let X and Y be topological spaces, $f : X \to Y$ a continuous function and $F \subseteq Y$ a Souslin-F set. Show that $f^{-1}[F]$ is a Souslin-F set in X.

(l) Let X be any perfectly normal topological space (e.g., any metrizable space); let $\mathcal{G}$ be the set of open subsets of X, $\mathcal{F}$ the set of closed subsets, and $\mathcal{B}$ the Borel σ-algebra. Show that $\mathcal{S}(\mathcal{G}) = \mathcal{S}(\mathcal{F}) = \mathcal{S}(\mathcal{B})$.

(m) Let us say that a Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S^*}$ is **regular** if $E_\sigma \subseteq E_\tau$ whenever σ, $\tau \in S^*$, $\#(\tau) \le \#(\sigma)$ and $\sigma(i) \le \tau(i)$ for every $i < \#(\sigma)$. Let $\mathcal{E}$ be a family of sets such that $E \cup F$ and $E \cap F$ belong to $\mathcal{E}$ for all E, $F \in \mathcal{E}$. Show that every member of $\mathcal{S}(\mathcal{E})$ can be expressed as the kernel of a regular Souslin scheme in $\mathcal{E}$. (*Hint*: if $\langle E_\sigma\rangle_{\sigma\in S^*}$ is any Souslin scheme in $\mathcal{E}$ with kernel A, set $F_\sigma = \bigcap_{\tau\subseteq\sigma} E_\tau$, $G_\sigma = \bigcup_{\tau\le\sigma} F_\tau$, where $\tau \le \sigma$ if $\tau(i) \le \sigma(i)$ for $i < \#(\tau) = \#(\sigma)$; show that A is the kernel of $\langle F_\sigma\rangle_{\sigma\in S^*}$ and of $\langle G_\sigma\rangle_{\sigma\in S^*}$, using an idea from 421M for the latter.)

>(n) Let X be a Hausdorff topological space and $\langle K_\sigma\rangle_{\sigma\in S^*}$ a Souslin scheme in which every K_σ is a compact subset in X. Show that $\bigcup_{\phi\in K}\bigcap_{n\ge1} K_{\phi\restriction n}$ is compact for any compact $K \subseteq \mathbb{N}^{\mathbb{N}}$.

421Y Further exercises (a) Let X be a topological space, Y a Hausdorff space and $f : X \to Y$ a continuous function. Let $\mathcal{K}$ be the family of closed countably compact subsets of X. Show that for any $\mathcal{E} \subseteq \mathcal{K}$ such that $E \cap F \in \mathcal{E}$ for all E, $F \in \mathcal{E}$,

$$\{f[A] : A \in \mathcal{S}(\mathcal{E})\} = \mathcal{S}(\{f[E] : E \in \mathcal{E}\}).$$

(b) Let $\mathcal{E}$ be a family of sets and F any set. Show that

$$\begin{aligned}\mathcal{S}(\mathcal{E} \cup \{F\}) = \{F\} \cup \{A \cap F : A \in \mathcal{S}(\mathcal{E})\} \cup \{B \cup F : B \in \mathcal{S}(\mathcal{E})\}\\ \cup \{(A \cap F) \cup B : A,\, B \in \mathcal{S}(\mathcal{E})\}.\end{aligned}$$

(c) Let X be a topological space, and $\mathcal{B}\mathfrak{a}$ its Baire σ-algebra. Show that $\mathcal{S}(\mathcal{B}\mathfrak{a})$ is just the family of sets expressible as $f^{-1}[B]$ where f is a continuous function from X to some metrizable space Y and $B \subseteq Y$ is Souslin-F.

(d) Let X be a set, $\mathcal{E}$ a family of subsets of X, and Σ the smallest σ-algebra of subsets of X including $\mathcal{E}$ and closed under Souslin's operation. Show that if $\#(\mathcal{E}) \le \mathfrak{c}$ then $\#(\Sigma) \le \mathfrak{c}$. (*Hint*: define $\langle\mathcal{E}_\xi\rangle_{\xi<\omega_1}$ by setting $\mathcal{E}_\xi = \mathcal{S}(\{X \setminus E : E \in \mathcal{E} \cup \bigcup_{\eta<\xi}\mathcal{E}_\xi\})$ for each ξ. Show that $\#(\mathcal{E}_\xi) \le \mathfrak{c}$ for every ξ and that $\Sigma = \bigcup_{\xi<\omega_1}\mathcal{E}_\xi$.)

(e) Let X be a compact space and A a Souslin-F set in X. Show that there is a family $\langle F_\xi\rangle_{\xi<\omega_1}$ of Borel sets such that $X \setminus A = \bigcup_{\xi<\omega} F_\xi$ and whenever $B \subseteq X \setminus A$ is a Souslin-F set there is a $\xi < \omega_1$ such that $B \subseteq F_\xi$. (*Hint*: take $F_\xi = \{x : r(T_x) \le \xi\} \setminus A$ as in 421Ob, and apply 421Q.)

421 Notes and comments In 111G, I defined the Borel sets of $\mathbb{R}$ to be the members of the smallest σ-algebra containing every open set. In 114E, I defined a set to be Lebesgue measurable if it behaves in the right way with respect to Lebesgue outer measure. The latter formulation, at least, provides some sort of testing principle to determine whether a set is Lebesgue measurable. But the definition of 'Borel set' does not. The only tool so far available for proving that a set $E \subseteq \mathbb{R}$ is *not* Borel is to find a σ-algebra containing all open sets and not containing E; conversely, the only method we have for proving properties of Borel sets is to show that a property is possessed by every member of some σ-algebra containing every open set. The revolutionary insight of Souslin 1917 was a construction which could build every Borel set from rational intervals. (See 421Xd.) For fundamental reasons, no construction of this kind can provide all Borel sets without also producing other sets, and to actually characterize the Borel σ-algebra a further idea is needed (423Fa); but the class of analytic sets, being those constructible by Souslin's operation from rational intervals (or open sets, or closed sets, or Borel sets – the operation is robust under such variations), turns out to have remarkable properties which make it as important in modern real analysis as the Borel algebra itself.

The guiding principle of 'descriptive set theory' is that the properties of a set may be analysed in the light of a construction for that set. Thus we can think of a closed set $F \subseteq \mathbb{R}$ as

$$\mathbb{R} \setminus \textstyle\bigcup_{(q,q')\in I}\,]q, q'[$$

where $I \subseteq \mathbb{Q} \times \mathbb{Q}$. The principle can be effective because we often have such descriptions in terms of objects fundamentally simpler than the set being described. In the formula above, for instance, $\mathbb{Q} \times \mathbb{Q}$ is simpler than the set F, being a countable set with a straightforward description from $\mathbb{N}$. The set $\mathcal{P}(\mathbb{Q}\times\mathbb{Q})$ is relatively complex; but a single subset I of $\mathbb{Q}\times\mathbb{Q}$ can easily be coded as a single subset of $\mathbb{N}$ (taking some more or less natural enumeration of $\mathbb{Q}^2$ as a sequence $\langle(q_n,q'_n)\rangle_{n\in\mathbb{N}}$, and matching I with $\{n:(q_n,q'_n)\in I\}$). So, subject to an appropriate coding, we have a description of closed subsets of $\mathbb{R}$ in terms of subsets of $\mathbb{N}$. At the most elementary level, this shows that there are at most $\mathfrak{c}$ closed subsets of $\mathbb{R}$. But we can also set out to analyse such operations as intersection, union, closure in terms of these descriptions. The details are complex, and I shall go no farther along this path until Chapter 56 in Volume 5; but investigations of this kind are at the heart of some of the most exciting developments of twentieth-century real analysis.

The particular descriptive method which concerns us in the present section is Souslin's operation. Starting from a relatively simple class $\mathcal{E}$, we proceed to the larger class $\mathcal{S}(\mathcal{E})$. The most fundamental property of $\mathcal{S}$ is 421D: $\mathcal{S}\mathcal{S}(\mathcal{E})=\mathcal{S}(\mathcal{E})$. This means, for instance, that if $\mathcal{E}\subseteq\mathcal{S}(\mathcal{F})$ and $\mathcal{F}\subseteq\mathcal{S}(\mathcal{E})$, then $\mathcal{S}(\mathcal{E})$ will be equal to $\mathcal{S}(\mathcal{F})$; consequently, different classes of sets will often have the same Souslin closures, as in 421Xd-421Xg. After a little practice you will find that it is often easy to see when two classes $\mathcal{E}$ and $\mathcal{F}$ are at the same level in this sense; but watch out for traps like the class of compact subsets of $\mathbb{N}^{\mathbb{N}}$ (421Xe) and odd technical questions (421Xf).

Souslin's operation, and variations on it, will be the basis of much of the next chapter; it has dramatic applications in general topology and functional analysis as well as in real analysis and measure theory. An important way of looking at the kernel of a Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S^*}$ is to regard it as the projection on the second coordinate of the corresponding set $R=\bigcap_{k\ge1}\bigcup_{\sigma\in\mathbb{N}^k}I_\sigma\times E_\sigma$ (421Ce). We find that many other sets $R\subseteq\mathbb{N}^{\mathbb{N}}\times X$ will also have projections in $\mathcal{S}(\mathcal{E})$ (421G, 421H). Let me remark that it is essential here that the first coordinate should be of the right type. In one sense, indeed, $\mathbb{N}^{\mathbb{N}}$ is the only thing that will do; but its virtue transfers to analytic spaces, as we shall see in 423M-423O below. We shall often want to deal with members of $\mathcal{S}(\mathcal{E})$ which are most naturally defined in terms of some such auxiliary space.

I have moved into slightly higher gear for 421N-421Q because these are not essential for most of the work of the next chapter. From the point of view of this section 421P is very striking but the significance of 421Q is unlikely to be apparent. It becomes important in contexts in which the condition

$$\forall\,\phi,\,\psi\in\mathbb{N}^{\mathbb{N}}\;\exists\,n\ge1,\;\bigcap_{1\le i\le n}A_{\phi\restriction i}\cap B_{\psi\restriction i}=\emptyset$$

is satisfied for natural reasons. I will expand on these in the next two sections. In the meantime, I offer 421Ye as an example of what 421O and 421Q together can tell us.

422 K-analytic spaces

I introduce K-analytic spaces, defined in terms of usco-compact relations. The first step is to define the latter (422A) and give their fundamental properties (422B-422E). I reach K-analytic spaces themselves in 422F, with an outline of the most important facts about them in 422G-422K.

422A Definition Let X and Y be Hausdorff spaces. A relation $R\subseteq X\times Y$ is **usco-compact** if

(α) $R[\{x\}]$ is a compact subset of Y for every $x\in X$,

(β) $R^{-1}[F]$ is a closed subset of X for every closed set $F\subseteq Y$.

(Relations satisfying condition (β) are sometimes called 'upper semi-continuous'.)

422B The following elementary remark will be useful.

Lemma Let X and Y be Hausdorff spaces and $R\subseteq X\times Y$ an usco-compact relation. If $x\in X$ and H is an open subset of Y including $R[\{x\}]$, there is an open set $G\subseteq X$, containing x, such that $R[G]\subseteq H$.

proof Set $G=X\setminus R^{-1}[Y\setminus H]$. Because $Y\setminus H$ is closed, so is $R^{-1}[Y\setminus H]$, and G is open. Of course $R[G]\subseteq H$, and $x\in G$ because $R[\{x\}]\subseteq H$.

422C Proposition Let X and Y be Hausdorff spaces. Then a subset R of $X\times Y$ is an usco-compact relation iff whenever $\mathcal{F}$ is an ultrafilter on $X\times Y$, containing R, such that the first-coordinate image $\pi_1[[\mathcal{F}]]$ of $\mathcal{F}$ has a limit in X, then $\mathcal{F}$ has a limit in R.

proof Recall that, writing $\pi_1(x,y)=x$ and $\pi_2(x,y)=y$ for $(x,y)\in X\times Y$,

$$\pi_1[[\mathcal{F}]]=\{A:A\subseteq X,\,\pi_1^{-1}[A]\in\mathcal{F}\}=\{A:A\subseteq X,\,A\times Y\in\mathcal{F}\}$$

(2A1Ib), and that $\mathcal{F} \to (x,y)$ iff $\pi_1[[\mathcal{F}]] \to x$ and $\pi_2[[\mathcal{F}]] \to y$ (3A3Ic).

(a) Suppose that R is usco-compact and that $\mathcal{F}$ is an ultrafilter on $X \times Y$, containing R, such that $\pi_1[[\mathcal{F}]]$ has a limit $x \in X$. **?** If $\mathcal{F}$ has no limit in R, then, in particular, it does not converge to (x,y) for any $y \in R[\{x\}]$; that is, $\pi_2[[\mathcal{F}]]$ does not converge to any point of $R[\{x\}]$, that is, every point of $R[\{x\}]$ belongs to an open set not belonging to $\pi_2[[\mathcal{F}]]$. Because $R[\{x\}]$ is compact, it is covered by a finite union of open sets not belonging to $\pi_2[[\mathcal{F}]]$; but as $\pi_2[[\mathcal{F}]]$ is an ultrafilter (2A1N), there is an open set $H \supseteq R[\{x\}]$ such that $Y \setminus H \in \pi_2[[\mathcal{F}]]$.

Now 422B tells us that there is an open set G containing x such that $R[G] \subseteq H$. In this case, $G \in \pi_1[[\mathcal{F}]]$ so $G \times Y \in \mathcal{F}$; at the same time, $X \times (Y \setminus H) \in \mathcal{F}$. So

$$R \cap (G \times Y) \cap (X \times (Y \setminus H)) \in \mathcal{F}.$$

But this is empty, by the choice of G; which is intolerable. **X**

Thus $\mathcal{F}$ has a limit in R, as required.

(b) Now suppose that R has the property described.

(i) Let $x \in X$, and suppose that $\mathcal{G}$ is an ultrafilter on Y containing $R[\{x\}]$. Set $h(y) = (x,y)$ for $y \in Y$; then $\mathcal{F} = h[[\mathcal{G}]]$ is an ultrafilter on $X \times Y$ containing R. The image $\pi_1[[\mathcal{F}]]$ is just the principal filter generated by $\{x\}$, so certainly converges to x; accordingly $\mathcal{F}$ must converge to some point $(x,y) \in R$, and $\mathcal{G} = \pi_2[[\mathcal{F}]]$ converges to $y \in R[\{x\}]$. As $\mathcal{G}$ is arbitrary, $R[\{x\}]$ is compact (2A3R).

(ii) Let $F \subseteq Y$ be closed, and $x \in \overline{R^{-1}[F]} \subseteq X$. Consider

$$\mathcal{E} = \{R, X \times F\} \cup \{G \times Y : G \subseteq X \text{ is open, } x \in G\}.$$

Then $\mathcal{E}$ has the finite intersection property. **P** If $G_0, \dots, G_n$ are open sets containing x, then $R^{-1}[F]$ meets $G_0 \cap \dots \cap G_n$ in z say, and now $(z,y) \in R \cap (X \times F) \cap \bigcap_{i \le n}(G_i \times Y)$ for some $y \in F$. **Q** Let $\mathcal{F}$ be an ultrafilter on $X \times Y$ including $\mathcal{E}$ (4A1Ia). Because $G \times Y \in \mathcal{E} \subseteq \mathcal{F}$ for every open set G containing x, $\pi_1[[\mathcal{F}]] \to x$, so $\mathcal{F}$ converges to some point (x,y) of R. Because $X \times F$ is a closed set belonging to $\mathcal{E} \subseteq \mathcal{F}$, $y \in F$ and $x \in R^{-1}[F]$. As x is arbitrary, $R^{-1}[F]$ is closed; as F is arbitrary, R satisfies condition (β) of 422A, and is usco-compact.

422D Lemma (a) Let X and Y be Hausdorff spaces. If $R \subseteq X \times Y$ is an usco-compact relation, then R is closed in $X \times Y$.

(b) Let X and Y be Hausdorff spaces. If $R \subseteq X \times Y$ is an usco-compact relation and $R' \subseteq R$ is a closed set, then R' is usco-compact.

(c) Let X and Y be Hausdorff spaces. If $f : X \to Y$ is a continuous function, then its graph is an usco-compact relation.

(d) Let $\langle X_i \rangle_{i \in I}$ and $\langle Y_i \rangle_{i \in I}$ be families of Hausdorff spaces, and $R_i \subseteq X_i \times Y_i$ an usco-compact relation for each i. Set $X = \prod_{i \in I} X_i$, $Y = \prod_{i \in I} Y_i$ and

$$R = \{(x,y) : x \in X,\, y \in Y,\, (x(i), y(i)) \in R_i \text{ for every } i \in I\}.$$

Then R is usco-compact in $X \times Y$.

(e) Let X and Y be Hausdorff spaces, and $R \subseteq X \times Y$ an usco-compact relation. Then (i) $R[K]$ is a compact subset of Y for any compact subset K of X (ii) $R[L]$ is a Lindelöf subset of Y for any Lindelöf subset L of X.

(f) Let X, Y and Z be Hausdorff spaces, and $R \subseteq X \times Y$, $S \subseteq Y \times Z$ usco-compact relations. Then the composition

$$S \circ R = \{(x,z) : \text{there is some } y \in Y \text{ such that } (x,y) \in R \text{ and } (y,z) \in S\}$$

is usco-compact in $X \times Z$.

(g) Let X and Y be Hausdorff spaces and Y_0 any subset of Y. Then a relation $R \subseteq X \times Y_0$ is usco-compact when regarded as a relation between X and Y_0 iff it is usco-compact when regarded as a relation between X and Y.

(h) Let Y be a Hausdorff space and $R \subseteq \mathbb{N}^{\mathbb{N}} \times Y$ an usco-compact relation. Set

$$R' = \{(\alpha, y) : \alpha \in \mathbb{N}^{\mathbb{N}},\, y \in Y \text{ and there is a } \beta \le \alpha \text{ such that } (\beta, y) \in R\}.$$

Then R' is usco-compact.

proof (a) If $(x,y) \in \overline{R}$, there is an ultrafilter $\mathcal{F}$ containing R and converging to (x,y) (4A2Bc). By 422C, $\mathcal{F}$ must have a limit in R; but as $X \times Y$ is Hausdorff, this limit must be (x,y), and $(x,y) \in R$. As (x,y) is arbitrary, R is closed.

(b) It is obvious that R' will satisfy the condition of 422C if R does.

(c) $f[\{x\}]=\{f(x)\}$ is surely compact for every $x\in X$, and $f^{-1}[F]$ is closed for every closed set $F\subseteq Y$ because f is continuous.

(d) For $i\in I$, $x\in X$, $y\in Y$ set $\phi_i(x,y)=(x(i),y(i))$. If $\mathcal{F}$ is an ultrafilter on $X\times Y$ containing R such that $\pi_1[[\mathcal{F}]]$ has a limit in X, then

$$\pi_1\phi_i[[\mathcal{F}]]=\psi_i\pi_1[[\mathcal{F}]]$$

has a limit in X_i for every $i\in I$, writing $\psi_i(x)=x(i)$ for $i\in I$ and $x\in X$. But $\phi_i[[\mathcal{F}]]$ contains R_i, so has a limit $(x_0(i),y_0(i))$ in $X_i\times Y_i$, for each i. Accordingly (x_0,y_0) is a limit of $\mathcal{F}$ in $X\times Y$ (3A3Ic). As $\mathcal{F}$ is arbitrary, R is usco-compact.

(e) For the moment, let L be any subset of X. Let $\mathcal{H}$ be a family of open sets in Y covering $R[L]$. Let $\mathcal{G}$ be the family of those open sets $G\subseteq X$ such that $R[G]$ can be covered by finitely many members of $\mathcal{H}$. Then $\mathcal{G}$ covers L. **P** If $x\in L$, then $R[\{x\}]$ is a compact subset of $R[L]\subseteq\bigcup\mathcal{H}$, so there is a finite set $\mathcal{H}'\subseteq\mathcal{H}$ covering $R[\{x\}]$. Now there is an open set G containing x such that $R[G]\subseteq\bigcup\mathcal{H}'$, by 422B. **Q**

(i) If L is compact, then there must be a finite subfamily $\mathcal{G}'$ of $\mathcal{G}$ covering L; now $R[L]\subseteq R[\bigcup\mathcal{G}']$ is covered by finitely many members of $\mathcal{H}$. As $\mathcal{H}$ is arbitrary, $R[L]$ is compact.

(ii) If L is Lindelöf, then there must be a countable subfamily $\mathcal{G}'$ of $\mathcal{G}$ covering L; now $R[L]\subseteq R[\bigcup\mathcal{G}']$ is covered by countably many members of $\mathcal{H}$. As $\mathcal{H}$ is arbitrary, $R[L]$ is Lindelöf.

(f) If $x\in X$ then $R[\{x\}]\subseteq Y$ is compact, so $(SR)[\{x\}]=S[R[\{x\}]]$ is compact, by (e-i). If $F\subseteq Z$ is closed then $S^{-1}[F]\subseteq Y$ is closed so $(SR)^{-1}[F]=R^{-1}[S^{-1}[F]]$ is closed.

(g)(i) Suppose that R is usco-compact when regarded as a subset of $X\times Y_0$. Set $S=\{(y,y):y\in Y_0\}$; by (c), S is usco-compact when regarded as a subset of $Y_0\times Y$, so by (f) $R=SR$ is usco-compact when regarded as a subset of $X\times Y$.

(ii) If R is usco-compact when regarded as a subset of $X\times Y$, and $x\in X$, then $R[\{x\}]$ is a subset of Y_0 which is compact for the topology of Y, therefore for the subspace topology of Y_0. If $F\subseteq Y_0$ is closed for the subspace topology, it is of the form $F'\cap Y_0$ for some closed $F'\subseteq Y$, so $R^{-1}[F]=R^{-1}[F']$ is closed in X. As x and F are arbitrary, R is usco-compact in $X\times Y_0$.

(h) Set $S=\{(\alpha,\beta):\beta\le\alpha\in\mathbb{N}^{\mathbb{N}}\}$. Then S is usco-compact in $\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$. **P** $S[\{\alpha\}]=\{\beta:\beta\le\alpha\}$ is a product of finite sets, so is compact, for every $\alpha\in\mathbb{N}^{\mathbb{N}}$. If $F\subseteq\mathbb{N}^{\mathbb{N}}$ is closed and $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is a convergent sequence in $S^{-1}[F]$ with limit $\alpha\in\mathbb{N}^{\mathbb{N}}$, then for every $n\in\mathbb{N}$ there is a $\beta_n\in F$ such that $\beta_n\le\alpha_n$. For any $i\in\mathbb{N}$, $\sup_{n\in\mathbb{N}}\beta_n(i)\le\sup_{n\in\mathbb{N}}\alpha_n(i)$ is finite, so $\{\beta_n:n\in\mathbb{N}\}$ is relatively compact and $\langle\beta_n\rangle_{n\in\mathbb{N}}$ has a cluster point β say; now

$$\beta(i)\le\limsup_{n\to\infty}\beta_n(i)\le\limsup_{n\to\infty}\alpha_n(i)=\alpha(i)$$

for every $i\in\mathbb{N}$, so $\beta\le\alpha$. Also, of course, $\beta\in F$, so $\alpha\in S^{-1}[F]$. As $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $S^{-1}[F]$ is closed; as F is arbitrary, S is usco-compact. **Q**

Now $R'=S\circ R$ is usco-compact, by (f) above.

422E The following lemma is actually very important in the structure theory of K-analytic spaces (see 422Yb). It will be useful to us in 423C below.

Lemma Let X and Y be Hausdorff spaces, and $R\subseteq X\times Y$ an usco-compact relation. If X is regular, so is R (in its subspace topology).

proof ? Suppose, if possible, otherwise; that there are a closed set $F\subseteq R$ and an $(x,y)\in R\setminus F$ which cannot be separated from F by open sets (in R). If G, H are open sets containing x, y respectively, then $R\cap(G\times H)$, $R\setminus(\overline{G}\times\overline{H})$ are disjoint relatively open sets in R, so the latter cannot include F; that is, $F\cap(\overline{G}\times\overline{H})\ne\emptyset$ whenever G, H are open, $x\in G$ and $y\in H$. Accordingly there is an ultrafilter $\mathcal{F}$ on $X\times Y$ such that $F\cap(\overline{G}\times\overline{H})\in\mathcal{F}$ whenever $G\subseteq X$ and $H\subseteq Y$ are open sets containing x, y respectively. In this case $R\in\mathcal{F}$, and $\overline{G}\in\pi_1[[\mathcal{F}]]$ for every open set G containing x. Because the topology of X is regular, every open set containing x includes $\overline{G}$ for some smaller open set G containing x, and belongs to $\pi_1[[\mathcal{F}]]$; thus $\pi_1[[\mathcal{F}]]\to x$ in X. Because R is usco-compact, $\mathcal{F}$ has a limit in R, which must be of the form (x,y'). Because $F\in\mathcal{F}$ is closed (in R), $(x,y')\in F$. But also $y'\in\overline{H}$ for every open set H containing y, since $X\times\overline{H}$ is a closed set belonging to $\mathcal{F}$; because the topology of Y is Hausdorff, y' must be equal to y, and $(x,y)\in F$, which is absurd. **X**

422F Definition (FROLÍK 61) Let X be a Hausdorff space. Then X is **K-analytic** if there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = X$.

If X is a Hausdorff space, we call a subset of X **K-analytic** if it is a K-analytic space in its subspace topology.

422G Theorem (a) Let X be a Hausdorff space. Then a subset A of X is K-analytic iff there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$.

(b) $\mathbb{N}^{\mathbb{N}}$ is K-analytic.

(c) Compact Hausdorff spaces are K-analytic.

(d) If X and Y are Hausdorff spaces and $R \subseteq X \times Y$ is an usco-compact relation, then $R[A]$ is K-analytic whenever $A \subseteq X$ is K-analytic. In particular, a Hausdorff continuous image of a K-analytic Hausdorff space is K-analytic.

(e) A product of countably many K-analytic Hausdorff spaces is K-analytic.

(f) A closed subset of a K-analytic Hausdorff space is K-analytic.

(g) A K-analytic Hausdorff space is Lindelöf, so a regular K-analytic Hausdorff space is completely regular.

proof (a) A is K-analytic iff there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times A$ with projection A. But a subset of $\mathbb{N}^{\mathbb{N}} \times X$ with projection A is usco-compact in $\mathbb{N}^{\mathbb{N}} \times A$ iff it is usco-compact in $\mathbb{N}^{\mathbb{N}} \times X$, by 422Dg.

(b) The identity function from $\mathbb{N}^{\mathbb{N}}$ to itself is an usco-compact relation, by 422Dc.

(c) If X is compact, then $R = \mathbb{N}^{\mathbb{N}} \times X$ is an usco-compact relation (because $R[\{\phi\}] = X$ is compact for every $\phi \in \mathbb{N}^{\mathbb{N}}$, while $R^{-1}[F]$ is either $\mathbb{N}^{\mathbb{N}}$ or $\emptyset$ for every closed $F \subseteq X$), so $X = R[\mathbb{N}^{\mathbb{N}}]$ is K-analytic.

(d) By (a), there is an usco-compact relation $S \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $S[\mathbb{N}^{\mathbb{N}}] = A$. Now $R \circ S \subseteq \mathbb{N}^{\mathbb{N}} \times Y$ is usco-compact, by 422Df, and $RS[\mathbb{N}^{\mathbb{N}}] = R[A]$.

In particular, if X itself is K-analytic and $f : X \to Y$ is a continuous surjection, f is an usco-compact relation (422Dc), so $Y = f[X]$ is K-analytic.

(e) Let $\langle X_i \rangle_{i \in I}$ be a countable family of K-analytic Hausdorff spaces with product X. If $I = \emptyset$ then $X = \{\emptyset\}$ is compact, therefore K-analytic. Otherwise, choose for each $i \in I$ an usco-compact relation $R_i \subseteq \mathbb{N}^{\mathbb{N}} \times X_i$ such that $R_i[\mathbb{N}^{\mathbb{N}}] = X_i$. Set

$$R = \{(\phi, x) : \phi \in (\mathbb{N}^{\mathbb{N}})^I,\ x \in X,\ (\phi(i), x(i)) \in R_i \text{ for every } i \in I\}.$$

By 422Dd, R is an usco-compact relation in $(\mathbb{N}^{\mathbb{N}})^I \times X$, and it is easy to see that $R[(\mathbb{N}^{\mathbb{N}})^I] = X$. But $(\mathbb{N}^{\mathbb{N}})^I \cong \mathbb{N}^{\mathbb{N} \times I}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, because I is countable, so we can identify R with a relation in $\mathbb{N}^{\mathbb{N}} \times X$ which is still usco-compact, and X is K-analytic.

(f) Let X be a K-analytic Hausdorff space and F a closed subset. Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = X$. Set $R' = R \cap (\mathbb{N}^{\mathbb{N}} \times F)$. Then R' is a closed subset of R, so is usco-compact (422Da). By (a), $F = R'[\mathbb{N}^{\mathbb{N}}]$ is K-analytic.

(g) Let X be a K-analytic Hausdorff space. $\mathbb{N}^{\mathbb{N}}$ is Lindelöf (4A2Ub), and there is an usco-compact relation R such that $R[\mathbb{N}^{\mathbb{N}}] = X$, so that X is Lindelöf, by 422D(e-ii). 4A2H(b-i) now tells us that if X is regular it is completely regular.

422H Theorem (a) If X is a Hausdorff space, then any K-analytic subset of X is Souslin-F in X.

(b) If X is a K-analytic Hausdorff space, then a subset of X is K-analytic iff it is Souslin-F in X.

(c) For any Hausdorff space X, the family of K-analytic subsets of X is closed under Souslin's operation.

proof (a) If $A \subseteq X$ is K-analytic, there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$, by 422Ga. By 422Da, R is a closed set; so A is Souslin-F by 421J.

(b) Now suppose that X itself is K-analytic, and that $A \subseteq X$ is Souslin-F in X. Then there is a closed set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$ (421J, in the other direction). $\mathbb{N}^{\mathbb{N}} \times X$ is K-analytic (422Gb, 422Ge), and R is closed, therefore itself K-analytic (422Gf); so its continuous image A is K-analytic, by 422Gd.

(c)(i) The first step is to show that the union of a sequence of K-analytic subsets of X is K-analytic. **P** Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence of K-analytic sets, with union A. For each $n \in \mathbb{N}$, let $R_n \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R_n[\mathbb{N}^{\mathbb{N}}] = A_n$. In $(\mathbb{N} \times \mathbb{N}^{\mathbb{N}}) \times X$ let R be the set

$$\{((n, \phi), x) : n \in \mathbb{N},\ (\phi, x) \in R_n\}.$$

If $(n, \phi) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, then $R[\{(n, \phi)\}] = R_n[\{\phi\}]$ is compact; if $F \subseteq X$ is closed, then

$$R^{-1}[F] = \{(n, \phi) : n \in \mathbb{N},\, \phi \in R_n^{-1}[F]\}$$

is closed in $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$. So R is usco-compact, and of course

$$R[\mathbb{N} \times \mathbb{N}^{\mathbb{N}}] = \bigcup_{n\in\mathbb{N}} R_n[\mathbb{N}^{\mathbb{N}}] = A.$$

As $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ is K-analytic (in fact, homeomorphic to $\mathbb{N}^{\mathbb{N}}$), A is K-analytic. **Q**

(ii) Now suppose that $\langle A_\sigma\rangle_{\sigma\in S^*}$ is a Souslin scheme consisting of K-analytic sets with kernel A. Then $X' = \bigcup_{\sigma\in S^*} A_\sigma$ is K-analytic, by (i). By (a), every A_σ is Souslin-F when regarded as a subset of X'. But since the family of Souslin-F subsets of X' is closed under Souslin's operation, by 421D, A also is Souslin-F in X'. By (b) of this theorem, A is K-analytic. As $\langle A_\sigma\rangle_{\sigma\in S^*}$ is arbitrary, we have the result.

422I It seems that for the measure-theoretic results of §432, at least, the following result (the 'First Separation Theorem') is not essential. However I do not think it possible to get a firm grasp on K-analytic and analytic spaces without knowing some version of it, so I present it here. It is most often used through the forms in 422J and 422Xd below.

Lemma Let X be a Hausdorff space. Let $\mathcal{E}$ be a family of subsets of X such that (i) $\bigcup_{n\in\mathbb{N}} E_n$ and $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{E}$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{E}$ (ii) whenever x, y are distinct points of X, there are disjoint E, $F \in \mathcal{E}$ such that $x \in \operatorname{int} E$ and $y \in \operatorname{int} F$. Then whenever A, B are disjoint non-empty K-analytic subsets of X, there are disjoint E, $F \in \mathcal{E}$ such that $A \subseteq E$ and $B \subseteq F$.

proof (a) We need to know that if K, L are disjoint non-empty compact subsets of X, there are disjoint E, $F \in \mathcal{E}$ such that $K \subseteq \operatorname{int} E$ and $L \subseteq \operatorname{int} F$. **P** For any point $(x, y) \in K \times L$, we can find disjoint E_{xy}, $F_{xy} \in \mathcal{E}$ such that $x \in \operatorname{int} E$ and $y \in \operatorname{int} F$. Because L is compact and non-empty, there is for each $x \in K$ a non-empty finite set $I_x \subseteq L$ such that $L \subseteq \bigcup_{y\in I_x} \operatorname{int} F_{xy}$. Set $E_x = \bigcap_{y\in I_x} E_{xy}$, $F_x = \bigcup_{y\in I_x} F_{xy}$; then E_x, F_x are disjoint members of $\mathcal{E}$, $x \in \operatorname{int} E_x$ and $L \subseteq \operatorname{int} F_x$. Because K is compact and not empty, there is a non-empty finite set $J \subseteq K$ such that $K \subseteq \bigcup_{x\in J} \operatorname{int} E_x$. Set $E = \bigcup_{x\in J} E_x$, $F = \bigcap_{x\in J} F_x$; then E, $F \in \mathcal{E}$, $E \cap F = \emptyset$, $K \subseteq \operatorname{int} E$ and $L \subseteq \operatorname{int} F$, as required. **Q**

(b) Let Q, $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be usco-compact relations such that $Q[\mathbb{N}^{\mathbb{N}}] = A$ and $R[\mathbb{N}^{\mathbb{N}}] = B$. For each $\sigma \in S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$, set

$$I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}, \quad A_\sigma = Q[I_\sigma], \quad B_\sigma = R[I_\sigma],$$

so that $A = A_\emptyset$ and $A_\sigma = \bigcup_{i\in\mathbb{N}} A_{\sigma^\frown<i>}$ for every σ.

(c) Write T for the set of pairs

$$\{(\sigma, \tau) : \sigma,\, \tau \in S \text{ and there are disjoint } E,\, F \in \mathcal{E} \text{ such that } A_\sigma \subseteq E \text{ and } B_\tau \subseteq F\}.$$

If σ, $\tau \in S$ are such that $(\sigma^\frown<i>, \tau^\frown<j>) \in T$ for every i, $j \in \mathbb{N}$, then $(\sigma, \tau) \in T$. **P** For each i, $j \in \mathbb{N}$ take disjoint E_{ij}, $F_{ij} \in \mathcal{E}$ such that

$$A_{\sigma^\frown<i>} \subseteq E_{ij}, \quad B_{\tau^\frown<j>} \subseteq F_{ij}.$$

Then $E = \bigcup_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} E_{ij}$, $F = \bigcap_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} F_{ij}$ are disjoint and belong to $\mathcal{E}$, and $A_\sigma \subseteq E$, $B_\tau \subseteq F$. So $(\sigma, \tau) \in T$. **Q**

(d) ? Now suppose, if possible, that there are no disjoint E, $F \in \mathcal{E}$ such that $A \subseteq E$ and $B \subseteq F$; that is, that $(\emptyset, \emptyset) \notin T$. By (c), used repeatedly, we can find sequences $\langle\phi(i)\rangle_{i\in\mathbb{N}}$, $\langle\psi(i)\rangle_{i\in\mathbb{N}}$ such that $(\phi\restriction n, \psi\restriction n) \notin T$ for every $n \in \mathbb{N}$. Set $K = Q[\{\phi\}]$, $L = R[\{\psi\}]$. These are compact (because R is usco-compact) and disjoint (because $K \subseteq A$ and $L \subseteq B$). By (a), there are disjoint E, $F \in \mathcal{E}$ such that $K \subseteq \operatorname{int} E$ and $L \subseteq \operatorname{int} F$.

By 422B, there are open sets U, $V \subseteq \mathbb{N}^{\mathbb{N}}$ such that

$$\phi \in U, \quad Q[U] \subseteq \operatorname{int} E, \quad \psi \in V, \quad R[V] \subseteq \operatorname{int} F.$$

But now there is some $n \in \mathbb{N}$ such that $I_{\phi\restriction n} \subseteq U$ and $I_{\psi\restriction n} \subseteq V$, in which case

$$A_{\phi\restriction n} \subseteq E, \quad B_{\psi\restriction n} \subseteq F,$$

and $(\phi\restriction n, \psi\restriction n) \in T$, contrary to the choice of ϕ and ψ. **X**

This contradiction shows that the lemma is true.

422J Corollary Let X be a Hausdorff space and A, B disjoint K-analytic subsets of X. Then there is a Borel set which includes A and is disjoint from B.

proof Apply 422I with $\mathcal{E}$ the Borel σ-algebra of X.

***422K** I give the next step in the theory of 'constituents' begun in 421N-421Q.

Theorem Let X be a Hausdorff space.

(i) Suppose that X is regular. Let $A \subseteq X$ be a K-analytic set. Then there is a non-decreasing family $\langle E_\xi \rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every Souslin-F subset of X disjoint from A is included in some E_ξ.

(ii) Suppose that X is regular. Let $A \subseteq X$ be a Souslin-F set. Then there is a non-decreasing family $\langle E_\xi \rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every K-analytic subset of $X \setminus A$ is included in some E_ξ.

(iii) Let $A \subseteq X$ be a K-analytic set. Then there is a non-decreasing family $\langle E_\xi \rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every K-analytic subset of $X \setminus A$ is included in some E_ξ.

proof (a) The first two parts depend on the following fact: if X is regular, $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ is usco-compact, $\langle F_\sigma \rangle_{\sigma\in S^*}$ is a Souslin scheme consisting of closed sets with kernel B, and $R[\mathbb{N}^{\mathbb{N}}] \cap B = \emptyset$, then for any ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ there is an $n \geq 1$ such that $\overline{R[I_{\phi\restriction n}]} \cap \bigcap_{1\leq i\leq n} F_{\psi\restriction i}$ is empty, where I write $I_\sigma = \{\theta : \sigma \subseteq \theta \in \mathbb{N}^{\mathbb{N}}\}$ for $\sigma \in S^* = \bigcup_{n\geq 1} \mathbb{N}^n$. **P** We know that $K = R[\{\phi\}]$ is a compact set disjoint from the closed set $\bigcap_{n\geq 1} F_{\psi\restriction n}$. So there is some $m \geq 1$ such that $K \cap F = \emptyset$ where $F = \bigcap_{1\leq i\leq m} F_{\psi\restriction i}$. Because X is regular, there are disjoint open sets G, $H \subseteq X$ such that $K \subseteq G$ and $F \subseteq H$ (4A2F(h-ii)). Now $R^{-1}[X \setminus G]$ is a closed set not containing ϕ, so there is some n such that $R[I_{\phi\restriction n}] \subseteq G$. Of course we can take $n \geq m$, and in this case

$$\overline{R[I_{\phi\restriction n}]} \cap \bigcap_{1\leq i\leq n} F_{\psi\restriction i} \subseteq \overline{G} \cap F = \emptyset,$$

as required. **Q**

(b)(i) Suppose that $A \subseteq X$ is K-analytic. Then there is an usco-compact set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$, and R is closed (422Da), so that A is the kernel of the Souslin scheme $\langle \overline{R[I_\sigma]} \rangle_{\sigma\in S^*}$ (421I). For $x \in X$ set $T_x = \{\sigma : \sigma \in S^*, x \in \overline{R[I_\sigma]}\}$, as in 421Ng, and let $r(T_x) < \omega_1$ be the rank of the tree T_x. Then $E_\xi = \{x : x \in X \setminus A, r(T_x) \leq \xi\}$ is a Borel set for every $\xi < \omega_1$, by 421Ob. Now suppose that $B \subseteq X \setminus A$ is a Souslin-F set. Then it is the kernel of a Souslin scheme $\langle F_\sigma \rangle_{\sigma\in S^*}$ consisting of closed sets. If ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ then by (a) above there is an $n \geq 1$ such that $\overline{R[I_{\phi\restriction n}]} \cap \bigcap_{1\leq i\leq n} F_{\psi\restriction i}$ is empty. By 421Q, there must be some $\xi < \omega_1$ such that $B \subseteq E_\xi$. So $\langle E_\xi \rangle_{\xi<\omega_1}$ is a suitable family.

(ii) The other part is almost the same. Suppose that $A \subseteq X$ is Souslin-F. Then it is the kernel of a Souslin scheme $\langle F_\sigma \rangle_{\sigma\in S^*}$ consisting of closed sets. For $x \in X$ set

$$T_x = \bigcup_{n\geq 1} \{\sigma : \sigma \in \mathbb{N}^n, x \in \bigcap_{1\leq i\leq n} F_{\sigma\restriction i}\},$$

and let $r(T_x) < \omega_1$ be the rank of the tree T_x. Then $E_\xi = \{x : x \in X \setminus A, r(T_x) \leq \xi\}$ is a Borel set for every $\xi < \omega_1$. Now let $B \subseteq X \setminus A$ be a K-analytic set. There is an usco-compact set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = B$, and B is the kernel of the Souslin scheme $\langle \overline{R[I_\sigma]} \rangle_{\sigma\in S^*}$. If ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ then by (a) above there is an $n \geq 1$ such that $\bigcap_{1\leq i\leq n} F_{\psi\restriction i} \cap \overline{R[I_{\phi\restriction n}]}$ is empty. So there must be some $\xi < \omega_1$ such that $B \subseteq E_\xi$. Thus here again $\langle E_\xi \rangle_{\xi<\omega_1}$ is a suitable family.

(c) If X is not regular, we still have a version of the result in (a), as follows: if R, $S \subseteq \mathbb{N}^{\mathbb{N}} \times X$ and $R[\mathbb{N}^{\mathbb{N}}] \cap S[\mathbb{N}^{\mathbb{N}}] = \emptyset$, then for any ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ there is an $n \geq 1$ such that $\overline{R[I\phi\restriction n]} \cap \overline{S[I_{\psi\restriction n}]}$ is empty. **P** This time, $R[\{\phi\}]$ and $S[\{\psi\}]$ are disjoint compact sets, so there are disjoint open sets G, H with $R[\{\phi\}] \subseteq G$ and $S[\{\psi\}] \subseteq H$ (4A2F(h-i)). Now $R[I_{\phi\restriction n}] \subseteq G$ and $S[I_{\psi\restriction n}] \subseteq H$ for all n large enough. **Q**

Now the argument of (b-i), with $F_\sigma = S[I_\sigma]$, gives part (iii).

422X Basic exercises (a) Let X and Y be Hausdorff spaces and X_0 a closed subset of X. Show that a relation $R \subseteq X_0 \times Y$ is usco-compact when regarded as a relation between X_0 and Y iff it is usco-compact when regarded as a relation between X and Y.

(b) Show that a locally compact Hausdorff space is K-analytic iff it is Lindelöf iff it is σ-compact.

>(c) Prove 422Hc from first principles, without using 421D. (*Hint*: if $\langle R_\sigma \rangle_{\sigma\in S^*}$ is a Souslin scheme of usco-compact relations in $\mathbb{N}^{\mathbb{N}} \times X$,

$$\{((\phi, \langle \psi_\sigma \rangle_{\sigma\in S^*}), x) : (\psi_{\phi\restriction n}, x) \in R_{\phi\restriction n} \text{ for every } n \geq 1\}$$

is usco-compact in $(\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{S^*}) \times X$.)

(d) Let X be a completely regular Hausdorff space and A, B disjoint K-analytic subsets of X. Show that there is a Baire set including A and disjoint from B.

(e) Let X be a K-analytic Hausdorff space. (i) Show that every Baire subset of X is K-analytic. (*Hint*: apply 136Xi to the family of K-analytic subsets of X.) (ii) Show that if X is regular, it is perfectly normal iff it is hereditarily Lindelöf iff every open subset of X is K-analytic.

(f) Let X be a Hausdorff space in which every open set is K-analytic. Show that every Borel set is K-analytic.

422Y Further exercises (a) Let X be a completely regular Hausdorff space, and βX its Stone-Čech compactification. Show that X is K-analytic iff it is a Souslin-F set in βX.

(b) Show that a Hausdorff space is K-analytic iff it is a continuous image of a $\mathrm{K}_{\sigma\delta}$ set in a compact Hausdorff space, that is, a set expressible as $\bigcap_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}K_{mn}$ where every K_{mn} is compact. (*Hint*: Write $\mathcal{K}^*$ for the class of Hausdorff continuous images of $\mathrm{K}_{\sigma\delta}$ subsets of compact Hausdorff spaces. (i) Show that $\mathbb{N}^{\mathbb{N}}$ is a $\mathrm{K}_{\sigma\delta}$ set in $Y^{\mathbb{N}}$, where Y is the one-point compactification of $\mathbb{N}$. (ii) Show that if X is a compact Hausdorff space and $R\subseteq\mathbb{N}^{\mathbb{N}}\times X$ is closed, then $R\in\mathcal{K}^*$. (iii) Show that if X is a compact Hausdorff space, then every Souslin-F subset of X belongs to $\mathcal{K}^*$. (iv) Show that if X is a regular K-analytic Hausdorff space, then $X\in\mathcal{K}^*$. (v) Show that if X is any Hausdorff space and $R\subseteq\mathbb{N}^{\mathbb{N}}\times X$ is an usco-compact relation, then $R\in\mathcal{K}^*$. See Jayne 76.)

(c) Let X be a normal space and $\mathcal{C}$ the family of countably compact closed subsets of X. Let A, B be disjoint sets obtainable from $\mathcal{C}$ by Souslin's operation. (For instance, if X itself is countably compact, A and B could be disjoint Souslin-F sets.) Show that there is a Borel set including A and disjoint from B. (*Hint*: adapt the proof of 422I.)

(d) Let X be a Hausdorff space and $\langle A_n\rangle_{n\in\mathbb{N}}$ a sequence of K-analytic subsets of X such that $\bigcap_{n\in\mathbb{N}}A_n=\emptyset$. Show that there is a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ of Borel sets such that $A_n\subseteq E_n$ for every $n\in\mathbb{N}$ and $\bigcap_{n\in\mathbb{N}}E_n=\emptyset$. (*Hint*: for each $n\in\mathbb{N}$ choose an usco-compact $R_n\subseteq\mathbb{N}^{\mathbb{N}}\times X$ with projection A_n. Consider the set $T=\{\langle\sigma_n\rangle_{n\in\mathbb{N}}:\exists$ Borel E_n, $R_n[I_{\sigma_n}]\subseteq E_n\ \forall\, n$, $\bigcap_{n\in\mathbb{N}}E_n=\emptyset\}$.)

(e) Explain how to prove 422J from 421Q, without using 422I.

(f) Let X be a set and $\mathfrak{S}$, $\mathfrak{T}$ two Hausdorff topologies on X such that $\mathfrak{S}\subseteq\mathfrak{T}$ and $(X,\mathfrak{T})$ is K-analytic. Show that $\mathfrak{S}$ and $\mathfrak{T}$ yield the same K-analytic subspaces of X.

(g) Let X be a Hausdorff space, $\mathcal{K}$ the family of K-analytic subsets of X, Y a set and $\mathcal{H}$ a family of subsets of Y containing $\emptyset$. Show that $R[X]\in\mathcal{S}(\mathcal{H})$ for every $R\in\mathcal{S}(\{K\times H:K\in\mathcal{K},\,H\in\mathcal{H}\})$.

422 Notes and comments In a sense, this section starts at the deep end of its topic. 'Descriptive set theory' originally developed in the context of the real line and associated spaces, and this remains the centre of the subject. But it turns out that some of the same arguments can be used in much more general contexts, and in particular greatly illuminate the theory of Radon measures on Hausdorff spaces. I find that a helpful way to look at K-analytic spaces is to regard them as a common generalization of compact Hausdorff spaces and Souslin-F subsets of $\mathbb{R}$; if you like, any theorem which is true of both these classes has a fair chance of being true of all K-analytic spaces. In the next section we shall come to the special properties of the more restricted class of 'analytic' spaces, which are much closer to the separable metric spaces of the original theory.

The phrase 'usco-compact' is neither elegant nor transparent, but is adequately established and (in view of the frequency with which it is needed) seems preferable to less concise alternatives. If we think of a relation $R\subseteq X\times Y$ as a function $x\mapsto R[\{x\}]$ from x to $\mathcal{P}Y$, then an usco-compact relation is one which takes compact values and is 'upper semi-continuous' in the sense that $\{x:R[\{x\}]\subseteq H\}$ is open for every open set $H\subseteq Y$; just as a real-valued function is upper semi-continuous if $\{x:f(x)<\alpha\}$ is open for every α.

This is not supposed to be a book on general topology, and in my account of the topological properties of K-analytic spaces I have concentrated on facts which are useful when proving that spaces are K-analytic, on the assumption that these will be valuable when we seek to apply the results of §432 below. Other properties are mentioned only when they are relevant to the measure-theoretic results which are my real concern, and readers already acquainted with this area may be startled by some of my omissions. For a proper treatment of the subject, I refer you to Rogers 80. As usual, however, I take technical details seriously in the material I do cover. I hope you will not find that such results as 422Dg and 422Ga try your patience too far. I think a moment's thought will persuade you that it is of the highest importance that K-analyticity (like compactness) is an intrinsic property. In contrast, the property of being 'Souslin-F', like the property of being closed, depends on the surrounding space. A completely regular Hausdorff space is compact iff it must be a closed set in any surrounding Hausdorff space iff it is closed in its Stone-Čech compactification; and it is K-analytic iff it must be a Souslin-F set in any surrounding Hausdorff space iff it is a Souslin-F set in its Stone-Čech compactification (422Ya).

For regular spaces, 422K gives us another version of the First Separation Theorem. But this one is simultaneously more restricted in its scope (it does not seem to have applications to Baire σ-algebras, for instance) and very much more powerful in its application. When all Borel sets are Souslin-F, as in the next section, it tells us something very important about the cofinal structure of the Souslin-F subsets of the complement of a K-analytic set.

423 Analytic spaces

We come now to the original class of K-analytic spaces, the 'analytic' spaces. I define these as continuous images of $\mathbb{N}^{\mathbb{N}}$ (423A), but move as quickly as possible to their characterization as K-analytic spaces with countable networks (423C), so that many other fundamental facts (423E-423G) can be regarded as simple corollaries of results in §422. I give two versions of Lusin's theorem on injective images of Borel sets (423I), and a form of the von Neumann-Jankow measurable selection theorem (423N). I end with notes on constituents of coanalytic sets (423P-423Q).

423A Definition A Hausdorff space is **analytic** or **Souslin** if it is either empty or a continuous image of $\mathbb{N}^{\mathbb{N}}$.

423B Proposition (a) A Polish space is analytic.

(b) A Hausdorff continuous image of an analytic Hausdorff space is analytic.

(c) A product of countably many analytic Hausdorff spaces is analytic.

(d) A closed subset of an analytic Hausdorff space is analytic.

(e) An analytic Hausdorff space has a countable network consisting of analytic sets.

proof (a) Let X be a Polish space. If $X = \emptyset$, we can stop. Otherwise, let ρ be a metric on X, inducing its topology, under which X is complete. For $\sigma \in S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ choose $X_\sigma \subseteq X$ as follows. $X_\emptyset = X$. Given that X_σ is a closed non-empty subset of X, where $\sigma \in S$, then X_σ is separable, because X is separable and metrizable (4A2P(a-iv)), and we can choose a sequence $\langle x_{\sigma i}\rangle_{i\in\mathbb{N}}$ in X_σ such that $\{x_{\sigma i} : i \in \mathbb{N}\}$ is dense in X_σ. Set $X_{\sigma^\frown<i>} = X_\sigma \cap B(x_{\sigma i}, 2^{-n})$ for each $i \in \mathbb{N}$, where $B(x,\delta) = \{y : \rho(y,x) \le \delta\}$, and continue. Note that because $\{x_{\sigma i} : i \in \mathbb{N}\}$ is dense in X_σ, $X_\sigma = \bigcup_{i\in\mathbb{N}} X_{\sigma^\frown<i>}$, for every $\sigma \in S$.

For each $\phi \in \mathbb{N}^{\mathbb{N}}$, $\langle X_{\phi\restriction n}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-empty closed sets, and $\operatorname{diam}(X_{\phi\restriction n+1}) \le 2^{-n+1}$ for every n. Because X is complete under ρ, $\bigcap_{n\in\mathbb{N}} X_{\phi\restriction n}$ is a singleton $\{f(\phi)\}$ say. ($f(\phi)$ is the limit of the Cauchy sequence $\langle x_{\phi\restriction n,\phi(n)}\rangle_{n\in\mathbb{N}}$.) Thus we have a function $f : \mathbb{N}^{\mathbb{N}} \to X$. f is continuous because $\rho(f(\psi), f(\phi)) \le 2^{-n+1}$ whenever $\phi\restriction n+1 = \psi\restriction n+1$ (since in this case both $f(\psi)$ and $f(\phi)$ belong to $X_{\phi\restriction n+1}$). f is surjective because, given $x \in X$, we can choose $\langle\phi(i)\rangle_{i\in\mathbb{N}}$ inductively so that $x \in X_{\phi\restriction n}$ for every n; at the inductive step, we have $x \in X_{\phi\restriction n} = \bigcup_{i\in\mathbb{N}} X_{(\phi\restriction n)^\frown<i>}$, so we can take $\phi(n)$ such that $x \in X_{(\phi\restriction n)^\frown<\phi(n)>} = X_{\phi\restriction n+1}$.

Thus X is a continuous image of $\mathbb{N}^{\mathbb{N}}$, as claimed.

(b) If X is an analytic Hausdorff space and Y is a Hausdorff continuous image of X, then either X is a continuous image of $\mathbb{N}^{\mathbb{N}}$ and Y is a continuous image of $\mathbb{N}^{\mathbb{N}}$, or $X = \emptyset$ and $Y = \emptyset$.

(c) Let $\langle X_i\rangle_{i\in I}$ be a countable family of analytic Hausdorff spaces, with product X. Then X is Hausdorff (3A3Id). If $I = \emptyset$ then $X = \{\emptyset\}$ is a continuous image of $\mathbb{N}^{\mathbb{N}}$, therefore analytic. If there is some $i \in I$ such that $X_i = \emptyset$, then $X = \emptyset$ is analytic. Otherwise, we have for each $i \in I$ a continuous surjection $f_i : \mathbb{N}^{\mathbb{N}} \to X_i$. Setting $f(\phi) = \langle f_i(\phi(i))\rangle_{i\in I}$ for $\phi \in (\mathbb{N}^{\mathbb{N}})^I$, $f : (\mathbb{N}^{\mathbb{N}})^I \to X$ is a continuous surjection. But $(\mathbb{N}^{\mathbb{N}})^I \cong \mathbb{N}^{\mathbb{N}\times I}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, so X is analytic.

(d) Let X be an analytic Hausdorff space and F a closed subset of X. Then F is Hausdorff in its subspace topology (4A2F(a-i)). If $X = \emptyset$ then $F = \emptyset$ is analytic. Otherwise, there is a continuous surjection $f : \mathbb{N}^{\mathbb{N}} \to X$. Now $H = f^{-1}[F]$ is a closed subset of the Polish space $\mathbb{N}^{\mathbb{N}}$, therefore Polish in its induced topology (4A2Qd). By (a), H is analytic, so its continuous image $F = f[H]$ also is analytic, by (b).

(e) Let X be an analytic Hausdorff space. If it is empty then of course it has a countable network consisting of analytic sets. Otherwise, there is a continuous surjection $f : \mathbb{N}^{\mathbb{N}} \to X$. For $\sigma \in S$ set $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$; then

$\{I_\sigma : \sigma \in S\}$ is a base for the topology of $\mathbb{N}^\mathbb{N}$, so $\{f[I_\sigma] : \sigma \in S\}$ is a network for the topology of X (see the proof of 4A2Nd). But I_σ is homeomorphic to $\mathbb{N}^\mathbb{N}$, so $f[I_\sigma]$ is analytic, for every $\sigma \in S$, and $\{f[I_\sigma] : \sigma \in S\}$ is a countable network consisting of analytic sets.

423C Theorem A Hausdorff space is analytic iff it is K-analytic and has a countable network.

proof (a) Let X be an analytic Hausdorff space. By 423Be, it has a countable network. If $X = \emptyset$ then surely it is K-analytic. Otherwise, X is a continuous image of $\mathbb{N}^\mathbb{N}$. But $\mathbb{N}^\mathbb{N}$ is K-analytic (422Gb), so X also is K-analytic, by 422Gd.

(b) Now suppose that X is a K-analytic Hausdorff space and has a countable network.

(i) If $X \subseteq \mathbb{N}^\mathbb{N}$ then X is analytic. **P** Let $R \subseteq \mathbb{N}^\mathbb{N} \times X$ be an usco-compact relation such that $R[\mathbb{N}^\mathbb{N}] = X$. Then R is still usco-compact when regarded as a subset of $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ (422Dg), so is closed in $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ (422Da). But $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \cong \mathbb{N}^\mathbb{N}$ is analytic, so R is in itself an analytic space (423Bd), and its continuous image X is analytic, by 423Bb. **Q**

(ii) Now suppose that X is regular. By 4A2Ng, X has a countable network $\mathcal{E}$ consisting of closed sets. Adding $\emptyset$ to $\mathcal{E}$ if need be, we may suppose that $\mathcal{E} \ne \emptyset$. Let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{E}$. For each $n \in \mathbb{N}$, let $\langle F_{ni} \rangle_{i\in\mathbb{N}}$ be a sequence running over $\{E_n\} \cup \{E : E \in \mathcal{E},\ E \cap E_n = \emptyset\}$. Now consider the relation

$$R = \{(\phi, x) : \phi \in \mathbb{N}^\mathbb{N},\ x \in \bigcap_{n\in\mathbb{N}} F_{n,\phi(n)}\} \subseteq \mathbb{N}^\mathbb{N} \times X.$$

(α) R is closed in $\mathbb{N}^\mathbb{N} \times X$. **P** Because every F_{ni} is closed,

$$(\mathbb{N}^\mathbb{N} \times X) \setminus R = \bigcup_{i,n\in\mathbb{N}} \{(\phi, x)\colon\ \phi(n) = i \text{ and } x \notin F_{ni}\}$$

is open. **Q**

(β) $R[\mathbb{N}^\mathbb{N}] = X$. **P** For every $n \in \mathbb{N}$,

$$X \setminus E_n = \bigcup\{E : E \in \mathcal{E},\ E \subseteq X \setminus E_n\}$$

because $\mathcal{E}$ is a network and E_n is closed, so $\bigcup_{i\in\mathbb{N}} F_{ni} = X$. So, given $x \in X$, we can find for each n a $\phi(n)$ such that $x \in F_{n,\phi(n)}$, and $(\phi, x) \in R$. **Q**

(γ) R is the graph of a function. **P?** Suppose that we have (ϕ, x) and (ϕ, y) in R where $x \ne y$. Because the topology of X is Hausdorff, there is an $n \in \mathbb{N}$ such that $x \in E_n$ and $y \notin E_n$. But in this case $x \in E_n \cap F_{n,\phi(n)}$, so $F_{n,\phi(n)} = E_n$, while $y \in F_{n,\phi(n)} \setminus E_n$, so $F_{n,\phi(n)} \ne E_n$; which is absurd. **XQ**

(δ) Set $A = R^{-1}[X]$, so that R is the graph of a function from A to X; in recognition of its new status, give it a new name f. Then f is continuous. **P** Suppose that $\phi \in A$ and that $x = f(\phi) \in G$, where $G \subseteq X$ is open. Then there is an $n \in \mathbb{N}$ such that $x \in E_n \subseteq G$. In this case $x \in F_{n,\phi(n)}$, because $(\phi, x) \in R$, so $F_{n,\phi(n)} = E_n$. Now if $\psi \in A$ and $\psi(n) = \phi(n)$, we must have

$$f(\psi) \in F_{n,\psi(n)} = E_n \subseteq G.$$

Thus $f^{-1}[G]$ includes a neighbourhood of ϕ in A. As ϕ and G are arbitrary, f is continuous. **Q**

(ϵ) At this point recall that X is K-analytic. It follows that $\mathbb{N}^\mathbb{N} \times X$ is K-analytic (422Ge), so that its closed subset R is K-analytic (422Gf) and A, which is a continuous image of R, is K-analytic (422Gd). But now A is a K-analytic subset of $\mathbb{N}^\mathbb{N}$, so is analytic, by (i) just above. And, finally, X is a continuous image of A, so is analytic.

(iii) Thus any regular K-analytic space with a countable network is analytic. Now suppose that X is an arbitrary K-analytic Hausdorff space with a countable network. Let $R \subseteq \mathbb{N}^\mathbb{N} \times X$ be an usco-compact relation such that $R[\mathbb{N}^\mathbb{N}] = X$. Then R is a closed subset of $\mathbb{N}^\mathbb{N} \times X$, so is itself a K-analytic space with a countable network. But it is also regular, by 422E. So R is analytic and its continuous image X is analytic.

This completes the proof.

423D Corollary (a) An analytic Hausdorff space is hereditarily Lindelöf.

(b) In a regular analytic Hausdorff space, closed sets are zero sets and the Baire and Borel σ-algebras coincide.

(c) A compact subset of an analytic Hausdorff space is metrizable.

(d) A metrizable space is analytic iff it is K-analytic.

proof (a)-(c) These are true just because there is a countable network (4A2Nb, 4A3Kb, 4A2Na, 4A2Qh).

(d) Let X be a metrizable space. If X is analytic, of course it is K-analytic. If X is K-analytic, it is Lindelöf (422Gg) therefore separable (4A2Pd) and has a countable network (4A2P(a-iii)), so is analytic.

423E Theorem (a) For any Hausdorff space X, the family of subsets of X which are analytic in their subspace topologies is closed under Souslin's operation.

(b) Let $(X, \mathfrak{T})$ be an analytic Hausdorff space. For a subset A of X, the following are equiveridical:

(i) A is analytic;

(ii) A is K-analytic;

(iii) A is Souslin-F;

(iv) A can be obtained by Souslin's operation from the family of Borel subsets of X.

In particular, all Borel sets in X are analytic.

proof (a) Let X be a Hausdorff space and $\mathcal{A}$ the family of analytic subsets of X. Let $\langle A_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{A}$ with kernel A. Then every A_σ is K-analytic, so A is K-analytic, by 422Hc. Also every A_σ has a countable network, so $A' = \bigcup_{\sigma\in S^*} A_\sigma$ has a countable network (4A2Nc); as $A \subseteq A'$, A also has a countable network (4A2Na) and is analytic.

(b) Because X has a countable network, so does A. So 423C tells us at once that (i)$\Leftrightarrow$(ii). In particular, X is K-analytic, so 422Hb tells us that (ii)$\Leftrightarrow$(iii). Of course (iii)$\Rightarrow$(iv).

Now suppose that $G \subseteq X$ is open. Then $G \in \mathcal{A}$. **P** If $X = \emptyset$ then $G = \emptyset$ is open. Otherwise, there is a continuous surjection $f : \mathbb{N}^\mathbb{N} \to X$. Set $H = f^{-1}[G]$, so that $H \subseteq \mathbb{N}^\mathbb{N}$ is open and $G = f[H]$. Being an open set in a metric space, H is F_σ (4A2Lc), so, in particular, is Souslin-F; but $\mathbb{N}^\mathbb{N}$ is analytic, so H is analytic and its continuous image G is analytic. **Q**

We have already seen that closed subsets of X belong to $\mathcal{A}$ (423Bd). Because $\mathcal{A}$ is closed under Souslin's operation, it contains every Borel set, by 421F. It therefore contains every set obtainable by Souslin's operation from Borel sets, and (iv)$\Rightarrow$(i).

Remark See also 423Yb below.

423F Proposition Let $(X, \mathfrak{T})$ be an analytic Hausdorff space.

(a) A set $E \subseteq X$ is Borel iff both E and $X \setminus E$ are analytic.

(b) If $\mathfrak{S}$ is a coarser (= smaller) Hausdorff topology on X, then $\mathfrak{S}$ and $\mathfrak{T}$ have the same Borel sets.

proof (a) If E is Borel, then E and $X \setminus E$ are analytic, by 423Eb. If E and $X \setminus E$ are analytic, they are K-analytic (423Eb) and disjoint, so there is a Borel set $F \supseteq E$ which is disjoint from $X \setminus E$ (422J); but now of course $F = E$, so E must be Borel.

(b) Because the identity map from $(X, \mathfrak{T})$ to $(X, \mathfrak{S})$ is continuous, $\mathfrak{S}$ is an analytic topology (423Bb) and every $\mathfrak{S}$-Borel set is $\mathfrak{T}$-Borel. If $E \subseteq X$ is $\mathfrak{T}$-Borel, then it and its complement are $\mathfrak{T}$-analytic, therefore $\mathfrak{S}$-analytic (423Bb), and E is $\mathfrak{S}$-Borel by (a).

423G Lemma Let X and Y be analytic Hausdorff spaces and $f : X \to Y$ a Borel measurable function.

(a) (The graph of) f is an analytic set.

(b) $f[A]$ is an analytic set in Y for any analytic set (in particular, any Borel set) $A \subseteq X$.

(c) $f^{-1}[B]$ is an analytic set in X for any analytic set (in particular, any Borel set) $B \subseteq Y$.

proof (a) Let $\mathcal{E}$ be a countable network for the topology of Y. Set

$$R = \bigcap_{E\in\mathcal{E}} (X \times \overline{E}) \cup ((X \setminus f^{-1}[\overline{E}]) \times Y).$$

Then R is a Borel set in $X \times Y$. But also R is the graph of f. **P** If $f(x) = y$, then surely $y \in \overline{E}$ whenever $x \in f^{-1}[\overline{E}]$, so $(x, y) \in R$. On the other hand, if $x \in X$, $y \in Y$ and $f(x) \neq y$, there are disjoint open sets G, $H \subseteq Y$ such that $f(x) \in G$ and $y \in H$; now there is an $E \in \mathcal{E}$ such that $f(x) \in E \subseteq G$, so that $f(x) \in \overline{E}$ but $y \notin \overline{E}$, and $(x, y) \notin R$. **Q**

Because $X \times Y$ is analytic (423Bc), R is analytic (423Eb).

(b) If $A \subseteq X$ is analytic, then $A \times Y$ and $R \cap (A \times Y)$ are analytic (423Ea), so $f[A] = R[A]$, which is a continuous image of $R \cap (A \times Y)$, is analytic.

(c) Similarly, if $B \subseteq Y$ is analytic, then $f^{-1}[B]$ is a continuous image of $R \cap (X \times B)$, so is analytic.

423H Lemma Let $(X,\mathfrak{T})$ be an analytic Hausdorff space, and $\langle E_n\rangle_{n\in\mathbb{N}}$ any sequence of Borel sets in X. Then the topology $\mathfrak{T}'$ generated by $\mathfrak{T}\cup\{E_n : n\in\mathbb{N}\}$ is analytic.

proof If $X=\emptyset$ this is trivial. Otherwise, there is a continuous surjection $f:\mathbb{N}^{\mathbb{N}}\to X$. Set $F_n=f^{-1}[E_n]$ for each n; then F_n is a Borel subset of $\mathbb{N}^{\mathbb{N}}$, so there is a Polish topology $\mathfrak{S}'$ on $\mathbb{N}^{\mathbb{N}}$, finer than the usual topology, for which every F_n is open, by 4A3I. But now f is continuous for $\mathfrak{S}'$ and $\mathfrak{T}'$, so $\mathfrak{T}'$ is analytic, by 423Ba and 423Bb. (Of course $\mathfrak{T}'$ is Hausdorff, because it is finer than $\mathfrak{T}$.)

423I Theorem Let X be a Polish space, $E\subseteq X$ a Borel set, Y a Hausdorff space and $f:E\to Y$ an injective function.

(a) If f is continuous, then $f[E]$ is Borel.

(b) If Y has a countable network (e.g., is an analytic space or a separable metrizable space), and f is Borel measurable, then $f[E]$ is Borel.

proof (a)(i) Since there is a finer Polish topology on X for which E is closed (4A3I), therefore Polish in the subspace topology (4A2Qd), and f will still be continuous for this topology, we may suppose that $E=X$.

(ii) Let $\langle U_n\rangle_{n\in\mathbb{N}}$ run over a base for the topology of X (4A2P(a-i)). For each pair m, $n\in\mathbb{N}$ such that $U_m\cap U_n$ is empty, $f[U_m]$ and $f[U_n]$ are analytic sets in Y (423Eb, 423Bb) and are disjoint (because f is injective), so there is a Borel set H_{mn} including $f[U_m]$ and disjoint from $f[U_n]$ (422J). Set

$$E_n=\overline{f[U_n]}\cap\bigcap\{H_{nm}\setminus H_{mn} : m\in\mathbb{N},\, U_m\cap U_n=\emptyset\}$$

for each $n\in\mathbb{N}$; then E_n is a Borel set in Y including $f[U_n]$. Note that if $U_m\cap U_n$ is empty, then $E_m\cap E_n\subseteq(H_{mn}\setminus H_{nm})\cap(H_{nm}\setminus H_{mn})$ is also empty.

Fix a metric ρ on X, inducing its topology, for which X is complete, and for $k\in\mathbb{N}$ set

$$F_k=\bigcup\{E_n : n\in\mathbb{N},\, \operatorname{diam}U_n\le 2^{-k}\},$$

so that F_k is Borel. Let $F=\bigcap_{k\in\mathbb{N}}F_k$; then F also is a Borel subset of Y.

The point is that $F=f[X]$. **P** (i) If $x\in X$, then for every $k\in\mathbb{N}$ there is an $n\in\mathbb{N}$ such that

$$x\in U_n\subseteq\{y:\rho(y,x)\le 2^{-k-1}\};$$

now $\operatorname{diam}U_n\le 2^{-k}$, so

$$f(x)\in f[U_n]\subseteq E_n\subseteq F_k.$$

As k is arbitrary, $f(x)\in F$; as x is arbitrary, $f[X]\subseteq F$. (ii) If $y\in F$, then for each $k\in\mathbb{N}$ we can find an $n(k)$ such that $y\in E_{n(k)}$ and $\operatorname{diam}U_{n(k)}\le 2^{-k}$. Since $\overline{f[U_{n(k)}]}\supseteq E_{n(k)}$ is not empty, nor is $U_{n(k)}$, and we can choose $x_k\in U_{n(k)}$. Indeed, for any j, $k\in\mathbb{N}$, $E_{n(j)}\cap E_{n(k)}$ contains y, so is not empty, and $U_{n(j)}\cap U_{n(k)}$ cannot be empty; but this means that there is some x in the intersection, and

$$\rho(x_j,x_k)\le\rho(x_j,x)+\rho(x,x_k)\le\operatorname{diam}U_{n(j)}+\operatorname{diam}U_{n(k)}\le 2^{-j}+2^{-k}.$$

This means that $\langle x_k\rangle_{k\in\mathbb{N}}$ is a Cauchy sequence. But X is supposed to be complete, so $\langle x_k\rangle_{k\in\mathbb{N}}$ has a limit x say.

? If $f(x)\ne y$, then (because Y is Hausdorff) there is an open set H containing $f(x)$ such that $y\notin\overline{H}$. Now f is continuous, so there is a $\delta>0$ such that $f(x')\in H$ whenever $\rho(x',x)\le\delta$. There is a $k\in\mathbb{N}$ such that $2^{-k}+\rho(x_k,x)\le\delta$. If $x'\in U_{n(k)}$, then

$$\rho(x',x)\le\rho(x',x_k)+\rho(x_k,x)\le\delta;$$

thus $f[U_{n(k)}]\subseteq H$, and

$$E_{n(k)}\subseteq\overline{f[U_{n(k)}]}\subseteq\overline{H}.$$

But $y\in E_{n(k)}\setminus\overline{H}$. **X**

Thus $y=f(x)$ belongs to $f[X]$; as y is arbitrary, $F\subseteq f[X]$. **Q** Accordingly $f[X]=F$ is a Borel subset of Y, as claimed.

(b) By 4A2Nf, there is a countable family $\mathcal{V}$ of open sets in Y such that whenever y, y' are distinct points of Y there are disjoint V, $V'\in\mathcal{V}$ such that $y\in V$ and $y'\in V'$. Let $\mathfrak{S}'$ be the topology generated by $\mathcal{V}$; then $\mathfrak{S}'$ is Hausdorff. For each $V\in\mathcal{V}$, $f^{-1}[V]$ is a Borel set in X, so there is a Polish topology $\mathfrak{T}'$ on X, finer than the original topology, for which every $f^{-1}[V]$ is open (4A3I again). Now f is continuous for $\mathfrak{T}'$ and $\mathfrak{S}'$ (4A2B(a-ii)), and E is $\mathfrak{T}'$-Borel, so $f[E]$ is a $\mathfrak{S}'$-Borel set in Y, by (a). Since $\mathfrak{S}'$ is coarser than the original topology $\mathfrak{S}$ on Y, $f[E]$ is also $\mathfrak{S}$-Borel.

423J Lemma If X is an uncountable analytic Hausdorff space, it has subsets homeomorphic to $\{0,1\}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.

proof (a) Let $f:\mathbb{N}^{\mathbb{N}}\to X$ be a continuous surjection. Write $S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$, $I_\sigma=\{\phi:\sigma\subseteq\phi\in\mathbb{N}^{\mathbb{N}}\}$ for $\sigma\in S$,

$$T=\{\sigma:\sigma\in S,\ f[I_\sigma]\text{ is uncountable}\}.$$

Then if $\sigma\in T$ there are τ, $\tau'\in T$, both extending σ, such that $f[I_\tau]\cap f[I_{\tau'}]=\emptyset$. **P** Set

$$A=\bigcup\{f[I_\tau]:\tau\in S\setminus T\}.$$

Then A is a countable union of countable sets, so is countable. There must therefore be distinct points x, y of $f[I_\sigma]\setminus A$; express x as $f(\phi)$ and y as $f(\psi)$ where ϕ and ψ belong to I_σ. Because X is Hausdorff, there are disjoint open sets G, H such that $x\in G$ and $y\in H$. Because f is continuous, there are m, $n\in\mathbb{N}$ such that $I_{\phi\restriction m}\subseteq f^{-1}[G]$ and $I_{\psi\restriction n}\subseteq f^{-1}[H]$. Of course both $\tau=\phi\restriction m$ and $\tau'=\psi\restriction n$ must extend σ, and they belong to T because $x\in f[I_\tau]\setminus A$ and $y\in f[I_{\tau'}]\setminus A$. **Q**

(b) We can therefore choose inductively a family $\langle\tau(\upsilon)\rangle_{\upsilon\in S_2}$ in T, where $S_2=\bigcup_{n\in\mathbb{N}}\{0,1\}^n$, such that

$$\tau(\emptyset)=\emptyset,$$

$$\tau(\upsilon^\frown{<}i{>})\supseteq\tau(\upsilon)\text{ whenever }\upsilon\in S_2,\ i\in\{0,1\},$$

$$f[I_{\tau(\upsilon^\frown<0>)}]\cap f[I_{\tau(\upsilon^\frown<1>)}]=\emptyset\text{ for every }\upsilon\in S_2.$$

Note that $\#(\tau(\upsilon))\ge\#(\upsilon)$ for every $\upsilon\in S_2$. For each $z\in\{0,1\}^{\mathbb{N}}$, $\langle\tau(z\restriction n)\rangle_{n\in\mathbb{N}}$ is a sequence in S in which each term strictly extends its predecessor, so there is a unique $g(z)\in\mathbb{N}^{\mathbb{N}}$ such that $\tau(z\restriction n)\subseteq g(z)$ for every n. Now $g(z')\restriction n=g(z)\restriction n$ whenever $z\restriction n=z'\restriction n$, so g and $fg:\{0,1\}^{\mathbb{N}}\to X$ are continuous. If w, z are distinct points of $\{0,1\}^{\mathbb{N}}$, there is a first n such that $w(n)\ne z(n)$, in which case $fg(w)\in f[I_{\tau(w\restriction n)^\frown<w(n)>}]$ and $fg(z)\in f[I_{\tau(w\restriction n)^\frown<z(n)>}]$ are distinct. So $fg:\{0,1\}^{\mathbb{N}}\to X$ is a continuous injection, therefore a homeomorphism between $\{0,1\}^{\mathbb{N}}$ and its image, because $\{0,1\}^{\mathbb{N}}$ is compact (3A3Dd).

(c) Thus X has a subspace homeomorphic to $\{0,1\}^{\mathbb{N}}$. Now $\{0,1\}^{\mathbb{N}}$ has a subspace homeomorphic to $\mathbb{N}$. **P** For instance, setting $d_n(n)=1$, $d_n(i)=0$ for $i\ne n$, $D=\{d_n:n\in\mathbb{N}\}$ is homeomorphic to $\mathbb{N}$. **Q** Now $D^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and is a subspace of $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}\cong\{0,1\}^{\mathbb{N}\times\mathbb{N}}\cong\{0,1\}^{\mathbb{N}}$, so $\{0,1\}^{\mathbb{N}}$ has a subspace homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Accordingly X also has a subspace homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

423K Corollary Any uncountable Borel set in any analytic Hausdorff space has cardinal $\mathfrak{c}$.

proof If X is an analytic space and $E\subseteq X$ is an uncountable Borel set, then E is analytic (423E), so includes a copy of $\{0,1\}^{\mathbb{N}}$ and must have cardinal at least $\#(\{0,1\}^{\mathbb{N}})=\mathfrak{c}$. On the other hand, E is also a continuous image of $\mathbb{N}^{\mathbb{N}}$, so has cardinal at most $\#(\mathbb{N}^{\mathbb{N}})=\mathfrak{c}$.

423L Proposition Let X be an uncountable analytic Hausdorff space. Then it has a non-Borel analytic subset.

proof (a) I show first that there is an analytic set $A\subseteq\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$ such that every analytic subset of $\mathbb{N}^{\mathbb{N}}$ is a vertical section of A. **P** Let $\mathcal{U}$ be a countable base for the topology of $\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$, containing $\emptyset$, and $\langle U_n\rangle_{n\in\mathbb{N}}$ an enumeration of $\mathcal{U}$. Write

$$M=(\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}})\setminus\bigcup_{m,n\in\mathbb{N}}(\{x:x(m)=n\}\times U_n).$$

Then M is a closed subset of $(\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}})\times\mathbb{N}^{\mathbb{N}}$, therefore analytic (423Ba, 423Bd), so its continuous image

$$A=\{(x,z):\text{there is some }y\text{ such that }(x,y,z)\in M\}$$

is analytic (423Bb).

Now let E be any analytic subset of $\mathbb{N}^{\mathbb{N}}$. By 423E, E is Souslin-F; by 421J, there is a closed set $F\subseteq\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$ such that $E=\{z:\exists\,y,\ (y,z)\in F\}$. Let $\langle x(m)\rangle_{m\in\mathbb{N}}$ be a sequence running over $\{n:n\in\mathbb{N},\ U_n\cap F=\emptyset\}$, so that

$$F=(\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}})\setminus\bigcup_{m\in\mathbb{N}}U_{x(m)}=\{(y,z):(x,y,z)\in M\}.$$

Now

$$\begin{aligned}\{z:(x,z)\in A\}&=\{z:\text{ there is some }y\text{ such that }(x,y,z)\in M\}\\&=\{z:\text{ there is some }y\text{ such that }(y,z)\in F\}=E,\end{aligned}$$

and E is a vertical section of A, as required. **Q**

(b) It follows that there is a non-Borel analytic set $B \subseteq \mathbb{N}^{\mathbb{N}}$. **P** Take A from (a) above, and try

$$B = \{x : (x, x) \in A\}.$$

Because B is the inverse image of A under the continuous map $x \mapsto (x, x)$, it is analytic (423Gc). **?** If B were a Borel set, then $B' = \mathbb{N}^{\mathbb{N}} \setminus B$ would also be Borel, therefore analytic (423E), and there would be an $x \in \mathbb{N}^{\mathbb{N}}$ such that $B' = \{y : (x, y) \in A\}$. But in this case

$$x \in B \iff (x, x) \in A \iff x \in B',$$

which is a difficulty you may have met before. **XQ**

(c) Now return to our arbitrary uncountable analytic Hausdorff space X. By 423J, X has a subset Z homeomorphic to $\mathbb{N}^{\mathbb{N}}$. By (b), Z has an analytic subset A which is not Borel in Z, therefore cannot be a Borel subset of X.

423M I devote a few paragraphs to an important method of constructing selectors.

Theorem Let X be an analytic Hausdorff space, Y a set, and $\mathcal{C} \subseteq \mathcal{P}Y$. Write T for the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathcal{C})$, where $\mathcal{S}$ is Souslin's operation, and $\mathcal{V}$ for $\mathcal{S}(\{F \times C : F \subseteq X \text{ is closed}, C \in \mathcal{C}\})$. If $W \in \mathcal{V}$, then $W[X] \in \mathcal{S}(\mathcal{C})$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$.

proof Write $\mathcal{F}$ for $\{F \times C : F \subseteq X \text{ is closed}, C \in \mathcal{C}\}$.

(a) Consider first the case in which $X = \mathbb{N}^{\mathbb{N}}$ and all the horizontal sections $W^{-1}[\{y\}]$ of W are closed. Let $\mathcal{E}$ be the family of closed subsets of Y. For $\sigma \in S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ set $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$. Then $W \cap (I_\sigma \times Y) \in \mathcal{V}$. **P** Because Souslin's operation is idempotent (421D), $\mathcal{S}(\mathcal{V}) = \mathcal{V}$. The set $\{V : V \cap (I_\sigma \times Y) \in \mathcal{V}\}$ is therefore closed under Souslin's operation (apply 421Cc to the identity map from $I_\sigma \times Y$ to $X \times Y$, or otherwise); since it includes $\mathcal{F}$, it is the whole of $\mathcal{V}$, and contains W. **Q**

By 421G, $W[I_\sigma] = (W \cap (I_\sigma \times Y))[\mathbb{N}^{\mathbb{N}}]$ belongs to $\mathcal{S}(\mathcal{C}) \subseteq$ T for every σ. In particular, $W[\mathbb{N}^{\mathbb{N}}] = W[I_\emptyset] \in \mathcal{S}(\mathcal{C})$. Define $\langle Y_\sigma \rangle_{\sigma\in S}$ in T inductively, as follows. $Y_\emptyset = W[\mathbb{N}^{\mathbb{N}}]$. Given that $Y_\sigma \in$ T and that $Y_\sigma \subseteq W[I_\sigma]$, set

$$Y_{\sigma^\frown<j>} = Y_\sigma \cap W[I_{\sigma^\frown<j>}] \setminus \bigcup_{i<j} W[I_{\sigma^\frown<i>}]$$

for every $j \in \mathbb{N}$. Continue.

At the end of the induction, we have

$$\bigcup_{j\in\mathbb{N}} Y_{\sigma^\frown<j>} = Y_\sigma \cap \bigcup_{j\in\mathbb{N}} W[I_{\sigma^\frown<j>}] = Y_\sigma \cap W[I_\sigma] = Y_\sigma$$

for every $\sigma \in S$, while $\langle Y_{\sigma^\frown<j>} \rangle_{j\in\mathbb{N}}$ is always disjoint. So for each $y \in Y_\emptyset = W[\mathbb{N}^{\mathbb{N}}]$ we have a unique $f(y) \in \mathbb{N}^{\mathbb{N}}$ such that $y \in Y_{f(y)\restriction n}$ for every n. Since $f^{-1}[I_\sigma] = Y_\sigma \in$ T for every $\sigma \in S$, f is T-measurable (4A3Db). Also $(f(y), y) \in W$ for every $y \in W[\mathbb{N}^{\mathbb{N}}]$. **P** For each $n \in \mathbb{N}$, $y \in Y_{f(y)\restriction n} = W[I_{f(y)\restriction n}]$, so there is an $x_n \in \mathbb{N}^{\mathbb{N}}$ such that $x_n\restriction n = f(y)\restriction n$ and $(x_n, y) \in W$. But this means that $f(y) = \lim_{n\to\infty} x_n$; since we are supposing that the horizontal sections of W are closed, $(f(y), y) \in W$. **Q**

Thus the theorem is true if $X = \mathbb{N}^{\mathbb{N}}$ and W has closed horizontal sections.

(b) Now suppose that $X = \mathbb{N}^{\mathbb{N}}$ and that $W \subseteq \mathbb{N}^{\mathbb{N}} \times Y$ is any set in $\mathcal{V}$. Then there is a Souslin scheme $\langle F_\sigma \times C_\sigma \rangle_{\sigma\in S^*}$ in $\mathcal{F}$ with kernel W; of course I mean you to suppose that $F_\sigma \subseteq \mathbb{N}^{\mathbb{N}}$ is closed and $C_\sigma \in \mathcal{C}$ for every σ. Set

$$\tilde{W} = \bigcap_{k\ge1} \bigcup_{\sigma\in\mathbb{N}^k} I_\sigma \times F_\sigma \times C_\sigma \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times Y.$$

Then W is the projection of $\tilde{W}$ onto the last two coordinates, by 421Ce. If $y \in Y$, then

$$\{(\phi, \psi) : (\phi, \psi, y) \in \tilde{W}\} = \bigcap_{k\ge1} \bigcup \{I_\sigma \times F_\sigma : \sigma \in \mathbb{N}^k,\, y \in C_\sigma\}$$

is closed in $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. (If J is any subset of $\mathbb{N}^k$, then

$$(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}) \setminus \bigcup_{\sigma\in J} I_\sigma \times F_\sigma = \bigcup_{\sigma\in J} I_\sigma \times (\mathbb{N}^{\mathbb{N}} \setminus F_\sigma) \cup \bigcup_{\sigma\in\mathbb{N}^k\setminus J} I_\sigma \times \mathbb{N}^{\mathbb{N}}$$

is open.) Also $I_\sigma \times F_\sigma$ is a closed subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ for every σ, and $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. We can therefore apply (a) to $\tilde{W}$, regarded as a subset of $(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}) \times Y$, to see that $W[\mathbb{N}^{\mathbb{N}}] = \tilde{W}[\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}] \in \mathcal{S}(\mathcal{C})$ and that there is a T-measurable function $h = (g, f) : W[\mathbb{N}^{\mathbb{N}}] \to \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $(g(y), f(y), y) \in \tilde{W}$ for every $y \in W[\mathbb{N}^{\mathbb{N}}]$. Now, of course, $f : W[\mathbb{N}^{\mathbb{N}}] \to \mathbb{N}^{\mathbb{N}}$ is T-measurable and $(f(y), y) \in W$ for every $y \in W[\mathbb{N}^{\mathbb{N}}]$.

(c) Finally, suppose only that X is an analytic Hausdorff space and that $W \in \mathcal{V}$. If X is empty, so is Y, and the result is trivial. Otherwise, there is a continuous surjection $h : \mathbb{N}^{\mathbb{N}} \to X$. Set $\tilde{h}(\phi, y) = (h(\phi), y)$ for $\phi \in \mathbb{N}^{\mathbb{N}}$ and $y \in Y$; then $\tilde{h} : \mathbb{N}^{\mathbb{N}} \times Y \to X \times Y$ is a continuous surjection, and $\tilde{W} = \tilde{h}^{-1}[W]$ is the kernel of a Souslin scheme in

$$\{\tilde{h}^{-1}[F \times C] : F \subseteq \mathbb{N}^{\mathbb{N}} \text{ is closed}, C \in \mathcal{C}\} = \{h^{-1}[F] \times C : F \subseteq \mathbb{N}^{\mathbb{N}} \text{ is closed}, C \in \mathcal{C}\}$$

by 421Cb. So we can apply (b) to see that $W[X] = \tilde{W}[\mathbb{N}^{\mathbb{N}}] \in \mathcal{S}(\mathcal{C})$ and there is a T-measurable $g : W[X] \to \mathbb{N}^{\mathbb{N}}$ such that $(g(y), y) \in \tilde{W}$ for every $y \in Y$. Finally $f = hg : W[X] \to X$ is T-measurable and $(f(y), y) \in W$ for every $y \in Y$. This completes the proof.

423N The expression

$$\mathcal{V} = \mathcal{S}(\{F \times C : F \subseteq X \text{ is closed, } C \in \mathcal{C}\})$$

in 423M is a new formulation, and I had better describe one of the basic cases in which we can use the result.

Corollary Let X be an analytic Hausdorff space and Y any topological space. Let T be the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y. If $W \in \mathcal{S}(\mathcal{B}(X \times Y))$, then $W[X] \in \mathrm{T}$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$.

proof (a) Suppose to begin with that $X = \mathbb{N}^{\mathbb{N}}$. In 423M, set $\mathcal{C} = \mathcal{B}(Y)$. Then every open subset and every closed subset of $X \times Y$ belongs to $\mathcal{V}$ as defined in 423M. **P** For $\sigma \in S = \bigcup_{k \in \mathbb{N}} \mathbb{N}^k$, set $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$. If $V \subseteq X \times Y$ is open, set

$$H_\sigma = \bigcup\{H : H \subseteq Y \text{ is open, } I_\sigma \times H_\sigma \subseteq V\}$$

for each $\sigma \in S$. Because $\{I_\sigma : \sigma \in S\}$ is a base for the topology of $\mathbb{N}^{\mathbb{N}}$, $V = \bigcup_{\sigma \in S} I_\sigma \times H_\sigma \in \mathcal{V}$.

As for the complement of V, we have

$$\begin{aligned}
(\mathbb{N}^{\mathbb{N}} \times Y) \setminus V &= \bigcap_{\substack{k \in \mathbb{N} \\ \sigma \in \mathbb{N}^k}} (\mathbb{N}^{\mathbb{N}} \times Y) \setminus (I_\sigma \times H_\sigma) \\
&= \bigcap_{\substack{k \in \mathbb{N} \\ \sigma \in \mathbb{N}^k}} ((\mathbb{N}^{\mathbb{N}} \setminus I_\sigma) \times Y) \cup (\mathbb{N}^{\mathbb{N}} \times (Y \setminus H_\sigma)) \\
&= \bigcap_{\substack{k \in \mathbb{N} \\ \sigma \in \mathbb{N}^k}} \bigcup_{\substack{\tau \in \mathbb{N}^k \\ \tau \neq \sigma}} (I_\tau \times Y) \cup (\mathbb{N}^{\mathbb{N}} \times (Y \setminus H_\sigma))
\end{aligned}$$

which again belongs to $\mathcal{V}$, because $\mathcal{V}$ is closed under countable unions and intersections and contains $I_\tau \times Y$ and $\mathbb{N}^{\mathbb{N}} \times (Y \setminus H_\sigma)$ for all σ, $\tau \in S$. **Q**

By 421F, $\mathcal{V}$ contains every Borel subset of $\mathbb{N}^{\mathbb{N}} \times Y$, so includes $\mathcal{S}(\mathcal{B}(\mathbb{N}^{\mathbb{N}} \times Y))$. So in this case we can apply 423M directly to get the result.

(b) Now suppose that X is any analytic space. If X is empty, the result is trivial. Otherwise, let $h : \mathbb{N}^{\mathbb{N}} \to X$ be a continuous surjection. Set $\tilde{h}(\phi, y) = (h(\phi), y)$ for $\phi \in \mathbb{N}^{\mathbb{N}}$ and $y \in Y$, so that $\tilde{h} : \mathbb{N}^{\mathbb{N}} \times Y \to X \times Y$ is continuous. Set $\tilde{W} = \tilde{h}^{-1}[W]$. If $V \in \mathcal{B}(X \times Y)$ then $\tilde{h}^{-1}[V] \in \mathcal{B}(\mathbb{N}^{\mathbb{N}} \times Y)$ (4A3Cd), so $\tilde{W} \in \mathcal{S}(\mathcal{B}(\mathbb{N}^{\mathbb{N}} \times Y))$ (421Cc). By (a), $\tilde{W}[\mathbb{N}^{\mathbb{N}}] \in \mathrm{T}$ and there is a T-measurable function $g : \tilde{W}[\mathbb{N}^{\mathbb{N}}] \to \mathbb{N}^{\mathbb{N}}$ such that $(g(y), y) \in \tilde{W}$ for every $y \in \tilde{W}[\mathbb{N}^{\mathbb{N}}]$. It is now easy to check that $W[X] = \tilde{W}[\mathbb{N}^{\mathbb{N}}] \in \mathrm{T}$ (this is where we need to know that h is surjective), that $f = hg : W[X] \to X$ is T-measurable, and that $(f(y), y) \in W$ for every $y \in W[X]$, as required.

Remark This is a version of the **von Neumann-Jankow selection theorem**.

423O Corollary Let X and Y be analytic Hausdorff spaces, A an analytic subset of X and $f : A \to Y$ a Borel measurable function. Let T be the σ-algebra of subsets of Y generated by the Souslin-F subsets of Y. Then $f[A] \in \mathrm{T}$ and there is a T-measurable function $g : f[A] \to A$ such that fg is the identity on $f[A]$.

proof In this context, every Borel subset of Y is Souslin-F (423Eb), so every member of $\mathcal{S}(\mathcal{B}(Y))$ is Souslin-F (421D) and $\mathrm{T} = \mathcal{S}(\mathcal{B}(Y))$. If we think of f as a subset of $X \times Y$, it is analytic (423Ga), therefore Souslin-F in $X \times Y$; now we can use 423N to find a T-measurable function $g : f[A] \to A$ such that $(g(y), y) \in f$, that is, $f(g(y)) = y$, for every $y \in f[A]$.

***423P Constituents of coanalytic sets: Theorem** Let X be a Hausdorff space, and $A \subseteq X$ an analytic subset of X. Then there is a non-decreasing family $\langle E_\xi\rangle_{\xi<\omega_1}$ of Borel subsets of X, with union $X \setminus A$, such that every analytic subset of $X \setminus A$ is included in some E_ξ.

proof Put 422K(iii) and 423C together.

***423Q Remarks (a)** Let A be an analytic set in an analytic space X and $\langle E_\xi\rangle_{\xi<\omega_1}$ a family of Borel sets as in 423P. There is nothing unique about the E_ξ. But if $\langle E'_\xi\rangle_{\xi<\omega_1}$ is another such family, then every E'_ξ is an analytic subset of $X \setminus A$, by 423E, so is included in some E_η; and, similarly, every E_ξ is included in some E'_η. We therefore have a function $f:\omega_1 \to \omega_1$ such that $E'_\xi \subseteq E_{f(\xi)}$ and $E_\xi \subseteq E'_{f(\xi)}$ for every $\xi < \omega$. If we set $C = \{\xi : \xi < \omega_1,\ f(\eta) < \xi$ for every $\eta < \xi\}$, then C is a closed cofinal set in ω_1 (4A1Bc), and $\bigcup_{\eta<\xi} E_\eta = \bigcup_{\eta<\xi} E'_\eta$ for every $\xi \in C$. If $X \setminus A$ is itself analytic, that is, if A is a Borel set, then we shall have to have $X \setminus A = E_\xi = E'_\xi$ for some $\xi < \omega_1$.

Another way of expressing the result in 423P is to say that if we write $\mathcal{I} = \{B : B \subseteq X \setminus A$ is analytic$\}$, then $\{E : E \in \mathcal{I},\ E$ is Borel$\}$ is cofinal with $\mathcal{I}$ (this is the First Separation Theorem) and $\operatorname{cf}\mathcal{I} \le \omega_1$.

(b) It is a remarkable fact that, in some models of set theory, we can have non-Borel coanalytic sets in Polish spaces such that all their constituents are countable (Jech 78, p. 529, Cor. 2). (Note that, by (a), this is the same thing as saying that $X \setminus A$ is uncountable but all its Borel subsets are countable.) But in 'ordinary' cases we shall have, for every Borel subset E of $X \setminus A$, an uncountable Borel subset of $(X \setminus A) \setminus E$; so that for any family $\langle G_\xi\rangle_{\xi<\omega_1}$ of Borel constituents of $X \setminus A$, there must be uncountably many uncountable G_ξ. To see that this happens at least sometimes, take any non-Borel analytic subset A_0 of $\mathbb{N}^\mathbb{N}$ (423L), and consider $A = A_0 \times \mathbb{N}^\mathbb{N} \subseteq (\mathbb{N}^\mathbb{N})^2$. Then A is analytic (423B). If $E \subseteq (\mathbb{N}^\mathbb{N})^2 \setminus A$ is Borel, then $\pi_1[E] = \{x : (x,y) \in E\}$ is an analytic subset of $\mathbb{N}^\mathbb{N} \setminus A$, so is not the whole of $\mathbb{N}^\mathbb{N} \setminus A_0$ (by 423Fa). Taking any $x \in (\mathbb{N}^\mathbb{N} \setminus A_0) \setminus \pi_1[E]$, $\{x\} \times \mathbb{N}^\mathbb{N}$ is an uncountable Borel subset of $((\mathbb{N}^\mathbb{N})^2 \setminus A) \setminus E$. For an alternative construction, see 423Ye.

***423R Coanalytic and PCA sets** In the case of Polish spaces, we can go a great deal farther. I mention only some fragments which will be used in Volume 5. Let X be a Polish space.

(a) A subset A of X is **coanalytic** or $\mathbf{\Pi}^1_1$ if $X \setminus A$ is analytic, and **PCA** or $\mathbf{\Sigma}^1_2$ if there is a coanalytic set $R \subseteq \mathbb{N}^\mathbb{N} \times X$ such that $R[\mathbb{N}^\mathbb{N}] = A$. Generally, for $n \ge 1$, $A \subseteq X$ is $\mathbf{\Pi}^1_n$ if $X \setminus A$ is $\mathbf{\Sigma}^1_n$, and $\mathbf{\Sigma}^1_{n+1}$ if there is a $\mathbf{\Pi}^1_n$ set $R \subseteq \mathbb{N}^\mathbb{N} \times X$ such that $R[\mathbb{N}^\mathbb{N}] = A$. If A is both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$, we say that A is $\mathbf{\Delta}^1_n$.

(b) Analytic subsets of X are Souslin-F (423Eb). Applying 421P to the Borel σ-algebra of X, we see that a subset of X which is either analytic or coanalytic can be expressed as the union of at most ω_1 Borel sets. It follows that every PCA set $A \subseteq X$ can be expressed as the union of at most ω_1 Borel sets. **P** Let $R \subseteq \mathbb{N}^\mathbb{N} \times X$ be a coanalytic set such that $A = R[\mathbb{N}^\mathbb{N}]$. Express R as $\bigcup_{\xi<\omega_1} R_\xi$ where every R_ξ is a Borel subset of the Polish space $\mathbb{N}^\mathbb{N} \times X$. For each $\xi < \omega_1$, R_ξ is analytic (423Eb) so $A_\xi = R_\xi[\mathbb{N}^\mathbb{N}]$ is analytic (423Bb) and can be expressed as $\bigcup_{\eta<\omega_1} E_{\xi\eta}$ where every $E_{\xi\eta}$ is a Borel subset of X. Now $A = \bigcup_{\xi,\eta<\omega_1} E_{\xi\eta}$ is the union of at most ω_1 Borel sets. **Q**

(c) A subset of X is Borel iff it is $\mathbf{\Delta}^1_1$, that is, is both analytic and coanalytic (423Eb, 423Fa). Since the intersection and union of a sequence of analytic subsets of X are analytic (423Ea), the union and intersection of a sequence of coanalytic subsets of X are coanalytic. If Y is another Polish space and $h : X \to Y$ is Borel measurable, then $h^{-1}[A]$ is analytic for every analytic $A \subseteq Y$ (423Gc), so $h^{-1}[B]$ is coanalytic in X for every coanalytic $B \subseteq Y$. If Y is a G_δ subset of X, and $B \subseteq Y$ is coanalytic in Y (remember that Y, with its subspace topology, is Polish, by 4A2Qd), then B is coanalytic in X, because $X \setminus B = (X \setminus Y) \cup (Y \setminus B)$ is the union of two analytic sets.

(d) If X and Y are Polish spaces, $A \subseteq Y$ is PCA and $f : X \to Y$ is Borel measurable, then $f^{-1}[A]$ is a PCA set in X. **P** Let $R \subseteq \mathbb{N}^\mathbb{N} \times X$ be a coanalytic set such that $R[\mathbb{N}^\mathbb{N}] = A$. Set $g(\phi, x) = (\phi, f(x))$ for $\phi \in \mathbb{N}^\mathbb{N}$ and $x \in X$; writing $\mathcal{B}(X)$ for the Borel σ-algebra of X, etc., g is $(\mathcal{B}(\mathbb{N}^\mathbb{N})\widehat{\otimes}\mathcal{B}(X), \mathcal{B}(\mathbb{N}^\mathbb{N})\widehat{\otimes}\mathcal{B}(Y))$-measurable (use 4A3Bc). But $\mathcal{B}(\mathbb{N}^\mathbb{N})\widehat{\otimes}\mathcal{B}(X) = \mathcal{B}(\mathbb{N}^\mathbb{N} \times X)$ and $\mathcal{B}(\mathbb{N}^\mathbb{N})\widehat{\otimes}\mathcal{B}(Y) = \mathcal{B}(\mathbb{N}^\mathbb{N} \times Y)$ (4A3Ga), so g is Borel measurable. By (c), $S = g^{-1}[R]$ is a coanalytic set in $\mathbb{N}^\mathbb{N} \times X$. Now

$$S[\mathbb{N}^\mathbb{N}] = \{x : \exists\, \phi,\ (\phi, x) \in g^{-1}[R]\} = \{x : \exists\, \phi,\ (\phi, f(x)) \in R\} = f^{-1}[A],$$

so $f^{-1}[A]$ is PCA. **Q**

(e) For a fuller account of this material, see Kechris 95 or Kuratowski 66.

423S Proposition Let $(X,\mathfrak{T})$ be an analytic Hausdorff space, and Σ a countably generated σ-subalgebra of the Borel σ-algebra $\mathcal{B}(X,\mathfrak{T})$ which separates the points of X. Then $\Sigma=\mathcal{B}(X,\mathfrak{T})$.

proof Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ which σ-generates Σ. If x, $y\in X$ are distinct, the set $\{E:E\subseteq X, x\in E\iff y\in E\}$ is a σ-algebra of subsets of X not including Σ, so there is an $n\in\mathbb{N}$ such that just one of x, y belongs to E_n, and the topology $\mathfrak{S}$ generated by $\{E_n:n\in\mathbb{N}\}\cup\{X\setminus E_n:n\in\mathbb{N}\}$ is Hausdorff. Write $\mathfrak{T}\vee\mathfrak{S}$ for the topology generated by $\mathfrak{T}\cup\mathfrak{S}$, that is, the topology generated by $\mathfrak{T}\cup\{E_n:n\in\mathbb{N}\}\cup\{X\setminus E_n:n\in\mathbb{N}\}$. By 423H, $\mathfrak{T}\vee\mathfrak{S}$ is analytic; because both $\mathfrak{T}$ and $\mathfrak{S}$ are Hausdorff topologies coarser than $\mathfrak{T}\vee\mathfrak{S}$, the Borel σ-algebras $\mathcal{B}(X,\mathfrak{T})$, $\mathcal{B}(X,\mathfrak{T}\vee\mathfrak{S})$ and $\mathcal{B}(X,\mathfrak{S})$ are all the same (423Fb). Next, $\mathfrak{S}$ is second-countable, therefore hereditarily Lindelöf (4A2O), with a subbase included in Σ, so $\mathcal{B}(X,\mathfrak{S})\subseteq\Sigma$ (4A3Da) and $\mathcal{B}(X,\mathfrak{T})$ must be equal to Σ.

423X Basic exercises >(a) For a Hausdorff space X, show that the following are equiveridical: (i) X is analytic; (ii) X is a continuous image of a Polish space; (iii) X is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$.

(b) Write out a direct proof of 423Ea, not quoting 423C or 421D.

>(c) Let X be a set, $\mathfrak{S}$ a Hausdorff topology on X and $\mathfrak{T}$ an analytic topology on X such that $\mathfrak{S}\subseteq\mathfrak{T}$. Show that $\mathfrak{S}$ and $\mathfrak{T}$ have the same analytic sets. (*Hint*: 423F.)

(d) Let X be an analytic Hausdorff space. (i) Show that its Borel σ-algebra $\mathcal{B}(X)$ is countably generated as σ-algebra. (*Hint*: use 4A2Nf and 423Fb.) (ii) Show that there is an analytic subset Y of $\mathbb{R}$ such that $(X,\mathcal{A}_X)$ is isomorphic to $(Y,\mathcal{A}_Y)$, where $\mathcal{A}_X$, $\mathcal{A}_Y$ are the families of Souslin-F subsets of X, Y respectively. (*Hint*: show that there is an injective Borel measurable function from X to $\mathbb{R}$ (cf. 343E), and use 423G.) (iii) Show that $(X,\mathcal{B}(X))$ is isomorphic to $(Y,\mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y. (*Hint*: 423Fa.) (iv) Let T_X, T_Y be the σ-algebras generated by $\mathcal{A}_X$, $\mathcal{A}_Y$ respectively. Show that (X,T_X) and (Y,T_Y) are isomorphic.

(e) Let $\mathfrak{S}$ be the right-facing Sorgenfrey topology on $\mathbb{R}$ (415Xc). Show that $\mathfrak{S}$ has the same Borel sets as the usual topology $\mathfrak{T}$ on $\mathbb{R}$. (*Hint*: $\mathfrak{S}$ is hereditarily Lindelöf (419Xf) and has a base consisting of $\mathfrak{T}$-Borel sets.) Show that $\mathfrak{S}$ is not analytic.

(f) Let X be an analytic Hausdorff space and Y any topological space. Let T be the σ-algebra of subsets of Y generated by the Souslin-F sets. Show that if $W\subseteq X\times Y$ is Souslin-F, then $W[X]\in\mathrm{T}$ and there is a T-measurable function $f:W[X]\to Y$ such that $(f(y),y)\in W$ for every $y\in W[X]$. (*Hint*: start with $X=\mathbb{N}^{\mathbb{N}}$, as in 423N.)

(g) Let X and Y be Hausdorff spaces, and $R\subseteq X\times Y$ an analytic set such that $R^{-1}[Y]=X$. Show that there is a function $g:X\to Y$, measurable with respect to the σ-algebra generated by the Souslin-F subsets of X, such that $(x,g(x))\in R$ for every $x\in X$.

>(h)(i) Show that the family of analytic subsets of $[0,1]$ has cardinal $\mathfrak{c}$. (*Hint*: 421Xc.) (ii) Show that the σ-algebra T of subsets of $[0,1]$ generated by the analytic sets has cardinal $\mathfrak{c}$. (*Hint*: 421Xh.) (iii) Show that there is a set $A\subseteq[0,1]$ which does not belong to T.

(i) Let $X=Y=[0,1]$. Give Y the usual topology, and give X the topology corresponding to the one-point compactification of the discrete topology on $[0,1[$, that is, a set $G\subseteq X$ is open if either $1\notin G$ or G is cofinite. Show that the identity map $f:X\to Y$ is a Borel measurable bijection, but that f^{-1} is not measurable for the σ-algebra of subsets of Y generated by the Souslin-F sets.

423Y Further exercises (a) Show that a space with a countable network is **hereditarily separable** (that is, every subset is separable), therefore countably tight.

(b) Show that if X is a Hausdorff space with a countable network, then every analytic subset of X is obtainable by Souslin's operation from the open subsets of X.

(c) Let $(X,\mathfrak{T})$ and $(Y,\mathfrak{S})$ be analytic Hausdorff spaces and $f:X\to Y$ a Borel measurable function. (i) Show that there is a zero-dimensional separable metrizable topology $\mathfrak{S}'$ on Y with the same Borel sets and the same analytic sets as $\mathfrak{S}$. (*Hint*: 423Xd.) (ii) Show that there is a zero-dimensional separable metrizable topology $\mathfrak{T}'$ on X, with the same Borel sets and the same analytic sets as $\mathfrak{T}$, such that f is continuous for the topologies $\mathfrak{T}'$ and $\mathfrak{S}'$. (iii) Explain how to elaborate these ideas to deal with any countable family of analytic spaces and Borel measurable functions between them.

(d) Let X be an analytic Hausdorff space, Y a Hausdorff space with a countable network, and $f : X \to Y$ a Borel measurable surjection. Let T be the σ-algebra of subsets of Y generated by the Souslin-F sets in Y. Show that there is a T-measurable function $g : Y \to X$ such that fg is the identity on Y.

(e) Set $S^* = \bigcup_{n\ge 1} \mathbb{N}^n$ and consider $\mathcal{P}(S^*)$ with its usual topology. Let $\mathcal{T} \subseteq \mathcal{P}S^*$ be the set of trees (421N); show that $\mathcal{T}$ is closed, therefore a compact metrizable space. Set $F_\sigma = \{T : \sigma \in T \in \mathcal{T}\}$ for $\sigma \in S^*$, and let A be the kernel of the Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S^*}$. Show that the constituents of $\mathcal{T} \setminus A$ for this scheme are just the sets $G_\xi = \{T : r(T) = \xi\}$, where r is the rank function of 421Ne. Show by induction on ξ that all the G_ξ is non-empty, so that A is not a Borel set. Show that $\#(G_\xi) = \mathfrak{c}$ for $1 \le \xi < \omega_1$. Show that if X is any topological space and $B \subseteq X$ is a Souslin-F set, there is a Borel measurable function $f : X \to \mathcal{T}$ such that $B = f^{-1}[A]$.

423 Notes and comments We have been dealing, in this section and the last, with three classes of topological space: the class of analytic spaces, the class of K-analytic spaces and the class of spaces with countable networks. The first is more important than the other two put together, and I am sure many people would find it more comfortable, if more time-consuming, to learn the theory of analytic spaces thoroughly first, before proceeding to the others. This was indeed my own route into the subject. But I think that the theory of K-analytic spaces has now matured to the point that it can stand on its own, without constant reference to its origin as an extension of descriptive set theory on the real line; and that our understanding of analytic spaces is usefully advanced by seeing how easily their properties can be deduced from the fact that they are K-analytic spaces with countable networks.

As in §422, I have made no attempt to cover the general theory of analytic spaces, nor even to give a balanced introduction. I have tried instead to give a condensed account of the principal methods for showing that spaces are analytic, with some of the ideas which can be applied to make them more accessible to the imagination (423J, 423Xc-423Xd, 423Yb-423Yc). Lusin's theorem 423I does not mention 'analytic' sets in its statement, but it depends essentially on the separation theorem 422J, so cannot really be put with the other results on Polish spaces in 4A2Q. You must of course know that not all analytic sets are Borel (423L) and that not all sets are analytic (423Xh). For further information about this fascinating subject, see Rogers 80, Kechris 95 and Moschovakis 80.

'Selection theorems' appear everywhere in mathematics. The axiom of choice is a selection theorem; it says that whenever $R \subseteq X \times Y$ is a relation and $R[X] = Y$, there is a function $f : Y \to X$ such that $(f(y), y) \in R$ for every $y \in Y$. The Lifting Theorem (§341) asks for a selector which is a Boolean homomorphism. In general topology we look for continuous selectors. In measure theory, naturally, we are interested in measurable selectors, as in 423M-423O. Any selection theorem will have expressions either as a theorem on right inverses of functions, as in 423O, or as a theorem on selectors for relations, as in 423M-423N. In the language here, however, we get better theorems by examining relations, because the essence of the method is that we can find measurable functions into analytic spaces, and the relations of 423M can be very far from being analytic, even when there is a natural topology on the space Y. The value of these results will become clearer in §433, when we shall see that the σ-algebras T of 423M-423O are often included in familiar σ-algebras. Typical applications are in 433F-433G below.

424 Standard Borel spaces

This volume is concerned with topological measure spaces, and it will come as no surprise that the topological properties of Polish spaces are central to the theory. But even from the point of view of unadorned measure theory, not looking for topological structures on the underlying spaces, it turns out that the Borel algebras of Polish spaces have a very special position. It will be useful later on to be able to refer to some fundamental facts concerning them.

424A Definition Let X be a set and Σ a σ-algebra of subsets of X. We say that (X, Σ) is a **standard Borel space** if there is a Polish topology on X for which Σ is the algebra of Borel sets.

Warning! Many authors reserve the phrase 'standard Borel space' for the case in which X is uncountable. I have seen the phrase 'Borel space' used for what I call a 'standard Borel space'.

424B Proposition (a) If (X, Σ) is a standard Borel space, then Σ is countably generated as σ-algebra of sets.

(b) If $\langle(X_i, \Sigma_i)\rangle_{i\in I}$ is a countable family of standard Borel spaces, then $(\prod_{i\in I} X_i, \widehat{\bigotimes}_{i\in I}\Sigma_i)$ (definition: 254E) is a standard Borel space.

(c) Let (X, Σ) and (Y, T) be standard Borel spaces and $f : X \to Y$ a (Σ, T)-measurable surjection. Then

(i) if $E \in \Sigma$ is such that $f[E] \cap f[X \setminus E] = \emptyset$, then $f[E] \in \mathrm{T}$;

(ii) $\mathrm{T} = \{F : F \subseteq Y,\ f^{-1}[F] \in \Sigma\}$;

(iii) if f is a bijection it is an isomorphism.

(d) Let (X,Σ) and (Y,T) be standard Borel spaces and $f : X \to Y$ a (Σ,T)-measurable injection. Then $Z = f[X] \in \mathrm{T}$ and f is an isomorphism between (X,Σ) and (Z,T_Z), where T_Z is the subspace σ-algebra.

proof (a) Let $\mathfrak{T}$ be a Polish topology on X such that Σ is the algebra of Borel sets. Then $\mathfrak{T}$ has a countable base $\mathcal{U}$, which generates Σ (4A3Da/4A3E).

(b) For each $i \in I$ let $\mathfrak{T}_i$ be a Polish topology on X_i such that Σ_i is the algebra of $\mathfrak{T}_i$-Borel sets. Then $X = \prod_{i\in I} \mathfrak{T}_i$, with the product topology $\mathfrak{T}$, is Polish (4A2Qc). By 4A3Dc/4A3E, $\Sigma = \widehat{\bigotimes}_{i\in I}\Sigma_i$ is just the Borel σ-algebra of X, so (X,Σ) is a standard Borel space.

(c) Let $\mathfrak{T}$, $\mathfrak{S}$ be Polish topologies on X, Y respectively for which Σ and T are the Borel σ-algebras. Then f is Borel measurable.

(i) By 423Eb and 423Gb, $f[E]$ and $f[X \setminus E]$ are analytic subsets of Y. But they are complementary, so they are Borel sets, by 423Fa.

(ii) $f^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$, just because f is measurable. On the other hand, if $F \subseteq Y$ and $E = f^{-1}[F] \in \Sigma$, then $F = f[E] \in \mathrm{T}$ by (i).

(iii) follows at once.

(d) Give X and Y Polish topologies for which Σ, T are the Borel σ-algebras. By 423Ib, $f[E] \in \mathrm{T}$ for every $E \in \Sigma$; in particular, $Z = f[X]$ belongs to T. Also $f^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}_Z$, so f is an isomorphism between (X,Σ) and (Z,T_Z).

424C Theorem Let (X,Σ) be a standard Borel space.

(a) If X is countable then $\Sigma = \mathcal{P}X$.

(b) If X is uncountable then (X,Σ) is isomorphic to $(\mathbb{N}^{\mathbb{N}},\mathcal{B}(\mathbb{N}^{\mathbb{N}}))$, where $\mathcal{B}(\mathbb{N}^{\mathbb{N}})$ is the algebra of Borel subsets of $\mathbb{N}^{\mathbb{N}}$.

proof Let $\mathfrak{T}$ be a Polish topology on X such that Σ is its Borel σ-algebra.

(a) Every singleton subset of X is closed, so must belong to Σ. If X is countable, every subset of X is a countable union of singletons, so belongs to Σ.

(b) (Rao & Srivastava 94) The strategy of the proof is to find Borel sets $Z \subseteq X$, $W \subseteq \mathbb{N}^{\mathbb{N}}$ such that $(Z,\Sigma_Z) \cong (\mathbb{N}^{\mathbb{N}},\mathcal{B}(\mathbb{N}^{\mathbb{N}}))$ and $(W,\mathcal{B}(W)) \cong (X,\Sigma)$ (writing Σ_Z, $\mathcal{B}(W)$ for the subspace σ-algebras), and use a form of the Schröder-Bernstein theorem.

(i) By 423J, X has a subset Z homeomorphic to $\mathbb{N}^{\mathbb{N}}$; let $h : \mathbb{N}^{\mathbb{N}} \to Z$ be a homeomorphism. By 424Bd, h is an isomorphism between $(\mathbb{N}^{\mathbb{N}},\mathcal{B}(\mathbb{N}^{\mathbb{N}}))$ and (Z,Σ_Z).

(ii) Let $\langle U_n\rangle_{n\in\mathbb{N}}$ run over a base for the topology of X. Define $g : X \to \{0,1\}^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ by setting $g(x) = \langle\chi U_n(x)\rangle_{n\in\mathbb{N}}$ for every $x \in \mathbb{N}$. Then g is injective, because X is Hausdorff. Also g is Borel measurable, by 4A3D(c-ii). By 424Bd, g is an isomorphism between (X,Σ) and $(W,\mathcal{B}(W))$, where $W = g[X]$ belongs to $\mathcal{B}(\mathbb{N}^{\mathbb{N}})$.

(iii) We have $Z \in \Sigma$, $W \in \mathcal{B}(\mathbb{N}^{\mathbb{N}})$ such that $(Z,\Sigma_Z) \cong (\mathbb{N}^{\mathbb{N}},\mathcal{B}(\mathbb{N}^{\mathbb{N}}))$ and $(W,\mathcal{B}(W)) \cong (X,\Sigma)$. By 344D, $(X,\Sigma) \cong (\mathbb{N}^{\mathbb{N}},\mathcal{B}(\mathbb{N}^{\mathbb{N}}))$, as claimed.

424D Corollary (a) If (X,Σ) and (Y,T) are standard Borel spaces and $\#(X) = \#(Y)$, then (X,Σ) and (Y,T) are isomorphic.

(b) If (X,Σ) is an uncountable standard Borel space then $\#(X) = \#(\Sigma) = \mathfrak{c}$.

proof These follow immediately from 424C, if we recall that $\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) = \mathfrak{c}$ (4A3Fb).

424E Proposition Let X be a set and Σ a σ-algebra of subsets of X; suppose that (X,Σ) is countably separated in the sense that there is a countable set $\mathcal{E} \subseteq \Sigma$ separating the points of X. If $A \subseteq X$ is such that (A,Σ_A) is a standard Borel space, where Σ_A is the subspace σ-algebra, then $A \in \Sigma$.

proof Give A a Polish topology $\mathfrak{T}$ such that Σ_A is the Borel σ-algebra of A, and let $\mathfrak{S}$ be the topology on X generated by $\mathcal{E}\cup\{X\setminus E : E \in \mathcal{E}\}$. Then $\mathfrak{S}$ is second-countable (4A2Oa), so has a countable network (4A2Oc), and is Hausdorff because $\mathcal{E}$ separates the points of X. The identity map from A to X is Borel measurable for $\mathfrak{T}$ and $\mathfrak{S}$, so 423Ib tells us that A is $\mathfrak{S}$-Borel; but of course the $\mathfrak{S}$-Borel σ-algebra is just the σ-algebra generated by $\mathcal{E}$ (4A3Da), so is included in Σ.

424F Corollary Let X be a Polish space and $A \subseteq X$ any set which is not Borel. Let $\mathcal{B}(A)$ be the Borel σ-algebra of A. Then $(A, \mathcal{B}(A))$ is not a standard Borel space.

424G Proposition Let (X, Σ) be a standard Borel space. Then (E, Σ_E) is a standard Borel space for every $E \in \Sigma$, writing Σ_E for the subspace σ-algebra.

proof Let $\mathfrak{T}$ be a Polish topology on X for which Σ is the Borel σ-algebra. Then there is a Polish topology $\mathfrak{T}' \supseteq \mathfrak{T}$ for which E is closed (4A3I), therefore itself Polish in the subspace topology $\mathfrak{T}'_E$ (4A2Qd). But $\mathfrak{T}'$ and $\mathfrak{T}$ have the same Borel sets (423Fb), so Σ_E is just the Borel σ-algebra of E for $\mathfrak{T}'_E$, and (E, Σ_E) is a standard Borel space.

***424H** For the full strength of a theorem in §448 we need a remarkable result concerning group actions on Polish spaces.

Theorem (Becker & Kechris 96) Let G be a Polish group, $(X, \mathfrak{T})$ a Polish space and $\bullet$ a Borel measurable action of G on X. Then there is a Polish topology $\mathfrak{T}'$ on X, yielding the same Borel sets as $\mathfrak{T}$, such that the action is continuous for $\mathfrak{T}'$ and the given topology of G.

proof (a) Fix on a right-translation-invariant metric ρ on G defining the topology of G (4A5Q), and let D be a countable dense subset of G; write e for the identity of G. Let Z be the set of 1-Lipschitz functions from G to $[0,1]$, that is, functions $f : G \to [0,1]$ such that $|f(g) - f(h)| \le \rho(g,h)$ for all g, $h \in G$. Then Z, with the topology of pointwise convergence inherited from the product topology of $[0,1]^G$, is a compact metrizable space. **P** It is a closed subset of $[0,1]^G$, so is a compact Hausdorff space. Writing $q(f) = f \restriction D$ for $f \in Z$, $q : Z \to [0,1]^D$ is injective, because D is dense and every member of Z is continuous; but this means that Z is homeomorphic to $q[Z]$, which is metrizable, by 4A2Pc. **Q**

We see also that the Borel σ-algebra of Z is the σ-algebra generated by sets of the form $W_{g\alpha} = \{f : f(g) < \alpha\}$ where $g \in G$ and $\alpha \in [0,1]$. **P** This σ-algebra contains every set of the form $\{f : f \in Z, \alpha < f(g) < \beta\}$, where $g \in G$ and α, $\beta \in \mathbb{R}$; since these sets generate the topology of Z, the σ-algebra they generate is the Borel σ-algebra of Z, by 4A3Da. **Q**

(b) There is a continuous action of G on Z defined by setting

$$(g \bullet_r f)(h) = f(hg)$$

for $f \in Z$ and g, $h \in G$. **P** (i) If $f \in Z$ and g, h_1, $h_2 \in G$, then

$$|(g \bullet_r f)(h_1) - (g \bullet_r f)(h_2)| = |f(h_1 g) - f(h_2 g)| \le \rho(h_1 g, h_2 g) = \rho(h_1, h_2)$$

because ρ is right-translation-invariant. So $g \bullet_r f \in Z$ for every $f \in Z$, $g \in G$. (ii) As in 4A5Cc-4A5Cd, $\bullet_r$ is an action of G on Z. (iii) Suppose that $g_0 \in G$, $f_0 \in Z$, $h \in G$ and $\epsilon > 0$. Set

$$V = \{g : g \in G,\, \rho(hg, hg_0) < \tfrac{1}{2}\epsilon\},$$

$$W = \{f : f \in Z,\, |f(hg_0) - f_0(hg_0)| < \tfrac{1}{2}\epsilon\}.$$

Then V is an open set in G containing g_0 (because $g \mapsto \rho(hg, hg_0)$ is continuous) and W is an open set in Z containing f_0. If $g \in V$ and $f \in W$,

$$\begin{aligned}|(g \bullet_r f)(h) - (g_0 \bullet_r f_0)(h)| &= |f(hg) - f_0(hg_0)| \\ &\le |f(hg) - f(hg_0)| + |f(hg_0) - f_0(hg_0)| \\ &\le \rho(hg, hg_0) + \frac{1}{2}\epsilon \le \epsilon.\end{aligned}$$

As f_0, g_0, ϵ are arbitrary, the map $(g, f) \mapsto (g \bullet_r f)(h)$ is continuous; as h is arbitrary, the map $(g, f) \mapsto g \bullet_r f$ is continuous. **Q**

(c) Let $\mathcal{B}(X)$ be the Borel σ-algebra of X. For $x \in X$ and $B \in \mathcal{B}(X)$, set

$$P_B(x) = \{g : g \in G,\, g \bullet x \in B\},$$

$$Q_B(x) = \bigcup\{V : V \subseteq G \text{ is open, } V \setminus P_B(x) \text{ is meager}\},$$

$$f_B(x)(g) = \inf(\{1\} \cup \{\rho(g,h) : h \in G \setminus Q_B(x)\})$$

for $g \in G$. It is easy to check that

$$f_B(x)(g') \le \rho(g,g') + f_B(x)(g')$$

for all g, $g' \in G$, so that every $f_B(x)$ belongs to Z.

Every $P_B(x)$ is a Borel set, because $\bullet$ is Borel measurable, so has the Baire property in X (4A3Rb). Because X is a Baire space (4A2Ma), $Q_B(x) \subseteq \overline{P_B(x)}$ (4A3Ra).

Let $\mathcal{V}$ be a countable base for the topology of G containing G.

(d) For each $B \in \mathcal{B}(X)$, the map $f_B : X \to Z$ is Borel measurable. **P** Because the Borel σ-algebra of Z is generated by the sets $W_{g\alpha}$ of (a) above, it is enough to show that

$$\{x : f_B(x) \in W_{g\alpha}\} = \{x : f_B(x)(g) < \alpha\}$$

always belongs to $\mathcal{B}(X)$, because $\{W : f_B^{-1}[W] \in \mathcal{B}\}$ is surely a σ-algebra of subsets of Z. But if $\alpha > 1$ this set is X, while if $\alpha \le 1$ it is

$$\begin{aligned}\{x : \text{ there is some } h \in X \setminus Q_B(x) \text{ such that } \rho(g,h) < \alpha\} \\ = \{x : \text{ there is some } V \in \mathcal{V} \text{ such that } V \setminus Q_B(x) \ne \emptyset \\ \text{and } \rho(g,h) < \alpha \text{ for every } h \in V\} \\ = \bigcup_{V \in \mathcal{V}'} \{x : V \not\subseteq Q_B(x)\}\end{aligned}$$

(where $\mathcal{V}' = \{V : V \in \mathcal{V},\ \rho(g,h) < \alpha \text{ for every } h \in V\}$)

$$= \bigcup_{V \in \mathcal{V}'} \{x : V \setminus P_B(x) \text{ is not meager}\}.$$

But for any fixed $V \in \mathcal{V}$,

$$\{x : V \setminus P_B(x) \text{ is not meager}\} = X \setminus \{x : W[\{x\}] \text{ is meager}\}$$

where

$$W = \{(y,g) : y \in X,\ g \in V,\ g\bullet y \in X \setminus B\}$$

is a Borel subset of $X \times G$, because $\bullet$ is supposed to be Borel measurable; and therefore $W \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(G)$, writing $\mathcal{B}(G)$ for the Borel σ-algebra of G (4A3G). Now $\mathcal{B}(G) \subseteq \widehat{\mathcal{B}}(G)$, the Baire-property algebra of G (4A3Rb), so $W \in \mathcal{B}(X)\widehat{\otimes}\widehat{\mathcal{B}}(G)$. By 4A3Rc, the quotient algebra $\widehat{\mathcal{B}}/\mathcal{M}$ has a countable order-dense set, where $\mathcal{M}$ is the σ-ideal of meager sets, so 4A3Sa tells us that $\{x : W[\{x\}] \text{ is meager}\}$ is a Borel subset of X. Accordingly $\{x : V \setminus P_B(x)$ is not meager$\}$ is Borel for every V, and $\{x : f_B(x)(g) < \alpha\}$ is Borel. **Q**

(e) If $g \in G$, $B \in \mathcal{B}(X)$ and $x \in X$, then $g\bullet_r f_B(x) = f_B(g\bullet x)$. **P**

$$P_B(g\bullet x) = \{h : h\bullet(g\bullet x) \in B\} = \{h : hg \in P_B(x)\} = P_B(x)g^{-1}.$$

Because the map $h \mapsto hg^{-1} : G \to G$ is a homeomorphism, $Q_B(g\bullet x) = Q_B(x)g^{-1}$; because it is an isometry,

$$\begin{aligned}f_B(g\bullet x)(h) &= \min(1, \rho(h, X \setminus Q_B(g\bullet x))) = \min(1, \rho(h, X \setminus Q_B(x)g^{-1})) \\ &= \min(1, \rho(hg, X \setminus Q_B(x))) = f_B(x)(hg) = (g\bullet_r f_B(x))(h)\end{aligned}$$

for every $h \in G$. **Q**

(f) Let $\langle B_{0m}\rangle_{m\in\mathbb{N}}$ run over a base for $\mathfrak{T}$ containing X. We can now find a countable set $\mathcal{E} \subseteq \mathcal{B}(X)$ such that (i) the topology $\mathfrak{T}^*$ generated by $\mathcal{E}$ is a Polish topology finer than $\mathfrak{T}$ (ii) f_E is $\mathfrak{T}^*$-continuous for every $E \in \mathcal{E}$ (iii) $X \in \mathcal{E}$. **P** Let $\mathcal{W}$ be a countable base for the topology of Z. Enumerate $\mathbb{N} \times \mathbb{N} \times \mathcal{W}$ as $\langle(k_n, m_n, W_n)\rangle_{n\in\mathbb{N}}$ in such a way that $k_n \le n$ for every n. Having chosen Borel sets $B_{ij} \subseteq X$ for $i \le n$, $j \in \mathbb{N}$ in such a way that the topology $\mathfrak{S}_n$ generated by $\{B_{ij} : i \le n,\ j \in \mathbb{N}\}$ is a Polish topology finer than $\mathfrak{T}$, consider the set

$$C_n = \{x : f_{B_{k_n,m_n}}(x) \in W_n\}.$$

This is $\mathfrak{T}$-Borel, therefore $\mathfrak{S}_n$-Borel, so by 4A3H there is a Polish topology $\mathfrak{S}_{n+1} \supseteq \mathfrak{S}_n$ such that $C_n \in \mathfrak{S}_{n+1}$. Let $\langle B_{n+1,m}\rangle_{m\in\mathbb{N}}$ run over a base for $\mathfrak{S}_{n+1}$; by 423Fb, every $B_{n+1,m}$ belongs to $\mathcal{B}(X)$. Continue.

Let $\mathfrak{T}^*$ be the topology generated by $\mathcal{E} = \{B_{ij} : i, j \in \mathbb{N}\}$. By 4A2Qf, $\mathfrak{T}^*$ is Polish, because it is the topology generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{S}_n$. If $W \in \mathcal{W}$ and $E \in \mathcal{E}$ there are i, $j \in \mathbb{N}$ such that $B = B_{ij}$ and an $n \in \mathbb{N}$ such that $(i, j, W) = (k_n, m_n, W_n)$; now

$$f_E^{-1}[W] = C_n \in \mathfrak{S}_{n+1} \subseteq \mathfrak{T}^*.$$

As W is arbitrary, f_E is $\mathfrak{T}^*$-continuous. Also

$$X \in \{B_{0m} : m \in \mathbb{N}\} \subseteq \mathcal{E},$$

as required. **Q**

(g) Define $\theta : X \to Z^{\mathcal{E}}$ by setting $\theta(x)(E) = f_E(x)$ for $x \in X$ and $E \in \mathcal{E}$. Then θ is injective. **P** Suppose that x, $y \in X$ and that $x \neq y$. For every $g \in G$,

$$g^{-1}\bullet(g\bullet x) = x \neq y = g^{-1}\bullet(g\bullet y),$$

so $g\bullet x \neq g\bullet y$ and there is some $m \in \mathbb{N}$ such that $g\bullet x \in B_{0m}$ while $g\bullet y \notin B_{0m}$, that is, $g \in P_{B_{0m}}(x) \setminus P_{B_{0m}}(y)$. Thus $G = \bigcup_{m\in\mathbb{N}} P_{B_{0m}}(x) \setminus P_{B_{0m}}(y)$; by Baire's theorem, there is some $m \in \mathbb{N}$ such that $P_{B_{0m}}(x) \setminus P_{B_{0m}}(y)$ is non-meager. Because $Q_{B_{0m}}(x)\triangle P_{B_{0m}}(x)$ and $Q_{B_{0m}}(y)\triangle P_{B_{0m}}(y)$ are both meager,

$$\{g : f_{B_{0m}}(x)(g) > 0\} = Q_{B_{0m}}(x) \neq Q_{B_{0m}}(y) = \{g : f_{B_{0m}}(y)(g) > 0\},$$

and

$$\theta(x)(B_{0m}) = f_{B_{0m}}(x) \neq f_{B_{0m}}(y) = \theta(y)(B_{0m}).$$

So $\theta(x) \neq \theta(y)$. **Q** Because f_E is $\mathfrak{T}^*$-continuous for every $E \in \mathcal{E}$, θ is $\mathfrak{T}^*$-continuous.

(h) Let $\mathfrak{T}'$ be the topology on X induced by θ; that is, the topology which renders θ a homeomorphism between X and $\theta[X]$. Because $\theta[X] \subseteq Z^{\mathcal{E}}$ is separable and metrizable, $\mathfrak{T}'$ is separable and metrizable. Because θ is $\mathfrak{T}^*$-continuous, $\mathfrak{T}' \subseteq \mathfrak{T}^*$.

(i) The action of G on X is continuous for the given topology $\mathfrak{S}$ on G and $\mathfrak{T}'$ on X. **P** For any $E \in \mathcal{E}$,

$$(g, x) \mapsto \theta(g\bullet x)(E) = f_E(g\bullet x) = g\bullet_r f_E(x)$$

((e) above) is $\mathfrak{S} \times \mathfrak{T}'$-continuous because the action of G on Z is continuous ((b) above) and $f_E : X \to Z$ is $\mathfrak{T}'$-continuous (by the definition of $\mathfrak{T}'$). But this means that $(g, x) \mapsto \theta(g\bullet x)$ is $\mathfrak{S} \times \mathfrak{T}'$-continuous, so that $(g, x) \mapsto g\bullet x$ is $(\mathfrak{S} \times \mathfrak{T}', \mathfrak{T}')$-continuous. **Q**

(j) Let σ be a complete metric on G defining the topology $\mathfrak{S}$, and τ a complete metric on X defining the topology $\mathfrak{T}^*$. (We do not need to relate σ to ρ in any way beyond the fact that they both give rise to the same topology $\mathfrak{S}$.) For $E \in \mathcal{E}$, $V \in \mathcal{V}$ and $n \in \mathbb{N}$ let S_{EVn} be the set of those $\phi \in Z^{\mathcal{E}}$ such that *either* $\phi(E)(g) = 0$ for every $g \in V$ *or* there is an $F \in \mathcal{E}$ such that $F \subseteq E$, $\operatorname{diam}_\tau(F) \leq 2^{-n}$ and $\phi(F)(g) > 0$ for some $g \in V$. Then S_{EVn} is the union of a closed set and an open set, so is a G_δ set in $Z^{\mathcal{E}}$ (4A2C(a-i)). Consequently

$$Y = \{\phi : \phi \in Z^{\mathcal{E}}, \, \phi(X) = \chi G\} \cap \overline{\theta[X]} \cap \bigcap_{E\in\mathcal{E}, V\in\mathcal{V}, n\in\mathbb{N}} S_{EVn}$$

is a G_δ subset of $Z^{\mathcal{E}}$, being the intersection of countably many G_δ sets.

(k) $\overline{\theta[X]} \subseteq Y$. **P** Let $x \in X$. (i) $P_X(x) = G$ so $Q_X(x) = G$ and $\theta(x)(X) = f_X(x) = \chi G$. (ii) Of course $\theta(x) \in \overline{\theta[X]}$. (iii) Suppose that $E \in \mathcal{E}$, $V \in \mathcal{V}$ and $n \in \mathbb{N}$. If $Q_E(x) \cap V = \emptyset$ then $\theta(x)(E)(g) = f_E(x)(g) = 0$ for every $g \in V$, and $\theta(x)(E) \in S_{EVn}$. Otherwise, $V \cap P_E(x)$ is non-meager. But

$$E = \bigcup\{F : F \in \mathcal{E}, \, F \subseteq E, \, \operatorname{diam}_\tau(F) \leq 2^{-n}\},$$

so

$$P_E(x) = \bigcup\{P_F(x) : F \in \mathcal{E}, \, F \subseteq E, \, \operatorname{diam}_\tau(F) \leq 2^{-n}\};$$

because $\mathcal{E}$ is countable, this is a countable union and there is an $F \in \mathcal{E}$ such that $F \subseteq E$, $\operatorname{diam}_\tau(F) \leq 2^{-n}$ and $P_F(x) \cap V$ is non-meager. In this case $Q_F(x) \cap V$ is non-empty and $\theta(x)(F) = f_F(x)$ is non-zero at some point of V; thus again $\theta(x) \in S_{EVn}$. As E, V and n are arbitrary, we have the result. **Q**

(l) (The magic bit.) $Y \subseteq \theta[X]$. **P** Take any $\phi \in Y$. Choose $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$ and $\langle \tilde{g}_n\rangle_{n\in\mathbb{N}}$ in G as follows. $E_0 = X$ and $V_0 = G$. Given that $\phi(E_n)$ is non-zero at some point of V_n, then, because $\phi \in S_{E_nV_nn}$, there is an $E_{n+1} \in \mathcal{E}$ such that $E_{n+1} \subseteq E_n$, $\operatorname{diam}_\tau(E_{n+1}) \leq 2^{-n}$ and $\phi(E_{n+1})$ is non-zero at some point of V_n; say $\tilde{g}_n \in V_n$

is such that $\phi(E_{n+1})(\tilde{g}_n) > 0$. Now we can find a $V_{n+1} \in \mathcal{V}$ such that $\tilde{g}_n \in V_{n+1} \subseteq V_n$ and $\operatorname{diam}_\sigma(V_{n+1}) \le 2^{-n}$. Continue.

We are supposing also that $\phi \in \overline{\theta[X]}$, so we have a sequence $\langle x_i \rangle_{i\in\mathbb{N}}$ in X such that $\langle \theta(x_i)\rangle_{i\in\mathbb{N}} \to \phi$, that is, $\langle f_E(x_i)\rangle_{i\in\mathbb{N}} \to \phi(E)$ for every $E \in \mathcal{E}$. In particular,

$$\lim_{i\to\infty} f_{E_{n+1}}(x_i)(\tilde{g}_n) = \phi(E_{n+1})(\tilde{g}_n) > 0$$

for every n. Let $\langle i_n \rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $f_{E_{n+1}}(x_{i_n})(\tilde{g}_n) > 0$ for every n. Then

$$\tilde{g}_n \in V_{n+1} \cap Q_{E_{n+1}}(x_{i_n}) \subseteq V_{n+1} \cap \overline{P_{E_{n+1}}(x_{i_n})};$$

there is therefore some $g_n \in V_{n+1} \cap P_{E_{n+1}}(x_{i_n})$, so that $g_n \bullet x_{i_n} \in E_{n+1}$.

$\langle V_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of sets with σ-diameters converging to 0, so $\langle g_n \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for the complete metric σ. Similarly, $\langle g_n \bullet x_{i_n}\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for the complete metric τ, because $\operatorname{diam}_\tau(E_{n+1}) \le 2^{-n}$. We therefore have $g \in G$, $y \in X$ such that $\langle g_n \rangle_{n\in\mathbb{N}} \to g$ for $\mathfrak{S}$ and $\langle g_n \bullet x_{i_n}\rangle_{n\in\mathbb{N}} \to y$ for $\mathfrak{T}^*$. In this case, $\langle g_n \bullet x_{i_n}\rangle_{n\in\mathbb{N}} \to y$ for the coarser topology $\mathfrak{T}'$, while $\langle g_n^{-1}\rangle_{n\in\mathbb{N}} \to g^{-1}$ for $\mathfrak{S}$. Because the action is $(\mathfrak{S}\times\mathfrak{T}',\mathfrak{T}')$-continuous,

$$\langle x_{i_n}\rangle_{n\in\mathbb{N}} = \langle g_n^{-1} \bullet (g_n \bullet x_{i_n})\rangle_{n\in\mathbb{N}} \to g^{-1}\bullet y$$

for $\mathfrak{T}'$. But θ is continuous for $\mathfrak{T}'$, by the definition of $\mathfrak{T}'$, so

$$\theta(g^{-1}\bullet y) = \lim_{n\to\infty} \theta(x_{i_n}) = \phi,$$

and $\phi \in \theta[X]$. As ϕ is arbitrary, $Y \subseteq \theta[X]$. **Q**

(m) Thus $\theta[X] = Y$ is a G_δ set in the compact metric space $Z^{\mathcal{E}}$, and is a Polish space in its induced topology (4A2Qd). But this means that $(X, \mathfrak{T}')$, which is homeomorphic to $\theta[X]$, is also Polish.

(n) I have still to check that $\mathfrak{T}'$ has the same Borel sets as $\mathfrak{T}$. But $\mathfrak{T}$, $\mathfrak{T}^*$ and $\mathfrak{T}'$ are all Polish topologies and $\mathfrak{T}^*$ is finer than both the other two. By 423Fb, $\mathfrak{T}^*$ has the same Borel sets as either of the others.

This completes the proof.

424X Basic exercises >(a) Let $\langle (X_i, \Sigma_i)\rangle_{i\in I}$ be a countable family of standard Borel spaces, and (X, Σ) their direct sum, that is, $X = \{(x, i) : i \in I,\ x \in X_i\}$, $\Sigma = \{E : E \subseteq X,\ \{x : (x, i) \in E\} \in \Sigma_i \text{ for every } i\}$. Show that (X, Σ) is a standard Borel space.

>(b) Let (X, Σ) be a standard Borel space and T a countably generated σ-subalgebra of Σ. Show that there is an analytic Hausdorff space Z such that T is isomorphic to the Borel σ-algebra of Z. (*Hint*: by 4A3I, we can suppose that X is a Polish space and T is generated by a sequence of open-and-closed sets, corresponding to a continuous function from X to $\{0,1\}^{\mathbb{N}}$.)

>(c) Let (X, Σ) be a standard Borel space and T_1, T_2 two countably generated σ-subalgebras of Σ which separate the same points, in the sense that if x, $y \in X$ then there is an $E \in \mathrm{T}_1$ such that $x \in E$ and $y \in X \setminus E$ iff there is an $E' \in \mathrm{T}_2$ such that $x \in E'$ and $y \in X \setminus E'$. Show that $\mathrm{T}_1 = \mathrm{T}_2$. (*Hint*: 424Xb, 423Fb.) In particular, if T_1 separates the points of X then $\mathrm{T}_1 = \Sigma$.

(d) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that $\mathfrak{A}$ is isomorphic to the Borel σ-algebra of an analytic Hausdorff space iff it is isomorphic to a countably generated σ-subalgebra of the Borel σ-algebra of $[0,1]$.

>(e) Let U be a separable Banach space. Show that its Borel σ-algebra is generated, as σ-algebra, by the sets of the form $\{u : h(u) \le \alpha\}$ as h runs over the dual U^* and α runs over $\mathbb{R}$.

>(f) Let X be a compact metrizable space. Show that the Borel σ-algebra of $C(X)$ (the Banach space of continuous real-valued functions on X) is generated, as σ-algebra, by the sets $\{u : u \in C(X),\ u(x) \ge \alpha\}$ as x runs over X and α runs over $\mathbb{R}$.

(g) Let (X, Σ) be a standard Borel space, Y any set, and T a σ-algebra of subsets of Y. Write T^* for the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathrm{T})$, where $\mathcal{S}$ is Souslin's operation. Let $W \in \mathcal{S}(\Sigma\widehat{\otimes}\mathrm{T})$. Show that $W[X] \in \mathcal{S}(\mathrm{T})$ and that there is a (T^*, Σ)-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$. (*Hint*: 423M.)

(h) Let (X,Σ) be a standard Borel space, Y any set, and T a countably generated σ-algebra of subsets of Y. Let $f : X \to Y$ be a (Σ, T)-measurable function, and write T^* for the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathrm{T})$, where $\mathcal{S}$ is Souslin's operation. Show that there is a (T^*, Σ)-measurable function $g : f[X] \to X$ such that gf is the identity on X. (*Hint*: start with the case in which T separates the points of Y, so that the graph of f belongs to $\Sigma\widehat{\otimes}\mathrm{T}$.)

>(i) Show that 424Xc and 424Xh are both false if we omit the phrase 'countably generated' from the hypotheses. (*Hint*: consider (i) the countable-cocountable algebra of $\mathbb{R}$ (ii) the split interval.)

(j) Let (X,Σ,μ) be a σ-finite measure space in which Σ is countably generated. Let $\mathcal{A}$ be the set of atoms A of the Boolean algebra Σ such that $\mu A > 0$, and set $H = X \setminus \bigcup\mathcal{A}$. Show that the subspace measure on H is atomless.

424Y Further exercises (a) Let $(X,\mathfrak{T})$ be a Polish space and $\mathcal{F}$ the family of closed subsets of X. Let Σ be the σ-algebra of subsets of $\mathcal{F}$ generated by the sets $\mathcal{E}_H = \{F : F \in \mathcal{F},\ F \cap H \ne \emptyset\}$ as H runs over the open subsets of X. (i) Show that $(\mathcal{F},\Sigma)$ is a standard Borel space. (*Hint*: take a complete metric ρ defining the topology of X. Set $S^* = \bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ and choose a family $\langle U_\sigma\rangle_{\sigma\in S^*}$ of open sets in X such that $U_\emptyset = X$, $\operatorname{diam} U_\sigma \le 2^{-n}$ whenever $\#(\sigma) = n+1$, $U_\sigma = \bigcup_{i\in\mathbb{N}} U_{\sigma^\frown<i>}$ for every σ, $\overline{U}_{\sigma^\frown<i>} \subseteq U_\sigma$ for every σ, i. Define $f : \mathcal{F} \to \{0,1\}^{S^*}$ by setting $f(F)(\sigma) = 1$ if $F \cap U_\sigma \ne \emptyset$, 0 otherwise. Show that $Z = f[\mathcal{F}]$ is a Borel set and that f is an isomorphism between Σ and the Borel σ-algebra of Z.) (This is the **Effros Borel structure** on $\mathcal{F}$.) (ii) Show that $[X]^n \in \Sigma$ for every $n \in \mathbb{N}$.

(b) Let (X,Σ) be a standard Borel space. Let $\mathbb{T}$ be the family of Polish topologies on X for which Σ is the Borel σ-algebra. Show that any sequence in $\mathbb{T}$ has an upper bound in $\mathbb{T}$, and that any sequence with a lower bound has a least upper bound.

(c) Let (X,Σ) be a standard Borel space. Say that $C \subseteq X$ is **coanalytic** if its complement belongs to $\mathcal{S}(\Sigma)$. Show that for any such C the partially ordered set $\Sigma \cap \mathcal{P}C$ has cofinality 1 if $C \in \Sigma$ and cofinality ω_1 otherwise. (*Hint*: 423P.)

(d) Let $I^\parallel$ be the split interval. Show that there is a σ-algebra Σ of subsets of $I^\parallel$ such that $(I^\parallel,\Sigma)$ is a standard Borel space and $\{(x,y) : x,\ y \in I^\parallel,\ x \le y\} \in \Sigma\widehat{\otimes}\Sigma$.

(e) Let (X,Σ) be a standard Borel space. Show that if X is uncountable, Σ has a countably generated σ-subalgebra not isomorphic either to Σ or to $\mathcal{P}I$ for any set I.

(f) Let (X,Σ,μ) be a σ-finite countably separated perfect measure space (definition: 342K/451Ad). Show that there is a standard Borel space (Y,T) such that $Y \in \Sigma$, $\mathrm{T} \subseteq \Sigma$ and μ is inner regular with respect to T.

424 Notes and comments In this treatise I have generally indulged my prejudice in favour of 'complete' measures. Consequently Borel σ-algebras, as such, have taken subordinate roles. But important parts of the theory of Lebesgue measure, and Radon measures on Polish spaces in general, are associated with the fact that these are completions of measures defined on standard Borel spaces. Moreover, such spaces provide a suitable framework for a large part of probability theory. Of course they become deficient in contexts where we need to look at uncountable independent families of random variables, and there are also difficulties with σ-subalgebras, even countably generated ones, since these can correspond to the Borel algebras of general analytic spaces, which will not always be standard Borel structures (424F, 423L). 424Xf and 424Ya suggest the ubiquity of standard Borel structures; the former shows that they are not always presented as countably generated algebras, while the latter is an example in which we have to make a special construction in order to associate a topology with the algebra. The theory is of course dominated by the results of §423, especially 423Fb and 423I.

I include 424H in this section because there is no other convenient place for it, but I have an excuse: the idea of 'Borel measurable action' can, in this context, be described entirely in terms of σ-algebras, since the Borel algebra of $G\times X$ is just the σ-algebra product of the Borel algebras of the factors (as in 424Bb). Of course for the theorem as expressed here we do need to know that G has a Polish group structure; but X could be presented just as a standard Borel space. The result is a dramatic expression of the fact that, given a standard Borel space (X,Σ), we have a great deal of freedom in defining a corresponding Polish topology on X.

*425 Realization of automorphisms

In §344 I presented some results on the representation of a countable semigroup of Boolean homomorphisms in a measure algebra by a semigroup of functions on the measure space underlying the algebra. §424 provides us with the tools needed for a remarkable extension, in the case of the Lebesgue measure algebra, to groups of cardinal ω_1 (Theorem 425D). The expression of the ideas is made smoother by using the language of group actions (4A5B-4A5C).

425A I begin with what amounts to a special case of the main theorem, with some refinements which will be useful elsewhere.

Proposition (a) Let (X,Σ) and (Y,T) be non-empty standard Borel spaces, and $\mathcal{I}$, $\mathcal{J}$ σ-subalgebras of Σ, T respectively; write $\mathfrak{A}=\Sigma/\mathcal{I}$ and $\mathfrak{B}=\mathrm{T}/\mathcal{J}$ for the quotient algebras. For $E\in\Sigma$, $F\in\mathrm{T}$ write Σ_E, T_F for the subspace σ-algebras on E, F respectively.

(a) If $\pi:\mathfrak{A}\to\mathfrak{B}$ is a sequentially order-continuous Boolean homomorphism, there is a (T,Σ)-measurable $f:Y\to X$ which represents π in the sense that $\pi E^\bullet=f^{-1}[E]^\bullet$ for every $E\in\Sigma$.

(b) If $\pi:\mathfrak{A}\to\mathfrak{B}$ is a Boolean isomorphism, there are $G\in\mathcal{I}$, $H\in\mathcal{J}$ and a bijection $h:Y\setminus H\to X\setminus G$ which is an isomorphism between $(Y\setminus H,\mathrm{T}_{Y\setminus H})$ and $(X\setminus G,\Sigma_{X\setminus G})$, and represents π in the sense that $\pi E^\bullet=h^{-1}[E\setminus G]^\bullet$ for every $E\in\Sigma$.

(c) If $\pi:\mathfrak{A}\to\mathfrak{A}$ is a Boolean automorphism, there is a bijection $h:X\to X$ which is an automorphism of (X,Σ) and represents π in the sense of (a).

(d) If $\#(X)=\#(Y)=\mathfrak{c}$, $\mathfrak{A}$ and $\mathfrak{B}$ are ccc, and $\pi:\mathfrak{A}\to\mathfrak{B}$ is a Boolean isomorphism, there is a bijection $h:Y\to X$ which is an isomorphism between (Y,T) and (X,Σ), and represents π in the sense of (a).

proof (a)(i) If either $\mathfrak{A}$ or $\mathfrak{B}$ is $\{0\}$, so is the other, and we can take f to be a constant function. So henceforth let us suppose that $X\notin\mathcal{I}$ and $Y\notin\mathcal{J}$.

(ii) If $X\subseteq\mathbb{N}$ and $\Sigma=\mathcal{P}X$, set $Z=\{n:n\in X,\,\{n\}\notin\mathcal{I}\}$. For $n\in Z$, set $a_n=\{n\}^\bullet$ and $b_n=\pi a_n$, and choose $F_n\in\mathrm{T}$ such that $F_n^\bullet=b_n$. Since $\sup_{n\in Z}a_n=1$ in $\mathfrak{A}$, $\sup_{n\in Z}b_n=1$ in $\mathfrak{B}$ and $Y\setminus\bigcup_{n\in Z}F_n\in\mathcal{J}$. Define $f:Y\to X$ by saying that

$$\begin{aligned} f(y)&=\min\{n:n\in Z,\,y\in F_n\}\text{ if }y\in\bigcup_{n\in Z}F_n,\\ &=\min\ \text{otherwise}.\end{aligned}$$

Then f represents π in the required sense.

(iii) If $X=\{0,1\}^{\mathbb{N}}$ and Σ is its Borel σ-algebra, then for each $n\in\mathbb{N}$ set $E_n=\{x:x(n)=1\}$ and $e_n=E_n^\bullet$ and choose $F_n\in\mathrm{T}$ such that $F_n^\bullet=\pi e_n$. Set $f(y)=\langle\chi F_n(y)\rangle_{n\in\mathbb{N}}$ for $y\in Y$. Then $f^{-1}[E_n]^\bullet=\pi E_n^\bullet$ for every n; as $\{E:f^{-1}[E]^\bullet=\pi E^\bullet\}$ is a σ-subalgebra of Σ containing every E_n, it is the whole of Σ, and again f represents π.

(iv) By 424C, any standard Borel space is isomorphic to either that in (iii) or one of those in (ii), so these cases together are sufficient to prove the general result.

(b) By (a), we have $f:Y\to X$ and $g:X\to Y$ representing π, π^{-1} respectively. Now we see that $gf:Y\to Y$ represents $\pi\pi^{-1}:\mathfrak{B}\to\mathfrak{B}$, that is, $F\triangle(gf)^{-1}[F]\in\mathcal{J}$ for every $F\in\mathrm{T}$. Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence in T which separates the points of Y, and set $H_0=\bigcup_{n\in\mathbb{N}}F_n\triangle(gf)^{-1}[F_n]$; then $H_0\in\mathcal{J}$ and $g(f(y))=y$ for every $y\in Y\setminus H_0$.

Similarly, there is a $G_0\in\mathcal{I}$ such that $f(g(x))=x$ for every $x\in X\setminus G_0$. Set $G=G_0\cup g^{-1}[H_0]$ and $H=H_0\cup f^{-1}[G_0]$. Then $g[X\setminus G]\subseteq Y\setminus H$. **P** If $x\in X\setminus G$ then $g(x)\notin H_0$; moreover, $f(g(x))=x\notin G_0$ so $g(x)\notin f^{-1}[G_0]$ and $g(x)\notin H$. **Q** Similarly, $f(y)\in X\setminus G$ for every $y\in Y\setminus H$. As $f(g(x))=x$ for $x\in X\setminus G$ and $g(f(y))=y$ for $y\in Y\setminus H$, $h=f\restriction Y\setminus H$ is a bijection with inverse $g\restriction X\setminus G$. Because f is (T,Σ)-measurable, h is $(\mathrm{T}_{Y\setminus H},\Sigma_{X\setminus G})$-measurable; because g is (Σ,T)-measurable, h^{-1} is $(\Sigma_{X\setminus G},\mathrm{T}_{Y\setminus H})$-measurable, and h is an isomorphism between $(Y\setminus H,\mathrm{T}_{Y\setminus H})$ and $(X\setminus G,\Sigma_{X\setminus G})$. Finally, if $E\in\Sigma$,

$$\pi(E^\bullet)=\pi((E\setminus G)^\bullet)=(f^{-1}[E\setminus G])^\bullet=(h^{-1}[E\setminus G])^\bullet,$$

so h represents π in the sense declared.

(c) We can repeat the proof of (b) with an additional idea. The point is that there is an element E_0 of $\mathcal{I}$ with maximal cardinality. **P** Because (G,Σ_G) is a standard Borel space (424G), $\#(G)$ is either $\mathfrak{c}$ or countable (424Db), for every $G\in\mathcal{I}$. If $\mathcal{I}$ contains arbitrarily large finite sets, it must contain an infinite set, because it is a σ-algebra. So the supremum $\sup\{\#(G):G\in\mathcal{I}\}$ is attained. **Q**

Now, in (b), take $(Y,\mathrm{T}) = (X,\Sigma)$ and $\mathfrak{B} = \mathfrak{A}$, and choose f, H_0, g and G_0 as before; but this time set $G = (G_0 \cup E_0) \cup g^{-1}[H_0 \cup E_0]$ and $H = (H_0 \cup E_0) \cup f^{-1}[G_0 \cup E_0]$. The same arguments as before tell us that $h_0 = f{\restriction}Y \setminus H$ is an isomorphism between $(Y \setminus H, \mathrm{T}_{Y\setminus H})$ and $(X \setminus G, \Sigma_{X\setminus G})$ representing π. We now, however, have G, $H \in \mathcal{I}$ and both include E_0; so we must have $\#(G) = \#(E_0) = \#(H)$. By 424Da, there is an isomorphism h_1 between (H, Σ_H) and (G, Σ_G). So if we set

$$\begin{aligned} h(y) &= h_0(y) \text{ if } y \in X \setminus H,\\ &= h_1(y) \text{ if } y \in H, \end{aligned}$$

then $h : X \to X$ is a bijection which is an automorphism of (X,Σ), and

$$h^{-1}[E]^\bullet = (H \cap h^{-1}[E])^\bullet = (h^{-1}[E \cap G])^\bullet = (h_0^{-1}[E \cap G])^\bullet = \pi E^\bullet$$

for every $E \in \Sigma$, as required.

(d) An adaptation of the ideas of (c) works in this case too. First note that as $\#(X) = \mathfrak{c}$, $(X,\Sigma) \cong (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and there is a partition of X into $\mathfrak{c}$ members of Σ all of cardinal $\mathfrak{c}$. As $\mathfrak{A}$ is ccc, all but countably many of these must belong to $\mathcal{I}$, and we have an $E_0 \in \mathcal{I}$ with $\#(E_0) = \mathfrak{c}$. Similarly, there is an $F_0 \in \mathcal{J}$ with $\#(F_0) = \mathfrak{c}$.

Now choose f, H_0, g and G_0 as in (b); but this time set $G = (G_0 \cup E_0) \cup g^{-1}[H_0 \cup F_0]$ and $H = (H_0 \cup F_0) \cup f^{-1}[G_0 \cup E_0]$. Again, $h_0 = f{\restriction}Y \setminus H$ is an isomorphism between $(Y \setminus H, \mathrm{T}_{Y\setminus H})$ and $(X \setminus G, \Sigma_{X\setminus G})$ representing π. As $\#(G) = \mathfrak{c} = \#(H)$, there is an isomorphism h_1 between (H, T_H) and (G, Σ_G). So if we set

$$\begin{aligned} h(y) &= h_0(y) \text{ if } y \in Y \setminus H,\\ &= h_1(y) \text{ if } y \in H, \end{aligned}$$

we get a bijection $h : Y \to X$ which is an isomorphism between (Y,T) and (X,Σ), and represents π, just as in (c).

425B Lemma Let G be a group, G_0 a subgroup of G, H another group, and X, Z sets; let $\bullet_r$ be the right shift action of H on Z^H (4A5C(c-ii)). Suppose we are given a group homomorphism $\theta : G \to H$, an injective function $f : \mathbb{N} \times Z^H \to X$ and an action $\bullet_0$ of G_0 on X such that $\pi\bullet_0 f(n,z) = f(n, \theta(\pi)\bullet_r z)$ whenever $n \in \mathbb{N}$ and $z \in Z^H$.

(a) If $\#(X \setminus f[\mathbb{N} \times Z^H]) \le \#(Z)$, there is an action $\bullet$ of G on X extending $\bullet_0$.

(b) Suppose moreover that H is countable, X and Z are Polish spaces, and f is Borel measurable when $\mathbb{N} \times Z^H$ is given the product topology. If $x \mapsto \pi\bullet_0 x$ is Borel measurable for every $\pi \in G_0$, then $\bullet$ can be chosen in such a way that $x \mapsto \psi\bullet x$ is Borel measurable for every $\psi \in G$.

proof (a)(i) Let $D \subseteq H$ be a selector for the left cosets of the subgroup $\theta[G_0]$, so that every member of H is uniquely expressible as $\psi\theta(\pi)$ where $\psi \in D$ and $\pi \in G_0$. Observe that if $\pi \in G_0$ and $x \in f[\mathbb{N} \times Z^H]$, then $\pi^{-1}\bullet_0 x \in f[\mathbb{N} \times Z^H]$; consequently, setting $Y = X \setminus f[\mathbb{N} \times Z^H]$, $\pi\bullet_0 y \in Y$ for every $y \in Y$, because $\pi^{-1}\bullet_0(\pi\bullet_0 y) = y$. Let $g_0 : Y \to Z$ be any injection, and define $g : Y \to Z^H$ by setting $g(y)(\psi\theta(\pi)) = g_0(\pi\bullet_0 y)$ whenever $\psi \in D$, $\pi \in G_0$ and $y \in Y$. In this case, $\theta(\pi)\bullet_r g(y) = g(\pi\bullet_0 y)$ whenever $y \in Y$ and $\pi \in G_0$. **P** Take any $\psi \in D$ and $\phi \in G_0$. Then

$$\begin{aligned} (\theta(\pi)\bullet_r g(y))(\psi\theta(\phi)) &= g(y)(\psi\theta(\phi)\theta(\pi)) = g(y)(\psi\theta(\phi\pi))\\ &= g_0(\phi\pi\bullet_0 y) = g_0(\phi\bullet_0(\pi\bullet_0 y)) = g(\pi\bullet_0 y)(\psi\theta(\phi)). \end{aligned}$$

As $D\theta[G_0] = H$, this shows that $\theta(\pi)\bullet_r g(y) = g(\pi\bullet_0 y)$. **Q**

Note that as $g(y)(\psi) = g_0(y)$ whenever $\psi \in D$ and $y \in Y$, g also is injective.

(ii) Now define $h : \mathbb{N} \times Z^H \to X$ by setting

$$\begin{aligned} h(n,z) &= g^{-1}(z) \text{ if } n = 0 \text{ and } z \in g[Y]\\ &\qquad\qquad \text{matching } \{0\} \times g[Y] \text{ with } Y,\\ &= f(n-1,z) \text{ if } n \ge 1 \text{ and } z \in g[Y]\\ &\qquad\qquad \text{matching } (\mathbb{N} \setminus \{0\}) \times g[Y] \text{ with } f[\mathbb{N} \times g[Y]],\\ &= f(n,z) \text{ if } n \in \mathbb{N} \text{ and } z \notin g[Y]\\ &\qquad\qquad \text{matching } \mathbb{N} \times (Z^H \setminus g[Y]) \text{ with } f[\mathbb{N} \times (Z^H \setminus g[Y])]. \end{aligned}$$

Clearly h is a bijection. If $n \in \mathbb{N}$, $z \in Z^H$ and $\pi \in G_0$, then $h(n, \theta(\pi)\bullet_r z) = \pi\bullet_0 h(n,z)$. **P** If $z \in g[Y]$, then

$$\theta(\pi)\bullet_r z = \theta(\pi)\bullet_r g(g^{-1}(z)) = g(\pi\bullet_0 g^{-1}(z)) \in g[Y],$$

so

$$h(0, \theta(\pi)\bullet_r z) = \pi\bullet_0 g^{-1}(z) = \pi\bullet_0 h(0, z),$$

while if $n \ge 1$, then

$$h(n, \theta(\pi)\bullet_r z) = f(n-1, \theta(\pi)\bullet_r z) = \pi\bullet_0 f(n-1, z) = \pi\bullet_0 h(n, z).$$

On the other hand, if $n \in \mathbb{N}$ and $z \in Z^H \setminus g[Y]$, then $\theta(\pi)\bullet_r z \notin g[Y]$, because $\theta(\pi^{-1})\bullet_r(\theta(\pi)\bullet_r z) \notin g[Y]$; so

$$h(n, \theta(\pi)\bullet_r z) = f(n, \theta(\pi)\bullet_r z) = \pi\bullet_0 f(n, z) = \pi\bullet_0 h(n, z). \text{ } \mathbf{Q}$$

(iii) Try

$$\psi\bullet x = h(n, \theta(\psi)\bullet_r z)$$

whenever $x \in X$, $h^{-1}(x) = (n, z)$ and $\psi \in G$. If $x \in X$, ψ, $\psi' \in G$ and $\pi \in G_0$, express $h^{-1}(x)$ as (n, z); then, writing ι for the identity of G,

$$\begin{aligned} \iota\bullet x &= h(n, \theta(\iota)\bullet_r z) = h(n, z) = x, \\ \psi'\psi\bullet x &= h(n, \theta(\psi'\psi)\bullet_r z) = h(n, \theta(\psi')\bullet_r(\theta(\psi)\bullet_r z)) = \psi'\bullet h(n, \theta(\psi)\bullet_r z) = \psi'\bullet(\psi\bullet x), \\ \pi\bullet x &= h(n, \theta(\pi)\bullet_r z) = \pi\bullet_0 h(n, z) = \pi\bullet_0 x, \end{aligned}$$

so we have an action of G on X extending $\bullet_0$, as required.

(b) Under the topological hypotheses, we follow the same line of argument, but taking time for checks at each stage. Because Z, and therefore $\mathbb{N} \times Z^H$, are Polish spaces, and f is a measurable injection, $f[\mathbb{N} \times Z^H]$ and Y are Borel sets (423Ib). In (i), we must of course take g_0 to be Borel measurable; since all the functions $x \mapsto \pi\bullet_0 x$ are Borel measurable, all the functions $y \mapsto g(y)(\psi)$ will be Borel measurable, and $g : Y \to Z^H$ will be Borel measurable. Consequently all the sets $\{n\} \times g[Y]$ will be Borel, and h will be Borel measurable, therefore a Borel isomorphism between $\mathbb{N} \times Z^H$ and X. Finally, because $z \mapsto \theta(\psi)\bullet_r z : Z^H \to Z^H$ is Borel measurable for every $\psi \in G$, $(n, z) \mapsto (n, \theta(\psi)\bullet_r z)$ is Borel measurable for every n, and $x \mapsto \psi\bullet x$ is Borel measurable, for every $\psi \in G$. So $\bullet$ is measurable in the required sense.

425C Master actions (a) For each $R \subseteq \mathbb{N}^2$, consider the family F_R of injective functions f from countable ordinals to $\mathbb{N}$ such that

for every $\beta \in \operatorname{dom} f$, $f(\beta)$ is the unique member of $\mathbb{N}$ such that $R[\{f(\beta)\}] = f[\beta]$.

If f, $g \in F_R$ have domains α, α' respectively where $\alpha \le \alpha'$, then $f = g{\restriction}\alpha$. **P** If $\beta < \alpha$ is such that $f{\restriction}\beta = g{\restriction}\beta$, then both $f(\beta)$ and $g(\beta)$ are the unique $m \in \mathbb{N}$ such that $R[\{m\}] = f[\beta]$. **Q** Consequently $f_R = \bigcup F_R$ is the unique maximal element of F_R. (Compare 2A1B.)

For a countable ordinal α, let $\mathcal{R}_\alpha$ be the set of those $R \subseteq \mathbb{N}^2$ such that $\alpha \le \operatorname{dom} f_R$. Note that if $f : \alpha \to \mathbb{N}$ is injective, then there is an $R \in \mathcal{R}_\alpha$ such that $f = f_R{\restriction}\alpha$ (set $R = \{(f(\beta), f(\gamma)) : \gamma < \beta < \alpha\} \cup ((\mathbb{N} \setminus f[\alpha]) \times \mathbb{N})$).

(b) Let ✠ be the family of group operations $\star$ on $\mathbb{N}$. We are going to need the natural Borel structure on ✠ corresponding to the identification of each $\star \in$ ✠, which is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, with the set

$$\{(i, j, k) : i \star j = k\} \subseteq \mathbb{N}^3.$$

So we can think of ✠ as the set of subsets $\star$ of $\mathbb{N}^3$ such that

for all i, $j \in \mathbb{N}$ there is just one $k \in \mathbb{N}$ such that $(i, j, k) \in \star$,

if i, j, k, l, m, $n \in \mathbb{N}$, and (i, j, l), (l, k, n), (j, k, m) belong to $\star$, then $(i, m, n) \in \star$,

$(0, i, i)$, $(i, 0, i) \in \star$ for every $i \in \mathbb{N}$,

for every $i \in \mathbb{N}$ there is a $j \in \mathbb{N}$ such that $(i, j, 0)$ and $(j, i, 0)$ belong to $\star$.

For $\star \in$ ✠, let $\bullet_r^\star$ be the corresponding right shift action of $\mathbb{N}$ on $\mathbb{R}^{\mathbb{N}}$, so that $(m\bullet_r^\star z)(i) = z(i \star m)$ whenever $z \in \mathbb{R}^{\mathbb{N}}$ and i, $m \in \mathbb{N}$.

(c) Let G be a group, of cardinal ω_1, with identity ι; let $\langle\pi_\alpha\rangle_{\alpha<\omega_1}$ enumerate G, with $\pi_0 = \iota$. Let $F \subseteq \omega_1$ be the set of those α such that $G_\alpha = \{\pi_\beta : \beta < \alpha\}$ is a subgroup of G. For $\alpha \in F$, set

$$\mathcal{S}_\alpha=\{(R,\star):R\in\mathcal{R}_\alpha,\,\star\in\maltese\text{ and }f_R(\beta)\star f_R(\gamma)=f_R(\delta)\\ \text{whenever }\beta,\,\gamma,\,\delta<\alpha\text{ and }\pi_\beta\pi_\gamma=\pi_\delta\};$$

so that if $(R,\star)\in\mathcal{S}_\alpha$, $f_R\restriction\alpha$ codes a group homomorphism from G_α to $(\mathbb{N},\star)$.

(d) For $\alpha\in F$, set

$$\mathcal{M}_\alpha=\{(R,\star,z):(R,\star)\in\mathcal{S}_\alpha,\,z\in\mathbb{R}^{\mathbb{N}}\}.$$

Then we have an action $\bullet'_\alpha$ of G_α on $\mathcal{M}_\alpha$ defined by saying that

$$\pi_\beta\bullet'_\alpha(R,\star,z)=(R,\star,f_R(\beta)\bullet^\star_r z)$$

whenever $(R,\star)\in\mathcal{S}_\alpha$, $z\in\mathbb{R}^{\mathbb{N}}$ and $\beta<\alpha$. **P** $\bullet'_\alpha$ is well-defined as a function on $G_\alpha\times\mathcal{M}_\alpha$ because $f_R(\beta)=f_R(\gamma)$ whenever $(R,\star)\in\mathcal{S}_\alpha$ and $\pi_\beta=\pi_\gamma$. If $\beta,\,\gamma,\,\delta<\alpha$ and $\pi_\delta=\pi_\beta\pi_\gamma$, then

$$\begin{aligned}\pi_\beta\bullet'_\alpha(\pi_\gamma\bullet'_\alpha(R,\star,z))&=\pi_\beta\bullet'_\alpha(R,\star,f_R(\gamma)\bullet^\star_r z)\\&=(R,\star,f_R(\beta)\bullet^\star_r(f_R(\gamma)\bullet^\star_r z))=(R,\star,f_R(\delta)\bullet^\star_r z)\end{aligned}$$

(because $(f_R(\beta),f_R(\gamma),f_R(\delta))\in\star$)

$$\begin{aligned}&=\pi_\delta\bullet'_\alpha(R,\star,z),\\ \iota\bullet'_\alpha(R,\star,z)&=\pi_0\bullet'_\alpha(R,\star,z)=(R,\star,f_R(0)\bullet^\star_r z)=(R,\star,z).\ \mathbf{Q}\end{aligned}$$

(e) If $\alpha,\,\beta\in F$ and $\alpha\le\beta$, then it is elementary to check that $\mathcal{R}_\beta\subseteq\mathcal{R}_\alpha$, $\mathcal{S}_\beta\subseteq\mathcal{S}_\alpha$, $\mathcal{M}_\beta\subseteq\mathcal{M}_\alpha$ and $\pi\bullet'_\beta(R,\star,z)=\pi\bullet'_\alpha(R,\star,z)$ whenever $(R,\star,z)\in\mathcal{M}_\beta$ and $\pi\in G_\alpha$. If $\beta\in F$ and $\beta=\sup(\beta\cap F)$, then $\mathcal{R}_\beta=\bigcap_{\alpha\in\beta\cap F}\mathcal{R}_\alpha$ and $\mathcal{M}_\beta=\bigcap_{\alpha\in\beta\cap F}\mathcal{M}_\alpha$.

(f)(i) For $\alpha<\omega_1$, $\mathcal{R}_\alpha$ belongs to the Borel σ-algebra $\mathcal{B}(\mathcal{P}\mathbb{N}^2)$ when $\mathcal{P}(\mathbb{N}^2)$ is given its usual compact Hausdorff topology (4A2Ud), and moreover $\{(R,m):R\subseteq\mathbb{N}^2,\,(\beta,m)\in f_R\}\in\mathcal{B}(\mathcal{P}\mathbb{N}^2\times\mathbb{N})$ whenever $m\in\mathbb{N}$ and $\beta<\alpha$. **P** Induce on α. We start with $\mathcal{R}_0=\mathcal{P}(\mathbb{N}^2)$. For the inductive step to a successor ordinal $\alpha+1$, given $R\subseteq\mathbb{N}^2$ and $m\in\mathbb{N}$, then

$$\begin{aligned}(\alpha,m)\in f_R\iff&R\in\mathcal{R}_\alpha\text{ and }m\text{ is the unique member of }\mathbb{N}\\&\qquad\text{such that }R[\{m\}]=f_R[\alpha]\\\iff&R\in\mathcal{R}_\alpha,\,R[\{m\}]\ne R[\{n\}]\text{ for every }n\ne m,\\&\qquad\forall\,i\in\mathbb{N},\,(m,i)\in R\iff\exists\,\beta<\alpha,\,(\beta,i)\in f_R.\end{aligned}$$

So $\{(R,m):(\alpha,m)\in f_R\}$ is

$$\begin{aligned}(\mathcal{R}_\alpha\times\mathbb{N})\cap\bigcap_{n\in\mathbb{N}}\bigcup_{i\in\mathbb{N}}&\{(R,m):n=m\text{ or }(m,i)\in R\,\&\,(n,i)\notin R\text{ or }(m,i)\notin R\,\&\,(n,i)\in R\}\\\cap\bigcap_{i\in\mathbb{N}}\bigcup_{\beta<\alpha}&\{(R,m):(m,i)\notin R\text{ or }(\beta,i)\in f_R\}\\\cap\bigcap_{i\in\mathbb{N}}\bigcap_{\beta<\alpha}&\{(R,m):(m,i)\in R\text{ or }(\beta,i)\notin f_R\},\end{aligned}$$

and is a Borel set. Now

$$\mathcal{R}_{\alpha+1}=\bigcup_{m\in\mathbb{N}}\{R:(\alpha,m)\in f_R\}\in\mathcal{B}(\mathcal{P}\mathbb{N}^2).$$

For the inductive step to a countable limit ordinal $\alpha>0$, $\mathcal{R}_\alpha=\bigcap_{\gamma<\alpha}\mathcal{R}_\gamma$. **Q**

(ii) $\maltese$ is a Borel subset of $\mathcal{P}(\mathbb{N}^3)$, so $\mathcal{S}_\alpha$ is a Borel subset of $\mathcal{P}(\mathbb{N}^2)\times\mathcal{P}(\mathbb{N}^3)$, and $\mathcal{M}_\alpha$ is a Borel subset of $\mathcal{P}(\mathbb{N}^2)\times\mathcal{P}(\mathbb{N}^3)\times\mathbb{R}^{\mathbb{N}}$, for every $\alpha\in F$. Next, $(R,\star,z)\mapsto\pi\bullet'_\alpha(R,\star,z):\mathcal{M}_\alpha\to\mathcal{M}_\alpha$ is Borel measurable whenever $\alpha\in F$ and $\pi\in G_\alpha$. **P** Let $\beta<\alpha$ be such that $\pi=\pi_\beta$. If $E_0\subseteq\mathcal{P}(\mathbb{N}^2)\times\mathcal{P}(\mathbb{N}^3)$ is a Borel set, and $E=\{(R,\star,z):(R,\star,z)\in\mathcal{M}_\alpha,\,(R,\star)\in E_0\}$, then

$$\{(R,\star,z):\pi\bullet'_\alpha(R,\star,z)\in E\}=E$$

is Borel; if $n\in\mathbb{N}$ and $E_1\subseteq\mathbb{R}$ is a Borel set, and $E=\{(R,\star,z):(R,\star,z)\in\mathcal{M}_\alpha,\,z(n)\in E_1\}$, then

$$\begin{aligned}\{(R,\star,z):\pi\bullet'_\alpha(R,\star,z)\in E\}&=\{(R,\star,z):(f_R(\beta)\bullet^\star_r z)(n)\in E_1\}\\&=\bigcup_{i,j\in\mathbb{N}}\{(R,\star,z):f_R(\beta)=i,\,(n,i,j)\in\star,\,z(j)\in E_1\}\end{aligned}$$

is Borel. Since $\mathcal{B}(\mathcal{P}\mathbb{N}^2\times\mathcal{P}\mathbb{N}^3\times\mathbb{R}^{\mathbb{N}})$ is the product σ-algebra

$$\mathcal{B}(\mathcal{P}\mathbb{N}^2)\widehat{\otimes}\mathcal{B}(\mathcal{P}\mathbb{N}^3)\widehat{\otimes}\widehat{\bigotimes}_{\mathbb{N}}\mathcal{B}(\mathbb{R}),$$

$(R,\star,z)\mapsto\pi\bullet'_\alpha(R,\star,z)$ is Borel measurable. **Q**

425D Törnquist's theorem (Törnquist 11) Let (X,Σ) be a standard Borel space and $\mathcal{I}$ a σ-ideal of Σ containing an uncountable set. Let $\mathfrak{A}$ be the quotient algebra $\Sigma/\mathcal{I}$, and $G\subseteq\operatorname{Aut}\mathfrak{A}$ a subgroup of cardinal at most ω_1. Then there is an action $\bullet$ of G on X which represents G in the sense that $\pi\bullet E=\{\pi\bullet x:x\in E\}$ belongs to Σ, and $(\pi\bullet E)^\bullet=\pi(E^\bullet)$, for every $E\in\Sigma$ and $\pi\in G$.

proof (a) It may help if I try to describe the line of argument I mean to follow. The important case is when G has an enumeration $\langle\pi_\alpha\rangle_{\alpha<\omega_1}$; let F be the set of those $\alpha<\omega_1$ such that $G_\alpha=\{\pi_\beta:\beta<\alpha\}$ is a subgroup of G. For each $\pi\in G$, choose $g_\pi:X\to X$ representing π. For $\alpha\in F$, set

$$Y_\alpha=\{x:x\in X,\,g_\phi(g_\pi(x))=g_{\pi\phi}(x)\in X\setminus M\text{ for all }\pi,\,\phi\in G_\alpha\},$$

where M is an uncountable member of $\mathcal{I}$. Choose $\langle\bullet_\alpha\rangle_{\alpha\in F}$ inductively such that $\bullet_\alpha$ is an action of G_α on X, $\pi\bullet_\alpha x=g_{\pi^{-1}}(x)$ whenever $\pi\in G_\alpha$ and $x\in Y_\alpha$, and $\pi\bullet_\alpha x=\pi\bullet_\beta x$ whenever $\beta<\alpha$, $\pi\in G_\beta$ and $x\in X$. At the end of the construction, set $\bullet=\bigcup_{\alpha\in F}\bullet_\alpha$.

It is straightforward to show that $X\setminus Y_\alpha$ always belongs to $\mathcal{I}$ (part (c) of the proof); consequently $\bullet$ will represent G. The non-trivial part of the proof is in the extension of a given action of G_α to an action of G_β where β is the next element of F above α, and this is where we shall need 425B-425C.

Now for the details.

(b) Give X a Polish topology for which Σ is the Borel σ-algebra $\mathcal{B}(X)$. For nearly the whole of the proof (down to the end of (g) below), suppose that $\#(G)=\omega_1$, that $X=\{0,1\}^{\mathbb{N}}$ and that $\Sigma=\mathcal{B}(X)$ is the Borel σ-algebra of X. Enumerate G as $\langle\pi_\alpha\rangle_{\alpha<\omega_1}$ starting with π_0 equal to the identity ι. We need to know that, setting $G_\alpha=\{\pi_\beta:\beta<\alpha\}$, the set $F=\{\alpha:\alpha<\omega_1,\,G_\alpha\text{ is a subgroup of }G\}$ is a closed cofinal subset of ω_1. **P** This is elementary. If $\alpha\in\overline{F}$, then $\{G_\beta:\beta\in F,\,\beta\le\alpha\}$ is a non-empty upwards-directed family of subgroups of G, so $G_\alpha=\bigcup_{\beta\in F,\beta\le\alpha}G_\beta$ is a subgroup of G, and $\alpha\in F$. If $\alpha<\omega_1$, let $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in ω_1 such that $\alpha_0=\max(1,\alpha)$ and, for each $n\in\mathbb{N}$,

— for all β, $\gamma<\alpha_n$ there is a $\delta<\alpha_{n+1}$ such that $\pi_\delta=\pi_\beta\pi_\gamma$,

— for every $\beta<\alpha_n$ there is a $\delta<\alpha_{n+1}$ such that $\pi_\delta=\pi_\beta^{-1}$.

Setting $\alpha^*=\sup_{n\in\mathbb{N}}\alpha_n$, we see that $\alpha\le\alpha^*\in F$. So F is cofinal with ω_1. **Q**

Of course $1=\min F$ and $G_1=\{\iota\}$, because we started with $\pi_0=\iota$.

(c) Next, for every $\pi\in\operatorname{Aut}\mathfrak{A}$, we can choose a Borel measurable $g_\pi:X\to X$ representing π in the sense that $\pi E^\bullet=g_\pi^{-1}[E]^\bullet$ for every $E\in\mathcal{B}(X)$ (425Ac). Of course when $\pi=\iota$ we take $g_\iota(x)=x$ for every $x\in X$. Fix an uncountable $M\in\mathcal{I}$, and for $\alpha\in F$ set

$$Y_\alpha=\{x:x\in X,\,g_\phi(g_\pi(x))=g_{\pi\phi}(x)\in X\setminus M\text{ for all }\pi,\,\phi\in G_\alpha\},$$

as declared in (a). If $\alpha\in F$ and $\pi\in G_\alpha$, $X\setminus Y_\alpha\in\mathcal{I}$ and $g_\pi\restriction Y_\alpha$ is a permutation of Y_α. **P** (Compare the proof of 344B.) There is a sequence $\langle E_k\rangle_{k\in\mathbb{N}}$ in $\mathcal{B}(X)$ separating the points of X. So, for any ψ, $\phi\in G_\alpha$, the set

$$\{x:g_\psi g_\phi(x)\ne g_{\phi\psi}(x)\}=\bigcup_{k\in\mathbb{N}}g_\phi^{-1}[g_\psi^{-1}[E_k]]\triangle g_{\phi\psi}^{-1}[E_k]$$

is Borel, and moreover, transferring the formulae to the quotient algebra,

$$\begin{aligned}\{x:g_\psi g_\phi(x)\ne g_{\phi\psi}(x)\}^\bullet&=\sup_{k\in\mathbb{N}}g_\phi^{-1}[g_\psi^{-1}[E_k]]^\bullet\triangle g_{\phi\psi}^{-1}[E_k]^\bullet\\&=\sup_{k\in\mathbb{N}}\phi(\psi E_k^\bullet)\triangle(\phi\psi)E_k^\bullet=0,\end{aligned}$$

so $\{x:g_\psi g_\phi(x)\ne g_{\phi\psi}(x)\}\in\mathcal{I}$. Accordingly

$$X \setminus Y_\alpha = \bigcup_{\psi,\phi\in G_\alpha}\{x : g_\psi g_\phi(x) \ne g_{\phi\psi}(x)\} \cup \bigcup_{\psi\in G_\alpha} g_\psi^{-1}[M]$$

belongs to $\mathcal{I}$.

If $\pi \in G_\alpha$ and $x \in Y_\alpha$, then

$$g_\psi g_\phi g_\pi(x) = g_\psi g_{\pi\phi}(x) = g_{\pi\phi\psi}(x) = g_{\phi\psi} g_\pi(x)$$

and $g_{\pi\phi\psi}(x) \notin M$, for all ϕ, $\psi \in G_\alpha$; so $g_\pi(x) \in Y_\alpha$. Similarly, $g_{\pi^{-1}}[Y_\alpha] \subseteq Y_\alpha$. Moreover,

$$g_\pi g_{\pi^{-1}}(x) = g_\iota(x) = x = g_{\pi^{-1}} g_\pi(x)$$

for every $x \in Y_\alpha$, so that $g_\pi \restriction Y_\alpha$ must be a permutation of Y_α. **Q**

(d) From the group G and the enumeration $\langle\pi_\alpha\rangle_{\alpha<\omega_1}$, construct ✠ and families $\langle f_R\rangle_{R\in\mathcal{P}\mathbb{N}^2}$, $\langle\mathcal{R}_\alpha\rangle_{\alpha<\omega_1}$, $\langle\bullet_r^\star\rangle_{\star\in✠}$ and $\langle(\mathcal{S}_\alpha, \mathcal{M}_\alpha, \bullet'_\alpha)\rangle_{\alpha\in F}$ as in 425C. Let h be a Borel isomorphism from $\mathbb{R}\times\mathbb{N}\times\mathcal{M}_1$ to M (424Da again). Fix a family $\langle t_\delta\rangle_{\delta\in F}$ of distinct members of $\mathbb{R}$, and set $J_\alpha = \mathbb{R}\setminus\{t_\delta : \delta < \alpha\}$ for $\alpha \in F$.

(e) (The key. Some readers may wish at this point to provide themselves with coffee and a large scratch-pad.) Let $\alpha < \beta < \gamma$ be members of F, and suppose that $\bullet_0$ is an action of G_α on X such that

$x \mapsto \pi\bullet_0 x$ is Borel measurable for every $\pi \in G_\alpha$,

$\pi\bullet_0 x = g_{\pi^{-1}}(x)$ whenever $x \in Y_\alpha$ and $\pi \in G_\alpha$,

$\pi\bullet_0 h(t, n, q) = h(t, n, \pi\bullet'_\beta q)$ whenever $t \in J_\alpha$, $n \in \mathbb{N}$, $q \in \mathcal{M}_\beta$ and $\pi \in G_\alpha$.

Then there is an action $\bullet_1$ of G_β on X such that

$x \mapsto \pi\bullet_1 x$ is Borel measurable for every $\pi \in G_\beta$,

$\pi\bullet_1 x = g_{\pi^{-1}}(x)$ whenever $x \in Y_\beta$ and $\pi \in G_\beta$,

$\pi\bullet_1 h(t, n, q) = h(t, n, \pi\bullet'_\gamma q)$ whenever $t \in J_\beta$, $n \in \mathbb{N}$, $q \in \mathcal{M}_\gamma$ and $\pi \in G_\beta$,

$\pi\bullet_1 x = \pi\bullet_0 x$ whenever $\pi \in G_\alpha$ and $x \in X$.

P Choose $\star \in$ ✠ such that there is an injective group homomorphism θ from G_β to $(\mathbb{N}, \star)$, and set $f'(\delta) = \theta(\pi_\delta)$ for $\delta < \beta$; let $R \in \mathcal{R}_\beta$ be such that $f_R\restriction\beta = f'$. (In the normal case, when $\beta \ge \omega$, we can choose f' first, as an arbitrary bijection from β to $\mathbb{N}$, and use this to define θ, $\star$ and R. If $\beta < \omega$, the first step is to take an injective group homomorphism from G_β to a countably infinite group, e.g., $G_\beta \times \mathbb{Z}$.) Then $(R, \star) \in \mathcal{S}_\beta$. Define $h_0 : \mathbb{N}\times\mathbb{R}^{\mathbb{N}} \to M$ by setting $h_0(n, z) = h(t_\alpha, n, (R, \star, z))$ for $n \in \mathbb{N}$ and $z \in \mathbb{R}^{\mathbb{N}}$. Then h_0 is injective and Borel measurable, and if $\pi \in G_\alpha$ then

$$h_0(n, \theta(\pi)\bullet_r^\star z) = h(t_\alpha, n, (R, \star, \theta(\pi)\bullet_r^\star z)) = h(t_\alpha, n, (R, \star, f_R(\delta)\bullet_r^\star z))$$

(where $\delta < \alpha$ is such that $\pi = \pi_\delta$)

$$= h(t_\alpha, n, \pi\bullet'_\alpha(R, \star, z)) = h(t_\alpha, n, \pi\bullet'_\beta(R, \star, z))$$
$$= \pi\bullet_0 h(t_\alpha, n, (R, \star, z)) = \pi\bullet_0 h_0(n, z)$$

for every $n \in \mathbb{N}$ and $z \in \mathbb{R}^{\mathbb{N}}$. Set

$$V = h[J_\beta\times\mathbb{N}\times\mathcal{M}_\gamma] \subseteq M \subseteq X\setminus Y_\beta, \quad X' = X\setminus(Y_\beta\cup V).$$

Note that if $\pi \in G_\alpha$, $t \in J_\beta$, $n \in \mathbb{N}$ and $q \in \mathcal{M}_\gamma$, then

$$\pi\bullet_0 h(t, n, q) = h(t, n, \pi\bullet'_\beta q) = h(t, n, \pi\bullet'_\gamma q) \in V;$$

thus V is invariant under the action $\bullet_0$. The same is true of Y_β, because $g_{\pi^{-1}}\restriction Y_\beta$ is a permutation of Y_β for every $\pi \in G_\alpha$, therefore also of X'. Note that as $t_\alpha \notin J_\beta$, $h_0[\mathbb{N}\times\mathbb{R}^{\mathbb{N}}] \subseteq X'$, while there is certainly a Borel measurable injection from $X'\setminus h_0[\mathbb{N}\times\mathbb{R}^{\mathbb{N}}]$ into $\mathbb{R}$. So 425Bb tells us that there is an action $\hat\bullet_1$ of G_β on X', extending $\bullet_0\restriction G_\alpha\times X'$, such that $x \mapsto \pi\hat\bullet_1 x : X' \to X'$ is Borel measurable for every $\pi \in G_\beta$.

We can therefore define $\bullet_1$ by setting

$$\begin{aligned}\pi\bullet_1 x &= g_{\pi^{-1}}(x) \text{ if } x \in Y_\beta,\\ &= h(t, n, \pi\bullet'_\gamma q) \text{ whenever } t \in J_\beta,\ n \in \mathbb{N},\ q \in \mathcal{M}_\gamma \text{ and } x = h(t, n, q),\\ &= \pi\hat\bullet_1 x \text{ if } x \in X'\end{aligned}$$

for every $\pi \in G_\beta$. It is easy to check that $\bullet_1$ is a function from $G_\beta \times X$ to X extending $\bullet_0$, and that $x \mapsto \pi\bullet_1 x$ is Borel measurable for every $\pi \in G_\beta$. If π, $\phi \in G_\beta$ and $x \in X$, then

$$\begin{aligned}\pi\bullet_1(\phi\bullet_1 x) &= \pi\bullet_1 g_{\phi^{-1}}(x) = g_{\pi^{-1}}g_{\phi^{-1}}(x) = g_{(\pi\phi)^{-1}}(x) = (\pi\phi)\bullet_1 x \text{ if } x \in Y_\beta,\\ &= \pi\bullet_1 h(t,n,\phi\bullet'_\gamma q) = h(t,n,\pi\bullet'_\gamma(\phi\bullet'_\gamma q)) = h(t,n,(\pi\phi)\bullet'_\gamma q)\\ &= (\pi\phi)\bullet_1 h(t,n,q) = (\pi\phi)\bullet_1 x\\ &\qquad\text{whenever } t \in J_\beta,\ n \in \mathbb{N},\ q \in \mathcal{M}_\gamma \text{ and } x = h(t,n,q),\\ &= \pi\hat{\bullet}_1(\phi\hat{\bullet}_1 x) = (\pi\phi)\hat{\bullet}_1 x = (\pi\phi)\bullet_1 x \text{ if } x \in X'.\end{aligned}$$

So $\bullet_1$ is an action of G_β on X, as required. **Q**

(f) Accordingly we can build $\langle\bullet_\alpha\rangle_{\alpha\in F}$ inductively, as follows. The inductive hypothesis will be that, for each $\alpha \in F$,

$\bullet_\alpha$ is an action of G_α on X,
$x \mapsto \pi\bullet_\alpha x$ is Borel measurable for every $\pi \in G_\alpha$,
$\pi\bullet_\alpha x = g_{\pi^{-1}}(x)$ whenever $x \in Y_\alpha$ and $\pi \in G_\alpha$,
$\pi\bullet_\alpha h(t,n,q) = h(t,n,\pi\bullet'_\beta q)$ whenever $t \in J_\alpha$, $n \in \mathbb{N}$, $q \in \mathcal{M}_\beta$, $\beta \in F$, $\beta > \alpha$ and $\pi \in G_\alpha$,
$\bullet_\delta = \bullet_\alpha\restriction G_\delta \times X$ whenever $\delta \in F \cap \alpha$.

The induction starts with $G_1 = \{\iota\}$ and $\iota\bullet_1 x = x$ for every $x \in X$.

Given $\alpha \in F$ and $\bullet_\alpha$, let β be the next element of F above α and γ the next element of F above β. By (d), we have an action $\bullet_\beta$ of G_β on X such that

$x \mapsto \pi\bullet_\beta x$ is Borel measurable for every $\pi \in G_\beta$,
$\pi\bullet_\beta x = g_{\pi^{-1}}(x)$ whenever $x \in Y_\beta$ and $\pi \in G_\beta$,
$\pi\bullet_\beta h(t,n,q) = h(t,n,\pi\bullet'_\gamma q)$ whenever $t \in J_\beta$, $n \in \mathbb{N}$, $q \in \mathcal{M}_\gamma$ and $\pi \in G_\alpha$,
$\bullet_\alpha = \bullet_\beta\restriction G_\alpha \times X$.

It follows at once that if $\delta \in F$ and $\delta \le \alpha$,

$$\pi\bullet_\delta x = \pi\bullet_\alpha x = \pi\bullet_\beta x$$

whenever $\pi \in G_\delta$ and $x \in X$; on the other side, if $\delta \in F$ and $\delta \ge \gamma$,

$$\pi\bullet_\beta h(t,n,q) = h(t,n,\pi\bullet'_\gamma q) = h(t,n,\pi\bullet'_\delta q)$$

whenever $t \in J_\beta$, $n \in \mathbb{N}$, $q \in \mathcal{M}_\delta$ and $\pi \in G_\beta$. So the induction continues to the next step.

If $\alpha \in F$ and $\alpha = \sup(F \cap \alpha)$, then $G_\alpha = \bigcup_{\beta\in F\cap\alpha} G_\beta$, so we have an action $\bullet_\alpha = \bigcup_{\beta\in F\cap\alpha}\bullet_\beta$ of G_α on X; and it is elementary to check that the inductive hypothesis is satisfied at the new level.

(g) At the end of the induction, $\bullet = \bigcup_{\alpha\in F}\bullet_\alpha$ will be an action of G on X such that $x \mapsto \pi\bullet x$ is Borel measurable for every $\pi \in G$. Moreover, $\pi E^\bullet = (\pi\bullet E)^\bullet$ for every Borel set $E \subseteq X$ and $\pi \in G$. **P** Let $\alpha \in F$ be such that $\pi \in G_\alpha$. Then

$$\pi(E^\bullet) = \pi((E \cap Y_\alpha)^\bullet)$$

(because $X \setminus Y_\alpha \in \mathcal{I}$)

$$= (g_\pi^{-1}[E \cap Y_\alpha])^\bullet = (Y_\alpha \cap g_\pi^{-1}[E \cap Y_\alpha])^\bullet = (g_{\pi^{-1}}[E \cap Y_\alpha])^\bullet$$

(because $g_\pi\restriction Y_\alpha$ is a permutation with inverse $g_{\pi^{-1}}\restriction Y_\alpha$)

$$= (\pi\bullet(E \cap Y_\alpha))^\bullet \subseteq (\pi\bullet E)^\bullet.$$

(Because $x \mapsto \pi\bullet x$ is a Borel measurable permutation of X, $\pi\bullet E$ is certainly a Borel set.) Since equally we must have

$$\pi((X \setminus E)^\bullet) \subseteq (\pi\bullet(X \setminus E))^\bullet,$$

while $\pi(E^\bullet) \cup \pi((X \setminus E)^\bullet) = 1_{\mathfrak{A}}$ and $(\pi\bullet E)^\bullet \cap (\pi\bullet(X \setminus E))^\bullet = 0_{\mathfrak{A}}$, both the inclusions here are equalities, and $\pi E^\bullet = (\pi\bullet E)^\bullet$. **Q**

(h) As for the elementary case in which G is countable, we can use arguments already presented, as follows. For each $\pi \in G$, choose g_π representing π. This time, go straight to $Y = \{x : g_{\pi\phi}(x) = g_\phi g_\pi(x) \text{ for all } \pi,\ \phi \in G\}$; as in (c), $X \setminus Y \in \mathcal{I}$ and $g_\pi\restriction Y$ is a permutation of Y for every $\pi \in G$. So if we set

$$\begin{aligned}\pi\bullet x &= g_{\pi^{-1}}(x) \text{ for } x \in Y,\\ &= x \text{ for } x \in X \setminus Y,\end{aligned}$$

$\bullet$ will be an appropriate action of G on X.

425E Scholium The theorem here applies to groups of cardinal at most ω_1. So it is worth noting that in the context of 425D the whole group $\mathrm{Aut}\,\mathfrak{A}$ has cardinal at most $\mathfrak{c}$. **P** Since Σ is countably σ-generated, there is a countable set $D \subseteq \mathfrak{A}$ countably σ-generating $\mathfrak{A}$. If π, $\phi \in \mathrm{Aut}\,\mathfrak{A}$ and $\pi\restriction D = \phi\restriction D$, then $\pi = \phi$; so $\#(\mathrm{Aut}\,\mathfrak{A}) \le \#(\mathfrak{A}^D)$. As $\#(\mathfrak{A}) \le \#(\Sigma) \le \mathfrak{c}$ (424Db), $\#(\mathrm{Aut}\,\mathfrak{A})$ is at most $\mathfrak{c}$ (4A1A(c-ii)). **Q**

We therefore have a corollary of 425D, as follows:

> Suppose the continuum hypothesis is true. Let (X, Σ) be a standard Borel space and $\mathcal{I}$ a σ-ideal of subsets of Σ containing an uncountable set. Then there is an action $\bullet$ of $\mathrm{Aut}(\Sigma/\mathcal{I})$ on X such that $\pi E^\bullet = (\pi\bullet E)^\bullet$ whenever $E \in \Sigma$ and $\pi \in \mathrm{Aut}(\Sigma/\mathcal{I})$.

425X Basic exercises (a) (Cf. 382Xc.) Let (X, Σ) be a standard Borel space, $\mathcal{I}$ a σ-ideal of subsets of X and $\mathfrak{A} = \Sigma/\mathcal{I}$ the quotient algebra. (i) Show that every member of $\mathrm{Aut}\,\mathfrak{A}$ has a separator (definition: 382Aa). (ii) Show that if G is a countably full subgroup of $\mathrm{Aut}\,\mathfrak{A}$, then every member of G is expressible as the product of at most three involutions belonging to G.

(b) Let (X, Σ) be a standard Borel space and set $\mathfrak{A} = \Sigma/[X]^{\le\omega}$. (i) Show that the Boolean algebra $\mathfrak{A}$ is homogeneous (definition: 316N). (ii) Show that $\mathrm{Aut}\,\mathfrak{A}$ is simple. (*Hint*: 382Yc.)

(c) Let (X, Σ) be a standard Borel space and $\mathcal{I}$ a σ-subalgebra of Σ with associated quotient algebra $\mathfrak{A} = \Sigma/\mathcal{I}$. Suppose that G is a countable semigroup of sequentially order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Show that there is a family $\langle f_\pi\rangle_{\pi\in G}$ of (Σ, Σ)-measurable functions from X to itself such that (α) $\pi E^\bullet = f_\pi^{-1}[E]^\bullet$ for every $\pi \in G$ and $E \in \Sigma$ (β) $f_{\pi\phi} = f_\phi f_\pi$ for all π, $\phi \in G$.

(d) Let X be a set, Σ a σ-algebra of subsets of X, $\mathcal{I}$ a σ-ideal of Σ and $\mathfrak{A}$ the quotient $\Sigma/\mathcal{I}$. Suppose that (X, Σ) is countably separated in the sense that there is a countable subset of Σ separating the points of X. Let G be a countable subsemigroup of the semigroup of Boolean homomorphisms from $\mathfrak{A}$ to itself such that for every $\pi \in G$ there is a (Σ, Σ)-measurable $g : X \to X$ such that $\pi E^\bullet = g^{-1}[E]^\bullet$ for every $E \in \Sigma$. Show that there is a family $\langle f_\pi\rangle_{\pi\in G}$ of (Σ, Σ)-measurable functions from X to itself such that $f_\pi^{-1}[E]^\bullet = \pi E^\bullet$ and $f_{\pi\phi} = f_\phi f_\pi$ whenever π, $\phi \in G$ and $E \in \Sigma$. (*Hint*: 344B.)

(e) Show that the set ✠ in 425Cb is a G_δ set in the compact metrizable space $\mathcal{P}(\mathbb{N}^3)$.

(f) Give an example of a standard Borel space (X, Σ), a σ-ideal $\mathcal{I}$ of Σ, a finite subgroup G of $\mathrm{Aut}(\Sigma/\mathcal{I})$, a subgroup H of G and an action $\bullet_0$ of H on X such that $\pi E^\bullet = (\pi\bullet_0 E)^\bullet$ whenever $\pi \in H$ and $x \in X$, but there is no action $\bullet$ of G on X, extending $\bullet_0$, such that $\pi E^\bullet = (\pi\bullet E)^\bullet$ whenever $\pi \in G$ and $x \in X$. (*Hint*: $\#(X) = 6$, $\#(H) = 2$.)

(g) Let $I^\parallel$ be the split interval with its usual topology and measure, and $\mathfrak{A}$ its measure algebra. Let G be a subgroup of $\mathrm{Aut}\,\mathfrak{A}$ of cardinal at most ω_1. Show that there is an action $\bullet$ of G on $I^\parallel$ such that $\pi\bullet E$ is a Borel set and $(\pi\bullet E)^\bullet = \pi E^\bullet$ for every $E \in \mathcal{B}(I^\parallel)$ and $\pi \in G$.

425Y Further exercises (a) (Törnquist 11) Let X be a set, Σ a σ-algebra of subsets of X, $\mathcal{I}$ a σ-ideal of Σ and $\mathfrak{A}$ the quotient $\Sigma/\mathcal{I}$. Suppose that $(X, \Sigma, \mathcal{I})$ is countably separated in the sense that there is a countable subset of Σ separating the points of X, and complete in the sense that $A \in \mathcal{I}$ whenever $A \subseteq B \in \mathcal{I}$. Let $G \subseteq \mathrm{Aut}\,\mathfrak{A}$ be a group of size at most ω_1 such that for every $\pi \in G$ there is a (Σ, Σ)-measurable function $g : X \to X$ such that $\pi E^\bullet = g^{-1}[E]^\bullet$ for every $E \in \Sigma$. Show that if $\mathcal{I}$ contains a set of cardinal $\mathfrak{c}$ there is an action $\bullet$ of G on X such that $\pi E^\bullet = (\pi\bullet E)^\bullet$ for every $\pi \in G$ and $E \in \Sigma$.

(b) Let (X, Σ, μ) be a countably separated perfect complete strictly localizable measure space, $\mathfrak{A}$ its measure algebra and G a subgroup of $\mathrm{Aut}\,\mathfrak{A}$ of cardinal at most ω_1. Show that there is an action $\bullet$ of G on X such that $\pi\bullet E \in \Sigma$ and $(\pi\bullet E)^\bullet = \pi(E^\bullet)$ whenever $\pi \in G$ and $E \in \Sigma$.

425Z Problems (a) Suppose that (X, Σ) is an uncountable standard Borel space and $\mathcal{I}$ the ideal $[X]^{\le\omega}$ of countable subsets of X. Which subgroups G of $\mathrm{Aut}(\Sigma/\mathcal{I})$ can be represented by actions of G on X?

(b) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of Lebesgue measure μ on $[0, 1]$, and G a semigroup of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself with $\#(G) = \omega_1$. Must there be a family $\langle f_\pi\rangle_{\pi\in G}$ of inverse-measure-preserving functions from $[0, 1]$ to itself such that $f_{\pi\phi} = f_\phi f_\pi$ for all π, $\phi \in G$ and f_π represents π, in the sense of 425A, for every $\pi \in G$? (See 344B.)

(c) Let $(\mathfrak{B}_{\omega_1},\bar\nu_{\omega_1})$ be the measure algebra of the usual measure on $\{0,1\}^{\omega_1}$, and G a group of measure-preserving automorphisms of $\mathfrak{B}_{\omega_1}$ with $\#(G)=\omega_1$. Must there be a family $\langle f_\pi\rangle_{\pi\in G}$ of inverse-measure-preserving functions from $\{0,1\}^{\omega_1}$ to itself such that $f_{\pi\phi}=f_\phi f_\pi$ for all π, $\phi\in G$ and f_π represents π for every $\pi\in G$? (See 344E.)

425 Notes and comments I have starred this section because (apart from 425A) it deals with a very special topic. From the point of view of measure theory, 425D applies only to copies of Borel measures on $\mathbb{R}$ (and not quite all of those), and the limitation to groups of cardinal ω_1 means that we need to assume the continuum hypothesis, or at least $\mathfrak{p}=\mathfrak{c}$ (535Yd[1]), to get a theorem we really want. However the result connects naturally with an important theme from Chapter 34, and the general question of simultaneous representation of many automorphisms has significant implications for the ergodic theory treated in Chapter 38 and §494 of this volume.

What makes 425D difficult is the ambitious target: we want to represent the automorphisms in G by a consistent family of Borel measurable functions. We know from Chapter 34 that we can hope to handle countable groups, so it is natural to start by expressing G as an inductive limit of a family $\langle G_\alpha\rangle_{\alpha\in F}$, as in parts (a)-(b) of the proof of 425D, and to try to define the action of G from actions of the G_α. Since any $\pi\in G$ must eventually determine a Borel measurable function on X, we are going to have to freeze its action at some point; we could afford to change it once or twice, or even countably often, as the induction continued, but sometime we must stop tinkering, and really we want to have $\pi{\bullet}x=\pi{\bullet}_\alpha x$ whenever $x\in X$ and $\pi\in G_\alpha$. In this case, ${\bullet}_\beta$ will have to be a direct extension of ${\bullet}_\alpha$ whenever $\beta>\alpha$. We are in a context in which arbitrary actions are not always extensible (425Xf), and something like Lemma 425B is going to be needed. This demands a plentiful supply of copies of shift actions, at the very least including representations of all the shift actions on X^{G_α}, which will have to be built in from the very beginning. The trouble is that these have to be assembled in a way which will give us Borel measurable functions. Now while X can be taken to be a fixed Polish space, the groups G_α are more or less arbitrary countable groups. They can all be represented by group structures on $\mathbb{N}$, but if we go by that road, we seem to need to choose injective functions from countable ordinals into $\mathbb{N}$. (Of course each G_α comes with a bijection between it and the ordinal α.) No direct enumeration of these is going to lead to Borel structures. Instead, we have to look at the whole set of group structures on $\mathbb{N}$, the set ✠ of 425Cb, and nearly all injective functions from countable ordinals to $\mathbb{N}$, coded by subsets of $\mathbb{N}^2$, as in 425Ca. Fortunately it does not matter that there is a great deal of redundancy in this coding; it can all be fitted naturally into a standard Borel structure, and we just need to include, in the hypotheses of 425D, a negligible set of size $\mathfrak{c}$. When $\mathcal{I}\subseteq[X]^{\le\omega}$, the problem changes (425Za).

In order to ensure that each of the actions ${\bullet}_\alpha$ correctly represents the action of G_α on $\mathfrak{A}$, we can use essentially the same method as that of 344B (part (c) of the proof of 425D). This means, of course, that 425B has to be applied to a carefully chosen fragment of X, the set X' of part (d) of the proof of 425D. There is an awkward shift here between the representation of an automorphism π by the function g_π, where I follow the conventions used in Chapter 34 and 425A here with the contravariant formula $\pi(E^\bullet)=(g_\pi^{-1}[E])^\bullet$, and the representation in the statement of this theorem, with the covariant formula $\pi E^\bullet=(\pi{\bullet}E)^\bullet$. The latter is forced by the rule of 4A5Ba that $(\pi\phi){\bullet}x=\pi{\bullet}(\phi{\bullet}x)$. By allowing 'reverse actions', in which $(\pi\phi){\bullet}x=\phi{\bullet}(\pi{\bullet}x)$, we could escape this conflict, and in the formulae of parts (d)-(e) of the proof of 425D we should have $\pi{\bullet}_\alpha x=g_\pi(x)$ for $x\in Y_\alpha$ and $\pi\in G_\alpha$. This would be essential if we wanted to use the formulae here on semigroups of Boolean homomorphisms, as in §344, so that the g_π were no longer injective on conegligible sets. But there seem to be more substantial obstacles (425Zb).

The proof of 425D appeals repeatedly to the special properties of standard Borel spaces. But conceivably enough of it can be applied to the Baire σ-algebras of powers of $\{0,1\}$ to give a similar result for other important probability spaces (425Zc). If we change the rules, and assume that $\mathcal{I}$ is an ideal of $\mathcal{P}X$ as well as of Σ, we can dispense with the ideas of 425C and work directly from 425Ba, using copies of $\mathbb{N}\times X^{G_\alpha}$ inside a negligible set M (425Ya); this gives us an approach to use on complete measure spaces (425Yb, 535Yd).

Another way of looking at 425D is to think of it as a kind of lifting theorem. Let Φ be the group of (Σ,Σ)-bimeasurable $\mathcal{I}$-invariant permutations of X. Then each $f\in\Phi$ induces an automorphism f° of $\mathfrak{A}$ defined by saying that $f^\circ(E^\bullet)=(f[E])^\bullet$ for each $E\in\Sigma$. (I am using the push-forward rather than the pull-back representation here.) 425Ac is enough to show that the group homomorphism $f\mapsto f^\circ:\Phi\to\operatorname{Aut}\mathfrak{A}$ is surjective. Under the conditions of 425D, the action $\bullet$ corresponds to a group homomorphism $\theta:G\to\Phi$ such that $(\theta\pi)^\circ=\pi$ for every $\pi\in G$; and subject to the continuum hypothesis, we have a lifting for the whole of $\operatorname{Aut}\mathfrak{A}$. The word 'lifting' in this context should remind you of the Lifting Theorem of measure theory (341K). That theorem demands a complete measure space, and does not ordinarily apply to Borel measures. However, subject to the continuum hypothesis, there is a lifting theorem applicable to a variety of non-complete measure spaces, including any (X,Σ,μ) where (X,Σ) is a standard Borel space and μ is σ-finite (535E(b-i) of Volume 5). I am not sure that it is really helpful to think of

[1] Later editions only.

425E and 535E together; certainly the manoeuvres of 425C have no analogues in the lifting theorems of measure theory. But the correspondence is striking and suggests directions of enquiry which may be worth exploring.

Chapter 43

Topologies and measures II

The first chapter of this volume was 'general' theory of topological measure spaces; I attempted to distinguish the most important properties a topological measure can have – inner regularity, τ-additivity – and describe their interactions at an abstract level. I now turn to rather more specialized investigations, looking for features which offer explanations of the behaviour of the most important spaces, radiating outwards from Lebesgue measure.

In effect, this chapter consists of three distinguishable parts and two appendices. The first three sections are based on ideas from descriptive set theory, in particular Souslin's operation (§431); the properties of this operation are the foundation for the theory of two classes of topological space of particular importance in measure theory, the K-analytic spaces (§432) and the analytic spaces (§433). The second part of the chapter, §§434-435, collects miscellaneous results on Borel and Baire measures, looking at the ways in which topological properties of a space determine properties of the measures it carries. In §436 I present the most important theorems on the representation of linear functionals by integrals; if you like, this is the inverse operation to the construction of integrals from measures in §122. The ideas continue into §437, where I discuss spaces of signed measures representing the duals of spaces of continuous functions, and topologies on spaces of measures. The first appendix, §438, looks at a special topic: the way in which the patterns in §§434-435 are affected if we assume that our spaces are not unreasonably complex in a rather special sense defined in terms of measures on discrete spaces. Finally, I end the chapter with a further collection of examples, mostly to exhibit boundaries to the theorems of the chapter, but also to show some of the variety of the structures we are dealing with.

431 Souslin's operation

I begin the chapter with a short section on Souslin's operation (§421). The basic facts we need to know are that (in a complete locally determined measure space) the family of measurable sets is closed under Souslin's operation (431A), and that the kernel of a Souslin scheme can be approximated from within in measure (431D).

431A Theorem Let (X,Σ,μ) be a complete locally determined measure space. Then Σ is closed under Souslin's operation.

proof (I follow the notation of 421A-421B.) Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in Σ with kernel A. Write $S=\bigcup_{k\in\mathbb{N}}\mathbb{N}^k$. If $F\in\Sigma$ and $\mu F<\infty$, then $A\cap F\in\Sigma$. **P** For each $\sigma\in S$, set

$$A_\sigma=\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}},\phi\supseteq\sigma}\bigcap_{n\ge1}E_{\phi\restriction n},$$

and let G_σ be a measurable envelope of $A_\sigma\cap F$. Because $A_\sigma\subseteq E_\sigma$ (writing $E_\emptyset=X$), we may suppose that $G_\sigma\subseteq E_\sigma\cap F$. Now, for any $\sigma\in S$,

$$A_\sigma\cap F=\bigcup_{i\in\mathbb{N}}A_{\sigma^\frown<i>}\cap F\subseteq\bigcup_{i\in\mathbb{N}}G_{\sigma^\frown<i>},$$

so

$$H_\sigma=G_\sigma\setminus\bigcup_{i\in\mathbb{N}}G_{\sigma^\frown<i>}$$

is negligible.

Set $H=\bigcup_{\sigma\in S}H_\sigma$, so that H is negligible. Take any $x\in G_\emptyset\setminus H$. Choose $\langle\phi(i)\rangle_{i\in\mathbb{N}}$ inductively, as follows. Given that $\sigma=\langle\phi(i)\rangle_{i<k}$ has been chosen and $x\in G_\sigma$, then $x\notin H_\sigma$, so there must be some $j\in\mathbb{N}$ such that $x\in G_{\sigma^\frown<j>}$; set $\phi(k)=j$, and continue. Now

$$x\in\bigcap_{k\ge1}G_{\phi\restriction k}\subseteq\bigcap_{k\ge1}E_{\phi\restriction k}\subseteq A.$$

Thus we see that $G_\emptyset\setminus H\subseteq A$; as $G_\emptyset\subseteq F$, $G_\emptyset\setminus H\subseteq A\cap F$. On the other hand, $A\cap F\subseteq G_\emptyset$. Because H is negligible and μ is complete, $A\cap F\in\Sigma$. **Q**

Because μ is locally determined, it follows that $A\in\Sigma$. As $\langle E_\sigma\rangle_{\sigma\in S^*}$ is arbitrary, Σ is closed under Souslin's operation.

431B Corollary If $(X,\mathfrak{T},\Sigma,\mu)$ is a complete locally determined topological measure space, every Souslin-F set in X (definition: 421K) is measurable.

431C Corollary Let X be a set and θ an outer measure on X. Let μ be the measure defined by Carathéodory's method, and Σ its domain. Then Σ is closed under Souslin's operation.

proof Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in Σ with kernel A. Take any $C\subseteq X$ such that $\theta C<\infty$. Then $\theta_C=\theta\restriction\mathcal{P}C$ is an outer measure on C; let μ_C be the measure on C defined from θ_C by Carathéodory's method, and Σ_C its domain. If $\sigma\in S^*$ and $D\subseteq C$ then

$$\begin{aligned}\theta_C(D\cap C\cap E_\sigma)+\theta_C(D\setminus(C\cap E_\sigma))&=\theta(D\cap E_\sigma)+\theta(D\setminus E_\sigma)\\&=\theta D=\theta_C D;\end{aligned}$$

as D is arbitrary, $C\cap E_\sigma\in\Sigma_C$. μ_C is a complete totally finite measure, so 431A tells us that the kernel of the Souslin scheme $\langle C\cap E_\sigma\rangle_{\sigma\in S^*}$ belongs to Σ_C. But this is just $C\cap A$ (applying 421Cb to the identity map from C to X). So

$$\theta(C\cap A)+\theta(C\setminus A)=\theta_C(C\cap A)+\theta_C(C\setminus A)=\theta_C C=\theta C.$$

As C is arbitrary, $A\in\Sigma$ (113D). As $\langle E_\sigma\rangle_{\sigma\in S^*}$ is arbitrary, we have the result.

431D Theorem Let (X,Σ,μ) be a complete locally determined measure space, and $\langle E_\sigma\rangle_{\sigma\in S^*}$ a Souslin scheme in Σ with kernel A. Then

$$\begin{aligned}\mu A&=\sup\{\mu(\bigcup_{\phi\in K}\bigcap_{n\ge1}E_{\phi\restriction n}):K\subseteq\mathbb{N}^{\mathbb{N}}\text{ is compact}\}\\&=\sup\{\mu(\bigcup_{\phi\le\psi}\bigcap_{n\ge1}E_{\phi\restriction n}):\psi\in\mathbb{N}^{\mathbb{N}}\},\end{aligned}$$

writing $\phi\le\psi$ if $\phi(i)\le\psi(i)$ for every $i\in\mathbb{N}$.

proof (a) By 431A, A is measurable. For $K\subseteq\mathbb{N}^{\mathbb{N}}$, set $H_K=\bigcup_{\phi\in K}\bigcap_{n\ge1}E_{\phi\restriction n}$. Of course $H_K\subseteq A$, and we know from 421M (or otherwise) that $H_K\in\Sigma$ if K is compact. So surely $\mu A\ge\mu H_K$ for every compact $K\subseteq\mathbb{N}^{\mathbb{N}}$. If $\psi\in\mathbb{N}^{\mathbb{N}}$, then $\{\phi:\phi\le\psi\}=\prod_{i\in\mathbb{N}}(\psi(i)+1)$ is compact. We therefore have

$$\begin{aligned}\mu A&\ge\sup\{\mu(\bigcup_{\phi\in K}\bigcap_{n\ge1}E_{\phi\restriction n}):K\subseteq\mathbb{N}^{\mathbb{N}}\text{ is compact}\}\\&\ge\sup\{\mu(\bigcup_{\phi\le\psi}\bigcap_{n\ge1}E_{\phi\restriction n}):\psi\in\mathbb{N}^{\mathbb{N}}\}.\end{aligned}$$

So what I need to prove is that

$$\mu A\le\sup\{\mu(\textstyle\bigcup_{\phi\le\psi}\bigcap_{n\ge1}E_{\phi\restriction n}):\psi\in\mathbb{N}^{\mathbb{N}}\}.$$

(b) Fix on a set $F\in\Sigma$ of finite measure. For $\sigma\in S=\bigcup_{k\in\mathbb{N}}\mathbb{N}^k$ set

$$A_\sigma=\textstyle\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}},\phi\supseteq\sigma}\bigcap_{n\ge1}E_{\phi\restriction n}.$$

We need to know that A_σ belongs to Σ; this follows from 431A, because writing $E'_\tau=E_\tau$ if $\tau\subseteq\sigma$ or $\sigma\subseteq\tau$, $\emptyset$ otherwise,

$$A_\sigma=\textstyle\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}}\bigcap_{n\ge1}E'_{\phi\restriction n}\in\mathcal{S}(\Sigma)=\Sigma,$$

writing $\mathcal{S}$ for Souslin's operation, as in §421.

Let $\epsilon>0$, and take a family $\langle\epsilon_\sigma\rangle_{\sigma\in S}$ of strictly positive real numbers such that $\sum_{\sigma\in S}\epsilon_\sigma\le\epsilon$. For each $\sigma\in S$ we have $A_\sigma=\bigcup_{i\in\mathbb{N}}A_{\sigma^\frown<i>}$, so there is an $m_\sigma\in\mathbb{N}$ such that

$$\mu(F\cap A_\sigma\setminus\textstyle\bigcup_{i\le m_\sigma}A_{\sigma^\frown<i>})\le\epsilon_\sigma.$$

Define $\psi\in\mathbb{N}^{\mathbb{N}}$ by saying that

$$\psi(k)=\max\{m_\sigma:\sigma\in\mathbb{N}^k,\ \sigma(i)\le\psi(i)\text{ for every }i<k\}$$

for each $k\in\mathbb{N}$. Set

$$H=\textstyle\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}},\phi\le\psi}\bigcap_{n\ge1}E_{\phi\restriction n}.$$

(c) Set

$$G=\bigcup_{\sigma\in S}F\cap A_\sigma\setminus\bigcup_{i\le m_\sigma}A_{\sigma^\frown<i>},$$

so that $\mu G\le\epsilon$, by the choice of the ϵ_σ and the m_σ. Then $F\cap A\setminus G\subseteq H$. **P** If $x\in F\cap A\setminus G$, choose $\langle\phi(i)\rangle_{i\in\mathbb{N}}$ inductively, as follows. Given that $\phi(i)\le\psi(i)$ for $i<k$ and $x\in A_\sigma$, where $\sigma=\langle\phi(i)\rangle_{i<k}$, then $x\notin A_\sigma\setminus\bigcup_{j\le m_\sigma}A_{\sigma^\frown<j>}$, so there must be some $j\le m_\sigma$ such that $x\in A_{\sigma^\frown<j>}$; set $\phi(k)=j$; because $\sigma\in\prod_{i<k}(\psi(i)+1)$, $j\le m_\sigma\le\psi(k)$, and the induction continues. At the end of the induction, $\phi\le\psi$ and

$$x\in\bigcap_{n\ge1}A_{\phi\restriction n}\subseteq\bigcap_{n\ge1}E_{\phi\restriction n}\subseteq H.\ \mathbf{Q}$$

(d) It follows that

$$\mu(A\cap F)\le\mu G+\mu H\le\epsilon+\mu H.$$

As F and ϵ are arbitrary, and μ is semi-finite,

$$\mu A\le\sup\{\mu(\bigcup_{\phi\le\psi}\bigcap_{n\ge1}E_{\phi\restriction n}):\psi\in\mathbb{N}^{\mathbb{N}}\},$$

and the proof is complete.

431E Corollary If $(X,\mathfrak{T},\Sigma,\mu)$ is a topological measure space and $E\subseteq X$ is a Souslin-F set with finite outer measure, then $\mu^*E=\sup\{\mu F:F\subseteq E\text{ is closed}\}$.

proof Let $\tilde\mu$ be the c.l.d. version of μ (213E). Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme of closed sets with kernel E. Then 213Fb and 431D tell us that

$$\mu^*E=\tilde\mu E=\sup_{K\subseteq\mathbb{N}^{\mathbb{N}}\text{ is compact}}\mu F_K,$$

where $F_K=\bigcup_{\phi\in K}\bigcap_{n\ge1}E_{\phi\restriction n}$ for $K\subseteq\mathbb{N}^{\mathbb{N}}$. But every F_K is closed, by 421M. So $\mu^*E\le\sup_{F\subseteq E\text{ is closed}}\mu F$; as the reverse inequality is trivial, we have the result.

***431F** Two further versions of the ideas in 431A will be useful. The first is topological.

Theorem Let X be any topological space, and $\widehat{\mathcal{B}}$ its Baire-property algebra.

(a) For any $A\subseteq X$, there is a Baire-property envelope of A, that is, a set $E\in\widehat{\mathcal{B}}$ such that $A\subseteq E$ and $E\setminus F$ is meager whenever $A\subseteq F\in\widehat{\mathcal{B}}$.

(b) $\widehat{\mathcal{B}}$ is closed under Souslin's operation.

proof (a) By 4A3Ra, there is an open set $H\subseteq X$ such that $A\setminus H$ is meager and $H\cap G$ is empty whenever $G\subseteq X$ is open and $A\cap G$ is meager. Set $E=A\cup H$; then $E\supseteq A$ and $E\triangle H=A\setminus H$ is meager, so $E\in\widehat{\mathcal{B}}$. If $A\subseteq F\in\widehat{\mathcal{B}}$, let G be an open set such that $G\triangle(X\setminus F)$ is meager. Then $G\cap A\subseteq G\cap F$ is meager, so $G\cap H$ is empty and $E\setminus F\subseteq(E\triangle H)\cup(G\triangle(X\setminus F))$ is meager. Thus E is a Baire-property envelope of A.

(b) Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\widehat{\mathcal{B}}$ with kernel A. Write

$$S=\bigcup_{k\in\mathbb{N}}\mathbb{N}^k=S^*\cup\{\emptyset\}.$$

For each $\sigma\in S$, set

$$A_\sigma=\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}},\phi\supseteq\sigma}\bigcap_{n\ge1}E_{\phi\restriction n},$$

and let G_σ be a Baire-property envelope of A_σ as described in (a). Because $A_\sigma\subseteq E_\sigma$ (writing $E_\emptyset=X$), we may suppose that $G_\sigma\subseteq E_\sigma$. Now, for any $\sigma\in S$,

$$A_\sigma=\bigcup_{i\in\mathbb{N}}A_{\sigma^\frown<i>}\subseteq\bigcup_{i\in\mathbb{N}}G_{\sigma^\frown<i>},$$

so

$$H_\sigma=G_\sigma\setminus\bigcup_{i\in\mathbb{N}}G_{\sigma^\frown<i>}$$

is meager.

Set $H=\bigcup_{\sigma\in S}H_\sigma$, so that H is meager. Take any $x\in G_\emptyset\setminus H$. Choose $\langle\phi(i)\rangle_{i\in\mathbb{N}}$ inductively, as follows. Given that $\sigma=\langle\phi(i)\rangle_{i<k}$ has been chosen and $x\in G_\sigma$, then $x\notin H_\sigma$, so there must be some $j\in\mathbb{N}$ such that $x\in G_{\sigma^\frown<j>}$; set $\phi(k)=j$, and continue. Now

$$x\in\bigcap_{k\ge1}G_{\phi\restriction k}\subseteq\bigcap_{k\ge1}E_{\phi\restriction k}\subseteq A.$$

Thus we see that $G_\emptyset \setminus H \subseteq A$. On the other hand, $A \subseteq G_\emptyset$, so $G_\emptyset \triangle A$ is meager and $A \in \widehat{\mathcal{B}}$. As $\langle E_\sigma \rangle_{\sigma \in S^*}$ is arbitrary, $\widehat{\mathcal{B}}$ is closed under Souslin's operation.

***431G** The second relies on a countable chain condition to give the same envelope property.

Theorem Let X be a set, Σ a σ-algebra of subsets of X and $\mathcal{I} \subseteq \Sigma$ a σ-ideal of subsets of X. If $\Sigma/\mathcal{I}$ is ccc then Σ is closed under Souslin's operation.

proof (a) As before, the essential fact is that for every $A \subseteq X$ there is an $E \in \Sigma$ such that $A \subseteq E$ and $F \in \mathcal{I}$ whenever $F \in \Sigma$ and $F \subseteq E \setminus A$. **P** Let $\mathcal{E}$ be a maximal disjoint family of members of $\Sigma \setminus \mathcal{I}$ disjoint from A. Because $\Sigma/\mathcal{I}$ is ccc, $\mathcal{E}$ is countable (316C), so $E = X \setminus \bigcup \mathcal{E}$ belongs to Σ; now it is easy to see that this E serves. **Q**

In this case I will call E a 'measurable envelope' of A.

(b) Now we can argue as in 431A or 431F. Let $\langle E_\sigma \rangle_{\sigma \in S^*}$ be a Souslin scheme in Σ with kernel A; for $\sigma \in S$, set

$$A_\sigma = \bigcup_{\phi \in \mathbb{N}^{\mathbb{N}}, \sigma \subseteq \phi} \bigcap_{n \ge 1} E_{\phi \restriction n},$$

and let $G_\sigma \subseteq E_\sigma$ be a measurable envelope of A_σ. Setting

$$H = \bigcup_{\sigma \in S} (G_\sigma \setminus \bigcup_{i \in \mathbb{N}} G_{\sigma^\frown<i>}),$$

$H \in \mathcal{I}$ and $G_\emptyset \triangle A \subseteq H$, so $A \in \Sigma$. As $\langle E_\sigma \rangle_{\sigma \in S^*}$ is arbitrary, Σ is closed under Souslin's operation.

431X Basic exercises (a) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined topological measure space, Y a topological space, and $f : X \to Y$ a measurable function. Let $\mathcal{B}(Y)$ be the Borel algebra of Y. Show that $f^{-1}[B] \in \Sigma$ for every $B \in \mathcal{S}(\mathcal{B}(Y))$.

(b) Let X be a topological space and μ a semi-finite topological measure on X which is inner regular with respect to the Souslin-F sets. Show that μ is inner regular with respect to the closed sets.

(c) Let (X, Σ, μ) be a complete locally determined measure space, and $\langle E_\sigma \rangle_{\sigma \in S^*}$ a regular Souslin scheme in Σ (definition: 421Xm) with kernel A. Show that

$$\mu A = \sup_{\phi \in \mathbb{N}^{\mathbb{N}}} \mu(\bigcap_{n \ge 1} E_{\phi \restriction n}).$$

>(d) Let (X, Σ, μ) be a measure space with locally determined negligible sets (definition: 213I), and $\langle E_\sigma \rangle_{\sigma \in S^*}$ a Souslin scheme in Σ with kernel A. Show that

$$\mu^* A = \sup_{\psi \in \mathbb{N}^{\mathbb{N}}} \mu(\bigcup_{\phi \le \psi} \bigcap_{n \ge 1} E_{\phi \restriction n}).$$

>(e) Let (X, Σ, μ) be a semi-finite measure space, and $\langle E_\sigma \rangle_{\sigma \in S^*}$ a Souslin scheme in Σ with kernel A. Show that

$$\mu_* A = \sup_{\psi \in \mathbb{N}^{\mathbb{N}}} \mu(\bigcup_{\phi \le \psi} \bigcap_{n \ge 1} E_{\phi \restriction n}).$$

431Y Further exercises (a) Let (X, Σ, μ) be a complete measure space with the measurable envelope property (213Xl). Show that Σ is closed under Souslin's operation.

(b) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X such that $\mathcal{I} \subseteq \Sigma$. Suppose that for every $A \subseteq X$ there is an $F \in \Sigma$ such that $A \subseteq F$ and $F \setminus E \in \mathcal{I}$ whenever $A \subseteq E \in \Sigma$. Show that Σ is closed under Souslin's operation.

(c) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ an ω_1-saturated σ-ideal of Σ; suppose that $A \in \Sigma$ whenever $A \subseteq B \in \mathcal{I}$. Show that Σ is closed under Souslin's operation.

431 Notes and comments From the point of view of measure theory, the most important property of Souslin's operation, after its idempotence, is the fact that (for many measure spaces) the family of measurable sets is closed under the operation (431A). The proof I give here is based on the concept of measurable envelope, which can be used in other cases of great interest (431Yb, 431Yc). But for some applications it is also very important to know that if A is the kernel of a Souslin scheme $\langle E_\sigma \rangle_{\sigma \in S^*}$, then A can be approximated from inside by sets of the form $H = \bigcup_{\phi \le \psi} \bigcap_{n \ge 1} E_{\phi \restriction n}$ (431D, 431Xd), which belong to the σ-algebra generated by the E_σ (421M). A typical application of this idea is when every E_σ is a Borel subset of $\mathbb{R}$; then we find not only that A is Lebesgue measurable (indeed, measured by every Radon measure on $\mathbb{R}$) but that (for any given Radon measure μ) the Souslin scheme itself provides Borel subsets H of A of measure approximating the measure of A.

Let me repeat that the essence of descriptive set theory is that we are not satisfied merely to know that a set of a certain type exists. We want also to know how to build it, because we expect that an explicit construction will be valuable later on. For instance, the construction given in 431D shows that if the Souslin scheme consists of closed compact sets, the sets H will be compact (421Xn).

I mention 431B as a typical application of 431A, even though it is both obvious and obviously less than what can be said. The algebras Σ of this section are algebras closed under Souslin's operation. In a complete locally determined topological measure space, the algebra Σ of measurable sets includes the open sets (by definition), therefore the Borel algebra $\mathcal{B}$, therefore $\mathcal{S}(\mathcal{B})$; but now we can take the algebra $\mathcal{A}_1$ generated by $\mathcal{S}(\mathcal{B})$, and $\mathcal{A}_1$ and $\mathcal{S}(\mathcal{A}_1)$ will also be included in Σ, so that Σ will included the algebra $\mathcal{A}_2$ generated by $\mathcal{S}(\mathcal{A}_1)$, and so on. (Note that $\mathcal{S}(\mathcal{A}_1)$ includes the σ-algebra generated by $\mathcal{A}_1$, by 421F, so I do not need to mention that separately.) We have to run through all the countable ordinals before we can be sure of getting to the smallest algebra $\mathcal{A}_{\omega_1} = \bigcup_{\xi<\omega_1} \mathcal{A}_\xi$ which contains every open set and is closed under Souslin's operation, and we shall then have $\mathcal{A}_{\omega_1} \subseteq \Sigma$.

The result in 431D is one of the special features of measures. (A similar result, based on rather different hypotheses, is in 432K.) But the argument of 431A can be applied in many other cases; see 431Yb-431Yc. A striking one is 431F, which will be useful in Volume 5.

432 K-analytic spaces

I describe the basic measure-theoretic properties of K-analytic spaces (§422). I start with 'elementary' results (432A-432C), assembling ideas from §§421, 422 and 431. The main theorem of the section is 432D, one of the leading cases of the general extension theorem 416P. An important corollary (432G) gives a sufficient condition for the existence of pull-back measures. I briefly mention 'capacities' (432J-432L).

432A Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete locally determined Hausdorff topological measure space. Then every K-analytic subset of X is measurable.

proof If $A \subseteq X$ is K-analytic, it is Souslin-F (422Ha), therefore measurable (431B).

432B Theorem Let X be a K-analytic Hausdorff space, and μ a semi-finite topological measure on X. Then

$$\mu X = \sup\{\mu K : K \subseteq X \text{ is compact}\}.$$

proof If $\gamma < \mu X$, there is an $E \in \operatorname{dom}\mu$ such that $\gamma < \mu E < \infty$; set $\nu F = \mu(E \cap F)$ for every Borel set $F \subseteq X$, so that ν is a totally finite Borel measure on X, and $\nu X > \gamma$. Let $\hat\nu$ be the completion of ν. Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = X$. Set $F_\sigma = \overline{R[I_\sigma]}$ for $\sigma \in S^* = \bigcup_{k\ge1} \mathbb{N}^k$, where $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$. Because R is closed in $\mathbb{N}^{\mathbb{N}} \times X$ (422Da), X is the kernel of the Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S^*}$ (421I). By 431D, there is a compact $L \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\hat\nu(\bigcup_{\phi\in L}\bigcap_{n\in\mathbb{N}} F_{\phi\restriction n}) \ge \gamma$. But, by 421I, this is just $\hat\nu(R[L])$; and $R[L]$ is compact, by 422D(e-i). So $\mu R[L]$ is defined, with $\mu R[L] \ge \nu R[L] = \hat\nu R[L]$, and we have a compact subset of X of measure at least γ. As γ is arbitrary, the theorem is proved.

432C Proposition Let X be a Hausdorff space such that all its open sets are K-analytic, and μ a Borel measure on X.

(a) If μ is semi-finite, it is tight.

(b) If μ is locally finite, its completion is a Radon measure on X.

proof (a) By 422Hb, every open subset of X is Souslin-F. Applying 421F to the family $\mathcal{E}$ of closed subsets of X, we see that every Borel subset of X is Souslin-F, therefore K-analytic (422Ha). Now suppose that $E \subseteq X$ is a Borel set. Then the subspace measure μ_E is a semi-finite Borel measure on the K-analytic space E, so by 432B $\mu E = \sup_{K\subseteq E \text{ is compact}} \mu K$. As E is arbitrary, μ is inner regular with respect to the compact sets; but we are supposing that X is Hausdorff, so these are all closed, and μ is tight.

(b) Because X is Lindelöf (422Gg), μ is σ-finite (411Ge), therefore semi-finite. So (a) tells us that μ is tight. By 416F, its c.l.d. version is a Radon measure. But (because μ is σ-finite) this is just its completion (213Ha).

432D Theorem (Aldaz & Render 00) Let X be a K-analytic Hausdorff space and μ a locally finite measure on X which is inner regular with respect to the closed sets. Then μ has an extension to a Radon measure on X. In particular, μ is τ-additive.

proof The point is that if $\mu E > 0$ then there is a compact $K \subseteq E$ such that $\mu^* K > 0$. **P** Write Σ for the domain of μ. Take $\gamma < \mu E$. Because X is Lindelöf (422Gg), μ is σ-finite (411Ge), therefore semi-finite. Let $E' \subseteq E$ be such that $\gamma < \mu E' < \infty$. Because μ is inner regular with respect to the closed sets, there is a closed set $F \subseteq E$ such that $\mu F > \gamma$. F is K-analytic (422Gf); let $R \subseteq \mathbb{N}^{\mathbb{N}} \times F$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = F$. For $\sigma \in S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ set

$$A_\sigma = \{x : (\phi, x) \in R \text{ for some } \phi \in \mathbb{N}^{\mathbb{N}} \text{ such that } \phi(i) \le \sigma(i) \text{ for every } i < \#(\sigma)\}.$$

Then $\langle A_{\sigma^\frown<i>}\rangle_{i\in\mathbb{N}}$ is a non-decreasing sequence with union A_σ, so

$$\mu^* A_\sigma = \sup_{i\in\mathbb{N}} \mu^* A_{\sigma^\frown<i>}$$

for every $\sigma \in S$ (132Ae). We can therefore find a sequence $\psi \in \mathbb{N}^{\mathbb{N}}$ such that

$$\mu^* A_{\psi\restriction n} > \gamma$$

for every $n \in \mathbb{N}$. Set

$$K = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}},\ \phi(i) \le \psi(i) \text{ for every } i \in \mathbb{N}\};$$

then $K = \prod_{n\in\mathbb{N}}(\psi(n) + 1)$ is compact, so $R[K]$ is compact (422D(e-i)).

? Suppose, if possible, that $\mu^* R[K] < \gamma$. Then there is an $H \in \Sigma$ such that $R[K] \subseteq H \subseteq F$ and $\mu(F\setminus H) > \mu F-\gamma$. Because μ is inner regular with respect to the closed sets, there is a closed set $F' \in \Sigma$ such that $F' \subseteq F \setminus H$ and $\mu F' > \mu F - \gamma$. Since $R[K] \cap F' = \emptyset$, $K \cap R^{-1}[F'] = \emptyset$. $R^{-1}[F']$ is closed, because R is usco-compact, so there is some n such that

$$L = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}},\ \phi\restriction n = \phi'\restriction n \text{ for some } \phi' \in K\}$$

does not meet $R^{-1}[F']$ (4A2F(h-vi)), and $R[L] \cap F' = \emptyset$. But L is just $\{\phi : \phi(i) \le \psi(i) \text{ for every } i < n\}$, so $R[L] = A_{\psi\restriction n}$, and

$$\gamma < \mu^* A_{\psi\restriction n} \le \mu(F \setminus F') < \gamma,$$

which is absurd. **X**

Thus $\mu^* R[K] \ge \gamma$. As $\gamma > 0$, we have the result. **Q**

Now the theorem follows at once from 416P(ii)$\Rightarrow$(i).

432E Corollary Let X be a K-analytic Hausdorff space, and μ a locally finite quasi-Radon measure on X. Then μ is a Radon measure.

proof By 432D, μ has an extension to a Radon measure μ'. But of course μ and μ' must coincide, by 415H or otherwise.

432F Corollary Let X be a K-analytic Hausdorff space, and ν a locally finite Baire measure on X. Then ν has an extension to a Radon measure on X; in particular, it is τ-additive. If the topology of X is regular, the extension is unique.

proof Because X is Lindelöf (422Gg), ν is σ-finite, therefore semi-finite; by 412D, it is inner regular with respect to the closed sets. So 432D tells us that it has an extension to a Radon measure on X. Since the extension is τ-additive, so is ν.

If X is regular, then it must be completely regular (4A2H(b-i)), and the family $\mathcal{G}$ of cozero sets is a base for the topology closed under finite unions. If μ, μ' are Radon measures extending ν, they agree on $\mathcal{G}$, and must be equal, by 415H(iv).

432G Corollary Let X be a K-analytic Hausdorff space, Y a Hausdorff space and ν a locally finite measure on Y which is inner regular with respect to the closed sets. Let $f : X \to Y$ be a continuous function such that $f[X]$ has full outer measure in Y. Then there is a Radon measure μ on X such that f is inverse-measure-preserving for μ and ν. If ν is Radon, it is precisely the image measure μf^{-1}.

proof (a) Write T for the domain of ν, and set $\Sigma_0 = \{f^{-1}[F] : f \in \mathrm{T}\}$, so that Σ_0 is a σ-algebra of subsets of X, and we have a measure μ_0 on X defined by setting $\mu_0 f^{-1}[F] = \nu F$ whenever $F \in \mathrm{T}$ (234F).

(b) If $E \in \Sigma_0$ and $\gamma < \mu_0 E$, there is an $F \in \mathrm{T}$ such that $E = f^{-1}[F]$. Now there is a closed set $F' \subseteq F$ such that $\nu F' \ge \gamma$. Because f is continuous, $f^{-1}[F']$ is closed, and we have $f^{-1}[F'] \subseteq E$ and $\mu_0 f^{-1}[F'] \ge \gamma$. As E and γ are arbitrary, μ_0 is inner regular with respect to the closed sets.

If $x \in X$, then (because ν is locally finite) there is an open set $H \subseteq Y$ such that $f(x) \in H$ and $\nu^* H < \infty$; as f is of course inverse-measure-preserving for μ_0 and ν, $\mu_0^* f^{-1}[H] \le \nu^* H$ (234B(f-i)) is finite, while $f^{-1}[H]$ is an open set containing x. Thus μ_0 is locally finite.

(c) By 432D, there is a Radon measure μ on X extending μ_0. Because f is inverse-measure-preserving for μ_0 and ν, it is surely inverse-measure-preserving for μ and ν.

The image measure μf^{-1} extends ν, so must be locally finite; it is therefore a Radon measure (418I). So if ν itself is a Radon measure, it must be identical with μf^{-1}, by 416Eb.

432H Corollary Suppose that X is a set and that $\mathfrak{S}$, $\mathfrak{T}$ are Hausdorff topologies on X such that $(X, \mathfrak{T})$ is K-analytic and $\mathfrak{S} \subseteq \mathfrak{T}$. Then the totally finite Radon measures on X are the same for $\mathfrak{S}$ and $\mathfrak{T}$.

proof Write f for the identity function on X regarded as a continuous function from $(X, \mathfrak{T})$ to $(X, \mathfrak{S})$. If μ is a totally finite $\mathfrak{T}$-Radon measure on X, then $\mu = \mu f^{-1}$ is $\mathfrak{S}$-Radon, by 418I again. If ν is a totally finite $\mathfrak{S}$-Radon measure on X, then 432G tells us that it is of the form $\mu = \mu f^{-1}$ for some $\mathfrak{T}$-Radon measure μ, that is, is itself $\mathfrak{T}$-Radon.

432I Corollary Let X be a K-analytic Hausdorff space, and $\mathcal{U}$ a subbase for the topology of X. Let (Y, T, ν) be a complete totally finite measure space and $\phi : Y \to X$ a function such that $\phi^{-1}[U] \in \mathrm{T}$ for every $U \in \mathcal{U}$. Then there is a Radon measure μ on X such that $\int f d\mu = \int f\phi\, d\nu$ for every bounded continuous $f : X \to \mathbb{R}$.

proof (a) Let $\nu\phi^{-1}$ be the image measure on X, and Σ_0 its domain. Then Σ_0 is a σ-algebra of subsets of X including $\mathcal{U}$. So if x, y are distinct points of X, there are disjoint open sets U, $V \in \Sigma_0$ containing x, y respectively. **P** Because X is Hausdorff, there are disjoint open sets U_0 and V_0 such that $x \in U_0$ and $y \in V_0$. Because Σ_0 is closed under finite intersections and includes the subbase $\mathcal{U}$, it includes a base for the topology of X (4A2B(a-i)), and there are open sets U, $V \in \Sigma_0$ such that $x \in U \subseteq U_0$ and $y \in V \subseteq V_0$. **Q**

Every cozero subset of X belongs to Σ_0. **P** If $G \subseteq X$ is a cozero set, there is a sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ of closed subsets of X with union G. For each n, F_n and $X \setminus G$ are disjoint K-analytic subsets of X (422Gf), so there is an $E_n \in \Sigma_0$ such that $F_n \subseteq E_n \subseteq G$ (422I). Now $G = \bigcup_{n\in\mathbb{N}} E_n$ belongs to Σ_0. **Q**

(b) It follows that the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$ of X is included in Σ_0. So $\mu_0 = \nu\phi^{-1} \restriction \mathcal{B}\mathfrak{a}(X)$ is a Baire measure on X. By 432F, μ_0 has an extension to a Radon measure μ on X.

If $f \in C_b(X)$, then f is μ_0-integrable; since ϕ is inverse-measure-preserving for ν and μ_0, $\int f\phi\, d\nu$ is defined and equal to $\int f d\mu_0$ (235G). Similarly $\int f d\mu = \int f d\mu_0$. So $\int f d\mu = \int f\phi\, d\nu$, as required.

432J Capacitability The next theorem is not exactly measure theory as studied in most of this treatise; but it is clearly very close to the other ideas of this section, and it has important applications to measure theory in the narrow sense.

Definitions Let $(X, \mathfrak{T})$ be a topological space.

(a) A **Choquet capacity** on X is a function $c : \mathcal{P}X \to [0, \infty]$ such that

(i) $c(A) \le c(B)$ whenever $A \subseteq B \subseteq X$;

(ii) $\lim_{n\to\infty} c(A_n) = c(A)$ whenever $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A;

(iii) $c(K) = \inf\{c(G) : G \supseteq K \text{ is open}\}$ for every compact set $K \subseteq X$.

(b) A Choquet capacity c on X is **outer regular** if $c(A) = \inf\{c(G) : G \supseteq A \text{ is open}\}$ for every $A \subseteq X$.

(c) If P is any lattice, an order-preserving function $c : P \to \,]-\infty, \infty]$ is **submodular** if $c(p \wedge q) + c(p \vee q) \le c(p) + c(q)$ for all p, $q \in P$. The phrase ‘strongly subadditive’ is used by many authors in similar contexts.

432K Theorem (Choquet 55) Let X be a Hausdorff space and c a Choquet capacity on X. If $A \subseteq X$ is K-analytic, then $c(A) = \sup\{c(K) : K \subseteq A \text{ is compact}\}$.

proof Take $\gamma < c(A)$. Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = A$; for $\sigma \in S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ set

$$A_\sigma = \{x : (\phi, x) \in R \text{ for some } \phi \in \mathbb{N}^\mathbb{N} \text{ such that } \phi(i) \le \sigma(i) \text{ for every } i < \#(\sigma)\}.$$

Then $\langle A_{\sigma^\frown<i>}\rangle_{i\in\mathbb{N}}$ is a non-decreasing sequence with union A_σ, so

$$c(A_\sigma) = \sup_{i\in\mathbb{N}} c(A_{\sigma^\frown<i>})$$

for every $\sigma \in S$. We can therefore find a sequence $\psi \in \mathbb{N}^\mathbb{N}$ such that $c(A_{\psi\restriction n}) > \gamma$ for every $n \in \mathbb{N}$. Set

$$K = \{\phi : \phi \in \mathbb{N}^\mathbb{N},\ \phi(i) \le \psi(i) \text{ for every } i \in \mathbb{N}\};$$

then $K = \prod_{n\in\mathbb{N}}(\psi(n)+1)$ is compact, so $R[K]$ is compact (422D(e-i)).

? Suppose, if possible, that $c(R[K]) < \gamma$. Then, by (iii) of 432J, there is an open set $G \supseteq R[K]$ such that $c(G) < \gamma$. Set $F = X \setminus G$, so that F is closed and $K \cap R^{-1}[F] = \emptyset$. $R^{-1}[F]$ is closed, because R is usco-compact, so there is some n such that

$$L = \{\phi : \phi \in \mathbb{N}^\mathbb{N},\ \phi\restriction n = \phi'\restriction n \text{ for some } \phi' \in K\}$$

does not meet $R^{-1}[F]$ (4A2F(h-vi) again), and $R[L] \cap F = \emptyset$, that is, $R[L] \subseteq G$. But L is just $\{\phi : \phi(i) \le \psi(i)$ for every $i < n\}$, so $R[L] = A_{\psi\restriction n}$, and

$$\gamma < c(A_{\psi\restriction n}) \le c(G) < \gamma,$$

which is absurd. **X**

Thus $c(R[K]) \ge \gamma$. As γ is arbitrary and $R[K]$ is compact, we have the result.

432L Proposition Let $(X, \mathfrak{T})$ be a topological space.

(a) Let $c_0 : \mathfrak{T} \to [0,\infty]$ be a functional such that

$c_0(G) \le c_0(H)$ whenever G, $H \in \mathfrak{T}$ and $G \subseteq H$;

c_0 is submodular;

$c_0(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} c_0(G_n)$ for every non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{T}$.

Then c_0 has a unique extension to an outer regular Choquet capacity c on X, and c is submodular.

(b) Suppose that X is regular. Let $\mathcal{K}$ be the family of compact subsets of X, and $c_1 : \mathcal{K} \to [0,\infty]$ a functional such that

c_1 is submodular;

$c_1(K) = \inf_{G\in\mathfrak{T}, G\supseteq K} \sup_{L\in\mathcal{K}, L\subseteq G} c_1(L)$ for every $K \in \mathcal{K}$.

Then c_1 has a unique extension to an outer regular Choquet capacity c on X such that

$$c(G) = \sup\{c(K) : K \subseteq G \text{ is compact}\} \text{ for every open } G \subseteq X,$$

and c is submodular.

proof (a) For $A \subseteq X$, set $c(A) = \inf\{c_0(G) : A \subseteq G \in \mathfrak{T}\}$. Then $c : \mathcal{P}X \to [0,\infty]$ extends c_0 because c_0 is order-preserving. Conditions (i) and (iii) of 432J are obviously satisfied. As for (ii), let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of subsets of X with union A. Then $\lim_{n\to\infty} c(A_n)$ is defined and not greater than $c(A)$. If the limit is infinite, then certainly it is equal to $c(A)$. Otherwise, take $\epsilon > 0$. For each $n \in \mathbb{N}$, choose an open set $G_n \supseteq A_n$ such that $c_0(G_n) \le c(A_n) + 2^{-n}\epsilon$. Set $H_n = \bigcup_{i\le n} G_i$ for $n \in \mathbb{N}$. Then $c_0(H_n) \le c(A_n) + 2\epsilon - 2^{-n}\epsilon$ for every n. **P** Induce on n. If $n = 0$ then $H_0 = G_0$ and the result is immediate. For the inductive step to $n+1$, we have

$$c_0(H_{n+1}) + c_0(H_n \cap G_{n+1}) \le c_0(H_n) + c_0(G_{n+1})$$

(because c_0 is submodular)

$$\le c(A_n) + 2\epsilon - 2^{-n}\epsilon + c(A_{n+1}) + 2^{-n-1}\epsilon$$

(using the inductive hypothesis)

$$\le c_0(H_n \cap G_{n+1}) + 2\epsilon - 2^{-n-1}\epsilon + c(A_{n+1});$$

as $c_0(H_n \cap G_{n+1}) \le c_0(G_{n+1})$ is finite, $c_0(H_{n+1}) \le 2\epsilon - 2^{-n-1}\epsilon + c(A_{n+1})$, as required. **Q**

Set $H = \bigcup_{n\in\mathbb{N}} H_n$. As $\langle H_n\rangle_{n\in\mathbb{N}}$ is non-decreasing,

$$c(A) \le c_0(H) = \lim_{n\to\infty} c_0(H_n) \le \lim_{n\to\infty} c(A_n) + 2\epsilon.$$

As ϵ is arbitrary, $c(A) \le \lim_{n\to\infty} c(A_n)$ and the final condition of 432J is satisfied.

Of course c is the only Choquet capacity extending c_0 and outer regular with respect to the open sets.

As for the submodularity of c, if A, $B \subseteq X$ and $\epsilon > 0$, there are open sets $G \supseteq A$ and $H \supseteq B$ such that $c_0(G) + c_0(H) \leq c(G) + c(H) + \epsilon$; so that

$$\begin{aligned} c(A \cup B) + c(A \cap B) &\leq c_0(G \cup H) + c_0(G \cap H) \\ &\leq c_0(G) + c_0(H) \leq c(A) + c(B) + \epsilon. \end{aligned}$$

As ϵ is arbitrary, $c(A \cup B) + c(A \cap B) \leq c(A) + c(B)$, as required.

(b)(i) The key fact is this: if G, $H \in \mathfrak{T}$, K, $L \in \mathcal{K}$, $K \subseteq G \cup H$ and $L \subseteq G \cap H$, then there are K_1, $L_1 \in \mathcal{K}$ such that $K_1 \subseteq G$, $L_1 \subseteq H$, $K \subseteq K_1 \cup L_1$ and $L \subseteq K_1 \cap L_1$. **P** Because $(X, \mathfrak{T})$ is regular, there is an open set G_1 such that $K \setminus H \subseteq G_1$ and $\overline{G}_1 \subseteq G$ (4A2F(h-ii)). Set $K_1 = (K \cap \overline{G}_1) \cup L$, $L_1 = (K \setminus G_1) \cup L$; these work. **Q**

(ii) Define $c_0 : \mathfrak{T} \to [0, \infty]$ by setting $c_0(G) = \sup_{K \in \mathcal{K}, K \subseteq G} c_1(K)$ for open $G \subseteq X$. Then c_0 satisfies the conditions of (a). **P** The first and third are elementary. As for the second, if G, $H \in \mathfrak{T}$ and $\gamma < c_0(G \cup H) + c_0(G \cap H)$, there are K, $L \in \mathcal{K}$ such that $K \subseteq G \cup H$, $L \subseteq G \cap H$ and $\gamma \leq c_1(K) + c_2(L)$. Now (i) tells us that there are compact sets $K_1 \subseteq G$ and $L_1 \subseteq H$ such that $K \subseteq K_1 \cup L_1$ and $L \subseteq K_1 \cap L_1$, in which case

$$\begin{aligned} c_0(G) + c_0(H) &\geq c_1(K_1) + c_1(L_1) \geq c_1(K_1 \cup L_1) + c_1(K_1 \cap L_1) \\ &\geq c_1(K) + c_1(L) \geq \gamma. \end{aligned}$$

As γ is arbitrary, $c_0(G) + c_0(H) \geq c_0(G \cup H) + c_0(G \cap H)$, as required. **Q**

(iii) We therefore have a submodular outer regular Choquet capacity $c : \mathcal{P}X \to [0, \infty]$ defined by setting $c(A) = \inf_{G \in \mathfrak{T}, A \subseteq G} c_0(G)$ for every $A \subseteq X$. From the second condition on c_1, we see that c extends c_1. Clearly c satisfies the two regularity conditions, and is the only extension of c_1 which does so.

432X Basic exercises (a) Put 422Xf, 431Xb and 432D together to prove 432C.

(b) Let X be a K-analytic Hausdorff space, and μ a measure on X which is outer regular with respect to the open sets. Show that $\mu X = \sup_{K \subseteq X \text{ is compact}} \mu^* K$. (*Hint*: see the proof of 432D.)

>(c) Let X be a K-analytic Hausdorff space, and μ a semi-finite topological measure on X. Show that if *either* μ is inner regular with respect to the closed sets *or* X is regular and μ is a τ-additive Borel measure, then μ is tight.

(d) Use 422Gf, 432B and 416C to prove 432E.

>(e) Suppose that X is a set and that $\mathfrak{S}$, $\mathfrak{T}$ are Hausdorff topologies on X such that $(X, \mathfrak{T})$ is K-analytic and $\mathfrak{S} \subseteq \mathfrak{T}$. Let $(Z, \mathfrak{U}, \mathrm{T}, \nu)$ be a Radon measure space and $f : Z \to X$ a function which is almost continuous for $\mathfrak{U}$ and $\mathfrak{S}$. Show that f is almost continuous for $\mathfrak{U}$ and $\mathfrak{T}$. (*Hint*: it is enough to consider totally finite ν; show that νf^{-1} is $\mathfrak{T}$-Radon, so is inner regular for $\{K : \mathfrak{T}_K = \mathfrak{S}_K\}$, writing $\mathfrak{T}_K$ for the subspace topology induced by $\mathfrak{T}$ on K.)

(f) Let X be a topological space and μ a locally finite measure on X which is inner regular with respect to the closed sets. Show that μ^* is a Choquet capacity.

(g) Let X be a topological space and F a closed subset of X. Define $c : \mathcal{P}X \to \{0, 1\}$ by setting $c(A) = 1$ if A meets F, 0 otherwise. Show that c is a Choquet capacity on X.

(h) Let X and Y be Hausdorff spaces, and $R \subseteq X \times Y$ an usco-compact relation. Show that if c is a Choquet capacity on Y, then $A \mapsto c(R[A])$ is a Choquet capacity on X.

(i) Use 432K and 432Xf to shorten the proof of 432D.

(j) Let P be a lattice, and $c : P \to [0, \infty[$ a submodular order-preserving functional. For p, $q \in P$ set $\rho(p, q) = 2c(p \vee q) - c(p) - c(q)$. Show that ρ is a pseudometric.

432Y Further exercises (a) Show that there are a K-analytic Hausdorff space X and a probability measure μ on X such that (i) μ is inner regular with respect to the Borel sets (ii) the domain of μ includes a base for the topology of X (iii) every compact subset of X is negligible. Show that there is no extension of μ to a topological measure on X.

(b) Let X, Y be Hausdorff spaces, $R \subseteq X \times Y$ an usco-compact relation and μ a Radon probability measure on X such that $\mu_* R^{-1}[Y] = 1$. Show that there is a Radon probability measure on Y such that $\nu_* R[A] \geq \mu_* A$ for every $A \subseteq X$.

432 Notes and comments The measure-theoretic properties of K-analytic spaces can largely be summarised in the slogan 'K-analytic spaces have lots of compact sets'. I said above that it is sometimes helpful to think of K-analytic spaces as an amalgam of compact Hausdorff spaces and Souslin-F subsets of $\mathbb{R}$. For the former, it is obvious that they have many compact subsets; for the latter, it is not obvious, but is of course one of their fundamental properties, deducible from 422De. 432B and the proof of 432D (repeated in 432K) are typical manifestations of the phenomenon. The real point of these theorems is that we can extend a Borel or Baire measure to a Radon measure with no prior assumption of τ-additivity (432F). A Radon measure must be τ-additive just because it is tight. A (locally finite) Borel or Baire measure must be τ-additive whenever the measurable open sets are K-analytic.

The condition 'every open set is K-analytic' in 432C is of course a very strong one in the context of compact Hausdorff spaces (422Xe). But for analytic spaces it is automatically satisfied (423Eb), and that is the side on which the principal applications of 432C appear.

The results which I call corollaries of 432D can mostly be proved by more direct methods (see 432Xd), but the line I choose here seems to be the most powerful technique. Indeed it can be used to deal with 432C as well (432Xa).

In §434 I will discuss 'universally measurable' sets in topological spaces. In fact K-analytic sets are universally measurable in a particularly strong sense (432A). The point here is that K-analyticity is intrinsic; a K-analytic space is measurable whenever embedded as a subspace of a (complete locally determined) Hausdorff topological measure space.

The theorems here touch on two phenomena of particular importance. First, in 432G we have an example of 'pulling back' a measure, that is, we have a measure ν on a set Y and a function $f : X \to Y$ and seek a Radon measure μ on X such that f is inverse-measure-preserving, or, even better, such that $\nu = \mu f^{-1}$. There was a similar result in 418L. In both cases we have to suppose that f is continuous and (in effect) that ν is a Radon measure. (This is not part of the hypotheses of 432G, but of course it is an easy consequence of them, using 432B.) In 418L, we need a special hypothesis to ensure that there are enough compact subsets of X to carry an appropriate Radon measure; in 432G, this is an automatic result of assuming that X is K-analytic. Both 418L and 432G can be regarded as consequences of Henry's theorem (416N). The difficulty arises from the requirement that μ should be a Radon measure; if we do not insist on this there is a much simpler solution, since we need suppose only that $f[X]$ has full outer measure (234F).

The next theme I wish to mention is a related one, the investigation of comparable topologies. If $\mathfrak{S}$ and $\mathfrak{T}$ are (Hausdorff) topologies on a set X, and $\mathfrak{S}$ is coarser than $\mathfrak{T}$ (so that $(X, \mathfrak{S})$ is a continuous image of $(X, \mathfrak{T})$), then 418I tells us that any totally finite $\mathfrak{T}$-Radon measure is $\mathfrak{S}$-Radon. We very much want to know when the reverse is true, so that the (totally finite) Radon measures for the two topologies are the same. 432H provides one of the important cases in which this occurs. The hypothesis '$(X, \mathfrak{T})$ is K-analytic' generalizes the alternative '$(X, \mathfrak{T})$ is compact'; in the latter case, $\mathfrak{S} = \mathfrak{T}$, so that the result is, from our point of view here, trivial. (But from the point of view of elementary general topology, of course, it is one of the pivots of the theory of compact Hausdorff spaces.) In a similar vein we have a variety of important topological consequences of the same hypotheses (422Yf, 423Fb).

The paragraphs 432J-432L may appear to be no more that a minor extension of ideas already set out. I ought therefore to say plainly that the topological and measure theory of K-analytic spaces have co-evolved with the notion of capacity, and that 432K ('K-analytic spaces are capacitable') is one of the cornerstones of a theory of which I am giving only a minuscule part. For a idea of the vitality and scope of this theory, see Dellacherie 80.

433 Analytic spaces

We come now to the special properties of measures on 'analytic' spaces, that is, continuous images of $\mathbb{N}^{\mathbb{N}}$, as described in §423. I start with a couple of facts about spaces with countable networks.

433A Proposition Let $(X, \mathfrak{T})$ be a topological space with a countable network, and μ a localizable topological measure on X which is inner regular with respect to the Borel sets. Then μ has countable Maharam type.

proof Let $\tilde{\mu}$ be the c.l.d. version of μ (213E). Then the measure algebra $\mathfrak{A}$ of $\tilde{\mu}$ can be identified with the measure algebra of μ (322D(b-iii)). Also $\tilde{\mu}$ is complete, locally determined and localizable, so every subset of X has a measurable envelope with respect to $\tilde{\mu}$ (213J, 213L). Let $\tilde{\Sigma}$ be the domain of $\tilde{\mu}$, and $\mathcal{E}$ a countable network for $\mathfrak{T}$. For each $E \in \mathcal{E}$, let $F_E \in \tilde{\Sigma}$ be a measurable envelope of E.

Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{F_E^\bullet : E \in \mathcal{E}\}$, and set $\mathrm{T} = \{F : F \in \tilde{\Sigma},\, F^\bullet \in \mathfrak{B}\}$. Because $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$, T is a σ-subalgebra of $\tilde{\Sigma}$. Now $\mathfrak{T} \subseteq \mathrm{T}$. **P** If $G \in \mathfrak{T}$, set $\mathcal{E}_0 = \{E : E \in \mathcal{E},\, E \subseteq G\}$. Set $F = \bigcup_{E\in\mathcal{E}_0} F_E$, so that $F \in \mathrm{T}$ and $G \subseteq F$. For each $E \in \mathcal{E}_0$, $F_E \setminus G$ is negligible, so $F \setminus G$ is negligible, and $G^\bullet = F^\bullet \in \mathfrak{B}$, so $G \in \mathrm{T}$. **Q**

It follows that T includes the Borel σ-algebra of X. Because μ is inner regular with respect to the Borel sets, $\mathfrak{B}$ is order-dense in $\mathfrak{A}$, and $\mathfrak{B} = \mathfrak{A}$. Thus the countable set $\{F_E^\bullet : E \in \mathcal{E}\}$ τ-generates $\mathfrak{A}$, and the Maharam type of $\mathfrak{A}$, which is the Maharam type of μ, is countable.

433B Lemma If $(X, \mathfrak{T})$ is a Hausdorff space with a countable network, then any topological measure on X is countably separated in the sense of 343D.

proof By 4A2Nf, there is a countable family of open sets separating the points of X.

433C Theorem Let X be an analytic Hausdorff space, and μ a Borel measure on X.

(a) If μ is semi-finite, it is tight.

(b) If μ is locally finite, its completion is a Radon measure on X.

proof X is K-analytic (423C); moreover, every open subset of X is again analytic (423Eb). So 432C gives the result at once.

Remark Compare 256C.

433D Theorem Let X and Y be analytic Hausdorff spaces, ν a totally finite Radon measure on Y and $f : X \to Y$ a Borel measurable function such that $f[X]$ is of full outer measure for ν. Then there is a Radon measure μ on X such that $\nu = \mu f^{-1}$.

proof By 423Ga, the graph R of f is an analytic set in $X \times Y$, therefore K-analytic. Set $\pi_1(x, y) = x$, $\pi_2(x, y) = y$ for $(x, y) \in R$, so that π_1 and π_2 are continuous. Now $\pi_2[R] = f[X]$ has full outer measure, so by 432G there is a Radon measure λ on R such that $\nu = \lambda\pi_2^{-1}$. Next, because π_1 is continuous, the image $\mu = \lambda\pi_1^{-1}$ is a Radon measure on X, by 418I. But $\pi_2 = f\pi_1$, so

$$\mu f^{-1} = (\lambda\pi_1^{-1})f^{-1} = \lambda(f\pi_1)^{-1} = \lambda\pi_2^{-1} = \nu,$$

as required.

433E Proposition Let (X, Σ, μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Let $(Y, \mathfrak{S})$ be an analytic Hausdorff space and $f : X \to Y$ a measurable function. Then f is almost continuous.

proof Take $E \in \Sigma$ and $\gamma < \mu E$. Then there is an $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$. For Borel sets $H \subseteq Y$, set $\nu H = \mu(F \cap f^{-1}[H])$. Then ν is a totally finite Borel measure on Y, so is tight (433C); let $K \subseteq Y$ be a compact set such that $\nu K > \gamma$, so that $\mu(F \cap f^{-1}[K]) > \gamma$. The subspace measure on $L = F \cap f^{-1}[K]$ is still inner regular with respect to the (relatively) closed sets (412Pc), and $f{\restriction}L$ is still measurable; but $f{\restriction}L$ is a function from L to K, and K is metrizable, by 423Dc. So $f{\restriction}L$ is almost continuous, by 418J, and there is a set $F' \subseteq L$, of measure at least γ, such that $f{\restriction}F'$ is continuous.

As E and γ are arbitrary, f is almost continuous.

Remark Compare 418Yg.

433F I give some simple corollaries of the von Neumann-Jankow selection theorem (423M-423O).

Proposition Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be analytic Hausdorff spaces, and $f : X \to Y$ a Borel measurable surjection. Let ν be a complete locally determined topological measure on Y, and T its domain. Then there is a T-measurable function $g : Y \to X$ such that gf is the identity on X.

proof By 423O we know that there is a function $g : Y \to X$ such that fg is the identity and g is T_1-measurable, where T_1 is the σ-algebra generated by the Souslin-F subsets of Y. But T contains every Souslin-F subset of Y, by 431B, therefore includes T_1, and g is actually T-measurable.

433G Proposition Let $(X,\mathfrak{T})$ be an analytic Hausdorff space, (Y,T,ν) a complete locally determined measure space, and $f:X\to Y$ a surjection. Suppose that there is some countable family $\mathcal{F}\subseteq\mathrm{T}$ such that $\mathcal{F}$ separates the points of Y (that is, whenever y, y' are distinct points of Y there is a member of $\mathcal{F}$ containing one and not the other) and $f^{-1}[F]$ is a Borel subset of X for every $F\in\mathcal{F}$. Then there is a T-measurable function $g:Y\to X$ such that fg is the identity on Y.

proof Set $\mathcal{A}=\mathcal{F}\cup\{Y\setminus F:F\in\mathcal{F}\}$. The topology $\mathfrak{T}_1$ on X generated by $\mathfrak{T}\cup\{f^{-1}[A]:A\in\mathcal{A}\}$ is still analytic (423H). If we take $\mathfrak{S}$ to be the topology on Y generated by $\mathcal{A}$, then $\mathfrak{S}$ is Hausdorff and f is $(\mathfrak{T}_1,\mathfrak{S})$-continuous, so $\mathfrak{S}$ is analytic (423Bb).

Because $\mathfrak{S}$ is generated by a countable subset $\mathcal{A}$ of T, it is second-countable, and $\mathfrak{S}\subseteq\mathrm{T}$ (4A3Da/4A3E). So ν is a topological measure with respect to $\mathfrak{S}$. By 433F, there is a function $g:Y\to X$, measurable for T and the topology $\mathfrak{T}_1$, such that gf is the identity on X; and of course g is still measurable for T and the coarser original topology $\mathfrak{T}$ on X.

433H Proposition Let X be an analytic Hausdorff space, and (Y,T,ν) a complete locally determined measure space. Suppose that $W\subseteq X\times Y$ belongs to $\mathcal{S}(\mathcal{B}(X)\widehat{\otimes}\mathrm{T})$, where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Then $W[X]\in\mathrm{T}$ and there is a T-measurable function $g:W[X]\to X$ such that $(g(y),y)\in W$ for every $y\in W[X]$.

proof Set $\mathcal{V}=\mathcal{S}(\{E\times F:E\subseteq X\text{ is closed, }F\in\mathrm{T}\})$. Then $\mathcal{V}$ contains $H\times Y$ for every Souslin-F subset H of X (421Cb), and therefore for every $H\in\mathcal{B}(X)$ (423Eb); by 421F, it follows that $\mathcal{V}$ includes $\mathcal{B}(Y)\widehat{\otimes}\mathrm{T}$ and therefore $W\in\mathcal{S}(\mathcal{V})=\mathcal{V}$ (421D). By 423M, $W[X]\in\mathcal{S}(\mathrm{T})$, which by 431A is just T, and there is a T-measurable function which is a selector for W^{-1}.

433I Because analytic spaces have countable networks (423C), and their compact subsets are therefore metrizable (423Dc), their measure theory is very close to that of $\mathbb{R}$ or $[0,1]$ or $\{0,1\}^{\mathbb{N}}$. I give some simple manifestations of this principle.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of analytic Hausdorff spaces, and for each $i\in I$ let μ_i be a Radon probability measure on X_i. Let λ be the ordinary product measure on $X=\prod_{i\in I}X_i$, as defined in §254.

(a) If I is countable then λ is a Radon measure.

(b) If every μ_i is strictly positive, then λ is a quasi-Radon measure.

proof (a) In this case, X is analytic (423Bc), therefore hereditarily Lindelöf (423Da). Let Λ be the domain of λ and $\mathfrak{T}$ the topology of X. Then $\Lambda\cap\mathfrak{T}$ is a base for $\mathfrak{T}$; by 4A3Da, $\mathfrak{T}\subseteq\Lambda$. By 417Sb, λ is the τ-additive product measure on X; by 417Q, this is a Radon measure.

(b) By (a), the ordinary product measure on $\prod_{i\in J}X_i$ is a Radon measure for every finite set $J\subseteq I$. So 417Sc tells us that λ is the τ-additive product measure on X; by 417O, this is a quasi-Radon measure.

433J Proposition Let X be an analytic Hausdorff space, and T a countably generated σ-subalgebra of the Borel σ-algebra $\mathcal{B}(X)$ of X. Then any locally finite measure with domain T has an extension to a Radon measure on X.

proof Let μ_0 be a locally finite measure with domain T.

(a) Consider first the case in which μ_0 is totally finite. Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence in T generating T as σ-algebra. Define $f:X\to\{0,1\}^{\mathbb{N}}$ by setting

$$f(x)(n)=\chi F_n(x)\text{ for }n\in\mathbb{N},\ x\in X.$$

Then f is T-measurable (use 418Bd), so we have a Borel measure ν_0 on $\{0,1\}^{\mathbb{N}}$ defined by setting $\nu_0E=\mu_0f^{-1}[E]$ for every Borel set $E\subseteq\{0,1\}^{\mathbb{N}}$. Now the completion ν of ν_0 is a Radon measure (433C). Also $f[X]$ must be analytic, by 423Gb, because f is $\mathcal{B}(X)$-measurable. So ν measures $f[X]$ (432A), and

$$\nu f[X]=\nu_0^*f[X]=\nu_0\{0,1\}^{\mathbb{N}},$$

that is, $f[X]$ is ν-conegligible. By 433D, there is a Radon measure μ on X such that $\nu=\mu f^{-1}$.

Because every F_n is expressible as $f^{-1}[E]$ for some $E\in\mathcal{B}(\{0,1\}^{\mathbb{N}})$, so is every member of T. If $F\in\mathrm{T}$, take $H\in\mathcal{B}(\{0,1\}^{\mathbb{N}})$ such that $F=f^{-1}[H]$; then

$$\mu F=\nu H=\nu_0H=\mu_0F.$$

Thus μ extends μ_0 and $\mu\restriction\Sigma$ will serve.

(b) In general, because X is Lindelöf and μ_0 is locally finite, μ_0 is σ-finite. Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a partition of X into members of T such that $\mu_0 X_n$ is finite for every n, and set $\mu_0^{(n)}F=\mu_0(F\cap X_n)$ for every n and every $F\in\mathrm{T}$; then every $\mu_0^{(n)}$ has an extension to a Radon measure $\mu^{(n)}$. Let μ be the sum $\sum_{n=0}^\infty\mu^{(n)}$ (234G[1]). Of course μ extends $\mu_0=\sum_{n=0}^\infty\mu_0^{(n)}$. Because μ_0 is locally finite, so is μ, and μ is a Radon measure (416De).

433K I turn now to a brief mention of 'standard Borel spaces'. From the point of view of this chapter, it is natural to regard the following results as simple corollaries of theorems about Polish spaces. But, as remarked in §424, there are cases in which a standard Borel space is presented without any specific topology being attached; and in any case it is interesting to look at the ways in which we can express these ideas as theorems about σ-algebras rather than about topological spaces.

Proposition Let (X,Σ) be a standard Borel space and T a countably generated σ-subalgebra of Σ. Then any σ-finite measure with domain T has an extension to Σ.

proof Let μ_0 be a σ-finite measure with domain T.

(a) If μ_0 is totally finite, give X a Polish topology for which Σ is the Borel σ-algebra of X, and use 433J.

(b) In general, let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a partition of X into members of T such that $\mu_0X_n<\infty$ for every n, and set $\mu_0^{(n)}F=\mu_0(F\cap X_n)$ for every n and every $F\in\mathrm{T}$; then every $\mu_0^{(n)}$ has an extension to a measure $\mu^{(n)}$ with domain Σ, and we can set $\mu=\sum_{n=0}^\infty\mu^{(n)}$.

433L Proposition Let $\langle(X_n,\Sigma_n,\mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of probability spaces such that (X_n,Σ_n) is a standard Borel space for every n. Suppose that for each $n\in\mathbb{N}$ we are given an inverse-measure-preserving function $f_n:X_{n+1}\to X_n$. Then we can find a standard Borel space (X,Σ), a probability measure μ with domain Σ, and inverse-measure-preserving functions $g_n:X\to X_n$ such that $f_ng_{n+1}=g_n$ for every n.

proof For each n, choose a Polish topology $\mathfrak{T}_n$ on X_n such that Σ_n is the algebra of $\mathfrak{T}_n$-Borel sets. Let $\hat\mu_n$ be the completion of μ_n; then $\hat\mu_n$ is a Radon measure (433C). Every f_n is inverse-measure-preserving for $\hat\mu_{n+1}$ and $\hat\mu_n$, by 234Ba[2], and almost continuous, by 418J.

By 418Q, we have a Radon measure $\hat\mu$ on

$$X=\{x:x\in\prod_{n\in\mathbb{N}}X_n,\ f_n(x(n+1))=x(n)\text{ for every }n\in\mathbb{N}\}$$

such that the continuous maps $x\mapsto x(n)=g_n(x):X\to X_n$ are inverse-measure-preserving for every n. Now X is a Borel subset of $Z=\prod_{n\in\mathbb{N}}X_n$. **P** For each $n\in\mathbb{N}$, let $\mathcal{U}_n$ be a countable base for $\mathfrak{T}_n$. Then

$$Z\setminus X=\bigcup_{n\in\mathbb{N}}\bigcup_{U,V\in\mathcal{U}_n,U\cap V=\emptyset}\{z:z(n)\in U,\ f_n(z(n+1))\in V\}$$

is a countable union of Borel sets because $\{z:z(n)\in U\}$ is open and $\{z:z(n+1)\in f_n^{-1}[V]\}$ is Borel whenever $n\in\mathbb{N}$ and U, $V\in\mathcal{U}_n$. So $Z\setminus X$ and X are Borel sets. **Q**

Accordingly (X,Σ) is a standard Borel space, where Σ is the Borel σ-algebra of X (424G). So if we take $\mu=\hat\mu\restriction\Sigma$, we shall have a suitable measure on X.

433X Basic exercises (a) Let $(X,\mathfrak{T})$ be a topological space with a countable network, and μ a topological measure on X which is inner regular with respect to the Borel sets and has the measurable envelope property (213Xl). Show that μ has countable Maharam type.

(b) Show that an effectively locally finite measure on a hereditarily Lindelöf space (in particular, on any analytic Hausdorff space) is σ-finite.

(c) Let $X\subseteq[0,1]$ be a set of Lebesgue outer measure 1 and inner measure 0. Show that the subspace measure on X is a totally finite Borel measure which is not tight.

(d) Let X be a Hausdorff space and μ a locally finite measure on X, inner regular with respect to the Borel sets, such that $\operatorname{dom}\mu$ includes a base for the topology of X. Suppose that $Y\subseteq X$ is an analytic set of full outer measure. Show that μ has a unique extension to a Radon measure $\tilde\mu$ on X, and that Y is $\tilde\mu$-conegligible.

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[1] Formerly 112Ya.

[2] Formerly 235Hc.

(e) Let (X,Σ) be a standard Borel space. (i) Show that any semi-finite measure with domain Σ is a compact measure (definition: 342Ac, or 451Ab below), therefore perfect. (*Hint*: if X is given a suitable topology, the measure is tight.) (ii) Show that any measure with domain including Σ is countably separated.

>(f) (i) Let (X,Σ,μ) and (Y,T,ν) be atomless probability spaces such that (X,Σ) and (Y,T) are standard Borel spaces. Show that (X,Σ,μ) and (Y,T,ν) are isomorphic. (*Hint*: by 344I, their completions are isomorphic; by 344H, they have negligible sets of cardinal $\mathfrak{c}$; show that any isomorphism between the completions is (Σ,T)-measurable on a conegligible set; use 424Da to match residual negligible sets.) (ii) Let X be a Polish space and μ an atomless Radon measure on X. Show that there is a Borel isomorphism between X and $[0,1]$ which matches μ to Lebesgue measure on $[0,1]$.

(g) Let X be $[0,1]\times\{0,1\}$, with its usual topology, and $I^{\|}$ the split interval (419L); define $f:X\to I^{\|}$ by setting $f(t,0)=t^-$, $f(t,1)=t^+$ for $t\in[0,1]$. (i) Give $I^{\|}$ its usual Radon measure ν (343J, 419Lc). Show that there is no Radon measure λ on X such that $\nu=\lambda f^{-1}$. (ii) Let μ be the product Radon probability measure on X, starting from Lebesgue measure on $[0,1]$ and the usual fair-coin measure on $\{0,1\}$. Show that f is inverse-measure-preserving for μ and ν. Show that f is not almost continuous.

433Y Further exercises (a) Find a Hausdorff topological space X with a countable network and a semi-finite Borel measure on X which does not have countable Maharam type.

(b) Let X be an analytic Hausdorff space and μ an atomless Radon measure on X. Show that (X,μ) is isomorphic to Lebesgue measure on some measurable subset of $\mathbb{R}$. (*Hint*: 344I.)

(c) Let $(X,\mathfrak{T})$ be a Polish space without isolated points, and μ a σ-finite topological measure on X. Show that there is a conegligible meager set. (*Hint*: Show that every non-empty open set is uncountable. Find a countable dense negligible set and a negligible G_δ set including it.)

(d) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of analytic Hausdorff spaces and for each $n\in\mathbb{N}$ let μ_n be a Borel probability measure on X_n. Suppose that for each $n\in\mathbb{N}$ we are given an inverse-measure-preserving function $f_n:X_{n+1}\to X_n$. Show that we can find a standard Borel space (X,Σ), a probability measure μ with domain Σ, and inverse-measure-preserving functions $g_n:X\to X_n$ such that $f_ng_{n+1}=g_n$ for every n.

433 Notes and comments The measure-theoretic results of 433C-433E are of much the same type as those in §432. A characteristic difference is that Borel measurable functions between analytic spaces behave in many ways like continuous functions. (Compare 433D and 432G.) You may feel that 423Yc offers some explanation of this. For any question which refers to the Borel algebra of an analytic space X, or to the class of its analytic subsets, we can expect to be able to suppose that X is separable and metrizable (see 423Xd), and that any single Borel measurable function appearing is continuous. (424H is a particularly remarkable instance of this principle.)

433I here amounts to spelling out a special case of ideas already treated in 417S. As this territory is relatively unfamiliar, I give detailed examples (423Xi, 433Xc, 433Xg, 439A, 439K) to show that the theorems of this section are not generally valid for compact Hausdorff spaces (the archetype of K-analytic spaces which need not be analytic), nor for separable metric spaces (the archetypical spaces with countable network). They really do depend on the particular combination of properties possessed by analytic spaces.

For large parts of probability theory, standard Borel spaces provide an adequate framework, and have a number of advantages; some of the technical problems concerning measurability which loom rather large in this treatise disappear in such contexts. Many authors accordingly give them great prominence. I myself believe that the simplifications are an entrapment rather than a liberation, that sooner or later everyone has to leave the comfortable environment of Borel algebras on Polish spaces, and that it is better to be properly equipped with a suitable general theory when one does. But it is surely important to know what the simplifications are, and the results in 433K-433L will I hope show at least that there are wonderful ideas here, even if my own presentation tends to leave them on one side.

434 Borel measures

What one might call the fundamental question of topological measure theory is the following.

What kinds of measures can arise on what kinds of topological space?

Of course this question has inexhaustible ramifications, corresponding to all imaginable properties of measures and topologies and connexions between them. The challenge I face here is that of identifying particular ideas as being more important than others, and the chief difficulty lies in the bewildering variety of topological properties which have been studied, any of which may have implications for the measure theory of the spaces involved. In this section and the next I give a sample of what is known, necessarily biased and incomplete. I try however to include the results which are most often applied and enough others for the proofs to contain, between them, most of the non-trivial arguments which have been found effective in this area.

In 434A I set out a crude classification of Borel measures on topological spaces. For compact Hausdorff spaces, at least, the first question is whether they carry Borel measures which are not, in effect, Radon measures; this leads us to the definition of 'Radon' space (434C) which is also of interest in the context of general Hausdorff spaces. I give a brief account of the properties of Radon spaces (434F, 434Nd). I look also at two special topics: 'quasi-dyadic' spaces (434O-434Q) and a construction of Borel product measures by integration of sections (434R).

In the study of Radon spaces we find ourselves looking at 'universally measurable' subsets of topological spaces (434D-434E). These are interesting in themselves, and also interact with constructions from earlier parts of this treatise (434S-434T). Three further classes of topological space, defined in terms of the types of topological measure which they carry, are the 'Borel-measure-compact', 'Borel-measure-complete' and 'pre-Radon' spaces; I discuss them briefly in 434G-434J. They provide useful methods for deciding whether Hausdorff spaces are Radon (434K).

434A Types of Borel measures In §411 I introduced the following properties which a Borel measure may or may not have:

(i) inner regularity with respect to closed sets;

(ii) inner regularity with respect to zero sets;

(iii) tightness (that is, inner regularity with respect to closed compact sets);

(iv) τ-additivity.

These are of course interrelated. (ii)$\Rightarrow$(i) just because zero sets are closed, and (iii)$\Rightarrow$(iv) by 411E; in a Hausdorff space, (iii)$\Rightarrow$(i); and for an effectively locally finite measure on a regular topological space, (iv)$\Rightarrow$(i) (414Mb).

On a regular Hausdorff space, therefore, we can divide totally finite Borel measures into four classes:

(A) measures which are not inner regular with respect to the closed sets;

(B) measures which are inner regular with respect to the closed sets, but not τ-additive nor tight;

(C) measures which are τ-additive and inner regular with respect to the closed sets, but not inner regular with respect to the compact sets;

(D) tight measures;

and each of the classes (B)-(D) can be further subdivided into those which are completion regular (B_1, C_1, D_1) and those which are not (B_0, C_0, D_0). Examples may be found in 434Xf (type A), 411Q and 439K (type B_0), 439J (type B_1), 415Xc and 434Xa (type C_1) and 434Xb (type D_0), while Lebesgue measure itself is of type D_1, and any direct sum of spaces of types D_0 and C_1 will have type C_0. (The space in 439J depends for its construction on supposing that there is a cardinal which is not measure-free. It seems that no convincing example of a space of class B_1, that is, a completion regular, non-τ-additive Borel probability measure on a completely regular Hausdorff space, is known which does not depend on some special axiom beyond ordinary ZFC. For one of the obstacles to finding such a space, see 434Q.)

Note that a totally finite Borel measure μ on a regular Hausdorff space can be extended to a quasi-Radon measure iff μ is of class C or D (415M), and that in this case the quasi-Radon measure must be just the completion $\hat{\mu}$ of μ. $\hat{\mu}$ will be of the same type, on the classification here, as μ; in particular, $\hat{\mu}$ will be a Radon measure iff μ is of class D (416F).

434B Compact, analytic and K-analytic spaces For any class of topological spaces, we can enquire which of the seven types of measure described above can be realized by measures on spaces of that class. The enquiry is limited only by our enterprise and diligence in seeking out new classes of topological space. For the spaces studied in §§432-433, however, we have something worth repeating here. On a K-analytic Hausdorff space, a semi-finite Borel measure which is inner regular with respect to the closed sets is tight (432B, 432D); consequently classes B and C of 434A cannot appear, and we are left with only the types A, D_0 and D_1, all of which appear on compact

Hausdorff spaces (434Xb, 434Xf). On an analytic Hausdorff space we have further simplifications: every semi-finite Borel measure is tight (433Ca), and (if X is regular) every closed set is a zero set (423Db). Thus on an analytic regular Hausdorff space only type D_1, of the seven types in 434A, can appear. (If the topology is not regular, we may also get measures of type D_0; see 434Ya.)

434C Radon spaces: Definition For K-analytic Hausdorff spaces, therefore, we have a large gap between the 'bad' measures of class A and the 'good' measures of class D; furthermore, we have an important class of spaces in which type A cannot appear. It is natural to enquire further into the spaces in which every (totally finite) Borel measure is of class D, and (given that no exact description can be found) we are led, as usual, to a definition. A Hausdorff space X is **Radon** if every totally finite Borel measure on X is tight.

434D Universally measurable sets Before going farther with the study of Radon spaces it will be useful to spend a couple of paragraphs on the following concept. Let X be a topological space.

(a) I will say that a subset E of X is **universally measurable** (in X) if it is measured by the completion of every totally finite Borel measure on X; that is, for every totally finite Borel measure μ on X there is a Borel set $F \subseteq X$ such that $E \triangle F$ is μ-negligible.

(b) A subset of X is universally measurable iff it is measured by every complete locally determined topological measure on X. **P** (i) Suppose that $A \subseteq X$ is universally measurable and that μ is a complete locally determined topological measure on X. Let $F \subseteq X$ be such that μF is defined and finite. Then we have a totally finite Borel measure ν on X defined by setting $\nu E = \mu(F \cap E)$ for every Borel set $E \subseteq X$. Now there are Borel sets E, $B \subseteq X$ such that $A \triangle E \subseteq B$ and $\nu B = 0$. In this case, $(A \cap F) \triangle (E \cap F) \subseteq B \cap F$ and $\mu(B \cap F) = 0$, so that (because μ is complete) $A \cap F$ is measured by μ. Because F is arbitrary and μ is locally determined, A is measured by μ. (ii) Suppose that $A \subseteq X$ is measured by every complete locally determined topological measure on X. Then, in particular, it is measured by the completion of any totally finite Borel measure, so is universally measurable. **Q**

(c) The family Σ_{um} of universally measurable subsets of X is a σ-algebra closed under Souslin's operation and including the Borel σ-algebra. (For it is the intersection of the domains of a family of complete totally finite measures, and all these are σ-algebras including the Borel σ-algebra and closed under Souslin's operation, by 431A.) In particular, Souslin-F sets are universally measurable, so (if X is Hausdorff) K-analytic and analytic sets are (422Ha, 423C).

(d) Note that a function $f : X \to \mathbb{R}$ is Σ_{um}-measurable iff it is μ-virtually measurable for every totally finite Borel measure μ on X (122Q, 212Fa). Generally, if Y is another topological space, I will say that $f : X \to Y$ is **universally measurable** if $f^{-1}[H] \in \Sigma_{\mathrm{um}}$ for every open set $H \subseteq Y$; that is, if f is $(\Sigma_{\mathrm{um}}, \mathcal{B}(Y))$-measurable, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y.

(e) In fact, if $f : X \to Y$ is universally measurable, then it is $(\Sigma_{\mathrm{um}}, \Sigma_{\mathrm{um}}^{(Y)})$-measurable, where $\Sigma_{\mathrm{um}}^{(Y)}$ is the algebra of universally measurable subsets of Y. **P** Take $F \in \Sigma_{\mathrm{um}}^{(Y)}$ and a totally finite Borel measure μ on X. If $\hat{\mu}$ is the completion of μ, then the image measure $\nu = \hat{\mu} f^{-1}$ is a complete totally finite topological measure on Y, so measures F, and $f^{-1}[F] \in \operatorname{dom} \hat{\mu}$. As μ is arbitrary, $f^{-1}[F] \in \Sigma_{\mathrm{um}}$; as F is arbitrary, f is $(\Sigma_{\mathrm{um}}, \Sigma_{\mathrm{um}}^{(Y)})$-measurable. **Q**

(f) It follows that if Z is a third topological space and $f : X \to Y$, $g : Y \to Z$ are universally measurable, then $gf : X \to Z$ is universally measurable.

434E Universally Radon-measurable sets A companion idea is the following. Let X be a Hausdorff space.

(a) I will say that a subset E of X is **universally Radon-measurable** if it is measured by every Radon measure on X.

(b) The family Σ_{uRm} of universally Radon-measurable subsets of X is a σ-algebra closed under Souslin's operation and including the algebra Σ_{um} of universally measurable subsets of X (and, *a fortiori*, including the Borel σ-algebra). (Use 434Db and the idea of 434Dc.)

(c) If Y is another topological space, I will say that a function $f : X \to Y$ is **universally Radon-measurable** if $f^{-1}[H] \in \Sigma_{\mathrm{uRm}}$ for every open set $H \subseteq Y$. A function $f : X \to \mathbb{R}$ is Σ_{uRm}-measurable iff it is μ-virtually measurable for every totally finite tight Borel measure μ on X. (Compare 434Dd.)

434F Elementary properties of Radon spaces: Proposition Let X be a Hausdorff space.

(a) The following are equiveridical:

(i) X is a Radon space;

(ii) every semi-finite Borel measure on X is tight;

(iii) if μ is a locally finite Borel measure on X, its c.l.d. version $\tilde{\mu}$ is a Radon measure;

(iv) whenever μ is a totally finite Borel measure on X, and $G \subseteq X$ is an open set with $\mu G > 0$, then there is a compact set $K \subseteq G$ such that $\mu K > 0$;

(v) whenever μ is a non-zero totally finite Borel measure on X, there is a Radon subspace Y of X such that $\mu^* Y > 0$.

(b) If $Y \subseteq X$ is a subspace which is a Radon space in its induced topology, then Y is universally measurable in X.

(c) If X is a Radon space and $Y \subseteq X$, then Y is Radon iff it is universally measurable in X iff it is universally Radon-measurable in X. In particular, all Borel subsets and all Souslin-F subsets of X are Radon spaces.

(d) The family of Radon subspaces of X is closed under Souslin's operation and set difference.

proof (a)(i)$\Rightarrow$(ii) Let μ be a semi-finite Borel measure on X, $E \subseteq X$ a Borel set and $\gamma < \mu E$. Because μ is semi-finite, there is a Borel set H of finite measure such that $\mu(E \cap H) > \gamma$. Set $\nu F = \mu(F \cap H)$ for every Borel set $F \subseteq X$; then ν is a totally finite Borel measure on X, and $\nu E > \gamma$. Because X is a Radon space, there is a compact set $K \subseteq E$ such that $\nu K \geq \gamma$, and now $\mu K \geq \gamma$. As μ, E and γ are arbitrary, (ii) is true.

(ii)$\Rightarrow$(i) and **(i)$\Rightarrow$(v)** are trivial.

(v)$\Rightarrow$(iv) Assume (v), and let μ be a totally finite Radon measure on X and G a non-negligible open set. Set $\nu E = \mu(E \cap G)$ for every Borel set $E \subseteq X$. Then ν is a non-zero totally finite Borel measure on X, so there is a Radon subspace Y of X such that $\nu^* Y > 0$. The subspace measure ν_Y on Y is a Borel measure on Y, so is tight. Since $\nu_Y(Y \setminus G) = \nu(X \setminus G) = 0$, $\nu_Y(Y \cap G) > 0$ and there is a compact set $K \subseteq Y \cap G$ such that $\nu_Y K > 0$. Now $\mu K > 0$. As μ and G are arbitrary, (iv) is true.

not-(i)$\Rightarrow$not-(iv) If X is not Radon, there is a totally finite Borel measure μ on X which is not tight. By 416F(iii), there is an open set $G \subseteq X$ such that

$$\mu G > \sup_{K \subseteq G \text{ is compact}} \mu K = \gamma$$

say. Let $\mathcal{K}$ be the family of compact subsets of G. By 215B(v), there is a non-decreasing sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}$ such that $\mu(K \setminus F) = 0$ for every $K \in \mathcal{K}$, where $F = \bigcup_{n \in \mathbb{N}} K_n$. Observe that

$$\mu F = \lim_{n \to \infty} \mu K_n \leq \gamma < \mu G.$$

Now set $\nu E = \mu(E \cap G \setminus F)$ for every Borel set $E \subseteq X$. Then ν is a Borel measure on X, and $\nu G > 0$. If $K \subseteq G$ is compact, then $\nu K = \mu(K \setminus F) = 0$. So ν and G witness that (iii) is false.

(i)$\Rightarrow$(iii) The point is that $\tilde{\mu}$ is tight. **P** If $\tilde{\mu} E > \gamma$, then, because $\tilde{\mu}$ is semi-finite, there is a set $E' \subseteq E$ such that $\gamma < \tilde{\mu} E' < \infty$; now there is a Borel set $H \subseteq E'$ such that $\mu H = \tilde{\mu} E'$ (213Fc). Setting $\nu F = \mu(H \cap F)$ for every Borel set F, ν is a totally finite Borel measure on X and $\nu H > \gamma$, so there is a compact set $K \subseteq H$ such that $\nu K \geq \gamma$. Since $\mu K < \infty$, $\tilde{\mu} K = \mu K \geq \gamma$ (213Fa), while $K \subseteq E$. As E and γ are arbitrary, $\tilde{\mu}$ is tight. **Q**

On the other hand, every point of X belongs to an open set of finite measure for μ, which is still of finite measure for $\tilde{\mu}$ (213Fa again). So $\tilde{\mu}$ is locally finite; since it is surely complete and locally determined, it is a Radon measure.

(iii)$\Rightarrow$(i) Assume (iii), and let μ be a totally finite Borel measure on X. Then its c.l.d. version $\tilde{\mu}$ is tight. But $\tilde{\mu}$ extends μ (213Hc), so μ also is tight. As μ is arbitrary, X is a Radon space.

(b) Let μ be a totally finite Borel measure on X, and $\hat{\mu}$ its completion; let $\epsilon > 0$. Let μ_Y be the subspace measure on Y, so that μ_Y is a totally finite Borel measure on Y, and is tight. There is a compact set $K \subseteq Y$ such that $\nu K \geq \mu_Y Y - \epsilon$. But this means that

$$\mu^* Y = \mu_Y Y \leq \mu_Y K + \epsilon = \mu^*(K \cap Y) + \epsilon = \mu K + \epsilon \leq \mu_* Y + \epsilon.$$

As ϵ is arbitrary, $\mu_* Y = \mu^* Y$, and Y is measured by $\hat{\mu}$ (413Ef); as μ is arbitrary, Y is universally measurable.

(c) (i) If Y is Radon, it is universally measurable, by (b). (ii) If Y is universally measurable, it is universally Radon-measurable, by 434Eb. (iii) Suppose that Y is universally Radon-measurable, and that ν is a totally finite Borel measure on Y. For Borel sets $E \subseteq X$, set $\mu E = \nu(E \cap Y)$. Then μ is a totally finite Borel measure on X, so its c.l.d. version $\tilde{\mu}$ is a Radon measure on X, by (a-iii). We are supposing that Y is universally Radon-measurable, so, in particular, it must be measured by $\tilde{\mu}$. We have

$$\tilde{\mu}(X \setminus Y) = \sup_{K \subseteq X \setminus Y \text{ is compact}} \tilde{\mu}K = \sup_{K \subseteq X \setminus Y \text{ is compact}} \mu K$$

(213Ha, because μ is totally finite)

$$= \sup_{K \subseteq X \setminus Y \text{ is compact}} \nu(K \cap Y) = 0,$$

and Y is $\tilde{\mu}$-conegligible.

Now suppose that $E \subseteq Y$ is a (relatively) Borel subset of Y. Then E is of the form $F \cap Y$ where F is a Borel subset of X, so that

$$\nu E = \mu F = \tilde{\mu}F = \tilde{\mu}(Y \cap F) = \tilde{\mu}E$$
$$= \sup_{K \subseteq E \text{ is compact}} \mu K = \sup_{K \subseteq E \text{ is compact}} \nu K.$$

As E is arbitrary, ν is tight; as ν is arbitrary, Y is a Radon space.

By 434Dc, it follows that all Borel subsets and all Souslin-F subsets of X are Radon spaces.

(d) The first step is to note that if $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence of Radon subspaces of X with union E, then E is Radon; this is immediate from (a-v) above.

Now let $\langle E_\sigma \rangle_{\sigma\in S}$ be a Souslin scheme, consisting of Radon subsets of X, with kernel A. We know that $E = \bigcup_{\sigma\in S} E_\sigma$ is a Radon space. Every E_σ is universally measurable in E, by (b), so A also is (434Dc), and must be Radon, by (c). Thus the family of Radon subspaces of X is closed under Souslin's operation.

If E and F are Radon subsets of X, then $E \cup F$ is Radon, and, just as above, F is universally measurable in $E \cup F$. But this means that $E \setminus F = (E \cup F) \setminus F$ is universally measurable in $E \cup F$, so that $E \setminus F$ is Radon.

434G Just as we can address the question 'when can we be sure that every Borel measure is of class D?' in terms of the definition of 'Radon' space (434C), we can form other classes of topological space by declaring that the Borel measures they support must be of certain kinds. Three definitions which lead to interesting patterns of ideas are the following.

Definitions (a) A topological space X is **Borel-measure-compact** (Gardner & Pfeffer 84) if every totally finite Borel measure on X which is inner regular with respect to the closed sets is τ-additive, that is, X carries no measure of class B in the classification of 434A.

(b) A topological space X is **Borel-measure-complete** (Gardner & Pfeffer 84) if every totally finite Borel measure on X is τ-additive. (If X is regular and Hausdorff, this amounts to saying that X carries no measures of classes A or B in the classification of 434A.)

(c) A Hausdorff space X is **pre-Radon** (also called '**hypo-radonian**', '**semi-radonian**') if every τ-additive totally finite Borel measure on X is tight. (If X is regular, this amounts to saying that X carries no measure of class C in the classification of 434A.)

434H Proposition Let X be a topological space and $\mathcal{B}$ its Borel σ-algebra.

(a) The following are equiveridical:

(i) X is Borel-measure-compact;

(ii) every semi-finite Borel measure on X which is inner regular with respect to the closed sets is τ-additive;

(iii) every effectively locally finite Borel measure on X which is inner regular with respect to the closed sets has an extension to a quasi-Radon measure;

(iv) every totally finite Borel measure on X which is inner regular with respect to the closed sets has a support;

(v) if μ is a non-zero totally finite Borel measure on X, inner regular with respect to the closed sets, and $\mathcal{G}$ is an open cover of X, then there is some $G \in \mathcal{G}$ such that $\mu G > 0$.

(b) If X is Lindelöf (in particular, if X is a K-analytic Hausdorff space), it is Borel-measure-compact.

(c) If X is Borel-measure-compact and $A \subseteq X$ is a Souslin-F set, then A is Borel-measure-compact in its subspace topology. In particular, any Baire subset of X is Borel-measure-compact.

proof (a)(i)⇒(ii) Assume (i), and let μ be a semi-finite Borel measure on X which is inner regular with respect to the closed sets. Let $\mathcal{G}$ be an upwards-directed family of open sets with union G^*, and $\gamma < \mu G^*$. Because μ is semi-finite, there is an $H \in \mathcal{B}$ such that $\mu H < \infty$ and $\mu(H \cap G^*) \ge \gamma$. Set $\nu E = \mu(E \cap H)$ for every $E \in \mathcal{B}$; then ν is a totally finite Borel measure on X. For any $E \in \mathcal{B}$,

$$\nu E = \mu(E \cap H) = \sup\{\mu F : F \subseteq E \cap H \text{ is closed}\} \le \sup\{\nu F : F \subseteq E \text{ is closed}\},$$

so ν is inner regular with respect to the closed sets, and must be τ-additive. Now

$$\gamma \le \nu G^* = \sup_{G \in \mathcal{G}} \nu G \le \sup_{G \in \mathcal{G}} \mu G.$$

As γ and $\mathcal{G}$ is arbitrary, μ is τ-additive.

(ii)⇒(iii) Assume (ii), and let μ be an effectively locally finite Borel measure on X which is inner regular with respect to the closed sets. Then it is semi-finite (411Gd), therefore τ-additive. By 415L, it has an extension to a quasi-Radon measure on X.

(iii)⇒(i) If (iii) is true and μ is a totally finite Borel measure on X which is inner regular with respect to the closed sets, then μ has an extension to a quasi-Radon measure, which is τ-additive, so μ also is τ-additive (411C).

(i)⇒(iv) Use 411Nd.

(iv)⇒(v) Suppose that (iv) is true, that μ is a non-zero totally finite Borel measure on X which is inner regular with respect to the closed sets, and that $\mathcal{G}$ is an open cover of X. If F is the support of μ, then $\mu F > 0$ so $F \ne \emptyset$; there must be some $G \in \mathcal{G}$ meeting F, and now $\mu G > 0$.

not-(i)⇒not-(v) Suppose that there is a totally finite Borel measure μ on X, inner regular with respect to the closed sets, which is not τ-additive. Let $\mathcal{G}$ be an upwards-directed family of open sets such that $\mu G^* > \gamma$, where $G^* = \bigcup \mathcal{G}$ and $\gamma = \sup_{G \in \mathcal{G}} \mu G$. Let $\langle G_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence in $\mathcal{G}$ such that $\mu(G \setminus G_0^*)$ for every $G \in \mathcal{G}$, where $G_0^* = \bigcup_{n \in \mathbb{N}} G_n$ (215B(v)). Then $\mu G_0^* \le \gamma$, so there is a closed set $F \subseteq G^* \setminus G_0^*$ such that $\mu F > 0$.

Let ν be the Borel measure on X defined by setting $\nu E = \mu(E \cap F)$ for every $E \in \mathcal{B}$. As in the argument for (i)⇒(ii), ν is inner regular with respect to the closed sets. Consider $\mathcal{H} = \mathcal{G} \cup \{X \setminus F\}$; this is an open cover of X. If $G \in \mathcal{G}$ then $\nu G \le \mu(G \setminus G_0^*) = 0$, so $\nu H = 0$ for every $H \in \mathcal{H}$; thus ν and $\mathcal{H}$ witness that (v) is false.

(b) Use (a-v) and 422Gg.

(c) Let μ be a Borel measure on A which is inner regular with respect to the closed sets, that is to say, the relatively closed sets in A. Let ν be the corresponding Borel measure on X, defined by setting $\nu E = \mu(A \cap E)$ for every $E \in \mathcal{B}$. Let $\hat{\nu}$ be the completion of ν. Putting 431D and 421M together, we see that $\hat{\nu} A = \sup\{\hat{\nu} F : F \subseteq A$ is closed in $X\}$, that is, $\nu X = \sup\{\mu F : F \subseteq A$ is closed in $X\}$. But this means that if $E \in \mathcal{B}$ and $\gamma < \nu E$, there is a closed set F in X such that $F \subseteq A$ and $\mu(E \cap F) > \gamma$; now there is a relatively closed set $F' \subseteq A$ such that $F' \subseteq E \cap F$ and $\mu F' \ge \gamma$, and as F' must be relatively closed in F it is closed in X, while $\nu F' \ge \gamma$. Since E and γ are arbitrary, ν is inner regular with respect to the closed sets, and will be τ-additive.

Now suppose that $\mathcal{G}$ is an upwards-directed family of relatively open subsets of A. Set $\mathcal{H} = \{H : H \subseteq X$ is open, $H \cap A \in \mathcal{G}\}$. Then $\mathcal{H}$ is upwards-directed, so

$$\mu(\bigcup \mathcal{G}) = \nu(\bigcup \mathcal{H}) = \sup_{H \in \mathcal{H}} \nu H = \sup_{G \in \mathcal{G}} \mu G.$$

As μ and $\mathcal{G}$ are arbitrary, A is Borel-measure-compact.

By 421L, it follows that any Baire subset of X is Borel-measure-compact.

434I Proposition Let X be a topological space.

(a) The following are equiveridical:

(i) X is Borel-measure-complete;

(ii) every semi-finite Borel measure on X is τ-additive;

(iii) every totally finite Borel measure on X has a support;

(iv) whenever μ is a totally finite Borel measure on X there is a base $\mathcal{U}$ for the topology of X such that $\mu(\bigcup\{U : U \in \mathcal{U}, \mu U = 0\}) = 0$.

(b) If X is regular, it is Borel-measure-complete iff every effectively locally finite Borel measure on X has an extension to a quasi-Radon measure.

(c) If X is Borel-measure-complete, it is Borel-measure-compact.

(d) If X is Borel-measure-complete, so is every subspace of X.

(e) If X is hereditarily Lindelöf (for instance, if X is separable and metrizable, see 4A2P(a-iii)), it is Borel-measure-complete, therefore Borel-measure-compact.

proof (a)(i)⇒(ii) Use the argument of (i)⇒(ii) of 434Ha; this case is simpler, because we do not need to check that the auxiliary measure ν is inner regular.

(ii)⇒(i) is trivial.

(i)⇒(iv) If X is Borel-measure-complete and μ is a totally finite Borel measure on X, take $\mathcal{U}$ to be the family of all open subsets of X. This is surely a base for the topology, and setting $\mathcal{U}_0 = \{U : U \in \mathcal{U}, \mu U = 0\}$, $\mathcal{U}_0$ is upwards-directed so $\mu(\bigcup \mathcal{U}_0) = \sup_{U \in \mathcal{U}_0} \mu U = 0$, as required.

(iv)⇒(iii) Assume (iv), and let μ be a totally finite Borel measure on X. Take a base $\mathcal{U}$ as in (iv), so that $\mu(\bigcup \mathcal{U}_0) = 0$, where $\mathcal{U}_0$ is the family of negligible members of $\mathcal{U}$. Set $F = X \setminus \bigcup \mathcal{U}_0$, so that F is a conegligible closed set. If $G \subseteq X$ is an open set meeting F, there is a member U of $\mathcal{U}$ such that $U \subseteq G$ and $U \cap F \neq \emptyset$; now $U \notin \mathcal{U}$ so

$$\mu(G \cap F) = \mu G \geq \mu U > 0.$$

As G is arbitrary, F is self-supporting and is the support of μ.

(iii)⇒(i) Assume (iii), and let μ be a totally finite Borel measure on X. Let $\mathcal{G}$ be an upwards-directed family of open sets with union G^*. Set $\gamma = \sup_{G \in \mathcal{G}} \mu G$. Let $\langle G_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence in $\mathcal{G}$ such that $\mu(G \setminus G_0^*)$ for every $G \in \mathcal{G}$, where $G_0^* = \bigcup_{n \in \mathbb{N}} G_n$ (215B(v)). Then $\mu G_0^* \leq \gamma$. Let ν be the Borel measure on X defined by setting $\mu E = \mu(E \cap G^* \setminus G_0^*)$ for every $E \in \mathcal{B}$. Then ν has a support F say. Now $\nu G = 0$ for every $G \in \mathcal{G}$, so $F \cap G = \emptyset$ for every $G \in \mathcal{G}$, and $F \cap G^* = \emptyset$; but this means that

$$\mu(G^* \setminus G_0^*) = \nu X = \nu F = \mu(F \cap G^* \setminus G_0^*) = 0.$$

Accordingly $\mu G^* = \gamma$. As μ and $\mathcal{G}$ are arbitrary, X is Borel-measure-complete.

(b) If X is Borel-measure-complete and μ is an effectively locally finite Borel measure on X, then μ is τ-additive, by (a-ii), so extends to a quasi-Radon measure on X, by 415Cb. If effectively locally finite Borel measures on X extend to quasi-Radon measures, then any totally finite Borel measure is τ-additive, by 411C, and X is Borel-measure-complete.

(c) Immediate from the definitions.

(d) If $Y \subseteq X$ and μ is a totally finite Borel measure on Y, let ν be the Borel measure on X defined by setting $\nu E = \mu(E \cap Y)$ for every Borel set $E \subseteq X$. Then ν is τ-additive. So if $\mathcal{G}$ is an upwards-directed family of relatively open subsets of Y, and we set $\mathcal{H} = \{H : H \subseteq X \text{ is open}, H \cap Y \in \mathcal{G}\}$, we shall get

$$\mu(\bigcup \mathcal{G}) = \nu(\bigcup \mathcal{H}) = \sup_{H \in \mathcal{H}} \nu H = \sup_{G \in \mathcal{G}} \mu G.$$

As μ and $\mathcal{G}$ are arbitrary, Y is Borel-measure-complete.

(e) If μ is a totally finite Borel measure on X and $\mathcal{G}$ is a non-empty upwards-directed family of open subsets of X with union G^*, then there is a sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{G}$ with union G^*, by 4A2H(c-i). Because $\mathcal{G}$ is upwards-directed, there is a non-decreasing sequence $\langle G'_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{G}$ such that $G'_n \supseteq G_n$ for every $n \in \mathbb{N}$, so that

$$\mu G^* = \lim_{n \to \infty} \mu G'_n \leq \sup_{G \in \mathcal{G}} \mu G.$$

As μ and $\mathcal{G}$ are arbitrary, X is Borel-measure-complete.

434J Proposition Let X be a Hausdorff space.

(a) The following are equiveridical:

(i) X is pre-Radon;

(ii) every effectively locally finite τ-additive Borel measure on X is tight;

(iii) whenever μ is a non-zero totally finite τ-additive Borel measure on X, there is a compact set $K \subseteq X$ such that $\mu K > 0$;

(iv) whenever μ is a totally finite τ-additive Borel measure on X, $\mu X = \sup_{K \subseteq X \text{ is compact}} \mu K$;

(v) whenever μ is a locally finite effectively locally finite τ-additive Borel measure on X, the c.l.d. version of μ is a Radon measure on X.

(b) If X is pre-Radon, then every locally finite quasi-Radon measure on X is a Radon measure.

(c) If X is regular and every totally finite quasi-Radon measure on X is a Radon measure, then X is pre-Radon.

(d) If X is pre-Radon, then any universally Radon-measurable subspace (in particular, any Borel subset or Souslin-F subset) of X is pre-Radon.

(e) If $A \subseteq X$ is pre-Radon in its subspace topology, it is universally Radon-measurable in X.

(f) If X is K-analytic (for instance, if it is compact), it is pre-Radon.

(g) If X is completely regular and Čech-complete (for instance, if it is locally compact (4A2Gk), or metrizable and complete under a metric inducing its topology (4A2Md)), it is pre-Radon.

(h) If $X = \prod_{i \in I} X_i$ where $\langle X_i \rangle_{i \in I}$ is a countable family of pre-Radon Hausdorff spaces, then X is pre-Radon.

(i) If every point of X belongs to a pre-Radon open subset of X, then X is pre-Radon.

proof (a)(i)⇒(ii) Suppose that X is pre-Radon, that μ is an effectively locally finite τ-additive Borel measure on X, that $E \subseteq X$ is Borel, and that $\gamma < \mu E$. Because μ is semi-finite, there is a Borel set $H \subseteq X$ of finite measure such that $\mu(H \cap E) > \gamma$. Set $\nu F = \mu(F \cap H)$ for every Borel set $F \subseteq X$; then ν is a totally finite Borel measure on X, and is τ-additive by 414Ea. Now $\nu E > \gamma$, so there is a compact set $K \subseteq E$ such that $\gamma \le \nu K \le \mu K$. As E is arbitrary, μ is tight.

(ii)⇒(iii) is trivial.

(iii)⇒(iv) Assume (iii), and let μ be a totally finite τ-additive Borel measure on X. Let $\mathcal{K}$ be the family of compact subsets of X and set $\alpha = \sup_{K \in \mathcal{K}} \mu K$. **?** Suppose, if possible, that $\mu X > \alpha$. Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{K}$ such that $\sup_{n \in \mathbb{N}} \mu K_n = \alpha$, and set $L = \bigcup_{n \in \mathbb{N}} K_n$; then

$$\mu L = \lim_{n \to \infty} \mu(\textstyle\bigcup_{i \le n} K_i) = \alpha.$$

Set $\nu E = \mu(E \setminus L)$ for every Borel set $E \subseteq X$. Then ν is a non-zero totally finite Borel measure on X, and is τ-additive, by 414Ea again. So there is a $K \in \mathcal{K}$ such that $\nu K > 0$. But now there is an $n \in \mathbb{N}$ such that $\nu K + \mu K_n > \alpha$, and in this case $K \cup K_n \in \mathcal{K}$ and

$$\mu(K \cup K_n) = \mu(K \setminus K_n) + \mu K_n \ge \nu K + \mu K_n > \alpha,$$

which is impossible. **X** So $\mu X = \alpha$, as required.

(iv)⇒(i) Assume (iv), and let μ be a totally finite τ-additive Borel measure on X. Suppose that $E \subseteq X$ is Borel and that $\gamma < \mu E$. By (iii), there is a compact set $K \subseteq X$ such that $\mu K > \mu X - \mu E + \gamma$, so that $\mu(E \cap K) > \gamma$. Consider the subspace measure μ_K on K. By 414K, this is τ-additive, so inner regular with respect to the closed subsets of K (414Mb). There is therefore a relatively closed subset F of K such that $F \subseteq K \cap E$ and $\mu_K F \ge \gamma$; but now F is a compact subset of E and $\mu F \ge \gamma$. As E and γ are arbitrary, μ is tight. As μ is arbitrary, X is pre-Radon.

(ii)⇒(v) Assume (ii), and let μ be a locally finite effectively locally finite τ-additive Borel measure on X. Then μ is tight, so by 416F(ii) its c.l.d. version is a Radon measure.

(v)⇒(iv) Assume (v), and let μ be a totally finite τ-additive Borel measure on X. Then the c.l.d. version $\tilde{\mu}$ of μ is a Radon measure; but $\tilde{\mu}$ extends μ (213Hc), so

$$\sup_{K \subseteq X \text{ is compact}} \mu K = \sup_{K \subseteq X \text{ is compact}} \tilde{\mu} K = \tilde{\mu} X = \mu X.$$

(b) Let μ be a locally finite quasi-Radon measure on X. By (a-ii), μ is tight; by 416C, μ is a Radon measure.

(c) Let μ be a totally finite τ-additive Borel measure on X. Because X is regular, the c.l.d. version $\tilde{\mu}$ or μ is a quasi-Radon measure (415Cb), therefore a Radon measure; but $\tilde{\mu}$ extends μ (213Hc again), so μ, like $\tilde{\mu}$, must be tight. As μ is arbitrary, X is pre-Radon.

(d) Let A be a universally Radon-measurable subset of X, and μ a totally finite τ-additive Borel measure on A. Set $\nu E = \mu(E \cap A)$ for every Borel set $E \subseteq X$; then ν is a totally finite τ-additive Borel measure on X. So its c.l.d. version (that is, its completion $\hat{\nu}$, by 213Ha) is a Radon measure on X, by (a-v). Now $\hat{\nu}$ measures A, so

$$\mu A = \nu^* A = \hat{\nu} A = \sup\{\hat{\nu} K : K \subseteq A \text{ is compact}\} = \sup\{\mu K : K \subseteq A \text{ is compact}\}.$$

By (a-iv), A is pre-Radon.

(e) Let μ be a totally finite Radon measure on X. Then the subspace measure μ_A is τ-additive (414K), so its restriction ν to the Borel σ-algebra of A is still τ-additive. Because A is pre-Radon,

$$\begin{aligned} \mu^* A = \mu_A A = \nu A &= \sup\{\nu K : K \subseteq A \text{ is compact}\} \\ &= \sup\{\mu K : K \subseteq A \text{ is compact}\} = \mu_* A, \end{aligned}$$

and μ measures A (413Ef). As μ is arbitrary, A is universally Radon-measurable.

(f) Put 432B and (a-iv) together.

(g) If we identify X with a G_δ set in a compact Hausdorff space Z, then Z is pre-Radon, by (f), so X is pre-Radon, by (d).

(h) Let μ be a totally finite τ-additive Borel measure on X, and $\epsilon > 0$. Let $\langle \epsilon_i \rangle_{i \in I}$ be a family of strictly positive real numbers such that $\sum_{i \in I} \epsilon_i \le \epsilon$ (4A1P). For each $i \in I$ and Borel set $F \subseteq X_i$, set $\mu_i F = \mu \pi_i^{-1}[F]$, where $\pi_i(x) = x(i)$ for $x \in X$; because $\pi_i : X \to X_i$ is continuous, μ_i is a totally finite τ-additive Borel measure on X_i. Because X_i is pre-Radon, we can find a compact set $K_i \subseteq X_i$ such that $\mu_i(X_i \setminus K_i) \le \epsilon_i$, by (a-iv). Now $K = \prod_{i \in I} K_i$ is compact (3A3J), and $X \setminus K \subseteq \bigcup_{i \in I} \pi_i^{-1}[X_i \setminus K_i]$, so

$$\mu(X \setminus K) \le \sum_{i \in I} \mu_i(X_i \setminus K_i) \le \sum_{i \in I} \epsilon_i \le \epsilon.$$

As ϵ and μ are arbitrary, X satisfies the condition of (a-iv), and is pre-Radon.

(i) Let $\mathcal{G}$ be a cover of X by pre-Radon open sets. Let μ be a non-zero totally finite τ-additive Borel measure on X. Then $\mu X = \sup\{\mu(\bigcup \mathcal{G}_0) : \mathcal{G}_0 \subseteq \mathcal{G}$ is finite$\}$, so there is some $G \in \mathcal{G}$ such that $\mu G > 0$. Now the subspace measure μ_G is a non-zero totally finite τ-additive Borel measure on G, so there is a compact set $K \subseteq G$ such that $\mu_G K > 0$, in which case $\mu K > 0$. As μ is arbitrary, X is pre-Radon, by (a-iii).

434K I return to criteria for deciding whether Hausdorff spaces are Radon.

Proposition (a) A Hausdorff space is Radon iff it is Borel-measure-complete and pre-Radon.

(b) An analytic Hausdorff space is Radon. In particular, any compact metrizable space is Radon and any Polish space is Radon.

(c) ω_1 and $\omega_1 + 1$, with their order topologies, are not Radon.

(d) For a set I, $[0,1]^I$ is Radon iff $\{0,1\}^I$ is Radon iff I is countable.

(e) A hereditarily Lindelöf K-analytic Hausdorff space is Radon; in particular, the split interval (343J, 419L) is Radon.

proof (a) Put the definitions 434C, 434Gb and 434Gc together, recalling that a tight measure is necessarily τ-additive (411E).

(b) 433Cb.

(c) Dieudonné's measure (411Q) is a Borel measure on ω_1 which is not tight, so ω_1 is certainly not a Radon space; as it is an open set in $\omega_1 + 1$, and the subspace topology inherited from $\omega_1 + 1$ is the order topology of ω_1 (4A2S(a-iii)), $\omega_1 + 1$ cannot be Radon (434Fc).

(d) If I is countable, then $\{0,1\}^I$ and $[0,1]^I$ are compact metrizable spaces, so are Radon. If I is uncountable, then $\omega_1 + 1$, with its order topology, is homeomorphic to a closed subset of $\{0,1\}^I$. **P** Set $\kappa = \#(I)$. For $\xi \le \omega_1$, $\eta < \kappa$ set $x_\xi(\eta) = 1$ if $\eta < \xi$, 0 if $\xi \le \eta$. The map $\xi \mapsto x_\xi : \omega_1 + 1 \to \{0,1\}^\kappa$ is injective because $\kappa \ge \omega_1$, and is continuous because all the sets $\{\xi : x_\xi(\eta) = 0\} = (\omega_1 + 1) \cap (\eta + 1)$ are open-and-closed in $\omega_1 + 1$. Since $\omega_1 + 1$ is compact in its order topology (4A2S(a-i)), it is homeomorphic to its image in $\{0,1\}^\kappa \cong \{0,1\}^I$. **Q**

By 434Fc, $\{0,1\}^I$ cannot be a Radon space. Since $\{0,1\}^I$ is a closed subset of $[0,1]^I$, $[0,1]^I$ also is not a Radon space.

(e) Suppose that X is a hereditarily Lindelöf K-analytic Hausdorff space. Then it is Borel-measure-complete by 434Ie and pre-Radon by 434Jf, so by (a) here it is Radon.

Since the split interval is compact and Hausdorff and hereditarily Lindelöf (419La), it is a Radon space.

434L It is worth noting an elementary special property of metric spaces.

Proposition If (X, ρ) is a metric space, then any quasi-Radon measure on X is inner regular with respect to the totally bounded subsets of X.

proof Let μ be a quasi-Radon measure on X and Σ its domain. Suppose that $E \in \Sigma$ and $\gamma < \mu E$. Then there is an open set G of finite measure such that $\mu(E \cap G) > \gamma$; set $\delta = \mu(E \cap G) - \gamma$. For $n \in \mathbb{N}$, $I \subseteq X$ set $H(n, I) = \bigcup_{x \in I}\{y : \rho(y, x) < 2^{-n}\}$. Then $\{H(n, I) : I \in [X]^{<\omega}\}$ is an upwards-directed family of open sets covering X. Because μ is τ-additive, there is a finite set $I_n \subseteq X$ such that $\mu(G \setminus H(n, I_n)) \le 2^{-n-1}\delta$. Consider $F = \bigcap_{n \in \mathbb{N}} H(n, I_n)$. This is totally bounded and $\mu(G \setminus F) \le \delta$, so $E \cap F$ is totally bounded and $\mu(E \cap F) \ge \gamma$. As E and γ are arbitrary, μ is inner regular with respect to the totally bounded sets.

434M I turn next to a couple of ideas depending on countable compactness.

Lemma Let X be a countably compact topological space and $\mathcal{E}$ a non-empty family of closed subsets of X with the finite intersection property. Then there is a Borel probability measure μ on X, inner regular with respect to the closed sets, such that $\mu F = 1$ for every $F \in \mathcal{E}$.

proof (a) By Zorn's lemma, $\mathcal{E}$ is included in a maximal family $\mathcal{E}^*$ of closed subsets of X with the finite intersection property.

(i) If $F \subseteq X$ is closed and $F \cap F_0 \cap \ldots \cap F_n \neq \emptyset$ for every $F_0, \ldots, F_n \in \mathcal{E}^*$, then $\mathcal{E}^* \cup \{F\}$ has the finite intersection property, so $F \in \mathcal{E}^*$.

(ii) If F, $F' \in \mathcal{E}^*$, then $F \cap F' \cap F_0 \cap \ldots \cap F_n \neq \emptyset$ for all $F_0, \ldots, F_n \in \mathcal{E}^*$, so $F \cap F' \in \mathcal{E}^*$.

(iii) If $F \subseteq X$ is closed and $F \cap F' \in \mathcal{E}^*$ for every $F' \in \mathcal{E}^*$, then $F \cap F_0 \cap \ldots \cap F_n \in \mathcal{E}^*$ for every $F_0, \ldots, F_n \in \mathcal{E}^*$ (because $F_0 \cap \ldots \cap F_n \in \mathcal{E}^*$, by (ii)), so $F \in \mathcal{E}^*$.

(iv) If $\langle F_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathcal{E}^*$, with intersection F, and $F' \in \mathcal{E}^*$, then $F' \cap \bigcap_{i \le n} F_i \neq \emptyset$ for every $n \in \mathbb{N}$. Because X is countably compact, $F' \cap F \neq \emptyset$ (4A2G(f-ii)). As F' is arbitrary, $F \in \mathcal{E}^*$, by (iii). Thus $\mathcal{E}^*$ is closed under countable intersections.

(b) Set

$$\Sigma = \{E : E \subseteq X, \text{ there is an } F \in \mathcal{E}^* \text{ such that either } F \subseteq E \text{ or } F \cap E = \emptyset\},$$

and define $\hat{\mu} : \Sigma \to \{0, 1\}$ by saying that $\hat{\mu}E = 1$ if there is some $F \in \mathcal{E}^*$ such that $F \subseteq E$, 0 otherwise. Then $\hat{\mu}$ is a probability measure on X. **P**

(i) $\emptyset \in \Sigma$ because $\mathcal{E}^* \supseteq \mathcal{E}$ is not empty.

(ii) $X \setminus E \in \Sigma$ whenever $E \in \Sigma$ because the definition of Σ is symmetric between E and $X \setminus E$.

(iii) If $\langle E_n \rangle_{n \in \mathbb{N}}$ is any sequence in Σ with union E, then either there are $n \in \mathbb{N}$ and $F \in \mathcal{E}^*$ such that $F \subseteq E_n \subseteq E$ and $E \in \Sigma$, or for every $n \in \mathbb{N}$ there is an $F_n \in \mathcal{E}^*$ such that $F_n \cap E_n = \emptyset$. In this case $F = \bigcap_{n \in \mathbb{N}} F_n \in \mathcal{E}^*$, by (a-iv), and $E \cap F = \emptyset$, so again $E \in \Sigma$. Thus Σ is a σ-algebra of subsets of X.

(iv) $\hat{\mu}\emptyset = 0$ because $\emptyset$ cannot belong to $\mathcal{E}^*$.

(v) If $\langle E_n \rangle_{n \in \mathbb{N}}$ is any disjoint sequence in Σ with union E, then either there is some n such that $\hat{\mu}E_n = 1$, in which case $\hat{\mu}E_i = 0$ for every $i \neq n$ (because any two members of $\mathcal{E}^*$ must meet) and $\hat{\mu}E = 1 = \sum_{i=0}^{\infty} \hat{\mu}E_i$, or $\hat{\mu}E_i = 0$ for every i, in which case, just as in (iii), $\hat{\mu}E = 0 = \sum_{i=0}^{\infty} \hat{\mu}E_i$. Thus $\hat{\mu}$ is a measure.

(vi) Because $\mathcal{E}^* \neq \emptyset$, $\hat{\mu}X = 1$. Thus $\hat{\mu}$ is a probability measure. **Q**

(c) If $F \subseteq X$ is a closed set, then either F itself belongs to $\mathcal{E}^*$, so $F \in \Sigma$, or there is some $F' \in \mathcal{E}^*$ such that $F \cap F' = \emptyset$, in which case again $F \in \Sigma$. So Σ contains every closed set, therefore every Borel set, and $\hat{\mu}$ is a topological measure. By construction, $\hat{\mu}$ is inner regular with respect to $\mathcal{E}^*$ and therefore with respect to the closed sets. Finally, if $F \in \mathcal{E}$ then $F \in \mathcal{E}^*$, so $\hat{\mu}F = 1$. We may therefore take μ to be the restriction of $\hat{\mu}$ to the Borel σ-algebra of X, and μ will be a Borel measure on X, inner regular with respect to the closed sets, such that $\mu E = 1$ for every $E \in \mathcal{E}$.

434N Proposition (a) Let X be a Borel-measure-compact topological space. Then closed countably compact subsets of X are compact.

(b) Let X be a Borel-measure-complete topological space. Then countably compact subsets of X are compact.

(c) Let X be a Hausdorff Borel-measure-complete topological space. Then compact subsets of X are countably tight.

(d) In particular, any Radon compact Hausdorff space is countably tight.

proof (a) Let C be a closed countably compact subset of X. Let $\mathcal{F}$ be an ultrafilter on C. Let $\mathcal{E}$ be the family of closed subsets of C belonging to $\mathcal{F}$. Then $\mathcal{E}$ has the finite intersection property, so by 434M there is a Borel probability measure μ on C, inner regular with respect to the closed sets, such that $\mu E = 1$ for every $E \in \mathcal{E}$. Let ν be the Borel measure on X defined by setting $\nu H = \mu(C \cap H)$ for every Borel set $H \subseteq X$. Then ν is also inner regular with respect to the closed sets (of either C or X); because X is Borel-measure-compact, ν has a support F (434H(a-iv)). Since $\nu F = \nu X = 1$, $F \cap C \neq \emptyset$; take $x \in F \cap C$. If G is any open set (in X) containing x, then $\mu(C \setminus G) = \nu(X \setminus G) < 1$, so $C \setminus G \notin \mathcal{F}$ and $C \cap G \in \mathcal{F}$. As G is arbitrary, $\mathcal{F} \to x$; as $\mathcal{F}$ is arbitrary, C is compact.

(b) Repeat the argument of (a). Let C be a countably compact subset of X and $\mathcal{F}$ an ultrafilter on C. Let $\mathcal{E}$ be the family of relatively closed subsets of C belonging to $\mathcal{F}$. Then there is a Borel probability measure μ on C such that $\mu E = 1$ for every $E \in \mathcal{E}$. Let ν be the Borel measure on X defined by setting $\nu H = \mu(C \cap H)$ for every Borel set $H \subseteq X$. Because X is Borel-measure-complete, ν has a support F (434I(a-iii)). Since $\nu F = \nu X = 1$, $F \cap C \neq \emptyset$;

take $x \in F \cap C$. If G is any open set containing x, then $\nu(X \setminus G) < 1$, so $C \setminus G \notin \mathcal{F}$ and $C \cap G \in \mathcal{F}$. As G is arbitrary, $\mathcal{F} \to x$; as $\mathcal{F}$ is arbitrary, C is compact.

(c) Again let C be a (countably) compact subset of X. Take $A \subseteq C$, and set $C_0 = \bigcup\{\overline{B} : B \in [A]^{\le\omega}\}$. Then C_0 is countably compact. **P** If $\langle y_n\rangle_{n\in\mathbb{N}}$ is any sequence in C_0, it has a cluster point $y \in C$. For each $n \in \mathbb{N}$ there is a countable set $B_n \subseteq A$ such that $y_n \in \overline{B_n}$. Now $B = \bigcup_{n\in\mathbb{N}} B_n$ is a countable subset of A, and $y \in \overline{B} \subseteq C_0$, so y is a cluster point of $\langle y_n\rangle_{n\in\mathbb{N}}$ in C_0. As $\langle y_n\rangle_{n\in\mathbb{N}}$ is arbitrary, C_0 is countably compact. **Q**

By (b), C_0 is compact, therefore closed, and must include $\overline{A}$. Thus every point of $\overline{A}$ is in the closure of some countable subset of A. As A is arbitrary, C is countably tight.

(d) Finally, a compact Radon Hausdorff space is Borel-measure-complete (434Ka) and countably compact, therefore countably tight.

434O Quasi-dyadic spaces I wish now to present a result in an entirely different direction. Measures of type B_1 in the classification of 434A (completion regular, but not τ-additive) seem to be hard to come by. The next theorem shows that on a substantial class of spaces they cannot appear. First, we need a definition.

Definition A topological space X is **quasi-dyadic** if it is expressible as a continuous image of a product of separable metrizable spaces.

I give some elementary results to indicate what kind of spaces we have here.

434P Proposition (a) A continuous image of a quasi-dyadic space is quasi-dyadic.

(b) Any product of quasi-dyadic spaces is quasi-dyadic.

(c) A space with a countable network is quasi-dyadic.

(d) A Baire subset of a quasi-dyadic space is quasi-dyadic.

(e) If X is any topological space, a countable union of quasi-dyadic subspaces of X is quasi-dyadic.

proof (a) Immediate from the definition.

(b) Again immediate; if X_i is a continuous image of $\prod_{j\in J_i} Y_{ij}$, where Y_{ij} is a separable metrizable space for every $i \in I$ and $j \in J_i$, then $\prod_{i\in I} X_i$ is a continuous image of $\prod_{i\in I,j\in J_i} Y_{ij}$.

(c) Let $\mathcal{E}$ be a countable network for the topology of X. On X let $\sim$ be the equivalence relation in which $x \sim y$ if they belong to just the same members of $\mathcal{E}$; let Y be the space $X/\sim$ of equivalence classes, and $\phi : X \to Y$ the canonical map. Y has a separable metrizable topology with base $\{\phi[E] : E \in \mathcal{E}\} \cup \{\phi[X \setminus E] : E \in \mathcal{E}\}$. Let I be any set such that $\#(\{0,1\}^I) \ge \#(X)$, and for each $y \in Y$ let $f_y : \{0,1\}^I \to y$ be a surjection. Then we have a continuous surjection $f : Y \times \{0,1\}^I \to X$ given by saying that $f(y,z) = f_y(z)$ for $y \in Y$ and $z \in \{0,1\}^I$.

(d) Let $\langle Y_i\rangle_{i\in I}$ be a family of separable metrizable spaces with product Y and $f : Y \to X$ a continuous surjection. If $W \subseteq Y$ is a Baire set, it is determined by coordinates in a countable subset of I (4A3Nb), so can be regarded as $W' \times \prod_{i\in I\setminus J} Y_i$, where $J \subseteq I$ is countable and $W' \subseteq \prod_{i\in J} Y_i$; as $\prod_{i\in J} Y_i$ and W' are separable metrizable spaces (4A2Pa), W can be thought of as a product of separable metrizable spaces. Now the set $\{E : E \subseteq X,\ f^{-1}[E]$ is a Baire set in $Y\}$ is a σ-algebra containing every zero set in X, so contains every Baire set. Thus every Baire subset of X is a continuous image of a Baire subset of Y, and is therefore quasi-dyadic.

(e) If $E_n \subseteq X$ is quasi-dyadic for each $n \in \mathbb{N}$, then $Z = \mathbb{N} \times \prod_{n\in\mathbb{N}} E_n$ is quasi-dyadic, and $f : Z \to \bigcup_{n\in\mathbb{N}} E_n$ is a continuous surjection, where $f(n, \langle x_i\rangle_{i\in\mathbb{N}}) = x_n$. So $\bigcup_{n\in\mathbb{N}} E_n$ is quasi-dyadic.

434Q Theorem (Fremlin & Grekas 95) A semi-finite completion regular topological measure on a quasi-dyadic space is τ-additive.

proof ? Suppose, if possible, otherwise.

(a) The first step is the standard reduction to the case in which $\mu X = 1$ and X is covered by open sets of zero measure. In detail: suppose that X is a quasi-dyadic space and μ_0 is a semi-finite completion regular topological measure on X which is not τ-additive. Let $\mathcal{G}$ be an upwards-directed family of open sets in X such that $\mu_0(\bigcup\mathcal{G})$ is strictly greater than $\sup_{G\in\mathcal{G}} \mu_0 G = \gamma$ say. Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{G}$ such that $\lim_{n\to\infty} \mu_0 G_n = \gamma$, and set $H_0 = \bigcup_{n\in\mathbb{N}} G_n$, so that $\mu_0 H_0 = \gamma$; take a closed set $Z \subseteq \bigcup\mathcal{G}$ such that $\gamma < \mu_0 Z < \infty$. Set $\mu_1 E = \mu_0(E \cap Z \setminus H_0)$ for every Borel set $E \subseteq X$. Then μ_1 is a non-zero totally finite Borel measure on X, and is completion regular. **P** If $E \subseteq X$ is a Borel set and $\epsilon > 0$, there is a zero set $F \subseteq E \cap Z \setminus H_0$ such that $\mu_0 F \ge \mu_0(E \cap X \setminus H_0) - \epsilon$, and now $\mu_1 F \ge \mu_1 E - \epsilon$. **Q** Note that $\mu_1(X \setminus Z) = \mu_1 G = 0$ for every $G \in \mathcal{G}$.

For Borel sets $E \subseteq X$, set $\mu E = \mu_1 E/\mu_1 X$; then μ is a completion regular Borel probability measure on X, and $\mathcal{G} \cup \{X \setminus Z\}$ is a cover of X by open negligible sets.

(b) Now let $\langle Y_i \rangle_{i \in I}$ be a family of separable metrizable spaces such that there is a continuous surjection $f : Y \to X$, where $Y = \prod_{i \in I} Y_i$. For each $i \in I$ let $\mathcal{B}_i$ be a countable base for the topology of Y_i. For $J \subseteq I$ let $\mathcal{C}(J)$ be the family of all non-empty open cylinders in Y expressible in the form

$$\{s : s(i) \in B_i \ \forall \ i \in K\},$$

where K is a finite subset of J and $B_i \in \mathcal{B}_i$ for each $i \in K$; thus $\mathcal{C}(I)$ is a base for the topology of Y. Set $\mathcal{C}_0(J) = \{U : U \in \mathcal{C}(J),\ \mu^* f[U] = 0\}$ for each $J \subseteq I$. Note that (because every $\mathcal{B}_i$ is countable) $\mathcal{C}(J)$ and $\mathcal{C}_0(J)$ are countable for every countable subset J of I. It is easy to see that $\mathcal{C}(J) \cap \mathcal{C}(K) = \mathcal{C}(J \cap K)$ for all J, $K \subseteq I$, because if $U \in \mathcal{C}(I)$ it belongs to $\mathcal{C}(J)$ iff its projection onto X_i is the whole of X_i for every $i \notin J$.

For each negligible set $E \subseteq X$, let $\langle F_n(E) \rangle_{n \in \mathbb{N}}$ be a family of zero sets, subsets of $X \setminus E$, such that $\sup_{n \in \mathbb{N}} \mu F_n(E) = 1$. Then each $f^{-1}[F_n(E)]$ is a zero set in Y, so there is a countable set $M(E) \subseteq I$ such that all the sets $f^{-1}[F_n(E)]$ are determined by coordinates in $M(E)$ (4A3Nc). Let $\mathcal{J}$ be the family of countable subsets J of I such that $M(f[U]) \subseteq J$ for every $U \in \mathcal{C}_0(J)$; then $\mathcal{J}$ is cofinal with $[I]^{\le\omega}$, that is, every countable subset of I is included in some member of $\mathcal{J}$. **P** If we start from any countable subset J_0 of I and set

$$J_{n+1} = J_n \cup \bigcup\{M(f[U]) : U \in \mathcal{C}_0(J_n)\}$$

for each $n \in \mathbb{N}$, then every J_n is countable, and $\bigcup_{n \in \mathbb{N}} J_n \in \mathcal{J}$, because $\langle J_n \rangle_{n \in \mathbb{N}}$ is non-decreasing, so $\mathcal{C}_0(\bigcup_{n \in \mathbb{N}} J_n) = \bigcup_{n \in \mathbb{N}} \mathcal{C}_0(J_n)$. **Q**

(c) For each $J \in \mathcal{J}$, set

$$Q_J = \bigcap\{\bigcup_{n \in \mathbb{N}} F_n(f[U]) : U \in \mathcal{C}_0(J)\}.$$

Then $\mu Q_J = 1$ and $f^{-1}[Q_J]$ is determined by coordinates in J, while $f^{-1}[Q_J] \cap U = \emptyset$ whenever $U \in \mathcal{C}_0(J)$.

If $G \subseteq X$ is an open set, then $G \cap Q_J = \emptyset$ whenever $J \in \mathcal{J}$ and there is a negligible Baire set $Q \supseteq G$ such that $f^{-1}[Q]$ is determined by coordinates in J. **P** Set $H = \pi_J^{-1}[\pi_J[f^{-1}[G]]]$, where $\pi_J : Y \to \prod_{i \in J} Y_i$ is the canonical map; then H is a union of members of $\mathcal{C}(J)$, because $f^{-1}[G]$ is open in Y and $\pi_J[f^{-1}[G]]$ is open in $\prod_{i \in J} Y_i$. Also, because $f^{-1}[Q]$ is determined by coordinates in J, $H \subseteq f^{-1}[Q]$, so $f[H] \subseteq Q$ and $\mu^* f[H] = 0$; thus all the members of $\mathcal{C}(J)$ included in H actually belong to $\mathcal{C}_0(J)$, and $H \cap f^{-1}[Q_J] = \emptyset$. But this means that $f^{-1}[G] \cap f^{-1}[Q_J] = \emptyset$ and (because f is a surjection) $G \cap Q_J = \emptyset$, as claimed. **Q** In particular, if G is a negligible open set in X, then $G \cap Q_J = \emptyset$ whenever $J \in \mathcal{J}$ and $J \supseteq M(G)$.

(d) If $J \in \mathcal{J}$, there are s, $s' \in f^{-1}[Q_J]$ such that $s{\restriction}J = s'{\restriction}J$ and $f(s)$, $f(s')$ can be separated by open sets in X. **P** Start from any $x \in Q_J$ and take a negligible open set G containing x (recall that our hypothesis is that X is covered by negligible open sets). For each $n \in \mathbb{N}$ let $h_n : X \to \mathbb{R}$ be a continuous function such that $F_n(G) = h_n^{-1}[\{0\}]$. We know that $G \cap Q_J \ne \emptyset$, while $G \subseteq X \setminus (\bigcup_{n \in \mathbb{N}} F_n(G) \cap Q_J)$, which is a negligible Baire set; by (c), $f^{-1}[X \setminus (\bigcup_{n \in \mathbb{N}} F_n(G) \cap Q_J)]$ is not determined by coordinates in J, and there must be some n such that $f^{-1}[F_n(G) \cap Q_J]$ is not determined by coordinates in J. Accordingly there must be s, $s' \in Y$ such that $s{\restriction}J = s'{\restriction}J$, $s \in f^{-1}[F_n(G) \cap Q_J]$ and $s' \notin f^{-1}[F_n(G) \cap Q_J]$. Now $s \in f^{-1}[Q_J]$, which is determined by coordinates in J; since $s{\restriction}J = s'{\restriction}J$, $s' \in f^{-1}[Q_J]$ and $s' \notin f^{-1}[F_n(G)]$. Accordingly $h_n(f(s)) = 0 \ne h_n(f(s'))$ and $f(s)$, $f(s')$ can be separated by open sets. **Q**

(e) We are now ready to embark on the central construction of the argument. We may choose inductively, for ordinals $\xi < \omega_1$, sets $J_\xi \in \mathcal{J}$, negligible open sets G_ξ, $G'_\xi \subseteq X$, points s_ξ, $s'_\xi \in Y$ and sets U_ξ, V_ξ, $V'_\xi \in \mathcal{C}(I)$ such that

$J_\eta \subseteq J_\xi$, U_η, V_η, V'_η all belong to $\mathcal{C}(J_\xi)$ and $G_\eta \cap Q_{J_\xi} = \emptyset$ whenever $\eta < \xi < \omega_1$ (using the results of (b) and (c) to choose J_ξ);

$s_\xi{\restriction}J_\xi = s'_\xi{\restriction}J_\xi$, $s_\xi \in f^{-1}[Q_{J_\xi}]$ and $f(s_\xi)$ and $f(s'_\xi)$ can be separated by open sets in X (using (d) to choose s_ξ, s'_ξ);

G_ξ, G'_ξ are disjoint negligible open sets containing $f(s_\xi)$, $f(s'_\xi)$ respectively (choosing G_ξ, G'_ξ);

$U_\xi \in \mathcal{C}(J_\xi)$, V_ξ, $V'_\xi \in \mathcal{C}(I \setminus J_\xi)$, $s_\xi \in U_\xi \cap V_\xi \subseteq f^{-1}[G_\xi]$, $s'_\xi \in U_\xi \cap V'_\xi \subseteq f^{-1}[G'_\xi]$ (choosing U_ξ, V_ξ, V'_ξ, using the fact that $s_\xi{\restriction}J_\xi = s'_\xi{\restriction}J_\xi$).

On completing this construction, take for each $\xi < \omega_1$ a finite set $K_\xi \subseteq J_{\xi+1}$ such that U_ξ, V_ξ and V'_ξ all belong to $\mathcal{C}(K_\xi)$. By the Δ-system Lemma (4A1Db), there is an uncountable $A \subseteq \omega_1$ such that $\langle K_\xi \rangle_{\xi\in A}$ is a Δ-system with root K say. For $\xi \in A$, express U_ξ as $\tilde{U}_\xi \cap U'_\xi$ where $\tilde{U}_\xi \in \mathcal{C}(K)$ and $U'_\xi \in \mathcal{C}(K_\xi \setminus K)$. Then there are only countably many possibilities for $\tilde{U}_\xi$, so there is an uncountable $B \subseteq A$ such that $\tilde{U}_\xi$ is constant for $\xi \in B$; write $\tilde{U}$ for the constant value. Let $C \subseteq B$ be an uncountable set, not containing $\min A$, such that $K_\xi \setminus K$ does not meet J_η whenever ξ, $\eta \in C$ and $\eta < \xi$ (4A1Eb). Let $D \subseteq C$ be such that D and $C \setminus D$ are both uncountable.

Note that $K \subseteq K_\eta \subseteq J_\xi$ whenever η, $\xi \in A$ and $\eta < \xi$, so that $K \subseteq J_\xi$ for every $\xi \in C$. Consequently U'_ξ, V_ξ and V'_ξ all belong to $\mathcal{C}(K_\xi \setminus K)$ for every $\xi \in C$.

(f) Consider the open set

$$G = \bigcup_{\xi\in D} G_\xi \subseteq X.$$

At this point the argument divides.

case 1 Suppose $\mu^*(G \cap f[\tilde{U}]) > 0$. Then there is a Baire set $Q \subseteq G$ such that $\mu^*(Q \cap f[\tilde{U}]) > 0$. Let $J \subseteq I$ be a countable set such that $f^{-1}[Q]$ is determined by coordinates in J. Let $\gamma \in C \setminus D$ be so large that $K_\xi \setminus K$ does not meet J for any $\xi \in A$ with $\xi \geq \gamma$. Then $Q \cap Q_{J_\gamma} \cap f[\tilde{U}]$ is not empty; take $s \in \tilde{U} \cap f^{-1}[Q \cap Q_{J_\gamma}]$. Because the $K_\xi \setminus K$ are disjoint from each other and from $J \cup J_\gamma$ for $\xi > \gamma$, we may modify s to form s' such that $s' \restriction J \cup J_\gamma = s \restriction J \cup J_\gamma$ and $s' \in U'_\xi \cap V'_\xi$ for every $\xi > \gamma$; now $s' \in \tilde{U}$ (because $K \subseteq J_\gamma$), so $s' \in \tilde{U} \cap U'_\xi \cap V'_\xi \subseteq f^{-1}[G'_\xi]$ and $f(s') \notin G_\xi$ whenever $\xi > \gamma$. On the other hand, if $\xi \in D$ and $\xi < \gamma$, $G_\xi \cap Q_{J_\gamma} = \emptyset$, while $s' \in f^{-1}[Q_{J_\gamma}]$ (because $f^{-1}[Q_{J_\gamma}]$ is determined by coordinates in J_γ), so again $f(s') \notin G_\xi$.

Thus $f(s') \notin G$. But $s' \restriction J = s \restriction J$ so $f(s) \in Q \subseteq G$; which is impossible.

This contradiction disposes of the possibility that $\mu^*(G \cap f[\tilde{U}]) > 0$.

case 2 Suppose that $\mu^*(G \cap f[\tilde{U}]) = 0$. In this case there is a negligible Baire set $Q \supseteq G \cap f[\tilde{U}]$. Let $J \subseteq I$ be a countable set such that $f^{-1}[Q]$ is determined by coordinates in J. Let $\gamma < \omega_1$ be such that $J \cap J_\gamma = J \cap \bigcup_{\xi<\omega_1} J_\xi$ and $J \cap K_\xi \setminus K = \emptyset$ for every $\xi \geq \gamma$. Take $\xi \in D$ such that $\xi \geq \gamma$. Then

$$\tilde{U} \cap U'_\xi \cap V_\xi \subseteq f^{-1}[G_\xi] \cap \tilde{U} \subseteq f^{-1}[G \cap f[\tilde{U}]] \subseteq f^{-1}[Q],$$

so $\tilde{U} \subseteq f^{-1}[Q]$, because $U'_\xi \cap V_\xi$ is a non-empty member of $\mathcal{C}(I \setminus J)$. But this means that $\mu^* f[\tilde{U}] = 0$ and $\mu^* f[U_\xi] = 0$. On the other hand, we have $s_\xi \in U_\xi \cap f^{-1}[Q_{J_\xi}]$, so $U_\xi \notin \mathcal{C}_0(J_\xi)$ and $\mu^* f[U_\xi] > 0$. **X**

Thus this route also is blocked and we must abandon the original hypothesis that there is a quasi-dyadic space with a semi-finite completion regular topological measure which is not τ-additive.

434R There is a useful construction of Borel product measures which can be fitted in here.

Proposition Let X and Y be topological spaces with Borel measures μ and ν; write $\mathcal{B}(X)$, $\mathcal{B}(Y)$ for the Borel σ-algebras of X and Y respectively. If *either* X is first-countable *or* ν is τ-additive and effectively locally finite, there is a Borel measure λ_B on $X \times Y$ defined by the formula

$$\lambda_B W = \sup_{F\in\mathcal{B}(Y),\nu F<\infty} \int \nu(W[\{x\}] \cap F)\mu(dx)$$

for every Borel set $W \subseteq X \times Y$. Moreover

(i) if μ is semi-finite, then λ_B agrees with the c.l.d. product measure λ on $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$, and the c.l.d. version $\tilde{\lambda}_B$ of λ_B extends λ;

(ii) if ν is σ-finite, then $\lambda_B W = \int \nu W[\{x\}]\mu(dx)$ for every Borel set $W \subseteq X \times Y$;

(iii) if both μ and ν are τ-additive and effectively locally finite, then λ_B is just the restriction of the τ-additive product measure $\tilde{\lambda}$ (417D, 417G) to the Borel σ-algebra of $X \times Y$; in particular, λ_B is τ-additive.

proof (a) The point is that $x \mapsto \nu(W[\{x\}] \cap F)$ is lower semi-continuous whenever $W \subseteq X \times Y$ is open and $\nu F < \infty$. **P** Of course $W[\{x\}]$ is always open, so ν always measures $W[\{x\}] \cap F$. Take any $\alpha \in \mathbb{R}$ and set $G = \{x : x \in X, \nu(W[\{x\}] \cap F) > \alpha\}$; let $x_0 \in G$.

(α) Suppose that X is first-countable. Let $\langle U_n \rangle_{n\in\mathbb{N}}$ be a non-increasing sequence running over a base of open neighbourhoods of x_0. For each $n \in \mathbb{N}$, set

$$V_n = \bigcup\{V : V \subseteq Y \text{ is open}, U_n \times V \subseteq W\}.$$

Then $\langle V_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with union $W[\{x\}]$, so there is an $n \in \mathbb{N}$ such that $\nu(V_n \cap F) > \alpha$. Now $V_n \subseteq W[\{x\}]$ for every $x \in U_n$, so $U_n \subseteq G$.

(β) Suppose that ν is τ-additive and effectively locally finite. Set

$$\mathcal{V} = \{V : V \subseteq Y \text{ is open, } U \times V \subseteq W \text{ for some open set } U \text{ containing } x_0\}.$$

Then $\mathcal{V}$ is an upwards-directed family of open sets with union $W[\{x_0\}]$, so there is a $V \in \mathcal{V}$ such that $\nu(V \cap F) > \alpha$ (414Ea). Let U be an open set containing x_0 such that $U \times V \subseteq W$; then $V \subseteq W[\{x\}]$ for every $x \in U$, so $U \subseteq G$.

(γ) Thus in either case we have an open set containing x_0 and included in G. As x_0 is arbitrary, G is open; as α is arbitrary, $x \mapsto \nu(W[\{x\}] \cap F)$ is lower semi-continuous. **Q**

(b) It follows that $x \mapsto \nu(W[\{x\}] \cap F)$ is Borel measurable whenever $W \subseteq X \times Y$ is a Borel set and $\nu F < \infty$. **P** Let $\mathcal{W}$ be the family of sets $W \subseteq X \times Y$ such that $W[\{x\}]$ is a Borel set for every $x \in X$ and $x \mapsto \nu(W[\{x\}] \cap F)$ is Borel measurable. Then every open subset of $X \times Y$ belongs to $\mathcal{W}$ (by (a) above), $W \setminus W' \in \mathcal{W}$ whenever W, $W' \in \mathcal{W}$ and $W' \subseteq W$, and $\bigcup_{n\in\mathbb{N}} W_n \in \mathcal{W}$ whenever $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}$. By the Monotone Class Theorem (136B), $\mathcal{W}$ includes the σ-algebra generated by the open sets, that is, the Borel σ-algebra of $X \times Y$. **Q**

(c) It is now easy to check that $W \mapsto \int \nu(W[\{x\}] \cap F)\mu(dx)$ is a Borel measure on $X \times Y$ whenever $\nu F < \infty$, and therefore that λ_B, as defined here, is a Borel measure.

(d) Now suppose that μ is semi-finite, and that $W \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. Then

$$\lambda W = \sup_{\mu E<\infty,\nu F<\infty} \lambda(W \cap (E \times F))$$

(by the definition of 'c.l.d. product measure', 251F)

$$= \sup_{\mu E<\infty,\nu F<\infty} \int_E \nu(W[\{x\}] \cap F)\mu(dx)$$

(by Fubini's theorem, 252C, applied to the product of the subspace measures μ_E and ν_F)

$$= \sup_{\nu F<\infty} \int \nu(W[\{x\}] \cap F)\mu(dx)$$

(by 213B, because μ is semi-finite)

$$= \lambda_B W.$$

(e) If, on the other hand, ν is σ-finite, let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of sets of finite measure covering Y; then

$$\begin{aligned}\lambda_B W &= \sup_{\nu F<\infty} \int \nu(W[\{x\}] \cap F)\mu(dx) \geq \sup_{n\in\mathbb{N}} \int \nu(W[\{x\}] \cap F_n)\mu(dx)\\ &= \int \sup_{n\in\mathbb{N}} \nu(W[\{x\}] \cap F_n)\mu(dx) = \int \nu W[\{x\}]\mu(dx) \geq \lambda_B W\end{aligned}$$

for any Borel set $W \subseteq X$.

(f) If both μ and ν are τ-additive and effectively locally finite, so that we have a τ-additive product measure $\tilde\lambda$, then Fubini's theorem for such measures (417H) tells us that $\lambda_B W = \tilde\lambda W$ at least when $W \subseteq X \times Y$ is a Borel set and $\tilde\lambda W$ is finite. If W is any Borel subset of $X \times Y$, then, as in (d),

$$\begin{aligned}\lambda_B W &= \sup_{\mu E<\infty,\nu F<\infty} \int_E \nu(W[\{x\}] \cap F)\mu(dx)\\ &= \sup_{\mu E<\infty,\nu F<\infty} \tilde\lambda(W \cap (E \times F)) = \tilde\lambda W\end{aligned}$$

by 417C(iii).

Remark The case in which X is first-countable is due to Johnson 82.

***434S** The concept of 'universally measurable' set enables us to extend a number of ideas from earlier sections. First, recall a problem from the very beginning of measure theory on the real line: the composition of Lebesgue measurable functions need not be Lebesgue measurable (134Ib), while the composition of a Borel measurable function with a Lebesgue measurable function is measurable (121Eg). In fact we can replace 'Borel measurable' by 'universally measurable', as follows.

Proposition Let (X, Σ, μ) be a complete locally determined measure space, Y and Z topological spaces, $f : X \to Y$ a measurable function and $g : Y \to Z$ a universally measurable function. Then $gf : X \to Z$ is measurable. In particular, $f^{-1}[F] \in \Sigma$ for every universally measurable set $F \subseteq Y$.

proof Let $H \subseteq Z$ be an open set and $E \in \Sigma$ a set of finite measure. Let μ_E be the subspace measure on E. Then the image measure $\nu = \mu_E(f{\restriction}E)^{-1}$ is a complete totally finite topological measure on Y, so its domain contains $g^{-1}[H]$, and

$$E \cap (gf)^{-1}[H] = (f{\restriction}E)^{-1}[g^{-1}[H]] \in \operatorname{dom}\mu_E \subseteq \Sigma.$$

As E is arbitrary and μ is locally determined, $(gf)^{-1}[H] \in \Sigma$; as H is arbitrary, gf is measurable.

Applying this to $g = \chi F$, we see that $f^{-1}[F] \in \Sigma$ for every universally measurable $F \subseteq Y$.

***434T** The next remark concerns the concept $[\![u \in E]\!]$ of §364.

Proposition Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra. Write Σ_{um} for the algebra of universally measurable subsets of $\mathbb{R}$.

(a) For any $u \in L^0 = L^0(\mathfrak{A})$, we have a sequentially order-continuous Boolean homomorphism $E \mapsto [\![u \in E]\!] : \Sigma_{\mathrm{um}} \to \mathfrak{A}$ defined by saying that

$$\begin{aligned}[\![u \in E]\!] &= \sup\{[\![u \in F]\!] : F \subseteq E \text{ is Borel}\} = \sup\{[\![u \in K]\!] : K \subseteq E \text{ is compact}\}\\ &= \inf\{[\![u \in F]\!] : F \supseteq E \text{ is Borel}\} = \inf\{[\![u \in G]\!] : G \supseteq E \text{ is open}\}\end{aligned}$$

for every $E \in \Sigma_{\mathrm{um}}$.

(b) For any $u \in L^0$ and universally measurable function $h : \mathbb{R} \to \mathbb{R}$ we have a corresponding element $\bar h(u)$ of L^0 defined by the formula

$$[\![\bar h(u) \in E]\!] = [\![u \in h^{-1}[E]]\!] \text{ for every } E \in \Sigma_{\mathrm{um}},\ u \in L^0.$$

proof We can regard $(\mathfrak{A}, \bar\mu)$ as the measure algebra of a complete strictly localizable measure space (X, Σ, μ) (322O), in which case L^0 can be identified with $L^0(\mu)$ (364Ic[3]). Write $\mathcal{B}$ for the Borel σ-algebra of $\mathbb{R}$.

(a) Let $f : X \to \mathbb{R}$ be a Σ-measurable function representing u. Then $f^{-1}[E] \in \Sigma$ for every $E \in \Sigma_{\mathrm{um}}$, by 434S. Setting $\phi E = (f^{-1}[E])^\bullet$, $\phi : \Sigma_{\mathrm{um}} \to \mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism.

Now

$$\phi E = \sup\{[\![u \in K]\!] : K \subseteq E \text{ is compact}\}$$

for every $E \in \Sigma_{\mathrm{um}}$. **P** If $H \in \Sigma$ and $\mu H < \infty$, then (writing μ_H for the subspace measure on H) the image measure $\mu_H(f{\restriction}H)^{-1}$ is a complete topological measure, and its restriction ν to the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$ is a totally finite Borel measure. Now E is measured by the completion $\hat\nu$ of ν, which is a Radon measure (256C), so for any $\epsilon > 0$ there are a compact $K \subseteq E$ and a Borel $F \supseteq E$ such that $\nu F = \hat\nu E \le \nu K + \epsilon$. In this case,

$$[\![u \in K]\!] = (f^{-1}[K])^\bullet \subseteq \phi E \subseteq (f^{-1}[F])^\bullet = [\![u \in F]\!],$$

using the formula of 364Ib, while

$$\begin{aligned}\bar\mu(H^\bullet \cap \phi E) &\le \bar\mu(H^\bullet \cap [\![u \in F]\!]) = \mu(H \cap f^{-1}[F])\\ &= \nu F \le \nu K + \epsilon = \bar\mu(H^\bullet \cap [\![u \in K]\!]) + \epsilon.\end{aligned}$$

As ϵ is arbitrary,

$$H^\bullet \cap \phi E = \sup\{H^\bullet \cap [\![u \in K]\!] : K \subseteq E \text{ is compact}\};$$

as H is arbitrary, and $(\mathfrak{A}, \bar\mu)$ is semi-finite, $\phi E = \sup\{[\![u \in K]\!] : K \subseteq E \text{ is compact}\}$. **Q**

Applying this to $\mathbb{R} \setminus E$, we see that $\phi E = \inf\{[\![u \in G]\!] : G \supseteq E \text{ is open}\}$. Of course it follows at once that

[3] Formerly 364Jc.

$$[\![u \in E]\!] = \sup\{[\![u \in F]\!] : F \subseteq E \text{ is Borel}\} = \inf\{[\![u \in F]\!] : F \supseteq E \text{ is Borel}\}.$$

We can therefore identify the sequentially order-continuous Boolean homomorphism ϕ with $E \mapsto [\![u \in E]\!]$, as described.

(b) Once again identifying u with $f^\bullet$ where f is Σ-measurable, we see that hf is Σ-measurable (by 434S), so we have a corresponding element $(hf)^\bullet$ of L^0. If $E \in \Sigma_{\mathrm{um}}$, then

$$[\![(hf)^\bullet \in E]\!] = ((hf)^{-1}[E])^\bullet = (f^{-1}[h^{-1}[E]])^\bullet = [\![u \in h^{-1}[E]]\!],$$

using 434De to check that $h^{-1}[E] \in \Sigma_{\mathrm{um}}$, so that we can identify $(hf)^\bullet$ with $\bar{h}(u)$, as described.

434X Basic exercises >(a) Let $A \subseteq [0,1]$ be any non-measurable set. Show that the subspace measure on A is completion regular and τ-additive but not tight.

>(b) Let X be any Hausdorff space with a point x such that $\{x\}$ is not a G_δ set; for instance, $X = \omega_1 + 1$ and $x = \omega_1$, or $X = \{0,1\}^I$ for any uncountable set I and x any point of X. Show that setting $\mu E = \chi E(x)$ we get a tight Borel measure on X which is not completion regular.

>(c) Let X be a topological space. (i) Show that if $A \subseteq X$ is universally measurable in X, then $A \cap Y$ is universally measurable in Y for any set $Y \subseteq X$. (ii) Show that if $Y \subseteq X$ is universally measurable in X, and $A \subseteq Y$ is universally measurable in Y, then A is universally measurable in X. (iii) Suppose that X is the product of a countable family $\langle X_i \rangle_{i \in I}$ of topological spaces, and $E_i \subseteq X_i$ is a universally measurable set for each $i \in I$. Show that $\prod_{i \in I} E_i$ is universally measurable in X.

(d) Let X be an analytic Hausdorff space. (i) Suppose that Y is a topological space and W is a Borel subset of $X \times Y$. Show that $W[X]$ is a universally measurable subset of Y. (*Hint*: 423N.) (ii) Let A be a subset of X. Show that the following are equiveridical: (α) A is universally measurable in X; (β) $f^{-1}[A]$ is Lebesgue measurable for every Borel measurable function $f : [0,1] \to X$; (γ) $f^{-1}[A]$ is measured by the usual measure on $\{0,1\}^{\mathbb{N}}$ for every continuous function $f : \{0,1\}^{\mathbb{N}} \to X$.

(e) Let X be a Hausdorff space. (i) Show that, for $A \subseteq X$, the following are equiveridical: (α) A is universally Radon-measurable in X; (β) A is measured by every atomless Radon probability measure on X; (γ) $A \cap K$ is universally Radon-measurable in K for every compact $K \subseteq X$. (ii) Show that if $A \subseteq X$ is universally Radon-measurable in X, then $A \cap Y$ is universally Radon-measurable in Y for any set $Y \subseteq X$. (iii) Show that if $Y \subseteq X$ is universally Radon-measurable in X, and $A \subseteq Y$ is universally Radon-measurable in Y, then A is universally Radon-measurable in X. (iv) Show that if $\mathcal{G}$ is an open cover of X, and $A \subseteq X$ is such that $A \cap G$ is universally Radon-measurable (in G or in X) for every $G \in \mathcal{G}$, then A is universally Radon-measurable in X. (v) Show that if Y is another Hausdorff space, and $\Sigma^{(X)}_{\mathrm{uRm}}$, $\Sigma^{(Y)}_{\mathrm{uRm}}$ are the algebras of universally Radon-measurable subsets of X, Y respectively, then every continuous function from X to Y is $(\Sigma^{(X)}_{\mathrm{uRm}}, \Sigma^{(Y)}_{\mathrm{uRm}})$-measurable. (vi) Suppose that X is the product of a countable family $\langle X_i \rangle_{i \in I}$ of topological spaces, and $E_i \subseteq X_i$ is a universally Radon-measurable set for each $i \in I$. Show that $\prod_{i \in I} E_i$ is universally Radon-measurable in X.

>(f) (i) Let μ_0 be Dieudonné's measure on ω_1. Give $\omega_1 + 1 = \omega_1 \cup \{\omega_1\}$ its compact Hausdorff order topology, and define a Borel measure μ on $\omega_1 + 1$ by setting $\mu E = \mu_0(E \cap \omega_1)$ for every Borel set $E \subseteq \omega_1 + 1$. Show that μ is a complete probability measure and is neither τ-additive nor inner regular with respect to the closed sets. (ii) Show that the universally measurable subsets of $\omega_1 + 1$ are just its Borel sets. (*Hint*: 4A3J, 411Q.) (iii) Show that every totally finite τ-additive topological measure on $\omega_1 + 1$ has a countable support. (iv) Show that every subset of $\omega_1 + 1$ is universally Radon-measurable.

(g)(i) Show that there is a set $X \subseteq [0,1]$ such that $K \cap X$ and $K \setminus X$ are both of cardinal $\mathfrak{c}$ for every uncountable compact set $K \subseteq [0,1]$. (*Hint*: 4A3Fa, 423K.) (ii) Show that if we give X its subspace topology, then every subset of X is universally Radon-measurable, but not every subset is universally measurable. (*Hint*: every compact subset of X is countable, so every Radon measure on X is purely atomic, but X has full outer Lebesgue measure in $[0,1]$.)

(h) Show that a Hausdorff space X is Radon iff (α) every compact subset of X is Radon (β) for every non-zero totally finite Borel measure μ on X there is a compact subset K of X such that $\mu K > 0$. (*Hint*: 434F(a-v).)

>(i)(i) Let X and Y be K-analytic Hausdorff spaces and $f : X \to Y$ a continuous surjection. Suppose that $F \subseteq Y$ and that $f^{-1}[F]$ is universally Radon-measurable in X. Show that F is universally Radon-measurable in Y. (*Hint*: 432G.) (ii) Let X and Y be analytic Hausdorff spaces and $f : X \to Y$ a Borel measurable surjection. Suppose that $F \subseteq Y$ and that $f^{-1}[F]$ is universally Radon-measurable in X. Show that F is universally Radon-measurable in Y. (*Hint*: 433D.)

(j) Show that if X is a perfectly normal space then it is Borel-measure-compact iff it is Borel-measure-complete.

(k) Let X be a Radon Hausdorff space. (i) Show that $X \times Y$ is Borel-measure-compact whenever Y is Borel-measure-compact. (ii) Show that $X \times Y$ is Borel-measure-complete whenever Y is Borel-measure-complete.

(l) Show that if we give $\omega_1 + 1$ its order topology, it is Borel-measure-compact but not Borel-measure-complete or pre-Radon, and its open subset ω_1 is not Borel-measure-compact.

(m) Show that $\mathbb{R}$, with the right-facing Sorgenfrey topology, is Borel-measure-complete and Borel-measure-compact, but not Radon or pre-Radon.

(n) Let X be a topological space. (i) Show that the family of Borel-measure-complete subsets of X is closed under Souslin's operation. (ii) Show that the union of a sequence of Borel-measure-compact subsets of X is Borel-measure-compact. (iii) Show that if X is Hausdorff then the family of pre-Radon subsets of X is closed under Souslin's operation. (*Hint*: in (i) and (iii), start by showing that the family under consideration is closed under countable unions.)

(o) Show that $]0,1[^{\omega_1}$ is not pre-Radon.

(p) Let X be a separable metrizable space. Show that the following are equiveridical: (i) X is a Radon space; (ii) X is a pre-Radon space; (iii) there is a metric on X, defining the topology of X, such that X is universally Radon-measurable in its completion; (iv) whenever Y is a separable metrizable space and X' is a subset of Y such that there is a Borel isomorphism between X and X', then X' is universally measurable in Y; (v) X is a Radon space under any separable metrizable topology giving rise to the same Borel sets as the original topology.

>(q) Show that a K-analytic Hausdorff space is Radon iff all its compact subsets are Radon. (*Hint*: 432B, 434Xh.)

(r) Suppose that X is a K-analytic Hausdorff space such that every Radon measure on X is completion regular. Show that X is a Radon space.

(s) Let X and Y be topological spaces, and suppose that Y has a countable network. (i) Show that if X is Borel-measure-complete, then $X \times Y$ is Borel-measure-complete. (ii) Show that if X and Y are Radon Hausdorff spaces, then $X \times Y$ is Radon.

(t) Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be a sequence of topological spaces; write $X = \prod_{n\in\mathbb{N}} X_n$ and $Z_n = \prod_{i\le n} X_i$ for each n. (i) Show that if every Z_n is Borel-measure-complete, so is X. (ii) Show that if every Z_n is Hausdorff and pre-Radon, so is X. (iii) Show that if every Z_n is Hausdorff and Radon, so is X.

(u)(i) Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be a sequence of Radon Hausdorff spaces such that $\prod_{i\le n} K_i$ is Radon whenever $n \in \mathbb{N}$ and $K_i \subseteq X_i$ is compact for every $i \le n$. Show that $X = \prod_{n\in\mathbb{N}} X_n$ is Radon. (ii) Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be a sequence of Radon Hausdorff spaces with countable networks. Show that $\prod_{n\in\mathbb{N}} X_n$ is Radon.

(v) Show that if, in 434R, ν is σ-finite, then $\int g\,d\lambda_B = \iint g(x,y)\nu(dy)\mu(dx)$ for every λ_B-integrable function $g : X \times Y \to \mathbb{R}$.

(w) Show that the product measure construction of 434R is 'associative' and 'distributive' in the sense that (under appropriate hypotheses) the product measures on $(X \times Y) \times Z$ and $X \times (Y \times Z)$ agree, and those on $\bigcup_{i,j\in\mathbb{N}}(X_i \times Y_j)$ and $(\bigcup_{i\in\mathbb{N}} X_i) \times (\bigcup_{j\in\mathbb{N}} Y_j)$ agree.

>(x) Show that the product measure construction of 434R is not 'commutative'; indeed, taking $\mu = \nu$ to be Dieudonné's measure on ω_1, show that the Borel measures λ_1, λ_2 on ω_1^2 defined by setting

$$\lambda_1 W = \int \nu W[\{\xi\}]\mu(d\xi), \quad \lambda_2 W = \int \mu W^{-1}[\{\eta\}]\nu(d\eta)$$

are different.

(y) Read through §271, looking for ways to apply the concept '$\Pr(\boldsymbol{X} \in E)$' for random variables $\boldsymbol{X}$ and universally measurable sets E.

(z) Let Σ_{um} be the algebra of universally measurable subsets of $\mathbb{R}$, and μ the restriction of Lebesgue measure to Σ_{um}. Show that μ is translation-invariant, but has no translation-invariant lifting. (*Hint*: 345F.)

434Y Further exercises (a) Set $X = \mathbb{N} \setminus \{0, 1\}$. For m, $p \in X$ set $U_{mp} = m + p\mathbb{N}$; show that $\{U_{mp} : m, p \in X$ are coprime$\}$ is a base for a connected Hausdorff topology on X. (*Hint*: $pq \in \overline{U}_{mp}$ for every $q \ge 1$. See Steen & Seebach 78, ex. 60.) Show that X is a second-countable analytic Hausdorff space and carries a Radon measure which is not completion regular.

(b) Let (X, Σ, μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a topological space with a countable network consisting of universally measurable sets, and that $f : X \to Y$ is measurable. Show that f is almost continuous.

(c) Let X be a Hausdorff space. Let Σ be the family of those subsets E of X such that $f^{-1}[E]$ has the Baire property in Z whenever Z is a compact Hausdorff space and $f : Z \to X$ is continuous. Show that Σ is a σ-algebra of subsets of X closed under Souslin's operation. Show that every member of Σ is universally Radon-measurable.

(d) Let X be a completely regular Hausdorff space. Show that the following are equiveridical: (α) X is Radon; (β) X is a universally measurable subset of its Stone-Čech compactification; (γ) whenever Y is a Hausdorff space and X' is a subspace of Y which is homeomorphic to X, then X' is universally measurable in Y.

(e) Let X be a completely regular Hausdorff space. Show that the following are equiveridical: (α) X is pre-Radon; (β) X is a universally Radon-measurable subset of its Stone-Čech compactification; (γ) whenever Y is a Hausdorff space and X' is a subspace of Y which is homeomorphic to X, then X' is universally Radon-measurable in Y.

(f) Set $X = \omega_1 + 1$, with its order topology, and let Σ be the σ-algebra of subsets of X generated by the countable sets and the set Ω of limit ordinals in X. Show that there is a unique probability measure μ on X with domain Σ such that $\mu\xi = \mu\Omega = 0$ for every $\xi < \omega_1$. Show that μ is inner regular with respect to the Borel sets, is defined on a base for the topology of the compact Hausdorff space X, but has no extension to a topological measure on X.

(g) Let X be a metrizable space without isolated points, and μ a σ-finite Borel measure on X. Show that there is a conegligible meager set. (*Hint*: there is a dense set $D \subseteq X$ such that $\{\{d\} : d \in D\}$ is σ-metrically-discrete.)

(h) Give an example of a Hausdorff uniform space $(X, \mathcal{W})$ with a quasi-Radon probability measure which is not inner regular with respect to the totally bounded sets.

(i) Show that $\beta\mathbb{N}$ is not countably tight, therefore not Borel-measure-complete.

(j) Let X be a quasi-dyadic space. Suppose that $\langle(G_\xi, H_\xi)\rangle_{\xi<\omega_1}$ is a family of pairs of disjoint non-empty open sets. Show that there is an uncountable $A \subseteq \omega_1$ such that $\bigcap_{\xi\in B} G_\xi \cap \bigcap_{\xi\in A\setminus B} H_\xi$ is non-empty for every $B \subseteq A$. (*Hint*: start by supposing that X is itself a product of separable metrizable spaces, and that every G_ξ, H_ξ is an open cylinder set; use the Δ-system Lemma.)

(k)(i) Show that the split interval is not quasi-dyadic. (ii) Show that $\mathbb{R}$, with the Sorgenfrey right-facing topology, is not quasi-dyadic. (iii) Show that ω_1 and $\omega_1 + 1$, with their order topologies, are not quasi-dyadic. (*Hint*: 434Yj.)

(l) Show that a perfectly normal quasi-dyadic space is Borel-measure-compact.

(m) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of first-countable spaces, and μ_n a Borel probability measure on X_n for each n. For $n \in \mathbb{N}$ set $Z_n = \prod_{i<n} X_i$, and let λ_n be the product Borel measure on Z_n constructed by repeatedly using the method of 434R (cf. 434Xw). Show that there is a unique Borel measure λ on $Z = \prod_{n\in\mathbb{N}} X_n$ such that all the canonical maps from Z to Z_n are inverse-measure-preserving. Show that, for any n, λ can be identified with the product of λ_n and a suitable product measure on $\prod_{i\ge n} X_i$.

(n) (Aldaz 97) A topological space X is **countably metacompact** if whenever $\mathcal{G}$ is a countable open cover of X then there is a point-finite open cover $\mathcal{H}$ of X refining G. (i) Show that X is countably metacompact iff whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed sets with empty intersection in X then there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open sets, with empty intersection, such that $F_n \subseteq G_n$ for every n. (ii) Let X be any topological space and $\nu : \mathcal{P}X \to [0,1]$ a finitely additive functional such that $\nu X = 1$. Show that there is a finitely additive $\nu' : \mathcal{P}X \to [0,1]$ such that $\nu F \le \nu' F = \inf\{\nu' G : G \supseteq F \text{ is open}\}$ for every closed $F \subseteq X$. (*Hint*: 413Q.) (iii) Show that if X is countably metacompact and μ is any Borel probability measure on X, there is a Borel probability measure μ' on X, inner regular with respect to the closed sets, such that $\mu F \le \mu' F$ for every closed set $F \subseteq X$; so that μ and μ' agree on the Baire σ-algebra of X.

(o) Let X be a totally ordered set with its order topology. Show that any τ-additive Borel probability measure on X has countable Maharam type. (*Hint*: $\{]-\infty, x]^\bullet : x \in X\}$ generates the measure algebra.)

(p) (Oxtoby 70) Let μ be an atomless strictly positive Radon probability measure on $\mathbb{N}^\mathbb{N}$. (i) Show that if $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is any sequence in $[0,1]$ such that $\sum_{n=0}^\infty \alpha_n = 1$, then there is a partition $\langle U_n\rangle_{n\in\mathbb{N}}$ of $\mathbb{N}^\mathbb{N}$ into open sets such that $\mu U_n = \alpha_n$ for every n. (ii) Show that if ν is any other atomless strictly positive Radon probability measure on $\mathbb{N}^\mathbb{N}$, there is a homeomorphism $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\nu = \mu f^{-1}$.

(q) If X is a topological space and ρ is a metric on X, X is **σ-fragmented** by ρ if for every $\epsilon > 0$ there is a countable cover $\mathcal{A}$ of X such that whenever $\emptyset \ne B \subseteq A \in \mathcal{A}$ there is a non-empty relatively open subset of B of ρ-diameter at most ϵ. Now suppose that X is a Hausdorff space which is σ-fragmented by a metric ρ such that (i) X is complete under ρ (ii) the topology generated by ρ is finer than the given topology on X. Show that X is a pre-Radon space.

(r) Let X be a Hausdorff space and μ an atomless strictly localizable tight Borel measure on X. Show that μ is σ-finite. (*Hint*: Fremlin n05.)

(s) If X is a topological space, a set $A \subseteq X$ is **universally capacitable** if $c(A) = \sup\{c(K) : K \subseteq A \text{ is compact}\}$ for every Choquet capacity c on X. (i) Show that if X is a Hausdorff space and π_1, $\pi_2 : X \times X \to X$ are the coordinate maps, then we have a Choquet capacity c on $X \times X$ defined by saying that $c(A) = 0$ if $A \subseteq X \times X$ and there is a Borel set $E \subseteq X$ including $\pi_1[A]$ and disjoint from $\pi_2[A]$, and $c(A) = 1$ for other $A \subseteq X \times X$. (ii) Show that there is a universally measurable subset of $\mathbb{R}$ which is not universally capacitable. (*Hint*: 423L.)

(t) Let X be a Hausdorff space such that there is a countable family $\mathcal{A}$ of universally Radon-measurable subsets of X which separates the points of X in the sense that whenever $I \in [X]^2$ there is an $A \in \mathcal{A}$ such that $\#(I \cap A) = 1$. Show that two Radon probability measures on X which agree on $\mathcal{A}$ are identical.

434Z Problems (a) Must every Radon compact Hausdorff space be sequentially compact?

(b) Must a Hausdorff continuous image of a Radon compact Hausdorff space be Radon?

434 Notes and comments I said that the fundamental question of topological measure theory is 'which measures can appear on which topological spaces'? In this section I have concentrated on Borel measures, classified according to the scheme laid out in §411. (Of course there are other kinds of classification. One of the most interesting is the Maharam classification of Chapter 33: we can ask what measure algebras can appear from topological measures on a given topological space. I will return to this idea in §531 of Volume 5; for the moment I pass it by, with only 434Yo to give a taste.) We can ask this question from either of two directions. The obvious approach is to ask, for a given class of topological spaces, which types of measure can appear. But having discovered that (for instance) there are several types of topological space on which all (totally finite) Borel measures are tight, we can use this as a definition of a class of topological spaces, and ask the ordinary questions about this class. Thus we have 'Radon', 'Borel-measure-complete', 'Borel-measure-compact' and 'pre-Radon' spaces (434C, 434G). I have given precedence to the first partly to honour the influence of Schwartz 73 and partly because a compact Hausdorff space is always Borel-measure-compact and pre-Radon (434Hb, 434Jf) and is Borel-measure-complete iff it is Radon (434Ka). In effect, 'Borel-measure-complete' means 'Borel measures are quasi-Radon' (434Ib), 'pre-Radon' means 'quasi-Radon measures are Radon' (434Jb), and 'Radon' means 'Borel measures are Radon' (434F(a-iii)). These slogans have to be interpreted with care; but it is true that a Hausdorff space is Radon iff it is both Borel-measure-complete and pre-Radon (434Ka).

The concept of 'Radon' space is in fact one of the important contributions of measure theory to general topology, offering a variety of challenging questions. One which has attracted some attention is the problem of determining when products of Radon spaces are Radon. Uncountable products hardly ever are (434K); for countable products it is enough to understand products of finitely many compact spaces (434Xu); but the product of two compact spaces already seems to lead us into undecidable questions (438Xq, Wage 80). Two more very natural questions are in 434Z. One of the obstacles to the investigation is the rather small number of Radon compact Hausdorff spaces which are known. I should remark that if the continuum hypothesis (for instance) is true, then every compact Hausdorff space in which countably compact sets are closed is sequentially compact (Ismail & Nyikos 80, or Fremlin 84, 24Nc), so that in this case we have a quick answer to 434Za.

You will recognise the construction of 434M as a universal version of Dieudonné's measure (411Q). 'Tightness' is of great interest for other reasons (Engelking 89), and here is very helpful in giving quick proofs that spaces are not Radon (434Yi).

A large proportion of the definitions in general topology can be regarded as different abstractions from the concept of metrizability. Countable tightness is an obvious example; so is 'first-countability' (434R). In quite a different direction we have 'metacompactness' (438J, 434Yn). The construction of the product measure in 434R is an obvious idea, as soon as you have seen Fubini's theorem, but it is not obvious just when it will work.

'Quasi-dyadic' spaces are a relatively recent invention; I introduce them here only as a vehicle for the argument of 434Q. Of course a dyadic space is quasi-dyadic; for basic facts on dyadic spaces, see 4A2D, 4A5T and Engelking 89, §3.12 and 4.5.9-4.5.11.

435 Baire measures

Imitating the programme of §434, I apply a similar analysis to Baire measures, starting with a simple-minded classification (435A). This time the central section (435D-435H) is devoted to 'measure-compact' spaces, those on which all (totally finite) Baire measures are τ-additive.

435A Types of Baire measures In 434A I looked at a list of four properties which a Borel measure may or may not possess: inner regularity with respect to closed sets, inner regularity with respect to zero sets, tightness (that is, inner regularity with respect to closed compact sets), and τ-additivity. Since every (semi-finite) Baire measure is inner regular with respect to the zero sets (412D), only two of the four are important considerations for Baire measures: tightness and τ-additivity. On the other hand, there is a new question we can ask. Given a Baire measure on a topological space, when can it be extended to a Borel measure? And in the case of a positive answer, we can ask whether the extension is unique, and whether we can find extensions to Borel measures satisfying the properties considered in 434A.

We already have some information on this. If X is a completely regular space, and μ is a τ-additive effectively locally finite Baire measure on X, then μ has a (unique) extension to a τ-additive Borel measure (415N). While if μ is tight, the extension will also be tight (cf. 416C). Perhaps I should remark immediately that while there can be only one τ-additive Borel measure extending μ, there might be another Borel measure, not τ-additive, also extending μ; see 435Xa. Of course if there is any completion regular Borel measure extending μ, there is only one; moreover, if μ is σ-finite, and there is a completion regular Borel measure extending μ, this is the only Borel measure extending μ. (For every Borel set will be measured by the completion of μ.)

A possible division of Baire measures is therefore into classes

(E) measures which are not τ-additive,

(F) measures which are τ-additive, but not tight,

(G) tight measures,

and within these classes we can distinguish measures with no extension to a Borel measure (type E_0), measures with more than one extension to a Borel measure (types E_1, F_1 and G_1), measures with exactly one extension to a Borel measure which is not completion regular (types E_2, F_2 and G_2) and measures with an extension to a completion regular Borel measure (types E_3, F_3 and G_3). For examples, see 439M and 439O (E_0), 439N (E_2), 439J (E_3), 435Xc (F_1), 435Xd (F_2), 415Xc and 434Xa (F_3), 435Xa (G_1), 435Xb (G_2) and the restriction of Lebesgue measure to the Baire subsets of $\mathbb{R}$ (G_3); other examples may be constructed as direct sums of these.

A separate question we can ask of a Baire measure is whether it can be extended to a Radon measure. For this there is a straightforward criterion (435B), which shows that (at least for totally finite measures on completely regular spaces) only the types F_1 and F_2 are divided by this question. (If a Baire measure μ can be extended to a Radon measure, it is surely τ-additive. If μ is tight, it satisfies the criteria of 435B, so has an extension to a Radon measure. If μ has an extension to a completion regular Borel measure μ_1 *and* has an extension to a Radon measure μ_2, then the completion $\hat{\mu}$ of μ extends μ_1, while μ_2 extends $\hat{\mu}$; so μ_1 is the restriction of μ_2 to the Borel sets and

$\mu_2 = \hat{\mu}_1 = \hat{\mu}$ and μ, like μ_2, is tight, by 412Hb or otherwise. Thus no measure of type F_3 can be extended to a Radon measure.)

As with the classification of Borel measures that I offered in §434, any restriction on the topology of the underlying space may eliminate some of these possibilities. For instance, because a semi-finite Baire measure is inner regular with respect to the closed sets, we can have no (semi-finite) measure of classes E or F on a compact Hausdorff space. On a locally compact Hausdorff space we can have no effectively locally finite Baire measure of class F (435Xe), while on a K-analytic Hausdorff space we can have no locally finite Baire measure of class E (432F). In a metrizable space, or a regular space with a countable network (e.g., a regular analytic Hausdorff space), the Baire and Borel σ-algebras coincide (4A3Kb), so we can have no measures of type E_0, E_1, F_1 or G_1.

435B Theorem Let X be a Hausdorff space and μ a locally finite Baire measure on X. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) for every non-negligible Baire set $E \subseteq X$ there is a compact set $K \subseteq E$ such that $\mu^*K > 0$.

If μ is totally finite, we can add

(iii) $\sup\{\mu^*K : K \subseteq X \text{ is compact}\} = \mu X$.

proof Because μ is inner regular with respect to the closed sets (412D), this is just a special case of 416P.

435C Theorem (Mařík 57) Let X be a normal countably paracompact space. Then any semi-finite Baire measure on X has an extension to a semi-finite Borel measure which is inner regular with respect to the closed sets.

proof (a) Let ν be a semi-finite Baire measure on X. Let $\mathcal{K}$ be the family of those closed subsets of X which are included in zero sets of finite measure, and set $\phi_0 K = \nu^* K$ for $K \in \mathcal{K}$. Then $\mathcal{K}$ and ϕ_0 satisfy the conditions of 413I, that is,

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$,

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection.

P The first three are trivial.

(α) Take K, $L \in \mathcal{K}$ with $L \subseteq K$, and set $\gamma = \sup\{\phi_0 K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\}$. (i) If $K' \subseteq K \setminus L$ is closed, then (because X is normal) there is a zero set F including K' and disjoint from L (4A2F(d-iv)), so

$$\phi_0 K' + \phi_0 L = \nu^*((K' \cup L) \cap F) + \nu^*((K' \cup L) \setminus F) = \nu^*(K' \cup L) \le \nu^* K.$$

As K' is arbitrary, $\gamma + \phi_0 L \le \phi_0 K$. (ii) Let $\epsilon > 0$. Let F_0 be a zero set of finite measure including K. Because ν is inner regular with respect to the zero sets (412D), there is a zero set $F \subseteq F_0 \setminus L$ such that $\nu F \ge \nu_*(F_0 \setminus L) - \epsilon$ (413Ee), so that $\nu(F_0 \setminus F) \le \nu^* L + \epsilon$ (413Ec). Set $K' = K \cap F$. Then

$$\nu^* K = \nu^*(K \setminus F) + \nu^*(K \cap F) \le \nu(F_0 \setminus F) + \nu^* K' \le \nu^* L + \epsilon + \gamma.$$

As ϵ is arbitrary, $\nu^* K \le \nu^* L + \gamma$.

(β) If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, then (because X is countably paracompact) there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open sets such that $K_n \subseteq G_n$ for every n and $\bigcap_{n\in\mathbb{N}} G_n = \emptyset$ (4A2Ff). Because X is normal, there are zero sets F_n such that $K_n \subseteq F_n \subseteq G_n$ for each n (4A2F(d-iv) again), so that $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. We may suppose that F_0 has finite measure. In this case,

$$\lim_{n\to\infty} \nu^* K_n \le \lim_{n\to\infty} \nu\Bigl(\bigcap_{i\le n} F_i\Bigr) = 0.$$

Thus $\mathcal{K}$ and ϕ_0 satisfy the conditions (α) and (β) as well. **Q**

(b) By 413I, there is a complete locally determined measure μ on X, extending ϕ_0 and inner regular with respect to $\mathcal{K}$. If $F \subseteq X$ is closed, then $F \cap K \in \mathcal{K}$ for every $K \in \mathcal{K}$, so $F \in \operatorname{dom}\mu$ (413F(ii)); accordingly μ is a topological measure, and because ν also is inner regular with respect to $\mathcal{K}$, μ must extend ν. So the restriction of μ to the Borel sets is a Borel extension of ν which is inner regular with respect to the closed sets.

Remark If X is normal, but not countably paracompact, the result may fail; see 439O. I have stated the result in terms of 'countable paracompactness', but the formally distinct 'countable metacompactness' is also sufficient

(435Ya). If we are told that the Baire measure is τ-additive and effectively locally finite, we have a much stronger result (415M).

435D Just as with the 'Radon' spaces of §434, we can look at classes of topological spaces defined by the behaviour of the Baire measures they carry. The class which has aroused most interest is the following.

Definition A completely regular topological space X is **measure-compact** (sometimes called **almost Lindelöf**) if every totally finite Baire measure on X is τ-additive, that is, has an extension to a quasi-Radon measure on X (415N).

435E The following lemma will make our path easier.

Lemma Let X be a completely regular topological space and ν a totally finite Baire measure on X. Suppose that $\sup_{G\in\mathcal{G}}\nu G=\nu X$ whenever $\mathcal{G}$ is an upwards-directed family of cozero sets with union X. Then ν is τ-additive.

proof Let $\mathcal{G}$ be an upwards-directed family of open Baire sets such that $G^*=\bigcup\mathcal{G}$ also is a Baire set, and $\epsilon>0$. Because ν is inner regular with respect to the zero sets, there is a zero set $F\subseteq G^*$ such that $\nu F\ge\nu G^*-\epsilon$. Let $\mathcal{G}'$ be the family of cozero sets included in members of $\mathcal{G}$; because X is completely regular, so that the cozero sets are a base for its topology, $\bigcup\mathcal{G}'=G^*$, and of course $\mathcal{G}'$ is upwards-directed. Now

$$\mathcal{H}=\{G\cup(X\setminus F):G\in\mathcal{G}'\}$$

is an upwards-directed family of cozero sets with union X, so there is a $G_0\in\mathcal{G}'$ such that $\nu(G_0\cup(X\setminus F))\ge\nu X-\epsilon$. In this case

$$\sup_{G\in\mathcal{G}}\nu G\ge\nu G_0\ge\nu X-\epsilon-\nu(X\setminus F)=\nu F-\epsilon\ge\nu G^*-2\epsilon.$$

As $\mathcal{G}$ and ϵ are arbitrary, ν is τ-additive.

435F Elementary facts (a) If X is a completely regular space which is not measure-compact, there are a Baire probability measure μ on X and a cover of X by μ-negligible cozero sets. **P** There is a totally finite Baire measure ν on X which is not τ-additive. By 435E, there is an upwards-directed family $\mathcal{G}$ of cozero sets, covering X, such that $\sup_{G\in\mathcal{G}}\nu G<\nu X$. Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}$ such that $\sup_{n\in\mathbb{N}}\nu G_n=\sup_{G\in\mathcal{G}}\nu G$. Then $\gamma=\nu(X\setminus\bigcup_{n\in\mathbb{N}}G_n)>0$. Set

$$\mu H=\frac{1}{\gamma}\nu(H\setminus\bigcup_{n\in\mathbb{N}}G_n)$$

for Baire sets $H\subseteq X$; then μ is a Baire probability measure and $\mathcal{G}$ is a cover of X by μ-negligible cozero sets. **Q**

(b) Regular Lindelöf spaces are measure-compact. (For if a Lindelöf space can be covered by negligible open sets, it can be covered by countably many negligible open sets, so is itself negligible.) In particular, compact Hausdorff spaces, indeed all regular K-analytic Hausdorff spaces (422Gg), are measure-compact.

Note that regular Lindelöf spaces are normal and paracompact (4A2H(b-i)), so their measure-compactness is also a consequence of 435C and 434Hb.

(c) An open subset of a measure-compact space need not be measure-compact (435Xi(i)). A continuous image of a measure-compact space need not be measure-compact (435Xi(ii)). $\mathbb{N}^{\mathfrak{c}}$ is not measure-compact (439P). The product of two measure-compact spaces need not be measure-compact (439Q).

(d) If X is a measure-compact completely regular space it is Borel-measure-compact. **P** Let μ be a non-zero totally finite Borel measure on X and $\mathcal{G}$ an open cover of X. Let ν be the restriction of μ to the Baire σ-algebra of X, so that ν is τ-additive. Let $\mathcal{U}$ be the set of cozero sets $U\subseteq X$ included in members of $\mathcal{G}$; because the family of cozero sets is a base for the topology of X, $\bigcup\mathcal{U}=X$, and there is some $U\in\mathcal{U}$ such that $\nu U>0$. This means that there is some $G\in\mathcal{G}$ such that $\mu G>0$. By 434H(a-v), X is Borel-measure-compact. **Q**

435G Proposition A Souslin-F subset of a measure-compact completely regular space is measure-compact.

proof (a) Let X be a measure-compact completely regular space, $\langle F_\sigma\rangle_{\sigma\in S}$ a Souslin scheme consisting of closed subsets of X with kernel A, ν a totally finite Baire measure on A, and $\mathcal{G}$ an upwards-directed family of (relatively) cozero subsets of A covering A. Let ν_1 be the Baire measure on X defined by setting $\nu_1H=\nu(A\cap H)$ for every

Baire subset H of X. Because X is measure-compact, ν_1 has an extension to a quasi-Radon measure μ on X. Let μ_A be the subspace measure on A.

(b) By 431B, A is measured by μ. In fact $\mu A = \nu A$. **P** The construction of μ given in 415K-415N ensures that $\mu F = \nu_1^* F$ for every closed set F, and this is in any case a consequence of the facts that μ is τ-additive and $\operatorname{dom}\nu_1$ includes a base for the topology. For each $\sigma \in S$, in particular, $\mu F_\sigma = \nu_1^* F_\sigma$; let $F'_\sigma \supseteq F_\sigma$ be a Baire set such that $\nu_1 F'_\sigma = \nu_1^* F_\sigma$. Then

$$\mu F'_\sigma = \nu_1 F'_\sigma = \nu_1^* F_\sigma = \mu F_\sigma$$

and $\mu(F'_\sigma \setminus F_\sigma) = 0$ for every $\sigma \in S$. Let A' be the kernel of the Souslin scheme $\langle F'_\sigma \rangle_{\sigma \in S}$. Then $A \subseteq A'$ and

$$\mu(A' \setminus A) \le \sum_{\sigma \in S} \mu(F'_\sigma \setminus F_\sigma) = 0,$$

so $\mu A = \mu A'$. On the other hand, writing $\hat\nu_1$ for the completion of ν_1, A' is measured by $\hat\nu_1$, by 431A, so that (because μ extends ν_1)

$$\mu A = \mu A' = \mu^* A' \le \nu_1^* A' = (\nu_1)_* A' \le \mu_* A' = \mu A'.$$

Thus $\mu A = \nu_1^* A'$. But of course

$$\nu A = \nu_1 X = \nu_1^* A = \nu_1^* A',$$

so that $\mu A = \nu A$. **Q**

Since we surely have

$$\mu X = \nu_1 X = \nu A,$$

we see that $\mu(X \setminus A) = 0$.

(c) It follows that $\mu F = \nu F$ for every (relatively) zero set $F \subseteq A$. **P** There is a closed set $F' \subseteq X$ such that $F = A \cap F'$. Now if $H \subseteq X$ is a Baire set including F', $H \cap A$ is a (relatively) Baire set including F, so $\nu F \le \nu(H \cap A) = \nu_1 H$; as H is arbitrary, $\nu F \le \nu_1^* F'$. But $\nu_1^* F' = \mu F'$, as remarked in (b) above, and $\mu(X \setminus A) = 0$, so

$$\mu F = \mu F' = \nu_1^* F' \ge \nu F.$$

On the other hand, $A \setminus F$ is (relatively) cozero, so there is a non-decreasing sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ of (relatively) zero subsets of A with union $A \setminus F$, and

$$\mu(A \setminus F) = \lim_{n\to\infty} \mu F_n \ge \lim_{n\to\infty} \nu F_n = \nu(A \setminus F).$$

Since we already know that $\mu A = \nu A$, it follows that

$$\mu F = \mu A - \mu(A \setminus F) \le \nu A - \nu(A \setminus F) = \nu F,$$

and $\mu F = \nu F$. **Q**

(d) The set

$$\{E : E \subseteq A \text{ is a (relative) Baire set}, \mu E = \nu E\}$$

therefore contains every (relatively) zero set, and by the Monotone Class Theorem (136C) contains every (relatively) Baire set. What this means is that μ actually extends ν; so the subspace measure $\mu_A = \mu \restriction \mathcal{P}A$ also extends ν. But μ_A is a quasi-Radon measure (415B), therefore τ-additive, and ν must also be τ-additive.

435H Corollary A Baire subset of a measure-compact completely regular space is measure-compact.

proof Put 435G and 421L together.

435X Basic exercises >(a) Give $\omega_1 + 1$ its order topology. (i) Show that its Baire σ-algebra Σ is just the family of sets $E \subseteq \omega_1 + 1$ such that either E or its complement is a countable subset of ω_1. (ii) Show that there is a unique Baire probability measure ν on $\omega_1 + 1$ such that $\nu\{\xi\} = 0$ for every $\xi < \omega_1$. (iii) Show that ν is τ-additive. (iv) Show that there is exactly one Radon measure on $\omega_1 + 1$ extending ν, but that the measure μ of 434Xf is another Borel measure also extending ν.

>(b) Let I be a set of cardinal ω_1, endowed with its discrete topology, and $X = I \cup \{\infty\}$ its one-point compactification (3A3O). Let μ be the Dirac measure on X concentrated at ∞. (i) Show that every subset of X is a Borel set. (ii) Show that $\{\infty\}$ is not a zero set. (iii) Let ν be the restriction of μ to the Baire σ-algebra of X. Show that ν is tight. Show that μ is the unique Borel measure extending ν (*hint*: you will need 419G), but is not completion regular. (iv) Show that the subspace measure ν_I on I is the countable-cocountable measure on I, and is not a Baire measure, nor has any extension to a Baire measure on I. (v) Show that X is measure-compact.

(c) On $\mathbb{R}^{\omega_1}$ let μ be the Baire measure defined by saying that $\mu E = 1$ if $\chi\omega_1 \in E$, 0 otherwise. (i) Show that μ is τ-additive, but not tight. (*Hint*: 4A3P.) (ii) Show that the map $\xi \mapsto \chi\xi : \omega_1 + 1 \to \mathbb{R}^{\omega_1}$ is continuous, so that μ has more than one extension to a Borel measure. (iii) Show that μ has an extension to a Radon measure.

(d) Set $X = \omega_1 + 1$ with the topology $\mathcal{P}\omega_1 \cup \{X \setminus A : A \subseteq \omega_1 \text{ is countable}\}$. Let μ be the Baire measure on X defined by saying that, for Baire sets $E \subseteq X$, $\mu E = 1$ if $\omega_1 \in E$, 0 otherwise. (i) Show that a function $f : X \to \mathbb{R}$ is continuous iff $\{\xi : \xi \in X,\ f(\xi) \neq f(\omega_1)\}$ is countable; show that X is completely regular and Hausdorff. (ii) Show that μ is τ-additive. (iii) Show that every subset of X is Borel. (iv) Show that the only Borel measure extending μ is the Dirac measure concentrated at ω_1, and that this is a Radon measure. (v) Show that all compact subsets of X are finite, so that μ is not tight. *(vi) Show that X is Lindelöf.

(e) Let X be a locally compact Hausdorff space and μ an effectively locally finite τ-additive Baire measure on X. Show that μ is tight. (*Hint*: the relatively compact cozero sets cover X; use 414Ea and 412D.)

>(f) Let X be a completely regular space and μ a totally finite τ-additive Borel measure on X. Let μ_0 be the restriction of μ to the Baire σ-algebra of X. Show that $\mu F = \mu_0^* F$ for every closed set $F \subseteq X$.

(g) Show that if a semi-finite Baire measure ν on a normal countably paracompact space is extended to a Borel measure μ by the construction in 435C, then the measure algebra of ν becomes embedded as an order-dense subalgebra of the measure algebra of μ, so that $L^1(\mu)$ can be identified with $L^1(\nu)$.

(h) Show that a Borel-measure-compact normal countably paracompact space is measure-compact.

(i)(i) Show that $\omega_1 + 1$ is measure-compact, in its order topology, but that its open subset ω_1 is not (cf. 434Xl). (ii) Show that a discrete space of cardinal ω_1 is measure-compact, but that it has a continuous image which is not measure-compact.

(j) Let X be a metacompact completely regular space and ν a totally finite strictly positive Baire measure on X. Show that X is Lindelöf, so that ν has an extension to a quasi-Radon measure on X. (*Hint*: if $\mathcal{H}$ is a point-finite open cover of X, not containing $\emptyset$, then for each $H \in \mathcal{H}$ choose a non-empty cozero set $G_H \subseteq H$; show that $\{H : \nu G_H \geq \delta\}$ is finite for every $\delta > 0$.)

(k) A completely regular space X is **strongly measure-compact** (Moran 69) if $\mu X = \sup\{\mu^* K : K \subseteq X \text{ is compact}\}$ for every totally finite Baire measure μ on X. (i) Show that a completely regular Hausdorff space X is strongly measure-compact iff every totally finite Baire measure on X has an extension to a Radon measure iff X is measure-compact and pre-Radon. (ii) Show that a Souslin-F subset of a strongly measure-compact completely regular space is strongly measure-compact. (iii) Show that a discrete space of cardinal ω_1 is strongly measure-compact. (iv) Show that a countable product of strongly measure-compact completely regular spaces is strongly measure-compact. (v) Show that $\mathbb{N}^{\omega_1}$ is not strongly measure-compact. (*Hint*: take a non-trivial probability measure on $\mathbb{N}$ and consider its power on $\mathbb{N}^{\omega_1}$.) (vi) Show that if X and Y are completely regular spaces, X is measure-compact and Y is strongly measure-compact then $X \times Y$ is measure-compact.

(l) (T.D.Austin) Let X be a topological space, μ an atomless Baire probability measure on X and $\hat{\mu}$ its completion. Show that there is a continuous function $f : X \to [0,1]$ which is inverse-measure-preserving for $\hat{\mu}$ and Lebesgue measure on $[0,1]$. (*Hint*: Check the case $X = [0,1]$ first. For the general case, let Z be the set of continuous functions from X to $[0,1]$ with the complete metric induced by $\| \|_\infty$, and set $\alpha(f) = \max\{\mu f^{-1}[\{t\}] : t \in [0,1]\}$ for $f \in Z$. Show that $\operatorname{int}\{f : \alpha(f) \leq \epsilon\}$ is dense in Z for every $\epsilon > 0$, so that there is an $f \in Z$ such that μf^{-1} is atomless.)

(m) Let X be a normal space and μ a complete σ-finite topological probability measure on X which is inner regular with respect to the closed sets. (i) Let ν be the restriction of μ to the Baire σ-algebra of X. Show that μ and ν have isomorphic measure algebras. (ii) Show that if μ is an atomless probability measure there is a continuous $f : X \to [0,1]$ which is inverse-measure-preserving for μ and Lebesgue measure.

(n) Let X be a topological space and $\mathcal{G}$ the family of cozero sets in X. Show that a functional $\psi : \mathcal{G} \to [0, \infty[$ can be extended to a Baire measure on X iff ψ is modular (definition: 413Xq) and $\lim_{n\to\infty} \psi G_n = 0$ whenever $\langle G_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{G}$ with empty intersection. (*Hint*: if ψ satisfies the conditions, first check that $\psi\emptyset = 0$ and that $\psi G \le \psi H$ whenever $G \subseteq H$; now apply 413I with $\phi K = \inf\{\psi G : K \subseteq G \in \mathcal{G}\}$ for zero sets K.)

(o) Let X be a countably compact topological space and μ a totally finite Baire measure on X. Show that μ has an extension to a Borel measure which is inner regular with respect to the closed sets. (*Hint*: 413O.)

435Y Further exercises (a) Show that a normal countably metacompact space (434Yn) is countably paracompact.

(b) Let X be a completely regular Hausdorff space and βX its Stone-Čech compactification. Show that X is measure-compact iff whenever ν is a Radon measure on βX such that $\nu X = 0$, there is a ν-negligible Baire subset of βX including X.

435 Notes and comments The principal reason for studying Baire measures is actually outside the main line of this chapter. For a completely regular Hausdorff space X, write $C_b(X)$ for the M-space of bounded continuous real-valued functions on X. Then $C_b(X)^* = C_b(X)^\sim$ is an L-space (356N), and inside $C_b(X)^*$ we have the bands generated by the tight, smooth and sequentially smooth functionals (see 437A and 437F below), all identifiable, if we choose, with spaces of 'signed Baire measures'. Wheeler 83 argues convincingly that for the questions a functional analyst naturally asks, these Baire measures are often an effective aid.

From the point of view of the arguments in this section, the most fundamental difference between 'Baire' and 'Borel' measures lies in their action on subspaces. If X is a topological space and A is a subset of X, then any Borel or Baire measure μ on A provides us with a measure μ_1 of the same type on X, setting $\mu_1 E = \mu(A \cap E)$ for the appropriate sets E. In the other direction, if μ is a Borel measure on X, then the subspace measure μ_A is a Borel measure on A, because the Borel σ-algebra of A is just the subspace σ-algebra derived from the Borel algebra of X (4A3Ca). But if μ is a Baire measure on X, it does not follow that μ_A is a Baire measure on A; this is because (in general) not every continuous function $f : A \to [0, 1]$ has a continuous extension to X, so that not every zero set in A is the intersection of A with a zero set in X (see 435Xb). The analysis of those pairs (X, A) for which the Baire σ-algebra of A is just the subspace algebra derived from the Baire sets in X is a challenging problem in general topology which I pass by here. For the moment I note only that avoiding it is the principal technical problem in the proof of 435G.

I do not know if I ought to apologise for 'countably tight' spaces (434N), 'first-countable' spaces (434R), 'metacompact' spaces (438J), 'normal countably paracompact' spaces (435C), 'quasi-dyadic' spaces (434O) and 'sequential' spaces (436F). General topology is notorious for invoking arcane terminology to stretch arguments to their utmost limit of generality, and even specialists may find their patience tried by definitions which seem to have only one theorem each. In 438J, for instance, it is obvious that the original result concerned metrizable spaces (438H), and you may well feel at first that the extension is a baroque over-elaboration. On the other hand, there are (if you look for them) some very interesting metacompact spaces (Engelking 89, §5.3), and metacompactness has taken its place in the standard lists. In this book I try to follow a rule of introducing a class of topological spaces only when it is both genuinely interesting, from the point of view of general topology, and also a support for an idea which is interesting from the point of view of measure theory.

436 Representation of linear functionals

I began this treatise with the three steps which make measure theory, as we know it, possible: a construction of Lebesgue measure, a definition of an integral from a measure, and a proof of the convergence theorems. I used what I am sure is the best route: Lebesgue measure from Lebesgue outer measure, and integrable functions from simple functions. But of course there are many other paths to the same ends, and some of them show us slightly different aspects of the subject. In this section I come – rather later than many authors would – to an account of a procedure for constructing measures from integrals.

I start with three fundamental theorems, the first and third being the most important. A positive linear functional on a truncated Riesz space of functions is an integral iff it is sequentially smooth (436D); a smooth linear functional corresponds to a quasi-Radon measure (436H); and if X is a compact Hausdorff space, any positive linear functional on $C(X)$ corresponds to a Radon measure (436J-436K).

436A Definition Let X be a set, U a Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a positive linear functional. I say that f is **sequentially smooth** if whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$, then $\lim_{n\to\infty} f(u_n) = 0$.

If (X, Σ, μ) is a measure space and U is a Riesz subspace of the space of real-valued μ-integrable functions defined everywhere on X, then $\int d\mu : U \to \mathbb{R}$ is sequentially smooth, by Fatou's Lemma or Lebesgue's Dominated Convergence Theorem.

Remark It is essential to distinguish between 'sequentially smooth', as defined here, and 'sequentially order-continuous', as in 313Hb or 355G. In the context here, a positive linear operator $f : U \to \mathbb{R}$ is sequentially order-continuous if $\lim_{n\to\infty} f(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that 0 is the greatest lower bound for $\{u_n : n \in \mathbb{N}\}$ in U; while f is sequentially smooth if $\lim_{n\to\infty} f(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that 0 is the greatest lower bound for $\{u_n : n \in \mathbb{N}\}$ in $\mathbb{R}^X$. So there can be sequentially smooth functionals which are not sequentially order-continuous, as in 436Xi. A sequentially order-continuous positive linear functional is of course sequentially smooth.

436B Definition Let X be a set. I will say that a Riesz subspace U of $\mathbb{R}^X$ is **truncated** (or satisfies **Stone's condition**) if $u \wedge \chi X \in U$ for every $u \in U$.

In this case, $u \wedge \gamma\chi X \in U$ for every $\gamma \ge 0$ and $u \in U$ (being $-u^-$ if $\gamma = 0$, $\gamma(\gamma^{-1}u \wedge \chi X)$ otherwise).

436C Lemma Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Write $\mathcal{K}$ for the family of sets of the form $\{x : x \in X, u(x) \ge 1\}$ as u runs over U. Let $f : U \to \mathbb{R}$ be a sequentially smooth positive linear functional, and μ a measure on X such that μK is defined and equal to $\inf\{f(u) : \chi K \le u \in U\}$ for every $K \in \mathcal{K}$. Then $\int u\,d\mu$ exists and is equal to $f(u)$ for every $u \in U$.

proof It is enough to deal with the case $u \ge 0$, since $U = U^+ - U^+$ and both f and $\int$ are linear. Note that if $v \in U$, $K \in \mathcal{K}$ and $v \le \chi K$, then $v \le w$ whenever $\chi K \le w \in U$, so $f(v) \le \mu K$. For k, $n \in \mathbb{N}$ set

$$K_{nk} = \{x : u(x) \ge 2^{-n}k\}, \quad u_{nk} = u \wedge 2^{-n}k\chi X.$$

Then, for $k \ge 1$,

$$K_{nk} = \{x : \frac{2^n}{k}u \ge 1\} \in \mathcal{K},$$

$$2^n(u_{n,k+1} - u_{nk}) \le \chi K_{nk} \le 2^n(u_{nk} - u_{n,k-1}).$$

So

$$2^n f(u_{n,k+1} - u_{nk}) \le \mu K_{nk} \le 2^n f(u_{nk} - u_{n,k-1}),$$

and

$$f(u_{n,4^n+1} - u_{n1}) \le \sum_{k=1}^{4^n} 2^{-n}\mu K_{nk} \le f(u_{n,4^n} \le f(u).$$

But setting $w_n = u_{n,4^n+1} - u_{n1}$, $\langle w_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of functions in U and $\sup_{n\in\mathbb{N}} w_n(x) = u(x)$ for every x, so $\lim_{n\to\infty} f(u - w_n) = 0$ and $\lim_{n\to\infty} f(w_n) = f(u)$. Also, setting $v_n = \sum_{k=1}^{4^n} 2^{-n}\chi K_{nk}$, we have $w_n \le v_n \le u$ and $f(w_n) \le \int v_n \le f(u)$ for each n, so

$$\int u = \lim_{n\to\infty} \int v_n = f(u)$$

by B.Levi's theorem.

436D Theorem Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Let $f : U \to \mathbb{R}$ be a positive linear functional. Then the following are equiveridical:

(i) f is sequentially smooth;

(ii) there is a measure μ on X such that $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in U$.

proof I remarked in 436A that (ii)$\Rightarrow$(i) is a consequence of Fatou's Lemma. So the argument here is devoted to proving that (i)$\Rightarrow$(ii).

(a) Let $\mathcal{K}$ be the family of sets $K \subseteq X$ such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ for some sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U, taking the infimum in $\mathbb{R}^X$, so that $(\inf_{n\in\mathbb{N}} u_n)(x) = \inf_{n\in\mathbb{N}} u_n(x)$ for every $x \in X$. Then $\mathcal{K}$ is closed under finite unions and countable intersections. **P** (i) If K, $K' \in \mathcal{K}$ take sequences $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle u'_n\rangle_{n\in\mathbb{N}}$ in U such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ and

$\chi K' = \inf_{n\in\mathbb{N}} u'_n$; then $\chi(K \cup K') = \inf_{m,n\in\mathbb{N}} u_m \vee u'_n$, so $K \cup K' \in \mathcal{K}$. (ii) If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$, then for each $n \in \mathbb{N}$ we can choose a sequence $\langle u_{ni}\rangle_{i\in\mathbb{N}}$ in U such that $\chi K_n = \inf_{i\in\mathbb{N}} u_{ni}$; now $\chi(\bigcap_{n\in\mathbb{N}} K_n) = \inf_{n,i\in\mathbb{N}} u_{ni}$, so $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$. **Q**

Note that $\emptyset \in \mathcal{K}$ because $0 \in U$.

(b) We need to know that if $u \in U$ then $K = \{x : u(x) \geq 1\}$ belongs to $\mathcal{K}$. **P** Set

$$u_n = 2^n((u \wedge \chi X) - (u \wedge (1 - 2^{-n})\chi X)).$$

Because U is truncated, every u_n belongs to U, and it is easy to check that $\inf_{n\in\mathbb{N}} u_n = \chi K$. **Q** It follows that

$$\{x : u(x) \geq \alpha\} = \{x : \frac{1}{\alpha}u(x) \geq 1\} \in \mathcal{K}$$

whenever $u \in U$ and $\alpha > 0$.

(c) For $K \in \mathcal{K}$, set $\phi_0 K = \inf\{f(u) : u \in U,\ u \geq \chi K\}$. Then ϕ_0 satisfies the conditions of 413I. **P** I have already checked (†) and (‡) of 413I.

(α) Fix K, $L \in \mathcal{K}$ with $L \subseteq K$. Set $\gamma = \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$.

(i) Suppose that $K' \in \mathcal{K}$ is included in $K \setminus L$, and $\epsilon > 0$. Let $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle u'_n\rangle_{n\in\mathbb{N}}$ be sequences in U such that $\chi L = \inf_{n\in\mathbb{N}} u_n$ and $\chi K' = \inf_{n\in\mathbb{N}} u'_n$, and let $u \in U$ be such that $u \geq \chi K$ and $f(u) \leq \phi_0 K + \epsilon$. Set $v_n = u \wedge \inf_{i\leq n} u_i$, $v'_n = u \wedge \inf_{i\leq n} u'_i$ for each n. Then $\langle v_n \wedge v'_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U with infimum $\chi L \wedge \chi K' = 0$, so there is an n such that $f(v_n \wedge v'_n) \leq \epsilon$. In this case

$$\begin{aligned}\phi_0 L + \phi_0 K' &\leq f(v_n) + f(v'_n) = f(v_n + v'_n)\\ &= f(v_n \vee v'_n) + f(v_n \wedge v'_n) \leq f(u) + \epsilon \leq \phi_0 K + 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\phi_0 L + \phi_0 K' \leq \phi_0 K$. As K' is arbitrary, $\phi_0 L + \gamma \leq \phi_0 K$.

(ii) Next, given $\epsilon \in]0, 1[$, there are u, $v \in U$ such that $u \geq \chi K$, $v \geq \chi L$ and $f(v) \leq \phi_0 L + \epsilon$. Consider

$$K' = \{x : x \in K,\ \min(1, u(x)) - v(x) \geq \epsilon\} \subseteq K \setminus L.$$

By (b), $K' \in \mathcal{K}$. If $w \in U$ and $w \geq \chi K'$, then $v(x) + w(x) \geq 1 - \epsilon$ for every $x \in K$, so

$$\phi_0 K \leq \frac{1}{1-\epsilon} f(v + w) \leq \frac{1}{1-\epsilon}(\phi_0 L + \epsilon + f(w)).$$

As w is arbitrary,

$$(1 - \epsilon)\phi_0 K \leq \phi_0 L + \epsilon + \phi_0 K' \leq \phi_0 L + \epsilon + \gamma.$$

As ϵ is arbitrary, $\phi_0 K \leq \phi_0 L + \gamma$ and we have equality, as required by (α) in 413I.

(β) Now suppose that $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection. For each $n \in \mathbb{N}$ let $\langle u_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence in U with infimum χK_n in $\mathbb{R}^X$. Set $v_n = \inf_{i,j\leq n} u_{ji}$ for each n; then $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U with infimum $\inf_{n\in\mathbb{N}} \chi K_n = 0$, so $\inf_{n\in\mathbb{N}} f(v_n) = 0$. But

$$v_n \geq \inf_{j\leq n} \chi K_j = \chi K_n, \quad \phi_0 K_n \leq f(v_n)$$

for every n, so $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$, as required by ($\beta$) of 413I. **Q**

(d) By 413I, there is a complete locally determined measure μ on X, inner regular with respect to $\mathcal{K}$, extending ϕ_0. By 436C, $f(u) = \int u\,d\mu$ for every $u \in U$, as required.

436E Proposition Let X be any topological space, and $C_b = C_b(X)$ the space of bounded continuous real-valued functions on X. Then there is a one-to-one correspondence between totally finite Baire measures μ on X and sequentially smooth positive linear functionals $f : C_b \to \mathbb{R}$, given by the formulae

$$f(u) = \int u\,d\mu \text{ for every } u \in C_b,$$

$$\mu Z = \inf\{f(u) : \chi Z \leq u \in C_b\} \text{ for every zero set } Z \subseteq X.$$

proof (a) If μ is a totally finite Baire measure on X, then every continuous bounded real-valued function is integrable, and $f = \int d\mu$ is a sequentially smooth positive linear operator on C_b, by Fatou's Lemma, as usual.

(b) If $f : C_b \to \mathbb{R}$ is a sequentially smooth positive linear operator, then 436D tells us that there is a measure μ_0 on X such that $\int u\,d\mu_0$ is defined and equal to $f(u)$ for every $u \in C_b$. By the construction in 436D, or otherwise, we may suppose that μ_0 is complete, so that every $u \in C_b$ is Σ-measurable, where Σ is the domain of μ_0. It follows by the definition of the Baire σ-algebra $\mathcal{B}\mathfrak{a}$ of X (4A3K) that $\mathcal{B}\mathfrak{a} \subseteq \Sigma$, so that $\mu = \mu_0 \restriction \Sigma$ is a Baire measure; of course we still have $f(u) = \int u\,d\mu$ for every $u \in C_b$. Also, if $Z \subseteq X$ is a zero set, $\mu Z = \inf\{f(u) : \chi Z \le u \in C_b\}$. **P** Express Z as $\{x : v(x) = 0\}$ where $v : X \to [0,1]$ is continuous. Set

$$u_n = (\chi X - 2^n v)^+$$

for $n \in \mathbb{N}$; then $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in C_b and $\langle u_n(x) \rangle_{n\in\mathbb{N}} \to (\chi Z)(x)$ for every $x \in X$, so

$$\begin{aligned}\mu Z &\le \inf\{\int u\,d\mu : \chi Z \le u \in C_b\} = \inf\{f(u) : \chi Z \le u \in C_b\}\\ &\le \inf_{n\in\mathbb{N}} f(u_n) = \lim_{n\to\infty} \int u_n d\mu = \mu Z. \ \mathbf{Q}\end{aligned}$$

(c) The argument of (b) shows that if two totally finite Baire measures give the same integrals to every member of C_b, then they must agree on all zero sets. By the Monotone Class Theorem (136C) they agree on the σ-algebra generated by the zero sets, that is, $\mathcal{B}\mathfrak{a}$, and are therefore equal. Thus the operator $\mu \mapsto \int d\mu$ from the set of totally finite Baire measures on X to the set of sequentially smooth positive linear operators on C_b is a bijection, and if $f = \int d\mu$ then $\mu Z = \inf\{f(u) : \chi Z \le u \in C_b\}$ for every zero set Z, as required.

436F Corresponding to 434R, we have the following construction for product Baire measures, applicable to a slightly larger class of spaces.

Proposition Let X be a sequential space, Y a topological space, and μ, ν totally finite Baire measures on X, Y respectively. Then there is a Baire measure λ on $X \times Y$ such that

$$\lambda W = \int \nu W[\{x\}]\mu(dx), \quad \int f d\lambda = \iint f(x,y)\nu(dy)\mu(dx)$$

for every Baire set $W \subseteq X \times Y$ and every bounded continuous function $f : X \times Y \to \mathbb{R}$.

proof (a) $\phi(f) = \iint f(x,y)dydx$ is defined in $\mathbb{R}$ for every bounded continuous function $f : X \times Y \to \mathbb{R}$. **P** For each $x \in X$, $g(x) = \int f(x,y)dy$ is defined because $y \mapsto f(x,y)$ is continuous. If $\langle x_n \rangle_{n\in\mathbb{N}}$ is any sequence in X converging to $x \in X$, then

$$g(x) = \int f(x,y)dy = \int \lim_{n\to\infty} f(x_n,y)dy = \lim_{n\to\infty} \int f(x_n,y)dy = \lim_{n\to\infty} g(x_n)$$

by Lebesgue's Dominated Convergence Theorem. So g is sequentially continuous; because X is sequential, g is continuous (4A2Kd). So $\iint f(x,y)dydx = \int g(x)dx$ is defined in $\mathbb{R}$. **Q**

(b) Of course ϕ is a positive linear functional on $C_b(X \times Y)$, and B.Levi's theorem shows that it is sequentially smooth. By 436E, there is a Baire measure λ on $X \times Y$ such that $\int f d\lambda = \phi(f)$ for every $f \in C_b(X \times Y)$.

(c) If $W \subseteq X \times Y$ is a zero set, there is a non-increasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $C_b(X \times Y)$ such that $\chi W = \inf_{n\in\mathbb{N}} f_n$. So B.Levi's theorem tells us that

$$\int \nu W[\{x\}]dx = \lim_{n\to\infty} \int f_n(x,y)dydx = \lim_{n\to\infty} \int f_n d\lambda = \lambda W.$$

Now the Monotone Class Theorem (136B) tells us that

$$\{W : W \subseteq X \times Y \text{ is Baire}, \int \nu W[\{x\}]dx \text{ exists} = \lambda W\}$$

includes the σ-algebra generated by the zero sets, that is, contains every Baire set in $X \times Y$. So λ has the required properties.

436G Definition Let X be a set, U a Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a positive linear functional. I say that f is **smooth** if whenever A is a non-empty downwards-directed family in U such that $\inf_{u\in A} u(x) = 0$ for every $x \in X$, then $\inf_{u\in A} f(u) = 0$.

Of course a smooth functional is sequentially smooth. If $(X, \mathfrak{T}, \Sigma, \mu)$ is an effectively locally finite τ-additive topological measure space and U is a Riesz subspace of $\mathbb{R}^X$ consisting of integrable continuous functions, then $\int d\mu : U \to \mathbb{R}$ is smooth, by 414Bb. Corresponding to the remark in 436A, note that an order-continuous positive linear functional must be smooth, but that a smooth positive linear functional need not be order-continuous.

436H Theorem Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Let $f : U \to \mathbb{R}$ be a positive linear functional. Then the following are equiveridical:

(i) f is smooth;

(ii) there are a topology $\mathfrak{T}$ and a measure μ on X such that μ is a quasi-Radon measure with respect to $\mathfrak{T}$, $U \subseteq C(X)$ and $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in U$;

(iii) writing $\mathfrak{S}$ for the coarsest topology on X for which every member of U is continuous, there is a measure μ on X such that μ is a quasi-Radon measure with respect to $\mathfrak{S}$, and $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in U$.

proof As remarked in 436G, in a fractionally more general context, (ii)$\Rightarrow$(i) is a consequence of 414B. Of course (iii)$\Rightarrow$(ii). So the argument here is devoted to proving that (i)$\Rightarrow$(iii). Except for part (b) it is a simple adaptation of the method of 436D.

(a) Let $\mathcal{K}$ be the family of sets $K \subseteq X$ such that $\chi K = \inf A$ in $\mathbb{R}^X$ for some non-empty set $A \subseteq U$. Then $\mathcal{K}$ is closed under finite unions. **P** If K, $K' \in \mathcal{K}$ take A, $A' \subseteq U$ such that $\chi K = \inf A$, $\chi K' = A'$; then $\chi(K \cup K') = \inf\{u \vee u' : u \in A,\ u' \in A'\}$, so $K \cup K' \in \mathcal{K}$. **Q**

Note that $\emptyset \in \mathcal{K}$ because $0 \in U$.

As in part (b) of the proof of 436D, $\{x : u(x) \ge \alpha\} \in \mathcal{K}$ whenever $\alpha > 0$ and $u \in U$.

(b) Every member of $\mathcal{K}$ is closed for $\mathfrak{S}$, being of the form $\{x : u(x) \ge 1$ for every $u \in A\}$ for some $A \subseteq U$. We need to know that if $K \in \mathcal{K}$ and $G \in \mathfrak{S}$, then $K \setminus G \in \mathcal{K}$. **P** Take a non-empty set $B \subseteq U$ such that $\chi K = \inf B$. Because $\mathcal{K}$ is closed under finite unions and arbitrary intersections, $\mathfrak{S}_K = \{G : G \subseteq X,\ K \setminus G \in \mathcal{K}\}$ is a topology on X. (i) If $u \in U$ and $G = \{x : u(x) > 0\}$, then $\chi(K \setminus G) = \inf\{(v - ku)^+ : v \in B,\ k \in \mathbb{N}\}$ so $K \setminus G \in \mathcal{K}$ and $G \in \mathfrak{S}_K$. (ii) If $u \in U$ and $\alpha > 0$, then $\{x : u(x) > \alpha\} = \{x : (u - u \wedge \alpha\chi X)(x) > 0\}$ belongs to $\mathfrak{S}_K$, by (i). (iii) If $u \in U$ and $\alpha > 0$, set $G = \{x : u(x) < \alpha\}$. Then

$$K \setminus G = K \cap \{x : u(x) \ge \alpha\} \in \mathcal{K},$$

so $G \in \mathfrak{S}_K$. (iv) Thus every member of U^+ is $\mathfrak{S}_K$-continuous (2A3Bc), so every member of U is $\mathfrak{S}_K$-continuous (2A3Be), and $\mathfrak{S} \subseteq \mathfrak{S}_K$, that is, $K \setminus G \in \mathcal{K}$ for every $G \in \mathfrak{S}$. **Q**

(c) For $K \in \mathcal{K}$, set $\phi_0 K = \inf\{f(u) : u \in U,\ u \ge \chi K\}$. Then ϕ_0 satisfies the conditions of 415K. **P** I have already checked (†) and (‡) of 415K.

(α) Fix K, $L \in \mathcal{K}$ with $L \subseteq K$. Set $\gamma = \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$.

(i) Suppose that $K' \in \mathcal{K}$ is included in $K \setminus L$ and $\epsilon > 0$. Set $A = \{u : \chi L \le u \in U\}$, $A' = \{u : \chi K' \le u \in U\}$, so that $\chi L = \inf A$ and $\chi K' = \inf A'$, and let $v \in U$ be such that $v \ge \chi K$ and $f(v) \le \phi_0 K + \epsilon$. Then $\{u \wedge u' : u \in A,\ u' \in A'\}$ is a downwards-directed family with infimum 0 in $\mathbb{R}^X$, so (because f is smooth) there are $u \in A$, $u' \in A'$ such that $f(u \wedge u') \le \epsilon$. In this case

$$\phi_0 L + \phi_0 K' \le f(v \wedge u) + f(v \wedge u') = f(v \wedge (u \vee u')) + f(v \wedge u \wedge u') \le \phi_0 K + 2\epsilon.$$

As ϵ is arbitrary, $\phi_0 L + \phi_0 K' \le \phi_0 K$. As K' is arbitrary, $\phi_0 L + \gamma \le \phi_0 K$.

(ii) Next, given $\epsilon \in]0,1[$, there are u, $v \in U$ such that $u \ge \chi K$, $v \ge \chi L$ and $f(v) \le \phi_0 L + \epsilon$. Consider

$$K' = \{x : x \in K,\ \min(1, u(x)) - v(x) \ge \epsilon\}.$$

By the last remark in (a), $K' \in \mathcal{K}$. If $w \in U$ and $w \ge \chi K'$, then $v(x) + w(x) \ge 1 - \epsilon$ for every $x \in K$, so

$$\phi_0 K \le \tfrac{1}{1-\epsilon} f(v + w) \le \tfrac{1}{1-\epsilon}(\phi_0 L + \epsilon + f(w)).$$

As w is arbitrary,

$$(1 - \epsilon)\phi_0 K \le \phi_0 L + \epsilon + \phi_0 K' \le \phi_0 L + \epsilon + \gamma.$$

As ϵ is arbitrary, $\phi_0 K \le \phi_0 L + \gamma$ and we have equality, as required by (α) in 415K.

(β) Now suppose that $\mathcal{K}'$ is a non-empty downwards-directed subset of $\mathcal{K}$ with empty intersection. Set

$$A = \bigcup_{K \in \mathcal{K}'} \{u : \chi K \le u \in U\}.$$

Then A is a downwards-directed subset of U and $\inf A = 0$ in $\mathbb{R}^X$. Because f is smooth,

$$0 = \inf_{u \in A} f(u) = \inf_{K \in \mathcal{K}'} \phi_0 K.$$

Thus (β) of 415K is satisfied.

(γ) If $K \in \mathcal{K}$ and $\phi_0 K > 0$, take $u \in U$ such that $u \ge \chi K$, and consider $G = \{x : u(x) > \frac{1}{2}\}$. Then $K \subseteq G$, while $G \subseteq \{x : 2u(x) \ge 1\}$, so

$$\sup\{\phi_0 K' : K' \in \mathcal{K},\, K' \subseteq G\} \le 2f(u) < \infty.$$

Thus ϕ_0 satisfies (γ) of 415K. **Q**

(d) By 415K, there is a quasi-Radon measure μ on X extending ϕ_0. By 436C, $f(u) = \int u\,d\mu$ for every $u \in U$.

Remark It is worth noting explicitly that μ, as constructed here, is inner regular with respect to the family $\mathcal{K}$ of sets $K \subseteq X$ such that $\chi K = \inf A$ for some set $A \subseteq U$.

436I Lemma Let X be a topological space. Let $C_0 = C_0(X)$ be the space of continuous functions $u : X \to \mathbb{R}$ which 'vanish at infinity' in the sense that $\{x : |u(x)| \ge \epsilon\}$ is compact for every $\epsilon > 0$.

(a) C_0 is a norm-closed solid linear subspace of $C_b = C_b(X)$, so is a Banach lattice in its own right.

(b) $C_0^* = C_0^\sim$ is an L-space (definition: 354M).

(c) If $A \subseteq C_0$ is a non-empty downwards-directed set such that $\inf_{u\in A} u(x) = 0$ for every $x \in X$, then $\inf_{u\in A} \|u\|_\infty = 0$.

proof (a)(i) If $u \in C_0$, then $K = \{x : |u(x)| \ge 1\}$ is compact, so $\|u\|_\infty \le \sup(\{1\} \cup \{|u(x)| : x \in K\})$ is finite, and $u \in C_b$.

(ii) If u, $v \in C_0$ and $\alpha \in \mathbb{R}$ and $w \in C_b$ and $|w| \le |u|$, then for any $\epsilon > 0$

$$\{x : |u(x) + v(x)| \ge \epsilon\} \subseteq \{x : |u(x)| \ge \frac{1}{2}\epsilon\} \cup \{x : |v(x)| \ge \frac{1}{2}\epsilon\},$$

$$\{x : |\alpha u(x)| \ge \epsilon\} \subseteq \{x : |u(x)| \ge \tfrac{\epsilon}{1+|\alpha|}\},$$

$$\{x : |w(x)| \ge \epsilon\} \subseteq \{x : |u(x)| \ge \epsilon\}$$

are closed relatively compact sets, so are compact, and $u+v$, αu, w belong to C_0. Thus C_0 is a solid linear subspace of C_b.

(iii) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in C_0 which $\|\,\|_\infty$-converges to $u \in C_b$, then for any $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $\|u - u_n\|_\infty \le \frac{1}{2}\epsilon$, so that

$$\{x : |u(x)| \ge \epsilon\} \subseteq \{x : |u_n(x)| \ge \frac{1}{2}\epsilon\}$$

is compact, and $u \in C_0$. Thus C_0 is norm-closed in C_b.

(iv) Being a norm-closed Riesz subspace of a Banach lattice, C_0 is itself a Banach lattice.

(b) By 356Dc, $C_0^* = C_0^\sim$ is a Banach lattice. Now $\|f + g\| = \|f\| + \|g\|$ for all non-negative f, $g \in C_0^*$. **P** Of course $\|f + g\| \le \|f\| + \|g\|$. On the other hand, for any $\epsilon > 0$ there are u, $v \in C_0$ such that $\|u\|_\infty \le 1$, $\|v\|_\infty \le 1$ and $|f(u)| \ge \|f\| - \epsilon$, $|g(v)| \ge \|g\| - \epsilon$. Set $w = |u| \vee |v|$; then $w \in C_0$ and

$$\|w\|_\infty = \max(\|u\|_\infty, \|v\|_\infty) \le 1.$$

So

$$\|f + g\| \ge (f + g)(w) \ge f(|u|) + g(|v|) \ge |f(u)| + |g(v)| \ge \|f\| + \|g\| - 2\epsilon.$$

As ϵ is arbitrary, $\|f + g\| \ge \|f\| + \|g\|$. **Q**

So C_0^* is an L-space.

(c) Let $\epsilon > 0$. For $u \in A$ set $K_u = \{x : u(x) \ge \epsilon\}$. Then $\{K_u : u \in A\}$ is a downwards-directed family of closed compact sets with empty intersection, so there must be some $u \in A$ such that $K_u = \emptyset$, and $\|u\|_\infty \le \epsilon$. As ϵ is arbitrary, we have the result.

Remark (c) is a version of **Dini's theorem**.

436J Riesz Representation Theorem (first form) Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space, and $C_k = C_k(X)$ the space of continuous real-valued functions on X with compact support. If $f : C_k \to \mathbb{R}$ is any positive linear functional, there is a unique Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k$.

proof (a) The point is that f is smooth. **P** Suppose that $A \subseteq C_k$ is non-empty and downwards-directed and that $\inf A = 0$ in $\mathbb{R}^X$. Fix $u_0 \in A$ and set $K = \overline{\{x : u_0(x) > 0\}}$, so that K is compact. Because X is locally compact, there is an open relatively compact set $G \supseteq K$. Now there is a continuous function $u_1 : X \to [0, 1]$ such that $u_1(x) = 1$ for $x \in K$ and $u_1(x) = 0$ for $x \in X \setminus G$ (4A2F(h-iii)). Because G is relatively compact, $u_1 \in C_k$.

Take any $\epsilon > 0$. By 436Ic, there is a $v \in A$ such that $\|v\|_\infty \le \epsilon$. Now there is a $v' \in A$ such that $v' \le v \wedge u_0$, so that $v'(x) \le \epsilon$ for every $x \in K$ and $v'(x) = 0$ for $x \notin K$. In this case $v' \le \epsilon u_1$, and

$$\inf_{u \in A} f(u) \le f(v') \le \epsilon f(u_1).$$

As ϵ is arbitrary, $\inf_{u \in A} f(u) = 0$; as A is arbitrary, f is smooth. **Q**

(b) Note that because $\mathfrak{T}$ is locally compact, it is the coarsest topology on X for which every function in C_k is continuous (4A2G(e-ii)). Also C_k is a truncated Riesz subspace of $\mathbb{R}^X$. So 436H tells us that there is a quasi-Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k$. And μ is locally finite. **P** If $x_0 \in X$, then (as in (a) above) there is a $u_1 \in C_k^+$ such that $u_1(x_0) = 1$; now $G = \{x : u_1(x) > \frac{1}{2}\}$ is an open set containing x_0, and $\mu G \le 2f(u_1)$ is finite. **Q**

By 416G, or otherwise, μ is a Radon measure.

(c) By 416E(b-v), μ is unique.

436K Riesz Representation Theorem (second form) Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space. If $f : C_0(X) \to \mathbb{R}$ is any positive linear functional, there is a unique totally finite Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_0 = C_0(X)$.

proof (a) As noted in 436Ib, $C_0^* = C_0^\sim$, so f is $\|\,\|_\infty$-continuous. $C_k(X)$ is a linear subspace of C_0, and $f{\restriction}C_k(X)$ is a positive linear functional; so by 436J there is a unique Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k(X)$. Now μ is totally finite. **P** By 414Ab,

$$\mu X = \sup\{f(u) : u \in C_k(X),\ 0 \le u \le \chi X\} \le \|f\| < \infty. \text{ **Q**}$$

(b) Accordingly $\int u\,d\mu$ is defined for every $u \in C_b(X)$, and in particular for every $u \in C_0$. Next, $C_k = C_k(X)$ is norm-dense in C_0. **P** If $u \in C_0^+$, then $u_n = (u - 2^{-n}\chi X)^+$ belongs to C_k and $\|u - u_n\|_\infty \le 2^{-n}$ for every $n \in \mathbb{N}$, so $u \in \overline{C}_k$; accordingly $C_0 = C_0^+ - C_0^+$ is included in $\overline{C}_k$. **Q** Since $\int d\mu$, regarded as a linear functional on C_0, is positive, therefore continuous, and agrees with f on C_k, it must be identical to f. Thus $f(u) = \int u\,d\mu$ for every $u \in C_0$.

(c) Because there is only one Radon measure giving the right integrals to members of C_k (436J), μ is unique.

***436L** The results here, by opening a path between measure theory and the study of linear functionals on spaces of continuous functions, provide an enormously powerful tool for the analysis of dual spaces $C(X)^*$ and their relatives. I will explore some of these ideas in the next section. Here I will give only a sample pair of facts to show how measure theory can tell us things about Banach lattices which seem difficult to reach by other methods.

Proposition Let X be a topological space; write C_b for $C_b(X)$. Suppose that U is a norm-closed linear subspace of C_b^* such that the functional $u \mapsto f(u \times v) : C_b \to \mathbb{R}$ belongs to U whenever $f \in U$ and $v \in C_b$. Then U is a band in the L-space C_b^*.

proof (a) Let $e = \chi X$ be the standard order unit of C_b, and if $f \in C_b^*$ and u, $v \in C_b$ write $f_v(u)$ for $f(u \times v)$. By 356Na, $C_b^* = C_b^\sim$ is an L-space.

(b) I show first that U is a Riesz subspace of $C_b^\sim$. **P** If $f \in U$ and $\epsilon > 0$, there is a $v \in C_b$ such that $|v| \le e$ and $f(v) \ge |f|(e) - \epsilon$ (356B). Now $f_v \le |f|$ and

$$\||f| - f_v\| = (|f| - f_v)(e) \le \epsilon$$

(356Nb), while $f_v \in U$. As ϵ is arbitrary, $|f| \in \overline{U} = U$; as f is arbitrary, U is a Riesz subspace of C_b^* (352Ic). **Q**

(c) Now suppose that X is a compact Hausdorff space. Then U is a solid linear subspace of $C_b^\sim = C(X)^\sim$. **P** Suppose that $f \in U$ and that $0 \le g \le f$. Let $\epsilon > 0$. By either 436J or 436K, there are Radon measures μ, ν on

X such that $f(u)=\int u\,d\mu$ and $g(u)=\int u\,d\nu$ for every $u\in C(X)$. By 416Ea, $\nu\le\mu$ in the sense of 234P, so ν is an indefinite-integral measure over μ (415Oa, or otherwise); let $w:X\to[0,1]$ be such that $\int_E w\,d\mu=\nu E$ for every $E\in\operatorname{dom}\nu$. There is a continuous function $v:X\to\mathbb{R}$ such that $\int|w-v|d\mu\le\epsilon$ (416I), and now

$$|g(u)-f_v(u)|=|\int u\,d\nu-\int u\times v\,d\mu|=|\int u\times(w-v)d\mu|$$

(235K)

$$\le\|u\|_\infty\int|w-v|d\mu\le\epsilon\|u\|_\infty$$

for every $u\in C(X)$, so $\|g-f_v\|\le\epsilon$, while $f_v\in U$. As ϵ is arbitrary, $g\in U$; as f and g are arbitrary (and U is a Riesz subspace of $C(X)^*$), U is a solid linear subspace of $C(X)^*$. **Q**

Since every norm-closed solid linear subspace of an L-space is a band (354Eg), it follows that (provided X is a compact Hausdorff space) U is actually a band.

(d) For the general case, let Z be the set of all Riesz homomorphisms $z:C_b\to\mathbb{R}$ such that $z(e)=1$. Then Z is a weak*-closed subset of the unit ball of C_b^* so is a compact Hausdorff space. We have a Banach lattice isomorphism $T:C_b\to C(Z)$ given by the formula $(Tu)(z)=z(u)$ for $u\in C_b$, $z\in Z$ (see the proofs of 353M and 354K). But note also that T is multiplicative (353Pd), and $T':C(Z)^*\to C_b^*$ is a Banach lattice isomorphism. Let V be $(T')^{-1}[U]\subseteq C(Z)^*$; then V is a closed linear subspace of $C(Z)^*$. If $g\in V$ and v, $w\in C(Z)$, then

$$g(v\times w)=(T'g)(T^{-1}v\times T^{-1}w),$$

so g_w, defined in $C(Z)^*$ by the convention of (a) above, is just $(T')^{-1}((T'g)_{T^{-1}w})$, and belongs to V. By (b), V is a band in $C(Z)^*$ so U is a band in C_b^*, as required.

***436M Corollary** Let $\mathfrak{A}$ be a Boolean algebra, and $M(\mathfrak{A})$ the L-space of bounded finitely additive functionals on $\mathfrak{A}$ (362B). Let $U\subseteq M(\mathfrak{A})$ be a norm-closed linear subspace such that $a\mapsto\nu(a\cap b)$ belongs to U whenever $\nu\in U$ and $b\in\mathfrak{A}$. Then U is a band in $M(\mathfrak{A})$.

proof (a) Let Z be the Stone space of $\mathfrak{A}$, so that $C(Z)$ is the M-space $L^\infty(\mathfrak{A})$ (363A), and we have an L-space isomorphism $T:M(\mathfrak{A})\to C(Z)^*$ defined by saying that $(T\nu)(\chi\widehat{a})=\nu a$ whenever $\nu\in M(\mathfrak{A})$, $a\in\mathfrak{A}$ and $\widehat{a}$ is the open-and-closed subset of Z corresponding to a (363K). Now $V=T[U]$ is a norm-closed linear subspace of $C(Z)^*$.

(b) V satisfies the condition of 436L. **P** Suppose that $f\in V$ and $v\in C(Z)$; set $f_v(u)=f(u\times v)$ for $u\in C(Z)$, and $\nu=T^{-1}f\in U$. (i) If v is of the form $\chi\widehat{b}$, where $b\in\mathfrak{A}$, then $\nu_b\in U$, where $\nu a=\nu(a\cap b)$ for $a\in\mathfrak{A}$. Now $T\nu_b\in V$ and

$$(T\nu_b)(\chi\widehat{a})=\nu_b a=\nu(a\cap b)=f(\chi(\widehat{a\cap b}))=f(\chi\widehat{a}\times\chi\widehat{b})=f_v(\chi\widehat{a})$$

for every $a\in\mathfrak{A}$, so $f_v=T\nu_b$ belongs to U. (ii) If v is of the form $\sum_{i=0}^n\alpha_i\chi\widehat{b}_i$, where $b_0,\dots,b_n\in\mathfrak{A}$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$, then $f_v=\sum_{i=0}^n\alpha_i f_{v_i}$, where $v_i=\chi\widehat{b}_i$ for each i; as $f_{v_i}\in V$ for each i, $f_v\in V$, because V is a linear subspace. (iii) In general, given $v\in C(Z)=L^\infty(\mathfrak{A})$ and $\epsilon>0$, there is a $w\in C(Z)$, expressible in the form of (ii), such that $\|v-w\|_\infty\le\epsilon$ (363C). In this case $f_w\in V$, by (ii), while

$$|f_v(u)-f_w(u)|=|f(u\times(v-w))|\le\|f\|\|u\|_\infty\epsilon$$

for every $u\in C(Z)$, and $\|f_v-f_w\|\le\epsilon\|f\|$. As ϵ is arbitrary and V is closed, $f_v\in V$. **Q**

(c) By 436L, V is a band in $C(Z)^*$; as T is a Riesz space isomorphism, U is a band in $M(\mathfrak{A})$.

436X Basic exercises >(a) Let (X,Σ,μ_0) be a measure space, and U the set of μ_0-integrable Σ-measurable real-valued functions defined everywhere on X. For $u\in U$ set $f(u)=\int u\,d\mu_0$. Show that U and f satisfy the conditions of 436D, and that the measure μ constructed from f by the procedure there is just the c.l.d. version of μ_0.

(b) Let μ and ν be two complete locally determined measures on a set X, and suppose that $\int f\,d\mu=\int f\,d\nu$ for every function $f:X\to\mathbb{R}$ for which either integral is defined in $\mathbb{R}$. Show that $\mu=\nu$.

>(c) Let X be a set, U a truncated Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a sequentially smooth positive linear functional. For $A \subseteq X$ set

$$\theta A = \inf\{\sup_{n\in\mathbb{N}} f(u_n) : \langle u_n\rangle_{n\in\mathbb{N}} \text{ is a non-decreasing sequence in } U^+,$$
$$\lim_{n\to\infty} u_n(x) = 1 \text{ for every } x \in A\},$$

taking $\inf\emptyset = \infty$ if need be. Show that θ is an outer measure on X. Let μ_0 be the measure defined from θ by Carathéodory's method. Show that $f(u) = \int u\,d\mu_0$ for every $u \in U$. Show that the measure μ constructed in 436D is the c.l.d. version of μ_0.

(d) Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Let $\tau : U \to [0, \infty[$ be a seminorm such that (i) $\tau(u) \le \tau(v)$ whenever $|u| \le |v|$ (ii) $\lim_{n\to\infty} \tau(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$. Show that for any $u_0 \in U^+$ there is a measure μ on x such that $\int u\,d\mu$ is defined, and less than or equal to $\tau(u)$, for every $u \in U$, and $\int u_0 d\mu = \tau(u_0)$. (*Hint*: put the Hahn-Banach theorem together with 436D.)

(e)(i) Let X be any topological space. Show that every positive linear functional on $C(X)$ is sequentially smooth (compare 375A), so corresponds to a totally finite Baire measure on X. (ii) Let X be a regular Lindelöf space. Show that every positive linear functional on $C(X)$ is smooth, so corresponds to a totally finite quasi-Radon measure on X. (iii) Let X be a K-analytic Hausdorff space. Show that every positive linear functional on $C(X)$ corresponds to at least one totally finite Radon measure on X.

(f) Let X be a completely regular space. Show that it is measure-compact iff every sequentially smooth positive linear functional on $C_b(X)$ is smooth. (*Hint*: 436Xj.)

(g) A completely regular topological space X is called **realcompact** if every Riesz homomorphism from $C(X)$ to $\mathbb{R}$ is of the form $u \mapsto \alpha u(x)$ for some $x \in X$ and $\alpha \ge 0$. (i) Show that, for any topological space X, any Riesz homomorphism from $C(X)$ to $\mathbb{R}$ is representable by a Baire measure on X which takes at most two values. (ii) Show that a completely regular space X is realcompact iff every $\{0, 1\}$-valued Baire measure on X is of the form $E \mapsto \chi E(x)$. (iii) Show that a completely regular space X is realcompact iff every purely atomic totally finite Baire measure on X is τ-additive. (iii) Show that a measure-compact completely regular space is realcompact. (iv) Show that the discrete topology on $[0, 1]$ is realcompact. (*Hint*: if ν is a Baire measure taking only the values 0 and 1, set $x_0 = \sup\{x : \nu[x, 1] = 1\}$.) (v) Show that any product of realcompact completely regular spaces is realcompact. (vi) Show that a Souslin-F subset of a realcompact completely regular space is realcompact. (For realcompact spaces which are not measure-compact, see 439Xn.)

(h) In 436F, suppose that μ and ν are τ-additive. Let $\tilde\mu$ and $\tilde\nu$ be the corresponding quasi-Radon measures (415N), and $\tilde\lambda$ the quasi-Radon product of $\tilde\mu$ and $\tilde\nu$ (417N). Show that λ is the restriction of $\tilde\lambda$ to the Baire σ-algebra of $X \times Y$.

>(i) For $u \in C([0, 1])$ let $f(u)$ be the Lebesgue integral of u. Show that f is smooth (therefore sequentially smooth) but not sequentially order-continuous (therefore not order-continuous). (*Hint*: enumerate $\mathbb{Q} \cap [0, 1]$ as $\langle q_n\rangle_{n\in\mathbb{N}}$, and set $u_n(x) = \min_{i\le n} 2^{i+2}|x - q_i|$ for $n \in \mathbb{N}$, $x \in [0, 1]$; show that $\inf_{n\in\mathbb{N}} u_n = 0$ in $C([0, 1])$ but $\lim_{n\to\infty} f(u_n) > 0$.)

(j) In 436E, show that μ is τ-additive iff f is smooth.

(k) Suppose that X is a set, U is a truncated Riesz subspace of $\mathbb{R}^X$ and $f : U \to \mathbb{R}$ is a smooth positive linear functional. For $A \subseteq X$ set

$$\theta A = \inf\{\sup_{u\in B} f(u) : B \text{ is an upwards-directed family in } U^+$$
$$\text{such that } \sup_{u\in B} u(x) = 1 \text{ for every } x \in A\},$$

taking $\inf\emptyset = \infty$ if need be. Show that θ is an outer measure on X. Let μ_0 be the measure defined from θ by Carathéodory's method. Show that $f(u) = \int u\,d\mu_0$ for every $u \in U$. Show that the measure μ constructed in 436H is the c.l.d. version of μ_0.

(l) Let X be a completely regular topological space and f a smooth positive linear functional on $C_b(X)$. Show that there is a unique totally finite quasi-Radon measure μ on X such that $f(u)=\int u\,d\mu$ for every $u\in C_b(X)$.

>(m) For $u\in C([0,1])$ let $f(u)$ be the Riemann integral of u (134K). Show that the Radon measure on $[0,1]$ constructed by the method of 436J is just Lebesgue measure on $[0,1]$. Explain how to construct Lebesgue measure on $\mathbb{R}$ from an appropriate version of the Riemann integral on $\mathbb{R}$.

(n) Let X be a topological space. Let $f:C_b(X)\to\mathbb{R}$ be a linear functional. (i) Show that the following are equiveridical: (α) f is **tight**, that is, for every $\epsilon>0$ there is a closed compact $K\subseteq X$ such that $|f(u)|\le\epsilon$ whenever $0\le u\le\chi(X\setminus K)$ (β) there is a tight totally finite Borel measure μ on X such that $|f(u)|\le\int|u|\,d\mu$ for every $u\in C_b(X)$. (*Hint*: show that a positive tight functional is smooth.) (ii) Show that the set of tight functionals on $C_b(X)$ is a band in $C_b(X)^\sim$.

>(o) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be locally compact Radon measure spaces. (i) Show that the function $x\mapsto\int w(x,y)\nu(dy)$ belongs to $C_k(X)$ for every $w\in C_k(X\times Y)$, so we have a positive linear functional $h:C_k(X\times Y)\to\mathbb{R}$ defined by setting

$$h(w)=\iint w(x,y)\nu(dy)\mu(dx)$$

for $w\in C_k(X\times Y)$. (ii) Show that the corresponding Radon measure on $X\times Y$ is just the product Radon measure as defined in 417P/417R.

>(p) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space; identify $L^\infty(\mathfrak{A})$ with $C(Z)$, as in 363A. Let ν be a non-negative finitely additive functional on $\mathfrak{A}$, f the corresponding positive linear functional on $L^\infty(\mathfrak{A})$ (363K), and μ the corresponding Radon measure on Z (416Qb). Show that $f(u)=\int u\,d\mu$ for every $u\in L^\infty(\mathfrak{A})$.

(q) Let X be a locally compact Hausdorff space. Show that a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $C_0(X)$ converges to 0 for the weak topology on $C_0(X)$ iff $\sup_{n\in\mathbb{N}}\|u_n\|_\infty$ is finite and $\lim_{n\to\infty}f_n(x)=0$ for every $x\in X$.

(r) Let X be a non-empty compact Hausdorff space, and $\phi:X\to X$ a continuous function. Show that there is a Radon probability measure μ on X such that ϕ is inverse-measure-preserving for μ. (*Hint*: let $\mathcal{F}$ be a non-principal ultrafilter on X, x_0 any point of X, and define μ by the formula $\int ud\mu=\lim_{n\to\mathcal{F}}\frac{1}{n+1}\sum_{k=0}^{n}u(\phi^k(x_0))$ for every $u\in C(X)$. Use 416E(b-v) to show that $\mu\phi^{-1}=\mu$.)

(s) Let X be a locally compact Hausdorff space. (i) Write $M_{\mathrm{R}}^{\infty+}$ for the set of all Radon measures on X. For $\mu\in M_{\mathrm{R}}^{\infty+}$, let $S\mu$ be the corresponding functional on $C_k(X)$, defined by setting $(S\mu)(u)=\int u\,d\mu$ for every $u\in C_k(X)$. Show that $S(\mu+\nu)=S\mu+S\nu$ and $S(\alpha\mu)=\alpha S\mu$ whenever μ, $\nu\in M_{\mathrm{R}}^{\infty+}$ and $\alpha\ge0$, where addition and scalar multiplication of measures are defined as in 234G and 234Xf. (ii) Write M_{R}^{+} for the set of totally finite Radon measures on X. For $\mu\in M_{\mathrm{R}}^{+}$, let $T\mu$ be the corresponding functional on $C_b(X)$, defined by setting $(T\mu)(u)=\int u\,d\mu$ for every $u\in C_b(X)$. Show that $T(\mu+\nu)=T\mu+T\nu$ and $T(\alpha\mu)=\alpha T\mu$ whenever μ, $\nu\in M_{\mathrm{R}}^{+}$ and $\alpha\ge0$.

436Y Further exercises (a) Let X be a set, U a Riesz subspace of $\mathbb{R}^X$, and $f:U\to\mathbb{R}$ a sequentially smooth positive linear functional. (i) Write U_σ for the set of functions from X to $[0,\infty]$ expressible as the supremum of a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U^+ such that $\sup_{n\in\mathbb{N}}f(u_n)<\infty$. Show that there is a functional $f_\sigma:U_\sigma\to[0,\infty[$ such that $f_\sigma(u)=\sup_{n\in\mathbb{N}}f(u_n)$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in U^+ with supremum $u\in U_\sigma$. (Compare 122I.) (ii) Show that $u+v\in U_\sigma$ and $f_\sigma(u+v)=f_\sigma(u)+f_\sigma(v)$ for all u, $v\in U_\sigma$. (iii) Suppose that u, $v\in U_\sigma$, $u\le v$ and $u(x)=v(x)$ whenever $v(x)$ is finite. Show that $f_\sigma(u)=f_\sigma(v)$. (*Hint*: take non-decreasing sequences $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ with suprema u, v. Consider $\langle f(v_k-u_n-\delta v_n)^+\rangle_{n\in\mathbb{N}}$ where $k\in\mathbb{N}$, $\delta>0$.) (iv) Let V be the set of functions $v:X\to\mathbb{R}$ such that there are u_1, $u_2\in U_\sigma$ such that $v(x)=u_1(x)-u_2(x)$ whenever $u_1(x)$, $u_2(x)$ are both finite. Show that V is a linear subspace of $\mathbb{R}^X$ and that there is a linear functional $g:V\to\mathbb{R}$ defined by setting $g(v)=f_\sigma(u_1)-f_\sigma(u_2)$ whenever $v=u_1-u_2$ in the sense of the last sentence. (v) Show that V is a Riesz subspace of $\mathbb{R}^X$. (vi) Show that if $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in V and $\gamma=\sup_{n\in\mathbb{N}}g(v_n)$ is finite, then there is a $v\in V$ such that $g(v)=\sup_{n\in\mathbb{N}}g(v\wedge v_n)=\gamma$. (This is a version of the **Daniell integral**.)

(b) Develop further the theory of 436Ya, finding a version of Lebesgue's Dominated Convergence Theorem, a concept of 'negligible' subset of X, and an L-space of equivalence classes of 'integrable' functions.

(c) Let X be a countably compact topological space. Show that every positive linear functional on $C_b(X)$ is sequentially smooth, so corresponds to a totally finite Baire measure on X.

(d) Let X be a sequential space, Y a topological space, μ a semi-finite Baire measure on X and ν a σ-finite Baire measure on Y. Let $\tilde{\mu}$ be the c.l.d. version of μ. Show that there is a semi-finite Baire measure λ on $X \times Y$ such that

$$\lambda W = \int \nu W[\{x\}]\tilde{\mu}(dx), \qquad \int f d\lambda = \iint f(x,y)\nu(dy)\tilde{\mu}(dx)$$

for every Baire set $W \subseteq X \times Y$ and every non-negative continuous function $f : X \times Y \to \mathbb{R}$. Show that the c.l.d. version of λ extends the c.l.d. product measure of μ and ν.

(e) Let $X_0, \ldots, X_n$ be sequential spaces and μ_i a totally finite Baire measure on X_i for each i. (i) Show that if $f : X_0 \times \ldots \times X_n \to \mathbb{R}$ is a bounded separately continuous function, then

$$\phi(f) = \int \ldots \int f(x_0, \ldots, x_n)\mu_n(dx_n) \ldots \mu_0(dx_0)$$

is defined, so that we have a corresponding Baire product measure on $X_0 \times \ldots \times X_n$. (ii) Show that this product is associative.

(f) Let X and Y be compact Hausdorff spaces. (i) Show that there is a unique bilinear operator $\phi : C(X)^* \times C(Y)^* \to C(X \times Y)^*$ which is separately continuous for the weak* topologies and such that $\phi(\delta_x, \delta_y) = \delta_{(x,y)}$ for all $x \in X$ and $y \in Y$, setting $\delta_x(f) = f(x)$ for $f \in C(X)$ and $x \in X$. (ii) Show that if μ and ν are Radon measures on X, Y respectively with Radon measure product λ, then $\phi(\int d\mu, \int d\nu) = \int d\lambda$. (iii) Show that $\|\phi\| \le 1$ (definition: 253Ab).

(g) Let X be any topological space. (i) Let C_k be the set of continuous functions $u : X \to \mathbb{R}$ such that $\overline{\{x : u(x) \ne 0\}}$ is compact, and $f : C_k \to \mathbb{R}$ a positive linear functional. Show that there is a tight quasi-Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k$. (ii) Let $\tilde{C}_k$ be the set of continuous functions $u : X \to \mathbb{R}$ such that $\{x : u(x) \ne 0\}$ is relatively compact, and $f : \tilde{C}_k \to \mathbb{R}$ a positive linear functional. Show that there is a tight quasi-Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k$.

436 Notes and comments From the beginning, integration has been at the centre of measure theory. My own view – implicit in the arrangement of this treatise, from Chapter 11 onward – is that 'measure' and 'integration' are not quite the same thing. I freely acknowledge that my treatment of 'integration' is distorted by my presentation of it as part of measure theory; on the other side of the argument, I hold that regarding 'measure' as a concept subsidiary to 'integral', as many authors do, seriously interferes with the development of truly penetrating intuitions for the former. But it is undoubtedly true that every complete locally determined measure can be derived from its associated integral (436Xb). Moreover, it is clearly of the highest importance that we should be able to recognise integrals when we see them; I mean, given a linear functional on a linear space of functions, then if it can be expressed as an integral with respect to a measure this is something we need to know at once. And thirdly, investigation of linear functionals frequently leads us to measures of great importance and interest.

Concerning the conditions in 436D, an integral must surely be 'positive' (because measures in this treatise are always non-negative) and 'sequentially smooth' (because measures are supposed to be countably additive). But it is not clear that we are forced to restrict our attention to Riesz subspaces of $\mathbb{R}^X$, and even less clear that they have to be 'truncated'. In 439I below I give an example to show that this last condition is essential for 436D and 436H as stated. However it is not necessary for large parts of the theory. In many cases, if $U \subseteq \mathbb{R}^X$ is a Riesz subspace which is not truncated, we can take an element $e \in U^+$ and look at $Y = \{x : e(x) > 0\}$, $V_e = \{u : u(x) = 0$ for every $x \in X \setminus Y\} \cong W_e$, where $W_e = \{u/e : u \in V_e\}$ is a truncated Riesz subspace of $\mathbb{R}^Y$. But there are applications in which this approach is unsatisfactory and a more radical revision of the basic theory of integration, as in 436Ya, is useful.

I have based the arguments of this section on the inner measure constructions of §413. Of course it is also possible to approach them by means of outer measures (436Xc, 436Xk).

I emphasize the exercise 436Xo because it is prominent in 'Bourbakist' versions of the theory of Radon measures, in which (following Bourbaki 65 rather than Bourbaki 69) Radon measures are regarded as linear functionals on spaces of continuous functions. By 436J, this is a reasonably effective approach as long as we are interested only in locally compact spaces, and there are parts of the theory of topological groups (notably the duality theory of §445 below) in which it even has advantages. The construction of 436Xo shows that we can find a direct approach to the tensor product of linear functionals which does not require any attempt to measure sets rather than integrate functions. I trust that the prejudices I am expressing will not be taken as too sweeping a disparagement of such methods. Practically all correct arguments in mathematics (and not a few incorrect ones) are valuable in some way, suggesting new possibilities for investigation. In particular, one of the challenges of measure theory (not faced in this treatise) is that of devising effective theories of vector-valued measures. Typically this is much easier with Riemann-type integrals, and any techniques for working directly with these should be noted.

436L revisits ideas from Chapter 35, and the result would be easier to find if it were in §356. I include it here as an example of the way in which familiar material from measure theory (in particular, the Radon-Nikodým theorem) can be drafted to serve functional analysis. I should perhaps remark that there are alternative routes which do not use measure theory explicitly, and while longer are (in my view) more illuminating.

437 Spaces of measures

Once we have started to take the correspondence between measures and integrals as something which operates in both directions, we can go a very long way. While 'measures', as dealt with in this treatise, are essentially positive, an 'integral' can be thought of as a member of a linear space, dual in some sense to a space of functions. Since the principal spaces of functions are Riesz spaces, we find ourselves looking at dual Riesz spaces as discussed in §356; while the corresponding spaces of measures are close to those of §362. Here I try to draw these ideas together with an examination of spaces $U_\sigma^\sim$ and $U_\tau^\sim$ of sequentially smooth and smooth functionals, and the matching spaces M_σ and M_τ of countably additive and τ-additive measures (437A-437I). Because a (sequentially) smooth functional corresponds to a countably additive measure, which can be expected to integrate many more functions than those in the original Riesz space (typically, a space of continuous functions), we find that relatively large spaces of bounded measurable functions can be canonically embedded into the biduals $(U_\sigma^\sim)^*$ and $(U_\tau^\sim)^*$ (437C, 437H, 437I).

The guiding principles of functional analysis encourage us not only to form linear spaces, but also to examine linear space topologies, starting with norm and weak topologies. The theory of Banach lattices described in §354, particularly the theory of M- and L-spaces, is an important part of the structure here. In addition, our spaces $U_\sigma^\sim$ have natural weak* topologies which can be regarded as topologies on spaces of measures; these are the 'vague' topologies of 437J, which have already been considered, in a special case, in §285.

It turns out that on the positive cone of M_τ, at least, the vague topology may have an alternative description directly in terms of the behaviour of the measures on open sets (437L). This leads us to a parallel idea, the 'narrow' topology on non-negative additive functionals (437Jd). The second half of the section is devoted to the elementary properties of narrow topologies (437K-437N), with especial reference to compact sets in these topologies (437P, 437Rf, 437T). Seeking to identify narrowly compact sets, we come to the concept of 'uniform tightness' (437O). Bounded uniformly tight sets are narrowly relatively compact (437P); in 'Prokhorov spaces' (437U) the converse is true. I end the section with a list of the best-known Prokhorov spaces (437V).

437A Smooth and sequentially smooth duals Let X be a set, and U a Riesz subspace of $\mathbb{R}^X$. Recall that $U^\sim$ is the Dedekind complete Riesz space of order-bounded linear functionals on U, that $U_c^\sim$ is the band of differences of sequentially order-continuous positive linear functionals, and that $U^\times$ is the band of differences of order-continuous positive linear functionals (356A). A functional $f \in (U^\sim)^+$ is 'sequentially smooth' if $\inf_{n\in\mathbb{N}} f(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$, and 'smooth' if $\inf_{u\in A} f(u) = 0$ whenever $A \subseteq U$ is a non-empty downwards-directed set and $\inf_{u\in A} u(x) = 0$ for every $x \in X$ (436A, 436G).

(a) Set $U_\sigma^\sim = \{f : f \in U^\sim, |f| \text{ is sequentially smooth}\}$, the **sequentially smooth dual** of U. Then $U_\sigma^\sim$ is a band in $U^\sim$. **P** (i) If $f \in U_\sigma^\sim$, $g \in U^\sim$ and $|g| \le |f|$, then

$$|g|(u_n) \le |f|(u_n) \to 0$$

whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n\to\infty} u_n(x) = 0$ for every x, so $|g|$ is sequentially smooth and $g \in U_\sigma^\sim$. Thus $U_\sigma^\sim$ is a solid subset of $U^\sim$. (ii) If f, $g \in U_\sigma^\sim$ and $\alpha \in \mathbb{R}$, then

$$|f+g|(u_n) \le |f|(u_n) + |g|(u_n) \to 0, \quad |\alpha f|(u_n) = |\alpha||f|(u_n) \to 0$$

whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n\to\infty} u_n(x) = 0$ for every x, so $|f+g|$ and $|\alpha f|$ are sequentially smooth. Thus $U_\sigma^\sim$ is a Riesz subspace of $U^\sim$. (iii) Now suppose that $B \subseteq (U_\sigma^\sim)^+$ is an upwards-directed set with supremum $g \in U^\sim$, and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$. Then, given $\epsilon > 0$, there is an $f \in B$ such that $(g - f)(u_0) \le \epsilon$ (355Ed), so that $g(u_n) \le f(u_n) + \epsilon$ for every n, and

$$\limsup_{n\to\infty} g(u_n) \le \epsilon + \lim_{n\to\infty} f(u_n) \le \epsilon.$$

As ϵ and $\langle u_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $g \in U_\sigma^\sim$; as B is arbitrary, $U_\sigma^\sim$ is a band (352Ob). **Q**

As remarked in 436A, sequentially order-continuous positive linear functionals are sequentially smooth, so $U_c^\sim \subseteq U_\sigma^\sim$.

(b) Set $U_\tau^\sim = \{f : f \in U^\sim, |f| \text{ is smooth}\}$, the **smooth dual** of U. Then $U_\tau^\sim$ is a band in $U^\sim$. **P** (i) Suppose that f, $g \in U_\tau^\sim$, $\alpha \in \mathbb{R}$, $h \in U^\sim$ and $|h| \le |f|$. If $A \subseteq U$ is a non-empty downwards-directed set and $\inf_{u\in A} u(x) = 0$ for every $x \in X$, and $\epsilon > 0$, then there are u_0, $u_1 \in A$ such that $|f|(u_0) \le \epsilon$ and $|g|(u_1) \le \epsilon$, and a $u \in A$ such that $u \le u_0 \wedge u_1$. In this case

$$|h|(u) \le |f|(u) \le \epsilon,$$

$$|f + g|(u) \le |f|(u) + |g|(u) \le 2\epsilon,$$

$$|\alpha f|(u) = |\alpha||f|(u) \le |\alpha|\epsilon.$$

As A and ϵ are arbitrary, h, $f+g$ and αf all belong to $U_\tau^\sim$; so that $U_\tau^\sim$ is a solid Riesz subspace of $U^\sim$. (ii) Now suppose that $B \subseteq (U_\tau^\sim)^+$ is an upwards-directed set with supremum $g \in U^\sim$, and that $A \subseteq U$ is a non-empty downwards-directed set such that $\inf_{u\in A} u(x) = 0$ for every $x \in X$. Fix any $u_0 \in A$. Then, given $\epsilon > 0$, there is an $f \in B$ such that $(g - f)(u_0) \le \epsilon$, so that $g(u) \le f(u) + \epsilon$ whenever $u \in A$ and $u \le u_0$. But $A_0 = \{u : u \in A,\ u \le u_0\}$ is also a downwards-directed set, and $\inf_{u\in A_0} u(x) = 0$ for every $x \in X$, so

$$\inf_{u\in A} g(u) \le \epsilon + \inf_{u\in A_0} f(u) \le \epsilon.$$

As ϵ and A are arbitrary, $g \in U_\tau^\sim$; as B is arbitrary, $U_\tau^\sim$ is a band. **Q**

Just as $U_c^\sim \subseteq U_\sigma^\sim$, $U^\times \subseteq U_\tau^\sim$.

437B Signed measures Collecting these ideas together with those of §§362-363, we are ready to approach 'signed measures'. Recall that if X is a set and Σ is a σ-algebra of subsets of X, we can identify $L^\infty = L^\infty(\Sigma)$, as defined in §363, with the space $\mathcal{L}^\infty = \mathcal{L}^\infty(\Sigma)$ of bounded Σ-measurable real-valued functions (363H). Now, because $\mathcal{L}^\infty$ is sequentially order-closed in $\mathbb{R}^X$, sequentially smooth functionals on $\mathcal{L}^\infty$ are actually sequentially order-continuous, so $(\mathcal{L}^\infty)_\sigma^\sim = (\mathcal{L}^\infty)_c^\sim$. Next, we can identify $(L^\infty)_c^\sim$ with the space M_σ of countably additive functionals, or 'signed measures', on Σ (363K); if $\nu \in M_\sigma$, the corresponding member of $(L^\infty)_c^\sim$ is the unique order-bounded (or norm-continuous) linear functional f on L^∞ such that $f(\chi E) = \nu E$ for every $E \in \Sigma$. If $\nu \ge 0$, so that ν is a totally finite measure with domain Σ, then of course f, when interpreted as a functional on $\mathcal{L}^\infty$, must be just integration with respect to ν.

The identification between $(L^\infty)_c^\sim$ and M_σ described in 363K is an L-space isomorphism. So it tells us, for instance, that if we are willing to use the symbol $⨍$ for the duality between L^∞ and the space of bounded finitely additive functionals on Σ, as in 363L, then we can write

$$⨍ u\,d(\mu + \nu) = ⨍ u\,d\mu + ⨍ u\,d\nu$$

for every $u \in \mathcal{L}^\infty$ and all μ, $\nu \in M_\sigma$.

437C Theorem Let X be a set and U a Riesz subspace of $\ell^\infty(X)$, the M-space of bounded real-valued functions on X, containing the constant functions.

(a) Let Σ be the smallest σ-algebra of subsets of X with respect to which every member of U is measurable. Let $M_\sigma = M_\sigma(\Sigma)$ be the L-space of countably additive functionals on Σ (326I[4], 362B). Then there is a Banach lattice isomorphism $T : M_\sigma \to U_\sigma^\sim$ defined by saying that $(T\mu)(u) = \int u\,d\mu$ whenever $\mu \in M_\sigma^+$ and $u \in U$.

(b) We now have a sequentially order-continuous norm-preserving Riesz homomorphism S, embedding the M-space $\mathcal{L}^\infty = \mathcal{L}^\infty(\Sigma)$ of bounded real-valued Σ-measurable functions on X (363Hb) into the M-space $(U_\sigma^\sim)^\sim = (U_\sigma^\sim)^* = (U_\sigma^\sim)^\times$, defined by saying that $(Sv)(T\mu) = \int v\,d\mu$ whenever $v \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$. If $u \in U$, then $(Su)(f) = f(u)$ for every $f \in U_\sigma^\sim$.

proof (a)(i) The norm $\| \|_\infty$ is an order-unit norm on U (354Ga), so $U^* = U^\sim$ is an L-space (356N), and the band $U_\sigma^\sim$ (437Aa) is an L-space in its own right (354O).

(ii) As noted in 437B, we have a Banach lattice isomorphism $T_0 : M_\sigma \to (\mathcal{L}^\infty)_c^\sim$ defined by saying that $(T_0\mu)(u) = \int u\,d\mu$ whenever $u \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$. If we set $T\mu = T_0\mu\restriction U$, then T is a positive linear operator from

[4]Formerly 326E.

M_σ to $U^\sim$, just because U is a linear subspace of $\mathcal{L}^\infty$; and since $T\mu \in U^\sim_\sigma$ for every $\mu \in M_\sigma^+$, T is an operator from M_σ to $U^\sim_\sigma$. Now every $f \in (U^\sim_\sigma)^+$ is of the form $T\mu$ for some $\mu \in M_\sigma^+$. **P** By 436D, there is some measure λ such that $\int u\,d\lambda = f(u)$ for every $u \in U$. Completing λ if necessary, we see that we may suppose that every member of U is $(\operatorname{dom}\lambda)$-measurable, that is, that $\Sigma \subseteq \operatorname{dom}\lambda$; take $\mu = \lambda\restriction\Sigma$. **Q** So T is surjective.

(iii) Write $\mathcal{K}$ for the family of sets $K \subseteq X$ such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ for some sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U. (See the proof of 436D.) We need to know the following. (α) $\mathcal{K} \subseteq \Sigma$. ($\beta$) If $K \in \mathcal{K}$, then there is a non-increasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U such that $\chi K = \inf_{n\in\mathbb{N}} u_n$. (For if $\langle u'_n\rangle_{n\in\mathbb{N}}$ is any sequence in U such that $\chi K = \inf_{n\in\mathbb{N}} u'_n$, we can set $u_n = \inf_{i\le n} u'_i$ for each i.) (γ) The σ-algebra of subsets of X generated by $\mathcal{K}$ is Σ. **P** Let T be the σ-algebra of subsets of X generated by $\mathcal{K}$. $\mathrm{T} \subseteq \Sigma$ because $\mathcal{K} \subseteq \Sigma$. If $u \in U$ and $\alpha > 0$ then $\{x : u(x) \ge \alpha\} \in \mathcal{K}$ (see part (b) of the proof of 436D). So every member of U^+, therefore every member of U, is T-measurable, and $\Sigma \subseteq \mathrm{T}$. **Q**

(iv) T is injective. **P** If μ_1, $\mu_2 \in M_\sigma$ and $T\mu_1 = T\mu_2$, set $\nu_i = \mu_i + \mu_1^- + \mu_2^-$ for each i, so that ν_i is non-negative and $T\nu_1 = T\nu_2$. If $K \in \mathcal{K}$ then there is a non-increasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ in $\mathbb{R}^X$, so

$$\nu_1 K = \inf_{n\in\mathbb{N}} \int u_n d\nu_1 = \inf_{n\in\mathbb{N}} \int u_n d\nu_2 = \nu_2 K.$$

Now $\mathcal{K}$ contains X and is closed under finite intersections and ν_1 and ν_2 agree on $\mathcal{K}$. By the Monotone Class Theorem (136C), ν_1 and ν_2 agree on the σ-algebra generated by $\mathcal{K}$, which is Σ; so $\nu_1 = \nu_2$ and $\mu_1 = \mu_2$. **Q**

Thus T is a linear space isomorphism between M_σ and $U^\sim_\sigma$.

(v) As noted in (ii), $T[M_\sigma^+] = (U^\sim_\sigma)^+$; so T is a Riesz space isomorphism.

(vi) Now if $\mu \in M_\sigma$,

$$\|T\mu\| = |T\mu|(\chi X)$$

(356Nb)

$$= (T|\mu|)(\chi X)$$

(because T is a Riesz homomorphism)

$$= |\mu|(X) = \|\mu\|$$

(362Ba). So T is norm-preserving and is an L-space isomorphism, as claimed.

(b)(i) By 356Pb, $(U^\sim_\sigma)^* = (U^\sim_\sigma)^\sim = (U^\sim_\sigma)^\times$ is an M-space.

(ii) We have a canonical map $S_0 : \mathcal{L}^\infty \to ((\mathcal{L}^\infty)^\sim_c)^\times$ defined by saying that $(S_0 v)(h) = h(v)$ for every $v \in \mathcal{L}^\infty$ and $h \in (\mathcal{L}^\infty)^\sim_c$; and by 356F, S_0 is a Riesz homomorphism. If $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{L}^\infty$ with infimum 0, then $\inf_{n\in\mathbb{N}}(S_0 v_n)(h) = \inf_{n\in\mathbb{N}} h(v_n) = 0$ for every $h \in ((\mathcal{L}^\infty)^\sim_c)^+$, so $\inf_{n\in\mathbb{N}} S_0 v_n = 0$ (355Ee); as $\langle v_n\rangle_{n\in\mathbb{N}}$ is arbitrary, S_0 is sequentially order-continuous (351Gb).

Also S_0 is norm-preserving. **P** (α) If $h \in (\mathcal{L}^\infty)^\sim_c$ and $v \in \mathcal{L}^\infty$, then

$$|(S_0 v)(h)| = |h(v)| \le \|h\|\|v\|_\infty,$$

so $\|S_0 v\| \le \|v\|_\infty$. ($\beta$) If $v \in \mathcal{L}^\infty$ and $0 \le \gamma < \|v\|_\infty$, take $x \in X$ such that $|v(x)| > \gamma$, and define $h_x \in (\mathcal{L}^\infty)^\sim_c$ by setting $h_x(w) = w(x)$ for every $w \in \mathcal{L}^\infty$; then $\|h_x\| = 1$ and

$$|(S_0 v)(h_x)| = |h_x(v)| = |v(x)| \ge \gamma,$$

so $\|S_0 v\| \ge \gamma$. As γ is arbitrary, $\|S_0 v\| \ge \|v\|_\infty$ and $\|S_0 v\| = \|v\|_\infty$. **Q**

(iii) Now $T_0 : M_\sigma \to (\mathcal{L}^\infty)^\sim_c$ and $T : M_\sigma \to U^\sim_\sigma$ are both norm-preserving Riesz space isomorphisms, so $T_0 T^{-1} : U^\sim_\sigma \to (\mathcal{L}^\infty)^\sim_c$ is another, and its adjoint $S_1 : ((\mathcal{L}^\infty)^\sim_c)^* \to (U^\sim_\sigma)^*$ must also be a norm-preserving Riesz space isomorphism. So if we set $S = S_1 S_0$, S will be a norm-preserving sequentially order-continuous Riesz homomorphism from $\mathcal{L}^\infty$ to $(U^\sim_\sigma)^\times = (U^\sim_\sigma)^*$.

(iv) Setting the construction out in this way tells us a lot about the properties of the operator S, but undeniably leaves it somewhat obscure. So let us start again from $v \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$, and seek to calculate $(Sv)(T\mu)$. We have

$$(Sv)(T\mu) = (S_1 S_0 v)(T\mu) = (S_0 v)(T_0 T^{-1} T\mu)$$

(because S_1 is the adjoint of $T_0 T^{-1}$)

$$= (S_0 v)(T_0\mu) = (T_0\mu)(v) = \int v\,d\mu,$$

as claimed.

If $u \in U$, then $(T\mu)(u) = (T_0\mu)(u)$ for every $\mu \in M_\sigma$, so if $f \in U^\sim_\sigma$ then

$$(Su)(f) = (S_1S_0u)(f) = (S_0u)(T_0T^{-1}f) = (T_0T^{-1}f)(u) = (TT^{-1}f)(u) = f(u).$$

This completes the proof.

437D Remarks What is happening here is that the canonical Riesz homomorphism $u \mapsto \hat{u}$ from U to $(U^\sim_\sigma)^*$ (356F) has a natural extension to $\mathcal{L}^\infty(\Sigma)$. The original homomorphism $u \mapsto \hat{u}$ is not, as a rule, sequentially order-continuous, just because $U^\sim_\sigma$ is generally larger than $U^\sim_c$; but the extension to $\mathcal{L}^\infty$ is sequentially order-continuous. If you like, it is sequential smoothness which is carried over to the extension, and because the embedding of $\mathcal{L}^\infty$ in $\mathbb{R}^X$ is sequentially order-continuous, a sequentially smooth operator on $\mathcal{L}^\infty$ is sequentially order-continuous.

In the statement of 437C, I have used the formulae $(T\mu)(u) = \int u\,d\mu$ and $(Sv)(T\mu) = \int v\,d\mu$ on the assumption that $\mu \in M^+_\sigma$, so that μ is actually a measure on the definition used in this treatise, and $\int d\mu$ is the ordinary integral as constructed in §122. Since the functions u and v are bounded, measurable and defined everywhere, we can choose to extend the notion of integration to signed measures, as in 363L, in which case the formulae $(T\mu)(u) = ⨍ u\,d\mu$ and $(Sv)(T\mu) = ⨍ v\,d\mu$ become meaningful, and true, for all $\mu \in M_\sigma$, $u \in U$ and $v \in \mathcal{L}^\infty$.

In fact the ideas here can be pushed farther, as in 437Ib, 437Xf and 437Yd.

437E Corollary Let X be a completely regular Hausdorff space, and $\mathcal{B}\mathfrak{a} = \mathcal{B}\mathfrak{a}(X)$ its Baire σ-algebra. Then we can identify $C_b(X)^\sim_\sigma$ with the L-space $M_\sigma(\mathcal{B}\mathfrak{a})$ of countably additive functionals on $\mathcal{B}\mathfrak{a}$, and we have a norm-preserving sequentially order-continuous Riesz homomorphism S from $\mathcal{L}^\infty(\mathcal{B}\mathfrak{a})$ to $(C_b(X)^\sim_\sigma)^*$ defined by setting $(Sv)(f) = \int v\,d\mu_f$ for every $v \in \mathcal{L}^\infty$ and $f \in (C_b(X)^\sim_\sigma)^+$, where μ_f is the Baire measure associated with f.

proof Apply 437C with $U = C_b(X)$ (cf. 436E).

437F Proposition Let X be a topological space and $\mathcal{B} = \mathcal{B}(X)$ its Borel σ-algebra. Let M_σ be the L-space of countably additive functionals on $\mathcal{B}$.

(a) Write $M_\tau \subseteq M_\sigma$ for the set of differences of τ-additive totally finite Borel measures. Then M_τ is a band in M_σ, so is an L-space in its own right.

(b) Write $M_t \subseteq M_\tau$ for the set of differences of totally finite Borel measures which are tight (that is, inner regular with respect to the closed compact sets). Then M_t is a band in M_σ, so is an L-space in its own right.

proof (a)(i) Let μ_1, μ_2 be totally finite τ-additive Borel measures on X, $\alpha \ge 0$, and $\mu \in M_\sigma$ such that $0 \le \mu \le \mu_1$. Then $\mu_1 + \mu_2$, $\alpha\mu_1$ and μ are totally finite τ-additive Borel measures. **P** They all belong to M_σ, that is, are totally finite Borel measures. Now let $\mathcal{G}$ be a non-empty upwards-directed family of open sets in X with union H, and $\epsilon > 0$. Then there are G_1, $G_2 \in \mathcal{G}$ such that $\mu_1G_1 \ge \mu_1H - \epsilon$ and $\mu_2G_2 \ge \mu_2H - \epsilon$, and a $G \in \mathcal{G}$ such that $G \supseteq G_1 \cup G_2$. In this case,

$$(\mu_1 + \mu_2)(G) \ge (\mu_1 + \mu_2)(H) - 2\epsilon,$$

$$(\alpha\mu_1)(G) \ge (\alpha\mu_1)(H) - \alpha\epsilon$$

and

$$\mu G = \mu H - \mu(H \setminus G) \ge \mu H - \mu_1(H \setminus G) \ge \mu H - \epsilon.$$

As $\mathcal{G}$ and ϵ are arbitrary, $\mu_1 + \mu_2$, $\alpha\mu_1$ and μ are all τ-additive. **Q**

It follows that M_τ is a solid linear subspace of M_σ.

(ii) Now suppose that $B \subseteq M^+_\tau$ is non-empty and upwards-directed and has a supremum ν in M_σ. Then $\nu \in M_\tau$. **P** If $\mathcal{G}$ is a non-empty upwards-directed family of open sets with union H, then

$$\nu H = \sup_{\mu \in B} \mu H$$

(362Be)

$$= \sup_{\mu \in B, G \in \mathcal{G}} \mu G = \sup_{G \in \mathcal{G}} \nu G;$$

as $\mathcal{G}$ is arbitrary, ν is τ-additive and belongs to M_τ. **Q**

As B is arbitrary, M_τ is a band in M_σ. By 354O, it is itself an L-space.

(b) We can use the same arguments. Suppose that μ_1, $\mu_2 \in M_\sigma^+$ are tight, $\alpha \geq 0$, and $\mu \in M_\sigma$ is such that $0 \leq \mu \leq \mu_1$. If $E \in \mathcal{B}$ and $\epsilon > 0$, there are closed compact sets K_1, $K_2 \subseteq E$ such that $\mu_1 K_1 \geq \mu_1 E - \epsilon$ and $\mu_2 K_2 \geq \mu_2 E - \epsilon$. Set $K = K_1 \cup K_2$, so that K also is a closed compact subset of E. Then

$$(\mu_1 + \mu_2)(K) \geq (\mu_1 + \mu_2)(E) - 2\epsilon,$$

$$(\alpha\mu_1)(K) \geq (\alpha\mu_1)(E) - \alpha\epsilon$$

and

$$\mu K = \mu E - \mu(E \setminus K) \geq \mu E - \mu_1(E \setminus K) \geq \mu E - \epsilon.$$

As $\mathcal{G}$ and ϵ are arbitrary, $\mu_1 + \mu_2$, $\alpha\mu_1$ and μ are all tight; as μ_1, μ_2, μ and α are arbitrary, M_t is a solid linear subspace of M_σ.

Now suppose that $B \subseteq M_t^+$ is non-empty and upwards-directed and has a supremum ν in M_σ. Take any $E \in \mathcal{B}$ and $\epsilon > 0$. Then there is a $\mu \in B$ such that $\mu E \geq \nu E - \epsilon$; there is a closed compact set $K \subseteq E$ such that $\mu K \geq \mu E - \epsilon$; and now $\nu K \geq \nu E - 2\epsilon$. As E and ϵ are arbitrary, ν is tight; as B is arbitrary, M_t is a band in M_σ, and is in itself an L-space.

437G Definitions Let X be a topological space. A **signed Baire measure** on X will be a countably additive functional on the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$, which by the Jordan decomposition theorem (231F) is expressible as the difference of two totally finite Baire measures; a **signed Borel measure** will be a countably additive functional on the Borel σ-algebra $\mathcal{B}(X)$, that is, the difference of two totally finite Borel measures; a **signed τ-additive Borel measure** will be a member of the L-space M_τ as described in 437F, that is, the difference of two τ-additive totally finite Borel measures; and a **signed tight Borel measure** will be a member of the L-space M_t as described in 437F, that is, the difference of two tight totally finite Borel measures.

437H Theorem Let X be a set and U a Riesz subspace of $\ell^\infty(X)$ containing the constant functions. Let $\mathfrak{T}$ be the coarsest topology on X rendering every member of U continuous, and $\mathcal{B}$ the corresponding Borel σ-algebra.

(a) Let M_τ be the L-space of signed τ-additive Borel measures on X. Then we have a Banach lattice isomorphism $T : M_\tau \to U_\tau^\sim$ defined by saying that $(T\mu)(u) = \int u\,d\mu$ whenever $\mu \in M_\tau^+$ and $u \in U$.

(b) We now have a sequentially order-continuous norm-preserving Riesz homomorphism S, embedding the M-space $\mathcal{L}^\infty = \mathcal{L}^\infty(\mathcal{B})$ of bounded Borel measurable functions on X into $(U_\tau^\sim)^\sim = (U_\tau^\sim)^* = (U_\tau^\sim)^\times$, defined by saying that $(Sv)(T\mu) = \int v\,d\mu$ whenever $v \in \mathcal{L}^\infty$ and $\mu \in M_\tau^+$. If $u \in U$, then $(Su)(f) = f(u)$ for every $f \in U_\tau^\sim$.

proof The proof follows the same lines as that of 437C.

(a)(i) As before, the norm $\| \|_\infty$ is an order-unit norm on U, so $U^* = U^\sim$ is an L-space, and the band $U_\tau^\sim$ (437Ab) is an L-space in its own right, like M_τ (437F).

(ii) Let M_σ be the L-space of all countably additive functionals on $\mathcal{B}$, so that M_τ is a band in M_σ. Let $T_0 : M_\sigma \to (\mathcal{L}^\infty)_c^\sim$ be the canonical Banach lattice isomorphism defined by saying that $(T_0\mu)(u) = \int u\,d\mu$ whenever $u \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$. If we set $T\mu = T_0\mu{\restriction}U$ for $\mu \in M_\tau$, then T is a positive linear operator from M_τ to $U^\sim$, just because U is a Riesz subspace of $\mathcal{L}^\infty$; and since $T\mu \in U_\tau^\sim$ for every $\mu \in M_\tau^+$ (436H), T is an operator from M_τ to $U_\tau^\sim$. Now every $f \in (U_\tau^\sim)^+$ is of the form $T\mu$ for some $\mu \in M_\tau^+$. **P** By 436H, there is a quasi-Radon measure λ such that $\int u\,d\lambda = f(u)$ for every $u \in U$; set $\mu = \lambda{\restriction}\mathcal{B}$. **Q** So T is surjective.

(iii) Let $\mathcal{K}$ be the family of subsets K of X such that $\chi K = \inf A$ in $\mathbb{R}^X$ for some non-empty subset A of U. Then $\mathcal{K}$ is just the family of closed sets for $\mathfrak{T}$. **P** As noted in part (b) of the proof of 436H, every member of $\mathcal{K}$ is closed, and $K \setminus G \in \mathcal{K}$ whenever $K \in \mathcal{K}$ and $G \in \mathfrak{T}$; but as, in the present case, $X \in \mathcal{K}$, every closed set belongs to $\mathcal{K}$. **Q**

(iv) T is injective. **P** If μ_1, $\mu_2 \in M_\tau$ and $T\mu_1 = T\mu_2$, set $\nu_i = \mu_i + \mu_1^- + \mu_2^-$ for each i, so that ν_i is non-negative and $T\nu_1 = T\nu_2$. If $K \in \mathcal{K}$, set $A = \{u : u \in U,\ u \geq \chi K\}$, so that $\chi K = \inf A$ in $\mathbb{R}^X$, and A is downwards-directed. By 414Bb,

$$\nu_1 K = \inf_{u\in A}\int u\,d\nu_1 = \inf_{u\in A}\int u\,d\nu_2 = \nu_2 K.$$

Now $\mathcal{K}$ contains X and is closed under finite intersections and ν_1 and ν_2 agree on $\mathcal{K}$. By the Monotone Class Theorem, ν_1 and ν_2 agree on the σ-algebra generated by $\mathcal{K}$, which is $\mathcal{B}$; so $\nu_1=\nu_2$ and $\mu_1=\mu_2$. **Q**

Thus T is a linear space isomorphism between M_τ and $U_\tau^\sim$.

(v) As noted in (ii), $T[M_\tau^+]=(U_\tau^\sim)^+$; so T is a Riesz space isomorphism.

(vi) Now if $\mu\in M_\tau$,

$$\|T\mu\| = |T\mu|(\chi X) = (T|\mu|)(\chi X) = |\mu|(X) = \|\mu\|.$$

So T is norm-preserving and is an L-space isomorphism, as claimed.

(b)(i) By 356Pb, $(U_\tau^\sim)^* = (U_\tau^\sim)^\sim = (U_\tau^\sim)^\times$ is an M-space.

(ii) Because $T_0: M_\sigma\to(\mathcal{L}^\infty)_c^\sim$ is a Banach lattice isomorphism, and M_τ is a band in M_σ, $W=T_0[M_\tau]$ is a band in $(\mathcal{L}^\infty)_c^\sim$. We therefore have a Riesz homomorphism $S_0:\mathcal{L}^\infty\to W^\times$ defined by writing $(S_0v)(h)=h(v)$ for $v\in\mathcal{L}^\infty$, $h\in W$ (356F). Just as in (b-ii) of the proof of 437C, S_0 is sequentially order-continuous and norm-preserving. (We need to observe that h_x in the second half of the argument there always belongs to W; this is because $h_x=T_0(\delta_x)$, where $\delta_x\in M_\tau$ is defined by setting $\delta_x(E)=\chi E(x)$ for every Borel set E.)

(iii) Now $T_0: M_\sigma\to(\mathcal{L}^\infty)_c^\sim$ and $T: M_\tau\to U_\tau^\sim$ are both norm-preserving Riesz space isomorphisms, so $T_0T^{-1}: U_\tau^\sim\to W$ is another, and its adjoint $S_1: W^*\to(U_\tau^\sim)^*$ must also be a norm-preserving Riesz space isomorphism. So if we set $S=S_1S_0$, S will be a norm-preserving sequentially order-continuous Riesz homomorphism from $\mathcal{L}^\infty$ to $(U_\tau^\sim)^\times=(U_\tau^\sim)^*$.

(iv) If $v\in\mathcal{L}^\infty$ and $\mu\in M_\tau^+$,

$$\begin{aligned}(Sv)(T\mu) &= (S_1S_0v)(T\mu) = (S_0v)(T_0T^{-1}T\mu)\\ &= (S_0v)(T_0\mu) = (T_0\mu)(v) = \int v\,d\mu;\end{aligned}$$

if $u\in U$ and $f\in U_\tau^\sim$, then $(T\mu)(u)=(T_0\mu)(u)$ for every $\mu\in M_\tau$, so

$$(Su)(f) = (S_1S_0u)(f) = (S_0u)(T_0T^{-1}f) = (T_0T^{-1}f)(u) = (TT^{-1}f)(u) = f(u).$$

437I Proposition Let X be a locally compact Hausdorff space, $\mathcal{B}=\mathcal{B}(X)$ its Borel σ-algebra, and $\mathcal{L}^\infty(\mathcal{B})$ the M-space of bounded Borel measurable real-valued functions on X.

(a) Let M_t be the L-space of signed tight Borel measures on X. Then we have a Banach lattice isomorphism $T: M_t\to C_0(X)^*$ defined by saying that $(T\mu)(u)=\int u\,d\mu$ whenever $\mu\in M_t^+$ and $u\in C_0(X)$.

(b) Let Σ_{uRm} be the algebra of universally Radon-measurable subsets of X (definition: 434E), and $\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ the M-space of bounded Σ_{uRm}-measurable real-valued functions on X. Then we have a norm-preserving sequentially order-continuous Riesz homomorphism $S:\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})\to C_0(X)^{**}$ defined by saying that $(Sv)(T\mu)=\int v\,d\mu$ whenever $v\in\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ and $\mu\in M_t^+$; and $(Su)(f)=f(u)$ for every $u\in C_0(X)$, $f\in C_0(X)^*$.

proof (a) The point is just that in this context M_t is equal to M_τ, as defined in 437F-437H (416H), while $C_0(X)^*=C_0(X)_\tau^\sim$ (see part (a) of the proof of 436J), and the topology of X is completely regular, so we just have a special case of 437Ha.

(b)(i) As in 437Hb, we have a sequentially order-continuous Riesz homomorphism $S_0:\mathcal{L}^\infty(\mathcal{B})\to C_0(X)^{**}$ defined by saying that $(S_0v)(T\nu)=\int v\,d\nu$ whenever $v\in\mathcal{L}^\infty(\mathcal{B})$ and $\nu\in M_t^+$.

(ii) If $v\in\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$, then

$$\sup\{S_0w: w\in\mathcal{L}^\infty(\mathcal{B}),\ w\le v\} = \inf\{S_0w: w\in\mathcal{L}^\infty(\mathcal{B}),\ w\ge v\}$$

in $C_0(X)^{**}$. **P** Set

$$A=\{w: w\in\mathcal{L}^\infty(\mathcal{B}),\ w\le v\},\quad B=\{w: w\in\mathcal{L}^\infty(\mathcal{B}),\ w\ge v\}.$$

Because the constant functions belong to $\mathcal{L}^\infty(\mathcal{B})$, A and B are both non-empty; of course $w\le w'$ and $S_0w\le S_0w'$ for every $w\in A$ and $w'\in B$; because $C_0(X)^{**}$ is Dedekind complete, $\phi=\sup S_0[A]$ and $\psi=\inf S_0[B]$ are both defined in $C_0(X)^{**}$, and $\phi \le \psi$. If $f \ge 0$ in $C_0(X)^*$, then there is a $\nu \in M_t^+$ such that $T\nu = f$. Since v is ν-virtually measurable (see 434Ec), there are (bounded) Borel measurable functions w, w' such that $w \le v \le w'$ and $w = w'$ ν-a.e., that is, $w \in A$, $w' \in B$ and

$$(S_0 w)(f) = \int w\,d\nu = \int w' d\nu = (S_0 w')(f).$$

But as

$$(S_0 w)(f) \le \phi(f) \le \psi(f) \le (S_0 w')(f),$$

$\phi(f) = \psi(f)$; as f is arbitrary, $\phi = \psi$. **Q**

(iii) We can therefore define $S : \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}}) \to C_0(X)^{**}$ by setting

$$Sv = \sup\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \le v\} = \inf\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \ge v\}$$

for every $v \in \mathcal{L}^\infty$. The argument in (ii) tells us also that $(Sv)(T\nu) = \int v\,d\nu$ for every $\nu \in M_t^+$; that is, that $(Sv)(T\mu) = \int v\,d\mu$ for every Radon measure μ on X.

(iv) Now S is a norm-preserving sequentially order-continuous Riesz homomorphism. **P** (Compare 355F.) (α) The non-trivial part of this is actually the check that S is additive. But the formula

$$Sv = \sup\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \le v\}$$

ensures that $Sv_1 + Sv_2 \le S(v_1 + v_2)$ for all v_1, $v_2 \in \mathcal{L}^\infty$, while the formula

$$Sv = \inf\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \ge v\}$$

ensures that $Sv_1 + Sv_2 \ge S(v_1 + v_2)$ for all v_1, v_2. (β) It is easy to check that $S(\alpha v) = \alpha Sv$ whenever $v \in \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ and $\alpha > 0$, so that S is linear. (γ) If $v_1 \wedge v_2 = 0$ in $\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$,

$$\begin{aligned} Sv_1 \wedge Sv_2 &= \sup\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \le v_1\} \wedge \sup\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \le v_2\} \\ &= \sup\{S_0 w_1 \wedge S_0 w_2 : w_1,\ w_2 \in \mathcal{L}^\infty(\mathcal{B}),\ w_1 \le v_1,\ w_2 \le v_2\} \end{aligned}$$

(352Ea)

$$= \sup\{S_0(w_1 \wedge w_2) : w_1,\ w_2 \in \mathcal{L}^\infty(\mathcal{B}),\ w_1 \le v_1,\ w_2 \le v_2\}$$

(because S_0 is a Riesz homomorphism)

$$= 0.$$

So S is a Riesz homomorphism (352G(iv)). (δ) Now suppose that $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ with infimum 0 in $\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$. Then $\inf_{n\in\mathbb{N}} v_n(x) = 0$ for every $x \in X$, so $\inf_{n\in\mathbb{N}} \int v_n d\nu = 0$ for every $\nu \in M_t^+$ and $\inf_{n\in\mathbb{N}}(Sv_n)(f) = 0$ for every $f \in (C_0(X)^*)^+$. So $\inf_{n\in\mathbb{N}} Sv_n = 0$; as $\langle v_n\rangle_{n\in\mathbb{N}}$ is arbitrary, S is sequentially order-continuous. (ϵ) If $v \in \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$, then $|v| \le \|v\|_\infty \chi X$, so

$$\|Sv\| \le \|v\|_\infty \|S(\chi X)\| = \|v\|_\infty.$$

On the other hand, for any $x \in X$ we have the Dirac measure δ_x on X concentrated at x, with the matching functional $h_x \in C_0(X)^*$, and

$$\|Sv\| = \||Sv|\| = \|S|v|\| \ge (S|v|)(h_x) = \int |v| d\delta_x = |v(x)|,$$

so $\|Sv\| \ge \|v\|_\infty$; thus S is norm-preserving. **Q**

Remark As in 437D, we can write $(Sv)(T\mu) = \fint v\,d\mu$ whenever $v \in \mathcal{L}^\infty(\mathcal{B})$ and $u \in U$ and $\mu \in M_\tau$ (in 437H) or $v \in \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ and $u \in C_0(X)$ and $\mu \in M_t$ (in 437I).

437J Vague and narrow topologies We are ready for another look at 'vague' topologies on spaces of measures. Let X be a topological space.

(a) Let Σ be an algebra of subsets of X. I will say that Σ **separates zero sets** if whenever F, $F' \subseteq X$ are disjoint zero sets then there is an $E \in \Sigma$ such that $F \subseteq E$ and $E \cap F' = \emptyset$.

(b) If Σ is any algebra of subsets of X, we can identify the Banach algebra and Banach lattice $L^\infty(\Sigma)$, as defined in §363, with the $\|\,\|_\infty$-closed linear subspace of $\ell^\infty(X)$ generated by $\{\chi E : E \in \Sigma\}$ (363C, 363Ha). If we do this, then $C_b(X) \subseteq L^\infty(\Sigma)$ iff Σ separates zero sets. **P** (i) Suppose that $C_b(X) \subseteq L^\infty(\Sigma)$ and that F_1, $F_2 \subseteq X$ are disjoint zero sets. Let u_1, $u_2 : X \to \mathbb{R}$ be continuous functions such that $F_i = u_i^{-1}[\{0\}]$ for both i; then $|u_1(x)| + |u_2(x)| > 0$ for every x; set $v = \frac{|u_1|}{|u_1|+|u_2|}$, so that $v : X \to [0,1]$ is continuous, $v(x) = 0$ for $x \in F_1$ and $v(x) = 1$ for $x \in F_2$. Now $v \in C_b(X) \subseteq L^\infty(\Sigma)$, so there is a $w \in S(\Sigma)$, the linear subspace of $L^\infty(\Sigma)$ generated by $\{\chi E : E \in \Sigma\}$, such that $\|v - w\|_\infty < \frac{1}{2}$ (363C). Set $E = \{x : w(x) \le \frac{1}{2}\}$; then $E \in \Sigma$ and $F_1 \subseteq E \subseteq X \setminus F_2$. As F_1 and F_2 are arbitrary, Σ separates zero sets.

(ii) Now suppose that Σ separates zero sets, that $u : X \to [0,1]$ is continuous, and that $n \ge 1$ is an integer. For $i \le n$, set $F_i = \{x : x \in X,\, u(x) \le \frac{i}{n}\}$, $F_i' = \{x : x \in X,\, u(x) \ge \frac{i+1}{n}\}$. Then F_i and F_i' are disjoint zero sets so there is an $E_i \in \Sigma$ such that $F_i' \subseteq E_i \subseteq X \setminus F_i$. Set $w = \frac{1}{n}\sum_{i=1}^n \chi E_i \in S(\Sigma)$. If $x \in X$, let $j \le n$ be such that $\frac{j}{n} \le u(x) < \frac{j+1}{n}$; then for $i \le n$

$$i < j \Longrightarrow x \in F_i' \Longrightarrow x \in E_i \Longrightarrow x \notin F_i \Longrightarrow i \le j,$$

and $w(x) = \frac{1}{n}\#(\{i : i \le n,\, x \in E_i\})$ is either $\frac{j}{n}$ or $\frac{j+1}{n}$. Thus $|w(x) - u(x)| \le \frac{1}{n}$. As x is arbitrary, $\|u - w\|_\infty \le \frac{1}{n}$; as n is arbitrary, $u \in L^\infty(\Sigma)$. As $L^\infty(\Sigma)$ is a linear subspace of $\ell^\infty(X)$, this is enough to show that $C_b(X) \subseteq L^\infty(\Sigma)$. **Q**

(c) It follows that if Σ is an algebra of subsets of X separating the zero sets, and $\nu : \Sigma \to \mathbb{R}$ is a bounded additive functional, we can speak of $⨍ u\,d\nu$ for any $u \in C_b(X)$; $⨍ d\nu$ is the unique norm-continuous linear functional on $L^\infty(\Sigma)$ such that $⨍ \chi E\,d\nu = \nu E$ for every $E \in \Sigma$ (363L). The map $\nu \mapsto ⨍ d\nu$ is a Banach lattice isomorphism from the L-space $M(\Sigma)$ of bounded additive functionals on Σ to $L^\infty(\Sigma)^* = L^\infty(\Sigma)^\sim$ (363K). We therefore have a positive linear operator $T : M(\Sigma) \to C_b(X)^*$ defined by setting $(T\nu)(u) = ⨍ u\,d\nu$ for every $\nu \in M(\Sigma)$ and $u \in C_b(X)$. Except in the trivial case $X = \emptyset$, $\|T\| = 1$ (if $x \in X$, we have $\delta_x \in M(\Sigma)$ defined by setting $\delta_x(E) = \chi E(x)$ for $E \in \Sigma$, and $\|T(\delta_x)\| = 1$).

The **vague topology** on $M(\Sigma)$ is now the topology generated by the functionals $\nu \mapsto ⨍ u\,d\nu$ as u runs over $C_b(X)$; that is, the coarsest topology on $M(\Sigma)$ such that the canonical map $T : M(\Sigma) \to C_b(X)^*$ is continuous for the weak* topology of $C_b(X)^*$. Because the functionals $\nu \mapsto |⨍ u\,d\nu|$ are seminorms on $M(\Sigma)$, the vague topology is a locally convex linear space topology.

(d) There is a variant of the vague topology which can be applied directly to spaces of (non-negative) totally finite measures. Let $\tilde{M}^+$ be the set of all non-negative real-valued additive functionals defined on algebras of subsets of X which contain every open set. The **narrow topology** on $\tilde{M}^+$ is that generated by sets of the form

$$\{\nu : \nu \in \tilde{M}^+,\, \nu G > \alpha\}, \quad \{\nu : \nu \in \tilde{M}^+,\, \nu X < \alpha\}$$

for open sets $G \subseteq X$ and real numbers α. (See Topsøe 70b, 8.1.)

Observe that $\nu \mapsto \nu X : \tilde{M}^+ \to [0,\infty[$ is continuous for the narrow topology, and if $G \subseteq X$ is open then $\nu \mapsto \nu G$ is lower semi-continuous for the narrow topology. Writing P_{top} for the set of topological probability measures on X, then the narrow topology on P_{top} is generated by sets of the form $\{\mu : \mu \in P_{\mathrm{top}},\, \mu G > \alpha\}$ for open sets $G \subseteq X$.

Writing $\tilde{M}^+_\sigma$ for the set of totally finite topological measures on X, then $\nu \mapsto \nu E : \tilde{M}^+_\sigma \to [0,\infty[$ is Borel measurable, for the narrow topology on $\tilde{M}^+_\sigma$, for every Borel set $E \subseteq X$ (because the family of sets E for which $\nu \mapsto \nu E$ is Borel measurable is a Dynkin class containing the open sets). Similarly, $\nu \mapsto \int u\,d\nu : \tilde{M}^+_\sigma \to \mathbb{R}$ is Borel measurable for every bounded measurable function $u : X \to \mathbb{R}$, being the limit of a sequence of linear combinations of Borel measurable functions.

(e) Vague topologies, being linear space topologies, are necessarily associated with uniformities (3A4Ad), therefore completely regular (4A2Ja). In the very general context of (c) here, in which we have a space $M(\Sigma)$ of all finitely additive functionals on an algebra Σ, we do not expect the vague topology to be Hausdorff. But if we look at particular subspaces, such as the space $M_\sigma(\mathcal{B}\mathfrak{a}(X))$ of signed Baire measures, or the space M_τ of signed τ-additive Borel measures on a completely regular space X, we may well have a Hausdorff vague topology (437Xg).

Similarly, the narrow topology on $\tilde{M}^+$ is rarely Hausdorff. But on important subspaces we can get Hausdorff topologies. In particular, if X is Hausdorff, then the narrow topology on the space M^+_{R} of totally finite Radon measures on X is Hausdorff (437R(a-ii)).

(f) It will be useful to know that if $u : X \to \mathbb{R}$ is bounded and lower semi-continuous, then $\nu \mapsto ⨍ u\,d\nu : \tilde{M}^+ \to \mathbb{R}$ is lower semi-continuous for the narrow topology. **P** (i) Perhaps I should start by explaining why $⨍ u\,d\nu$ is always defined; this is because the algebra T generated by the open sets is always a subalgebra of $\operatorname{dom}\nu$, and $\{x : u(x) > \alpha\} \in \mathrm{T}$ for every α, so $u \in L^\infty(\mathrm{T})$ (363Ha). (ii) Now suppose for a moment that $u \ge 0$. If $\nu_0 \in \tilde{M}^+$ and $\gamma < \int u\,d\nu_0$, let $\epsilon > 0$ be such that $\gamma + \epsilon(1 + \nu_0 X) < \int u\,d\nu_0$, let $n \ge 1$ be such that $\|u\|_\infty \le n\epsilon$, and for $i \le n$ set $G_i = \{x : u(x) > i\epsilon\}$. Then

$$\epsilon\sum_{i=1}^n \chi G_i \le u \le \epsilon(\chi X + \sum_{i=1}^n \chi G_i),$$

$$\int u\,d\nu_0 \le \epsilon(\nu_0 X + \sum_{i=1}^n \nu_0 G_i),$$

$$V = \{\nu : \nu \in \tilde{M}^+, \sum_{i=0}^n \nu G_i > \sum_{i=0}^n \nu_0 G_i - 1\}$$

is a neighbourhood of ν_0 in $\tilde{M}^+$, and

$$\int u\,d\nu \ge \epsilon\sum_{i=1}^n \nu G_i > \gamma$$

for every $\nu \in V$. As ν_0 and γ are arbitrary, $\nu \mapsto \int u\,d\nu$ is lower semi-continuous. (iii) In general, u is expressible as the sum of a constant function and a non-negative lower semi-continuous function; as $\nu \mapsto \nu X$ is continuous, $\nu \mapsto \int u\,d\nu$ is the sum of two lower semi-continuous functions and is lower semi-continuous. **Q**

Of course it follows at once that if $u : X \to \mathbb{R}$ is bounded and continuous, then $\nu \mapsto \int u\,d\nu$ is continuous for the narrow topology; that is, the vague topology is coarser than the narrow topology in contexts in which both make sense.

(g) With the more liberal definitions I use when considering integrals with respect to σ-additive measures, we have another version of the same idea. If $u : X \to [0, \infty]$ is a lower semi-continuous function, then $\nu \mapsto \int u\,d\nu : \tilde{M}^+_\sigma \to [0, \infty]$ is lower semi-continuous for the narrow topology. **P** It is the supremum of the lower semi-continuous functions $\nu \mapsto \int (u \wedge n\chi X)d\nu$. **Q**

(h) Let X and Y be topological spaces, $\phi : X \to Y$ a continuous function, and $\tilde{M}^+(X)$, $\tilde{M}^+(Y)$ the spaces of functionals described in (d). For a functional ν defined on a subset of $\mathcal{P}X$, define $\nu\phi^{-1}$ by saying that $(\nu\phi^{-1})(F) = \nu(\phi^{-1}[F])$ whenever $F \subseteq Y$ and $\phi^{-1}[F] \in \operatorname{dom}\nu$. Then $\nu\phi^{-1} \in \tilde{M}^+(Y)$ whenever $\nu \in \tilde{M}^+(X)$, and the map $\nu \mapsto \nu\phi^{-1} : \tilde{M}^+(X) \to \tilde{M}^+(Y)$ is continuous for the narrow topologies (use 4A2B(a-ii)).

(i) I am trying to maintain the formal distinctions between 'quasi-Radon measure' and 'τ-additive effectively locally finite Borel measure inner regular with respect to the closed sets', and between 'Radon measure' and 'tight locally finite Borel measure'. There are obvious problems in interpreting the sum and difference of measures with different domains, which are readily soluble (see 234G and 416De) but in the context of this section are unilluminating. If, however, we take M^+_{qR} to be the set of totally finite quasi-Radon measures on X, and X is completely regular, we have a canonical embedding of M^+_{qR} into a cone in the L-space $C_b(X)^*$; more generally, even if our space X is not completely regular, the map $\mu \mapsto \mu\restriction\mathcal{B}(X) : M^+_{\mathrm{qR}} \to M_\sigma(\mathcal{B}(X))$ is still injective, and we can identify M^+_{qR} with a cone in the L-space M_τ of signed τ-additive Borel measures (often the whole positive cone of M_τ, as in 415M). Similarly, when X is Hausdorff, we can identify totally finite Radon measures with tight totally finite Borel measures (416F). The definition in 437Jd makes it plain that these identifications are homeomorphisms for the narrow topology,

It is even possible to extend these ideas to measures which are not totally finite (437Yi), though there may be new difficulties (415Ya).

(j) For a different kind of narrow topology, adapted to the space of all Radon measures on a Hausdorff space, see 495O below.

437K Proposition Let X be a topological space, and $\tilde{M}^+$ the set of all non-negative real-valued additive functionals defined on algebras of subsets of X containing every open set.

(a) We have a function $T : \tilde{M}^+ \to C_b(X)^*$ defined by the formula $(T\nu)(u) = \int u\,d\nu$ whenever $\nu \in \tilde{M}^+$ and $u \in C_b(X)$.

(b) T is continuous for the narrow topology $\mathfrak{S}$ on $\tilde{M}^+$ and the weak* topology on $C_b(X)^*$.

(c) Suppose now that X is completely regular, and that $W \subseteq \tilde{M}^+$ is a family of τ-additive totally finite topological measures such that two members of W which agree on the Borel σ-algebra are equal. Then $T\restriction W$ is a homeomorphism between W, with the narrow topology, and $T[W]$, with the weak* topology.

proof (a) We have only to assemble the operators of 437Jc, noting that if an algebra of subsets of X contains every open set then it certainly separates the zero sets (indeed, it actually contains every zero set).

(b) As already noted in 437Jf, $\nu \mapsto (T\nu)(u) = \int u\,d\nu$ is $\mathfrak{S}$-continuous for every $u \in C_b(X)$. Since the weak* topology on $C_b(X)^*$ is the coarsest topology on $C_b(X)^*$ for which all the functionals $f \mapsto f(u)$ are continuous, T is continuous.

(c)(i) Write $\mathfrak{T}$ for the topology on W induced by T, that is, the family of sets of the form $W \cap T^{-1}[V]$ where $V \subseteq C_b(X)^*$ is weak*-open. If $G \subseteq X$ is open, then $A = \{u : u \in C_b(X),\ 0 \le u \le \chi G\}$ is upwards-directed and has supremum χG, so $\mu G = \sup_{u\in A} \int u\,d\mu$ for every $\mu \in W$ (414Ba). Accordingly $\{\mu : \mu \in W,\ \mu G > \alpha\} = \bigcup_{u\in A}\{\mu : (T\mu)(u) > \alpha\}$ belongs to $\mathfrak{T}$ for every $\alpha \in \mathbb{R}$. Also, of course, $\{\mu : \mu X < \alpha\} = \{\mu : (T\mu)(\chi X) < \alpha\} \in \mathfrak{T}$ for every α. So if $\mathfrak{S}'$ is the narrow topology on W, $\mathfrak{S}' \subseteq \mathfrak{T}$. Putting this together with (b), we see that $\mathfrak{S}' = \mathfrak{T}$.

(ii) Now the same formulae show that $T\restriction W$ is injective. **P** Suppose that μ_1, $\mu_2 \in W$ and that $T\mu_1 = T\mu_2$. Then $\mu_1 G = \mu_2 G$ for every open set $G \subseteq X$. By the Monotone Class Theorem, μ_1 and μ_2 agree on all Borel sets; but our hypothesis is that this is enough to ensure that $\mu_1 = \mu_2$. **Q**

Since $T : W \to T[W]$ is continuous and open, it is a homeomorphism.

437L Corollary Let X be a completely regular topological space, and M_τ the space of signed τ-additive Borel measures on X. Then the narrow and vague topologies on M_τ^+ coincide.

437M Theorem (Ressel 77) For a topological space X, write $M_{\text{qR}}^+(X)$ for the space of totally finite quasi-Radon measures on X, $P_{\text{qR}}(X)$ for the space of quasi-Radon probability measures on X, and $M_\tau(X)$ for the L-space of signed τ-additive Borel measures on X.

(a) Let X and Y be topological spaces. If $\mu \in M_{\text{qR}}^+(X)$ and $\nu \in M_{\text{qR}}^+(Y)$, write $\mu \times \nu$ for their τ-additive product measure on $X \times Y$ (417G). Then $(\mu, \nu) \mapsto \mu \times \nu$ is continuous for the narrow topologies on $M_{\text{qR}}^+(X)$, $M_{\text{qR}}^+(Y)$ and $M_{\text{qR}}^+(X \times Y)$.

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, with product X. If $\langle \mu_i\rangle_{i\in I}$ is a family of probability measures such that $\mu_i \in P_{\text{qR}}(X_i)$ for each i, write $\prod_{i\in I}\mu_i$ for its τ-additive product on X. Then $\langle \mu_i\rangle_{i\in I} \mapsto \prod_{i\in I}\mu_i$ is continuous for the narrow topology on $P_{\text{qR}}(X)$ and the product of the narrow topologies on $\prod_{i\in I} P_{\text{qR}}(X_i)$.

(c) Let X and Y be topological spaces.

(i) We have a unique bilinear operator $\psi : M_\tau(X) \times M_\tau(Y) \to M_\tau(X \times Y)$ such that $\psi(\mu, \nu)$ is the restriction of the τ-additive product of μ and ν to the Borel σ-algebra of $X \times Y$ whenever μ, ν are totally finite Borel measures on X, Y respectively.

(ii) $\|\psi\| \le 1$ (definition: 253Ab).

(iii) ψ is separately continuous for the vague topologies on $M_\tau(X)$, $M_\tau(Y)$ and $M_\tau(X \times Y)$.

(d) In (c), suppose that X and Y are compact and Hausdorff. If $B \subseteq M_\tau(X)$ and $B' \subseteq M_\tau(Y)$ are norm-bounded, then $\psi\restriction B \times B'$ is continuous for the vague topologies.

proof (a)(i) I ought to note that we need 417N to assure us that $\mu \times \nu \in M_{\text{qR}}^+(X \times Y)$ whenever $\mu \in M_{\text{qR}}^+(X)$ and $\nu \in M_{\text{qR}}^+(Y)$.

(ii) If $W \subseteq X \times Y$ is open and $\alpha \in \mathbb{R}$, then $Q = \{(\mu, \nu) : (\mu \times \nu)(W) > \alpha\}$ is open in $M_{\text{qR}}^+(X) \times M_{\text{qR}}^+(Y)$. **P** Suppose that $(\mu_0, \nu_0) \in Q$. Because $\mu_0 \times \nu_0$ is τ-additive, there is a subset $W' \subseteq W$, expressible in the form $\bigcup_{i\le n} G_i \times H_i$ where $G_i \subseteq X$ and $H_i \subseteq Y$ are open for every i, such that

$$\alpha < (\mu_0 \times \nu_0)(W') = \int \nu_0 W'[\{x\}]\mu_0(dx)$$

(417C(iv)). Set $u(x) = \nu_0 W'[\{x\}]$ for $x \in X$, so that u is lower semi-continuous (417Ba). Let $\eta > 0$ be such that $\int u\,d\mu_0 > \alpha + (1 + 2\mu_0 X)\eta$, and set $E_i = \{x : u(x) > \eta i\}$ for $i \in \mathbb{N}$, so that $\eta\sum_{i=1}^\infty \mu_0 E_i > \int u\,d\mu_0 - \eta\mu_0 X$. Because every E_i is open, there is a neighbourhood U of μ_0 in $M_{\text{qR}}^+(X)$ such that

$$\int u\,d\mu_0 - \eta\mu_0 X \le \eta\textstyle\sum_{i=1}^\infty \mu E_i \le \displaystyle\int u\,d\mu = (\mu \times \nu_0)(W')$$

for every $\mu \in U$; shrinking U if necessary, we can arrange at the same time that $\mu X < \mu_0 X + 1$ for every $\mu \in U$. Next, observe that $\mathcal{H} = \{W'[\{x\}] : x \in X\} \subseteq \{\bigcup_{i\in I} H_i : I \subseteq \{0, \dots, n\}\}$ is finite, so there is a neighbourhood V of ν_0 in $M_{\text{qR}}^+(Y)$ such that $\nu H \ge \nu_0 H - \eta$ for every $H \in \mathcal{H}$ and $\nu \in V$. If $\mu \in U$ and $\nu \in V$, we have

$$\begin{aligned}(\mu \times \nu)(W) \ge (\mu \times \nu)(W') &= \int \nu W'[\{x\}]\mu(dx) \ge \int u(x) - \eta\,\mu(dx)\\ &= \int u\,d\mu - \eta\mu X \ge \int u\,d\mu_0 - \eta\mu_0 X - \eta(1 + \mu_0 X) > \alpha.\end{aligned}$$

As μ_0 and ν_0 are arbitrary, Q is open. **Q**

(iii) Since $(\mu\times\nu)(X\times Y)=\mu X\cdot\nu Y$, the sets $\{(\mu,\nu):(\mu\times\nu)(X\times Y)<\alpha\}$ are also open for every $\alpha\in\mathbb{R}$. So $(\mu,\nu)\mapsto\mu\times\nu$ is continuous (4A2B(a-ii) again).

(b) Similarly, we can refer to 417O to check that $\prod_{i\in I}\mu_i\in P_{\mathrm{qR}}(X)$ whenever $\mu_i\in P_{\mathrm{qR}}(X_i)$ for each i. For finite sets I, the result is a simple induction on $\#(I)$, using 417Db and part (a) just above. For infinite I, let $W\subseteq X$ be an open set and $\alpha\in\mathbb{R}$, and consider

$$Q=\{\langle\mu_i\rangle_{i\in I}:\mu_i\in P_{\mathrm{qR}}(X_i)\text{ for each }i,\ (\textstyle\prod_{i\in I}\mu_i)(W)>\alpha\}.$$

If $\langle\mu_i\rangle_{i\in I}\in Q$, then there is an open set $W'\subseteq W$, determined by coordinates in a finite set $J\subseteq I$, such that $(\prod_{i\in I}\mu_i)(W')>\alpha$. Setting $V=\{x\restriction J:x\in W'\}$, we have $(\prod_{i\in J}\mu_i)(V)>\alpha$. Now we can find open sets U_i in $P_{\mathrm{qR}}(X_i)$, for $i\in J$, such that $(\prod_{i\in J}\nu_i)(V)>\alpha$ whenever $\nu_i\in U_i$ for $i\in J$. If now $\langle\nu_i\rangle_{i\in I}\in\prod_{i\in I}P_{\mathrm{qR}}(X_i)$ is such that $\nu_i\in U_i$ for every $i\in J$,

$$(\textstyle\prod_{i\in I}\nu_i)(W)\ge(\prod_{i\in I}\nu_i)(W')=(\prod_{i\in J}\nu_i)(V)>\alpha,$$

so $\prod_{i\in I}\nu_i\in Q$. As $\langle\mu_i\rangle_{i\in I}$ is arbitrary, Q is open.

As W and α are arbitrary, $\langle\mu_i\rangle_{i\in I}\mapsto\prod_{i\in I}\mu_i$ is continuous.

(c)(i) Start by writing $\psi(\mu,\nu)=(\mu\times\nu)\restriction\mathcal{B}(X\times Y)$ for $\mu\in M_\tau^+(X)$ and $\nu\in M_\tau^+(Y)$, where $\mathcal{B}(X\times Y)$ is the Borel σ-algebra of $X\times Y$. If μ, μ_1, $\mu_2\in M_\tau^+(X)$ and ν, ν_1, $\nu_2\in M_\tau^+(Y)$ and $\alpha\ge0$, then

$$\psi(\mu_1+\mu_2,\nu)=\psi(\mu_1,\nu)+\psi(\mu_2,\nu).$$

P On each side of the equation we have a τ-additive Borel measure, and the two measures agree on the standard base $\mathcal{W}$ for the topology of $X\times Y$ consisting of products of open sets; since $\mathcal{W}$ is closed under finite intersections, they agree on the algebra generated by $\mathcal{W}$ and therefore on all open sets and therefore (using the Monotone Class Theorem yet again) on all Borel sets. **Q** Similarly,

$$\psi(\mu,\nu_1+\nu_2)=\psi(\mu,\nu_1)+\psi(\mu,\nu_2),\quad \psi(\alpha\mu,\nu)=\psi(\mu,\alpha\nu)=\alpha\psi(\mu,\nu)$$

whenever $\mu\in M_\tau^+(X)$, ν, ν_1, $\nu_2\in M_\tau^+(Y)$ and $\alpha\in\mathbb{R}$. Now if μ_1', $\mu_2'\in M_\tau^+(X)$ and ν_1', $\nu_2'\in M_\tau^+(Y)$ are such that $\mu_1-\mu_2=\mu_1'-\mu_2'$ and $\nu_1-\nu_2=\nu_1'-\nu_2'$, we shall have

$$\begin{aligned}\psi(\mu_1,\nu_1)&-\psi(\mu_1,\nu_2)-\psi(\mu_2,\nu_1)+\psi(\mu_2,\nu_2)\\&=\psi(\mu_1,\nu_1+\nu_2')-\psi(\mu_1+\mu_2',\nu_2')+\psi(\mu_2',\nu_2')\\&\qquad-\psi(\mu_1,\nu_1'+\nu_2)+\psi(\mu_1+\mu_2',\nu_1')-\psi(\mu_2',\nu_1')\\&\qquad-\psi(\mu_2,\nu_1+\nu_2')+\psi(\mu_2+\mu_1',\nu_2')-\psi(\mu_1',\nu_2')\\&\qquad+\psi(\mu_2,\nu_1'+\nu_2)-\psi(\mu_2+\mu_1',\nu_1')+\psi(\mu_1',\nu_1')\\&=\psi(\mu_2',\nu_2')-\psi(\mu_2',\nu_1')-\psi(\mu_1',\nu_2')+\psi(\mu_1',\nu_1').\end{aligned}$$

We can therefore extend ψ to an operator on $M_\tau(X)\times M_\tau(Y)$ by setting

$$\psi(\mu_1-\mu_2,\nu_1-\nu_2)=\psi(\mu_1,\nu_1)-\psi(\mu_1,\nu_2)-\psi(\mu_2,\nu_1)+\psi(\mu_2,\nu_2)$$

whenever μ_1, $\mu_2\in M_\tau^+(X)$ and ν_1, $\nu_2\in M_\tau^+(Y)$, and it is straightforward to check that ψ is bilinear.

(ii) If $\mu\in M_\tau(X)$, then $\|\mu\|=\mu^+(X)+\mu^-(X)$, where μ^+ and μ^- are evaluated in the Riesz space $M_\tau(X)$. Now if $\nu\in M_\tau(Y)$,

$$\begin{aligned}|\psi(\mu,\nu)|&=|\psi(\mu^+,\nu^+)-\psi(\mu^+,\nu^-)-\psi(\mu^-,\nu^+)+\psi(\mu^-,\nu^-)|\\&\le\psi(\mu^+,\nu^+)+\psi(\mu^+,\nu^-)+\psi(\mu^-,\nu^+)+\psi(\mu^-,\nu^-),\end{aligned}$$

so

$$\begin{aligned}\|\psi(\mu,\nu)\|&=|\psi(\mu,\nu)|(X\times Y)\\&\le\psi(\mu^+,\nu^+)(X\times Y)+\psi(\mu^+,\nu^-)(X\times Y)\\&\qquad+\psi(\mu^-,\nu^+)(X\times Y)+\psi(\mu^-,\nu^-)(X\times Y)\\&=\mu^+(X)\cdot\nu^+(Y)+\mu^+(X)\cdot\nu^-(Y)+\mu^-(X)\cdot\nu^+(Y)+\mu^-(X)\cdot\nu^-(Y)\\&=\|\mu\|\|\nu\|.\end{aligned}$$

As μ and ν are arbitrary, $\|\psi\| \le 1$.

(iii) Fix $\nu \in M_\tau^+(Y)$ and $w \in C_b(X \times Y)^+$, and consider the map $\mu \mapsto \int w\,d\psi(\mu,\nu) : M_\tau(X) \to \mathbb{R}$. Note first that if $\mu \in M_\tau^+(X)$,

$$\int w\,d\psi(\mu,\nu) = \int w\,d(\mu \times \nu) = \iint w(x,y)\nu(dy)\mu(dx) = \iint w(x,y)\nu(dy)\mu(dx)$$

(417H). Since both sides of this equation are linear in μ, we have

$$\int w\,d\psi(\mu,\nu) = \iint w(x,y)\nu(dy)\mu(dx)$$

for every $\mu \in M_\tau(X)$. Now $x \mapsto \int w(x,y)\nu(dy)$ is continuous. **P** By 417Bc, it is lower semi-continuous; but if $\alpha \ge \|w\|_\infty$ and $w' = \alpha\chi(X \times Y) - w$, then $x \mapsto \int w'(x,y)\nu(dy)$ is lower semi-continuous, so

$$x \mapsto \alpha\nu Y - \int w'(x,y)\nu(dy) = \int w(x,y)\nu(dy)$$

is also upper semi-continuous, therefore continuous. **Q** It follows at once that $\mu \mapsto \iint w(x,y)\nu(dy)\mu(dx)$ is continuous for the vague topology on $M_\tau(X)$. The argument has supposed that w and ν are positive; but taking positive and negative parts as usual, we see that $\mu \mapsto \int w\,d\psi(\mu,\nu)$ is vaguely continuous for every $w \in C_b(X \times Y)$ and $\nu \in M_\tau(Y)$. As w is arbitrary, $\mu \mapsto \psi(\mu,\nu)$ is vaguely continuous, for every ν. Similarly, $\nu \mapsto \psi(\mu,\nu)$ is vaguely continuous for every μ, and ψ is separately continuous.

(d) Now suppose that X and Y are compact. Let W be the linear subspace of $C(X \times Y)$ generated by $\{u \otimes v : u \in C(X),\, v \in C(Y)\}$, writing $(u \otimes v)(x,y) = u(x)v(y)$ as in 253B. Then W is a subalgebra of $C(X \times Y)$ separating the points of $X \times Y$ and containing the constant functions, so is $\| \|_\infty$-dense in $C(X \times Y)$ (281E). Now

$$(\mu,\nu) \mapsto \int u \otimes v\,d\psi(\mu,\nu) = \int u\,d\mu \cdot \int v\,d\nu$$

is continuous whenever $u \in C(X)$ and $v \in C(Y)$, so

$$(\mu,\nu) \mapsto \int w\,d\psi(\mu,\nu)$$

is continuous whenever $w \in W$.

Next suppose that $B \subseteq M_\tau(X)$ and $B' \subseteq M_\tau(Y)$ are bounded. Let $\gamma \ge 0$ be such that $\|\mu\| \le \gamma$ for every $\mu \in B$ and $\|\nu\| \le \gamma$ for every $\nu \in B$. If $w \in C(X \times Y)$ and $\epsilon > 0$, there is a $w' \in W$ such that $\|w - w'\|_\infty \le \epsilon$. In this case

$$\begin{aligned} |\int w\,d\psi(\mu,\nu) - \int w'\,d\psi(\mu,\nu)| &\le \|w - w'\|_\infty \|\psi(\mu,\nu)\| \\ &\le \epsilon\|\mu\|\|\nu\| \le \gamma^2\epsilon \end{aligned}$$

whenever $\mu \in B$ and $\nu \in B'$. As ϵ is arbitrary, the function $(\mu,\nu) \mapsto \int w\,d\psi(\mu,\nu)$ is uniformly approximated on $B \times B'$ by vaguely continuous functions, and is therefore itself vaguely continuous on $B \times B'$.

437N One of the standard constructions of Radon measures is as image measures. It leads naturally to maps between spaces of Radon measures, and of course we wish to know whether they are continuous.

Proposition (a) Let X and Y be Hausdorff spaces, and $\phi : X \to Y$ a continuous function. Let $M_{\mathrm{R}}^+(X)$, $M_{\mathrm{R}}^+(Y)$ be the spaces of totally finite Radon measures on X and Y respectively. Write $\tilde\phi(\mu)$ for the image measure $\mu\phi^{-1}$ for $\mu \in M_{\mathrm{R}}^+(X)$.

(i) $\tilde\phi : M_{\mathrm{R}}^+(X) \to M_{\mathrm{R}}^+(Y)$ is continuous for the narrow topologies on $M_{\mathrm{R}}^+(X)$ and $M_{\mathrm{R}}^+(Y)$.

(ii) $\tilde\phi(\mu + \nu) = \tilde\phi(\mu) + \tilde\phi(\nu)$ and $\tilde\phi(\alpha\mu) = \alpha\tilde\phi(\mu)$ for all μ, $\nu \in M_{\mathrm{R}}^+(X)$ and $\alpha \ge 0$.

(b) If Y is a Hausdorff space, X a subset of Y, and $\phi : X \to Y$ the identity map, then $\tilde\phi$ is a homeomorphism between $M_{\mathrm{R}}^+(X)$ and $\{\nu : \nu \in M_{\mathrm{R}}^+(Y),\, \nu(Y \setminus X) = 0\}$.

proof (a)(i) All we have to do is to recall from 418I that $\mu\phi^{-1} \in M_{\mathrm{R}}^+(Y)$ for every $\mu \in M_{\mathrm{R}}^+(X)$, and observe that

$$\{\mu : (\mu\phi^{-1})(H) > \alpha\} = \{\mu : \mu\phi^{-1}[H] > \alpha\}, \quad \{\mu : (\mu\phi^{-1})(Y) < \alpha\} = \{\mu : \mu X < \alpha\}$$

are narrowly open in $M_{\mathrm{R}}^+(X)$ for every open set $H \subseteq Y$ and $\alpha \in \mathbb{R}$.

(ii) As usual, since all the measures here are Radon measures, it is enough to check that $\tilde\phi(\mu + \nu)(E) = \tilde\phi(\mu)(E) + \tilde\phi(\nu)(E)$ and $\tilde\phi(\alpha\mu)(E) = \alpha\tilde\phi(\mu)(E)$ for every Borel set $E \subseteq X$, and this is easy.

(b) First note that if $\mu \in M_{\mathrm{R}}^+(X)$, then certainly $\tilde\phi(\mu)(Y \setminus X) = 0$; while if $\nu \in M_{\mathrm{R}}^+(Y)$ and $\nu(Y \setminus X) = 0$, then $\mu = \nu{\restriction}\mathcal{P}X$ is a Radon measure on X (416Rb) and $\nu = \tilde\phi(\mu)$. Thus $\tilde\phi$ is a continuous bijection from $M_{\mathrm{R}}^+(X)$ to

$\{\nu : \nu \in M_{\mathrm{R}}^+(Y),\, \nu(Y \setminus X) = 0\}$. Now if $G \subseteq X$ is relatively open and $\alpha \in \mathbb{R}$, there is an open set $H \subseteq Y$ such that $G = H \cap X$, so that

$$\{\mu : \mu \in M_{\mathrm{R}}^+(X),\, \mu G > \alpha\} = \{\mu : \tilde{\phi}(\mu)(H) > \alpha\}$$

is the inverse image of a narrowly open set in $M_{\mathrm{R}}^+(Y)$; and of course

$$\{\mu : \mu \in M_{\mathrm{R}}^+(X),\, \mu X < \alpha\} = \{\mu : \tilde{\phi}(\mu)(Y) < \alpha\}$$

is also the inverse image of an open set. So $\tilde{\phi}$ is a homeomorphism between $M_{\mathrm{R}}^+(X)$ and $\{\nu : \nu \in M_{\mathrm{R}}^+(Y),\, \nu(Y \setminus X) = 0\}$.

437O Uniform tightness Let X be a topological space. If ν is a bounded additive functional on an algebra of subsets of X, I say that it is **tight** if

$$\nu E \in \overline{\{\nu K : K \subseteq E,\, K \in \operatorname{dom}\nu,\, K \text{ is closed and compact}\}}$$

for every $E \in \operatorname{dom}\nu$, and that a set A of tight functionals is **uniformly tight** if every member of A is tight and for every $\epsilon > 0$ there is a closed compact set $K \subseteq X$ such that νK is defined and $|\nu E| \le \epsilon$ whenever $\nu \in A$ and $E \in \operatorname{dom}\nu$ is disjoint from K.

437P Proposition Let X be a topological space.

(a) Let M_{qR}^+ be the set of totally finite quasi-Radon measures on X. Suppose that $A \subseteq M_{\mathrm{qR}}^+$ is uniformly totally finite (that is, $\{\mu X : \mu \in A\}$ has a finite upper bound) and for every $\epsilon > 0$ there is a closed compact $K \subseteq X$ such that $\mu(X \setminus K) \le \epsilon$ for every $\mu \in A$. Then A is relatively compact in M_{qR}^+ for the narrow topology.

(b) Suppose now that X is Hausdorff, and that M_{R}^+ is the set of Radon measures on X. If $A \subseteq M_{\mathrm{R}}^+$ is uniformly totally finite and uniformly tight, then it is relatively compact in M_{R}^+ for the narrow topology.

proof (a)(i) I show first that the closure $\overline{A}$ of A in M_{qR}^+ has the same two properties. **P** Because $\mu \mapsto \mu X$ is continuous for the narrow topology, $\{\mu X : \mu \in \overline{A}\} \subseteq \overline{\{\mu X : \mu \in A\}}$ is bounded. If $\epsilon > 0$, there is a closed compact set $K \subseteq X$ such that $\mu(X \setminus K) \le \epsilon$ for every $\mu \in A$. In this case $\{\mu : \mu \in M_{\mathrm{qR}}^+,\, \mu(X \setminus K) \le \epsilon\} = M_{\mathrm{qR}}^+ \setminus \{\mu : \mu(X \setminus K) > \epsilon\}$ is closed in M_{qR}^+, so includes $\overline{A}$. As ϵ is arbitrary, we have the result. **Q**

(ii) Now let $\mathcal{F}$ be an ultrafilter on M_{qR}^+ containing $\overline{A}$.

(α) For Borel sets $E \subseteq X$, set $\theta E = \lim_{\nu \to \mathcal{F}} \nu E$; this is defined in $\mathbb{R}$ because $\sup_{\nu \in \overline{A}} \nu X$ is finite. θ is a non-negative additive functional on the Borel σ-algebra of X. The family $\mathcal{K}$ of closed compact subsets of X is a compact class closed under finite unions and countable intersections, so 413S tells us that there is a complete measure μ on X such that $\mu X \le \theta X$, $\mathcal{K} \subseteq \operatorname{dom}\mu$, $\mu K \ge \theta K$ for every $K \in \mathcal{K}$, and μ is inner regular with respect to $\mathcal{K}$.

If $F \subseteq X$ is closed, then $F \cap K \in \mathcal{K}$ for every $K \in \mathcal{K}$, so μ measures F (412Ja). Thus μ is a topological measure. Because μ is tight, it is τ-additive (411E). So μ is a complete totally finite τ-additive measure which is inner regular with respect to the closed sets, and is a quasi-Radon measure.

(β) Given $\epsilon > 0$, there is a $K \in \mathcal{K}$ such that $\nu(X \setminus K) \le \epsilon$ for every $\nu \in \overline{A}$, so $\theta(X \setminus K) \le \epsilon$ and

$$\theta X - \epsilon \le \theta K \le \mu K \le \mu X \le \theta X.$$

As ϵ is arbitrary, $\mu X = \theta X = \lim_{\nu \to \mathcal{F}} \nu X$.

(γ) If $G \subseteq X$ is open and $\epsilon > 0$, there is a $K \in \mathcal{K}$ such that $\nu(X \setminus K) \le \epsilon$ for every $\nu \in \overline{A}$, so $\theta(X \setminus K) \le \epsilon$ and

$$\mu G = \mu X - \mu(X \setminus G) \le \theta X - \mu(K \setminus G) \le \theta X - \theta(K \setminus G)$$

(because $K \setminus G \in \mathcal{K}$)

$$\le \theta X - \theta(X \setminus G) + \theta(X \setminus K) \le \theta G + \epsilon.$$

As ϵ is arbitrary,

$$\mu G \le \theta G = \lim_{\nu \to \mathcal{F}} \nu G.$$

(δ) Putting (β) and (γ) together, we see that $\mathcal{F} \to \mu$ for the narrow topology on M^+_{qR}; it follows that $\mu \in \overline{A}$. Thus every ultrafilter on M^+_{qR} containing $\overline{A}$ has a limit in $\overline{A}$, and $\overline{A}$ is compact. Accordingly A is relatively compact in M^+_{qR}, as claimed.

(b) We know from the proof of (a) that the closure $\overline{A}$ of A in M^+_{qR} is compact, so it will be enough to show that $\overline{A} \subseteq M^+_{\mathrm{R}}$. If $\mu \in \overline{A}$ and $E \in \operatorname{dom}\mu$, then for every $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu(X \setminus K) \le \epsilon$, by (a-i) above; also there is a closed set $F \subseteq E$ such that $\mu(E \setminus F) \le \epsilon$. But now $F \cap K$ is compact and $\mu(E \setminus (F \cap K)) \le 2\epsilon$. As E and ϵ are arbitrary, μ is inner regular with respect to the compact sets, so is a Radon measure.

437Q Two metrics So far, as elsewhere in this volume, I have been writing about topologies with as few restrictions on their nature as possible. Of course the repeated invocation of L-spaces in the first part of the section indicates that there are norms and their associated metrics about, and when the underlying set X is metrizable we rather hope that the constructions of 437J will lead to metrizable topologies on the spaces of measures considered there. I offer two definitions which seem to give us interesting paths to explore.

(a)(i) If X is a set and μ, ν are bounded additive functionals defined on algebras of subsets of X, then $\mu - \nu : \operatorname{dom}\mu \cap \operatorname{dom}\nu \to \mathbb{R}$ is bounded and additive, and we can set

$$\rho_{\mathrm{tv}}(\mu, \nu) = |\mu - \nu|(X) = \sup_{E,F \in \operatorname{dom}\mu \cap \operatorname{dom}\nu} (\mu - \nu)(E) - (\mu - \nu)(F).$$

In this generality, ρ_{tv} is not even a pseudometric, but if we have a class M of totally finite measures on X all of which are inner regular with respect to a subset $\mathcal{K}$ of $\bigcap_{\mu \in M} \operatorname{dom}\mu$, then we have

$$\rho_{\mathrm{tv}}(\mu, \nu) = \sup_{K,L \in \mathcal{K}} (\mu K - \mu L) - (\nu K - \nu L)$$

for all μ, $\nu \in M$, and $\rho_{\mathrm{tv}} \restriction M \times M$ is a pseudometric on M. If moreover M is such that distinct members of M differ on $\mathcal{K}$ (as when $\mathcal{K}$ is the family of closed sets in a topological space X and $M = M^+_{\mathrm{qR}}(X)$, or when $\mathcal{K}$ the family of compact sets in a Hausdorff space X and $M = M^+_{\mathrm{R}}(X)$), then ρ_{tv} gives us a metric on M. In such a case I will call $\rho_{\mathrm{tv}} \restriction M \times M$ the **total variation metric** on M. (Compare the 'total variation norms' of 362B.)

(ii) Note that if $\Sigma \subseteq \operatorname{dom}\mu \cap \operatorname{dom}\nu$ is a σ-algebra then

$$|\int u\,d\mu - \int u\,d\nu| \le \|u\|_\infty \rho_{\mathrm{tv}}(\mu, \nu)$$

whenever $u \in \mathcal{L}^\infty(\Sigma)$. So if, for instance, X is a topological space and $M \subseteq M^+_{\mathrm{qR}}(X)$, then $u \mapsto \int u\,d\mu$ will be continuous for the total variation metric on M whenever $u : X \to \mathbb{R}$ is a bounded universally measurable function.

(iii) It is of course worth knowing when to expect a complete metric. When our set M can be identified with the positive cone of a band in some L-space M_σ of countably additive functions, as in 437F, then we naturally have a complete metric, because bands in L-spaces are closed subspaces (354Bd). In particular, for any Hausdorff space X, $M^+_{\mathrm{R}}(X)$ can be identified with the positive cone of the L-space of tight Borel measures on X, so is complete. See also 437Xo.

(iv) There is an obvious variation on ρ_{tv} as defined here: the function

$$(\mu, \nu) \mapsto \sup_{E \in \operatorname{dom}\mu \cap \operatorname{dom}\nu} |\mu E - \nu E|,$$

which will be a metric on nearly all occasions when ρ_{tv} is a metric, and will then be uniformly equivalent to ρ_{tv}. But the more complex formulation gives a better match to the Riesz norm metric of the leading examples.

(b) Suppose that (X, ρ) is a metric space. Write M^+_{qR} for the set of totally finite quasi-Radon measures on X. For μ, $\nu \in M^+_{\mathrm{qR}}$ set

$$\rho_{\mathrm{KR}}(\mu, \nu) = \sup\{|\int u\,d\mu - \int u\,d\nu| : u : X \to [-1, 1] \text{ is 1-Lipschitz}\}.$$

Then ρ_{KR} is a metric on M^+_{qR}. **P** It is immediate from the form of the definition that ρ_{KR} is a pseudometric. If μ, $\nu \in M^+_{\mathrm{qR}}$ are different, there is an open set G such that $\mu G \ne \nu G$ (415G(iii)); suppose that $\mu G < \nu G$. Set $u(x) = \rho(x, X \setminus G)$ for $x \in X$. There must be an $n \in \mathbb{N}$ such that $\mu G < \nu F_n$ where $F_n = \{x : u(x) \ge 2^{-n}\}$. In this case, setting $u' = u \wedge 2^{-n}\chi X$,

$$\int u' d\mu \le 2^{-n}\mu G < 2^{-n}\nu F_n \le \int u' d\nu,$$

so $\rho_{\mathrm{KR}}(\mu, \nu) \ge 2^{-n}(\nu F_n - \mu G) > 0$. As μ and ν are arbitrary, ρ_{KR} is a metric. **Q**

Remark ρ_{KR} here is taken from Bogachev 07, §8.3, where it is called the 'Kantorovich-Rubinshtein metric'. For its principal properties, see 437R(g)-(h) below. A variation of this construction will be used in 457L; see also 437Xs.

437R Theorem Let X be a topological space; write $M^+_{\mathrm{qR}} = M^+_{\mathrm{qR}}(X)$ for the set of totally finite quasi-Radon measures on X, and if X is Hausdorff write $M^+_{\mathrm{R}} = M^+_{\mathrm{R}}(X)$ for the set of totally finite Radon measures on X, both endowed with their narrow topologies.

(a)(i) If X is regular then M^+_{qR} is Hausdorff.

(ii) If X is Hausdorff then M^+_{R} is Hausdorff.

(b) If X has a countable network then M^+_{qR} has a countable network.

(c) Suppose that X is separable.

(i) If X is a T_1 space, then M^+_{qR} is separable.

(ii) If X is Hausdorff, M^+_{R} is separable.

(d) If X is a K-analytic Hausdorff space, so is $M^+_{\mathrm{qR}} = M^+_{\mathrm{R}}$.

(e) If X is an analytic Hausdorff space, so is $M^+_{\mathrm{qR}} = M^+_{\mathrm{R}}$.

(f)(i) If X is compact, then for any real $\gamma \ge 0$ the sets $\{\mu : \mu \in M^+_{\mathrm{qR}},\, \mu X \le \gamma\}$ and $\{\mu : \mu \in M^+_{\mathrm{qR}},\, \mu X = \gamma\}$ are compact.

(ii) If X is compact and Hausdorff, then for any real $\gamma \ge 0$ the sets $\{\mu : \mu \in M^+_{\mathrm{R}},\, \mu X \le \gamma\}$ and $\{\mu : \mu \in M^+_{\mathrm{R}},\, \mu X = \gamma\}$ are compact. In particular, the set P_{R} of Radon probability measures on X is compact.

(g) Suppose that X is metrizable and ρ is a metric on X inducing its topology.

(i) The metric ρ_{KR} on M^+_{qR} (437Qb) induces the narrow topology on M^+_{qR}.

(ii) If (X, ρ) is complete then $M^+_{\mathrm{qR}} = M^+_{\mathrm{R}}$ is complete under ρ_{KR}.

(h) If X is Polish, so is $M^+_{\mathrm{qR}} = M^+_{\mathrm{R}}$.

proof (a)(i) (Cf. 437Qb.) Take distinct μ_0, $\mu_1 \in M^+_{\mathrm{qR}}$. If $\mu_0 X \ne \mu_1 X$ then they can be separated by open sets of the form $\{\mu : \mu X < \alpha\}$, $\{\mu : \mu X > \alpha\}$. Otherwise, set $\gamma = \mu_0 X = \mu_1 X$. There is certainly an open set G such that $\mu_0 G \ne \mu_1 G$ (415H); suppose that $\mu_0 G < \mu_1 G$. Because μ_1 is inner regular with respect to the closed sets, there is a closed set $F \subseteq G$ such that $\mu_0 G < \mu_1 F$. Now consider

$$\mathcal{H} = \{H : H \text{ is open}, \overline{H} \subseteq G\}.$$

Then $\mathcal{H}$ is upwards-directed; because X is regular, $\bigcup \mathcal{H} = G$; because μ_1 is quasi-Radon,

$$\sup_{H\in\mathcal{H}} \mu_1 H \ge \sup_{H\in\mathcal{H}} \mu_1(H \cap F) = \mu_1 F > \mu_0 G$$

and there is an $H \in \mathcal{H}$ such that $\mu_1 H > \mu_0 G$. Now

$$\mu_1 H + \mu_0(X \setminus \overline{H}) \ge \mu_1 H + \gamma - \mu_0 G > \gamma.$$

Let α, β be such that $\mu_1 H > \alpha$, $\mu_0(X \setminus \overline{H}) > \beta$ and $\alpha + \beta > \gamma$. Then

$$\{\mu : \mu \in M^+_{\mathrm{qR}},\, \mu(X \setminus \overline{H}) > \beta\}, \quad \{\mu : \mu \in M^+_{\mathrm{qR}},\, \mu H > \alpha,\, \mu X < \alpha + \beta\}$$

are disjoint open sets containing μ_0, μ_1 respectively, so again we have separation.

(ii) We can use the same ideas. Take distinct μ_0, $\mu_1 \in M^+_{\mathrm{R}}$. If $\mu_0 X \ne \mu_1 X$ then μ_0 and μ_1 can be separated by open sets of the form $\{\mu : \mu X < \alpha\}$, $\{\mu : \mu X > \alpha\}$. Otherwise, set $\gamma = \mu_0 X = \mu_1 X$, and take an open set G such that $\mu_0 G \ne \mu_1 G$; suppose that $\mu_0 G < \mu_1 G$. Then $\mu_0(X \setminus G) + \mu_1 G > \gamma$. Because μ_0 and μ_1 are inner regular with respect to the compact sets, there are compact sets $K_0 \subseteq X \setminus G$, $K_1 \subseteq G$ such that $\mu_0 K_0 + \mu_1 K_1 > \gamma$. Now there are disjoint open sets H_0, H_1 such that $K_i \subseteq H_i$ for both i (4A2F(h-i)), in which case $\mu_0 H_0 + \mu_1 H_1 > \gamma$. Take $\alpha_0 < \mu_0 H_0$ and $\alpha_1 < \mu_1 H_1$ such that $\alpha_0 + \alpha_1 > \gamma$. In this case, $\{\mu : \mu H_0 > \alpha_0\}$ and $\{\mu : \mu H_1 > \alpha_1,\, \mu X < \alpha_0 + \alpha_1\}$ are disjoint open sets containing μ_0, μ_1 respectively.

(b) Let $\mathcal{A}$ be a countable network for the topology of X; replacing $\mathcal{A}$ by $\{\bigcup \mathcal{A}_0 : \mathcal{A}_0 \in [\mathcal{A}]^{<\omega}\}$ if necessary, we can suppose that $\mathcal{A}$ is closed under finite unions. Let $\mathcal{D}$ be the family of sets of the form

$$\{\mu : \mu \in M^+_{\mathrm{qR}},\, \mu X < \gamma,\, \mu^* A_i > \gamma_i \text{ for } i \le n\}$$

where $n \in \mathbb{N}$, $A_0, \dots, A_n \in \mathcal{A}$ and $\gamma, \gamma_0, \dots, \gamma_n \in \mathbb{Q}$. Then $\mathcal{D}$ is countable. If $V \subseteq M^+_{\mathrm{qR}}$ is an open set and $\mu_0 \in V$, there must be open sets $G_0, \dots, G_n \subseteq X$ and $\gamma, \gamma_0, \dots, \gamma_n \in \mathbb{Q}$ such that

$$\mu_0 \in \{\mu : \mu X < \gamma,\, \mu G_i > \gamma_i \text{ for every } i \le n\} \subseteq V.$$

For each $i \le n$, $\{A : A \in \mathcal{A},\, A \subseteq G_i\}$ is a countable upwards-directed set with union G_i, so there is a non-decreasing sequence $\langle A_{ij}\rangle_{j\in\mathbb{N}}$ in $\mathcal{A}$ with union G_i, and there must be a $j_i \in \mathbb{N}$ such that $\mu_0^* A_{ij_i} > \gamma_i$ (132Ae). Now

$$\{\mu : \mu X < \gamma,\, \mu^* A_{ij_i} > \gamma_i \text{ for every } i \le n\}$$

belongs to $\mathcal{D}$, contains μ and is included in V. As μ and V are arbitrary, $\mathcal{D}$ is a countable network for the topology of M_{qR}^+.

(c)(i) If X is empty, then M_{qR}^+ is a singleton, and we can stop. Otherwise, let D be a countable dense subset of X. Set $D' = \{\sum_{i=0}^n \alpha_i \delta_{x_i} : x_0, \dots, x_n \in D,\, \alpha_0, \dots, \alpha_n \in \mathbb{Q} \cap [0,\infty[\}$, writing δ_x for the Dirac measure concentrated at x for each $x \in X$. Because X is T_1, $D' \subseteq M_{\mathrm{qR}}^+$. In fact D' is dense in M_{qR}^+. **P** Take any $\mu \in M_{\mathrm{qR}}^+$, a finite family $\mathcal{G}$ of open subsets of X, and $\epsilon > 0$. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{G}$, and $\mathcal{A}$ the set of atoms of $\mathcal{E}$. For each $E \in \mathcal{A}$ choose $x_E \in D \cap \bigcap\{G : E \subseteq G \in \mathcal{G}\}$ and $\alpha_E \in \mathbb{Q} \cap [0,\infty[$ such that $|\alpha_E - \mu E| \le \frac{\epsilon}{\#(\mathcal{A})}$. Try $\nu = \sum_{E\in\mathcal{A}} \alpha_E \delta_{x_E} \in D'$. If $G \in \mathcal{G}$, then

$$\begin{aligned}\mu G = \sum_{E\in\mathcal{A}, E\subseteq G} \mu E &\le \sum_{E\in\mathcal{A}, x_E\in G} \mu E \\ &\le \epsilon + \sum_{E\in\mathcal{A}, x_E \in G} \alpha_E = \epsilon + \nu G;\end{aligned}$$

while

$$\nu X = \textstyle\sum_{E\in\mathcal{A}} \alpha_E \le \epsilon + \sum_{E\in\mathcal{A}} \mu E = \epsilon + \mu X.$$

As μ, $\mathcal{G}$ and ϵ are arbitrary, D' is dense in M_{qR}^+. **Q** So M_{qR}^+ is separable.

(ii) If X is Hausdorff, use the same construction; in this case $D' \subseteq M_{\mathrm{R}}^+$, so M_{R}^+ also is separable.

(d) Most of the argument will be devoted to proving that the set P_{R} of Radon probability measures on X is K-analytic in its narrow topology.

(i) We are supposing that there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = X$ (422F). Set $R_1 = \{(\alpha, x) : \text{there is a } \beta \le \alpha \text{ such that } (\beta, \alpha) \in R\}$; then R_1 is also usco-compact (422Dh).

Set

$$\begin{aligned}\tilde{R} &= \{(\boldsymbol{\alpha}, \mu) : \boldsymbol{\alpha} \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}},\, \mu \in P_{\mathrm{R}},\, \mu R_1[\{\boldsymbol{\alpha}(n)\}] \ge 1 - 2^{-n} \text{ for every } n \in \mathbb{N}\} \\ &\subseteq (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \times P_{\mathrm{R}}.\end{aligned}$$

(Of course $R_1[\{\boldsymbol{\alpha}(n)\}]$ is compact, therefore universally measurable, whenever $\boldsymbol{\alpha} \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ and $n \in \mathbb{N}$.)

(ii) $\tilde{R}[(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}] = P_{\mathrm{R}}$. **P** $\mathbb{N}^{\mathbb{N}} \times X$ is K-analytic (422Ge), while R is a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$ (422Da), so is itself K-analytic (422Gf). Let $\pi_1 : R \to \mathbb{N}^{\mathbb{N}}$ and $\pi_2 : R \to X$ be the coordinate maps. If $\mu \in P_{\mathrm{R}}$, there is a Radon probability measure λ on R such that $\mu = \lambda\pi_2^{-1}$ (432G). For each $n \in \mathbb{N}$ let $L_n \subseteq R$ be a compact set such that $\lambda L_n > 1 - 2^{-n}$; then $\pi_1[L_n]$ is a non-empty compact subset of $\mathbb{N}^{\mathbb{N}}$. Define $\boldsymbol{\alpha}$ by setting

$$\boldsymbol{\alpha}(n)(m) = \sup\{\beta(m) : \beta \in \pi_1[L_n]\}$$

for m, $n \in \mathbb{N}$. Then

$$\mu R_1[\{\boldsymbol{\alpha}(n)\}] \ge \mu R[\pi_1[L_n]] \ge \mu\pi_2[L_n] \ge \lambda L_n \ge 1 - 2^{-n}$$

for every $n \in \mathbb{N}$, so $(\boldsymbol{\alpha}, \mu) \in \tilde{R}$ and $\mu \in \tilde{R}[(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}]$. **Q**

(iii) $\tilde{R}[\{\boldsymbol{\alpha}\}]$ is a compact subset of P_{R} for every $\boldsymbol{\alpha} \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. **P** Since $R_1[\{\boldsymbol{\alpha}(n)\}]$ is compact for every n, $\tilde{R}[\{\boldsymbol{\alpha}\}]$ is uniformly tight, therefore relatively compact in M_{R}^+, by 437Pb. On the other hand, $\{\mu : \mu \in M_{\mathrm{R}}^+,\, \mu X = 1\}$ and $\{\mu : \mu \in M_{\mathrm{R}}^+,\, \mu(X \setminus R_1[\{\boldsymbol{\alpha}(n)\}]) \le 1 - 2^{-n}\}$ are closed for every n, so $\tilde{R}[\{\boldsymbol{\alpha}\}]$ is closed, therefore compact. **Q**

(iv) If $F \subseteq P_{\mathrm{R}}$ is closed, then $\tilde{R}^{-1}[F]$ is closed in $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. **P** Let $\langle \boldsymbol{\alpha}_k \rangle_{k\in\mathbb{N}}$ be a sequence in $\tilde{R}^{-1}[F]$ converging to $\boldsymbol{\alpha}$ in $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. For each $k \in \mathbb{N}$ choose $\mu_k \in F$ such that $(\boldsymbol{\alpha}_k, \mu_k) \in \tilde{R}$. For n, $k \in \mathbb{N}$ set $L_{nk} = \{\boldsymbol{\alpha}(n)\} \cup \{\boldsymbol{\alpha}_l(n) : l \ge k\}$. Then L_{nk} is a compact subset of $\mathbb{N}^{\mathbb{N}}$, so $R_1[L_{nk}]$ is a compact subset of X. **?** If $x \in \bigcap_{k\in\mathbb{N}} R_1[L_{nk}] \setminus R_1[\{\boldsymbol{\alpha}(n)\}]$, then for every $k \in \mathbb{N}$ there is an $l_k \ge k$ such that $(\boldsymbol{\alpha}_{l_k}(n), x) \in R_1$; but $R_1^{-1}[\{x\}]$ is closed in $\mathbb{N}^{\mathbb{N}}$, so contains $\lim_{k\to\infty} \boldsymbol{\alpha}_{l_k}(n) = \boldsymbol{\alpha}(n)$, and $x \in R_1[\{\boldsymbol{\alpha}(n)\}]$. **X** Thus $\bigcap_{k\in\mathbb{N}} R_1[L_{nk}] = R_1[\{\boldsymbol{\alpha}(n)\}]$.

For any n and k, $\mu_l R_1[L_{nk}] \ge \mu_l R_1[\{\boldsymbol{\alpha}_l(n)\}] \ge 1-2^{-n}$ for every $l \ge k$. In the first place, taking $k=0$, $\{\mu_l : l \in \mathbb{N}\}$ is uniformly tight, therefore relatively compact and $\langle \mu_l \rangle_{l\in\mathbb{N}}$ has a cluster point μ say, which must belong to F. Now, for any n,

$$\mu R_1[\{\boldsymbol{\alpha}(n)\}] = \inf_{k\in\mathbb{N}} \mu R_1[L_{nk}] \ge \inf_{k\in\mathbb{N}, l\ge k} \mu_l R_1[L_{nk}]$$

(because $R_1[L_{nk}]$ is compact, therefore closed, and $\mu \in \overline{\{\mu_l : l \ge k\}}$, for each k)

$$\ge \inf_{k\in\mathbb{N}, l\ge k} \mu_l R_1[\{\boldsymbol{\alpha}_l(n)\}] \ge 1-2^{-n}.$$

So $(\boldsymbol{\alpha}, \mu) \in \tilde{R}$ and $\boldsymbol{\alpha} \in \tilde{R}^{-1}[F]$. As $\langle \boldsymbol{\alpha}_k \rangle_{k\in\mathbb{N}}$ is arbitrary, $\tilde{R}^{-1}[F]$ is closed. **Q**

(v) Thus $\tilde{R} \subseteq (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \times P_{\mathrm{R}}$ is usco-compact. Since $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, like $\mathbb{N}^{\mathbb{N}}$, is Polish (4A2Ub, 4A2Qc), $\tilde{R}[(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}]$ is K-analytic (422Gd). But we saw in (ii) that $\tilde{R}[(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}] = P_{\mathrm{R}}$. So P_{R} is K-analytic.

(vi) Now observe that $(\alpha, \mu) \mapsto \alpha\mu : [0,\infty[\times P_{\mathrm{R}} \to M_{\mathrm{R}}^+$ is continuous. **P** We have only to note that

$$(\alpha, \mu) \mapsto (\alpha\mu)(X) = \alpha$$

is continuous, and that

$$\{(\alpha, \mu) : (\alpha\mu)(G) > \gamma\} = \bigcup_{\beta>0} \{(\alpha, \mu) : \alpha > \beta,\ \mu G > \tfrac{\gamma}{\beta}\}$$

is open for every open $G \subseteq X$ and $\gamma \ge 0$. **Q** Since $[0,\infty[$ and P_{R} are K-analytic, and (except in the trivial case $X = \emptyset$) every member of M_{R}^+ is expressible as a non-negative multiple of a probability measure, M_{R}^+ is K-analytic (using 422Ge and 422Gd again).

(vii) Finally, $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$ by 432E.

(e) Put (d), (b) and 423C together.

(f)(i) Because X is compact, every quasi-Radon measure on X is tight, and M_{qR}^+ itself is uniformly tight; by 437Pa, $\{\mu : \mu \in M_{\mathrm{qR}}^+,\ \mu X \le \gamma\}$ is relatively compact in M_{qR}^+. But as it is also closed in M_{qR}^+, it is actually compact. The same argument applies to $\{\mu : \mu \in M_{\mathrm{qR}}^+,\ \mu X = \gamma\}$.

(ii) Use the same idea, but with 437Pb in place of 437Pa.

(g)(i) Write $\mathfrak{T}_{\mathrm{KR}}$ for the topology generated by ρ_{KR}.

($\boldsymbol{\alpha}$) If $\mu \in M_{\mathrm{qR}}^+$ and $\mu X > \alpha$, then $\nu X > \alpha$ whenever $\nu \in M_{\mathrm{qR}}^+$ and $\rho_{\mathrm{KR}}(\mu, \nu) < \mu X - \alpha$, just because χX is a 1-Lipschitz function; so $\{\mu : \mu \in M_{\mathrm{qR}}^+,\ \mu X > \alpha\} \in \mathfrak{T}_{\mathrm{KR}}$ for every $\alpha \in \mathbb{R}$.

If $G \subseteq X$ is open, $\alpha \ge 0$ and $\mu \in M_{\mathrm{qR}}^+$ is such that $\mu G > \alpha$, there is a $\delta \in \,]0,1]$ such that $\mu F > \alpha + \delta$, where $F = \{x : \rho(x, X \setminus G) \ge \delta\}$. Let u be a 1-Lipschitz function such that $\delta\chi F \le u \le \delta\chi G$. If $\nu \in M_{\mathrm{qR}}^+$ and $\rho_{\mathrm{KR}}(\mu, \nu) \le \delta^2$, then

$$\delta\nu G \ge \int u\,d\nu \ge \int u\,d\mu - \delta^2 \ge \delta\mu F - \delta^2 > \delta\alpha$$

and $\nu G > \alpha$. This shows that $\{\mu : \mu \in M_{\mathrm{qR}}^+,\ \mu G > \alpha\} \in \mathfrak{T}_{\mathrm{KR}}$. As G and α are arbitrary, $\mathfrak{T}_{\mathrm{KR}}$ is finer than the narrow topology.

($\boldsymbol{\beta}$) Suppose that $\mu \in M_{\mathrm{qR}}^+$ and $\epsilon > 0$; let $\delta > 0$ be such that $\delta(3\delta + 6\mu X + 7) \le \epsilon$. Then there is a totally bounded closed set $F \subseteq X$ such that $\mu(X \setminus F) \le \delta$ (434L). Set $G = \{x : \rho(x, F) < \delta\}$. Let $x_0, \dots, x_n \in X$ be such that $F \subseteq \bigcup_{i\le n} B(x_i, \delta)$; then $G \subseteq \bigcup_{i\le n} B(x_i, 2\delta)$ and there are $v_0, \dots, v_n \in C_b(X)^+$ such that $\chi G \le \sum_{i=0}^n v_i(x) \le \chi X$ and $\{x : v_i(x) > 0\} \subseteq B(x_i, 3\delta)$ for every $i \le n$. Let $w \in C_b(X)$ be such that $\chi(X \setminus G) \le w \le \chi(X \setminus F)$. By the choice of F, $\int w\,d\mu \le \delta$.

If $u : X \to [-1,1]$ is 1-Lipschitz then

$$|u - \textstyle\sum_{i=0}^n u(x_i) v_i| \le 3\delta\chi X + 2w.$$

P If $x \in G$, then

$$|u(x)-\sum_{i=0}^n u(x_i)v_i(x)| = |\sum_{i=0}^n (u(x)-u(x_i))v_i(x)|$$

(because $\sum_{i=0}^n v_i(x)=1$)

$$\le \sum_{i=0}^n |u(x)-u(x_i)|v_i(x) \le \sum_{i=0}^n 3\delta v_i(x)$$

(because whenever $v_i(x)>0$, $|u(x)-u(x_i)|\le\rho(x,x_i)\le 3\delta$)

$$\le 3\delta.$$

If $x\in X\setminus G$, then

$$|u(x)-\sum_{i=0}^n u(x_i)v_i(x)| \le |u(x)|+\sum_{i=0}^n |u(x_i)|v_i(x) \le 2 = 2w(x). \text{ **Q**}$$

So if $\nu\in M_{\mathrm{qR}}^+$,

$$|\int u\,d\nu - \sum_{i=0}^n u(x_i)\int v_i d\nu| \le 3\delta\nu X + 2\int w\,d\nu.$$

By 437Jf or 437L, there is a neighbourhood V of μ for the narrow topology in M_{qR}^+ such that if $\nu\in V$ then $\nu X\le\mu X+\delta$, $\int w\,d\nu\le 2\delta$ and $|\int v_i d\mu - \int v_i d\nu| \le \frac{\delta}{n+1}$ for every $i\le n$. So if $\nu\in V$ and $u:X\to[-1,1]$ is 1-Lipschitz, we shall have

$$\begin{aligned}|\int u\,d\nu - \int u\,d\mu| &\le 3\delta(\nu X+\mu X) + 2(\int w\,d\nu + \int w\,d\mu)\\ &\qquad + \sum_{i=0}^n |\int v_i d\nu - \int v_i d\mu|\\ &\le 3\delta(2\mu X+\delta)+6\delta+\sum_{i=0}^n \frac{\delta}{n+1} \le \delta(6\mu X+3\delta+7)\le\epsilon.\end{aligned}$$

Thus $\{\nu:\rho_{\mathrm{KR}}(\nu,\mu)\le\epsilon\}\supseteq V$ is a neighbourhood of μ for the narrow topology; as μ and ϵ are arbitrary, the narrow topology is finer than $\mathfrak{T}_{\mathrm{KR}}$, and the two topologies are equal.

(ii) If X is ρ-complete then $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$ by 434Jg and 434Jb. Now suppose that $\langle\mu_n\rangle_{n\in\mathbb{N}}$ is a ρ_{KR}-Cauchy sequence in M_{R}^+.

(α) For every $\epsilon\in\,]0,1]$ there is a compact $K\subseteq X$ such that $\mu_n(X\setminus U(K,\epsilon))\le\epsilon$ for every $n\in\mathbb{N}$, where $U(K,\epsilon)=\{x:\rho(x,y)<\epsilon$ for some $y\in K\}$. **P** Take $m\in\mathbb{N}$ such that $\rho_{\mathrm{KR}}(\mu_m,\mu_n)\le\frac{1}{2}\epsilon^2$ for every $n\ge m$. Let $K\subseteq X$ be a compact set such that $\mu_n(X\setminus K)\le\frac{1}{2}\epsilon$ for every $n\le m$. Set $G=U(K,\epsilon)$. There is a 1-Lipschitz function $u:X\to[0,\epsilon]$ such that $\epsilon\chi(X\setminus G)\le u\le\chi(X\setminus K)$. If $n\le m$, then of course $\mu_n(X\setminus G)\le\epsilon$. If $n\ge m$, then

$$\begin{aligned}\mu_n(X\setminus G) &\le \frac{1}{\epsilon}\int u\,d\mu_n \le \frac{1}{\epsilon}\big(\rho_{\mathrm{KR}}(\mu_n,\mu_m)+\int u\,d\mu_m\big)\\ &\le \frac{1}{\epsilon}\big(\frac{\epsilon^2}{2}+\epsilon\mu_m(X\setminus K)\big) \le \frac{\epsilon}{2}+\mu_m(X\setminus K)\le\epsilon.\end{aligned}$$

So we have an appropriate K. **Q**

(β) $\{\mu_n:n\in\mathbb{N}\}$ is uniformly totally finite and uniformly tight. **P** Since $|\mu_m X-\mu_n X|\le\rho_{\mathrm{KR}}(\mu_m,\mu_n)$ for all m, $n\in\mathbb{N}$, $\{\mu_n X:n\in\mathbb{N}\}$ is bounded. Of course all the μ_n are tight. Now take any $\epsilon\in\,]0,1]$. For each $m\in\mathbb{N}$, (i) tells us that there is a compact set $K_m\subseteq X$ such that $\mu_n(X\setminus U(K_m,2^{-m}\epsilon))\le 2^{-m}\epsilon$ for every $n\in\mathbb{N}$. Set $E=\bigcap_{m\in\mathbb{N}}U(K_m,2^{-m}\epsilon)$, $K=\overline{E}$. Then E and K are totally bounded; because (X,ρ) is complete, K is compact. And

$$\mu_n(X\setminus K)\le\sum_{m=0}^\infty \mu_n(X\setminus U(K_m,2^{-m}\epsilon))\le 2\epsilon$$

for every $n\in\mathbb{N}$. As ϵ is arbitrary, $\{\mu_n:n\in\mathbb{N}\}$ is uniformly tight. **Q**

(γ) By 437Pb, $\langle\mu_n\rangle_{n\in\mathbb{N}}$ has a cluster point μ in M_{R}^+ for the narrow topology. Now, for $m\in\mathbb{N}$,

$$\rho_{\mathrm{KR}}(\mu,\mu_m)=\sup\{|\int u\,d\mu-\int u\,d\mu_m|:u:X\to[-1,1]\text{ is 1-Lipschitz}\}$$

$$\le\sup\{|\int u\,d\mu_n-\int u\,d\mu_m|:u:X\to[-1,1]\text{ is 1-Lipschitz, }n\ge m\}$$

(437Jf again)

$$\le\sup_{n\ge m}\rho_{\mathrm{KR}}(\mu_n,\mu_m),$$

and $\lim_{n\to\infty}\rho_{\mathrm{KR}}(\mu,\mu_m)=0$.

(δ) Thus every ρ_{KR}-Cauchy sequence in M_{R}^+ has a limit in M_{R}^+, and M_{R}^+ is complete.

(h) Put (g-ii) and (c-ii) together.

437S The sets of measures we have been considering have generally been convex, if addition and multiplication by non-negative scalars are defined as in 234G and 234Xf. We can therefore look for extreme points, in the hope that they will have straightforward characterizations, as in the following.

Proposition Let X be a Hausdorff space, and P_{R} the set of Radon probability measures on X. Then the extreme points of P_{R} are just the Dirac measures on X.

proof (a) Suppose that $x\in X$, and that δ_x is the Dirac measure on X concentrated at x. If μ_1, $\mu_2\in P_{\mathrm{R}}$ are such that $\delta_x=\frac{1}{2}(\mu_1+\mu_2)$, then we must have $\mu_1E\le 2\mu E$ for every Borel set E; in particular, $\mu_1(X\setminus\{x\})=0$ and $\mu_1\{x\}=1$, that is, $\mu_1=\delta_x$. Similarly, $\mu_2=\delta_x$; as μ_1 and μ_2 are arbitrary, δ_x is an extreme point of P_{R}.

(b) Suppose that μ is an extreme point of P_{R}. Let K be the support of μ. **?** If K has more than one point, take distinct x, $y\in K$. As X is Hausdorff, there are disjoint open sets G, H such that $x\in G$ and $y\in H$. Set $E=G\cap K$, $\alpha=\mu E$. Because K is the support of μ, $\alpha>0$. But similarly $\mu(H\cap K)>0$ and $\alpha<1$. Let μ_1, μ_2 be the indefinite-integral measures defined over μ by $\frac{1}{\alpha}\chi E$ and $\frac{1}{\beta}\chi(X\setminus E)$ respectively. Then both are Radon probability measures on X (416S), so belong to P_{R}. Now $\mu F=\alpha\mu_1F+(1-\alpha)\mu_2F$ for every Borel set F; as μ and $\alpha\mu_1+(1-\alpha)\mu_2$ are both Radon measures, they coincide; as neither μ_1 nor μ_2 is equal to μ, μ is not extreme in P_{R}. **X**

Thus $K=\{x\}$ for some $x\in X$. But this means that $\mu\{x\}=1$ and $\mu(X\setminus\{x\})=0$, so $\mu=\delta_x$ is of the declared form.

437T We now have a language in which to express a fundamental result in the theory of dynamical systems.

Theorem Let X be a non-empty compact Hausdorff space, and $\phi:X\to X$ a continuous function. Write Q_ϕ for the set of Radon probability measures on X for which ϕ is inverse-measure-preserving. Then Q_ϕ is convex and not empty, and is compact for the narrow topology.

proof (a) Write M_{R}^+ for the set of totally finite Radon measures on X, and let $\tilde\phi:M_{\mathrm{R}}^+\to M_{\mathrm{R}}^+$ be the function corresponding to $\phi:X\to X$ as described in 437N. Now, for $\mu\in M_{\mathrm{R}}^+$, $\mu\in Q_\phi$ iff $\mu X=1$ and $\mu(\phi^{-1}[E])=\mu E$ whenever μ measures E, that is, iff the image measure $\mu\phi^{-1}=\tilde\phi(\mu)$ extends μ. But as $\tilde\phi(\mu)$ and μ are Radon measures, $\mu\in Q_\phi$ iff $\mu X=1$ and $\tilde\phi(\mu)=\mu$.

Since $\tilde\phi$ is continuous (and M_{R}^+ is Hausdorff, see 437Ra), Q_ϕ is closed for the narrow topology. By 437Pb/437R(f-ii), it is compact. Because $\tilde\phi$ respects addition and scalar multiplication, Q_ϕ is convex.

(b) To see that Q_ϕ is not empty, take any $x_0\in X$ and a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Define $f:C(X)\to\mathbb{R}$ by setting $f(u)=\lim_{n\to\mathcal{F}}\frac{1}{n+1}\sum_{i=0}^n u(\phi^i(x_0))$ for every $u\in C(X)$. Then f is a positive linear functional and $f(\chi X)=1$. So there is a $\mu\in M_{\mathrm{R}}^+$ such that $f(u)=\int u\,d\mu$ for every $u\in C(X)$.

If $u\in C(X)$, then $f(u)=f(u\phi)$. **P**

$$|f(u\phi)-f(u)|=|\lim_{n\to\mathcal{F}}\frac{1}{n+1}\sum_{i=0}^n u(\phi^{i+1}(x_0))-u(\phi^i(x_0))|$$

$$=|\lim_{n\to\mathcal{F}}\frac{1}{n+1}u(\phi^{n+1}(x_0))-u(x_0)|$$

$$\le\lim_{n\to\mathcal{F}}\frac{1}{n+1}|u(\phi^{n+1}(x_0))-u(x_0)|\le\lim_{n\to\mathcal{F}}\frac{2\|u\|_\infty}{n+1}=0.\ \mathbf{Q}$$

On the other hand,

$$\int u\,d(\tilde\phi(\mu))=\int u\phi\,d\mu$$

$$=f(u\phi)=f(u)=\int u\,d\mu \quad (235G)$$

for every $u\in C(X)$. By the uniqueness of the representation of f as an integral, $\mu=\tilde\phi(\mu)$. Of course $\mu X=f(\chi X)=1$ so $\mu\in Q_\phi$, as required.

437U In important cases, the narrowly compact subsets of $M^+_{\mathrm{R}}(X)$ are exactly the bounded uniformly tight sets. Once again, it is worth introducing a word to describe when this happens.

Definition Let X be a Hausdorff space and $P_{\mathrm{R}}(X)$ the set of Radon probability measures on X. X is a **Prokhorov space** if every subset of $P_{\mathrm{R}}(X)$ which is compact for the narrow topology is uniformly tight.

437V Theorem (a) Compact Hausdorff spaces are Prokhorov spaces.
(b) A closed subspace of a Prokhorov Hausdorff space is a Prokhorov space.
(c) An open subspace of a Prokhorov Hausdorff space is a Prokhorov space.
(d) The product of a countable family of Prokhorov Hausdorff spaces is a Prokhorov space.
(e) Any G_δ subset of a Prokhorov Hausdorff space is a Prokhorov space.
(f) Čech-complete spaces are Prokhorov spaces.
(g) Polish spaces are Prokhorov spaces.

proof (a) This is trivial; on a compact Hausdorff space the set of all Radon probability measures is uniformly tight.

(b) Let X be a Prokhorov Hausdorff space, Y a closed subset of X, and $A\subseteq P_{\mathrm{R}}(Y)$ a narrowly compact set. Taking ϕ to be the identity map from Y to X, and defining $\tilde\phi:M^+_{\mathrm{R}}(Y)\to M^+_{\mathrm{R}}(X)$ as in 437N, $\tilde\phi[A]$ is narrowly compact in $P_{\mathrm{R}}(X)$, so is uniformly tight. For any $\epsilon>0$, there is a compact set $K\subseteq X$ such that $\tilde\phi(\mu)(X\setminus K)\le\epsilon$ for every $\mu\in A$. Now $K\cap Y$ is a compact subset of Y and $\mu(Y\setminus(K\cap Y))\le\epsilon$ for every $\mu\in A$. As ϵ is arbitrary, A is uniformly tight in $P_{\mathrm{R}}(Y)$.

(c) Let X be a Prokhorov Hausdorff space, Y an open subset of X, and $A\subseteq P_{\mathrm{R}}(Y)$ a narrowly compact set. Once again, take ϕ to be the identity map from Y to X, so that $\tilde\phi[A]\subseteq P_{\mathrm{R}}(X)$ is narrowly compact and uniformly tight in $P_{\mathrm{R}}(X)$.

? Suppose, if possible, that A is not uniformly tight in $P_{\mathrm{R}}(Y)$. Then there is an $\epsilon>0$ such that $A_K=\{\mu:\mu\in A,\ \mu(Y\setminus K)\ge 5\epsilon\}$ is non-empty for every compact set $K\subseteq Y$. Note that $A_K\subseteq A_{K'}$ whenever $K\supseteq K'$, so $\{A_K:K\subseteq Y$ is compact$\}$ has the finite intersection property, and there is an ultrafilter $\mathcal{F}$ on $P_{\mathrm{R}}(Y)$ containing every A_K. Because A is narrowly compact, there is a $\lambda\in P_{\mathrm{R}}(Y)$ such that $\mathcal{F}\to\lambda$. Let $K^*\subseteq Y$ be a compact set such that $\lambda(Y\setminus K^*)\le\epsilon$.

As $\tilde\phi[A]$ is uniformly tight, there is a compact set $L\subseteq X$ such that $\mu(Y\setminus L)=\tilde\phi(\mu)(X\setminus L)\le\epsilon$ for every $\mu\in A$. Now K^* and $L\setminus Y$ are disjoint compact sets in the Hausdorff space X, so there are disjoint open sets G, $H\subseteq X$ such that $K^*\subseteq G$ and $L\setminus Y\subseteq H$ (4A2F(h-i) again). Set $K=L\setminus H\supseteq L\cap G$; then K is a compact subset of Y. As $A_K\in\mathcal{F}$, there must be a $\mu\in A_K$ such that $\mu Y\le\lambda Y+\epsilon$ and $\mu(G\cap Y)\ge\lambda(G\cap Y)-\epsilon$. Accordingly

$$\begin{aligned}\mu(Y\setminus L)&\le\epsilon,\\ \mu(Y\setminus G)&=\mu Y-\mu(G\cap Y)\le\lambda Y+\epsilon-\lambda(G\cap Y)+\epsilon\\ &=\lambda(Y\setminus G)+2\epsilon\le\lambda(Y\setminus K^*)+2\epsilon\le 3\epsilon,\\ \mu((Y\setminus L)\cup(Y\setminus G))&=\mu(Y\setminus(L\cap G))\ge\mu(Y\setminus K)\ge 5\epsilon,\end{aligned}$$

which is impossible. **X**

Thus A is uniformly tight. As A is arbitrary, Y is a Prokhorov space.

(d) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of Prokhorov Hausdorff spaces with product X. Let $A\subseteq P_{\mathrm{R}}(X)$ be a narrowly compact set. Let $\epsilon>0$. For each $n\in\mathbb{N}$ let $\pi_n:X\to X_n$ be the canonical map and $\tilde\pi_n:M^+_{\mathrm{R}}(X)\to M^+_{\mathrm{R}}(X_n)$ the associated function. Then $\tilde\pi_n[A]$ is narrowly compact in $P_{\mathrm{R}}(X_n)$, therefore uniformly tight, and there is a compact set $K_n \subseteq X_n$ such that $(\tilde{\pi}_n\mu)(X_n \setminus K_n) \le 2^{-n-1}\epsilon$ for every $\mu \in A$. Set $K = \prod_{n\in\mathbb{N}} K_n$, so that K is a compact subset of X and $X \setminus K = \bigcup_{n\in\mathbb{N}} \pi_n^{-1}[X_n \setminus K_n]$. If $\mu \in A$, then

$$\mu(X \setminus K) \le \sum_{n=0}^{\infty} \mu\pi_n^{-1}[X_n \setminus K_n] \le \sum_{n=0}^{\infty} 2^{-n-1}\epsilon = \epsilon.$$

As ϵ is arbitrary, A is uniformly tight; as A is arbitrary, X is a Prokhorov space.

(e) Let X be a Prokhorov Hausdorff space and Y a G_δ subset of X. Express Y as $\bigcap_{n\in\mathbb{N}} Y_n$ where every $Y_n \subseteq X$ is open. Set $Z = \{z : z \in \prod_{n\in\mathbb{N}} Y_n,\ z(m) = z(n)$ for all $m, n \in \mathbb{N}\}$. Because X is Hausdorff, Z is a closed subspace of $\prod_{n\in\mathbb{N}} Y_n$ homeomorphic to Y. Putting (c), (d) and (b) together, Z and Y are Prokhorov spaces.

(f) Put (a), (e) and the definition of 'Čech-complete' together.

(g) This is a special case of (f) (4A2Md).

437X Basic exercises (a) Let X be a set, U a Riesz subspace of $\mathbb{R}^X$ and $f \in U^\sim$. (i) Show that $f \in U^\sim_\sigma$ iff $\lim_{n\to\infty} f(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$. (*Hint*: show that in this case, if $0 \le v_n \le u_n$, we can find $k(n)$ such that $|f(v_n \vee u_{k(n)}) - f(v_n)| \le 2^{-n}$.) (ii) Show that $f \in U^\sim_\tau$ iff $\inf_{u\in A} |f(u)| = 0$ whenever $A \subseteq U$ is a non-empty downwards-directed set and $\inf_{u\in A} u(x) = 0$ for every $x \in X$. (*Hint*: given $\epsilon > 0$, set $B = \{v : f(v) \ge \inf_{u\in A} f^+(u) - \epsilon,\ \exists\, w \in A,\ v \ge w\}$ and show that B is a downwards-directed set with infimum 0 in $\mathbb{R}^X$.)

(b) Let X be a set, U a Riesz subspace of $\ell^\infty(X)$ containing the constant functions, and Σ the smallest σ-algebra of subsets of X with respect to which every member of U is measurable. Let μ and ν be two totally finite measures on X with domain Σ, and f, g the corresponding linear functionals on U. Show that $f \wedge g = 0$ in $U^\sim$ iff there is an $E \in \Sigma$ such that $\mu E = \nu(X \setminus E) = 0$. (*Hint*: 326M[5].)

>(c) Let $I \subseteq \mathbb{R}$ be a interval with at least two points. (i) Show that if $g : I \to \mathbb{R}$ is of bounded variation on every compact subinterval of I, there is a unique signed tight Borel measure μ_g on I such that $\mu_g[a, b] = \lim_{x\downarrow b} g(x) - \lim_{x\uparrow a} g(x)$ whenever $a \le b$ in I, counting $\lim_{x\uparrow a} g(x)$ as $g(a)$ if $a = \min I$, and $\lim_{x\downarrow b} g(x)$ as $g(b)$ if $b = \max I$. (ii) Show that if $h : I \to \mathbb{R}$ is another function of bounded variation on every compact subinterval, then $\mu_h = \mu_g$ iff $\{x : h(x) \ne g(x)\}$ is countable iff $\{x : h(x) = g(x)\}$ is dense in I. (iii) Show that if ν is any signed Baire measure on I there is a g of bounded variation on every compact subinterval such that $\nu = \mu_g$.

(d)(i) Show that S, in 437C, is the unique sequentially order-continuous positive linear operator from $\mathcal{L}^\infty$ to $(U^\sim_\sigma)^*$ which extends the canonical embedding of U in $(U^\sim_\sigma)^*$. (ii) Show that S, in 437H, is the unique sequentially order-continuous positive linear operator from $\mathcal{L}^\infty$ to $(U^\sim_\tau)^*$ which extends the canonical embedding of U in $(U^\sim_\tau)^*$ and is 'τ-additive' in the sense that whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets with union H then $S(\chi H) = \sup_{G\in\mathcal{G}} S(\chi G)$ in $(U^\sim_\tau)^*$.

(e) Let X and Y be completely regular topological spaces and $\phi : X \to Y$ a continuous function. Define $T : C_b(Y) \to C_b(X)$ by setting $T(v) = v\phi$ for every $v \in C_b(Y)$, and let $T' : C_b(X)^* \to C_b(Y)^*$ be its adjoint. (i) Show that T' is a norm-preserving Riesz homomorphism. (ii) Show that $T'[C_b(X)^\sim_\sigma] \subseteq C_b(Y)^\sim_\sigma$, and that if $f \in C_b(X)^\sim_\sigma$ corresponds to a Baire measure μ on X, then $T'f$ corresponds to the Baire measure $\mu\phi^{-1}\restriction\mathcal{B}\mathfrak{a}(Y)$. (iii) Show that $T'[C_b(X)^\sim_\tau] \subseteq C_b(Y)^\sim_\tau$, and that if $f \in C_b(X)^\sim_\tau$ corresponds to a Borel measure μ on X, then $T'f$ corresponds to the Borel measure $\mu\phi^{-1}\restriction\mathcal{B}(Y)$. (iv) Write $\mathcal{L}^\infty_X$ and $\mathcal{L}^\infty_Y$ for the M-spaces of bounded real-valued Borel measurable functions on X, Y respectively, and $S_X : \mathcal{L}^\infty_X \to (C_b(X)^\sim_\tau)^*$, $S_Y : \mathcal{L}^\infty_Y \to (C_b(Y)^\sim_\tau)^*$ for the canonical Riesz homomorphisms as constructed in 437Hb. Show that if $T'' : (C_b(Y)^\sim_\tau)^* \to (C_b(X)^\sim_\tau)^*$ is the adjoint of $T'\restriction C_b(X)^\sim_\tau$, then $T''S_Y(v) = S_X(v\phi)$ for every $v \in \mathcal{L}^\infty_Y$.

(f) Let X be a topological space, $\mathcal{L}^\infty(\Sigma_{\mathrm{um}})$ the space of bounded universally measurable real-valued functions on X, and M_σ the space of countably additive functionals on the Borel σ-algebra of X. Show that we have a sequentially order-continuous Riesz homomorphism $S : \mathcal{L}^\infty(\Sigma_{\mathrm{um}}) \to M^*_\sigma$ defined by the formula $(Sv)(\mu) = \int v\,d\mu$ whenever $v \in \mathcal{L}^\infty(\Sigma_{\mathrm{um}})$ and $\mu \in M^+_\sigma$.

(g) Let X be a completely regular topological space. Show that the vague topology on the space M_τ of differences of τ-additive totally finite Borel measures on X is Hausdorff.

[5] Formerly 326I.

>(h) Let X and Y be topological spaces, and $\phi : X \to Y$ a continuous function. Write $M_\#(X)$ for any of $M(\mathcal{B}\mathfrak{a}(X))$, $M_\sigma(\mathcal{B}\mathfrak{a}(X))$, $M(\mathcal{B}(X))$, $M_\sigma(\mathcal{B}(X))$, $M_\tau(X)$ or $M_t(X)$, where $M_\tau(X) \subseteq M_\sigma(\mathcal{B}(X))$ is the space of signed τ-additive Borel measures and $M_t(X) \subseteq M_\tau(X)$ is the space of signed tight Borel measures; and $M_\#(Y)$ for the corresponding space based on Y. Show that there is a positive linear operator $\tilde{\phi} : M_\#(X) \to M_\#(Y)$ defined by saying that $\tilde{\phi}(\mu)(E) = \mu\phi^{-1}[E]$ whenever $\mu \in M_\#(X)$ and E belongs to $\mathcal{B}\mathfrak{a}(Y)$ or $\mathcal{B}(Y)$, as appropriate, and that $\tilde{\phi}$ is continuous for the vague topologies on $M_\#(X)$ and $M_\#(Y)$.

>(i) Let X be a zero-dimensional compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. (i) Show that $\mathcal{E}$ separates zero sets. (ii) Show that the vague topology on $M(\mathcal{E})$ is just the topology of pointwise convergence induced by the usual topology of $\mathbb{R}^{\mathcal{E}}$. (iii) Writing M_t for the space of signed tight Borel measures on X, show that $\mu \mapsto \mu{\restriction}\mathcal{E} : M_t \to M(\mathcal{E})$ is a Banach lattice isomorphism between the L-spaces M_t and $M(\mathcal{E})$, and is also a homeomorphism when M_t and $M(\mathcal{E})$ are given their vague topologies.

(j) (i) Let X be a topological space, and Σ an algebra of subsets of X containing every open set; let $M(\Sigma)^+$ be the set of non-negative real-valued additive functionals on Σ, endowed with its narrow topology, E a member of Σ, and ∂E its boundary. Show that $\nu \mapsto \nu E : M(\Sigma)^+ \to [0, \infty[$ is continuous at $\nu_0 \in M(\Sigma)^+$ iff $\nu_0(\partial E) = 0$. (ii) Let X be a completely regular topological space, and Σ a σ-algebra of subsets of X including the Baire σ-algebra. Write M_σ for the L-space of countably additive functionals on Σ. Let $\mathcal{F}$ be a filter on the positive cone M_σ^+ and μ a member of M_σ^+. Show that $\mathcal{F} \to \mu$ for the vague topology on M_σ iff $\mu E = \lim_{\nu\to\mathcal{F}} \nu E$ whenever $E \in \Sigma$ and $\mu(\partial E) = 0$.

(k) Let X be a compact Hausdorff space, M_{R}^+ the set of Radon measures on X and P_{R} the set of Radon probability measures on X. (i) Show that M_{R}^+, with its narrow topology and its natural convex structure, can be identified with the positive cone of $C(X)^*$ with its weak* topology. (ii) Show that P_{R}, with its narrow topology and its natural convex structure, can be identified with $\{f : f \in C(X)^*,\ f \ge 0,\ f(\chi X) = 1\}$ with its weak* topology.

(l) In 437Mc, show that $|\psi(\mu, \nu)| = \psi(|\mu|, |\nu|)$ for every $\mu \in M_\tau(X)$ and $\nu \in M_\tau(Y)$.

(m) Let X be a topological space, Y a regular topological space and $M_{\mathrm{qR}}^+(X)$, $M_{\mathrm{qR}}^+(Y)$ the spaces of totally finite quasi-Radon measures on X, Y respectively. For a continuous $\phi : X \to Y$ define $\tilde{\phi} : M_{\mathrm{qR}}^+(X) \to M_{\mathrm{qR}}^+(Y)$ by saying that ϕ is inverse-measure-preserving for μ and $\tilde{\phi}(\mu)$ for every $\mu \in M_{\mathrm{qR}}^+(X)$ (418Hb). Show that $\tilde{\phi}$ is continuous for the narrow topologies.

>(n) Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a countable family of Radon probability spaces, and Q the set of Radon probability measures μ on $X = \prod_{i\in I} X_i$ such that the image of μ under the map $x \mapsto x(i)$ is μ_i for every $i \in I$. Show that Q is uniformly tight and is compact for the narrow topology on the set of totally finite topological measures on X.

(o) Let X be any topological space, and M_{qR}^+ the space of totally finite quasi-Radon measures on X. Show that M_{qR}^+ is complete in the total variation metric.

(p) Let X and Y be topological spaces, and $\rho_{\mathrm{tv}}^{(X)}$, $\rho_{\mathrm{tv}}^{(Y)}$, $\rho_{\mathrm{tv}}^{(X\times Y)}$ the total variation metrics on the spaces $M_{\mathrm{qR}}^+(X)$, $M_{\mathrm{qR}}^+(Y)$ and $M_{\mathrm{qR}}^+(X \times Y)$ of quasi-Radon measures. Let μ_1, μ_2 be totally finite quasi-Radon measures on X, ν_1, ν_2 totally finite quasi-Radon measures on Y, and $\mu_1 \times \nu_1$, $\mu_2 \times \nu_2$ the quasi-Radon product measures. Show that

$$\rho_{\mathrm{tv}}^{(X\times Y)}(\mu_1 \times \nu_1, \mu_2 \times \nu_2) \le \rho_{\mathrm{tv}}^{(X)}(\mu_1, \mu_2) \cdot \nu_2 Y + \mu_1 X \cdot \rho_{\mathrm{tv}}^{(Y)}(\nu_1, \nu_2).$$

(q)(i) Show that the set $M_\sigma^+(\mathcal{B}(X))$ of totally finite Borel probability measures on X is T_0 in its narrow topology for any topological space X. (ii) Give $X = \omega_1 + 1$ its order topology. Show that the narrow topology on $M_\sigma^+(\mathcal{B}(X))$ is not T_1. (*Hint*: consider interpretations of Dieudonné's measure on ω_1 and the Dirac measure concentrated at ω_1 as Borel measures on X.)

(r) Let X be any topological space and $\tilde{M}^+$ the set of non-negative additive functionals defined on subalgebras of $\mathcal{P}X$ containing every open set. For μ, $\nu \in \tilde{M}^+$ define $\mu + \nu \in \tilde{M}^+$ by setting $(\mu + \nu)(E) = \mu E + \nu E$ for $E \in \operatorname{dom}\mu \cap \operatorname{dom}\nu$. (i) Show that addition on $\tilde{M}^+$ is continuous for the narrow topology. (ii) Show that $(\alpha, \mu) \mapsto \alpha\mu : [0, \infty[\times \tilde{M}^+ \to \tilde{M}^+$ is continuous for the narrow topology on $\tilde{M}^+$. (iii) Writing $\tilde{P}$ for $\{\mu : \mu \in \tilde{M}^+,\ \mu X = 1\}$, and δ_x for the Dirac measure concentrated at x for each $x \in X$, show that the convex hull of $\{\delta_x : x \in X\}$ is dense in $\tilde{P}$ for the narrow topology. (iv) Suppose that A and B are uniformly tight subsets of $\tilde{M}^+$ and $\gamma \ge 0$. Show that $A \cup B$, $A + B = \{\mu + \nu : \mu \in A,\ \nu \in B\}$ and $\{\alpha\mu : \mu \in A,\ 0 \le \alpha \le \gamma\}$ are uniformly tight.

(s) Let (X,ρ) be a metric space, Σ a subalgebra of $\mathcal{P}X$ containing all the open sets, and $M=M(\Sigma)$ the set of bounded finitely additive functionals on Σ. For μ, $\nu\in M$ set

$$\rho_{\mathrm{KR}}(\mu,\nu)=\sup\{|\textstyle\int u\,d\mu-\int u\,d\nu|:u:X\to[-1,1]\text{ is 1-Lipschitz}\},$$

$$\rho_{\mathrm{LP}}(\mu,\nu)=\inf\{\epsilon:\epsilon>0,\ \nu F-\mu U(F;\epsilon)\le\epsilon\text{ and }\mu F-\nu U(F;\epsilon)\le\epsilon$$
$$\text{for every non-empty closed }F\subseteq X\},$$

where $U(F;\epsilon)=\{x:\rho(x,F)<\epsilon\}$ for non-empty subsets F of X. (i) Show that ρ_{KR} and ρ_{LP} are pseudometrics on M. (ii) Show that if μ, $\nu\in M$ and $\rho_{\mathrm{LP}}(\mu,\nu)=\delta$ then $\min(1,\delta^2)\le\rho_{\mathrm{KR}}(\mu,\nu)\le 2\delta(1+\delta+|\nu|X)$. (iii) Show that ρ_{KR} and ρ_{LP} induce the same topology on M and the same uniformity on $\{\mu:\mu\in M,\ |\mu|X\le\gamma\}$, for any $\gamma\ge 0$. (*Hint*: Bogachev 07, 8.10.43. ρ_{LP} is the **Lévy-Prokhorov** pseudometric.)

(t) Let X be a Hausdorff space, and P_{R} the set of Radon probability measures on X with its narrow topology. For $x\in X$ let δ_x be the Dirac measure on X concentrated at x. Show that $x\mapsto\delta_x$ is a homeomorphism between X and its image in P_{R}.

(u) Let X be a Hausdorff space and P'_{R} the set of Radon measures μ on X such that $\mu X\le 1$. Show that the extreme points of P'_{R} are the Dirac measures on X, as in 437S, together with the zero measure.

(v) Let X be a T_0 topological space, and P_{qR} the set of quasi-Radon probability measures on X. Show that the extreme points of P_{qR} are just the Dirac measures on X concentrated on points x such that $\{x\}$ is closed.

>(w) Let X be a Prokhorov Hausdorff space, and A a set of totally finite Radon measures on X which is compact for the narrow topology. Show that A is uniformly tight. (*Hint*: (i) $\gamma=\sup_{\mu\in A}\mu X$ is finite; (ii) for any $\epsilon>0$ the set $\{\frac{1}{\mu X}\mu:\mu\in A,\ \mu X\ge\epsilon\}$ is narrowly compact, therefore uniformly tight; for any $\epsilon>0$ the set $\{\mu:\mu\in A,\ \mu X\ge\epsilon\}$ is uniformly tight.)

(x) Give ω_1 its order topology, and let M_t be the L-space of signed tight Borel measures on ω_1. (i) Show that ω_1 is a Prokhorov space. (ii) For $\xi<\omega_1$, define $\mu_\xi\in M_t$ by setting $\mu_\xi(E)=\chi E(\xi)-\chi E(\xi+1)$ for every Borel set $E\subseteq\omega_1$. Show that $\{\mu_\xi:\xi<\omega_1\}$ is relatively compact in M_t for the vague topology, but is not uniformly tight. (Compare 437Yy below.)

437Y Further exercises (a) Let X be a set and U a Riesz subspace of $\mathbb{R}^X$. Give formulae for the components of a given element of $U^\sim$ in the bands $U^\sim_\sigma$, $(U^\sim_\sigma)^\perp$, $U^\sim_\tau$ and $(U^\sim_\tau)^\perp$. (*Hint*: 356Yb.)

(b) Let X be a compact Hausdorff space. Show that the dual $C(X;\mathbb{C})^*$ of the complex linear space of continuous functions from X to $\mathbb{C}$ can be identified with the space of 'complex tight Borel measures' on X, that is, the space of functionals $\mu:\mathcal{B}(X)\to\mathbb{C}$ expressible as a complex linear combination of tight totally finite Borel measures; explain how this may be identified, as Banach space, with the complexification of the L-space M_t of signed tight Borel measures as described in 354Yl. Show that the complex Banach space $\mathcal{L}^\infty_{\mathbb{C}}(\mathcal{B}(X))$ is canonically embedded in $C(X;\mathbb{C})^{**}$.

(c) Write μ_c for counting measure on $[0,1]$, and μ_L for Lebesgue measure; write $\mu_c\times\mu_L$ for the product measure on $[0,1]^2$, and μ for the direct sum of μ_c and $\mu_c\times\mu_L$. Show that the L-space $C([0,1])^\sim$ is isomorphic, as L-space, to $L^1(\mu)$. (*Hint*: every Radon measure on $[0,1]$ has countable Maharam type.)

(d) Let X be a set, U a Riesz subspace of $\ell^\infty(X)$ containing the constant functions, and Σ the smallest σ-algebra of subsets of X with respect to which every member of U is measurable. Write $\tilde\Sigma$ for the intersection of the domains of the completions of the totally finite measures with domain Σ. Show that there is a unique sequentially order-continuous norm-preserving Riesz homomorphism from $\mathcal{L}^\infty(\tilde\Sigma)$ to $(U^\sim_\sigma)^*\cong M^*_\sigma$ such that $(Su)(f)=f(u)$ whenever $u\in U$ and $f\in U^\sim_\sigma$.

(e) Show that in 437Ib the operator $S:\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})\to C_0(X)^{**}$ is multiplicative if $C_0(X)^{**}$ is given the Arens multiplication described in 4A6O based on the ordinary multiplication $(u,v)\mapsto u\times v$ on $C_0(X)$.

(f) Explain how to express the proof of 285L(iii)$\Rightarrow$(ii) as (α) a proof that if the characteristic functions of a sequence $\langle\nu_n\rangle_{n\in\mathbb{N}}$ of Radon probability measures on $\mathbb{R}^r$ converge pointwise to a characteristic function, then $\{\nu_n:n\in\mathbb{N}\}$ is uniformly tight (β) the observation that any subalgebra of $C_b(\mathbb{R}^r)$ which separates the points of $\mathbb{R}^r$ and contains the constant functions will define the vague topology on any vaguely compact set of measures.

(g) Let X be a topological space. (i) Let $\mathcal{B}a$ be the Baire σ-algebra of X, $M_\sigma(\mathcal{B}a)$ the space of signed Baire measures on X, and $u : X \to \mathbb{R}$ a bounded Baire measurable function. Show that we have a linear functional $\mu \mapsto \int u\,d\mu : M_\sigma(\mathcal{B}\mathfrak{a}) \to \mathbb{R}$ agreeing with ordinary integration with respect to non-negative measures. Show that this functional is Baire measurable with respect to the vague topology on $M_\sigma(\mathcal{B}\mathfrak{a})$. (ii) Let M_τ be the space of signed τ-additive Borel measures on X, and $u : X \to \mathbb{R}$ a bounded Borel measurable function. Show that we have a linear functional $\mu \mapsto \int u\,d\mu : M_\tau \to \mathbb{R}$ agreeing with ordinary integration with respect to non-negative measures. Show that this functional is Borel measurable with respect to the vague topology on M_τ.

(h) Let X be a topological space, μ_0 a totally finite τ-additive topological measure on X, and $u : X \to \mathbb{R}$ a bounded function which is continuous μ_0-a.e. Let $\tilde{M}^+_\sigma$ be the set of totally finite topological measures on X, with its narrow topology. Show that $\nu \mapsto \overline{\int} u\,d\nu : \tilde{M}^+_\sigma \to \mathbb{R}$ is continuous at μ_0.

(i) Let X be a Hausdorff space, and $M^{\infty+}_{\mathrm{R}}$ the set of all Radon measures on X. Define addition and scalar multiplication (by positive scalars) on $M^{\infty+}_{\mathrm{R}}$ as in 234G and 234Xf, and $\le$ by the formulae of 234P. (i) Show that there is a Dedekind complete Riesz space V such that the positive cone of V is isomorphic to $M^{\infty+}_{\mathrm{R}}$. (ii) Show that every principal band in V is an L-space. (iii) Show that if X is first-countable then V is perfect.

(j) For a topological space X let $M_\tau(X)$ be the L-space of signed τ-additive Borel measures on X, and $\psi : M_\tau(X) \times M_\tau(X) \to M_\tau(X \times X)$ the canonical bilinear operator (437M); give $M_\tau(X)$ and $M_\tau(X \times X)$ their vague topologies. (i) Show that if $X = [0,1]$ then ψ is not continuous. (ii) Show that $X = \mathbb{N}$ and B is the unit ball of $M_\tau(X)$ then $\psi{\restriction}B \times B$ is not continuous.

(k) Let X and Y be topological spaces, and $\psi : M_\tau(X) \times M_\tau(Y) \to M_\tau(X \times Y)$ the bilinear map of 437Mc. Write $M_t(X)$, etc., for the spaces of signed tight Borel measures. (i) Show that $\psi(\mu,\nu) \in M_t(X \times Y)$ for every $\mu \in M_t(X)$, $\nu \in M_t(Y)$. (ii) Show that if $B \subseteq M_t(X)$, $B' \subseteq M_t(Y)$ are norm-bounded and uniformly tight, then $\psi{\restriction}B \times B'$ is continuous for the vague topologies.

(l) Let X be a topological space, and $\tilde{M}$ the space of bounded additive functionals defined on subalgebras of $\mathcal{P}X$ containing every open set. For $\nu \in \tilde{M}$, say that $|\nu|(E) = \sup\{\nu F - \nu(E \setminus F) : F \in \operatorname{dom}\nu,\ F \subseteq E\}$ for $E \in \operatorname{dom}\nu$. Show that a set $A \subseteq \tilde{M}$ is uniformly tight in the sense of 437O iff every member of A is tight and $\{|\nu| : \nu \in A\}$ is uniformly tight.

(m) Let X be a completely regular space and P_{qR} the space of quasi-Radon probability measures on X. Let $B \subseteq P_{\mathrm{qR}}$ be a non-empty set. Show that the following are equiveridical: (i) B is relatively compact in P_{qR} for the narrow topology; (ii) whenever $A \subseteq C_b(X)$ is non-empty and downwards-directed and $\inf_{u\in A} u(x) = 0$ for every $x \in A$, then $\inf_{u\in A}\sup_{\mu\in B}\int u\,d\mu = 0$; (iii) whenever $\mathcal{G}$ is an upwards-directed family of open sets with union X, then $\sup_{G\in\mathcal{G}}\inf_{\mu\in B}\mu G = 1$.

(n) (i) Let X be a regular topological space, and M^+_{qR} the space of totally finite quasi-Radon measures on X, with its narrow topology. Show that M^+_{qR} is regular. (ii) Find a second-countable Hausdorff space X such that the space P_{qR} of quasi-Radon probability measures on X is not Hausdorff in its narrow topology.

(o) Let (X,ρ) and (Y,σ) be metric spaces, and give $M^+_{\mathrm{qR}}(X)$ and $M^+_{\mathrm{qR}}(Y)$ the corresponding metrics ρ_{KR}, σ_{KR} as in 437Rg. For a continuous function $\phi : X \to Y$, let $\tilde{\phi} : M^+_{\mathrm{qR}}(X) \to M^+_{\mathrm{qR}}(Y)$ be the map described in 437Xm. (i) Show that if ϕ is γ-Lipschitz, where $\gamma \ge 1$, then $\tilde{\phi}$ is γ-Lipschitz. (ii) (J.Pachl) Show that if ϕ is uniformly continuous, then $\tilde{\phi}$ is uniformly continuous on any uniformly totally finite subset of $M^+_{\mathrm{qR}}(X)$. (iii) Show that if (X,ρ) is $\mathbb{R}$ with its usual metric, then ρ_{KR} is not uniformly equivalent to Lévy's metric as described in 274Yc[6]. (For a discussion of various metrics related to ρ_{KR}, see Bogachev 07, 8.10.43-8.10.48.)

(p) Let (X,ρ) be a metric space. For $f \in C_b(X)^\sim_\sigma$, set $\|f\|_{\mathrm{KR}} = \sup\{|f(u)| : u \in C_b(X),\ \|u\|_\infty \le 1,\ u$ is 1-Lipschitz$\}$. (i) Show that $\|\ \|_{\mathrm{KR}}$ is a norm on $C_b(X)^\sim_\sigma$. (ii) Let (X',ρ') and (X'',ρ'') be metric spaces, and ρ the ℓ^1-product metric on $X = X' \times X''$ defined by saying that $\rho((x',x''),(y',y'')) = \rho'(x',y') + \rho''(x'',y'')$. Identifying the spaces $M_\tau(X')$, $M_\tau(X'')$ and $M_\tau(X)$ of signed τ-additive Borel measures with subspaces of $C_b(X')^\sim_\sigma$, $C_b(X'')^\sim_\sigma$ and $C_b(X')^\sim_\sigma$, as in 437E-437H, show that the bilinear map $\psi : M_\tau(X') \times M_\tau(X'') \to M_\tau(X)$ described in 437Mc has norm 1 when $M_\tau(X')$, $M_\tau(X'')$ and $M_\tau(X)$ are given the appropriate norms $\|\ \|_{\mathrm{KR}}$.

[6]Formerly 274Ya.

(q) Let X be a topological space and $\tilde{M}^+$ the set of non-negative real-valued additive functionals defined on algebras of subsets of X containing every open set, endowed with its narrow topology. Show that the weight $w(\tilde{M}^+)$ of $\tilde{M}^+$ is at most $\max(\omega, w(X))$.

(r) Let X be a Čech-complete completely regular Hausdorff space and P_{R} the set of Radon probability measures on X, with its narrow topology. Show that P_{R} is Čech-complete.

(s) Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, and $\mathcal{F}$ an ultrafilter on I. Let $(\mathfrak{A}, \bar{\mu}) = \prod_{i\in I}(\mathfrak{A}_i, \bar{\mu}_i)|\mathcal{F}$ be the reduced product as defined in 328C. For each $i \in I$, let (Z_i, ν_i) be the Stone space of $(\mathfrak{A}_i, \bar{\mu}_i)$; give $W = \{(z, i) : i \in I,\, z \in Z_i\}$ its disjoint union topology, and let βW be the Stone-Čech compactification of W. For each $i \in I$, define $\phi_i : Z_i \to W \subseteq \beta W$ by setting $\phi_i(z) = (z, i)$ for $z \in Z_i$, and let $\nu_i\phi_i^{-1}$ be the image measure on βW. Let ν be the limit $\lim_{i\to\mathcal{F}} \nu_i\phi_i^{-1}$ for the narrow topology on the space of Radon probability measures on βW, and Z its support. Show that (Z, ν) can be identified with the Stone space of $(\mathfrak{A}, \bar{\mu})$.

(t) (i) Show that there are a continuous $\phi : \{0,1\}^{\mathbb{N}} \to [0,1]$ and a positive linear operator $T : C(\{0,1\}^{\mathbb{N}}) \to C([0,1])$ such that $T(u\phi) = u$ for every $u \in C([0,1])$. (*Hint*: if $I_\sigma = \{x : \sigma \subseteq x \in \{0,1\}^{\mathbb{N}}$ for $\sigma \in \bigcup_{n\in\mathbb{N}}\{0,1\}^n$, arrange that $\{t : T(\chi I_\sigma)(t) > 0\}$ is always an interval of length $(\frac{2}{3})^{\#(\sigma)}$.) (ii) Show that there are a continuous $\tilde{\phi} : (\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ and a positive linear operator $\tilde{T} : C((\{0,1\}^{\mathbb{N}})^{\mathbb{N}}) \to C([0,1]^{\mathbb{N}})$ such that $\tilde{T}(h\tilde{\phi}) = h$ for every $h \in C([0,1]^{\mathbb{N}})$. (*Hint*: if, in (i), $(Tg)(t) = \int g\,d\nu_t$ for $t \in [0,1]$ and $g \in C(\{0,1\}^{\mathbb{N}})$, take $\nu_{\boldsymbol{t}}$ to be the product measure $\prod_{n\in\mathbb{N}} \nu_{t_n}$ for $\boldsymbol{t} = \langle t_n\rangle_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$.)

(u) Let X be a separable metrizable space and $P = P_{\mathrm{R}}(X)$ the set of Radon probability measures on X, with its narrow topology. Show that there is a family $\langle f_\mu\rangle_{\mu\in P}$ of functions from $[0,1]$ to X such that (i) $(\mu, t) \mapsto f_\mu(t)$ is Borel measurable (ii) writing μ_L for Lebesgue measure on $[0,1]$, $\mu = \mu_L f_\mu^{-1}$ for every $\mu \in P$ (iii) whenever $\langle\mu_n\rangle_{n\in\mathbb{N}}$ is a sequence in P converging to $\mu \in P$, there is a countable set $A \subseteq [0,1]$ such that $f_\mu(t) = \lim_{n\to\infty} f_{\mu_n}(t)$ for every $t \in [0,1] \setminus A$. (*Hint*: first consider the cases $X = [0,1]$ and $X = \{0,1\}^{\mathbb{N}}$, then use 437Yt to deal with $[0,1]^{\mathbb{N}}$ and its subspaces. See Bogachev 07, §8.5.)

(v) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $\mathfrak{A}^f$ the ideal of elements of $\mathfrak{A}$ of finite measure. For $a \in \mathfrak{A}^f$ and $u \in L^0 = L^0(\mathfrak{A})$, let ν_{au} be the totally finite Radon measure on $\mathbb{R}$ defined by saying that $\nu_{au}(E) = \bar{\mu}(a \cap [\![u \in E]\!])$ (definition: 364G, 434T) for Borel sets $E \subseteq \mathbb{R}$. For $a \in \mathfrak{A}^f$ and u, $v \in L^0$ set $\bar{\rho}_a(u, v) = \rho_{KR}(\nu_{au}, \nu_{av})$, where ρ_{KR} is the metric on $M_{\mathrm{R}}^+ = M_{\mathrm{R}}^+(\mathbb{R})$ defined from the usual metric on $\mathbb{R}$. (i) Show that the family $\mathrm{P} = \{\bar{\rho}_a : a \in \mathfrak{A}^f\}$ of pseudometrics defines the topology of convergence in measure on L^0 (definition: 367L). (ii) Show that if $(\mathfrak{A}, \bar{\mu})$ is semi-finite then the uniformity $\mathcal{U}$ defined from P is metrizable iff $(\mathfrak{A}, \bar{\mu})$ is σ-finite and $\mathfrak{A}$ has countable Maharam type. (iii) Show that if $(\mathfrak{A}, \bar{\mu})$ is semi-finite then L^0 is complete under $\mathcal{U}$ (definition: 3A4F) iff $\mathfrak{A}$ is purely atomic. (*Hint*: if $(\mathfrak{A}, \bar{\mu})$ is an atomless probability algebra and $\langle c_n\rangle_{n\in\mathbb{N}}$ is an independent sequence of elements of $\mathfrak{A}$ of measure $\frac{1}{2}$, show that $\langle\nu_{a,\chi c_n}\rangle_{n\in\mathbb{N}}$ is convergent in M_{R}^+ for every $a \in \mathfrak{A}$, so $\langle\chi c_n\rangle_{n\in\mathbb{N}}$ is $\mathcal{U}$-Cauchy.)

(w) Let X be a non-empty compact metrizable space, and $\phi : X \to X$ a continuous function. Show that there is an $x \in X$ such that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n u(\phi^i(x))$ is defined for every $u \in C(X)$. (*Hint*: 372H[7], 4A2Pe.)

(x)(i) Let X be a metrizable space, and A a narrowly compact subset of the set of Radon probability measures on X. Show that there is a separable subset Y of X which is conegligible for every measure in A. (ii) Show that a metrizable space is Prokhorov iff all its closed separable subspaces are Prokhorov.

(y) I say that a completely regular Hausdorff space X is **strongly Prokhorov** if every vaguely compact subset of the space $M_t(X)$ of signed tight Borel measures on X is uniformly tight. (i) Check that a strongly Prokhorov completely regular Hausdorff space is Prokhorov. (ii) Show that a closed subspace of a strongly Prokhorov completely regular Hausdorff space is strongly Prokhorov. (iii) Show that the product of a countable family of strongly Prokhorov completely regular Hausdorff spaces is strongly Prokhorov. (iv) Show that a G_δ subset of a strongly Prokhorov metrizable space is strongly Prokhorov. (v) Show that if (X, ρ) is a complete metric space then X is strongly Prokhorov.

[7] Formerly 372I.

(z) Let X be a regular Hausdorff topological space and C a non-empty narrowly compact set of totally finite topological measures on X, all inner regular with respect to the closed sets. Set $c(A) = \sup_{\mu\in C}\mu^*A$ for $A \subseteq X$. Show that $c : \mathcal{P}X \to [0, \infty[$ is a Choquet capacity.

437 Notes and comments The ramifications of the results here are enormous. For completely regular topological spaces X, the theorems of §436 give effective descriptions of the totally finite Baire, quasi-Radon and Radon measures on X as linear functionals on $C_b(X)$ (436E, 436Xl, 436Xn). This makes it possible, and natural, to integrate the topological measure theory of X into functional analysis, through the theory of $C_b(X)^*$. (See Wheeler 83 for an extensive discussion of this approach.) For the rest of this volume we shall never be far away from such considerations. In 437C-437I I give only a sample of the results, heavily slanted towards the abstract theory of Riesz spaces in Chapter 35 and the first part of Chapter 36.

Note that while the constructions of the dual spaces $U^\sim$, $U^\sim_c$ and $U^\times$ are 'intrinsic' to a Riesz space U, in that we can identify these functions as soon as we know the linear and order structure of U, the spaces $U^\sim_\sigma$ and $U^\sim_\tau$ are definable only when U is presented as a Riesz subspace of $\mathbb{R}^X$. In the same way, while the space $M_\sigma(\Sigma)$ of countably additive functionals on a σ-algebra Σ depends only on the Boolean algebra structure, the spaces M_τ here (not to be confused with the space of completely additive functionals considered in 362B) depend on the topology as well as the Borel algebra. (For an example in which radically different topologies give rise to the same Borel algebra, see Juhász Kunen & Rudin 76.)

You may have been puzzled by the shift from 'quasi-Radon' measures in 436H to 'τ-additive' measures in 437H; somewhere the requirement of inner regularity has got lost. The point is that the topologies being considered here, being defined by declaring certain families of functions continuous, are (completely) regular; so that τ-additive measures are necessarily inner regular with respect to the closed sets (414Mb).

The theory of 'vague' and 'narrow' topologies in 437J-437V here hardly impinges on the questions considered in §§274 and 285, where vague topologies first appeared. This is because the earlier investigation was dominated by the very special position of the functions $x \mapsto e^{iy\cdot x}$ (what we shall in §445 come to call the 'characters' of the additive groups of $\mathbb{R}$ or $\mathbb{R}^r$). One idea which does appear essentially in the proof of 285L, and has a natural interpretation in the general theory, is that of a 'uniformly tight' family of Radon measures (437O). In 445Yh below I set out a generalization of 285L to abelian locally compact groups.

In §461 I will return to the general theory of extreme points in compact convex sets. Here I remark only that it is never surprising that extreme points should be special in some way, as in 437S and 461Q-461R; but the precise ways in which they are special are often unexpected. A good deal of work has been done on relationships between the topological properties of a topological space X and the space P_{R} of Radon probability measures on X with the narrow topology. Here I give only a sample of basic facts in 437R and 437Yq-437Yr. Having observed that $M^+_{\mathrm{qR}}(X)$ is metrizable whenever X is (437Rg), it is natural to seek ways of defining a metric on $M^+_{\mathrm{qR}}(X)$ from a metric on X. The Kantorovich-Rubinstein metric ρ_{KR} I have chosen here is only one of many possibilities; compare the Lévy metric of 274Yc and the metric ρ_{W} of 457K below. Note that it can make a difference whether we look at quasi-Radon (or τ-additive) probability measures, or at general Borel measures (438Yl).

The terms 'vague' and 'narrow' both appear in the literature on this topic, and I take the opportunity to use them both, meaning slightly different things. Vague topologies, in my usage, are linear space topologies on linear spaces of functionals; narrow topologies are topologies on spaces of (finitely additive) measures, which are not linear spaces, though we can in some cases define addition and multiplication by non-negative scalars (437N, 437Xr, 437Yi). I must warn you that this distinction is not standard. I see that the word 'narrow' appears above a good deal oftener than the word 'vague', which is in part a reflection of a simple prejudice against signed measures; but from the point of view of this treatise as a whole, it is more natural to work with a concept well adapted to measures with variable domains, even if we are considering questions (like compactness of sets of measures) which originate in linear analysis. I should mention also that the definition in 437Jc includes a choice. The duality considered there uses the space $C_b(X)$; for locally compact X, we have the rival spaces $C_0(X)$ and $C_k(X)$ (see 436J and 436K), and there are occasions when one of these gives a more suitable topology on a space of measures (as in 495Xl below).

The elementary theory of uniform tightness and Prokhorov spaces (437O-437V) is both pretty and useful. The emphasis I give it here, however, is partly because it provides the background to a remarkable construction by D.Preiss (439S below), showing that $\mathbb{Q}$ is not a Prokhorov space.

438 Measure-free cardinals

At several points in §418, and again in §434, we had theorems about separable metrizable spaces in which the proofs undoubtedly needed some special property of these spaces (e.g., the fact that they are Lindelöf), but left it unclear whether something more general could be said. When we come to investigate further, asking (for instance) whether complete metric spaces in general are Radon (438H), we find ourselves once again approaching the Banach-Ulam problem, already mentioned at several points in previous volumes, and in particular in 363S. It seems to be undecidable, in ordinary set theory with the axiom of choice, whether or not every discrete space is Radon in the sense of 434C. On the other hand it is known that discrete spaces of cardinal at most ω_{ω_1} (for instance) are indeed always Radon. While as a rule I am deferring questions of this type to Volume 5, this particular phenomenon is so pervasive that I think it is worth taking a section now to clarify it.

The central definition is that of 'measure-free cardinal' (438A), and the basic results are 438B-438D. In particular, 'small' infinite cardinals are measure-free (438C). From the point of view of measure theory, a metrizable space whose weight is measure-free is almost separable, and most of the results in §418 concerning separable metrizable spaces can be extended (438E-438G). In fact 'measure-free weight' exactly determines whether a metrizable space is measure-compact (438J, 438Xm) and whether a complete metric space is Radon (438H). If $\mathfrak{c}$ is measure-free, some interesting spaces of functions are Radon (438T). I approach these last spaces through the concept of 'hereditary weak θ-refinability' (438K), which enables us to do most of the work without invoking any special axiom.

438A Measure-free cardinals: Definition A cardinal κ is **measure-free** or **of measure zero** if whenever μ is a probability measure with domain $\mathcal{P}\kappa$ then there is a $\xi < \kappa$ such that $\mu\{\xi\} > 0$. In 363S I discussed some statements equiveridical with the assertion 'every cardinal is measure-free'.

438B It is worth getting some basic facts out into the open immediately.

Lemma Let (X, Σ, μ) be a semi-finite measure space and $\langle E_i \rangle_{i \in I}$ a point-finite family of subsets of X such that $\#(I)$ is measure-free and $\bigcup_{i \in J} E_i \in \Sigma$ for every $J \subseteq I$. Set $E = \bigcup_{i \in I} E_i$.

(a) $\mu E = \sup_{J \subseteq I \text{ is finite}} \mu(\bigcup_{i \in J} E_i)$.

(b) If $\langle E_i \rangle_{i \in I}$ is disjoint, then $\mu E = \sum_{i \in I} \mu E_i$. In particular, if $\Sigma = \mathcal{P}X$ and $A \subseteq X$ has measure-free cardinal, then $\mu A = \sum_{x \in A} \mu\{x\}$.

(c) If μ is σ-finite, then $L = \{i : i \in I,\ \mu E_i > 0\}$ is countable and $\bigcup_{i \in I \setminus L} E_i$ is negligible.

proof (a)(i) The first step is to show, by induction on n, that the result is true if $\mu X < \infty$ and every E_i is negligible and $\#(\{i : i \in I,\ x \in E_i\}) \le n$ for every $x \in X$. If $n = 0$ this is trivial, since every E_i must be empty. For the inductive step to $n \ge 1$, define $\nu : \mathcal{P}I \to [0, \infty[$ by setting $\nu J = \mu(\bigcup_{i \in J} E_i)$ for every $J \subseteq I$. Then ν is a measure on I. **P** Write $F_J = \bigcup_{i \in J} E_j$ for $J \subseteq I$. (α) If J, $K \subseteq I$ are disjoint, then for $i \in I$ set $E'_i = E_i \cap F_K$ for $i \in J$, $\emptyset$ for $i \in I \setminus J$. In this case, $\langle E'_i \rangle_{i \in I}$ is a family of negligible subsets of X, $\bigcup_{i \in J'} E'_i = F_{J' \cap J} \cap F_K$ is measurable for every $J' \subseteq I$, and $\#(\{i : x \in E'_i\}) \le n - 1$ for every $x \in X$; so the inductive hypothesis tells us that

$$\mu(\bigcup_{i \in I} E'_i) = \sup_{J' \subseteq I \text{ is finite}} \mu(\bigcup_{i \in J'} \mu E'_i) = 0,$$

that is, $F_J \cap F_K$ is negligible. But this means that

$$\nu(J \cup K) = \mu F_{J \cup K} = \mu(F_J \cup F_K) = \mu F_J + \mu F_K = \nu J + \nu K.$$

As J and K are arbitrary, ν is additive. (β) If $\langle J_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\mathcal{P}I$, then

$$\nu(\bigcup_{n \in \mathbb{N}} J_n) = \mu(\bigcup_{n \in \mathbb{N}} F_{J_n}) = \lim_{n \to \infty} \mu(\bigcup_{m \le n} F_{J_m})$$
$$= \lim_{n \to \infty} \sum_{m=0}^{n} \nu J_m = \sum_{n=0}^{\infty} \nu J_n,$$

so ν is countably additive and is a measure. **Q**

At the same time, $\nu\{i\} = \mu E_i = 0$ for every i. Because $\#(I)$ is measure-free, $\nu I = 0$. **P?** Otherwise, let $f : I \to \kappa = \#(I)$ be any bijection and set $\lambda A = \frac{1}{\nu I} \nu f^{-1}[A]$ for every $A \subseteq \kappa$; then λ is a probability measure with domain $\mathcal{P}\kappa$ which is zero on singletons, and κ is not measure-free. **XQ** But this means just that $\mu(\bigcup_{i \in I} E_i) = 0$. Thus the induction proceeds.

(ii) **?** Now suppose, if possible, that the general result is false. For finite sets $J \subseteq I$ set $F_J = \bigcup_{i\in J} E_i$, as before, and consider $\mathcal{E} = \{F_J : J \in [\kappa]^{<\omega}\}$ (see 3A1J for this notation). Then $\mathcal{E}$ is closed under finite unions and $\gamma = \sup_{H\in\mathcal{E}} \mu H$ is finite, because it is less than μE; let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{E}$ such that $\mu(H \setminus H^*) = 0$ for every $H \in \mathcal{E}$, where $H^* = \bigcup_{n\in\mathbb{N}} H_n$ and $\mu H^* = \gamma$ (215Ab).

Because μ is semi-finite, there is an $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$. For each $n \in \mathbb{N}$, set

$$Y_n = \{x : x \in F \setminus H^*, \#(\{i : x \in E_i\}) \le n\}.$$

Then there is some $n \in \mathbb{N}$ such that $\mu^* Y_n > 0$. Let ν be the subspace measure on Y_n, so that ν is non-zero and totally finite. Now $\langle E_i \cap Y_n\rangle_{i\in I}$ is a family of negligible subsets of Y_n, $\bigcup_{i\in J} E_i \cap Y_n = Y_n \cap \bigcup_{i\in J} E_i$ is measured by ν for every $J \subseteq I$, and $\#(\{i : x \in E_i \cap Y_n\}) \le n$ for every $x \in Y_n$. But this contradicts (i) above. **X**

This proves (a).

(b) If $\langle E_i\rangle_{i\in I}$ is disjoint, then

$$\sup_{J\in[I]^{<\omega}} \mu(\textstyle\bigcup_{i\in J} E_i) = \sup_{J\in[I]^{<\omega}} \sum_{i\in J} \mu E_i = \sum_{i\in I} \mu E_i.$$

Setting $I = A$, $E_x = \{x\}$ for $x \in A$, we get the special case.

(c) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of measurable sets of finite measure covering X. For each n, set $L_n = \{i : i \in I, \mu(E_i \cap X_n) \ge 2^{-n}\}$. **?** If L_n is infinite, take a sequence $\langle i_k\rangle_{k\in\mathbb{N}}$ of distinct elements in L_n, and consider $G_m = \bigcup_{k\ge m} E_{i_k}$ for $m \in \mathbb{N}$; then every G_m has measure at least 2^{-n}, G_0 has finite measure, $\langle G_m\rangle_{m\in\mathbb{N}}$ is non-increasing, and $\bigcap_{m\in\mathbb{N}} G_m$ is empty, because $\langle E_{i_k}\rangle_{k\in\mathbb{N}}$ is point-finite. But this is impossible. **X**

Thus every L_n is finite and $L = \bigcup_{n\in\mathbb{N}} L_n$ is countable.

Now (a), applied to $\langle E'_i\rangle_{i\in I}$ where $E'_i = E_i$ if $i \in I \setminus L$, $\emptyset$ if $i \in L$, tells us that $\bigcup_{i\in I\setminus L} E_i$ is negligible.

438C I do not think we are ready for the most interesting set-theoretic results concerning measure-free cardinals. But the following facts may help to make sense of the general pattern.

Theorem (ULAM 30) (a) ω is measure-free.

(b) If κ is a measure-free cardinal and $\kappa' \le \kappa$ is a smaller cardinal, then κ' is measure-free.

(c) If $\langle\kappa_\xi\rangle_{\xi<\lambda}$ is a family of measure-free cardinals, and λ also is measure-free, then $\kappa = \sup_{\xi<\lambda} \kappa_\xi$ is measure-free.

(d) If κ is a measure-free cardinal so is κ^+.

(e) The following are equiveridical:

(i) $\mathfrak{c}$ is not measure-free;

(ii) there is a semi-finite measure space $(X, \mathcal{P}X, \mu)$ which is not purely atomic;

(iii) there is a measure μ on $[0, 1]$ extending Lebesgue measure and measuring every subset of $[0, 1]$.

(f) If $\kappa \ge \mathfrak{c}$ is a measure-free cardinal then 2^κ is measure-free.

proof (a) This is trivial.

(b) If μ is a probability measure with domain $\mathcal{P}\kappa'$, set $\nu A = \mu(\kappa' \cap A)$ for every $A \subseteq \kappa$. Then ν is a probability measure with domain $\mathcal{P}\kappa$, so there is a $\xi < \kappa$ such that $\nu\{\xi\} > 0$; evidently $\xi < \kappa'$ and $\mu\{\xi\} > 0$.

(c) Let μ be a probability measure on κ with domain $\mathcal{P}\kappa$. Define $f : \kappa \to \lambda$ by setting $f(\alpha) = \min\{\xi : \alpha < \kappa_\xi\}$ for $\alpha < \kappa$. Then the image measure μf^{-1} is a probability measure on λ with domain $\mathcal{P}\lambda$, so there is a $\xi < \lambda$ such that $\mu f^{-1}[\{\xi\}] > 0$. Now $\mu\kappa_\xi > 0$. Applying 438Bb to $A = \kappa_\xi$, we see that there is an $\alpha < \kappa_\xi$ such that $\mu\{\alpha\} > 0$. As μ is arbitrary, κ is measure-free.

(d) By (a) and (b), we need consider only the case $\kappa \ge \omega$. **?** Suppose, if possible, that μ is a probability measure with domain $\mathcal{P}\kappa^+$ such that $\mu\{\alpha\} = 0$ for every $\alpha < \kappa^+$. For each $\alpha < \kappa^+$, choose an injection $f_\alpha : \alpha \to \kappa$. For $\beta < \kappa^+$, $\xi < \kappa$ set $A(\beta, \xi) = \{\alpha : \beta < \alpha < \kappa^+, f_\alpha(\beta) = \xi\}$. Then $\kappa^+ \setminus \bigcup_{\xi<\kappa} A(\beta, \xi) = \beta + 1$ has cardinal at most κ, which is measure-free, so $\mu(\beta + 1) = 0$ and $\mu(\bigcup_{\xi<\kappa} A(\beta, \xi)) > 0$. Also $\langle A(\beta, \xi)\rangle_{\xi<\kappa}$ is disjoint. There is therefore a $\xi_\beta < \kappa$ such that $\mu A(\beta, \xi_\beta) > 0$, by 438Bb. Now $\kappa^+ > \max(\omega, \kappa)$, so there must be an $\eta < \kappa$ such that $B = \{\beta : \xi_\beta = \eta\}$ is uncountable. In this case, however, $\langle A(\beta, \eta)\rangle_{\beta\in B}$ is an uncountable family of sets of measure greater than zero, and cannot be disjoint, because μ is totally finite (215B(iii)); but if $\alpha \in A(\beta, \eta) \cap A(\beta', \eta)$, where $\beta \ne \beta'$, then $f_\alpha(\beta) = f_\alpha(\beta') = \eta$, which is impossible, because f_α is supposed to be injective. **X**

So there is no such measure μ, and κ^+ is measure-free.

(e)(i)⇒(ii) Suppose that $\mathfrak{c}$ is not measure-free; let μ be a probability measure with domain $\mathcal{P}\mathfrak{c}$ such that $\mu\{\xi\} = 0$ for every $\xi < \mathfrak{c}$. Then μ is atomless. **P?** Suppose, if possible, that $A \subseteq \mathfrak{c}$ is an atom for μ. Let $f : \mathfrak{c} \to \mathcal{P}\mathbb{N}$ be a bijection. For each $n \in \mathbb{N}$, set $E_n = \{\xi : n \in f(\xi)\}$. Set $D = \{n : \mu(A \cap E_n) = \mu A\}$. Because A is an atom, $\mu(A \cap E_n) = 0$ for every $n \in \mathbb{N} \setminus D$. This means that $B = \bigcap_{n \in D} E_n \setminus \bigcup_{n \in \mathbb{N} \setminus D} E_n$ has measure $\mu A > 0$; but $f(\xi) = D$ for every $\xi \in B$, so $\#(B) \leq 1$, and $\mu\{\xi\} > 0$ for some ξ, contrary to hypothesis. **XQ**

So (ii) is true.

(ii)⇒(iii) Suppose that there is a semi-finite measure space $(X, \mathcal{P}X, \mu)$ which is not purely atomic. Then there is a non-negligible set $E \subseteq X$ which does not include any atom; let $F \subseteq E$ be a set of non-zero finite measure. If we take ν to be $\frac{1}{\mu F}\mu_F$, where μ_F is the subspace measure on F, then ν is an atomless probability measure with domain $\mathcal{P}F$. Consequently there is a function $g : F \to [0, 1]$ which is inverse-measure-preserving for ν and Lebesgue measure (343Cb). But this means that the image measure νg^{-1} is a measure defined on every subset of $[0, 1]$ which extends Lebesgue measure.

not-(i)⇒not-(iii) Conversely, if $\mathfrak{c}$ is measure-free, then any probability measure on $[0, 1]$ measuring every subset must give positive measure to some singleton, and cannot extend Lebesgue measure.

(f) We are supposing that $\kappa \geq \mathfrak{c}$ is measure-free, so, in particular, $\mathfrak{c}$ is measure-free. Let μ be a probability measure with domain $\mathcal{P}(2^\kappa)$. By (e), it cannot be atomless; let $E \subseteq 2^\kappa$ be an atom. Let $f : 2^\kappa \to \mathcal{P}\kappa$ be a bijection, and for $\xi < \kappa$ set $E_\xi = \{\alpha : \alpha < 2^\kappa, \xi \in f(\alpha)\}$; set $D = \{\xi : \xi < \kappa, \mu(E \cap E_\xi) = \mu E\}$. Note that $\mu(E \cap E_\xi)$ must be zero for every $\xi \in \kappa \setminus D$, so that $E \cap \{\alpha : \xi \in D \triangle f(\alpha)\}$ is always negligible. Consider

$$A_\xi = \{\alpha : \alpha \in E, \xi = \min(D \triangle f(\alpha))\}$$

for $\xi < \kappa$. Then $\langle A_\xi \rangle_{\xi < \kappa}$ is a disjoint family of negligible sets, so its union A is negligible, by 438Bb, because κ is measure-free. But $E \setminus A \subseteq f^{-1}[\{D\}]$ has at most one element, and is not negligible; so $\mu\{\alpha\} > 0$ for some α. As μ is arbitrary, 2^κ is measure-free.

Remark This extends the result of 419G, which used a different approach to show that ω_1 is measure-free.

We see from (d) above that $\omega_2, \omega_3, \dots$ are all measure-free; so, by (c), ω_ω also is; generally, if κ is any measure-free cardinal, so is ω_κ (438Xa). I ought to point out that there are more powerful arguments showing that any cardinal which is not measure-free must be enormous (see 541L in Volume 5). In this context, however, $\mathfrak{c} = 2^\omega$ can be 'large', at least in the absence of an axiom like the continuum hypothesis to locate it in the hierarchy $\langle \omega_\xi \rangle_{\xi \in \mathrm{On}}$; it is generally believed that it is consistent to suppose that $\mathfrak{c}$ is not measure-free.

438D I turn now to the contexts in which measure-free cardinals behave as if they were 'small'.

Proposition Let (X, Σ, μ) be a σ-finite measure space, Y a metrizable space with measure-free weight, and $f : X \to Y$ a measurable function. Then there is a closed separable set $Y_0 \subseteq Y$ such that $f^{-1}[Y_0]$ is conegligible; that is, there is a conegligible measurable set $X_0 \subseteq X$ such that $f[X_0]$ is separable.

proof Let $\mathcal{U}$ be a σ-disjoint base for the topology of Y (4A2L(g-ii)); express it as $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ where each $\mathcal{U}_n$ is a disjoint family of open sets. If $n \in \mathbb{N}$, $\#(\mathcal{U}_n) \leq w(Y)$ (4A2Db) is a measure-free cardinal (438Cb), and $\langle f^{-1}[U] \rangle_{U \in U_n}$ is a disjoint family in Σ such that $\bigcup_{u \in \mathcal{V}} f^{-1}[U] = f^{-1}[\bigcup \mathcal{V}]$ is measurable for every $\mathcal{V} \subseteq \mathcal{U}_n$; so 438Bc tells us that there is a countable set $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that

$$f^{-1}[\bigcup(\mathcal{U}_n \setminus \mathcal{V}_n)] = \bigcup_{U \in \mathcal{U}_n \setminus \mathcal{V}_n} f^{-1}[U]$$

is negligible. Set

$$Y_0 = Y \setminus \bigcup_{n \in \mathbb{N}} \bigcup(\mathcal{U}_n \setminus \mathcal{V}_n).$$

Then $f^{-1}[Y \setminus Y_0] = \bigcup_{n \in \mathbb{N}} f^{-1}[\bigcup(\mathcal{U}_n \setminus \mathcal{V}_n)]$ is negligible. On the other hand,

$$\{U \cap Y_0 : U \in \mathcal{U}\} \subseteq \{\emptyset\} \cup \{V \cap Y_0 : V \in \bigcup_{n \in \mathbb{N}} \mathcal{V}_n\}$$

is countable, and is a base for the subspace topology of Y_0 (4A2B(a-vi)); so Y_0 is second-countable and must be separable (4A2Oc).

Thus we have an appropriate Y_0. Now $X_0 = f^{-1}[Y_0]$ is conegligible and measurable and $f[X_0] \subseteq Y_0$ is separable (4A2P(a-iv)).

438E Proposition (cf. 418B) Let (X, Σ, μ) be a complete locally determined measure space.

(a) If Y is a topological space, Z is a metrizable space, $w(Z)$ is measure-free, and $f : X \to Y$, $g : X \to Z$ are measurable functions, then $x \mapsto (f(x), g(x)) : X \to Y \times Z$ is measurable.

(b) If $\langle Y_n\rangle_{n\in\mathbb{N}}$ is a sequence of metrizable spaces, with product Y, $w(Y_n)$ is measure-free for every $n\in\mathbb{N}$, and $f_n:X\to Y_n$ is measurable for every $n\in\mathbb{N}$, then $x\mapsto f(x)=\langle f_n(x)\rangle_{n\in\mathbb{N}}:X\to\prod_{n\in\mathbb{N}}Y_n$ is measurable.

proof (a)(i) Consider first the case in which μ is totally finite. Then there is a conegligible set $X_0\subseteq X$ such that $g[X_0]$ is separable (438D). Applying 418Bb to $f\restriction X_0$ and $g\restriction X_0$, we see that $x\mapsto(f(x),g(x)):X_0\to Y\times g[Z_0]$ is measurable. As μ is complete, it follows that $x\mapsto(f(x),g(x)):X\to Y\times Z$ is measurable.

(ii) In the general case, take any open set $W\subseteq Y\times Z$ and any measurable set $F\subseteq X$ of finite measure. Set $Q=\{x:(f(x),g(x))\in W\}$. By (i), applied to $f\restriction F$ and $g\restriction F$, $F\cap Q\in\Sigma$; as F is arbitrary and μ is locally determined, $Q\in\Sigma$; as W is arbitrary, $x\mapsto(f(x),g(x))$ is measurable.

(b) As in (a), it is enough to consider the case in which μ is totally finite. In this case, we have for each $n\in\mathbb{N}$ a conegligible set X_n such that $f_n[X_n]$ is separable. Set $X'=\bigcap_{n\in\mathbb{N}}X_n$; then 418Bd tells us that $f\restriction X'$ is measurable, so that f is measurable.

438F Proposition (cf. 418J) Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a metrizable space, $w(Y)$ is measure-free and $f:X\to Y$ is measurable. Then f is almost continuous.

proof Take $E\in\Sigma$ and $\gamma<\mu E$. Then there is a measurable set $F\subseteq E$ such that $\gamma<\mu F<\infty$. Let $F_0\subseteq F$ be a measurable set such that $F\setminus F_0$ is negligible and $f[F_0]$ is separable (438D). By 412Pc, the subspace measure on F_0 is still inner regular with respect to the closed sets, so $f\restriction F_0$ is almost continuous (418J), and there is a measurable set $H\subseteq F_0$, of measure at least γ, such that $f\restriction H$ is continuous. As E and γ are arbitrary, f is almost continuous.

438G Corollary (cf. 418K) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space and Y a metrizable space such that $w(Y)$ is measure-free. Then a function $f:X\to Y$ is measurable iff it is almost continuous.

438H Now let us turn to questions which arose in §434.

Proposition A complete metric space is Radon iff its weight is measure-free.

proof Let (X,ρ) be a complete metric space, and $\kappa=w(X)$ its weight.

(a) If κ is measure-free, let μ be any totally finite Borel measure on X. Applying 438D to the identity map from X to itself, we see that there is a closed separable conegligible subspace X_0. Now X_0 is complete, so is a Polish space, and by 434Kb is a Radon space. The subspace measure μ_{X_0} is therefore tight (that is, inner regular with respect to the compact sets); as X_0 is conegligible, it follows at once that μ also is. As μ is arbitrary, X is a Radon space.

(b) If κ is not measure-free, take any σ-disjoint base $\mathcal{U}$ for the topology of X. Express $\mathcal{U}$ as $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ where every $\mathcal{U}_n$ is disjoint. Then $\kappa\le\#(\mathcal{U})$ and there is a probability measure ν on $\mathcal{U}$, with domain $\mathcal{PU}$, such that $\nu\{U\}=0$ for every $U\in\mathcal{U}$. Let $n\in\mathbb{N}$ be such that $\nu\mathcal{U}_n>0$. For each $U\in\mathcal{U}_n$ choose $x_U\in U$. For Borel sets $E\subseteq X$ set $\mu E=\nu\{U:U\in\mathcal{U}_n,\,x_U\in E\}$; then μ is a Borel measure on X and $\mu(\bigcup\mathcal{U}_n)=\nu\mathcal{U}_n>0$, while $\mu(\bigcup\mathcal{V})=\nu\mathcal{V}=0$ for every finite $\mathcal{V}\subseteq\mathcal{U}_n$. Thus μ is not τ-additive and cannot be tight, and X is not a Radon space.

438I Proposition Let X be a metrizable space and $\langle F_\xi\rangle_{\xi<\kappa}$ a non-decreasing family of closed subsets of X, where κ is a measure-free cardinal. Then

$$\mu(\bigcup_{\xi<\kappa}F_\xi)=\sum_{\xi<\kappa}\mu(F_\xi\setminus\bigcup_{\eta<\xi}F_\eta)$$

for every semi-finite Borel measure μ on X.

proof (a) I had better begin by remarking that $H_\xi=\bigcup_{\eta<\xi}F_\eta$ is an F_σ set for every ordinal $\xi\le\kappa$, by 4A2Ld and 4A2Ka. So, setting $E_\xi=F_\xi\setminus H_\xi$, $\sum_{\xi<\kappa}\mu E_\xi$ is defined.

(b) I show by induction on ζ that $\mu H_\zeta=\sum_{\xi<\zeta}\mu E_\xi$ for every $\zeta\le\kappa$. The induction starts trivially with $\mu H_0=0$. The inductive step to a successor ordinal $\zeta+1$ is also immediate, as $H_{\zeta+1}=H_\zeta\cup E_\zeta$. For the inductive step to a limit ordinal ζ of countable cofinality, let $\langle\zeta_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in ζ with supremum ζ; then

$$\mu H_\zeta=\sup_{n\in\mathbb{N}}\mu H_{\zeta_n}=\sup_{n\in\mathbb{N}}\sum_{\xi<\zeta_n}\mu E_\xi=\sum_{\xi<\zeta}\mu E_\xi,$$

as required.

(c) So we are left with the case in which ζ is a limit ordinal of uncountable cofinality. In this case, $\mu(E\cap H_\zeta)\le\sum_{\xi<\zeta}\mu E_\xi$ whenever μE is finite. **P** Let $\mathcal{U}$ be a σ-disjoint base for the topology of X (4A2L(g-ii)), and express $\mathcal{U}$ as $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ where each $\mathcal{U}_n$ is disjoint. For $n\in\mathbb{N}$, $\xi\le\zeta$ set

$$\mathcal{V}_{n\xi}=\{U:U\in\mathcal{U}_n,\,U\cap H_\xi\ne\emptyset\},\quad V_{n\xi}=\bigcup\mathcal{V}_{n\xi}.$$

Define $\phi_n:V_{n\zeta}\to\zeta$ by saying that $\phi_n(x)=\min(\{\xi:\xi<\zeta,\,x\in V_{n\xi}\})$. Then, for any $D\subseteq\zeta$,

$$\phi_n^{-1}[D]=\bigcup_{\xi\in D}\bigcup(\mathcal{V}_{n\xi}\setminus\bigcup_{\eta<\xi}\mathcal{V}_{n\eta})$$

is a union of members of $\mathcal{U}_n$, so is open. We therefore have a measure ν_n on $\mathcal{P}\zeta$ defined by saying that $\nu_nD=\mu(E\cap\phi_n^{-1}[D])$ for every $D\subseteq\zeta$. At this point, recall that we are supposing that κ is measure-free, so $\#(\zeta)$ also is measure-free (438Cb) and $\nu_n\zeta=\sum_{\xi<\zeta}\nu_n\{\xi\}=\sup_{\xi<\zeta}\nu_n\xi$ (438Bb). Interpreting this in X, we have $\mu(E\cap V_{n\zeta})=\sup_{\xi<\zeta}\mu(E\cap V_{n\xi})$.

This is true for every $n\in\mathbb{N}$. So there is a countable set $C\subseteq\zeta$ such that $\mu(E\cap V_{n\zeta})=\sup_{\xi\in C}\mu(E\cap V_{n\xi})$ for every $n\in\mathbb{N}$. Because $\operatorname{cf}\zeta>\omega$, there is an $\alpha<\zeta$ such that $C\subseteq\alpha$, and $\mu(E\cap V_{n\zeta})=\mu(E\cap V_{n\alpha})$, that is, $E\cap V_{n\zeta}\setminus V_{n\alpha}$ is negligible, for every $n\in\mathbb{N}$.

Now note that F_α is closed. So

$$\begin{aligned}H_\zeta\setminus F_\alpha&\subseteq\bigcup\{U:U\in\mathcal{U},\,H_\zeta\cap U\ne\emptyset,\,U\cap F_\alpha=\emptyset\}\\&=\bigcup_{n\in\mathbb{N}}V_{n\zeta}\setminus V_{n,\alpha+1},\end{aligned}$$

and $E\cap H_\zeta\setminus F_\alpha$ is negligible. Accordingly, using the inductive hypothesis,

$$\mu(E\cap H_\zeta)\le\mu F_\alpha=\mu H_{\alpha+1}\le\sum_{\xi\le\alpha}\mu E_\xi\le\sum_{\xi<\zeta}\mu E_\xi,$$

as claimed. **Q**

Because μ is semi-finite, and E is arbitrary, $\mu H_\zeta\le\sum_{\xi<\zeta}E_\xi$; but the reverse inequality is trivial, so we have equality, and the induction proceeds in this case also.

(d) At the end of the induction we have $\mu H_\kappa=\sum_{\xi<\kappa}\mu E_\xi$, as stated.

438J So far we have been looking at metrizable spaces, the obvious first step. But it turns out that the concept of 'metacompactness' leads to generalizations of some of the results above.

Proposition (Moran 70, Haydon 74) Let X be a metacompact space with measure-free weight.

(a) X is Borel-measure-compact.

(b) If X is normal, it is measure-compact.

(c) If X is perfectly normal (for instance, if it is metrizable), it is Borel-measure-complete.

proof (a) ? If X is not Borel-measure-compact, there are a non-zero totally finite Borel measure μ on X and a cover $\mathcal{G}$ of X by negligible open sets (434H(a-v)). Let $\mathcal{H}$ be a point-finite open cover of X refining $\mathcal{G}$. By 4A2Dc, $\#(\mathcal{H})$ is at most $\max(\omega,w(X))$, so is measure-free, by 438C. Because μ is a Borel measure, $\bigcup\mathcal{H}'$ is measurable for every $\mathcal{H}'\subseteq\mathcal{H}$; $\mu H=0$ for every $H\in\mathcal{H}$; while $\mu(\bigcup\mathcal{H})=\mu X>0$. But this contradicts 438Ba. **X**

(b) Now suppose that X is normal, and that μ is a totally finite Baire measure on X. Because a normal metacompact space is countably paracompact (4A2F(g-iii)), μ has an extension to a Borel measure μ_1 which is inner regular with respect to the closed sets, by Mařík's theorem (435C). Now μ_1 is τ-additive, by (a) above, so μ also is (411C). As μ is arbitrary, X is measure-compact.

(c) Since on a perfectly normal space the Baire and Borel measures are the same, X is Borel-measure-complete iff it is measure-compact, and we can use (b).

Remark The arguments here can be adapted in various ways, and in particular the hypotheses can be weakened; see 438Yd-438Yf.

438K Hereditarily weakly θ-refinable spaces A topological space X is **hereditarily weakly θ-refinable** (also called **hereditarily σ-relatively metacompact**, **hereditarily weakly submetacompact**) if for every family $\mathcal{G}$ of open subsets of X there is a σ-isolated family $\mathcal{A}$ of subsets of X, refining $\mathcal{G}$, such that $\bigcup\mathcal{A}=\bigcup\mathcal{G}$.

438L Lemma (a) Any subspace of a hereditarily weakly θ-refinable topological space is hereditarily weakly θ-refinable.

(b) A hereditarily metacompact space (e.g., any metrizable space, see 4A2Lb) is hereditarily weakly θ-refinable.

(c) A hereditarily Lindelöf space is hereditarily weakly θ-refinable.

(d) A topological space with a σ-isolated network is hereditarily weakly θ-refinable.

proof (a) If X is hereditarily weakly θ-refinable, Y is a subspace of X, and $\mathcal{H}$ is a family of open subsets of Y, set $\mathcal{G}=\{G : G\subseteq X \text{ is open}, G\cap Y\in\mathcal{H}\}$. Then there is a σ-isolated family $\mathcal{A}$, refining $\mathcal{G}$, with union $\bigcup\mathcal{G}$; and $\{A\cap Y : A\in\mathcal{A}\}$ is σ-isolated (4A2B(a-viii)), refines $\mathcal{H}$, and has union $\bigcup\mathcal{H}$. As $\mathcal{H}$ is arbitrary, Y is hereditarily weakly θ-refinable.

(b) If X is hereditarily metacompact, and $\mathcal{G}$ is any family of open sets in X with union W, then $\mathcal{G}$ is an open cover of the metacompact space W, so has a point-finite open refinement $\mathcal{H}$ with the same union. For each $x\in W$, set $\mathcal{H}_x=\{H : x\in H\in\mathcal{H}\}$, $V_x=\bigcap\mathcal{H}_x$, so that $\mathcal{H}_x$ is a non-empty finite set and V_x is an open set containing x. For $n\ge 1$, set $A_n=\{x : x\in W, \#(\mathcal{H}_x)=n\}$; then for any distinct x, $y\in A_n$, either $\mathcal{H}_x=\mathcal{H}_y$ and $V_x=V_y$, or $\#(\mathcal{H}_x\cup\mathcal{H}_y)>n$ and $V_x\cap V_y\cap A_n=\emptyset$. This means that $\mathcal{A}_n=\{V_x\cap A_n : x\in A_n\}$ is a partition of A_n into relatively open sets, and is an isolated family. Also, $\mathcal{A}_n$ is a refinement of $\mathcal{H}$ and therefore of $\mathcal{G}$; so $\bigcup_{n\ge 1}\mathcal{A}_n$ is a σ-isolated refinement of $\mathcal{G}$, and its union is $\bigcup_{n\ge 1}A_n=W$. As $\mathcal{G}$ is arbitrary, X is hereditarily weakly θ-refinable.

(c) If X is hereditarily Lindelöf and $\mathcal{G}$ is a family of open subsets of X, there is a countable $\mathcal{G}_0\subseteq\mathcal{G}$ with the same union; now $\mathcal{G}_0$, being countable, is σ-isolated. As $\mathcal{G}$ is arbitrary, X is hereditarily weakly θ-refinable.

(d) If X has a σ-isolated network $\mathcal{A}$, and $\mathcal{G}$ is any family of open subsets of X, then

$$\mathcal{E}=\{A : A\in\mathcal{A}, A \text{ is included in some member of } \mathcal{G}\}$$

is a σ-isolated family (4A2B(a-viii) again), refining $\mathcal{G}$, with union $\bigcup\mathcal{G}$.

438M Proposition (Gardner 75) If X is a hereditarily weakly θ-refinable topological space with measure-free weight, it is Borel-measure-complete.

proof Let μ be a Borel probability measure on X, and $\mathcal{G}$ the family of μ-negligible open sets. Let $\mathcal{A}$ be a σ-isolated family refining $\mathcal{G}$ with union $\bigcup\mathcal{G}$. Express $\mathcal{A}$ as $\bigcup_{n\in\mathbb{N}}\mathcal{A}_n$ where each $\mathcal{A}_n$ is an isolated family; for each $n\in\mathbb{N}$, set $X_n=\bigcup\mathcal{A}_n$ and let μ_n be the subspace measure on X_n. Then $\mathcal{A}_n$ is a disjoint family of relatively open μ_n-negligible sets; as $\#(\mathcal{A}_n)\le w(X_n)\le w(X)$ (4A2D) is measure-free, and μ_n is a totally finite Borel measure on X_n,

$$\mu^*X_n=\mu_nX_n=\mu_n(\bigcup\mathcal{A}_n)=0,$$

by 438Bb. Now $\mu(\bigcup\mathcal{G})=\mu^*(\bigcup_{n\in\mathbb{N}}X_n)=0$. As μ is arbitrary, X is Borel-measure-complete (434I(a-iv)).

438N For the next few paragraphs, I will use the following notation. Let X be a topological space and $\mathcal{G}$ a family of subsets of X. Then $\mathcal{J}(\mathcal{G})$ will be the family of subsets of X expressible as $\bigcup\mathcal{A}$ for some σ-isolated family $\mathcal{A}$ refining $\mathcal{G}$. Observe that X is hereditarily weakly θ-refinable iff $\bigcup\mathcal{G}$ belongs to $\mathcal{J}(\mathcal{G})$ for every family $\mathcal{G}$ of open subsets of X.

(a) $\mathcal{J}(\mathcal{G})$ is always a σ-ideal of subsets of X. **P** If $\mathcal{A}$ is a σ-isolated family of subsets of X, refining $\mathcal{G}$, and B is any set, then $\{B\cap A : A\in\mathcal{A}\}$ is still σ-isolated and still refines $\mathcal{G}$; so any subset of a member of $\mathcal{J}(\mathcal{G})$ belongs to $\mathcal{J}(\mathcal{G})$. If $\langle\mathcal{A}_n\rangle_{n\in\mathbb{N}}$ is a sequence of σ-isolated families refining $\mathcal{G}$, then $\bigcup_{n\in\mathbb{N}}\mathcal{A}_n$ is σ-isolated and refines $\mathcal{G}$; so the union of any sequence in $\mathcal{J}(\mathcal{G})$ belongs to $\mathcal{J}(\mathcal{G})$. **Q**

(b) If $\mathcal{H}$ refines $\mathcal{G}$, then $\mathcal{J}(\mathcal{H})\subseteq\mathcal{J}(\mathcal{G})$. **P** All we need to remember is that any family refining $\mathcal{H}$ also refines $\mathcal{G}$. **Q**

(c) If X and Y are topological spaces, $A\subseteq X$, $f:A\to Y$ is continuous, and $\mathcal{H}$ is a family of subsets of Y, set $\mathcal{G}=\{f^{-1}[H] : H\in\mathcal{H}\}$. Then $\mathcal{J}(\mathcal{G})\supseteq\{f^{-1}[B] : B\in\mathcal{J}(\mathcal{H})\}$. **P** If $B\in\mathcal{J}(\mathcal{H})$, there is a σ-isolated family $\mathcal{D}$ of subsets of Y, refining $\mathcal{H}$, and with union B. Now $\mathcal{A}=\{f^{-1}[D] : D\in\mathcal{D}\}$ refines $\mathcal{G}$ and has union $f^{-1}[B]$. We can express $\mathcal{D}$ as $\bigcup_{n\in\mathbb{N}}\mathcal{D}_n$, where each $\mathcal{D}_n$ is an isolated family; set $\mathcal{A}_n=\{f^{-1}[D] : D\in\mathcal{D}_n\}$, so that $\mathcal{A}=\bigcup_{n\in\mathbb{N}}\mathcal{A}_n$. For each n, $\mathcal{A}_n$ is disjoint, because $\mathcal{D}_n$ is. Moreover, if $D\in\mathcal{D}_n$, then $D=H\cap\bigcup\mathcal{D}_n$ for some open set $H\subseteq Y$, so that $f^{-1}[D]=f^{-1}[H]\cap\bigcup\mathcal{A}_n$ is relatively open in $\bigcup\mathcal{A}_n$; this shows that $\mathcal{A}_n$ is an isolated family. Accordingly $\mathcal{A}$ is σ-isolated and witnesses that $f^{-1}[B]\in\mathcal{J}(\mathcal{G})$. As B is arbitrary, we have the result. **Q**

(d) If X is a topological space, $\mathcal{G}$ is a family of subsets of X, and $\langle D_i\rangle_{i\in I}$ is an isolated family in $\mathcal{J}(\mathcal{G})$, then $\bigcup_{i\in I} D_i \in \mathcal{J}(\mathcal{G})$. **P** For each $i\in I$, let $\langle\mathcal{A}_{ni}\rangle_{n\in\mathbb{N}}$ be a sequence of isolated families, all refining $\mathcal{G}$, such that $D_i = \bigcup_{n\in\mathbb{N}}\bigcup\mathcal{A}_{in}$. Set $\mathcal{A}_n = \bigcup_{i\in I}\mathcal{A}_{in}$ for each n. Then $\mathcal{A}_n$ refines $\mathcal{G}$, and $\bigcup_{n\in\mathbb{N}}\bigcup\mathcal{A}_n = \bigcup_{i\in I} D_i$. It is easy to check that every $\mathcal{A}_n$ is isolated, so that $\bigcup_{n\in\mathbb{N}}\mathcal{A}_n$ witnesses that $\bigcup_{i\in I} D_i$ belongs to $\mathcal{J}(\mathcal{G})$. **Q**

438O Lemma Give $\mathbb{R}$ the topology $\mathfrak{S}$ generated by the closed intervals $]-\infty,t]$ for $t\in\mathbb{R}$, and let $r\ge 1$. Then $\mathbb{R}^r$, with the product topology corresponding to $\mathfrak{S}$, is hereditarily weakly θ-refinable.

proof Induce on r. Write $\le$ for the usual partial order of $\mathbb{R}^r$, and $]-\infty,x]$ for $\{y:y\le x\}$; set $V_A=\bigcup_{x\in A}\,]-\infty,x]$ for $A\subseteq\mathbb{R}^r$. The sets $]-\infty,x]$, as x runs over $\mathbb{R}^r$, form a base for the topology of $\mathbb{R}^r$.

The induction starts easily because $\mathfrak{S}$ itself is hereditarily Lindelöf. **P** If $\mathcal{G}\subseteq\mathfrak{S}$, set

$$A=\{x:x\in\mathbb{R},\text{ there is some }G\in\mathcal{G}\text{ such that }]-\infty,x]\subseteq G\}.$$

Then A has a countable cofinal set D, so that there is a corresponding countable subset of $\mathcal{G}$ with the same union as $\mathcal{G}$. **Q** By 438Lc, $\mathfrak{S}$ is hereditarily weakly θ-refinable.

For the inductive step to $r+1$, where $r\ge 1$, let $\mathcal{G}$ be a family of open subsets of $\mathbb{R}^{r+1}$, and set

$$A=\{x:x\in\mathbb{R}^r,\text{ there is some }G\in\mathcal{G}\text{ such that }]-\infty,x]\subseteq G\}.$$

For each $k\le r$, $q\in\mathbb{Q}$ set $K_k=(r+1)\setminus\{k\}$ and let $B_{kq}\subseteq\mathbb{R}^{K_k}$ be the set $\{z:z^\frown\langle q\rangle\in V_A\}$, writing $z^\frown\langle q\rangle$ for that member x of $\mathbb{R}^{r+1}$ such that $x\restriction K_k=z$ and $x(k)=q$. (I am thinking of members of $\mathbb{R}^{r+1}$ as functions from $r+1=\{0,\dots,r\}$ to $\mathbb{R}$.) Set $A_{kq}=\{x:x\in V_A,\ x(k)=q\}$, $\mathcal{G}_{kq}=\{]-\infty,x]:x\in A_{kq}\}$. Then, in the notation of 438N, $V_{A_{kq}}\in\mathcal{J}(\mathcal{G})$. **P** Set $f(x)=x\restriction K_k$ for each $x\in V_{A_{kq}}$, so that $f:V_{A_{kq}}\to\mathbb{R}^{K_k}$ is continuous. For $x\in A_{kq}$, $]-\infty,x]=f^{-1}[\,]-\infty,f(x)]\,]$. Now $\mathbb{R}^{K_k}$ is hereditarily weakly θ-refinable, by the inductive hypothesis, so if we set $\mathcal{H}_{kq}=\{]-\infty,f(x)]:x\in A_{kq}\}$, $\bigcup\mathcal{H}_{kq}\in\mathcal{J}(\mathcal{H}_{kq})$ and (by 438Nc)

$$V_{A_{kq}}=f^{-1}[\bigcup\mathcal{H}_{kq}]\in\mathcal{J}(\mathcal{G}_{kq})\subseteq\mathcal{J}(\mathcal{G}).\ \mathbf{Q}$$

Accordingly $W\in\mathcal{J}(\mathcal{G})$, where $W=\bigcup_{k\le r,q\in\mathbb{Q}}V_{A_{kq}}$, by 438Na.

Now consider $V_A\setminus W$. If x, $x'\in V_A\setminus W$ and $x\le x'$ then $x=x'$. **P?** Otherwise, there are a $k\le n$ and a $q\in\mathbb{Q}$ such that $x(k)\le q\le x'(k)$. In this case, setting $y\restriction K_k=x\restriction K_k$ and $y(k)=q$, we have $y\in A_{kq}$ and $x\in V_{A_{kq}}$. **XQ**

But this means that the subspace topology of $V_A\setminus W$ is discrete, so that $\{\{x\}:x\in V_A\setminus W\}$ is an isolated family covering $V_A\setminus W$ and refining $\mathcal{G}$; thus $V_A\setminus W\in\mathcal{J}(\mathcal{G})$ and $\bigcup\mathcal{G}=V_A$ belongs to $\mathcal{J}(\mathcal{G})$. As $\mathcal{G}$ is arbitrary, $\mathbb{R}^{r+1}$ is hereditarily weakly θ-refinable and the induction proceeds.

438P Lemma Let X be a Polish space, and $\tilde C^{\Downarrow}=\tilde C^{\Downarrow}(X)$ the family of functions $\omega:\mathbb{R}\to X$ such that $\lim_{s\uparrow t}\omega(s)$ and $\lim_{s\downarrow t}\omega(s)$ are defined in X for every $t\in\mathbb{R}$.

(a) For $A\subseteq B\subseteq\mathbb{R}$ and $f\in X^B$, set

$$\begin{aligned}\operatorname{jump}_A(f,\epsilon)=\sup\{n:\ &\text{there is an }I\in[A]^n\text{ such that }\rho(f(s),f(t))>\epsilon\\ &\text{whenever }s<t\text{ are successive elements of }I\}.\end{aligned}$$

Now a function $\omega\in X^{\mathbb{R}}$ belongs to $\tilde C^{\Downarrow}$ iff $\operatorname{jump}_{[-n,n]}(\omega,\epsilon)$ is finite for every $n\in\mathbb{N}$ and $\epsilon>0$.

(b) If $\omega\in\tilde C^{\Downarrow}$ then ω is continuous at all but countably many points of $\mathbb{R}$.

(c) If $\omega\in\tilde C^{\Downarrow}$ then $\omega[\,[-n,n]\,]$ is relatively compact in X for every $n\in\mathbb{N}$.

proof Fix on a complete metric ρ inducing the topology of X.

(a)(i) Suppose that $\omega\in\tilde C^{\Downarrow}$, $n\in\mathbb{N}$ and $\epsilon>0$. For every $t\in[-n,n]$, there is a $\delta_t>0$ such that $\rho(\omega(s),\omega(s'))\le\epsilon$ whenever either $t<s\le s'\le t+\delta_t$ or $t-\delta_t\le s\le s'<t$. Now there is an $m\ge1$ such that whenever s, $s'\in[-n,n]$ and $|s-s'|\le\frac{2n}{m}$ there is a $t\in[-n,n]$ such that both s and s' belong to $[t-\delta_t,t+\delta_t]$. Suppose now that $-n\le t_0<t_1<\ldots<t_{3m}\le n$. Then there must be an $i<m$ such that $t_{3i+3}-t_{3i}\le\frac{2n}{m}$. Let t be such that both t_{3i} and t_{3i+3} belong to $[t-\delta_t,t+\delta_t]$. There is at least one j such that $3i\le j\le 3i+2$ and $t\notin[t_j,t_{j+1}]$; in which case $\rho(\omega(t_j),\omega(t_{j+1}))\le\epsilon$. So $\operatorname{jump}_{[-n,n]}(\omega,\epsilon)\le 3m$. As n and ϵ are arbitrary, the condition is satisfied.

(ii) Suppose that ω satisfies the condition. If $t\in\mathbb{R}$ and $\epsilon>0$, take $n\ge|t|+1$ and $m=\operatorname{jump}_{[-n,n]}(\omega,\epsilon)$; then there must be a $\delta>0$ such that $\operatorname{diam}\{\omega(s):t<s\le t+\delta\}\le 2\epsilon$, since otherwise we should be able to find

$t_0 > t_1 > \ldots > t_m > t$ such that $t_0 = t+1$ and $\rho(\omega(t_{i+1}),\omega(t_i)) > \epsilon$ for $i < m$. Because X is ρ-complete, $\lim_{s\downarrow t}\omega(s)$ is defined. Similarly, $\lim_{s\uparrow t}\omega(s)$ is defined; as t is arbitrary, $\omega \in \tilde{C}^{\Downarrow}$.

(b) For $k \in \mathbb{N}$, set set $A_k = \{t : t \in \mathbb{R}, \limsup_{s\to t}\rho(\omega(s),\omega(t)) > 2^{-k+1}\}$. Then $\#(A_k \cap [-n,n[) \le \mathrm{jump}_{[-n,n]}(\omega, 2^{-k})$ for every $n \in \mathbb{N}$. **P** If $t_0,\ldots,t_m \in A_k$ and $-n \le t_0 < \ldots < t_m < n$, then we can choose $s_0,\ldots,s_m$ such that $s_0 = t_0$,

$$s_{i-1} < s_i < t_{i+1}, \quad \rho(\omega(s_i),\omega(s_{i-1})) > 2^{-k}$$

whenever $1 \le i \le m$, interpreting t_{m+1} as n. Now $\{s_i : i \le m\}$ witnesses that $\mathrm{jump}_{[-n,n]}(\omega, 2^{-k}) > m$. **Q**

By (a), $A_k \cap [-n,n[$ is finite for every n, and

$$\{t : \omega \text{ is discontinuous at } t\} = \bigcup_{k\in\mathbb{N}} A_k$$

is countable.

(c) If $\langle t_k\rangle_{k\in\mathbb{N}}$ is any monotonic sequence in $\mathbb{R}$ with limit t, $\langle\omega(t_k)\rangle_{k\in\mathbb{N}}$ is convergent to one of $\lim_{s\uparrow t}\omega(s)$, $\lim_{s\downarrow t}\omega(s)$. But this means that if $\langle t_k\rangle_{k\in\mathbb{N}}$ is any sequence in $[-n,n]$, $\langle\omega(t_k)\rangle_{k\in\mathbb{N}}$ has a subsequence which is convergent in X; by 4A2Le, $\omega[\,[-n,n]\,]$ is relatively compact in X.

438Q Theorem Let X be a Polish space, and $\tilde{C}^{\Downarrow} = \tilde{C}^{\Downarrow}(X)$ the family of functions $\omega : \mathbb{R} \to X$ such that $\lim_{s\uparrow t}\omega(s)$ and $\lim_{s\downarrow t}\omega(s)$ are defined in X for every $t \in \mathbb{R}$.

(a) $\tilde{C}^{\Downarrow}$, with its topology of pointwise convergence inherited from the product topology of $X^{\mathbb{R}}$, is K-analytic.

(b) $\tilde{C}^{\Downarrow}$ is hereditarily weakly θ-refinable.

proof Fix on a complete metric ρ inducing the topology of X.

(a) By 4A2Qg, X can be regarded as a G_δ set in a compact metrizable space Z.

(i) Give the space $\mathcal{C} = \mathcal{C}(Z)$ of closed subsets of Z its Fell topology; then $\mathcal{C}$ is compact and metrizable (4A2T(b-iii), 4A2Tf). Let $\mathcal{K}$ be the family of compact subsets of X, that is, the set of those $K \in \mathcal{C}$ which are included in X. Then $\mathcal{K}$ is a G_δ set in $\mathcal{C}$. **P** $Z \setminus X$ is an F_σ set in Z, so is expressible as the union of a sequence $\langle L_n\rangle_{n\in\mathbb{N}}$ of compact sets; now $\mathcal{K} = \bigcap_{n\in\mathbb{N}}\{K : K \in \mathcal{C},\ K \cap L_n = \emptyset\}$ is G_δ, by the definition of the Fell topology (4A2T(a-ii)). **Q**

(ii) For $n \in \mathbb{N}$, set

$$Q_n = \{\omega : \omega \in Z^{\mathbb{R}},\ \omega[\,[-n,n]\,] \text{ is a relatively compact subset of } X\}.$$

Then Q_n is K-analytic. **P** Set

$$R_n = \{(K,\omega) : K \in \mathcal{K},\ \omega \in Z^{\mathbb{R}},\ \omega[\,[-n,n]\,] \subseteq K\}.$$

Then

$$R_n = \bigcap_{t\in[-n,n]}\{(K,\omega) : K \in \mathcal{K},\ \omega \in Z^{\mathbb{R}},\ \omega(t) \in K\}$$

is a closed set in $\mathcal{K} \times Z^{\mathbb{R}}$, by 4A2T(e-i). Since $\mathcal{K}$ is Polish and $Z^{\mathbb{R}}$ is compact, R_n is K-analytic (423Ba, 423C, 422Ge, 422Gf). But Q_n is the projection of R_n onto the second coordinate, so it too is K-analytic (422Gd). **Q**

(iii) Next, for m, $k \in \mathbb{N}$, defining the function $\mathrm{jump}_{[-n,n]}$ from ρ as in 438P,

$$V_{mk} = \{\omega : \omega \in Q_n,\ \mathrm{jump}_{[-n,n]}(\omega, 2^{-k}) \le m\}$$

is relatively closed in Q_n, therefore K-analytic, and

$$Q'_n = Q_n \cap \bigcap_{k\in\mathbb{N}}\bigcup_{m\in\mathbb{N}} V_{mk}$$

is K-analytic (422Hc again).

(iv) Consequently $Q = \bigcap_{n\in\mathbb{N}} Q'_n$ is K-analytic. But $Q = \tilde{C}^{\Downarrow}$. **P** For n, $k \in \mathbb{N}$, $\tilde{C}^{\Downarrow} \subseteq Q_n$ by 438Pc, and $\tilde{C}^{\Downarrow} \subseteq \bigcup_{m\in\mathbb{N}} V_{mk}$ by 438Pa. So $\tilde{C}^{\Downarrow} \subseteq Q$. Conversely, if $\omega \in Q$, then surely $\omega(t) \in X$ for every $t \in \mathbb{R}$, and $\mathrm{jump}_{[-n,n]}(\omega, 2^{-k})$ is finite for all n, $k \in \mathbb{N}$; so $\omega \in \tilde{C}^{\Downarrow}$ by 438Pa in the other direction. **Q**

Accordingly $\tilde{C}^{\Downarrow}$ is K-analytic, as claimed.

(b) Let $\mathcal{G}$ be a family of open sets in $\tilde{C}^{\Downarrow}$.

(i) In the notation of 438N, I seek to show that $\bigcup \mathcal{G}$ belongs to $\mathcal{J}(\mathcal{G})$. Of course it will be enough to consider the case in which $\bigcup \mathcal{G}$ is non-empty. The following elementary remarks will be useful.

(α) If $D \subseteq \bigcup \mathcal{G}$, and $\mathcal{E}$ is a partition of D into relatively open sets such that $\mathcal{E}$ refines $\mathcal{G}$, then $D \in \mathcal{J}(\mathcal{G})$.

(β) If $\langle D_i \rangle_{i \in I}$ is any family in $\mathcal{J}(\mathcal{G})$, and $\langle H_i \rangle_{i \in I}$ is a family of open sets, and $D \subseteq \{\omega : \#(\{i : \omega \in H_i\}) = 1\}$, then $D \cap \bigcup_{i \in I} H_i \cap D_i$ belongs to $\mathcal{J}(\mathcal{G})$. **P** $\langle D \cap H_i \cap D_i \rangle_{i \in I}$ is an isolated family in $\mathcal{J}(\mathcal{G})$; use 438Nd. **Q**

(ii) Let $\mathcal{I}$ be the family of non-empty open intervals in $\mathbb{R}$ with rational endpoints, and $\mathcal{U}$ a countable base for the topology of X. Write Q for the family of all finite sequences

$$\boldsymbol{q} = (I_0, U_0, V_0, W_0, I_1, U_1, V_1, W_1, \dots, I_n, U_n, V_n, W_n)$$

where $I_0, I_1, \dots, I_n$ are disjoint members of $\mathcal{I}$, all the U_i, V_i, W_i belong to $\mathcal{U}$, and, for each $i \le n$, any pair of U_i, V_i, W_i are either equal or disjoint. Fix $\boldsymbol{q} = (I_0, \dots, W_n) \in Q$ for the moment.

(iii) Set $T_{\boldsymbol{q}} = \prod_{i \le n} I_i$. For $\tau \in T_{\boldsymbol{q}}$, write $F_{\boldsymbol{q}\tau}$ for the set of those $\omega \in \tilde{C}^{\Downarrow}$ such that, for every $i \le n$, $\omega(s) \in U_i$ for $s \in I_i \cap \left]-\infty, \tau(i)\right[$, $\omega(\tau(i)) \in V_i$ and $\omega(s) \in W_i$ for $s \in I_i \cap \left]\tau(i), \infty\right[$. Set $\Omega_{\boldsymbol{q}} = \bigcup\{F_{\boldsymbol{q}\tau} : \tau \in T_{\boldsymbol{q}}\}$, and for $\tau \in T_{\boldsymbol{q}}$ set $H_{\boldsymbol{q}\tau} = \{\omega : \omega \in \Omega_{\boldsymbol{q}},\, \omega(\tau(i)) \in V_i$ for every $i \le n\}$. Finally, set

$$S_{\boldsymbol{q}} = \{\tau : \tau \in T_{\boldsymbol{q}} \text{ and } H_{\boldsymbol{q}\tau} \text{ is included in some member of } \mathcal{G}\}.$$

(iv) If $T \subseteq S_{\boldsymbol{q}}$ then $H = \bigcup_{\tau \in T} H_{\boldsymbol{q}\tau}$ belongs to $\mathcal{J}(\mathcal{G})$. **P** Induce on $\#(L(T))$, where

$$L(T) = \{i : i \le n, \text{ there are } \tau,\, \tau' \in T \text{ such that } \tau(i) \ne \tau'(i)\}.$$

If $L(T) = \emptyset$, then $\#(T) \le 1$, so H is either empty or included in some member of $\mathcal{G}$, and the induction starts. For the inductive step to $\#(L(T)) = k \ge 1$, consider three cases.

case 1 Suppose there is a $j \in L(T)$ such that $U_j = V_j = W_j$. Then $\omega(t) \in V_j$ whenever $\omega \in \Omega_{\boldsymbol{q}}$ and $t \in I_j$. Fix any $t^* \in I_j$ and for $\tau \in T_{\boldsymbol{q}}$ define $\tau^* \in T_{\boldsymbol{q}}$ by setting $\tau^*(j) = t^*$, $\tau^*(i) = \tau(i)$ for $i \ne j$; then $H_{\boldsymbol{q}\tau^*} = H_{\boldsymbol{q}\tau}$. Set $T^* = \{\tau^* : \tau \in T\}$; then $L(T^*) = L(T) \setminus \{j\}$, so $\#(L(T^*)) < \#(L(T))$, while $T^* \subseteq S_{\boldsymbol{q}}$. By the inductive hypothesis, $H = \bigcup_{\tau \in T^*} H_{\boldsymbol{q}\tau}$ belongs to $\mathcal{J}(\mathcal{G})$.

case 2 Suppose there is a $j \in L(T)$ such that $U_j \ne V_j$ and $V_j \ne W_j$. Then $V_j \cap (U_j \cup W_j) = \emptyset$. For $s \in I_j$ set $T^*_s = \{\tau : \tau \in T,\, \tau(j) = s\}$. Then $\#(L(T^*_s)) < \#(L(T))$ so, by the inductive hypothesis, $H^*_s \in \mathcal{J}(\mathcal{G})$, where $H^*_s = \bigcup_{\tau \in T^*_s} H_{\boldsymbol{q}\tau}$. But, for $\tau \in T$ and $\omega \in H_{\boldsymbol{q}\tau}$, $\omega(s) \in V_j$ iff $s = \tau(j)$; so $H^*_s = \{\omega : \omega \in H,\, \omega(s) \in V_j\}$ and $\langle H^*_s \rangle_{s \in I_j}$ is a partition of H into relatively open sets. By (i-β), $H \in \mathcal{J}(\mathcal{G})$.

case 3 Otherwise, $L(T) = J \cup J'$ where $J = \{i : i \in L(T),\, U_i = V_i\}$ and $J' = \{i : i \in L(T),\, V_i = W_i\}$ are disjoint. For $\omega \in H$ and $i \in J$ we see that there is a largest $t \in I_i$ such that $\omega(t) \in V_i$; set $\phi_i(\omega) = -t$. Similarly, if $\omega \in H$, $i \in J'$ there is a smallest $t \in I_i$ such that $\omega(t) \in V_i$; in this case, set $\phi_i(\omega) = t$. Observe that, for $i \in J$ and $s \in \mathbb{R}$,

$$\begin{aligned}\{\omega : \omega \in H,\, \phi_i(\omega) \le s\} &= \emptyset \text{ if } s < -t \text{ for every } t \in I_i,\\ &= H \text{ if } -t < s \text{ for every } t \in I_i,\\ &= \{\omega : \omega \in H,\, \omega(-s) \in V_i\} \text{ if } -s \in I_i,\end{aligned}$$

so is always relatively open in H, and $\phi_i : H \to \mathbb{R}$ is continuous if $\mathbb{R}$ is given the left-facing topology $\mathfrak{S}$ of Lemma 438O. Similarly, for $i \in J'$, $s \in \mathbb{R}$,

$$\begin{aligned}\{\omega : \omega \in H,\, \phi_i(\omega) \le s\} &= \emptyset \text{ if } s < t \text{ for every } t \in I_i,\\ &= H \text{ if } t < s \text{ for every } t \in I_i,\\ &= \{\omega : \omega \in H,\, \omega(s) \in V_i\} \text{ if } s \in I_i.\end{aligned}$$

So in this case also ϕ_i is continuous.

Accordingly, giving $\mathbb{R}^{L(T)}$ the product topology corresponding to $\mathfrak{S}$, we have a continuous map $\phi : H \to \mathbb{R}^{L(T)}$ defined by setting $\phi(\omega) = \langle \phi_i(\omega) \rangle_{i \in L(T)}$ for $\omega \in H$. For $\tau \in T$, set $\tilde{\tau}(i) = -\tau(i)$ if $i \in J$, $\tau(i)$ if $i \in J'$, and $\tilde{H}_\tau = \left]-\infty, \tilde{\tau}\right] \subseteq \mathbb{R}^{L(T)}$. Then

$$\begin{aligned} H_{\boldsymbol{q}\tau} &= \{\omega : \omega \in H,\, \omega(\tau(i)) \in V_i \text{ for every } i \le n\} \\ &= \{\omega : \omega \in H,\, \omega(\tau(i)) \in V_i \text{ for every } i \in L(T)\} \end{aligned}$$

(because if $\omega \in H$, $i \le n$ and $i \notin L(T)$ then there is some $\tau' \in T$ such that $\omega \in H_{\boldsymbol{q}\tau'}$, so that $\omega(\tau'(i)) \in V_i$ and therefore $\omega(\tau(i)) \in V_i$)

$$= \{\omega : \omega \in H,\, \phi_i(\omega) \le \tilde\tau(i) \text{ for every } i \in L(T)\} = \phi^{-1}[\tilde H_\tau].$$

Set $\tilde{\mathcal{G}} = \{\tilde H_\tau : \tau \in T\}$. Because $\mathbb{R}^{L(T)}$ is hereditarily weakly θ-refinable (438O), and $\tilde{\mathcal{G}}$ is a family of open subsets of $\mathbb{R}^{L(T)}$, $\bigcup\tilde{\mathcal{G}} \in \mathcal{J}(\tilde{\mathcal{G}})$. By 438Nc, $H = \bigcup_{\tau\in T} H_{\boldsymbol{q}\tau} = \phi^{-1}[\bigcup\tilde{\mathcal{G}}]$ belongs to $\mathcal{J}(\{H_{\boldsymbol{q}\tau} : \tau \in T\}$ and therefore to $\mathcal{J}(\mathcal{G})$, by 438Nb.

Thus in all three cases the induction proceeds. **Q**

(v) This means that, for any $\boldsymbol{q} \in Q$, $Y_{\boldsymbol{q}} = \bigcup\{H_{\boldsymbol{q}\tau} : \tau \in S_{\boldsymbol{q}}\}$ belongs to $\mathcal{J}(\mathcal{G})$. Since Q is countable, $Y \in \mathcal{J}(\mathcal{G})$, where $Y = \bigcup_{\boldsymbol{q}\in Q} Y_{\boldsymbol{q}}$. But $\bigcup\mathcal{G} \subseteq Y$. **P** If $\omega \in G \in \mathcal{G}$, there are $t_0 < \ldots < t_n$ and $V_i' \in \mathcal{I}$, for $i \le n$, such that

$$\omega \in \{\omega' : \omega' \in \tilde C^{\Vert\!\lfloor}, \, \omega'(t_i) \in V_i' \text{ for every } i \le n\} \subseteq G.$$

Set $x_i = \omega(t_i)$, $x_i^- = \lim_{s\uparrow t_i}\omega(s)$, $x_i^+ = \lim_{s\downarrow t_i}\omega(s)$ for $i \le n$; let U_i, V_i, $W_i \in \mathcal{U}$ be such that $x_i^- \in U_i$, $x_i \in V_i \subseteq V_i'$, $x_i^+ \in W_i$ and any pair of U_i, V_i, W_i are either equal or disjoint; and let $I_0, \ldots, I_n \in \mathcal{I}$ be disjoint and such that $t_i \in I_i$, $\omega(s) \in U_i$ for $s \in I_i \cap]-\infty, t_i[$ and $\omega(s) \in W_i$ for $s \in I_i \cap]t_i, \infty[$ for each $i \le n$. Then, setting $\boldsymbol{q} = (I_0, \ldots, W_n)$ and $\tau(i) = t_i$ for $i \le n$,

$$\omega \in F_{\boldsymbol{q}\tau} \subseteq H_{\boldsymbol{q}\tau} \subseteq G,$$

so that $\tau \in S_{\boldsymbol{q}}$ and $\omega \in Y_{\boldsymbol{q}} \subseteq Y$. **Q**

As $\mathcal{G}$ is arbitrary, $\tilde C^{\Vert\!\lfloor}$ is hereditarily weakly θ-refinable.

438R Corollary (a) Let $I^{\Vert}$ be the split interval (419L). Then any countable power of $I^{\Vert}$ is a hereditarily weakly θ-refinable compact Hausdorff space.

(b) Let Y be the 'Helly space', the space of non-decreasing functions from $[0,1]$ to itself with the topology of pointwise convergence inherited from the product topology on $[0,1]^{[0,1]}$ (Kelley 55, Ex. 5M). Then Y is a hereditarily weakly θ-refinable compact Hausdorff space.

proof These are both (homeomorphic to) subspaces of the space $\tilde C^{\Vert\!\lfloor}$ of Proposition 438Q, if we take X there to be $\mathbb{R}$. To see this, argue as follows. For (a), observe that we have a function $f : I^{\Vert} \to \tilde C^{\Vert\!\lfloor}$ defined by setting $f(t^-)(s) = f(t^+)(s) = 1$ if $s < t$, $f(t^-)(s) = f(t^+)(s) = 0$ if $s > t$, and $f(t^-)(t) = 0$, $f(t^+)(t) = 1$, and that f is a homeomorphism between $I^{\Vert}$ and its image. Next, for any $L \subseteq \mathbb{N}$, we can define $g : (I^{\Vert})^L \to \tilde C^{\Vert\!\lfloor}$ by setting

$$\begin{aligned} g(\mathbf{t})(s) &= f(t_n)(s - 2n) \text{ if } n \in L \text{ and } 2n \le s \le 2n+1, \\ &= 0 \text{ if } s \in \mathbb{R} \setminus \bigcup_{n\in L}[2n, 2n+1] \end{aligned}$$

for $\mathbf{t} = \langle t_n\rangle_{n\in L} \in (I^{\Vert})^L$; it is easy to check that g is a homeomorphism between $(I^{\Vert})^L$ and its image in $\tilde C^{\Vert\!\lfloor}$. As for (b), if we take $g(y)$ to be the extension of the function $y : [0,1] \to [0,1]$ to the function which is constant on each of the intervals $]-\infty, 0]$ and $[1, \infty[$, then $g : Y \to \tilde C^{\Vert\!\lfloor}$ is a homeomorphism between Y and its image $g[Y]$.

Since both $(I^{\Vert})^L$ and Y are compact, they are homeomorphic to closed subsets of $\tilde C^{\Vert\!\lfloor}$, and are hereditarily weakly θ-refinable (438La).

***438S Càllàl functions** To support some of the theory of Lévy processes which I will present in §455, I give a further consequence of 438Q.

Proposition Let X be a Polish space. Let $C^{\Vert\!\lfloor} = C^{\Vert\!\lfloor}(X)$ be the set of càllàl functions (definition: 4A2A) from $[0, \infty[$ to X, with its topology of pointwise convergence inherited from the product topology of $X^{[0,\infty[}$.

(a)(i) If $\omega \in C^{\Vert\!\lfloor}$, then ω is continuous at all but countably many points of $[0, \infty[$.

(ii) If ω, $\omega' \in C^{\Vert\!\!\!\text{L}}$, D is a dense subset of $[0,\infty[$ containing every point at which ω is discontinuous, and $\omega'\restriction D = \omega\restriction D$, then $\omega' = \omega$.

(b) $C^{\Vert\!\!\!\text{L}}$ is hereditarily weakly θ-refinable.

(c) $C^{\Vert\!\!\!\text{L}}$ is K-analytic.

proof Fix a complete metric ρ on X defining its topology. Let $\tilde{C}^{\Vert\!\!\!\text{L}} \subseteq X^{\mathbb{R}}$ be the space of 438P-438Q.

(a) If $X = \emptyset$ the results are trivial. Otherwise, fix $x_0 \in X$, and for $\omega \in X^{[0,\infty[}$ define $\tilde{\omega} \in X^{\mathbb{R}}$ to be that extension of ω which takes the value x_0 everywhere on $]-\infty, 0[$.

(i) If $\omega \in C^{\Vert\!\!\!\text{L}}$, then $\tilde{\omega} \in \tilde{C}^{\Vert\!\!\!\text{L}}$; so the result follows from 438Pb.

(ii) If $t \in D$, $\omega'(t)$ is certainly equal to $\omega(t)$. Next,

$$\omega'(0) = \lim_{s\downarrow 0}\omega'(s) = \lim_{s\in D, s\downarrow 0}\omega'(s) = \lim_{s\in D, s\downarrow 0}\omega(s) = \omega(0).$$

If $t \in]0,\infty[\setminus D$, then ω is continuous at t, so

$$\lim_{s\uparrow t}\omega'(s) = \lim_{s\in D, s\uparrow t}\omega'(s) = \lim_{s\in D, s\uparrow 0}\omega(s) = \omega(t),$$

$$\lim_{s\downarrow t}\omega'(s) = \lim_{s\in D, s\downarrow t}\omega'(s) = \lim_{s\in D, s\downarrow 0}\omega(s) = \omega(t).$$

Since $\omega'(t)$ must be either $\lim_{s\uparrow t}\omega'(s)$ or $\lim_{s\downarrow t}\omega'(s)$, it is again equal to $\omega(t)$. So $\omega' = \omega$.

(b) Since $C^{\Vert\!\!\!\text{L}}$ is homeomorphic to a subspace of $\tilde{C}^{\Vert\!\!\!\text{L}}$, it is hereditarily weakly θ-refinable (438Qb, 438La).

(c) Set $\tilde{Q} = \{\tilde{\omega} : \omega \in C^{\Vert\!\!\!\text{L}}\}$; then $\tilde{Q} \subseteq \tilde{C}^{\Vert\!\!\!\text{L}}$ is homeomorphic to $C^{\Vert\!\!\!\text{L}}$. But $\tilde{Q}$ is a Souslin-F set in $\tilde{C}^{\Vert\!\!\!\text{L}}$. **P** If $\omega \in C^{\Vert\!\!\!\text{L}}$ then it belongs to $\tilde{Q}$ iff $\omega(t) = x_0$ for every $t < 0$. $\lim_{t\downarrow 0}\omega(t) = \omega(0)$, and $\omega(t) \in \{\lim_{s\uparrow t}\omega(s), \lim_{s\downarrow t}\omega(s)\}$ for every $t > 0$. Now

$$\{\omega : \omega(t) = x_0 \text{ for every } t < 0\}$$

is closed, while

$$\{\omega : \omega \in \tilde{C}^{\Vert\!\!\!\text{L}}, \, \omega(0) = \lim_{t\downarrow 0}\omega(t)\} = \{\omega : \omega \in \tilde{C}^{\Vert\!\!\!\text{L}}, \, \omega(0) = \lim_{i\to\infty}\omega(2^{-i})\}$$

(because $\lim_{t\downarrow 0}\omega(t)$ is defined for every $\omega \in \tilde{C}^{\Vert\!\!\!\text{L}}$)

$$= \bigcap_{k\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\bigcap_{i\ge m}\{\omega : \omega \in \tilde{C}^{\Vert\!\!\!\text{L}}, \, \rho(\omega(2^{-i}), \omega(0)) \le 2^{-k}\}$$

is Souslin-F. As for the other condition, note that for $t > 0$ and $\omega \in \tilde{C}^{\Vert\!\!\!\text{L}}$, $\omega(t)$ belongs to $\{\lim_{s\uparrow t}\omega(s), \lim_{s\downarrow t}\omega(s)\}$ iff for every $\epsilon > 0$ there are distinct rational numbers q, $q' \in [t-\epsilon, t+\epsilon]$ such that $\omega(q)$ and $\omega(q')$ belong to $B(\omega(t), \epsilon)$. Let $\mathcal{U}$ be a countable base for the topology of X and $\mathcal{I}$ a countable base for the topology of $]0,\infty[$ not containing $\emptyset$; then for $\omega \in \tilde{C}^{\Vert\!\!\!\text{L}}$, $\omega(t) \in \{\lim_{s\uparrow t}\omega(s), \lim_{s\downarrow t}\omega(s)\}$ for every $t > 0$ if and only if

for every $U \in \mathcal{U}$ and $I \in \mathcal{I}$ either $I \cap \omega^{-1}[U] = \emptyset$ or $I \cap \mathbb{Q} \cap \omega^{-1}[\overline{U}]$ has at least two members.

Since, for $U \in \mathcal{U}$ and $I \in \mathcal{I}$,

$$\{\omega : \omega \in \tilde{C}^{\Vert\!\!\!\text{L}}, \, I \cap \omega^{-1}[U] = \emptyset\} = \bigcap_{t\in I}\{\omega : \omega \in \tilde{C}^{\Vert\!\!\!\text{L}}, \, \omega(t) \notin U\}$$

is closed in $\tilde{C}^{\Vert\!\!\!\text{L}}$, and

$$\{\omega : \omega \in \tilde{C}^{\Vert\!\!\!\text{L}}, \, I \cap \mathbb{Q} \cap \omega^{-1}[\overline{U}] \text{ has at least two members}\}$$
$$= \bigcup_{\substack{q,q'\in I\cap\mathbb{Q} \\ q<q'}}\{\omega : \omega(q), \, \omega(q') \in \overline{U}\}$$

is F_σ in $\tilde{C}^{\Vert\!\!\!\text{L}}$, while $\mathcal{I}$ and $\mathcal{U}$ are countable,

$$\{\omega : \omega \in \tilde{C}^{\Vert\!\!\!\text{L}}, \, \omega(t) \in \{\lim_{s\uparrow t}\omega(s), \lim_{s\downarrow t}\omega(s)\} \text{ for every } t > 0\}$$

is Souslin-F in $\tilde{C}^{\Vert\!\!\!\text{L}}$. Taking the intersection, we see that $\tilde{Q}$ is Souslin-F. **Q**

Accordingly $\tilde{Q}$ and $C^{\Vert\!\!\!\text{L}}$ are K-analytic (422Hb).

438T Proposition Assume that $\mathfrak{c}$ is measure-free. Then $(I^{\|})^{\mathbb{N}}$, the Helly space (438Rb) and the spaces $\tilde{C}^{\mathbb{1}}(X)$, $C^{\mathbb{1}}(X)$ of 438Q and 438S, for any Polish space X, are all Radon spaces.

proof By 438Q-438S, they are K-analytic and hereditarily weakly θ-refinable, also they have weight at most $w(X^{\mathbb{R}}) \le \mathfrak{c}$. They are therefore pre-Radon (434Jf), Borel-measure-complete (438M) and Radon (434Ka).

438U In 434R I described a construction of product measures. In accordance with my general practice of examining the measure algebra of any new measure, I give the following result.

Proposition Let X and Y be topological spaces with σ-finite Borel measures μ, ν respectively. Suppose that *either* X is first-countable *or* ν is τ-additive and effectively locally finite. Write λ for the Borel measure on $X \times Y$ defined by the formula

$$\lambda W = \int \nu W[\{x\}]\mu(dx) \text{ for every Borel set } W \subseteq X \times Y$$

as in 434R(ii). If *either* the weight of X *or* the Maharam type of ν is a measure-free cardinal, then for every Borel set $W \subseteq X \times Y$ there is a set $W' \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ such that $\lambda(W \triangle W') = 0$; consequently, the measure algebra of λ can be identified with the localizable measure algebra free product of the measure algebras of μ and ν.

proof (a) Write $(\mathfrak{B}, \bar{\nu})$ for the measure algebra of ν. With its measure-algebra topology, this is metrizable (323Gb). Let $\langle Y_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of Borel sets of finite measure in Y with union Y.

(b) For the moment (down to the end of (e) below) fix on an open set $W \subseteq X \times Y$. For $x \in X$, set $f(x) = W[\{x\}]^\bullet$ in $\mathfrak{B}$. Then $f : X \to \mathfrak{B}$ is Borel measurable.

P **(i)** Let $H \subseteq \mathfrak{B}$ be an open set. For k, $n \in \mathbb{N}$ set

$$E_{nk} = \{x : x \in X,\ 2^{-n}k \le \nu(Y_n \cap W[\{x\}]) < 2^{-n}(k+1)\}.$$

Just as in part (a) of the proof of 434R, the function $x \mapsto \nu(Y_n \cap W[\{x\}])$ is lower semi-continuous, so E_{nk} is a Borel set. Set

$$G_{nk} = \bigcup\{G : G \subseteq X \text{ is open, } G \cap E_{nk} \subseteq f^{-1}[H]\};$$

then $E = \bigcup_{n,k\in\mathbb{N}}(G_{nk} \cap E_{nk})$ is a Borel set included in $f^{-1}[H]$.

(ii) The point is that $E = f^{-1}[H]$. To see this, take any x such that $f(x) \in H$. Then there are $b \in \mathfrak{B}$, $\epsilon > 0$ such that $\bar{\nu}b < \infty$ and $c \in H$ whenever $c \in \mathfrak{B}$ and $\bar{\nu}(b \cap (c \triangle f(x))) \le 5\epsilon$. Since $b = \sup_{n\in\mathbb{N}} b \cap Y_n^\bullet$, there is an $n \in \mathbb{N}$ such that $\bar{\nu}(b \setminus Y_n^\bullet) \le \epsilon$ and $2^{-n} \le \epsilon$. In this case $c \in H$ whenever $c \in \mathfrak{B}$ and $\bar{\nu}(Y_n^\bullet \cap (c \triangle f(x))) \le 4\epsilon$; thus

$$\{x' : \nu(Y_n \cap (W[\{x'\}] \triangle W[\{x\}])) \le 4\epsilon\} \subseteq f^{-1}[H].$$

Let $k \in \mathbb{N}$ be such that $2^{-n}k \le \nu(Y_n \cap W[\{x\}]) < 2^{-n}(k+1)$, that is, $x \in E_{nk}$. Again using the ideas of part (a) of the proof of 434R, there are an open set G containing x and an open set $V \subseteq Y$ such that $G \times V \subseteq W$ and $\nu(Y_n \cap V) \ge 2^{-n}(k-1)$. Now if $x' \in G \cap E_{nk}$, $V \subseteq W[\{x'\}] \cap W[\{x\}]$, so

$$\begin{aligned}\nu(Y_n \cap (W[\{x'\}] \triangle W[\{x\}])) &\le \nu(Y_n \cap W[\{x'\}]) + \nu(Y_n \cap W[\{x\}]) - 2\nu(Y_n \cap V)\\ &\le 2^{-n}((k+1) + (k+1) - 2(k-1)) = 4 \cdot 2^{-n} \le 4\epsilon.\end{aligned}$$

But this means that $G \cap E_{nk} \subseteq f^{-1}[H]$, so $G \subseteq G_{nk}$ and $x \in G \cap E_{nk} \subseteq E$. As x is arbitrary, $f^{-1}[H] \subseteq E$ and $E = f^{-1}[H]$.

(iii) Thus $f^{-1}[H]$ is a Borel set. As H is arbitrary, f is Borel measurable. **Q**

(c) We need to know also that if $\mathcal{H}$ is a disjoint family of open subsets of $\mathfrak{B}$ all meeting $f[X]$, then

$$\#(\mathcal{H}) \le \max(\omega, \min(w(X), \tau(\mathfrak{B})))$$

P Repeat the ideas of (b) above, setting

$$G_{nk}^{(H)} = \bigcup\{G : G \subseteq X \text{ is open, } G \cap E_{nk} \subseteq f^{-1}[H]\}$$

for $H \in \mathcal{H}$ and k, $n \in \mathbb{N}$, so that $f^{-1}[H] = \bigcup_{n,k\in\mathbb{N}} G_{nk}^{(H)} \cap E_{nk}$. For fixed n and k the family $\langle G_{nk}^{(H)} \cap E_{nk}\rangle_{H\in\mathcal{H}}$ is disjoint, so can have at most $w(E_{nk}) \le w(X)$ non-empty members (4A2D again). But this means that

$$\mathcal{H} = \bigcup_{n,k\in\mathbb{N}}\{H : G_{nk}^{(H)} \cap E_{nk} \ne \emptyset\}$$

has cardinal at most $\max(\omega, w(X))$.

On the other hand, there is a set $B \subseteq \mathfrak{B}$, of cardinal $\tau(\mathfrak{B})$, which τ-generates $\mathfrak{B}$. The algebra $\mathfrak{B}_0$ generated by B has cardinal at most $\max(\omega, \#(B))$ (331Gc), and $\mathfrak{B}_0$ is topologically dense in $\mathfrak{B}$ (323J), so every member of $\mathcal{H}$ meets $\mathfrak{B}_0$, and

$$\#(\mathcal{H}) \le \#(\mathfrak{B}_0) \le \max(\omega, \tau(\mathfrak{B})).$$

Putting these together, we have the result. **Q**

In particular, under the hypotheses above, $\#(\mathcal{H})$ is measure-free whenever $\mathcal{H}$ is a disjoint family of open subsets of $\mathfrak{B}$ all meeting $f[X]$.

(d) The next step is to observe that there is a conegligible Borel set $Z \subseteq X$ such that $f[Z]$ is separable. **P** Let $\mathcal{H}$ be a σ-disjoint base for the topology of $\mathfrak{B}$; express it as $\bigcup_{n\in\mathbb{N}} \mathcal{H}_n$ where each $\mathcal{H}_n$ is disjoint. Let $\langle X_m\rangle_{m\in\mathbb{N}}$ be a cover of X by Borel sets of finite measure. For $n \in \mathbb{N}$ consider $\mathcal{H}'_n = \{H : H \in \mathcal{H}_n,\ H \cap f[X] \ne \emptyset\}$. For $m \in \mathbb{N}$, we have a totally finite measure ν_{nm} with domain $\mathcal{P}\mathcal{H}'_n$ defined by saying

$$\nu_{nm}\mathcal{E} = \mu(X_m \cap f^{-1}(\bigcup \mathcal{E}))$$

for every $\mathcal{E} \subseteq \mathcal{H}'_n$. Since $\mathcal{H}'_n$ has measure-free cardinal, by (c), there must be a countable set $\mathcal{E}_{nm} \subseteq \mathcal{H}'_n$ such that $\nu_{nm}(\mathcal{H}'_n \setminus \mathcal{E}_{nm}) = 0$. Set

$$Z = X \setminus \bigcup_{m,n\in\mathbb{N}} (X_m \cap f^{-1}[\bigcup(\mathcal{H}'_n \setminus \mathcal{E}_{nm})]);$$

then Z is conegligible. If $x \in Z$ and $f(x) \in H \in \mathcal{H}_n$, then there is some $m \in \mathbb{N}$ such that $x \in X_m$, while $H \in \mathcal{H}'_n$, so H must belong to $\mathcal{E}_{nm}$. But this means that $\{f[Z] \cap H : H \in \mathcal{H}\}$, which is a base for the topology of $f[Z]$, is just $\{f[Z] \cap H : H \in \bigcup_{m,n\in\mathbb{N}} \mathcal{E}_{mn}\}$, and is countable. So $f[Z]$ is separable (4A2Oc), as required. **Q**

(e) 418T(a-ii) now tells us that there is a set $W' \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ such that $f(x) = W'[\{x\}]^\bullet$ for every $x \in Z$, so that $\nu(W[\{x\}] \triangle W'[\{x\}]) = 0$ for almost every x, that is, $\lambda(W \triangle W') = 0$. And this is true for every open set $W \subseteq X \times Y$.

(f) Now let $\mathcal{W}$ be the family of those Borel sets $W \subseteq X \times Y$ for which there is a $W' \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ such that $\lambda(W \triangle W') = 0$. This is a σ-algebra containing every open set, so is the whole Borel σ-algebra, as required.

Since the c.l.d. product measure λ_0 on $X \times Y$ is just the completion of its restriction to $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ (251K), and λ_0 and λ agree on $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ (by Fubini's theorem), the embedding $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y) \subsetneqq \mathcal{B}(X \times Y)$ induces an isomorphism between the measure algebras of λ and λ_0. As remarked in 325Eb, because μ and ν are strictly localizable, the latter may be identified with 'the' localizable measure algebra free product of the measure algebras of μ and ν.

Remark The hypothesis on the weight of X can be slightly weakened; see 438Yg. 439L below shows that some restriction on (X, μ) and (Y, ν) is necessary.

438X Basic exercises (a) Show that a cardinal κ is measure-free iff $M_\sigma = M_\tau$, where M_σ, M_τ are the spaces of countably additive and completely additive functionals on the algebra $\mathcal{P}\kappa$ (362B).

(b) Let (X, Σ, μ) be a localizable measure space with magnitude (definition: 332Ga) which is either finite or a measure-free cardinal. Show that any absolutely continuous countably additive functional $\nu : \Sigma \to \mathbb{R}$ is truly continuous. (*Hint*: 363S.)

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra with measure-free cellularity. Show that any countably additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is completely additive.

(d) Let U be a Dedekind complete Riesz space such that any disjoint order-bounded family in U^+ has measure-free cardinal. Show that $U^\sim_c = U^\times$.

(e) Let (X, Σ, μ) be a complete locally determined measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Suppose that the magnitude of ν is either finite or a measure-free cardinal. Show that μ is strictly localizable.

(f) Let (X_1, Σ_1, μ_1), (X_2, Σ_2, μ_2), $(Y_1, \mathrm{T}_1, \nu_1)$ and $(Y_2, \mathrm{T}_2, \nu_2)$ be measure spaces, and λ_1, λ_2 the c.l.d. product measures on $X_1 \times Y_1$, $X_2 \times Y_2$ respectively; suppose that $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ are inverse-measure-preserving functions, and that $h(x, y) = (f(x), g(y))$ for $x \in X_1$, $y \in Y_1$. Show that if μ_2 and ν_2 are both strictly localizable, with magnitudes which are either finite or measure-free cardinals, then h is inverse-measure-preserving. (Compare 251L.)

>(g) Show that if κ is a measure-free cardinal, so is ω_κ. (*Hint*: show by induction on ordinals ξ that if $\#(\xi)$ is measure-free, then so is ω_ξ.)

>(h) Let (X,Σ,μ) be a complete locally determined measure space, (Y,ρ) a complete metric space with measure-free weight, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of measurable functions from X to Y. Show that $\{x : \lim_{n\to\infty} f_n(x)$ is defined in $Y\}$ is measurable. (Cf. 418C.)

(i) Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces with c.l.d. product $(X\times Y,\Lambda,\lambda)$. Give $L^0(\nu)$ the topology of convergence in measure. Suppose that $f : X \to L^0(\nu)$ is measurable and there is a conegligible set $X_0 \subseteq X$ such that $w(f[X_0])$ is measure-free. Show that there is an $h \in \mathcal{L}^0(\lambda)$ such that $f(x) = h_x^\bullet$ for every $x \in X$, where $h_x(y) = h(x,y)$ for $(x,y) \in \operatorname{dom} h$. (Cf. 418S.)

(j) Let (Y,T,ν) be a σ-finite measure space, and $(\mathfrak{B},\bar\nu)$ its measure algebra, with its usual topology; assume that the Maharam type of $\mathfrak{B}$ is measure-free. Let (X,Σ,μ) be a σ-finite measure space and Λ the domain of the c.l.d. product measure λ on $X\times Y$. Show that if $f : X \to \mathfrak{B}$ is measurable, then there is a $W \in \Lambda$ such that $f(x) = W[\{x\}]^\bullet$ for every $x \in X$. (Cf. 418T(b-ii).)

(k) Let (X,Σ,μ) be a complete locally determined measure space and V a normed space such that $w(V)$ is measure-free. (i) Show that the space $\mathcal{L}$ of measurable functions from X to V is a linear space, setting $(f+g)(x) = f(x)+g(x)$, etc. (ii) Show that if V is a Riesz space with a Riesz norm then $\mathcal{L}$ is a Riesz space under the natural operations.

>(l) Let X be a topological space and $\mathcal{G}$ a point-finite open cover of X such that $\#(\mathcal{G})$ is measure-free. Suppose that $E \subseteq X$ is such that $E\cap G$ is universally measurable for every $G \in \mathcal{G}$. Show that E is universally measurable. (Compare 434Xe(iv).)

>(m) Show that for a metrizable space X, the following are equiveridical: (i) X is Borel-measure-compact; (ii) X is Borel-measure-complete; (iii) X is measure-compact; (iv) $w(X)$ is measure-free.

(n) Let X be a topological space and $\mathcal{G}$ a family of open subsets of X. Show that the following are equiveridical: (i) there is a σ-isolated family $\mathcal{A}$ of sets, refining $\mathcal{G}$, such that $\bigcup\mathcal{A} = \bigcup\mathcal{G}$; (ii) there is a sequence $\mathcal{H}_n$ of families of open sets, all refining $\mathcal{G}$, such that for every $x \in \bigcup\mathcal{G}$ there is an $n \in \mathbb{N}$ such that $\{H : x \in H \in \mathcal{H}_n\}$ is finite and not empty.

(o) Let Y be the Helly space. (i) Show that Y is a compact convex subset of $\mathbb{R}^{[0,1]}$ with its usual topology. (ii) Show that there is a natural one-to-one correspondence between the split interval $I^{\parallel}$ and the set of extreme points of Y, matching $t^- \in I^{\parallel}$ with the function $\chi[0,t[$ and t^+ with $\chi[0,t]$. (iii) Let P_{R} be the set of Radon probability measures on $I^{\parallel}$ with its narrow topology (437J). Show that there is a natural homeomorphism $\phi : P_{\mathrm{R}} \to Y$ defined by setting $\phi(\mu)(t) = \mu[0^-,t^-]$ for $\mu \in P_{\mathrm{R}}$, $t \in [0,1]$.

(p) Give $\mathbb{R}$ the right-facing Sorgenfrey topology (415Xc). Show that any countable power of $\mathbb{R}$ is hereditarily weakly θ-refinable.

>(q) Let $I^{\parallel}$ be the split interval. Show that $I^{\parallel}\times I^{\parallel}$ is a Radon space iff $\mathfrak{c}$ is measure-free. (*Hint*: $\{(\alpha^+,(1-\alpha)^+) : \alpha \in [0,1]\}$ is a discrete Borel subset of cardinal $\mathfrak{c}$.)

(r) Give $\mathbb{R}$ the right-facing Sorgenfrey topology. Show that the following are equiveridical: (i) $\mathfrak{c}$ is measure-free; (ii) $\mathbb{R}^{\mathbb{N}}$, with the corresponding product topology, is Borel-measure-complete; (iii) $\mathbb{R}^2$, with the product topology, is Borel-measure-compact. (Compare 439Q.)

(s) Suppose that $\mathfrak{c}$ is measure-free. Let $X \subseteq \mathbb{R}^{\mathbb{R}}$ be the set of functions of bounded variation on $\mathbb{R}$, with the topology of pointwise convergence inherited from the product topology of $\mathbb{R}^{\mathbb{R}}$. Show that X is a Radon space. (*Hint*: X is an F_σ subset of the space $\tilde C^{\Downarrow\!\!\Uparrow}$ of 438Q.)

438Y Further exercises (a) Let (X,Σ,μ) be a probability space and $\langle E_i\rangle_{i\in I}$ a point-finite family of measurable sets such that $\nu J = \mu(\bigcup_{i\in J} E_i)$ is defined for every $J \subseteq I$. Show directly that ν is a uniformly exhaustive Maharam submeasure on $\mathcal{P}I$, and use the Kalton-Roberts theorem to prove 438Ba.

(b) Suppose that $\mathfrak{c}$ is measure-free, but that $\kappa > \mathfrak{c}$ is not measure-free. Show that there is a non-principal ω_1-complete ultrafilter on κ. (*Hint*: part (b) of the proof of 451Q.)

(c) Show that if X is a metrizable space and $\min(\mathfrak{c}, w(X))$ is measure-free, then every σ-finite Borel measure on X has countable Maharam type.

(d) Let X be a metacompact T_1 space. Show that X is Borel-measure-compact iff every closed discrete subspace has measure-free cardinal.

(e) Let X be a topological space such that every subspace of X is metacompact and has measure-free cellularity. Show that X is Borel-measure-complete.

(f) Let X be a normal metacompact Hausdorff space. Show that it is measure-compact iff every closed discrete subspace has measure-free cardinal.

(g) In 438U, show that it would be enough to suppose that every discrete subset of X has measure-free cardinal.

(h) Suppose that $\mathfrak{c}$ is measure-free. Let D be any subset of $\mathbb{R}$ and $X \subseteq \mathbb{R}^D$ the set of functions of bounded variation on D, with the topology of pointwise convergence inherited from the product topology of $\mathbb{R}^D$. Show that X is a Radon space.

(i) Let X be a totally ordered set with its order topology. Show that if $c(X)$ is measure-free then every σ-finite Borel measure on X has countable Maharam type. (Cf. 434Yo.)

(j) Suppose that X is a normal metacompact Hausdorff space which is not realcompact. Show that there are a closed discrete subset D of X and a non-principal ω_1-complete ultrafilter on D. (*Hint*: in 435C, if we start with a $\{0,1\}$-valued Baire measure we obtain a $\{0,1\}$-valued Borel measure; in the proof of 438Ba, if μ is $\{0,1\}$-valued then ν is $\{0,1\}$-valued.)

(k) Let Z be a regular Hausdorff space, T a Dedekind complete totally ordered space with least and greatest elements a, b, and $x : T \to Z$ a function such that $\lim_{s\uparrow t} x(s)$ and $\lim_{s\downarrow t} x(s)$ are defined in Z for every $t \in T$ (except $t = a$ in the first case and $t = b$ in the second). Show that $x[T]$ is relatively compact in Z.

(l) Let (X, ρ) be a metric space, and P_{Bor} the set of Borel probability measures on X. For μ, $\nu \in P_{\mathrm{Bor}}$ set $\bar\rho_{\mathrm{KR}}(\mu, \nu) = \sup\{|\int u\,d\mu - \int u\,d\nu| : u : X \to [-1, 1]$ is 1-Lipschitz$\}$. (i) Show that $\bar\rho_{\mathrm{KR}}$ is a metric on P_{Bor}. (ii) Let $\mathfrak{T}_{\mathrm{KR}}$ be the topology it induces on P_{Bor}. Show that $\mathfrak{T}_{\mathrm{KR}}$ is finer than the narrow topology on P_{Bor}. (iii) Show that the following are equiveridical: (α) the narrow topology on P_{Bor} is metrizable; (β) $\mathfrak{T}_{\mathrm{KR}}$ is the narrow topology on P_{Bor}; (γ) $w(X)$ is measure-free. (Cf. 437Rg, 437Yp.)

438 Notes and comments Since the axiom 'every cardinal is measure-free' is admissible – that is, will not lead to a paradox unless one is already latent in the Zermelo-Fraenkel axioms for set theory – it is tempting, in the context of this section, to assume it; so that 'every complete metric space is Radon' becomes a theorem, along with 'every measurable function from a quasi-Radon measure space to a metrizable space is almost continuous' (438G), '$U^\sim_c = U^\times$ for every Dedekind complete Riesz space U' (438Xd), 'metacompact spaces are Borel-measure-compact' (438J), 'the sum of two measurable functions from a complete probability space to a normed space is measurable' (438Xk) and 'the Helly space is Radon' (438T). Undoubtedly the consequent mathematical universe is tidier. In my view, the tidiness is the tidiness of poverty. Apart from anything else, it leads us to neglect such questions as 'is every measurable function from a Radon measure space to a metrizable space almost continuous?', which have answers in ZFC (451T).

From the point of view of measure theory, the really interesting question is whether $\mathfrak{c}$ is measure-free. It is not quite clear from the results above why this should be so; 438T is a very small part of the story. There is a larger hint in 438Ce-438Cf: if $\mathfrak{c}$ is measure-free, but $\kappa > \mathfrak{c}$ is not measure-free, then the witnessing measures will be purely atomic. I will return to this point in §543 of Volume 5. For a general exploration of universes in which $\mathfrak{c}$ is *not* measure-free, see §544 and Fremlin 93. For fragments of what happens if we suppose that we have an atom for a measure which witnesses that κ is not measure-free, see 438Yb and the notes on normal filters in 4A1I-4A1L.

There are many further applications of 438Q besides those in 438R and 438Xp-438Xs. But the most obvious candidate, the space $C(\mathbb{R})$ of continuous real-valued functions on $\mathbb{R}$, although indeed it is a Borel subset of the potentially Radon space of 438Q, is in fact Radon whether or not $\mathfrak{c}$ is measure-free (454Sa). As soon as we start using any such special axiom as '$\mathfrak{c} = \omega_1$' or '$\mathfrak{c}$ is measure-free', we must make a determined effort to check, through such examples as 438Xq, that our new theorems do indeed depend on something more than ZFC.

439 Examples

As in Chapter 41, I end this chapter with a number of examples, exhibiting some of the boundaries around the results in the rest of the chapter, and filling in a gap with basic facts about Lebesgue measure (439E). The first three examples (439A) are measures defined on σ-subalgebras of the Borel σ-algebra of $[0,1]$ which have no extensions to the whole Borel algebra. The next part of the section (439B-439G) deals with 'universally negligible' sets; I use properties of these to show that Hausdorff measures are generally not semi-finite (439H), closing some unfinished business from §§264, and that smooth linear functionals may fail to be representable by integrals in the absence of Stone's condition (439I). In 439J-439R I set out some examples relevant to §§434-435, filling out the classification schemes of 434A and 435A, with spaces which just miss being Radon (439K) or measure-compact (439N, 439P, 439Q). In 439S I present the canonical example of a non-Prokhorov topological space, answering an obvious question from §437.

439A Example Let $\mathcal{B}$ be the Borel σ-algebra of $[0,1]$. There is a probability measure ν defined on a σ-subalgebra T of $\mathcal{B}$ which has no extension to a measure on $\mathcal{B}$.

first construction Let $A \subseteq [0,1]$ be an analytic set which is not Borel (423L). Let $\mathcal{I}$ be the family of sets of the form $E \cup F$ where E, F are Borel sets, $E \subseteq A$ and $F \subseteq [0,1] \setminus A$. Then $\mathcal{I}$ is a σ-ideal of $\mathcal{B}$ not containing $[0,1]$. Set $\mathrm{T} = \mathcal{I} \cup \{[0,1] \setminus H : H \in \mathcal{I}\}$, and define $\nu : \mathrm{T} \to \{0,1\}$ by setting $\nu H = 0$, $\nu([0,1] \setminus H) = 1$ for every $H \in \mathcal{I}$; then ν is a probability measure (cf. countable-cocountable measures (211R) or Dieudonné's measure (411Q)).

? If $\mu : \mathcal{B} \to [0,1]$ is a measure extending ν, then its completion $\hat{\mu}$ measures A (432A). Also $\hat{\mu}$ is a Radon measure (433Cb). Now every compact subset of A belongs to $\mathcal{I}$, so

$$\hat{\mu}A = \sup\{\hat{\mu}K : K \subseteq A \text{ is compact}\} = \sup\{\nu K : K \subseteq A \text{ is compact}\} = 0.$$

Similarly $\hat{\mu}([0,1] \setminus A) = 0$, which is absurd. **X**

second construction This time, let $\mathcal{I}$ be the family of meager Borel sets in $[0,1]$. As before, let T be $\mathcal{I} \cup \{[0,1] \setminus E : E \in \mathcal{I}\}$, and set $\nu E = 0$, $\nu([0,1] \setminus E) = 1$ for $E \in \mathcal{I}$. **?** If μ is a Borel measure extending ν, then $\mu([0,1] \setminus \mathbb{Q}) = 1$, and μ is tight (that is, inner regular with respect to the compact sets), so there is a closed subset F of $[0,1] \setminus \mathbb{Q}$ such that $\mu F > 0$. But F is nowhere dense, so $\nu F = 0$. **X**

third construction[8] There is a function $f : [0,1] \to \{0,1\}^{\mathfrak{c}}$ which is $(\mathcal{B}, \mathcal{B}\mathfrak{a})$-measurable, where $\mathcal{B}\mathfrak{a}$ is the Baire σ-algebra of $\{0,1\}^{\mathfrak{c}}$, and such that $f[\,[0,1]\,]$ meets every non-empty member of $\mathcal{B}\mathfrak{a}$. **P** Set $X = C([0,1])^{\mathbb{N}}$ with the product of the norm topologies, so that X is an uncountable Polish space (4A2Pe, 4A2Qc), and $([0,1], \mathcal{B})$ is isomorphic to $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ is the Borel σ-algebra of X (424Da). Define $g : X \to \{0,1\}^{[0,1]}$ by saying that $g(\langle u_i \rangle_{i \in \mathbb{N}})(t) = 1$ iff $\lim_{i \to \infty} u_i(t) = 1$. For each $t \in [0,1]$,

$$\{\langle u_i \rangle_{i \in \mathbb{N}} : \lim_{i \to \infty} u_i(t) = 1\} = \bigcap_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \{\langle u_i \rangle_{i \in \mathbb{N}} : |u_i(t) - 1| \le 2^{-m} \text{ for every } i \ge n\}$$

is a Borel subset of X, so g is $(\mathcal{B}(X), \mathcal{B}\mathfrak{a}(\{0,1\}^{[0,1]}))$-measurable, where $\mathcal{B}\mathfrak{a}(\{0,1\}^{[0,1]})$ is the Baire σ-algebra of $\{0,1\}^{[0,1]}$ (4A3Ne). If $E \in \mathcal{B}\mathfrak{a}(\{0,1\}^{[0,1]})$ is non-empty, there is a countable set $I \subseteq [0,1]$ such that E is determined by coordinates in I (4A3Nb), so that $E \supseteq \{w : w \restriction I = z\}$ for some $z \in \{0,1\}^I$. Now we can find a sequence $\langle u_i \rangle_{i \in \mathbb{N}}$ in $C([0,1])$ such that $\lim_{i \to \infty} u_i(t) = z(t)$ for every $t \in I$ (if $I \subseteq \{t_j : j \in \mathbb{N}\}$, take u_i such that $|u_i(t_j) - z(t_j)| \le 2^{-i}$ whenever $j \le i$), and in this case $g(\langle u_i \rangle_{i \in \mathbb{N}}) \in E$.

Because $(X, \mathcal{B}(X)) \cong ([0,1], \mathcal{B})$ and $(\{0,1\}^{[0,1]}, \mathcal{B}\mathfrak{a}(\{0,1\}^{[0,1]})) \cong (\{0,1\}^{\mathfrak{c}}, \mathcal{B}\mathfrak{a})$, we can copy g to a function f with the required properties. **Q**

In particular, $f[\,[0,1]\,]$ has full outer measure for the usual measure $\nu_{\mathfrak{c}}$ on $\{0,1\}^{\mathfrak{c}}$, because $\nu_{\mathfrak{c}}$ is completion regular (415E). Setting $\mathrm{T} = \{f^{-1}[H] : H \in \mathcal{B}\mathfrak{a}\}$, we have a measure ν with domain T such that f is inverse-measure-preserving for ν and $\nu_{\mathfrak{c}}$ (234F[9]). The map $H^\bullet \mapsto f^{-1}[H]^\bullet$ from the measure algebra of $\nu_{\mathfrak{c}}$ to the measure algebra of ν is measure-preserving; since it is surely surjective, the measure algebras are isomorphic, and ν has Maharam type $\mathfrak{c}$.

However, any probability measure on the whole algebra $\mathcal{B}$ has countable Maharam type (433A), so cannot extend ν.

Remark Compare 433J-433K.

[8] I am grateful to M.Laczkovich and D.Preiss for showing this to me.

[9] Formerly 132G.

439B Definition Let X be a Hausdorff space. I will call X **universally negligible** if there is no Borel probability measure μ defined on X such that $\mu\{x\} = 0$ for every $x \in X$. A subset of X will be 'universally negligible' if it is universally negligible in its subspace topology.

439C Proposition Let X be a Hausdorff space.

(a) If A is a subset of X, the following are equiveridical:

(i) A is universally negligible;

(ii) $\mu^*A = 0$ whenever μ is a Borel probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$;

(iii) $\mu^*A = 0$ whenever μ is a σ-finite topological measure on X such that $\mu\{x\} = 0$ for every $x \in A$;

(iv) for every σ-finite topological measure μ on X there is a countable set $B \subseteq A$ such that $\mu^*A = \mu B$;

(v) A is a Radon space and every compact subset of A is scattered.

In particular, countable subsets of X are universally negligible.

(b) The family of universally negligible subsets of X is a σ-ideal.

(c) Suppose that Y is a universally negligible Hausdorff space and $f : X \to Y$ a Borel measurable function such that $f^{-1}[\{y\}]$ is universally negligible for every $y \in Y$. Then X is universally negligible.

(d) If the topology on X is discrete, X is universally negligible iff $\#(X)$ is measure-free.

proof (a)(i)⇒(iii) If A is universally negligible and μ is a σ-finite topological measure on X such that $\mu\{x\} = 0$ for every $x \in A$, let μ_A be the subspace measure on A. **?** If $\mu^*A = \alpha > 0$, then (because μ is σ-finite) there is a measurable set $E \subseteq X$ such that $\gamma = \mu^*(E \cap A)$ is finite and non-zero. The subspace measure $\mu_{E\cap A}$ is a topological measure on $E \cap A$; set $\nu F = \gamma^{-1}\mu_{E\cap A}(E \cap F)$ for relatively Borel sets $F \subseteq A$; then ν is a Borel probability measure on A which is zero on singletons. **X** So $\mu^*A = 0$.

(iii)⇒(iv) If (iii) is true and μ is a σ-finite topological measure on X, set $B = \{x : x \in X,\ \mu\{x\} > 0\}$. Because μ is σ-finite, B must be countable, therefore measurable, and if we set $\nu E = \mu(E \setminus B)$ for every Borel set $E \subseteq X$, ν is a σ-finite Borel measure on X and $\nu\{x\} = 0$ for every $x \in X$. By (iii), $\nu^*A = 0$, that is, there is a Borel set $E \supseteq A$ such that $\mu(E \setminus B) = 0$; in which case

$$\mu^*A \le \mu(E \setminus (B \setminus A)) = \mu(E \setminus B) + \mu(A \cap B) = \mu(A \cap B) \le \mu^*A,$$

so $\mu^*A = \mu(A \cap B)$. As μ is arbitrary, (iv) is true.

(iv)⇒(ii) is trivial.

not-(i)⇒not-(ii) If A is not universally negligible, let μ be a Borel probability measure on A which is zero on singletons. Set $\nu E = \mu(E \cap A)$ for any Borel set $E \subseteq X$; then ν is a Borel probability measure on X which is zero on singletons, and $\nu^*A = 1$.

(i)⇒(v) Suppose that A is universally negligible. Let μ be a totally finite Borel measure on A. Applying (i)⇒(iv) with $X = A$, we see that there is a countable set $B \subseteq A$ such that $\mu B = \mu A$; but this means that μ is inner regular with respect to the finite subsets of B, which of course are compact. As μ is arbitrary, A is a Radon space. **?** Suppose, if possible, that A has a compact set K which is not scattered. In this case there is a continuous surjection $f : K \to [0, 1]$ (4A2G(j-iv)). Now there is a Radon probability measure ν on K such that f is inverse-measure-preserving for ν and Lebesgue measure on $[0, 1]$ and induces an isomorphism of the measure algebras, so that ν is atomless (418L). Accordingly we have a Borel probability measure μ on A defined by setting $\mu E = \nu(K \cap E)$ for every relatively Borel set $E \subseteq A$, and $\mu\{x\} = 0$ for every $x \in A$, so A is not universally negligible. **X** Thus all compact subsets of A are scattered, and (v) is true.

(v)⇒(i) Now suppose that (v) is true and that that μ is a Borel probability measure on A. Then μ has an extension to a Radon measure $\tilde{\mu}$ (434F(a-iii)). Let $K \subseteq A$ be a non-empty compact set which is self-supporting for $\tilde{\mu}$ (416Dc). K is scattered, so has an isolated point $\{x\}$; because K is self-supporting, $\mu\{x\} = \tilde{\mu}\{x\} > 0$. As μ is arbitrary, A is universally negligible.

(b) This is immediate from (a-ii).

(c) Let μ be a Borel probability measure on X. Then $F \mapsto \nu f^{-1}[F]$ is a Borel probability measure on Y. Because Y is universally negligible, there must be a $y \in Y$ such that $\mu f^{-1}[\{y\}] > 0$. Set $E = f^{-1}[\{y\}]$ and let μ_E be the subspace measure on E. Then μ_E is a non-zero totally finite Borel measure on E. Since E is supposed to be universally negligible, there must be some $x \in E$ such that $0 < \mu_E\{x\} = \mu\{x\}$.

(d) This is just a re-phrasing of the definition in 438A.

439D Remarks (a) The following will be useful when interpreting the definition in 439B. Let X be a hereditarily Lindelöf Hausdorff space and μ a topological probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$. Then μ is atomless.

P Suppose that $\mu H > 0$. Write

$$\mathcal{G} = \{G : G \subseteq X \text{ is open, } \mu(G \cap H) = 0\}.$$

Then there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\bigcup \mathcal{G}_0 = \bigcup \mathcal{G}$ (4A2H(c-i)), so

$$\mu(H \cap \bigcup \mathcal{G}) = \mu(H \cap \bigcup \mathcal{G}_0) = 0,$$

and $\mu(H \setminus \bigcup \mathcal{G}) > 0$. Because μ is zero on singletons, $H \setminus \bigcup \mathcal{G}$ has at least two points x, y say. Now there are disjoint open sets G_0, G_1 containing x, y respectively, and neither belongs to $\mathcal{G}$, so $H \cap G_0$, $H \cap G_1$ are disjoint subsets of H of positive measure. Thus H is not an atom. As H is arbitrary, μ is atomless. **Q**

(b) The obvious applications of (a) are when X is separable and metrizable; but, more generally, we can use it on any Hausdorff space with a countable network, e.g., on any analytic space.

439E Lemma (a) Let E, $B \subseteq \mathbb{R}$ be such that E is measurable and $\mu_L E$, $\mu_L^* B$ are both greater than 0, where μ_L is Lebesgue measure. Then $E - B = \{x - y : x \in E,\ y \in B\}$ includes a non-trivial interval.

(b) If $A \subseteq \mathbb{R}$ and $\mu_L^* A > 0$, then $A + \mathbb{Q}$ is of full outer measure in $\mathbb{R}$.

proof (a) By 223B or 261Da, there are $a \in E$, $b \in B$ such that

$$\lim_{\delta \downarrow 0} \frac{1}{2\delta} \mu(E \cap [a - \delta, a + \delta]) = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \mu^*(B \cap [b - \delta, b + \delta]) = 1.$$

Let $\gamma > 0$ be such that

$$\mu_L(E \cap [a - \delta, a + \delta]) > \frac{3}{2}\delta, \quad \mu_L^*(B \cap [b - \delta, b + \delta]) > \frac{3}{2}\delta$$

whenever $0 < \delta \le \gamma$. Now suppose that $0 < \delta \le \gamma$. Then

$$\begin{aligned}\mu_L((E + b) \cap [a + b, a + b + \delta]) &= \mu_L(E \cap [a, a + \delta]) \\ &\ge \mu_L(E \cap [a - \delta, a + \delta]) - \delta > \frac{1}{2}\delta,\end{aligned}$$

and similarly

$$\begin{aligned}\mu_L^*((B + a + \delta) \cap [a + b, a + b + \delta]) &= \mu_L^*(B \cap [b - \delta, b]) \\ &\ge \mu_L^*(B \cap [b - \delta, b + \delta]) - \delta > \frac{1}{2}\delta.\end{aligned}$$

But this means that $(E + b) \cap (B + a + \delta)$ cannot be empty. If $u \in (E + b) \cap (B + a + \delta)$, then $u - b \in E$, $u - a - \delta \in B$ so

$$a - b + \delta = (u - b) - (u - a - \delta) \in E - B.$$

As δ is arbitrary, $E - B$ includes the interval $]a - b, a - b + \gamma]$.

(b) ? Suppose, if possible, otherwise; that there is a measurable set $E \subseteq \mathbb{R}$ such that $\mu_L E > 0$ and $E \cap (A + \mathbb{Q}) = \emptyset$. Then $E - A$ does not meet $\mathbb{Q}$ and cannot include any non-trivial interval. **X**

Remark There will be a dramatic generalization of (a) in 443Db.

439F Proposition Let κ be the least cardinal of any set of non-zero Lebesgue outer measure in $\mathbb{R}$.

(a) There is a set $X \subseteq [0, 1]$ of cardinal κ and full outer Lebesgue measure.

(b) If (Z, T, ν) is any atomless complete locally determined measure space and $A \subseteq Z$ has cardinal less than κ, then $\nu^* A = 0$.

(c) (Grzegorek 81) There is a universally negligible set $Y \subseteq [0, 1]$ of cardinal κ.

proof (a) Take any set $A \subseteq \mathbb{R}$ such that $\#(A) = \kappa$ and $\mu_L^* A > 0$, where μ_L is Lebesgue measure. Set $B = A + \mathbb{Q}$. Then $(\mu_L)_*(\mathbb{R} \setminus B) = 0$, by 439Eb. Set $X = [0, 1] \cap B$; then $\mu_L^* X = 1$ while $\#(X) \le \#(B) = \kappa$. By the definition of κ, $\#(X)$ must be exactly κ.

(b) ? Otherwise, by 412Jc, there is a set $F \subseteq Z$ such that $\nu F < \infty$ and $\nu^*(F \cap A) > 0$. By 343Cc, there is a function $f : F \to [0, \nu F]$ which is inverse-measure-preserving for the subspace measure ν_F and Lebesgue measure on $[0, \nu F]$. But $f[A \cap F]$ has cardinal less than κ, so $\mu_L f[A \cap F] = 0$ and

$$0 < \nu^*(A \cap F) \le \nu f^{-1}[f[A \cap F]] = 0,$$

which is absurd. **X**

(c)(i) Enumerate X as $\langle x_\xi \rangle_{\xi<\kappa}$. For each $\xi < \kappa$, $A_\xi = \{x_\eta : \eta \le \xi\}$ has cardinal less than κ, so is Lebesgue negligible; let $\langle I_{\xi n} \rangle_{n\in\mathbb{N}}$ be a sequence of intervals covering A_ξ with $\sum_{n=0}^\infty \mu_L I_{\xi n} < \frac{1}{2}$. Enlarging the intervals slightly if necessary, we may suppose that every $I_{\xi n}$ has rational endpoints; let $\langle J_m \rangle_{m\in\mathbb{N}}$ enumerate the family of intervals in $\mathbb{R}$ with rational endpoints.

Set

$$C_{mn} = \{\xi : \xi < \kappa,\, I_{\xi n} = J_m\}$$

for each m, $n \in \mathbb{N}$.

(ii) If ν is an atomless totally finite measure on κ which measures every C_{mn}, then $\nu\kappa = 0$. **P** Note first that (by (b), applied to the completion of ν) $\nu^*\xi = 0$ for every $\xi < \kappa$. Let λ be the (c.l.d.) product of μ_X, the subspace measure on X, with ν. Set

$$B = \bigcup_{m,n\in\mathbb{N}}((X \cap J_m) \times C_{mn}) \subseteq X \times \kappa.$$

Then B is measured by λ, so, by Fubini's theorem,

$$\int \nu B[\{x\}]\mu_X(dx) = \int \mu_X B^{-1}[\{\xi\}]\nu(d\xi)$$

(252D).

Now look at the sectional measures $\nu B[\{x\}]$, $\mu_X B^{-1}[\{\xi\}]$. (Because B is actually a countable union of measurable rectangles, these are always defined.) For any $x \in X$, there is an $\eta < \kappa$ such that $x = x_\eta$, and now

$$\begin{aligned} B[\{x\}] &= \{\xi : \text{ there are } m,\, n \in \mathbb{N} \text{ such that } x \in J_m \text{ and } \xi \in C_{mn}\} \\ &= \{\xi : \text{ there are } m,\, n \in \mathbb{N} \text{ such that } x \in J_m \text{ and } I_{\xi n} = J_m\} \\ &= \{\xi : \text{ there is an } n \in \mathbb{N} \text{ such that } x \in I_{\xi n}\} \supseteq \kappa \setminus \eta \end{aligned}$$

by the choice of the $I_{\xi n}$. But as $\nu^*\eta = 0$, this means that $\nu B[\{x\}] = \nu\kappa$.

On the other hand, if $\xi < \kappa$, then

$$\begin{aligned} B^{-1}[\{\xi\}] &= \{x : \text{ there are } m,\, n \in \mathbb{N} \text{ such that } x \in J_m \text{ and } \xi \in C_{mn}\} \\ &= \{x : \text{ there are } m,\, n \in \mathbb{N} \text{ such that } x \in J_m \text{ and } I_{\xi n} = J_m\} \\ &= \{x : \text{ there is an } n \in \mathbb{N} \text{ such that } x \in I_{\xi n}\} = X \cap \bigcup_{n\in\mathbb{N}} I_{\xi n}, \end{aligned}$$

so that

$$\mu_X B^{-1}[\{\xi\}] \le \sum_{n=0}^\infty \mu_L I_{\xi n} \le \frac{1}{2}.$$

Returning to the integrals, we have

$$\nu\kappa = \int \nu B[\{x\}]\mu_X(dx) = \int \mu_X B^{-1}[\{\xi\}]\nu(d\xi) \le \frac{1}{2}\nu\kappa,$$

so that $\nu\kappa$ must be 0, as claimed. **Q**

(iii) Now there is an injective function $g : \kappa \to [0, 1]$ such that $g[C_{mn}]$ is relatively Borel in $g[\kappa]$ for every m, $n \in \mathbb{N}$. **P** Define $h : \kappa \to \{0, 1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ by setting

$$\begin{aligned} h(\xi)(m, n, k) &= 1 \text{ if } \xi \in C_{mn} \text{ and } x_\xi \in J_k, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then h is injective (because if $\xi \ne \eta$ then $x_\xi \ne x_\eta$, so there is some k such that $x_\xi \in J_k$ and $x_\eta \notin J_k$), and

$$h[C_{mn}] = h[\kappa] \cap \{w : \text{ there is some } k \text{ such that } w(m, n, k) = 1\}$$

is relatively Borel in $h[\kappa]$ for every m, $n \in \mathbb{N}$. But now recall that $\{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set $C \subseteq [0,1]$ (4A2Uc). If $\phi : \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \to C$ is any homeomorphism, then ϕh has the required properties. **Q**

(iv) Set $Y = g[\kappa]$. Because g is injective, $\#(Y) = \kappa$. Also Y is universally negligible. **P** Suppose that $\tilde{\nu}$ is a Borel measure on Y which is zero on singletons. Then it is atomless, because Y is separable and metrizable (439D). So its copy $\nu = \tilde{\nu}(g^{-1})^{-1}$ on κ is atomless. Because $g[C_{mn}]$ is a Borel subset of Y, ν measures C_{mn} for all m, $n \in \mathbb{N}$, so $\tilde{\nu}Y = \nu\kappa = 0$, by (ii) above. **Q**

439G Corollary A metrizable continuous image of a universally negligible metrizable space need not be universally negligible.

proof Take X and Y from 439Fa and 439Fc above, and let $f : X \to Y$ be any bijection. Let Γ be the graph of f. The projection map $(x, y) \mapsto y : \Gamma \to Y$ is continuous and injective, so Γ is universally negligible, by 439Cc. On the other hand, the projection map $(x, y) \mapsto x : \Gamma \to X$ is continuous and surjective, and X is surely not universally negligible, since it is not Lebesgue negligible.

439H Corollary One-dimensional Hausdorff measure on $\mathbb{R}^2$ is not semi-finite.

proof Let μ_{H1} be one-dimensional Hausdorff measure on $\mathbb{R}^2$. Let X, Γ be the sets described in 439F and the proof of 439G.

(a) $\mu^*_{H1}\Gamma > 0$. **P** The first-coordinate map $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is 1-Lipschitz, so, writing μ_L for Lebesgue measure on $\mathbb{R}$,

$$1 = \mu^*_L X = \mu^*_L \pi_1[\Gamma] \le \mu^*_{H1}\Gamma$$

by 264G/471J and 264I. **Q**

(b) If $E \subseteq \mathbb{R}^2$ and $\mu_{H1}E < \infty$, then $\mu_{H1}(E \cap \Gamma) = 0$, because $E \cap \Gamma$ is universally negligible (439Cb) and μ_{H1} is a topological measure (264E/471Da) which is zero on singletons.

(c) ? Suppose, if possible, that Γ is not measured by μ_{H1}. Then there is a set $A \subseteq \mathbb{R}^2$ such that $\mu^*_{H1}A < \mu^*_{H1}(A \cap \Gamma) + \mu^*_{H1}(A \setminus \Gamma)$ (264C/471A, 264Fb/471Dc). Let E be a Borel set including A such that $\mu_{H1}E = \mu^*_{H1}A$ (264Fa/471Db); then $\mu_{H1}(E \cap \Gamma) = 0$, so

$$\mu^*_{H1}(A \cap \Gamma) + \mu^*_{H1}(A \setminus \Gamma) \le \mu_{H1}(E \cap \Gamma) + \mu^*_{H1}A = \mu^*_{H1}A. \quad \mathbf{X}$$

(d) Since Γ is measurable, not negligible, and meets every measurable set of finite measure in a negligible set, it is purely infinite, and μ_{H1} is not semi-finite.

439I Example There are a set X, a Riesz subspace U of $\mathbb{R}^X$ and a smooth positive linear functional $h : U \to \mathbb{R}$ which is not expressible as an integral.

proof Take X and Y from 439F. Replacing Y by $Y \setminus \{0\}$ if need be, we may suppose that $0 \notin Y$. Let $f : X \to Y$ be any bijection.

Let U be the Riesz subspace $\{u \times f : u \in C_b\} \subseteq \mathbb{R}^X$, where C_b is the space of bounded continuous functions from X to $\mathbb{R}$. Because f is strictly positive, $u \mapsto u \times f : C_b \to U$ is a bijection, therefore a Riesz space isomorphism; moreover, for a non-empty set $A \subseteq C_b$, $\inf_{u \in A} u(x) = 0$ for every $x \in X$ iff $\inf_{u \in A} u(x)f(x) = 0$ for every $x \in X$. We therefore have a smooth linear functional $h : U \to \mathbb{R}$ defined by setting $h(u \times f) = \int u \, d\mu_X$ for every $u \in C_b$, where μ_X is the subspace measure on X induced by Lebesgue measure. (By 415B, μ_X is quasi-Radon, so the integral it defines on C_b is smooth, as noted in 436H.)

? But suppose, if possible, that h is the integral with respect to some measure ν on X. Since $f \in U$, it must be T-measurable, where T is the domain of the completion $\hat{\nu}$ of ν. Note that $\hat{\nu}\{x\} = 0$ for every $x \in X$. **P** Set $u_n(y) = \max(0, 1 - 2^n|y - x|)$ for $y \in X$. Then

$$f(x)\hat{\nu}\{x\} = \lim_{n\to\infty} \int u_n \times f \, d\nu = \lim_{n\to\infty} h(u_n \times f)$$
$$= \lim_{n\to\infty} \int u_n \, d\mu_X = \mu_X\{x\} = 0,$$

so $\hat{\nu}\{x\} = 0$. **Q**

For Borel sets $E \subseteq [0,1]$ set $\lambda E = \hat{\nu} f^{-1}[E]$. Then the completion $\hat{\lambda}$ of λ is a Radon measure on $[0,1]$ (433Cb or 256C). If $t \in [0,1]$ then $f^{-1}[\{t\}]$ contains at most one point, so $\hat{\lambda}\{t\} = \lambda\{t\} = 0$. But Y is supposed to be universally negligible, so $\lambda^* Y = \hat{\lambda}^* Y = 0$ (439Ca), that is, there is a Borel set $E \supseteq Y$ with $\lambda E = 0$; in which case $\nu X = \hat{\nu} f^{-1}[E] = 0$, which is impossible. **X**

Thus h is not an integral, despite being a smooth linear functional on a Riesz subspace of $\mathbb{R}^X$.

Remark This example is adapted from Fremlin & Talagrand 78.

439J Example Assume that there is some cardinal κ which is not measure-free. Give κ its discrete topology, and let μ be a probability measure with domain $\mathcal{P}\kappa$ such that $\mu\{\xi\} = 0$ for every $\xi < \kappa$. Now every subset of κ is open-and-closed, so μ is simultaneously a Baire probability measure and a completion regular Borel probability measure. Of course it is not τ-additive. In the classification schemes of 434A and 435A, we have a measure which is of type B_1 as a Borel measure and type E_3 as a Baire measure.

439K Example There is a first-countable compact Hausdorff space which is not Radon.

proof The construction starts from a compact metrizable space $(Z, \mathfrak{S})$ with an atomless Radon probability measure μ. The obvious candidate is $[0,1]$ with Lebesgue measure; but for technical convenience in a later application I will instead use $Z = \{0,1\}^{\mathbb{N}}$ with its usual product topology and measure (254J).

(a) There is a topology $\mathfrak{T}_{\mathfrak{c}}$ on Z such that

(α) $\mathfrak{S} \subseteq \mathfrak{T}_{\mathfrak{c}}$;

(β) every point of Z belongs to a countable set which is compact and open for $\mathfrak{T}_{\mathfrak{c}}$;

(γ) if $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence of $\mathfrak{T}_{\mathfrak{c}}$-closed sets with empty intersection, then $\bigcap_{n\in\mathbb{N}} \overline{F}_n^{\mathfrak{S}}$ is countable, where I write $\overline{F}^{\mathfrak{S}}$ for the $\mathfrak{S}$-closure of F.

P(i) $\mathfrak{T}_{\mathfrak{c}}$ will be the last in a family $\langle \mathfrak{T}_\xi \rangle_{\xi\le\mathfrak{c}}$ of topologies. We must begin by enumerating Z as $\langle z_\xi \rangle_{\xi<\mathfrak{c}}$ and taking a family $\langle\langle I_{\xi n}\rangle_{n\in\mathbb{N}}\rangle_{\xi<\mathfrak{c}}$ running over $([Z]^{\le\omega})^{\mathbb{N}}$ with cofinal repetitions, where $[Z]^{\le\omega}$ is the family of countable subsets of Z. (This can be done because $\#([Z]^{\le\omega}) = \mathfrak{c}$, by 2A1Hb.) Together with $\langle \mathfrak{T}_\xi \rangle_{\xi\le\mathfrak{c}}$ we choose simultaneously families $\langle x_\xi \rangle_{\xi<\mathfrak{c}}$, $\langle y_\xi \rangle_{\xi<\mathfrak{c}}$ of points in Z, and the inductive hypothesis will be

$\mathfrak{T}_\xi$ is a topology on $X_\xi = \{x_\eta : \eta < \xi\} \cup \{y_\eta : \eta < \xi\}$ finer than the topology on X_ξ induced by $\mathfrak{S}$;

if $\eta < \xi \le \mathfrak{c}$, then $X_\eta \in \mathfrak{T}_\xi$ and $\mathfrak{T}_\eta$ is the subspace topology on X_η induced by $\mathfrak{T}_\xi$;

every point of X_ξ belongs to a countable set which is compact and open for $\mathfrak{T}_\xi$.

The induction starts with $X_0 = \emptyset$, $\mathfrak{T}_0 = \{\emptyset\}$.

(ii) *Inductive step to a successor ordinal* Suppose that we have found X_ξ and $\mathfrak{T}_\xi$ where $\xi < \mathfrak{c}$.

(α) Start by picking $y_\xi \in Z \setminus X_\xi$ such that $y_\xi = z_\xi$ if $z_\xi \notin X_\xi$. Examine the sequence $\langle I_{\xi n} \rangle_{n\in\mathbb{N}}$. If *either* $\bigcup_{n\in\mathbb{N}} I_{\xi n} \not\subseteq X_\xi$ *or* $\bigcap_{n\in\mathbb{N}} \overline{I}_{\xi n}^{\mathfrak{S}}$ is countable, take x_ξ to be any point of $Z \setminus (X_\xi \cup \{y_\xi\})$ and set $K_m = \emptyset$ for every m before proceeding to (γ) below.

(β) If $I_{\xi n} \subseteq X_\xi$ for every n and $\bigcap_{n\in\mathbb{N}} \overline{I}_{\xi n}^{\mathfrak{S}}$ is uncountable, it must have cardinal $\mathfrak{c}$, by 423K, so cannot be included in $X_\xi \cup \{y_\xi\}$. Take any $x_\xi \in \bigcap_{n\in\mathbb{N}} \overline{I}_{\xi n}^{\mathfrak{S}} \setminus (X_\xi \cup \{y_\xi\})$. Let $\langle t_m \rangle_{m\in\mathbb{N}}$ be a sequence in Z such that $t_m \restriction m = x_\xi \restriction m$ for every $m \in \mathbb{N}$ and $t_m \in I_{\xi n}$ whenever $r \in \mathbb{N}$, $n \le 2r$ and $m = r^2 + n$. (Thus, for each n, $t_m \in I_{\xi n}$ for infinitely many m, while $\langle t_m \rangle_{m\in\mathbb{N}} \to x_\xi$ in the ordinary sense.) By the inductive hypothesis, we can find countable sets $K_m \subseteq X_\xi$, compact and open for $\mathfrak{T}_\xi$, such that $t_m \in K_m$ for each m. Because $\{t : t \in X_\xi,\ t\restriction m = x_\xi \restriction m\}$ is open-and-closed for $\mathfrak{T}_\xi$ and contains t_m, we may suppose that $t \restriction m = x_\xi \restriction m$ for every $t \in K_m$.

(γ) Let $\mathfrak{T}_{\xi+1}$ be the topology on $X_{\xi+1} = X_\xi \cup \{x_\xi, y_\xi\}$ generated by

$$\mathfrak{T}_\xi \cup \{\{y_\xi\}\} \cup \{L_n : n \in \mathbb{N}\},$$

where $L_n = \{x_\xi\} \cup \bigcup_{m\ge n} K_m$ for each n.

(δ) Because $\mathfrak{T}_\xi \subseteq \mathfrak{T}_{\xi+1}$, X_ξ will be open in $X_{\xi+1}$. Because the K_m are always $\mathfrak{T}_\xi$-open, and x_ξ, y_ξ are distinct points of $Z \setminus X_\xi$, the topology on X_ξ induced by $\mathfrak{T}_{\xi+1}$ is just $\mathfrak{T}_\xi$. Consequently (by the inductive hypothesis) the topology on X_η induced by $\mathfrak{T}_{\xi+1}$ is $\mathfrak{T}_\eta$ for every $\eta \le \xi$. We have $t\restriction n = x_\xi \restriction n$ for every $t \in L_n$, so $\mathfrak{T}_{\xi+1}$ is finer than the usual topology on $X_{\xi+1}$.

If $x \in X_\xi$, then there is a countable $\mathfrak{T}_\xi$-open $\mathfrak{T}_\xi$-compact set containing x, which is still $\mathfrak{T}_{\xi+1}$-open and $\mathfrak{T}_{\xi+1}$-compact. Of course $\{y_\xi\}$ is a countable $\mathfrak{T}_{\xi+1}$-open $\mathfrak{T}_{\xi+1}$-compact set containing y_ξ. As for x_ξ, L_0 is surely countable and $\mathfrak{T}_{\xi+1}$-open. To see that it is $\mathfrak{T}_{\xi+1}$-compact, observe that any ultrafilter containing L_0 either contains every L_n, and converges to x_ξ, or contains some K_m and converges to a point of K_m.

Thus the induction proceeds at successor stages.

(iii) *Inductive step to a limit ordinal* If $\xi \le \mathfrak{c}$ is a non-zero limit ordinal, then we have $X_\xi = \bigcup_{\eta<\xi} X_\eta$, and can take $\mathfrak{T}_\xi$ to be the topology generated by $\bigcup_{\eta<\xi}\mathfrak{T}_\eta$. It is easy to check that this works (because the topologies $\mathfrak{T}_\eta$ are consistent with each other).

(iv) At the end of the induction, we have $X_\mathfrak{c} = Z$ because $z_\xi \in X_{\xi+1} \subseteq X_\mathfrak{c}$ for every ξ. The final topology $\mathfrak{T}_\mathfrak{c}$ on Z will have the properties (α) and (β) required. **?** Now suppose, if possible, that $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence of $\mathfrak{T}_\mathfrak{c}$-closed sets with empty intersection, and that $\bigcap_{n\in\mathbb{N}} \overline{F}_n^{\mathfrak{S}}$ is uncountable. For each $n \in \mathbb{N}$, let $J_n \subseteq F_n$ be a countable $\mathfrak{S}$-dense set. Then there is some $\zeta < \mathfrak{c}$ such that $\bigcup_{n\in\mathbb{N}} J_n \subseteq X_\zeta$ (because $\operatorname{cf}\mathfrak{c} > \omega$, see 4A1A(c-iii)). Let $\xi \ge \zeta$ be such that $J_n = I_{\xi n}$ for every $n \in \mathbb{N}$. Then in the construction of $\mathfrak{T}_{\xi+1}$ we must be in case (β) of (ii) above. Taking $\langle t_m\rangle_{m\in\mathbb{N}}$ as described there, we have $\langle t_m\rangle_{m\in\mathbb{N}} \to x_\xi$ for $\mathfrak{T}_{\xi+1}$, and therefore for $\mathfrak{T}_\mathfrak{c}$. But for any $n \in \mathbb{N}$, $t_m \in J_n \subseteq F_n$ for infinitely many m, so $x_\xi \in F_n$. Thus $x_\xi \in \bigcap_{n\in\mathbb{N}} F_n$; but this is impossible. **X**

So we have a topology of the type required. **Q**

(b) There is a probability measure ν on Z, extending the usual measure μ, such that with respect to $\mathfrak{T}_\mathfrak{c}$ ν is a topological measure inner regular with respect to the closed sets, but is not τ-additive.

P Let $\mathcal{K}$ be the family of $\mathfrak{T}_\mathfrak{c}$-closed subsets of Z. For $F \in \mathcal{K}$, set $\phi F = \mu\overline{F}^{\mathfrak{S}}$.

(i) If E, F are disjoint $\mathfrak{T}_\mathfrak{c}$-closed sets, then $\overline{E}^{\mathfrak{S}} \cap \overline{F}^{\mathfrak{S}}$ must be countable (take $F_{2n} = E$, $F_{2n+1} = F$ in (a-γ)). So

$$\begin{aligned}\phi(E\cup F) &= \mu\overline{E\cup F}^{\mathfrak{S}} = \mu\overline{E}^{\mathfrak{S}} + \mu\overline{F}^{\mathfrak{S}} - \mu(\overline{E}^{\mathfrak{S}} \cap \overline{F}^{\mathfrak{S}})\\ &= \mu\overline{E}^{\mathfrak{S}} + \mu\overline{F}^{\mathfrak{S}} = \phi E + \phi F.\end{aligned}$$

(ii) If E, $F \in \mathcal{K}$, $E \subseteq F$ and $\epsilon > 0$, there is an $\mathfrak{S}$-open set $G \supseteq \overline{E}^{\mathfrak{S}}$ such that $\mu G \le \mu\overline{E}^{\mathfrak{S}} + \epsilon$. Now $F \setminus G \in \mathcal{K}$ and

$$\phi F = \mu(\overline{F}^{\mathfrak{S}} \cap G) + \mu(\overline{F}^{\mathfrak{S}} \setminus G) \le \mu G + \phi(F\setminus G) \le \phi E + \phi(F\setminus G) + \epsilon.$$

Putting this together with (i), we see that

$$\phi F = \phi E + \sup\{\phi E' : E' \in \mathcal{K},\ E' \subseteq F \setminus E\}.$$

(iii) If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, then

$$\lim_{n\to\infty}\phi F_n = \lim_{n\to\infty}\mu\overline{F}_n^{\mathfrak{S}} = \mu(\bigcap_{n\in\mathbb{N}}\overline{F}_n^{\mathfrak{S}}) = 0.$$

(iv) Thus $\mathcal{K}$ and ϕ satisfy all the conditions of 413I, and there is a measure ν, extending ϕ, which is defined on every member of $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, and therefore is (for $\mathfrak{T}_\mathfrak{c}$) a topological measure inner regular with respect to the closed sets.

If we write $\mathcal{V}$ for the family of $\mathfrak{T}_\mathfrak{c}$-compact $\mathfrak{T}_\mathfrak{c}$-open countable subsets of Z, then for any $K \in \mathcal{V}$

$$\nu K = \phi K = \mu K = 0,$$

while $\mathcal{V}$ is upwards-directed and has union Z; so that ν is not τ-additive. **Q**

(c) So far we seem to have very little more than is provided by ω_1 with the order topology and Dieudonné's measure. The point of doing all this work is the next step. Set $X = Z \times \{0,1\}$ and give X the topology $\mathfrak{T}$ generated by

$$\{G\times\{0,1\} : G \in \mathfrak{S}\} \cup \{H\times\{1\} : H \in \mathfrak{T}_\mathfrak{c}\} \cup \{X\setminus(K\times\{1\}) : K \text{ is } \mathfrak{T}_\mathfrak{c}\text{-compact}\}.$$

(i) $\mathfrak{T}$ is Hausdorff. **P** If w, z are distinct points of X, then either their first coordinates differ and they are separated by sets of the form $G_0\times\{0,1\}$, $G_1\times\{0,1\}$ where G_0, G_1 belong to $\mathfrak{S}$, or they are of the form $(x,1)$, $(x,0)$ and are separated by open sets of the form $K\times\{1\}$, $X\setminus(K\times\{1\})$ for some set K which is compact and open for $\mathfrak{T}_\mathfrak{c}$. **Q**

(ii) $\mathfrak{T}$ is compact. **P** Let $\mathcal{F}$ be an ultrafilter on X. Writing $\pi_1(x,0) = \pi_1(x,1) = x$ for $x \in Z$, $\pi_1[[\mathcal{F}]]$ is $\mathfrak{S}$-convergent, to x_0 say. If $K \times \{1\} \in \mathcal{F}$ for some $\mathfrak{T}_{\mathfrak{c}}$-compact set K, then $\mathcal{F}$ is $\mathfrak{T}$-convergent to $(x,1)$; otherwise, it is $\mathfrak{T}$-convergent to $(x,0)$ (using 4A2B(a-iv)). **Q**

(iii) $\mathfrak{T}$ is first-countable. **P** If $x \in Z$, then $\{(x,0),(x,1)\} = \pi_1^{-1}[\{x\}]$ is a G_δ set in X because $\{x\}$ is a G_δ set in Z and π_1 is continuous (4A2C(a-iii)). Now $\{(x,0)\}$ and $\{(x,1)\}$ are relatively open in $\{(x,0),(x,1)\}$, so are G_δ sets in X (4A2C(a-iv)). Thus singletons are G_δ sets. Because $\mathfrak{T}$ is compact and Hausdorff, it is first-countable (4A2Kf). **Q**

(iv) $(X,\mathfrak{T})$ is not a Radon space. **P** $Z \times \{1\}$ is an open subset of X, homeomorphic to Z with the topology $\mathfrak{T}_{\mathfrak{c}}$. But the measure ν of (b) above (or, if you prefer, its restriction to the $\mathfrak{T}_{\mathfrak{c}}$-Borel algebra) witnesses that $\mathfrak{T}_{\mathfrak{c}}$ is not a Radon topology, so $\mathfrak{T}$ also cannot be a Radon topology, by 434Fc. **Q**

Remark Aficionados will recognise $\mathfrak{T}_{\mathfrak{c}}$ as a kind of 'JKR-space', derived from the construction in Juhász Kunen & Rudin 76.

439L Example Suppose that κ is a cardinal which is not measure-free; let μ be a probability measure with domain $\mathcal{P}\kappa$ which is zero on singletons. Give κ its discrete topology, so that μ is a Borel measure and κ is first-countable. Let ν be the restriction of the usual measure on $Y = \{0,1\}^\kappa$ to the algebra $\mathcal{B}$ of Borel subsets of Y, so that ν is a τ-additive probability measure, and λ the product measure on $\kappa \times Y$ constructed by the method of 434R. Then

$$W = \{(\xi,y) : \xi < \kappa,\, y(\xi) = 1\} = \bigcup_{\xi<\kappa} \{\xi\} \times \{y : y(\xi) = 1\}$$

is open in $\kappa \times Y$.

If $W' \in \mathcal{P}\kappa \widehat{\otimes} \mathcal{B}$ then $\lambda(W \triangle W') = \frac{1}{2}$. **P** There is a countable set $\mathcal{E} \subseteq \mathcal{B}$ such that W' belongs to the σ-algebra generated by $\{A \times E : A \subseteq \kappa,\, E \in \mathcal{E}\}$ (331Gd). For $J \subseteq \kappa$, write $\pi_J(y) = y\restriction J$ for $y \in Y$, let ν_J be the usual measure on $\{0,1\}^J$ and T_J its domain, and let T'_J be the family of sets $E \subseteq Y$ such that there are $H,\, H' \in \mathrm{T}_J$ such that $\pi_J^{-1}[H] \subseteq E \subseteq \pi_J^{-1}[H']$ and $\nu_J(H' \setminus H) = 0$. Then $\mathrm{T}'_J \subseteq \mathrm{T}'_K$ whenever $J \subseteq K \subseteq \kappa$, and every set measured by ν belongs to T'_J for some countable J (254Ob). There is therefore a countable set $J \subseteq \kappa$ such that $\mathcal{E} \subseteq \mathrm{T}'_J$. Also, of course, T'_J is a σ-algebra of subsets of Y.

The set

$$\{V : V \subseteq \kappa \times Y,\, V[\{\xi\}] \in \mathrm{T}'_J \text{ for every } \xi < \kappa\}$$

is a σ-algebra of subsets of $\kappa \times Y$ containing $A \times E$ whenever $A \subseteq \kappa$ and $E \in \mathcal{E}$, so contains W'. But this means that if $\xi \in \kappa \setminus J$, $W[\{\xi\}]$ and $W'[\{\xi\}]$ are stochastically independent, and $\nu(W[\{\xi\}] \triangle W'[\{\xi\}]) = \frac{1}{2}$. Since $\mu(\kappa \setminus J) = 1$,

$$\lambda(W \triangle W') = \int \nu(W[\{\xi\}] \triangle W'[\{\xi\}])\mu(d\xi) = \frac{1}{2},$$

as claimed. **Q**

In particular, $W^\bullet$ in the the measure algebra of λ cannot be represented by a member of $\mathcal{P}\kappa \widehat{\otimes} \mathcal{B}$.

439M Example There is a first-countable locally compact Hausdorff space X with a Baire probability measure μ which is not τ-additive and has no extension to a Borel measure. In the classification of 435A, μ is of type E_0.

proof Let Ω be the set of non-zero countable limit ordinals, and for each $\xi \in \Omega$ let $\langle \theta_\xi(i) \rangle_{i\in\mathbb{N}}$ be a strictly increasing sequence of ordinals with supremum ξ. Set $X = \omega_1 \times (\omega + 1)$, and define a topology $\mathfrak{T}$ on X by saying that $G \subseteq X$ is open iff

$\{\xi : (\xi,n) \in G\}$ is open in the order topology of ω_1 for every $n \in \mathbb{N}$,

whenever $\xi \in \Omega$ and $(\xi,\omega) \in G$ then there is some $n < \omega$ such that $(\eta,i) \in G$ whenever $n \le i < \omega$ and $\theta_\xi(i) < \eta \le \xi$.

This is finer than the product of the order topologies, so is Hausdorff. For every $\xi < \omega_1$ and $n \in \mathbb{N}$, $(\xi+1) \times \{n\}$ is a countable compact open set containing (ξ,n); for every $\xi \in \omega_1 \setminus \Omega$, $\{(\xi,\omega)\}$ is a countable compact open set containg (ξ,ω); and for every $\xi \in \Omega$,

$$\{(\xi,\omega)\} \cup \{(\eta,i) : i < \omega,\, \theta_\xi(i) < \eta \le \xi\}$$

is a countable compact open subset of X containing (ξ,ω). Thus $\mathfrak{T}$ is locally compact, and every singleton subset of X is G_δ, so $\mathfrak{T}$ is first-countable (4A2Kf).

If $f : X \to \mathbb{R}$ is continuous, then for every $n \in \mathbb{N}$ there is a $\zeta_n < \omega_1$ such that f is constant on $\{(\xi, n) : \zeta_n \le \xi < \omega_1\}$ (4A2S(b-iii)). Setting $\zeta = \sup_{n\in\mathbb{N}} \zeta_n$, f must be constant on $\{(\xi, \omega) : \xi \in \Omega,\ \xi > \zeta\}$. **P** If ξ, $\eta \in \Omega \setminus (\zeta + 1)$, then $f(\xi, \omega) = \lim_{i\to\infty} f(\theta_\xi(i) + 1, i)$ and $f(\eta, \omega) = \lim_{i\to\infty} f(\theta_\eta(i) + 1, i)$. But there is some n such that both $\theta_\xi(i)$ and $\theta_\eta(i)$ are greater than ζ for every $i \ge n$, so that $f(\theta_\xi(i) + 1, i) = f(\theta_\eta(i) + 1, i)$ for every $i \ge n$ and $f(\xi, \omega) = f(\eta, \omega)$. **Q**

Writing Σ for the family of subsets E of X such that $\{\xi : \xi \in \Omega,\ (\xi, \omega) \in E\}$ is either countable or cocountable in Ω, Σ is a σ-algebra of subsets of X such that every continuous function is Σ-measurable, so every Baire set belongs to Σ. We therefore have a Baire measure μ_0 on X defined by saying that $\mu_0 E = 0$ if $E \cap (\Omega \times \{\omega\})$ is countable, 1 otherwise. $\{(\xi + 1) \times (\omega + 1) : \xi < \omega_1\}$ is a cover of X by negligible open-and-closed sets, so μ_0 is not τ-additive.

? Suppose, if possible, that μ were a Borel measure on X extending μ_0. Then we must have $\mu(\omega_1 \times \{n\}) = \mu_0(\omega_1 \times \{n\}) = 0$ for every $n \in \mathbb{N}$, so $\mu(\omega_1 \times \{\omega\}) = 1$. Let λ be the subspace measure on $\omega_1 \times \{\omega\}$ induced by μ. If $A \subseteq \omega_1$, $(A \times \{\omega\}) \cup (\omega_1 \times \omega)$ is an open set, so λ is defined on every subset of $\omega_1 \times \{\omega\}$; and if $\xi < \omega_1$, then $\mu_0((\xi + 1) \times (\omega + 1)) = 0$, so λ is zero on singletons. And this contradicts Ulam's theorem (419G, 438Cd). **X**

439N Example Give ω_1 its order topology.

(i) ω_1 is a normal Hausdorff space which is not measure-compact.

(ii) There is a Baire probability measure μ_0 on ω_1 which is not τ-additive and has a unique extension to a Borel measure, which is not completion regular; that is, μ_0 is of type E_2 in the classification of 435A.

proof (a) As noted in 4A2Rc, order topologies are always normal and Hausdorff.

(b) Let μ be Dieudonné's measure on ω_1, and μ_0 its restriction to the Baire σ-algebra. Then μ is the only Borel measure extending μ_0. **P** Let ν be any Borel measure extending μ_0. Every set $[0, \xi] = [0, \xi + 1[$, where $\xi < \omega_1$, is open-and-closed, so

$$\nu[0, \xi] = \mu_0[0, \xi] = \mu[0, \xi] = 0;$$

also, of course, $\nu\omega_1 = 1$. Let $F \subseteq \omega_1$ be any closed set. If F is countable, then it is included in some initial segment $[0, \xi]$, so $\nu F = \mu F = 0$. Now suppose that F is uncountable. Set $G = \omega_1 \setminus F$. For each $\xi \in F$, set $\zeta_\xi = \min\{\eta : \xi < \eta \in F\}$ and $G_\xi =]\xi, \zeta_\xi[$. Then $\langle G_\xi \rangle_{\xi\in F}$ is a disjoint family of open sets. By 438Bb and 419G/438Cd,

$$\nu(\bigcup_{\xi\in F} G_\xi) = \sum_{\xi\in F} \nu G_\xi = 0.$$

But now

$$1 = \nu\omega_1 = \nu F + \nu\,[0, \min F[+ \nu(\bigcup_{\xi\in F} G_\xi) = \nu F = \mu F.$$

Thus μ and ν agree on the family $\mathcal{E}$ of closed sets. By the Monotone Class Theorem (136C), they agree on the σ-algebra generated by $\mathcal{E}$, which is their common domain; so they are equal. **Q**

(c) I have already remarked in 411Q-411R that μ and μ_0 are not τ-additive and μ is not completion regular. So of course ω_1 is not measure-compact.

439O In 439M I described a Baire measure with no extension to a Borel measure. In view of Mařík's theorem (435C), it is natural to ask whether this can be done with a normal space. This leads us into relatively deep water, and the only examples known need special assumptions.

Example Assume Ostaszewski's $\clubsuit$. Then there is a normal Hausdorff space with a Baire probability measure μ which is not τ-additive and not extendable to a Borel measure. (In the classification of 435A, μ is of type E_0.)

proof (a) $\clubsuit$ implies that there is a family $\langle C_\xi \rangle_{\xi<\omega_1}$ of sets such that (i) $C_\xi \subseteq \xi$ for every $\xi < \omega_1$ (ii) $C_\xi \cap \eta$ is finite whenever $\eta < \xi < \omega_1$ (iii) for any uncountable sets A, $B \subseteq \omega_1$ there is a $\xi < \omega_1$ such that $A \cap C_\xi$ and $B \cap C_\xi$ are both infinite (4A1N). For $A \subseteq \omega_1$, set $A' = \{\xi : \xi < \omega_1,\ A \cap C_\xi \text{ is infinite}\}$; then $A' \cap B'$ is non-empty whenever A, $B \subseteq \omega_1$ are uncountable. But this means that $A' \cap B'$ is actually uncountable for uncountable A, B, since $A' \cap B' \setminus \gamma \supseteq (A \setminus \gamma)' \cap (B \setminus \gamma)'$ is non-empty for every $\gamma < \omega_1$.

Set $X = \omega_1 \times \mathbb{N}$. For $x = (\xi, n) \in X$, say that

$$\begin{aligned} I_x &= C_\xi \times \{n - 1\} \text{ if } n \ge 1, \\ &= \emptyset \text{ otherwise .} \end{aligned}$$

(b) Define a topology $\mathfrak{T}$ on X by saying that a set $G \subseteq X$ is open iff $I_x \setminus G$ is finite for every $x \in G$.

The form of the construction ensures that $\mathfrak{T}$ is T_1. In fact, $I_x \cap I_y$ is finite whenever $x \neq y$ in X. **P** Express x as (ξ, m) and y as (η, n) where $\eta \le \xi$. If either $m = 0$ or $n = 0$ or $m \neq n$, $I_x \cap I_y = \emptyset$. If $n \ge 1$ and $\eta < \xi$, then

$$I_x \cap I_y \subseteq (C_\xi \cap \eta) \times \{n-1\}$$

is finite. Similarly, $I_x \cap I_y$ is finite if $m \ge 1$ and $\xi < \eta$. **Q** Consequently $\{x\} \cup J$ is closed for every $x \in X$, $J \subseteq I_x$.

Observe that $(\xi+1) \times \mathbb{N}$ is open and closed for every $\xi < \omega_1$, again because $C_\eta \cap (\xi+1)$ is finite whenever $\eta \in \Omega$ and $\eta > \xi$, while $C_\xi \subseteq \xi$ for every ξ.

(c) The next step is to understand the uncountable closed subsets of X. First, if $F \subseteq X$ is closed and $n \in \mathbb{N}$, then $F^{-1}[\{n\}]'$, as defined in (a), is a subset of $F^{-1}[\{n+1\}]$, since if $\xi \in F^{-1}[\{n\}]'$ then $I_{(\xi,n+1)} \cap F$ is infinite. If F is uncountable, there is some $n \in \mathbb{N}$ such that $F^{-1}[\{n\}]$ is uncountable, so that (inducing on m) $F^{-1}[\{m\}]$ is uncountable for every $m \ge n$. Finally, this means that if E, $F \subseteq X$ are uncountable closed sets, there is an $m \in \mathbb{N}$ such that $E^{-1}[\{m\}]$ and $F^{-1}[\{m\}]$ are both uncountable, so that $E^{-1}[\{m\}]' \cap F^{-1}[\{m\}]'$ is non-empty and $E \cap F$ is non-empty.

(d) It follows that X is normal. **P** Let E and F be disjoint closed sets in X. By (c), at least one of them is countable; let us take it that $E \subseteq \zeta \times \mathbb{N}$ where $\zeta < \omega_1$. Enumerate the open-and-closed set $W = (\zeta+1) \times \mathbb{N}$ as $\langle x_n \rangle_{n \in \mathbb{N}}$. Choose $\langle U_n \rangle_{n \in \mathbb{N}}$, $\langle V_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. $U_0 = E$, $V_0 = F \cap W$. If $x_n \in U_n$, then $U_{n+1} = U_n \cup (I_{x_n} \setminus V_n)$ and $V_{n+1} = V_n$; if $x_n \notin U_n$, then $U_{n+1} = U_n$ and $V_{n+1} = V_n \cup \{x_n\} \cup (I_{x_n} \setminus U_n)$. An easy induction shows that, for every n, (α) $U_n \cap V_n = \emptyset$ (β) $U_n \cup V_n \subseteq W$ (γ) $I_x \cap (U_n \cup V_n)$ is finite for every $x \in X \setminus (U_n \cup V_n)$ (δ) $I_x \cap V_n$ is finite for every $x \in U_n$ (ϵ) $I_x \cap U_n$ is finite for every $x \in V_n$.

At the end of the induction, set $G = \bigcup_{n \in \mathbb{N}} U_n$, $H = \bigcup_{n \in \mathbb{N}} V_n \cup (X \setminus W)$. Then $E \subseteq G$, $F \subseteq H$ and $G \cap H = \emptyset$. If $x \in G$, it is of the form x_n for some n, in which case $x_n \in U_n$ (because $x_n \notin V_{n+1}$) and $I_x \setminus U_{n+1} = I_x \cap V_n$ is finite; thus G is open. If $x \in H \cap W$, again it is of the form x_n where this time $x_n \notin U_n$, so that $I_x \setminus V_{n+1} = I_x \cap U_n$ is finite; so H is open.

Thus E and F are separated by open sets; since E and F are arbitrary, X is normal. **Q**

Being T_1 (see (b)), X is also Hausdorff.

(e) Because disjoint closed sets in X cannot both be uncountable ((c) above), any bounded continuous function on X must be constant on a cocountable set. (Compare 4A2S(b-iii).) The countable-cocountable measure μ_0 is therefore a Baire measure on X (cf. 411R). But it has no extension to a Borel measure. **P** The point is that if A is any subset of ω_1, and $n \in \mathbb{N}$, then

$$(A \times \{n\}) \cup (\omega_1 \times n), \quad \omega_1 \times n$$

are both open, so $A \times \{n\}$ is Borel; accordingly every subset of X is a Borel set. But ω_1 is measure-free (419G, 438Cd), so there can be no Borel probability measure on X which is zero on singletons. **Q**

Of course μ_0 is not τ-additive, because $\{(\xi+1) \times \mathbb{N} : \xi < \omega_1\}$ is a cover of X by open-and-closed negligible sets.

Remark Thus in Mařík's theorem we really do need 'countably paracompact' as well as 'normal', at least if we want a theorem valid in ZFC.

Observe that any example of this phenomenon must involve a **Dowker space**, that is, a normal Hausdorff space which is not countably paracompact. The one here is based on de Caux 76. Such spaces are hard to come by in ZFC if we do not allow ourselves to use special principles like ♣. 'Real' Dowker spaces have been described by Rudin 71 and Balogh 96; for a survey, see Rudin 84. I do not know if either of these can be adapted to provide a ZFC example to replace the one above.

439P Example (cf. Moran 68) $\mathbb{N}^{\mathfrak{c}}$ is not Borel-measure-compact, therefore not Borel-measure-complete, measure-compact or Radon.

proof Consider the topology $\mathfrak{T}_{\mathfrak{c}}$ on $Z = \{0,1\}^{\mathbb{N}}$, as constructed in 439K. Then $(Z, \mathfrak{T}_{\mathfrak{c}})$ is homeomorphic to a closed subset of $\mathbb{N}^Z \times \{0,1\}^{\mathbb{N}}$, where in this product the second factor $\{0,1\}^{\mathbb{N}}$ is given its usual topology $\mathfrak{S}$. **P** For each $x \in Z$, let L_x be a $\mathfrak{T}_{\mathfrak{c}}$-open $\mathfrak{T}_{\mathfrak{c}}$-compact subset of Z. The first thing to observe is that if $x \in Z$, and we write $V_{xm} = \{y : y \in Z, y \restriction m = x \restriction m\}$ for each $m \in \mathbb{N}$, then $\mathcal{U}_x = \{L_x \cap V_{xm} : m \in \mathbb{N}\}$ is a downwards-directed family of compact open neighbourhoods of x with intersection $\{x\}$, so is a base of neighbourhoods of x (4A2Gd); thus $\mathcal{U} = \{L_x : x \in Z\} \cup \mathfrak{S}$ generates $\mathfrak{T}_{\mathfrak{c}}$. Now, for $x \in Z$, define $\phi_x : Z \to \mathbb{N}$ by setting

$$\begin{aligned} \phi_x(y) &= 0 \text{ if } y \in L_x, \\ &= m+1 \text{ if } y \in V_{xm} \setminus (L_x \cup V_{x,m+1}). \end{aligned}$$

Then every ϕ_x is $\mathfrak{T}_\mathfrak{c}$-continuous, so we have a $\mathfrak{T}_\mathfrak{c}$-continuous function $\phi : Z \to \mathbb{N}^Z \times \{0,1\}^\mathbb{N}$ defined by setting $\phi(y) = (\langle\phi_z(y)\rangle_{z\in Z}, y)$ for $y \in Z$. Because every element of $\mathcal{U}$ is of the form $\phi^{-1}[H]$ for some open set $H \subseteq \mathbb{N}^Z \times \{0,1\}^\mathbb{N}$, Z is homeomorphic to its image $\phi[Z]$.

Now suppose that $(w,z) \in \overline{\phi[Z]}$. In this case, there is a filter $\mathcal{G}$ containing $\phi[Z]$ which converges to (w,z) (4A2Bc). Let $\mathcal{F}$ be an ultrafilter on Z including $\{\phi^{-1}[A] : A \in \mathcal{G}\}$; then $\phi[[\mathcal{F}]]$ includes $\mathcal{G}$ so converges to (w,z), and $\mathcal{F} \to z$ for $\mathfrak{S}$. **?** If z is not the $\mathfrak{T}_\mathfrak{c}$-limit of $\mathcal{F}$, then $\mathcal{F}$ can have no $\mathfrak{T}_\mathfrak{c}$-limit, and can contain no $\mathfrak{T}_\mathfrak{c}$-compact set (2A3R). In particular, $L_z \notin \mathcal{F}$; but in this case $V_{zm} \setminus L_z \in \mathcal{F}$ for every m, so that $\{(v,y) : v(x) > m\} \in \phi[[\mathcal{F}]]$ for every m, and $w(x) > m$ for every m, which is impossible. **X** Thus $\mathcal{F} \to z$, and (as ϕ is continuous) $(w,z) = \phi(z)$.

This shows that $\phi[Z]$ is closed, so we have the required homeomorphism between Z and a closed subset of $\mathbb{N}^Z \times \{0,1\}^\mathbb{N}$. **Q**

Of course $\mathbb{N}^Z \times \{0,1\}^\mathbb{N}$ is a closed subset of $\mathbb{N}^Z \times \mathbb{N}^\mathbb{N} \cong \mathbb{N}^\mathfrak{c}$. So Z is homeomorphic to a closed subset of $\mathbb{N}^\mathfrak{c}$. But Z, with $\mathfrak{T}_c$, carries a Borel probability measure ν which is inner regular with respect to the closed sets and is not τ-additive (439Kb). So $(Z,\mathfrak{T}_\mathfrak{c})$ is not Borel-measure-compact. By 434Hc, $\mathbb{N}^\mathfrak{c}$ is not Borel-measure-compact. By 434Ic, $\mathbb{N}^\mathfrak{c}$ is not Borel-measure-complete; by 434Ka, it is not Radon; by 435Fd, it is not measure-compact.

439Q Example Let X be the real line with the **right-facing Sorgenfrey topology**, generated by sets of the form $[a,b[$ where $a < b$ in $\mathbb{R}$. Then X is measure-compact but X^2 is not.

proof (a) Note that every set $[a,b[$ is open-and-closed in X, so that the topology is zero-dimensional, therefore completely regular; and it is finer than the usual topology of $\mathbb{R}$, so is Hausdorff.

X is Lindelöf. **P** Let $\mathcal{G}$ be an open cover of X. For each $q \in \mathbb{Q}$, set

$$A_q = \{a : a \in \,]-\infty,q[\text{ and there is some } G \in \mathcal{G} \text{ such that } [a,q[\, \subseteq G\}.$$

Then $\bigcup_{q\in\mathbb{Q}} A_q = \mathbb{R}$. For each $q \in \mathbb{Q}$, there is a countable set $A'_q \subseteq A_q$ such that $\inf A'_q = \inf A_q$ in $[-\infty,\infty]$ and A'_q contains $\min A_q$ if A_q has a least element. Now, for each pair (a,q) where $q \in \mathbb{Q}$ and $a \in A'_q$, choose $G_{aq} \in \mathcal{G}$ such that $[a,q[\, \subseteq G_{aq}$. It is easy to see that $\bigcup\{G_{aq} : a \in A'_q\} \supseteq A_q$, so that the countable family $\{G_{aq} : q \in \mathbb{Q},\, a \in A'_q\}$ covers X. As $\mathcal{G}$ is arbitrary, X is Lindelöf. **Q**

It follows that X is measure-compact (435Fb).

(b) Let $\mathfrak{S}$ be the usual topology on $\mathbb{R}^2$, and $\mathfrak{T}$ the product topology on X^2.

(i) Whenever G, H are disjoint $\mathfrak{T}$-open sets, there is an $\mathfrak{S}$-Borel set E such that $G \subseteq E \subseteq X^2 \setminus H$. **P** For $n \in \mathbb{N}$, set

$$A_n = \{(a,b) : [a,a+2^{-n}[\, \times [b,b+2^{-n}[\, \subseteq G\}.$$

? Suppose, if possible, that there is a point $(x,y) \in \overline{A}_n^{\mathfrak{S}} \cap H$, where I write $\overline{}^{\mathfrak{S}}$ to denote closure for the topology $\mathfrak{S}$. Let $\delta > 0$ be such that $[x,x+2\delta[\, \times [y,y+2\delta[\, \subseteq H$ and $2\delta < 2^{-n}$. Then there must be $(a,b) \in A_n$ such that $|a-x| \le \delta$ and $|b-y| \le \delta$. In this case, $a \le x+\delta < a+2^{-n}$ and $b \le y+\delta < b+2^{-n}$, so $(x+\delta,y+\delta) \in G$; while δ was chosen so that $(x+\delta,y+\delta)$ would belong to H. **X**

Accordingly $E = \bigcup_{n\in\mathbb{N}} \overline{A}_n^{\mathfrak{S}}$ is an $\mathfrak{S}$-Borel set disjoint from H. But $G = \bigcup_{n\in\mathbb{N}} A_n$, so $G \subseteq E$. **Q**

(ii) Consequently every $\mathfrak{T}$-continuous real-valued function is $\mathfrak{S}$-Borel measurable. **P** If $f : X^2 \to \mathbb{R}$ is $\mathfrak{T}$-continuous and $\alpha \in \mathbb{R}$, then there is an $\mathfrak{S}$-Borel set E_α such that

$$\{(x,y) : f(x,y) < \alpha\} \subseteq E_\alpha \subseteq \{(x,y) : f(x,y) \le \alpha\}.$$

But this means that $\{(x,y) : f(x,y) < \alpha\} = \bigcup_{n\in\mathbb{N}} E_{\alpha-2^{-n}}$ is $\mathfrak{S}$-Borel. **Q**

(iii) It follows that every $\mathfrak{T}$-Baire set is $\mathfrak{S}$-Borel. We therefore have a $\mathfrak{T}$-Baire probability measure ν on X^2 defined by setting

$$\nu E = \mu_L\{t : t \in [0,1],\ (t,1-t) \in E\}$$

for every $\mathfrak{T}$-Baire subset of X^2, where μ_L is Lebesgue measure on $\mathbb{R}$. In this case every point (x,y) of X^2 belongs to a $\mathfrak{T}$-open set of zero measure for ν. **P** Set $K = \{(t,1-t) : t \in [0,1]\}$. Then K is $\mathfrak{S}$-closed, therefore $\mathfrak{T}$-closed, and $\nu(X^2 \setminus K) = 0$, so if $(x,y) \notin K$ then we can stop. If $(x,y) \in K$, then $[x,x+1[\, \times [y,y+1[$ is a $\mathfrak{T}$-open $\mathfrak{T}$-closed set meeting K in the single point (x,y), so is a negligible $\mathfrak{T}$-neighbourhood of (x,y). **Q**

Thus ν is not τ-additive and X^2 is not measure-compact.

Remark Contrast this with 438Xr.

439R Example There are first-countable completely regular Hausdorff spaces X, Y with Baire probability measures μ, ν such that the Baire measures λ, λ' on $X \times Y$ defined by the formulae

$$\int f d\lambda = \iint f(x,y)\nu(dy)\mu(dx), \quad \int f d\lambda' = \iint f(x,y)\mu(dx)\nu(dy)$$

(436F) are different.

proof Let X, Y be disjoint stationary subsets of ω_1 (4A1Cd). Give each the topology induced by the order topology of ω_1. Let $\tilde{\mu}$ be Dieudonné's measure on ω_1, and $\tilde{\mu}_X$, $\tilde{\mu}_Y$ the subspace measures induced on X and Y by $\tilde{\mu}$; let μ and ν be the restrictions of $\tilde{\mu}_X$, $\tilde{\nu}_Y$ to the Baire σ-algebras of X, Y respectively. Then

$$\mu X = \tilde{\mu}_X X = \tilde{\mu}^* X = 1$$

because X meets every cofinal closed set in ω_1; similarly, $\nu Y = 1$.

Set

$$W = \{(x,y) : x \in X,\, y \in Y,\, x < y\} = \{(x,y) : x \in X,\, y \in Y,\, x \le y\}.$$

Then W is open-and-closed in $X \times Y$ (use 4A2Rl), so that $f = \chi W$ is continuous. But

$$\iint f(x,y)\nu(dy)\mu(dx) = \int \nu\{y : y \in Y,\, x < y\}\mu(dx) = 1,$$

$$\iint f(x,y)\mu(dx)\nu(dy) = \int \mu\{x : x \in X,\, x < y\}\nu(dy) = 0.$$

Remark Contrast this with 434Xx and 439Yi.

439S The results of 437V leave open the question of which familiar spaces, beyond Čech-complete spaces, can be Prokhorov. In fact rather few are. The basis of any further investigation must be the following result.

Theorem (Preiss 73) $\mathbb{Q}$ is not a Prokhorov space.

proof (a) There is a non-decreasing sequence $\langle X_k \rangle_{k\in\mathbb{N}}$ of non-empty compact subsets of $X = \mathbb{Q} \cap [0,1]$, with union X, such that whenever $k \in \mathbb{N}$, $x \in X_k$ and $\delta > 0$, then $X_{k+1} \cap [x-\delta, x+\delta]$ is infinite. **P** Start by enumerating X as $\langle q_k \rangle_{k\in\mathbb{N}}$. Set $X_0 = \{q_0\}$. Given that $X_k \subseteq X$ is compact, then for each $m \in \mathbb{N}$ let $\mathcal{E}_m$ be a finite cover of X_k by open intervals of length at most 2^{-m} all meeting X_k, and let I_{km} be a finite subset of $X \setminus X_k$ meeting every member of $\mathcal{E}_m$; set $X_{k+1} = X_k \cup \{q_{k+1}\} \cup \bigcup_{m\in\mathbb{N}} I_{km}$. If $\mathcal{H}$ is any cover of X_{k+1} by open sets in $\mathbb{R}$, then there is a finite $\mathcal{H}_0 \subseteq \mathcal{H}$ covering X_k. There must be an $m \in \mathbb{N}$ such that $[x - 2^{-m}, x + 2^{-m}] \subseteq \bigcup \mathcal{H}_0$ for every $x \in X_k$ (2A2Ed), so that $I_{kl} \subseteq \bigcup \mathcal{H}_0$ for every $l \ge m$, and $X_{k+1} \setminus \bigcup \mathcal{H}_0$ is finite; accordingly there is a finite $\mathcal{H}_1 \subseteq \mathcal{H}$ covering X_{k+1}. As $\mathcal{H}$ is arbitrary, X_{k+1} is compact, and the induction can proceed. If $x \in X_k$ and $\delta > 0$, then for every $m \in \mathbb{N}$ there is an $x' \in X_{k+1} \setminus X_k$ such that $|x' - x| \le 2^{-m}$, so that $[x-\delta, x+\delta] \cap X_{k+1}$ must be infinite. **Q**

(b) If $\langle \epsilon_k \rangle_{k\in\mathbb{N}}$ is any sequence in $]0,\infty[$, and $F \subseteq [0,1]$ is a countable closed set, then there is an $x^* \in X \setminus F$ such that $\rho(x^*, X_k) < \epsilon_k$ for every $k \in \mathbb{N}$. **P** We can suppose that $\lim_{k\to\infty} \epsilon_k = 0$. Define $\langle H_k \rangle_{k\in\mathbb{N}}$ inductively, as follows. $H_0 = \mathbb{R}$. Given H_k, set $H_{k+1} = H_k \cap \{x : \rho(x, X_k \cap H_k) < \epsilon_k\}$, where $\rho(x, A) = \inf_{y\in A} |x - y|$ for $x \in \mathbb{R}$, $A \subseteq \mathbb{R}$. Observe that every H_k is an open subset of $\mathbb{R}$ and that $X_k \cap H_k \subseteq H_{k+1} \subseteq H_k$ for every k; consequently, setting $E = \bigcap_{k\in\mathbb{N}} H_k$, E is a G_δ subset of $\mathbb{R}$ and $X_k \cap H_k \subseteq E$ for every k. In particular, $E \cap X$ contains q_0 and is not empty. Next, for each k, $\rho(x, E \cap X_k) < \epsilon_k$ for every $x \in H_{k+1}$ and therefore for every $x \in E$; accordingly $E \cap X$ is dense in E. Moreover, if $x \in E \cap X$, there is a $k \in \mathbb{N}$ such that $x \in X_k$; we must have $x \in H_k$, and in this case H_{k+1} is a neighbourhood of x. So every neighbourhood of x contains infinitely many points of $H_{k+1} \cap X_{k+1} \subseteq E \cap X$. Thus $E \cap X$ has no isolated points; it follows that E has no isolated points. By 4A2Mc and 4A2Me, E is uncountable.

There is therefore a point $z \in E \setminus F$. Let $m \in \mathbb{N}$ be such that $\rho(z, F) \ge \epsilon_m$. As $z \in H_{m+1}$, there is an $x^* \in H_m \cap X_m$ such that $|z - x^*| < \epsilon_m$ and $x^* \notin F$. Let $k \in \mathbb{N}$. If $k \ge m$ then certainly $\rho(x^*, X_k) = 0 < \epsilon_k$. If $k < m$ then $x^* \in H_{k+1}$ so $\rho(x^*, X_k) \le \rho(x^*, H_k \cap X_k) < \epsilon_k$. So we have a suitable x^*. **Q**

(c) For n, $k \in \mathbb{N}$ set

$$G_{kn} = \{x : x \in \mathbb{R} \setminus X_k,\, \rho(x, X_n) > 2^{-k}\}.$$

Then G_{kn} is an open subset of $\mathbb{R}$. Let A be the set of Radon probability measures μ on X such that $\mu(G_{kn} \cap X) \le 2^{-n}$ for all n, $k \in \mathbb{N}$.

(d) Write $\tilde{A}$ for the set of Radon probability measures μ on $[0,1]$ such that $\mu(G_{kn} \cap [0,1]) \le 2^{-n}$ for all k, $n \in \mathbb{N}$. Then $\tilde{A}$ is a narrowly closed subset of the set of Radon probability measures on $[0,1]$, which is itself narrowly compact (437R(f-ii)). Also $\mu([0,1] \setminus X) = 0$ for every $\mu \in \tilde{A}$. **P** Let $K \subseteq [0,1] \setminus X$ be compact, and $n \in \mathbb{N}$. Then K and X_n

are disjoint compact sets, so there is some $k \in \mathbb{N}$ such that $|x-y| > 2^{-k}$ for every $x \in X_n$ and $y \in K$. In this case $K \subseteq G_{kn}$ so $\mu K \le 2^{-n}$. As n is arbitrary, $\mu K = 0$; as K is arbitrary, $\mu([0,1] \setminus X) = 0$. **Q**

A is compact in the narrow topology. **P** The identity map $\phi : X \to [0,1]$ induces a map $\tilde{\phi} : M_{\mathrm{R}}^+(X) \to M_{\mathrm{R}}^+([0,1])$ which is a homeomorphism between $M_{\mathrm{R}}^+(X)$ and $\{\mu : \mu \in M_{\mathrm{R}}^+([0,1]),\ \mu([0,1] \setminus X) = 0\}$ (437Nb). The definition of A makes it plain that it is $\tilde{\phi}^{-1}[\tilde{A}]$; since $\tilde{A} \subseteq \{\mu : \mu \in M_{\mathrm{R}}^+([0,1]),\ \mu([0,1] \setminus X) = 0\}$, $\tilde{\phi}\restriction A$ is a homeomorphism between A and $\tilde{A}$, and A is compact. **Q**

(e) A, regarded as a subset of $M_{\mathrm{R}}^+(X)$, is not uniformly tight. **P** Let $K \subseteq X$ be compact. Consider the set C of those $w \in [0,1]^X$ such that $w(x) = 0$ for every $x \in K$, $\sum_{x \in X} w(x) \le 1$ and $\sum_{x \in G_{kn} \cap X} w(x) \le 2^{-n}$ for all k, $n \in \mathbb{N}$. Then C is a compact subset of $[0,1]^X$. If $D \subseteq C$ is any non-empty upwards-directed set, then $\sup D$, taken in $[0,1]^X$, belongs to C. By Zorn's Lemma, C has a maximal member w say. **?** Suppose, if possible, that $\sum_{x \in X} w(x) = \gamma < 1$. For each $n \in \mathbb{N}$, let $L_n \subseteq X$ be a finite set such that $\sum_{x \in L_n} w(x) \ge \gamma - 2^{-n-1}$, and $m_n \in \mathbb{N}$ such that $L_n \subseteq X_{m_n}$. By (b), there is an $x^* \in X \setminus K$ such that $\rho(x^*, X_n) < 2^{-m_n}$ for every $n \in \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $x^* \in X_r$ and $\gamma + 2^{-r} \le 1$, and set $w'(x^*) = w(x^*) + 2^{-r}$, $w'(x) = w(x)$ for every $x \in X \setminus \{x^*\}$. Then certainly $w' \in [0,1]^X$ and $\sum_{x \in X} w'(x) \le 1$. If k, $n \in \mathbb{N}$ and $x^* \notin G_{kn}$, then $\sum_{x \in G_{kn} \cap X} w'(x) = \sum_{x \in G_{kn} \cap X} w(x) \le 2^{-n}$. If $x^* \in G_{kn}$, then $n < r$ and $2^{-k} < \rho(x^*, X_n) < 2^{-m_n}$, so $m_n < k$ and $L_n \subseteq X_k$ and

$$\textstyle\sum_{x \in G_{kn} \cap X} w(x) \le \sum_{x \in X \setminus X_k} w(x) \le \sum_{x \in X \setminus L_n} w(x) \le 2^{-n-1},$$

$$\textstyle\sum_{x \in G_{kn} \cap X} w'(x) \le 2^{-n-1} + 2^{-r} \le 2^{-n}.$$

Thus $w' \in C$ and w was not maximal. **X**

Accordingly $\sum_{x \in X} w(x) = 1$ and the point-supported measure μ defined by w is a probability measure on X. By the definition of C, $\mu \in A$ and $\mu(X \setminus K) = 1$. As K is arbitrary, A cannot be uniformly tight. **Q**

(f) Thus A witnesses that $X = \mathbb{Q} \cap [0,1]$ is not a Prokhorov space. Since X is a closed subset of $\mathbb{Q}$, 437Vb tells us that $\mathbb{Q}$ is not a Prokhorov space.

439X Basic exercises (a) (i) Show that there is a set $A \subseteq [0,1]$ such that $\mu_L^* A = 1$, where μ_L is Lebesgue measure, and every member of $[0,1]$ is uniquely expressible as $a + q$ where $a \in A$, $q \in \mathbb{Q}$. (*Hint*: 134B.) (ii) Define $f : [0,1] \to A$ by setting $f(x) = a$ when $x \in a + \mathbb{Q}$. Show that the image measure $\mu_L f^{-1}$ takes only the values 0 and 1. (Aldaz 95. Compare 342Xg.)

(b) Let X be a Radon Hausdorff space and A a subset of X. Show that A is universally negligible iff $\mu A = 0$ for every atomless Radon measure on X.

>(c) Let X be a Hausdorff space. Show that a set $A \subseteq X$ is universally negligible iff $\mu A = 0$ whenever μ is a complete locally determined topological measure on X such that $\mu\{x\} = 0$ for every $x \in X$.

(d) Let X be a Hausdorff space. Show that any universally negligible subset of X is universally measurable in the sense of 434D.

(e)(i) Show that there is an analytic set $A \subseteq \mathbb{R}$ such that for any Borel subset E of $\mathbb{R} \setminus A$ there is an uncountable Borel subset of $\mathbb{R} \setminus (A \cup E)$. (*Hint*: 423Qb, part (c) of the proof of 423L.) (ii) Show that A is universally measurable, but there is no Borel set E such that $A \triangle E$ is universally negligible.

(f) Show that a first-countable compact Hausdorff space is universally negligible iff it is scattered iff it is countable.

(g) Show that the product of two universally negligible Hausdorff spaces is universally negligible.

(h) Let us say that a Hausdorff space X is **universally τ-negligible** if there is no τ-additive Borel probability measure on X which is zero on singletons. (i) Show that if X is a Hausdorff space and $A \subseteq X$, then A is universally τ-negligible iff $\mu^* A = 0$ for every τ-additive Borel probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$. (ii) Show that if X is a regular Hausdorff space, then a subset A of X is universally τ-negligible iff $\mu A = 0$ for every atomless quasi-Radon measure on X. (iii) Show that if X is a completely regular Hausdorff space, it is universally τ-negligible iff whenever μ is an atomless Radon measure on a space Z, and $X' \subseteq Z$ is homeomorphic to X, then $\mu X' = 0$. (iv) Show that a Hausdorff space X is universally negligible iff it is Borel-measure-complete and universally τ-negligible. (v) Show that if X is a Hausdorff space, Y is a universally τ-negligible Hausdorff space, and $f : X \to Y$

is a continuous function such that $f^{-1}[\{y\}]$ is universally τ-negligible for every $y \in Y$, then X is universally τ-negligible. (vi) Show that the product of two universally τ-negligible Hausdorff spaces is universally τ-negligible. (vi) Show that a scattered Hausdorff space (in particular, any discrete space) is universally τ-negligible. (vii) Show that a compact Hausdorff space is universally τ-negligible iff it is scattered.

>(i) Let κ be the smallest cardinal of any subset of $\mathbb{R}$ which is not Lebesgue negligible. Show that if (Z, T, ν) is any complete locally determined atomless measure space and $A \subseteq Z$ has cardinal less than κ, then $\nu A = 0$.

(j) Let $(X, \leq)$ be any well-ordered set and μ a non-zero σ-finite measure on X such that every singleton is negligible. Show that $\{(x, y) : x \leq y\}$ is not measured by the (c.l.d.) product measure on $X \times X$. (*Hint*: Reduce to the case in which μ is complete and totally finite, $X = \zeta$ is an ordinal and $\mu\xi = 0$ for every $\xi < \zeta$. You will probably need 251Q.)

>(k) Show that 439Fc, or any of the examples of 439A, can be regarded as an example of a probability space (X, μ) and a function $f : X \to [0, 1]$ such that there is no extension of μ to a measure ν such that f is $\operatorname{dom}\nu$-measurable; and accordingly can provide an example of a probability space (X, μ) with a countable totally ordered family $\mathcal{A}$ of subsets of X such that there is no extension of μ to a measure measuring every member of $\mathcal{A}$. Contrast with 214P, 214Xm-214Xn and 214Yb.

(l) Show that 1-dimensional Hausdorff measure on $\mathbb{R}^2$ is not inner regular with respect to the closed sets. (*Hint*: 439H.)

(m) Show that $\mathbb{N}^I$ is not pre-Radon for any uncountable set I. (*Hint*: 417Xq.)

(n)(i) Suppose that X is a completely regular Hausdorff space and there is a continuous function f from X to a separable metrizable space Z such that $f^{-1}[\{z\}]$ is Lindelöf for every $z \in Z$. Show that X is realcompact (definition: 436Xg). (ii) Show that the spaces X of 439K and X^2 of 439Q are realcompact. (iii) Show that $\mathfrak{c}$ with its discrete topology, and $\mathbb{N}^{\mathfrak{c}}$ with the product topology, are realcompact.

(o) Show that the one-point compactification of the space $(Z, \mathfrak{T}_{\mathfrak{c}})$ described in 439K is a scattered compact Hausdorff space with an atomless Borel probability measure.

(p) Let X be a Polish space, $A \subseteq X$ an analytic set which is not Borel (423Qb, 423Ye), and $\langle E_\xi \rangle_{\xi<\omega_1}$ a family of Borel constituents of $X \setminus A$ (423P). Suppose that $x_\xi \in E_\xi \setminus \bigcup_{\eta<\xi} E_\eta$ for every $\xi < \omega_1$. Show that $\{x_\xi : \xi < \omega_1\}$ is universally negligible. Hence show that any probability measure with domain $\mathcal{P}\omega_1$ is point-supported.

439Y Further exercises (a) Show that a subset A of $\mathbb{R}$ is universally negligible iff $f[A]$ is Lebesgue negligible for every continuous injective function $f : \mathbb{R} \to \mathbb{R}$. (*Hint*: if ν is an atomless Borel probability measure on $\mathbb{R}$, set $f(x) = x + \nu[0, x]$ for $x \geq 0$, and show that $\mu_L f[E] = \mu E + \nu E$ for every Borel set $E \subseteq [0, \infty[$.)

(b) For this exercise only, let us say that a 'universally negligible measurable space' is a pair (X, Σ) where X is a set and Σ a σ-algebra of subsets of X containing every countable subset of X such that there is no probability measure μ with domain Σ such that $\mu\{x\} = 0$ for every $x \in X$. (i) Let X be a set, Σ a σ-algebra of subsets of X containing all countable subsets of X, $A \subseteq X$ and Σ_A the subspace σ-algebra. Show that (A, Σ_A) is universally negligible iff $\mu^* A = 0$ whenever μ is a probability measure with domain Σ which is zero on singletons. Show that if (X, Σ) is universally negligible so is (A, Σ_A). (ii) Let X and Y be sets, Σ and T σ-algebras of subsets of X and Y containing all appropriate countable sets, and $f : X \to Y$ a (Σ, T)-measurable function. Suppose that (Y, T) and $(f^{-1}[\{y\}], \Sigma_{f^{-1}[\{y\}]})$ are universally negligible for every $y \in Y$. Show that (X, Σ) is universally negligible. (iii) Let X be a set and Σ a σ-algebra of subsets of X containing all countable subsets of X. Show that the set of those $A \subseteq X$ such that (A, Σ_A) is universally negligible is a σ-ideal of subsets of X.

(c) Let X be an analytic Hausdorff space and A an analytic subset of X. Show that $X \setminus A$ is universally negligible iff all the constituents of $X \setminus A$ (for any Souslin scheme defining A) are countable.

(d)(i) Let X be a metrizable space such that $f[X]$ is Lebesgue negligible for every continuous function $f : X \to \mathbb{R}$. Show that X is universally negligible. (ii) Let X be a completely regular Hausdorff space such that $f[X]$ is Lebesgue negligible for every continuous function $f : X \to \mathbb{R}$. Show that X is universally τ-negligible.

(e) Let $\mathfrak{T}_{\mathfrak{c}}$ be the topology on $\{0,1\}^{\mathbb{N}}$ constructed in the proof of 439K. (i) Show that it is normal and countably paracompact. (ii) Show that any $\mathfrak{T}_{\mathfrak{c}}$-zero set is an $\mathfrak{S}$-Borel set.

(f) Show that there is no atomless Borel probability measure on ω_1 endowed with its order topology. (*Hint*: 411R, 439N.)

(g) Show that the space of 439O is locally compact and locally countable, therefore first-countable.

(h) Show that the Sorgenfrey right-facing topology on $\mathbb{R}$ is hereditarily Lindelöf, but that its square is not Lindelöf.

(i) Show that if ω_1 is given its order topology, and $f : \omega_1^2 \to \mathbb{R}$ is continuous, then there is a $\zeta < \omega_1$ such that f is constant on $(\omega_1 \setminus \zeta)^2$. (*Hint*: 4A2S(b-iii).) Show that if μ and ν are Baire probability measures on ω_1, then the Baire probability measures $\mu \times \nu$, $\nu \times \mu$ on ω_1^2 defined by the formulae of 436F coincide.

(j) Let $X \subseteq [0,1]$ be a dense set with no uncountable compact subset. Show that X is not a Prokhorov space.

(k)(i) A pair $(\langle a_\xi \rangle_{\xi<\omega_1}, \langle b_\xi \rangle_{\xi<\omega_1})$ of families of subsets of $\mathbb{N}$ is a **Hausdorff gap** if $a_\xi \setminus a_\eta$, $a_\xi \setminus b_\xi$ and $b_\eta \setminus b_\xi$ are finite whenever $\xi \le \eta < \omega_1$, $a_\eta \setminus a_\xi$ and $b_\xi \setminus b_\eta$ are infinite whenever $\xi < \eta < \omega_1$, and moreover $\{\xi : \xi < \eta, a_\xi \subseteq b_\eta \cup n\}$ is finite for every $\eta < \omega_1$. (For a construction of a Hausdorff gap, see Fremlin 84, 21L.) Show that in this case there is no $c \subseteq \mathbb{N}$ such that $a_\xi \setminus c$ and $c \setminus b_\xi$ are finite for every $\xi < \omega_1$, and that $\{a_\xi : \xi < \omega_1\} \cup \{b_\xi : \xi < \omega_1\}$ is universally negligible in $\mathcal{P}\mathbb{N}$. (ii) Let $\phi : (\mathcal{P}\mathbb{N})^{\mathbb{N}} \to \mathcal{P}\mathbb{N}$ be a homeomorphism. For $0 < \xi < \omega_1$ let $\langle \theta(\xi,n) \rangle_{n\in\mathbb{N}}$ be a sequence running over ξ. Set $a_0 = \emptyset$ and for $0 < \xi < \omega_1$ set $a_\xi = \phi(\langle a_{\theta(\xi,n)} \rangle_{n\in\mathbb{N}})$. Show that $\{a_\xi : \xi < \omega_1\}$ is universally negligible.

439 Notes and comments I give three separate constructions in 439A because the phenomenon here is particularly important. For two chapters I have, piecemeal, been offering theorems on the extension of measures. The principal ones so far seem to be 413O, 415L, 416N, 417C, 417E and 435C, and I have used methods reflecting my belief that the essential feature on which each such theorem depends is inner regularity of an appropriate kind. I think we should simultaneously seek to develop an intuition for measures which do *not* extend, and those in 439A are especially significant because they refer to the Borel algebra of the unit interval, which in so many other contexts is comfortably clear of the obstacles which beset more exotic structures.

Note that because the Borel σ-algebra of $\mathbb{R}$ is countably generated, the examples here are examples of measures which cannot be extended to measure every member of a countable family of sets. Recall that in 214P I showed that measures can be extended to measure the sets in arbitrary *well-ordered* families.

Outside the context of Polish spaces, the terms 'universally measurable' and 'universally negligible' are not properly settled. I have tried to select definitions which lead to a reasonable pattern. At least a universally negligible subset of a Hausdorff space is universally measurable (439Xd), and both concepts can be expressed in terms of sets with σ-algebras, as in 439Yb. It is important to notice, in 439B, that I write '$\mu\{x\} = 0$ for every $x \in X$', not 'μ is atomless'. For instance, Dieudonné's measure shows that ω_1, with its order topology, is not universally negligible on the definition here; but it is easy to show that there is no atomless Borel probability measure on ω_1 (439Yf). In many cases, of course, we do not need to make this distinction (439D).

The cardinal κ of 439F (the 'uniformity' of the Lebesgue null ideal) is one of a large family of cardinals which will be examined in Chapter 52 in the next volume.

In some of the arguments above (439J, 439L, 439O) I appeal to (different) principles ('there is a cardinal which is not measure-free', ♣) which are not theorems according to the rules I follow in this book. Such examples would in some ways fit better into Volume 5, where I mean to investigate such principles properly. I include the examples here because they do at least exhibit bounds on what can be proved in ZFC. I should not want anyone to waste her time trying to show, for instance, that all completion regular Borel measures are τ-additive. Nevertheless, the absence of a 'real' counter-example (obviously we want a probability measure on a completely regular Hausdorff space) remains in my view a significant gap. It remains conceivable that there is a mathematical world in which no such space exists. Clearly the discovery of such a world is likely to require familiarity with the many worlds already known, and I am not going to embark on any such exploration in this volume. On the other hand, it is also very possible that all we need is a bit of extra ingenuity to construct a counter-example in ZFC. In this section we have two examples of successes of this kind. In 439F-439H, for instance, we have results which were long known as consequences of the continuum hypothesis; the particular insight of Grzegorek 81 was the observation that they depended on determinate properties of the cardinal κ of 439F, and that its indeterminate position between ω_1 and

$\mathfrak{c}$ was unimportant. In 439K I show how a re-working of ideas in Juhász Kunen & Rudin 76, where a similar example was constructed (for an entirely different purpose) assuming the continuum hypothesis, provides us with an interesting space (a first-countable non-Radon compact Hausdorff space) in ZFC. Let me emphasize that these ideas were originally set out in a framework supported by an extra axiom, where some technical details were easier and the prize aimed at (a non-Lindelöf hereditarily separable space) more important.

The examples in 439K-439R are mostly based on constructions more or less familiar from general topology. I have already mentioned the origins of 439K. 439M is related to the Tychonoff and Dieudonné planks (Steen & Seebach 78, §§86-89). 439N and 439Q revisit yet again ω_1 and the Sorgenfrey line. 439O is adapted from one of the standard constructions of Dowker spaces. Products of disjoint stationary sets (439R) have also been used elsewhere.

Chapter 44

Topological groups

Measure theory begins on the real line, which is of course a group; and one of the most fundamental properties of Lebesgue measure is its translation-invariance (134A). Later we come to the standard measure on the unit circle (255M), and counting measure on the integers is also translation-invariant, if we care to notice; moreover, Fourier series and transforms clearly depend utterly on the fact that shift operators don't disturb the measure-theoretic structures we are building. Yet another example appears in the usual measure on $\{0,1\}^I$, which is translation-invariant if we identify $\{0,1\}^I$ with the group $\mathbb{Z}_2^I$ (345Ab). Each of these examples is special in many other ways. But it turns out that a particular combination of properties which they share, all being locally compact Hausdorff spaces with group operations for which multiplication and inversion are continuous, is the basis of an extraordinarily powerful theory of invariant measures.

As usual, I have no choice but to move rather briskly through a wealth of ideas. The first step is to set out a suitably general existence theorem, assuring us that every locally compact Hausdorff topological group has non-trivial invariant Radon measures, that is, 'Haar measures' (441E). As remarkable as the existence of Haar measures is their (essential) uniqueness (442B); the algebra, topology and measure theory of a topological group are linked in so many ways that they form a peculiarly solid structure. I investigate a miscellany of facts about this structure in §443, including the basic theory of the modular functions linking left-invariant measures with right-invariant measures.

I have already mentioned that Fourier analysis depends on the translation-invariance of Lebesgue measure. It turns out that substantial parts of the abstract theory of Fourier series and transforms can be generalized to arbitrary locally compact groups. In particular, convolutions (§255) appear again, even in non-abelian groups (§444). But for the central part of the theory, a transform relating functions on a group X to functions on its 'dual' group $\mathcal{X}$, we do need the group to be abelian. Actually I give only the foundation of this theory: if X is an abelian locally compact Hausdorff group, it is the dual of its dual (445U). (In 'ordinary' Fourier theory, where we are dealing with the cases $X=\mathcal{X}=\mathbb{R}$ and $X=S^1$, $\mathcal{X}=\mathbb{Z}$, this duality is so straightforward that one hardly notices it.) But on the way to the duality theorem we necessarily see many of the themes of Chapter 28 in more abstract guises.

A further remarkable fact is that any Haar measure has a translation-invariant lifting (447J). The proof demands a union between the ideas of the ordinary Lifting Theorem (§341) and some of the elaborate structure theory which has been developed for locally compact groups (§446).

For the last two sections of the chapter, I look at groups which are not locally compact, and their actions on appropriate spaces. For a particularly important class of group actions, Borel measurable actions of Polish groups on Polish spaces, we have a natural necessary and sufficient condition for the existence of an invariant measure (448P), complementing the result for locally compact spaces in 441C. In a slightly different direction, we can look at those groups, the 'amenable' groups, for which all actions (on compact Hausdorff spaces) have invariant measures. This again leads to some very remarkable ideas, which I sketch in §449.

441 Invariant measures on locally compact spaces

I begin this chapter with the most important theorem on the existence of invariant measures: every locally compact Hausdorff group has left and right Haar measures (441E). I derive this as a corollary of a general result concerning invariant measures on locally compact spaces (441C), which has other interesting consequences (441H).

441A Group actions I recall a fundamental definition from group theory.

(a) If G is a group and X is a set, an **action** of G on X is a function $(a,x)\mapsto a{\bullet}x : G\times X\to X$ such that

$$(ab){\bullet}x = a{\bullet}(b{\bullet}x) \text{ for all } a,\, b\in G,\ x\in X,$$

$$e{\bullet}x = x \text{ for every } x\in X$$

where e is the identity of G (4A5B). In this context I write

$$a{\bullet}A=\{a{\bullet}x : x\in A\}$$

for $a\in G$, $A\subseteq X$. If f is a function defined on a subset of X, then $(a{\bullet}f)(x)=f(a^{-1}{\bullet}x)$ whenever $a\in G$ and $x\in X$ and $a^{-1}{\bullet}x\in\operatorname{dom} f$ (4A5C(c-i)).

(b) If a group G acts on a set X, a measure μ on X is G-**invariant** if $\mu(a^{-1}\bullet E)$ is defined and equal to μE whenever $a \in G$ and μ measures E.

(Of course this is the same thing as saying that $\mu(a\bullet E) = \mu E$ for every $a \in G$ and measurable set E; I use the formula with a^{-1} so as to match my standard practice when a is actually a function from X to X.)

441B It will be useful later to be able to quote the following elementary results.

Lemma Let X be a topological space, G a group, and $\bullet$ an action of G on X such that $x \mapsto a\bullet x$ is continuous for every $a \in G$.

(a) If μ is a quasi-Radon measure on X such that $\mu(a\bullet U) \le \mu U$ for every open set $U \subseteq X$ and every $a \in G$, then μ is G-invariant.

(b) If μ is a Radon measure on X such that $\mu(a\bullet K) \le \mu K$ for every compact set $K \subseteq X$ and every $a \in G$, then μ is G-invariant.

proof Note first that the maps $x \mapsto a\bullet x$ are actually homeomorphisms (4A5Bd), so that $a\bullet U$ and $a\bullet K$ will be open, or compact, as U and K are. Next, the inequality $\le$ in the hypotheses is an insignificant refinement; since we must also have

$$\mu U = \mu(a^{-1}\bullet a\bullet U) \le \mu(a\bullet U)$$

in (a),

$$\mu K = \mu(a^{-1}\bullet a\bullet K) \le \mu(a\bullet K)$$

in (b), we always have equality here.

Now fix $a \in G$, and set $T_a(x) = a\bullet x$ for $x \in X$. Then T_a is a homeomorphism, so the image measure μT_a^{-1} will be quasi-Radon, or Radon, if μ is. In (a), we are told that μT_a^{-1} agrees with μ on the open sets, while in (b) we are told that they agree on the compact sets; so in both cases we have $\mu = \mu T_a^{-1}$, by 415H(iii) or 416E(b-ii). Consequently we have $\mu T_a^{-1}[E] = \mu E$ whenever μ measures E. As a is arbitrary, μ is G-invariant.

441C Theorem (Steinlage 75) Let X be a non-empty locally compact Hausdorff space and G a group acting on X. Suppose that

(i) $x \mapsto a\bullet x$ is continuous for every $a \in G$;

(ii) every orbit $\{a\bullet x : a \in G\}$ is dense;

(iii) whenever K and L are disjoint compact subsets of X there is a non-empty open subset U of X such that, for every $a \in G$, at most one of K, L meets $a\bullet U$.

Then there is a non-zero G-invariant Radon measure μ on X.

proof (a) $\bigcup_{a\in G} a\bullet U = X$ for every non-empty open $U \subseteq X$. **P** If $x \in X$, then the orbit of x must meet U, so there is a $a \in G$ such that $a\bullet x \in U$; but this means that $x \in a^{-1}\bullet U$. **Q**

Fix some point z_0 of X and write $\mathcal{V}$ for the set of open sets containing z_0. Then if K, L are disjoint compact subsets of X there is a $U \in \mathcal{V}$ such that, for every $a \in G$, at most one of K, L meets $a\bullet U$. **P** By hypothesis, there is a non-empty open set V such that, for every $a \in G$, at most one of K, L meets $a\bullet V$. Now there is an $b \in G$ such that $b\bullet z_0 \in V$; set $U = b^{-1}\bullet V$. Because b^{-1} acts on X as a homeomorphism, $U \in \mathcal{V}$; and if $a \in G$, then $a\bullet U = (ab^{-1})\bullet V$ can meet at most one of K and L. **Q**

(b) If $U \in \mathcal{V}$ and $A \subseteq X$ is any relatively compact set, then $\{a\bullet U : a \in G\}$ is an open cover of X, so there is a finite set $I \subseteq G$ such that $\overline{A} \subseteq \bigcup_{a\in I} a\bullet U$. Write $\lceil A : U \rceil$ for $\min\{\#(I) : I \subseteq G,\ A \subseteq \bigcup_{a\in I} a\bullet U\}$.

(c) The following facts are now elementary.

(i) If $U \in \mathcal{V}$ and A, $B \subseteq X$ are relatively compact, then

$$0 \le \lceil A : U \rceil \le \lceil A \cup B : U \rceil \le \lceil A : U \rceil + \lceil B : U \rceil,$$

and $\lceil A : U \rceil = 0$ iff $A = \emptyset$.

(ii) If U, $V \in \mathcal{V}$ and V is relatively compact, and $A \subseteq X$ also is relatively compact, then

$$\lceil A : U \rceil \le \lceil A : V \rceil \, \lceil V : U \rceil.$$

P If $A \subseteq \bigcup_{a\in I} a\bullet V$ and $V \subseteq \bigcup_{b\in J} b\bullet U$, then $A \subseteq \bigcup_{a\in I, b\in J} (ab)\bullet U$. **Q**

(iii) If $U \in \mathcal{V}$, $A \subseteq X$ is relatively compact and $b \in G$, then $\lceil b\bullet A : U\rceil = \lceil A : U\rceil$. **P** If $I \subseteq G$ and $A \subseteq \bigcup_{a\in I} a\bullet U$, then $b\bullet A \subseteq \bigcup_{a\in I}(ba)\bullet U$, so $\lceil b\bullet A : U\rceil \le \#(I)$; as I is arbitrary, $\lceil b\bullet A : U\rceil \le \lceil A : U\rceil$. On the other hand, the same argument shows that

$$\lceil A : U\rceil \le \lceil b^{-1}\bullet b\bullet A : U\rceil = \lceil A : U\rceil,$$

so we must have equality. **Q**

(d) Fix a relatively compact $V_0 \in \mathcal{V}$. (This is the first place where we use the hypothesis that X is locally compact.) For every $U \in \mathcal{V}$ and every relatively compact set $A \subseteq X$ write

$$\lambda_U A = \frac{\lceil A : U\rceil}{\lceil V_0 : U\rceil}.$$

Then (c) tells us immediately that

(i) if A, $B \subseteq X$ are relatively compact,

$$0 \le \lambda_U A \le \lambda_U(A \cup B) \le \lambda_U A + \lambda_U B;$$

(ii) $\lambda_U A \le \lceil A : V_0\rceil$ for every relatively compact $A \subseteq X$;

(iii) $\lambda_U(b\bullet A) = \lambda_U A$ for every relatively compact $A \subseteq X$ and every $b \in G$;

(iv) $\lambda_U V_0 = 1$.

(e) Now for the point of the hypothesis (iii) of the theorem. If K, L are disjoint compact subsets of X, there is a $V \in \mathcal{V}$ such that $\lambda_U(K \cup L) = \lambda_U K + \lambda_U L$ whenever $U \in \mathcal{V}$ and $U \subseteq V$. **P** By (a), there is a $V \in \mathcal{V}$ such that any translate $a\bullet V$ can meet at most one of K and L. Take any $U \in \mathcal{V}$ included in V. Let $I \subseteq G$ be such that $\bigcup_{a\in I} a\bullet U \supseteq K \cup L$ and $\#(I) = \lceil K \cup L : U\rceil$. Then

$$I' = \{a : a \in I,\, K \cap a\bullet U \ne \emptyset\}, \quad I'' = \{a : a \in I,\, L \cap a\bullet U \ne \emptyset\}$$

are disjoint. $K \subseteq \bigcup_{a\in I'} a\bullet U$, so $\lceil K : U\rceil \le \#(I')$, and similarly $\lceil L : U\rceil \le \#(I'')$. But this means that

$$\lceil K : U\rceil + \lceil L : U\rceil \le \#(I') + \#(I'') \le \#(I) = \lceil K \cup L : U\rceil,$$

$$\lambda_U K + \lambda_U L \le \lambda_U(K \cup L).$$

As we already know that $\lambda_U(K \cup L) \le \lambda_U K + \lambda_U L$, we must have equality, as claimed. **Q**

(f) Now let $\mathcal{F}$ be an ultrafilter on $\mathcal{V}$ containing all sets of the form $\{U : U \in \mathcal{V}, U \subseteq V\}$ for $V \in \mathcal{V}$. If $A \subseteq X$ is relatively compact, $0 \le \lambda_U A \le \lceil A : V_0\rceil$ for every $U \in \mathcal{V}$, so $\lambda A = \lim_{U\to\mathcal{F}} \lambda_U A$ is defined in $[0, \lceil A : V_0\rceil]$. From (d-i) and (d-iii) we see that

$$0 \le \lambda(b\bullet A) = \lambda A \le \lambda(A \cup B) \le \lambda A + \lambda B$$

for all relatively compact A, $B \subseteq X$ and $b \in G$. From (d-iv) we see that $\lambda V_0 = 1$. Moreover, from (e) we see that if K, $L \subseteq X$ are disjoint compact sets,

$$\{U : U \in \mathcal{V},\, \lambda_U(K \cup L) = \lambda_U K + \lambda_U L\} \in \mathcal{F},$$

so $\lambda(K \cup L) = \lambda K + \lambda L$.

(g) By 416M, there is a Radon measure μ on X such that

$$\mu K = \inf\{\lambda L : L \subseteq X \text{ is compact, } K \subseteq \operatorname{int} L\}$$

for every compact set $K \subseteq X$. Now μ is G-invariant. **P** Take $b \in G$. If K, $L \subseteq X$ are compact and $K \subseteq \operatorname{int} L$, then $b\bullet K \subseteq \operatorname{int} b\bullet L$, because $x \mapsto b\bullet x$ is a homeomorphism; so

$$\mu(b\bullet K) \le \lambda(b\bullet L) = \lambda L.$$

As L is arbitrary, $\mu(b\bullet K) \le \mu K$. As b and K are arbitrary, μ is G-invariant, by 441Bb. **Q**

(h) Finally, $\mu V_0 \ge \lambda V_0 \ge 1$, so μ is non-zero.

441D The hypotheses of 441C are deliberately drawn as widely as possible. The principal application is the one for which the chapter is named.

Definition If G is a topological group, a **left Haar measure** on G is a non-zero quasi-Radon measure μ on G which is invariant for the left action of G on itself, that is, $\mu(aE) = \mu E$ whenever μ measures E and $a \in G$.

Similarly, a **right Haar measure** is a non-zero quasi-Radon measure μ such that $\mu(Ea) = \mu E$ for every $E \in \operatorname{dom}\mu$, $a \in G$.

(My reasons for requiring 'quasi-Radon' here will appear in §§442 and 443.)

441E Theorem A locally compact Hausdorff topological group has left and right Haar measures, which are both Radon measures.

proof Both the left and right actions of G on itself satisfy the conditions of 441C. **P** In both cases, condition (i) is just the (separate) continuity of multiplication, and (ii) is trivial, as every orbit is the whole of G. As for (iii), let us take the left action first. Given disjoint compact subsets K, L of G, then $M = \{y^{-1}z : y \in K, z \in L\}$ is a compact subset of G not containing the identity e. Because the topology is Hausdorff, M is closed and $X \setminus M$ is a neighbourhood of e. Because multiplication and inversion are continuous, there are open neighbourhoods V, V' of e such that $uv^{-1} \in G \setminus M$ whenever $u \in V$ and $v \in V'$. Set $U = V \cap V'$; then U is a non-empty open set in G.

? Suppose, if possible, that there is a $a \in G$ such that aU meets both K and L. Take $y \in K \cap aU$ and $z \in L \cap aU$. Then $a^{-1}y \in U \subseteq V$ and $a^{-1}z \in U \subseteq V'$, so

$$y^{-1}z = (a^{-1}y)^{-1}a^{-1}z \in G \setminus M;$$

but also $y^{-1}z \in M$. **X**

Thus aU meets at most one of K, L for any $a \in G$. As K and L are arbitrary, condition (iii) of 441C is satisfied.

For the right action, we use the same ideas, but vary the formulae. Set $M = \{yz^{-1} : y \in K, z \in L\}$, and choose V and V' such that $uv^{-1} \in X \setminus M$ for $u \in V$, $v \in V'$. Then if $a \in G$, $y \in K$ and $z \in L$, $za(ya)^{-1} \in M$ and one of za, ya does not belong to $U = V \cap V'$, that is, one of z, y does not belong to $Ua^{-1} = a \bullet U$. **Q**

Then 441C provides us with non-zero left and right Haar measures on G, and also tells us that they are Radon measures.

441F A different type of example is provided by locally compact metric spaces.

Definition If (X, ρ) is any metric space, its **isometry group** is the set of permutations $g : X \to X$ which are **isometries**, that is, $\rho(g(x), g(y)) = \rho(x, y)$ for all x, $y \in X$.

441G The topology of an isometry group Let (X, ρ) be a metric space and G the isometry group of X.

(a) Give G the topology of pointwise convergence inherited from the product topology of X^X. Then G is a Hausdorff topological group and the action of G on X is continuous. **P** If $x \in X$, g_0, $h_0 \in G$ and $\epsilon > 0$, then $V = \{g : \rho(gh_0(x), g_0h_0(x)) \le \frac{1}{2}\epsilon\}$ is a neighbourhood of g_0 and $V' = \{h : \rho(h(x), h_0(x)) \le \frac{1}{2}\epsilon\}$ is a neighbourhood of h_0. If $g \in V$ and $h \in V'$ then

$$\rho(gh(x), g_0h_0(x)) \le \rho(gh(x), gh_0(x)) + \rho(gh_0(x), g_0h_0(x)) \le \rho(h(x), h_0(x)) + \frac{1}{2}\epsilon \le \epsilon.$$

As g_0, h_0 and ϵ are arbitrary, the function $(g, h) \mapsto gh(x)$ is continuous; as x is arbitrary, multiplication on G is continuous. As for inversion, suppose that $g_0 \in G$, $\epsilon > 0$ and $x \in X$. Then $V = \{g : \rho(gg_0^{-1}(x), x) \le \epsilon\}$ is a neighbourhood of g_0, and if $g \in V$ then

$$\rho(g^{-1}(x), g_0^{-1}(x)) = \rho(x, gg_0^{-1}(x)) \le \epsilon.$$

Because g_0, ϵ and x are arbitrary, inversion on G is continuous, and G is a topological group. Because X is Hausdorff, so is G.

To see that the action is continuous, take $g_0 \in G$, $x_0 \in X$ and $\epsilon > 0$. Then $V = \{g : g \in G, \rho(g(x_0), g_0(x_0)) \le \frac{1}{2}\epsilon\}$ is a neighbourhood of g_0. If $g \in V$ and $x \in U(x_0, \frac{1}{2}\epsilon)$, then

$$\rho(g(x), g_0(x_0)) \le \rho(g(x), g(x_0)) + \rho(g(x_0), g_0(x_0)) \le \rho(x, x_0) + \frac{1}{2}\epsilon \le \epsilon.$$

As g_0, x_0 and ϵ are arbitrary, $(g, x) \mapsto g(x) : G \times X \to X$ is continuous. **Q**

(b) If X is compact, so is G. **P** By Tychonoff's theorem (3A3J), X^X is compact. Suppose that $g \in X^X$ belongs to the closure of G in X^X. For any x, $y \in X$, the set $\{f : f \in X^X, \rho(f(x), f(y)) = \rho(x, y)\}$ is closed and includes G, so contains g; thus g is an isometry. **?** If $g[X] \ne X$, take $x \in X \setminus g[X]$ and set $x_n = g^n(x)$ for every $n \in \mathbb{N}$. Because g is continuous and X is compact, $g[X]$ is closed and there is some $\delta > 0$ such that $U(x, \delta) \cap g[X] = \emptyset$. But this means that

$$\rho(x_m,x_n)=\rho(g^m(x),g^m(x_{n-m}))=\rho(x,x_{n-m})\ge\delta$$

whenever $m<n$, so that $\langle x_n\rangle_{n\in\mathbb{N}}$ can have no cluster point in X; which is impossible, because X is supposed to be compact. **X** This shows that g is surjective and belongs to G. As g is arbitrary, G is closed in X^X, therefore compact. **Q**

441H Theorem If (X,ρ) is a non-empty locally compact metric space with isometry group G, then there is a non-zero G-invariant Radon measure on X.

proof (a) Fix any $x_0\in X$, and set $Z=\overline{\{g(x_0):g\in G\}}$; then Z is a closed subset of X, so is in itself locally compact. Let H be the isometry group of Z.

(b) We need to know that $g{\restriction}Z\in H$ for every $g\in G$. **P** Because $g:X\to X$ is a homeomorphism,

$$g[Z]=\overline{\{gg'(x_0):g'\in G\}}=Z,$$

so $g{\restriction}Z$ is a permutation of Z, and of course it is an isometry, that is, belongs to H. **Q**

(c) Now Z and H satisfy the conditions of 441C.

P(i) is true just because all isometries are continuous.

(ii) Take $z\in Z$ and let U be a non-empty relatively open subset of Z. Then $U=Z\cap V$ for some open set $V\subseteq X$; as $Z\cap V\ne\emptyset$, there must be a $g_0\in G$ such that $g_0(x_0)\in V$. At the same time, there is a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in G such that $z=\lim_{n\to\infty}h_n(x_0)$. Now

$$\rho(g_0(x_0),g_0h_n^{-1}(z))=\rho(h_n(x_0),z)\to 0$$

as $n\to\infty$, so there is some n such that $g_0h_n^{-1}(z)\in V$; of course $g_0h_n^{-1}(z)$ also belongs to U, while $g_0h_n^{-1}{\restriction}Z$ belongs to H, by (b) above. As U is arbitrary, the H-orbit of z is dense in Z; as z is arbitrary, H satisfies condition (ii) of 441C.

(iii) Given that K and L are disjoint compact subsets of Z, there must be a $\delta>0$ such that $\rho(y,z)\ge\delta$ for every $y\in K$, $z\in L$. Let U be the relatively open ball $\{z:z\in Z,\,\rho(z,x_0)<\frac{1}{2}\delta\}$. Then for any $h\in H$, $\rho(y,z)<\delta$ for any y, $z\in h[U]$, so $h[U]$ cannot meet both K and L. **Q**

(d) 441C therefore provides us with a non-zero H-invariant Radon measure ν on Z. Setting $\mu E=\nu(E\cap Z)$ whenever $E\subseteq X$ and $E\cap Z\in\operatorname{dom}\nu$, it is easy to check that μ is G-invariant (using (b) again) and is a Radon measure on Z.

441I Remarks (a) Evidently there is a degree of overlap between the cases above. In an abelian group, for instance, the left and right group actions necessarily give rise to the same invariant measures. If we take $X=\mathbb{R}^2$, it has a group structure (addition) for which we have invariant measures (e.g., Lebesgue measure); these are just the translation-invariant measures. But 441H tells us that we also have measures which are invariant under all isometries (rotations and reflections as well as translations); from where we now stand, there is no surprise remaining in the fact that Lebesgue measure is invariant under this much larger group. (Though if you look back at Chapter 26, you will see that a bare-handed proof of this takes a certain amount of effort.) If we turn next to the unit sphere $\{x:\|x\|=1\}$ in $\mathbb{R}^3$, we find that there is no useful group structure, but it is a compact metric space, so carries invariant measures, e.g., two-dimensional Hausdorff measure.

(b) The arguments of 441C leave open the question of how far the invariant measures constructed there are unique. Of course any scalar multiple of an invariant measure will again be invariant. It is natural to give a special place to invariant probability measures, and call them 'normalized'; whenever we have a non-zero totally finite invariant measure we shall have an invariant probability measure. Counting measure on any set will be invariant under any action of any group, and it is natural to say that these measures also are 'normalized'; when faced with a finite set with two or more elements, we have to choose which normalization seems most reasonable in the context.

(c) We shall see in 442B that Haar measures (with a given handedness) are necessarily scalar multiples of each other. In 442Ya, 443Ud and 443Xy we have further situations in which invariant measures are essentially unique. If, in 441C, there are non-trivial G-invariant subsets of X, we do not expect such a result. But there are interesting cases in which the question seems to be open.

441J Of course we shall be much concerned with integration with respect to invariant measures. The results we need are elementary corollaries of theorems already dealt with at length, but it will be useful to have them spelt out.

Proposition Let X be a set, G a group acting on X, and μ a G-invariant measure on X. If f is a real-valued function defined on a subset of X, and $a \in G$, then $\int f(x)\mu(dx) = \int f(a\bullet x)\mu(dx)$ if either integral is defined in $[-\infty, \infty]$.

proof Apply 235G[1] to the inverse-measure-preserving functions $x \mapsto a\bullet x$ and $x \mapsto a^{-1}\bullet x$.

441K Theorem Let X be a set, G a group acting on X, and μ a G-invariant measure on X with measure algebra $\mathfrak{A}$.

(a) We have an action of G on $\mathfrak{A}$ defined by setting $a\bullet E^\bullet = (a\bullet E)^\bullet$ whenever $a \in G$ and μ measures E.

(b) We have an action of G on $L^0 = L^0(\mu)$ defined by setting $a\bullet f^\bullet = (a\bullet f)^\bullet$ for every $a \in G$, $f \in \mathcal{L}^0(\mu)$.

(c) For $1 \le p \le \infty$ the formula of (b) defines actions of G on $L^p = L^p(\mu)$, and $\|a\bullet u\|_p = \|u\|_p$ for every $u \in L^p$, $a \in G$.

proof (a) If E, $F \in \operatorname{dom}\mu$ and $E^\bullet = F^\bullet$, then (because $x \mapsto a^{-1}\bullet x$ is inverse-measure-preserving) $(a\bullet E)^\bullet = (a\bullet F)^\bullet$. So the given formula does define a function from $G \times \mathfrak{A}$ to $\mathfrak{A}$. It is now easy to check that it is an action.

(b) Let $f \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$, $a \in G$. Set $\phi_a(x) = a^{-1}\bullet x$ for $x \in X$, so that $\phi_a : X \to X$ is inverse-measure-preserving. Then $a\bullet f = f\phi_a$ belongs to $\mathcal{L}^0$. If f, $g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, then $f\phi_a =_{\text{a.e.}} g\phi_a$, so $(a\bullet f)^\bullet = (a\bullet g)^\bullet$. This shows that the given formula defines a function from $G \times L^0$ to L^0, and again it is easy to see that it is an action.

(c) If $f \in \mathcal{L}^p(\mu)$ then

$$\int |a\bullet f|^p d\mu = \int |f(a^{-1}\bullet x)|^p d\mu = \int |f|^p d\mu$$

by 441J. So $a\bullet f \in \mathcal{L}^p(\mu)$ and $\|a\bullet f^\bullet\|_p = \|f^\bullet\|_p$. Thus we have a function from $G \times L_p$ to L^p, and once more it must be an action.

441L Proposition Let X be a locally compact Hausdorff space and G a group acting on X in such a way that $x \mapsto a\bullet x$ is continuous for every $a \in G$. If μ is a Radon measure on X, then μ is G-invariant iff $\int f(x)\mu(dx) = \int f(a\bullet x)\mu(dx)$ for every $a \in G$ and every continuous function $f : X \to \mathbb{R}$ with compact support.

proof For $a \in G$, set $T_a(x) = a\bullet x$ for every $x \in X$. Then $\nu_a = \mu T_a^{-1}$ is a Radon measure on X. If $f \in C_k(X)$, then

$$\int f d\nu_a = \int f T_a d\mu$$

by 235G. Now

$$\begin{aligned} \mu \text{ is } G\text{-invariant} &\iff \nu_a = \mu \text{ for every } a \in G \\ &\iff \int f d\nu_a = \int f d\mu \text{ for every } a \in G,\ f \in C_k(X) \end{aligned}$$

(416E(b-v))

$$\iff \int f T_a d\mu = \int f d\mu \text{ for every } a \in G,\ f \in C_k(X)$$

as claimed.

441X Basic exercises >(a) Let X be a set. (i) Show that there is a one-to-one correspondence between actions $\bullet$ of the group $\mathbb{Z}$ on X and permutations $f : X \to X$ defined by the formula $n\bullet x = f^n(x)$. (ii) Show that if $f : X \to X$ is a permutation, a measure μ on X is $\mathbb{Z}$-invariant for the corresponding action iff f and f^{-1} are both inverse-measure-preserving. (iii) Show that if X is a compact Hausdorff space and $\bullet$ is a continuous action of $\mathbb{Z}$ on X, then there is a $\mathbb{Z}$-invariant Radon probability measure on X. (*Hint*: 437T.)

(b) Let (X, T, ν) be a measure space and G a group acting on X. Set $\Sigma = \{E : E \subseteq X,\ g\bullet E \in \mathrm{T} \text{ for every } g \in G\}$, and for $E \in \Sigma$ set

[1] Formerly 235I.

$$\mu E=\sup\{\sum_{i=0}^{n}\nu(g_i{\bullet}F_i):n\in\mathbb{N},F_0,\dots,F_n\text{ are disjoint subsets of }E$$
$$\text{belonging to }\Sigma,\ g_i\in G\text{ for each }i\le n\}$$

(cf. 112Yd). Show that μ is a G-invariant measure on X.

(c) Let X be a topological space and G a group acting on X such that (α) all the maps $x\mapsto a{\bullet}x$ are continuous (β) all the orbits of G are dense. Show that any non-zero G-invariant quasi-Radon measure on X is strictly positive.

>(d) Let G be a compact Hausdorff topological group. (i) Show that its conjugacy classes are closed. (ii) Show that if K, $L\subseteq G$ are disjoint compact sets then $\{ac^{-1}da^{-1}:a\in G,\,c\in K,\,d\in L\}$ is a compact set not containing e, so that there is a neighbourhood U of e such that whenever $c^{-1}d\in U$ and $a\in G$ then either $aca^{-1}\notin K$ or $ada^{-1}\notin L$. (iii) Show that every conjugacy class of G carries a Radon probability measure which is invariant under the conjugacy action of G.

(e) Let $(G,\cdot)$ be a topological group. (i) On G define a binary operation $\diamond$ by saying that $x\diamond y=y\cdot x$ for all x, $y\in G$. Show that $(G,\diamond)$ is a topological group isomorphic to $(G,\cdot)$, and that any element of G has the same inverse for either group operation. (ii) Suppose that μ is a left Haar measure on $(G,\cdot)$. Show that μ is a right Haar measure on $(G,\diamond)$. (iii) Set $\phi(a)=a^{-1}$ for $a\in G$. Show that if μ is a left Haar measure on $(G,\cdot)$ then the image measure $\mu\phi^{-1}$ is a right Haar measure on $(G,\cdot)$. (iv) Show that $(G,\cdot)$ has a left Haar measure iff it has a right Haar measure. (v) Show that $(G,\cdot)$ has a left Haar probability measure iff it has a totally finite left Haar measure iff it has a right Haar probability measure. (iv) Show that $(G,\cdot)$ has a σ-finite left Haar measure iff it has a σ-finite right Haar measure.

>(f)(i) For Lebesgue measurable $E\subseteq\mathbb{R}\setminus\{0\}$, set $\nu E=\int_E\frac{1}{|x|}dx$. Show that ν is a (two-sided) Haar measure if $\mathbb{R}\setminus\{0\}$ is given the group operation of multiplication. (ii) For Lebesgue measurable $E\subseteq\mathbb{C}\setminus\{0\}$, identified with $\mathbb{R}^2\setminus\{0\}$, set $\nu E=\int_E\frac{1}{|z|^2}\mu(dz)$, where μ is two-dimensional Lebesgue measure. Show that ν is a (two-sided) Haar measure on $\mathbb{C}\setminus\{0\}$ if we take complex multiplication for the group operation. (*Hint*: 263D.)

>(g)(i) Show that Lebesgue measure is a (two-sided) Haar measure on $\mathbb{R}^r$, for any $r\ge 1$, if we take addition for the group operation. (ii) Show that the usual measure on $\{0,1\}^I$ is a two-sided Haar measure on $\{0,1\}^I$, for any set I, if we give $\{0,1\}^I$ the group operation corresponding to its identification with $\mathbb{Z}_2^I$. (iii) Describe the corresponding Haar measure on $\mathcal{P}I$ when $\mathcal{P}I$ is given the group operation $\triangle$.

(h) Let G be a locally compact Hausdorff topological group. (i) Show that any (left) Haar measure on G must be strictly positive. (ii) Show that G has a totally finite (left) Haar measure iff it is compact. (iii) Show that G has a σ-finite (left) Haar measure iff it is σ-compact.

>(i)(i) Let G and H be topological groups with left Haar measures μ and ν. Show that the quasi-Radon product measure on $G\times H$ (417N) is a left Haar measure on $G\times H$. (ii) Let $\langle G_i\rangle_{i\in I}$ be a family of topological groups, and suppose that each G_i has a left Haar probability measure (as happens, for instance, if each G_i is compact). Show that the quasi-Radon product measure on $\prod_{i\in I}G_i$ (417O) is a left Haar measure on $\prod_{i\in I}G_i$.

(j)(i) Show that any (left) Haar measure on a topological group, as defined in 441D, must be locally finite. (ii) Show that any (left) Haar measure on a locally compact Hausdorff group must be a Radon measure.

(k) Let $r\ge 1$ be an integer, and set $X=\{x:x\in\mathbb{R}^r,\,\|x\|=1\}$. Let G be the group of orthogonal $r\times r$ real matrices, so that G acts transitively on X. Show that (when given its natural topology as a subset of $\mathbb{R}^{r^2}$) G is a compact Hausdorff topological group. Let μ be a left Haar measure on G, and x any point of X; set $\phi_x(T)=Tx$ for $T\in G$. Show that the image measure $\mu\phi_x^{-1}$ is a G-invariant measure on X, independent of the choice of x.

(l) Let X be a non-abelian Hausdorff topological group with a left Haar probability measure μ. Let λ be the quasi-Radon product measure on X^2. Show that $\lambda\{(x,y):xy=yx\}\le\frac{5}{8}$. (*Hint*: if Z is the centre of X, X/Z is not cyclic, so $\mu Z\le\frac{1}{4}$.)

(m) Let X be a compact metric space, and $g : X \to X$ any isometry. Show that g is surjective. (*Hint*: if $x \in X$, then $\rho(g^m x, g^n x) \ge \rho(x, g[X])$ for any $m < n$.)

(n) Let (X, ρ) be a locally compact metric space, and $\mathcal{C}$ the set of closed subsets of X with its Fell topology (4A2T). Show that if G is the isometry group of X with its topology of pointwise convergence, then $(g, F) \mapsto g[F]$ is a continuous action of G on $\mathcal{C}$.

(o) Let (X, ρ) be a metric space and $\mathcal{K}$ the family of compact subsets of X with the topology induced by the Vietoris topology on the space of closed subsets of X (4A2T). Show that if G is the isometry group of X with its topology of pointwise convergence, then $(g, K) \mapsto g[K]$ is a continuous action of G on $\mathcal{K}$.

>(p) Let (X, ρ) be a metric space, and G its isometry group with the topology $\mathfrak{T}$ of pointwise convergence. (i) Show that if X is compact, $\mathfrak{T}$ can be defined by the metric $(g, h) \mapsto \max_{x \in X} \rho(g(x), h(x))$. (ii) Show that if $\{y : \rho(y, x) \le \gamma\}$ is compact for every $x \in X$ and $\gamma > 0$, then G is locally compact. (iii) Show that if X is separable then G is metrizable. (iv) Show that if (X, ρ) is complete then G is complete under its bilateral uniformity. (v) Show that if X is separable and (X, ρ) is complete then G is Polish.

>(q) Give $\mathbb{N}$ the zero-one metric ρ. Let G be the isometry group of $\mathbb{N}$ (that is, the group of all permutations of $\mathbb{N}$) with its topology of pointwise convergence. (i) Show that G is a G_δ subset of $\mathbb{N}^{\mathbb{N}}$, so is a Polish group. (ii) Show that if we set $\Delta(g, h) = \min\{n : n \in \mathbb{N}, g(n) \ne h(n)\}$ and $\sigma(g, h) = 1/(1 + \Delta(g^{-1}, h^{-1}))$ for distinct g, $h \in G$, then σ is a right-translation-invariant metric on G inducing its topology. (iii) Show that there is no complete right-translation-invariant metric on G inducing its topology. (*Hint*: any such metric must have the same Cauchy sequences as σ.) (iv) Show that G is not locally compact.

>(r) Let $r \ge 1$ be an integer, and S_{r-1} the sphere $\{x : x \in \mathbb{R}^r, \|x\| = 1\}$. (i) Show that every isometry ϕ from S_{r-1} to itself corresponds to an orthogonal $r \times r$ matrix T. (*Hint*: $T = \langle \phi(e_i) \,\boldsymbol{.}\, e_j \rangle_{i,j<r}$.) (ii) Show that the topology of pointwise convergence on the isometry group of S_{r-1} corresponds to the topology on the set of $r \times r$ matrices regarded as a subset of $\mathbb{R}^{r^2}$.

(s) Let X be a locally compact metric space and G a subgroup of the isometry group of X. Show that for every $x \in X$ there is a non-zero G-invariant Radon measure on $\overline{\{g(x) : g \in G\}}$.

(t) Let $r \ge 1$ be an integer, and $X = [0, 1[^r$. Let G be the set of $r \times r$ matrices with integer coefficients and determinant ± 1, and for $A \in G$, $x \in X$ say that $A \bullet x = \begin{pmatrix} <\eta_1> \\ \dots \\ <\eta_r> \end{pmatrix}$ where $\begin{pmatrix} \eta_1 \\ \dots \\ \eta_r \end{pmatrix} = Ax$ and $<\alpha>$ is the fractional part of α for each $\alpha \in \mathbb{R}$. (i) Show that $\bullet$ is an action of G on X, and that Lebesgue measure on X is G-invariant. (ii) Show that if X is given the compact Hausdorff topology corresponding to the bijection $\alpha \mapsto (\cos 2\pi\alpha, \sin 2\pi\alpha)$ from X to the unit circle in $\mathbb{R}^2$, and G is given its discrete topology, the action is continuous.

441Y Further exercises (a) Let (X, ρ) be a metric space, and $\mathcal{C}$ the family of non-empty closed subsets of X, with its Hausdorff metric $\tilde{\rho}$ (4A2T). Show that if G is the isometry group of X, $(g, F) \mapsto g[F]$ is an action of G on $\mathcal{C}$.

(b) Take $1 \le s \le r \in \mathbb{N}$. Let $\mathcal{C}$ be the set of closed subsets of $\mathbb{R}^r$ with its Fell topology. Let $\mathcal{C}_s \subseteq \mathcal{C}$ be the set of s-dimensional linear subspaces of $\mathbb{R}^r$. Show that $\mathcal{C}_s$ is a closed subset of $\mathcal{C}$, therefore a compact metrizable space in its own right, and that the group G of orthogonal $r \times r$ matrices acts transitively on $\mathcal{C}_s$, so that there is a G-invariant Radon probability measure on $\mathcal{C}_s$.

(c) For w, $z \in \mathbb{C} \setminus \{0\}$ set $\rho(w, z) = |\operatorname{Ln}(\frac{w}{z})|$, where $\operatorname{Ln} v = \ln |v| + i \arg v$ for non-zero complex numbers v. (i) Show that ρ is a metric on $\mathbb{C} \setminus \{0\}$. (ii) Show that the 2-dimensional Hausdorff measure $\mu_{H2}^{(\rho)}$ derived from ρ (471A) is a Haar measure for the multiplicative group $\mathbb{C} \setminus \{0\}$. (iii) Show that $\mu_{H2}^{(\rho)} = \frac{4}{\pi}\nu$, where ν is the measure of 441Xf(ii). (*Hint*: 264I.)

(d) Let X be the group of real $r \times r$ orthogonal matrices, where $r \ge 2$ is an integer. Give X the Euclidean metric, regarding it as a subset of $\mathbb{R}^{r^2}$. (i) Show that the left and right actions of X on itself are distance-preserving. (ii) Show that $\frac{r(r-1)}{2}$-dimensional Hausdorff measure on X is a two-sided Haar measure.

(e) Let $X=\mathrm{SO}(3)$ be the set of real 3×3 orthogonal matrices with determinant 1. Give X the metric corresponding to its embedding in 9-dimensional Euclidean space. (i) Show that X can be parametrized as the set of matrices

$$\phi\begin{pmatrix}z\\ \alpha\\ \theta\end{pmatrix}=\begin{pmatrix}z & -\cos\theta\sqrt{1-z^2} & \sin\theta\sqrt{1-z^2}\\ \cos\alpha\sqrt{1-z^2} & z\cos\alpha\cos\theta-\sin\alpha\sin\theta & -z\cos\alpha\sin\theta-\sin\alpha\cos\theta\\ \sin\alpha\sqrt{1-z^2} & z\sin\alpha\cos\theta+\cos\alpha\sin\theta & -z\sin\alpha\sin\theta+\cos\alpha\cos\theta\end{pmatrix}$$

with $z\in[-1,1]$, $\alpha\in[-\pi,\pi]$ and $\theta\in[-\pi,\pi]$. (ii) Show that if T is the 9×3 matrix which is the derivative of ϕ at $\begin{pmatrix}z\\ \alpha\\ \theta\end{pmatrix}$, then $T^\top T=\begin{pmatrix}\frac{2}{\sqrt{1-z^2}} & 0 & 0\\ 0 & 2 & 2z\\ 0 & 2z & 2\end{pmatrix}$ has constant determinant, so that if μ is Lebesgue measure on $[-1,1]\times[-\pi,\pi]^2$ then the image measure $\mu\phi^{-1}$ is a Haar measure on X. (*Hint*: 441Yd, 265E.) (iii) Show that if $A\in X$ corresponds to a rotation through an angle $\gamma(A)\in[0,\pi]$ then its trace $\mathrm{tr}(A)$ (that is, the sum of its diagonal entries) is $1+2\cos\gamma(A)$. (*Hint*: $\mathrm{tr}(AB)=\mathrm{tr}(BA)$ for any square matrices A and B of the same size.) (iv) Show that if X is given its Haar probability measure then $\cos\gamma(A)$ has expectation $-\frac{1}{2}$.

(f) Let $(X,\mathcal{W})$ be a locally compact Hausdorff uniform space, and suppose that G is a group acting on X 'uniformly equicontinuously'; that is, for every $W\in\mathcal{W}$ there is a $V\in\mathcal{W}$ such that $(a\bullet x,a\bullet y)\in W$ whenever $(x,y)\in V$ and $a\in G$. Show that there is a non-zero G-invariant Radon measure on X.

(g) In 441Xk, show that $\mu\phi_x^{-1}$ is a scalar multiple of Hausdorff $(r-1)$-dimensional measure on X.

(h) For any topological spaces X and Y, and any set G of functions from X to Y, the **compact-open** topology on G is the topology generated by sets of the form $\{g:g\in G,\ g[K]\subseteq H\}$, where $K\subseteq X$ is compact and $H\subseteq Y$ is open. Show that if (X,ρ) is a metric space and G is the isometry group of X, then the topology of pointwise convergence on G is its compact-open topology.

(i) Let X be a compact Hausdorff space and G the group of all homeomorphisms from X to itself. (i) Let P be the family of all continuous pseudometrics on X (see 4A2Jg). For $\rho\in\mathrm{P}$ and g, $h\in G$, set $\tau_\rho(g,h)=\max_{x\in X}\rho(g(x),h(x))$. Show that every τ_ρ is a right-translation-invariant pseudometric on G, and that G with the topology generated by $\{\tau_\rho:\rho\in\mathrm{P}\}$ is a topological group. (ii) Show that this is the compact-open topology as defined in 441Yh. (iii) Show that if X is metrizable then G is Polish.

(j) Let X be a compact metric space and G the isometry group of X. Show that every G-orbit in X is closed.

(k) Let T be any set, and ρ the $\{0,1\}$-valued metric on T. Let X be the set of partial orders on T, regarded as a subset of $\mathcal{P}(T\times T)$. Show that X is compact (cf. 418Xv). Let G be the group of isometries of T with its topology of pointwise convergence. Set $\pi\bullet x=\{(t,u):(\pi^{-1}(t),\pi^{-1}(u))\in x\}$ for $\pi\in G$ and $x\in X$. Show that $\bullet$ is a continuous action of G on X. Show that there is a strictly positive G-invariant Radon probability measure μ on X.

(l) Let X be a set, G a group acting on X, and μ a G-invariant measure on X with measure algebra $\mathfrak{A}$. Show that if τ is any rearrangement-invariant extended Fatou norm on $L^0(\mathfrak{A})$ then the formula of 441Kb defines a norm-preserving action of G on the Banach space L^τ.

(m) (M.Elekes & T.Keleti) Let X be a set, G a group acting on X, Σ a σ-algebra of subsets of X such that $g\bullet E\in\Sigma$ whenever $E\in\Sigma$ and $g\in G$, and H an element of Σ. Suppose that μ is a measure, with domain the subspace σ-algebra Σ_H, such that $\mu(g\bullet E)=\mu E$ whenever $E\in\Sigma_H$ and $g\in G$ are such that $g\bullet E\subseteq H$. (i) Show that $\sum_{n=0}^\infty\mu E_n=\sum_{n=0}^\infty\mu E'_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ and $\langle E'_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ_H for which there are sequences $\langle g_n\rangle_{n\in\mathbb{N}}$, $\langle g'_n\rangle_{n\in\mathbb{N}}$ in G such that $\langle g_n\bullet E_n\rangle_{n\in\mathbb{N}}$ and $\langle g'_n\bullet E'_n\rangle_{n\in\mathbb{N}}$ are partitions of the same subset of X. (ii) Show that there is a G-invariant measure with domain Σ which extends μ.

(n) Let $\bullet_X$ be an action of a group G on a set X, μ a G-invariant measure on X, $(\mathfrak{A},\bar\mu)$ its measure algebra and $\bullet_{\mathfrak{A}}$ the induced action on $\mathfrak{A}$. Set $Z=X^G$; define $\phi:X\to Z$ by setting $\phi(x)=\langle g^{-1}\bullet x\rangle_{g\in G}$ for $x\in X$; let ν be the image measure $\mu\phi^{-1}$, and $(\mathfrak{B},\bar\nu)$ its measure algebra. Let $\bullet_Z$ be the left shift action of G on Z; show that ν is $\bullet_Z$-invariant, so that there is an induced action $\bullet_{\mathfrak{B}}$ on $\mathfrak{B}$. Show that $(\mathfrak{A},\bar\mu,\bullet_{\mathfrak{A}})$ and $(\mathfrak{B},\bar\nu,\bullet_{\mathfrak{B}})$ are isomorphic.

(o) Let X be a topological space, G a topological group and $\bullet$ a continuous action of G on X. Let M_{qR}^+ be the set of totally finite quasi-Radon measures on X. As in 4A5B-4A5C, write $a\bullet E = \{a\bullet x : x \in E\}$ for $a \in G$ and $E \subseteq X$, and $(a\bullet f)(x) = f(a^{-1}\bullet x)$ for $a \in G$, $x \in X$ and a real-valued function f defined at $a^{-1}\bullet x$. (i) Show that we have an action $\bullet$ of G on M_{qR}^+ defined by saying that $(a\bullet\nu)(E) = \nu(a^{-1}\bullet E)$ whenever $a \in G$, $\nu \in M_{\mathrm{qR}}^+$ and $E \subseteq X$ are such that ν measures $a^{-1}\bullet E$. (ii) Show that this action is continuous if we give M_{qR}^+ its narrow topology. (iii) Show that if $\nu \in M_{\mathrm{qR}}^+$, $f \in \mathcal{L}^1(\nu)$ is non-negative and $f\nu$ is the corresponding indefinite-integral measure, then $a\bullet(f\nu)$ is the indefinite-integral measure $(a\bullet f)(a\bullet\nu)$ for every $a \in G$.

(p) Let X be a set, G a group acting on X and μ a totally finite G-invariant measure on X with domain Σ. Suppose there is a probability measure ν on G, with domain T, such that $(a, x) \mapsto a^{-1}\bullet x : G \times X \to X$ is $(\mathrm{T}\widehat{\otimes}\Sigma, \Sigma)$-measurable and ν is invariant under the left action of G on itself. Let $u \in L^0(\mu)$ be such that $a\bullet u = u$ for every $a \in G$. Show that there is an $f \in \mathcal{L}^0(\mu)$ such that $f^\bullet = u$ and $a\bullet f = f$ for every $a \in G$. (*Hint*: if $u = g^\bullet$ where $g : X \to \mathbb{R}$ is Σ-measurable and μ-integrable, try $f(x) = \int g(a^{-1}\bullet x)\nu(da)$ when this is defined.)

(q) Let X be a topological space, G a compact Hausdorff group, $\bullet$ a continuous action of G on X, and μ a G-invariant quasi-Radon measure on X. Let $u \in L^0(\mu)$ be such that $a\bullet u = u$ for every $a \in G$. Show that there is an $f \in \mathcal{L}^0(\mu)$ such that $f^\bullet = u$ and $a\bullet f = f$ for every $a \in G$.

441 Notes and comments The proof I give of 441C is essentially the same as the proof of 441E in Halmos 50, §58.

In part (f) of the proof of 441C I use an ultrafilter, relying on a fairly strong consequence of the Axiom of Choice. In this volume, as in the last, I generally employ the axiom of choice without stopping to consider whether it is really needed. But Haar measure, at least, is so important that I point out now that it can be built with much weaker principles. For a construction of Haar measure not dependent on choice, see 561G in Volume 5. I ought to remark that the argument there leads us to a translation-invariant linear functional rather than a measure, and that while there is still a version of the Riesz representation theorem (564I), we may get something less than a proper countably additive measure if we do not have countable choice. Moreover, in the absence of the full axiom of choice, we may find that we have fewer locally compact topological groups than we expect.

While Haar measure is surely the pre-eminent application of the theory here, I think that some of the other consequences of 441C (441H, 441Xd, 441Xk, 441Yb, 441Yf) are sufficiently striking to justify the trouble involved in the extra generality. I ought to remark that there are important examples of invariant measures which have nothing to do with 441C. Some of these will appear in §449; for the moment I note only 441Xa.

Federer 69, §2.7, develops a general theory of 'covariant' measures μ ('relatively invariant' in Halmos 50) such that $\mu(a\bullet E) = \psi(a)\mu E$ for appropriate sets $E \subseteq X$ and $a \in G$, where $\psi : G \to]0, \infty[$ is a homomorphism; for instance, taking μ to be Lebesgue measure on $\mathbb{R}^r$, we have $\mu T[E] = |\det T|\mu E$ for every linear space isomorphism $T : \mathbb{R}^r \to \mathbb{R}^r$ and every measurable set E (263A). The theory I have described here can deal only with the subgroup of isometric linear isomorphisms (that is, the orthogonal group). Covariant measures arise naturally in many other contexts, such as 443T below.

Hausdorff measures, being defined in terms of metrics, are necessarily invariant under isometries, so appear naturally in this context, starting with 264I. There are interesting challenges both in finding suitable metrics and in establishing exact constants, as in 441Yc-441Yd.

It is worth pausing over the topology of an isometry group, as described in 441G. It is quite surprising that such an elementary idea should give us a topological group at all. I offer some exercises (441Xp-441Xr, 441Yh) to help you relate the construction to material which may be more familiar. It is of course a 'weak' topology, except when the underlying space is compact (441Xp(i)). See also 441Yi. These groups are rarely locally compact, and you may find them pushed out of your mind by the extraordinary theory which you will see developed in the next hundred pages; but in the last two sections of this chapter, and in §493, they will become leading examples.

442 Uniqueness of Haar measures

Haar measure has an extraordinary wealth of special properties, and it will be impossible for me to cover them all properly in this chapter. But surely the second thing to take on board, after the existence of Haar measures on locally compact Hausdorff groups (441E), is the fact that they are, up to scalar multiples, unique. This is the content of 442B. We find also that while left and right Haar measures can be different (442Xf), they are not only direct mirror images of each other (442C) – as is, I suppose, to be expected – but even more closely related (442F, 442H, 442L). Investigating this relation, we are led naturally to the 'modular function' of a group (442I).

442A Lemma Let X be a topological group and μ a left Haar measure on X.

(a) μ is strictly positive and locally finite.

(b) If $G \subseteq X$ is open and $\gamma < \mu G$, there are an open set H and an open neighbourhood U of the identity such that $HU \subseteq G$ (writing HU for $\{xy : x \in H,\, y \in U\}$) and $\mu H \geq \gamma$.

(c) If X is locally compact and Hausdorff, μ is a Radon measure.

proof (a)(i) ? If $G \subseteq X$ were a non-empty open set such that $\mu G = 0$, then we should have $\mu(xG) = 0$ for every $x \in X$, so that X would be covered by negligible open sets; but as μ is supposed to be τ-additive, $\mu X = 0$. **X**

(ii) Because μ is effectively locally finite, there is some non-empty open set G such that $\mu G < \infty$; but now $\{xG : x \in X\}$ is a cover of X by open sets of finite measure.

(b) Let $\mathcal{U}$ be the family of open sets containing the identity e, and $\mathcal{H}$ the family of open sets H such that $HU \subseteq G$ for some $U \in \mathcal{U}$. Because $\mathcal{U}$ is downwards-directed, $\mathcal{H}$ is upwards-directed; because $e \in U$ for every $U \in \mathcal{U}$, $\bigcup \mathcal{H} \subseteq G$. If $x \in G$, then $x^{-1}G \in \mathcal{U}$, and there is a $U \in \mathcal{U}$ such that $UU \subseteq x^{-1}G$; but now $xUU \subseteq G$, so $xU \in \mathcal{H}$. Thus $\bigcup \mathcal{H} = G$. Because μ is τ-additive, there is an $H \in \mathcal{H}$ such that $\mu H \geq \gamma$.

(c) Use (a) and 416G.

442B Theorem Let X be a topological group. If μ and ν are left Haar measures on X, they are multiples of each other.

proof (a) Let $\mathcal{G}$ be the family of non-empty open sets G such that μG and νG are both finite; because μ and ν are locally finite (442Aa), $\mathcal{G}$ is a base for the topology of X. Note that $G \cup H \in \mathcal{G}$ for all $G, H \in \mathcal{G}$. Set $\mathcal{U} = \{U : U \in \mathcal{G},\, U = U^{-1},\, e \in U\}$, where e is the identity of G; then $\mathcal{U}$ is a base of neighbourhoods of e (4A5Ec). Let $\mathcal{F}$ be the filter on $\mathcal{U}$ generated by the sets $\{U : U \in \mathcal{U},\, U \subseteq V\}$ as V runs over $\mathcal{U}$.

(b) (The key.) If $G \in \mathcal{G}$ and $0 < \epsilon < 1$, there is a $V_1 \in \mathcal{U}$ such that

$$(1-\epsilon)\frac{\mu G}{\nu G} \leq \frac{\mu U}{\nu U}$$

whenever $U \in \mathcal{U}$ and $U \subseteq V_1$. **P** By 442Aa, μG and νG are both non-zero. By 442Ab, there are an open set H and a neighbourhood V_1 of e such that $HV_1 \subseteq G$ and $\mu H \geq (1-\epsilon)\mu G$; shrinking V_1 if need be, we may suppose that $V_1 \in \mathcal{U}$. Take any $U \in \mathcal{U}$ such that $U \subseteq V_1$, so that $HU \subseteq G$. Consider the product quasi-Radon measure λ of μ and ν on $X \times X$ (417R), and the set $W = \{(x, y) : x, y \in G,\, x^{-1}y \in U\}$. Because the function $(x, y) \mapsto x^{-1}y$ is continuous (4A5Eb), W is open. Consequently

$$\int \mu\{x : (x,y) \in W\}\nu(dy) = \lambda W = \int \nu\{y : (x,y) \in W\}\mu(dx)$$

(417C(iv)). But we see that if $x \in H$ then $xU \subseteq G$, so $(x, y) \in W$ whenever $y \in xU$, and

$$\int \nu\{y : (x,y) \in W\}\mu(dx) \geq \int_H \nu(xU)\mu(dx) = \mu H \cdot \nu U.$$

On the other hand,

$$\begin{aligned}\int \mu\{x : (x,y) \in W\}\nu(dy) &\leq \int_G \mu\{x : x^{-1}y \in U\}\nu(dy) = \int_G \mu\{x : y^{-1}x \in U\}\nu(dy) \\ &= \int_G \mu(yU)\nu(dy) = \mu U \cdot \nu G,\end{aligned}$$

so that

$$(1-\epsilon)\mu G \cdot \nu U \leq \mu H \cdot \nu U \leq \lambda W \leq \mu U \cdot \nu G.$$

Dividing both sides by $\nu U \cdot \nu G$, we have the result. **Q**

(c) In the same way, there is a $V_2 \in \mathcal{V}$ such that

$$(1-\epsilon)\frac{\nu G}{\mu G} \le \frac{\nu U}{\mu U}$$

whenever $U \in \mathcal{U}$ and $U \subseteq V_2$. So if $U \in \mathcal{U}$ and $U \subseteq V_1 \cap V_2$, we have

$$(1-\epsilon)\frac{\mu G}{\nu G} \le \frac{\mu U}{\nu U} \le \frac{1}{1-\epsilon}\frac{\mu G}{\nu G}.$$

As ϵ is arbitrary,

$$\lim_{U\to\mathcal{F}} \frac{\mu U}{\nu U} = \frac{\mu G}{\nu G}.$$

And this is true for every $G \in \mathcal{G}$.

(d) So if we set $\alpha = \lim_{U\to\mathcal{F}} \frac{\mu U}{\nu U}$, we shall have $\mu G = \alpha\nu G$ for every $G \in \mathcal{G}$. Now μ and $\alpha\nu$ are quasi-Radon measures agreeing on the base $\mathcal{G}$, which is closed under finite unions, so are identical, by 415H(iv).

442C Proposition Let X be a topological group and μ a left Haar measure on X. Setting $\nu E = \mu(E^{-1})$ whenever $E \subseteq X$ is such that $E^{-1} = \{x^{-1} : x \in E\}$ is measured by μ, ν is a right Haar measure on X.

proof Set $\phi(x) = x^{-1}$ for $x \in X$. Then ϕ is a homeomorphism, so the image measure $\nu = \mu\phi^{-1}$ is a quasi-Radon measure. It is non-zero because $\nu X = \mu X$. If $E \in \operatorname{dom}\nu$ and $x \in X$, then

$$\nu(Ex) = \mu(x^{-1}E^{-1}) = \mu E^{-1} = \nu E.$$

So ν is a right Haar measure.

442D Remark Clearly all the arguments of 442A-442C must be applicable to right Haar measures; that is, any right Haar measure must be locally finite and strictly positive; two right Haar measures on the same group must be multiples of each other; and if X carries a right Haar measure ν then $E \mapsto \nu E^{-1}$ will be a left Haar measure on X. (If you are unhappy with such a bold appeal to the symmetry between 'left' and 'right' in topological groups, write the reflected version of 442C out in full, and use it to reflect 442A-442B.)

Thus we may say that a topological group **carries Haar measures** if it has either a left or a right Haar measure. These can, of course, be the same; in fact it takes a certain amount of exploration to find a group in which they are different (e.g., 442Xf).

442E Lemma Let X be a topological group, μ a left Haar measure on X and ν a right Haar measure on X. If G, $H \subseteq X$ are open, then

$$\mu G \cdot \nu H = \int_H \nu(xG^{-1})\mu(dx).$$

proof Let λ be the quasi-Radon product measure of μ and ν on $X \times X$. The sets $W_1 = \{(x,y) : y^{-1}x \in G,\, x \in H\}$ and $W_2 = \{(x,y) : x \in G,\, yx \in H\}$ are both open, so 417C(iv) tells us that

$$\begin{aligned}
\int_H \nu(xG^{-1})\mu(dx) &= \int \nu\{y : x \in H,\, y^{-1}x \in G\}\mu(dx) = \lambda W_1\\
&= \int \mu\{x : x \in H,\, y^{-1}x \in G\}\nu(dy) = \int \mu(H \cap yG)\nu(dy)\\
&= \int \mu(y^{-1}H \cap G)\nu(dy) = \int \mu\{x : x \in G,\, yx \in H\}\nu(dy)\\
&= \lambda W_2 = \int_G \nu\{y : yx \in H\}\mu(dx)\\
&= \int_G \nu(Hx^{-1})\mu(dx) = \mu G \cdot \nu H.
\end{aligned}$$

442F Domains of Haar measures 442B tells us, in part, that any two left Haar measures on a topological group must have the same domain and the same negligible sets; similarly, any two right Haar measures have the same domain and the same negligible sets. In fact left and right Haar measures agree on both.

Proposition Let X be a topological group which carries Haar measures. If μ is a left Haar measure and ν is a right Haar measure on X, then they have the same domains and the same null ideals.

proof (a) Suppose that $F \subseteq X$ is a closed set such that $\mu F = 0$. Then $\nu F = 0$. **P?** Otherwise, there is an open set H such that $\nu H < \infty$ and $\nu(F \cap H) > 0$. Let G be any open set such that $0 < \mu G < \infty$. By 442E,

$$\mu G \cdot \nu H = \int_H \nu(xG^{-1})\mu(dx).$$

But also

$$\mu G \cdot \nu(H \setminus F) = \int_{H \setminus F} \nu(xG^{-1})\mu(dx) = \int_H \nu(xG^{-1})\mu(dx) = \mu G \cdot \nu H,$$

so $\mu G \cdot \nu(H \cap F) = 0$. **XQ**

(b) It follows that

$$\nu E = \sup_{F \subseteq E \text{ is closed}} \nu F = 0$$

whenever E is a Borel set such that $\mu E = 0$. Now take any $E \in \operatorname{dom} \mu$. Set

$$\mathcal{G} = \{G : G \subseteq X \text{ is open}, \, \mu G < \infty, \, \nu G < \infty\}.$$

Because both μ and ν are locally finite, $\mathcal{G}$ covers X. If $G \in \mathcal{G}$, there are Borel sets E', E'' such that $E' \subseteq E \cap G \subseteq E''$ and $\mu(E'' \setminus E') = 0$. In this case $\nu(E'' \setminus E') = 0$ so $E \cap G \in \operatorname{dom} \nu$. Because ν is complete, locally determined and τ-additive, $E \in \operatorname{dom} \nu$ (414I). If $\mu E = 0$, it follows that

$$\nu E = \sup_{F \subseteq E \text{ is closed}} \nu F = 0$$

just as above.

(c) Thus ν measures E whenever μ measures E, and E is ν-negligible whenever it is μ-negligible. I am sure you will have no difficulty in believing that all the arguments above, in particular that of 442E, can be re-cast to show that $\operatorname{dom} \nu \subseteq \operatorname{dom} \mu$; alternatively, apply the result in the form just demonstrated to the left Haar measure ν' and the right Haar measure μ', where

$$\nu' E = \nu E^{-1}, \quad \mu' E = \mu E^{-1}$$

as in 442C.

442G Corollary Let X be a topological group and μ a left Haar measure on X with domain Σ. Then, for $E \subseteq X$ and $a \in X$,

$$E \in \Sigma \iff E^{-1} \in \Sigma \iff Ea \in \Sigma,$$

$$\mu E = 0 \iff \mu E^{-1} = 0 \iff \mu(Ea) = 0.$$

proof Apply 442F with $\nu E = \mu E^{-1}$.

442H Remark From 442F-442G we see that if X is any topological group which carries Haar measures, there is a distinguished σ-algebra Σ of subsets of X, which we may call the algebra of **Haar measurable sets**, which is the domain of any Haar measure on X. Similarly, there is a σ-ideal $\mathcal{N}$ of $\mathcal{P}X$, the ideal of **Haar negligible sets**[2], which is the null ideal for any Haar measure on X. Both Σ and $\mathcal{N}$ are translation-invariant and also invariant under the inversion operation $x \mapsto x^{-1}$.

If we form the quotient $\mathfrak{A} = \Sigma/\mathcal{N}$, then we have a fixed Dedekind complete Boolean algebra which is the **Haar measure algebra** of the group X in the sense that any Haar measure on X, whether left or right, has measure algebra based on $\mathfrak{A}$. If $a \in X$, the maps $x \mapsto ax$, $x \mapsto xa$, $x \mapsto x^{-1}$ give rise to Boolean automorphisms of $\mathfrak{A}$.

For a member of Σ, we have a notion of 'σ-finite' which is symmetric between left and right (442Xd). We do not in general have a corresponding two-sided notion of 'finite measure' (442Xg(i)); but of course we can if we wish speak of a set as having 'finite left Haar measure' or 'finite right Haar measure' without declaring which Haar measure we are thinking of. It is the case, however, that if the group X itself has finite left Haar measure, it also has finite right Haar measure; see 442Ic-d below.

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[2]**Warning!** do not confuse with the 'Haar null' sets described in 444Ye below.

442I The modular function Let X be a topological group which carries Haar measures.

(a) There is a group homomorphism $\Delta : X \to]0, \infty[$ defined by the formula

$$\mu(Ex) = \Delta(x)\mu E \text{ whenever } \mu \text{ is a left Haar measure on } X \text{ and } E \in \operatorname{dom}\mu.$$

P Fix on a left Haar measure $\tilde{\mu}$ on X. For $x \in X$, let μ_x be the function defined by saying

$$\mu_x E = \tilde{\mu}(Ex) \text{ whenever } E \subseteq X,\ Ex \in \operatorname{dom}\mu,$$

that is, for every Haar measurable set $E \subseteq X$. Because the function $\phi_x : X \to X$ defined by setting $\phi_x(y) = yx^{-1}$ is a homeomorphism, $\mu_x = \tilde{\mu}\phi_x^{-1}$ is a quasi-Radon measure on X; and

$$\mu_x(yE) = \tilde{\mu}(yEx) = \tilde{\mu}(Ex) = \mu_x E$$

whenever μ_x measures E, so μ_x is a left Haar measure on X. By 442B, there is a $\Delta(x) \in]0, \infty[$ such that $\mu_x = \Delta(x)\tilde{\mu}$; because $\tilde{\mu}$ surely takes at least one value in $]0, \infty[$, $\Delta(x)$ is uniquely defined.

If μ is any other left Haar measure on X, then $\mu = \alpha\tilde{\mu}$ for some $\alpha > 0$, so that

$$\mu(Ex) = \alpha\tilde{\mu}(Ex) = \alpha\Delta(x)\tilde{\mu}E = \Delta(x)\mu E.$$

Thus $\Delta : X \to]0, \infty[$ has the property asserted in the formula offered.

To see that Δ is a group homomorphism, take any x, $y \in X$ and a Haar measurable set E such that $0 < \tilde{\mu}E < \infty$, and observe that

$$\Delta(xy)\tilde{\mu}E = \tilde{\mu}(Exy) = \Delta(y)\tilde{\mu}(Ex) = \Delta(y)\Delta(x)\tilde{\mu}E,$$

so that $\Delta(xy) = \Delta(y)\Delta(x) = \Delta(x)\Delta(y)$. **Q**

Δ is called the **left modular function** of X.

(b) We find now that $\nu(xE) = \Delta(x^{-1})\nu E$ whenever ν is a right Haar measure on X, $x \in X$ and $E \subseteq X$ is Haar measurable. **P** Let μ be the left Haar measure derived from ν, so that $\mu E = \nu E^{-1}$ whenever E is Haar measurable. If $x \in X$ and $E \in \operatorname{dom}\nu$, then

$$\nu(xE) = \mu(E^{-1}x^{-1}) = \Delta(x^{-1})\mu E^{-1} = \Delta(x^{-1})\nu E. \text{ **Q**}$$

Thus we may call $x \mapsto \Delta(x^{-1}) = \frac{1}{\Delta(x)}$ the **right modular function** of X.

(c) If X is abelian, then obviously $\Delta(x) = 1$ for every $x \in X$, because $\mu(Ex) = \mu(xE) = \mu E$ whenever $x \in X$, μ is a left Haar measure on X and E is Haar measurable. Equally, if any (therefore every) left (or right) Haar measure μ on X is totally finite, then $\mu(Xx) = \mu(xX) = \mu X$, so again $\Delta(x) = 1$ for every $x \in X$. This will be the case, in particular, for any compact Hausdorff topological group (recall that by 441E any such group carries Haar measures), because its Haar measures are locally finite, therefore totally finite.

Groups in which $\Delta(x) = 1$ for every x are called **unimodular**.

(d) From the definition of Δ, we see that a topological group carrying Haar measures is unimodular iff every left Haar measure is a right Haar measure.

(e) In particular, if a group has any totally finite (left or right) Haar measure, its left and right Haar measures are the same, and it has a unique Haar probability measure, which we may call its **normalized Haar measure**.

In the other direction, any group with its discrete topology is unimodular, since counting measure is a two-sided Haar measure.

442J Proposition For any topological group carrying Haar measures, its left modular function is continuous.

proof Let X be a topological group carrying Haar measures, with left modular function Δ.

(a) If $\epsilon > 0$, there is an open set U_ϵ containing the identity e of X such that $\Delta(x) \le 1 + \epsilon$ for every $x \in U$. **P** Take any left Haar measure μ on X, and an open set G such that $0 < \mu G < \infty$. By 442Ab, there are an open set H and a neighbourhood U_ϵ of the identity such that $HU_\epsilon \subseteq G$ and $\mu G \le (1 + \epsilon)\mu H$. If $x \in U_\epsilon$, then $Hx \subseteq G$, so $\Delta(x) = \frac{\mu(Hx)}{\mu H} \le 1 + \epsilon$. **Q**

(b) Now, given $x_0 \in X$ and $\epsilon > 0$, $V = \{x : x^{-1}x_0 \in U_\epsilon,\ x_0^{-1}x \in U_\epsilon\}$ is an open set containing x_0. If $x \in V$, then

$$\Delta(x) = \Delta(x_0)\Delta(x_0^{-1}x) \le (1 + \epsilon)\Delta(x_0),$$

$$\Delta(x_0) = \Delta(x)\Delta(x^{-1}x_0) \le (1+\epsilon)\Delta(x),$$

so

$$\frac{1}{1+\epsilon}\Delta(x_0) \le \Delta(x) \le (1+\epsilon)\Delta(x).$$

As ϵ is arbitrary, Δ is continuous at x_0; as x_0 is arbitrary, Δ is continuous.

442K Theorem Let X be a topological group and μ a left Haar measure on X. Let Δ be the left modular function of X.

(a) $\mu(E^{-1}) = \int_E \Delta(x^{-1})\mu(dx)$ for every $E \in \operatorname{dom}\mu$.

(b)(i) $\int f(x^{-1})\mu(dx) = \int \Delta(x^{-1})f(x)\mu(dx)$ whenever f is a real-valued function such that either integral is defined in $[-\infty,\infty]$;

(ii) $\int f(x)\mu(dx) = \int \Delta(x^{-1})f(x^{-1})\mu(dx)$ whenever f is a real-valued function such that either integral is defined in $[-\infty,\infty]$.

(c) $\int f(xy)\mu(dx) = \Delta(y^{-1})\int f(x)\mu(dx)$ whenever $y \in X$ and f is a real-valued function such that either integral is defined in $[-\infty,\infty]$.

proof (a)(i) Setting $\nu_1 E = \mu E^{-1}$ for Haar measurable sets $E \subseteq X$, we know that ν_1 is a right Haar measure, so 442E tells us that

$$\mu G \cdot \nu_1 H = \int_H \nu_1(xG^{-1})\mu(dx) = \int_H \mu(Gx^{-1})\mu(dx) = \mu G \int_H \Delta(x^{-1})\mu(dx)$$

for all open sets G, $H \subseteq X$. Since there is an open set G such that $0 < \mu G < \infty$, $\mu H^{-1} = \int_H \Delta(x^{-1})\mu(dx)$ for every open set $H \subseteq X$.

(ii) Now let ν_2 be the indefinite-integral measure defined by setting $\nu_2 E = \int \Delta(x^{-1})\chi E(x)\mu(dx)$ whenever this is defined in $[0,\infty]$ (234J[3]). Then ν_2 is effectively locally finite. **P** If $\nu_2 E > 0$, then $\mu E > 0$, so there is an $n \in \mathbb{N}$ such that $\mu(E \cap H) > 0$, where H is the open set $\{x : \Delta(x^{-1}) < n\}$. Now there is an open set $G \subseteq H$ such that $\mu G < \infty$ and $\mu(E \cap G) > 0$, in which case $\nu_2 G \le n\mu G < \infty$ and $\nu_2(E \cap G) > 0$. **Q**

Accordingly ν_2 is a quasi-Radon measure (415Ob). Since it agrees with the quasi-Radon measure ν_1 on open sets, by (i), the two are equal; that is, $\mu E^{-1} = \int_E \Delta(x^{-1})\mu(dx)$ for every $E \in \operatorname{dom}\mu$.

(b)(i) Apply 235E with $X = Y$, $\Sigma = \mathrm{T} = \operatorname{dom}\mu$, $\mu = \nu$ and $\phi(x) = x^{-1}$, $J(x) = \Delta(x^{-1})$, $g(x) = \Delta(x^{-1})f(x)$ for $x \in X$. From (a) we have

$$\int J \times \chi(\phi^{-1}[F])d\mu = \int_{F^{-1}} \Delta(x^{-1})\mu(dx) = \mu F = \nu F$$

for every $F \in \mathrm{T}$ (using 442G to see that $F^{-1} \in \Sigma$). So we get

$$\begin{aligned}\int f(x^{-1})\mu(dx) &= \int \Delta(x^{-1})g(x^{-1})\mu(dx) = \int J \times g\phi\, d\mu \\ &= \int g\, d\nu = \int \Delta(x^{-1})f(x)\mu(dx)\end{aligned}$$

if any of the integrals is defined in $[-\infty,\infty]$.

(ii) Set $\overleftrightarrow{f}(x) = f(x^{-1})$ whenever this is defined (4A5C(c-ii)), and apply (i) to $\overleftrightarrow{f}$.

(c) Similarly, apply 235E with $\mu = \nu$, $\phi(x) = xy$, $J(x) = \Delta(y)$ for every $x \in X$; then

$$\int J \times \chi(\phi^{-1}[F])d\mu = \Delta(y)\mu(Fy^{-1}) = \mu F$$

for every $F \in \operatorname{dom}\mu$, so

$$\int f(xy)\mu(dx) = \Delta(y^{-1})\int J \times f\phi\, d\mu = \Delta(y^{-1})\int f(x)\mu(dx).$$

442L Corollary Let X be a group carrying Haar measures. If μ is a left Haar measure on X and ν is a right Haar measure, then each is an indefinite-integral measure over the other.

proof Let $\overleftrightarrow{\mu}$ be the right Haar measure defined by setting $\overleftrightarrow{\mu}E = \mu E^{-1}$ for every Haar measurable $E \subseteq X$. Then $\overleftrightarrow{\mu}E = \int_E \Delta(x^{-1})\mu(dx)$ for every $E \in \operatorname{dom}\mu = \operatorname{dom}\overleftrightarrow{\mu}$, so $\overleftrightarrow{\mu}$ is an indefinite-integral measure over μ; because ν is a multiple of $\overleftrightarrow{\mu}$, it also is an indefinite-integral measure over μ. Similarly, or because Δ is strictly positive, μ is an indefinite-integral measure over ν.

[3] Formerly 234B.

442X Basic exercises >(a) Let X and Y be topological groups with (left) Haar probability measures μ and ν, and $\phi : X \to Y$ a continuous surjective group homomorphism. Show that ϕ is inverse-measure-preserving for μ and ν.

(b) (i) Let X and Y be two topological groups carrying Haar measures. Show that the product topological group $X \times Y$ (4A5G) carries Haar measures. (ii) Let $\langle X_i \rangle_{i \in I}$ be any family of topological groups carrying totally finite Haar measures. Show that the product group $\prod_{i\in I} X_i$ carries a totally finite Haar measure. (*Hint*: 417O.)

(c) Let X be a subgroup of the group $(\mathbb{R}, +)$. Show that X carries Haar measures iff it is either discrete (so that counting measure is a Haar measure on X) or of full outer Lebesgue measure (so that the subspace measure on X is a Haar measure). (*Hint*: if G has a Haar measure ν and is not discrete, then $\nu(G \cap [\alpha, \beta]) = (\beta - \alpha)\nu(G \cap [0, 1])$ whenever $\alpha \le \beta$.) In particular, $\mathbb{Q}$ does not carry Haar measures.

(d) Let X be a topological group carrying Haar measures; let Σ be the algebra of Haar measurable subsets of X. Let μ and ν be any Haar measures on X (either left or right). Show that a set $E \in \Sigma$ can be covered by a sequence of sets of finite measure for μ iff it can be covered by a sequence of sets of finite measure for ν.

(e) Let X be a topological group carrying Haar measures and $\mathfrak{A}$ its Haar measure algebra (in the sense of 442H). Show that we have left, right and conjugacy actions of X on $\mathfrak{A}$ given by the formulae $z{\bullet}E^\bullet = (zE)^\bullet$, $z{\bullet}E^\bullet = (Ez^{-1})^\bullet$ and $z{\bullet}E^\bullet = (zEz^{-1})^\bullet$ for every Haar measurable $E \subseteq X$ and every $z \in X$.

>(f) On $\mathbb{R}^2$ define a binary operation $*$ by setting $(\xi_1, \xi_2) * (\eta_1, \eta_2) = (\xi_1 + \eta_1, \xi_2 + e^{\xi_1}\eta_2)$. (i) Show that $*$ is a group operation under which $\mathbb{R}^2$ is a locally compact topological group. (ii) Show that Lebesgue measure μ is a right Haar measure for $*$. (iii) Let ν be the indefinite-integral measure on $\mathbb{R}^2$ defined by setting $\nu E = \int_E e^{-\xi_1} d\xi_1 d\xi_2$ for Lebesgue measurable sets $E \subseteq \mathbb{R}^2$. Show that ν is a left Haar measure for $*$. (*Hint*: 263D.) (iv) Thus $(\mathbb{R}^2, *)$ is not unimodular. (v) Show that the left modular function of $(\mathbb{R}^2, *)$ is $(\xi_1, \xi_2) \mapsto e^{-\xi_1}$.

>(g) Let X be any topological group which is not unimodular. (i) Show that there is an open subset of X which is of finite measure for all left Haar measures on X and of infinite measure for all right Haar measures. (*Hint*: the modular function is unbounded.) (ii) Let μ be a left Haar measure on X and ν a right Haar measure. Show that $L^0(\mu) = L^0(\nu)$ and $L^\infty(\mu) = L^\infty(\mu)$, but that $L^p(\mu) \ne L^p(\nu)$ for any $p \in [1, \infty[$.

(h) Let X and Y be topological groups carrying Haar measures, with left modular functions Δ_X and Δ_Y respectively. Show that the left modular function of $X \times Y$ is $(x, y) \mapsto \Delta_X(x)\Delta_Y(y)$.

(i) Let X be any topological group and $\Delta : X \to]0, \infty[$ a group homomorphism such that $\{x : \Delta(x) \le 1 + \epsilon\}$ is a neighbourhood of the identity in X for every $\epsilon > 0$. Show that Δ is continuous.

(j) Let X be a topological group with a right Haar measure ν and left modular function Δ. Show that $\nu E^{-1} = \int_E \Delta(x)\nu(dx)$ for every Haar measurable set $E \subseteq X$.

442Y Further exercises (a) In 441Yb, show that the only G-invariant Radon measures on $\mathcal{C}_s$ are multiples of Hausdorff $s(r-s)$-dimensional measure on $\mathcal{C}_s$. (*Hint*: G itself is $\frac{r(r-1)}{2}$-dimensional (cf. 441Yd), and for any $C \in \mathcal{C}_s$ the stabilizer of C is $\frac{s(s-1)}{2} + \frac{(r-s)(r-s-1)}{2}$-dimensional. See Federer 69, 3.2.28.)

(b) Let $r \ge 1$, and let X be the group of non-singular $r \times r$ real matrices. Regarding X as an open subset of $\mathbb{R}^{r^2}$, show that a two-sided Haar measure μ can be defined on X by setting $\mu E = \int_E \frac{1}{|\det A|^r}\mu_L(dA)$, where μ_L is Lebesgue measure on $\mathbb{R}^{r^2}$; so that X is unimodular.

(c) Show that there is a set $A \subseteq [0, 1]$, of Lebesgue outer measure 1, such that no countable set of translates of A covers any set of Lebesgue measure greater than 0. (*Hint*: let $\langle F_\xi \rangle_{\xi < \mathfrak{c}}$ run over the uncountable closed subsets of $[0, 1]$ with cofinal repetitions (4A3Fa), and enumerate the countable subsets of $\mathbb{R}$ as $\langle I_\xi \rangle_{\xi < \mathfrak{c}}$. Choose inductively x_ξ, $x'_\xi \in F_\xi$ such that $x_\xi \notin \bigcup_{\eta,\zeta<\xi} x'_\eta - I_\zeta$, $x'_\xi \notin \bigcup_{\eta,\zeta\le\xi} x_\eta + I_\zeta$; set $A = \{x_\xi : \xi < \mathfrak{c}\}$.) Show that we can extend Lebesgue measure on $\mathbb{R}$ to a translation-invariant measure for which A is negligible. (*Hint*: 417A.)

442Z Problem Let X be a compact Hausdorff space, and G the group of autohomeomorphisms of X. Suppose that G acts transitively on X. Does it follow that there is at most one G-invariant Radon probability measure on X?

442 Notes and comments Haar measure dominates the theory of locally compact topological groups for two reasons: it is ubiquitous (the existence theorem, 441E) and essentially defined by the group structure (the uniqueness theorem, 442B). I have tried to show that these are rather different results by setting the theorems out with different hypotheses. I presented the existence theorem as a special case of 441C, which demands a locally compact space and a group, but allows them to be different. In the uniqueness theorem (roughly following HALMOS 50) I demand a group with an invariant quasi-Radon measure, but do not (at this point) ask for any hypothesis of compactness. In fact it will become apparent in the next section that this is a somewhat spurious generality; 442B and 442I here can be deduced from the traditional forms in which the group is assumed to be locally compact and Hausdorff. From the point of view of the topological measure theory to which this volume is devoted, however, I think the small extra labour involved in tracing through the arguments without relying on the Riesz Representation Theorem is instructive. For instance, it emphasizes interesting features of the domains and null ideals of Haar measures (442H).

There is however a more serious question concerning the uniqueness theorem. I do not know whether it really belongs to the theory of topological groups, as described here, or to the theory of group actions along with 441C. The trouble is that I know of no example of a Hausdorff space X and a transitive group G of homeomorphisms of X such that X carries G-invariant Radon measures which are not multiples of each other (see 442Z). 443U and 443Xy below eliminate the simplest possibilities. We do need to put some restriction on the measures; for instance, counting measure on $\mathbb{R}$ is translation-invariant, but has nothing to do with Lebesgue measure. There are also proper translation-invariant extensions of Lebesgue measure (442Yc); for far-reaching elaborations of this idea see HEWITT & ROSS 63, §16.

443 Further properties of Haar measure

I devote a section to filling in some details of the general theory of Haar measures before turning to the special topics dealt with in the rest of the chapter. The first question concerns the left and right shift operators acting on sets, on elements of the measure algebra, on measurable functions and on function spaces. All these operations can be regarded as group actions, and, if appropriate topologies are assigned, they are continuous actions (443C, 443G). As an immediate consequence of this I give an important result about product sets $\{ab : a \in A,\, b \in B\}$ in a topological group carrying Haar measures (443D).

The second part of the section revolves around a basic structure theorem: all the Haar measures considered here can be reduced to Haar measures on locally compact Hausdorff groups (443L). The argument involves two steps: the reduction to the Hausdorff case, which is elementary, and the completion of a Hausdorff topological group. Since a group carries more than one natural uniform structure we must take care to use the correct one, which in this context is the 'bilateral' uniformity (443H-443I, 443K). On the way I pick up an essential fact about the approximation of Haar measurable sets by Borel sets (443J). Finally, I give Halmos' theorem that Haar measures are completion regular (443M) and a note on the complementary nature of the meager and null ideals for atomless Haar measure (443O).

In the third part of the section I turn to the special properties of quotient groups of locally compact groups and the corresponding actions, following A.Weil. If X is a locally compact Hausdorff group and Y is a closed subgroup of X, then Y is again a locally compact Hausdorff group, so has Haar measures and a modular function; at the same time, we have a natural action of X on the set of left cosets of Y. It turns out that there is an invariant Radon measure for this action if and only if the modular function of Y matches that of X (443R). In this case we can express a left Haar measure of X as an integral of measures supported by the cosets of Y (443Q). When Y is a normal subgroup, so that X/Y is itself a locally compact Hausdorff group, we can relate the modular functions of X and X/Y (443T). We can apply these results whenever we have a continuous transitive action of a compact group on a compact space (443U).

443A Haar measurability I recall and expand on some facts already noted in 442H. Let X be a topological group carrying Haar measures.

(a) All Haar measures on X, whether left or right, have the same domain Σ, which I call the algebra of 'Haar measurable' sets, and the same null ideal $\mathcal{N}$, which I call the ideal of 'Haar negligible' sets. The corresponding quotient algebra $\mathfrak{A} = \Sigma/\mathcal{N}$, the 'Haar measure algebra', is the Boolean algebra underlying the measure algebra of any Haar measure. Because Haar measures are (by the definition in 441D) quasi-Radon, therefore complete and strictly localizable (415A), Σ_G is closed under Souslin's operation (431A) and $\mathfrak{A}$ is Dedekind complete (322Be). Recall that any semi-finite measure on $\mathfrak{A}$, and in particular any Haar measure on X, gives rise to the same measure-algebra topology and uniformity on $\mathfrak{A}$ (324H), so we may speak of 'the' topology and uniformity of $\mathfrak{A}$.

Because Σ is the domain of a left Haar measure, $xE \in \Sigma$ whenever $E \in \Sigma$ and $x \in X$; because Σ is the domain of a right Haar measure, $Ex \in \Sigma$ whenever $E \in \Sigma$ and $x \in X$. Similarly, xE and Ex are Haar negligible whenever E is Haar negligible and $x \in X$. Moreover, $E^{-1} = \{x^{-1} : x \in E\}$ is Haar measurable or Haar negligible whenever E is.

Note that Σ and $\mathcal{N}$ are invariant in the strong sense that if $\phi : X \to X$ is any group automorphism which is also a homeomorphism, then $\Sigma = \{\phi[E] : E \in \Sigma\}$ and $\mathcal{N} = \{\phi[E] : E \in \mathcal{N}\}$. **P** If μ is a left Haar measure on X, let ν be the image measure $\mu\phi^{-1}$. Because ϕ is a homeomorphism, ν is a non-zero quasi-Radon measure. If ν measures E and $a \in X$, then

$$\nu(aE) = \mu\phi^{-1}[aE] = \mu((\phi^{-1}a)(\phi^{-1}[E])) = \mu\phi^{-1}[E] = \nu E.$$

So ν is again a left Haar measure, and has domain Σ and null ideal $\mathcal{N}$. But $\operatorname{dom}\nu = \{E : \phi^{-1}[E] \in \Sigma\} = \{\phi[E] : E \in \Sigma\}$ and $\nu^{-1}[\{0\}] = \{E : \phi^{-1}[E] \in \mathcal{N}\} = \{\phi[E] : E \in \mathcal{N}$. **Q**

(b) We even have a symmetric notion of 'measurable envelope' in X: for any $A \subseteq X$, there is a Haar measurable set $E \supseteq A$ such that $\mu(E \cap F) = \mu^*(A \cap F)$ for any Haar measurable $F \subseteq X$ and any Haar measure μ on X. **P** Start with a fixed Haar measure μ_0. Then A has a measurable envelope E for μ_0, by 213J and 213L. Now to say that 'E is a measurable envelope for A' means just that (i) $A \subseteq E \in \Sigma$ (ii) if $F \in \Sigma$ and $F \subseteq E \setminus A$ then $F \in \mathcal{N}$, so E is also a measurable envelope for A for any other Haar measure on X. **Q**

In this context I will call E a **Haar measurable envelope** of A.

(c) Similarly, we have a notion of **full outer Haar measure**: a subset A of X is of full outer Haar measure if X is a Haar measurable envelope of A, that is, $A \cap E \neq \emptyset$ whenever E is a Haar measurable set which is not Haar negligible, that is, A is of full outer measure for any (left or right) Haar measure on X.

(d) For any Haar measure μ on X, we can identify $L^\infty(\mu)$ with $L^\infty(\mathfrak{A})$ (363I) and $L^0(\mu)$ with $L^0(\mathfrak{A})$ (364Ic). Thus these constructions are independent of μ. The topology of convergence in measure of L^0 is determined by its Riesz space structure (367T); so that this also is independent of the particular Haar measure we may select. Of course the same is true of the norm of L^∞. Note however that the spaces L^p, for $1 \le p < \infty$, are different for left and right Haar measures on any group which is not unimodular (442Xg), and that even for left Haar measures μ the norm on $L^p(\mu)$ changes if μ is re-normalized.

(e) When it seems appropriate, I will use the phrases **Haar measurable function**, meaning a function measurable with respect to the σ-algebra of Haar measurable sets, and **Haar almost everywhere**, meaning 'on the complement of a Haar negligible set'. Note that we can identify $L^0(\mathfrak{A})$ with the set of equivalence classes in the space $\mathcal{L}^0$, where $\mathcal{L}^0$ is the space of Haar measurable real-valued functions defined Haar-a.e. in X, and $f \sim g$ if $f = g$ Haar-a.e. In the language of §241, $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ for any Haar measure μ on X.

(f) Because $E^{-1} \in \Sigma$ for every $E \in \Sigma$, and $E^{-1} \in \mathcal{N}$ whenever $E \in \mathcal{N}$, we have a canonical automorphism $a \mapsto \overleftrightarrow{a} : \mathfrak{A} \to \mathfrak{A}$ defined by writing $(E^\bullet)^\leftrightarrow = (E^{-1})^\bullet$ for every $E \in \Sigma$. Being an automorphism, this must be a homeomorphism for the measure-algebra topology of $\mathfrak{A}$ (324G). In the same way, if $f \in \mathcal{L}^0$ then $\overleftrightarrow{f} \in \mathcal{L}^0$, where $\overleftrightarrow{f}(x) = f(x^{-1})$ whenever this is defined (4A5C(c-ii)), since $\{x : x \in \operatorname{dom}\overleftrightarrow{f},\ \overleftrightarrow{f}(x) > \alpha\} = \{x : x \in \operatorname{dom} f,\ f(x) > \alpha\}^{-1}$ belongs to Σ for every α; and we can define an f-algebra automorphism $u \mapsto \overleftrightarrow{u} : L^0 \to L^0$ by saying that $(f^\bullet)^\leftrightarrow = (\overleftrightarrow{f})^\bullet$ for $f \in \mathcal{L}^0$. If we identify L^0 with $L^0(\mathfrak{A})$ rather than with a set of equivalence classes in $\mathcal{L}^0$, then we can define the map $u \mapsto \overleftrightarrow{u}$ as the Riesz homomorphism associated with the Boolean homomorphism $a \mapsto \overleftrightarrow{a} : \mathfrak{A} \to \mathfrak{A}$, as in 364P. Note that $\|\overleftrightarrow{u}\|_\infty = \|u\|_\infty$ for every $u \in L^\infty$, but that (unless X is unimodular) other L^p spaces are not invariant under the involution $^\leftrightarrow$ (442Xg again).

(g) If X carries any totally finite (left or right) Haar measure, it is unimodular, and has a unique, two-sided, Haar probability measure (442Ie). (In particular, this is the case whenever X is compact.) For such groups we have L^p-spaces, for $1 \le p \le \infty$, defined by the group structure, with canonical norms.

443B Lemma Let X be a topological group and μ a left Haar measure on X. If $E \subseteq X$ is measurable and $\mu E < \infty$, then for any $\epsilon > 0$ there is a neighbourhood U of the identity e such that $\mu(E \triangle xEy) \le \epsilon$ whenever x, $y \in U$.

proof Set $\delta=\frac{\min(1,\epsilon)}{10+3\mu E}>0$. Write $\mathcal{U}$ for the family of open neighbourhoods of e. Because μ is effectively locally finite, there is an open set G_0 of finite measure such that $\mu(E\setminus G_0)\le\delta$. Let $F\subseteq G_0\setminus E$ be a closed set such that $\mu F\ge\mu(G_0\setminus E)-\delta$, and set $G=G_0\setminus F$, so that $\mu(G\setminus E)\le\delta$ and $\mu(E\setminus G)\le\delta$. For $U\in\mathcal{U}$ set $H_U=\operatorname{int}\{x:UxU\subseteq G\}$. Then $\mathcal{H}=\{H_U:U\in\mathcal{U}\}$ is upwards-directed, and has union G, because if $x\in G$ there is a $U\in\mathcal{U}$ such that $UxUU\subseteq G$, so that $x\in H_U$. So there is a $V\in\mathcal{U}$ such that $\mu(G\setminus H_V)\le\delta$. Recall that the left modular function Δ of X is continuous (442J). So there is a $U\in\mathcal{U}$ such that $U\subseteq V$ and $|\Delta(y)-1|\le\delta$ for every $y\in U$.

Now suppose that x, $y\in U$. Set $E_1=E\cap H_V$. Then $xE_1y\subseteq G$, so

$$\mu(E_1\cup xE_1y)\le\mu G\le\mu E+\delta.$$

On the other hand,

$$\mu E_1\ge\mu E-\mu(E\setminus G)-\mu(G\setminus H_V)\ge\mu E-2\delta,$$

$$\mu(xE_1y)=\Delta(y)\mu E_1\ge(1-\delta)(\mu E-2\delta)\ge\mu E-(2+\mu E)\delta.$$

So

$$\mu(E\cap xEy)\ge\mu(E_1\cap xE_1y)=\mu E_1+\mu(xE_1y)-\mu(E_1\cup xE_1y)\ge\mu E-(5+\mu E)\delta.$$

At the same time,

$$\mu(xEy)=\Delta(y)\mu E\le(1+\delta)\mu E.$$

So

$$\mu(E\triangle xEy)=\mu E+\mu(xEy)-2\mu(E\cap xEy)\le(10+3\mu E)\delta\le\epsilon,$$

as required.

443C Theorem Let X be a topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra. Then we have continuous actions of X on $\mathfrak{A}$ defined by writing

$$x\bullet_l E^\bullet=(xE)^\bullet,\quad x\bullet_r E^\bullet=(Ex^{-1})^\bullet,\quad x\bullet_c E^\bullet=(xEx^{-1})^\bullet$$

for Haar measurable sets $E\subseteq X$ and $x\in X$.

proof (a) The functions $\bullet_l$, $\bullet_r$ and $\bullet_c$ are all well defined because the maps $E\mapsto xE$, $E\mapsto Ex^{-1}$ and $E\mapsto xEx^{-1}$ are all Boolean automorphisms of the algebra Σ of Haar measurable sets preserving the ideal of Haar negligible sets (442G). It is elementary to check that they are actions of X on $\mathfrak{A}$.

Fix a left Haar measure μ on X and let $\bar\mu$ be the corresponding measure on $\mathfrak{A}$. Then the topology of $\mathfrak{A}$ is defined by the pseudometrics ρ_a, for $\bar\mu a<\infty$, where $\rho_a(b,c)=\bar\mu(a\cap(b\triangle c))$.

(b) Now suppose that $x_0\in X$, $b_0\in\mathfrak{A}$, $\bar\mu a<\infty$ and $\epsilon>0$. Let E, $F_0\in\Sigma$ be such that $E^\bullet=a$ and $F_0^\bullet=b_0$; set $\delta=\frac{1}{4}\epsilon>0$. Note that

$$\mu(x_0^{-1}E\cap F_0)\le\mu(x_0^{-1}E)=\mu E<\infty.$$

Let U be a neighbourhood of the identity e such that

$$\mu(E\triangle yE)\le\delta,\quad\mu((x_0^{-1}E\cap F_0)\triangle y(x_0^{-1}E\cap F_0))\le\delta$$

whenever $y\in U$ (443B). Set

$$a'=x_0^{-1}\bullet_l a=(x_0^{-1}E)^\bullet.$$

Now suppose that $x\in Ux_0\cap x_0U^{-1}$ and that $\rho_{a'}(b,b_0)\le\delta$. Then $\rho_a(x\bullet_l b,x_0\bullet_l b_0)\le\epsilon$. **P** Let $F\in\Sigma$ be such that $F^\bullet=b$. Then xx_0^{-1} and $x^{-1}x_0$ both belong to U, so

$$\begin{aligned}\rho_a(x\bullet_l b, x_0\bullet_l b_0) &= \mu(E\cap(xF\triangle x_0F_0))\\ &= \mu(E\cap xF)+\mu(E\cap x_0F_0)-2\mu(E\cap xF\cap x_0F_0)\\ &= \mu(x^{-1}E\cap F)+\mu(E\cap x_0F_0)-2\mu(x^{-1}x_0(x_0^{-1}E\cap F_0)\cap F)\\ &\le \mu(x_0^{-1}E\cap F)+\mu(x^{-1}E\triangle x_0^{-1}E)+\mu(x_0^{-1}E\cap F_0)\\ &\qquad -2\mu(x_0^{-1}E\cap F_0\cap F)+2\mu(x^{-1}x_0(x_0^{-1}E\cap F_0)\triangle(x_0^{-1}E\cap F_0))\\ &\le \mu(x_0^{-1}E\cap(F\triangle F_0))+\mu(E\triangle xx_0^{-1}E)+2\delta\\ &\le \delta+\delta+2\delta=4\delta=\epsilon.\ \mathbf{Q}\end{aligned}$$

As x_0, b_0, a and ϵ are arbitrary, $\bullet_l$ is continuous.

(c) The same arguments, using a right Haar measure to provide pseudometrics defining the topology of $\mathfrak{A}$, show that $\bullet_r$ is continuous. (Or use the method of 443X(d-ii).)

(d) Accordingly the map $(x,y,a)\mapsto x\bullet_l(y\bullet_r a)$ is continuous. So

$$(x,a)\mapsto x\bullet_l(x\bullet_r a)=x\bullet_c a$$

is continuous.

443D Proposition Let X be a topological group carrying Haar measures. If $E\subseteq X$ is Haar measurable but not Haar negligible, and $A\subseteq X$ is not Haar negligible, then

(a) there are x, $y\in X$ such that $A\cap xE$, $A\cap Ey$ are not Haar negligible;

(b) EA and AE both have non-empty interior;

(c) $E^{-1}E$ and EE^{-1} are neighbourhoods of the identity.

proof (a)(i) Let μ be any left Haar measure on X, and for Borel sets $F\subseteq X$ set

$$\nu F=\sup\{\mu(F\cap IE): I\subseteq X \text{ is finite}\}.$$

It is easy to check that ν is an effectively locally finite τ-additive Borel measure, inner regular with respect to the closed sets, because $\{IE: I\in[X]^{<\omega}\}$ is upwards-directed and μ is quasi-Radon. Moreover,

$$\nu(xF)=\sup_{I\in[X]^{<\omega}}\mu(xF\cap IE)=\sup_{I\in[X]^{<\omega}}\mu(F\cap x^{-1}IE)=\nu F$$

for every Borel set $F\subseteq X$ and every $x\in X$. Accordingly the c.l.d. version $\tilde\nu$ of ν is a left-translation-invariant quasi-Radon measure on X (415Cb); since $\nu E>0$, $\tilde\nu$ is non-zero and is itself a left Haar measure. Consequently A is not $\tilde\nu$-negligible. Let H be a measurable envelope of A for Haar measure (443Ab). Then H is not Haar negligible, so there is a closed set $F\subseteq H$ which is not Haar negligible, and $\nu F=\tilde\nu F>0$. Thus there is an $x\in X$ such that

$$0<\mu(F\cap xE)\le\mu(H\cap xE)=\mu^*(A\cap xE),$$

and $A\cap xE$ is not Haar negligible.

(ii) Applying the same arguments, but starting with a right Haar measure μ, we see that there is a $y\in X$ such that $A\cap Ey$ is not Haar negligible.

(b) Let μ be a left Haar measure on X, and F a Haar measurable envelope of A. The function $x\mapsto(xE^{-1})^\bullet: X\to\mathfrak{A}$ is continuous, where $\mathfrak{A}$ is the Haar measure algebra of X (443C), so

$$H=\{x:\mu^*(A\cap xE^{-1})>0\}=\{x:F^\bullet\cap(xE^{-1})^\bullet\ne 0\}$$

is open. Now

$$H\subseteq\{x:A\cap xE^{-1}\ne\emptyset\}=AE,$$

so $H\subseteq\operatorname{int}AE$; and E^{-1} is Haar measurable and not Haar negligible, so $H\ne\emptyset$, by (a). Thus $\operatorname{int}AE\ne\emptyset$.

Similarly, using a right Haar measure (or observing that $EA=(A^{-1}E^{-1})^{-1}$), we see that EA has non-empty interior.

(c) Again taking a left Haar measure μ, μ is semi-finite, so there is an $F\subseteq E$ such that $0<\mu F<\infty$. By 443B, there is a neighbourhood U of the identity such that $\mu(F\triangle xFy)<\mu F$ for all x, $y\in U$. In particular, if $x\in U$, $\mu(F\setminus xF)<\mu F$, so $F\cap xF\ne\emptyset$ and $x\in FF^{-1}\subseteq EE^{-1}$; at the same time, $\mu(F\setminus Fx)<\mu F$, $F\cap Fx\ne\emptyset$ and $x\in F^{-1}F\subseteq E^{-1}E$. So $E^{-1}E$ and EE^{-1} both include U and are neighbourhoods of the identity.

443E Corollary Let X be a Hausdorff topological group carrying Haar measures. Then the following are equiveridical:

(i) X is locally compact;

(ii) every Haar measure on X is a Radon measure;

(iii) there is some compact subset of X which is not Haar negligible.

proof (i)⇒(ii) Haar measures are locally finite quasi-Radon measures (441D, 442Aa), so on locally compact Hausdorff spaces must be Radon measures (416G).

(ii)⇒(iii) is obvious, just because Haar measures are non-zero and any Radon measure is tight (that is, inner regular with respect to the closed compact sets).

(iii)⇒(i) If $K \subseteq X$ is a compact set which is not Haar negligible, then KK is a compact set with non-empty interior, so X is locally compact (4A5Eg).

443F Later in the chapter we shall need the following straightforward fact.

Lemma Let X be a topological group carrying Haar measures, and Y an open subgroup of X. If μ is a left Haar measure on X, then the subspace measure μ_Y is a left Haar measure on Y. Consequently a subset of Y is Haar measurable or Haar negligible, when regarded as a subset of the topological group Y, iff it is Haar measurable or Haar negligible when regarded as a subset of the topological group X.

proof By 415B, μ_Y is a quasi-Radon measure; because μ is strictly positive, μ_Y is non-zero, and it is easy to check that it is left-translation-invariant. So it is a Haar measure on Y. The rest follows at once from 442H/443A.

443G We can repeat the ideas of 443C in terms of function spaces, as follows.

Theorem Let X be a topological group with a left Haar measure μ. Let Σ be the domain of μ, $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ the space of Σ-measurable real-valued functions defined almost everywhere in X, and $L^0 = L^0(\mu)$ the corresponding space of equivalence classes (§241).

(a) $a\bullet_l f$, $a\bullet_r f$ and $a\bullet_c f$ (definitions: 4A5C(c-ii)) belong to $\mathcal{L}^0$ for every $f \in \mathcal{L}^0$ and $a \in X$.

(b) If $a \in X$, then $\operatorname{ess\,sup}|a\bullet_l f| = \operatorname{ess\,sup}|a\bullet_r f| = \operatorname{ess\,sup}|f|$ for every $f \in \mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$, where $\operatorname{ess\,sup}|f|$ is the essential supremum of $|f|$ (243Da). For $1 \le p < \infty$, $\|a\bullet_l f\|_p = \|f\|_p$ and $\|a\bullet_r f\|_p = \Delta(a)^{-1/p}\|f\|_p$ for every $f \in \mathcal{L}^p = \mathcal{L}^p(\mu)$, where Δ is the left modular function of X.

(c) We have shift actions of X on L^0 defined by setting

$$a\bullet_l f^\bullet = (a\bullet_l f)^\bullet, \quad a\bullet_r f^\bullet = (a\bullet_r f)^\bullet, \quad a\bullet_c f^\bullet = (a\bullet_c f)^\bullet$$

for $a \in X$ and $f \in \mathcal{L}^0$. If $^\leftrightarrow$ is the reversal operator on L^0 defined in 443Af, we have

$$a\bullet_l \overset{\leftrightarrow}{u} = (a\bullet_r u)^\leftrightarrow, \quad a\bullet_c \overset{\leftrightarrow}{u} = (a\bullet_c u)^\leftrightarrow$$

for every $a \in X$ and $u \in L^0$.

(d) If we give L^0 its topology of convergence in measure these three actions, and also the reversal operator $^\leftrightarrow$, are continuous.

(e) For $1 \le p \le \infty$ the formulae of (c) define actions of X on $L^p = L^p(\mu)$, and $\|a\bullet_l u\|_p = \|u\|_p$ for every $u \in L^p$, $a \in X$; interpreting $\Delta(a)^{-1/\infty}$ as 1 if necessary, $\|a\bullet_r u\|_p = \Delta(a)^{-1/p}\|u\|_p$ whenever $u \in L^p$ and $a \in X$.

(f) For $1 \le p < \infty$ these actions are continuous.

proof (a) Let $f \in \mathcal{L}^0$. Then $F = \operatorname{dom} f$ is conegligible, so $aF = \operatorname{dom} a\bullet_l f$ and $Fa^{-1} = \operatorname{dom} a\bullet_r f$ are conegligible (442G). For any $\alpha \in \mathbb{R}$, set $E_\alpha = \{x : x \in F,\ f(x) < \alpha\}$; then $\{x : (a\bullet_l f)(x) < \alpha\} = aE_\alpha$ and $\{x : (a\bullet_r f)(x) < \alpha\} = E_\alpha a^{-1}$ are measurable, so $a\bullet_l f$ and $a\bullet_r f$ are measurable. Thus $a\bullet_l f$ and $a\bullet_r f$ belong to $\mathcal{L}^0$. It follows at once that $a\bullet_c f = a\bullet_l(a\bullet_r f)$ belongs to $\mathcal{L}^0$.

(b)(i) For $\alpha \ge 0$,

$$\begin{aligned}\operatorname{ess\,sup}|f| \le \alpha &\iff |f(x)| \le \alpha \text{ for almost all } x\\ &\iff |(a\bullet_l f)(x)| \le \alpha \text{ for almost all } x\\ &\iff |(a\bullet_r f)(x)| \le \alpha \text{ for almost all } x\end{aligned}$$

because the null ideal of μ is invariant under both left and right translations. So $\operatorname{ess\,sup}|f| = \operatorname{ess\,sup}|a\bullet_l f| = \operatorname{ess\,sup}|a\bullet_r f|$.

(ii) For $1 \le p < \infty$,

$$\|a\bullet_l f\|_p^p = \int |(a\bullet_l f)(x)|^p \mu(dx) = \int |f(a^{-1}x)|^p \mu(dx) = \int |f(x)|^p \mu(dx)$$

(441J)

$$= \|f\|_p^p,$$

$$\|a\bullet_r f\|_p^p = \int |(a\bullet_r f)(x)|^p \mu(dx) = \int |f(xa)|^p \mu(dx) = \Delta(a^{-1}) \int |f(x)|^p \mu(dx)$$

(442Kc)

$$= (\Delta(a)^{-1/p} \|f\|_p)^p.$$

(c)(i) I have already checked that $a\bullet_l f$, $a\bullet_r f$ and $a\bullet_c f$ belong to $\mathcal{L}^0$ whenever $f \in \mathcal{L}^0$ and $a \in X$. If f, $g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, let E be the conegligible set $\{x : x \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(x) = g(x)\}$; then aE and Ea^{-1} and aEa^{-1} are conegligible and

$$(a\bullet_l f)(x) = (a\bullet_l g)(x) \text{ for every } x \in aE, \quad (a\bullet_r f)(x) = (a\bullet_r g)(x) \text{ for every } x \in Ea^{-1},$$

$$(a\bullet_c f)(x) = (a\bullet_c g)(x) \text{ for every } x \in aEa^{-1},$$

so $a\bullet_l f =_{\text{a.e.}} a\bullet_l g$, $a\bullet_r f =_{\text{a.e.}} a\bullet_r g$ and $a\bullet_c f =_{\text{a.e.}} a\bullet_c g$. Accordingly the formulae given define functions $\bullet_l$, $\bullet_r$ and $\bullet_c$ from $X \times L^0$ to L^0. They are actions just because the original $\bullet_l$, $\bullet_r$ and $\bullet_c$ are actions of X on $\mathcal{L}^0$ (4A5Cc-4A5Cd).

(ii) If $f \in \mathcal{L}^0$, then

$$(a\bullet_l \overset{\leftrightarrow}{f})(x) = \overset{\leftrightarrow}{f}(a^{-1}x) = f(x^{-1}a) = (a\bullet_r f)(x^{-1}) = (a\bullet_r f)^{\leftrightarrow}(x)$$

when any of these is defined, which is almost everywhere, so $a\bullet_l u = (a\bullet_r u)^{\leftrightarrow}$ for every $u \in L^0$. Similarly,

$$(a\bullet_c \overset{\leftrightarrow}{f})(x) = \overset{\leftrightarrow}{f}(a^{-1}xa) = f(a^{-1}x^{-1}a) = (a\bullet_c f)(x^{-1}) = (a\bullet_c f)^{\leftrightarrow}(x)$$

and $a\bullet_c \overset{\leftrightarrow}{u} = (a\bullet_c u)^{\leftrightarrow}$.

(d)(i) In 367T there is a description of convergence in measure on L^0 in terms of its Riesz space structure. As $\leftrightarrow$ is a Riesz space automorphism of L^0, it must also be a homeomorphism for the topology of convergence in measure.

(ii) To see that $\bullet_l$ is continuous, it will be convenient to work with the space $\mathcal{L}^0_\Sigma$ of Haar measurable real-valued functions defined on the whole of X. I will use a characterization of convergence in measure from 245F: a subset W of L^0 is open iff whenever $f_0^\bullet \in W$ there are a set E of finite measure and an $\epsilon > 0$ such that $f^\bullet \in W$ whenever $\mu\{x : x \in E, |f(x) - f_0(x)| > \epsilon\} \le \epsilon$. Now if E is a measurable set of finite measure, $f \in \mathcal{L}^0_\Sigma$ and $\epsilon > 0$, there is a neighbourhood U of the identity e of X such that $\mu\{x : x \in E, |f(ax) - f(x)| \ge \epsilon\} \le \epsilon$ for every $a \in U$. **P** Let $m \ge 1$ be such that $\mu\{x : x \in E, |f(x)| \ge m\epsilon\} \le \frac{1}{2}\epsilon$. For $-m \le k < m$, set $E_k = \{x : x \in E, k\epsilon \le f(x) < (k+1)\epsilon\}$. By 443B, there is a neighbourhood U of e such that

$$\mu(E_k \triangle a^{-1}E_k) \le \frac{\epsilon}{4m}$$

whenever $a \in U$ and $-m \le k < m$. Now, for $a \in U$,

$$\{x : x \in E, |f(ax) - f(x)| \ge \epsilon\} \subseteq \{x : x \in E, |f(x)| \ge m\epsilon\} \cup \textstyle\bigcup_{k=-m}^{m-1} (E_k \triangle a^{-1}E_k)$$

has measure at most

$$\frac{\epsilon}{2} + 2m\frac{\epsilon}{4m} = \epsilon. \textbf{ Q}$$

(iii) Let E be a measurable set of finite measure, $a_0 \in X$, $f_0 \in \mathcal{L}^0_\Sigma$ and $\epsilon > 0$. Set $\delta = \epsilon/3 > 0$. Note that $\mu(a_0^{-1}E) = \mu E$ is finite. Let U be a neighbourhood of e such that

$$\mu\{x : x \in a_0^{-1}E, |f_0(yx) - f_0(x)| \ge \delta\} \le \delta, \quad \mu(yE \triangle E) \le \delta$$

whenever $y \in U$.

Now suppose that $a \in Ua_0 \cap a_0U^{-1}$ and that $f \in \mathcal{L}^0_\Sigma$ is such that $\mu\{x : x \in a_0^{-1}E, |f(x) - f_0(x)| \ge \delta\} \le \delta$. In this case,

$$\begin{aligned}
&\{x : x \in E,\, |f(a^{-1}x) - f_0(a_0^{-1}x)| \ge \epsilon\} \\
&\qquad \subseteq \{x : x \in E,\, |f(a^{-1}x) - f_0(a^{-1}x)| \ge \delta\} \\
&\qquad\qquad \cup \{x : x \in E,\, |f_0(a^{-1}x) - f_0(a_0^{-1}x)| \ge \delta\} \\
&\qquad \subseteq (E \triangle aa_0^{-1}E) \cup \{x : x \in aa_0^{-1}E,\, |f(a^{-1}x) - f_0(a^{-1}x)| \ge \delta\} \\
&\qquad\qquad \cup a_0\{w : w \in a_0^{-1}E,\, |f_0(a^{-1}a_0w) - f_0(w)| \ge \delta\} \\
&\qquad \subseteq (E \triangle aa_0^{-1}E) \cup a\{w : w \in a_0^{-1}E,\; |f(w) - f_0(w)| \ge \delta\} \\
&\qquad\qquad \cup a_0\{w : w \in a_0^{-1}E,\, |f_0(a^{-1}a_0w) - f_0(w)| \ge \delta\}
\end{aligned}$$

has measure at most $3\delta = \epsilon$ because aa_0^{-1}, e and $a^{-1}a_0$ all belong to U. Because E and ϵ are arbitrary, the function $(a, u) \mapsto a{\bullet_l}u$ is continuous at $(a_0, f_0^\bullet)$; as a_0 and f_0 are arbitrary, $\bullet_l$ is continuous.

(iv) Now

$$(a, u) \mapsto a{\bullet_r}u = (a{\bullet_l}\overset{\leftrightarrow}{u})^{\leftrightarrow}$$

must also be continuous. It follows at once that $\bullet_c$ is continuous, since $a{\bullet_c}u = a{\bullet_l}(a{\bullet_r}u)$.

(e) follows at once from (b) and (c).

(f) Fix $p \in [1, \infty[$.

(i) If $u \in L^p$ and $\epsilon > 0$, there is a neighbourhood U of e such that $\|u - y{\bullet_l}(z{\bullet_r}u)\|_p \le \epsilon$ whenever y, $z \in U$. **P** When u is of the form $(\chi E)^\bullet$, where $\mu E < \infty$, we have

$$y{\bullet_l}(z{\bullet_r}u) = \chi(yEz^{-1})^\bullet, \quad \|u - y{\bullet_l}(z{\bullet_r}u)\|_p = \mu(E \triangle yEz^{-1})^{1/p},$$

so the result is immediate from 443B. If $u = \sum_{i=0}^n \alpha_i(\chi E_i)^\bullet$, where every E_i has finite measure, then, setting $u_i = (\chi E_i)^\bullet$ for each i,

$$\|u - y{\bullet_l}(z{\bullet_r}u)\|_p \le \sum_{i=0}^n |\alpha_i| \|u_i - y{\bullet_l}(z{\bullet_r}u_i)\|_p \le \epsilon$$

whenever y and z are close enough to e. In general, there is a v of this form such that $\|u - v\|_p \le \frac{1}{4}\epsilon$. If we take a neighbourhood U of e such that $\|v - y{\bullet_l}(z{\bullet_r}v)\|_p \le \frac{1}{4}\epsilon$ and $\Delta(z)^{-1/p} \le 2$ whenever y, $z \in U$, then

$$\|y{\bullet_l}(z{\bullet_r}u) - y{\bullet_l}(z{\bullet_r}v)\|_p = \Delta(z)^{-1/p}\|u - v\|_p \le \frac{1}{2}\epsilon$$

whenever $z \in U$, so

$$\|u - y{\bullet_l}(z{\bullet_r}u)\|_p \le \|u - v\|_p + \|v - y{\bullet_l}(z{\bullet_r}v)\|_p + \|y{\bullet_l}(z{\bullet_r}v) - y{\bullet_l}(z{\bullet_r}u)\|_p \le \epsilon$$

whenever y, $z \in U$. **Q**

(ii) Now suppose that $u_0 \in L^p$, a_0, $b_0 \in X$ and $\epsilon > 0$. Set $v_0 = a_0{\bullet_l}(b_0{\bullet_r}u_0)$ and $\delta = \epsilon/(1 + 2\Delta(b_0)^{-1/p}) > 0$. Let U be a neighbourhood of e such that $\Delta(y)^{-1/p} \le 2$ and $\|v_0 - y{\bullet_l}(z{\bullet_r}v_0)\|_p \le \delta$ whenever y, $z \in U$. If $a \in Ua_0$, $b \in Ub_0$ and $\|u - u_0\|_p \le \delta$, then

$$\begin{aligned}
\|a{\bullet_l}(b{\bullet_r}u) - v_0\|_p &\le \|a{\bullet_l}(b{\bullet_r}u) - a{\bullet_l}(b{\bullet_r}u_0)\|_p + \|a{\bullet_l}(b{\bullet_r}u_0) - v_0\|_p \\
&= \Delta(b)^{-1/p}\|u - u_0\|_p + \|aa_0^{-1}{\bullet_l}(bb_0^{-1}{\bullet_r}v_0) - v_0\|_p \\
&\le \Delta(bb_0^{-1})^{-1/p}\Delta(b_0)^{-1/p}\delta + \delta \le \delta(1 + 2\Delta(b_0)^{-1/p}) = \epsilon.
\end{aligned}$$

As ϵ is arbitrary, $(a, b, u) \mapsto a{\bullet_l}(b{\bullet_r}u)$ is continuous at (a_0, b_0, u_0).

As in (c), this is enough to show that $\bullet_l$, $\bullet_r$ and $\bullet_c$ are all continuous actions.

Remark I have written this out for a left Haar measure μ, since the spaces $L^p(\mu)$ depend on this; if ν is a right Haar measure, and X is not unimodular, then $L^p(\mu) \ne L^p(\nu)$ for $1 \le p < \infty$. But recall that the topology of convergence in measure on L^0 is the same for all Haar measures (443Ad), so (c) above, and the case $p = \infty$ of (b) and (d), are two-sided; they belong to the theory of the Haar measure algebra.

443H Theorem Let X be a topological group carrying Haar measures. Then there is a neighbourhood of the identity which is totally bounded for the bilateral uniformity on X.

proof Let μ be a left Haar measure on X. Let V_0 be a neighbourhood of the identity e such that $\mu V_0 < \infty$ (442Aa). Let V be a neighbourhood of e such that $VV \subseteq V_0$ and $V^{-1} = V$.

? Suppose, if possible, that V is not totally bounded for the bilateral uniformity on X. By 4A5Oa, one of the following must occur:

case 1 There is an open neighbourhood U of e such that $V \not\subseteq IU$ for any finite set $I \subseteq X$. In this case, we may choose a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in V inductively such that $x_n \notin x_i U$ whenever $i < n$. Let U_1 be an open neighbourhood of e such that $U_1 \subseteq V$ and $U_1 U_1^{-1} \subseteq U$; then $\langle x_n U_1 \rangle_{n\in\mathbb{N}}$ is disjoint. Since $\mu(x_n U_1) = \mu U_1 > 0$ for every n (by the other clause in 442Aa), $\mu(\bigcup_{n\in\mathbb{N}} x_n U_1) = \infty$; but $x_n U_1 \subseteq V_0$ for every n, so this is impossible.

case 2 There is an open neighbourhood U of e such that $V \not\subseteq UI$ for any finite set $I \subseteq X$. So we may choose a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in V inductively such that $x_n \notin Ux_i$ whenever $i < n$. Let U_1 be an open neighbourhood of e such that $U_1 \subseteq V$ and $U_1 U_1^{-1} \subseteq U$; then $\langle U_1^{-1} x_n \rangle_{n\in\mathbb{N}}$ is disjoint, so $\langle x_n^{-1} U_1 \rangle_{n\in\mathbb{N}}$ is also disjoint. Since $\mu(x_n^{-1} U_1) = \mu U_1 > 0$ for every n, $\mu(\bigcup_{n\in\mathbb{N}} x_n^{-1} U_1) = \infty$; but $x_n^{-1} U_1 \subseteq V_0$ for every n, so this also is impossible. **X**

Thus V is totally bounded for the bilateral uniformity on X, and we have the required totally bounded neighbourhood of e.

443I Corollary Let X be a topological group. If $A \subseteq X$ is totally bounded for the bilateral uniformity of X, it has finite outer measure for any (left or right) Haar measure on X.

proof If μ is a Haar measure on X, let U be an open neighbourhood of the identity e of finite measure. There is a finite set I such that $A \subseteq IU \cap UI$ (4A5Oa again), so that $\mu^* A \le \#(I)\mu U$ is finite.

443J Proposition Let X be a topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra.

(a) There is an open-and-closed subgroup Y of X such that, for any Haar measure μ on X, Y can be covered by countably many open sets of finite measure.

(b)(i) If $E \subseteq X$ is any Haar measurable set, there are an F_σ set $E' \subseteq E$ and a G_δ set $E'' \supseteq E$ such that $E'' \setminus E'$ is Haar negligible.

(ii) Every Haar negligible set is included in a Haar negligible Borel set, and for every Haar measurable set E there is a Borel set F such that $E \triangle F$ is Haar negligible.

(iii) The Haar measure algebra $\mathfrak{A}$ of X may be identified with $\mathcal{B}/\mathcal{I}$, where $\mathcal{B}$ is the Borel σ-algebra of X and $\mathcal{I}$ is the ideal of Haar negligible Borel sets.

(iv) Every member of $L^0(\mathfrak{A})$ can be identified with the equivalence class of some Borel measurable function from X to $\mathbb{R}$. Every member of $L^\infty(\mathfrak{A})$ can be identified with the equivalence class of a bounded Borel measurable function from X to $\mathbb{R}$.

proof (a) Let V be an open neighbourhood of the identity which is totally bounded for the bilateral uniformity of X (443H); we may suppose that $V^{-1} = V$. Set $Y = V \cup VV \cup VVV \cup VVVV \cup \ldots$. Then Y is an open subgroup of X, therefore also closed (4A5Ek). By 4A5Ob, every power of V is totally bounded, so Y is a countable union of totally bounded sets. If μ is any left Haar measure on X, then any totally bounded set has finite outer measure for μ (443I). Thus Y is a countable union of sets of finite measure for μ. The same argument applies to right Haar measures, so Y is a subgroup of the required form.

(b)(i) Let $E \subseteq X$ be a Haar measurable set, and fix a left Haar measure μ on X. Take the open subgroup Y of (a), and index the set of its left cosets as $\langle Y_i \rangle_{i\in I}$; because any translate of a totally bounded set is totally bounded (4A5Ob again), each Y_i is an open set expressible as $\bigcup_{n\in\mathbb{N}} H_{in}$, where every H_{in} is a totally bounded open set, so that μH_{in} is finite.

For $i \in I$ and m, $n \in \mathbb{N}$ there is a closed set $F_{imn} \subseteq E \cap H_{im}$ such that $\mu F_{imn} \ge \mu(E \cap H_{im}) - 2^{-n}$. Set $F_{mn} = \bigcup_{i\in I} F_{imn}$ for each m, $n \in \mathbb{N}$; then F_{mn} is closed (4A2Bb). So $E' = \bigcup_{m,n\in\mathbb{N}} F_{mn}$ is F_σ. For each $i \in I$,

$$(E \setminus E') \cap Y_i \subseteq \textstyle\bigcup_{m\in\mathbb{N}} \big(E \cap H_{im} \setminus \bigcup_{n\in\mathbb{N}} F_{imn}\big)$$

is negligible; thus $\{G : G \subseteq X \text{ is open}, \ \mu(G \cap E \setminus E') = 0\}$ covers X and $E \setminus E'$ must be negligible (414Ea).

In the same way, there is an F_σ set $F^* \subseteq X \setminus E$ such that $(X \setminus E) \setminus F^*$ is negligible; now $E'' = X \setminus F^*$ is G_δ and $E'' \setminus E$ is negligible, so $E'' \setminus E'$ also is. (I am speaking here as if 'negligible' meant 'μ-negligible'. But of course this is the same thing as the 'Haar negligible' of the statement of the proposition.)

(ii), (iii), (iv) follow at once.

443K Theorem Let X be a Hausdorff topological group carrying Haar measures. Then the completion $\widehat{X}$ of X under its bilateral uniformity is a locally compact Hausdorff group, and X is of full outer Haar measure in $\widehat{X}$. Any (left or right) Haar measure on X is the subspace measure corresponding to a Haar measure (of the same chirality) on $\widehat{X}$.

proof (a) By 443H and 4A5N, $\widehat{X}$ is a locally compact Hausdorff group in which X is embedded as a dense subgroup.

(b) Let μ be a left Haar measure on X. Then there is a Radon measure λ on $\widehat{X}$ such that μ is the subspace measure λ_X. **P** For Borel sets $E \subseteq \widehat{X}$, set $\nu E = \mu(X \cap E)$. Then ν is a Borel measure, and it is τ-additive because μ is. Any point of $\widehat{X}$ has a compact neighbourhood V in $\widehat{X}$; now V must be totally bounded for the bilateral uniformity of $\widehat{X}$ (4A2Je), so $V \cap X$ is totally bounded for the bilateral uniformity of X (4A5Ma), and $\nu V = \mu(V \cap X)$ is finite (443I). Thus ν is locally finite. If $\nu E > 0$, there is an open set $H \subseteq X$ such that $\mu H < \infty$ and $\mu(H \cap X \cap E) > 0$, because μ is effectively locally finite; now there is an open set $G \subseteq \widehat{X}$ such that $H = X \cap G$, so that $\nu G < \infty$ and $\nu(G \cap E) > 0$. Thus ν is effectively locally finite.

By 416H, the c.l.d. version λ of ν is a Radon measure on $\widehat{X}$. Since $\lambda K = \nu K = 0$ whenever $K \subseteq \widehat{X} \setminus X$ is compact, X is of full outer measure for λ. Accordingly

$$\lambda_X(G \cap X) = \lambda G = \nu G = \mu(G \cap X)$$

for every open set $G \subseteq \widehat{X}$, and $\lambda_X = \mu$, because they are quasi-Radon measures agreeing on the open sets (415B, 415H(iii)). **Q**

(c) Continuing the argument of (b), λ is a left Haar measure on $\widehat{X}$. **P** Let $G \subseteq \widehat{X}$ be open, and $z \in \widehat{X}$. If $K \subseteq zG$ is compact, then $z^{-1}K \subseteq G$, and $\{w : w \in \widehat{X},\ wK \subseteq G\}$ is a non-empty open set (4A5Ei), so meets X. Take $x \in X$ such that $xK \subseteq G$; then

$$\lambda G = \mu(X \cap G) \ge \mu(X \cap xK) = \mu(x(X \cap K)) = \mu(X \cap K) = \lambda K.$$

As K is arbitrary, $\lambda(zG) \le \lambda G$. By 441Ba, λ is invariant under the left action of $\widehat{X}$ on itself, that is, is a left Haar measure. **Q**

We know that X is of full outer measure for λ, so this shows that it has full outer Haar measure in $\widehat{X}$.

(d) The same arguments, looking at Gz and Kz^{-1} in (c), show that if μ is a right Haar measure on X it is the subspace measure λ_X for a right Haar measure λ on $\widehat{X}$.

443L Corollary Let X be any topological group with a Haar measure μ. Then we can find Z, λ and ϕ such that

(i) Z is a locally compact Hausdorff topological group;

(ii) λ is a Haar measure on Z;

(iii) $\phi : X \to Z$ is a continuous homomorphism, inverse-measure-preserving for μ and λ;

(iv) μ is inner regular with respect to $\{\phi^{-1}[K] : K \subseteq Z \text{ is compact}\}$;

(v) if $E \subseteq X$ is Haar measurable, we can find a Haar measurable set $F \subseteq Z$ such that $\phi^{-1}[F] \subseteq E$ and $E \setminus \phi^{-1}[F]$ is Haar negligible;

(vi) a set $G \subseteq X$ is an open set in X iff it is of the form $\phi^{-1}[H]$ for some open set $H \subseteq Z$;

(vii) a set $G \subseteq X$ is a regular open set in X iff it is of the form $\phi^{-1}[H]$ for some regular open set $H \subseteq Z$;

(viii) a set $A \subseteq X$ is nowhere dense in X iff $\phi[A]$ is nowhere dense in Z.

proof (a) Let $Y \subseteq X$ be the closure of $\{e\}$, where e is the identity of X. Then Y is a closed normal subgroup of X, and if $\phi : X \to X/Y$ is the quotient map, every open (or closed) subset of X is of the form $\phi^{-1}[H]$ for some open (or closed) set $H \subseteq Y$ (4A5Kb).

Consider the image measure $\nu = \mu\phi^{-1}$ on X/Y. This is quasi-Radon. **P** Because μ is a complete τ-additive topological measure, so is ν. If $F \in \operatorname{dom}\nu$ and $\nu F > 0$, there is an open set $G \subseteq X$ such that $\mu G < \infty$ and $\mu(G \cap \phi^{-1}[F]) > 0$; now $G = \phi^{-1}[H]$ for some open set $H \subseteq X/Y$, and $\nu H = \mu G$ is finite, while $\nu(H \cap F) = \mu(G \cap \phi^{-1}[F]) > 0$. Thus ν is effectively locally finite (therefore semi-finite). Again, if $F \in \operatorname{dom}\nu$ and $\nu F > \gamma$, there is a closed set $E \subseteq \phi^{-1}[F]$ such that $\mu E \ge \gamma$; now E is expressible as $\phi^{-1}[H]$ for some closed set $H \subseteq X/Y$; because ϕ is surjective, $H \subseteq F$, and $\nu H = \mu E \ge \gamma$. Thus ν is inner regular with respect to the closed sets. Finally, suppose that $F \subseteq X/Y$ is such that $F \cap F' \in \operatorname{dom}\nu$ whenever $\nu F' < \infty$. If $E \subseteq X$ is a closed set of finite measure, it is of the form $\phi^{-1}[F']$ where $\nu F' = \mu E < \infty$, so $F' \cap F \in \operatorname{dom}\nu$ and $E \cap \phi^{-1}[F] \in \operatorname{dom}\mu$; by 412Ja, we can conclude that $\phi^{-1}[F] \in \operatorname{dom}\mu$ and $F \in \operatorname{dom}\nu$. Thus ν is locally determined and is a quasi-Radon measure. **Q**

We find also that ν is a left Haar measure. **P** If $z \in X/Y$ and $F \in \operatorname{dom}\nu$, express z as $\phi(x)$ where $x \in X$; then $\phi^{-1}[zF] = x\phi^{-1}[F]$, so

$$\nu(zF) = \mu(x\phi^{-1}[F]) = \mu\phi^{-1}[F] = \nu F. \textbf{ Q}$$

(b) Thus X/Y is a topological group with a left Haar measure ν. Because Y is closed, X/Y is Hausdorff (4A5J(b-ii-α)). We can therefore form its completion $Z = \widehat{X/Y}$, a locally compact Hausdorff group, and find a left Haar measure λ on Z such that ν is the corresponding subspace measure on X/Y, which is of full outer measure for λ (443K). The embedding $X/Y \subsetneqq Z$ is therefore inverse-measure-preserving for ν and λ, so that ϕ, regarded as a map from X to Z, is inverse-measure-preserving for μ and λ. Also, of course, $\phi : X \to Z$ is a continuous homomorphism.

If $E \subseteq X$ and $\mu E > \gamma$, there is a closed set $E' \subseteq E$ such that $\mu E' > \gamma$. Now E' is of the form $\phi^{-1}[F]$ where $F \subseteq X/Y$ is closed and $\nu F = \mu E' > \gamma$. Next, F is of the form $(X/Y)\cap F'$ where $F' \subseteq Z$ is closed and $\lambda F' = \nu F > \gamma$. So there is a compact set $K \subseteq F'$ such that $\lambda K \ge \gamma$, and we have

$$\phi^{-1}[K] \subseteq \phi^{-1}[F'] = \phi^{-1}[F] \subseteq E, \quad \mu\phi^{-1}[K] = \nu(K \cap (X/Y)) = \lambda K \ge \gamma.$$

As E and γ are arbitrary, μ is inner regular with respect to $\{\phi^{-1}[K] : K \subseteq Z \text{ is compact}\}$. So (i)-(iv) are true.

(c) If $E \subseteq X$ is Haar measurable, then by 443J(b-i) there is an F_σ set $E' \subseteq E$ such that $E \setminus E'$ is Haar negligible. Now there are an F_σ set $H \subseteq X/Y$ such that $E' = \phi^{-1}[H]$ and an F_σ set $F \subseteq Z$ such that $H = (X/Y) \cap F$, in which case $E' = \phi^{-1}[F]$, and $E \setminus \phi^{-1}[F]$ is Haar negligible. This deals with (v).

(d) Concerning (vi)-(viii), we just have to put 4A2B and 4A5Kb together. If G, $A \subseteq X$, then

$$\begin{aligned} G \text{ is open in } X &\iff \text{there is an open } V \subseteq X/Y \text{ such that } G = \phi^{-1}[V] \\ &\iff \text{there are a } V \subseteq X/Y \text{ and an open } H \subseteq Z \\ &\qquad\quad \text{such that } G = \phi^{-1}[V] \text{ and } V = (X/Y) \cap H \\ &\iff \text{there is an open } H \subseteq Z \text{ such that } G = \phi^{-1}[H]; \end{aligned}$$

$$G \text{ is a regular open set in } X \iff \phi[G] \text{ is a regular open subset of } X/Y$$

(4A5K(b-iii), 4A2B(f-iii))

$$\iff \text{there is a regular open } H \subseteq Z \text{ such that } \phi[G] = (X/Y) \cap H$$

(4A2B(j-ii))

$$\iff \text{there is a regular open } H \subseteq Z \text{ such that } G = \phi^{-1}[H];$$

$$A \text{ is nowhere dense in } X \iff \phi[A] \text{ is nowhere dense in } X/Y$$

(4A5K(iv)-(v))

$$\iff \phi[A] \text{ is nowhere dense in } Z$$

(4A2B(j-i)).

443M Theorem (Halmos 50) Let X be a topological group and μ a Haar measure on X. Then μ is completion regular.

proof (a) Suppose first that μ is a left Haar measure and that X is locally compact and Hausdorff. In this case any self-supporting compact set $K \subseteq X$ is a zero set. **P** For each $n \in \mathbb{N}$, there is an open neighbourhood U_n of the identity such that $\mu(K \triangle xK) \le 2^{-n}$ for every $x \in U_n$ (443B); we may suppose that $\overline{U}_{n+1} \subseteq U_n$ for each n. Each set U_nK is open (4A5Ed), so $\bigcap_{n\in\mathbb{N}} U_nK$ is a G_δ set. **?** If $K \ne \bigcap_{n\in\mathbb{N}} U_nK$, there is an $x \in \bigcap_{n\in\mathbb{N}} U_nK \setminus K$. For each $n \in \mathbb{N}$, there are $y_n \in U_n$, $z_n \in K$ such that $x = y_nz_n$. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Then $z = \lim_{n\to\mathcal{F}} z_n$ is defined in K, so

$$xz^{-1} = \lim_{n\to\mathcal{F}} xz_n^{-1} = \lim_{n\to\mathcal{F}} y_n$$

is defined in X; because $y_n \in \overline{U}_i$ for every $i \le n$,

$$xz^{-1} \in \bigcap_{i\in\mathbb{N}} \overline{U}_i = \bigcap_{i\in\mathbb{N}} U_i.$$

Consequently $\mu(xz^{-1}K \triangle K) = 0$; because μ is left-translation-invariant, $\mu(K \setminus zx^{-1}K) = 0$. But as $x \notin K$, $z \in K \setminus zx^{-1}K$ and $K \cap (X \setminus zx^{-1}K)$ is non-empty. And $zx^{-1}K$ is closed, so $X \setminus zx^{-1}K$ is open and K is not self-supporting, contrary to hypothesis. **X**

Thus $K = \bigcap_{n\in\mathbb{N}} U_n K$ is a G_δ set. Being a compact G_δ set in a completely regular Hausdorff space, it is a zero set (4A2F(h-v)). **Q**

Since μ is surely inner regular with respect to the compact self-supporting sets (414F), it is inner regular with respect to the zero sets, and is completion regular.

(b) Now suppose that μ is a left Haar measure on an arbitrary topological group X. By 443L, we can find a locally compact Hausdorff topological group Z, a continuous homomorphism $\phi : X \to Z$ and a left Haar measure λ on Z such that ϕ is inverse-measure-preserving for μ and λ and μ is inner regular with respect to $\{\phi^{-1}[K] : K \subseteq Z$ is compact$\}$. Now if $E \in \operatorname{dom}\mu$ and $\gamma < \mu E$, there is a compact set $K \subseteq Z$ such that $\phi^{-1}[K] \subseteq E$ and $\nu K > \gamma$. Next, there is a zero set $L \subseteq K$ such that $\nu L \ge \gamma$; in which case $\phi^{-1}[L] \subseteq E$ is a zero set and $\mu\phi^{-1}[L] \ge \gamma$. Thus μ is inner regular with respect to the zero sets and is completion regular.

(c) Finally, if μ is a right Haar measure on a topological group X, let $\overleftrightarrow{\mu}$ be the corresponding left Haar measure, setting $\overleftrightarrow{\mu}E = \mu E^{-1}$ for Haar measurable sets E. Then $\overleftrightarrow{\mu}$ is inner regular with respect to the zero sets; because $x \mapsto x^{-1} : X \to X$ is a homeomorphism, so is μ.

443N I give a simple result showing how the measure-theoretic properties of groups carrying Haar measures have topological consequences which might not be expected.

Proposition Let X be a topological group carrying Haar measures (for instance, X might be any locally compact Hausdorff group).

(i) Let G be a regular open subset of X. Then G is a cozero set.

(ii) Let F be a nowhere dense subset of X. Then F is included in a nowhere dense zero set.

proof (a) Suppose to begin with that X is locally compact, σ-compact and Hausdorff. Let μ be a left Haar measure on X; then μ is σ-finite, because X is covered by a sequence of compact sets, which must all have finite measure.

(i) Write $\mathcal{U}$ for the family of open neighbourhoods of the identity e of X. For each $U \in \mathcal{U}$, set $H_U = \operatorname{int}\{x : xU \subseteq G\}$; then $\{H_U : U \in \mathcal{U}\}$ is an upwards-directed family of open sets with union G, as in the proofs of 442Ab and 442B, so $G^\bullet = \sup_{U\in\mathcal{U}} H_U^\bullet$ in the measure algebra $\mathfrak{A}$ of μ. Because μ is σ-finite, $\mathfrak{A}$ is ccc (322G) and there is a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{U}$ such that $G^\bullet = \sup_{n\in\mathbb{N}} H_{U_n}^\bullet$ (316E). In this case, $G \setminus \bigcup_{n\in\mathbb{N}} H_{U_n}$ is negligible, so must have empty interior.

By 4A5S, there is a closed normal subgroup Y of X, included in $\bigcap_{n\in\mathbb{N}} U_n$, such that X/Y is metrizable. Let $\pi : X \to X/Y$ be the canonical map.

For each $n \in \mathbb{N}$, $Q_n = \pi[H_{U_n}]$ is open (4A5J(a-i)), and

$$H_{U_n} \subseteq \pi^{-1}[Q_n] = H_{U_n}Y \subseteq H_{U_n}U_n \subseteq G.$$

So

$$\overline{G} = \overline{\bigcup_{n\in\mathbb{N}} H_{U_n}} = \overline{\bigcup_{n\in\mathbb{N}} \pi^{-1}[Q_n]}.$$

Setting $Q = \operatorname{int}\overline{\bigcup_{n\in\mathbb{N}} Q_n}$, and using 4A5J(a-i) and 4A2B(f-ii), we see that

$$\pi^{-1}[Q] = \operatorname{int}\overline{\bigcup_{n\in\mathbb{N}} \pi^{-1}[Q_n]} = \operatorname{int}\overline{G} = G$$

(this is where I use the hypothesis that G is a regular open set). But Q, being an open set in a metrizable space, is a cozero set (4A2Lc), so $G = \pi^{-1}[Q]$ is a cozero set (4A2C(b-iv)), as required by (i).

(ii) Now consider the nowhere dense set $F \subseteq X$. This time, let $\mathcal{G}$ be a maximal disjoint family of cozero subsets of $X \setminus F$. Then $\mathcal{G}$ is countable, again because μ is σ-finite, and $\bigcup \mathcal{G}$ is dense, because the topology of X is completely regular. So $X \setminus \bigcup \mathcal{G}$ is a nowhere dense zero set including F.

(b) Next, suppose that X is any locally compact Hausdorff topological group. Then X has a σ-compact open subgroup X_0 (4A5El). By (a), any regular open set in X_0 is a cozero set in X_0. The same applies to all the (left) cosets of X_0, because these are homeomorphic to X_0.

If C is any coset of X_0, then $G \cap C$ is a regular open set in C, so is a cozero set in C. But as the left cosets of X_0 form a partition of X into open sets, G is also a cozero set in X (4A2C(b-vii)).

Similarly, $F \cap C$ is nowhere dense in C for every left coset C of X_0, so is included in a nowhere dense zero set in C, and the union of these will be a nowhere dense zero set in X including F.

(c) Now suppose that X is any group carrying Haar measures. Let Z, λ and $\phi : X \to Z$ be as in 443L. Then G is expressible as $\phi^{-1}[H]$ for some regular open set $H \subseteq Z$ (443L(vii)); by (b), H is a cozero set, so G also is (4A2C(b-iv) again). As for F, $\phi[F]$ is nowhere dense in Z, by 443L(viii). Let $F' \supseteq \phi[F]$ be a nowhere dense zero set; then $\phi^{-1}[F'] \supseteq F$ is a zero set, and $\phi[\phi^{-1}[F']] \subseteq F'$ is nowhere dense, so $\phi^{-1}[F']$ is nowhere dense.

443O An expected result, well known for Lebesgue measure, but which seems to need a little attention for the non-metrizable case, is the following.

Proposition Let X be a topological group and μ a left Haar measure on X. Then the following are equiveridical:

(i) μ is not purely atomic;

(ii) μ is atomless;

(iii) there is a non-negligible nowhere dense subset of X;

(iv) μ is inner regular with respect to the nowhere dense sets;

(v) there is a conegligible meager subset of X;

(vi) there is a negligible comeager subset of X.

If X is Hausdorff, we can add

(vii) the topology of X is not discrete.

proof Write Σ for the domain of μ.

(a)(i)$\Rightarrow$(ii) If μ is not purely atomic, let $E \in \Sigma$ be a non-negligible set not including any atom. If $F \in \Sigma$ is any other non-negligible set, then there is an $x \in X$ such that $F \cap xE$ is not negligible, by 443Da. Now the subspace measures on $F \cap xE$ and $x^{-1}F \cap E$ are isomorphic, and the latter is atomless, so $F \cap xE$ is not an atom and F is not an atom. As F is arbitrary, μ is atomless.

(b)(iii)$\Rightarrow$(iv) The argument is similar. Suppose that A is a non-negligible nowhere dense subset of X; then $E = \overline{A}$ is a non-negligible closed nowhere dense set. If $F \in \Sigma$ is non-negligible, there is an $x \in X$ such that $F \cap xE$ is non-negligible; as $y \mapsto xy : X \to X$ is a homeomorphism, xE and $F \cap xE$ are nowhere dense. Thus every non-negligible measurable set includes a nowhere dense non-negligible measurable set; as the family of nowhere dense sets is an ideal, μ is inner regular with respect to the nowhere dense sets (412Aa).

(c)(iv)$\Rightarrow$(i) Suppose that μ is inner regular with respect to the nowhere dense sets. Let U be an open neighbourhood of the identity e of X with finite measure (442Aa once more), and V an open neighbourhood of e such that $VV^{-1}V \subseteq U$. Then $\mu V > 0$, by the other half of 442Aa.

? If there is a μ-atom $E \subseteq V$, let $F_0 \subseteq E$ be a non-negligible measurable nowhere dense set, $F_1 \subseteq F_0$ a non-negligible closed set and $F \subseteq F_1$ a non-empty self-supporting closed set (414F). Because $y \mapsto xy$ is a measure-preserving automorphism, xF is a self-supporting closed set, and an atom for μ, for every $x \in X$. So if x, $y \in X$ and $xF \cap yF$ is non-negligible, then $xF \setminus yF$ is negligible and $xF \subseteq yF$; similarly, $yF \subseteq xF$, so $xF = yF$. Now no finite number of translates of the nowhere dense set F can cover the non-empty open set V, while $V \subseteq \bigcup_{x \in X} xF$, so we must have a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in X such that $x_n F \cap V \setminus \bigcup_{i<n} x_i F \neq \emptyset$ for every $n \in \mathbb{N}$. In this case, $x_i F \cap x_n F$ is negligible whenever $i < n$, so

$$\mu(\bigcup_{n \in \mathbb{N}} x_n F) = \sum_{n \in \mathbb{N}} \mu(x_n F) = \infty.$$

However $V \cap x_n F \neq \emptyset$, so $x_n \in VF^{-1} \subseteq VV^{-1}$ and $x_n F \subseteq VV^{-1}V \subseteq U$ for every $n \in \mathbb{N}$; and U is supposed to have finite measure. **X**

Accordingly V does not include any atom and μ cannot be purely atomic.

(d)(iv)$\Rightarrow$(v) Suppose that μ is inner regular with respect to the nowhere dense sets. Let $\mathcal{G}$ be a maximal disjoint family of open sets of non-zero finite measure. Then $\operatorname{int}(X \setminus \bigcup \mathcal{G})$ is negligible, because μ is effectively locally finite, so must be empty, and $X \setminus \bigcup \mathcal{G}$ is nowhere dense. For each $G \in \mathcal{G}$, let $\langle F_{Gn} \rangle_{n \in \mathbb{N}}$ be a sequence of nowhere dense measurable subsets of G such that $\mu G = \lim_{n \to \infty} \mu F_{Gn}$. For $n \in \mathbb{N}$, set $A_n = \bigcup_{G \in \mathcal{G}} F_{Gn}$. Then A_n is nowhere dense. **P** If $H \subseteq X$ is open and not empty, either $H \cap A_n = \emptyset$ or there is a $G \in \mathcal{G}$ such that $H \cap G \neq \emptyset$, in which case $H \cap G \setminus \overline{F}_{Gn} \subseteq H \setminus A_n$ is open and non-empty. **Q** So $D = (X \setminus \bigcup \mathcal{G}) \cup \bigcup_{n \in \mathbb{N}} \overline{A}_n$ is meager. **?** If D is not conegligible, let H be an open set of finite measure such that $H \setminus D$ is non-negligible. As $H \setminus D \subseteq \bigcup \mathcal{G}$, there is a $G \in \mathcal{G}$ such that $\mu(H \cap G \setminus D) > 0$; but $H \cap G \setminus D \subseteq G \setminus \bigcup_{n\in\mathbb{N}} F_{Gn}$. **X**

Thus D is a conegligible meager set.

(e)(v)⇔(vi) The complement of a witness for (v) witnesses (vi), and conversely.

(f)(v)⇒(iii) is elementary, since μ is non-zero.

(g)(ii)⇒(iii) Suppose that μ is atomless. I proceed through an expanding series of special cases, as in 443N.

(α) Suppose to begin with that X is locally compact, Hausdorff and σ-compact. In this case X has a closed negligible normal subgroup Y such that X/Y is separable and metrizable. **P** Since μ is atomless, $\{e\}$ must be negligible. Since μ is locally finite and inner regular with respect to the closed sets, there must be a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ of open neighbourhoods of e such that $\inf_{n\in\mathbb{N}} \mu U_n = 0$. By 4A5S again, there is a closed normal subgroup $Y \subseteq \bigcap_{n\in\mathbb{N}} U_n$ such that X/Y is metrizable, and of course Y is negligible. Since the canonical map from X onto X/Y is continuous (4A5J(a-i) again), X/Y is σ-compact, therefore separable (4A2Hd, 4A2Pd). **Q**

Write $\pi : X \to X/Y$ for the canonical map. Let $D \subseteq X/Y$ be a countable dense set, and consider $\pi^{-1}[D]$. This is a countable union of translates of Y, so is negligible; let $F \subseteq X \setminus \pi^{-1}[D]$ be a closed non-negligible set. Then $\pi[F]$ does not meet D. Because π is an open mapping (4A5Ja once more), $\operatorname{int} F = \emptyset$ and F is nowhere dense.

Thus in this case we have a non-negligible nowhere dense set, as required.

(β) Now suppose just that X is locally compact and Hausdorff. In this case it has an open σ-compact subgroup X_0 say. The subspace measure μ_{X_0} on X_0 is a left Haar measure on X_0 (443F), and is atomless; by (α), there is a nowhere dense set $F \subseteq X_0$ such that $0 < \mu_{X_0} F = \mu F$. So in this case too we have a non-negligible nowhere dense set.

(γ) For the general case, let Z, λ and $\phi : X \to Z$ be as in 443L. Then λ is atomless. **P** 443L(v) implies that the measure-preserving Boolean homomorphism from the measure algebra of λ to the measure algebra of μ induced by ϕ is surjective, therefore an isomorphism; so both measure algebras are atomless and λ is atomless (322Bg). **Q**

By (β), there is a nowhere dense non-negligible subset H of Z; replacing H by its closure, if necessary, we may suppose that H is closed. Set $F = \phi^{-1}[H]$; then $F \subseteq X$ is closed and non-negligible because ϕ is continuous and inverse-measure-preserving. Since $\phi[F] \subseteq H$ is nowhere dense, so is F (443L(viii)). Thus we have the required non-negligible nowhere dense set in the general case also.

(h) Now suppose that X is Hausdorff. If μ is atomless, then $\mu\{x\} = 0$ for every x, so $\{x\}$ is never open and the topology is not discrete. If μ has an atom E, let $F \subseteq E$ be a closed self-supporting set of non-zero measure; then F also is an atom, so cannot have two disjoint non-empty relatively open sets, and must be a singleton. Thus we have an x_0 such that $\mu\{x_0\} > 0$; as μ is left-translation-invariant, $\mu\{x\} = \mu\{x_0\}$ for every $x \in X$. We know also that there is an open set G of non-zero finite measure, which must be finite; so every singleton subset of G is open. It follows that every singleton subset of X is open, and X has its discrete topology. Thus (ii)⇔(vii) when X is Hausdorff.

443P Quotient spaces I come now to the relationship between the modular functions of §442, normal subgroups and quotient spaces.

Lemma Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X. Let $Z = X/Y$ be the set of left cosets of Y in X with the quotient topology and $\pi : X \mapsto Z$ the canonical map, so that Z is a locally compact Hausdorff space and we have a continuous action of X on Z defined by writing $a{\bullet}\pi x = \pi(ax)$ for $a, x \in X$ (4A5J(b-iii)). Let ν be a left Haar measure on Y and write $C_k(X)$, $C_k(Z)$ for the spaces of continuous real-valued functions with compact supports on X, Z respectively.

(a) We have a positive linear operator $T : C_k(X) \to C_k(Z)$ defined by writing

$$(Tf)(\pi x) = \int_Y f(xy)\nu(dy)$$

for every $f \in C_k(X)$ and $x \in X$. If $f > 0$ in $C_k(X)$ then $Tf > 0$ in $C_k(Z)$. If $h \ge 0$ in $C_k(Z)$ then there is an $f \ge 0$ in $C_k(X)$ such that $Tf = h$.

(b) If $a \in X$ and $f \in C_k(X)$, then $T(a{\bullet_l}f)(z) = (Tf)(a^{-1}{\bullet}z)$ for every $z \in Z$.

(c) Now suppose that a belongs to the normalizer of Y (that is, $aYa^{-1} = Y$). In this case, we can define $\psi(a) \in \,]0,\infty[$ by the formula

$$\nu(aFa^{-1}) = \psi(a)\nu F \text{ for every } F \in \operatorname{dom}\nu,$$

and

$$T(a\bullet_r f)(\pi x) = \psi(a) \cdot (Tf)(\pi(xa))$$

for every $x \in X$ and $f \in C_k(X)$.

proof I should begin by remarking that because Y is a closed subgroup of a locally compact Hausdorff group, it is itself a locally compact Hausdorff group, so does have a left Haar measure, which is a Radon measure.

(a)(i) The first thing to check is that if $f \in C_k(X)$ then Tf is well-defined as a member of $\mathbb{R}^Z$. **P** (α) If $x \in X$, then $y \mapsto f(xy) : Y \to \mathbb{R}$ is a continuous function with compact support, so $\int f(xy)\nu(dy)$ is defined in $\mathbb{R}$. (β) If x_1, $x_2 \in X$ and $\pi x_1 = \pi x_2$, then $x_1^{-1}x_2 \in Y$, and

$$\int f(x_2 y)\nu(dy) = \int f(x_1(x_1^{-1}x_2 y))\nu(dy) = \int f(x_1 y)\nu(dy),$$

applying 441J to the function $y \mapsto f(x_1 y)$ and the left action of Y on itself. Thus we can safely write $(Tf)(\pi x) = \int f(xy)\nu(dy)$ for every $x \in X$, and Tf will be a real-valued function on Z. **Q**

(ii) Now Tf is continuous for every $f \in C_k(X)$. **P** Given $z_0 \in Z$, take $x_0 \in X$ such that $z = \pi x_0$. We have an $h \in C_k(X)^+$ such that for every $\epsilon > 0$ there is an open set U_ϵ containing x_0 such that $|f(x_0 y) - f(xy)| \le \epsilon h(y)$ whenever $x \in U_\epsilon$ and $y \in X$ (4A5Pb). In this case,

$$|(Tf)(\pi x) - (Tf)(\pi x_0)| = |\int f(xy) - f(x_0 y)\nu(dy)| \le \epsilon \int_Y h(y)\nu(dy)$$

for every $x \in U_\epsilon$, so that $|(Tf)(z) - (Tf)(z_0)| \le \epsilon \int_Y h\,d\nu$ for every $z \in \pi[U_\epsilon]$. Since each $\pi[U_\epsilon]$ is an open neighbourhood of z_0 (4A5J(a-i), as always), Tf is continuous at z_0; as z_0 is arbitrary, Tf is continuous. **Q**

(iii) Since

$$\begin{aligned}\{z : (Tf)(z) \ne 0\} &= \{\pi x : \int f(xy)\nu(dy) \ne 0\} \subseteq \{\pi x : f(xy) \ne 0 \text{ for some } y \in Y\}\\ &= \{\pi x : f(x) \ne 0\} \subseteq \pi[\overline{\{x : f(x) \ne 0\}}]\end{aligned}$$

is relatively compact, $Tf \in C_k(Z)$ for every $f \in C_k(X)$.

(iv) The formula for Tf makes it plain that $T : C_k(X) \to C_k(Z)$ is a positive linear operator.

(v) If $f \in C_k(X)^+$ and $x \in X$ are such that $f(x) > 0$, then $\{y : y \in Y,\, f(xy) > 0\}$ is a non-empty open set in Y; because ν is strictly positive,

$$(Tf)(\pi x) = \int f(xy)\nu(dy) > 0.$$

In particular, $Tf > 0$ if $f > 0$. Moreover, if $z \in Z$ there is an $f \in C_k(X)^+$ such that $(Tf)(z) > 0$. Now

$$\{\{z : (Tf)(z) > 0\} : f \in C_k(X)^+\}$$

is an upwards-directed family of open subsets of Z, so if $L \subseteq Z$ is any compact set there is an $f \in C_k(X)^+$ such that $(Tf)(z) > 0$ for every $z \in L$.

(vi) Now suppose that $h \in C_k(Z)^+$. By (v), there is an $f_0 \in C_k(X)^+$ such that $(Tf_0)(z) > 0$ whenever $z \in \overline{\{w : h(w) \ne 0\}}$. Setting $h'(z) = h(z)/(Tf_0)(z)$ when $h(z) \ne 0$, 0 for other $z \in Z$, we have $h' \in C_k(Z)$ and $h = h' \times Tf_0$. Set

$$f(x) = f_0(x)h'(\pi x) \ge 0$$

for every $x \in X$. Because h' and π are continuous, $f \in C_k(X)$. For any $x \in X$,

$$\begin{aligned}(Tf)(\pi x) &= \int f_0(xy)h'(\pi(xy))\nu(dy) = h'(\pi x)\int f_0(xy)\nu(dy)\\ &= h'(\pi x)(Tf_0)(\pi x) = h(\pi x).\end{aligned}$$

Thus $Tf = h$.

(b) If $z = \pi x$, then $a^{-1}\bullet z = \pi(a^{-1}x)$, so

$$(Tf)(a^{-1}\bullet z) = \int f(a^{-1}xy)\nu(dy) = \int (a\bullet_l f)(xy)\nu(dy) = T(a\bullet_l f)(z).$$

(c) Define $\phi : Y \to Y$ by writing $\phi(y) = a^{-1}ya$ for $y \in Y$. Because ϕ is a homeomorphism, the image measure $\nu\phi^{-1}$ is a Radon measure on Y; because ϕ is a group automorphism, $\nu\phi^{-1}$ is a left Haar measure. (If $F \in \operatorname{dom}\nu\phi^{-1}$ and $y \in Y$, then

$$\nu\phi^{-1}[yF] = \nu(ayFa^{-1}) = \nu(Fa^{-1}) = \nu(aFa^{-1}) = \nu\phi^{-1}[F].)$$

$\nu\phi^{-1}$ must therefore be a multiple of ν; say $\nu\phi^{-1} = \psi(a)\nu$.

If $g \in C_k(Y)$, then

$$\int g(a^{-1}ya)\nu(dy) = \int g\phi\,d\nu = \int g\,d(\nu\phi^{-1}) = \psi(a)\int g\,d\nu.$$

Now take $f \in C_k(X)$. Then

$$T(a\bullet_r f)(\pi x) = \int (a\bullet_r f)(xy)\nu(dy) = \int f(xya)\nu(dy)$$
$$= \int f(xa(a^{-1}ya))\nu(dy) = \psi(a)\int f(xay)\nu(dy)$$

(using the remark above with $g(y) = f(xay)$)

$$= \psi(a)(Tf)(\pi(xa))$$

for every $x \in X$, as claimed.

443Q Theorem Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X. Let $Z = X/Y$ be the set of left cosets of Y in X with the quotient topology, and $\pi : X \to Z$ the canonical map, so that Z is a locally compact Hausdorff space and we have a continuous action of X on Z defined by writing $a\bullet\pi x = \pi(ax)$ for a, $x \in X$. Let ν be a left Haar measure on Y. Suppose that λ is a non-zero X-invariant Radon measure on Z.

(a) For each $z \in Z$, we have a Radon measure ν_z on X defined by the formula

$$\nu_z E = \nu(Y \cap x^{-1}E)$$

whenever $\pi x = z$ and the right-hand side is defined. In this case, for a real-valued function f defined on a subset of X,

$$\int f\,d\nu_z = \int f(xy)\nu(dy)$$

whenever either side is defined in $[-\infty, \infty]$.

(b) We have a left Haar measure μ on X defined by the formulae

$$\int f\,d\mu = \iint f\,d\nu_z\lambda(dz)$$

for every $f \in C_k(X)$, and

$$\mu G = \int \nu_z G\,\lambda(dz)$$

for every open set $G \subseteq X$.

(c) If $D \subseteq Z$, then $D \in \operatorname{dom}\lambda$ iff $\pi^{-1}[D] \subseteq X$ is Haar measurable, and $\lambda D = 0$ iff $\pi^{-1}[D]$ is Haar negligible.

(d) If $\nu Y = 1$, then λ is the image measure $\mu\pi^{-1}$.

(e) Suppose now that X is σ-compact. Then $\mu E = \int \nu_z E\,\lambda(dz)$ for every Haar measurable set $E \subseteq X$. If $f \in \mathcal{L}^1(\mu)$, then $\int f\,d\mu = \iint f\,d\nu_z\lambda(dz)$.

(f) Still supposing that X is σ-compact, take $f \in \mathcal{L}^1(\mu)$, and for $a \in X$ set $f_a(y) = f(ay)$ whenever $y \in Y$ and $ay \in \operatorname{dom} f$. Then $Q_f = \{a : a \in X,\ f_a \in \mathcal{L}^1(\nu)\}$ is μ-conegligible, and the function $a \mapsto f_a^\bullet : Q_f \to L^1(\nu)$ is almost continuous.

proof (a) First, we do have a function ν_z depending only on z, because if $z = \pi x_1 = \pi x_2$ then $x_2^{-1}x_1 \in Y$, so

$$\nu(Y \cap x_1^{-1}E) = \nu(x_2^{-1}x_1(Y \cap x_1^{-1}E)) = \nu(Y \cap x_2^{-1}E)$$

whenever either side is defined. Of course ν_z, being the image of the Radon measure ν under the continuous map $y \mapsto xy : Y \to X$ whenever $\pi x = z$, is always a Radon measure on X (418I). We also have

$$\int_Y f(xy)\nu(dy) = \int_X f\,d\nu_{\pi x}$$

whenever $x \in X$ and f is a real-valued function such that either side is defined in $[-\infty, \infty]$, by 235J.

I remark here that if $z \in Z$ then the coset $C = \pi^{-1}[\{z\}]$ is ν_z-conegligible, because if $\pi x = z$ then $Y = Y \cap x^{-1}C$.

(b)(i) Let $T : C_k(X) \to C_k(Z)$ be the positive linear operator of 443P; that is,

$$(Tf)(z) = \int f(xy)\nu(dy) = \int f\,d\nu_z$$

whenever $f \in C_k(X)$, $x \in X$ and $z = \pi x$. Then we have a positive linear functional $\theta : C_k(X) \to \mathbb{R}$ defined by setting $\theta(f) = \int Tf\,d\lambda$ for every $f \in C_k(X)$. By the Riesz Representation Theorem (436J), there is a Radon measure μ on X defined by saying that $\int f\,d\mu = \theta(f)$ for every $f \in C_k(X)$. Note that μ is non-zero. **P** Because λ is non-zero, there is some $h \in C_k(Z)^+$ such that $\int h\,d\lambda \neq 0$; now there is some $f \in C_k(X)$ such that $Tf = h$, by 443Pa, and $\int f\,d\mu \neq 0$. **Q**

(ii) μ is a left Haar measure. **P** If $f \in C_k(X)$ and $a \in X$, then we have $T(a\bullet_l f)(z) = (Tf)(a^{-1}\bullet z)$ for every $z \in Z$, by 443Pb. So

$$\int a\bullet_l f d\mu = \int T(a\bullet_l f)d\lambda = \int Tf(a^{-1}\bullet z)\lambda(dz) = \int Tf(z)\lambda(dz)$$

(by 441J or 441L, because λ is X-invariant)

$$= \int f\,d\mu.$$

By 441L in the other direction, μ is invariant under the left action of X on itself, that is, is a left Haar measure. **Q**

(iii) If $G \subseteq X$ is open then $\mu G = \int \nu_z G\,\lambda(dz)$. **P** Set $A = \{f : f \in C_k(X),\, 0 \le f \le \chi G\}$. Then $\mu G = \sup_{f\in A}\int f\,d\mu$ and

$$\nu_z G = \sup_{f\in A}\int f\,d\nu_z = \sup_{f\in A}(Tf)(z)$$

for every $z \in Z$, by 414Ba, because ν_z is τ-additive. But as Tf is continuous for every $f \in A$, and λ is τ-additive, we also have

$$\int \nu_z G\,\lambda(dz) = \sup_{f\in A}\int Tf\,d\lambda = \sup_{f\in A}\int f\,d\mu = \mu G.\ \mathbf{Q}$$

(c)(i) Let $\mathcal{A}$ be the family of those sets $A \subseteq X$ such that μA and $\int \nu_z(A)\lambda(dz)$ are defined in $[0,\infty]$ and equal. Then $\bigcup_{n\in\mathbb{N}} A_n$ belongs to $\mathcal{A}$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$, and $A \setminus B \in \mathcal{A}$ whenever A, $B \in \mathcal{A}$, $B \subseteq A$ and $\mu A < \infty$. Moreover, if $A \in \mathcal{A}$ and $\mu A = 0$, then every subset of A belongs to $\mathcal{A}$, since A must be ν_z-negligible for λ-almost every z. We also know from (b) that every open set belongs to $\mathcal{A}$.

Applying the Monotone Class Theorem (136B) to $\{A : A \in \mathcal{A},\, A \subseteq G\}$, we see that if $E \subseteq X$ is a Borel set included in an open set G of finite measure, then $E \in \mathcal{A}$. So if E is a relatively compact Haar measurable set, $E \in \mathcal{A}$ (using 443J(b-i), or otherwise).

(ii) If $D \in \operatorname{dom}\lambda$ then $\pi^{-1}[D] \in \operatorname{dom}\mu$. **P** Let $K \subseteq X$ be compact. Then $\pi[K] \subseteq Z$ is compact, so there are Borel sets F_1, $F_2 \subseteq Z$ such that $F_1 \subseteq D \cap \pi[K] \subseteq F_2$ and $F_2 \setminus F_1$ is λ-negligible. Now $\nu_z(K \cap \pi^{-1}[F_2 \setminus F_1]) = 0$ whenever $z \notin F_2 \setminus F_1$, by the remark added to the proof of (a) above, so

$$\mu(K \cap \pi^{-1}[F_2 \setminus F_1]) = \int \nu_z(K \cap \pi^{-1}[F_2 \setminus F_1])\lambda(dz) = 0.$$

Since

$$K \cap \pi^{-1}[F_1] \subseteq K \cap \pi^{-1}[D] \subseteq K \cap \pi^{-1}[F_2],$$

$K \cap \pi^{-1}[D] \in \operatorname{dom}\mu$. As K is arbitrary, $\pi^{-1}[D]$ is measured by μ, so is Haar measurable. **Q**

If $\lambda D = 0$ then the same arguments show that $\mu(K \cap \pi^{-1}[D]) = 0$ for every compact $K \subseteq X$, so that $\mu\pi^{-1}[D] = 0$ and $\pi^{-1}[D]$ is Haar negligible.

(iii) Now suppose that $D \subseteq Z$ is such that $\pi^{-1}[D] \in \operatorname{dom}\mu$. Let $L \subseteq Z$ be compact. Then there is a relatively compact open set $G \subseteq X$ such that $\pi[G] \supseteq L$ (because $\{\pi[G] : G \subseteq X$ is open and relatively compact$\}$ is an upwards-directed family of open sets covering Z). In this case,

$$\int \nu_z(G \cap \pi^{-1}[D \cap L])\lambda(dz) = \mu(G \cap \pi^{-1}[D] \cap \pi^{-1}[L])$$

is well-defined, by (i). But if $z = \pi x$ then

$$\begin{aligned} \nu_z(G \cap \pi^{-1}[D \cap L]) &= 0 \text{ if } z \notin D \cap L, \\ &= \nu_z G = \nu(Y \cap x^{-1}G) > 0 \text{ if } z \in D \cap L, \end{aligned}$$

because if $z \in L$ then $\pi x \in \pi[G]$ and $Y \cap x^{-1}G \neq \emptyset$. So

$$D \cap L = \{z : \nu_z(G \cap \pi^{-1}[D \cap L]) > 0\}$$

is measured by λ. As L is arbitrary, and λ is a Radon measure, $D \in \operatorname{dom}\lambda$.

(iv) If $\pi^{-1}[D]$ is Haar negligible, then, in (iii) above, we shall have $\int \nu_z(G \cap \pi^{-1}[D \cap L])\lambda(dz) = 0$, so that $\lambda(D \cap L) = 0$; as L is arbitrary, $\lambda D = 0$, by 412Ib or 412Jc.

(d) If $\nu Y = 1$, then, for any open set $H \subseteq Z$, $\nu_z \pi^{-1}[H] = 1$ if $z \in H$, 0 otherwise. So

$$\mu\pi^{-1}[H] = \int \nu_z(\pi^{-1}[H])\lambda(dz) = \lambda H.$$

Thus λ and the image measure $\mu\pi^{-1}$ agree on the open sets and, being Radon measures (418I again), must be equal (416E(b-iii)).

(e) If X is actually σ-compact, then (c)(i) of this proof tells us that $\mu E = \int \nu_z E\,\lambda(dz)$ for every Haar measurable set $E \subseteq X$, since E is the union of an increasing sequence of relatively compact measurable sets. Consequently $\int f\,d\mu = \iint f\,d\nu_z \lambda(dz)$ for every μ-simple function f. Now suppose that f is a non-negative μ-integrable function. Then there is a non-decreasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of non-negative μ-simple functions converging to f everywhere in $\operatorname{dom} f$. If we set $A = \{z : \nu_z^*(X \setminus \operatorname{dom} f) > 0\}$, then

$$\int \nu_z(X \setminus \operatorname{dom} f)\lambda(dz) = \mu(X \setminus \operatorname{dom} f) = 0,$$

so $\lambda A = 0$. Since $\int f\,d\nu_z = \lim_{n\to\infty} \int f_n d\nu_z$ for every $z \in Z \setminus A$,

$$\begin{aligned}\int f\,d\mu &= \lim_{n\to\infty} \int f_n\,d\mu \\ &= \lim_{n\to\infty} \iint f_n d\nu_z \lambda(dz) = \iint f\,d\nu_z \lambda(dz).\end{aligned}$$

Applying this to the positive and negative parts of f, we see that the same formula is valid for any μ-integrable function f.

(f)(i) If $f \in \mathcal{L}^1(\mu)$ and $a \in X$, then

$$\int f_a d\nu = \int f(ay)\nu(dy) = \int f\,d\nu_{\pi a}$$

if any of these are defined. So if f, $g \in \mathcal{L}^1(\mu)$, $\|f_a - g_a\|_1 = \int |f - g| d\nu_{\pi a}$ if either is defined.

(ii) Let Φ be the set of all almost continuous functions from μ-conegligible subsets of X to $L^1(\nu)$, where $L^1(\nu)$ is given its norm topology. (In terms of the definition in 411M, a member ϕ of Φ is to be almost continuous with respect to the subspace measure on $\operatorname{dom}\phi$.) If $\phi \in \Phi$ and ψ is a function from a conegligible subset of X to $L^1(\nu)$ which is equal almost everywhere to ϕ, then $\psi \in \Phi$. If $\langle \phi_n \rangle_{n\in\mathbb{N}}$ is a sequence in Φ converging μ-almost everywhere to ψ, then $\psi \in \Phi$ (418F).

(iii) For $f \in \mathcal{L}^1(\mu)$, set $\phi_f(a) = f_a^\bullet$ whenever this is defined in $L^1(\nu)$. Set $M = \{f : f \in \mathcal{L}^1(\mu),\ \phi_f \in \Phi\}$. If $\langle f^{(n)} \rangle_{n\in\mathbb{N}}$ is a sequence in M, $f \in \mathcal{L}^1(\mu)$ and $\|f^{(n)} - f\|_1 \le 4^{-n}$ for every n, then $f \in M$. **P** Set $g = \sum_{n=0}^\infty 2^n |f^{(n)} - f|$, defined on

$$\{x : x \in \operatorname{dom} f \cap \bigcap_{n\in\mathbb{N}} \operatorname{dom} f^{(n)},\ \textstyle\sum_{n=0}^\infty 2^n |f^{(n)}(x) - f(x)| < \infty\};$$

then $g \in \mathcal{L}^1(\mu)$. Now

$$D = \{z : z \in Z,\ g \text{ is } \nu_z\text{-integrable}\}$$

is λ-conegligible, by (e), and $E = \{a : a \in X,\ \pi a \in D\}$ is μ-conegligible, by (c).

If $a \in E$, then

$$|f_a^{(n)} - f_a| \le 2^{-n} g\ \ \nu_{\pi a}\text{-a.e.}$$

for every $n \ge 1$. So

$$\begin{aligned}\|\phi_f(a) - \phi_{f^{(n)}}(a)\|_1 &= \int_Y |f_a - f_a^{(n)}| d\nu = \int_X |f - f^{(n)}| d\nu_{\pi a} \\ &\le 2^{-n} \int g\,d\nu_{\pi a} \to 0\end{aligned}$$

as $n \to \infty$. Thus $\phi_f = \lim_{n\to\infty} \phi_{f^{(n)}}$ almost everywhere, and $\phi \in \Phi$, by (ii). **Q**

(iv) $C_k(X) \subseteq M$. **P** If $f \in C_k(X)$ and $a_0 \in X$, then there is an $h \in C_k(X)^+$ such that for every $\epsilon > 0$ there is an open set G_ϵ containing a_0 such that $|f(a_0y) - f(ay)| \le \epsilon h(y)$ whenever $a \in G_\epsilon$ and $y \in X$ (4A5Pb again). In this case,

$$\|\phi_f(a_0) - \phi_f(a)\|_1 = \int |f(a_0y) - f(ay)|\nu(dy) \le \epsilon\int_Y h\,d\nu$$

whenever $a \in G_\epsilon$. As ϵ is arbitrary, ϕ_f is continuous at a_0; as a_0 is arbitrary, ϕ_f is continuous, and $f \in M$. **Q**

(v) Now take any $f \in \mathcal{L}^1(\mu)$. Then for each $n \in \mathbb{N}$ we can find $f^{(n)} \in C_k(X)$ such that $\|f^{(n)} - f\|_1 \le 4^{-n}$ (416I), so $f \in M$, by (iii). This completes the proof.

443R Theorem Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X. Let $Z = X/Y$ be the set of left cosets of Y in X with the quotient topology, so that Z is a locally compact Hausdorff space and we have a continuous action of X on Z defined by writing $a\bullet(xY) = axY$ for a, $x \in X$. Let Δ_X be the left modular function of X and Δ_Y the left modular function of Y. Then the following are equiveridical:

(i) there is a non-zero X-invariant Radon measure λ on Z;

(ii) Δ_Y is the restriction of Δ_X to Y.

proof Fix a left Haar measure ν on Y, and let $T : C_k(X) \to C_k(Z)$ be the corresponding linear operator as defined in 443Pa.

(a)(i)$\Rightarrow$(ii) Suppose that λ is a non-zero X-invariant Radon measure on Z. Construct a left Haar measure μ on X as in 443Q. In the notation of part (b-i) of the proof of 443Q, we have

$$\int f d\mu = \iint f d\nu_z \lambda(dz) = \int Tf d\lambda$$

for every $f \in C_k(X)$.

Suppose that $a \in Y$. In this case, a surely belongs to the normalizer of Y, and, in the language of 443Pc, we have $\nu(aFa^{-1}) = \psi(a)\nu F$ for every $F \in \operatorname{dom}\nu$. But as

$$\nu(aFa^{-1}) = \nu(Fa^{-1}) = \Delta_Y(a^{-1})\nu F,$$

we must have $\psi(a) = \Delta_Y(a^{-1})$.

Fix some $f > 0$ in $C_k(X)$. We have

$$T(a\bullet_r f)(\pi x) = \psi(a)\cdot(Tf)(\pi(xa)) = \psi(a)\int f(xay)\nu(dy) = \psi(a)\int f(xy)\nu(dy)$$

(because $a \in Y$)

$$= \psi(a)\cdot(Tf)(\pi x)$$

for every x, so that (using 442Kc) we have

$$\Delta_X(a^{-1})\int f\,d\mu = \int a\bullet_r f d\mu = \int T(a\bullet_r f)d\lambda$$
$$= \psi(a)\int Tf\,d\lambda = \psi(a)\int f\,d\mu = \Delta_Y(a^{-1})\int f\,d\mu.$$

As $\int f\,d\mu > 0$, we must have $\Delta_X(a^{-1}) = \Delta_Y(a^{-1})$; as a is arbitrary, $\Delta_Y = \Delta_X\restriction Y$, as required by (ii).

(b)(ii)$\Rightarrow$(i) Now suppose that $\Delta_Y = \Delta_X\restriction Y$. This time, start with a left Haar measure μ on X.

(α) (The key.) If $f \in C_k(X)$ is such that $Tf \ge 0$ in $C_k(Z)$, then $\int f\,d\mu \ge 0$. **P** There is an $h \in C_k(Z)^+$ such that $h(z) = 1$ whenever $z \in Z$ and $(Tf)(z) \ne 0$; by 443Pa, we can find a $g \in C_k(X)^+$ such that $Tg = h$. Now observe that $x \mapsto (Tf)(\pi x)$ is a non-negative continuous real-valued function on X, so

$$0 \le \int_X g(x)(Tf)(\pi x)\mu(dx)$$
$$= \int_X g(x)\int_Y f(xy)\nu(dy)\mu(dx) = \int_Y\int_X g(x)f(xy)\mu(dx)\nu(dy)$$

(by 417Ha or 417Hb, because $(x, y) \mapsto g(x)f(xy) : X \times Y \to \mathbb{R}$ is a continuous function with compact support)

$$= \int_Y \Delta_X(y^{-1}) \int_X g(xy^{-1}) f(x) \mu(dx) \nu(dy)$$

(applying 442Kc to the function $x \mapsto g(xy^{-1})f(x)$)

$$= \int_X f(x) \int_Y \Delta_X(y^{-1}) g(xy^{-1}) \nu(dy) \mu(dx)$$

(because $(x, y) \mapsto \Delta_X(y^{-1})g(xy^{-1})f(x)$ is continuous and has compact support)

$$= \int_X f(x) \int_Y \Delta_Y(y^{-1}) g(xy^{-1}) \nu(dy) \mu(dx)$$

(because $\Delta_X \restriction Y = \Delta_Y$, by hypothesis)

$$= \int_X f(x) \int_Y g(xy) \nu(dy) \mu(dx)$$

(applying 442K(b-ii) to the function $y \mapsto g(xy)$)

$$= \int_X f(x)(Tg)(\pi x) \mu(dx) = \int_X f(x) \mu(dx)$$

because $(Tg)(\pi x) = 1$ whenever $f(x) \neq 0$. **Q**

(β) Applying this to f and $-f$, we see that $\int f\,d\mu = 0$ whenever $Tf = 0$, so that $\int f\,d\mu = \int g\,d\mu$ whenever f, $g \in C_k(X)$ and $Tf = Tg$. Accordingly (because T is surjective) we have a functional $\theta : C_k(Z) \to \mathbb{R}$ defined by saying that $\theta(Tf) = \int f\,d\mu$ whenever $f \in C_k(X)$, and θ is positive and linear. By the Riesz Representation Theorem again, there is a Radon measure λ on Z such that $\theta(h) = \int h\,d\lambda$ for every $h \in C_k(Z)$.

If $a \in X$ and $h \in C_k(Z)$ take $f \in C_k(X)$ such that $Tf = h$. Then, for any $x \in X$,

$$(Tf)(a\bullet\pi x) = (Tf)(\pi(ax)) = \int f(axy)\nu(dy) = \int (a^{-1}\bullet_l f)(xy)\nu(dy) = T(a^{-1}\bullet_l f)(\pi x).$$

So

$$\int h(a\bullet z)\lambda(dz) = \int (Tf)(a\bullet z)\lambda(dz) = \int T(a^{-1}\bullet_l f)(z)\lambda(dz)$$
$$= \int a^{-1}\bullet_l f\,d\mu = \int f\,d\mu = \int h(z)\lambda(dz).$$

By 441L again, λ is X-invariant. Also λ is non-zero because there is surely some f such that $\int f\,d\mu \neq 0$. So we have the required non-zero X-invariant Radon measure on Z.

443S Applications This theorem applies in a variety of cases. Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X.

(a) If Y is a normal subgroup of X, then $\Delta_Y = \Delta_X \restriction Y$. **P** X/Y has a group structure under which it is a locally compact Hausdorff group (4A5J(b-ii)). It therefore has a left Haar measure, which is surely X-invariant in the sense of 443R. **Q**

Note that in this context any of the invariant measures λ of 443Q must be left Haar measures on the quotient group.

(b) If Y is compact, then $\Delta_Y = \Delta_X \restriction Y$. **P** Both Δ_Y and $\Delta_X \restriction Y$ are continuous homomorphisms from Y to $]0, \infty[$; since the only compact subgroup of $]0, \infty[$ is $\{1\}$, they are both constant with value 1. **Q** So 443R tells us that we have an X-invariant Radon measure λ on X/Y. Since Y has a Haar probability measure, λ will be the image of a left Haar measure under the canonical map (443Qd).

(c) If, in (b), Y is a normal subgroup, then we find that, for $W \in \operatorname{dom}\lambda$ and $x \in X$,

$$\lambda(W \cdot \pi x) = \mu(\pi^{-1}[W] \cdot x) = \Delta_X(x)\lambda W,$$

so that $\Delta_{X/Y}\pi = \Delta_X$, writing $\pi : X \to X/Y$ for the canonical map. This is a special case of 443T below, because (in the terminology there) $\psi(a) = \nu Y/\nu Y = 1$ for every $a \in X$.

(d) If Y is open, $\Delta_Y = \Delta_X \restriction Y$. **P** If μ is a left Haar measure on X, then the subspace measure μ_Y is a left Haar measure on Y (443F). There is an open set $G \subseteq Y$ such that $0 < \mu G < \infty$, and now $\mu(Gy) = \Delta_X(y)\mu G = \Delta_Y(y)\mu G$ for every $y \in Y$. **Q** This time, X/Y is discrete, so counting measure is an X-invariant Radon measure on X/Y.

443T Theorem Let X be a locally compact Hausdorff topological group and Y a closed normal subgroup of X; let $Z = X/Y$ be the quotient group, and $\pi : X \to Z$ the canonical map. Write Δ_X, Δ_Z for the left modular functions of X, Z respectively. Define $\psi : X \to]0, \infty[$ by the formula

$$\nu(aFa^{-1}) = \psi(a)\nu F \text{ whenever } F \in \operatorname{dom}\nu \text{ and } a \in X,$$

where ν is a left Haar measure on Y (cf. 443Pc). Then

$$\Delta_Z(\pi a) = \psi(a)\Delta_X(a)$$

for every $a \in X$.

proof Let $T : C_k(X) \to C_k(Z)$ be the map defined in 443P, and λ a left Haar measure on Z; then, as in 443Qb, we have a left Haar measure μ on X defined by the formula $\int f\,d\mu = \int Tf\,d\lambda$ for every $f \in C_k(X)$. Fix on some $f > 0$ in $C_k(X)$ and $a \in X$, and set $w = \pi a$. By 443Pc, we have

$$T(a\bullet_r f)(z) = \psi(a)(Tf)(\pi(xa)) = \psi(a)(Tf)(zw)$$

whenever $\pi x = z$. So

$$\Delta_X(a^{-1}) \int f\,d\mu = \int a\bullet_r f d\mu$$

(442Kc once more)

$$= \int T(a\bullet_r f) d\lambda = \psi(a) \int (Tf)(zw)\lambda(dz)$$

$$= \psi(a)\Delta_Z(w^{-1}) \int Tf(z)\lambda(dz) = \psi(a)\Delta_Z(w^{-1}) \int f\,d\mu.$$

Thus $\Delta_X(a^{-1}) = \psi(a)\Delta_Z(w^{-1})$; because both Δ_X and Δ_Z are multiplicative,

$$\Delta_Z(\pi a) = \Delta_Z(w) = \psi(a)\Delta_X(a).$$

443U Transitive actions All the results from 443P onwards have been expressed in terms of groups acting on quotient groups. But the same structures can appear if we start from a group action. To simplify the hypotheses, I give the following result for compact groups only.

Theorem Let X be a compact Hausdorff topological group, Z a non-empty compact Hausdorff space, and $\bullet$ a transitive continuous action of X on Z. Write $\pi_z(x) = x\bullet z$ for $z \in Z$ and $x \in X$.

(a) For every $z \in Z$, $Y_z = \{x : x \in X, x\bullet z = z\}$ is a compact subgroup of X. If we give the set X/Y_z of left cosets of Y_z in X its quotient topology, we have a homeomorphism $\phi_z : X/Y_z \to Z$ defined by the formula $\phi_z(xY_z) = x\bullet z$ for every $x \in X$.

(b) Let μ be a Haar probability measure on X. Then the image measure $\mu\pi_z^{-1}$ is an X-invariant Radon probability measure on Z, and $\mu\pi_w^{-1} = \mu\pi_z^{-1}$ for all w, $z \in Z$.

(c) Every non-zero X-invariant Radon measure on Z is of the form $\mu\pi_z^{-1}$ for a Haar measure μ on X and some (therefore any) $z \in Z$.

(d) There is a strictly positive X-invariant Radon probability measure on Z, and any two non-zero X-invariant Radon measures on Z are scalar multiples of each other.

(e) Take any $z \in Z$, and let ν be the Haar probability measure of Y_z. If μ is a Haar measure on X, then

$$\mu E = \int \nu(Y_z \cap x^{-1}E)\mu(dx)$$

whenever $E \subseteq X$ is Haar measurable.

proof (a) Because $\bullet$ is an action of X on Z, Y_z is always a subgroup; because $\bullet$ is continuous, Y_z is closed, therefore compact. Given $z \in Z$, then for x, $y \in X$ we have

$$x\bullet z = y\bullet z \iff x^{-1}y \in Y_z \iff xY_z = yY_z.$$

So the formula given for ϕ_z defines an injection from Z/Y_z to Z, which is surjective because $\bullet$ is transitive. To see that ϕ_z is continuous, take any open set $H \subseteq Z$. Then

$$\{x : xY_z \in \phi_z^{-1}[H]\} = \{x : x\bullet z \in H\} = \pi_z^{-1}[H]$$

is open in X (because $\bullet$ is continuous), so $\phi_z^{-1}[H]$ is open in X/Y_z. Because X/Y_z is compact and ϕ_z is a bijection, ϕ_z is a homeomorphism (3A3Dd).

(b) Because X is compact, therefore unimodular (442Ic), we can speak of 'Haar measures' on X without specifying 'left' or 'right'. If μ is the Haar probability measure on X, then the image measure $\mu\pi_z^{-1}$ is a Radon probability measure on Z (418I once more). To see that the measures $\mu\pi_z^{-1}$ are X-invariant, take any Borel set $H \subseteq Z$ and $y \in X$, and consider

$$\begin{aligned}(\mu\pi_z^{-1})(y^{-1}\bullet H) &= \mu\{x : x\bullet z \in y^{-1}\bullet H\} = \mu\{x : yx\bullet z \in H\} = \mu\{x : yx \in \pi_z^{-1}[H]\} \\ &= \mu(y^{-1}\pi_z^{-1}[H]) = \mu(\pi_z^{-1}[H]) = (\mu\pi_z^{-1})(H).\end{aligned}$$

By 441B, this is enough to ensure that $\mu\pi_z^{-1}$ is invariant.

If w, $z \in Z$ and $H \subseteq Z$ is a Borel set, then there is a $y \in X$ such that $y\bullet w = z$, and now

$$\pi_w^{-1}[H] = \{x : x\bullet w \in H\} = \{x : (xy^{-1})\bullet z \in H\} = \{x : xy^{-1} \in \pi_z^{-1}[H]\} = (\pi_z^{-1}[H])y.$$

But μ is a two-sided Haar measure, so

$$(\mu\pi_w^{-1})(H) = \mu(\pi_w^{-1}[H]) = \mu((\pi_z^{-1}[H])y) = \mu(\pi_z^{-1}[H]) = (\mu\pi_z^{-1})(H).$$

Thus $\mu\pi_w^{-1}$ and $\mu\pi_z^{-1}$ agree on the Borel sets and must be equal (416Eb).

(c) Now we come to the interesting bit. Suppose that λ is a non-zero X-invariant Radon measure on Z. Take any $z \in Z$ and consider the Radon measure λ' on X/Y_z got by setting $\lambda' H = \lambda\phi_z[H]$ whenever $H \subseteq X/Y_z$ and $\phi_z[H]$ is measured by λ. In this case, if $x \in X$ and $H \subseteq X/Y_z$ is measured by λ',

$$\begin{aligned}\lambda'(x\bullet H) &= \lambda\{\phi_z(x\bullet w) : w \in H\} = \lambda\{\phi_z(x\bullet yY_z) : y \in X,\, yY_z \in H\} \\ &= \lambda\{\phi_z(xyY_z) : y \in X,\, yY_z \in H\} = \lambda\{xy\bullet z : y \in X,\, yY_z \in H\} \\ &= \lambda\{x\bullet(y\bullet z) : y \in X,\, yY_z \in H\} = \lambda(x\bullet\phi_z[H]) = \lambda\phi_z[H]\end{aligned}$$

(because λ is X-invariant)

$$= \lambda' H.$$

So λ' is X-invariant.

Now let ν be the Haar probability measure on the compact Hausdorff group Y_z. By 443Qb, we have a (left) Haar measure μ on X defined by the formula $\mu G = \int \nu_w G\,\lambda'(dw)$ for every open $G \subseteq X$, where $\nu_{xY_z} G = \nu(Y_z \cap x^{-1}G)$ for every $y \in X$ and every open $G \subseteq X$. Let $H \subseteq Z$ be an open set. Then for any $x \in X$,

$$\begin{aligned}\nu_{xY_z}(\pi_z^{-1}[H]) &= \nu\{y : y \in Y_z,\, xy \in \pi_z^{-1}[H]\} = \nu\{y : y \in Y_z,\, xy\bullet z \in H\} \\ &= \nu\{y : y \in Y_z,\, x\bullet(y\bullet z) \in H\} = \nu\{y : y \in Y_z,\, x\bullet z \in H\} \\ &= \nu Y_z = 1 \text{ if } x\bullet z \in H, \\ &= \nu\emptyset = 0 \text{ otherwise.}\end{aligned}$$

So

$$\mu(\pi_z^{-1}[H]) = \lambda'\{xY_z : x \in X,\, x\bullet z \in H\} = \lambda'\phi_z^{-1}[H] = \lambda H.$$

As H is arbitrary, the image measure $\mu\pi_z^{-1}$ agrees with λ on the open subsets of Z; as they are both Radon measures, $\mu\pi_z^{-1} = \lambda$, as required.

(d) This is now easy. X carries a non-zero Haar measure, so by (b) there is an X-invariant Radon probability measure on Z. If λ_1 and λ_2 are non-zero X-invariant Radon measures on Z, then they are of the form $\mu_1\pi_w^{-1}$ and $\mu_2\pi_z^{-1}$ where μ_1 and μ_2 are Haar measures on X and w, $z \in Z$. By (b) again, $\mu_1\pi_w^{-1} = \mu_1\pi_z^{-1}$, and since μ_1 and μ_2 are multiples of each other (442B), so are λ_1 and λ_2.

To see that the invariant probability measure λ on X is strictly positive, take any non-empty open set $H \subseteq Z$. Then Z is covered by the open sets $x\bullet H$, as x runs over X. Because Z is compact, it is covered by finitely many of these, so at least one of them has non-zero measure. But they all have the same measure as H, so $\lambda H > 0$.

(e) Write $\theta : X \to X/Y_z$ for the canonical map. For $w \in X/Y_z$ we have a Radon measure ν_w on X defined by setting $\nu_w E = \nu(Y_z \cap x^{-1}E)$ whenever $\theta x = w$ and the right-hand side is defined (443Qa). By (a)-(b) above, or otherwise, there is a non-zero X-invariant Radon measure λ on X/Y_z; re-scaling if necessary, we may suppose that $\lambda(X/Y_z) = \mu X$. By 443Qe, we have a Haar measure μ' on X defined by setting $\mu' E = \int \nu_w E\,\lambda(dw)$ for every Haar measurable E; since

$$\mu' X = \int \nu_w X\,\lambda(dw) = \int \nu Y_z\,\lambda(dw) = \lambda(X/Y_z) = \mu X,$$

$\mu' = \mu$. Moreover, $\lambda = \mu'\theta^{-1}$ (443Qd). So

$$\begin{aligned}\mu E = \mu' E &= \int \nu_w E\,\lambda(dw)\\ &= \int \nu_{\theta x} E\,\mu'(dx) = \int \nu(Y_z \cap x^{-1}E)\mu(dx)\end{aligned}$$

for every Haar measurable $E \subseteq X$.

443X Basic exercises >(a) Let X be a topological group, μ a left Haar measure on X and λ the corresponding quasi-Radon product measure on $X \times X$. (i) Show that the maps $(x, y) \mapsto (y, x)$, $(x, y) \mapsto (x, xy)$, $(x, y) \mapsto (y^{-1}x, y)$ are automorphisms of the measure space $(X \times X, \lambda)$. (*Hint*: use 417C(iv) to show that they preserve the measures of open sets.) (ii) Show that the maps $(x, y) \mapsto (y^{-1}, xy)$, $(x, y) \mapsto (yx, x^{-1})$, $(x, y) \mapsto (y^3x, x^{-1}y^{-2})$ are automorphisms of $(X \times X, \lambda)$. (*Hint*: express them as compositions of maps of the forms in (ii).)

(b) Let X be a topological group carrying Haar measures and $A \subseteq X$. (i) Show that A is self-supporting (definition: 411Na) for one Haar measure on X iff it is self-supporting for every Haar measure on X. (ii) Show that A has non-zero inner measure for one Haar measure on X iff it has non-zero inner measure for every Haar measure on X.

(c) Let X be a topological group, μ a Haar measure on X, and E, F measurable subsets of X. Show that $(x, y, w, z) \mapsto \mu(xEy \cap wFz) : X^4 \to [0, \infty]$ is lower semi-continuous.

(d) Let X be a topological group carrying Haar measures and $\mathfrak{A}$ its Haar measure algebra. (i) Show that we have a continuous action of $X \times X$ on $\mathfrak{A}$ defined by the formula $(x, y)\bullet E^\bullet = (xEy^{-1})^\bullet$ for x, $y \in X$ and Haar measurable sets $E \subseteq X$. (ii) Show that if $x \in X$ and $a \in \mathfrak{A}$ then $x\bullet_r a = (x\bullet_l \overset{\leftrightarrow}{a})^{\leftrightarrow}$, where $\overset{\leftrightarrow}{a}$ is as defined in 443Af.

(e) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Show that it is isomorphic to the measure algebra of a topological group with a Haar measure iff it is localizable and quasi-homogeneous in the sense of 374G-374H.

(f) Let X be a topological group with a left Haar measure μ. (i) Show that if Y is a subgroup of X such that $\mu_*Y > 0$, then Y is open. In particular, any non-negligible closed subgroup of X is open. (ii) Let Y be any subgroup of X which is not Haar negligible. Show that the subspace measure μ_Y is a left Haar measure on Y. Show that $\overline{Y}$ is a Haar measurable envelope of Y. (*Hint*: apply 443Db inside the topological group $\overline{Y}$.)

(g) Write out a version of 443G for right Haar measures.

(h) Let X be a topological group carrying Haar measures, and L^0 the space of equivalence classes of Haar measurable functions, as in 443A; let $u \mapsto \overset{\leftrightarrow}{u} : L^0 \to L^0$ be the operator of 443Af. Show that if μ is a left Haar measure on X and ν is a right Haar measure, $p \in [1, \infty]$ and $u \in L^0$, then $\overset{\leftrightarrow}{u} \in L^p(\nu)$ iff $u \in L^p(\mu)$.

(i) Let X be a topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra. Show that, in the language of 443C and 443G, $\chi(x\bullet_l a) = x\bullet_l \chi a$ and $\chi(x\bullet_r a) = x\bullet_r \chi a$ for every $x \in X$ and $a \in \mathfrak{A}$.

(j) Let X be a topological group carrying Haar measures. Show that X is totally bounded for its bilateral uniformity iff X is totally bounded for its right uniformity (definition: 4A5Ha) iff its Haar measures are totally finite.

(k) Let X be a topological group, μ a left Haar measure on X, and $A \subseteq X$ a set which is self-supporting for μ. Show that the following are equiveridical: (i) for every neighbourhood U of the identity e, there is a countable set $I \subseteq X$ such that $A \subseteq UI$; (ii) for every neighbourhood U of e, there is a countable set $I \subseteq X$ such that $A \subseteq IU$; (iii) for every neighbourhood U of e, there is a countable set $I \subseteq X$ such that $A \subseteq IUI$; (iv) A can be covered by countably many sets of finite measure for μ; (v) A can be covered by countably many open sets of finite measure for μ; (vi) A can be covered by countably many sets which are totally bounded for the bilateral uniformity on X.

>(l) Let X be a topological group carrying Haar measures. (i) Show that the following are equiveridical: (α) X is ccc; (β) X has a σ-finite Haar measure; (γ) every Haar measure on X is σ-finite. (ii) Show that if X is locally compact and Hausdorff, we can add (δ) X is σ-compact.

(m) Let X be a topological group carrying Haar measures. Show that every subset of X has a Haar measurable envelope which is a Borel set.

(n) In 443L, show that (i) $\phi[A]$ is Haar negligible in Z whenever A is Haar negligible in X (ii) $\Delta_X = \Delta_Z\phi$, where Δ_X, Δ_Z are the left modular functions of X, Z respectively (iii) $\phi[X]$ is dense in Z (iv) Z is unimodular iff X is unimodular.

>(o) Let X and Y be topological groups with left Haar measures μ and ν. Show that the c.l.d. and quasi-Radon product measures of μ and ν on $X\times Y$ coincide. (*Hint*: start with locally compact Hausdorff spaces, and show that a compact G_δ set in $X\times Y$ belongs to $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$, where $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ are the Borel σ-algebras of X and Y; now use 441Xi and 443L.)

(p) Let $\langle X_i\rangle_{i\in I}$ be a family of topological groups and $X=\prod_{i\in I}X_i$ their product. Suppose that each X_i has a Haar probability measure μ_i. Show that the ordinary and quasi-Radon product measures on X coincide.

(q) Let X be a locally compact Hausdorff group and Y a closed subgroup of X; write X/Y for the space of left cosets of Y in X, with its quotient topology. Show that if λ_1 and λ_2 are non-zero X-invariant Radon measures on X/Y, then each is a multiple of the other. (*Hint*: look at the Haar measures they define on X.)

>(r) Write S^1 for the circle group $\{s: s\in\mathbb{C}, |s|=1\}$, and set $X=S^1\times S^1$, where the first copy of S^1 is given its usual topology and the second copy is given its discrete topology, so that X is an abelian locally compact Hausdorff group. Set $E=\{(s,s): s\in S^1\}$. (i) Show that E is a closed Haar negligible subset of X. (ii) Set $Y=\{(1,s): s\in S^1\}$; check that Y is a closed normal subgroup of X, and that the quotient group X/Y can be identified with S^1 with its usual topology; let λ be the Haar probability measure of X/Y. Let ν be counting measure on Y. Show that, in the language of 443Q, $\nu_zE=1$ for every $z\in X/Y$, so that $\mu E\ne\int\nu_zE\,\lambda(dz)$. (iii) Setting $f=\chi E$, show that the map $a\mapsto f_a^\bullet$ described in 443Qf is not almost continuous.

(s)(i) In 443P, suppose that $G\subseteq X$ is an open set such that $GY=X$. Show that for every $h\in C_k(Z)^+$ there is an $f\in C_k(X)^+$ such that $Tf=h$ and $\overline{\{x: f(x)>0\}}\subseteq G$. (ii) In 443Pc, show that ψ is multiplicative. (iii) In 443R, suppose that there is an open set $G\subseteq X$ such that GY has finite measure for the left Haar measures of X. Show that Z has an X-invariant Radon probability measure. (*Hint*: Y is totally bounded for its right uniformity.)

(t) Let X be a locally compact Hausdorff group. Show that it has a closed normal subgroup Y such that Y and X/Y are both unimodular. (*Hint*: take $Y=\{x:\Delta(x)=1\}$.)

>(u) Let $X=\mathbb{R}^2$ be the example of 442Xf. (i) Let Y_1 be the subgroup $\{(\xi,0):\xi\in\mathbb{R}\}$. Describe the left cosets of Y_1 in X. Show that there is no non-trivial X-invariant Radon measure on the set X/Y_1 of these left cosets. Find a base $\mathcal{U}$ for the topology of X/Y_1 such that you can identify the sets $x\bullet U$, where $x\in X$ and $U\in\mathcal{U}$, with sufficient precision to explain why the hypothesis (iii) of 441C is not satisfied. (ii) Let Y_2 be the normal subgroup $\{(0,\xi):\xi\in\mathbb{R}\}$. Find the associated function $\psi: X\to\,]0,\infty[$ as described in 443Pc and 443T.

(v) Let X be a locally compact Hausdorff group and Y a compact normal subgroup of X. Show that X is unimodular iff the quotient group X/Y is unimodular. (*Hint*: the function ψ of 443T must be constant.)

(w) Take any integer $r\ge 1$, and let G be the isometry group of $\mathbb{R}^r$ with its topology of pointwise convergence (441G). (i) Show that G is metrizable and locally compact. (*Hint*: 441Xp.) (ii) Let $H\subseteq G$ be the set of translations. Show that H is an abelian closed normal subgroup of G, and that Lebesgue measure on $\mathbb{R}^r$ can be regarded as a Haar measure on H. (iii) Show that the quotient group G/H is compact. (iv) Show that G is unimodular. (*Hint*: the function ψ of 443T is constant.)

>(x) Set $X=\mathbb{R}^3$ with the operation

$$(\xi_1,\xi_2,\xi_3)*(\eta_1,\eta_2,\eta_3)=(\xi_1+\eta_1,\xi_2+e^{\xi_1}\eta_2,\xi_3+e^{-\xi_1}\eta_3).$$

(i) Show that (with the usual topology of $\mathbb{R}^3$) X is a topological group. (ii) Show that it is unimodular. (*Hint*: Lebesgue measure is a two-sided Haar measure.) (iii) Show that X has both a closed subgroup and a Hausdorff quotient group which are not unimodular.

>(y) Let (X,ρ) be a non-empty compact metric space such that the group G of isometries of X is transitive. Show that any two non-zero G-invariant Radon measures on X must be multiples of each other. (*Hint*: 441Gb, 443U.)

>(z) Show that 443G is equally valid if we take functions to be complex-valued rather than real-valued, and work with $L^p_{\mathbb{C}}$ rather than L^p.

443Y Further exercises (a) Let X be a topological group carrying Haar measures and $\mathfrak{A}$ its Haar measure algebra. Show that two principal ideals of $\mathfrak{A}$ are isomorphic (as Boolean algebras) iff they have the same cellularity.

(b) Let X be a topological group carrying Haar measures, and $E\subseteq X$ a Haar measurable set such that $E\cap U$ is not Haar negligible for any neighbourhood U of the identity. Show that for any $A\subseteq X$ the set $A'=\{x:x\in A,\ A\cap xE$ is Haar negligible$\}$ is Haar negligible.

(c) Let X be a locally compact Hausdorff group. Show that we have continuous shift actions $\bullet_l$, $\bullet_r$ and $\bullet_c$ of X on the Banach space $C_0(X)$ defined by formulae corresponding to those of 443G.

(d) Let X be a compact Hausdorff topological group and $\mathfrak{A}$ its Haar measure algebra. Let Y be a subgroup of X; for $y\in Y$, define $\hat{y}\in\operatorname{Aut}\mathfrak{A}$ by setting $\hat{y}(a)=y{\bullet_l}a$ for $a\in\mathfrak{A}$. Show that $\{\hat{y}:y\in Y\}$ is ergodic (definition: 395Ge) iff Y is dense in X.

(e) Let $\mathfrak{A}$ be a Boolean algebra, G a group, and $\bullet$ an action of G on $\mathfrak{A}$ such that $a\mapsto g{\bullet}a$ is a Boolean automorphism for every $g\in G$. (i) Show that we have a corresponding action of G on $L^\infty=L^\infty(\mathfrak{A})$ defined by saying that, for every $g\in G$, $g{\bullet}\chi a=\chi(g{\bullet}a)$ for $a\in\mathfrak{A}$ and $u\mapsto g{\bullet}u$ is a positive linear operator on L^∞. (ii) Show that if $\mathfrak{A}$ is Dedekind σ-complete, this action on L^∞ extends to an action on $L^0=L^0(\mathfrak{A})$ defined by saying that $[\![g{\bullet}u>\alpha]\!]=g{\bullet}[\![u>\alpha]\!]$ for $g\in G$, $u\in L^0$ and $\alpha\in\mathbb{R}$.

(f) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, G a topological group, and $\bullet$ a continuous action of G on $\mathfrak{A}$ (when $\mathfrak{A}$ is given its measure-algebra topology) such that $a\mapsto g{\bullet}a$ is a measure-preserving Boolean automorphism for every $g\in G$. (i) Show that the corresponding action of G on $L^0=L^0(\mathfrak{A})$, as defined in 443Ye, is continuous when L^0 is given the topology of convergence in measure, and induces continuous actions of G on $L^p=L^p(\mathfrak{A},\bar{\mu})$ for $1\le p<\infty$. (ii) Show that if we give the unit ball B of $L^\infty=L^\infty(\mathfrak{A})$ the topology induced by $\mathfrak{T}_s(L^\infty,L^1)$, then the action of G on L^0 induces a continuous action of G on B.

(g) Let X be a topological group with a left Haar measure μ, and $A\subseteq X$. Show that the following are equiveridical: (i) A is totally bounded for the bilateral uniformity of X (ii) there are non-empty open sets G, $H\subseteq X$ such that $\mu(AG)$, $\mu(A^{-1}H)$ are both finite.

(h) Give an example of a locally compact Hausdorff group, with left Haar measure μ, such that no open normal subgroup can be covered by a sequence of sets of finite measure for μ.

(i) Let X be a topological group. Let Σ be the family of subsets of X expressible in the form $\phi^{-1}[F]$ for some Borel subset F of a separable metrizable topological group Y and some continuous homomorphism $\phi:X\to Y$. Show that Σ is a σ-algebra of subsets of X and that multiplication, regarded as a function from $X\times X$ to X, is $(\Sigma\widehat{\otimes}\Sigma,\Sigma)$-measurable. Show that any compact G_δ set belongs to Σ. Show that if X is σ-compact, then Σ is the Baire σ-algebra of X.

(j) Let X be any Hausdorff topological group of cardinal greater than $\mathfrak{c}$. Let $\mathcal{B}$ be the Borel σ-algebra of X. Show that $(x,y)\mapsto xy$ is not $(\mathcal{P}X\widehat{\otimes}\mathcal{P}X,\mathcal{B})$-measurable.

(k) Let X be a topological group and μ a left Haar measure on X. Show that μ is inner regular with respect to the family of closed sets $F\subseteq X$ such that $F=\bigcap_{n\in\mathbb{N}}FU_n$ for some sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of the identity.

(l) Let X be a topological group carrying Haar measures. Let $E\subseteq X$ be a Haar measurable set such that $E\cap U$ is not Haar negligible for any neighbourhood U of the identity. Show that there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in E such that $x_{i_0}x_{i_1}\ldots x_{i_n}\in E$ whenever $n\in\mathbb{N}$ and $i_0<i_1<\ldots<i_n$ in $\mathbb{N}$. (See Plewik & Voigt 91.)

(m) Let X be a topological group and μ a Haar measure on X. Show that any closed self-supporting subset of X is a zero set.

(n) Find a compact Hausdorff space X with a strictly positive Radon measure such that there is a regular open set $G \subseteq X$ which is not a cozero set.

(o) Let X be a locally compact Hausdorff topological group which is not discrete (as topological space). (i) Show that there is a Haar negligible zero set containing the identity of X. (ii) Show that if X is σ-compact, it has a Haar negligible compact normal subgroup Y which is a zero set in X, so that X/Y is metrizable. (iii) Show that there is a Haar negligible set $A \subseteq X$ such that AA is not Haar measurable.

(p) Find a non-discrete locally compact Hausdorff topological group X such that if Y is a normal subgroup of X which is a zero set in X then Y is open.

(q) Show that there is a subgroup X of the additive group $\mathbb{R}^2$ such that X has full outer Lebesgue measure but $\{\xi : (\xi, 0) \in X\} = \mathbb{Q}$. Show that X carries Haar measures, but that its closed subgroup $X \cap (\mathbb{R} \times \{0\})$ does not.

(r) Show that if X is a locally compact Hausdorff group, Y a compact subgroup of X, $Z = X/Y$ the set of left cosets of Y and $\pi : X \to Z$ the canonical map, and Z is given its quotient topology, then $R = \{(\pi x, x) : x \in X\}$ is an usco-compact relation in $Z \times X$ and $R[L] = \pi^{-1}[L]$ is compact for every compact $L \subseteq Z$.

(s) Let G be the isometry group of $\mathbb{R}^r$, as in 443Xw. (i) Show that if we set $\rho(g, h) = \sup_{\|x\| \le 1} \|g(x) - h(x)\|$, then ρ is a metric on G defining its topology. (ii) Describe Haar measures on G (α) in terms of Hausdorff measure of an appropriate dimension for the metric ρ (β) in terms of a parametrization of G and Lebesgue measure on a suitable Euclidean space.

(t) Let $r \ge 1$ be an integer, and G the group of invertible affine transformations of $\mathbb{R}^r$, with the topology of pointwise convergence inherited from $(\mathbb{R}^r)^{\mathbb{R}^r}$. (i) Show that G is a locally compact Hausdorff topological group. (ii) Show that G is not unimodular, and find its modular functions.

(u) Let X be a topological group with a left Haar measure μ and left modular function Δ; suppose that X is not unimodular. Show that $\mu\{x : \alpha < \Delta(x) < \beta\} = \infty$ whenever $\alpha < \beta$ and $\{x : \alpha < \Delta(x) < \beta\}$ is non-empty.

443 Notes and comments Most of us, by the time we come to study measures on general topological groups, have come to trust our intuition concerning the behaviour of Lebesgue measure on $\mathbb{R}$, and the principal discipline imposed by the subject is the search for the true path between hopelessness and over-confidence when extending this intuition to the general setting. The biggest step is the loss of commutativity, especially as the non-abelian groups of elementary courses in group theory are mostly finite, and are therefore untrustworthy guides to the concerns of this chapter. Accordingly we find ourselves going rather slowly and carefully through the calculations in such results as 443C. Note, for instance, that in an abelian group the actions I call $\bullet_l$ and $\bullet_r$ are still different; $x \bullet_l E$ corresponds to $x + E = E + x$, but $x \bullet_r E$ corresponds to $E - x$. (The action $\bullet_c$ becomes trivial, of course.) When we come to translate the formulae of Fourier analysis in the next section, manoeuvres of this kind will often be necessary. In the present section, I have done my best to give results in 'symmetric' forms; you may therefore take it that when the words 'left' or 'right' appear in the statement of a proposition, there is some real need to break the symmetry. Subject to these remarks, such results as 443C and 443G are just a matter of careful conventional analysis. I see that I have used slightly different techniques in the two cases. Of course 443C can be thought of as a special case of 443Gc-443Gd, if we remember that $\chi : \mathfrak{A} \to L^0(\mathfrak{A})$ embeds $\mathfrak{A}$ topologically as a subspace of L^0 (367R).

443B, 443D and 443E belong to a different family; they deal with actual sets rather than with members of a measure algebra or a function space. I suppose it is 443D which will most often be quoted. Its corollary 443E deals with the obvious question of when Haar measures are tight.

Readers who have previously encountered the theory of Haar measures on locally compact groups will have been struck by how closely the more general theory here is able to follow it. The explanation lies in 443K-443L; my general Haar measures are really just subspace measures on subgroups of full Haar measure in locally compact groups. Knowing this, it is easy to derive all the results above from the locally compact theory. I use this method only in 443M-443O, because (following Halmos 50) I feel that from the point of view of pure measure theory the methods show themselves more clearly if we do not use ideas depending on compactness, but instead rely directly on τ-additivity. But I note that 443Xo and 443Yi also go faster with the aid of 443L.

Halmos 50 goes a little farther, with a theory of groups carrying translation-invariant measures for which the operation $(x,y) \mapsto xy$ is $(\Sigma\widehat{\otimes}\Sigma, \Sigma)$-measurable, where Σ is the domain of the measure. The essence of the theory here, and my reason for insisting on τ-additive measures, is that for these we have a suitable theory of product measures. If we start with quasi-Radon measures and use the quasi-Radon product measure, then multiplication is measurable just because it is continuous. In order to achieve similar results without either assuming metrizability or using the theory of τ-additive product measures, we have to restrict the measure to something smaller than the Borel σ-algebra, as in 443Yg. (See 443Yj.) 443J and 443Xo show that certain disconcerting features of general Radon measures (419C, 419E) cannot arise in the case of Haar measures.

When I come to look at subgroups and quotient groups, I do specialize to the locally compact case. One obstacle to generalization is the fact that a closed subgroup of a group carrying Haar measures need not itself carry Haar measures (443Yq). 443P and 443R are taken from Federer 69, who goes farther, with many other applications. I give a fairly detailed analysis of the relationship between a Haar measure on a group X and a corresponding X-invariant measure on a family of left cosets (443Q) for the sake of applications in §447. One of the challenges here is to distinguish clearly those results which apply to all locally compact groups from those which rely on σ-compactness or some such limitation. There is a standard example (443Xr) which provides a useful test in such questions. You may recognise this as a version of a fundamental example related to Fubini's theorem, given in 252K.

Most of the results of this section begin with a topological group X. But starting from §441 it is equally natural to start with a group action. If we have a topological group X acting continuously on a topological space Z, and X carries Haar measures, then we have a good chance of finding an invariant measure on Z. In the simplest case, in which X and Z are both compact and the action is transitive, there is a unique invariant Radon probability measure on Z (443U).

444 Convolutions

In this section, I look again at the ideas of §§255 and 257, seeking the appropriate generalizations to topological groups other than $\mathbb{R}$. Following Hewitt & Ross 63, I begin with convolutions of measures (444A-444E) before proceeding to convolutions of functions (444O-444V); in between, I mention the convolution of a function and a measure (444G-444M) and a general result concerning continuous group actions on quasi-Radon measure spaces (444F).

While I continue to give the results in terms of real-valued functions, the applications of the ideas here in the next section will be to complex-valued functions; so you may wish to keep the complex case in mind (444Xx).

444A Convolution of measures: Proposition If X is a topological group and λ and ν are two totally finite quasi-Radon measures on X, we have a quasi-Radon measure $\lambda * \nu$ on X defined by saying that

$$\begin{aligned}(\lambda * \nu)(E) &= (\lambda \times \nu)\{(x,y) : xy \in E\}\\ &= \int \nu(x^{-1}E)\lambda(dx) = \int \lambda(Ey^{-1})\nu(dy)\end{aligned}$$

for every $E \in \operatorname{dom}(\lambda * \nu)$, where $\lambda \times \nu$ is the quasi-Radon product measure on $X \times X$.

proof Set $\phi(x,y) = xy$ for $x, y \in X$. Then ϕ is continuous, while X, being a topological group, is regular (4A5Ha); so so 418Hb tells us that there is a unique quasi-Radon measure $\lambda * \nu$ on X such that ϕ is inverse-measure-preserving for $\lambda \times \nu$ and $\lambda * \nu$, that is, $(\lambda * \nu)(E) = (\lambda \times \nu)\{(x,y) : xy \in E\}$ whenever E is measured by $\lambda * \nu$.

As for the other formulae, Fubini's theorem (417Ha) tells us that

$$\begin{aligned}(\lambda * \nu)(E) &= (\lambda \times \nu)\phi^{-1}[E]\\ &= \int \nu(\phi^{-1}[E])[\{x\}]\lambda(dx) = \int \nu(x^{-1}E)\lambda(dx)\\ &= \int \lambda(\phi^{-1}[E])^{-1}[\{y\}]\nu(dy) = \int \lambda(Ey^{-1})\nu(dy)\end{aligned}$$

for any $E \in \operatorname{dom}(\lambda * \nu)$.

444B Proposition If X is a topological group, $\lambda_1 * (\lambda_2 * \lambda_3) = (\lambda_1 * \lambda_2) * \lambda_3$ for all totally finite quasi-Radon measures λ_1, λ_2 and λ_3 on X.

proof If $E \subseteq X$ is Borel, then

$$\begin{aligned}(\lambda_1 * (\lambda_2 * \lambda_3))(E) &= \int (\lambda_2 * \lambda_3)(x^{-1}E)\lambda_1(dx) \\ &= \int (\lambda_2 \times \lambda_3)\{(y,z) : xyz \in E\}\lambda_1(dx) \\ &= (\lambda_1 \times (\lambda_2 \times \lambda_3))\{(x,(y,z)) : xyz \in E\} \\ &= ((\lambda_1 \times \lambda_2) \times \lambda_3)\{((x,y),z) : xyz \in E\} \\ &= \int (\lambda_1 \times \lambda_2)\{(x,y) : xyz \in E\}\lambda_3(dz) \\ &= ((\lambda_1 * \lambda_2) * \lambda_3)(E).\end{aligned}$$

(For the central identification between $\lambda_1 \times (\lambda_2 \times \lambda_3)$ and $(\lambda_1 \times \lambda_2) \times \lambda_3$, observe that as both are quasi-Radon measures it is enough to check that they agree on sets of the type $G_1 \times (G_2 \times G_3) \cong (G_1 \times G_2) \times G_3$ where G_1, G_2 and G_3 are open, as in 417J.)

So $\lambda_1 * (\lambda_2 * \lambda_3)$ and $(\lambda_1 * \lambda_2) * \lambda_3$ agree on the Borel sets and must be identical.

444C Theorem Let X be a topological group and λ, ν two totally finite quasi-Radon measures on X. Then

$$\int f d(\lambda * \nu) = \int f(xy)(\lambda \times \nu)d(x,y) = \iint f(xy)\lambda(dx)\nu(dy) = \iint f(xy)\nu(dy)\lambda(dx)$$

for any $(\lambda * \nu)$-integrable real-valued function f. In particular, $(\lambda * \nu)(X) = \lambda X \cdot \nu X$.

proof If f is of the form χE, so that

$$\int f d(\lambda * \nu) = (\lambda * \nu)(E), \qquad \int f(xy)(\lambda \times \nu)d(x,y) = (\lambda \times \nu)\{(x,y) : xy \in E\},$$

$$\iint f(xy)\lambda(dx)\nu(dy) = \int \lambda(Ey^{-1})\nu(dy), \qquad \iint f(xy)\nu(dy)\lambda(dx) = \int \nu(x^{-1}E)\lambda(dx)$$

this is covered by the result in 444A. Now it is easy to run through the standard progression to the cases of (i) simple functions (ii) non-negative Borel measurable functions defined everywhere (iii) functions defined, and zero, almost everywhere (iv) non-negative integrable functions and (v) arbitrary integrable functions.

444D Proposition Let X be an abelian topological group. Then $\lambda * \nu = \nu * \lambda$ for all totally finite quasi-Radon measures λ, μ on X.

proof For any Borel set $E \subseteq X$,

$$\begin{aligned}(\lambda * \nu)(E) &= (\lambda \times \nu)\{(x,y) : xy \in E\} = (\nu \times \lambda)\{(y,x) : xy \in E\} \\ &= (\nu \times \lambda)\{(y,x) : yx \in E\} = (\nu * \lambda)(E).\end{aligned}$$

444E The Banach algebra of τ-additive measures (a) Let X be a topological group. Recall from 437Ab that we have a band $C_b(X)^\sim_\tau$ in the L-space $C_b(X)^\sim$ consisting of those order-bounded linear functionals $f : C_b(X) \to \mathbb{R}$ such that $|f|$ is smooth (equivalently, f^+ and f^- are both smooth); that is, such that $|f|$, f^+ and f^- can be represented by totally finite quasi-Radon measures on X. Because X is completely regular (4A5Ha again), $C_b(X)^\sim_\tau$ can be identified with the band M_τ of signed τ-additive Borel measures on X, that is, the set of those countably additive functionals ν defined on the Borel σ-algebra of X such that $|\nu|$ is τ-additive (437G).

(b) For any τ-additive totally finite Borel measures λ, ν on X we can define their convolution $\lambda*\nu$ by the formulae of 444A, that is,

$$(\lambda * \nu)(E) = \int \nu(x^{-1}E)\lambda(dx) = \int \lambda(Ey^{-1})\nu(dy)$$

for any Borel set $E \subseteq X$, if we note that the completions $\hat{\lambda}$, $\hat{\nu}$ of λ and ν are quasi-Radon measures (415Cb), so that $\lambda * \nu$, as defined by these formulae, is just the restriction of $\hat{\lambda} * \hat{\nu}$, as defined in 444A, to the Borel σ-algebra. Now the formulae make it obvious that the map $*$ is bilinear in the sense that

$$(\lambda_1 + \lambda_2) * \nu = \lambda_1 * \nu + \lambda_2 * \nu,$$

$$\lambda * (\nu_1 + \nu_2) = \lambda * \nu_1 + \lambda * \nu_2,$$

$$(\alpha\lambda) * \nu = \lambda * (\alpha\nu) = \alpha(\lambda * \nu)$$

for all totally finite τ-additive Borel measures λ, λ_1, λ_2, ν, ν_1, ν_2 and all $\alpha \ge 0$. Consequently, regarding elements of M_τ as functionals on the Borel σ-algebra, we have a bilinear operator $* : M_\tau \times M_\tau \to M_\tau$ defined by saying that

$$(\lambda_1 - \lambda_2) * (\nu_1 - \nu_2) = \lambda_1 * \nu_1 - \lambda_1 * \nu_2 - \lambda_2 * \nu_1 + \lambda_2 * \nu_2$$

for all λ_1, λ_2, ν_1, $\nu_2 \in M_\tau^+$.

(c) We see from 444B that $*$ is associative on M_τ^+, so it will be associative on the whole of M_τ. Observe that $*$ is positive in the sense that $\lambda * \nu \ge 0$ if λ, $\nu \ge 0$; so that

$$\begin{aligned}|\lambda * \nu| &= |\lambda^+ * \nu^+ - \lambda^+ * \nu^- - \lambda^- * \nu^+ + \lambda^- * \nu^-|\\ &\le \lambda^+ * \nu^+ + \lambda^+ * \nu^- + \lambda^- * \nu^+ + \lambda^- * \nu^-\\ &= |\lambda| * |\nu|\end{aligned}$$

for any λ, $\nu \in M_\tau$.

(d) If λ, $\nu \in M_\tau^+$ then $\|\lambda\| = \lambda X$ and $\|\nu\| = \nu X$ (362Ba), so

$$\|\lambda * \nu\| = (\lambda * \nu)(X) = (\lambda \times \nu)(X \times X) = \lambda X \cdot \nu X = \|\lambda\|\|\nu\|.$$

Generally, for any λ, $\nu \in M_\tau$,

$$\|\lambda * \nu\| = \||\lambda * \nu|\| \le \||\lambda| * |\nu|\| = \||\lambda|\|\||\nu|\| = \|\lambda\|\|\nu\|.$$

Thus M_τ is a Banach algebra under the operation $*$, as well as being an L-space. If X is abelian then M_τ will be a commutative algebra, by 444D.

444F In preparation for the next construction I give a general result extending ideas already touched on in 443C and 443G.

Theorem Let X be a topological space, G a topological group and $\bullet$ a continuous action of G on X. For $A \subseteq X$, $a \in G$ write $a\bullet A = \{a\bullet x : x \in A\}$. Let ν be a measure on X.

(a) If $f : X \to [0,\infty]$ is lower semi-continuous, then $a \mapsto \int a\bullet f\,d\nu : G \to [0,\infty]$ is lower semi-continuous. (See 4A5Cc for the definition of $a\bullet f$.) In particular, if $V \subseteq X$ is open, then $a \mapsto \nu(a\bullet V) : G \to [0,\infty]$ is lower semi-continuous.

(b) If $f : X \to \mathbb{R}$ is continuous, then $a \mapsto (a\bullet f)^\bullet : G \to L^0$ is continuous, if $L^0 = L^0(\nu)$ is given the topology of convergence in measure.

(c) If ν is σ-finite and $E \subseteq X$ is a Borel set, then $a \mapsto (a\bullet E)^\bullet : G \to \mathfrak{A}$ is Borel measurable, if the measure algebra $\mathfrak{A}$ of ν is given its measure-algebra topology.

(d) If ν is σ-finite and $f : X \to \mathbb{R}$ is Borel measurable, then $a \mapsto (a\bullet f)^\bullet : G \to L^0$ is Borel measurable.

(e) If ν is σ-finite, then

(i) $a \mapsto \nu(a\bullet E) : G \to [0,\infty]$ is Borel measurable for any Borel set $E \subseteq X$;

(ii) if $f : X \to \mathbb{R}$ is Borel measurable, then $Q = \{a : \int a\bullet f\,d\nu$ is defined in $[-\infty,\infty]\}$ is a Borel set, and $a \mapsto \int a\bullet f\,d\nu : Q \to [-\infty,\infty]$ is Borel measurable.

proof (a)(i) Note first that if $f : X \to [0,\infty]$ is Borel measurable, then, for each $a \in G$, $a\bullet f$ is the composition of f with the continuous function $x \mapsto a^{-1}\bullet x$, so is Borel measurable, and if f is finite-valued then $(a\bullet f)^\bullet$ is defined in $L^0 = L^0(\nu)$.

(ii) Suppose that $f : X \to [0,\infty]$ is lower semi-continuous, $\gamma \in [0,\infty[$, $a \in G$ and $\int a\bullet f > \gamma$. Let $\mathcal{U}$ be the set of open neighbourhoods of a in G. For $U \in \mathcal{U}$, $x \in X$ set

$$\phi_U(x) = \sup\{\inf_{b\in U, y\in V}(b\bullet f)(y) : V \text{ is an open neighbourhood of } x \text{ in } X\}.$$

Then ϕ_U is lower semi-continuous. **P** If $\phi_U(x) > \alpha$, there is an open neighbourhood V of x such that $\inf_{b\in U,y\in V}(b\bullet f)(y) > \alpha$; now $\phi_U(y) > \alpha$ for every $y \in V$. **Q** If $U' \subseteq U$ then $\phi_{U'} \ge \phi_U$, so $\{\phi_U : U \in \mathcal{U}\}$ is upwards-directed. Also $\sup_{U\in\mathcal{U}} \phi_U = a\bullet f$ in $[0,\infty]^X$. **P** Of course $\phi_U(x) \le (a\bullet f)(x)$ for every x. If $x \in X$ and $(a\bullet f)(x) > \alpha$, then

$\{y : f(y) > \alpha\}$ is an open set containing $a^{-1}\bullet x$, so (because $\bullet$ is continuous) there are a $U \in \mathcal{U}$ and an open neighbourhood V of x such that $f(b^{-1}\bullet y) > \alpha$ whenever $b \in U$ and $y \in V$; in which case $\phi_U(x) \ge \alpha$. As α is arbitrary, $\sup_{U\in\mathcal{U}} \phi_U(x) = (a\bullet f)(x)$. **Q**

By 414Ba, $\int a\bullet f\, d\nu = \sup_{U\in\mathcal{U}} \int \phi_U d\nu$, and there is a $U \in \mathcal{U}$ such that $\int \phi_U d\nu > \gamma$. Now suppose that $b \in U$; then $\phi_U(x) \le (b\bullet f)(x)$ for every x, so $\int b\bullet f\, d\nu > \gamma$. This shows that $\{a : \int a\bullet f\, d\nu > \gamma\}$ is an open set in G, so that $a \mapsto \int a\bullet f\, d\nu$ is lower semi-continuous.

(iii) If $V \subseteq X$ is open, then χV is lower semi-continuous, and $\chi(a\bullet V) = a\bullet(\chi V)$ for every $a \in G$. So $a \mapsto \nu(a\bullet V) = \int a\bullet(\chi V) d\nu$ is lower semi-continuous.

Thus (a) is true.

(b) Take any $a \in G$, $E \in \operatorname{dom}\nu$ such that $\nu E < \infty$ and $\epsilon > 0$. Let $\mathcal{U}$ be the family of open neighbourhoods of a in G, and for $U \in \mathcal{U}$ set

$$H_U = \operatorname{int}\{x : |(b\bullet f)(x) - (a\bullet f)(x)| \le \epsilon \text{ whenever } b \in U\}.$$

Then $\{H_U : U \in \mathcal{U}\}$ is upwards-directed. Also, it has union X. **P** If $x \in X$ then, because $(b, y) \mapsto f(b^{-1}\bullet y)$ is continuous, there are a $U \in \mathcal{U}$ and an open neighbourhood V of x such that $|(b\bullet f)(y) - (a\bullet f)(x)| \le \frac{1}{2}\epsilon$ whenever $b \in U$ and $y \in V$. But now $|(b\bullet f)(y) - (a\bullet f)(y)| \le \epsilon$ whenever $b \in U$ and $y \in V$, so that H_U includes V, which contains x. **Q**

So there is a $U \in \mathcal{U}$ such that $\nu(E \setminus H_U) \le \epsilon$ (414Ea). In this case, for any $b \in U$, we must have

$$\int_E \min(1, |b\bullet f - a\bullet f|) d\nu \le \epsilon(1 + \nu E).$$

As E and ϵ are arbitrary, $b \mapsto (b\bullet f)^\bullet$ is continuous at a; as a is arbitrary, it is continuous everywhere. Thus (b) is true.

(c)(i) Let us start by supposing that E is an open set and that ν is totally finite. In this case the function $a \mapsto \nu(a\bullet E)$ is lower semi-continuous, by (a) above, therefore Borel measurable. Now let $W \subseteq \mathfrak{A}$ be an open set, and write $H = \{a : a \in G, (a\bullet E)^\bullet \in W\}$. For $m, k \in \mathbb{N}$ set $H_{mk} = \{a : 2^{-m}k \le \nu(a\bullet E) < 2^{-m}(k+1)\}$, so that H_{mk} is Borel, and $U_{mk} = G \setminus \overline{H_{mk} \setminus H}$, so that U_{mk} is open. Let H' be $\bigcup_{m,k\in\mathbb{N}} H_{mk} \cap U_{mk}$; then H' is Borel and $H' \subseteq H$. In fact $H' = H$. **P** If $a \in H$, then W is an open set containing $(a\bullet E)^\bullet$. Let $\delta > 0$ be such that $a \in W$ whenever $\bar\nu(a \,\triangle\, (a\bullet E)^\bullet) \le \delta$, where $\bar\nu$ is the measure on $\mathfrak{A}$; let $m, k \in \mathbb{N}$ be such that $2^{-m} \le \frac{1}{4}\delta$ and $2^{-m}k \le \nu(a\bullet E) < 2^{-m}(k+1)$. If we take ν' to be the indefinite-integral measure over ν defined by $\chi(a\bullet E)$, then ν' is a quasi-Radon measure (415Ob), so (by (a) again) $U = \{b : \nu'(b\bullet E) > 2^{-m}(k-1)\}$ is an open set, and of course it contains a. If $b \in U \cap H_{mk}$, then

$$\begin{aligned}\nu((b\bullet E)\triangle(a\bullet E)) &= \nu(b\bullet E) + \nu(a\bullet E) - 2\nu((b\bullet E) \cap (a\bullet E)) \\ &\le 2^{-m}(k+1) + 2^{-m}(k+1) - 2\cdot 2^{-m}(k-1) \le 4\cdot 2^{-m} \le \delta,\end{aligned}$$

so $(b\bullet E)^\bullet \in W$ and $b \in H$. This shows that $U \cap (H_{mk} \setminus H) = \emptyset$ and $U \subseteq U_{mk}$ and $a \in U \cap H_{mk} \subseteq H'$. As a is arbitrary, $H = H'$. **Q** Thus H is a Borel subset of G. As W is arbitrary, the map $a \mapsto (a\bullet E)^\bullet$ is Borel measurable.

(ii) To extend this to a general σ-finite quasi-Radon measure ν, still supposing that E is open, let $h : X \to \mathbb{R}$ be a strictly positive integrable function (215B(viii)) and ν' the corresponding indefinite-integral measure. As in (i), this ν' also is a quasi-Radon measure. Since ν and ν' have the same domains and the same null ideals, the Boolean algebra $\mathfrak{A}$ is still the underlying algebra of the measure algebra of ν'; by 324H, the topologies on $\mathfrak{A}$ induced by the measures $\bar\nu$, $\bar\nu'$ are the same. So we can apply (i) to the measure ν' to see that $a \mapsto (a\bullet E)^\bullet : G \to \mathfrak{A}$ is still Borel measurable.

(iii) Next, suppose that E is expressible as $\bigcup_{i\le n} V_{2i} \setminus V_{2i+1}$ where each V_i is open. Then $a \mapsto (a\bullet E)^\bullet$ is Borel measurable. **P** Set $X' = X \times \{0, \dots, 2n+1\}$, with the product topology (giving $\{0, \dots, 2n+1\}$ its discrete topology) and define a measure ν' on X and an action of G on X' by setting

$$\nu' F = \sum_{i=0}^{2n+1} \nu\{x : (x, i) \in F\}$$

whenever $F \subseteq X'$ is such that $\{x : (x, i) \in F\} \in \operatorname{dom}\nu$ for every $i \le 2n+1$,

$$a\bullet(x, i) = (a\bullet x, i)$$

whenever $a \in G$, $x \in X$ and $i \le 2n+1$. Then $V = \{(x, i) : i \le 2n+1, x \in V_i\}$ is an open set in X', while ν' is a σ-finite quasi-Radon measure, as is easily checked; so, by (ii), the map $a \mapsto (a\bullet V)^\bullet : G \to \mathfrak{A}'$ is Borel measurable, where $\mathfrak{A}'$ is the measure algebra of ν'. On the other hand, we can identify $\mathfrak{A}'$ with the simple power $\mathfrak{A}^{2n+2}$ (322Lb), and the map

$$\langle c_i\rangle_{i\le 2n+1}\mapsto \sup_{i\le n} c_{2i}\setminus c_{2i+1}:\mathfrak{A}^{2n+2}\to\mathfrak{A}$$

is continuous, by 323B. So the map

$$a\mapsto (a\bullet E)^\bullet=\sup_{i\le n}(a\bullet V_{2i})^\bullet\setminus(a\bullet V_{2i+1})^\bullet$$

is the composition of a Borel measurable function with a continuous function, and is Borel measurable. **Q**

(iv) Now the family $\mathcal{E}$ of all those Borel sets $E\subseteq X$ such that $a\mapsto(a\bullet E)^\bullet$ is Borel measurable is closed under unions and intersections of monotonic sequences. **P** (α) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{E}$ with union E, then

$$(a\bullet E)^\bullet=\sup_{n\in\mathbb{N}}(a\bullet E_n)^\bullet=\lim_{n\to\infty}(a\bullet E_n)^\bullet$$

(323Ea) for every $a\in G$. So $a\mapsto(a\bullet E)^\bullet$ is the pointwise limit of a sequence of Borel measurable functions into a metrizable space (323Gb, because $(\mathfrak{A},\bar\nu)$ is σ-finite), and is Borel measurable, by 418Ba. Thus $E\in\mathcal{E}$. (β) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{E}$ the same argument applies, since this time

$$(a\bullet E)^\bullet=\inf_{n\in\mathbb{N}}(a\bullet E_n)^\bullet=\lim_{n\to\infty}(a\bullet E_n)^\bullet$$

(323Eb) for every $a\in G$. **Q**

Since $\mathcal{E}$ contains all sets of the form $\bigcup_{i\le n}V_i\cap F_i$ where every V_i is open and every F_i is closed, by (iii), $\mathcal{E}$ must be the whole Borel σ-algebra, by 4A3Cg(ii).

This completes the proof of (c).

(d)(i) We need the following extension of (c): if $\langle E_n\rangle_{n\in\mathbb{N}}$ is any sequence of Borel sets in X, then $a\mapsto\langle(a\bullet E_n)^\bullet\rangle_{n\in\mathbb{N}}:G\mapsto\mathfrak{A}^{\mathbb{N}}$ is Borel measurable. **P** I repeat the idea of (c-iii) above. On $X'=X\times\mathbb{N}$ define a measure ν' by setting

$$\nu'F=\sum_{n=0}^\infty\nu\{x:(x,n)\in F\}$$

whenever $F\subseteq X'$ is such that $\{x:(x,n)\in F\}\in\operatorname{dom}\nu$ for every $n\in\mathbb{N}$. As before, it is easy to check that ν' is a σ-finite quasi-Radon measure, if we give $\mathbb{N}$ its discrete topology and X' the product topology. As before, set $a\bullet(x,n)=(a\bullet x,n)$ for $a\in G$, $x\in X$ and $n\in\mathbb{N}$, to obtain a continuous action of G on X'. Applying (c) to this action, the map $a\mapsto(a\bullet E)^\bullet:G\to\mathfrak{A}'$ is Borel measurable, where $\mathfrak{A}'$ is the measure algebra of ν' and $E=\{(x,n):n\in\mathbb{N},\,x\in E_n\}$. But we can identify $\mathfrak{A}'$ (as Boolean algebra) with $\mathfrak{A}^{\mathbb{N}}$, by 322L, as before; so that if we re-interpret $a\mapsto(a\bullet E)^\bullet:G\to\mathfrak{A}'$ as $a\mapsto\langle(a\bullet E_n)^\bullet\rangle_{n\in\mathbb{N}}:G\to\mathfrak{A}^{\mathbb{N}}$ it is still Borel measurable. (As in (c-ii), this time using 323L, the measure-algebra topology of $\mathfrak{A}'$ matches the product topology on $\mathfrak{A}^{\mathbb{N}}$.) **Q**

(ii) Now suppose that $f:X\to[0,1]$ is Borel measurable. Define $\langle E_n\rangle_{n\in\mathbb{N}}$ inductively by the formula

$$E_n=\{x:x\in X,\,(f-\sum_{i<n}2^{-i-1}\chi E_i)(x)\ge 2^{-n-1}\}.$$

Then every E_n is a Borel set and $f=\sum_{n=0}^\infty 2^{-n-1}\chi E_n$. Next, observe that the function

$$\langle c_n\rangle_{n\in\mathbb{N}}\mapsto\sum_{n=0}^\infty 2^{-n-1}\chi c_n:\mathfrak{A}^{\mathbb{N}}\to L^0$$

is continuous, because each of the maps $c\mapsto 2^{-n-1}\chi c$ is continuous (367R), addition is continuous (245Da, 367Ma) and the series is uniformly summable. Accordingly we may think of the map $a\mapsto(a\bullet f)^\bullet$ as the composition of the continuous function $\langle c_n\rangle_{n\in\mathbb{N}}\mapsto\sum_{n=0}^\infty 2^{-n-1}\chi c_i$ with the Borel measurable function $a\mapsto\langle(a\bullet E_n)^\bullet\rangle_{n\in\mathbb{N}}$, and it is Borel measurable.

(iii) For general Borel measurable $f:X\to\mathbb{R}$, set $q(t)=\frac{1}{2}(1+\frac{t}{|t|+1})$, so that $q:\mathbb{R}\to\,]0,1[$ is a homeomorphism, and set $p=q^{-1}:\,]0,1[\,\to\mathbb{R}$. Then the function $\bar p:P\to L^0$ is continuous, where $P=\{u:u\in L^0,\,[\![u\in\,]0,1[\,]\!]=1\}$ (367S). But now

$$a\bullet f=p\circ q\circ(a\bullet f)=p\circ(a\bullet(q\circ f))$$

for every a, so that $a\mapsto(a\bullet f)^\bullet$ is the composition of the Borel map $a\mapsto(a\bullet(q\circ f))^\bullet$ with the continuous map $\bar p$, and is Borel measurable.

(e)(i) We need only recall that $\bar\nu:\mathfrak{A}\to\mathbb{R}$ is lower semi-continuous (323Cb), so that (applying (c) above)

$$a\mapsto\nu(a\bullet E)=\bar\nu(a\bullet E)^\bullet$$

is a composition of Borel measurable functions and is Borel measurable. (Of course there are much more direct arguments, using fragments of the proof above.)

(ii) The point is that the maps $u \mapsto u^+$, $u \mapsto u^- : L^0 \to (L^0)^+$ are continuous (245Db, 367S), while $u \mapsto \int u : (L^0)^+ \to [0,\infty]$ is lower semi-continuous (369Mb), therefore Borel measurable. Accordingly $a \mapsto \int (a\bullet f)^+ = \int ((a\bullet f)^\bullet)^+$ and $a \mapsto \int (a\bullet f)^-$ are Borel measurable functions from G to $[0,\infty]$, so that

$$Q = \{a : \min(\int (a\bullet f)^+, \int (a\bullet f)^-) < \infty\}$$

is a Borel set, and

$$a \mapsto \int a\bullet f = \int (a\bullet f)^+ - \int (a\bullet f)^- : Q \to [-\infty,\infty]$$

is Borel measurable.

444G Corollary Let X be a topological group and ν a σ-finite quasi-Radon measure on X.

(a) If $f : X \to \mathbb{R}$ is a Borel measurable function, then $\{x : \int f(y^{-1}x)\nu(dy)$ is defined in $[-\infty,\infty]\}$ is a Borel set in X and $x \mapsto \int f(y^{-1}x)\nu(dy)$ is Borel measurable.

(b) If f, $g : X \to \mathbb{R}$ are Borel measurable functions, then $\{x : \int f(xy^{-1})g(y)\nu(dy)$ is defined in $[-\infty,\infty]\}$ is a Borel set and $x \mapsto \int f(xy^{-1})g(y)\nu(dy)$ is Borel measurable.

(c) If ν is totally finite and $f : X \to \mathbb{R}$ is a bounded continuous function, then $x \mapsto \int f(y^{-1}x)\nu(dy) : X \to \mathbb{R}$ is continuous.

proof (a) Set $\overset{\leftrightarrow}{f}(x) = f(x^{-1})$ for $x \in X$ (4A5C(c-ii)); then $\overset{\leftrightarrow}{f}$ is Borel measurable. Let $\bullet_l$ be the left action of X on itself. Then, in the language of 444Fe, $Q = \{x : \int x\bullet_l \overset{\leftrightarrow}{f}\, d\nu$ is defined in $[-\infty,\infty]\}$ is a Borel set, and $x \mapsto \int x\bullet_l \overset{\leftrightarrow}{f}\, d\nu$ is Borel measurable. But

$$(x\bullet_l \overset{\leftrightarrow}{f})(y) = \overset{\leftrightarrow}{f}(x^{-1}y) = f(y^{-1}x)$$

for all x, y, so $\int x\bullet_l \overset{\leftrightarrow}{f}\, d\nu = \int f(y^{-1}x)\nu(dy)$ if either integral is defined.

(b)(i) Set $\overset{\leftrightarrow}{\nu}E = \nu E^{-1}$ when this is defined, writing $E^{-1} = \{x^{-1} : x \in E\}$ for $E \subseteq X$; that is, $\overset{\leftrightarrow}{\nu}$ is the image measure $\nu\phi^{-1}$, where $\phi(x) = x^{-1}$ for $x \in X$. Because ϕ is a homeomorphism, $\overset{\leftrightarrow}{\nu}$ is a quasi-Radon measure. By 235J, $\int h\, d\overset{\leftrightarrow}{\nu} = \int h(x^{-1})\nu(dx)$ for any real-valued function h for which either integral is defined in $[-\infty,\infty]$.

(ii) Now suppose that f and g are non-negative Borel measurable functions from X to $\mathbb{R}$. Then $\overset{\leftrightarrow}{g}$ also is a non-negative Borel measurable function. We know from 444Fd that $x \mapsto (x^{-1}\bullet_l f)^\bullet : X \to L^0(\overset{\leftrightarrow}{\nu})$ is a Borel measurable function; now multiplication in L^0 is continuous (245Dc), so the map $x \mapsto ((x^{-1}\bullet_l f) \times \overset{\leftrightarrow}{g})^\bullet$ is Borel measurable; since integration is lower semi-continuous on $(L^0)^+$, $x \mapsto \int (x^{-1}\bullet_l f) \times \overset{\leftrightarrow}{g}\, d\overset{\leftrightarrow}{\nu} : X \to [0,\infty]$ is Borel measurable. But

$$\int (x^{-1}\bullet_l f) \times \overset{\leftrightarrow}{g}\, d\overset{\leftrightarrow}{\nu} = \int f(xy)g(y^{-1})\overset{\leftrightarrow}{\nu}(dy) = \int f(xy^{-1})g(y)\nu(dy)$$

whenever any of the integrals is defined in $[-\infty,\infty]$, so this is the function we needed to know about.

(iii) For general Borel measurable functions f and g, we have

$$\begin{aligned}\int f(xy^{-1})g(y)\nu(dy) = \Bigl(\int f^+(xy^{-1})g^+(y)\nu(dy) + \int f^-(xy^{-1})g^-(y)\nu(dy)\Bigr)\\ - \Bigl(\int f^+(xy^{-1})g^-(y)\nu(dy) + \int f^-(xy^{-1})g^+(y)\nu(dy)\Bigr)\end{aligned}$$

exactly when the subtraction can be done in $[-\infty,\infty]$, so that $x \mapsto \int f(xy^{-1})g(y)\nu(dy)$ is a difference of Borel measurable functions and is Borel measurable, with Borel measurable domain.

(c) Continuing the argument of (a), if f is bounded and continuous and ν is totally finite then all the functions $x\bullet_l \overset{\leftrightarrow}{f}$ are bounded and continuous, so $\int x\bullet_l \overset{\leftrightarrow}{f}\, d\nu$ is defined and finite for every x. Next, the function $x \mapsto (x\bullet_l \overset{\leftrightarrow}{f})^\bullet : X \to L^0(\nu)$ is continuous for the topology of convergence in measure (444Fb), which agrees with the norm topology of $L^1(\nu)$ on $\|\,\|_\infty$-bounded sets (246Jb). It follows that

$$x \mapsto \int (x\bullet_l \overset{\leftrightarrow}{f})^\bullet = \int f(y^{-1}x)\nu(dy)$$

is continuous.

444H Convolutions of measures and functions I introduce some notation which I shall use for the rest of the section. Let X be a topological group. If f is a real-valued function defined on a subset of X, and ν is a measure on X, set

$$(\nu * f)(x) = \int f(y^{-1}x)\nu(dy)$$

whenever the integral is defined in $\mathbb{R}$.

444I Proposition Let X be a topological group and λ, ν two totally finite quasi-Radon measures on X.

(a) For any Borel measurable function $f : X \to \mathbb{R}$, $\nu * f$ is a Borel measurable function with a Borel domain.

(b) $\nu * f \in C_b(X)$ for every $f \in C_b(X)$.

(c) For any real-valued function f defined on a subset of X, $(\lambda * (\nu * f))(x) = ((\lambda * \nu) * f)(x)$ whenever the right-hand side is defined.

proof (a) This follows at once from 444Ga.

(b) This is just a restatement of 444Gc.

(c) If $((\lambda * \nu) * f)(x)$ is defined, then

$$((\lambda * \nu) * f)(x) = \int f(t^{-1}x)(\lambda * \nu)(dt) = \iint f((yz)^{-1}x)\nu(dz)\lambda(dy)$$

(444C)

$$= \iint f(z^{-1}y^{-1}x)\nu(dz)\lambda(dy)$$

$$= \int (\nu * f)(y^{-1}x)\lambda(dy) = (\lambda * (\nu * f))(x).$$

444J Convolutions of functions and measures Let X be a topological group carrying Haar measures; let Δ be its left modular function (442I). If f is a real-valued function defined on a subset of X, and ν is a measure on X, set

$$(f * \nu)(x) = \int f(xy^{-1})\Delta(y^{-1})\nu(dy)$$

whenever the integral is defined in $\mathbb{R}$. From 444Gb we see that if ν is a σ-finite quasi-Radon measure and f is Borel measurable, then $f * \nu$ is a Borel measurable function with a Borel domain. If f is non-negative and ν-integrable, write $f\nu$ for the corresponding indefinite-integral measure over ν (234J[4]).

444K Proposition Let X be a topological group with a left Haar measure μ. Let ν be a totally finite quasi-Radon measure on X. Then for any non-negative μ-integrable real-valued function f, $f\mu$ is a quasi-Radon measure; moreover, $\nu * f$ and $f * \nu$ are μ-integrable, and we have

$$(\nu * f)\mu = \nu * f\mu, \quad (f * \nu)\mu = f\mu * \nu.$$

In particular, $\int \nu * f\, d\mu = \int f * \nu\, d\mu = \nu X \cdot \int f d\mu$.

proof (a) $f\mu$ is a quasi-Radon measure by 415Ob.

(b) Suppose first that f is Borel measurable and defined everywhere on X, as well as being non-negative and μ-integrable.

(i) Let $E \subseteq X$ be a Borel set such that $\mu E < \infty$. The function $(x, y) \mapsto f(x^{-1}y)\chi E(y) : X \times X \to [0, \infty[$ is Borel measurable, so

$$(\nu * f\mu)(E) = \int (f\mu)(x^{-1}E)\nu(dx)$$

(444A)

[4]Formerly 234B.

$$= \iint f(y)\chi(x^{-1}E)(y)\mu(dy)\nu(dx)$$

(by the definition of $f\mu$, 234I[5])

$$= \iint f(x^{-1}y)\chi(x^{-1}E)(x^{-1}y)\mu(dy)\nu(dx)$$

(441J)

$$= \iint f(x^{-1}y)\chi E(y)\mu(dy)\nu(dx) = \iint f(x^{-1}y)\chi E(y)\nu(dx)\mu(dy)$$

(by Fubini's theorem, 417Ha, because $(x,y) \mapsto f(x^{-1}y)\chi E(y)$ is non-negative and Borel measurable and zero outside $X \times E$). So $\int f(x^{-1}y)\nu(dx)$ must be finite for μ-almost every $y \in E$. Because μ is complete and locally determined and inner regular with respect to the Borel sets of finite measure, $(\nu * f)(y) = \int f(x^{-1}y)\nu(dx)$ is defined in $\mathbb{R}$ for μ-almost every $y \in X$. So we have an indefinite-integral measure $(\nu * f)\mu$. Next, we have

$$\infty > \nu X \int f d\mu = (\nu * f\mu)(X) \ge (\nu * f\mu)(E)$$
$$= \iint f(x^{-1}y)\chi E(y)\nu(dx)\mu(dy) = \iint (\nu * f)(y)\chi E(y)\mu(dy) = ((\nu * f)\mu)(E)$$

for every Borel set E such that μE is finite. Again because μ is inner regular with respect to the Borel sets of finite measure, $\nu * f$ is μ-integrable and $(\nu * f)\mu$ is totally finite. Since $\nu * f\mu$ and $(\nu * f)(\mu)$ are totally finite quasi-Radon measures agreeing on open sets of finite measure for μ, and μ is locally finite (442Aa), 415H(iv) assures us that $\nu * f\mu = (\nu * f)(\mu)$.

(ii) Now consider $f * \nu$. This time, if $E \subseteq X$ is Borel and $\mu E < \infty$,

$$(f\mu * \nu)(E) = \int (f\mu)(Ey^{-1})\nu(dy) = \iint_{Ey^{-1}} f(x)\mu(dx)\nu(dy)$$
$$= \int \Delta(y^{-1}) \int_E f(xy^{-1})\mu(dx)\nu(dy)$$

(by 442Kc)

$$= \int_E \int \Delta(y^{-1})f(xy^{-1})\nu(dy)\mu(dx).$$

Once again, we see that $\int \Delta(y^{-1})f(xy^{-1})\nu(dy)$ is defined for μ-almost every $x \in E$; as E is arbitrary, $(f * \nu)(x)$ is defined in $\mathbb{R}$ for μ-almost every $x \in X$; and 444Gb tells us that $f * \nu$ is Borel measurable. As before,

$$\int_E f * \nu \, d\mu \le (f\mu * \nu)(X) = \int f d\mu \cdot \nu X < \infty$$

for Borel sets E with $\mu E < \infty$, so that $f * \nu$ is μ-integrable; as before, the quasi-Radon measures $(f * \nu)\mu$ and $f\mu * \nu$ agree on open sets of finite μ-measure, so coincide.

(c) Next, suppose that f is defined and zero μ-a.e. In this case there is a Borel set E such that $\mu E = 0$ and $f(x)$ is defined and equal to zero for every $x \in X \setminus E$ (443J(b-ii)). Set $g = \chi E$. Then $g\mu$ is the zero measure, so $(\nu * g)\mu = \nu * g\mu$, $(g * \nu)\mu = g\mu * \nu$ are all zero; that is, there is some μ-conegligible set F such that

$$0 = (\nu * g)(x) = \int \chi E(y^{-1}x)\nu(dy) = \nu(xE^{-1}),$$

$$0 = (g * \nu)(x) = \int \chi E(xy^{-1})\Delta(y^{-1})\nu(dy) = \int_{E^{-1}x} \Delta(y^{-1})\nu(dy)$$

for every $x \in F$. But now, for $x \in F$, we must have $\nu(xE^{-1}) = \nu(E^{-1}x) = 0$ (since Δ is strictly positive), so that

$$(\nu * f)(x) = \int f(y^{-1}x)\nu(dy) = \int_{xE^{-1}} f(y^{-1}x)\nu(dy) = 0,$$

because if $y \notin xE^{-1}$ then $y^{-1}x \notin E$ and $f(y^{-1}x) = 0$. Similarly,

$$(f * \nu)(x) = \int f(xy^{-1})\Delta(y^{-1})\nu(dy) = \int_{E^{-1}x} f(xy^{-1})\Delta(y^{-1})\nu(dy) = 0.$$

[5] Formerly 234A.

Thus $\nu * f$ and $f * \nu$ are defined, and zero, μ-almost everywhere.

(d) For an arbitrary non-negative μ-integrable function f, we can express it in the form $g + h$ where g is a non-negative μ-integrable Borel measurable function defined everywhere, and h is zero almost everywhere. In this case, $\nu * h^+$, $\nu * h^-$, $h^+ * \nu$ and $h^- * \nu$ are defined, and zero, μ-a.e., so $\nu * f =_{\text{a.e.}} \nu * g$ and $f * \nu =_{\text{a.e.}} g * \nu$. We therefore have

$$(\nu * f)\mu = (\nu * g)\mu = \nu * g\mu = \nu * f\mu, \quad (f * \nu)\mu = (g * \nu)\mu = g\mu * \nu = f\mu * \nu,$$

as required.

(e) Finally, we have

$$\int \nu * f d\mu = ((\nu * f)\mu)(X) = (\nu * f\mu)(X) = \nu X \cdot (f\mu)(X) = \nu X \cdot \int f d\mu,$$

$$\int f * \nu \, d\mu = ((f * \nu)\mu)(X) = (f\mu * \nu)(X) = (f\mu)(X) \cdot \nu X = \nu X \cdot \int f d\mu.$$

444L Corollary Let X be a topological group carrying Haar measures. Suppose that ν is a non-zero quasi-Radon measure on X and $E \subseteq X$ is a Haar measurable set such that $\nu(xE) = 0$ for every $x \in X$. Then E is Haar negligible.

proof Let μ be a left Haar measure on X. There is a non-zero totally finite quasi-Radon measure ν' on X such that $\nu'(xE) = 0$ for every $x \in X$. **P** Take any F such that $0 < \nu F < \infty$, and set $\nu' H = \nu(H \cap F)$ whenever this is defined. **Q** Let G be any Borel set such that $\mu G < \infty$, and set $f = \chi(G \cap E^{-1})$. Then f is μ-integrable, and

$$(\nu' * f)(x) = \int \chi(G \cap E^{-1})(y^{-1}x)\nu'(dy) = \nu'(xG^{-1} \cap xE) = 0$$

for every $x \in X$. By 444K, $\nu' * f\mu = (\nu' * f)\mu$ is the zero measure, and $(f\mu)(X) = 0$, that is, $\mu(G \cap E^{-1}) = 0$. As G is arbitrary, $\mu E^{-1} = 0$ and E is Haar negligible (442H).

444M Proposition Let X be a topological group and μ a left Haar measure on X. Let ν be a quasi-Radon measure on X and $p \in [1, \infty]$.

(a) Suppose that $\nu X < \infty$. Then we have a bounded positive linear operator $u \mapsto \nu * u : L^p(\mu) \to L^p(\mu)$, of norm at most νX, defined by saying that $\nu * f^\bullet = (\nu * f)^\bullet$ for every $f \in \mathcal{L}^p(\mu)$.

(b) Set $\gamma = \int \Delta(y)^{(1-p)/p}\nu(dy)$ if $p < \infty$, $\int \Delta(y)^{-1}\nu(dy)$ if $p = \infty$, where Δ is the left modular function of X. Suppose that $\gamma < \infty$. Then we have a bounded positive linear operator $u \mapsto u * \nu : L^p(\mu) \to L^p(\mu)$, of norm at most γ, defined by saying that $f^\bullet * \nu = (f * \nu)^\bullet$ for every $f \in \mathcal{L}^p(\mu)$.

proof I will write $\mathcal{L}^p$, L^p for $\mathcal{L}^p(\mu)$, $L^p(\mu)$. Note that if f_1, $f_2 \in \mathcal{L}^0(\mu)$ and $f_1 = f_2$ μ-a.e., then 444L tells us that $\nu * |f_1 - f_2|$ and $|f_1 - f_2| * \nu$ are both zero μ-almost everywhere, so that $\nu * f_1 =_{\text{a.e.}} \nu * f_2$ and $f_1 * \nu =_{\text{a.e.}} f_2 * \nu$, in the sense that there is a μ-conegligible set F such that $(\nu * f_1)\restriction F = (\nu * f_2)\restriction F$ and $(f_1 * \nu)\restriction F = (f_2 * \nu)\restriction F$. In particular, if we are told that $\nu * f_1$ belongs to $\mathcal{L}^p$, and that $f_1^\bullet = f_2^\bullet$ in $L^0(\mu)$, then we can be sure that $\nu * f_2 \in \mathcal{L}^p$ and $(\nu * f_2)^\bullet = (\nu * f_1)^\bullet$; and similarly for $f_1 * \nu$, $f_2 * \nu$.

(a) If $\nu X = 0$ the result is trivial. Multiplying ν by a positive scalar does not affect the inequalities we need, so we may suppose that $\nu X = 1$. If $f \geq 0$ is μ-integrable, then 444K tells us that $\nu * f$ is μ-integrable and that

$$\begin{aligned}\|\nu * f\|_1 &= \int_X (\nu * f)(x)\mu(dx) = ((\nu * f)\mu)(X) \\ &= (\nu * f\mu)(X) = \nu X \cdot (f\mu)(X) = \|f\|_1,\end{aligned}$$

using 444C or 444A for the penultimate equality. Since evidently $\nu * (f + g) = \nu * f + \nu * g$, $\nu * (\alpha f) = \alpha\nu * f$ at any point where the right-hand sides of the equations are defined in $\mathbb{R}$, we have a positive linear operator $T_1 : L^1 \to L^1$ defined by saying that $T_1 g^\bullet = (\nu * g)^\bullet$ for every μ-integrable Borel measurable function g, with $\|T_1\| = 1$.

Similarly, if $h : X \to \mathbb{R}$ is a bounded Borel measurable function, then $\nu * h$ also is a Borel measurable function, by 444Ga. Of course it is bounded, since

$$|(\nu * h)(x)| = |\int h(y^{-1}x)\nu(dy)| \leq \sup_{y \in X} |h(y)|$$

for every x. So we have a positive linear operator $T_\infty : L^\infty \to L^\infty$ defined by saying that $T_\infty h^\bullet = (\nu * h)^\bullet$ for every bounded Borel measurable function h. Moreover, if $u \in L^\infty$, there is a Borel measurable h such that $h^\bullet = u$ and $\sup_{y \in X} |h(y)| = \|u\|_\infty$, so that

$$\|T_\infty u\|_\infty \le \sup_{x\in X}|(\nu*h)(x)| \le \|u\|_\infty;$$

thus $\|T_\infty\|\le 1$.

Since T_1 and T_∞ agree on $L^1\cap L^\infty$, they have a common extension to a linear operator $T: L^1+L^\infty\to L^1+L^\infty$. By 371Gd, $\|Tu\|_p\le\|u\|_p$ whenever $p\in[1,\infty]$ and $u\in L^p$. (Strictly speaking, I am relying on the standard identifications of L^1, L^∞ and L^p with the corresponding subspaces of $L^0(\mathfrak{A})$, where $(\mathfrak{A},\bar\mu)$ is the measure algebra of μ. Of course the argument for 371Gd applies equally well in $L^0(\mu)$.) Now suppose that $f\in\mathcal{L}^p$. Then it is expressible as $g+h$ where $g\in\mathcal{L}^1$ and $h:X\to\mathbb{R}$ is a bounded Borel measurable function, so we shall have

$$\nu*f=\nu*g+\nu*h \text{ wherever the right-hand side is defined;}$$

accordingly $\nu*f$ is defined μ-a.e. and is measurable, and

$$\begin{aligned}\|\nu*f\|_p&=\|(\nu*g)^\bullet+(\nu*h)^\bullet\|_p=\|T_1g^\bullet+T_\infty h^\bullet\|_p\\&=\|Tf^\bullet\|_p\le\|f^\bullet\|_p=\|f\|_p,\end{aligned}$$

as required.

(b)(i) As in (a), the case $\nu X=0$ is trivial. Otherwise, because Δ is strictly positive, $\gamma>0$; again considering a scalar multiple of ν if necessary, we may suppose that $\gamma=1$. Note that ν is surely σ-finite.

(ii) If $p=1$, then $\nu X=\gamma=1$. If $f\in\mathcal{L}^1$ is non-negative, then, by 444K, as in (a) above,

$$\|f*\nu\|_1=((f*\nu)\mu)(X)=(f\mu*\nu)(X)=(f\mu)(X)\cdot\nu X=\|f\|_1.$$

For general μ-integrable f,

$$\|f*\nu\|_1\le\|f^+*\nu\|_1+\|f^-*\nu\|_1=\|f^+\|_1+\|f^-\|_1=\|f\|_1.$$

(iii) If $p=\infty$, then directly from the formula $(f*\nu)(x)=\int f(xy^{-1})\Delta(y^{-1})\nu(dy)$ we see that if $f:X\to\mathbb{R}$ is a bounded Borel measurable function then

$$|(f*\nu)(x)|\le\int\Delta(y)^{-1}\nu(dy)\cdot\sup_{y\in X}|f(y)|=\sup_{y\in X}|f(y)|$$

for every x. Since changing f on a μ-negligible set changes $f*\nu$ on a μ-negligible set, we can argue as in (a) above to see that $f^\bullet\mapsto(f*\nu)^\bullet$ defines a linear operator from L^∞ to itself of norm at most 1.

(iv) Now suppose that $1<p<\infty$. Set $q=\frac{p}{p-1}$, so that $\int\Delta(y^{-1})^{1/q}\nu(dy)=\gamma=1$. Suppose for the moment that $f\in\mathcal{L}^p$ is a non-negative Borel measurable function, and let $h:X\to\mathbb{R}$ be another non-negative Borel measurable function such that $\int h^q d\mu\le 1$. In this case

$$\begin{aligned}\int(f*\nu)\times h\,d\mu&=\iint h(x)f(xy^{-1})\Delta(y^{-1})\nu(dy)\mu(dx)\\&=\iint h(x)f(xy^{-1})\Delta(y^{-1})\mu(dx)\nu(dy)\end{aligned}$$

(by 417Ha, because $(x,y)\mapsto h(x)f(xy^{-1})\Delta(y^{-1})$ is Borel measurable and $\{x:h(x)\ne0\}$ is a countable union of sets of finite measure for μ, while ν is σ-finite)

$$=\iint h(xy)f(x)\mu(dx)\nu(dy)$$

by 442Kc, as usual, at least if the last integral is finite. But, for any $y\in X$,

$$\begin{aligned}\int h(xy)f(x)\mu(dx)&\le\|f\|_p\big(\int|h(xy)|^q\mu(dx)\big)^{1/q}\\&=\|f\|_p\big(\Delta(y^{-1})\int|h(x)|^q\mu(dx)\big)^{1/q}\le\|f\|_p\Delta(y^{-1})^{1/q}.\end{aligned}$$

So

$$\int(f*\nu)\times h\,d\mu=\iint h(xy)f(x)\mu(dx)\nu(dy)\le\int\|f\|_p\Delta(y^{-1})^{1/q}\nu(dy)=\|f\|_p.$$

Because μ (being a quasi-Radon measure) is semi-finite, this means that $f*\nu\in\mathcal{L}^p$ and that $\|f*\nu\|_p\le\|f\|_p$ (366D-366E, or 244Xe and 244Fa). (Once again, we need to know that every member of L^q can be represented by a Borel measurable function; this is a consequence of 443J or 412Xd.)

For general Borel measurable $f : X \to \mathbb{R}$ such that $\int |f|^p d\mu < \infty$, we know that from 444G that $f * \nu$ is Borel measurable, while $|f * \nu| \le |f| * \nu$ (and $f * \nu$ is defined wherever $|f| * \nu$ is finite), so that

$$\|f * \nu\|_p \le \||f| * \nu\|_p \le \||f|\|_p = \|f\|_p.$$

Finally, if $f \in \mathcal{L}^p$ is arbitrary, then there is a Borel measurable $g : X \to \mathbb{R}$ such that $f =_{\text{a.e.}} g$, so that $f * \nu =_{\text{a.e.}} g * \nu$ and

$$\|f * \nu\|_p = \|g * \nu\|_p \le \|g\|_p = \|f\|_p.$$

It follows at once that we have a bounded linear operator $f^\bullet \mapsto (f * \nu)^\bullet : L^p \to L^p$, of norm at most $1 = \gamma$.

444N The following lemma on exchanging the order of repeated integrals will be fundamental to the formulae in the rest of the section.

Lemma Let X be a topological group and μ a left Haar measure on X. Suppose that f, g, $h \in \mathcal{L}^0(\mu)$ (the space of measurable real-valued functions defined μ-a.e. in X) are non-negative. Then, writing $\int \dots d(x, y)$ to denote integration with respect to the quasi-Radon product measure $\mu \times \mu$,

$$\iint f(x)g(y)h(xy)dxdy = \iint f(x)g(y)h(xy)dydx = \int f(x)g(y)h(xy)d(x, y)$$

in $[0, \infty]$.

proof Following the standard pattern in results of this type, I deal with successively more complicated functions f, g and h. Evidently the situation is symmetric, so that it will be enough if I can show that $\iint f(x)g(y)h(xy)dxdy = \int f(x)g(y)h(xy)d(x, y)$.

(a) Suppose first that $f = \chi F$, $g = \chi G$ and $h = \chi H$, where F, G, H are Borel subsets of X. In this case

$$\iint f(x)g(y)h(xy)dxdy = \sup_{U,V \in \Sigma^f} \int_V \int_U f(x)g(y)h(xy)dxdy,$$

where Σ^f is the ideal of measurable sets of finite measure for μ. **P** For $y \in X$, $n \in \mathbb{N}$ and $U \in \Sigma^f$ write

$$q(y) = \int f(x)h(xy)dx = \mu(F \cap Hy^{-1}), \quad q_U(y) = \int_U f(x)h(xy)dx = \mu(U \cap F \cap Hy^{-1}),$$

$$q^{(n)}(y) = \min(n, q(y)), \quad q_U^{(n)}(y) = \min(n, q_U(y)).$$

Then every q_U is continuous, by 443C (with a little help from 323Cc), while $\sup_{U \in \Sigma^f} q_U(y) = q(y)$ for every y, because μ is semi-finite. Because μ is τ-additive and effectively locally finite, $(q^{(n)})^\bullet = \sup_{U \in \Sigma^f} (q_U^{(n)})^\bullet$ in $L^0(\mu)$ for every n (414Ab); because Σ^f is upwards-directed,

$$\begin{aligned}\int q(y)g(y)dy &= \sup_{n \in \mathbb{N}} \int q^{(n)}(y)g(y)dy \\ &= \sup_{n \in \mathbb{N}, U \in \Sigma^f} \int q_U^{(n)}(y)g(y)dy = \sup_{U \in \Sigma^f} \int q_U(y)g(y)dy,\end{aligned}$$

that is,

$$\iint f(x)g(y)h(xy)dxdy = \sup_{U \in \Sigma_f} \int \int_U f(x)g(y)h(xy)dxdy.$$

On the other hand, for any $U \in \Sigma^f$, we surely have

$$\int \int_U f(x)g(y)h(xy)dxdy = \sup_{V \in \Sigma^f} \int_V \int_U f(x)g(y)h(xy)dxdy,$$

again because μ is semi-finite. Putting these together, we have the result. **Q**

Looking at the other side of the equation, $\int f(x)g(y)h(xy)d(x, y) = (\mu \times \mu)W$, where $W = (F \times G) \cap \{(x, y) : xy \in H\}$ is a Borel set; so that

$$\begin{aligned}\iint f(x)g(y)h(xy)dxdy &= \sup_{U,V \in \Sigma^f} (\mu \times \mu)((U \times V) \cap W) \\ &= \sup_{U,V \in \Sigma^f} \int_{U \times V} f(x)g(y)h(xy)d(x, y)\end{aligned}$$

(417C(iii)). But now we can apply 417Ha to see that, for any U, $V \in \Sigma^f$,

$$\int_{U\times V} f(x)g(y)h(xy)d(x,y) = \int_V \int_U f(x)g(y)h(xy)dxdy.$$

Taking the supremum over U and V, we get

$$\int f(x)g(y)h(xy)d(x,y) = \iint f(x)g(y)h(xy)dxdy.$$

(b) Clearly both sides of our equation

$$\int f(x)g(y)h(xy)d(x,y) = \iint f(x)g(y)h(xy)dxdy$$

are additive in f, g and h separately (subtraction, of course, will be another matter, as I am allowing ∞ to appear without restriction); and also behave identically if f or g or h is multiplied by a non-negative scalar. So the identity will be valid if f, g and h are all finite sums of non-negative multiples of indicator functions of Borel sets. Moreover, by repeated use of B.Levi's theorem, we see that if $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ and $\langle h_n\rangle_{n\in\mathbb{N}}$ are non-decreasing sequences of such functions with suprema f, g and h, then

$$\begin{aligned}\int f(x)g(y)h(xy)d(x,y) &= \sup_{n\in\mathbb{N}} \int f_n(x)g_n(y)h_n(xy)d(x,y)\\ &= \sup_{n\in\mathbb{N}} \iint f_n(x)g_n(y)h_n(xy)dxdy = \iint f(x)g(y)h(xy)dxdy.\end{aligned}$$

So the identity is valid for all non-negative Borel functions f, g and h.

(c) Finally, suppose only that f, g and h are non-negative, measurable and defined almost everywhere. In this case, by 443J(b-iv), there are Borel measurable functions f_0, g_0 and h_0, non-negative, defined everywhere on X and equal almost everywhere to f, g and h respectively. Let E be the conegligible set

$$\{x : x \in \operatorname{dom} f \cap \operatorname{dom} h \cap \operatorname{dom} g,\ f(x) = f_0(x),\ g(x) = g_0(x),\ h(x) = h_0(x)\}.$$

We find that $\int f(x)h(xy)dx = \int f_0(x)h_0(xy)dx$ for every $y \in X$. **P** $E \cap Ey^{-1}$ is conegligible (see 443A), and $f(x)h(xy) = f_0(x)h_0(xy)$ for every $x \in E \cap Ey^{-1}$. **Q** Consequently

$$\iint f(x)g(y)h(xy)dxdy = \iint f_0(x)g_0(y)h_0(xy)dxdy.$$

Secondly, $f(x)g(y)h(xy) = f_0(x)g_0(y)h_0(xy)$ $(\mu\times\mu)$-a.e. **P** Set $W = \{(x,y) : x \in E,\ y \in E,\ xy \in E\}$. Fubini's theorem, applied to $(U \times V)\setminus W$ where U, $V \in \Sigma^f$, shows that W is conegligible; but of course $f(x)g(y)h(xy) = f_0(x)g_0(y)h_0(xy)$ whenever $(x,y) \in W$. **Q** Accordingly

$$\int f(x)g(y)h(xy)d(x,y) = \int f_0(x)g_0(y)h_0(xy)d(x,y).$$

Combining this with the result of (b), applied to f_0, g_0 and h_0, we see that once again

$$\int f(x)g(y)h(xy)d(x,y) = \iint f(x)g(y)h(xy)dxdy,$$

as required.

444O Convolutions of functions: Theorem Let X be a topological group and μ a left Haar measure on X. For f, $g \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$, write $(f*g)(x) = \int f(y)g(y^{-1}x)dy$ whenever this is defined in $\mathbb{R}$, taking the integral with respect to μ.

(a) Writing Δ for the left modular function of X,

$$\begin{aligned}(f*g)(x) &= \int f(y)g(y^{-1}x)dy = \int f(xy)g(y^{-1})dy\\ &= \int \Delta(y^{-1})f(y^{-1})g(yx)dy = \int \Delta(y^{-1})f(xy^{-1})g(y)dy\end{aligned}$$

whenever any of these integrals is defined in $\mathbb{R}$.

(b) If $f =_{\text{a.e.}} f_1$ and $g =_{\text{a.e.}} g_1$, then $f*g = f_1*g_1$.

(c)(i) $|(f*g)(x)| \le (|f|*|g|)(x)$ whenever either is defined in $\mathbb{R}$.

(ii)

$$((f_1+f_2)*g)(x) = (f_1*g)(x) + (f_2*g)(x),$$

$$(f*(g_1+g_2))(x) = (f*g_1)(x) + (f*g_2)(x),$$

$$((\alpha f) * g)(x) = (f * (\alpha g))(x) = \alpha(f * g)(x)$$

whenever the right-hand expressions are defined in $\mathbb{R}$.

(d) If f, g and h belong to $\mathcal{L}^0$ and any of

$$\int (|f| * |g|)(x)|h|(x)dx, \quad \iint |f(x)g(y)h(xy)|dxdy,$$

$$\iint |f(x)g(y)h(xy)|dydx, \quad \int |f(x)g(y)h(xy)|d(x,y)$$

is defined in $[0,\infty[$ (writing $\int \ldots d(x,y)$ for integration with respect to the quasi-Radon product measure $\mu \times \mu$ on $X \times X$), then

$$\int (f * g)(x)h(x)dx, \quad \iint f(x)g(y)h(xy)dxdy,$$

$$\iint f(x)g(y)h(xy)dydx, \quad \int f(x)g(y)h(xy)d(x,y)$$

are all defined, finite and equal, provided that in the expression $(f * g)(x)h(x)$ we interpret the product as 0 when $h(x) = 0$ and $(f * g)(x)$ is undefined.

(e) If f, g and h belong to $\mathcal{L}^0$, $f * g$ and $g * h$ are defined a.e. and $x \in X$ is such that either $(|f| * (|g| * |h|))(x)$ or $((|f| * |g|) * |h|)(x)$ is defined in $\mathbb{R}$, then $(f * (g * h))(x)$ and $((f * g) * h)(x)$ are defined and equal.

(f) If $a \in X$ and f, $g \in \mathcal{L}^0$,

$$a\bullet_l(f * g) = (a\bullet_l f) * g, \quad a\bullet_r(f * g) = f * (a\bullet_r g),$$

$$(a\bullet_r f) * g = \Delta(a^{-1})f * (a^{-1}\bullet_l g),$$

$$\overset{\leftrightarrow}{f} * \overset{\leftrightarrow}{g} = (g * f)^{\leftrightarrow}.$$

(g) If X is abelian then $f * g = g * f$ for all f and g.

proof (a) Use 441J and 442Kb to see that the four formulae for $f * g$ coincide.

(b) Setting

$$E = \{y : y \in \operatorname{dom} f \cap \operatorname{dom} f_1 \cap \operatorname{dom} g \cap \operatorname{dom} g_1,\ f(y) = f_1(y),\ g(y) = g_1(y)\},$$

E is conegligible. If $x \in X$, then $f(y)g(y^{-1}x) = f_1(y)g_1(y^{-1}x)$ for every $y \in E \cap xE^{-1}$, which is also conegligible, by 443A; so $(f * g)(x) = (f_1 * g_1)(x)$ if either is defined.

(c) These are all elementary.

(d) First consider non-negative f, g and h. The point is that, if any of the integrals is defined and finite,

$$\begin{aligned}\int (f * g)(x)h(x)dx &= \iint \Delta(x^{-1})f(x^{-1})g(xy)h(y)dxdy \\ &= \iint \Delta(x^{-1})f(x^{-1})g(xy)h(y)dydx\end{aligned}$$

(by 444N, recalling that $x \mapsto \Delta(x^{-1})f(x^{-1})$ will belong to $\mathcal{L}^0$ if f does, by 442J and 442H)

$$= \iint f(x)g(x^{-1}y)h(y)dydx = \iint f(x)g(y)h(xy)dydx$$

(substituting xy for y in the inner integral, as permitted by 441J). (The 'and finite' at the beginning of the last sentence is there because I have changed the rules since the last paragraph, and $f * g$ is not allowed to take the value ∞. So we have to take care that

$$\{y : h(y) > 0,\ \int f(x)g(x^{-1}y)dx = \infty\}$$

is negligible.) Now applying 444N again, we get

$$\begin{aligned}\int (f * g)(x)h(x)dx &= \iint f(x)g(y)h(xy)dxdy \\ &= \iint f(x)g(y)h(xy)dydx = \int f(x)g(y)h(xy)d(x,y)\end{aligned}$$

if any of these integrals is finite.

For the general case, the hypothesis on $|f|$, $|g|$ and $|h|$ is sufficient to ensure that the four expressions are equal for any combination of $f^{\pm}$, $g^{\pm}$ and $h^{\pm}$; adding and subtracting these combinations appropriately, we get the result.

(e) The point is that, for non-negative f, g and h,

$$((f*g)*h)(x)=\int(f*g)(z)h(z^{-1}x)dz=\int(f*g)(z)h'(z)dz$$

(setting $h'(z)=h(z^{-1}x)$)

$$=\iint f(y)g(z)h'(yz)dzdy$$

(using (d); to see that h' is measurable, refer to 443A as usual)

$$=\iint f(y)g(z)h(z^{-1}y^{-1}x)dzdy$$

$$=\int f(y)(g*h)(y^{-1}x)dy=(f*(g*h))(x)$$

at least as long as one of the expressions here is finite. (Note that, as in 255J, we need to suppose that $f*g$ and $g*h$ are defined a.e. when moving from $\int(f*g)(z)h(z^{-1}x)dz$ to $\iint f(y)g(z)h(z^{-1}y^{-1})dydz$ and from $\iint f(y)g(z)h(z^{-1}y^{-1}x)dzdy$ to $\int f(y)(g*h)(y^{-1}x)dy$, since in part (d) I am more tolerant of infinities in the repeated integrals than I was in the definition of $f*g$.) Once again, subject to the inner integrals implicit in the formulae $f*(g*h)$ and $(f*g)*h$ being adequately defined, we can use addition and subtraction to obtain the result for general f, g and h.

(f) These are immediate from the formulae in (a), using 442K if necessary.

(g) If X is abelian, then $\Delta(y)=1$ for every y, so

$$(g*f)(x)=\int g(y)f(y^{-1}x)dy=\int g(y)\Delta(y^{-1})f(xy^{-1})dy=(f*g)(x)$$

if either $(f*g)(x)$ or $(g*f)(x)$ is defined in $\mathbb{R}$.

444P Proposition Let X be a topological group and μ a left Haar measure on X.

(a) If $f\in\mathcal{L}^1(\mu)^+$ and $g\in\mathcal{L}^0(\mu)$ then $f*g$, as defined in 444O, is equal to $(f\mu)*g$ as defined in 444H.

(b) If $f\in\mathcal{L}^0(\mu)$ and $g\in\mathcal{L}^1(\mu)^+$ then $f*g=f*(g\mu)$.

proof Again, these are immediate from the formulae above:

$$(f*g)(x)=\int g(y^{-1}x)f(y)\mu(dy)=\int g(y^{-1}x)(f\mu)(dy)=(f\mu*g)(x),$$

$$(f*g)(x)=\int f(xy^{-1})\Delta(y^{-1})g(y)\mu(dy)=\int f(xy^{-1})\Delta(y^{-1})(g\mu)(dy)=(f*g\mu)(x)$$

whenever these are defined, using 235K, as usual, to calculate $\int\ldots d(f\mu)$, $\int\ldots d(g\mu)$. (Note that as we assume throughout that f and g are defined μ-almost everywhere, all the functions $y\mapsto g(y^{-1}x)$, $y\mapsto f(xy^{-1})$ are also defined μ-a.e., by the results set out in 443A.)

444Q Proposition Let X be a topological group and μ a left Haar measure on X.

(a) Let f, g be non-negative μ-integrable functions. Then, defining $f*g$ as in 444O, we have $f*g\in\mathcal{L}^1=\mathcal{L}^1(\mu)$ and

$$(f\mu)*(g\mu)=(f*g)\mu.$$

(b) For any f, $g\in\mathcal{L}^1$, $f*g\in\mathcal{L}^1$ and

$$\int f*g\,d\mu=\int fd\mu\int g\,d\mu,\quad \|f*g\|_1\le\|f\|_1\|g\|_1.$$

proof (a) Putting 444K and 444P together, $f\mu*g\mu=(f\mu*g)\mu$, so that $f*g=f\mu*g$ is μ-integrable, and

$$(f*g)\mu=(f\mu*g)\mu=f\mu*g\mu.$$

(b) Taking $h=\chi X$ in 444Od, we get $\int f*g\,d\mu=\int fd\mu\int g\,d\mu$. Now

$$\|f*g\|_1=\int|f*g|\le\int|f|*|g|=\int|f|\int|g|=\|f\|_1\|g\|_1.$$

444R Proposition Let X be a topological group and μ a left Haar measure on X. Take any $p \in [1, \infty]$.

(a) If $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^p(\mu)$, then $f * g \in \mathcal{L}^p(\mu)$ and $\|f * g\|_p \le \|f\|_1 \|g\|_p$.

(b) $f * \overleftrightarrow{g} = (g * \overleftrightarrow{f})^{\leftrightarrow}$ for all f, $g \in \mathcal{L}^0$. If X is unimodular then $\|\overleftrightarrow{f}\|_p = \|f\|_p$ for every $f \in \mathcal{L}^0$.

(c) Set $q = \infty$ if $p = 1$, $p/(p-1)$ if $1 < p < \infty$, 1 if $p = \infty$. If $f \in \mathcal{L}^p(\mu)$ and $g \in \mathcal{L}^q(\mu)$, then $f * \overleftrightarrow{g}$ is defined everywhere in X and is continuous, and $\|f * \overleftrightarrow{g}\|_\infty \le \|f\|_p \|g\|_q$. If X is unimodular, then $f * g \in C_b(X)$ and $\|f * g\|_\infty \le \|f\|_p \|g\|_q$ for every $f \in \mathcal{L}^p(\mu)$, $g \in \mathcal{L}^q(\mu)$.

Remark In the formulae above, interpret $\|g\|_\infty$ as $\|g^\bullet\|_\infty = \operatorname{ess\,sup} |g|$ for $g \in \mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$, and as ∞ for $g \in \mathcal{L}^0 \setminus \mathcal{L}^\infty$. Because μ is strictly positive, this agrees with the usual definition $\|g\|_\infty = \sup_{x \in X} |g(x)|$ when g is continuous and defined everywhere on X.

proof (a) If $f \ge 0$, then $f * g = (f\mu) * g$ belongs to $\mathcal{L}^p(\mu)$, and

$$\|f * g\|_p = \|f\mu * g\|_p \le (f\mu)(X)\|g\|_p = \|f\|_1 \|g\|_p,$$

by 444Pa and 444Ma. Generally, $f * g =_{\text{a.e.}} f^+ * g - f^- * g$ belongs to $\mathcal{L}^p(\mu)$ and

$$\|f * g\|_p \le \|f^+ * g\|_p + \|f^- * g\|_p \le (\|f^+\|_1 + \|f^-\|_1)\|g\|_p = \|f\|_1 \|g\|_p.$$

(b)(i) By 443A once more, $\overleftrightarrow{f} \in \mathcal{L}^0$ whenever $f \in \mathcal{L}^0$. For $x \in X$,

$$(f * \overleftrightarrow{g})(x) = \int f(y)\overleftrightarrow{g}(y^{-1}x)dy = \int \overleftrightarrow{f}(y^{-1})g(x^{-1}y)dy = (g * \overleftrightarrow{f})(x^{-1}) = (g * \overleftrightarrow{f})^{\leftrightarrow}(x)$$

if any of these are defined.

(ii) If X is unimodular, then, for any $f \in \mathcal{L}^0$,

$$\|\overleftrightarrow{f}\|_p^p = \int |f(x^{-1})|^p dx = \int \Delta(x^{-1})|f(x)|^p dx = \|f\|_p^p;$$

while $\operatorname{ess\,sup} |\overleftrightarrow{f}| = \operatorname{ess\,sup} |f|$ because E^{-1} is conegligible whenever $E \subseteq X$ is conegligible.

(c)(i) For any $x \in X$,

$$(f * \overleftrightarrow{g})(x) = \int f(y)g(x^{-1}y)dy = \int f \times (x \bullet_l g)$$

in the language of 443G and 444O. By 443Gb, $x \bullet_l g \in \mathcal{L}^q$, so $(f * \overleftrightarrow{g})(x) = \int f \times (x \bullet_l g)$ is defined.

(ii) If $p > 1$, so that $q < \infty$, then $x \mapsto (x \bullet_l g)^\bullet : X \to L^q$ is continuous (443Gf), so

$$x \mapsto (f * \overleftrightarrow{g})(x) = \int f^\bullet \times (x \bullet_l g)^\bullet$$

is continuous, because $f^\bullet \in L^p \cong (L^q)^*$. If $p = 1$, then

$$(f * \overleftrightarrow{g})(x) = \int f(xy)g(y)dy = \int (x^{-1} \bullet_l f) \times g$$

for every x; since $x \mapsto (x^{-1} \bullet_l f)^\bullet : X \to L^1$ is continuous, so is $f * \overleftrightarrow{g}$.

(iii) If X is unimodular then $f * g = f * \overleftrightarrow{\overleftrightarrow{g}}$ is continuous, because $\overleftrightarrow{g} \in \mathcal{L}^q$ by (b), and $\|f * g\|_\infty \le \|f\|_p \|\overleftrightarrow{g}\|_q = \|f\|_p \|g\|_q$.

444S Remarks Let X be a topological group and μ a left Haar measure on X.

(a) From 444Ob and 444Ra we see that we have a bilinear operator $(u, v) \mapsto u * v : L^1(\mu) \times L^p(\mu) \to L^p(\mu)$ defined by saying that $f^\bullet * g^\bullet = (f * g)^\bullet$ for every $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^p(\mu)$. Indeed, 444Ob tells us that $*$ can actually be regarded as a function from $L^1 \times L^p$ to $\mathcal{L}^p$. Putting 443Ge together with 444Oe and 444Of, we have

$$u * (v * w) = (u * v) * w,$$

$$a \bullet_l (u * w) = (a \bullet_l u) * w, \quad a \bullet_r (u * w) = u * (a \bullet_r w),$$

$$(a \bullet_r u) * w = \Delta(a^{-1}) u * (a^{-1} \bullet_l w)$$

whenever u, $v \in L^1$, $w \in L^p$ and $a \in X$.

Similarly, if the group is unimodular, and $\frac{1}{p} + \frac{1}{q} = 1$, the map $* : \mathcal{L}^p \times \mathcal{L}^q \to C_b(X)$ (444Rc) factors through a map from $L^p \times L^q$ to $C_b(X)$.

(b) In particular, $*: L^1 \times L^1 \to L^1$ is associative; evidently it is bilinear; and $\|u*v\|_1 \le \|u\|_1\|v\|_1$ for all u, $v \in L^1$. So L^1 is a normed algebra; since L^1 is $\| \|_1$-complete, it is a Banach algebra. By 444Qb, $\int u*v = \int u \int v$ for all u, $v \in L^1$. L^1 is commutative if X is abelian (444Og).

(c) Let $\mathcal{B}$ be the Borel σ-algebra of X and M_τ the Banach algebra of signed τ-additive Borel measures on X, as in 444E. If, for $f \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ and $E \in \mathcal{B}$, we write $(f\mu\restriction\mathcal{B})(E) = \int_E f d\mu$, then $f\mu\restriction\mathcal{B} \in M_\tau$; for $f \ge 0$, this is because the indefinite-integral measure $f\mu$ is a quasi-Radon measure, and in general it is because $f\mu\restriction\mathcal{B} = f^+\mu\restriction\mathcal{B} - f^-\mu\restriction\mathcal{B}$. For f, $g \in \mathcal{L}^1$, we have

$$f^\bullet = g^\bullet \text{ in } L^1 \Longrightarrow f =_{\text{a.e.}} g \Longrightarrow f\mu\restriction\mathcal{B} = g\mu\restriction\mathcal{B},$$

so we have an operator $T: L^1 \to M_\tau$ defined by setting $T(f^\bullet) = f\mu\restriction\mathcal{B}$ for $f \in \mathcal{L}^1$. It is easy to check that

$$(f+g)\mu\restriction\mathcal{B} = f\mu\restriction\mathcal{B} + g\mu\restriction\mathcal{B}, \quad (\alpha f)\mu\restriction\mathcal{B} = \alpha(f\mu\restriction\mathcal{B}),$$

$$(|f|\mu\restriction\mathcal{B})(E) = \int_E |f| = \sup_{F\in\mathcal{B}, F\subseteq E} \int_F f - \int_{E\setminus F} f = |f\mu\restriction\mathcal{B}|(E)$$

for f, $g \in \mathcal{L}^1$, $\alpha \in \mathbb{R}$ and $E \in \mathcal{B}$, so that T is a Riesz homomorphism; moreover,

$$\|f\mu\restriction\mathcal{B}\| = |f\mu\restriction\mathcal{B}|(X) = (|f|\mu\restriction\mathcal{B})(X) = \int |f| d\mu = \|f^\bullet\|_1$$

for $f \in \mathcal{L}^1$, so that T is norm-preserving. Finally, for non-negative f, $g \in \mathcal{L}^1$, we have

$$Tf^\bullet * Tg^\bullet = (f\mu\restriction\mathcal{B}) * (g\mu\restriction\mathcal{B}) = (f\mu * g\mu)\restriction\mathcal{B}$$

(444Eb, since the completions of $f\mu\restriction\mathcal{B}$, $g\mu\restriction\mathcal{B}$ are the quasi-Radon indefinite-integral measures $f\mu$, $g\mu$)

$$= (f*g)\mu\restriction\mathcal{B}$$

(444Qa)

$$= T(f*g)^\bullet = T(f^\bullet * g^\bullet).$$

Thus $Tu * Tv = T(u*v)$ for non-negative u, $v \in L^1$; by linearity, $Tu*Tv = T(u*v)$ for all u, $v \in L^1$, and T is an embedding of L^1 as a subalgebra of M_τ.

444T Proposition Let X be a topological group and μ a left Haar measure on X. Then for any $p \in [1,\infty[$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$ there is a neighbourhood U of the identity e in X such that $\|\nu*f - f\|_p \le \epsilon$ and $\|f*\nu - f\|_p \le \epsilon$ whenever ν is a quasi-Radon measure on X such that $\nu U = \nu X = 1$.

proof (a) To begin with, suppose that f is non-negative, continuous and bounded, and that $G = \{x : f(x) > 0\}$ has finite measure; set $M = \sup_{x\in X} f(x)$. Write $\mathcal{U}$ for the family of neighbourhoods of e. Take $\delta > 0$, $\eta \in]0,1]$ such that

$$(2\delta + (1+\delta)^p - 1)^{1/p}\|f\|_p \le \epsilon,$$

$$(1-\eta)^p \int((f - \eta\chi X)^+)^p d\mu - M^p\eta \ge (1-\delta)\int f^p d\mu, \quad (1-\eta)^{(1-p)/p} \le 1+\delta.$$

For each $U \in \mathcal{U}$, set

$$H_U = \operatorname{int}\{x : f(y) \ge f(x) - \eta \text{ for every } y \in xU^{-1} \cup U^{-1}x\}.$$

Then H_U is open and for every $x \in X$ there is a $U \in \mathcal{U}$ such that $|f(y) - f(x)| \le \frac{1}{2}\eta$ whenever $y \in xUU^{-1} \cup U^{-1}xU$, so that $x \in \operatorname{int} xU \subseteq H_U$. Thus $\{H_U : U \in \mathcal{U}\}$ is an upwards-directed family of open sets with union X, and there is a $U \in \mathcal{U}$ such that $\mu(G \setminus H_U) \le \eta$; moreover, because Δ is continuous, we can suppose that $\Delta(y^{-1}) \ge 1-\eta$ for every $y \in U$.

Now suppose that ν is a quasi-Radon measure on X such that $\nu U = \nu X = 1$. Then, for any $x \in H_U$,

$$(\nu * f)(x) = \int f(y^{-1}x)\nu(dy) = \int_U f(y^{-1}x)\nu(dy) \ge f(x) - \eta$$

because $x \in H_U$ and $y^{-1}x \in U^{-1}x$ whenever $y \in U$. Similarly,

$$(f*\nu)(x) = \int_U f(xy^{-1})\Delta(y^{-1})\nu(dy) \ge (f(x) - \eta)(1-\eta)$$

for every $x \in H_U$. Now this means that, setting $h_1 = \nu * f$, $h_2 = f*\nu$ we have $(f \wedge h_i)(x) \ge (f(x)-\eta)(1-\eta)$ for every $x \in H_U$ and both i. Accordingly

$$\begin{aligned}\int(f\wedge h_i)^p d\mu &\ge (1-\eta)^p\int_{G\cap H_U}((f-\eta\chi X)^+)^p d\mu\\ &\ge (1-\eta)^p\int((f-\eta\chi X)^+)^p d\mu-\int_{G\setminus H_U}f^p\\ &\ge (1-\eta)^p\int((f-\eta\chi X)^+)^p d\mu-M^p\eta\ge(1-\delta)\int f^p d\mu.\end{aligned}$$

Now, just because f and h_i are non-negative, and $p\ge 1$,

$$|f-h_i|^p+2(f\wedge h_i)^p\le f^p+h_i^p.$$

Also, writing

$$\gamma=\int\Delta(y)^{(1-p)/p}\nu(dy)=\int_U\Delta(y)^{(1-p)/p}\nu(dy)\le(1-\eta)^{(1-p)/p}\le 1+\delta,$$

we have

$$\|h_1\|_p=\|\nu*f\|_p\le\|f\|_p,\quad \|h_2\|_p=\|f*\nu\|_p\le\gamma\|f\|_p$$

(444M), so that $\int h_i^p d\nu\le(1+\delta)^p\int f^p d\mu$ for both i, and

$$\int|f-h_i|^p d\mu\le\int f^p d\mu+\int h_i^p d\mu-2\int(f\wedge h_i)^p d\mu\le(2\delta+(1+\delta)^p-1)\int f^p d\mu$$

for both i. But this means that

$$\max(\|f-f*\nu\|_p,\|f-\nu*f\|_p)\le(2\delta+(1+\delta)^p-1)^{1/p}\|f\|_p\le\epsilon.$$

As ν is arbitrary, we have found a suitable U.

(b) For any continuous bounded function f such that $\mu\{x:f(x)\ne 0\}<\infty$, we can find neighbourhoods U_1, U_2 of e such that

$$\|f^+-\nu*f^+\|_p\le\tfrac12\epsilon,\quad \|f^+-f^+*\nu\|_p\le\tfrac12\epsilon$$

whenever $\nu U_1=\nu X=1$,

$$\|f^--\nu*f^-\|_p\le\tfrac12\epsilon,\quad \|f^--f^-*\nu\|_p\le\tfrac12\epsilon$$

whenever $\nu U_2=\nu X=1$. So we shall have

$$\|f-\nu*f\|_p\le\epsilon,\quad \|f-f*\nu\|_p\le\epsilon$$

whenever $\nu(U_1\cap U_2)=\nu X=1$.

(c) For general $f\in\mathcal{L}^p(\mu)$, there is a bounded continuous function $g:X\to\mathbb{R}$ such that $\mu\{x:g(x)\ne0\}<\infty$ and $\|f-g\|_p\le\frac14\epsilon$ (415Pa). Now there is a neighbourhood U_1 of e such that

$$\|g-\nu*g\|_p\le\tfrac14\epsilon,\quad \|g-g*\nu\|_p\le\tfrac14\epsilon$$

whenever $\nu U_1=\nu X=1$. There is also a neighbourhood U_2 of e such that $\Delta(y^{-1})^{(1-p)/p}\le 2$ for every $y\in U_2$, so that

$$\|g*\nu-f*\nu\|_p\le 2\|g-f\|_p\le\tfrac12\epsilon$$

whenever $\nu U_2=\nu X=1$. Since we have

$$\|\nu*g-\nu*f\|_p\le\|g-f\|_p\le\tfrac14\epsilon$$

whenever $\nu X=1$, we get $\|f-\nu*f\|_p\le\epsilon$, $\|f=f*\nu\|_p\le\epsilon$ whenever $\nu(U_1\cap U_2)=\nu X=1$.

This completes the proof.

444U Corollary Let X be a topological group and μ a left Haar measure on X. For any Haar measurable $E\subseteq X$ such that $0<\mu E<\infty$, and any $f\in\bigcup_{1\le p<\infty}\mathcal{L}^p(\mu)$, write

$$f_E(x)=\frac{1}{\mu E}\int_{xE}f d\mu,\quad f'_E(x)=\frac{1}{\mu(Ex)}\int_{Ex}f d\mu$$

for $x\in X$. Then, for any $p\in[1,\infty[$, $f\in\mathcal{L}^p$ and $\epsilon>0$, there is a neighbourhood U of the identity in X such that $\|f_E-f\|_p\le\epsilon$ and $\|f'_E-f\|_p\le\epsilon$ whenever $E\subseteq U$ is a non-negligible Haar measurable set.

proof Take $\delta\in\,]0,1[$ such that $\delta(1-\delta)^{(1-p)/p}\|f\|_p\le\frac{1}{2}\epsilon$. By 444T, there is a neighbourhood U of the identity such that $\|f-f*\nu\|_p\le\frac{1}{2}\epsilon$, $\|f-\nu*f\|_p\le\epsilon$ whenever ν is a quasi-Radon measure on X such that $\nu U=\nu X=1$. Shrinking U if necessary, we may suppose also that $U=U^{-1}$, that $\mu U<\infty$ and that $|\Delta(x)-1|\le\delta$ for every $x\in U$, where Δ is the left modular function of X. If $E\subseteq U$ and $\mu E>0$, consider the quasi-Radon measures ν, ν', $\overset{\leftrightarrow}{\nu}$ and $\overset{\leftrightarrow}{\nu}'$ on X defined by setting

$$\nu F=\frac{1}{\mu E^{-1}}\int_{E\cap F}\Delta(x^{-1})\mu(dx),\quad \nu' F=\frac{\mu(E\cap F)}{\mu E},\quad \overset{\leftrightarrow}{\nu}F=\nu F^{-1},\quad \overset{\leftrightarrow}{\nu}'F=\nu' F^{-1}$$

whenever these are defined. (They are quasi-Radon measures because ν and ν' are totally finite indefinite-integral measures over μ and the map $x\mapsto x^{-1}$ is a homeomorphism.) Because $E\subseteq U=U^{-1}$, we have

$$\overset{\leftrightarrow}{\nu}U=\nu U^{-1}=\frac{1}{\mu E^{-1}}\int_E\Delta(x^{-1})\mu(dx)=1=\overset{\leftrightarrow}{\nu}X$$

by 442Ka, while

$$\overset{\leftrightarrow}{\nu}'X=\overset{\leftrightarrow}{\nu}'U=\nu' U^{-1}=1.$$

Now consider $f*\overset{\leftrightarrow}{\nu}$ and $\overset{\leftrightarrow}{\nu}'*f$. For any $x\in X$,

$$(f*\overset{\leftrightarrow}{\nu})(x)=\int f(xy^{-1})\Delta(y^{-1})\overset{\leftrightarrow}{\nu}(dy)=\int f(xy)\Delta(y)\nu(dy)$$

(because $\overset{\leftrightarrow}{\nu}$ is the image of ν under the map $y\mapsto y^{-1}$)

$$=\frac{1}{\mu E^{-1}}\int\chi E(y)\Delta(y^{-1})f(xy)\Delta(y)\mu(dy)$$

(noting that ν is an indefinite-integral measure over μ, and using 235K)

$$=\frac{1}{\mu E^{-1}}\int\chi E(x^{-1}y)f(y)\mu(dy)=\frac{1}{\mu E^{-1}}\int_{xE}f(y)\mu(dy)=\frac{\mu E}{\mu E^{-1}}f_E(x),$$

$$(\overset{\leftrightarrow}{\nu}'*f)(x)=\int f(y^{-1}x)\overset{\leftrightarrow}{\nu}'(dy)=\int f(yx)\nu'(dy)$$

(because $\overset{\leftrightarrow}{\nu}'$ is the image of ν' under the map $y\mapsto y^{-1}$)

$$=\frac{1}{\mu E}\int\chi E(y)f(yx)\mu(dy)=\frac{\Delta(x^{-1})}{\mu E}\int\chi E(yx^{-1})f(y)\mu(dy)$$

(by 442Kc)

$$=\frac{1}{\mu(Ex)}\int_{Ex}f(y)\mu(dy)=f'_E(x).$$

So

$$\|f-f_E\|_p\le\|f-f*\overset{\leftrightarrow}{\nu}\|_p+\Bigl|\frac{\mu E^{-1}}{\mu E}-1\Bigr|\|f*\overset{\leftrightarrow}{\nu}\|_p.$$

Now $\mu E^{-1}=\int_E\Delta(y^{-1})\mu(dy)$, so that

$$|\mu E-\mu E^{-1}|\le\int_E|\Delta(y^{-1})-1|\mu(dy)\le\delta\mu E,\quad \Bigl|\frac{\mu E^{-1}}{\mu E}-1\Bigr|\le\delta;$$

also

$$\|f*\overset{\leftrightarrow}{\nu}\|_p\le\|f\|_p\int\Delta(y)^{(1-p)/p}\nu(dy)\le\|f\|_p(1-\delta)^{(1-p)/p}$$

by 444Mb, so

$$\Bigl|\frac{\mu E^{-1}}{\mu E}-1\Bigr|\|f*\overset{\leftrightarrow}{\nu}\|_p\le\delta(1-\delta)^{(1-p)/p}\|f\|_p\le\frac{1}{2}\epsilon,$$

and $\|f-f_E\|_p\le\epsilon$.

On the other hand,

$$\|f-f'_E\|_p=\|f-\overset{\leftrightarrow}{\nu}'*f\|_p\le\epsilon;$$

as E is arbitrary, we have found a suitable U.

444V So far I have not emphasized the special properties of compact groups. But of course they are the centre of the subject, and for the sake of a fundamental theorem in §446 I give the following result.

Theorem Let X be a compact topological group and μ a left Haar measure on X.

(a) For any u, $v \in L^2 = L^2(\mu)$ we can interpret their convolution $u * v$ either as a member of the space $C(X)$ of continuous real-valued functions on X, or as a member of the space L^2.

(b) If $w \in L^2$, then $u \mapsto u * w$ is a compact linear operator whether regarded as a map from L^2 to $C(X)$ or as a map from L^2 to itself.

(c) If $w \in L^2$ and $w = \overset{\leftrightarrow}{w}$ (as defined in 443Af), then $u \mapsto u * w : L^2 \to L^2$ is a self-adjoint operator.

proof (a) Being compact, X is unimodular (442Ic). As noted in 444Sa, $*$ can be regarded as a bilinear operator from $L^2 \times L^2$ to $C_b(X) = C(X)$. Because μX must be finite, we now have a natural map $f \mapsto f^\bullet$ from $C(X)$ to L^2, so that we can think of $u * v$ as a member of L^2 for u, $v \in L^2$.

(b)(i) Evidently $u \mapsto u * w : L^2 \to C(X)$ is linear, for any $w \in L^2$.

(ii) Let B be the unit ball of L^2, and give it the topology induced by the weak topology $\mathfrak{T}_s(L^2, L^2)$, so that B is compact (4A4Ka). Let $\bullet_l$ be the left action of X on L^2 as in 443G and 444S.

If f, $g \in \mathcal{L}^2$ and $a \in X$, then

$$(f * g)(a) = \int f(x)g(x^{-1}a)dx = \int f(x)\overset{\leftrightarrow}{g}(a^{-1}x)dx = \int f \times a\bullet_l \overset{\leftrightarrow}{g}.$$

(Note that because X is unimodular, $\overset{\leftrightarrow}{g}$ and $a\bullet_l\overset{\leftrightarrow}{g}$ are square-integrable whenever g is.) So if u, $w \in L^2$ and $a \in X$, $(u * w)(a) = (u|a\bullet_l\overset{\leftrightarrow}{w})$. It follows that, for any $w \in L^2$, the function $(a, u) \mapsto (u * w)(a) : X \times B \to \mathbb{R}$ is continuous. **P** We know that $\bullet_l : X \times L^2 \to L^2$ is continuous when L^2 is given its norm topology (443Gf). Now $(u, v) \mapsto (u|v)$ is continuous, so $(a, u) \mapsto (u * w)(a) = (u|a\bullet_l\overset{\leftrightarrow}{w})$ must be continuous. **Q**

Because X is compact, this means that $u \mapsto u*w : B \to C(X)$ is continuous when $C(X)$ is given its norm topology and B is given the weak topology (4A2G(g-ii)). Because B is compact in the weak topology, $\{u * w : u \in B\}$ is compact in $C(X)$. But this implies that $u \mapsto u * w$ is a compact linear operator (definition: 3A5La).

(iii) Again because X is compact, μ is totally finite, so, for $f \in C(X)$, $\|f\|_2 \le \|f\|_\infty\sqrt{\mu X}$, and the natural map $f \mapsto f^\bullet : C(X) \to L^2$ is a bounded linear operator. Consequently the map $u \mapsto (u * w)^\bullet : L^2 \to L^2$ is a compact operator, by 4A4La.

(c) Now suppose that $w = \overset{\leftrightarrow}{w}$. In this case $(u * w|v) = (u|v * w)$ for all u, $v \in L^2$. **P** Express u, v and w as $f^\bullet$, $g^\bullet$ and $h^\bullet$ where f, g and h are square-integrable Borel measurable functions defined everywhere on X. We have

$$\begin{aligned}(u * w|v) &= \int (f * h)(x)g(x)dx = \iint f(y)h(y^{-1}x)g(x)dydx \\ &= \iint f(y)h(y^{-1}x)g(x)dxdy\end{aligned}$$

(because $(x, y) \mapsto f(y)h(y^{-1}x)g(x)$ is Borel measurable, μ is totally finite and $\iint |f(y)h(y^{-1}x)g(x)|dydx = (|u| * |w|||v|)$ is finite)

$$\begin{aligned}&= \iint f(y)\overset{\leftrightarrow}{h}(x^{-1}y)g(x)dxdy = \int f(y)(g * \overset{\leftrightarrow}{h})(y)dy \\ &= (u|v * \overset{\leftrightarrow}{w}) = (u|v * w). \quad \mathbf{Q}\end{aligned}$$

As u and v are arbitrary, this shows that $u \mapsto u * w : L^2 \to L^2$ is self-adjoint.

444X Basic exercises >(a) Let X be a Hausdorff topological group. Show that if λ and ν are totally finite Radon measures on X then $\lambda * \nu$ is the image measure $(\lambda \times \nu)\phi^{-1}$, where $\phi(x, y) = xy$ for x, $y \in X$, and in particular is a Radon measure.

>(b) Let X be a topological group and λ, ν two totally finite quasi-Radon measures on X. Writing $\operatorname{supp}\lambda$ for the support of λ, show that $\operatorname{supp}(\lambda * \nu) = \overline{(\operatorname{supp}\lambda)(\operatorname{supp}\nu)}$.

(c) Let X be a topological group and M^+_{qR} the family of totally finite quasi-Radon measures on X. Show that $(\lambda, \nu) \mapsto \lambda * \nu : M^+_{\mathrm{qR}} \times M^+_{\mathrm{qR}} \to M^+_{\mathrm{qR}}$ is continuous for the narrow topology on M^+_{qR}. (*Hint*: 437Ma, 437N.)

(d) Let X be a Hausdorff topological group. Show that X is abelian iff its Banach algebra of signed τ-additive Borel measures is commutative.

(e) Let X be a topological group, and M_τ its Banach algebra of signed τ-additive Borel measures. (i) Show that we have actions $\bullet_l$, $\bullet_r$ of X on M_τ defined by writing $(a\bullet_l\nu)(E) = \nu(aE)$, $(a\bullet_r\nu)(E) = \nu(Ea^{-1})$. (ii) Show that $(a\bullet_l\lambda) * \nu = a\bullet_l(\lambda * \nu)$, $\lambda * (a\bullet_r\nu) = a\bullet_r(\lambda * \nu)$ for all $a \in X$ and λ, $\nu \in M_\tau$.

(f) Let X be a compact Hausdorff topological group, and B a norm-bounded subset of the Banach algebra M_τ of signed τ-additive Borel measures on X. Show that $(\lambda, \nu) \mapsto \lambda * \nu : B \times B \to M_\tau$ is continuous for the vague topology on M_τ. (*Hint*: 437Md.)

(g) Let X be a topological group, and ν a totally finite quasi-Radon measure on X. Show that for any Borel sets E, $F \subseteq X$, the function $(g, h) \mapsto \nu(gE \cap Fh)$ is Borel measurable. (*Hint*: for Borel sets $W \subseteq X \times X$, set $\nu'W = \nu\{x : (x, x) \in W\}$. Consider the action of $X \times X$ on itself defined by writing $(g, h)\bullet(x, y) = (gx, yh^{-1})$.)

(h) Let X be a topological group and f a real-valued function defined on a subset of X. (i) Show that $a\bullet_r(\nu * f) = \nu * (a\bullet_r f)$ (definition: 4A5Cc) whenever $a \in X$ and ν is a measure on X. (ii) Show that if X carries Haar measures, then $a\bullet_l(f * \nu) = (a\bullet_l f) * \nu$ whenever $a \in X$ and ν is a measure on X.

(i) Let X be a topological group carrying Haar measures, $f : X \to \mathbb{R}$ a bounded continuous function and ν a totally finite quasi-Radon measure on X. Show that $f * \nu$ is continuous.

(j) Let X be a topological group carrying Haar measures, f a real-valued function defined on a subset of X, and λ, ν totally finite quasi-Radon measures on X. Show that $((f * \nu) * \lambda)(x) = (f * (\nu * \lambda))(x)$ whenever the right-hand side is defined. (See also 444Yj.)

(k) Let X be an abelian topological group carrying Haar measures. Show that $f * \nu = \nu * f$ for every measure ν on X and every real-valued function f defined on a subset of X.

>(l) Let X be a topological group and μ a left Haar measure on X. (i) Let ν be a totally finite quasi-Radon measure on X such that $x \mapsto \nu(xF)$ is continuous for every closed set $F \subseteq X$. Show that ν is truly continuous with respect to μ. (*Hint*: if $\mu F = 0$, apply 444K to $\nu * \chi F^{-1}$ to see that $\nu(xF) = 0$ for μ-almost every x.) (ii) Let ν be a totally finite Radon measure on X such that $x \mapsto \nu(xK)$ is continuous for every compact set $K \subseteq X$. Show that ν is truly continuous with respect to μ.

(m) Let X be a topological group carrying Haar measures, $E \subseteq X$ a Haar negligible set and ν a σ-finite quasi-Radon measure on X. Show that $\nu(xE) = \nu(Ex) = 0$ for Haar-a.e. $x \in X$.

(n) Let X be a topological group carrying Haar measures, and ν a non-zero totally finite quasi-Radon measure on X such that $\nu(xE) = 0$ whenever $x \in X$ and $\nu E = 0$. (i) Show that ν is strictly positive, so that X is ccc. (ii) Show that a subset of X is ν-negligible iff it is Haar negligible.

(o) Use the method of part (b) of the proof of 444M to prove part (a) there.

>(p) Let X be the group $S^1 \times S^1$, with the topology defined by giving the first coordinate the usual topology of S^1 and the second coordinate its discrete topology, so that X is a locally compact abelian group. Let μ be a Haar measure on X. (i) Find a Borel measurable function $f : X \times X \to \{0, 1\}$ such that $\iint f(x, y)\mu(dx)\mu(dy) \neq \iint f(x, y)\mu(dy)\mu(dx)$. (ii) Let ν be the Radon measure on X defined by setting $\nu E = \#(\{s : (s, s^{-1}) \in E\})$ if this is finite, ∞ otherwise. Define $g : X \to \{0, 1\}$ by setting $g(s, t) = 1$ if $s = t$, 0 otherwise. Show that $\iint g(xy)\nu(dy)\mu(dx) = \infty$, $\iint g(xy)\mu(dx)\nu(dy) = 0$. (iii) Find a closed set $F \subseteq X$ such that $x \mapsto \nu(xF)$ is not Haar measurable.

>(q) Let X be a Hausdorff topological group and for $a \in X$ write δ_a for the Dirac measure on X concentrated at a. (i) Show that $\delta_a * \delta_b = \delta_{ab}$ for all a, $b \in X$. (ii) Show that, in the notation of 444Xe, $\delta_a * \hat{\nu}$ is the completion of $a^{-1}\bullet_l\nu$ and $\hat{\nu} * \delta_a$ is the completion of $a\bullet_r\nu$ for every $a \in X$ and every totally finite τ-additive Borel measure ν on X with completion $\hat{\nu}$. (iii) Show that $\delta_a * f = a\bullet_l f$ for every $a \in X$ and every real-valued function f defined on a subset of X. (iv) Show that if X carries Haar measures, and has left modular function Δ, $f * \delta_a = \Delta(a^{-1})a^{-1}\bullet_r f$ for every $a \in X$ and every real-valued function f defined on a subset of X. (v) Use these formulae to relate 444Of to 444B.

(r) Let X be a topological group and μ a left Haar measure on X. Show that if f, $g \in \mathcal{L}^0(X)$ then $(f*g)^{\leftrightarrow} = \overset{\leftrightarrow}{g} * \overset{\leftrightarrow}{f}$.

(s) Let X be a locally compact Hausdorff topological group and μ a left Haar measure on X. Show that if f, $g : X \to \mathbb{R}$ are continuous functions with compact support, then $f*g$ is a continuous function with compact support.

(t) In 444Rc, show that $f * \overset{\leftrightarrow}{g}$ is uniformly continuous for the bilateral uniformity. (*Hint*: in 443Gf, $x \mapsto x \bullet_l u$ is uniformly continuous.)

(u) Let X be a topological group with a totally finite Haar measure μ. Show that (i) $(u * w|v) = (u|v * \overset{\leftrightarrow}{w})$ for any u, v, $w \in L^2 = L^2(\mu)$, where $\overset{\leftrightarrow}{w}$ and $u * v$ are defined as in 443Af and 444V (ii) the map $u \mapsto u * w : L^2 \to L^2$ is a compact linear operator for any $w \in L^2$. (*Hint*: for (ii), use 443L.)

(v) Let X be a topological group with a Haar probability measure μ. Show that $L^2(\mu)$ with convolution is a Banach algebra.

>(w)(i) Let X_1, X_2 be topological groups with totally finite quasi-Radon measures λ_i, ν_i on X_i for each i. Let $\lambda = \lambda_1 \times \lambda_2$, $\nu = \nu_1 \times \nu_2$ be the quasi-Radon product measure on the topological group $X = X_1 \times X_2$. Show that $\lambda * \nu = (\lambda_1 * \nu_1) \times (\lambda_2 * \nu_2)$. (ii) Let $\langle X_i \rangle_{i \in I}$ be a family of topological groups, and λ_i, ν_i quasi-Radon probability measures on X_i for each i. Let $\lambda = \prod_{i \in I} \lambda_i$, $\nu = \prod_{i \in I} \nu_i$ be the quasi-Radon product measures on the topological group $\prod_{i \in I} X_i$. Show that $\lambda * \nu = \prod_{i \in I} \lambda_i * \nu_i$.

>(x) Show that 444C, 444O, 444P, 444Qb and 444R-444U remain valid if we work with complex-valued, rather than real-valued, functions, and with $\mathcal{L}^p_{\mathbb{C}}$ and $L^p_{\mathbb{C}}$ rather than $\mathcal{L}^p$ and L^p.

(y) Let X be a topological group with a left Haar measure μ and left modular function Δ. Write $\boldsymbol{\Delta} \in L^0 = L^0(\mu)$ for the equivalence class of the function Δ. For $u \in L^0$ write u^* for $\overset{\leftrightarrow}{u} \times \overset{\leftrightarrow}{\boldsymbol{\Delta}}$. Show that (i) $(u^*)^* = u$ for every $u \in L^0$ (ii) $u \mapsto u^* : L^0 \to L^0$ is a Riesz space automorphism (iii) $u^* \in L^1$ for every $u \in L^1 = L^1(\mu)$ (iv) $u \mapsto u^* : L^1 \to L^1$ is an L-space automorphism (v) $u^* * v^* = (v * u)^*$ for all u, $v \in L^1$ (v) defining $T : L^1 \to M_\tau$ as in 444Sc, show that $Tu^* = \overset{\leftrightarrow}{Tu}$ (that is, $(Tu^*)(E) = (Tu)(E^{-1})$ for Borel sets E) for every $u \in L^1$.

444Y Further exercises (a) Find a subgroup X of $\{0,1\}^{\mathbb{N}}$ and quasi-Radon probability measures λ, ν on X and a set $A \subseteq X$ such that $(\lambda * \nu)^*(A) = 1$ but $(\lambda \times \nu)\{(x,y) : x, y \in X, x + y \in A\} = 0$.

(b) Let X be a **topological semigroup**, that is, a semigroup with a topology such that multiplication is continuous. (i) For totally finite τ-additive Borel measures λ, ν on X, show that there is a τ-additive Borel measure $\lambda * \nu$ defined by saying that $(\lambda * \nu)(E) = (\lambda \times \nu)\{(x,y) : xy \in E\}$ for every Borel set $E \subseteq X$. (ii) Show that in this context $(\lambda_1 * \lambda_2) * \lambda_3 = \lambda_1 * (\lambda_2 * \lambda_3)$. (iii) Show that $\int f d(\lambda * \nu) = \int f(xy) \lambda(dx) \nu(dy)$ whenever f is $(\lambda * \nu)$-integrable. (iv) Show that if the topology is Hausdorff and λ and ν are tight (that is, inner regular with respect to the compact sets) so is $\lambda * \nu$. (v) Show that we have a Banach algebra of signed τ-additive Borel measures on X, as in 444E.

(c) Let X be a topological group, and write $M_\tau^{(\mathbb{C})}$ for the complexification of the L-space M_τ of 444E, as described in 354Yl. Show that $M_\tau^{(\mathbb{C})}$, with the natural extension of the convolution operator of 444E, is a complex Banach algebra, and that we still have $|\lambda * \nu| \le |\lambda| * |\nu|$ for λ, $\nu \in M_\tau^{(\mathbb{C})}$.

(d) Find a locally compact Hausdorff topological group X, a Radon probability measure ν on X and an open set $G \subseteq X$ such that $\{(xGx^{-1})^\bullet : x \in X\}$ is not a separable subset of the measure algebra of ν.

(e) Let X be a metrizable group. We say that a subset A of X is **Haar null** if there are a universally Radon-measurable set $E \supseteq A$ and a non-zero Radon measure ν on X such that $\nu(xEy) = 0$ for every x, $y \in X$. (i) Show that the family of Haar null sets is a translation-invariant σ-ideal of subsets of X. (*Hint*: if $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence of universally Radon-measurable Haar null sets, we can find Radon probability measures ν_n concentrated on compact sets near the identity such that $\nu_n(xE_ny) = 0$ for every x, y and n; now construct an infinite convolution product $\nu = \nu_0 * \nu_1 * \ldots$ from the probability product of the ν_n and show that $\nu(xE_ny) = 0$ for every x, y and n.) (ii) Show that if X and Y are Polish groups, $\phi : X \to Y$ is a surjective continuous homomorphism and $B \subseteq Y$ is Haar null, then $\phi^{-1}[B]$ is Haar null in X. (iii) Show that if X is a locally compact Polish group then a subset of X is Haar null iff it is Haar negligible in the sense of 442H. (See SOLECKI 01.)

(f) Suppose that the continuum hypothesis is true. Let ν be Cantor measure on $\mathbb{R}$ (256Hc). Show that there is a set $A \subseteq \mathbb{R}$ such that $\nu(x+A)=0$ for every $x \in \mathbb{R}$, but A is not Haar negligible.

(g) Let X be a topological group and μ a left Haar measure on X. Let τ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mu)$ (§374). Show that if ν is any totally finite quasi-Radon measure on X, then we have a linear operator $f^\bullet \mapsto (\nu * f)^\bullet$ from L^τ to itself, of norm at most νX.

(h) Let X be a topological group with a left Haar measure μ, M_τ the Banach algebra of signed τ-additive Borel measures on X, and $p \in [1,\infty]$. (i) Show that we have a multiplicative linear operator T from M_τ to the Banach algebra $\mathsf{B}(L^p(\mu);L^p(\mu))$ defined by writing $(T\nu)(f^\bullet)=(\hat\nu * f)^\bullet$ whenever ν is a totally finite τ-additive Borel measure on X with completion $\hat\nu$ and $f \in \mathcal{L}^p(\mu)$. (*Hint*: Use 444K and 444B to show that $(\lambda*\nu)*f =_{\text{a.e.}} \lambda*(\nu*f)$ for enough λ, ν and f. See also 444Yj.) (ii) Show that $\|T\nu\| \le \|\nu\|$ for every $\nu \in M_\tau^+$. (iii) Give an example in which $\|T\nu\| < \|\nu\|$.

(i) Let X be a unimodular topological group with left Haar measure μ. Suppose that p, q, $r \in [1,\infty]$ are such that $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$, interpreting $\frac{1}{\infty}$ as 0. Show that if $f \in \mathcal{L}^p(\mu)$ and $g \in \mathcal{L}^q(\mu)$ then $f*g \in \mathcal{L}^r(\mu)$ and $\|f*g\|_r \le \|f\|_p\|g\|_q$. (*Hint*: 255Yl. Take care to justify any changes in order of integration.)

(j) Let X be a topological group carrying Haar measures. Investigate conditions under which the associative laws

$$\lambda*(\nu*f)=(\lambda*\nu)*f,\quad \lambda*(f*\nu)=(\lambda*f)*\nu,\quad f*(\lambda*\nu)=(f*\lambda)*\nu,$$

$$f*(g*\nu)=(f*g)*\nu,\quad f*(\nu*g)=(f*\nu)*g,\quad \nu*(f*g)=(\nu*f)*g$$

will be valid, where λ and ν are quasi-Radon measures on X and f, g are real-valued functions. Relate your results to 444Xq.

(k) Let X be a topological group with a left Haar measure μ and left modular function Δ. (i) Suppose that $f \in \mathcal{L}^0(\mu)$. Show that the following are equiveridical: (α) $f(yx)=\Delta(y)f(xy)$ for $(\mu\times\mu)$-almost every x, $y \in X$; (β) $(a\bullet_c f)^\bullet = \Delta(a^{-1})f^\bullet$ for every $a \in X$. (ii) Show that in this case $f(x)=0$ for almost every x such that $\Delta(x)\ne 1$. (iii) Suppose that $f \in \mathcal{L}^1(\mu)$. Show that the following are equiveridical: (α) $f(yx)=\Delta(y)f(xy)$ for $(\mu\times\mu)$-almost every x, $y \in X$; (β) $(f*g)^\bullet=(g*f)^\bullet$ for every $g \in \mathcal{L}^1(\mu)$.

(l) Let X be a topological group and μ a left Haar measure on X. Let τ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mu)$ such that $\tau\restriction L^\tau$ is an order-continuous norm. For a totally finite quasi-Radon measure ν on X, let $T_\nu : L^\tau \to L^\tau$ be the corresponding linear operator (444Yg). Show that for any $u \in L^\tau$ and $\epsilon>0$ there is a neighbourhood U of the identity in X such that $\tau(T_\nu u - u)\le\epsilon$ whenever $\nu U=\nu X=1$.

(m) Let X be a topological group with a left Haar measure μ. For $u \in L^2 = L^2(\mu)$, set $A_u=\{a\bullet_l u : a \in X\}$ (443G) in L^2, and $D=\{v*u : v \in L^1(\mu),\ v\ge 0,\ \int v = 1\}$. (i) Show that the closed convex hull of A_u in L^2 is the closure of D. (*Hint*: (α) use 444Od to show that if $w \in L^2$ and $(w'|w)\ge\gamma$ for every $w' \in A_u$, then $(w'|w)\ge\gamma$ for every $w' \in D$ (β) use 444U to show that $A_u \subseteq \overline{D}$.) (ii) Show that the closed linear subspace W_u generated by A_u is the closure of $\{v*u : v \in L^1\}$. (iii) Show that if $w \in L^2$ and $w \in A_u^\perp$, that is, $(u'|w)=0$ for every $u' \in A$, then $W_w \subseteq W_u^\perp$. (iv) Show that if X is σ-compact, then W_u is separable. (*Hint*: A_u is σ-compact, by 443Gf.) (v) Set $C=\{f^\bullet : f \in \mathcal{L}^2\cap C(X)\}$. Show that $C \cap W_u$ is dense in W_u. (*Hint*: $v*u \in C$ for many v, by 444Rc.) (vi) Show that if X is σ-compact, then W_u has an orthonormal basis in C. (vii) Show that L^2 has an orthonormal basis in C. (*Hint*: if X is σ-compact, take a maximal orthogonal family of subspaces W_u, find a suitable orthonormal basis of each, and use (iii) to see that these assemble to form a basis of L^2. For a general locally compact Hausdorff group, start with a σ-compact open subgroup, and then deal with its cosets. For a general topological group with a Haar measure, use 443L.) (Compare 416Yg.)

(n) Let X be a topological group with a left Haar measure μ. Let λ be the quasi-Radon product measure on $X\times X$. Let $\mathfrak{U}$ be the set of those $h \in \mathcal{L}^1(\lambda)$ such that $(X\times X)\setminus\{(x,y) : (x,y)\in\operatorname{dom} h,\ h(x,y)=0\}$ can be covered by a sequence of open sets of finite measure. (i) Show that if $h \in \mathfrak{U}$, then $(x,y)\mapsto h(y,y^{-1}x)$ belongs to $\mathfrak{U}$. (*Hint*: 443Xa.) (ii) Show that if $h \in \mathfrak{U}$, then $(Th)(x)=\int h(y,y^{-1}x)\mu(dy)$ is defined for almost every $x \in X$ and Th is μ-integrable, with $\|Th\|_1 \le \|h\|_1$. (*Hint*: 255Xj.) (iii) Show that if h_1, $h_2 \in \mathfrak{U}$ are equal λ-a.e. then $Th_1 = Th_2$ μ-a.e. (iv) Show that every member of $L^1(\lambda)$ can be represented by a member of $\mathfrak{U}$. (*Hint*: 443Xk.) (v) Show that if f, $g \in \mathcal{L}^1(\mu)$ and both are zero outside some countable union of open sets of finite measure, then $T(f\otimes g)=f*g$, where $(f \otimes g)(x, y) = f(x)g(y)$. (vi) Show that if we set $\tilde{T}(h^\bullet) = (Th)^\bullet$ for $h \in \mathfrak{U}$, then $\tilde{T} : L^1(\lambda) \to L^1(\mu)$ is the unique continuous linear operator such that $\tilde{T}(u \otimes v) = u * v$ for all $u, v \in L^1(\mu)$, where $u * v$ is defined in 444S and $\otimes : L^1(\mu) \times L^1(\mu) \to L^1(\lambda)$ is the canonical bilinear operator (253E).

(o) In 444Yn, suppose that $\mu X = 1$. (i) Show that the map $\tilde{T}$ belongs to the class $\mathcal{T}_{\bar{\lambda},\bar{\mu}}$ of §373. (ii) Show that if $p \in [1, \infty]$ then $\|Th\|_p \le \|h\|_p$ whenever $h \in \mathcal{U} \cap \mathcal{L}^p(\lambda)$.

(p) Rewrite this section in terms of right Haar measures instead of left Haar measures.

(q) Let X be a topological group and M_{qR}^+ the set of totally finite quasi-Radon measures on X. For $\nu \in M_{\mathrm{qR}}^+$, define $\overleftrightarrow{\nu} \in M_{\mathrm{qR}}^+$ by saying that $\overleftrightarrow{\nu}(E) = \nu E^{-1}$ whenever $E \subseteq X$ and ν measures E^{-1}. (i) Show that if $\lambda, \nu \in M_{\mathrm{qR}}^+$ then $\overleftrightarrow{\lambda} * \overleftrightarrow{\nu} = (\nu * \lambda)^{\leftrightarrow}$. (ii) Taking $\bullet_l$, $\bullet_r$ to be the left and right actions of X on itself, and defining corresponding actions of X on M_{qR}^+ as in 441Yo, show that $a\bullet_l(\lambda * \nu) = (a\bullet_l\lambda) * \nu$ and $a\bullet_r(\lambda * \nu) = \lambda * (a\bullet_r\nu)$ for $\lambda, \nu \in M_{\mathrm{qR}}^+$ and $a \in X$.

444 Notes and comments The aim of this section and the next is to work through ideas from the second half of Chapter 25, and Chapter 28, in forms natural in the context of general topological groups. (It is of course possible to go farther; see 444Yb. It is the glory and confusion of twentieth-century mathematics that it has no firm stopping points.) The move from $\mathbb{R}$ to an arbitrary topological group is a large one, and I think it is worth examining the various aspects of this leap as they affect the theorems here. The most conspicuous change, and the one which most greatly affects the forms of the results, is the loss of commutativity. We are forced to re-examine every formula to determine exactly which manipulations can still be justified. Multiplications must be written the right way round, and inversions especially must be watched. But while there are undoubtedly some surprises, we find that in fact (provided we take care over the definitions) the most important results survive. Of course I wrote the earlier results out with a view to what I expected to do here, but no dramatic manoeuvers are needed to turn the fundamental results 255G, 255H, 255J, 257B, 257E, 257F into the new versions 444Od, 444Qb, 444Oe, 444C, 444B, 444Qa. (The changed order of presentation is an indication of the high connectivity of the web here, not of any new pattern.) In fact what makes the biggest difference is not commutativity, as such, but unimodularity. In groups which are not unimodular we do have new phenomena, as in 444Mb and 444Of, and these lead to complications in the proofs of such results as 444U, even though the result there is exactly what one would expect.

In this section I ignore right Haar measures entirely. I do not even put them in the exercises. If you wish to take this theory farther, you may some day have to work out the formulae appropriate to right Haar measures. (You can check your results in Hewitt & Ross 63, 20.32.) But for the moment, I think that they are likely to be just a source of confusion. There is one point which you may have noticed. The theory of groups is essentially symmetric. In the definition of 'group' there is no distinction between left and right. In the formulae defining group actions, we do have such a distinction, because they must reflect the fact that we write $g\bullet x$ rather than $x\bullet g$. With $\bullet_l$ and $\bullet_r$, for instance (444Of), if we want them to be actions in the standard sense we have to put an $^{-1}$ into the definition of $\bullet_l$ but not into the definition of $\bullet_r$. But we still expect that, for instance, $\lambda * \nu$ and $\nu * \lambda$ will be related in some transparent way. However there is an exception to this rule in the definitions of $\nu * f$ and $f * \nu$ (444H, 444J). The modular function appears in the latter, so in fact the definition applies only in a more restricted class of groups. In abelian groups we assume that $f * \nu$ and $\nu * f$ will be equal, and they are (444Xk), but strictly speaking, on the definitions here, we can write $f * \nu = \nu * f$ only for abelian topological groups carrying Haar measures.

From the point of view of the proofs in this section, the principal change is that the Haar measures here are no longer assumed to be σ-finite. I am well aware that non-σ-finite measures are a minority interest, especially in harmonic analysis, but I do think it interesting that σ-finiteness is not relevant to the main results, and the techniques required to demonstrate this are very much in the spirit of this treatise (see, in particular, the proof of 444N, and the repeated applications of 443Jb). The basic difficulty is that we can no longer exchange repeated integrals, even of non-negative Borel measurable functions, quite automatically. Let me emphasize that the result in 444N is really rather special. If we try to generalize it to other measures or other types of function we encounter the usual obstacles (444Xp).

A difficulty of a different kind arises in the proof of 444Fc. Here I wish to show that the function $g \mapsto (g\bullet E)^\bullet : G \to \mathfrak{A}$ is Borel measurable for every Borel measurable set E. The first step is to deal with open sets E, and it would be nice if we could then apply the Monotone Class Theorem. But the difficulty is that even though the map $(a, b) \mapsto a \setminus b : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is continuous, it does not quite follow that the map

$$g \mapsto (g\bullet(E \setminus F))^\bullet = (g\bullet E)^\bullet \setminus (g\bullet F)^\bullet$$

is Borel measurable whenever $g \mapsto (g \bullet E)^\bullet$ and $g \mapsto (g \bullet F)^\bullet$ are, because the map $g \mapsto ((g \bullet E)^\bullet, (g \bullet F)^\bullet) : G \to \mathfrak{A} \times \mathfrak{A}$ might conceivably fail to be Borel measurable, if the metric space $\mathfrak{A}$ is not separable, that is, if the Maharam type of the measure ν is uncountable. Of course the difficulty is easily resolved by an extra twist in the argument.

I use different techniques for the two parts of 444M as an excuse to recall the ideas of §371; in fact part (a) is slightly easier than part (b) when proved by the method of the latter (444Xo).

444U is a kind of density theorem. Compared with the density theorems in §§223 and 261, it is a 'mean' rather than 'pointwise' density theorem; if E is concentrated near the identity, then $f_E^\bullet$ approximates $f^\bullet$ in L^p, but there is no suggestion that we can be sure that $f_E(x) \simeq f(x)$ for any particular xs unless we know much more about the set E. In fact this is to be expected from the form of the results concerning Lebesgue measure. The sets E considered in Volume 2 are generally intervals or balls, and even in such a general form as 223Ya we need a notion of scalar multiplication separate from the group operation.

445 The duality theorem

In this section I present a proof of the Pontryagin-van Kampen duality theorem (445U). As in Chapter 28, and for the same reasons, we need to use complex-valued functions; the relevant formulae in §§443 and 444 apply unchanged, and I shall not repeat them here, but you may wish to re-read parts of those sections taking functions to be complex- rather than real-valued. (It *is* possible to avoid complex-valued measures, which I relegate to the exercises.) The duality theorem itself applies only to abelian locally compact Hausdorff groups, and it would be reasonable, on first reading, to take it for granted that all groups here are of this type, which simplifies some of the proofs a little.

My exposition is based on that of Rudin 67. I start with the definition of 'dual group', including a description of a topology on the dual (445A), and the simplest examples (445B), with a mention of Fourier-Stieltjes transforms of measures (445C-445D). The elementary special properties of dual groups of groups carrying Haar measures are in 445E-445G; in particular, in these cases, the bidual of a group begins to make sense, and we can start talking about Fourier transforms of functions.

Serious harmonic analysis begins with the identification of the dual group with the maximal ideal space of L^1 (445H-445K). The next idea is that of 'positive definite' function (445L-445M). Putting these together, we get the first result here which asserts that the dual group of an abelian group X carrying Haar measures is sufficiently large to effectively describe functions on X (Bochner's theorem, 445N). It is now easy to establish that X can be faithfully embedded in its bidual (445O). We also have most of the machinery necessary to describe the correctly normalized Haar measure of the dual group, with a first step towards identifying functions whose Fourier transforms will have inverse Fourier transforms (the Inversion Theorem, 445P). This leads directly to the Plancherel Theorem, identifying the L^2 spaces of X and its dual (445R). At this point it is clear that the bidual $\mathfrak{X}$ cannot be substantially larger than X, since they must have essentially the same L^2 spaces. A little manipulation of shifts and convolutions in L^2 (445S-445T) shows that X must be dense in $\mathfrak{X}$, and a final appeal to local compactness shows that X is closed in $\mathfrak{X}$.

445A Dual groups Let X be any topological group.

(a) A **character** on X is a continuous group homomorphism from X to $S^1 = \{z : z \in \mathbb{C}, |z| = 1\}$. It is easy to see that the set $\mathcal{X}$ of all characters on X is a subgroup of the group $(S^1)^X$, just because S^1 is an abelian topological group. (If χ, $\theta \in \mathcal{X}$, then $x \mapsto \chi(x)\theta(x)$ is continuous, and

$$(\chi\theta)(xy) = \chi(xy)\theta(xy) = \chi(x)\chi(y)\theta(x)\theta(y) = \chi(x)\theta(x)\chi(y)\theta(y) = (\chi\theta)(x)(\chi\theta)(y).)$$

So $\mathcal{X}$ itself is an abelian group.

(b) Give $\mathcal{X}$ the topology of uniform convergence on subsets of X which are totally bounded for the bilateral uniformity on X (4A5Hb, 4A5O). (If $\mathcal{E}$ is the set of totally bounded subsets of X, then the topology of $\mathcal{X}$ is generated by the pseudometrics ρ_E, where $\rho_E(\chi, \theta) = \sup_{x \in E} |\chi(x) - \theta(x)|$ for $E \in \mathcal{E}$ and χ, $\theta \in \mathcal{X}$. It will be useful, in this formula, to interpret $\sup \emptyset = 0$, so that $\rho_\emptyset$ is the zero pseudometric. Note that $\mathcal{E}$ is closed under finite unions, so $\{\rho_E : E \in \mathcal{E}\}$ is upwards-directed, as in 2A3Fe.) Then $\mathcal{X}$ is a Hausdorff topological group. (If $x \in E \in \mathcal{E}$ and χ, χ_0, θ, $\theta_0 \in \mathcal{X}$,

$$\begin{aligned}|(\chi\theta)(x) - (\chi_0\theta_0)(x)| &= |\chi(x)(\theta(x) - \theta_0(x)) + \theta_0(x)(\chi(x) - \chi_0(x))| \\ &\le |\theta(x) - \theta_0(x)| + |\chi(x) - \chi_0(x)|,\end{aligned}$$

$$|\chi^{-1}(x) - \chi_0^{-1}(x)| = |\overline{\chi(x)} - \overline{\chi_0(x)}| = |\chi(x) - \chi_0(x)|,$$

so

$$\rho_E(\chi\theta, \chi_0\theta_0) \le \rho_E(\chi, \chi_0) + \rho_E(\theta, \theta_0), \quad \rho_E(\chi^{-1}, \chi_0^{-1}) = \rho_E(\chi, \chi_0).$$

If $\chi \ne \theta$ then there is an $x \in X$ such that $\chi(x) \ne \theta(x)$, and now $\{x\} \in \mathcal{E}$ and $\rho_{\{x\}}(\chi, \theta) > 0$.)

(c) Note that if X is locally compact, then its totally bounded sets are just its relatively compact sets (4A5Oe), so the topology of $\mathcal{X}$ is the topology of uniform convergence on compact subsets of X.

(d) If X is compact, then $\mathcal{X}$ is discrete. **P** X itself is totally bounded, so $U = \{\chi : |\chi(x) - 1| \le 1$ for every $x \in X\}$ is a neighbourhood of the identity ι in $\mathcal{X}$. But if $\chi \in U$ and $x \in X$ then $|\chi(x)^n - 1| \le 1$ for every $n \in \mathbb{N}$, so $\chi(x) = 1$. Thus $U = \{\iota\}$ and ι is an isolated point of $\mathcal{X}$; it follows that every point of $\mathcal{X}$ is isolated. **Q**

(e) If X is discrete then $\mathcal{X}$ is compact. **P** The only totally bounded sets in X are the finite sets, so the topology of $\mathcal{X}$ is just that induced by its embedding in $(S^1)^X$. On the other hand, every homomorphism from X to S^1 is continuous, so $\mathcal{X}$ is a closed set in $(S^1)^X$, which is compact by Tychonoff's theorem. **Q**

(f) I ought to remark that to most group theorists the word 'character' means something rather different. For a finite abelian group X with its discrete topology, the 'characters' on X, as defined in (a) above, are just the group homomorphisms from X to $\mathbb{C} \setminus \{0\}$, which in this context can be identified with the characters of the irreducible complex representations of X.

445B Examples (a) If $X = \mathbb{R}$ with addition, then $\mathcal{X}$ can also be identified with the additive group $\mathbb{R}$, if we write $\chi_y(x) = e^{iyx}$ for $x, y \in \mathbb{R}$.

P It is easy to check that every χ_y, so defined, is a character on $\mathbb{R}$, and that $y \mapsto \chi_y : \mathbb{R} \to \mathcal{X}$ is a homomorphism. On the other hand, if χ is a character, then (because it is continuous) there is a $\delta \ge 0$ such that $|\chi(x) - 1| \le 1$ whenever $|x| \le \delta$. $\chi(\delta)$ is uniquely expressible as $e^{i\alpha}$ where $|\alpha| \le \frac{\pi}{2}$. Set $y = \alpha/\delta$, so that $\chi(\delta) = \chi_y(\delta)$. Now $\chi(\frac{1}{2}\delta)$ must be one of the square roots of $\chi(\delta)$, so is $\pm\chi_y(\frac{1}{2}\delta)$; but as $|\chi(\frac{1}{2}\delta) - 1| \le 1$, it must be $+\chi_y(\frac{1}{2}\delta)$. Inducing on n, we see that $\chi(2^{-n}\delta) = \chi_y(2^{-n}\delta)$ for every $n \in \mathbb{N}$, so that $\chi(2^{-n}k\delta) = \chi_y(2^{-n}k\delta)$ for every $k \in \mathbb{Z}$, $n \in \mathbb{N}$; as χ and χ_y are continuous, $\chi = \chi_y$. Thus the map $y \mapsto \chi_y$ is surjective and is a group isomorphism between $\mathbb{R}$ and $\mathcal{X}$.

As for the topology of $\mathcal{X}$, $\mathbb{R}$ is a locally compact topological group, so the totally bounded sets are just the relatively compact sets (4A5Oe again), that is, the bounded sets in the usual sense (2A2F). Now a straightforward calculation shows that for any $\alpha \ge 0$ in $\mathbb{R}$ and $\epsilon \in \,]0, 2[$,

$$\rho_{[-\alpha,\alpha]}(\chi_y, \chi_z) \le \epsilon \iff \alpha|y - z| \le 2\arcsin\frac{\epsilon}{2},$$

so that the topology of $\mathcal{X}$ agrees with that of $\mathbb{R}$. **Q**

(b) Let X be the group $\mathbb{Z}$ with its discrete topology. Then we may identify its dual group $\mathcal{X}$ with S^1 itself, writing $\chi_\zeta(n) = \zeta^n$ for $\zeta \in S^1$, $n \in \mathbb{Z}$. **P** Once again, it is elementary to check that every χ_ζ is a character, and that $\zeta \mapsto \chi_\zeta$ is an injective group homomorphism from S^1 to $\mathcal{X}$. If $\chi \in \mathcal{X}$, set $\zeta = \chi(1)$; then $\chi = \chi_\zeta$. So $\mathcal{X} \cong S^1$. And because the only totally bounded sets in X are finite, $\chi \mapsto \chi_\zeta$ is continuous, therefore a homeomorphism. **Q**

(c) On the other hand, if $X = S^1$ with its usual topology, then we may identify its dual group $\mathcal{X}$ with $\mathbb{Z}$, writing $\chi_n(\zeta) = \zeta^n$ for $n \in \mathbb{Z}$, $\zeta \in S^1$. **P** The verification follows the same lines as in (a) and (b). As usual, the key step is to show that the map $n \mapsto \chi_n : \mathbb{Z} \to \mathcal{X}$ is surjective. We can do this by applying (a). If $\chi \in \mathcal{X}$, then $x \mapsto \chi(e^{ix})$ is a character of $\mathbb{R}$, so there is a $y \in \mathbb{R}$ such that $\chi(e^{ix}) = e^{iyx}$ for every $x \in \mathbb{R}$. In particular, $e^{2iy\pi} = \chi(1) = 1$, so $y \in \mathbb{Z}$, and $\chi = \chi_y$. Concerning the topology of $\mathcal{X}$, we know from 445Ad that it must be discrete, so that also matches the usual topology of $\mathbb{Z}$. **Q**

(d) Let $\langle X_j\rangle_{j\in J}$ be any family of topological groups, and X their product (4A5G). For each $j \in J$ let $\mathcal{X}_j$ be the dual group of X_j. Then the dual group of X can be identified with the subgroup $\mathcal{X}$ of $\prod_{j\in I} \mathcal{X}_j$ consisting of those $\chi \in \prod_{j\in J} \mathcal{X}_j$ such that $\{j : \chi(j)$ is not the identity$\}$ is finite; the action of $\mathcal{X}$ on X is defined by the formula

$$\chi{\bullet}x = \prod_{j\in J} \chi(j)(x(j)).$$

(This is well-defined because only finitely many terms in the product are not equal to 1.) If I is finite, so that $\mathcal{X} = \prod_{j\in I} \mathcal{X}_j$, the topology of $\mathcal{X}$ is the product topology.

P As usual, it is easy to check that $\bullet$, as defined above, defines an injective homomorphism from $\mathcal{X}$ to the dual group of X. If θ is any character on X, then for each $j \in I$ we have a continuous group homomorphism $\varepsilon_j : X_j \to X$ defined by setting $\varepsilon_j(\xi)(j) = \xi$, $\varepsilon_j(\xi)(k) = e_k$, the identity of X_k, for every $k \neq j$. Setting $\chi(j) = \theta\varepsilon_j$ for each j, we obtain $\chi \in \prod_{j\in I} \mathcal{X}_j$. Now there is a neighbourhood U of the identity of X such that $|\theta(x) - 1| \le 1$ for every $x \in U$, and we may suppose that U is of the form $\{x : x(j) \in G_j \text{ for every } j \in J\}$, where $J \subseteq I$ is finite and G_j is a neighbourhood of e_j for every $j \in J$. If $k \in I \setminus J$, $\varepsilon_k(\xi) \in U$ for every $\xi \in X_k$, so that $|\chi(k)(\xi) - 1| \le 1$ for every ξ, and $\chi(k)$ must be the identity character on X_k; this shows that $\chi \in \mathcal{X}$. If $x \in X$ and $x(j) = e_j$ for $j \in J$, then again $|\theta(x^n) - 1| \le 1$ for every $n \in \mathbb{N}$, so $\theta(x) = \chi\bullet x = 1$. For any $x \in X$, we can express it as a finite product $y \prod_{j\in J} \varepsilon_j(x(j))$ where $y(j) = e_j$ for every $j \in J$, so that

$$\theta(x) = \theta(y)\textstyle\prod_{j\in J} \theta\varepsilon_j(x(j)) = \prod_{j\in J} \chi(j)(x(j)) = \chi\bullet x.$$

Thus $\bullet$ defines an isomorphism between $\mathcal{X}$ and the dual group of X.

As for the topology of $\mathcal{X}$, a subset of X is totally bounded iff it is included in a product of totally bounded sets (4A5Od). If $E = \prod_{j\in I} E_j$ is such a product (and not empty), then for χ, $\theta \in \mathcal{X}$

$$\sup_{j\in I} \rho_{E_j}(\chi(j), \theta(j)) \le \rho_E(\chi, \theta) \le \textstyle\sum_{j\in I} \rho_{E_j}(\chi(j), \theta(j)),$$

so (if I is finite) the topology on $\mathcal{X}$ is just the product topology. **Q**

445C Fourier-Stieltjes transforms Let X be a topological group, and $\mathcal{X}$ its dual group. For any totally finite topological measure ν on X, we can form its 'characteristic function' or **Fourier-Stieltjes transform** $\hat{\nu} : \mathcal{X} \to \mathbb{C}$ by writing $\hat{\nu}(\chi) = \int \chi(x)\nu(dx)$. (This generalizes the 'characteristic functions' of §285.)

445D Theorem Let X be a topological group, and $\mathcal{X}$ its dual group. If λ and ν are totally finite quasi-Radon measures on X, then $(\lambda * \nu)^\wedge = \hat{\lambda} \times \hat{\nu}$.

proof If $\chi \in \mathcal{X}$, then, by 444C,

$$\begin{aligned}(\lambda * \nu)^\wedge(\chi) &= \int \chi\, d(\lambda * \nu) = \iint \chi(xy)\lambda(dx)\nu(dy)\\ &= \iint \chi(x)\chi(y)\lambda(dx)\nu(dy) = \int \chi(x)\lambda(dx) \cdot \int \chi(y)\nu(dy) = \hat{\lambda}(\chi)\hat{\nu}(\chi).\end{aligned}$$

445E Let us turn now to groups carrying Haar measures. I start with three welcome properties.

Proposition Let X be a topological group with a neighbourhood of the identity which is totally bounded for the bilateral uniformity on X, and $\mathcal{X}$ its dual group, with its dual group topology.

(a) The map $(\chi, x) \mapsto \chi(x) : \mathcal{X} \times X \to S^1$ is continuous.

(b) Let $\mathfrak{X}$ be the dual group of $\mathcal{X}$, again with its dual group topology, the topology of uniform convergence on totally bounded subsets of $\mathcal{X}$. Then we have a continuous homomorphism $x \mapsto \hat{x} : X \to \mathfrak{X}$ defined by setting $\hat{x}(\chi) = \chi(x)$ for $x \in X$ and $\chi \in \mathcal{X}$.

(c) For any totally finite quasi-Radon measure ν on X, its Fourier-Stieltjes transform $\hat{\nu} : \mathcal{X} \to \mathbb{C}$ is uniformly continuous.

Remark Note that the condition here is satisfied by any topological group X carrying Haar measures (443H).

proof Fix an open totally bounded set U_0 containing the identity.

(a) Let $\chi_0 \in \mathcal{X}$, $x_0 \in X$ and $\epsilon > 0$. Then x_0U_0 is totally bounded, so

$$V = \{\chi : |\chi(y) - \chi_0(y)| \le \frac{1}{2}\epsilon \text{ for every } y \in x_0U_0\}$$

is a neighbourhood of χ_0. Also

$$U = \{x : x \in x_0U_0,\ |\chi_0(x) - \chi_0(x_0)| \le \frac{1}{2}\epsilon\}$$

is a neighbourhood of x_0. And if $\chi \in V$, $x \in U$ we have

$$|\chi(x) - \chi_0(x_0)| \le |\chi(x) - \chi_0(x)| + |\chi_0(x) - \chi_0(x_0)| \le \epsilon.$$

As χ_0, x_0 and ϵ are arbitrary, $(\chi, x) \mapsto \chi(x)$ is continuous.

(b)(i) It is easy to check that $\hat{x}$, as defined above, is always a homomorphism from $\mathcal{X}$ to S^1, and that $x \mapsto \hat{x} : X \to (S^1)^{\mathcal{X}}$ is a homomorphism. Because $\rho_{\{x\}}$ is always one of the defining pseudometrics for the topology of $\mathcal{X}$ (445Ab), $\hat{x}$ is always continuous, so belongs to $\mathfrak{X}$.

(ii) To see that $\hat{}$ is continuous, I argue as follows. Take an open set $H \subseteq \mathfrak{X}$ and $x_0 \in X$ such that $\hat{x}_0 \in H$. Then there are a totally bounded set $F \subseteq \mathcal{X}$ and an $\epsilon > 0$ such that $\mathfrak{x} \in H$ whenever $\mathfrak{x} \in \mathfrak{X}$ and $\rho_F(\mathfrak{x}, \hat{x}_0) \le \epsilon$. Now $x_0 U_0$ is a totally bounded neighbourhood of x_0, so

$$V = \{\theta : \theta \in \mathcal{X},\ |\theta(y) - 1| \le \frac{1}{2}\epsilon \text{ for every } y \in x_0 U_0\}$$

is a neighbourhood of the identity in $\mathcal{X}$. There are therefore $\chi_0, \dots, \chi_n \in \mathcal{X}$ such that $F \subseteq \bigcup_{k \le n} \chi_k V$. Set

$$U = \{x : x \in x_0 U_0,\ |\chi_k(x) - \chi_k(x_0)| < \frac{1}{2}\epsilon \text{ for every } k \le n\}.$$

Then U is an open neighbourhood of x_0 in X.

If $x \in U$ and $\chi \in F$ then there is a $k \le n$ such that $\theta = \chi_k^{-1}\chi \in V$, so that

$$\begin{aligned} |\chi(x) - \chi(x_0)| &= |\chi_k(x)\theta(x) - \chi_k(x_0)\theta(x_0)| \\ &\le |\chi_k(x) - \chi_k(x_0)| + |\theta(x) - \theta(x_0)| \le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \end{aligned}$$

But this shows that $\rho_F(\hat{x}, \hat{x}_0) \le \epsilon$, so $\hat{x} \in H$.

So we have $x_0 \in U \subseteq \{x : \hat{x} \in H\}$. As x_0 is arbitrary, $\{x : \hat{x} \in H\}$ is open; as H is arbitrary, $x \mapsto \hat{x}$ is continuous.

(c) Let $\epsilon > 0$. Because ν is τ-additive, there are $x_0, \dots, x_n \in X$ such that $\nu(X \setminus \bigcup_{k \le n} x_k U_0) \le \frac{1}{3}\epsilon$. Set $E = \bigcup_{k \le n} x_k U_0$; then E is totally bounded. So

$$V = \{\theta : |\theta(x) - 1| \le \tfrac{\epsilon}{1 + 3\nu X} \text{ for every } x \in E\}$$

is a neighbourhood of the identity in $\mathcal{X}$. If χ, $\chi' \in \mathcal{X}$ are such that $\theta = \chi^{-1}\chi'$ belongs to V, then

$$\begin{aligned} |\hat{\nu}(\chi) - \hat{\nu}(\chi')| &\le \int |\chi(x) - \chi'(x)|\nu(dx) = \int |1 - \theta(x)|\nu(dx) \\ &\le 2\nu(X \setminus E) + \tfrac{\epsilon\nu E}{1 + 3\nu X} \le \epsilon. \end{aligned}$$

Since ϵ is arbitrary (and $\mathcal{X}$ is abelian), this is enough to show that $\hat{\nu}$ is uniformly continuous.

445F Fourier transforms of functions Let X be a topological group with a left Haar measure μ. For any μ-integrable complex-valued function f, define its **Fourier transform** $\hat{f} : \mathcal{X} \to \mathbb{C}$ by setting $\hat{f}(\chi) = \int f(x)\chi(x)\mu(dx)$ for every character χ of X. (Compare 283A. If f is real and non-negative, then $\hat{f} = (f\mu)^\wedge$ as defined in 445C, where $f\mu$ is the indefinite-integral measure, as in 444K.) Note that $\hat{f} = \hat{g}$ whenever $f =_{\text{a.e.}} g$, so we can equally well write $\hat{u}(\chi) = \hat{f}(\chi)$ whenever $u = f^\bullet$ in $L^1_{\mathbb{C}}(\mu)$.

445G Proposition Let X be a topological group with a left Haar measure μ. Then for any μ-integrable complex-valued functions f and g, $(f * g)^\wedge = \hat{f} \times \hat{g}$; that is, $(u * v)^\wedge = \hat{u} \times \hat{v}$ for all u, $v \in L^1_{\mathbb{C}}(\mu)$.

proof For any character χ on X,

$$\begin{aligned} \int \chi(x)(f * g)(x)dx &= \iint \chi(xy)f(x)g(y)dxdy \\ &= \iint \chi(x)\chi(y)f(x)g(y)dxdy = \hat{f}(\chi)\hat{g}(\chi) \end{aligned}$$

(using 444Od).

445H Theorem Let X be a topological group with a left Haar measure μ; let $\mathcal{X}$ be its dual group and let Φ be the set of non-zero multiplicative linear functionals on the complex Banach algebra $L^1_{\mathbb{C}} = L^1_{\mathbb{C}}(\mu)$ (444Sb). Then there is a one-to-one correspondence between $\mathcal{X}$ and Φ, defined by the formulae

$$\phi(f^\bullet) = \int f \times \chi \, d\mu = \hat{f}(\chi) \text{ for every } f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu),$$

$$\phi(a \bullet_l u) = \chi(a)\phi(u) \text{ for every } u \in L^1_{\mathbb{C}},\ a \in X,$$

for $\chi \in \mathcal{X}$ and $\phi \in \Phi$.

Remark I follow 443G in writing $a \bullet_l f^\bullet = (a \bullet_l f)^\bullet$, where $(a \bullet_l f)(x) = f(a^{-1}x)$ for $f \in \mathcal{L}^1_{\mathbb{C}}$ and a, $x \in X$, as in 4A5Cc.

proof (a) If $\chi \in \mathcal{X}$ then we can think of its equivalence class $\chi^\bullet$ as a member of $L^\infty_{\mathbb{C}} = L^\infty_{\mathbb{C}}(\mu)$, so that we can define $\phi_\chi \in (L^1_{\mathbb{C}})^*$ by writing $\phi_\chi(u) = \int \chi^\bullet \times u$ for every $u \in L^1_{\mathbb{C}}$; that is, $\phi_\chi(f^\bullet) = \int f \times \chi = \hat{f}(\chi)$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$. 445G tells us that ϕ_χ is multiplicative. To see that it is non-zero, recall that μ is strictly positive (442Aa) and that χ is continuous. Let G be an open set containing the identity e of X such that $|\chi(x) - 1| \le \frac{1}{2}$ for every $x \in G$; then $\mathcal{R}\mathrm{e}(\chi(x)) \ge \frac{1}{2}$ for every $x \in G$, so

$$|\int_G \chi(x)dx| \ge \mathcal{R}\mathrm{e} \int_G \chi(x)dx = \int_G \mathcal{R}\mathrm{e}(\chi(x))dx \ge \tfrac{1}{2}\mu G > 0.$$

Accordingly $\phi_\chi(\chi G)^\bullet \ne 0$ and $\phi_\chi \ne 0$. (I hope that no confusion will arise if I continue occasionally to write χE for the indicator function of a set E, even if the symbol χ is already active in the sentence.)

(b) Now suppose that ϕ is a non-zero multiplicative linear functional on $L^1_{\mathbb{C}}$. Fix on some $g_0 \in \mathcal{L}^1_{\mathbb{C}}$ such that $\phi(g_0^\bullet) = 1$. Let Δ be the left modular function of X. (If you are reading this proof on the assumption that X is abelian, then $a \bullet_r f = a^{-1} \bullet_l f$ and $\Delta \equiv 1$, so the argument below simplifies usefully.)

(i) For any $u \in L^1_{\mathbb{C}}$ and $a \in X$, $\phi(a^{-1} \bullet_l u) = \Delta(a)\phi(a \bullet_r u)$. **P** Let $f \in \mathcal{L}^1_{\mathbb{C}}$ be such that $f^\bullet = u$. Take any $\epsilon > 0$. Then for any sufficiently small open neighbourhood U of the identity, if we set $h = \frac{1}{\mu U}\chi U$, we shall have

$$\|(a \bullet_r f) * h - a \bullet_r f\|_1 \le \epsilon, \quad \|h * f - f\|_1 \le \epsilon$$

(444T, with 444P; see 444U). Setting $w = h^\bullet$, we have

$$\|(a \bullet_r u) * w - a \bullet_r u\|_1 \le \epsilon, \quad \|w * u - u\|_1 \le \epsilon,$$

$$|\phi(a^{-1} \bullet_l u) - \Delta(a)\phi(a \bullet_r u)| \le |\phi(a^{-1} \bullet_l u) - \phi(u * (a^{-1} \bullet_l w))|$$

(444Sa)

$$+ |\Delta(a)\phi((a \bullet_r u) * w) - \Delta(a)\phi(a \bullet_r u)|$$
$$= |\phi(a^{-1} \bullet_l u) - \phi(u)\phi(a^{-1} \bullet_l w)|$$
$$+ \Delta(a)|\phi((a \bullet_r u) * w) - \phi(a \bullet_r u)|$$
$$\le |\phi(a^{-1} \bullet_l u) - \phi(a^{-1} \bullet_l w)\phi(u)|$$
$$+ \Delta(a)\|(a \bullet_r u) * w - a \bullet_r u\|_1$$

(because $\|\phi\| \le 1$ in $(L^1_{\mathbb{C}})^*$, by 4A6F)

$$\le |\phi(a^{-1} \bullet_l u) - \phi((a^{-1} \bullet_l w) * u)| + \epsilon\Delta(a)$$
$$\le \|a^{-1} \bullet_l u - (a^{-1} \bullet_l w) * u\|_1 + \epsilon\Delta(a)$$
$$= \|a^{-1} \bullet_l (u - w * u)\|_1 + \epsilon\Delta(a)$$

(by another of the formulae in 444Sa)

$$= \|u - w * u\|_1 + \epsilon\Delta(a)$$

(443Ge)

$$\le (1 + \Delta(a))\epsilon.$$

As ϵ is arbitrary, we have the result. **Q**

(ii) For any u, $v \in L^1_{\mathbb{C}}$ and $a \in X$, $\phi(a \bullet_l u)\phi(v) = \phi(u)\phi(a \bullet_l v)$. **P**

$$\begin{aligned}\phi(a\bullet_l u)\phi(v) &= \phi((a\bullet_l u)*v) = \phi(a\bullet_l(u*v)) = \Delta(a^{-1})\phi(a^{-1}\bullet_r(u*v))\\ &= \Delta(a^{-1})\phi(u*a^{-1}\bullet_r v) = \phi(u)\Delta(a^{-1})\phi(a^{-1}\bullet_r v) = \phi(u)\phi(a\bullet_l v),\end{aligned}$$

using (i) for the third and sixth equalities, and 444Sa for the second and fourth. **Q**

(iii) Let v_0 be $g_0^\bullet$, so that $\phi(v_0) = 1$, and set $\chi(a) = \phi(a\bullet_l v_0)$ for every $a \in X$. Then if a, $b \in X$,

$$\chi(ab) = \phi(ab\bullet_l v_0) = \phi(a\bullet_l(b\bullet_l v_0))$$

(because $\bullet_l$ is an action of X on $L^1_{\mathbb{C}}$, as noted in 443Ge for L^1)

$$= \phi(a\bullet_l(b\bullet_l v_0))\phi(v_0) = \phi(b\bullet_l v_0)\phi(a\bullet_l v_0)$$

(by (ii) above)

$$= \chi(a)\chi(b).$$

So $\chi : X \to \mathbb{C}$ is a group homomorphism. Moreover, because $\bullet_l$ is continuous (443Gf), and ϕ also is continuous (indeed, of norm at most 1), χ is continuous. Finally,

$$|\chi(a)| \le \|a\bullet_l v_0\|_1 = \|v_0\|_1$$

for every $a \in X$, by 443Gb; it follows at once that $\{\chi(a)^n : n \in \mathbb{Z}\}$ is bounded, so that $|\chi(a)| = 1$, for every $a \in X$. Thus $\chi \in \mathcal{X}$. Moreover, for any $u \in L^1_{\mathbb{C}}$,

$$\phi(a\bullet_l u) = \phi(a\bullet_l u)\phi(v_0) = \phi(u)\phi(a\bullet_l v_0) = \chi(a)\phi(u).$$

(iv) Now $\phi = \phi_\chi$. **P** Because $\phi \in (L^1_{\mathbb{C}})^*$, there is some $h \in \mathcal{L}^\infty_{\mathbb{C}}(\mu)$ such that $\phi(f^\bullet) = \int h(x)f(x)dx$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$ (243Gb/243K; recall that by the rules of 441D, μ is suppose to be a quasi-Radon measure, therefore strictly localizable, by 415A). In this case, for any $f \in \mathcal{L}^1_{\mathbb{C}}$,

$$\phi(f^\bullet) = \phi(f*g_0)^\bullet = \int h(x)(f*g_0)(x)dx = \iint h(xy)f(x)g_0(y)dydx$$

(444Od)

$$\begin{aligned}&= \iint h(y)f(x)g_0(x^{-1}y)dydx = \int \phi(x\bullet_l g_0)^\bullet f(x)dx\\ &= \int \chi(x)f(x)dx = \phi_\chi(f^\bullet).\ \mathbf{Q}\end{aligned}$$

(c) Thus we see that the formulae announced do define a surjection from $\mathcal{X}$ onto Φ. We have still to confirm that it is injective. But if χ, θ are distinct members of $\mathcal{X}$, then $\{x : \chi(x) \ne \theta(x)\}$ is a non-empty open set, so has positive measure, because μ is strictly positive; because μ is semi-finite, they represent different linear functionals on $L^1_{\mathbb{C}}$, and $\phi_\chi \ne \phi_\theta$.

This completes the proof.

445I The topology of the dual group: Proposition Let X be a topological group with a left Haar measure μ, and $\mathcal{X}$ its dual group. For $\chi \in \mathcal{X}$, let $\chi^\bullet$ be its equivalence class in $L^0_{\mathbb{C}} = L^0_{\mathbb{C}}(\mu)$, and $\phi_\chi \in (L^1_{\mathbb{C}})^* = (L^1_{\mathbb{C}}(\mu))^*$ the multiplicative linear functional corresponding to χ, as in 445H. Then the maps $\chi \mapsto \chi^\bullet$ and $\chi \mapsto \phi_\chi$ are homeomorphisms between $\mathcal{X}$ and its images in $L^0_{\mathbb{C}}$ and $(L^1_{\mathbb{C}})^*$, if we give $L^0_{\mathbb{C}}$ the topology of convergence in measure (245A/245M) and $(L^1_{\mathbb{C}})^*$ the weak* topology (2A5Ig).

proof (a) Note that $\chi \mapsto \chi^\bullet$ is injective because μ is strictly positive, so that if χ, θ are distinct members of $\mathcal{X}$ then the non-empty open set $\{x : \chi(x) \ne \theta(x)\}$ has non-zero measure; and that $\chi \mapsto \phi_\chi$ is injective by 445H. So we have one-to-one correspondences between $\mathcal{X}$ and its images in $L^0_{\mathbb{C}}$ and $(L^1_{\mathbb{C}})^*$.

Write $\mathfrak{T}$ for the topology of $\mathcal{X}$ as defined in 445Ab, $\mathfrak{T}_m$ for the topology induced by its identification with its image in $L^0_{\mathbb{C}}$, and $\mathfrak{T}_w$ for the topology induced by its identification with its image in $(L^1_{\mathbb{C}})^*$. Let $\mathcal{E}$ be the family of non-empty totally bounded subsets of X and Σ^f the set of measurable sets of finite measure; for $E \in \mathcal{E}$, $F \in \Sigma^f$ and $f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$ set

$$\rho_E(\chi,\theta)=\sup_{x\in E}|\chi(x)-\theta(x)|,$$

$$\rho'_F(\chi,\theta)=\int_F\min(1,|\chi(x)-\theta(x)|)\mu(dx),$$

$$\rho''_f(\chi,\theta)=|\int f(x)\chi(x)\mu(dx)-\int f(x)\theta(x)\mu(dx)|$$

for χ, $\theta\in\mathcal{X}$. Then $\mathfrak{T}$ is generated by the pseudometrics $\{\rho_E:E\in\mathcal{E}\}$, $\mathfrak{T}_m$ is generated by $\{\rho'_F:F\in\Sigma^f\}$ and $\mathfrak{T}_w$ is generated by $\{\rho''_f:f\in\mathcal{L}^1_{\mathbb{C}}\}$.

(b) $\mathfrak{T}_m\subseteq\mathfrak{T}$. **P** Suppose that $F\subseteq X$ is a measurable set of finite measure, and $\epsilon>0$. There is a non-empty totally bounded open set $U\subseteq X$ (443H). Since $\{xU:x\in X\}$ is an open cover of X and μ is τ-additive, there are $y_0,\dots,y_n\in X$ such that $\mu(F\setminus\bigcup_{j\le n}y_jU)\le\frac{1}{3}\epsilon$; set $E=\bigcup_{j\le n}y_jU$. Then E is totally bounded, and $\rho'_F(\chi,\theta)\le\epsilon$ whenever $\rho_E(\chi,\theta)\le\frac{\epsilon}{1+3\mu E}$. As F and ϵ are arbitrary, the identity map $(\mathcal{X},\mathfrak{T})\to(\mathcal{X},\mathfrak{T}_m)$ is continuous (2A3H), that is, $\mathfrak{T}_m\subseteq\mathfrak{T}$. **Q**

(c) $\mathfrak{T}_w\subseteq\mathfrak{T}_m$. **P** If $f\in\mathcal{L}^1_{\mathbb{C}}$ and $\epsilon>0$ let $F\in\Sigma^f$, $M>0$ be such that $\int(|f|-M\chi F)^+d\mu\le\frac{1}{4}\epsilon$. If χ, $\theta\in\mathcal{X}$ and $\rho'_F(\chi,\theta)\le\frac{\epsilon}{4M}$, then

$$\begin{aligned}\rho''_f(\chi,\theta)&=\Bigl|\int(\chi-\theta)\times f\Bigr|\le\int|\chi-\theta|\times|f|\\&\le 2\int(|f|-M\chi F)^++M\int_F|\chi-\theta|\\&\le\frac{1}{2}\epsilon+2M\int_F\min(1,|\chi-\theta|)=\frac{1}{2}\epsilon+2M\rho'_F(\chi,\theta)\le\epsilon.\end{aligned}$$

As f and ϵ are arbitrary, this shows that $\mathfrak{T}_w\subseteq\mathfrak{T}_m$. **Q**

(d) Finally, $\mathfrak{T}\subseteq\mathfrak{T}_w$. **P** Fix $\chi\in\mathcal{X}$, $E\in\mathcal{E}$ and $\epsilon>0$. Let $u\in L^1_{\mathbb{C}}$ be such that $\phi_\chi(u)=1$, and represent u as $f^\bullet$ where $f\in\mathcal{L}^1_{\mathbb{C}}(\mu)$. Set $U=\{a:a\in X,\ \|a\bullet_l u-u\|_1<\frac{1}{4}\epsilon\}$; then U is an open neighbourhood of the identity e of X, because $a\mapsto a\bullet_l u$ is continuous (443Gf). Because E is totally bounded, there are $y_0,\dots,y_n\in X$ such that $E\subseteq\bigcup_{k\le n}y_kU$. Set $f_k=y_k\bullet_l f$, so that $f_k^\bullet=y_k\bullet_l u$ for each $k\le n$.

Now suppose that $\theta\in\mathcal{X}$ is such that

$$\rho''_f(\theta,\chi)\le\frac{\epsilon}{4},\quad\rho''_{f_k}(\theta,\chi)\le\frac{\epsilon}{4}\text{ for every }k\le n.$$

Take any $x\in E$. Then there is a $k\le n$ such that $x\in y_kU$, so that $y_k^{-1}x\in U$ and

$$\|x\bullet_l u-y_k\bullet_l u\|_1=\|y_k\bullet_l(y_k^{-1}x\bullet_l u-u)\|_1=\|y_k^{-1}x\bullet_l u-u\|_1\le\frac{\epsilon}{4}$$

(using 443Ge for the second equality). Now $\phi_\chi(x\bullet_l u)=\chi(x)$ (445H), so

$$\begin{aligned}|\phi_\theta(x\bullet_l u)-\chi(x)|&\le|\phi_\theta(x\bullet_l u-y_k\bullet_l u)|+|\phi_\theta(y_k\bullet_l u)-\phi_\chi(y_k\bullet_l u)|+|\phi_\chi(y_k\bullet_l u-x\bullet_l u)|\\&\le 2\|x\bullet_l u-y_k\bullet_l u\|_1+\rho''_{f_k}(\theta,\chi)\le\frac{3}{4}\epsilon.\end{aligned}$$

On the other hand,

$$|\theta(x)-\phi_\theta(x\bullet_l u)|=|\theta(x)||1-\phi_\theta(u)|=\rho''_f(\theta,\chi)\le\frac{\epsilon}{4}.$$

So $|\theta(x)-\chi(x)|\le\epsilon$. As x is arbitrary, $\rho_E(\theta,\chi)\le\epsilon$.

As χ, E and ϵ are arbitrary, this shows that $\mathfrak{T}\subseteq\mathfrak{T}_w$. **Q**

445J Corollary For any topological group X carrying Haar measures, its dual group $\mathcal{X}$ is locally compact and Hausdorff.

proof Let Φ be the set of non-zero multiplicative linear functionals on $L^1_{\mathbb{C}}=L^1_{\mathbb{C}}(\mu)$, for some left Haar measure μ on X, and give Φ its weak* topology. Then $\Phi\cup\{0\}\subseteq(L^1_{\mathbb{C}})^*$ is the set of all multiplicative linear functionals on $L^1_{\mathbb{C}}$, and is closed for the weak* topology, because

$$\{\phi : \phi \in (L^1_{\mathbb{C}})^*,\ \phi(u * v) = \phi(u)\phi(v)\}$$

is closed for all u, $v \in L^1_{\mathbb{C}}$. Because the unit ball of $(L^1_{\mathbb{C}})^*$ includes Φ (4A6F again), and is a compact Hausdorff space for the weak* topology (3A5F), so is $\Phi \cup \{0\}$. So Φ itself is an open subset of a compact Hausdorff space and is a locally compact Hausdorff space (3A3Bg). Since the topology on $\mathcal{X}$ can be identified with the weak* topology on Φ (445I), $\mathcal{X}$ also is locally compact and Hausdorff.

445K Proposition Let X be a topological group and μ a left Haar measure on X. Let $\mathcal{X}$ be the dual group of X, and write $C_0 = C_0(\mathcal{X};\mathbb{C})$ for the Banach algebra of continuous functions $h : \mathcal{X} \to \mathbb{C}$ such that $\{\chi : |h(\chi)| \ge \epsilon\}$ is compact for every $\epsilon > 0$.

(a) For any $u \in L^1_{\mathbb{C}} = L^1_{\mathbb{C}}(\mu)$, its Fourier transform $\hat{u}$ belongs to C_0.

(b) The map $u \mapsto \hat{u} : L^1_{\mathbb{C}} \to C_0$ is a multiplicative linear operator, of norm at most 1.

(c) Suppose that X is abelian. For $f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$, set $\tilde{f}(x) = \overline{f(x^{-1})}$ whenever this is defined. Then $\tilde{f} \in \mathcal{L}^1_{\mathbb{C}}$ and $\|\tilde{f}\|_1 = \|f\|_1$. For $u \in L^1_{\mathbb{C}}$, we may define $\tilde{u} \in L^1_{\mathbb{C}}$ by setting $\tilde{u} = \tilde{f}^\bullet$ whenever $u = f^\bullet$. Now $\hat{\tilde{u}}$ is the complex conjugate of $\hat{u}$, so $(u * \tilde{u})^\wedge = |\hat{u}|^2$.

(d) Still supposing that X is abelian, $\{\hat{u} : u \in L^1_{\mathbb{C}}\}$ is a norm-dense subalgebra of C_0, and $\|\hat{u}\|_\infty = r(u)$, the spectral radius of u (4A6G), for every $u \in L^1_{\mathbb{C}}$.

proof (a) As in 445H and 445J, let Φ be the set of non-zero multiplicative linear functionals on $L^1_{\mathbb{C}}$, so that $\Phi \cup \{0\}$ is compact for the weak* topology of $(L^1_{\mathbb{C}})^*$, and $\hat{u}(\chi) = \phi_\chi(u)$ for every $\chi \in \mathcal{X}$. By the definition of the weak* topology, $\phi \mapsto \phi(u)$ is continuous; since we can identify the weak* topology on Φ with the dual group topology of $\mathcal{X}$ (445I), $\hat{u}$ is continuous. Also, for any $\epsilon > 0$,

$$\{\chi : \chi \in \mathcal{X},\ |\hat{u}(\chi)| \ge \epsilon\} \cong \{\phi : \phi \in \Phi,\ |\phi(u)| \ge \epsilon\},$$

which is a closed subset of $\Phi \cup \{0\}$, therefore compact.

(b) It is immediate from the definition of $^\wedge$ that it is a linear operator from $L^1_{\mathbb{C}}$ to $\mathbb{C}^{\mathcal{X}}$, and therefore from $L^1_{\mathbb{C}}$ to C_0; it is multiplicative by 445G, and of norm at most 1 because all the multiplicative linear functionals $u \mapsto \hat{u}(\chi)$ must be of norm at most 1.

(c) Now suppose that X is abelian. If $f \in \mathcal{L}^1_{\mathbb{C}}$, then

$$\int \tilde{f}(x)dx = \overline{\int f(x^{-1})dx} = \overline{\int f(x)dx}$$

by 442Kb, so $\tilde{f} \in \mathcal{L}^1_{\mathbb{C}}$; the same formulae tell us that $\|\tilde{f}\|_1 = \|f\|_1$. If $f =_{\text{a.e.}} g$ then $\tilde{f} =_{\text{a.e.}} \tilde{g}$ (442G, or otherwise), so $\tilde{u}$ is well-defined. If $\chi \in \mathcal{X}$, and $u = f^\bullet$, then

$$\begin{aligned}\hat{\tilde{u}}(\chi) &= \int \tilde{f}(x)\chi(x)dx = \int \overline{f(x^{-1})}\chi(x)dx = \int \overline{f(x)}\chi(x^{-1})dx \\ &= \int \overline{f(x)\chi(x)}dx = \overline{\int f(x)\chi(x)dx} = \overline{\hat{u}(\chi)},\end{aligned}$$

so $\hat{\tilde{u}}$ is the complex conjugate of $\hat{u}$, and

$$(u * \tilde{u})^\wedge = \hat{u} \times \hat{\tilde{u}} = |\hat{u}|^2.$$

(d) To see that $A = \{\hat{u} : u \in L^1_{\mathbb{C}}\}$ is dense in C_0, we can use the Stone-Weierstrass theorem in the form 4A6B. A is a subalgebra of C_0; it separates the points (because the canonical map from $\mathcal{X}$ to $(L^1_{\mathbb{C}})^*$ is injective); if $\chi \in \mathcal{X}$, there is an $h \in A$ such that $h(\chi) \ne 0$ (because elements of $\mathcal{X}$ act on $L^1_{\mathbb{C}}$ as non-zero functionals); and the complex conjugate of any function in A belongs to A, by (c) above.

Accordingly A is dense in C_0, by 4A6B.

The calculation of $\|\hat{u}\|_\infty$ is an immediate consequence of the characterization of $r(u)$ as $\max\{|\phi(u)| : \phi \in \Phi\}$ (4A6K) and the identification of Φ with $\mathcal{X}$.

Remark This is the first point in this section where we really need to know whether or not our group is abelian.

445L Positive definite functions Let X be a group.

(a) A function $h : X \to \mathbb{C}$ is called **positive definite** if

$$\sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k h(x_k^{-1} x_j) \ge 0$$

for all $\zeta_0, \dots, \zeta_n \in \mathbb{C}$ and $x_0, \dots, x_n \in X$.

(b) Suppose that $h : X \to \mathbb{C}$ is positive definite. Then, writing e for the identity of X,

(i) $|h(x)| \le h(e)$ for every $x \in X$;

(ii) $h(x^{-1}) = \overline{h(x)}$ for every $x \in X$.

P If $\zeta \in \mathbb{C}$ and $x \in X$, take $n = 1$, $x_0 = e$, $x_1 = x$, $\zeta_0 = 1$ and $\zeta_1 = \zeta$ in the definition in (a) above, and observe that

$$(1 + |\zeta|^2) h(e) + \zeta h(x) + \bar{\zeta} h(x^{-1}) = h(e^{-1}e) + \zeta h(e^{-1}x) + \bar{\zeta} h(x^{-1}e) + \zeta\bar{\zeta} h(x^{-1}x) \ge 0.$$

Taking $\zeta = 0$, $x = e$ we get $h(e) \ge 0$. Taking $\zeta = 1$ we see that $h(x) + h(x^{-1})$ is real, and taking $\zeta = i$, we see that $h(x) - h(x^{-1})$ is purely imaginary; that is, $h(x^{-1}) = \overline{h(x)}$, for any x. Taking ζ such that $|\zeta| = 1$, $\zeta h(x) = -|h(x)|$ we get $2h(e) - 2|h(x)| \ge 0$, that is, $|h(x)| \le h(e)$ for every $x \in X$. **Q**

(c) If $h : X \to \mathbb{C}$ is positive definite and $\chi : X \to S^1$ is a homomorphism, then $h \times \chi$ is positive definite. **P** If $\zeta_0, \dots, \zeta_n \in \mathbb{C}$ and $x_0, \dots, x_n \in X$ then

$$\sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k (h \times \chi)(x_k^{-1} x_j) = \sum_{j,k=0}^{n} \zeta_j \chi(x_j) \overline{\zeta_k \chi(x_k)} h(x_k^{-1} x_j) \ge 0. \text{ } \mathbf{Q}$$

(d) If X is an abelian topological group and μ a Haar measure on X, then for any $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ the convolution $f * \tilde{f} : X \to \mathbb{C}$ is continuous and positive definite, where $\tilde{f}(x) = \overline{f(x^{-1})}$ whenever this is defined. **P** As in 444Rc, $f * \tilde{f}$ is defined everywhere on X and is continuous. (The definition of $\tilde{}$ has shifted since §444, but the argument there applies unchanged to the present situation.) Now, if $x_0, \dots, x_n \in X$ and $\zeta_0, \dots, \zeta_n \in \mathbb{C}$,

$$\begin{aligned}
\sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k (f * \tilde{f})(x_k^{-1} x_j) &= \sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k \int f(y) \tilde{f}(y^{-1} x_k^{-1} x_j) dy \\
&= \int \sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k f(y) \overline{f(x_j^{-1} x_k y)} dy \\
&= \int \sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k f(x_j y) \overline{f(x_j^{-1} x_k x_j y)} dy \\
&= \int \sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k f(x_j y) \overline{f(x_k y)} dy \\
&= \int |\sum_{j=0}^{n} \zeta_j f(x_j y)|^2 dy \ge 0.
\end{aligned}$$

So $f * \tilde{f}$ is positive definite. **Q**

445M Proposition Let X be a topological group and ν a quasi-Radon measure on X. If $h : X \to \mathbb{C}$ is a continuous positive definite function, then $\iint h(y^{-1}x) f(x) \overline{f(y)} \nu(dx) \nu(dy) \ge 0$ for every ν-integrable function f.

proof (a) Extend f, if necessary, to the whole of X; since the hypothesis implies that $\operatorname{dom} f$ is conegligible, this does not affect the integrals. Let λ be the product quasi-Radon measure on $X \times X$; because h is continuous (by hypothesis) and bounded (by 445L(b-i)), the function $(x, y) \mapsto h(y^{-1}x) f(x) \overline{f(y)}$ is λ-integrable, and (because $\{x : f(x) \ne 0\}$ can be covered by a sequence of sets of finite measure)

$$I = \iint h(y^{-1}x) f(x) \overline{f(y)} \nu(dx) \nu(dy) = \int h(y^{-1}x) f(x) \overline{f(y)} \lambda(d(x, y))$$

(417H).

(b) Let $\epsilon > 0$. Set $\gamma = \sup_{x \in X} |h(x)| = h(e)$ (445L(b-i)). Let $F \subseteq X$ be a non-empty measurable set of finite measure for ν such that $\gamma \int_{(X \times X) \setminus (F \times F)} |f(x) \overline{f(y)}| \lambda(d(x, y)) \le \frac{1}{2}\epsilon$ and f is bounded on F; say $|f(x)| \le M$ for every $x \in F$. Let $\delta > 0$ be such that

$$\delta(M^2 + 2M\gamma)(\nu F)^2 + 2M^2 \gamma \delta \le \tfrac{1}{2}\epsilon.$$

Let $\mathcal{G}$ be the set

$$\{G \times H : G,\, H \subseteq X \text{ are open}, |h(y^{-1}x) - h(y_1^{-1}x_1)| \le \delta \\ \text{whenever } x,\, x_1 \in G,\, y,\, y_1 \in H\}.$$

Because h is continuous, $\mathcal{G}$ is a cover of $X \times X$. Because λ is τ-additive, there is a finite set $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\lambda((F \times F) \setminus \bigcup \mathcal{G}_0) \le \delta$; we may suppose that $\mathcal{G}_0$ is non-empty. Set $W = (F \times F) \cap \bigcup \mathcal{G}_0$. Enumerate $\mathcal{G}_0$ as $\langle G_i \times H_i \rangle_{i \le n}$.

Let $\mathcal{F}$ be a finite partition of F into measurable sets such that $|f(x) - f(x')| \le \delta$ whenever x, x' belong to the same member of $\mathcal{F}$. Let $\mathcal{E}$ be the partition of F generated by $\mathcal{F} \cup \{F \cap G_j : j \le n\} \cup \{F \cap H_j : j \le n\}$. Enumerate $\mathcal{E}$ as $\langle E_k \rangle_{k \le m}$; for each $k \le m$ choose $x_k \in E_k$. Set $J = \{(j,k) : j \le m,\, k \le m,\, E_j \times E_k \subseteq W\}$; then $W = \bigcup_{(j,k) \in J} E_j \times E_k$.

(c) If $(j,k) \in J$, $x \in E_j$ and $y \in E_k$ then

$$|h(y^{-1}x)f(x)\overline{f(y)} - h(x_k^{-1}x_j)f(x_j)\overline{f(x_k)}| \le \delta(M^2 + 2M\gamma),$$

because there must be some $r \le n$ such that $E_j \times E_k \subseteq G_r \times H_r$, so that $|h(y^{-1}x) - h(x_k^{-1}x_j)| \le \delta$, while there are members of $\mathcal{F}$ including E_j and E_k, so that $|f(x) - f(x_j)| \le \delta$ and $|\overline{f(y)} - \overline{f(x_k)}| \le \delta$; at the same time,

$$|f(x)\overline{f(y)}| \le M^2, \quad |h(x_k^{-1}x_j)||\overline{f(y)}| \le M\gamma, \quad |h(x_k^{-1}x_j)||f(x_j)| \le M\gamma$$

because x, y, x_j and x_k all belong to F.

(d) Set $\zeta_j = f(x_j)\nu E_j$ for $j \le m$, so that $\bar{\zeta}_j = \overline{f(x_j)}\nu E_j$. Now consider

$$\begin{aligned}
&\Bigl|\int_W h(y^{-1}x)f(x)\overline{f(y)}\lambda(d(x,y)) - \sum_{(j,k)\in J} \zeta_j\bar{\zeta}_k h(x_k^{-1}x_j)\Bigr| \\
&\qquad = \Bigl|\sum_{(j,k)\in J} \int_{E_j \times E_k} h(y^{-1}x)f(x)\overline{f(y)} - h(x_k^{-1}x_j)f(x_j)\overline{f(x_k)}\lambda(d(x,y))\Bigr| \\
&\qquad \le \sum_{(j,k)\in J} \int_{E_j \times E_k} |h(y^{-1}x)f(x)\overline{f(y)} - h(x_k^{-1}x_j)f(x_j)\overline{f(x_k)}|\lambda(d(x,y)) \\
&\qquad \le \sum_{(j,k)\in J} \delta(M^2 + 2M\gamma)\nu E_j \nu E_k \\
&\qquad \le \delta(M^2 + 2M\gamma)\lambda W \le \delta(M^2 + 2M\gamma)(\nu F)^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Bigl|\int_{(X\times X)\setminus W} h(y^{-1}x)f(x)\overline{f(y)}\lambda(d(x,y))\Bigr| &\le \gamma \int_{(X\times X)\setminus(F\times F)} |f(x)\overline{f(y)}|\lambda(d(x,y)) \\
&\qquad + \gamma \int_{(F\times F)\setminus W} |f(x)\overline{f(y)}|\lambda(d(x,y)) \\
&\le \frac{1}{2}\epsilon + \gamma\delta M^2,
\end{aligned}$$

and

$$\begin{aligned}
\Bigl|\sum_{j\le m, k\le m, (j,k)\notin J} h(x_k^{-1}x_j)f(x_j)\tilde{f}(x_k)\Bigr| &\le \gamma M^2 \sum_{j\le m, k\le m, (j,k)\notin J} \nu E_j \nu E_k \\
&= \gamma M^2 \lambda((F \times F) \setminus W) \le \gamma M^2 \delta.
\end{aligned}$$

Putting these together,

$$|I - \sum_{j,k=0}^{m} \zeta_j\bar{\zeta}_k h(x_k^{-1}x_j)| \le \delta(M^2 + 2M\gamma)(\nu F)^2 + \frac{1}{2}\epsilon + \gamma\delta M^2 + \gamma M^2\delta \le \epsilon.$$

But $\sum_{j,k=0}^{m} \zeta_j\bar{\zeta}_k h(x_k^{-1}x_j) \ge 0$, because h is positive definite. As ϵ is arbitrary, $I \ge 0$, as required.

445N Bochner's theorem (Herglotz 1911, Bochner 33, Weil 40) Let X be an abelian topological group with a Haar measure μ, and $\mathcal{X}$ its dual group. Then for any continuous positive definite function $h : X \to \mathbb{C}$ there is a unique totally finite Radon measure ν on $\mathcal{X}$ such that

$$\int h \times f\,d\mu = \int \hat{f}\,d\nu \text{ for every } f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu),$$

$$h(x) = \int \chi(x)\nu(d\chi) \text{ for every } x \in X.$$

proof (a) If $h(e) = 0$, where e is the identity in X, then $h = 0$, by 445L(b-i), and we can take ν to be the zero measure. Otherwise, since multiplying h by a positive scalar leaves h positive definite and does not affect the result, we may suppose that $h(e) = 1$. For f, $g \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$ set

$$(f|g) = \iint f(x)\overline{g(y)}h(y^{-1}x)dxdy = \iint f(x)\overline{g(y^{-1})}h(yx)dxdy$$

(by 442Kb, since X is unimodular). Then, by 445M, $(f|f) \ge 0$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$. Also $(f_1 + f_2|g) = (f_1|g) + (f_2|g)$, $(\zeta f|g) = \zeta(f|g)$ and $(g|f) = \overline{(f|g)}$ for all f, g, f_1, $f_2 \in \mathcal{L}^1_{\mathbb{C}}$ and $\zeta \in \mathbb{C}$. **P** Only the last is anything but trivial, and for this we have

$$(g|f) = \iint g(x)\overline{f(y)}h(y^{-1}x)dxdy$$
$$= \iint g(x)\overline{f(y)}h(y^{-1}x)dydx$$

(by 417Ha, because $(x, y) \mapsto g(x)\overline{f(y)}h(y^{-1}x)$ is integrable for the product measure and zero off the square of a countable union of sets of finite measure)

$$= \iint g(x)\overline{f(y)h(x^{-1}y)}dydx$$

(using 445L(b-ii))

$$= \overline{\iint f(y)\overline{g(x)}h(x^{-1}y)dydx}$$
$$= \overline{(f|g)}. \textbf{ Q}$$

(b) If f, $g \in \mathcal{L}^1_{\mathbb{C}}$, $|(f|g)|^2 \le (f|f)(g|g)$. **P** (Really this is just Cauchy's inequality.) For any α, $\beta \in \mathbb{C}$,

$$|\alpha|^2(f|f) + 2\,\mathcal{R}\mathrm{e}(\alpha\bar{\beta}(f|g)) + |\beta|^2(g|g) = (\alpha f + \beta g|\alpha f + \beta g) \ge 0.$$

If $(f|f) = 0$ we have $2\,\mathcal{R}\mathrm{e}(\alpha(f|g)) + (g|g) \ge 0$ for every $\alpha \in \mathbb{C}$ so in this case $(f|g) = 0$; similarly $(f|g) = 0$ if $(g|g) = 0$; otherwise we can find non-zero α, β such that $|\alpha|^2 = (g|g)$, $|\beta|^2 = (f|f)$ and $\alpha\bar{\beta}(f|g) = -|\alpha\beta(f|g)|$, in which case the inequality simplifies to $|(f|g)| \le |\alpha\beta|$ and $|(f|g)|^2 \le (f|f)(g|g)$, as required. **Q**

(c) Now consider the functional $\psi \in (L^1_{\mathbb{C}})^* = (L^1_{\mathbb{C}}(\mu))^*$ corresponding to h, so that $\psi(f^\bullet) = \int h \times f\,d\mu$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$. Then $|\psi(f^\bullet)|^2 \le (f|f)$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$. **P** Let $\epsilon > 0$. Then there is an open neighbourhood U of e such that $U = U^{-1}$ and

$$|h(y^{-1}x) - h(e)| \le \epsilon \text{ whenever } x,\ y \in U,$$

$$\|a\bullet_l f - f\|_1 \le \epsilon \text{ for every } a \in U$$

where $(a\bullet_l f)(x) = f(a^{-1}x)$ whenever this is defined, as usual (443Gf). Shrinking U if need be, we may suppose that $\mu U < \infty$, and of course $\mu U > 0$. Set $g = \frac{1}{\mu U}\chi U \in \mathcal{L}^1_{\mathbb{C}}$. Then

$$
\begin{aligned}
|(f|g)-\psi(f^\bullet)| &= \frac{1}{\mu U}\Bigl|\int_U\int_X f(x)h(y^{-1}x)dxdy-\int_U\int_X f(x)h(x)dxdy\Bigr| \\
&= \frac{1}{\mu U}\Bigl|\int_U\int_X f(yx)h(x)dxdy-\int_U\int_X f(x)h(x)dxdy\Bigr| \\
&= \frac{1}{\mu U}\Bigl|\int_U\int_X ((y^{-1}\bullet_l f)(x)-f(x))h(x)dxdy\Bigr| \\
&\le \frac{1}{\mu U}\int_U\int_X |(y^{-1}\bullet_l f)(x)-f(x)||h(x)|dxdy \\
&\le \frac{1}{\mu U}\int_U \|y^{-1}\bullet_l f-f\|_1\mu(dy)\le\epsilon.
\end{aligned}
$$

Also

$$
\begin{aligned}
|(g|g)-1| &= \Bigl|\frac{1}{\mu U^2}\int_U\int_U(h(y^{-1}x)-1)dxdy\Bigr| \\
&\le \frac{1}{\mu U^2}\int_U\int_U|h(y^{-1}x)-1|dxdy\le\epsilon.
\end{aligned}
$$

So

$$\max(0,|\psi(f^\bullet)|-\epsilon)^2\le|(f|g)|^2\le(f|f)(g|g)\le(1+\epsilon)(f|f).$$

Letting $\epsilon\downarrow 0$ we have the result. **Q**

(d) If we look at $(f|f)$, however, and apply 444Od, we see that

$$
\begin{aligned}
(f|f) &= \iint f(x)\overline{f(y)}h(y^{-1}x)dxdy \\
&= \iint f(x)\overline{f(y^{-1})}h(yx)dxdy=\int h(x)(f*\tilde f)(x)dx,
\end{aligned}
$$

where $\tilde f(x)=\overline{f(x^{-1})}$ whenever this is defined; that is,

$$(f|f)=\psi(f*\tilde f)^\bullet.$$

(Note that $\tilde f\in\mathcal{L}^1_{\mathbb{C}}$ because X is unimodular, as in part (c) of the proof of 445K.) So (c) tells us that

$$|\psi(f^\bullet)|^2\le\psi(f*\tilde f)^\bullet$$

for every $f\in\mathcal{L}^1_{\mathbb{C}}$, that is, $|\psi(u)|^2\le\psi(u*\tilde u)$ for every $u\in L^1_{\mathbb{C}}$, defining $\tilde u$ as in 445Kc.

(e) In fact

$$|\psi(u)|\le\|\hat u\|_\infty$$

for every $u\in L^1_{\mathbb{C}}$. **P** Set $u_0=u$ and $u_{k+1}=u_k*\tilde u_k$ for every $k\in\mathbb{N}$. We need to know that $u_k=\tilde u_k$ for $k\ge1$. To see this, represent u_{k-1} as $f^\bullet$ where $f\in\mathcal{L}^1_{\mathbb{C}}$, so that $u_k=(f*\tilde f)^\bullet$. Now

$$
\begin{aligned}
(f*\tilde f)^\sim(x) &= \overline{(f*\tilde f)(x^{-1})}=\overline{\int f(x^{-1}y)\tilde f(y^{-1})dy} \\
&= \int\overline{f(x^{-1}y)}f(y)dy=\int\tilde f(y^{-1}x)f(y)dy=(f*\tilde f)(x)
\end{aligned}
$$

for every x, so $(f*\tilde f)^\sim=f*\tilde f$ and $\tilde u_k=u_k$. Accordingly $u_{k+1}=u_k*u_k$ for $k\ge1$ and we have $u_k=(u_1)^{2^{k-1}}$ for every $k\ge1$.

At the same time, we have $|\psi(u_k)|^2\le\psi(u_{k+1})$ for every k, by (d), so that, for $k\ge1$,

$$|\psi(u)|^{2^k}\le\psi(u_k)\le\|u_k\|_1=\|u_1^{2^{k-1}}\|_1,$$

$$|\psi(u)|\le\|u_1^{2^{k-1}}\|_1^{1/2^k}.$$

Letting $k\to\infty$, $|\psi(u)|\le\sqrt{r(u_1)}$, where $r(u_1)$ is the spectral radius of u_1.

At this point, recall that $r(u_1)=\|\hat{u}_1\|_\infty$ (445Kd), while $|\hat{u}|^2=\hat{u}_1$ (445Kc), so $r(u_1)=\|\hat{u}\|_\infty^2$ and $|\psi(u)|\le\|\hat{u}\|_\infty$. **Q**

(f) Now consider $\hat{}$ as a linear operator from $L^1_{\mathbb{C}}$ to $C_0=C_0(\mathcal{X};\mathbb{C})$, as in 445K. If $\hat{u}=0$ then $\psi(u)=0$, by (e), so setting $A=\{\hat{u}:u\in L^1_{\mathbb{C}}\}$ we have a linear functional $\psi_0:A\to\mathbb{C}$ defined by saying that $\psi_0(\hat{u})=\psi(u)$ for every $u\in L^1_{\mathbb{C}}$. By (e), $\|\psi_0\|\le 1$. Now ψ_0 has an extension to a bounded linear operator ψ_1, still of norm at most 1, from C_0 to $\mathbb{C}$ (3A5Ab).

(g) Suppose that $q\in C_0$ and $0\le q\le\mathbf{1}$, writing $\mathbf{1}$ for the constant function with value 1; set $\alpha=\psi_1(q)$. Then for any $\zeta\in\mathbb{C}$ and $\gamma\ge 0$ we have $|\zeta-\gamma\alpha|\le\max(|\zeta|,|\gamma|,|\zeta-\gamma|)$. **P** Let $\epsilon>0$. Set $V=\{x:|1-h(x)|<\epsilon\}$; then V is an open neighbourhood of e; set $f=\frac{1}{\mu V}\chi V$ and $u=f^\bullet$, so that

$$\|\hat{u}\|_\infty=r(u)\le\|u\|_1=1,$$

$$|1-\psi(u)|=|\frac{1}{\mu V}\int_V(1-h(x))dx|\le\epsilon.$$

Set $v=u*\tilde{u}$; then

$$\psi_1(\hat{v})=\psi(v)\ge|\psi(u)|^2\ge(1-\epsilon)^2,$$

using part (d) for the central inequality. But $\hat{v}=|\hat{u}|^2$, so that $0\le\hat{v}\le\mathbf{1}$ and $\psi_1(\hat{v})\le 1$.

Now consider $\|\zeta\hat{v}-\gamma q\|_\infty$. If $\chi\in\mathcal{X}$, then $\zeta\hat{v}(\chi)$ and $\gamma q(\chi)$ both lie in the triangle with vertices 0, ζ and γ, because $0\le\hat{v}\le\mathbf{1}$ and $0\le q\le\mathbf{1}$. So

$$|\zeta\hat{v}(\chi)-\gamma q(\chi)|\le\max(|\gamma|,|\zeta|,|\gamma-\zeta|).$$

As χ is arbitrary,

$$\|\zeta\hat{v}-\gamma q\|_\infty\le\max(|\gamma|,|\zeta|,|\gamma-\zeta|).$$

Accordingly

$$\begin{aligned}|\zeta-\gamma\alpha|&\le|\zeta-\zeta\psi_1(\hat{v})|+|\psi_1(\zeta\hat{v}-\gamma q)|\\&\le|\zeta|(1-(1-\epsilon)^2)+\|\zeta\hat{v}-\gamma q\|_\infty\le 2\epsilon|\zeta|+\max(|\zeta|,|\gamma|,|\zeta-\gamma|).\end{aligned}$$

As ϵ is arbitrary, we have the result. **Q**

Taking $\zeta=\gamma=1$ we see that $|1-\alpha|\le 1$, so that $\mathcal{R}\mathrm{e}\,\alpha\ge 0$. Taking $\zeta=\pm i$, we see that $|i\pm\gamma\alpha|\le\sqrt{1+\gamma^2}$ for every $\gamma\ge 0$, so that $\mathcal{I}\mathrm{m}\,\alpha=0$. Thus $\psi_1(q)\ge 0$; and this is true whenever $0\le q\le\mathbf{1}$ in C_0.

(h) It follows at once that $\psi_1(q)\ge 0$ whenever $q\ge 0$ in C_0. Applying the Riesz Representation Theorem, in the form 436K, to the restriction of ψ_1 to $C_0(\mathcal{X};\mathbb{R})$, we see that there is a totally finite Radon measure ν on $\mathcal{X}$ such that $\psi_1(q)=\int q\,d\nu$ for every real-valued $q\in C_0$; of course it follows that $\psi_1(q)=\int q\,d\nu$ for every $q\in C_0$. Unwrapping the definition of ψ_1, we see that

$$\int h(x)f(x)dx=\psi(f^\bullet)=\psi_1(\hat{f})=\int\hat{f}\,d\nu$$

for every $f\in\mathcal{L}^1_{\mathbb{C}}(\mu)$.

(i) For the second formula, argue as follows. Given $f\in\mathcal{L}^1_{\mathbb{C}}(\mu)$, consider the function $(x,\chi)\mapsto f(x)\chi(x):X\times\mathcal{X}\to\mathbb{C}$. Because $(\chi,x)\mapsto\chi(x)$ is continuous (445Ea), this is Λ-measurable, where Λ is the domain of the product quasi-Radon measure $\mu\times\nu$ on $X\times\mathcal{X}$. It is integrable because $\nu\mathcal{X}<\infty$ and $|\chi(x)|=1$ for every χ, x; moreover, it is zero off the set $\{x:f(x)\ne 0\}\times\mathcal{X}$, which is a countable union of products of sets of finite measure. Note also that because $\chi\mapsto\chi(x)$ is continuous and bounded for every $x\in X$, $h_1(x)=\int\chi(x)\nu(d\chi)$ is defined, and $|h_1(x)|\le\nu X$, for every $x\in X$. What is more, h_1 is continuous. **P** Let $\mathfrak{X}$ be the dual group of $\mathcal{X}$, and for $x\in X$ let $\hat{x}$ be the corresponding member of $\mathfrak{X}$. Then, in the language of 445C, applied to the topological group $\mathcal{X}$,

$$h_1(x)=\int\hat{x}\,d\nu=\hat{\nu}(\hat{x})$$

for every $x\in X$. But $\hat{\nu}:\mathfrak{X}\to\mathbb{C}$ is continuous, by 445Ec, and $x\mapsto\hat{x}:X\to\mathfrak{X}$ is continuous, by 445Eb; so h_1 also is continuous. **Q**

We may therefore apply Fubini's theorem (417H) to see that

$$\int f(x)h_1(x)\mu(dx) = \iint f(x)\chi(x)\nu(d\chi)\mu(dx) = \iint f(x)\chi(x)\mu(dx)\nu(d\chi)$$
$$= \int \hat{f}(\chi)\nu(d\chi) = \int f(x)h(x)\mu(dx).$$

Since this is true for every $f \in \mathcal{L}^1_{\mathbb{C}}$, $h_1 =_{\text{a.e.}} h$; since both are continuous, $h_1 = h$, as required.

(j) Finally, to see that ν is uniquely defined, note that $\{\hat{f} : f \in \mathcal{L}^1_{\mathbb{C}}\}$ is $\| \|_\infty$-dense in C_0 (445Kd), so 436K tells us that there can be at most one totally finite Radon measure ν on $\mathcal{X}$ such that $\int h \times f\,d\mu = \int \hat{f}\,d\nu$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$.

445O Proposition Let X be a Hausdorff abelian topological group carrying Haar measures. Then the map $x \mapsto \hat{x}$ from X to its bidual group $\mathfrak{X}$ is a homeomorphism between X and its image in $\mathfrak{X}$. In particular, the dual group $\mathcal{X}$ of X separates the points of X.

proof We already know that $\hat{}$ is continuous (445Eb) and that $\mathcal{X}$ is locally compact and Hausdorff (445J). Now let U be any neighbourhood of the identity e of X. Let $V \subseteq U$ be an open neighbourhood of e such that $VV^{-1} \subseteq U$ and $\mu V < \infty$. Then $f = \chi V \in \mathcal{L}^2_{\mathbb{C}}(\mu)$, so $f * \tilde{f}$ is positive definite and continuous (445Ld) and there is a totally finite Radon measure ν on $\mathcal{X}$ such that $(f * \tilde{f})(x) = \int \chi(x)\nu(d\chi)$ for every $x \in X$ (445N). Note that, writing $\mathfrak{e}$ for the identity of $\mathfrak{X}$ and $\hat{\nu} : \mathfrak{X} \to \mathbb{C}$ for the Fourier-Stieltjes transform of ν,

$$\hat{\nu}(\mathfrak{e}) = \int \mathfrak{e}(\chi)\nu(d\chi) = \int \chi(e)\nu(d\chi) = (f * \tilde{f})(e)$$
$$= \int f(y)\tilde{f}(y^{-1})\mu(dy) = \int |f(y)|^2\mu(dy) \neq 0.$$

Now $\hat{\nu}$ is continuous (445Ec), so $W = \{\mathfrak{x} : \hat{\nu}(\mathfrak{x}) \neq 0\}$ is a neighbourhood of $\mathfrak{e}$. If $x \in X$ and $\hat{x} \in W$, then

$$(f * \tilde{f})(x) = \int \chi(x)\nu(d\chi) = \int \hat{x}(\chi)\nu(d\chi) = \hat{\nu}(\hat{x}) \neq 0,$$

so there is some $y \in X$ such that $f(y)\tilde{f}(y^{-1}x) \neq 0$, that is, $f(y) \neq 0$ and $f(x^{-1}y) \neq 0$, that is, y and $x^{-1}y$ both belong to V; in which case $x \in VV^{-1} \subseteq U$.

Thus $U \supseteq \{x : \hat{x} \in W\}$. This means that, writing $\mathfrak{S}$ for $\{\{x : \hat{x} \in H\} : H \subseteq \mathfrak{X}$ is open$\}$, every neighbourhood of e for the original topology $\mathfrak{T}$ of X is a neighbourhood of e for $\mathfrak{S}$. But (it is easy to check) $(X, \mathfrak{S})$ is a topological group because $\mathfrak{X}$ is a topological group and $\hat{}$ is a homomorphism. So $\mathfrak{T} \subseteq \mathfrak{S}$ (4A5Fb). As we know already that $\mathfrak{S} \subseteq \mathfrak{T}$, the two topologies are equal.

It follows at once that if $\mathfrak{T}$ is Hausdorff, then (because $\mathfrak{S}$ is Hausdorff) the map $\hat{}$ is an injection and is a homeomorphism between X and its image in $\mathfrak{X}$.

445P The Inversion Theorem Let X be an abelian topological group and μ a Haar measure on X. Then there is a unique Haar measure λ on the dual group $\mathcal{X}$ of X such that whenever $f : X \to \mathbb{C}$ is continuous, μ-integrable and positive definite, then $\hat{f} : \mathcal{X} \to \mathbb{C}$ is λ-integrable and

$$f(x) = \int \hat{f}(\chi)\overline{\chi(x)}\lambda(d\chi)$$

for every $x \in X$.

proof (a) Write P for the set of μ-integrable positive definite continuous functions $h : X \to \mathbb{C}$. For $h \in P$, let ν_h be the corresponding totally finite Radon measure on $\mathcal{X}$ defined in 445N, so that

$$\int f \times h\,d\mu = \int \hat{f}\,d\nu_h$$

for every $f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$.

(b) The basis of the argument is the following fact. If $f \in \mathcal{L}^1_{\mathbb{C}}$ and h_1, $h_2 \in P$, then

$$\int \hat{f} \times \hat{\bar{h}}_2 d\nu_{h_1} = \int \hat{f} \times \hat{\bar{h}}_1 d\nu_{h_2}.$$

$$\mathbf{P} \int \hat{f} \times \hat{\bar{h}}_1 d\nu_{h_2} = \int (f * \bar{h}_1)^\wedge d\nu_{h_2}$$

(445G)

$$= \int h_2(x)(f * \bar{h}_1)(x)\mu(dx) = \int \bar{h}_2(x^{-1})(f * \bar{h}_1)(x)\mu(dx)$$

(by 445Lb)

$$= ((f * \bar{h}_1) * \bar{h}_2)(e) = ((f * \bar{h}_2) * \bar{h}_1)(e)$$

(because $*$ is associative and commutative, by 444Oe and 444Og)

$$= \int \hat{f} \times \hat{\hat{h}}_2 d\nu_{h_1}. \ \mathbf{Q}$$

Now because $\hat{\hat{h}}_1$ and $\hat{\hat{h}}_2$ are both bounded (by $\int |h_1|d\mu$ and $\int |h_2|d\mu$ respectively), and ν_{h_1} and ν_{h_2} are both totally finite measures, and $\{\hat{f} : f \in \mathcal{L}^1_{\mathbb{C}}(\mu)\}$ is $\| \|_\infty$-dense in $C_0 = C_0(\mathcal{X};\mathbb{C})$ (445Kd), we must have

$$\int p \times \hat{\hat{h}}_2 d\nu_{h_1} = \int p \times \hat{\hat{h}}_1 d\nu_{h_2}$$

for every $p \in C_0$.

(c) Let $\mathcal{K}$ be the family of compact subsets of $\mathcal{X}$. For $K \in \mathcal{K}$ set

$$P_K = \{h : h \in P,\, \hat{\hat{h}}(\chi) > 0 \text{ for every } \chi \in K\}.$$

Then P_K is non-empty. $\mathbf{P}$ Set

$$U = \{x : x \in X,\, |1 - \chi(x)| \le \tfrac{1}{2} \text{ for every } \chi \in K\}.$$

Then U is a neighbourhood of the identity e of X, by 445Eb. Let V be an open neighbourhood of e, of finite measure, such that $VV^{-1} \subseteq U$, set $g = \frac{1}{\mu V}\chi V$, and try $h = g * \tilde{g}$. Then h is continuous and positive definite (445Ld), real-valued and non-negative (because g and $\tilde{g}$ are), zero outside U (because $VV^{-1} \subseteq U$, as in the proof of 445O), and

$$\int h\, d\mu = \int g\, d\mu \cdot \int \tilde{g}\, d\mu = 1$$

(444Qb). Next,

$$\hat{\hat{h}} = \hat{h} = |\hat{g}|^2$$

(445Kc) is non-negative, and if $\chi \in K$ then

$$|1 - \hat{h}(\chi)| = |\int h(x) - h(x)\chi(x)\mu(dx)| \le \int h(x)|1 - \chi(x)|\mu(dx) \le \tfrac{1}{2}$$

because $|1 - \chi(x)| \le \frac{1}{2}$ if $x \in U$ and $h(x) = 0$ if $x \in X \setminus U$. So

$$\hat{\hat{h}}(\chi) = \hat{h}(\chi) \ge \tfrac{1}{2}$$

for every $\chi \in K$, and $h \in P_K$. $\mathbf{Q}$

(d) Because $\mathcal{K}$ is upwards-directed, $\{P_K : K \in \mathcal{K}\}$ is downwards-directed and generates a filter $\mathcal{F}$ on P. Let $C_k = C_k(\mathcal{X};\mathbb{C})$ be the space of continuous complex-valued functions on $\mathcal{X}$ with compact support. If $q \in C_k$, then

$$\phi(q) = \lim_{h\to\mathcal{F}} \int \frac{q}{\hat{\hat{h}}}\, d\nu_h$$

is defined in $\mathbb{C}$, where in the division $q/\hat{\hat{h}}$ we interpret $0/0$ as 0 if necessary. $\mathbf{P}$ Setting $K = \overline{\{\chi : q(\chi) \ne 0\}}$, we see in fact that for any h_1, $h_2 \in P_K$ we may define a function $p \in C_k$ by setting

$$\begin{aligned} p(\chi) &= \frac{q(\chi)}{\hat{\hat{h}}_1(\chi)\hat{\hat{h}}_2(\chi)} \text{ if } \chi \in K, \\ &= 0 \text{ if } q(\chi) = 0, \end{aligned}$$

so that

$$\int \frac{q}{\hat{\hat{h}}_1}\, d\nu_{h_1} = \int p \times \hat{\hat{h}}_2\, d\nu_{h_1} = \int p \times \hat{\hat{h}}_1\, d\nu_{h_2}$$

(by (b) above)

$$= \int \frac{q}{\hat{\hat{h}}_2}\,d\nu_{h_2}.$$

So this common value must be $\phi(q)$. **Q**

If q, $q' \in C_k$ and $\alpha \in \mathbb{C}$, then

$$\int \frac{q+q'}{\hat{\hat{h}}}\,d\nu_h = \int \frac{q}{\hat{\hat{h}}}\,d\nu_h + \int \frac{q'}{\hat{\hat{h}}}\,d\nu_h,$$

$$\int \frac{\alpha q}{\hat{\hat{h}}}\,d\nu_h = \alpha \int \frac{q}{\hat{\hat{h}}}\,d\nu_h$$

whenever $h \in P_K$, where $K = \overline{\{\chi : |q(\chi)| + |q'(\chi)| > 0\}}$; so $\phi(q+q') = \phi(q) + \phi(q')$ and $\phi(\alpha q) = \alpha\phi(q)$. Moreover, if $q \ge 0$, then $q/\hat{\hat{h}} \ge 0$ for every $h \in P_K$, so $\phi(q) \ge 0$.

(e) By the Riesz Representation Theorem (in the form 436J) there is a Radon measure λ on $\mathcal{X}$ such that $\int q\,d\lambda = \phi(q)$ for any continuous function q of compact support. (As in part (h) of the proof of 445N, the shift from real-valued q to complex-valued q is elementary.)

(f) Now λ is translation-invariant. **P** Take $\theta \in \mathcal{X}$ and $q \in C_k$. Set $K = \overline{\{\chi : q(\chi) \ne 0\}}$ and $L = \theta^{-1}K$, and take any $h \in P_K$. Set $h_1(x) = h \times \theta^{-1}$. Then h_1 is positive definite (445Lc); of course it is continuous and μ-integrable; and for any $\chi \in L$,

$$\hat{\hat{h}}_1(\chi) = \int \overline{h(x)}\theta(x)\chi(x)\mu(dx) = \hat{\hat{h}}(\theta\chi) > 0.$$

So $h_1 \in P_L$.

To relate ν_{h_1} to ν_h, observe that if $f \in \mathcal{L}^1_{\mathbb{C}}$ then

$$\hat{f}(\theta\chi) = \int f(x)\theta(x)\chi(x)\mu(dx) = (f \times \theta)^\wedge(\chi),$$

so

$$\int \hat{f}(\theta\chi)\nu_{h_1}(d\chi) = \int (f \times \theta)(x)h_1(x)\mu(dx) = \int f(x)h(x)\mu(dx) = \int \hat{f}(\chi)\nu_h(d\chi).$$

So we see that the equation

$$\int p(\theta\chi)\nu_{h_1}(d\chi) = \int p(\chi)\nu_h(d\chi)$$

is valid whenever p is of the form $\hat{f}$, for some $f \in \mathcal{L}^1_{\mathbb{C}}$, and therefore for every $p \in C_0$.

Set $q_1(\chi) = q(\theta\chi)$ for every $\chi \in \mathcal{X}$, so that $q_1 \in C_k$ and $L = \overline{\{\chi : q_1(\chi) \ne 0\}}$. Accordingly

$$\phi(q_1) = \int \frac{q_1(\chi)}{\hat{\hat{h}}_1(\chi)}\nu_{h_1}(d\chi) = \int \frac{q(\theta\chi)}{\hat{\hat{h}}(\theta\chi)}\nu_{h_1}(d\chi)$$
$$= \int \frac{q(\chi)}{\hat{\hat{h}}(\chi)}\nu_h(d\chi) = \phi(q).$$

So

$$\int q(\theta\chi)\lambda(d\chi) = \int q(\chi)\lambda(d\chi).$$

As q and θ are arbitrary, λ is translation-invariant (441L). **Q**

(g) Thus λ is either zero or a Haar measure on $\mathcal{X}$. I have still to confirm that

$$f(x) = \int \hat{f}(\chi)\overline{\chi(x)}\lambda(d\chi)$$

whenever f is continuous, positive definite and μ-integrable, and $x \in X$. But recall the formula from (b) above. If $q \in C_k$, $K = \overline{\{\chi : q(\chi) \ne 0\}}$ and $h \in P_K$, then we must have

$$\int q \times \hat{\hat{f}}\,d\lambda = \int \frac{q \times \hat{\hat{f}}}{\hat{\hat{h}}}\,d\nu_h = \int \frac{q \times \hat{\hat{h}}}{\hat{\hat{h}}}\,d\nu_f = \int q\,d\nu_f.$$

In particular, $\int q\times\hat{\hat{f}}d\lambda\ge 0$ whenever $q\ge 0$; since $\hat{\hat{f}}$ is continuous (445Ka), and λ, being a Radon measure, is strictly positive, this shows that $\hat{\hat{f}}\ge 0$. Also

$$\int\hat{\hat{f}}d\lambda=\sup\{\int q\times\hat{\hat{f}}d\lambda:q\in C_k,\,0\le q\le\mathbf{1}\}=\nu_f(\mathcal{X})<\infty,$$

so we have an indefinite-integral measure $\hat{\hat{f}}\lambda$; since this is a Radon measure (416S), and acts on C_k in the same way as ν_f, it is actually equal to ν_f (by the uniqueness guaranteed in 436J). In particular, for any $x\in X$,

$$f(x)=\int\chi(x)\nu_f(d\chi)=\int\chi(x)\hat{\hat{f}}(\chi)\lambda(d\chi)$$

by the second formula in 445N, and 235K. But

$$\hat{\hat{f}}(\chi)=\int\overline{f(x)}\chi(x)\mu(dx)=\int f(x^{-1})\chi(x)\mu(dx)$$

(by 445Lb)

$$=\int f(x)\chi(x^{-1})\mu(dx)$$

(because X is abelian, therefore unimodular)

$$=\hat{f}(\chi^{-1})$$

for every $\chi\in\mathcal{X}$. So

$$f(x)=\int\chi(x)\hat{f}(\chi^{-1})\lambda(d\chi)=\int\chi^{-1}(x)\hat{f}(\chi)\lambda(d\chi)$$

(because $\mathcal{X}$ is abelian)

$$=\int\hat{f}(\chi)\overline{\chi(x)}\lambda(d\chi).$$

(h) We should check that λ is non-zero and unique. But the construction in part (c) of the proof shows that there are many $f\in P$ such that $f(e)\ne 0$, and for any such f we have $f(e)=\int\hat{f}\,d\lambda$. This shows simultaneously that λ is non-zero, therefore a Haar measure; and as all the Haar measures on $\mathcal{X}$ are scalar multiples of each other, there is at most one suitable λ.

445Q Remark We can extract the following useful fact from part (g) of the proof above. If $h:X\to\mathbb{C}$ is μ-integrable, continuous and positive definite, then $\hat{\hat{h}}$ is non-negative and λ-integrable, and the Radon measure ν_h of 445N is just the indefinite-integral measure $\hat{\hat{h}}\lambda$.

Note also that λ is actually a Radon measure; of course it has to be, because $\mathcal{X}$ is locally compact and Hausdorff (445J).

445R The Plancherel Theorem Let X be an abelian topological group with a Haar measure μ, and $\mathcal{X}$ its dual group. Let λ be the Haar measure on $\mathcal{X}$ corresponding to μ (445P). Then there is a normed space isomorphism $T:L^2_{\mathbb{C}}(\mu)\to L^2_{\mathbb{C}}(\lambda)$ defined by setting $T(f^\bullet)=\hat{f}^\bullet$ whenever $f\in\mathcal{L}^1_{\mathbb{C}}(\mu)\cap\mathcal{L}^2_{\mathbb{C}}(\mu)$.

proof (a) Since $\hat{f}=\hat{g}$ whenever $f=_{\text{a.e.}}g$, the formula certainly defines an operator T from $L^1_{\mathbb{C}}(\mu)\cap L^2_{\mathbb{C}}(\mu)$ to $L^0_{\mathbb{C}}(\lambda)$, and of course it is linear.

If $f\in\mathcal{L}^1_{\mathbb{C}}(\mu)\cap\mathcal{L}^2_{\mathbb{C}}(\mu)$, $h=f*\tilde{f}$ is continuous and positive definite (445Ld) and integrable, and $\hat{h}=|\hat{f}|^2$ (445Kc). Now

$$\int|\hat{f}|^2\,d\lambda=\int\hat{h}\,d\lambda=h(e)$$

(445P)

$$= \int f(x)\tilde{f}(x^{-1})\mu(dx) = \int |f|^2 d\mu.$$

Thus $\|Tu\|_2 = \|u\|_2$ whenever $u \in L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$; since $L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$ is $\| \ \|_2$-dense in $L^2_{\mathbb{C}}(\mu)$ (244Ha/244Pb, or otherwise), we have a unique isometry $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ agreeing with the given formula on $L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$ (2A4I).

(b) The meat of the theorem is of course the proof that T is surjective. **P?** Suppose, if possible, that $W = T[L^2_{\mathbb{C}}(\mu)]$ is not equal to $L^2_{\mathbb{C}}(\lambda)$. Because T is linear, W is a linear subspace of $L^2_{\mathbb{C}}(\lambda)$; because T is an isometry, W is isometric to $L^2_{\mathbb{C}}(\mu)$, and in particular is complete, therefore closed in $L^2_{\mathbb{C}}(\lambda)$ (3A4Fd). There is therefore a non-zero continuous linear functional on $L^2_{\mathbb{C}}(\lambda)$ which is zero on W (3A5Ad), and this is of the form $u \mapsto \int u \times v$ for some $v \in L^2_{\mathbb{C}}(\lambda)$ (244J/244Pc). What this means is that there is a $g \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$ such that $g^\bullet \neq 0$ in $L^2_{\mathbb{C}}(\lambda)$ but $\int \hat{f} \times g \, d\lambda = 0$ for every $f \in \mathcal{L}^1_{\mathbb{C}}(\mu) \cap \mathcal{L}^2_{\mathbb{C}}(\mu)$.

Suppose that $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$ and that h is any μ-integrable continuous positive definite function on X. Then h is bounded (445Lb), so $|h|^2$ also is μ-integrable, and $f * \bar{h} \in \mathcal{L}^1_{\mathbb{C}}(\mu) \cap \mathcal{L}^2_{\mathbb{C}}(\mu)$ (444Ra). Accordingly

$$\int g \times \hat{f} \times \hat{\bar{h}} \, d\lambda = \int g \times (f * \bar{h})^\wedge d\lambda = 0.$$

Thus $\int g \times \hat{f} \, d\nu_h = 0$, where $\nu_h = \hat{\bar{h}}\lambda$ is the Radon measure on $\mathcal{X}$ corresponding to h constructed in 445N (see 445Q). And this is true for every $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$. But as $\{\hat{f} : f \in \mathcal{L}^1_{\mathbb{C}}(\mu)\}$ is $\| \ \|_\infty$-dense in $C_0(\mathcal{X};\mathbb{C})$ (445Kd), and g is ν_h-integrable (because $\hat{\bar{h}} \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$, by (a), so $\int |g \times \hat{\bar{h}}| d\lambda < \infty$), g must be zero ν_h-a.e., that is, $g \times \hat{\bar{h}} = 0$ λ-a.e. Now recall (from part (c) of the proof of 445P, for instance) that for every compact set $K \subseteq \mathcal{X}$ there is a μ-integrable continuous positive definite h such that $\hat{\bar{h}}(\chi) \neq 0$ for every $\chi \in K$, so $g = 0$ a.e. on K. Since λ is tight, $g = 0$ a.e. (412Jc), which is impossible. **XQ**

Thus T is surjective and we have the result.

445S While we do not have a direct definition of $\hat{f}$ for $f \in \mathcal{L}^2_{\mathbb{C}} \setminus \mathcal{L}^1_{\mathbb{C}}$, the map $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ does correspond to the map $f \mapsto \hat{f}$ in many ways. In particular, we have the following useful properties.

Proposition Let X be an abelian topological group with a Haar measure μ, $\mathcal{X}$ its dual group, λ the associated Haar measure on $\mathcal{X}$ and $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ the standard isometry described in 445R. Suppose that f_0, $f_1 \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ and g_0, $g_1 \in \mathcal{L}^2_{\mathbb{C}}(\nu)$ are such that $Tf_0^\bullet = g_0^\bullet$ and $Tf_1^\bullet = g_1^\bullet$, and take any $\theta \in \mathcal{X}$. Then

(a) setting $f_2 = \bar{f}_0$, $g_2(\chi) = \overline{g_0(\chi^{-1})}$ whenever this is defined, $Tf_2^\bullet = g_2^\bullet$;

(b) setting $f_3 = f_1 \times \theta$, $g_3(\chi) = g_1(\theta\chi)$ whenever this is defined, $Tf_3^\bullet = g_3^\bullet$;

(c) setting $f_4 = f_0 \times f_1 \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, $\hat{f}_4(\theta) = (g_0 * g_1)(\theta)$.

proof (a) We have isometries $R_1 : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\mu)$ and $R_2 : L^2_{\mathbb{C}}(\lambda) \to L^2_{\mathbb{C}}(\lambda)$ defined by setting $R_1 f^\bullet = (\bar{f})^\bullet$ for every $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ and $R_2 g^\bullet = (\tilde{g})^\bullet$ for every $g \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$, where $\tilde{g}(\chi) = \overline{g(\chi^{-1})}$ whenever this is defined. Now if $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, then

$$\hat{\bar{f}}(\chi) = \int \overline{f(x)}\chi(x)\mu(dx) = \overline{\int f(x)\chi^{-1}(x)\mu(dx)} = \tilde{\hat{f}}(\chi)$$

for every $\chi \in \mathcal{X}$. So $TR_1 u = R_2 T u$ for every $u \in L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$; as $L^1_{\mathbb{C}} \cap L^2_{\mathbb{C}}$ is dense in $L^2_{\mathbb{C}}$, $TR_1 = R_2 T$, which is what we need to know.

(b) This time, set $R_1 f^\bullet = (f \times \theta)^\bullet$ for every $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ and $R_2 g^\bullet = (\theta^{-1} \bullet_l g)^\bullet$ for $g \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$, where $(\theta \bullet_l g)(\chi) = g(\theta^{-1}\chi)$ whenever this is defined. Once again, $R_1 : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\mu)$ and $R_2 : L^2_{\mathbb{C}}(\lambda) \to L^2_{\mathbb{C}}(\lambda)$ are isometries. If $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, then

$$(f \times \theta)^\wedge(\chi) = \int f(x)\theta(x)\chi(x)\mu(dx) = \hat{f}(\theta\chi)$$

for every χ, so $TR_1 f^\bullet = R_2 T f^\bullet$; once again, this is enough to prove that $TR_1 = R_2 T$, as required.

(c) We have

$$(g_0 * g_1)(\theta) = \int g_0(\chi^{-1})g_1(\theta\chi)\lambda(d\chi) = \int \overline{g_2(\chi)}g_3(\chi)\lambda(d\chi) = (g_3^\bullet | g_2^\bullet)$$

(where $(\ | \ )$ is the standard inner product of $L^2_{\mathbb{C}}(\lambda)$)

$$= (Tf_3^\bullet|Tf_2^\bullet)$$

(using (a) and (b))

$$= (f_3^\bullet|f_2^\bullet)$$

(because linear isometries of Hilbert space preserve inner products, see 4A4Jc)

$$= \int f_3(x)\overline{f_2(x)}\mu(dx) = \int f_1(x)\theta(x)f_0(x)\mu(dx) = (f_0 \times f_1)\hat{}(\theta).$$

445T Corollary Let X be an abelian topological group with a Haar measure μ, and λ the corresponding Haar measure on the dual group $\mathcal{X}$ of X (445P). Then for any non-empty open set $H \subseteq \mathcal{X}$, there is an $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$ such that $\hat{f} \neq 0$ and $\hat{f}(\chi) = 0$ for $\chi \in \mathcal{X} \setminus H$.

proof Let V_1 and V_2 be non-empty open sets of finite measure such that $V_1V_2 \subseteq H$, and let g_1, g_2 be their indicator functions. Then there are f_1, $f_2 \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ such that $Tf_j^\bullet = g_j^\bullet$ for both j, where $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ is the isometry of 445R. In this case, by 445Sc, $(f_1 \times f_2)\hat{} = g_1 * g_2$. But it is easy to check that $g_1 * g_2$ is non-zero and zero outside H.

445U The Duality Theorem (Pontryagin 34, Kampen 35) Let X be a locally compact Hausdorff abelian topological group. Then the canonical map $x \mapsto \hat{x}$ from X to its bidual $\mathfrak{X}$ (445E) is an isomorphism between X and $\mathfrak{X}$ as topological groups.

proof By 445O, $\hat{}$ is a homeomorphism between X and its image $\hat{X} \subseteq \mathfrak{X}$. Accordingly $\hat{X}$ is itself, with its subspace topology, a locally compact topological group, and is closed in $\mathfrak{X}$ (4A5Mc).

? Suppose, if possible, that $\hat{X} \neq \mathfrak{X}$. Let μ be a Haar measure on X (441E) and λ the associated Haar measure on the dual $\mathcal{X}$ of X (445P). By 445T, there is a $g \in \mathcal{L}^1_{\mathbb{C}}(\lambda)$ such that $\hat{g}$ is zero on $\hat{X}$ but not zero everywhere, so that g is not zero a.e. We may suppose that g is defined everywhere on $\mathcal{X}$. In this case we have

$$0 = \hat{g}(\hat{x}) = \int g(\chi)\hat{x}(\chi)\lambda(d\chi) = \int g(\chi)\chi(x)\lambda(d\chi)$$

for every $x \in X$.

By 418J, $g : \mathcal{X} \to \mathbb{C}$ is almost continuous. Let $K \subseteq \{\chi : \chi \in \mathcal{X},\ g(\chi) \neq 0\}$ be a compact set such that $\int_K |g|d\lambda \geq \frac{3}{4}\int_{\mathcal{X}} |g|d\lambda$ and $g{\restriction}K$ is continuous. Set $q(\chi) = \overline{g(\chi)}/|g(\chi)|$ for $\chi \in K$. Now consider the linear span A of $\{\hat{x} : x \in X\}$ as a linear space of complex-valued functions on $\mathcal{X}$. Since $\widehat{xy} = \hat{x} \times \hat{y}$ for all x, $y \in X$, A is a subalgebra of $C_b = C_b(\mathcal{X};\mathbb{C})$; since $\widehat{x^{-1}} = \bar{\hat{x}}$ for every $x \in X$, $\bar{h} \in A$ for every $h \in A$; the constant function $\hat{e}$ belongs to A; and A separates the points of $\mathcal{X}$. By the Stone-Weierstrass theorem, in the form 281G, there is an $h \in A$ such that $|h(\chi) - q(\chi)| \leq \frac{1}{2}$ for every $\chi \in K$ and $|h(\chi)| \leq 1$ for every $\chi \in \mathcal{X}$. Of course $\int g \times h\,d\lambda = 0$ for every $h \in A$ because $\int g \times \hat{x}\,d\lambda = 0$ for every $x \in X$.

Now, however,

$$\begin{aligned}\int_K |g|d\lambda &= \int_K g \times q\,d\lambda \leq \Bigl|\int_K g \times h\,d\lambda\Bigr| + \int_K |g| \times |h - q|d\lambda \\ &\leq \Bigl|\int_{\mathcal{X}\setminus K} g \times h\,d\lambda\Bigr| + \frac{1}{2}\int_K |g|d\lambda \leq \int_{\mathcal{X}\setminus K} |g|d\lambda + \frac{1}{2}\int_K |g|d\lambda < \int_K |g|d\lambda,\end{aligned}$$

which is impossible. **X**

Thus $\hat{X} = \mathfrak{X}$ and the proof is complete.

445X Basic exercises (a) Consider the additive group $\mathbb{Q}$ with its usual topology. Show that its dual group can be identified with the additive group $\mathbb{R}$ with its usual topology.

(b) Let X be any topological group, and $\mathcal{X}$ its dual group. Show that if ν is a totally finite Radon measure on X, then its Fourier-Stieltjes transform $\hat{\nu} : \mathcal{X} \to \mathbb{C}$ is continuous.

(c) Let X be a topological group carrying Haar measures and $\mathcal{X}$ its dual group. For a totally finite quasi-Radon measure ν on $\mathcal{X}$ set $\hat{\nu}(x) = \int \chi(x)\nu(d\chi)$ for every $x \in X$. (i) Show that $\hat{\nu} : X \to \mathbb{C}$ is continuous. (ii) Show that for any totally finite quasi-Radon measure μ on X with Fourier-Stieltjes transform $\hat{\mu}$, $\int \hat{\mu}\,d\nu = \int \hat{\nu}\,d\mu$.

>(d) Let X be a group and h_1, $h_2 : X \to \mathbb{C}$ positive definite functions. Show that $h_1 + h_2$, αh_1 and $\bar{h}_1$ are also positive definite for any $\alpha \geq 0$.

(e)(i) Let X be a group, Y a subgroup of X and $h : Y \to \mathbb{C}$ a positive definite function. Set $h_1(x) = h(x)$ if $x \in Y$, $h_1(x) = 0$ if $x \in X \setminus Y$. Show that h_1 is positive definite. (ii) Let X and Y be groups, $\phi : X \to Y$ a group homomorphism and $h : Y \to \mathbb{C}$ a positive definite function. Show that $h\phi : X \to \mathbb{C}$ is positive definite.

(f) Let X be a topological group and $\mathcal{X}$ its dual group. (i) Let ν be any totally finite topological measure on $\mathcal{X}$ and set $h(x) = \int \chi(x)\nu(d\chi)$ for $x \in X$. Show that $h : X \to \mathbb{C}$ is positive definite. (ii) Let ν be any totally finite topological measure on X. Show that its Fourier transform $\hat{\nu} : \mathcal{X} \to \mathbb{C}$ is positive definite.

>(g) Let X be a topological group with a left Haar measure, and $h : X \to \mathbb{C}$ a bounded continuous function. Show that h is positive definite iff $\int h(x^{-1}y)f(y)\overline{f(x)}dxdy \geq 0$ for every integrable function f.

>(h) Let $\phi : \mathbb{R}^r \to \mathbb{C}$ be a function. Show that it is the characteristic function of a probability distribution on $\mathbb{R}^r$ iff it is continuous and positive definite and $\phi(0) = 1$.

>(i) Let X be a compact Hausdorff abelian topological group, and μ the Haar probability measure on X. Show that the corresponding Haar measure on the dual group $\mathcal{X}$ of X is just counting measure on $\mathcal{X}$.

>(j) Let X be an abelian group with its discrete topology, and μ counting measure on X. Let $\mathcal{X}$ be the dual group and λ the corresponding Haar measure on $\mathcal{X}$. Show that $\lambda\mathcal{X} = 1$.

(k) Let X be the topological group $\mathbb{R}$, and $\mu = \frac{1}{\sqrt{2\pi}}\mu_L$, where μ_L is Lebesgue measure. (i) Show that if we identify the dual group $\mathcal{X}$ of X with $\mathbb{R}$, writing $\chi(x) = e^{-i\chi x}$ for x, $\chi \in \mathbb{R}$, then the Haar measure on $\mathcal{X}$ corresponding to the Haar measure μ on X is μ itself. (ii) Show that if we change the action of $\mathbb{R}$ on itself by setting $\chi(x) = e^{-2\pi i\chi x}$, then the Haar measure on $\mathcal{X}$ corresponding to μ_L is μ_L.

(l) Let $X_0, \dots, X_n$ be abelian topological groups with Haar measures $\mu_0, \dots, \mu_n$, and let $X = X_0 \times \ldots \times X_n$ be the product group with its Haar measure $\mu = \mu_0 \times \ldots \times \mu_n$. For each $k \leq n$ let $\mathcal{X}_k$ be the dual group of X_k and λ_k the Haar measure on $\mathcal{X}_k$ corresponding to μ_k. Show that if we identify $\mathcal{X} = \mathcal{X}_0 \times \ldots \times \mathcal{X}_n$ with the dual group of X, then the Haar measure on $\mathcal{X}$ corresponding to μ is just the product measure $\lambda_0 \times \ldots \times \lambda_n$.

>(m) Let X be a compact Hausdorff abelian topological group, with dual group $\mathcal{X}$, and μ the Haar probability measure on X. (i) Show that $\int \chi\, d\mu = 0$ for every $\chi \in \mathcal{X}$ except the identity. (ii) Show that $\{\chi^\bullet : \chi \in \mathcal{X}\}$ is an orthonormal basis of the Hilbert space $L^2_{\mathbb{C}}(\mu)$. (*Hint*: $(u|\chi^\bullet) = (Tu|\hat{\chi}^\bullet)$ where T is the operator of 445R.)

(n) Let X be an abelian topological group with a Haar measure μ, λ the associated Haar measure on the dual group $\mathcal{X}$ and $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ the standard isometry. Suppose that u, $v \in L^2_{\mathbb{C}}(\mu)$. Suppose that f_0, $f_1 \in \mathcal{L}^2_{\mathbb{C}}(\mu)$, g_0, $g_1 \in \mathcal{L}^2_{\mathbb{C}}(\nu)$ are such that $Tf_0^\bullet = g_0^\bullet$ and $Tf_1^\bullet = g_1^\bullet$. Show that $(f_0 * f_1)(x) = \int g_0(\chi)g_1(\chi)\overline{\chi(x)}\lambda(d\chi)$ for any $x \in X$.

(o) Let X be a locally compact Hausdorff abelian topological group and $\mathcal{X}$ its dual group. Show that a function $h : \mathcal{X} \to \mathbb{C}$ is the Fourier-Stieltjes transform of a totally finite Radon measure on X iff it is continuous and positive definite.

(p)(i) Show that we can define a binary operation $+_{\text{2adic}}$ on $X = \{0,1\}^{\mathbb{N}}$ by setting $x +_{\text{2adic}} y = z$ whenever x, y, $z \in X$ and $\sum_{i=0}^k 2^i(x(i) + y(i) - z(i))$ is divisible by 2^{k+1} for every k. (ii) Show that if we give X its usual topology then $(X, +_{\text{2adic}})$ is an abelian topological group. (iii) Show that the usual measure on X is the Haar probability measure for this group operation. (iv) Show that $G = \{\zeta : \zeta \in \mathbb{C}, \exists\, n \in \mathbb{N}, \zeta^{2^n} = 1\}$ is a subgroup of $\mathbb{C}$. (v) Show that the dual of $(X, +_{\text{2adic}})$ is $\{\chi_\zeta : \zeta \in G\}$ where $\chi_\zeta(x) = \prod_{i=0}^\infty \zeta^{2^i x(i)}$ for $\zeta \in G$ and $x \in X$. (vi) Show that the functions f, $g : X \to X$ described in 388E are of the form $x \mapsto x \pm_{\text{2adic}} x_0$ for a certain $x_0 \in X$.

(q) Let X be a locally compact Hausdorff abelian topological group. Show that if two totally finite Radon measures on X have the same Fourier-Stieltjes transform, they are equal. (*Hint*: 281G.)

445Y Further exercises (a) Let X be any Hausdorff topological group. Let $\widehat{X}$ be its completion under its bilateral uniformity. Show that the dual groups of X and $\widehat{X}$ can be identified as groups. Show that they can be identified as topological groups if *either* X is metrizable *or* X has a totally bounded neighbourhood of the identity.

(b) Let $\langle X_j\rangle_{j\in I}$ be a countable family of topological groups, with product X; let $\mathcal{X}_j$ be the dual group of each X_j, and $\mathcal{X}$ the dual group of X. Show that the topology of $\mathcal{X}$ is generated by sets of the form $\mathcal{X}\cap\prod_{j\in I}H_j$ where $H_j\subseteq\mathcal{X}_j$ is open for each j.

(c) Let X be a real linear topological space, with addition as its group operation. Show that its dual group is just the set of functionals $x\mapsto e^{if(x)}$ where $f:X\to\mathbb{R}$ is a continuous linear functional. Hence show that there are abelian groups with trivial duals.

(d) Let X be the group of rotations of $\mathbb{R}^3$, that is, the group of orthogonal real 3×3 matrices with determinant 1, and give X its usual topology, corresponding to its embedding as a subspace of $\mathbb{R}^9$. Show that the only character on X is the constant function 1. (*Hint*: (i) show that two rotations through the same angle are conjugate in X; (ii) show that if $0<\theta\le\frac{\pi}{2}$ then the product of two rotations through the angle θ about orthogonal axes is not a rotation through an angle 2θ.)

(e) Let X be a finite abelian group, endowed with its discrete topology. Show that its dual is isomorphic to X. (*Hint*: X is a product of cyclic groups.)

(f) Show that if I is any uncountable set, then there is a quasi-Radon probability measure ν on the topological group $\mathbb{R}^I$ such that its Fourier-Stieltjes transform $\hat{\nu}$ is not continuous. (*Hint*: take ν to be a power of a suitably widely spread probability distribution on $\mathbb{R}$.)

(g) Let X be an abelian topological group with a Haar measure μ, and λ the associated Haar measure on the dual group $\mathcal{X}$ of X. Let $T:L^2_{\mathbb{C}}(\mu)\to L^2_{\mathbb{C}}(\lambda)$ be the standard isomorphism. Suppose that $f\in\mathcal{L}^2_{\mathbb{C}}(\mu)$ and $g\in\mathcal{L}^1_{\mathbb{C}}(\lambda)\cap\mathcal{L}^2_{\mathbb{C}}(\lambda)$ are such that $Tf^\bullet=g^\bullet$. Show that $f(x)=\int g(\chi)\overline{\chi(x)}\lambda(d\chi)$ for almost every $x\in X$. (*Hint*: look first at locally compact Hausdorff X.)

(h) Let X be a locally compact Hausdorff abelian topological group with dual group $\mathcal{X}$, P_{R} the set of Radon probability measures on X, $\langle\nu_n\rangle_{n\in\mathbb{N}}$ a sequence in P_{R} and ν a member of P_{R}. Show that the following are equiveridical: (i) $\langle\nu_n\rangle_{n\in\mathbb{N}}\to\nu$ for the narrow topology on P_{R}; (ii) $\lim_{n\to\infty}\hat{\nu}_n(\chi)=\hat{\nu}(\chi)$ for every $\chi\in\mathcal{X}$. (*Hint*: compare 285L. For the critical step, showing that $\{\nu_n:n\in\mathbb{N}\}$ is uniformly tight, use the formulae in 445N to show that there is an integrable $f:\mathcal{X}\to\mathbb{C}$ such that $0\le\check{f}\le\mathbf{1}$ and $\int\check{f}(x)\nu(dx)\ge 1-\frac{1}{2}\epsilon$.)

(i) Let X be a topological group carrying Haar measures and $\mathcal{X}$ its dual group. Let M_τ be the complex Banach space of signed totally finite τ-additive Borel measures on $\mathcal{X}$ (put the ideas of 437F and 437Yb together). Show that X separates the points of M_τ in the sense that if $\nu\in M_\tau$ is non-zero, there is an $x\in X$ such that $\int\chi(x)\nu(d\chi)\ne 0$, if the integral is appropriately interpreted. (*Hint*: use the method in the proof of 445U.)

(j) Let X be an abelian topological group carrying Haar measures. Let M_τ be the complex Banach space of signed totally finite τ-additive Borel measures on X. Show that the dual $\mathcal{X}$ of X separates the points of M_τ in the sense that if $\nu\in M_\tau$ is non-zero, there is a $\chi\in\mathcal{X}$ such that $\int\chi(x)\nu(dx)\ne 0$. (*Hint*: use 443L to reduce to the case in which X is locally compact and Hausdorff; now use 445U and 445Yi.)

(k) Let X be an abelian topological group and μ a Haar measure on X. Show that the spectral radius of any non-zero element of $L^1_{\mathbb{C}}(\mu)$ is non-zero. (*Hint*: 445Yj, 445Kd.)

(l) Show that for any integer $p\ge 2$ there is an operation $+_{\text{padic}}$ on $\{0,\ldots,p-1\}^{\mathbb{N}}$ with properties similar to those of the operation $+_{\text{2adic}}$ of 445Xp.

(m) Let μ be Lebesgue measure on $[0,\infty[$. (i) For f, $g\in\mathcal{L}^1(\mu)$ define $(f*g)(x)=\int_0^x f(y)g(x-y)\mu(dy)$ whenever the integral is defined. Show that $f*g\in\mathcal{L}^1(\mu)$. (ii) Show that we can define a bilinear operator $*$ on $L^1(\mu)$ by setting $f^\bullet*g^\bullet=(f*g)^\bullet$ for f, $g\in\mathcal{L}^1(\mu)$, and that under this multiplication $L^1(\mu)$ is a Banach algebra. (iii) Show that if $\phi:L^1(\mu)\to\mathbb{R}$ is a multiplicative linear operator then there is some $s\ge 0$ such that $\phi(f^\bullet)=\int_0^\infty f(x)e^{-sx}\mu(dx)$ for every $f\in\mathcal{L}^1(\mu)$.

445 Notes and comments I repeat that this section is intended to be a more or less direct attack on the duality theorem. At every point the clause 'let X be a locally compact Hausdorff abelian topological group' is present in spirit. The actual statement of each theorem involves some subset of these properties, purely in accordance with the principle of omission of irrelevant hypotheses, not because I expect to employ the results in any more general setting.

In 445Ab I describe a topology on the dual group in a context so abstract that we have rather a lot of choice. For groups carrying Haar measures, the alternative descriptions of the topology on the dual (445I) make it plain that this must be the first topology to study. By 445E it is already becoming fairly convincing. But it is not clear that there is any such pre-eminent topology in the general case.

Fourier-Stieltjes transforms hardly enter into the arguments of this section; I mention them mostly because they form the obvious generalization of the ideas in §285. But I note that the principal theorem of §285 (that sequential convergence of characteristic functions corresponds to sequential convergence of distributions, 285L) generalizes directly to the context here (445Yh).

I have tried to lay this treatise out in such a way that we periodically return to themes from past chapters at a higher level of sophistication. There seem to be four really important differences between this section and the previous treatment in Chapter 28. (i) The first is the obvious one; we are dealing with general locally compact Hausdorff abelian groups, rather than with $\mathbb{R}$ and S^1 and $\mathbb{Z}$. Of course this puts much heavier demands on our technique, and, to begin with, leaves our imaginations unfocused. (ii) The second concerns differentiation, or rather its absence; since we no longer have any differential structure on our groups, a substantial part of the theory evaporates, and we are forced to employ new tactics in the rest. (iii) The third concerns the normalization of the measure on the dual group. As soon as we know that $\mathcal{X}$ is a locally compact group (445J) we know that it carries Haar measures. The problem is to describe the particular one we need in appropriate terms. In the case of the dual pairs $(\mathbb{R},\mathbb{R})$ or $(S^1,\mathbb{Z})$, we have measures already presented (counting measure on $\mathbb{Z}$, Lebesgue measure on $]-\pi,\pi]$ and $\mathbb{R}$). (They are not in fact dual in the sense we need here, at least not if we use the simplest formulae for the duality, and have to be corrected in each case by a factor of 2π. See 445Xk.) But since we do have dual pairs already to hand, we can simultaneously develop theories of Fourier transforms and inverse Fourier transforms (for the pair $(S^1,\mathbb{Z})$ the inverse Fourier transform is just summation of trigonometric series), and the problem is to successfully match operations which have independent existences. (iv) The final change concerns an interesting feature of $\mathbb{Z}$ and $\mathbb{R}$. Repeatedly, in §§282-283, the formulae invoked symmetric limits $\lim_{n\to\infty}\sum_{k=-n}^{n}$ or $\lim_{a\to\infty}\int_{-a}^{a}$ to approach some conditionally convergent sum or integral. Elsewhere one sometimes deals with singularities by examining 'Cauchy principal values'; if $\int_{-1}^{1}f$ is undefined, try $\lim_{a\downarrow 0}(\int_{-1}^{-a}f+\int_{a}^{1}f)$. This particular method seems to disappear in the general context. But the general challenge of the subject remains the same: to develop a theory of the transform $u\mapsto\hat{u}$ which will apply to the largest possible family of objects u and will enable us to justify, in the widest possible contexts, the manipulations listed in the notes to §284. The calculations in 445S and 445Xn, treating 'shift' and 'convolution' in L^2, are typical.

In terms of the actual proofs of the results here, 'test functions' (284A) have gone, and in their place we take a lot more trouble over the Banach algebra L^1. This algebra is the key to one of the magic bits, which turns up in rather undignified corners in 445Kd and part (e) of the proof of 445N. Down to 445O, the dominating problem is that we do not know that the dual group $\mathcal{X}$ of a group X is large enough to tell us anything interesting about X. (After that, the problem reverses; we have to show that $\mathcal{X}$ is not too big.) We find that (under rather specially arranged circumstances) we are able to say something useful about the spectral radius of a member of L^1, and we use this to guarantee that it has a non-trivial Fourier transform. If we identify $\mathcal{X}$ with the maximal ideal space of L^1 (445H), then the Fourier transform on L^1 becomes the 'Gelfand map', a general construction of great power in the theory of commutative Banach algebras.

There is one similarity between the methods of this section and those of §284. In both cases we have isomorphisms between $L^2_{\mathbb{C}}(\mu)$ and $L^2_{\mathbb{C}}(\lambda)$ (the Plancherel Theorem), but cannot define the Fourier transform of a function in $\mathcal{L}^2_{\mathbb{C}}$ in any direct way; indeed, while the Fourier transform of a function in $\mathcal{L}^1_{\mathbb{C}}$, or even of a (totally finite) measure, can really be thought of as a (continuous) function, the transform of a function in L^2 is at best a member of L^2, not a function at all. We manoeuvre around this difficulty by establishing that our (genuine) Fourier transforms match dense subspaces isometrically. In §284 I used test functions, and in the present section I use $\mathcal{L}^1\cap\mathcal{L}^2$. Test functions are easier partly because the Fourier transform of a test function is again a test function, and all the formulae we need are easy to establish for such functions.

Searching for classes of functions which will be readily manageable in general locally compact abelian groups, we come to the 'positive definite' functions. The phrase is unsettling, since the functions themselves are in no obvious sense positive (nor even, as a rule, real-valued). Also their natural analogues in the theory of bilinear forms are commonly called 'positive semi-definite'. However, their Fourier transforms, whether regarded as measures (445N) or as functions (445Q), are positive, and, as a bonus, we get a characterization of the Fourier transforms of measures (445Xf, 445Xh), answering a question left hanging in 285Xr.

446 The structure of locally compact groups

I develop those fragments of the structure theory of locally compact Hausdorff topological groups which are needed for the main theorem of the next section. Theorem 446B here is of independent interest, being both itself important and with a proof which uses the measure theory of this chapter in an interesting way; but the rest of the section, from 446D on, is starred. Note that in this section, unlike the last, groups are not expected to be abelian.

446A Finite-dimensional representations (a) Definitions (i) For any $r \in \mathbb{N}$, write $M_r = M_r(\mathbb{R})$ for the space of $r \times r$ real matrices. If we identify it with the space $\mathrm{B}(\mathbb{R}^r;\mathbb{R}^r)$, where $\mathbb{R}^r$ is given its Euclidean norm, then M_r becomes a unital Banach algebra (4A6C), with identity I, the $r \times r$ identity matrix. Write $GL(r,\mathbb{R})$ for the group of invertible elements of M_r.

(ii) Let X be a topological group. A **finite-dimensional representation** of X is a continuous homomorphism from X to a group of the form $GL(r,\mathbb{R})$ for some $r \in \mathbb{N}$. If the homomorphism is injective the representation is called **faithful** (cf. 4A5Be).

(b) Observe that if X is any topological group and ϕ is a finite-dimensional representation with kernel Y, then X/Y has a faithful finite-dimensional representation ψ defined by writing $\psi(x^\bullet) = \phi(x)$ for every $x \in X$ (4A5La).

446B Theorem Let X be a compact Hausdorff topological group. Then for any $a \in X$, other than the identity, there is a finite-dimensional representation ϕ of X such that $\phi(a) \neq I$; and we can arrange that $\phi(x)$ is an orthogonal matrix for every $x \in X$.

proof (a) Let U be a symmetric neighbourhood of the identity e in X such that $a \notin UU$. Because X is completely regular (3A3Bb), there is a non-zero continuous function $h : X \to [0,\infty[$ such that $h(x) = 0$ for every $x \in X \setminus U$; replacing h by $x \mapsto h(x) + h(x^{-1})$ if necessary, we may suppose that $h(x) = h(x^{-1})$ for every x. Let μ be a (left) Haar measure on X (441E), and set $w = h^\bullet$ in $L^0(\mu)$.

(b) Define an operator T from $L^2 = L^2(\mu)$ to itself by setting

$$Tu = u * w \text{ for every } u \in L^2,$$

where $*$ is convolution; that is, $Tf^\bullet = (f * h)^\bullet$ for every $f \in \mathcal{L}^2 = \mathcal{L}^2(\mu)$. Then T is a compact self-adjoint operator on the real Hilbert space L^2 (444V).

(c) For any $z \in X$, define $S_z : L^2 \to L^2$ by setting $S_z u = z\bullet_l u$ for $u \in L^2$, where $\bullet_l$ is the left shift action, so that S_z is a norm-preserving linear operator (443Ge). Also S_z commutes with T. **P** (443Ge). ThenIf $f \in \mathcal{L}^2$, then

$$S_z T f^\bullet = (z\bullet_l (f * h))^\bullet = ((z\bullet_l f) * h)^\bullet$$

(444Of)

$$= T S_z f^\bullet. \text{ } \mathbf{Q}$$

(d) Now $S_a Tw \neq Tw$. **P** Set $g = h * h$, so that $Tw = g^\bullet$ and g, $a\bullet_l g$ are both continuous functions (444Rc). Then

$$(a\bullet_l g)(e) = g(a^{-1}) = \int h(y)h(y^{-1}a^{-1})dy = 0$$

because if $y \in U$ then $y^{-1}a^{-1} \notin U$, while

$$g(e) = \int h(y)h(y^{-1})dy = \int h(y)^2 dy > 0.$$

So the open set $\{x : g(x) \neq (a\bullet_l g)(x)\}$ is non-empty; because μ is strictly positive (442Aa), it is non-negligible, and

$$S_a Tw = (a\bullet_l g)^\bullet \neq g^\bullet = Tw. \text{ } \mathbf{Q}$$

(e) The closed linear subspace $\{u : S_a u = u\}$ therefore does not include $T[L^2]$. But the linear span of $\{Tv : v$ is an eigenvector of $T\}$ is dense in $T[L^2]$ (4A4M), so there is an eigenvector v^* of T such that $S_a Tv^* \ne Tv^*$. Let $\gamma \in \mathbb{R}$ be such that $Tv^* = \gamma v^*$; since $Tv^* \ne 0$, $\gamma \ne 0$, and $V = \{u : u \in L^2, Tu = \gamma u\}$ is finite-dimensional (4A4Lb).

(f) $S_z[V] \subseteq V$ for every $z \in X$. **P**

$$TS_z u = S_z Tu = S_z(\gamma u) = \gamma S_z u$$

for every $u \in V$. **Q**

We therefore have a map $z \mapsto S_z \restriction V : X \to \mathsf{B}(V;V)$. As observed in 443Gc, this is actually a semigroup homomorphism, and of course $S_e \restriction V$ is the identity of $\mathsf{B}(V;V)$, so $S_z \restriction V$ is always invertible, and we have a group homomorphism from X to the group of invertible elements of $\mathsf{B}(V;V)$. Taking an orthonormal basis $(v_1, \dots, v_r)$ of V, we have a homomorphism ϕ from X to $GL(r, \mathbb{R})$, defined by setting $\phi(z) = \langle (S_z v_i | v_j) \rangle_{1 \le i,j \le r}$ for every $z \in X$. Moreover, ϕ is continuous. **P** For any $u \in L^2$, $z \mapsto S_z u : X \to L^2$ is continuous, by 443Gf. But this means that all the maps $z \mapsto (S_z v_i | v_j)$ are continuous; since the topology of $GL(r, \mathbb{R})$ can be defined in terms of these functionals (see the formulae in 262H), ϕ is continuous. **Q**

Thus ϕ is a finite-dimensional representation of X. But V was chosen to contain v^*; of course $Tv^* \in V$, while $S_a Tv^* \ne Tv^*$; so that $\phi(a)$ is not the identity.

(g) Finally, $\phi(z)$ is an orthogonal matrix for every $z \in X$. **P** As observed in (c), S_z is norm-preserving, so $S_z \restriction V$ is again norm-preserving. By 4A4Jb, $(S_z v_i | S_z v_j) = (v_i | v_j)$ for $1 \le i, j \le r$, that is, $\phi(z)$ is orthogonal. **Q**

446C Corollary Let X be a compact Hausdorff topological group. Then for any neighbourhood U of the identity of X there is a finite-dimensional representation of X with kernel included in U.

proof Let Φ be the set of finite-dimensional representations of X. By 446B, $\bigcap_{\phi \in \Phi} \ker(\phi) = \{e\}$, where e is the identity of X. Because $X \setminus \operatorname{int} U$ is compact and disjoint from $\bigcap_{\phi \in \Phi} \ker(\phi)$ (and $\ker(\phi)$ is closed for every ϕ), there must be $\phi_0, \dots, \phi_n \in \Phi$ such that $\bigcap_{i \le n} \ker(\phi_i) \subseteq U$. For each $i \le n$, let $r_i \in \mathbb{N}$ be the integer such that ϕ_i is a continuous homomorphism from X to $GL(r_i, \mathbb{R})$. Set $r = \sum_{i=0}^n r_i$. Then we have a map $\phi : X \to GL(r, \mathbb{R})$ given by the formula

$$\phi(x) = \begin{pmatrix} \phi_0(x) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \phi_1(x) & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \phi_n(x) \end{pmatrix}$$

for every $x \in X$. It is easy to check that ϕ is a continuous homomorphism, and $\ker(\phi) = \bigcap_{i \le n} \ker(\phi_i) \subseteq U$. So we have an appropriate representation of X.

***446D Notation (a)** It will help to be clear on an elementary point of notation. If X is a group and A is a subset of X I will write $A^0 = \{e\}$, where e is the identity of X, and $A^{n+1} = AA^n$ for $n \in \mathbb{N}$, so that $A^3 = \{x_1 x_2 x_3 : x_1, x_2, x_3 \in A\}$, etc. Now we find that $A^{m+n} = A^m A^n$ and $A^{mn} = (A^m)^n$ for all $m, n \in \mathbb{N}$. Writing $A^{-1} = \{x^{-1} : x \in A\}$ as usual, we have $(A^r)^{-1} = (A^{-1})^r$. But note that if we also continue to write $A^{-1} = \{x^{-1} : x \in A\}$, then AA^{-1} is not in general equal to A^0; and that there is no simple relation between A^r, B^r and $(AB)^r$, unless X is abelian.

(b) In the rest of this section, I shall make extensive use of the following device. If X is a group with identity e, $e \in A \subseteq X$ and $n \in \mathbb{N}$, write $D_n(A) = \{x : x \in X, x^i \in A$ for every $i \le n\}$.

(i) $D_0(A) = X$.

(ii) $D_1(A) = A$.

(iii) $D_n(A) \subseteq D_m(A)$ whenever $m \le n$.

(iv) $D_{mn}(A) \subseteq D_m(D_n(A))$ for all $m, n \in \mathbb{N}$.

P If $x \in D_{mn}(A)$ then $(x^i)^j \in A$ whenever $j \le n$ and $i \le m$. **Q**

(v) If $r \in \mathbb{N}$ and $A^r \subseteq B$, then $D_n(A) \subseteq D_{nr}(B)$ for every $n \in \mathbb{N}$; in particular, $A \subseteq D_r(B)$.

P For $r = 0$ this is trivial. Otherwise, take $x \in D_n(A)$ and $i \le nr$. Then we can express i as $i_1 + \ldots + i_r$ where $i_j \le n$ for each j, so that

$$x^i = \prod_{j=1}^r x^{i_j} \in A^r \subseteq B. \textbf{ Q}$$

(vi) If $A = A^{-1}$ then $D_n(A) = D_n(A)^{-1}$ for every $n \in \mathbb{N}$.

(vii) If $D_m(A) \subseteq B$ where $m \in \mathbb{N}$, then $D_{mn}(A) \subseteq D_n(B)$ for every $n \in \mathbb{N}$.

P If $x \in D_{mn}(A)$ and $i \le n$, then $x^{ij} \in A$ for every $j \le m$, so $x^i \in D_m(A) \subseteq B$. **Q**

(c) In (b), if X is a topological group and A is closed, then every $D_n(A)$ is closed; if moreover A is compact, then $D_n(A)$ is compact for every $n \ge 1$. If A is a neighbourhood of e, then so is every $D_n(A)$, because the map $x \mapsto x^i$ is continuous for every $i \le n$.

***446E Lemma** Let X be a group, and $U \subseteq X$. Let $f : X \to [0, \infty[$ be a bounded function such that $f(x) = 0$ for $x \in X \setminus U$; set $\alpha = \sup_{x \in X} f(x)$. Let $A \subseteq X$ be a symmetric set containing e, and K a set including A^k, where $k \ge 1$. Define $g : X \to [0, \infty[$ by setting

$$g(x) = \tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} \sup\{f(yx) : y \in A^i\}$$

for $x \in X$. Then

(a) $f(x) \le g(x) \le \alpha$ for every $x \in X$, and $g(x) = 0$ if $x \notin K^{-1}U$.

(b) $|g(ax) - g(x)| \le \frac{j\alpha}{k}$ if $j \in \mathbb{N}$, $a \in A^j$ and $x \in X$.

(c) For any x, $z \in X$, $|g(x) - g(z)| \le \sup_{y \in K} |f(yx) - f(yz)|$.

proof (a) Of course

$$f(x) = \tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} f(ex) \le g(x) \le \tfrac{1}{k}\sum_{i=0}^{k-1} \alpha = \alpha$$

for every x. Suppose that $g(x) \ne 0$. Then there must be an $i < k$ and a $y \in A^i$ such that $f(yx) \ne 0$, so that $yx \in U$. But also, because $e \in A$, $y \in A^k \subseteq K$, so $x \in y^{-1}U \subseteq K^{-1}U$.

(b) Suppose first that $j = 1$, so that $a \in A$. If $\epsilon > 0$ there are $y_i \in A^i$, for $i < k$, such that $g(ax) \le \frac{1}{k}\sum_{i=0}^{k-1} f(y_i ax) + \epsilon$. Now $y_i a \in A^{i+1}$ for each i, so

$$g(x) \ge \tfrac{1}{k}\textstyle\sum_{i=0}^{k-2} f(y_i ax) \ge g(ax) - \epsilon - \frac{\alpha}{k}.$$

As ϵ is arbitrary, $g(x) \ge g(ax) - \frac{\alpha}{k}$. Similarly, as $a^{-1} \in A$ (because A is symmetric), $g(ax) \ge g(x) - \frac{\alpha}{k}$ and $|g(ax) - g(x)| \le \frac{\alpha}{k}$.

For the general case, induce on j. (If $j = 0$, then $a = e$ and the result is trivial.)

(c) Set $\gamma = \sup_{y \in K} |f(yx) - f(yz)|$. If $\epsilon > 0$, there are $y_i \in A^i$, for $i < k$, such that $g(x) \le \frac{1}{k}\sum_{i=0}^{k-1} f(y_i x) + \epsilon$. Now every y_i belongs to K, so

$$g(z) \ge \tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} f(y_i z) \ge \tfrac{1}{k}\sum_{i=0}^{k-1}(f(y_i x) - \gamma) \ge g(x) - \epsilon - \gamma.$$

As ϵ is arbitrary, $g(z) \ge g(x) - \gamma$; similarly, $g(x) \ge g(z) - \gamma$.

***446F Lemma** Let X be a locally compact Hausdorff topological group and $\langle A_n \rangle_{n \in \mathbb{N}}$ a sequence of closed symmetric subsets of X all containing the identity e of X. Suppose that for every neighbourhood W of e there is an $n_0 \in \mathbb{N}$ such that $A_n \subseteq W$ for every $n \ge n_0$. Let U be a compact neighbourhood of e and suppose that for each $n \in \mathbb{N}$ we have $k(n) \in \mathbb{N}$ such that $A_n^{k(n)} \subseteq U$ and $A_n^{k(n)+1} \not\subseteq U$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ and write Q for the limit $\lim_{n \to \mathcal{F}} A_n^{k(n)}$ in the space $\mathcal{C}$ of closed subsets of X with the Fell topology.

(i) If $Q^2 = Q$ then Q is a compact subgroup of X included in U and meeting the boundary of U.

(ii) If $Q^2 \ne Q$ then there are a neighbourhood W of e and an infinite set $I \subseteq \mathbb{N}$ such that for every $n \in I$ there are an $x \in A_n$ and an $i \le k(n)$ such that $x^i \notin W$.

proof (a) I ought to begin by explaining why the limit $\lim_{n \to \mathcal{F}} A_n^{k(n)}$ is defined; this is just because the Fell topology is always compact (4A2T(b-iii)) and when based on a locally compact Hausdorff space is Hausdorff (4A2T(e-ii)).

Because U is closed, $\{F : F \in \mathcal{C}, F \subseteq U\}$ is closed, by the definition of the Fell topology (4A2T(a-ii)); because every $A_n^{k(n)}$ is included in U, so is Q, and Q is compact. Because $x \mapsto x^{-1}$ is a homeomorphism of X, $F \mapsto F^{-1}$ is a homeomorphism of $\mathcal{C}$, and

$$Q^{-1} = \lim_{n\to\mathcal{F}}(A_n^{k(n)})^{-1} = \lim_{n\to\mathcal{F}}(A_n^{-1})^{k(n)} = \lim_{n\to\mathcal{F}} A_n^{k(n)} = Q.$$

And of course $e \in Q$ because $e \in A_n^{k(n)}$ for every n and $\{(x, F) : x \in F \in \mathcal{C}\}$ is closed in $X \times \mathcal{C}$ (4A2T(e-i)).

For each $n \in \mathbb{N}$, we have an $a_n \in A_n^{k(n)}$ and an $x_n \in A_n$ such that $a_n x_n \notin U$. Now $a = \lim_{n\to\mathcal{F}} a_n$ is defined (because U is compact), and belongs to Q. Also $\lim_{n\to\infty} x_n = e$, because every neighbourhood of e includes all but finitely many of the A_n, so $a = \lim_{n\to\mathcal{F}} a_n x_n \notin \operatorname{int} U$, and a belongs to the boundary of U. Thus Q meets the boundary of U.

(b) From (a) we see that if $Q^2 = Q$ then Q is a compact subgroup of X, included in U and meeting the boundary of U. So henceforth let us suppose that $Q^2 \ne Q$ and seek to prove (ii).

Let $w \in Q^2 \setminus Q$. Let $W_0 \subseteq U$ be an open neighbourhood of e such that $W_0 w W_0^2 \cap Q W_0^2 = \emptyset$ (4A5Ee).

(c) Fix a left Haar measure μ on X. Let $f : X \to [0, \infty[$ be a continuous function such that $\{x : f(x) > 0\} \subseteq W_0$ and $\int f(x)dx = 1$. Set $\alpha = \sup_{x\in X} f(x)$ and $\beta = \int f(x)^2 dx$, so that α is finite and $\beta > 0$. $W_0 U^2 W_0 \subseteq U^4$ is open and relatively compact, so has finite measure, and there is an $\eta > 0$ such that

$$2\eta(1 + \alpha\mu(W_0 U^2 W_0)) < \beta.$$

Let W be a neighbourhood of e such that $W \subseteq W_0$ and $|f(yax) - f(ybx)| \le \eta$ whenever $y \in (U^{-1})^2 \cup U$, $x \in X$ and $ab^{-1} \in W$ (4A5Pa).

(d) Express w as $w'w''$ where w', $w'' \in Q$. Then

$$\{n : A_n^{k(n)} \cap W_0 w' \ne \emptyset\}, \quad \{n : A_n^{k(n)} \cap w'' W_0 \ne \emptyset\},$$

$$\{n : A_n^{k(n)} \subseteq Q W_0\} = \{n : A_n^{k(n)} \cap (U \setminus Q W_0) = \emptyset\}$$

all belong to $\mathcal{F}$, by the definition of the Fell topology. Also

$$\{n : k(n) \ge 1\}$$

is cofinite in $\mathbb{N}$, because $A_n \subseteq U$ for all n large enough. Let I be the intersection of these four sets, so that I belongs to $\mathcal{F}$ and must be infinite.

(e) Let $n \in I$. **?** Suppose, if possible, that $x^i \in W$ for every $x \in A_n$ and $i \le k(n)$. (The rest of the proof will be a search for a contradiction.) Note that $k(n) \ge 1$.

Choose $x_j \in A_n$, for $j < 2k(n)$, such that the products $x_{2k(n)-1} x_{2k(n)-2} \ldots x_{k(n)}$, $x_{k(n)-1} \ldots x_0$ belong to $W_0 w'$, $w'' W_0$ respectively; set $\tilde{w} = x_{2k(n)-1} \ldots x_0$, so that

$$\tilde{w} \in A_n^{2k(n)} \cap W_0 w' w'' W_0 \subseteq W_0 w W_0.$$

Since $A_n^{k(n)} \subseteq Q W_0$ and $W_0 w W_0^2 \cap Q W_0^2$ is empty, $\tilde{w} W_0$ does not meet $A_n^{k(n)} W_0$.

(f) Define $g : X \to [0, \infty[$ by setting

$$g(x) = \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \sup\{f(yx) : y \in A_n^i\}.$$

Then $g(\tilde{w}x) f(x) = 0$ for every $x \in X$. **P** If $f(x) \ne 0$, then $x \in W_0$, so $\tilde{w}x \notin A_n^{k(n)} W_0$, and $g(\tilde{w}x) = 0$, by 446Ea. **Q** Accordingly

$$\int (g(x) - g(\tilde{w}x)) f(x) dx = \int g(x) f(x) dx \ge \beta$$

since $g \ge f$ (446Ea).

Set $y_0 = e$ and $y_{i+1} = x_i y_i$ for $i < 2k(n)$, so that $y_{2k(n)} = \tilde{w}$ and

$$y_i \in A_n^i \subseteq A_n^{2k(n)} = (A_n^{k(n)})^2 \subseteq U^2$$

for every $i \le 2k(n)$. Then

$$g(x) - g(\tilde{w}x) = \sum_{i=0}^{2k(n)-1} g(y_i x) - g(y_{i+1} x)$$

for every $x \in X$. Let $i < 2k(n)$ be such that

$$\int (g(y_i x) - g(y_{i+1} x)) f(x) dx \ge \frac{1}{2k(n)} \int (g(x) - g(\tilde{w}x)) f(x) dx \ge \frac{\beta}{2k(n)}.$$

Set $u = x_i$ and $v = y_i$, so that $u \in A_n$, $v \in U^2$ and

$$\int(g(x)-g(ux))f(v^{-1}x)dx=\int(g(vx)-g(uvx))f(x)dx$$
$$=\int(g(y_ix)-g(y_{i+1}x))f(x)dx\ge\frac{\beta}{2k(n)}.$$

(g) We have

$$\begin{aligned}\int(g(x)&-g(u^{k(n)}x))f(v^{-1}x)dx\\&=\sum_{j=0}^{k(n)-1}\int(g(u^jx)-g(u^{j+1}x))f(v^{-1}x)dx\\&=\sum_{j=0}^{k(n)-1}\int(g(x)-g(ux))f(v^{-1}u^{-j}x)dx\\&=k(n)\int(g(x)-g(ux))f(v^{-1}x)dx\\&\qquad+\sum_{j=0}^{k(n)-1}\int(g(x)-g(ux))(f(v^{-1}u^{-j}x)-f(v^{-1}x))dx,\end{aligned}$$

that is,

$$\begin{aligned}k(n)&\int(g(x)-g(ux))f(v^{-1}x)dx\\&=\int(g(x)-g(u^{k(n)}x))f(v^{-1}x)dx\\&\qquad-\sum_{j=0}^{k(n)-1}\int(g(x)-g(ux))(f(v^{-1}u^{-j}x)-f(v^{-1}x))dx.\end{aligned}$$

Set

$$\beta_1=\sum_{j=0}^{k(n)-1}\int(g(x)-g(ux))(f(v^{-1}u^{-j}x)-f(v^{-1}x))dx,$$

$$\beta_2=\int(g(x)-g(u^{k(n)}x))f(v^{-1}x)dx;$$

then

$$\beta_2-\beta_1=k(n)\int(g(x)-g(ux))f(v^{-1}x)dx\ge\tfrac{1}{2}\beta.$$

(h)($\boldsymbol{\alpha}$) Examine β_1. We know that, because $u\in A_n$, $|g(x)-g(ux)|\le\frac{\alpha}{k(n)}$ for every x (see 446Eb). On the other hand, we are supposing that $x^j\in W$ for every $j\le k(n)$ and every $x\in A_n$, so, in particular, $u^j\in W\subseteq W_0$ for every $j\le k(n)$. Also, as noted in (f), $v\in U^2$. So for any $j<k(n)$ we must have $|f(v^{-1}u^{-j}x)-f(v^{-1}x)|\le\eta$ for every $x\in X$, by the choice of W, while $f(v^{-1}u^{-j}x)-f(v^{-1}x)=0$ unless $x\in W_0U^2W_0$. So

$$\begin{aligned}|\beta_1|&\le\sum_{j=0}^{k(n)-1}\int|g(x)-g(ux)||f(v^{-1}u^{-j}x)-f(v^{-1}x)|dx\\&\le\sum_{j=0}^{k(n)-1}\frac{\alpha}{k(n)}\eta\mu(W_0U^2W_0)=\alpha\eta\mu(W_0U^2W_0).\end{aligned}$$

($\boldsymbol{\beta}$) Now consider β_2. As $u^{k(n)}\in W$, $|f(zu^{k(n)}x)-f(zx)|\le\eta$ for every $z\in U$ and $x\in X$, by the choice of W, so (because $A_n^{k(n)}\subseteq U$) $|g(u^{k(n)}x)-g(x)|\le\eta$ for every x (446Ec). Accordingly

$$|\beta_2|\le\eta\int f(v^{-1}x)dx=\eta\int f(x)dx=\eta.$$

(i) But this means that

$$\beta \le 2(|\beta_1| + |\beta_2|) \le 2\eta(1 + \alpha\mu(W_0 U^2 W_0)) < \beta,$$

which is absurd. **X**

(j) Thus for every $n \in I$ there are an $x \in A_n$ and an $i \le k(n)$ such that $x^i \notin W$, and (ii) is true. This completes the proof.

***446G 'Groups with no small subgroups' (a) Definition** Let X be a topological group. We say that X **has no small subgroups** if there is a neighbourhood U of the identity e of X such that the only subgroup of X included in U is $\{e\}$.

(b) If X is a Hausdorff topological group and U is a compact symmetric neighbourhood of the identity e such that the only subgroup of X included in U is $\{e\}$, then $\{D_n(U) : n \in \mathbb{N}\}$ is a base of neighbourhoods of e, where $D_n(U) = \{x : x \in X,\ x^i \in U \text{ for every } i \le n\}$. **P** By 446Dc, $\langle D_n(U)\rangle_{n\ge 1}$ is a non-increasing sequence of compact neighbourhoods of e, and if $x \in \bigcap_{n\in\mathbb{N}} D_n(U)$ then $x^i \in U$ for every $i \in \mathbb{N}$; as $U^{-1} = U$, U includes the subgroup $\{x^i : i \in \mathbb{Z}\}$, so $x = e$. Thus $\bigcap_{n\in\mathbb{N}} D_n(U) = \{e\}$ and $\{D_n(U) : n \in \mathbb{N}\}$ is a base of neighbourhoods of e (4A2Gd). **Q**

(c) In particular, a locally compact Hausdorff topological group with no small subgroups is metrizable (4A5Q).

***446H Lemma** Let X be a locally compact Hausdorff topological group. For $A \subseteq X$, $n \in \mathbb{N}$ set $D_n(A) = \{x : x^i \in A \text{ for every } i \le n\}$. If $U \subseteq X$ is a compact symmetric neighbourhood of the identity which does not include any subgroup of X other than $\{e\}$, then there is an $r \ge 1$ such that $D_{rn}(U)^n \subseteq U$ for every $n \in \mathbb{N}$.

proof ? Suppose, if possible, that for every $r \ge 1$ there is an $n_r \in \mathbb{N}$ such that $D_{rn_r}(U)^{n_r} \not\subseteq U$. Of course $n_r \ge 1$. Set $A_0 = U$ and $A_r = D_{rn_r}(U)$ for $r \ge 1$. Note that $D_{n_1}(U) \subseteq A_0$ but $D_{n_1}(U)^{n_1} \not\subseteq U$, so $A_0^{n_1} \not\subseteq U$. We therefore have, for every $r \in \mathbb{N}$, a k_r such that $A_r^{k(r)} \subseteq U$ but $A_r^{k(r)+1} \not\subseteq U$. Also, by 446Gb, every neighbourhood of e includes all but finitely many of the A_r. We can therefore apply 446F to the sequence $\langle A_r\rangle_{r\in\mathbb{N}}$. Of course $k(r) \ge 1$ for every r, while $k(r) < n_r$ for $r \ge 1$. Since U includes no non-trivial subgroup, (i) of 446F is impossible, and we are left with (ii). Let W, I be as declared there, so that $A_r \not\subseteq D_{k(r)}(W)$ for every $r \in I$. There must be some $m \ge 1$ such that $D_m(U) \subseteq W$ (446Gb). Take $r \in I$ such that $r \ge m$; then $rn_r \ge mk(r)$, so

$$A_r = D_{rn_r}(U) \subseteq D_{mk(r)}(U) \subseteq D_{k(r)}(D_m(U))$$

(446D(b-iv))

$$\subseteq D_{k(r)}(W),$$

which is impossible. **X**

***446I Lemma** Let X be a locally compact Hausdorff topological group and U a compact symmetric neighbourhood of the identity in X such that U does not include any subgroup of X other than $\{e\}$. For $n \in \mathbb{N}$, set $D_n(U) = \{x : x^i \in U \text{ for every } i \le n\}$, and let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Suppose that $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X such that $x_n \in D_n(U)$ for every $n \in \mathbb{N}$. Then we have a continuous homomorphism $q : \mathbb{R} \to X$ defined by setting $q(t) = \lim_{n\to\mathcal{F}} x_n^{i(n)}$ whenever $\langle i(n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n} = t$ in $\mathbb{R}$.

proof (a) If $\langle i(n)\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathbb{N}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n}$ is defined in $\mathbb{R}$, then $\lim_{n\to\mathcal{F}} x_n^{i(n)}$ is defined in X. **P** There is some $m \in \mathbb{N}$ such that $m > \lim_{n\to\mathcal{F}} \frac{i(n)}{n}$, so that $J = \{n : i(n) \le mn\} \in \mathcal{F}$; but if $n \in J$, then

$$x_n \in D_n(U) \subseteq D_{mn}(U^m) \subseteq D_{i(n)}(U^m),$$

by 446D(b-v), and $x_n^{i(n)} \in U^m$. But this means that $\mathcal{F}$ contains $\{n : x_n^{i(n)} \in U^m\}$; as U^m is compact, $\lim_{n\to\mathcal{F}} x_n^{i(n)}$ is defined in X. **Q**

More generally, if $\langle i(n)\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n}$ is defined in $\mathbb{R}$, then $\lim_{n\to\mathcal{F}} x_n^{i(n)}$ is defined in X. **P** At least one of $\{n : i(n) \ge 0\}$, $\{n : i(n) \le 0\}$ belongs to $\mathcal{F}$. In the former case, $\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} x_n^{\max(0,i(n))}$ is defined; in the latter case,

$$\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} (x_n^{\max(0,-i(n))})^{-1} = (\lim_{n\to\mathcal{F}} x_n^{\max(0,-i(n))})^{-1}$$

is defined. **Q**

(b) If V is any neighbourhood of e, there is a $\delta > 0$ such that $\lim_{n\to\mathcal{F}} x_n^{i(n)} \in V$ whenever $\langle i(n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} |\frac{i(n)}{n}| \le \delta$. **P** By 446Gb, there is an $m \ge 1$ such that $D_m(U) \subseteq V$. Take $\delta < \frac{1}{m}$. If $\lim_{n\to\mathcal{F}} |\frac{i(n)}{n}| \le \delta$, then $J = \{n : m|i(n)| \le n\} \in \mathcal{F}$. But if $n \in J$, then

$$x_n \in D_n(U) \subseteq D_{m|i(n)|}(U) \subseteq D_{|i(n)|}(D_m(U))$$

and $x_n^{|i(n)|} \in D_m(U)$; since $D_m(U)$, like U, is symmetric (446D(b-vi)), $x_n^{i(n)} \in D_m(U)$. This is true for every $n \in J$, so $\lim_{n\to\mathcal{F}} x_n^{i(n)} \in D_m(U) \subseteq V$, as required. **Q**

(c) It follows at once that if $\langle i(n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n} = 0$, then $\lim_{n\to\mathcal{F}} x_n^{i(n)} = e$. Consequently, if $\langle i(n)\rangle_{n\in\mathbb{N}}$, $\langle j(n)\rangle_{n\in\mathbb{N}}$ are sequences in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n}$ and $\lim_{n\to\mathcal{F}} \frac{j(n)}{n}$ both exist in $\mathbb{R}$ and are equal, then $\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} x_n^{j(n)}$. **P** Set $k(n) = i(n) - j(n)$. Then $\lim_{n\to\mathcal{F}} x_n^{k(n)} = e$ because $\lim_{n\to\mathcal{F}} \frac{k(n)}{n} = 0$. But now

$$\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} x_n^{j(n)} x_n^{k(n)} = \lim_{n\to\mathcal{F}} x_n^{j(n)}. \textbf{ Q}$$

(d) We do therefore have a function $q : \mathbb{R} \to X$ defined by the given formula. Now $q(s+t) = q(s)q(t)$ for all $s, t \in \mathbb{R}$. **P** Take sequences $\langle i(n)\rangle_{n\in\mathbb{N}}$, $\langle j(n)\rangle_{n\in\mathbb{N}}$ such that $s = \lim_{n\to\infty} \frac{i(n)}{n}$, $t = \lim_{n\to\infty} \frac{j(n)}{n}$; then $s+t = \lim_{n\to\infty} \frac{i(n)+j(n)}{n}$, so

$$q(s+t) = \lim_{n\to\mathcal{F}} x_n^{i(n)+j(n)} = \lim_{n\to\mathcal{F}} x_n^{i(n)} x_n^{j(n)} = q(s)q(t). \textbf{ Q}$$

(e) Thus q is a homomorphism. Finally, (b) shows that it is continuous at 0, so it must be continuous (4A5Fa).

***446J Lemma** Let X be a locally compact Hausdorff topological group with no small subgroups. Then there is a neighbourhood V of the identity e such that $x = y$ whenever x, $y \in V$ and $x^2 = y^2$.

proof Let U be a symmetric compact neighbourhood of e not including any subgroup of X except $\{e\}$. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of neighbourhoods of e comprising a base of neighbourhoods of e (446Gc), and with $U_0 = U$.

? Suppose, if possible, that for each $n \in \mathbb{N}$ there are distinct x_n, $y_n \in U_n$ such that $x_n^2 = y_n^2$. Set $a_n = x_n^{-1} y_n$; then

$$x_n^{-1} a_n x_n = x_n^{-2} y_n x_n = y_n^{-1} x_n = a_n^{-1}.$$

Accordingly

$$x_n^{-1} a_n^m x_n = (x_n^{-1} a_n x_n)^m = a_n^{-m}$$

for every $m \in \mathbb{N}$.

Since U includes no non-trivial subgroup, and $a_n \ne e$, there is a $k(n) \in \mathbb{N}$ such that $a_n^i \in U$ for $i \le k(n)$ and $a_n^{k(n)+1} \notin U$. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$; then $a = \lim_{n\to\mathcal{F}} a_n^{k(n)}$ is defined and belongs to U. Also, because $\langle x_n\rangle_{n\in\mathbb{N}}$ and $\langle y_n\rangle_{n\in\mathbb{N}}$ both converge to e, so does $\langle a_n\rangle_{n\in\mathbb{N}}$, and

$$a = \lim_{n\to\mathcal{F}} a_n^{k(n)+1} \in X \setminus \operatorname{int} U;$$

thus a cannot be e.

However, $x_n^{-1} a_n^{k(n)} x_n = a_n^{-k(n)}$ for each n, so

$$e^{-1} a e = \lim_{n\to\mathcal{F}} x_n^{-1} a_n^{k(n)} x_n = \lim_{n\to\mathcal{F}} (a_n^{k(n)})^{-1} = a^{-1}.$$

So $a = a^{-1}$ and $\{e, a\}$ is a non-trivial subgroup of X included in U, which is supposed to be impossible. **X**

Thus some U_n serves for V.

***446K Lemma** Let X be a locally compact Hausdorff topological group with no small subgroups. For $A \subseteq X$ set $D_n(A) = \{x : x^i \in A$ for every $i \le n\}$. Then there is a compact symmetric neighbourhood U of the identity e such that whenever V is a neighbourhood of e there are an $n_0 \in \mathbb{N}$ and a neighbourhood W of e such that whenever $n \ge n_0$, $x \in D_n(U)$, $y \in D_n(U)$ and $x^n y^n \in W$, then $xy \in D_n(V)$.

proof (a) Let U_0 be a compact symmetric neighbourhood of e such that (α) U_0^3 includes no subgroup of X other than $\{e\}$ (β) whenever x, $y \in U_0$ and $x^2 = y^2$, then $x = y$; such a neighbourhood exists by 446J. Let $r \ge 1$ be such that $D_{rn}(U_0)^n \subseteq U_0$ for every $n \in \mathbb{N}$ (446H). Let U be a compact symmetric neighbourhood of e such that $U^r \subseteq U_0$. In this case $D_n(U) \subseteq D_{rn}(U_0)$ for every n, by 446D(b-v). So $D_n(U)^n \subseteq U_0$ for every n.

(b) Fix a left Haar measure μ on X. Let $f : X \to [0, \infty[$ be a continuous function such that $\int f(x)dx = 1$ and $f(x) = 0$ for $x \in X \setminus U_0$. Set $\alpha = \sup_{x\in X} |f(x)|$, $\beta = \int f(x)^2 dx$, so that α is finite and $\beta > 0$. For $n \ge 1$, set

$$f_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \sup\{f(yx) : y \in D_n(U)^i\}$$

for $x \in X$. Because $D_n(U)^n \subseteq U_0$, we can apply 446E to see that, for each n,

(i) $f_n \ge f$,

(ii) $f_n(x) = 0$ if $x \notin U_0^2$,

(iii) $|f_n(ax) - f_n(x)| \le \frac{j\alpha}{n}$ if $j \in \mathbb{N}$, $a \in D_n(U)^j$ and $x \in X$,

(iv) for any x, $z \in X$, $|f_n(x) - f_n(z)| \le \sup_{y\in U_0} |f(yx) - f(yz)|$.

It follows that

(v) for any $\epsilon > 0$ there is a neighbourhood W of e such that $|f_n(ax) - f_n(bx)| \le \epsilon$ whenever a, b, $x \in X$, $n \in \mathbb{N}$ and $ab^{-1} \in W$

(4A5Pa again).

(c) It will help to have the following fact available. Suppose we are given sequences $\langle x_n\rangle_{n\in\mathbb{N}}$, $\langle y_n\rangle_{n\in\mathbb{N}}$ such that x_n and y_n belong to $D_n(U)$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n^n y_n^n = e$. Write

$$\gamma_n = \sup\{|f_n(y_n^j x) - f_n(x_n^{-j} x)| : j \le n,\ x \in X\}$$

for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \gamma_n = 0$. **P?** Otherwise, there is an $\eta > 0$ such that $J = \{n : \gamma_n > \eta\}$ is infinite. Let W be a neighbourhood of e such that $|f_n(ax) - f_n(bx)| \le \eta$ whenever $n \in \mathbb{N}$, x, a, $b \in X$ and $ab^{-1} \in W$ ((b-v) above); let W' be a neighbourhood of e such that $ab \in W$ whenever a, $b \in U$ and $ba \in W'$ (4A5Ej). Then for each $n \in J$ there must be a $j(n) \le n$ such that $y_n^{j(n)} x_n^{j(n)} \notin W$, while $x_n^{j(n)}$ and $y_n^{j(n)}$ both belong to U, so that $x^{j(n)} y^{j(n)} \notin W'$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ containing J. By 446I, there are continuous homomorphisms q, $\tilde{q}$ from $\mathbb{R}$ to X such that $q(t) = \lim_{n\to\mathcal{F}} x_n^{-i(n)}$, $\tilde{q}(t) = \lim_{n\to\mathcal{F}} y_n^{i(n)}$ whenever $\langle i(n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n} = t$ in $\mathbb{R}$. (Of course $x_n^{-1} \in D_n(U)$ for every n, by 446D(b-vi).) Setting $t_0 = \lim_{n\to\mathcal{F}} \frac{j(n)}{n} \in [0, 1]$,

$$q(-t_0)\tilde{q}(t_0) = \lim_{n\to\mathcal{F}} x_n^{j(n)} y_n^{j(n)} \notin \operatorname{int} W',$$

so $q(t_0) \ne \tilde{q}(t_0)$ and $q \ne \tilde{q}$. But

$$q(-1)\tilde{q}(1) = \lim_{n\to\mathcal{F}} x_n^n y_n^n = e,$$

so $q(1) = \tilde{q}(1)$. Now if $0 \le i(n) \le n$, then $x_n^{-i(n)} \in D_n(U)^n \subseteq U_0$; so if $0 \le t \le 1$, $q(t) \in U_0$. Similarly, $\tilde{q}(t) \in U_0$ whenever $t \in [0, 1]$. But recall that U_0 was chosen so that if x, $y \in U_0$ and $x^2 = y^2$ then $x = y$. In particular, since $q(\frac{1}{2})$ and $\tilde{q}(\frac{1}{2})$ both belong to U_0, and their squares $q(1)$, $\tilde{q}(1)$ are equal, $q(\frac{1}{2}) = \tilde{q}(\frac{1}{2})$. Repeating this argument, we see that $q(2^{-k}) = \tilde{q}(2^{-k})$ for every $k \in \mathbb{N}$, so that $q(2^{-k}i) = \tilde{q}(2^{-k}i)$ for every $k \in \mathbb{N}$, $i \in \mathbb{Z}$; since q and $\tilde{q}$ are supposed to be continuous, they must be equal; but $q(t_0) \ne \tilde{q}(t_0)$. **XQ**

(d) Now let V be any neighbourhood of e.

? Suppose, if possible, that for every neighbourhood W of e and $n_0 \in \mathbb{N}$ there are $n \ge n_0$ and x, $y \in D_n(U)$ such that $x^n y^n \in W$ but $xy \notin D_n(V)$. For $k \in \mathbb{N}$ choose $n_k \in \mathbb{N}$ and $\tilde{x}_k$, $\tilde{y}_k \in D_{n_k}(U)$ such that $\tilde{x}_k^{n_k} \tilde{y}_k^{n_k} \in D_k(U)$ but $\tilde{x}_k \tilde{y}_k \notin D_{n_k}(V)$, and $n_k > n_{k-1}$ if $k \ge 1$. Now we know that $\langle D_k(U)\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence constituting a base of neighbourhoods of e (446Gb), so $\lim_{k\to\infty} \tilde{x}_k^{n_k} \tilde{y}_k^{n_k} = e$. Set $J = \{n_k : k \in \mathbb{N}\}$. For $n = n_k$, set $x_n = \tilde{x}_k$, $y_n = \tilde{y}_k$; for $n \in \mathbb{N} \setminus J$, set $x_n = y_n = e$. Then x_n, $y_n \in D_n(U)$ for every $n \in \mathbb{N}$ and $\langle x_n^n y_n^n\rangle_{n\in\mathbb{N}} \to e$ as $n \to \infty$, while $x_n y_n \notin D_n(V)$ for $n \in J$.

We know from (c) that

$$\gamma_n = \sup\{|f_n(y_n^j x) - f_n(x_n^{-j}x)| : j \le n,\, x \in X\} \to 0$$

as $n \to \infty$; for future reference, take a sequence $\langle j(n)\rangle_{n\ge 1}$ such that $1 \le j(n) \le n$ for every $n \ge 1$, $\lim_{n\to\infty} \frac{j(n)}{n} = 0$ and $\lim_{n\to\infty} \frac{n\gamma_n}{j(n)} = 0$.

(e) Let $l \ge 1$ be such that $D_l(U_0^3) \subseteq V$ (446Gb again), and set $K = U_0^{2l+1}$. Then $D_{nl}(U_0^3) \subseteq D_n(V)$ for every n (446D(b-vii)). So $x_n y_n \notin D_{nl}(U_0^3)$ for $n \in J$. For each $n \in J$ choose $m(n) \le nl$ such that $(x_n y_n)^{m(n)} \notin U_0^3$; for $n \in \mathbb{N} \setminus J$, set $m(n) = 1$. Then $(x_n y_n)^{-m(n)} x \notin U_0^2$ for any $x \in U_0$ and $n \in J$, and $f_n((x_n y_n)^{-m(n)}x)f(x) = 0$ for any $x \in X$ and $n \in J$. So

$$|\int (f_n((x_n y_n)^{-m(n)}x) - f_n(x))f(x)dx| = \int f_n(x)f(x)dx \ge \beta$$

for $n \in J$, because $f_n \ge f$ ((b-i) above).

(f) Of course $m(n) > 0$ for every $n \in J$, therefore for every n. So we can set

$$g_n(x) = \frac{1}{m(n)}\sum_{i=0}^{m(n)-1} f((x_n y_n)^i x)$$

for $x \in X$ and $n \in \mathbb{N}$. Note that g_n, like f, is non-negative, and also that $\int g_n(x)dx = \int f(x)dx = 1$. We need to know that $g_n(x) = 0$ if $x \notin K$; this is because $(x_n y_n)^i \in D_n(U)^{2nl} \subseteq U_0^{2l}$ for every $i \le m(n)$, while $f(x) = 0$ if $x \notin U_0$, and U_0 is symmetric.

We also have

$$\begin{aligned}|g_n(ax) - g_n(x)| &\le \sup_{i<m(n)} |f((x_n y_n)^i ax) - f((x_n y_n)^i x)| \\ &\le \sup\{|f(wax) - f(wx)| : w \in U_0^{2l}\}\end{aligned}$$

for every $n \in \mathbb{N}$ and a, $x \in X$ (cf. 446Ec), so for every $\eta > 0$ there must be a neighbourhood W of e such that $|g_n(ax) - g(x)| \le \eta$ whenever $n \in \mathbb{N}$, $a \in W$ and $x \in X$, by 4A5Pa once more. Since also $g_n(ax) = g_n(x) = 0$ if $a \in U_0$ and $x \notin U_0^{-1}K$, and $U_0^{-1}K$ has finite measure, we see that for every $\eta > 0$ there is a neighbourhood W of e such that $\int |g_n(ax) - g(x)|dx \le \eta$ for every $a \in W$, $n \in \mathbb{N}$.

(g) Returning to the formula in (e), we see that, for any $n \in J$,

$$\begin{aligned}\beta &\le \Bigl|\int \bigl(f_n((x_n y_n)^{-m(n)}x) - f_n(x)\bigr)f(x)dx\Bigr| \\ &= \Bigl|\sum_{i=0}^{m(n)-1} \int \bigl(f_n((x_n y_n)^{-i-1}x) - f_n((x_n y_n)^{-i}x)\bigr)f(x)dx\Bigr| \\ &= \Bigl|\sum_{i=0}^{m(n)-1} \int \bigl(f_n((x_n y_n)^{-1}x) - f_n(x)\bigr)f((x_n y_n)^i x)dx\Bigr| \\ &= m(n)\Bigl|\int \bigl(f_n((x_n y_n)^{-1}x) - f_n(x)\bigr)g_n(x)dx\Bigr| \\ &\le ln\Bigl|\int \bigl(f_n(y_n^{-1}x_n^{-1}x) - f_n(x)\bigr)g_n(x)dx\Bigr| \\ &\le ln\Bigl|\int \bigl(f_n(y_n^{-1}x) - f_n(x) - f_n(y_n^{-1}x_n^{-1}x) + f_n(x_n^{-1}x)\bigr)g_n(x)dx\Bigr| \\ &\qquad + ln\Bigl|\int \bigl(2f_n(x) - f_n(x_n^{-1}x) - f_n(y_n^{-1}x)\bigr)g_n(x)dx\Bigr|.\end{aligned}$$

Next,

$$\begin{aligned}&j(n)\bigl(2f_n(x) - f_n(x_n^{-1}x) - f_n(y_n^{-1}x)\bigr) \\ &\quad = j(n)\bigl(f_n(x) - f_n(x_n^{-1}x)\bigr) - f_n(x) + f_n(x_n^{-j(n)}x) \\ &\qquad\quad + j(n)\bigl(f_n(x) - f_n(y_n^{-1}x)\bigr) - f_n(x) + f_n(y_n^{-j(n)}x) \\ &\qquad\quad + 2f_n(x) - f_n(x_n^{-j(n)}x) - f_n(y_n^{-j(n)}x)\end{aligned}$$

for every x, and finally

$$\int\big(2f_n(x)-f_n(x_n^{-j(n)}x)-f_n(y_n^{-j(n)}x)\big)g_n(x)dx$$

$$=\int\big(f_n(x)-f_n(y_n^{-j(n)}x)\big)g_n(x)dx-\int\big(f_n(y_n^{j(n)}x)-f_n(x)\big)g_n(x)dx$$
$$+\int\big(f_n(y_n^{j(n)}x)-f_n(x_n^{-j(n)}x)\big)g_n(x)dx$$
$$=\int\big(f_n(y_n^{j(n)}x)-f_n(x)\big)g_n(y_n^{j(n)}x)dx-\int\big(f_n(y_n^{j(n)}x)-f_n(x)\big)g_n(x)dx$$
$$+\int\big(f_n(y_n^{j(n)}x)-f_n(x_n^{-j(n)}x)\big)g_n(x)dx$$
$$=\int\big(f_n(y_n^{j(n)}x)-f_n(x)\big)\big(g_n(y_n^{j(n)}x)-g_n(x)\big)dx$$
$$+\int\big(f_n(y_n^{j(n)}x)-f_n(x_n^{-j(n)}x)\big)g_n(x)dx.$$

So if we write

$$\beta_{1n}=ln\int\Big(f_n(y_n^{-1}x)-f_n(x)-f_n(y_n^{-1}x_n^{-1}x)+f_n(x_n^{-1}x)\Big)g_n(x)dx,$$

$$\beta_{2n}=\frac{ln}{j(n)}\int\Big(j(n)(f_n(x)-f_n(x_n^{-1}x))-f_n(x)+f_n(x_n^{-j(n)}x)\Big)g_n(x)dx,$$

$$\beta_{3n}=\frac{ln}{j(n)}\int\Big(j(n)(f_n(x)-f_n(y_n^{-1}x))-f_n(x)+f_n(y_n^{-j(n)}x)\Big)g_n(x)dx,$$

$$\beta_{4n}=\frac{ln}{j(n)}\int\Big(f_n(y_n^{j(n)}x)-f_n(x)\Big)\Big(g_n(y_n^{j(n)}x)-g_n(x)\Big)dx,$$

$$\beta_{5n}=\frac{ln}{j(n)}\int\Big(f_n(y_n^{j(n)}x)-f_n(x_n^{-j(n)}x)\Big)g_n(x)dx$$

for $n\ge 1$, we have

$$|\beta_{1n}|+|\beta_{2n}|+|\beta_{3n}|+|\beta_{4n}|+|\beta_{5n}|\ge\beta$$

for every $n\in J$.

(h) Now $\beta_{1n}\to 0$ as $n\to\infty$. **P**

$$|\int\big(f_n(y_n^{-1}x)-f_n(x)-f_n(y_n^{-1}x_n^{-1}x)+f_n(x_n^{-1}x)\big)g_n(x)dx|$$
$$=|\int\big(f_n(y_n^{-1}x)-f_n(x)\big)g_n(x)dx-\int\big(f_n(y_n^{-1}x_n^{-1}x)-f_n(x_n^{-1}x)\big)g_n(x)dx|$$
$$=|\int\big(f_n(y_n^{-1}x)-f_n(x)\big)g_n(x)dx-\int\big(f_n(y_n^{-1}x)-f_n(x)\big)g_n(x_nx)dx|$$
$$=|\int\big(f_n(y_n^{-1}x)-f_n(x)\big)\big(g_n(x)-g_n(x_nx)\big)dx|$$
$$\le\frac{\alpha}{n}\int|g_n(x)-g_n(x_nx)|dx$$

by (b-iii). So

$$|\beta_{1n}|\le\alpha l\int|g_n(x)-g_n(x_nx)|dx\to 0$$

as $n\to\infty$, by (f), since surely $\langle x_n\rangle_{n\in\mathbb{N}}\to e$. **Q**

(i) Now look at β_{2n}. If we set

$$\gamma_n'=\sup\{\int|g_n(ax)-g(x)|dx:a\in D_n(U)^{j(n)}\},$$

then $\gamma'_n \to 0$ as $n \to \infty$. **P** Given $\epsilon > 0$, there is a neighbourhood W of e such that $\int |g_n(ax) - g_n(x)|dx \le \epsilon$ whenever $n \ge 1$ and $a \in W$, as noted at the end of (f). Let p be such that $D_p(U_0) \subseteq W$. Then, for all n large enough, $pj(n) \le n$, so that $D_n(U)^{pj(n)} \subseteq U_0$ and $D_n(U)^{j(n)} \subseteq D_p(U_0) \subseteq W$ (446D(b-v)) and $\int |g_n(ax) - g(x)|dx \le \epsilon$ for every $a \in D_n(U)^{j(n)}$. **Q**

We have

$$\begin{aligned}
&|\int \big(j(n)(f_n(x) - f_n(x_n^{-1}x)) - f_n(x) + f_n(x_n^{-j(n)}x)\big)g_n(x)dx| \\
&\quad = |\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x) - f_n(x_n^{-i}x) + f_n(x_n^{-i-1}x)\big)g_n(x)dx| \\
&\quad = |\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x)\big)g_n(x)dx - \int \big(f_n(x_n^{-i}x) - f_n(x_n^{-i-1}x)\big)g_n(x)dx| \\
&\quad = |\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x)\big)g_n(x)dx - \int \big(f_n(x) - f_n(x_n^{-1}x)\big)g_n(x_n^{i}x)dx| \\
&\quad = |\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x)\big)\big(g_n(x) - g_n(x_n^{i}x)\big)dx| \\
&\quad \le \sum_{i=0}^{j(n)-1} \int |f_n(x) - f_n(x_n^{-1}x)||g_n(x) - g_n(x_n^{i}x)|dx \le j(n)\frac{\alpha}{n}\gamma'_n.
\end{aligned}$$

So

$$|\beta_{2n}| \le l\alpha\gamma'_n \to 0$$

as $n \to \infty$. Similarly, $\langle \beta_{3n} \rangle_{n \ge 1} \to 0$.

(j) As for β_{4n}, we have

$$\int |f_n(y_n^{j(n)}x) - f_n(x)||g_n(y_n^{j(n)}x) - g_n(x)|dx \le \frac{\alpha j(n)}{n}\gamma'_n,$$

putting (b-iii) and the definition of γ'_n together. So

$$|\beta_{4n}| \le l\alpha\gamma'_n \to 0$$

as $n \to \infty$.

(k) We come at last to β_{5n}. Here, for every $n \ge 1$,

$$\begin{aligned}
|\int \big(f_n(y_n^{j(n)}x) - f_n(x_n^{-j(n)}x)\big)g_n(x)dx| &\le \int \big|f_n(y_n^{j(n)}x) - f_n(x_n^{-j(n)}x)\big|g_n(x)dx \\
&\le \gamma_n \int g_n(x)dx = \gamma_n
\end{aligned}$$

by the definition of γ_n in (c) above. So

$$|\beta_{5n}| \le l\frac{n}{j(n)}\gamma_n \to 0$$

by the choice of the $j(n)$.

(l) Thus $\beta_{in} \to 0$ as $n \to \infty$ for every i. But this is impossible, because $0 < \beta \le \sum_{i=1}^{5} |\beta_{in}|$ for every $n \in J$. **X**

This contradiction shows that we must be able to find a neighbourhood W of e and an $n_0 \in \mathbb{N}$ such that $xy \in D_n(V)$ whenever $n \ge n_0$, x, $y \in D_n(U)$ and $x^n y^n \in W$; as V is arbitrary, U has the property required.

***446L Definition** Let X be a topological group. A ***B*-sequence** in X is a non-increasing sequence $\langle V_n \rangle_{n \in \mathbb{N}}$ of closed neighbourhoods of the identity, constituting a base of neighbourhoods of the identity, such that there is some M such that for every $n \in \mathbb{N}$ the set $V_n V_n^{-1}$ can be covered by at most M left translates of V_n.

***446M Proposition** Let X be a locally compact Hausdorff topological group with no small subgroups. Then it has a B-sequence.

proof (a) For $A \subseteq X$ set $D_n(A) = \{x : x^i \in A \text{ for every } i \le n\}$. We know from 446K that there is a compact symmetric neighbourhood U of the identity e such that whenever V is a neighbourhood of e there are an $n_0 \in \mathbb{N}$ and a neighbourhood W of e such that whenever $n \ge n_0$, $x \in D_n(U)$, $y \in D_n(U)$ and $x^n y^n \in W$, then $xy \in D_n(V)$. Shrinking U if necessary, we may suppose that U includes no subgroup of X other than $\{e\}$, so that there is an $r \ge 1$ such that $D_{rn}(U)^n \subseteq U$ for every $n \in \mathbb{N}$ (446H).

Let V be a closed symmetric neighbourhood of e such that $V^{2r} \subseteq U$. Then $D_n(V)^2 \subseteq D_n(U)$ for every $n \in \mathbb{N}$. **P** $D_n(V) \subseteq D_{2rn}(U)$, by 446D(b-v), so

$$(D_n(V)^2)^n \subseteq D_{2rn}(U)^{2n} \subseteq U,$$

and $D_n(V)^2 \subseteq D_n(U)$ (446D(b-v) again). **Q** Take $n_0 \in \mathbb{N}$ and a neighbourhood W of e such that whenever $n \ge n_0$, x, $y \in D_n(U)$ and $x^n y^n \in W^{-1}W$, then $xy \in D_n(V)$.

(b) Let M be so large that U can be covered by M left translates of W. Then for any $n \ge n_0$, $D_n(V)D_n(V)^{-1} = D_n(V)^2$ can be covered by M left translates of $D_n(V)$.

P Let $z_0, \dots, z_{M-1}$ be such that $U \subseteq \bigcup_{i \le M} z_i W$. For each $i < M$, set $A_i = \{x : x \in D_n(U),\ x^n \in z_i W\}$; if $A_i \ne \emptyset$ choose $x_i \in A_i$; otherwise, set $x_i = e$.

For any $y \in D_n(V)^2$, $y \in D_n(U)$, so $y^n \in U$ and there is some $i < M$ such that $y \in A_i$. In this case x_i also belongs to A_i. Now $z_i^{-1}y^n$ and $z_i^{-1}x_i^n$ both belong to W, so $x_i^{-n}y^n$ belongs to $W^{-1}W$, and $x_i^{-1}y \in D_n(V)$, by the choice of W and n_0. But this means that $y \in x_i D_n(V)$. As y is arbitrary, $D_n(V)^2 \subseteq \bigcup_{i<M} x_i D_n(V)$ is covered by M left translates of $D_n(V)$. **Q**

(c) But this means that $\langle D_{n+n_0}(V)\rangle_{n\in\mathbb{N}}$ is a B-sequence in X. (It constitutes a base of neighbourhoods of e by 446Gb, as usual.)

***446N Proposition** Let X be a locally compact Hausdorff topological group with a faithful finite-dimensional representation. Then it has a B-sequence.

proof (a) Let $\phi : X \to GL(r, \mathbb{R})$ be a faithful finite-dimensional representation. Identifying M_r with the Banach algebra $\mathsf{B} = \mathsf{B}(\mathbb{R}^r; \mathbb{R}^r)$, where $\mathbb{R}^r$ is given the Euclidean norm, we see that $GL(r, \mathbb{R})$ is an open subset of B (4A6H). Note also that the operator norm $\|\ \|$ of B is equivalent to its 'Euclidean' norm corresponding to an identification with $\mathbb{R}^{r^2}$, that is, writing $\|T\|_{HS} = \sqrt{\sum_{i=1}^r \sum_{j=1}^r \tau_{ij}^2}$ if $T = \langle \tau_{ij}\rangle_{1\le i,j\le r}$, $\|\ \|_{HS}$ is equivalent to $\|\ \|$. (See the inequalities in 262H.) In particular, all the balls $B(T, \delta) = \{S : \|S - T\| \le \delta\}$ are closed for the Euclidean norm (4A2Lj). If we write μ_L for Lebesgue measure on B, identified with $\mathbb{R}^{r^2}$, and set $\gamma = \mu_L B(\mathbf{0}, 1)$, then $0 < \gamma < \infty$ (because $B(\mathbf{0}, 1)$ includes, and is included in, non-trivial Euclidean balls) and $\mu_L B(T, \delta) = \delta^{r^2}\gamma$ for every $T \in \mathsf{B}$ and $\delta \ge 0$ (using 263A, or otherwise).

(b) We need to recall a basic inequality concerning inversion in Banach algebras. If $T \in \mathsf{B}$ and $\|T - I\| \le \frac{1}{2}$, then T is invertible and

$$\|T^{-1} - I\| \le \tfrac{\|T-I\|}{1-\|T-I\|} \le 1$$

(4A6H), so $\|T^{-1}\| \le 2$.

(c) Now let V be a compact neighbourhood of the identity e of X. Let V_1 be a neighbourhood of e such that $(V_1V_1^{-1})^{-1}V_1V_1^{-1} \subseteq V$. For $\delta > 0$, set $U_\delta = \{x : x \in V,\ \|\phi(x) - I\| \le \delta\}$. Then each U_δ is a compact neighbourhood of e, because ϕ is continuous. Also

$$\bigcap_{\delta>0} U_\delta = \{x : x \in V,\ \phi(x) = I\} = \{e\}.$$

So $\{U_\delta : \delta > 0\}$ is a base of neighbourhoods of e (4A2Gd), and there is a $\delta_1 > 0$ such that $U_{\delta_1} \subseteq V_1$; of course we may suppose that $\delta_1 \le \frac{1}{8}$.

(d) If $\delta \le \frac{1}{2}$ and $x \in U_\delta U_\delta^{-1}$, then $\|\phi(x) - I\| \le 4\delta$. **P** Express x as yz^{-1} where y, $z \in U_\delta$. Then $\|\phi(z) - I\| \le \frac{1}{2}$, so $\|\phi(z^{-1})\| \le 2$ and

$$\|\phi(x) - I\| = \|(\phi(y) - \phi(z))\phi(z^{-1})\| \le 2\|\phi(y) - \phi(z)\| \le 4\delta. \ \mathbf{Q}$$

(e) Now if $\delta \le \delta_1$, $U_\delta U_\delta^{-1}$ can be covered by at most $m = 17^{r^2}$ left translates of U_δ. **P?** Suppose, if possible, otherwise. Then we can choose $x_0, \dots, x_m \in U_\delta U_\delta^{-1}$ such that $x_j \notin x_i U_\delta$ whenever $i < j \le m$. If $i < j \le m$, then

$$x_i^{-1} x_j \in (U_\delta U_\delta^{-1})^{-1} U_\delta U_\delta^{-1} \subseteq (V_1 V_1^{-1})^{-1} V_1 V_1^{-1} \subseteq V,$$

and $x_i^{-1} x_j \notin U_\delta$, so $\|\phi(x_i^{-1} x_j) - I\| > \delta$. Set $T_i = \phi(x_i)$ for each $i \le m$; then

$$\|T_i - I\| \le 4\delta \le \tfrac{1}{2}$$

for each i, by (d), while

$$\delta < \|\phi(x_i^{-1} x_j) - I\| = \|T_i^{-1} T_j - I\| \le \|T_i^{-1}\| \|T_j - T_i\| \le 2\|T_j - T_i\|$$

whenever $i < j \le m$. Write $B_i = B(T_i, \frac{\delta}{4})$ for each i; then all the B_i are disjoint. But also they are all included in $B(I, \frac{17\delta}{4})$, so we have

$$(17^{r^2} + 1)\bigl(\tfrac{\delta}{4}\bigr)^{r^2} \gamma \le \bigl(\tfrac{17\delta}{4}\bigr)^{r^2} \gamma,$$

which is impossible. **XQ**

(f) Accordingly, setting $W_n = U_{2^{-n}\delta_1}$, $\langle W_n \rangle_{n \in \mathbb{N}}$ is a B-sequence in X.

***446O Theorem** Let X be a locally compact Hausdorff topological group. Then it has an open subgroup Y which has a compact normal subgroup Z such that Y/Z has no small subgroups.

proof (a) Let U be a compact neighbourhood of the identity e of X. Then there are a subgroup Y_0 of X, included in U, and a neighbourhood W_0 of e such that every subgroup of X included in W_0 is also included in Y_0.

P(i) To begin with (down to the end of (iii)) let us suppose that X is metrizable. Let $\langle V_n \rangle_{n \in \mathbb{N}}$ be a non-increasing sequence of closed symmetric neighbourhoods of e running over a base of neighbourhoods of e, and such that $V_1^2 \subseteq V_0 \subseteq U$. For each $n \in \mathbb{N}$, set $A_n = \{x : x^i \in V_n \text{ for every } i \in \mathbb{N}\}$.

(ii)? Suppose, if possible, that $\bigcup_{k \in \mathbb{N}} A_n^k \not\subseteq U$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $k(n) \in \mathbb{N}$ be such that $A_n^{k(n)} \subseteq U$, $A_n^{k(n)+1} \not\subseteq U$. Then $\langle A_n \rangle_{n \in \mathbb{N}}$ and U satisfy the conditions of 446F (because $A_m \subseteq V_n$ whenever $m \ge n$, and $\{V_n : n \in \mathbb{N}\}$ is a base of neighbourhoods of e). Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ and set $Q = \lim_{n \to \mathcal{F}} A_n^{k(n)}$ in the space $\mathcal{C}$ of closed subsets of X with the Fell topology.

If W is any neighbourhood of e, there is an $n \in \mathbb{N}$ such that $V_n \subseteq W$, so that $x^i \in W$ whenever $x \in \bigcup_{m \ge n} A_m$ and $i \in \mathbb{N}$. Thus (ii) of 446F is not true, and Q must be a closed subgroup of X included in U and meeting the boundary of U.

By 446C, there are an $r \in \mathbb{N}$ and a continuous homomorphism $\phi : Q \to GL(r, \mathbb{R})$ such that the kernel Z of ϕ is included in $\operatorname{int} V_1$. Let $G \subseteq X$ be an open set including Z and with closure disjoint from $(X \setminus \operatorname{int} V_1) \cup \{x : x \in Q, \|\phi(x) - I\| \ge \frac{1}{6}\}$; such can be found because Z is compact and X is regular (4A2F(h-ii)). Then $Z \subseteq G$, and any subgroup Z' of Q included in $\overline{G}$ has $\|\phi(x)^i - I\| \le \frac{1}{6}$ for every $x \in Z'$ and $i \in \mathbb{N}$, so that $Z' \subseteq Z$, by 4A6N. Set $V = \overline{G}$.

Since $V \subseteq U$ and $A_n^{k(n)+1} \not\subseteq U$ for every n, we can find $j(n) \le k(n)$ such that $A_n^{j(n)} \subseteq V$ and $A_n^{j(n)+1} \not\subseteq V$ for every n. Set $Q' = \lim_{n \to \mathcal{F}} A_n^{j(n)}$. As before, (ii) of 446F cannot be true of Q', and Q' must be a closed subgroup of X meeting the boundary of V. Because $e \in A_n$, $A_n^{j(n)} \subseteq A_n^{k(n)}$ for every n, and $Q' \subseteq Q$, because $\{(E, F) : E \subseteq F\}$ is closed in $\mathcal{C}$ (4A2T(e-i)); also $Q' \subseteq V$, so $Q' \subseteq Z$. But Z does not meet the boundary of V. **X**

(iii) So there is some $n \in \mathbb{N}$ such that $A_n^k \subseteq U$ for every $k \in \mathbb{N}$. Because $A_n^{-1} = A_n$, $Y_0 = \bigcup_{k \in \mathbb{N}} A_n^k$ is a subgroup of X. Any subgroup of X included in V_n is a subset of A_n so is included in Y_0. Thus we have a pair Y_0, $W_0 = V_n$ of the kind required, at least when X is metrizable.

(iv) Now suppose that X is σ-compact. Let U_1 be a neighbourhood of e such that $U_1^2 \subseteq U$. Then there is a closed normal subgroup X_0 of X such that $X_0 \subseteq U_1$ and $X' = X/X_0$ is metrizable (4A5S). By (i)-(iii), there are a subgroup Y_0' of X', included in the image of U_1 in X', and a neighbourhood W_0' of the identity in X' such that any subgroup of X' included in W_0' must also be included in Y_0'. Write $\pi : X \to X'$ for the canonical homomorphism and consider $Y_0 = \pi^{-1}[Y_0']$, $W_0 = \pi^{-1}[W_0']$. Then W_0 is a neighbourhood of e and Y_0 is a subgroup of X included in

$$\pi^{-1}[\pi[U_1]] = U_1 X_0 \subseteq U_1^2 \subseteq U.$$

And if Z is any subgroup of X included in W_0, then $\pi[Z] \subseteq W_0'$ so $\pi[Z] \subseteq Y_0'$ and $Z \subseteq Y_0$. Thus in this case also we have the result.

(v) Finally, for the general case, observe that X has a σ-compact open subgroup X_1 (4A5El). So we can find a subgroup Y_0 of X_1, included in $U \cap X_1$, and a neighbourhood W_0 of the identity in X_1 such that any subgroup of X_1 included in W_0 is also included in Y_0. But of course Y_0 and W_0 also serve for X and U.

This completes the proof of (a). **Q**

(b) Of course $\overline{Y_0}$ is a subgroup of X (4A5Em); being included in U, it is compact. By 446C, there is a finite-dimensional representation $\phi : \overline{Y_0} \to GL(r, \mathbb{R})$, for some $r \in \mathbb{N}$, such that the kernel Z of ϕ is included in $\operatorname{int} W_0$. Let W_1 be a neighbourhood of e in X such that $\|\phi(x) - I\| \le \frac{1}{6}$ for every $x \in W_1 \cap \overline{Y_0}$, and set $W = W_1 Z \cap W_0$. Note that if $x \in W \cap \overline{Y_0}$, there is a $z \in Z$ such that $xz \in W_1 \cap \overline{Y_0}$, so that $\|\phi(x) - I\| = \|\phi(xz) - I\| \le \frac{1}{6}$. Of course $Z \subseteq \operatorname{int} W_1 Z$, so $Z \subseteq \operatorname{int} W$.

If Y' is a subgroup of $\overline{Y_0}$ included in W, then $\|\phi(x)^i - I\| \le \frac{1}{6}$ for every $i \in \mathbb{N}$ and $x \in Y'$, so $Y' \subseteq Z$. Consequently any subgroup of X included in W is a subgroup of Z, since by the choice of $\overline{Y_0}$ and W_0 it is a subgroup of $\overline{Y_0}$.

Now let Y be the normalizer of Z in X. Z is compact, so $G = \{x : xZx^{-1} \subseteq \operatorname{int} W\}$ is open (4A5Ei), and contains e; but also $G \subseteq Y$, because if $x \in G$ then xZx^{-1} is a subgroup of X included in W, and must be included in Z. Accordingly $Y = GY$ is open.

Since any subgroup of Y included in W is a subgroup of Z, we see that any subgroup of Y/Z included in the image of W is the trivial subgroup, and Y/Z has no small subgroups, as required.

***446P Corollary** Let X be a locally compact Hausdorff topological group. Then it has a chain $\langle X_\xi \rangle_{\xi \le \kappa}$ of closed subgroups, where κ is an infinite cardinal, such that

(i) X_0 is open,

(ii) $X_{\xi+1}$ is a normal subgroup of X_ξ for every $\xi < \kappa$,

(iii) X_ξ is compact for $\xi \ge 1$,

(iv) $X_\xi = \bigcap_{\eta<\xi} X_\eta$ for non-zero limit ordinals $\xi \le \kappa$,

(v) $X_\xi / X_{\xi+1}$ has a B-sequence for every $\xi < \kappa$,

(vi) $X_\kappa = \{e\}$, where e is the identity of X.

proof (a) By 446O, X has an open subgroup X_0 with a compact normal subgroup X_1 such that X_0/X_1 has no small subgroups. By 446M, X_0/X_1 has a B-sequence.

(b) Let Φ be the set of finite-dimensional representations of X_1; if we distinguish the trivial homomorphisms from X_1 to each $GL(r, \mathbb{R})$, Φ is infinite. Set $\kappa = \#(\Phi)$ and let $\langle \phi_\xi \rangle_{1 \le \xi < \kappa}$ run over Φ. For $1 \le \xi \le \kappa$, set

$$X_\xi = \{x : x \in X_1,\ \phi_\eta(x) = I \text{ for } 1 \le \eta < \xi\}.$$

Then $\langle X_\xi \rangle_{\xi \le \kappa}$ satisfies conditions (i)-(iv). As for (v), I have already checked the case $\xi = 0$, and if $1 \le \xi < \kappa$, then $\phi_\xi \restriction X_\xi$ is a finite-dimensional representation of X_ξ with kernel $X_{\xi+1}$, so $X_\xi / X_{\xi+1}$ has a faithful finite-dimensional representation (446Ab), and therefore has a B-sequence (446N).

Finally, $X_\kappa = \{e\}$ by 446B; if $x \in X_1$ and $x \ne e$, there is a $\phi \in \Phi$ such that $\phi(x) \ne \phi(e)$, so that $x \notin X_\kappa$.

446X Basic exercises (a) Let X be a locally compact Hausdorff abelian topological group. Show that for every element a of X, other than the identity, there is a two-dimensional representation ϕ of X such that $\phi(a) \ne I$. (*Hint*: 445O.)

(b) Let $\langle X_n \rangle_{n \in \mathbb{N}}$ be any sequence of groups with their discrete topologies, and X the product topological group $\prod_{n \in \mathbb{N}} X_n$. Show that X has a B-sequence. (*Hint*: set $V_n = \{x : x(i) = e(i) \text{ for } i < n\}$.)

446Y Further exercises (a) Let X be the countable group of all permutations of $\mathbb{N}$ which are products of an even number of transpositions. Give X its discrete topology, so that it is a locally compact topological group. Show that any finite-dimensional representation of X is trivial. (*Hint*: X is simple and has many commuting involutions.)

(b) Let X be a compact Hausdorff topological group, $f \in C(X)$ and $\epsilon > 0$. Show that there are a finite-dimensional representation $\phi : X \to GL(r, \mathbb{R})$ and a, $b \in \mathbb{R}^r$ such that $|f(x) - (\phi(x)(a)|b)| \le \epsilon$ for every $x \in X$.

(c) Let κ be an infinite cardinal, and $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ the measure algebra of the usual measure on $\{0,1\}^\kappa$. Give the group $\operatorname{Aut}_{\bar\nu_\kappa}(\mathfrak{B}_\kappa)$ of measure-preserving automorphisms of $\mathfrak{B}_\kappa$ its topology of pointwise convergence. Let X be a compact Hausdorff topological group of weight at most κ. Show that there is a continuous injective homomorphism from X to $\operatorname{Aut}_{\bar\nu_\kappa}(\mathfrak{B}_\kappa)$.

446 Notes and comments The ideas above are extracted from the structure theory for locally compact groups, as described in Montgomery & Zippin 55. (A brisker and sometimes neater, but less complete, exposition can be found in Kaplansky 71.) The full theory goes very much deeper into the analysis of groups with no small subgroups. One of the most important ideas, hidden away in 446I and part (c) of the proof of 446K, is that of 'one-parameter subgroup'; if X is a group with no small subgroups, there are enough continuous homomorphisms from $\mathbb{R}$ to X not only to provide a great deal of information on the topological group structure of X, but even to set up a differential structure (Kaplansky 71, §II.3). For our purposes here, however, all we need to know is that groups with no small subgroups have 'B-sequences' (446L-446M), which can form the basis of a theory corresponding to Vitali's theorem and Lebesgue's Density Theorem in $\mathbb{R}^r$ (447C-447D below).

There are four essential elements in the argument here. Working from the outside, the first step is 446O: starting from a locally compact Hausdorff group X, we can find an open subgroup X_0 of X and a compact normal subgroup X_1 of X_0 such that X_0/X_1 has no small subgroups. This depends on a subtle argument based on the first key lemma, the dichotomy in 446F, which in turn uses the 'smoothing' construction in 446E and a careful analysis of inequalities involving integrals. (Naturally enough, the translation-invariance of the Haar integral is a leitmotiv of this investigation.) Note the remarkable transition in 446H. The sets $D_n(U)$ are defined solely in terms of powers, while the sets $D_n(U)^n$ involve products. We are able to obtain information about products $x_1 \ldots x_n$ from information about the powers x_j^i for i, $j \le n$.

Next, we need to find a chain of closed subnormal subgroups of X_1, decreasing to $\{e\}$, such that the quotients all have faithful finite-dimensional representations (in this context, this means that they are isomorphic to compact subgroups of $GL(r, \mathbb{R})$). This step depends on the older ideas in 446B-446C, where we use the theory of compact operators on Hilbert spaces to show that a compact group has many representations as actions on finite-dimensional subspaces of its L^2 space. (Observe that in this section I revert to real-valued, rather than complex-valued, functions.) This can be thought of as a development of the result of 445O. If X is a locally compact abelian group, its characters separate its points (cf. 446Xa); if X is compact but not necessarily abelian, its finite-dimensional representations separate its points. (But if X is neither compact nor abelian, there are further difficulties; see 446Ya.)

The other two necessary facts are that both groups with no small subgroups, and groups with faithful finite-dimensional representations, have B-sequences. The latter is reasonably straightforward (446N); any complications are due entirely to the fact that the natural measure on $GL(r, \mathbb{R})$, inherited from $\mathbb{R}^{r^2}$, is not quite invariant under multiplication, so we have to manipulate some inequalities. For groups with no small subgroups (446M) we have much more to do. The proof I give here depends on a second key lemma, 446K, refining the methods of 446F; a slightly stronger version of this result is the basis of the analysis of one-parameter subgroups in the general theory (compare Montgomery & Zippin 55, §3.8).

447 Translation-invariant liftings

I devote a section to the main theorem of Ionescu Tulcea & Ionescu Tulcea 67: a group carrying Haar measures has a translation-invariant lifting (447J). The argument uses an inductive construction of the same type as that used in §341 for the ordinary Lifting Theorem. It depends on the structure theory for locally compact groups described in §446. On the way I describe a Vitali theorem for certain metrizable groups (447C), with a corresponding density theorem (447D).

447A Liftings and lower densities Let X be a group carrying Haar measures, Σ its algebra of Haar measurable sets and $\mathfrak{A}$ its Haar measure algebra (442H, 443A).

(a) Recall that a **lifting** of $\mathfrak{A}$ is either a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$, or a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that $E \triangle \phi E$ is Haar negligible for every $E \in \Sigma$ and $\phi E = \emptyset$ whenever E is Haar negligible (341A). Such a lifting θ or ϕ is **left-translation-invariant** if $\theta((xE)^\bullet) = x(\theta E^\bullet)$ or $\phi(xE) = x(\phi E)$ for every $E \in \Sigma$ and $x \in X$. (In the notation of 443C, a lifting $\theta : \mathfrak{A} \to \Sigma$ is left-translation-invariant if $\theta(x{\bullet_l}a) = x\theta a$ for every $x \in X$, $a \in \mathfrak{A}$.)

The language of 341A demanded a named measure; I spoke there of a lifting for a measure space (X, Σ, μ) or a measure μ. But (as noted in 341Lh) what the concept really depends on is a triple $(X, \Sigma, \mathcal{I})$, where Σ is an algebra of subsets of X and $\mathcal{I}$ is an ideal of Σ. Variations in the measure which do not affect the algebra of measurable sets or the null ideal are irrelevant. So, in the present context, we can speak of a 'lifting for Haar measure' without declaring which Haar measure we are using, nor even whether it is a left or right Haar measure.

(b) Now suppose that Σ_0 is a σ-subalgebra of Σ. In this case, a **partial lower density** on Σ_0 is a function $\underline{\phi}:\Sigma_0\to\Sigma$ such that $\underline{\phi}E=\underline{\phi}F$ whenever E, $F\in\Sigma_0$ and $E\triangle F$ is negligible, $E\triangle\underline{\phi}E$ is negligible for every $E\in\Sigma_0$, $\underline{\phi}\emptyset=\emptyset$ and $\underline{\phi}(E\cap F)=\underline{\phi}E\cap\underline{\phi}F$ for all E, $F\in\Sigma_0$. (See 341C-341D.) As in (a), such a function is **left-translation-invariant** if $xE\in\Sigma_0$ and $\underline{\phi}(xE)=x(\underline{\phi}E)$ for every $x\in X$ and $E\in\Sigma_0$.

447B Lemma Let X be a group carrying Haar measures and Y a subgroup of X. Write Σ_Y for the algebra of Haar measurable subsets E of X such that $EY=E$, and suppose that $\underline{\phi}:\Sigma_Y\to\Sigma_Y$ is a left-translation-invariant partial lower density. Then $G\subseteq\underline{\phi}(GY)$ for every open set $G\subseteq X$.

proof Of course GY is open (4A5Ed), so belongs to Σ_Y. Let $a\in G$ and let U be an open neighbourhood of the identity in X such that $U^{-1}Ua\subseteq G$. Then UaY is a non-empty open set, therefore not negligible (442Aa), and there is an $x\in UaY\cap\underline{\phi}(UaY)$. Express x as uay where $u\in U$ and $y\in Y$; then $ua=xy^{-1}\in Ua\cap\underline{\phi}(UaY)$, because $\underline{\phi}(UaY)\in\Sigma_Y$. So

$$a=u^{-1}ua\in u^{-1}\underline{\phi}(UaY)=\underline{\phi}(u^{-1}UaY)\subseteq\underline{\phi}(GY).$$

As a is arbitrary, $G\subseteq\underline{\phi}(GY)$.

447C Vitali's theorem Let X be a topological group with a left Haar measure μ, and $\langle V_n\rangle_{n\in\mathbb{N}}$ a B-sequence in X (definition: 446L). If $A\subseteq X$ is any set and K_x is an infinite subset of $\mathbb{N}$ for every $x\in A$, then there is a disjoint family $\mathcal{V}$ of sets such that $A\setminus\bigcup\mathcal{V}$ is negligible and every member of $\mathcal{V}$ is of the form xV_n for some $x\in A$ and $n\in K_x$.

proof (a) There is surely some r such that V_r is totally bounded for the bilateral uniformity on X (443H); replacing V_i by V_r for $i<r$ and K_x by $K_x\setminus r$ for each x, we may suppose that V_0 is totally bounded.

(b) Choose $\langle I_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Given $I_j\subseteq X$ for $j<n$, choose a set $I_n\subseteq A$ which is maximal subject to the conditions

$$n\in K_x\text{ for every }x\in I_n,$$

$$xV_n\cap yV_j=\emptyset\text{ whenever }x\in I_n,\ j<n\text{ and }y\in I_j,$$

$$xV_n\cap yV_n=\emptyset\text{ whenever }x,\ y\in I_n\text{ are distinct.}$$

On completing the induction, set $\mathcal{V}=\{xV_n:n\in\mathbb{N},\ x\in I_n\}$; this is a disjoint family.

(c) ? Suppose, if possible, that $A\setminus\bigcup\mathcal{V}$ is not negligible. By 415B, the subspace measure on $A\setminus\bigcup\mathcal{V}$ is τ-additive and has a non-empty support. Take any a belonging to this support and set $G=\operatorname{int}(aV_0)$; then G is totally bounded and $\delta=\mu^*(G\cap A\setminus\bigcup\mathcal{V})$ is non-zero. Let M be such that every $V_nV_n^{-1}$ can be covered by M left translates of V_n, so that $\mu(V_nV_n^{-1})\le M\mu V_n$ for every n. Set

$$J_n=I_n\cap GV_0^{-1},\quad E_n=J_nV_n,\quad \tilde{E}_n=E_nV_n^{-1}$$

for each n.

If $n\in\mathbb{N}$ and $x\in J_n$, then $xV_n\subseteq GV_0^{-1}V_0$. Accordingly (because $\langle xV_n\rangle_{x\in J_n}$ is disjoint)

$$\#(J_n)\mu V_n=\sum_{x\in J_n}\mu(xV_n)\le\mu(GV_0^{-1}V_0)<\infty$$

because $GV_0^{-1}V_0$ is totally bounded (4A5Ob). So J_n is finite and E_n is closed. Note that if $x\in I_n$ and $G\cap xV_n\ne\emptyset$, then $x\in GV_n^{-1}\subseteq GV_0^{-1}$ and $x\in J_n$; so $G\cap\bigcup\mathcal{V}=G\cap\bigcup_{n\in\mathbb{N}}E_n$. Also, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of subsets of $GV_0^{-1}V_0$; accordingly $\sum_{n=0}^{\infty}\mu E_n$ is finite, and there is an $m\in\mathbb{N}$ such that $M\sum_{n=m}^{\infty}\mu E_n<\delta$.

Observe next that, for any $x\in X$ and $n\in\mathbb{N}$,

$$\mu(xV_nV_n^{-1})=\mu(V_nV_n^{-1})\le M\mu V_n=M\mu(xV_n).$$

So

$$\mu\tilde{E}_n\le\sum_{x\in J_n}\mu(xV_nV_n^{-1})\le M\sum_{x\in J_n}\mu(xV_n)=M\mu E_n$$

for each n, and $\mu(\bigcup_{n\ge m})\tilde{E}_n<\delta$.

This means that $\bigcup_{n\ge m}\tilde{E}_n$ cannot include $A\cap G\setminus\bigcup\mathcal{V}$, and there is a z belonging to

$$A \cap G \setminus (\bigcup \mathcal{V} \cup \bigcup_{n \geq m} \tilde{E}_n) = A \cap G \setminus (\bigcup_{n \in \mathbb{N}} E_n \cup \bigcup_{n \geq m} \tilde{E}_n)$$
$$= A \cap G \setminus (\bigcup_{n < m} E_n \cup \bigcup_{n \geq m} \tilde{E}_n).$$

Now there must be a first $k \geq m$ such that $k \in K_z$ and $zV_k \subseteq G \setminus \bigcup_{n<m} E_n$. (This is where we use the hypothesis that $\{V_n : n \in \mathbb{N}\}$ is a base of neighbourhoods of the identity.) Since $z \in G \setminus J_k$, $z \notin I_k$, and there are $j \leq k$, $x \in I_j$ such that $zV_k \cap xV_j \neq \emptyset$. In this case, $x \in GV_0^{-1}$, so $x \in J_j$. Accordingly

$$z \in xV_jV_k^{-1} \subseteq xV_jV_j^{-1} \subseteq \tilde{E}_j$$

and $j < m$; but this means that $zV_k \cap xV_j \subseteq zV_k \cap E_j$ must be empty, which is impossible. **X**

(d) Thus $\mu(A \setminus \bigcup \mathcal{V}) = 0$, and $\mathcal{V}$ is an appropriate family.

447D Theorem Let X be a topological group with a left Haar measure μ, and $\langle V_n \rangle_{n \in \mathbb{N}}$ a B-sequence in X. Then for any Haar measurable set $E \subseteq X$,

$$\lim_{n \to \infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = \chi E(x)$$

for almost every $x \in X$.

proof (a) Let $\alpha < 1$, and set

$$A = \{x : x \in E, \liminf_{n \to \infty} \frac{\mu(E \cap xV_n)}{\mu V_n} < \alpha\}.$$

? Suppose, if possible, that A is not negligible. Then there is an open set G of finite measure such that $\mu^*(G \cap A) = \gamma > 0$. Let $\delta > 0$ be such that $\gamma > \alpha(\gamma + \delta) + \delta$. Take a Borel set F which is a measurable envelope of $G \cap A$ and a closed set $F_1 \subseteq G \setminus F$ such that $\mu F_1 \geq \mu(G \setminus F) - \delta$. Writing $H = G \setminus F_1$, we see that $H \cap A = G \cap A$ and

$$\mu H \leq \mu^*(H \cap A) + \delta = \gamma + \delta.$$

For each $x \in H \cap A$, set

$$K_x = \{n : xV_n \subseteq H,\ \mu(E \cap xV_n) \leq \alpha \mu V_n\}.$$

Then K_x is infinite. By Vitali's theorem in the form 447C, there is a disjoint family $\mathcal{V} \subseteq \{xV_n : x \in H \cap A,\ n \in K_x\}$ such that $(H \cap A) \setminus \bigcup \mathcal{V}$ is negligible. Since every member of $\mathcal{V}$ has non-zero measure, while μH is finite, $\mathcal{V}$ is countable. Now $\mu(\bigcup \mathcal{V}) \geq \mu^*(H \cap A)$, so $\mu(H \setminus \bigcup \mathcal{V}) \leq \delta$; also, because $\mu(E \cap V) \leq \alpha \mu V$ for every $V \in \mathcal{V}$, and $\mathcal{V}$ is disjoint,

$$\mu(E \cap \bigcup \mathcal{V}) \leq \alpha \mu(\bigcup \mathcal{V}) \leq \alpha \mu H$$

and

$$\gamma = \mu^*(A \cap H) \leq \mu(E \cap H) \leq \mu(E \cap \bigcup \mathcal{V}) + \delta \leq \alpha \mu H + \delta \leq \alpha(\gamma + \delta) + \delta,$$

which is impossible, by the choice of δ. **X**

(b) As α is arbitrary,

$$\liminf_{n \to \infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = 1$$

for almost every $x \in E$. Similarly,

$$\liminf_{n \to \infty} \frac{\mu(xV_n \setminus E)}{\mu V_n} = 1, \quad \limsup_{n \to \infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = 0$$

for almost every $x \in X \setminus E$, so

$$\lim_{n \to \infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = \chi E(x)$$

for almost every $x \in X$.

447E We need to recall some results from 443P-443R. Let X be a locally compact Hausdorff topological group, and Y a closed subgroup of X such that the modular function of Y is the restriction to Y of the modular function of X. Let μ be a left Haar measure on X and μ_Y a left Haar measure on Y.

(a) Writing $C_k(X)$ for the space of continuous real-valued functions on X with compact support, and X/Y for the set of left cosets of Y in X with the quotient topology, we have a linear operator $T : C_k(X) \to C_k(X/Y)$ defined by writing $(Tf)(x^\bullet) = \int_Y f(xy)\mu_Y(dy)$ whenever $x \in X$ and $f \in C_k(X)$ (443P); moreover, $T[C_k(X)^+] = C_k(X/Y)^+$ (443Pa), and we have an invariant Radon measure λ on X/Y such that $\int Tf d\lambda = \int f d\mu$ for every $f \in C_k(X)$ (see part (b) of the proof of 443R). Turning this structure round, we see from 443Qb that μ, μ_Y and λ here are related in exactly the same way as μ, ν and λ in 443Q. If Y is a normal subgroup of X, so that X/Y is the quotient group, λ is a left Haar measure. If Y is compact and μ_Y is the Haar probability measure on Y, then λ is the image measure $\mu\pi^{-1}$, where $\pi(x) = x^\bullet = xY$ for every $x \in X$ (443Qd).

(b) If $E \subseteq X$ and $EY = Y$, then E is Haar measurable iff $\tilde{E} = \{x^\bullet : x \in E\}$ belongs to the domain of λ, and E is Haar negligible iff $\tilde{E}$ is λ-negligible (443Qc).

(c) Now suppose that X is σ-compact. Then for any Haar measurable $E \subseteq X$, $\mu E = \int g\,d\lambda$ in $[0,\infty]$, where $g(x^\bullet) = \mu_Y(Y \cap x^{-1}E)$ is defined for almost every $x \in X$ (443Qe). In particular, E is Haar negligible iff $\mu_Y(Y \cap x^{-1}E) = 0$ for almost every $x \in X$.

(d) Again suppose that X is σ-compact. Then we can extend the operator T of part (a) to an operator from $\mathcal{L}^1(\mu)$ to $\mathcal{L}^1(\lambda)$ by writing $(Tf)(x^\bullet) = \int f(xy)\mu_Y(dy)$ whenever $f \in \mathcal{L}^1(\mu)$, $x \in X$ and the integral is defined, and $\int Tf d\lambda = \int f d\mu$ for every $f \in \mathcal{L}^1(\mu)$ (443Qe). If $f \in \mathcal{L}^1(\mu)$, and we set $f_x(y) = f(xy)$ for all those $x \in X$, $y \in Y$ for which $xy \in \operatorname{dom} f$, then $Q = \{x : f_x \in \mathcal{L}^1(\mu_Y)\}$ is μ-conegligible, and $x \mapsto f_x^\bullet : Q \to L^1(\mu_Y)$ is almost continuous (443Qf).

(e) If X is σ-compact, Y is compact and μ_Y is the Haar probability measure on Y, so that λ is the image measure $\mu\pi^{-1}$, then we can apply 235G to the formula in (d) to see that

$$\iint f(xy)\mu_Y(dy)\mu(dx) = \int (Tf)(x^\bullet)\mu(dx) = \int Tf\,d\lambda = \int f d\mu$$

for every μ-integrable function f, and therefore (because μ is σ-finite) for every function f such that $\int f d\mu$ is defined in $[-\infty,\infty]$. In particular, $\mu E = \int \nu(Y \cap x^{-1}E)\mu(dx)$ for every Haar measurable set $E \subseteq X$.

447F Lemma Let X be a σ-compact locally compact Hausdorff topological group and Y a closed subgroup of X such that the modular function of Y is the restriction to Y of the modular function of X. Let Z be a compact normal subgroup of Y such that the quotient group Y/Z has a B-sequence. Let Σ_Y be the σ-algebra of those Haar measurable subsets E of X such that $EY = E$, and Σ_Z the algebra of Haar measurable sets $E \subseteq X$ such that $EZ = E$. Let $\underline{\phi} : \Sigma_Y \to \Sigma_Y$ be a left-translation-invariant partial lower density. Then there is a left-translation-invariant partial lower density $\underline{\psi} : \Sigma_Z \to \Sigma_Z$ extending $\underline{\phi}$.

proof (a) Let μ be a left Haar measure on X, μ_Y a left Haar measure on Y and μ_Z the Haar probability measure on Z; then there is a left Haar measure ν on Y/Z such that $\int g(y)\mu_Y(dy) = \int (Tg)(u)\nu(du)$ for every $g \in C_k(Y)$, where $(Tg)(y^\bullet) = \int g(yz)\mu_Z(dz)$ for every $y \in Y$ (447Ea). We are supposing that Y/Z has a B-sequence $\langle V_n\rangle_{n\in\mathbb{N}}$. It follows that there is a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in $C_k(Y)^+$ such that (i) $\int h_n(y)\mu_Y(dy) = 1$ for every n (ii) whenever $F \subseteq Y$ is Haar measurable (regarded as a subset of Y, that is), and $FZ = Z$, then

$$\lim_{n\to\infty} \int \chi F(by)h_n(y)\mu_Y(dy) = \chi F(b)$$

for μ_Y-almost every $b \in Y$.

P Since any subsequence of $\langle V_n\rangle_{n\in\mathbb{N}}$ is a B-sequence, and Y/Z is locally compact, we may suppose that every V_n is compact. For each $n \in \mathbb{N}$, choose a non-negative $h'_n \in C_k(Y/Z)$ such that

$$\int h'_n d\nu = 1, \quad \int |h'_n - \frac{1}{\nu V_n}\chi V_n| d\nu \le 2^{-n}.$$

(This is possible by 416I, or otherwise.) Let $h_n \in C_k(Y)^+$ be such that $Th_n = h'_n$ (447Ea again); then $\int h_n d\mu_Y = \int h'_n d\nu = 1$. Now if $F \subseteq Y$ is Haar measurable and $FZ = Z$, there is a Haar measurable $\tilde{F} \subseteq Y/Z$ such that $F = \{y : y^\bullet \in \tilde{F}\}$ (447Eb). Take $b \in Y$ and $n \in \mathbb{N}$. Because

$$\int \chi F(byz)h_n(yz)\mu_Z(dz) = \int \chi\tilde{F}(b^\bullet y^\bullet)h_n(yz)\mu_Z(dz) = \chi\tilde{F}(b^\bullet y^\bullet)h'_n(y^\bullet)$$

for every $y \in Y$,

$$\int \chi F(by)h_n(y)\mu_Y(dy) = \iint \chi F(byz)h_n(yz)\mu_Z(dz)\mu_Y(dy)$$

(447Ee)

$$= \int \chi\tilde{F}(b^\bullet y^\bullet)h'_n(y^\bullet)\mu_Y(dy) = \int \chi\tilde{F}(b^\bullet u)h'_n(u)\nu(du)$$

because $y \mapsto y^\bullet$ is inverse-measure-preserving for μ_Y and ν (447Ee). So

$$\begin{aligned}|\chi F(b) - \int \chi F(by)h_n(y)\mu_Y(dy)| &= |\chi\tilde{F}(b^\bullet) - \int \chi\tilde{F}(b^\bullet u)h'_n(u)\nu(du)| \\ &\le |\chi\tilde{F}(b^\bullet) - \tfrac{1}{\nu V_n}\int \chi\tilde{F}(b^\bullet u)\chi V_n(u)\nu(du)| \\ &\qquad + \int |h'_n(u) - \tfrac{1}{\nu V_n}\chi V_n(u)|\nu(du) \\ &\le |\chi\tilde{F}(b^\bullet) - \tfrac{\nu(\tilde{F}\cap b^\bullet V_n)}{\nu V_n}| + 2^{-n}.\end{aligned}$$

Since

$$\{v : v \in Y/Z,\ \lim_{n\to\infty} \tfrac{\nu(\tilde{F}\cap vV_n)}{\nu V_n} \ne \chi\tilde{F}(v)\}$$

is ν-negligible (447D), its inverse image in Y is μ_Y-negligible, so $\chi F(b) = \lim_{n\to\infty} \int \chi F(by)h_n(y)\mu_Y(dy)$ for almost every b, as claimed. **Q**

(b) We find now that if $E \in \Sigma_Z$, then $\lim_{n\to\infty} \int \chi E(xy)h_n(y)\mu_Y(dy) = \chi E(x)$ for μ-almost every $x \in X$. **P** Set $E_x = Y \cap x^{-1}E$ for $x \in X$. Because X is σ-compact, we can express E as the union of a non-decreasing sequence $\langle F^{(k)}\rangle_{k\in\mathbb{N}}$ where each $F^{(k)}$ is Haar measurable and relatively compact; set $F^{(k)}_x = Y \cap x^{-1}F^{(k)}$ for each x. In this case, for any $k \in \mathbb{N}$, $Q_k = \{x : F^{(k)}_x \in \mathrm{dom}(\mu_Y)\}$ is conegligible, and $x \mapsto (\chi F^{(k)}_x)^\bullet : Q_k \to L^1(\mu_Y)$ is almost continuous (447Ed), so that $x \mapsto \int \chi F^{(k)}_x(y)h_n(y)\mu_Y(dy) : Q_k \to [0,1]$ is measurable, for each n. But this means that, setting $Q = \bigcap_{k\in\mathbb{N}} Q_k$,

$$x \mapsto \int \chi E_x(y)h_n(y)\mu_Y(dy) = \lim_{k\to\infty} \int \chi F^{(k)}_x(y)h_n(y)\mu_Y(dy) : Q \mapsto [0,1]$$

is measurable, for every $n \in \mathbb{N}$. Note that if $x \in X$ and $y \in Y$ then $F^{(k)}_{xy} = y^{-1}F^{(k)}_x$, so that $Q_kY = Q_k$ for every $k \in \mathbb{N}$, and $QY = Q$.

Now consider $\tilde{Q} = \{x : x \in Q,\ \lim_{n\to\infty} \int \chi E(xy)h_n(y)\mu_Y(dy) = \chi E(x)\}$. This is a Haar measurable subset of X. If $a \in Q$, then

$$Y \cap a^{-1}\tilde{Q} = \{y : y \in Y,\ \lim_{n\to\infty} \chi E(ays)h_n(s)\mu_Y(ds) = \chi E(ay)\}$$

is μ_Y-conegligible, by the choice of the h_n in (a) above. Because Q is μ-conegligible, $Q \setminus \tilde{Q}$ is μ-negligible (447Ec) and $\tilde{Q}$ is conegligible, as required. **Q**

(c) We are now ready for the formulae at the centre of this proof. For any Haar measurable set $E \subseteq X$, $n \in \mathbb{N}$ and $\gamma < 1$, set

$$\begin{aligned}\psi_{n\gamma}(E) = \bigcup\{G \cap \underline{\phi}F : {}& G \subseteq X \text{ is open, } F \in \Sigma_Y, \\ & \int \chi E(xy)h_n(y)\mu_Y(dy) \text{ is defined and at least } \gamma \text{ for every } x \in G \cap F\},\end{aligned}$$

$$\underline{\psi}E = \bigcap_{\gamma<1} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \psi_{m\gamma}(E).$$

The rest of the proof is devoted to checking that $\underline{\psi}\restriction\Sigma_Z$ is a left-translation-invariant partial lower density extending $\underline{\phi}$.

(d) I had better make one remark straight away. If $G \subseteq X$ is open, then $G \subseteq \underline{\phi}(GY)$ (447B). It follows that if $G \subseteq X$ is open, $F \in \Sigma_Y$ and $G \cap \underline{\phi}F \ne \emptyset$, then $G \cap F \ne \emptyset$. **P** If $a \in G \cap \underline{\phi}F$, then $a \in \underline{\phi}(GY) \cap \underline{\phi}F = \underline{\phi}(GY \cap F)$, so $GY\cap F\ne\emptyset$, that is, $G\cap F=G\cap FY^{-1}$ is non-empty. **Q** I mention this now because we need to know that the condition

$$\int\chi E(xy)h_n(y)\mu_Y(dy)\text{ is defined and at least }\gamma\text{ for every }x\in G\cap F$$

in the definition of $\psi_{n\gamma}(E)$ is never vacuously satisfied if $G\cap\underline{\phi}F\ne\emptyset$. In particular, $\psi_{n\gamma}(\emptyset)=\emptyset$ whenever $n\in\mathbb{N}$ and $0<\gamma<1$, so that $\underline{\psi}\emptyset=\emptyset$.

(e) If $E\subseteq X$ is Haar measurable, $n\in\mathbb{N}$ and $\epsilon>0$, then for almost every $a\in X$ there are an open set $G\subseteq X$ and an $F\in\Sigma_Y$ such that $a\in G\cap\underline{\phi}F$ and $\int|\chi E(xy)-\chi E(ay)|h_n(y)\mu_Y(dy)\le\epsilon$ whenever $x\in G\cap F$. **P** Let $\tilde\lambda$ be the invariant Radon measure on X/Y derived from μ and μ_Y as in 447Ea. Take $\delta>0$ such that $\delta(1+2\|h_n\|_\infty)\le\epsilon$. Because X is σ-compact, locally compact and Hausdorff, therefore Lindelöf and regular (4A2Hd, 3A3Bb), there is a sequence $\langle f_r\rangle_{r\in\mathbb{N}}$ of continuous functions from X to $[0,1]$ such that $\int|\chi E(x)-f_r(x)|\mu(dx)\le 2^{-r}$ for every r (415Pb). Set $g_r=|\chi E-f_r|$ for each r. For $f\in\mathcal{L}^1(\mu)$, define $\tilde Tf\in\mathcal{L}^1(\tilde\lambda)$ by writing $(\tilde Tf)(x^\bullet)=\int f(xy)\mu_Y(dy)$ whenever this is defined (447Ed); we have $\int\tilde Tfd\tilde\lambda=\int fd\mu$.

Set $Q=\{x:Y\cap x^{-1}E\in\operatorname{dom}\mu_Y\}$, so that $Q\in\Sigma_Y$ is conegligible (447Ec). For each r, set $\tilde F_r=\{u:u\in X/Y,$ $\tilde Tg_r(u)$ is defined and at least $\delta\}$; then

$$\tilde\lambda\tilde F_r\le\frac{1}{\delta}\int Tg_rd\tilde\lambda=\frac{1}{\delta}\int g_rd\mu\le 2^{-r}/\delta.$$

So $\bigcap_{r\in\mathbb{N}}\tilde F_r$ is $\tilde\lambda$-negligible. Set

$$\begin{aligned}F_r&=\{x:x\in X,\,x^\bullet\in\tilde F_r\}=\{x:\tilde Tg_r(x^\bullet)\ge\delta\}\\&=\{x:\int g_r(xy)\mu_Y(dy)\ge\delta\}=\{x:\int|\chi E(xy)-f_r(xy)|\mu_Y(dy)\ge\delta\}\end{aligned}$$

for each r; then $F_rY=F_r$, F_r is Haar measurable (447Eb) and $\bigcap_{r\in\mathbb{N}}F_r$ is μ-negligible (also by 447Eb). Since $(X\setminus F_r)\triangle\underline{\phi}(X\setminus F_r)$ is negligible for each r, $Q_1=Q\cap\bigcup_{r\in\mathbb{N}}\underline{\phi}(X\setminus F_r)\setminus F_r$ is conegligible. Note that $Q_1Y=Q_1$.

Suppose that $a\in Q_1$. Then there is an $r\in\mathbb{N}$ such that

$$a\in Q_1\cap\underline{\phi}(X\setminus F_r)\setminus F_r=(Q_1\setminus F_r)\cap\underline{\phi}(Q_1\setminus F_r).$$

Set $F=Q_1\setminus F_r\in\Sigma_Y$. Consider the function $x\mapsto\int f_r(xy)h_n(y)\mu_Y(dy)$. We chose h_n with compact support $L\subseteq Y$ say. If V is a compact neighbourhood of a in X, then f_r is uniformly continuous on VL for the right uniformity on X (4A5Ha, 4A2Jf). There is therefore an open neighbourhood U of the identity of X such that $|f_r(x')-f_r(x)|\le\delta$ whenever x, $x'\in VL$ and $x'x^{-1}\in U$; of course we may suppose that $G=Ua$ is a subset of V.

Take any $x\in G\cap F$. Then if $y\in L$, we have ay, xy both in VL, while $xy(ay)^{-1}\in U$, so that $|f_r(xy)-f_r(ay)|\le\delta$. Accordingly $|f_r(xy)-f_r(ay)|h_n(y)\le\delta h_n(y)$ for every $y\in Y$, and

$$\int|f_r(ay)-f_r(xy)|h_n(y)\mu_Y(dy)\le\delta.$$

At the same time, because both x and a belong to $F=Q_1\setminus F_r$,

$$\int|\chi E(ay)-f_r(ay)|h_n(y)\mu_Y(dy)\le\delta\|h_n\|_\infty,$$

$$\int|\chi E(xy)-f_r(xy)|h_n(y)\mu_Y(dy)\le\delta\|h_n\|_\infty.$$

Putting these together,

$$\int|\chi E(ay)-\chi E(xy)|h_n(y)\mu_Y(dy)\le\delta(1+2\|h_n\|_\infty)\le\epsilon.$$

Thus G and F witness that a has the property required; as a is any member of the conegligible set Q_1, we have the result. **Q**

(f) If $E\in\Sigma_Z$ then $E\triangle\underline{\psi}E$ is negligible. **P** By (e), applied in turn to every n and every ϵ of the form 2^{-i}, there is a conegligible set $Q_1\subseteq X$ such that whenever $a\in Q_1$, $n\in\mathbb{N}$ and $\epsilon>0$ there are an open set G containing a and an $F\in\Sigma_Y$ such that $a\in\underline{\phi}F$ and $\int|\chi E(xy)-\chi E(ay)|h_n(y)\mu_Y(dy)\le\epsilon$ for every $x\in G\cap F$. By (b), there is a conegligible set $Q_2\subseteq X$ such that $\lim_{n\to\infty}\int\chi E(ay)h_n(y)\mu_Y(dy)=\chi E(a)$ for every $a\in Q_2$.

Suppose that $a\in Q_1\cap Q_2\cap E$. Let $\gamma<1$; set $\epsilon=\frac{1}{2}(1-\gamma)$. Because $a\in Q_2$, there is an $n\in\mathbb{N}$ such that $\int\chi E(ay)h_m(y)\mu_Y(dy)\ge 1-\epsilon$ for every $m\ge n$. Take any $m\ge n$. Because $a\in Q_1$, there are an open set G and an $F\in\Sigma_Y$ such that $a\in G\cap\underline{\phi}F$ and $\int|\chi E(ay)-\chi E(xy)|h_m(y)\mu_Y(dy)\le\epsilon$ whenever $x\in G\cap F$. But now

$$\int\chi E(xy)h_m(y)\mu_Y(dy)\ge 1-2\epsilon=\gamma$$

for every $x \in G \cap F$, so $a \in \psi_{m\gamma}(E)$. This is true for every $m \ge n$; as γ is arbitrary, $a \in \underline{\psi}E$. As a is arbitrary, $Q_1 \cap Q_2 \cap E \subseteq \underline{\psi}E$.

Now suppose that $a \in Q_1 \cap Q_2 \cap \underline{\psi}E$. Then there is an $n \in \mathbb{N}$ such that $a \in \psi_{m,3/4}(E)$ for every $m \ge n$. There is an $m \ge n$ such that $|\int \chi E(ay)h_m(y)\mu_Y(dy) - \chi E(a)| \le \frac{1}{4}$. There are an open set G_1 and an $F_1 \in \Sigma_Y$ such that $a \in G_1 \cap \underline{\phi}F_1$ and $\int \chi E(xy)h_m(y)\mu_Y(dy) \ge \frac{3}{4}$ for every $x \in G_1 \cap F_1$. There are also an open set G_2 and an $F_2 \in \Sigma_Y$ such that $a \in G_2 \cap \underline{\phi}F_2$ and $\int |\chi E(ay) - \chi E(xy)|h_m(y)\mu_Y(dy) \le \frac{1}{4}$ for every $x \in G_2 \cap F_2$. Set $G = G_1 \cap G_2$, $F = F_1 \cap F_2$; then $a \in G \cap \underline{\phi}F$, so $G \cap F$ is not empty ((d) above). Take $x \in G \cap F$. Then

$$\int \chi E(xy)h_m(y)\mu_Y(dy) \ge \frac{3}{4},$$

$$\int |\chi E(ay) - \chi E(xy)|h_m(y)\mu_Y(dy) \le \frac{1}{4},$$

$$|\int \chi E(ay)h_m(y)\mu_Y(dy) - \chi E(a)| \le \frac{1}{4},$$

so $\chi E(a) \ge \frac{1}{4}$ and $a \in E$. This shows that $Q_1 \cap Q_2 \cap \underline{\psi}E \subseteq E$.

Accordingly $E \triangle \underline{\psi}E \subseteq X \setminus (Q_1 \cap Q_2)$ is negligible, as required. **Q**

In particular, $\underline{\psi}E$ is Haar measurable for every $E \in \Sigma_Z$.

(g) If $E \in \Sigma_Z$ then $\underline{\psi}E \in \Sigma_Z$. **P** We have just seen that $\underline{\psi}E$ is Haar measurable. Take $z \in Z$, $n \in \mathbb{N}$, $\gamma < 1$ and $a \in \psi_{n\gamma}(E)$. Then there are an open $G \subseteq X$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi}F$ and $\int \chi E(xy)h_n(y)\mu_Y(dy) \ge \gamma$ for every $x \in G \cap F$. Because F and $\underline{\phi}F$ belong to Σ_Y, $az \in \underline{\phi}F$. Of course Gz is an open set containing az. If $x \in Gz \cap F$, then $xz^{-1} \in G \cap F$ and $\int \chi E(xz^{-1}y)h_n(y)\mu_Y(dy) \ge \gamma$. But

$$\chi E(xz^{-1}y) = \chi E(xy \cdot y^{-1}z^{-1}y) = \chi E(xy)$$

for every $y \in Y$, because $Z \lhd Y$ (so $z' = y^{-1}z^{-1}y \in Z$) and we are supposing that $E \in \Sigma_Z$ (so $xyz' \in E$ iff $xy \in E$). So

$$\int \chi E(xy)h_n(y)\mu_Y(dy) = \int \chi E(xz^{-1}y)h_n(y)\mu_Y(dy) \ge \gamma.$$

As x is arbitrary, Gz and F witness that $az \in \psi_{n\gamma}E$.

This shows that, for any n and γ, $az \in \psi_{n\gamma}(E)$ whenever $a \in \psi_{n\gamma}(E)$ and $z \in Z$. It follows at once that $az \in \underline{\psi}E$ whenever $a \in \underline{\psi}E$ and $z \in Z$, as claimed. **Q**

(h) For any Haar measurable $E \subseteq X$ and $c \in X$, $\underline{\psi}(cE) = c\underline{\psi}E$. **P** Suppose that $n \in \mathbb{N}$, $\gamma < 1$ and $a \in \psi_{n\gamma}(E)$. Then there are an open set $G \subseteq X$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi}F$ and $\int \chi E(xy)h_n(y)\mu_Y(dy) \ge \gamma$ for every $x \in G \cap F$. Now cG is an open set containing ca, $cF \in \Sigma_Y$ and $\underline{\phi}(cF) = c\underline{\phi}F$ contains ca, and if $x \in cG \cap cF$ we have

$$\int \chi(cE)(xy)h_n(xy)\mu_Y(dy) = \int \chi E(c^{-1}xy)h_n(xy)\mu_Y(dy) \ge \gamma$$

because $c^{-1}x \in G \cap F$. But this means that cG, cF witness that $ca \in \psi_{n\gamma}(cE)$. Since a is arbitrary, $c\psi_{n\gamma}(E) \subseteq \psi_{n\gamma}(cE)$; as n and γ are arbitrary, $c\underline{\psi}E \subseteq \underline{\psi}(cE)$. Similarly, of course, $c^{-1}\underline{\psi}(cE) \subseteq \underline{\psi}E$, so in fact $\underline{\psi}(cE) = c\underline{\psi}E$, as claimed. **Q**

(i) If E_1, $E_2 \subseteq X$ are Haar measurable and $E_1 \setminus E_2$ is μ-negligible, $\underline{\psi}E_1 \subseteq \underline{\psi}E_2$. **P** Take $n \in \mathbb{N}$, $\gamma < 1$ and $a \in \psi_{n\gamma}(E_1)$. Then there are an open set $G \subseteq X$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi}F$ and $\int \chi E_1(xy)h_n(y)\mu_Y(dy) \ge \gamma$ for every $x \in G \cap F$. Let Q be the set of those $x \in X$ such that μ_Y measures $Y \cap x^{-1}(E_1 \setminus E_2)$ and $\int \chi(E_1 \setminus E_2)(xy)\mu_Y(dy) = 0$; then $QY = Q$ is conegligible (447Ec). Now $Q \cap F \in \Sigma_Y$, and $\underline{\phi}(Q \cap F) = \underline{\phi}F$ contains a. But if $x \in G \cap Q \cap F$, $\chi E_1(xy) \le \chi E_2(xy)$ for μ_Y-almost every y. So

$$\int \chi E_2(xy)h_n(y)\mu_Y(dy) \ge \int \chi E_1(xy)h_n(y)\mu_Y(dy) \ge \gamma.$$

Thus G and $Q \cap F$ witness that $a \in \psi_{n\gamma}(E_2)$. As a is arbitrary, $\psi_{n\gamma}(E_1) \subseteq \psi_{n\gamma}(E_2)$; as n and γ are arbitrary, $\underline{\psi}E_1 \subseteq \underline{\psi}E_2$. **Q**

In particular, (i) $\underline{\psi}E_1 = \underline{\psi}E_2$ whenever $E_1 \triangle E_2$ is negligible (ii) $\underline{\psi}E_1 \subseteq \underline{\psi}E_2$ whenever $E_1 \subseteq E_2$.

(j) If E_1, $E_2 \subseteq X$ are Haar measurable, $\underline{\psi}(E_1 \cap E_2) = \underline{\psi}E_1 \cap \underline{\psi}E_2$. **P** By (i), $\underline{\psi}(E_1 \cap E_2) \subseteq \underline{\psi}E_1 \cap \underline{\psi}E_2$. So take $a \in \underline{\psi}E_1 \cap \underline{\psi}E_2$. Let $\gamma < 1$. Set $\delta = \frac{1}{2}(1 + \gamma) < 1$. Then there are n_1, $n_2 \in \mathbb{N}$ such that $a \in \psi_{m\delta}(E_1)$ for every $m \ge n_1$ and $a \in \psi_{m\delta}(E_2)$ for every $m \ge n_2$. Set $n = \max(n_1, n_2)$ and take any $m \ge n$. Then there are open sets

G_1, $G_2 \subseteq X$ and F_1, $F_2 \in \Sigma_Y$ such that $a \in G_1 \cap G_2 \cap \underline{\phi}F_1 \cap \underline{\phi}F_2$, $\int \chi E_1(xy)h_m(y)\mu_Y(dy) \ge \delta$ for every $x \in G_1 \cap F_1$ and $\int \chi E_2(xy)h_m(y)\mu_Y(dy) \ge \delta$ for every $x \in G_2 \cap F_2$. Let $Q \in \Sigma_Y$ be the conegligible set of those $x \in X$ such that $\int \chi(E_1 \cap E_2)(xy)\mu_Y(dy)$ is defined. Set $G = G_1 \cap G_2$, $F = F_1 \cap F_2 \cap Q$; then G is open, $F \in \Sigma_Y$ and

$$\underline{\phi}F = \underline{\phi}(F_1 \cap F_2) = \underline{\phi}F_1 \cap \underline{\phi}F_2,$$

so that $a \in G \cap \underline{\phi}F$. Now take any $x \in G \cap F$. We have

$$\begin{aligned}\int(1-\chi(E_1\cap E_2))(xy)h_m(y)\mu_Y(dy)&\\ &\le \int(1-\chi E_1(xy))h_m(y)\mu(dy)+\int(1-\chi E_2(xy))h_m(y)\mu(dy)\\ &\le 2(1-\delta)=1-\gamma,\end{aligned}$$

and $\int \chi(E_1 \cap E_2)(xy)h_m(y)\mu_Y(dy) \ge \gamma$, because $\int h_m(y)\mu_Y(dy) = 1$.

As x is arbitrary, G and F witness that $a \in \psi_{m\gamma}(E_1 \cap E_2)$. And this is true for every $m \ge n$. As γ is arbitrary, $a \in \underline{\psi}(E_1 \cap E_2)$. As a is arbitrary, $\underline{\psi}E_1 \cap \underline{\psi}E_2 \subseteq \underline{\psi}(E_1 \cap E_2)$ and the two are equal. **Q**

(k) If $E \in \Sigma_Y$, $\underline{\psi}E = \underline{\phi}E$. **P** (i) Suppose $a \in \underline{\phi}E$, $n \in \mathbb{N}$ and $\gamma < 1$. Set $F = E \cap \underline{\phi}E \in \Sigma_Y$, $G = X$. Then $G \cap \underline{\phi}F = \underline{\phi}(E \cap \underline{\phi}E) = \underline{\phi}E$ contains a. Take any $x \in F$. Then

$$\int \chi E(xy)h_n(y)\mu_Y(dy) = \int h_n(y)\mu_Y(dy) = 1;$$

as x is arbitrary, $a \in \psi_{n\gamma}(E)$. As n and γ are arbitrary, $a \in \underline{\psi}E$; as a is arbitrary, $\underline{\phi}E \subseteq \underline{\psi}E$. (ii) Suppose $a \in \underline{\psi}E$. Then there must be some open $G \subseteq X$ and $F \in \Sigma_Y$ and $n \in \mathbb{N}$ such that $a \in G \cap \underline{\phi}F$ and $\int \chi E(xy)h_n(y)\mu_Y(dy) > 0$ for every $x \in G \cap F$. This surely implies that $G \cap F \subseteq EY = E$, so that $GY \cap F \subseteq E$. But $a \in G \subseteq \underline{\phi}(GY)$, by 447B, so

$$a \in \underline{\phi}(GY) \cap \underline{\phi}F = \underline{\phi}(GY \cap F) \subseteq \underline{\phi}E.$$

This shows that $\underline{\psi}E \subseteq \underline{\phi}E$. **Q**

(l) Thus we have assembled all the facts required to establish that $\underline{\psi}\restriction\Sigma_Z$ is a left-translation-invariant partial lower density extending $\underline{\phi}$.

447G Lemma Let X be a σ-compact locally compact Hausdorff topological group, and $\langle Y_n\rangle_{n\in\mathbb{N}}$ a non-increasing sequence of compact subgroups of X with intersection Y. Let Σ be the algebra of Haar measurable subsets of X; set $\Sigma_{Y_n} = \{E : E \in \Sigma,\ EY_n = E\}$ for each n, and $\Sigma_Y = \{E : E \in \Sigma,\ EY = E\}$. Suppose that for each $n \in \mathbb{N}$ we are given a left-translation-invariant partial lower density $\underline{\phi}_n : \Sigma_{Y_n} \to \Sigma_{Y_n}$, and that $\underline{\phi}_{n+1}$ extends $\underline{\phi}_n$ for every n. Then there is a left-translation-invariant partial lower density $\underline{\phi} : \Sigma_Y \to \Sigma_Y$ extending every $\underline{\phi}_n$.

proof (a) Fix a left Haar measure μ on X, and for each $n \in \mathbb{N}$ let ν_n be the Haar probability measure on Y_n (442Ie). As noted in 443Sb, the modular function of X must be equal to 1, and equal to the modular function of Y_n, everywhere in every Y_n.

(b) We need to know that for any $E \in \Sigma_Y$ there is an F in the σ-algebra Λ generated by $\bigcup_{n\in\mathbb{N}} \Sigma_{Y_n}$ such that $E\triangle F$ is negligible. **P** Because X is σ-compact and μ is a Radon measure (442Ac), there is a sequence $\langle K_i\rangle_{i\in\mathbb{N}}$ of compact sets such that $K_i \subseteq E$ for every i and $E \setminus \bigcup_{i\in\mathbb{N}} K_i$ is negligible. For each $i \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} K_iY_n = K_iY$ (4A5Eh), so is included in E. Set $F = \bigcup_{i\in\mathbb{N}}\bigcap_{n\in\mathbb{N}} K_iY_n$; then F belongs to the σ-algebra generated by $\bigcup_{n\in\mathbb{N}} \Sigma_{Y_n}$, and $K_i \subseteq F \subseteq E$ for every i, so $E\triangle F$ is negligible. **Q**

(c) For each $E \in \Sigma_Y$, $n \in \mathbb{N}$ set

$$g_{En}(x) = \nu_n(Y_n \cap x^{-1}E) \text{ whenever this is defined.}$$

By 447Ee, g_{En} is defined μ-almost everywhere and is Σ-measurable. In fact g_{En} is Σ_{Y_n}-measurable, because $g_{En}(xy) = g_{En}(x)$ whenever $x \in X$, $y \in Y_n$ and either is defined. If $F \in \Sigma_{Y_n}$ then $g_{En}(x)\chi F(x) = \nu_n(Y_n \cap x^{-1}(E\cap F))$ whenever this is defined, which is almost everywhere; so $\int_F g_{En}d\mu = \mu(E \cap F)$, by 447Ee. If E, $E' \in \Sigma_Y$ and $E\triangle E'$ is negligible, then $g_{En} =_{\text{a.e.}} g_{E'n}$, because $g_{E\triangle E',n} = 0$ a.e.

It follows that $\langle g_{En}\rangle_{n\in\mathbb{N}} \to \chi E$ μ-a.e. for every $E \in \Sigma_Y$. **P** Let $G \subseteq X$ be any non-empty relatively compact open set, and set $U = GY_0$, so that U also is a non-empty relatively compact open set, $UY_n = U$ for every n and $UY = U$. Set $\mu_U(F) = \dfrac{\mu F}{\mu U}$ whenever $F \in \Sigma$ and $F \subseteq U$, so that μ_U is a probability measure on U.

Writing $\Sigma_{Y_n}^{(U)}$, $\Sigma_Y^{(U)}$ and $\Lambda^{(U)}$ for the subspace σ-algebras on U generated by Σ_{Y_n}, Σ_Y and Λ, we see that if $F \in \Sigma_{Y_n}^{(U)}$ then

$$\int_F g_{En} d\mu_U = \frac{1}{\mu U}\int_F g_{En} d\mu = \mu_U(E \cap F).$$

So $g_{En}\restriction U$ is a conditional expectation of $\chi(E \cap U)$ on $\Sigma_{Y_n}^{(U)}$. By Lévy's martingale theorem (275I), $\langle g_{En}\rangle_{n\in\mathbb{N}}$ converges almost everywhere in U to a conditional expectation g of $\chi(E \cap U)$ on $\Lambda^{(U)}$, because of course $\Lambda^{(U)}$ is the σ-algebra of subsets of U generated by $\bigcup_{n\in\mathbb{N}} \Sigma_{Y_n}^{(U)}$. But as there is an $F \in \Lambda$ such that $E \triangle F$ is negligible, by (b) above, g must be equal to χE almost everywhere in U.

Thus $g_{En} \to \chi E$ almost everywhere in U and therefore almost everywhere in G. As G is arbitrary, $g_{En} \to \chi E$ almost everywhere in X, by 412Jb (applied to the family $\mathcal{K}$ of subsets of relatively compact open sets). **Q**

(d) Now we can use the method of 341G, as follows. For $E \in \Sigma_Y$, $k \geq 1$ and $n \in \mathbb{N}$ set

$$H_{kn}(E) = \{x : x \in \operatorname{dom}(g_{En}),\, g_{En}(x) \geq 1 - 2^{-k}\} \in \Sigma_{Y_n},$$

$$\tilde{H}_{kn}(E) = \underline{\phi}_n(H_{kn}(E)), \quad \underline{\phi}E = \bigcap_{k\geq 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n} \tilde{H}_{km}(E).$$

By the arguments of parts (e)-(i) of the proof of 341G, $\underline{\phi}$ is a lower density on Σ_Y extending every $\underline{\phi}_n$.

(e) To see that $\underline{\phi}$ is left-translation-invariant, we may argue as follows. Let $E \in \Sigma_Y$ and $a \in X$. Then, for any n,

$$g_{aE,n}(x) = \nu_n(Y_n \cap x^{-1}aE) = g_{En}(a^{-1}x)$$

for almost every x, so $aH_{kn}(E) \triangle H_{kn}(aE)$ is negligible, and

$$\tilde{H}_{kn}(aE) = \underline{\phi}_n(H_{kn}(aE)) = \underline{\phi}_n(aH_{kn}(E)) = a\underline{\phi}_n(H_{kn}(E)) = a\tilde{H}_{kn}(E),$$

for every k. Accordingly

$$\underline{\phi}(aE) = \bigcap_{k\geq 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n} \tilde{H}_{km}(aE) = a(\bigcap_{k\geq 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n} \tilde{H}_{km}(E)) = a\underline{\phi}(E),$$

as required.

447H Lemma Let X be a locally compact Hausdorff topological group, and Σ the algebra of Haar measurable sets in X. Then there is a left-translation-invariant lower density $\underline{\phi} : \Sigma \to \Sigma$.

proof (a) To begin with (down to the end of (c) below) let us suppose that X is σ-compact. By 446P, there is a family $\langle X_\xi\rangle_{\xi\leq\kappa}$ of closed subgroups of X, where κ is an infinite cardinal, such that

X_0 is an open subgroup of X,
for every $\xi < \kappa$, $X_{\xi+1}$ is a normal subgroup of X_ξ and $X_\xi/X_{\xi+1}$ has a B-sequence,
for every non-zero limit ordinal $\xi \leq \kappa$, $X_\xi = \bigcap_{\eta<\xi} X_\eta$,
X_1 is compact,
$X_\kappa = \{e\}$, where e is the identity of X.

Note that for every $\xi \leq \kappa$, the modular function Δ_ξ of X_ξ is just the restriction to X_ξ of the modular function Δ of X. **P** For $\xi = 0$ this is because X_0 is an open subgroup of X (443Sd). For $\xi \geq 1$, X_ξ is compact, so Δ_ξ and $\Delta\restriction X_\xi$ are both constant with value 1, as noted in 443Sb. **Q**

(b) For each $\xi \leq \kappa$, write Σ_ξ for the σ-algebra $\{E : E \in \Sigma,\, EX_\xi = E\}$. I seek to choose inductively a family $\langle\underline{\phi}_\xi\rangle_{\xi\leq\kappa}$ such that each $\underline{\phi}_\xi : \Sigma_\xi \to \Sigma_\xi$ is a left-translation-invariant partial lower density, and $\underline{\phi}_\xi$ extends $\underline{\phi}_\eta$ whenever $\eta < \xi$.

(i) *Start* Since X_0 is an open subgroup of X, every member of Σ_0 is open, and we can start the induction by setting $\underline{\phi}_0 E = E$ for every $E \in \Sigma_0$.

(ii) *Inductive step to a successor ordinal* If we have defined $\langle\underline{\phi}_\eta\rangle_{\eta\leq\xi}$, where $\xi < \kappa$, then $X_{\xi+1} \lhd X_\xi$ is compact and $X_\xi/X_{\xi+1}$ has a B-sequence. So the conditions of 447F are satisfied and $\underline{\phi}_\xi$ has an extension to a left-translation-invariant partial lower density $\underline{\phi}_{\xi+1} : \Sigma_{\xi+1} \to \Sigma_{\xi+1}$. Of course $\underline{\phi}_{\xi+1}$ extends $\underline{\phi}_\eta$ for every $\eta \leq \xi$ because $\underline{\phi}_\xi$ does.

(iii) *Inductive step to a limit ordinal of countable cofinality* If we have defined $\langle\underline{\phi}_\eta\rangle_{\eta<\xi}$, where $\xi \leq \kappa$ is a non-zero limit ordinal of countable cofinality, let $\langle\xi_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence with supremum ξ; we may suppose that $\xi_0 \geq 1$, so that every X_{ξ_n} is compact. Then $X_\xi = \bigcap_{n\in\mathbb{N}} X_{\xi_n}$, so by 447G there is a left-translation-invariant partial lower density $\underline{\phi} : \Sigma_\xi \to \Sigma_\xi$ extending every $\underline{\phi}_{\xi_n}$, and therefore extending $\underline{\phi}_\eta$ whenever $\eta < \xi$.

(iv) *Inductive step to a limit ordinal of uncountable cofinality* Suppose we have defined $\langle\underline{\phi}_\eta\rangle_{\eta<\xi}$, where $\xi\le\kappa$ is a limit ordinal of uncountable cofinality. Then for every $E\in\Sigma_\xi$ there are an $\eta<\xi$ and an $F\in\Sigma_\eta$ such that $E\triangle F$ is negligible. **P** (Cf. part (b) of the proof of 447G.) Because X is σ-compact, there are non-decreasing sequences $\langle K_i\rangle_{i\in\mathbb{N}}$, $\langle L_i\rangle_{i\in\mathbb{N}}$ of compact subsets of E, $X\setminus E$ respectively such that $E\setminus\bigcup_{i\in I}K_i$ and $(X\setminus E)\setminus\bigcup_{i\in\mathbb{N}}L_i$ are negligible. For each $i\in\mathbb{N}$, $K_iX_\xi\cap L_i\subseteq EX_\xi\setminus E$ is empty; by 4A5Eh again, there is an $\eta_i<\xi$ such that $K_iX_{\eta_i}\cap L_i$ is empty. Set $\eta=\sup_{i\in\mathbb{N}}\eta_i$, $F=\bigcup_{i\in\mathbb{N}}K_iX_\eta$; this works. **Q**

Accordingly we have a function $\underline{\phi}_\xi:\Sigma_\xi\to\Sigma_\xi$ defined by writing $\underline{\phi}_\xi(E)=\underline{\phi}_\eta(F)$ whenever $E\in\Sigma_\xi$, $\eta<\xi$, $F\in\Sigma_\eta$ and $E\triangle F$ is negligible. **P** If $\eta\le\eta'<\xi$ and $F\in\Sigma_\eta$, $F'\in\Sigma_{\eta'}$ are such that $E\triangle F$ and $E\triangle F'$ are both negligible, then $F\triangle F'$ is negligible so $\underline{\phi}_\eta(F)=\underline{\phi}_{\eta'}(F)=\underline{\phi}_{\eta'}(F')$. **Q** It is easy to check that $\underline{\phi}_\xi$ is a left-translation-invariant partial lower density (cf. part (A-d) of the proof of 341H), and of course it extends $\underline{\phi}_\eta$ for every $\eta<\xi$.

(c) On completing the induction, we see that $\Sigma_\kappa=\Sigma$, so that $\underline{\phi}_\kappa:\Sigma\to\Sigma$ is a left-translation-invariant lower density.

(d) For the general case, recall that X certainly has an open σ-compact subgroup Y say (4A5El). If Σ is the algebra of Haar measurable subsets of X, and T is the algebra of Haar measurable subsets of Y, then T is just $\Sigma\cap\mathcal{P}Y=\{E\cap Y:E\in\Sigma\}$, and the Haar negligible subsets of Y are just sets of the form $E\cap Y$ where E is a Haar negligible subset of X (443F).

Let $\underline{\psi}:\mathrm{T}\to\mathrm{T}$ be a left-translation-invariant lower density. For $E\in\Sigma$ set

$$\underline{\phi}E=\{x:x\in X,\ e\in\underline{\psi}(Y\cap x^{-1}E)\},$$

where e is the identity of X. It is easy to check that

$$\underline{\phi}\emptyset=\emptyset,$$

$$\underline{\phi}E=\underline{\phi}F \text{ if } E\triangle F \text{ is negligible,}$$

$$\underline{\phi}(E\cap F)=\underline{\phi}E\cap\underline{\phi}F \text{ for all } E,\,F\in\Sigma,$$

directly from the corresponding properties of $\underline{\psi}$. If $E\in\Sigma$ and $a\in X$, then

$$x\in\underline{\phi}E\iff e\in\underline{\psi}(Y\cap x^{-1}E)\iff e\in\underline{\psi}(Y\cap(ax)^{-1}aE)\iff ax\in\underline{\phi}(aE),$$

so $\underline{\phi}(aE)=a\underline{\phi}E$.

I have not yet checked that $E\triangle\underline{\phi}E$ is always negligible. But if $E\in\Sigma$, then

$$E\triangle\underline{\phi}E=\{x:e\in\underline{\psi}(Y\cap x^{-1}E)\triangle(Y\cap x^{-1}E)\},$$

so

$$\begin{aligned}(E\triangle\underline{\phi}E)\cap Y&=\{x:x\in Y,\ e\in\underline{\psi}(x^{-1}(E\cap Y))\}\triangle(E\cap Y)\\&=\{x:x\in Y,\ e\in x^{-1}\underline{\psi}(E\cap Y)\}\triangle(E\cap Y)=\underline{\psi}(E\cap Y)\triangle(E\cap Y)\end{aligned}$$

is negligible. Moreover, for any $a\in X$,

$$(E\triangle\underline{\phi}E)\cap aY=a((a^{-1}E\triangle\underline{\phi}(a^{-1}E))\cap Y)$$

because $\underline{\phi}$ is translation-invariant, so $(E\triangle\underline{\phi}E)\cap aY$ is negligible. Since $\{aY:a\in X\}$ is an open cover of X, $E\triangle\underline{\phi}E$ is negligible (412Jb again). In particular, $\underline{\phi}E\in\Sigma$. So $\underline{\phi}:\Sigma\to\Sigma$ is a left-translation-invariant lower density, as required.

447I Theorem (Ionescu Tulcea & Ionescu Tulcea 67) Let X be a locally compact Hausdorff topological group. Then it has a left-translation-invariant lifting for its Haar measures.

proof (Cf. 345B-345C.) Write Σ for the algebra of Haar measurable subsets of X, and let $\underline{\phi}:\Sigma\to\Sigma$ be a left-translation-invariant lower density (447H). Let $\phi_0:\Sigma\to\Sigma$ be any lifting such that $\phi_0E\supseteq\underline{\phi}E$ for every $E\in\Sigma$ (341Jb). For $E\in\Sigma$, set

$$\phi E=\{x:e\in\phi_0(x^{-1}E)\},$$

where e is the identity of X. It is easy to check that $\phi:\Sigma\to\mathcal{P}X$ is a Boolean homomorphism. Also

$$x\in\underline{\phi}E\Longrightarrow e\in\underline{\phi}(x^{-1}E)\Longrightarrow e\in\phi_0(x^{-1}E)\Longrightarrow x\in\phi E.$$

So ϕ is a lifting (341Ib). Finally, ϕ is left-translation-invariant by the argument used in (d) of the proof of 447H (and also in (e) of the proof of 345B).

447J Corollary Let X be any topological group carrying Haar measures. Then it has a left-translation-invariant lifting for its left Haar measures.

proof Let μ be a left Haar measure on X. By 443L, we have a locally compact Hausdorff topological group Z and a continuous homomorphism $f : X \to Z$, inverse-measure-preserving for μ and an appropriate left Haar measure ν on Z, such that for every E in the domain Σ of μ there is an F in the domain T of ν such that $f^{-1}[F] \subseteq E$ and $E \setminus f^{-1}[F]$ is negligible. Let ψ be a left-translation-invariant lifting for ν. Since $F^\bullet \mapsto f^{-1}[F]^\bullet$ is an isomorphism between the measure algebras of μ and ν, we have a lifting $\phi : \Sigma \to \Sigma$ given by saying that $\phi E = f^{-1}[\psi F]$ whenever $F \in \mathrm{T}$ and $E \triangle f^{-1}[F]$ is negligible (346D). Now ϕ is left-translation-invariant because f is a group homomorphism and ψ is left-translation-invariant.

447X Basic exercises >(a) Let $X = \mathbb{R} \times \{-1, 1\}$, with its usual topology, and define a multiplication on X by setting $(x, \delta)(y, \epsilon) = (x + \delta y, \delta\epsilon)$. Show that X is a locally compact topological group. Show that there is no lifting for the Haar measure algebra of X which is both left- and right-translation-invariant. (*Hint*: 345Xc.)

(b) Let X be a topological group carrying Haar measures which has a B-sequence. Show that it has a B-sequence $\langle V_n \rangle_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \mu(V_n V_n^{-1})/\mu V_n$ is finite for any Haar measure μ on X, whether left or right.

(c) Let X be a topological group with a left Haar measure μ, and $\langle V_n \rangle_{n\in\mathbb{N}}$ a B-sequence for X. Show that if $f \in \mathcal{L}^0(\mu)$ is locally integrable, then $f(x) = \lim_{n\to\infty} \frac{1}{\mu V_n} \int_{xV_n} f d\mu$ for almost every x.

447Y Further exercises (a) Describe a compact Hausdorff topological group such that its Haar measure has no lifting which is both left- and right-translation-invariant.

(b) Let (X, Σ, μ) be a measure space, with measure algebra $\mathfrak{A}$, and $\underline{\theta} : \mathfrak{A} \to \Sigma$ a lower density. Show that we have a function $q : L^\infty(\mathfrak{A})^+ \to L^\infty(\Sigma)^+$ such that $\{x : q(u)(x) > \alpha\} = \bigcup_{\beta>\alpha} \underline{\theta}[\![u > \beta]\!]$ for every $\alpha \ge 0$ and $u \in L^\infty(\mathfrak{A})^+$. Show that $q(u)^\bullet = u$, $q(\alpha u) = \alpha q(u)$, $q(u \wedge v) = q(u) \wedge q(v)$ and $q(\chi a) = \chi(\underline{\theta} a)$ for every u, $v \in L^\infty(\mathfrak{A})^+$, $\alpha \ge 0$ and $a \in \mathfrak{A}$.

447 Notes and comments The structure of the proof of 447I is exactly that of the proof of the ordinary Lifting Theorem in §341; the lifting is built from a lower density which is constructed inductively on a family of sub-σ-algebras. To get a translation-invariant lifting it is natural to look for a translation-invariant lower density, and a simple trick (already used in §345) ensures that this is indeed enough. The refinements we need here are dramatic but natural. To make the final lower density $\underline{\phi}$ (in 447H) translation-invariant, it is clearly sensible (if we can do it) to keep all the partial lower densities $\underline{\phi}_\xi$ translation-invariant. This means that their domains Σ_ξ should be translation-invariant. It does not quite follow that they have to be of the form $\Sigma_\xi = \{E : EX_\xi = X_\xi\}$ for closed subgroups X_ξ, but if we look at the leading example of $\{0, 1\}^I$ (345C) this also is a reasonable thing to try first. So now we have to consider what extra hypotheses will be needed to make the induction work. The inductive step to limit ordinals of uncountable cardinality remains elementary, at least if the X_ξ are compact (part (b-iv) of the proof of 447H). The inductive step to limit ordinals of countable cofinality (447G) is harder, but can be managed with ideas already presented. Indeed, compared with the version in 341G, we have the advantage of a formula for the auxiliary functions g_{En}, which is very helpful when we come to translation-invariance. We have to do something about the fact that we are no longer working with a probability space – that is the point of the μ_U in part (c) of the proof of 447G. (Another expression of the manoeuvre here is in 369Xq.)

Where we do need a new idea is in the inductive step to a successor ordinal. If $\Sigma_{\xi+1}$ is to be translation-invariant, it must be much bigger than the σ-algebra generated by $\Sigma_\xi \cup \{E\}$, as discussed in 341F. To make the step a small one (and therefore presumably easier), we want $X_{\xi+1}$ to be a large subgroup of X_ξ in some sense; as it turns out, a helpful approach is to ask for $X_{\xi+1}$ to be a normal subgroup of X_ξ and for $X_\xi/X_{\xi+1}$ to be small. At this point we have to know something of the structure theory of locally compact topological groups. The right place to start is surely the theory of compact Hausdorff groups. Such a group X actually has a continuous decreasing chain $\langle X_\xi \rangle_{\xi\le\kappa}$ of closed normal subgroups, from $X_0 = X$ to $X_\kappa = \{e\}$, such that all the quotients $X_\xi/X_{\xi+1}$ are Lie groups. I do not define 'Lie group' here, because for our purposes it is enough to know that the quotients have faithful finite-dimensional representations, and therefore have 'B-sequences' in the sense of 446L. Having identified this as a relevant property, it is not hard to repeat arguments from §221 and §261 to prove versions of Vitali's theorem and Lebesgue's Density Theorem in such groups (447C-447D). This will evidently provide translation-invariant lower densities for groups of this special type, just as Lebesgue lower density is a translation-invariant lower density on $\mathbb{R}^r$ (345B).

Of course we still have to find a way of combining this construction with a translation-invariant lower density on Σ_ξ to produce a translation-invariant lower density on $\Sigma_{\xi+1}$, and this is what I do in 447F. The argument I offer is essentially that of Ionescu Tulcea & Ionescu Tulcea 67, §7, and is the deepest part of this section.

For compact groups, these ideas are all we need, and indeed the step to a limit ordinal of countable cofinality is a little easier, since we have a Haar probability measure on the whole group. The next step, to general σ-compact locally compact groups, demands much deeper ideas from the structure theory, but from the point of view of the present section the modifications are minor. The subgroups X_ξ are now not always normal subgroups of X, which means that we have to be more careful in the description of the quotient spaces X/X_ξ (they must consist of *left* cosets), and we have to watch the modular functions of the X_ξ in order to be sure that there are invariant measures on the quotients. An extra obstacle at the beginning is that we may have to start the chain with a proper subgroup X_0 of X, but since X_0 can be taken to be open, it is pretty clear that this will not be serious, and in fact it gives no trouble (part (b-i) of the proof of 447H). For $\xi \geq 1$, the X_ξ are compact, so the inductive steps to limit ordinals are nearly the same.

The step to a general locally compact Hausdorff topological group (part (d) of the proof of 447H) is essentially elementary. And finally I note that the whole thing applies to general topological groups with Haar measures (447J), for the usual reasons. There is an implicit challenge here: find expressions of the arguments used in this section which will be valid in the more general context. The measure-theoretic part of such a programme might be achievable, but I do not see any hope of a workable structure theory to match that of §446 which does not use 443L or something like it.

448 Polish group actions

I devote this section to two quite separate theorems. The first is an interesting result about measures on Polish spaces which are invariant under actions of Polish groups. In contrast to §441, we no longer have a strong general existence theorem for such measures, but instead have a natural necessary and sufficient condition in terms of countable dissections: there is an invariant probability measure on X if and only if there is no countable dissection of X into Borel sets which can be rearranged, by the action of the group, into two copies of X (448P).

The principal ideas needed here have already been set out in §395, and in many of the proofs I allow myself to direct you to the corresponding arguments there rather than write the formulae out again. I do not think you need read through §395 before embarking on this section; I will try to give sufficiently detailed references so that you can take them one paragraph at a time, and many of the arguments referred to are in any case elementary. But unless you are already familiar with this topic, you will need a copy of §395 to hand to fully follow the proofs below.

The second theorem concerns the representation of group actions on measure algebras in terms of group actions on measure spaces. If we have a locally compact Polish group G (so that we do have Haar measures), and a Borel measurable action of G on the measure algebra of a Radon measure μ on a Polish space X, then it can be represented by a Borel measurable action of G on X (448S). The proof is mostly descriptive set theory based on §§423-424, but it also uses some interesting facts about L^0 spaces (448Q-448R).

448A Definitions (Compare 395A.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. For $a, b \in \mathfrak{A}$ I will say that an isomorphism $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ between the corresponding principal ideals belongs to the **countably full local semigroup generated by** G if there are a countable partition of unity $\langle a_i \rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle \pi_i \rangle_{i\in I}$ in G such that $\phi c = \pi_i c$ whenever $i \in I$ and $c \subseteq a_i$. If such an isomorphism exists I will say that a and b are **G-σ-equidecomposable**.

I write $a \preccurlyeq^\sigma_G b$ to mean that there is a $b' \subseteq b$ such that a and b' are G-σ-equidecomposable.

As in §395, I will say that a function f with domain $\mathfrak{A}$ is **G-invariant** if $f(\pi a) = f(a)$ whenever $a \in \mathfrak{A}$ and $\pi \in G$.

I have expressed these definitions, and most of the work below, in terms of abstract Dedekind σ-complete Boolean algebras. The applications I have in mind for this section are to σ-algebras of sets. If you have already worked through §395, the version here should come very easily; but even if you have not, I think that the extra abstraction clarifies some of the ideas.

448B I begin with results corresponding to 395B-395D; there is hardly any difference, except that we must now occasionally pause to check that a partition of unity is countable.

Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_σ for the countably full local semigroup generated by G.

(a) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_σ, then $\phi^{-1} : \mathfrak{A}_b \to \mathfrak{A}_a$ also belongs to G^*_σ.

(b) Suppose that a, b, a', $b' \in \mathfrak{A}$ and that $\phi : \mathfrak{A}_a \to \mathfrak{A}_{a'}$, $\psi : \mathfrak{A}_b \to \mathfrak{A}_{b'}$ belong to G^*_σ. Then $\psi\phi \in G^*_\sigma$; its domain is $\mathfrak{A}_c$ where $c = \phi^{-1}(b \cap a')$, and its set of values is $\mathfrak{A}_{c'}$ where $c' = \psi(b \cap a')$.

(c) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_σ, then $\phi\restriction\mathfrak{A}_c \in G^*_\sigma$ for any $c \subseteq a$.

(d) Suppose that a, $b \in \mathfrak{A}$ and that $\psi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism such that there are a countable partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle \phi_i\rangle_{i\in I}$ in G^*_σ such that $\psi c = \phi_i c$ whenever $i \in I$ and $c \subseteq a_i$. Then $\psi \in G^*_\sigma$.

proof (a) As 395Bb.

(b) As 395Bc.

(c) As 395Bd.

(d) For each $i \in I$, let $\langle a_{ij}\rangle_{j\in J(i)}$, $\langle \pi_{ij}\rangle_{j\in J(i)}$ witness that $\phi_i \in G^*_\sigma$; then $\langle a_i \cap a_{ij}\rangle_{i\in I,j\in J(i)}$ and $\langle \pi_{ij}\rangle_{i\in I,j\in J(i)}$ witness that $\psi \in G^*_\sigma$.

448C Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_σ for the countably full local semigroup generated by G.

(a) For a, $b \in \mathfrak{A}$, $a \preccurlyeq^\sigma_G b$ iff there is a $\phi \in G^*_\sigma$ such that $a \in \operatorname{dom}\phi$ and $\phi a \subseteq b$.

(b)(i) $\preccurlyeq^\sigma_G$ is transitive and reflexive;

(ii) if $a \preccurlyeq^\sigma_G b$ and $b \preccurlyeq^\sigma_G a$ then a and b are G-σ-equidecomposable.

(c) G-σ-equidecomposability is an equivalence relation on $\mathfrak{A}$.

(d) If $\langle a_i\rangle_{i\in I}$ and $\langle b_i\rangle_{i\in I}$ are countable families in $\mathfrak{A}$, of which $\langle b_i\rangle_{i\in I}$ is disjoint, and $a_i \preccurlyeq^\sigma_G b_i$ for every $i \in I$, then $\sup_{i\in I} a_i \preccurlyeq^\sigma_G \sup_{i\in I} b_i$.

proof The arguments of 395C apply unchanged, calling on 448B in place of 395B.

448D Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is an $a \ne 1$ such that a is G-σ-equidecomposable with 1;

(ii) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-σ-equidecomposable;

(iii) there are non-zero G-σ-equidecomposable a, b, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) there are G-σ-equidecomposable a, $b \in \mathfrak{A}$ such that $a \subset b$.

proof As 395D.

448E Definition (Compare 395E.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. I will say that G is **countably non-paradoxical** if the statements of 448D are false; that is, if one of the following equiveridical statements is true:

(i) if a is G-σ-equidecomposable with 1 then $a = 1$;

(ii) there is no disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-σ-equidecomposable;

(iii) there are no non-zero G-σ-equidecomposable a, b, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) if a, $b \in \mathfrak{A}$ are G-σ-equidecomposable and $a \subseteq b$ then $a = b$.

448F We now come to one of the points where we need to find a new path because we are looking at algebras which need not be Dedekind complete. Provided the original group G is *countable*, we can still follow the general line of §395, as follows.

Lemma (Compare 395G.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a countable subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of G.

(a) For any $a \in \mathfrak{A}$, $\operatorname{upr}(a, \mathfrak{C})$ (313S) is defined, and is given by the formula

$$\operatorname{upr}(a, \mathfrak{C}) = \sup\{\pi a : \pi \in G\}.$$

(b) If G^*_σ is the countably full local semigroup generated by G, then $\phi(c\cap a)=c\cap\phi a$ whenever $\phi\in G^*_\sigma$, $a\in\operatorname{dom}\phi$ and $c\in\mathfrak{C}$.

(c) $\operatorname{upr}(\phi a,\mathfrak{C})=\operatorname{upr}(a,\mathfrak{C})$ whenever $\phi\in G^*_\sigma$ and $a\in\operatorname{dom}\phi$; consequently, $\operatorname{upr}(a,\mathfrak{C})\subseteq\operatorname{upr}(b,\mathfrak{C})$ whenever $a\preccurlyeq^\sigma_G b$.

(d) If $a\preccurlyeq^\sigma_G b$ and $c\in\mathfrak{C}$ then $a\cap c\preccurlyeq^\sigma_G b\cap c$. So $a\cap c$ and $b\cap c$ are G-σ-equidecomposable whenever a and b are G-σ-equidecomposable and $c\in\mathfrak{C}$.

proof (a) As remarked in 395Ga, $\mathfrak{C}$ is order-closed. Because G is countable and $\mathfrak{A}$ is Dedekind σ-complete, $c^*=\sup\{\pi a:\pi\in G\}$ is defined in $\mathfrak{A}$ If $\phi\in G$, then

$$\phi c^*=\sup\{\phi\pi a:\pi\in G\}\subseteq c^*$$

because ϕ is order-continuous and $\phi\pi\in G$ for every $\pi\in G$. Similarly $\phi^{-1}c^*\subseteq c^*$ and $c^*\subseteq\phi c^*$. Thus $\phi c^*=c^*$; as ϕ is arbitrary, $c^*\in\mathfrak{C}$.

If $c\in\mathfrak{C}$, then

$$\begin{aligned}a\subseteq c&\iff \pi a\subseteq\pi c\text{ for every }\pi\in G\\&\iff\pi a\subseteq c\text{ for every }\pi\in G\\&\iff c^*\subseteq c,\end{aligned}$$

so $c^*=\inf\{c:a\subseteq c\in\mathfrak{C}\}$, taking the infimum in $\mathfrak{C}$, as required in the definition of $\operatorname{upr}(a,\mathfrak{C})$.

(b) Suppose that $\langle a_i\rangle_{i\in I}$, $\langle\pi_i\rangle_{i\in I}$ witness that $\phi\in G^*_\sigma$. Then

$$\phi(a\cap c)=\sup_{i\in I}\pi_i(a_i\cap a\cap c)=\sup_{i\in I}\pi_i(a_i\cap a)\cap c=c\cap\phi a.$$

(c) For $c\in\mathfrak{C}$,

$$a\subseteq c\iff a\cap c=a\iff\phi(a\cap c)=\phi a\iff c\cap\phi a=\phi a\iff\phi a\subseteq c.$$

(d) There is a $\phi\in G^*_\sigma$ such that $\phi a\subseteq b$; now

$$a\cap c\preccurlyeq^\sigma_G\phi(a\cap c)=c\cap\phi a\subseteq b\cap c.$$

448G With this support, we can now continue with the ideas of 395H-395L, adding at each step the hypothesis 'G is countable' to compensate for the weakening of the hypotheses '$\mathfrak{A}$ is Dedekind complete, G is fully non-paradoxical' to '$\mathfrak{A}$ is Dedekind σ-complete, G is countably non-paradoxical'.

Lemma (Compare 395H.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Write $\mathfrak{C}$ for the fixed-point subalgebra of G. Take any a, $b\in\mathfrak{A}$. Then $c_0=\sup\{c:c\in\mathfrak{C},\ a\cap c\preccurlyeq^\sigma_G b\}$ is defined in $\mathfrak{A}$ and belongs to $\mathfrak{C}$; $a\cap c_0\preccurlyeq^\sigma_G b$ and $b\setminus c_0\preccurlyeq^\sigma_G a$.

proof Let $\langle\pi_n\rangle_{n\in\mathbb{N}}$ be a sequence running over G. Define $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively, setting

$$a_n=(a\setminus\sup_{i<n}a_i)\cap\pi_n^{-1}(b\setminus\sup_{i<n}b_i),\quad b_n=\pi_n a_n.$$

Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_a$ and $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_b$, and $\sup_{n\in\mathbb{N}}a_n$ is G-σ-equidecomposable with $\sup_{n\in\mathbb{N}}b_n$. Set

$$a'=a\setminus\sup_{n\in\mathbb{N}}a_n,\quad b'=b\setminus\sup_{n\in\mathbb{N}}b_n,\quad c_0=1\setminus\operatorname{upr}(a',\mathfrak{C})\subseteq\sup_{n\in\mathbb{N}}a_n.$$

Then

$$a\cap c_0\subseteq\sup_{n\in\mathbb{N}}a_n\preccurlyeq^\sigma_G b.$$

Now $b'\subseteq c_0$. **P?** Otherwise, because $c_0=1\setminus\sup_{n\in\mathbb{N}}\pi_n a'$, there must be an $n\in\mathbb{N}$ such that $b'\cap\pi_n a'\ne0$. But in this case $d=a'\cap\pi_n^{-1}b'\ne0$, and we have

$$d\subseteq(a\setminus\sup_{i<n}a_i)\cap\pi_n^{-1}(b\setminus\sup_{i<n}b_i),$$

so that $d\subseteq a_n$, which is absurd. **XQ** Consequently

$$b\setminus c_0\subseteq b\setminus b'=\sup_{n\in\mathbb{N}}b_n\preccurlyeq^\sigma_G a.$$

Now take any $c\in\mathfrak{C}$ such that $a\cap c\preccurlyeq^\sigma_G b$, and consider $c'=c\setminus c_0$. Then $b'\cap c'=0$, that is, $b\cap c'=\sup_{n\in\mathbb{N}}b_n\cap c'$, which is G-σ-equidecomposable with $\sup_{n\in\mathbb{N}}a_n\cap c'=(a\setminus a')\cap c'$. But now

$$a\cap c'=a\cap c_0\cap c'\preccurlyeq^\sigma_G b\cap c'\preccurlyeq^\sigma_G(a\cap c')\setminus(a'\cap c');$$

because G is countably non-paradoxical, $a' \cap c'$ must be 0, that is, $c' \subseteq c_0$ and $c \subseteq c_0$. So c_0 has the required properties.

448H Lemma (Compare 395I.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of G. Suppose that a, $b \in \mathfrak{A}$ and that $\operatorname{upr}(a,\mathfrak{C}) = 1$. Then there are non-negative u, $v \in L^0(\mathfrak{C})$ such that

$$[\![u \ge n]\!] = \max\{c : c \in \mathfrak{C}, \text{ there is a disjoint family } \langle d_i\rangle_{i<n} \text{ such that } a \cap c \preccurlyeq^\sigma_G d_i \subseteq b \text{ for every } i < n\},$$

$$[\![v \le n]\!] = \max\{c : c \in \mathfrak{C}, \text{ there is a family } \langle d_i\rangle_{i<n} \text{ such that } d_i \preccurlyeq^\sigma_G a \text{ for every } i < n \text{ and } b \cap c \subseteq \sup_{i<n} d_i\}$$

for every $n \in \mathbb{N}$. Moreover, we have

(i) $[\![u \in \mathbb{N}]\!] = [\![v \in \mathbb{N}]\!] = 1$,

(ii) $[\![v > 0]\!] = \operatorname{upr}(b,\mathfrak{C})$,

(iii) $u \le v \le u + \chi 1$.

proof The argument of 395I applies unchanged, except that every $\preccurlyeq^\tau_G$ must be replaced with a $\preccurlyeq^\sigma_G$, and we use 448F and 448G in place of 395G and 395H. $\mathfrak{C}$ is Dedekind σ-complete because it is order-closed in the Dedekind σ-complete algebra $\mathfrak{A}$ (314Eb).

448I Notation (Compare 395J.) In the context of 448H, I will write $\lfloor b : a\rfloor$ for u, $\lceil b : a\rceil$ for v.

448J Lemma (Compare 395K-395L.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that a, a_1, a_2, b, b_1, $b_2 \in \mathfrak{A}$ and that

$$\operatorname{upr}(a,\mathfrak{C}) = \operatorname{upr}(a_1,\mathfrak{C}) = \operatorname{upr}(a_2,\mathfrak{C}) = 1.$$

Then

(a) $\lfloor 0 : a\rfloor = \lceil 0 : a\rceil = 0$ and $\lfloor 1 : a\rfloor \ge \chi 1$.

(b) If $b_1 \preccurlyeq^\sigma_G b_2$ then $\lfloor b_1 : a\rfloor \le \lfloor b_2 : a\rfloor$ and $\lceil b_1 : a\rceil \le \lceil b_2 : a\rceil$.

(c) $\lceil b_1 \cup b_2 : a\rceil \le \lceil b_1 : a\rceil + \lceil b_2 : a\rceil$.

(d) If $b_1 \cap b_2 = 0$, then $\lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor \le \lfloor b_1 \cup b_2 : a\rfloor$.

(e) If $c \in \mathfrak{C}$ is such that $a \cap c$ is a relative atom over $\mathfrak{C}$, then $c \subseteq [\![\lceil b : a\rceil - \lfloor b : a\rfloor = 0]\!]$.

(f) $\lfloor b : a_2\rfloor \ge \lfloor b : a_1\rfloor \times \lfloor a_1 : a_2\rfloor$, $\lceil b : a_2\rceil \le \lceil b : a_1\rceil \times \lceil a_1 : a_2\rceil$.

proof As in 395K-395L.

448K For the result corresponding to 395Mb, we again need to find a new approach; I deal with it by adding a further hypothesis to the list which has already accreted.

Definition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a countable subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. I will say that G has the ***σ*-refinement property** if for every $a \in \mathfrak{A}$ there is a $d \subseteq a$ such that $d \preccurlyeq^\sigma_G a \setminus d$ and $a' = a \setminus \operatorname{upr}(d,\mathfrak{C})$ is a relative atom over $\mathfrak{C}$, that is, every $b \subseteq a'$ is expressible as $a' \cap c$ for some $c \in \mathfrak{C}$.

(If we replace $\preccurlyeq^\sigma_G$ with $\preccurlyeq^\tau_G$, as used in §395, we see that 395Ma could be read as 'if $\mathfrak{A}$ is a Dedekind complete Boolean algebra, then any subgroup of $\operatorname{Aut}\mathfrak{A}$ has the τ-refinement property'.)

448L I give the principal case in which the 'σ-refinement property' just defined arises.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with countable Maharam type (definition: 331F). Then any countable subgroup of $\operatorname{Aut}\mathfrak{A}$ has the σ-refinement property.

proof (a) Let E be a countable subset of $\mathfrak{A}$ which τ-generates $\mathfrak{A}$, and $\mathfrak{E}$ the subalgebra of $\mathfrak{A}$ generated by E; then $\mathfrak{E}$ is countable (331Gc), and the smallest order-closed subset of $\mathfrak{A}$ including $\mathfrak{E}$ is a subalgebra of $\mathfrak{A}$ (313Fc), so must be $\mathfrak{A}$ itself.

(b) Suppose that $b \in \mathfrak{A} \setminus \{0\}$ and $\pi \in \operatorname{Aut}\mathfrak{A}$ are such that $b \cap \pi b = 0$. Then there is an $e \in \mathfrak{C}$ such that $b \cap e \setminus \pi e \neq 0$. **P?** Otherwise, set

$$D = \{d : d \in \mathfrak{A},\, b \cap d \setminus \pi d = 0\}.$$

Then $\mathfrak{C} \subseteq D$, but $b \notin D$. So D cannot be order-closed. **case 1** If $D_0 \subseteq D$ is a non-empty upwards-directed set with supremum $d_0 \notin D$, then $b \cap d_0 \setminus \pi d_0 \neq 0$, so there is a $d \in D_0$ such that $b \cap d \setminus \pi d_0 \neq 0$; but now $d \notin D$, which is impossible. **case 2** If $D_0 \subseteq D$ is a non-empty downwards-directed subset of D with infimum $d_0 \notin D$, then $b \cap d_0 \setminus \pi d_0 \neq 0$. But π is order-continuous, so there is a $d \in D_0$ such that $b \cap d_0 \setminus \pi d \neq 0$; and now $d \notin D$, which is impossible. Thus in either case we have a contradiction. **XQ**

(c) Now let G be a countable subgroup of $\operatorname{Aut}\mathfrak{A}$, with fixed-point subalgebra $\mathfrak{C}$, and let $\langle(\pi_n, e_n)\rangle_{n\in\mathbb{N}}$ be a sequence running over $G \times \mathfrak{C}$. Take any $a \in \mathfrak{A}$. For $k \in \mathbb{N}$ set

$$a_k = a \cap e_k \cap \pi_k^{-1}(a \setminus e_k),$$

$$a'_k = a_k \setminus \sup_{j<k} \operatorname{upr}(a'_j, \mathfrak{C}).$$

Then

$$a'_k \cap \pi_k a'_k \subseteq a_k \cap \pi_k a_k = 0$$

for every $k \in \mathbb{N}$, and whenever $j < k$ in $\mathbb{N}$ we have

$$a'_j \cap a'_k = 0, \quad \pi_j a'_j \cap a'_k = 0,$$

$$a'_j \cap \pi_k a'_k = \pi_k(\pi_k^{-1} a'_j \cap a'_k) = 0, \quad \pi_j a'_j \cap \pi_k a'_k = \pi_k(a'_k \cap \pi_k^{-1}\pi_j a'_j) = 0.$$

So, setting $d = \sup_{k\in\mathbb{N}} a'_k$ and $d' = \sup_{k\in\mathbb{N}} \pi_k a'_k$, d and d' are disjoint and G-σ-equidecomposable and included in a, and $d \preccurlyeq_G^{\sigma} a \setminus d$.

Consider $a' = a \setminus \operatorname{upr}(d, \mathfrak{C})$. Since $a'_k = a_k \setminus \sup_{j<k} \operatorname{upr}(a'_j, \mathfrak{C})$ for each k,

$$\operatorname{upr}(d, \mathfrak{C}) = \sup_{k\in\mathbb{N}} \operatorname{upr}(a'_k, \mathfrak{C}) = \sup_{k\in\mathbb{N}} \operatorname{upr}(a_k, \mathfrak{C}).$$

? Suppose, if possible, that a' is not a relative atom over $\mathfrak{C}$; that is, that there is a $b \subseteq a'$ such that $b \neq a' \cap c$ for any $c \in \mathfrak{C}$. Then, in particular, $b \neq a' \cap \operatorname{upr}(b, \mathfrak{C})$, and there is a $\pi \in G$ such that $b' = a' \cap \pi b \setminus b \neq 0$. Then $b' \cup \pi^{-1} b' \subseteq a$, while $b' \cap \pi^{-1} b' = 0$, so $\pi b' \cap b' = 0$. By (b), there is an $e \in \mathfrak{C}$ such that $b'' = b' \cap e \setminus \pi e \neq 0$. Let k be such that $\pi^{-1} = \pi_k$ and $e = e_k$, so that

$$b'' = b' \cap e_k \setminus \pi_k^{-1} e_k \subseteq a \cap e_k \cap \pi_k^{-1}(a \setminus e_k) = a_k.$$

(Because $\pi^{-1} b' \subseteq a$, $b' \subseteq \pi_k^{-1} a$.) Since also

$$b'' \cap \operatorname{upr}(a'_j, \mathfrak{C}) \subseteq a' \cap \operatorname{upr}(d, \mathfrak{C}) = 0$$

for every j, $b'' \subseteq a'_k \subseteq d$, which is impossible, because $b'' \subseteq a'$. **X**

Thus a' is a relative atom over $\mathfrak{C}$, as required.

448M Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. If G has the σ-refinement property, then for any $\epsilon > 0$ there is an $a^* \in \mathfrak{A}$ such that $\operatorname{upr}(a^*, \mathfrak{C}) = 1$ and $\lceil b : a^* \rceil \leq \lfloor b : a^* \rfloor + \epsilon \lfloor 1 : a^* \rfloor$ for every $b \in \mathfrak{A}$.

proof As part (b) of the proof of 395M.

448N Theorem (Compare 395N.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that G has the σ-refinement property of 448K. Then there is a function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{C})$ such that

(i) θ is additive, non-negative and sequentially order-continuous;

(ii) $\theta a = 0$ iff $a = 0$, $\theta 1 = \chi 1$;

(iii) $\theta(a \cap c) = \theta a \times \chi c$ for every $a \in \mathfrak{A}$, $c \in \mathfrak{C}$; in particular, $\theta c = \chi c$ for every $c \in \mathfrak{C}$;

(iv) if a, $b \in \mathfrak{A}$ are G-σ-equidecomposable, then $\theta a = \theta b$; in particular, θ is G-invariant.

proof The arguments of the proof of 395N apply here also, though we have to take things in a slightly different order. As in 395N, set

$$\theta_a(b) = \frac{\lceil b : a \rceil}{\lfloor 1 : a \rfloor} \in L^0(\mathfrak{C})$$

whenever $\operatorname{upr}(a, \mathfrak{C}) = 1$ and $b \in \mathfrak{A}$. This time, turn immediately to part (c) of the proof to see that if e_n is chosen (using 448M) such that $\operatorname{upr}(e_n, \mathfrak{C}) = 1$ and $\lceil b : e_n \rceil \le \lfloor b : e_n \rfloor + 2^{-n} \lfloor 1 : e_n \rfloor$ for every $b \in \mathfrak{A}$, then $\theta_{e_n} b \le \theta_a b + 2^{-n} \lceil 1 : a \rceil$ whenever $\operatorname{upr}(a, \mathfrak{C}) = 1$ and $b \in \mathfrak{A}$. So we can write

$$\theta b = \inf_{n \in \mathbb{N}} \theta_{e_n} b = \inf_{\operatorname{upr}(a, \mathfrak{C}) = 1} \theta_a b$$

for every $b \in \mathfrak{A}$, and we have a function $\theta : \mathfrak{A} \to L^0$ as before. The rest of the proof is unchanged, except that we have a simplification in (h), since we need consider only the case $\kappa = \omega$.

448O This concludes the adaptations we need from §395. I now return to the specific problem addressed in the present section. The first step is a variation on 448N.

Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable subgroup of $\operatorname{Aut} \mathfrak{A}$ with the σ-refinement property. Let $\mathfrak{C}$ be the fixed-point subalgebra of G. Then the following are equiveridical:

(i) there are a Dedekind σ-complete Boolean algebra $\mathfrak{D}$, not $\{0\}$, and a G-invariant sequentially order-continuous non-negative additive function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{D})$ such that $\theta 1 = \chi 1$;

(ii) if $a \in \mathfrak{A}$ and $1 \preccurlyeq^\sigma_G a$, then $\operatorname{upr}(1 \setminus a, \mathfrak{C}) \ne 1$;

(iii) if $a \in \mathfrak{A}$ and $1 \preccurlyeq^\sigma_G a$, then $1 \not\preccurlyeq^\sigma_G 1 \setminus a$.

proof (a)(i)⇒(iii) Take $\theta : \mathfrak{A} \to L^\infty(\mathfrak{D})$ as in (i). If $1 \preccurlyeq^\sigma_G a$, then there is a $b \subseteq a$ which is G-σ-equidecomposable with 1, so that $\theta b = \chi 1$, just as in 395N(v)/448N(iv). But this means that $\theta a = \chi 1$; so that $\theta(1 \setminus a) \ne \chi 1$ and $1 \not\preccurlyeq^\sigma_G 1 \setminus a$.

(b)not-(ii)⇒not-(iii) Suppose that (ii) is false; that there is an $a \in \mathfrak{A}$ such that $1 \preccurlyeq^\sigma_G a$ and $\operatorname{upr}(1 \setminus a, \mathfrak{C}) = 1$. Let G^*_σ be the countably full local semigroup generated by G; then there is a $\psi \in G^*_\sigma$ such that $\psi 1 \subseteq a$. Set $b_0 = 1 \setminus \psi 1$ and $b_n = \psi^n b_0$ for every $n \ge 1$; then

$$b_0 \cap b_n \subseteq b_0 \cap \psi 1 = 0$$

for every $n \ge 1$, so

$$b_m \cap b_n = \psi^m (b_0 \cap b_{n-m}) = 0$$

whenever $m < n$.

Let $\langle \pi_i \rangle_{i \in \mathbb{N}}$ be a sequence running over G; then

$$\sup_{i \in \mathbb{N}} \pi_i b_0 = \operatorname{upr}(b_0, \mathfrak{C}) \supseteq \operatorname{upr}(1 \setminus a, \mathfrak{C}) = 1.$$

Set

$$a_j = \pi_j b_0 \setminus \sup_{i<j} \pi_i b_0$$

for every $j \in \mathbb{N}$, so that $\langle a_j \rangle_{j \in \mathbb{N}}$ is a partition of unity in $\mathfrak{A}$. Define ψ_1, $\psi_2 : \mathfrak{A} \to \mathfrak{A}$ by setting

$$\psi_1 d = \sup_{i \in \mathbb{N}} \psi^{2i} \pi_i^{-1} (d \cap a_i), \quad \psi_2 d = \sup_{i \in \mathbb{N}} \psi^{2i+1} \pi_i^{-1} (d \cap a_i)$$

for every $d \in \mathfrak{A}$. Because $\psi^{2i} \pi_i^{-1} a_i \subseteq b_{2i}$ for every i, $\langle \psi^{2i} \pi_i^{-1} a_i \rangle_{i \in \mathbb{N}}$ is disjoint, so $\psi_1 \in G^*_\sigma$ (448Bd); similarly, $\psi_2 \in G^*_\sigma$. Thus

$$1 \preccurlyeq^\sigma_G \psi_1 1 \subseteq \sup_{i \in \mathbb{N}} b_{2i},$$

$$1 \preccurlyeq^\sigma_G \psi_2 1 \subseteq \sup_{i \in \mathbb{N}} b_{2i+1} \subseteq 1 \setminus \psi_1 1$$

and (iii) is false.

(c) For the rest of this proof I will suppose that (ii) is true and seek to prove (i).

Let $\mathcal{I}$ be the σ-ideal of $\mathfrak{A}$ generated by $\{1 \setminus a : a \in \mathfrak{A},\ 1 \preccurlyeq^\sigma_G a\}$. Then $1 \notin \mathcal{I}$. **P?** Otherwise, there is a sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ such that $1 \preccurlyeq^\sigma_G a_n$ for every n and $\sup_{n \in \mathbb{N}} 1 \setminus a_n = 1$. Choose $\psi_n \in G^*_\sigma$ such that $\psi_n 1 \subseteq a_n$, and set $c_n = \operatorname{upr}(1 \setminus \psi_n 1, \mathfrak{C})$ for each n, so that $\sup_{n \in \mathbb{N}} c_n = 1$. Set $c'_n = c_n \setminus \sup_{i<n} c_i$ for each n, and write

$$\psi d = \sup_{n \in \mathbb{N}} \psi_n (d \cap c'_n)$$

for each $d \in \mathfrak{A}$. Because every c'_n belongs to $\mathfrak{C}$, $\langle \psi_n c'_n \rangle_{n \in \mathbb{N}} = \langle c'_n \rangle_{n \in \mathbb{N}}$ is disjoint, and $\psi \in G^*_\sigma$.

By (ii), $c = \text{upr}(1 \setminus \psi 1, \mathfrak{C})$ is not 1; let n be such that $c' = c'_n \setminus c \neq 0$. Because $c' \subseteq c'_n$, $c' \setminus \psi_n 1 = c' \setminus \psi 1$; because $c' \in \mathfrak{C}$,

$$\begin{aligned} 0 \neq c' \cap c'_n \subseteq c' \cap \text{upr}(1 \setminus \psi_n 1, \mathfrak{C}) &= \text{upr}(c' \setminus \psi_n 1, \mathfrak{C}) \\ &= \text{upr}(c' \setminus \psi 1, \mathfrak{C}) = c' \cap \text{upr}(1 \setminus \psi 1, \mathfrak{C}) = 0 \end{aligned}$$

which is absurd. **XQ**

Let $\mathfrak{B}$ be the quotient Boolean algebra $\mathfrak{A}/\mathcal{I}$; then $\mathfrak{B}$ is Dedekind σ-complete and the canonical homomorphism $a \mapsto a^\bullet : \mathfrak{A} \to \mathfrak{B}$ is sequentially order-continuous (314C, 313Qb).

(d) Next, $\pi b \in \mathcal{I}$ whenever $b \in \mathcal{I}$ and $\pi \in G$. **P** The sets $\{a : 1 \preccurlyeq^\sigma_G a\}$ and $\{1 \setminus a : 1 \preccurlyeq^\sigma_G a\}$ are both invariant under the action of G, so $\mathcal{I}$ also must be invariant. **Q** We can therefore define, for each $\pi \in G$, a Boolean automorphism $\tilde{\pi} : \mathfrak{B} \to \mathfrak{B}$, setting $\tilde{\pi} a^\bullet = (\pi a)^\bullet$ for every $a \in \mathfrak{A}$. Because $(\pi\phi)^\sim = \tilde{\pi}\tilde{\phi}$ for all π, $\phi \in G$, $\tilde{G} = \{\tilde{\pi} : \pi \in G\}$ is a subgroup of $\text{Aut}\,\mathfrak{B}$; of course it is countable. Let $\mathfrak{D}$ be the fixed-point subalgebra of $\tilde{G}$ in $\mathfrak{B}$. Because $\mathfrak{B}$ is not $\{0\}$, nor is $\mathfrak{D}$.

(e) $\tilde{G}$ is countably non-paradoxical. **P** Suppose that b is $\tilde{G}$-σ-equidecomposable with 1 in $\mathfrak{B}$. Let $\langle b_n \rangle_{n \in \mathbb{N}}$ be a partition of unity in $\mathfrak{B}$ and $\langle \pi_n \rangle_{n \in \mathbb{N}}$ a sequence in G such that $\langle \tilde{\pi}_n b_n \rangle_{n \in \mathbb{N}}$ is disjoint and has supremum b. For each $n \in \mathbb{N}$, let $a_n \in \mathfrak{A}$ be such that $a_n^\bullet = b_n$. We have

$$(a_m \cap a_n)^\bullet = (\pi_m a_m \cap \pi_n a_n)^\bullet = 0$$

whenever $m \neq n$, so

$$d = \sup_{m \neq n}(a_m \cap a_n) \cup \sup_{m \neq n}(a_m \cap \pi_m^{-1}\pi_n a_n)$$

belongs to $\mathcal{I}$, while $\langle a_n \setminus d \rangle_{n \in \mathbb{N}}$, $\langle \pi_n(a_n \setminus d) \rangle_{n \in \mathbb{N}}$ are disjoint.

Because $\sup_{n \in \mathbb{N}} b_n = 1$ in $\mathfrak{B}$, $d' = 1 \setminus \sup_{n \in \mathbb{N}} a_n \in \mathcal{I}$. Because $\mathcal{I}$ is a σ-ideal and G is countable,

$$c^* = \text{upr}(d \cup d', \mathfrak{C}) = \sup_{\pi \in G} \pi(d \cup d')$$

belongs to $\mathcal{I}$, while $\{c^*\} \cup \{a_n \setminus c^* : n \in \mathbb{N}\}$ is a partition of unity in $\mathfrak{A}$.

Define $\psi \in G^*_\sigma$ by setting

$$\psi a = \sup_{n \in \mathbb{N}} \pi_n(a \cap a_n \setminus c^*) \cup (a \cap c^*)$$

for every $a \in \mathfrak{A}$. Then $1 \setminus \psi 1 \in \mathcal{I}$, by the definition of $\mathcal{I}$, so $(\psi 1)^\bullet = 1$ in $\mathfrak{B}$. But

$$\begin{aligned} (\psi 1)^\bullet &= \sup_{n \in \mathbb{N}} (\pi_n(a_n \setminus c^*))^\bullet = \sup_{n \in \mathbb{N}} (\pi_n a_n)^\bullet \\ &= \sup_{n \in \mathbb{N}} \tilde{\pi}_n b_n = b. \end{aligned}$$

So $b = 1$. As b is arbitrary, $\tilde{G}$ is countably non-paradoxical. **Q**

(f) $\tilde{G}$ has the σ-refinement property. **P** Let $b \in \mathfrak{B}$. Then there is an $a \in \mathfrak{A}$ such that $a^\bullet = b$. Because G is supposed to have the σ-refinement property, there is a $d \subseteq a$ such that $d \preccurlyeq^\sigma_G a \setminus d$ and $a \setminus \text{upr}(d, \mathfrak{C})$ is a relative atom over $\mathfrak{C}$. Set $e = d^\bullet \subseteq b$.

We know that there are a partition of unity $\langle d_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}_d$ and a sequence $\langle \pi_n \rangle_{n \in \mathbb{N}}$ in G such that $\pi_n d_n \subseteq a \setminus d$ for every n and $\langle \pi_n d_n \rangle_{n \in \mathbb{N}}$ is disjoint. Now $\langle d_n^\bullet \rangle_{n \in \mathbb{N}}$ is a partition of unity in $\mathfrak{B}_e$, $\tilde{\pi}_n d_n^\bullet \subseteq b \setminus e$ for every n, and $\langle \tilde{\pi}_n d_n^\bullet \rangle_{n \in \mathbb{N}}$ is disjoint; so $e \preccurlyeq^\sigma_{\tilde{G}} b \setminus e$.

Suppose that $b_0 \subseteq b \setminus \text{upr}(e, \mathfrak{D})$. Then it is expressible as $a_0^\bullet$ where $a_0 \subseteq a$ and

$$(a_0 \cap \pi d)^\bullet = b_0 \cap \tilde{\pi} e = 0$$

for every $\pi \in G$. So if we set $a_1 = a_0 \setminus \sup_{\pi \in G} \pi d$, we shall have $a_1 \subseteq a \setminus \text{upr}(d, \mathfrak{C})$ and $a_1^\bullet = b_0$. Now $a \setminus \text{upr}(d, \mathfrak{C})$ is supposed to be a relative atom over $\mathfrak{C}$, so $a_1 = a \cap c$ for some $c \in \mathfrak{C}$. In this case,

$$\tilde{\pi} c^\bullet = (\pi c)^\bullet = c^\bullet$$

for every $\pi \in G$, so $c^\bullet \in \mathfrak{D}$, while $b_0 = b \cap c^\bullet$. As b_0 is arbitrary, $b \setminus \text{upr}(e, \mathfrak{D})$ is a relative atom over $\mathfrak{D}$.

Thus e has both the properties required by the definition 448K. As b is arbitrary, $\tilde{G}$ has the σ-refinement property. **Q**

(g) 448N now tells us that there is a sequentially order-continuous non-negative additive functional $\theta_0 : \mathfrak{B} \to L^\infty(\mathfrak{D})$ such that $\theta_0 1 = \chi 1$ and $\theta_0(\tilde{\pi} b) = \theta_0 b$ whenever $b \in \mathfrak{B}$ and $\pi \in G$. If we set $\theta a = \theta_0 a^\bullet$ for $a \in \mathfrak{A}$, it is easy to see that θ has all the properties required by (i) of this theorem. Thus (ii)$\Rightarrow$(i), and the proof is complete.

448P At last we come to Polish spaces.

Theorem (Nadkarni 90, Becker & Kechris 96) Let G be a Polish group acting on a non-empty Polish space $(X,\mathfrak{T})$ with a Borel measurable action $\bullet$. For Borel sets E, $F\subseteq X$ say that $E\preccurlyeq^\sigma_G F$ if there are a countable partition $\langle E_i\rangle_{i\in I}$ of E into Borel sets, and a family $\langle g_i\rangle_{i\in I}$ in G, such that $g_i\bullet E_i\subseteq F$ for every i and $\langle g_i\bullet E_i\rangle_{i\in I}$ is disjoint. Then the following are equiveridical:

(i) there is a G-invariant Radon probability measure μ on X;

(ii) if $F\subseteq X$ is a Borel set such that $X\preccurlyeq^\sigma_G F$, then $\bigcap_{n\in\mathbb{N}}g_n\bullet F\ne\emptyset$ for any sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in G;

(iii) there are no disjoint Borel sets E, $F\subseteq X$ such that $X\preccurlyeq^\sigma_G E$ and $X\preccurlyeq^\sigma_G F$.

proof (a) Let us start with the easy parts.

(i)⇒(ii) Let μ be a G-invariant Radon probability measure on X, and suppose that $X\preccurlyeq^\sigma_G F$. Let $\langle E_i\rangle_{i\in I}$ be a countable partition of X into Borel sets and $\langle h_i\rangle_{i\in I}$ a family in G such that $\langle h_i\bullet E_i\rangle_{i\in I}$ is disjoint and $h_i\bullet E_i\subseteq F$ for every i. Then

$$\mu F\ge\sum_{i\in I}\mu(h_i\bullet E_i)=\sum_{i\in I}\mu E_i=\mu X,$$

so F is conegligible. Consequently $\bigcap_{n\in\mathbb{N}}g_n\bullet F$ must be conegligible and cannot be empty, for any sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in G. As F and $\langle g_n\rangle_{n\in\mathbb{N}}$ are arbitrary, (ii) is true.

(ii)⇒(iii) Assume (ii). **?** If there are disjoint E, F such that $X\preccurlyeq^\sigma_G E$ and $X\preccurlyeq^\sigma_G F$, then we have a countable partition $\langle E_i\rangle_{i\in I}$ of X into Borel sets and a family $\langle g_i\rangle_{i\in I}$ in G such that $g_i\bullet E_i\subseteq E$ for every $i\in I$. But there is an $x\in\bigcap_{i\in I}g_i^{-1}\bullet F$, by (ii). In this case there is a $j\in I$ such that $x\in E_j$ and $g_j\bullet x\in E\cap F$, which is impossible. **X** So (iii) must be true.

(b) For the rest of the proof, therefore, I shall assume (iii) and seek to prove (i).

Let $\mathfrak{T}$ be the topology of X, and $\mathcal{B}=\mathcal{B}(X)$ its Borel σ-algebra. For $g\in G$ define $\pi_g:\mathcal{B}\to\mathcal{B}$ by writing $\pi_gE=g\bullet E$ for every $E\in\mathcal{B}$. Then $\pi_{gh}=\pi_g\pi_h$ for all g, $h\in G$, so $\tilde G=\{\pi_g:g\in G\}$ is a subgroup of $\operatorname{Aut}\mathcal{B}$. Observe that for E, $F\in\mathcal{B}$, $E\preccurlyeq^\sigma_G F$, in the sense here, iff $E\preccurlyeq^\sigma_{\tilde G}F$ in the sense of 448A.

By the Becker-Kechris theorem (424H), there is a Polish topology $\mathfrak{T}_1$ on X, giving rise to the same Borel σ-algebra $\mathcal{B}$ as the original topology, for which the action of G is continuous. Let $\mathcal{U}$ be a countable base for $\mathfrak{T}_1$. (We are going to have three Polish topologies on X in this proof, so watch carefully.)

(c) For the time being (down to the end of (f) below) let us suppose that G, and therefore $\tilde G$, are countable. In this case, because $\mathcal{B}$ is countably generated, $\tilde G$ has the σ-refinement property, by 448L. We can therefore apply 448O to see that (iii) implies that

(i)$'$ there are a Dedekind σ-complete Boolean algebra $\mathfrak{D}$, not $\{0\}$, and a $\tilde G$-invariant sequentially order-continuous non-negative additive functional $\theta:\mathcal{B}\to L^\infty(\mathfrak{D})$ such that $\theta X=\chi 1$.

Express $\mathfrak{D}$ as $\Sigma/\mathcal{J}$ where Σ is a σ-algebra of subsets of a set Z and $\mathcal{J}$ is a σ-ideal of Σ (314N). Then we can identify $L^\infty(\mathfrak{D})$ with the quotient $\mathcal{L}^\infty/W$, where $\mathcal{L}^\infty$ is the space of bounded Σ-measurable real-valued functions on Z and W is the set $\{f:f\in\mathcal{L}^\infty,\,\{z:f(z)\ne0\}\in\mathcal{J}\}$ (363Hb). For each $E\in\mathcal{B}$, let $f_E\in\mathcal{L}^\infty$ be a representative of $\theta E\in L^\infty(\mathfrak{D})$; because $\theta(\pi E)=\theta E$ whenever $E\in\mathcal{B}$ and $\pi\in\tilde G$, we may suppose that $f_{\pi E}=f_E$ whenever $E\in\mathcal{B}$ and $\pi\in\tilde G$.

Let $\mathfrak{B}$ be the subalgebra of $\mathcal{B}$ generated by $\{\pi U:U\in\mathcal{U},\,\pi\in\tilde G\}$. Then $\mathfrak{B}$ is countable and $\pi E\in\mathfrak{B}$ for every $E\in\mathfrak{B}$, $\pi\in\tilde G$. By 4A3I, there is yet another Polish topology $\mathfrak{S}$ on X which is zero-dimensional and such that every member of $\mathfrak{B}$ is open-and-closed for $\mathfrak{S}$. Of course $\mathfrak{S}\supseteq\mathcal{U}$, so $\mathcal{B}$ is still the algebra of $\mathfrak{S}$-Borel sets (423Fb). Let $\mathcal{W}$ be a countable base for $\mathfrak{S}$ consisting of sets which are open-and-closed for $\mathfrak{S}$, and let $\mathfrak{B}_1$ be the subalgebra of $\mathcal{B}$ generated by $\mathcal{W}\cup\mathfrak{B}$; then $\mathfrak{B}_1$ is countable and consists of open-and-closed sets for $\mathfrak{S}$. Let $\langle W_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{W}$. Let ρ be a complete metric on X defining the topology $\mathfrak{S}$, and for m, $n\in\mathbb{N}$ set

$$W_{mn}=\bigcup\{W_i:i\le n,\,\operatorname{diam}_\rho(W_i)\le 2^{-m}\};$$

then for each $m\in\mathbb{N}$, $\langle W_{mn}\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{B}_1$ with union X.

(d) Consider the subsets of Z of the following types:

$$P_E=\{z:f_E(z)<0\},\text{ where }E\in\mathfrak{B}_1,$$

$$Q_{EF}=\{z:f_{E\cup F}(z)\ne f_E(z)+f_F(z)\},\text{ where }E,\,F\in\mathfrak{B}_1\text{ and }E\cap F=\emptyset,$$

$$R=\{z:f_X(z)\ne1\},$$

$$S_m = \{z : \sup_{n\in\mathbb{N}} f_{W_{mn}}(z) \neq 1\}, \text{ where } m \in \mathbb{N}.$$

Because

$$f_E^\bullet = \theta E \geq 0 \text{ for every } E \in \mathcal{B},$$

$$f_{E\cup F}^\bullet = \theta(E \cup F) = \theta E + \theta F = f_E^\bullet + f_F^\bullet \text{ whenever } E \cap F = \emptyset,$$

$$f_X^\bullet = \theta X = \chi 1,$$

$$\sup_{n\in\mathbb{N}} f_{W_{mn}}^\bullet = \sup_{n\in\mathbb{N}} \theta W_{mn} = \theta(\bigcup_{n\in\mathbb{N}} W_{mn}) = \theta X = \chi 1$$

for every $m \in \mathbb{N}$, all the sets P_E, Q_{EF}, R and S_m belong to $\mathcal{J}$. Since $\mathfrak{D} \neq \{0\}$, $Z \notin \mathcal{J}$; so there is a $z_0 \in Z$ not belonging to R or P_E or Q_{EF} or S_m whenever $m \in \mathbb{N}$ and E, $F \in \mathfrak{B}_1$ are disjoint.

Set $\nu E = f_E(z_0)$ for every $E \in \mathfrak{B}_1$. If E, $F \in \mathfrak{B}_1$ are disjoint, then $\nu(E \cup F) = \nu E + \nu F$ because $z_0 \notin Q_{EF}$; thus $\nu : \mathfrak{B}_1 \to \mathbb{R}$ is additive. If $E \in \mathfrak{B}_1$ then $\nu E \geq 0$ because $z_0 \notin P_E$, so ν is non-negative. $\nu X = 1$ because $z_0 \notin R$. For each $m \in \mathbb{N}$, $\sup_{n\in\mathbb{N}} \nu W_{mn} = 1$ because $z_0 \notin S_m$.

(e) For any $\epsilon > 0$ there is an $\mathfrak{S}$-compact set $K \subseteq X$ such that $\nu E \geq 1 - \epsilon$ whenever $E \in \mathfrak{B}_1$ and $E \supseteq K$. **P** For each $m \in \mathbb{N}$ we have a $k(m) \in \mathbb{N}$ such that $\nu W_{m,k(m)} \geq 1 - 2^{-m-1}\epsilon$. Set $K = \bigcap_{m\in\mathbb{N}} W_{m,k(m)}$. Because every $W_{m,k(m)}$ is $\mathfrak{S}$-closed, K is $\mathfrak{S}$-closed, therefore ρ-complete; because every $W_{m,k(m)}$ is a finite union of sets of diameter at most 2^{-m}, K is ρ-totally bounded, therefore $\mathfrak{S}$-compact (4A2Je). **?** Suppose, if possible, that $E \in \mathfrak{B}_1$ is such that $K \subseteq E$ and $\nu E < 1 - \epsilon$. For every $m \in \mathbb{N}$,

$$\nu(\bigcap_{i\leq m} W_{i,k(i)}) \geq 1 - \sum_{i=0}^m \nu(X \setminus W_{i,k(i)}) \geq 1 - \sum_{i=0}^m 2^{-i-1}\epsilon > 1 - \epsilon > \nu E$$

because ν is non-negative and finitely additive. So $\bigcap_{i\leq m} W_{i,k(i)} \setminus E$ must be non-empty. There is therefore an ultrafilter $\mathcal{F}$ on X containing $W_{i,k(i)} \setminus E$ for every $i \in \mathbb{N}$. Now for each i there must be a $j \leq k(i)$ such that $\operatorname{diam} W_j \leq 2^{-i}$ and $W_j \in \mathcal{F}$, so $\mathcal{F}$ is a ρ-Cauchy filter, and $\mathfrak{S}$-converges to x say. Because every $W_{i,k(i)}$ is $\mathfrak{S}$-closed, $x \in \bigcap_{i\in\mathbb{N}} W_{i,k(i)} = K$; because $E \in \mathfrak{S}$, $x \notin E$; but K is supposed to be included in E. **X**

Thus $\inf\{\nu E : K \subseteq E \in \mathfrak{B}_1\} \geq 1 - \epsilon$. As ϵ is arbitrary, we have the result. **Q**

(f) By 416O, there is an $\mathfrak{S}$-Radon measure μ on X extending ν. Because μ is just the completion of its restriction to $\mathcal{B}$, it is also $\mathfrak{T}$-Radon and $\mathfrak{T}_1$-Radon (433Cb).

Now μ is G-invariant. **P** Take any $g \in G$. Set $\mu_g E = \mu(g\bullet E)$ whenever $E \subseteq X$ and $g\bullet E \in \operatorname{dom}\mu$. The map $x \mapsto g\bullet x$ is a homeomorphism for $\mathfrak{T}_1$, so μ_g also is a $\mathfrak{T}_1$-Radon measure. (Setting $\phi(x) = g^{-1}\bullet x$, μ_g is the image measure $\mu\phi^{-1}$.) Again because $\mathfrak{T}$ and $\mathfrak{T}_1$ have the same Borel σ-algebras, μ_g is $\mathfrak{T}$-Radon. If $E \in \mathfrak{B}$, then E and $g\bullet E$ belong to $\mathfrak{B} \subseteq \mathfrak{B}_1$, so

$$\mu_g E = \mu(g\bullet E) = \nu(g\bullet E) = \nu(\pi_g E) = f_{\pi_g E}(z_0) = f_E(z_0)$$

(because $f_{\pi_g E} = f_E$, as declared in (c) above)

$$= \nu E = \mu E.$$

In particular, $\mu_g E = \mu E$ for every E in the algebra generated by $\mathcal{U}$. But μ_g and μ are both $\mathfrak{T}$-Radon measures, and $\mathcal{U}$ is a base for $\mathfrak{T}$, so $\mu_g = \mu$ (415H(iv)). As g is arbitrary, μ is G-invariant. **Q**

Thus we have found a G-invariant Radon probability measure, and (i) is true.

(g) Thus (iii)$\Rightarrow$(i) if G is countable. Now let us consider the general case. Because G is a Polish group, it has a countable dense subgroup H. (Take H to be the subgroup generated by any countable dense subset of G.) Of course there can be no disjoint E, $F \in \mathcal{B}$ such that $X \preccurlyeq_H^\sigma E$ and $X \preccurlyeq_H^\sigma F$, so there must be an H-invariant Radon probability measure μ on X, by the arguments of (b)-(f). (H need not be a Polish group in its subspace topology. But if we give it its discrete topology, then $x \mapsto h\bullet x$ is still a $\mathfrak{T}_1$-homeomorphism for every $h \in H$, so the action of H on X is still continuous if H is given its discrete topology and X is given $\mathfrak{T}_1$.)

Now μ is G-invariant. **P** For any $g \in G$, let μ_g be the Radon probability measure defined by setting $\mu_g E = \mu(g\bullet E)$ whenever this is defined. (As in (f) above, this formula does define a probability measure which is Radon for either $\mathfrak{T}$ or $\mathfrak{T}_1$.) Let $f : X \to \mathbb{R}$ be any bounded $\mathfrak{T}_1$-continuous function. Then

$$\int f d\mu_g = \int f(g^{-1}\bullet x)\mu(dx)$$

(applying 235G with $\phi(x) = g^{-1}\bullet x$). Now there is a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in H converging to g. In this case, because G is a topological group, $g^{-1} = \lim_{n\to\infty} h_n^{-1}$. Because the action of G on X is $\mathfrak{T}_1$-continuous, $g^{-1}\bullet x = \lim_{n\to\infty} h_n^{-1}\bullet x$, for $\mathfrak{T}_1$, for every $x \in X$. Because f is $\mathfrak{T}_1$-continuous, $f(g^{-1}\bullet x) = \lim_{n\to\infty} f(h_n^{-1}\bullet x)$ in $\mathbb{R}$ for every $x \in X$. By Lebesgue's Dominated Convergence Theorem,

$$\int f d\mu_g = \int f(g^{-1}\bullet x)\mu(dx) = \lim_{n\to\infty} f(h_n^{-1}\bullet x)\mu(dx) = \lim_{n\to\infty} \int f d\mu_{h_n} = \int f d\mu$$

because μ is H-invariant, so $\mu_{h_n} = \mu$ for every n. As f is arbitrary, $\mu_g = \mu$, by 415I. As g is arbitrary, μ is G-invariant. **Q**

Thus (iii)$\Rightarrow$(i) in all cases, and the proof is complete.

448Q I turn now to Mackey's theorem. I pave the way with a couple of lemmas which are of independent interest.

Lemma Let (X, Σ, μ) be a σ-finite measure space with countable Maharam type. Write $L^0(\Sigma)$ for the set of Σ-measurable functions from X to $\mathbb{R}$. Then there is a function $T : L^0(\mu) \to L^0(\Sigma)$ such that

(α) $u = (Tu)^\bullet$ for every $u \in L^0$,

(β) $(u, x) \mapsto (Tu)(x) : L^0 \times X \to \mathbb{R}$ is $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable,

where $\mathcal{B} = \mathcal{B}(L^0)$ is the Borel σ-algebra of L^0 with its topology of convergence in measure.

proof (a) Consider first the case in which μ is a probability measure.

(i) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ such that the measure algebra $\mathfrak{A}$ of μ is τ-generated by $\{E_n^\bullet : n \in \mathbb{N}\}$. For $n \in \mathbb{N}$ let Σ_n be the finite subalgebra of Σ generated by $\{E_i : i \le n\}$, and for $n \in \mathbb{N}$, $u \in L^\infty = L^\infty(\mu)$ and $x \in X$ set

$$(S_n u)(x) = \frac{1}{\mu E}\int_E u \text{ if } E \text{ is the atom of } \Sigma_n \text{ containing } x \text{ and } \mu E > 0,$$
$$= 0 \text{ if the atom of } \Sigma_n \text{ containing } x \text{ is negligible.}$$

Then $(u, x) \mapsto (S_n u)(x)$ is $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable, because $u \mapsto \int_E u : L^\infty \to \mathbb{R}$ is continuous (for the topology of convergence in measure) for every $E \in \Sigma$. So if we set $Su = \limsup_{n\to\infty} S_n u$ for $u \in L^\infty$, $(u, x) \mapsto (Su)(x)$ will be $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable.

On the other hand, if $f \in \mathcal{L}^\infty$, $S_n f^\bullet$ is a conditional expectation of f on Σ_n for each n. So Lévy's martingale theorem (275I) tells us that if $f \in \mathcal{L}^\infty$ then $\langle S_n f^\bullet\rangle_{n\in\mathbb{N}}$ converges a.e. to a conditional expectation g of f on the σ-algebra Σ_∞ generated by $\bigcup_{n\in\mathbb{N}} \Sigma_n$. But we chose $\langle E_n\rangle_{n\in\mathbb{N}}$ to generate $\mathfrak{A}$, so $\mathfrak{A} = \{E^\bullet : E \in \Sigma_\infty\}$. If now $E \in \Sigma$, there is an $F \in \Sigma_\infty$ such that $E\triangle F$ is negligible, so

$$\int_E g = \int_F g = \int_F f = \int_E f.$$

As E is arbitrary,

$$f =_{\text{a.e.}} g =_{\text{a.e.}} \limsup_{n\to\infty} S_n f^\bullet = S f^\bullet.$$

Turning this round, $(Su)^\bullet = u$ for every $u \in L^\infty$.

(ii) Now define $R : L^0 \to L^\infty$ by setting

$$Rf^\bullet = (\arctan f)^\bullet$$

for $f \in \mathcal{L}^0$ (see 241I). Then R is continuous for the topology of convergence in measure (245Dd), so $(u, x) \mapsto (SRu)(x) : L^0 \times X \to \mathbb{R}$ is $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable. Note that if $u \in L^0$, then $-\frac{\pi}{2} < (S_n Ru)(x) < \frac{\pi}{2}$ for every x and n, so $-\frac{\pi}{2} \le (SRu)(x) \le \frac{\pi}{2}$ for every x; also, if $u = f^\bullet$, then $-\frac{\pi}{2} < \arctan f(x) < \frac{\pi}{2}$ whenever $f(x)$ is defined, so $-\frac{\pi}{2} < (SRu)(x) < \frac{\pi}{2}$ for almost every x. If now we set

$$\tan_0 t = \tan t \text{ for } -\frac{\pi}{2} < t < \frac{\pi}{2},$$
$$= 0 \text{ for } t = \pm\frac{\pi}{2},$$

and $Tu = \tan_0 SRu$, we shall have $Tu \in L^0(\Sigma)$ and $(Tu)^\bullet = u$ for every $u \in L^0$, while $(u, x) \mapsto (Tu)(x)$ is $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable.

(b) For the general case, if $\mu X = 0$ the result is trivial, as we can just set $(Tu)(x) = 0$ for all u and x. So suppose otherwise. Let ν be a probability measure with the same domain and the same negligible sets as μ (215B(vii)). Then the measure algebra of ν, regarded as a Boolean algebra, is the same as that of μ, so ν also has countable Maharam type; similarly, $L^0 = L^0(\nu)$. Moreover, the topology of convergence in measure on L^0 is the same, whichever measure we take to define it (245Xm, 367T). So we can apply (a) to (X, Σ, ν).

448R Lemma Let (X,Σ,μ) be a σ-finite measure space with countable Maharam type.

(a) $L^0 = L^0(\mu)$, with its topology of convergence in measure, is a Polish space.

(b) Let $\mathfrak{A}$ be the measure algebra of μ, and $\mathfrak{A}^f$ the set $\{a : a \in \mathfrak{A}, \bar\mu a < \infty\}$. Then the Borel σ-algebra $\mathcal{B} = \mathcal{B}(L^0)$ is the σ-algebra of subsets of L^0 generated by sets of the form $\{u : \bar\mu(a \cap [\![u \in F]\!]) > \alpha\}$, where $a \in \mathfrak{A}^f$, $F \subseteq \mathbb{R}$ is Borel, and $\alpha \in \mathbb{R}$.

proof (a) By 245Eb, L^0 is metrizable, and complete when regarded as a linear topological space; so by 4A4Bj there is a metric on L^0, defining its topology, under which L^0 is complete. By 367Rb, L^0 is separable, so it is a Polish space.

(b) Write Υ for the σ-algebra of subsets of L^0 generated by sets of the form $\{u : \bar\mu(a \cap [\![u \in F]\!]) > \alpha\}$, where $a \in \mathfrak{A}^f$, $F \subseteq \mathbb{R}$ is Borel, and $\alpha \in \mathbb{R}$.

(i) If $a \in \mathfrak{A}^f$, $\alpha \in \mathbb{R}$ and $H \subseteq \mathbb{R}$ is open, then $U = \{u : \bar\mu(a \cap [\![u \in H]\!]) > \alpha\}$ is open in L^0. **P** If $u \in U$, there are a compact set $K \subseteq H$ and a $\delta > 0$ such that $\bar\mu(a \cap [\![u \in K]\!]) > \alpha + \delta$. Now there is an $\eta \in \,]0,1]$ such that $|\alpha - \beta| > \eta$ whenever $\alpha \in K$ and $\beta \in \mathbb{R} \setminus H$. In this case, $V = \{v : v \in L^0, \bar\mu(a \cap [\![|u-v| > \eta]\!]) \le \delta\}$ is a neighbourhood of u in L^0 (367L). If $v \in V$, then

$$[\![v \in H]\!] \supseteq [\![u \in K]\!] \cap [\![|u-v| \le \eta]\!],$$

$$\begin{aligned}\bar\mu(a \cap [\![v \in H]\!]) &\ge \bar\mu(a \cap [\![u \in K]\!]) - \bar\mu(a \cap [\![|u-v| > \eta]\!])\\ &\ge \bar\mu(a \cap [\![u \in K]\!]) - \delta > \alpha.\end{aligned}$$

Thus $V \subseteq U$ and U is a neighbourhood of u; as u is arbitrary, U is open. **Q**

(ii) Thus $u \mapsto \bar\mu(a \cap [\![u \in H]\!])$ is $\mathcal{B}$-measurable for every $a \in \mathfrak{A}^f$ and open $H \subseteq \mathbb{R}$. Now the set

$$\{F : F \subseteq \mathbb{R} \text{ is Borel, } u \mapsto \bar\mu(a \cap [\![u \in F]\!]) \text{ is } \mathcal{B}\text{-measurable for every } a \in \mathfrak{A}^f\}$$

is a Dynkin class containing all open sets, so is the Borel σ-algebra of $\mathbb{R}$ (136B), and $u \mapsto \bar\mu(a \cap [\![u \in F]\!])$ is $\mathcal{B}$-measurable for every $a \in \mathfrak{A}^f$ and Borel $F \subseteq \mathbb{R}$. Thus $\Upsilon \subseteq \mathcal{B}$.

(iii) In the other direction, we know that $\mathfrak{A}$ is separable, by 331O; let $\langle c_k\rangle_{k\in\mathbb{N}}$ run over a dense subset of $\mathfrak{A}$. We also know that there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}^f$ with supremum 1. Set

$$E_{nkqq'} = \{u : u \in L^0, \bar\mu(a_n \cap c_k \cap [\![u > q]\!]) > q'\} \in \Upsilon$$

for n, $k \in \mathbb{N}$ and q, $q' \in \mathbb{Q}$. If u, $v \in L^0$ are different, there are n, k, q and q' such that $E_{nkqq'}$ contains one of u, v and not the other. **P** Choose $q \in \mathbb{Q}$ such that $[\![u > q]\!] \ne [\![v > q]\!]$. Suppose for the moment that $c = [\![u > q]\!] \setminus [\![v > q]\!] \ne 0$. Let $n \in \mathbb{N}$ be such that $\bar\mu(a_n \cap c) > 0$. Let $k \in \mathbb{N}$ be such that $\bar\mu(a_n \cap (c \,\triangle\, c_k)) < \bar\mu(a_n \cap c)$. Then

$$\begin{aligned}\bar\mu(a_n \cap c_k \cap [\![v > q]\!]) &\le \bar\mu(a_n \cap c_k \setminus c)\\ &< \bar\mu(a_n \cap c) - \bar\mu(a_n \cap c \setminus c_k) \le \bar\mu(a_n \cap c_k \cap [\![u > q]\!]),\end{aligned}$$

so there is a $q' \in \mathbb{Q}$ such that $u \in E_{nkqq'}$ and $v \notin E_{nkqq'}$. Similarly, if $[\![v > q]\!] \not\subseteq [\![u > q]\!]$ there are n, $k \in \mathbb{N}$ and $q' \in \mathbb{Q}$ such that $v \in E_{nkqq'}$ and $u \notin E_{nkqq'}$. **Q**

By 423S, the σ-algebra generated by $\{E_{nkqq'} : n, k \in \mathbb{N}, q, q' \in \mathbb{Q}\}$ is the whole of $\mathcal{B}$, and Υ must be equal to $\mathcal{B}$, as claimed.

448S Mackey's theorem (Mackey 62) Let G be a locally compact Polish group, (X,Σ) a standard Borel space and μ a σ-finite measure with domain Σ. Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ with its measure-algebra topology. Let $\circ$ be a Borel measurable action of G on $\mathfrak{A}$ such that $a \mapsto g{\circ}a$ is a Boolean automorphism for every $g \in G$. Then we have a $(\mathcal{B}(G)\widehat{\otimes}\Sigma,\Sigma)$-measurable action $\bullet$ of G on X such that

$$g{\circ}E^\bullet = (g{\bullet}E)^\bullet$$

for every $g \in G$ and $E \in \Sigma$, writing $g{\bullet}E$ for $\{g{\bullet}x : x \in E\}$ as usual.

proof (a) To begin with (down to the end of (j) below) suppose that $X = \mathbb{R}$, with $\Sigma = \mathcal{B}(\mathbb{R})$ its Borel σ-algebra, and that μ is totally finite. The first thing to note is that for every $g \in G$ the automorphism $a \mapsto g{\circ}a$ can be represented by a Borel automorphism $f_g : \mathbb{R} \to \mathbb{R}$ such that $g{\circ}E^\bullet = f_g^{-1}[E]^\bullet$ for every $E \in \mathcal{B}(\mathbb{R})$ (425Ac). Of course f_g belongs to the space $L^0(\Sigma)$ of Σ-measurable functions from $\mathbb{R}$ to itself, so we can speak of its equivalence class $f_g^\bullet \in L^0(\mu)$. If we give $L^0(\mu)$ its topology of convergence in measure, it is a Polish space (448Ra).

The function $g \mapsto f_g^\bullet : G \to L^0(\mu)$ is Borel measurable. **P** If $E \in \mathcal{B}(\mathbb{R})$, $a \in \mathfrak{A}$ and $\alpha \in \mathbb{R}$, then, setting $b = E^\bullet$,

$$[\![f_g^\bullet \in E]\!] = f_g^{-1}[E]^\bullet = g \circ b$$

for every $g \in G$, so

$$\{g : \bar{\mu}(a \cap [\![f_g^\bullet \in E]\!]) > \alpha\} = \{g : \bar{\mu}(a \cap (g \circ b)) > \alpha\}$$

is a Borel set in G, because $\circ$ is Borel measurable and $\{c : \bar{\mu}(a \cap c) > \alpha\}$ is open in $\mathfrak{A}$. Thus $\mathcal{B}(G)$ contains the inverse images of the sets generating $\mathcal{B}(L^0(\mu))$ described in 448Rb, and therefore the inverse image of every set in $\mathcal{B}(L^0(\mu))$, as required. **Q**

(b) By 448Q, there is a function $T : L^0(\mu) \to L^0(\Sigma)$ such that $(Tu)^\bullet = u$ for every $u \in L^0(\mu)$ and $(u, x) \mapsto (Tu)(x)$ is $(\mathcal{B}(L^0(\mu)) \widehat{\otimes} \mathcal{B}(\mathbb{R}))$-measurable. Define $\phi : G \times \mathbb{R} \to \mathbb{R}$ by setting

$$\phi(g, x) = (Tf_g^\bullet)(x)$$

for $g \in G$ and $x \in \mathbb{R}$. Then ϕ is a composition of the Borel measurable functions $(g, x) \mapsto (f_g^\bullet, x)$ and $(u, x) \mapsto (Tu)(x)$, so is Borel measurable; and if $g \in G$ then $\phi(g, x) = f_g(x)$ for μ-almost every x, because $f_g =_{\text{a.e.}} Tf_g^\bullet$. Now

$$g \circ E^\bullet = (f_g^{-1}[E])^\bullet = \{x : \phi(g, x) \in E\}^\bullet$$

for every $g \in G$ and $E \in \mathcal{B}(\mathbb{R})$.

(c) Let λ be a Haar measure on G. Because G is a Polish space and λ is a Radon measure on G, λ is σ-finite (411Ge) and $L^0(\lambda)$, with its topology of convergence in measure, is a Polish space (448Ra again). For $x \in \mathbb{R}$, set $\phi_x(g) = \phi(g, x)$ for $g \in G$; then $\phi_x : G \to \mathbb{R}$ is Borel measurable. Set $\theta(x) = \phi_x^\bullet$ in $L^0(\lambda)$. Then $\theta : \mathbb{R} \to L^0(\lambda)$ is Borel measurable. **P** Again I use the characterization of the Borel σ-algebra of $L^0(\lambda)$ in 448Rb. Let $(\mathfrak{C}, \bar{\lambda})$ be the measure algebra of λ. If $c \in \mathfrak{C}$, $\bar{\lambda}c < \infty$, $E \subseteq \mathbb{R}$ is Borel, and $\alpha \in \mathbb{R}$, take a Borel set $F \subseteq G$ such that $c = F^\bullet$; then

$$\begin{aligned} \{x : \bar{\lambda}(c \cap [\![\theta(x) \in E]\!]) > \alpha\} &= \{x : \lambda(F \cap \phi_x^{-1}[E]) > \alpha\} \\ &= \{x : \lambda\{g : g \in F,\, \phi_x(g) \in E\} > \alpha\} \\ &= \{x : \lambda\{g : g \in F,\, \phi(g, x) \in E\} > \alpha\} \\ &= \{x : \lambda W^{-1}[\{x\}] > \alpha\} \end{aligned}$$

where $W = \{(g, x) : g \in F,\, x \in \mathbb{R},\, \phi(g, x) \in E\}$ is a Borel subset of $G \times \mathbb{R}$. But this means that $W \in \mathcal{B}(G) \widehat{\otimes} \mathcal{B}(\mathbb{R})$ (4A3Ga) and $x \mapsto \lambda W^{-1}[\{x\}]$ is Borel measurable (252P), so $\{x : \bar{\lambda}(c \cap [\![\theta(x) \in E]\!]) > \alpha\}$ is a Borel subset of $\mathbb{R}$. Thus the inverse image of every set in the generating family for the Borel σ-algebra of $L^0(\lambda)$ is a Borel set, and we have a Borel measurable function. **Q**

Let ν be the totally finite Borel measure on $L^0(\lambda)$ defined by setting $\nu F = \mu\theta^{-1}[F]$ for every Borel set $F \subseteq L^0(\lambda)$.

(d) If $E \subseteq \mathbb{R}$ is Borel, there is a set $A \subseteq L^0(\lambda)$ such that $E \triangle \theta^{-1}[A]$ is μ-negligible. **P** Let $\hat{\mu}$, $\hat{\nu}$ be the completions of μ and ν, so that θ is inverse-measure-preserving for $\hat{\mu}$ and $\hat{\nu}$ (234Ba). Because E is a Borel subset of $\mathbb{R}$, $\theta[E]$ is an analytic subset of $L^0(\lambda)$ (423Gb), therefore Souslin-F (423Eb); accordingly $\hat{\nu}$ measures $\theta[E]$ (431B). Let T_0 be the σ-algebra of subsets of $L^0(\lambda)$ generated by the Souslin-F subsets of $L^0(\lambda)$. By 423O, there is a T_0-measurable function $\theta' : \theta[E] \to E$ such that $\theta\theta'$ is the identity on $\theta[E]$. Now T_0 is included in the domain $\hat{\mathrm{T}}$ of $\hat{\nu}$, so θ' is $\hat{\mathrm{T}}$-measurable, and there is a Borel set $F_0 \subseteq \theta[E]$ such that $\theta' \restriction F_0$ is Borel measurable and $\theta[E] \setminus F_0$ is $\hat{\nu}$-negligible (212Fa). Since θ' is surely injective, $E_0 = \theta'[F_0]$ is a Borel subset of E (423Ib) and $\theta \restriction E_0$ is a bijection from E_0 to F_0 with inverse θ'. Note that

$$\mu(E \setminus \theta^{-1}[F_0]) \le \hat{\mu}(\theta^{-1}[\theta[E]] \setminus \theta^{-1}[F_0]) = \hat{\nu}(\theta[E] \setminus F_0) = 0.$$

Define $\psi : \mathbb{R} \to \mathbb{R}$ by setting

$$\begin{aligned} \psi(x) &= \theta'\theta(x) \text{ if } x \in \theta^{-1}[F_0], \\ &= x \text{ otherwise.} \end{aligned}$$

Then ψ is a Borel measurable function and $\theta\psi = \theta$, that is, $\phi_x^\bullet = \phi_{\psi(x)}^\bullet$ for every $x \in \mathbb{R}$. Consequently

$$\{(g, x) : g \in G,\, x \in \mathbb{R},\, \phi(g, x) \ne \phi(g, \psi(x))\}$$

has λ-negligible horizontal sections. Since it is a Borel set, it must have many μ-negligible vertical sections; let $g_0 \in G$ be such that $\{x : \phi(g_0, x) \ne \phi(g_0, \psi(x))\}$ is μ-negligible. By (b), we also have $\phi(g_0, x) = f_{g_0}(x)$ for μ-almost every x. So the Borel set $H = \{x : f_{g_0}(x) = \phi(g_0, x) = \phi(g_0, \psi(x))\}$ is μ-conegligible.

Set $A=\theta[E\cap\theta^{-1}[F_0]\cap H]$. Of course $A\subseteq F_0$. If $x\in\theta^{-1}[A]\setminus E$, then there is a $y\in E\cap\theta^{-1}[F_0]\cap H$ such that $\theta(y)=\theta(x)$; now $\psi(y)=\psi(x)$ and $x\ne y$, so

$$\phi(g_0,\psi(x))=\phi(g_0,\psi(y))=f_{g_0}(y)\ne f_{g_0}(x)$$

and $x\notin H$. Thus $\theta^{-1}[A]\setminus E\subseteq\mathbb{R}\setminus H$ is μ-negligible. On the other hand, $E\setminus\theta^{-1}[A]\subseteq E\setminus(\theta^{-1}[F_0]\cap H)$ is also μ-negligible. So $E\triangle\theta^{-1}[A]$ is μ-negligible, as required. **Q**

(e) There is a μ-conegligible Borel set $H\subseteq\mathbb{R}$ such that $\theta\restriction H$ is injective. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence of Borel sets in $\mathbb{R}$ such that whenever x, $y\in\mathbb{R}$ are distinct there is an n such that $x\in E_n$ and $y\notin E_n$. For each $n\in\mathbb{N}$ let $A_n\subseteq L^0(\lambda)$ be such that $E_n\triangle\theta^{-1}[A_n]$ is μ-negligible; let H be a μ-conegligible Borel set disjoint from $\bigcup_{n\in\mathbb{N}}(E_n\triangle\theta^{-1}[A_n])$. If x, $y\in H$ are distinct, there is an $n\in\mathbb{N}$ such that $x\in E_n$ and $y\notin E_n$; now $x\in\theta^{-1}[A_n]$ and $y\notin\theta^{-1}[A_n]$, so $\theta(x)\ne\theta(y)$. **Q**

(f) Of course $\theta[H]$ is now a Borel subset of $L^0(\lambda)$, and must be $\hat\nu$-conegligible. Let $\mathfrak{B}$ be the measure algebra of ν, and $\pi:\mathfrak{B}\to\mathfrak{A}$ the measure-preserving homomorphism defined by setting $\pi F^\bullet=\theta^{-1}[F]^\bullet$ for every Borel set F. If $E\subseteq\mathbb{R}$ is Borel, then $E^\bullet=\pi(\theta[E\cap H])^\bullet$ belongs to $\pi[\mathfrak{B}]$, so π is surjective and is an isomorphism.

(g) Recall that we have a continuous action $\bullet_l$ of G on $L^0(\lambda)$ defined as in 443G. If $g\in G$, then

$$g\bullet_l\theta(x)=\theta(\phi(g^{-1},x))=\theta(f_{g^{-1}}(x))$$

for μ-almost every $x\in\mathbb{R}$. **P** Consider the set $\{(h,x):\phi(g^{-1}h,x)=\phi(h,\phi(g^{-1},x))\}\subseteq G\times\mathbb{R}$. Because $h\mapsto g^{-1}h$ is continuous, it is Borel measurable, so $(h,x)\mapsto\phi(g^{-1}h,x)$ is Borel measurable; the same is true of $(h,x)\mapsto\phi(h,\phi(g^{-1},x))$, so $\{(h,x):\phi(g^{-1}h,x)=\phi(h,\phi(g^{-1},x))\}$ is a Borel set. For given $h\in G$ and $E\in\mathcal{B}(\mathbb{R})$, set $F=\{x:\phi(h,x)\in E\}$; then

$$\begin{aligned}\{x:\phi(g^{-1}h,x)\in E\}^\bullet&=(g^{-1}h)\circ E^\bullet=g^{-1}\circ(h\circ E^\bullet)=g^{-1}\circ\{x:\phi(h,x)\in E\}^\bullet\\&=g^{-1}\circ F^\bullet=\{x:\phi(g^{-1},x)\in F\}^\bullet=\{x:\phi(h,\phi(g^{-1},x))\in E\}^\bullet\end{aligned}$$

so $\{x:\phi(g^{-1}h,x)\in E\}\triangle\{x:\phi(h,\phi(g^{-1},x))\in E\}$ is μ-negligible. As E is arbitrary, $\phi(g^{-1}h,x)=\phi(h,\phi(g^{-1},x))$ for μ-almost every x.

This is true for every $h\in G$. So there is a μ-conegligible Borel set $H'\subseteq\mathbb{R}$ such that if $x\in H'$ then $\phi(g^{-1}h,x)=\phi(h,\phi(g^{-1},x))$ for λ-almost every h. But this means that if $x\in H'$ then

$$(g\bullet_l\phi_x)(h)=\phi_x(g^{-1}h)=\phi(g^{-1}h,x)=\phi(h,\phi(g^{-1},x))=\phi_{\phi(g^{-1},x)}(h)$$

for λ-almost every h, and

$$g\bullet_l\theta(x)=g\bullet_l\phi_x^\bullet=(g\bullet_l\phi_x)^\bullet=\phi^\bullet_{\phi(g^{-1},x)}=\theta(\phi(g^{-1},x)).$$

Thus $g\bullet_l\theta(x)=\theta(\phi(g^{-1},x))$ for almost every x. And of course we already know from (b) that $\phi(g^{-1},x)=f_{g^{-1}}(x)$ for almost every x. **Q**

(h) We have a function $\circ_l:G\times\mathfrak{B}\to\mathfrak{B}$ defined by setting

$$g\circ_l F^\bullet=(g\bullet_l F)^\bullet$$

for every Borel set $F\subseteq L^0(\lambda)$ and $g\in G$, writing $g\bullet_l F=\{g\bullet_l u:u\in F\}$ as in 4A5Bc. **P** Take any $g\in G$. By (g) just above, applied to g^{-1}, $g^{-1}\bullet_l\theta(x)=\theta(f_g(x))$ for μ-almost every x. Because the shift operator $u\mapsto g\bullet_l u:L^0(\lambda)\to L^0(\lambda)$ is a homeomorphism, it is a Borel automorphism, and $g\bullet_l F$ is a Borel set for every Borel set $F\subseteq L^0(\lambda)$. If $\nu F=0$, then

$$\begin{aligned}\nu(g\bullet_l F)&=\mu\{x:\theta(x)\in g\bullet_l F\}=\mu\{x:g^{-1}\bullet_l\theta(x)\in F\}\\&=\mu\{x:\theta(f_g(x))\in F\}=\mu(f_g^{-1}[\theta^{-1}[F]])=0\end{aligned}$$

because $\theta^{-1}[F]$ is μ-negligible and f_g represents an automorphism of the measure algebra $\mathfrak{A}$ of μ. It follows that $(g\bullet_l F_0)\triangle(g\bullet_l F_1)=g\bullet_l(F_0\triangle F_1)$ is ν-negligible and $(g\bullet_l F_0)^\bullet=(g\bullet_l F_1)^\bullet$ whenever $F_0^\bullet=F_1^\bullet$, which is what we need to know. **Q**

(i) For any $b\in\mathfrak{B}$ and $g\in G$, $g\circ\pi b=\pi(g\circ_l b)$. **P** Let $F\subseteq L^0(\lambda)$ be a Borel set such that $b=F^\bullet$, and set $E=\theta^{-1}[F]$, $a=E^\bullet=\pi b$. Then $g\circ_l b=(g\bullet_l F)^\bullet$, so

$$\begin{aligned}\pi(g\circ_l b)&=\{x:\theta(x)\in g\bullet_l F\}^\bullet=\{x:g^{-1}\bullet_l\theta(x)\in F\}^\bullet=\{x:\theta(f_g(x))\in F\}^\bullet\\&=\{x:f_g(x)\in E\}^\bullet=(f_g^{-1}[E])^\bullet=g\circ a=g\circ\pi b,\end{aligned}$$

as required. **Q**

(j) Now observe that because λ is a Haar measure, $\lambda G > 0$, so $L^0(\lambda) \neq \{0\}$, $L^0(\lambda)$ is uncountable and $\#(L^0(\lambda)) = \mathfrak{c} = \#(\mathbb{R})$ (423K). By 425Ad, there is a Borel isomorphism $\tilde{\theta} : \mathbb{R} \to L^0(\lambda)$ which represents π. Set

$$g\bullet x = \tilde{\theta}^{-1}(g\bullet_l \tilde{\theta}(x))$$

for $g \in G$ and $x \in \mathbb{R}$. Then $\bullet : G \times \mathbb{R} \to \mathbb{R}$ is a composition of the Borel measurable functions $(g, x) \mapsto (g, \tilde{\theta}(x))$, $(g, u) \mapsto g\bullet_l u$ and $u \mapsto \tilde{\theta}^{-1}(u)$, so is Borel measurable. Because $\Sigma = \mathcal{B}(\mathbb{R})$ and $\mathcal{B}(G \times \mathbb{R}) = \mathcal{B}(G)\widehat{\otimes}\mathcal{B}(\mathbb{R})$ (4A3Ga again), $\bullet$ is $(\mathcal{B}(G)\widehat{\otimes}\Sigma, \Sigma)$-measurable. If g, $h \in G$ and $x \in \mathbb{R}$,

$$gh\bullet x = \tilde{\theta}^{-1}(gh\bullet_l \tilde{\theta}(x)) = \tilde{\theta}^{-1}(g\bullet_l (h\bullet_l \tilde{\theta}(x))) = \tilde{\theta}^{-1}(g\bullet_l \tilde{\theta}(h\bullet x)) = g\bullet(h\bullet x),$$

and if e is the identity of G and $x \in \mathbb{R}$,

$$e\bullet x = \tilde{\theta}^{-1}(e\bullet_l \tilde{\theta}(x)) = \tilde{\theta}^{-1}\tilde{\theta}(x) = x.$$

Thus $\bullet$ is an action of G on $\mathbb{R}$. If $g \in G$ and $E \in \mathcal{B}(\mathbb{R})$, set $F = \tilde{\theta}[E]$. Then $F^\bullet = \pi^{-1}[E^\bullet]$, so

$$(\tilde{\theta}^{-1}[g\bullet_l F])^\bullet = \pi((g\bullet_l F)^\bullet) = \pi(g\circ_l F^\bullet) = g\circ \pi F^\bullet = g\circ E^\bullet.$$

As

$$g\bullet E = \{g\bullet x : x \in E\} = \{\tilde{\theta}^{-1}(g\bullet_l \tilde{\theta}(x)) : x \in E\} = \{\tilde{\theta}^{-1}(g\bullet_l u) : u \in F\} = \tilde{\theta}^{-1}[g\bullet_l F],$$

we see that

$$g\circ E^\bullet = (g\bullet E)^\bullet$$

as required in the statement of this theorem.

(k) Thus the result is true if $(X, \Sigma) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and μ is totally finite. As for non-totally-finite μ, there will always be a totally finite measure μ_1 with the same domain and the same null ideal (215B again), in which case the measure algebra of μ_1 will have the same Boolean algebra $\mathfrak{A}$, though with a different measure $\bar{\mu}_1$. However the measure-algebra topology of $\mathfrak{A}$ is unchanged (324H), so $\circ$ is still Borel measurable, and we can use the Borel measurable action of G on $\mathbb{R}$ found by the method of (a)-(j) above. Since we are assuming that (X, Σ) is a standard Borel space, this covers all the cases in which X is uncountable, by 424C-424D.

(l) We are left with the case of countable X. This is of course essentially trivial. $\Sigma = \mathcal{P}X$ and μ is a point-supported measure. Let Y be the set of atoms of μ, that is, the set $\{x : \mu\{x\} > 0\}$. Then we can identify the measure algebra $\mathfrak{A} = \mathcal{P}X/\mathcal{P}(X \setminus Y)$ with $\mathcal{P}Y$, in which case the equivalence class $E^\bullet$ of any $E \subseteq X$ becomes identified with $E \cap Y$. As in part (a) of the proof above, we can represent each automorphism $a \mapsto g\circ a : \mathfrak{A} \to \mathfrak{A}$ by a permutation $f_g : X \to X$, and we must have $f_g^{-1}[Y] = Y$. Try

$$\begin{aligned} g\bullet x &= f_g^{-1}(x) \text{ if } x \in Y, \\ &= x \text{ if } x \in X \setminus Y \end{aligned}$$

for every $g \in G$. If g, $h \in G$ and $x \in Y$,

$$\begin{aligned} \{gh\bullet x\} &= \{f_{gh}^{-1}(x)\} = gh\circ\{x\} = g\circ(h\circ\{x\}) = g\circ f_h^{-1}[\{x\}] \\ &= f_g^{-1}[f_h^{-1}[\{x\}]] = f_g^{-1}[\{f_h^{-1}(x)\}] = f_g^{-1}\{h\bullet x\} = \{g\bullet(h\bullet x)\} \end{aligned}$$

and $gh\bullet x = g\bullet(h\bullet x)$; if $x \in X \setminus Y$, then $gh\bullet x = x = g\bullet(h\bullet x)$. Of course $e\bullet x = x$ for every $x \in X$. So $\bullet$ is an action of G on X. To see that it is $(\mathcal{B}(G)\widehat{\otimes}\Sigma, \Sigma)$-measurable, note that the measure-algebra topology of $\mathfrak{A} \cong \mathcal{P}Y$ is the discrete topology. If $y \in Y$, then $\{g : g\bullet y = z\} = \{g : g\circ\{y\} = \{z\}\}$ is a Borel set for every $z \in X$; if $x \in X \setminus Y$, then $\{g : g\bullet x = z\}$ is either G or $\emptyset$ for every $z \in X$. So

$$\{(g, x) : g\bullet x \in W\} = \bigcup_{z \in W, x \in X} \{g : g\bullet x = z\} \times \{x\}$$

belongs to $\mathcal{B}(G)\widehat{\otimes}\Sigma$ for every subset W of X, and $\bullet$ is $(\mathcal{B}(G)\widehat{\otimes}\Sigma, \Sigma)$-measurable. Finally, if $g \in G$,

$$(g\bullet E)^\bullet = (g\bullet E) \cap Y = Y \cap f_g^{-1}[E] = (f_g^{-1}[E])^\bullet = g\circ E^\bullet$$

for every $E \subseteq X$. So again we have a suitable action of G on X.

This completes the proof.

448T Corollary Let G be a σ-compact locally compact Hausdorff group, X a Polish space, μ a σ-finite Borel measure on X, and $(\mathfrak{A},\bar\mu)$ the measure algebra of μ, with its measure-algebra topology. Let $\circ$ be a continuous action of G on $\mathfrak{A}$ such that $a\mapsto g{\circ}a$ is a Boolean automorphism for every $g\in G$. Then we have a Borel measurable action $\bullet$ of G on X such that

$$g{\circ}E^\bullet=(g{\bullet}E)^\bullet$$

for every $g\in G$ and $E\in\mathcal{B}(X)$.

proof We know that $\mathfrak{A}$ is separable (331O again) and metrizable (323Gb); let $\langle a_n\rangle_{n\in\mathbb{N}}$ run over a topologically dense subset of $\mathfrak{A}$ and $\langle U_n\rangle_{n\in\mathbb{N}}$ over a base for its topology. For each (m,n) such that $a_m\in U_n$, $V_{mn}=\{g:g{\circ}a_m\in U_n\}$ is a neighbourhood of the identity e of G. By 4A5S, there is a compact normal subgroup H of G such that $H\subseteq\bigcap\{V_{mn}:m,\,n\in\mathbb{N},\,a_m\in U_n\}$ and G/H is Polish. Now we have a continuous action $\bar\circ$ of G/H on $\mathfrak{A}$ such that $g^\bullet\bar\circ a=g{\circ}a$ for every $g\in G$ and $a\in\mathfrak{A}$. $\mathbf{P}$ If $g,\,h\in G$ are such that $g^\bullet=h^\bullet$, $m\in\mathbb{N}$, and $a_m\in U_n$, then $g^{-1}h\in V_{mn}$ so $g^{-1}h{\circ}a_m\in U_n$. As n is arbitrary, $g^{-1}h{\circ}a_m=a_m$; as $a\mapsto g^{-1}h{\circ}a$ is continuous, and m is arbitrary, $g^{-1}h{\circ}a=a$ and $h{\circ}a=g{\circ}a$ for every $a\in\mathfrak{A}$. This shows that the given formula defines a function $\bar\circ$ from $(G/H)\times\mathfrak{A}$ to $\mathfrak{A}$. It is easy to check that $\bar\circ$ is an action of G/H on $\mathfrak{A}$.

Now suppose that $v\in G/H$, $a\in\mathfrak{A}$ and U is a neighbourhood of $v\bar\circ a$ in $\mathfrak{A}$. Let $g\in G$ be such that $g^\bullet=v$; then $g{\circ}a=v\bar\circ a$, so there are open sets $V\subseteq G$ and $U'\subseteq\mathfrak{A}$ such that $g\in V$, $a\in U'$ and $h{\circ}b\in U$ whenever $h\in V$ and $b\in U'$. By 4A5Ja, $W=\{h^\bullet:h\in V\}$ is open in G/H; now $v\in W$ and $w\bar\circ b\in U$ whenever $w\in W$ and $b\in U'$. As v, a and U are arbitrary, $\bar\circ$ is continuous. $\mathbf{Q}$

There is therefore a Borel measurable action $\bar\bullet:(G/H)\times X\to X$ such that $v\bar\circ E^\bullet=(v\bar\bullet E)^\bullet$ whenever $v\in G/H$ and $E\in\mathcal{B}(X)$ (448S). Set $g{\bullet}x=g^\bullet\bar\bullet x$ for $g\in G$ and $x\in X$. It is elementary to check that $\bullet$ is an action of G on X. Also it is Borel measurable, because $(g,x)\mapsto(g^\bullet,x)$ is continuous, therefore Borel measurable, and $(g^\bullet,x)\mapsto g^\bullet\bar\bullet x$ is Borel measurable. If $g\in G$ and $E\in\mathcal{B}(X)$, then

$$g{\circ}E^\bullet=g^\bullet\bar\circ E^\bullet=(g^\bullet\bar\bullet E)^\bullet=(g{\bullet}E)^\bullet,$$

so $\bullet$ is an action of the kind we seek.

448X Basic exercises (a) Show that the results in 448Fb and 448Fd remain true if G is not assumed to be countable.

(b) In part (c) of the proof of 448O, show that $\mathcal{I}$ is just the set of those $d\in\mathfrak{A}$ such that $d\subseteq\operatorname{upr}(1\setminus a,\mathfrak{C})$ for some a such that $1\preccurlyeq^\sigma_G a$.

(c) Show that, in part (c) of the proof of 448P, we can if we wish take $Z=X$ and $\Sigma=\mathcal{B}$.

>(d) Let (X,Σ) be a standard Borel space and Σ_0 a countable subalgebra of Σ. Show that there is a sequence $\langle\langle E_{ni}\rangle_{i\in\mathbb{N}}\rangle_{n\in\mathbb{N}}$ of partitions of unity in Σ such that whenever $\nu:\Sigma\to\mathbb{R}$ is a finitely additive functional and $\nu X=\sum_{i=0}^\infty\nu E_{ni}$ for every $n\in\mathbb{N}$, then $\nu\restriction\Sigma_0$ is countably additive.

>(e) Set $X=[0,1]\setminus\mathbb{Q}$, $G=\mathbb{Q}$ and define $\bullet:G\times X\to X$ by requiring that $g{\bullet}x-g-x\in\mathbb{Z}$ for $g\in G$ and $x\in X$. Show that this is a Borel measurable action and that Lebesgue measure on X is G-invariant. Find a metric on X, inducing its topology, for which all the maps $x\mapsto g{\bullet}x$ are isometries.

>(f) Show that a Polish group carries Haar measures iff it is locally compact. (*Hint*: 443E.)

(g) Give $\mathbb{Z}^{\mathbb{N}}$ its usual (product) topology and abelian group structure. Show that it is a Polish group, and has no Haar measure.

>(h) Let (X,ρ) be a metric space, G a group and $\bullet$ an action of G on X such that $x\mapsto g{\bullet}x$ is an isometry for every $g\in G$. (i) Show that if μ is a G-invariant quasi-Radon probability measure on X then $\{g{\bullet}x:g\in G\}$ is totally bounded for every x in the support of μ. (ii) Show that if the action is transitive and there is a non-zero G-invariant quasi-Radon measure on X, then X is covered by totally bounded open sets. (iii) Suppose that X has measure-free weight (see §438; for instance, X could be separable). Show that if the action is transitive and there is a G-invariant topological probability measure on X then X is totally bounded.

(i) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, with the operation $\triangle$ and the measure-algebra topology. (i) Show that $\mathfrak{A}$ is a topological group. (ii) Show that if $\bar\mu$ is σ-finite and $\mathfrak{A}$ has countable Maharam type, it is a Polish group. (iii) Show that if $(\mathfrak{A},\bar\mu)$ is semi-finite and not purely atomic, then $\mathfrak{A}$ has no Haar measure.

(j) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$, with its measure metric, and $G=\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ the group of measure-preserving automorphisms on $\mathfrak{A}$. (i) Show that if we give G the topology induced by the topology of pointwise convergence on the isometry group of $\mathfrak{A}$, then it is a Polish group. (*Hint*: 441Xp(iv).) (ii) Show that if ν is a G-invariant topological probability measure on $\mathfrak{A}$, then $\nu\{0,1\}=1$.

448Y Further exercises (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$; let G^*_σ be the countably full local semigroup generated by G, and write H for the union of all the full local semigroups generated by countable subgroups of G (following the definition in 395A as written, without troubling about whether $\mathfrak{A}$ is Dedekind complete). (i) Show that $G^*_\sigma\subseteq H$. (ii) Find an example in which $H\ne G^*_\sigma$. (iii) Show that if $\mathfrak{A}$ is Dedekind complete then $G^*_\sigma=H$. (iv) Show that if $\mathfrak{A}$ is ccc then $G^*_\sigma=H$ is the full local semigroup generated by G.

(b) In 448N, show that θ is uniquely defined.

(c) Let (X,Σ) be a standard Borel space, Y any set, T a σ-algebra of subsets of Y and $\mathcal{J}$ a σ-ideal of subsets of T. Let $\theta:\Sigma\to L^\infty(\mathrm{T}/\mathcal{J})$ be a non-negative, sequentially order-continuous additive function. Show that there is a non-negative, sequentially order-continuous additive function $\phi:\Sigma\to L^\infty(\mathrm{T})$ such that (identifying $L^\infty(\mathrm{T}/\mathcal{J})$ with a quotient space of $L^\infty(\mathrm{T})$) $\theta E=(\phi E)^\bullet$ for every $E\in\Sigma$.

448 Notes and comments The keys to the first part of the section are in 448F, 448G and 448L. Even though we no longer have a Dedekind complete algebra, the fact that we are working with countable groups means that the suprema we actually need are defined. The final step, however, uses yet another idea. In a standard Borel space, given a finitely additive functional on the σ-algebra, we can sometimes confirm an adequate approximation to countable additivity by looking at only countably many sequences (448Xd). This enables us to pass from a G-invariant map $\theta:\mathfrak{A}\to L^\infty(\mathfrak{D})$ to a G-invariant Radon measure (parts (d)-(f) of the proof of 448P), without needing to know anything about the algebra $\mathfrak{D}$ except that it is Dedekind σ-complete. In particular (and in contrast to the corresponding step in 395P) we do not need to suppose that $\mathfrak{D}$ is a measurable algebra. I do not know whether there is a useful ergodicity condition which could be added to the hypotheses of 448O to ensure that $\mathfrak{D}$ there becomes $\{0,1\}$.

448P was proved in the case $G=\mathbb{Z}$ by Nadkarni 90; the extension to general Borel actions by Polish groups is due to Becker & Kechris 96. (See Nadkarni 90 for notes on the history of the problem, and Kechris 95 for the basic general theory of Polish groups and Borel actions.) It is a remarkable result, but its application is limited by the difficulty of determining whether either condition (ii) or condition (iii) is satisfied. Much commoner situations are those like 448Xe-448Xj, where either there is no invariant measure or we can find one easily.

The second main theorem makes no reference to the first. But it has something in common. It is an example of the power of descriptive set theory to dramatically extend a result on group actions, which is comparatively straightforward when the group in question is $\mathbb{Z}$, to arbitrary Polish groups. Nadkarni's theorem is not obvious, but it is a lot easier than the general result here. Mackey's theorem for countable groups also requires a little care, but is essentially covered (in usefully greater generality) by 344C. The descriptive set theory the theorem here relies on does not go as deep as the Becker-Kechris theorem, but in exchange it calls on a kind of lifting theorem quite different from those in Chapter 34. Looked at from the standpoint of Chapter 34, 448Q is a rank impossibility (see 341Xg); but the point is that we have abandoned the ordinary algebraic requirements on a lifting and replaced them by a strong measurability property.

Of course a lifting was used in 344C as well, but in a quite different way. There the hypotheses were adjusted to give a slightly more general context in which we could be sure that individual homomorphisms from the measure algebra to itself were representable by functions from the measure space to itself; and I relied indirectly on the lifting theorem 341K to set up the functions. For the context of the present section, this step was done in 425A with no mention of liftings, but using the classification of standard Borel spaces in 424D. In view of 424Yf, it is plain that we do not get much extra generality by using the argument through 341K. The real difference in 344B-344C is that we can deal with semigroups of homomorphisms as well as groups of automorphisms.

The proof of Mackey's theorem is based on there being a Haar measure on G, so that we can use Fubini's theorem (three times, in parts (c), (d) and (g) of the proof). There are non-locally-compact groups G for which a corresponding result is true (Kwiatowska & Solecki 11); it remains quite unclear when to expect this.

449 Amenable groups

I end this chapter with a brief introduction to 'amenable' topological groups. I start with the definition (449A) and straightforward results assuring us that there are many amenable groups (449C). At a slightly deeper level we have a condition for a group to be amenable in terms of a universal object constructible from the group, not invoking 'all compact Hausdorff spaces' (449E). I give some notes on amenable locally compact groups, concentrating on a long list of properties equivalent to amenability (449J), and a version of Tarski's theorem characterizing amenable discrete groups (449M). I end with Banach's theorem on extending Lebesgue measure in one and two dimensions.

449A Definition A topological group G is **amenable** if whenever X is a non-empty compact Hausdorff space and $\bullet$ is a continuous action of G on X, then there is a G-invariant Radon probability measure on X.

Warning: other definitions have been used, commonly based on conditions equivalent to amenability for locally compact Hausdorff groups, such as those listed in 449J(ii)-449J(xiv). In addition, many authors use the phrase 'amenable group' to mean a group which is amenable in its discrete topology. The danger of this to the non-specialist is that many theorems concerning amenable discrete groups do not generalize in the ways one might expect.

449B Lemma Let G be a topological group, X a locally compact Hausdorff space, and $\bullet$ a continuous action of G on X.

(a) Writing C_0 for the Banach space of continuous real-valued functions on X vanishing at ∞ (436I), the map $a \mapsto a^{-1}\bullet f : C_0 \to C_0$ (definition: 4A5Cc) is uniformly continuous for the right uniformity on G and the norm uniformity of C_0, for any $f \in C_0$.

(b) If μ is a G-invariant Radon measure on X and $1 \le p < \infty$, then $a \mapsto a^{-1}\bullet u : G \to L^p$ (definition: 441Kc) is uniformly continuous for the right uniformity on G and the norm uniformity of $L^p = L^p(\mu)$, for any $u \in L^p$.

proof (a)(i) Note first that if $a \in G$ and $f \in C_0$, then $x \mapsto a\bullet x : X \to X$ is a homeomorphism (4A5Bd), so $x \mapsto f(a\bullet x)$ belongs to C_0; but this is just the function $a^{-1}\bullet f$.

(ii) For any $\epsilon > 0$ and $f \in C_0$ there is a neighbourhood V of the identity e of G such that $\|f - a^{-1}\bullet_l f\|_\infty \le \epsilon$ for every $a \in V$. **P?** Suppose, if possible, otherwise. For each symmetric neighbourhood V of e set

$$Q_V = \{(a, x) : a \in V, \, x \in X, \, |f(x)| \ge \frac{\epsilon}{2}, \, |f(x) - f(a\bullet x)| \ge \epsilon\}.$$

We are supposing that there are $a \in V$ and $x \in X$ such that $|f(x) - f(a\bullet x)| \ge \epsilon$. If $|f(x)| \ge \frac{1}{2}\epsilon$ then $(a, x) \in Q_V$. Otherwise, $|f(a\bullet x)| \ge \frac{1}{2}\epsilon$, $a^{-1} \in V$ and $|f(a\bullet x) - f(a^{-1}a\bullet x)| \ge \epsilon$, so $(a^{-1}, a\bullet x) \in Q_V$. Thus Q_V is never empty. Since $Q_V \subseteq Q_{V'}$ whenever $V \subseteq V'$, there is an ultrafilter $\mathcal{F}$ on $G \times X$ such that $Q_V \in \mathcal{F}$ for every neighbourhood V of e. Setting $\pi_1(a, x) = a$ and $\pi_2(a, x) = x$ for $(a, x) \in G \times X$, we see that the image filter $\pi_1[[\mathcal{F}]]$ contains every neighbourhood of e, so converges to e, while $\pi_2[[\mathcal{F}]]$ contains the compact set $\{x : |f(x)| \ge \frac{1}{2}\epsilon\}$, so must have a limit x_0 in X. So $\mathcal{F} \to (e, x_0)$ in $G \times X$. Next, because the action is continuous, $\bullet[[\mathcal{F}]] \to e\bullet x_0 = x_0$, and there must be an $F \in \mathcal{F}$ such that $|f(x_0) - f(a\bullet x)| \le \frac{1}{3}\epsilon$ for every $(a, x) \in F$. Also, of course, there is an $F' \in \mathcal{F}$ such that $|f(x_0) - f(x)| \le \frac{1}{3}\epsilon$ whenever $(a, x) \in F'$. But now there is an $(a, x) \in Q_G \cap F \cap F'$, and we have

$$|f(x) - f(a\bullet x)| \ge \epsilon, \quad |f(x_0) - f(a\bullet x)| \le \tfrac{1}{3}\epsilon, \quad |f(x_0) - f(x)| \le \tfrac{1}{3}\epsilon$$

simultaneously, which is impossible. **XQ**

Now we find that if a, $b \in G$, $ab^{-1} \in V$ and $x \in X$, then

$$|(a^{-1}\bullet f)(x) - (b^{-1}\bullet f)(x)| = |f(a\bullet x) - f(b\bullet x)| = |f(ab^{-1}\bullet(b\bullet x)) - f(b\bullet x)| \le \epsilon.$$

As x is arbitrary, $\|a^{-1}\bullet f - b^{-1}\bullet f\|_\infty \le \epsilon$; as ϵ is arbitrary, $a \mapsto a^{-1}\bullet f$ is uniformly continuous for the right uniformity.

(b)(i) Suppose that $f : X \to \mathbb{R}$ is continuous and has compact support $K = \overline{\{x : f(x) \ne 0\}}$. Let $H \supseteq K$ be an open set of finite measure. Then $V_0 = \{a : a \in G, \, a\bullet x \in H \text{ for every } x \in K\}$ is a neighbourhood of e. **P** If we take a continuous function f_0 with compact support such that $\chi K \le f_0 \le \chi H$ (4A2G(e-i)), then $V_0 \supseteq \{a : \|f_0 - a^{-1}\bullet f_0\|_\infty < 1\}$, which is a neighbourhood of e by (a). **Q** Let $\epsilon > 0$. By (a) again, there is a symmetric neighbourhood V_1 of e such that $(\|f - a^{-1}\bullet f\|_\infty)^p \mu H \le \epsilon^p$ for every $a \in V_1$; we may suppose that $V_1 \subseteq V_0$. If $a \in V_1$, $f(x) = f(a\bullet x) = 0$ for every $x \in X \setminus H$, so that

$$\|f - a^{-1}\bullet f\|_p^p = \int_H |f - a^{-1}\bullet f|^p d\mu \le (\|f - a^{-1}\bullet f\|_\infty)^p \mu H \le \epsilon^p.$$

Now suppose that a, $b \in G$ and that $ab^{-1} \in V_1$. Then

$$\|a^{-1}\bullet f - b^{-1}\bullet f\|_p^p = \int |f(a\bullet x) - f(b\bullet x)|^p \mu(dx)$$
$$= \int |f(a\bullet(b^{-1}\bullet x)) - f(b\bullet(b^{-1}\bullet x))|^p \mu(dx)$$

(because μ is G-invariant, see 441L)

$$= \int |ba^{-1}\bullet f - f)|^p d\mu \le \epsilon^p,$$

and $\|a^{-1}\bullet f - b^{-1}\bullet f\|_p \le \epsilon$. As ϵ is arbitrary, $a \mapsto (a^{-1}\bullet f)^\bullet$ is uniformly continuous for the right uniformity.

(ii) In general, given $u \in L^p(\mu)$ and $\epsilon > 0$, there is an $f \in C_k(X)$ such that $\|u - f^\bullet\|_p \le \epsilon$ (416I). Let V be a neighbourhood of e such that $\|a^{-1}\bullet f - a^{-1}\bullet f\|_\infty \le \epsilon$ whenever $ab^{-1} \in V$; then $\|a^{-1}\bullet u - (a^{-1}\bullet f)^\bullet\|_p = \|u - f^\bullet\|_p$ (because μ is G-invariant), so

$$\|a^{-1}\bullet u - b^{-1}\bullet u\|_p \le \|a^{-1}\bullet u - a^{-1}\bullet f^\bullet\|_p + \|a^{-1}\bullet f - b^{-1}\bullet f\|_p + \|b^{-1}\bullet f^\bullet - b^{-1}\bullet u\|_p \le 3\epsilon$$

whenever $ab^{-1} \in V$ (using 441Kc). As ϵ is arbitrary, $a \mapsto a^{-1}\bullet u$ is uniformly continuous for the right uniformity. This completes the proof.

449C Theorem (a) Let G and H be topological groups such that there is a continuous surjective homomorphism from G onto H. If G is amenable, so is H.

(b) Let G be a topological group and suppose that there is a dense subset A of G such that every finite subset of A is included in an amenable subgroup of G. Then G is amenable.

(c) Let G be a topological group and H a normal subgroup of G. If H and G/H are both amenable, so is G.

(d) Let G be a topological group with two amenable subgroups H_0 and H_1 such that H_0 is normal and $H_0H_1 = G$. Then G is amenable.

(e) The product of any family of amenable topological groups is amenable.

(f) Any abelian topological group is amenable.

(g) Any compact Hausdorff topological group is amenable.

proof (a) Let $\phi : G \to H$ be a continuous surjective homomorphism. Let X be a non-empty compact Hausdorff space and $\bullet : H \times X \to X$ a continuous action. For $a \in G$ and $x \in X$, set $a\bullet_1 x = \phi(a)\bullet x$. Then $\bullet_1$ is a continuous action of G on X, so there is a G-invariant Radon probability measure μ on X. Because $\phi[G] = H$, μ is also H-invariant; as X and $\bullet$ are arbitrary, H is amenable.

(b)(i) Let X be a non-empty compact Hausdorff space and $\bullet$ a continuous action of G on X. Let $P = P_{\mathrm{R}}(X)$ be the set of Radon probability measures on X with the narrow topology (437Jd), so that P is a compact Hausdorff space (437R(f-ii)); recall that in this context the vague and narrow topologies coincide (437Kc). For $a \in G$ and $x \in X$, set $T_a(x) = a\bullet x$, so that $T_a : X \to X$ is a homeomorphism. For $a \in G$ and $\mu \in P$ write $a\bullet\mu$ for the image measure μT_a^{-1}, so that $a\bullet\mu \in P$ (418I); it is easy to check that $(a, \mu) \mapsto a\bullet\mu : G \times P \to P$ is an action. The point is that it is continuous. **P** Let $f \in C(X)$, $a_0 \in G$, $\mu_0 \in P$ and $\epsilon > 0$. By 449Ba, there is a neighbourhood V of a_0 in G such that $\|a^{-1}\bullet f - a_0^{-1}\bullet f\|_\infty \le \frac{1}{2}\epsilon$ for every $a \in V$. Next, there is a neighbourhood W of μ_0 in P such that $|\int a_0^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu_0| \le \frac{1}{2}\epsilon$ for every $\mu \in W$. But now, if $a \in V$ and $\mu \in W$,

$$|\int f d(a\bullet\mu) - \int f d(a_0\bullet\mu_0)| = |\int fT_a d\mu - \int fT_{a_0} d\mu_0|$$

(235J)

$$= |\int a^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu_0|$$
$$\le |\int a^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu| + |\int a_0^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu_0|$$
$$\le \|a^{-1}\bullet f - a_0^{-1}\bullet f\|_\infty + \frac{1}{2}\epsilon \le \epsilon.$$

As ϵ, a_0 and μ_0 are arbitrary, $(a, \mu) \mapsto \int f d(a\bullet\mu)$ is continuous; as f is arbitrary, $\bullet$ is continuous. **Q**

(ii) Because the topology of P is Hausdorff, it follows that $Q_a = \{\mu : \mu \in P,\ a\bullet\mu = \mu\}$ is closed in P for any $a \in G$, and that $G_\mu = \{a : a \in G,\ a\bullet\mu = \mu\}$ is closed in G for any $\mu \in P$. Now for any finite subset I of A there is an amenable subgroup H_I of G including I. The restriction of the action to $H_I \times X$ is a continuous action of H_I on X, so has an H_I-invariant Radon probability measure, and $\bigcap_{a\in I} Q_a \supseteq \bigcap_{a\in H_I} Q_a$ is non-empty. Because P is compact, there is a $\mu \in \bigcap_{a\in A} Q_a$. Since G_μ includes the dense set A, it is the whole of G, and μ is G-invariant. As X and $\bullet$ are arbitrary, G is amenable.

(c) Let X be a compact Hausdorff space and $\bullet$ a continuous action of G on X. Let P be the space of Radon probability measures on X with its narrow topology. Define $a\bullet\mu$, for $a \in G$ and $\mu \in P$, as in (b-i) above, so that this is a continuous action of G on P. Set $Q = \{\mu : \mu \in P,\ a\bullet\mu = \mu \text{ for every } a \in H\}$; then Q is a closed subset of P and, because H is amenable, is non-empty, since it is the set of H-invariant Radon probability measures on X. Next, $b\bullet\mu \in Q$ for every $\mu \in Q$ and $b \in G$. **P** If $a \in H$, then

$$a\bullet(b\bullet\mu) = (ab)\bullet\mu = (bb^{-1}ab)\bullet\mu = b\bullet((b^{-1}ab)\bullet\mu) = b\bullet\mu,$$

because H is normal, so $b^{-1}ab \in H$. As a is arbitrary, $b\bullet\mu \in Q$. **Q** Accordingly we have a continuous action of G on the compact Hausdorff space Q.

If $a \in H$ and $b \in G$, then $b\bullet\mu = (ba)\bullet\mu$ for every $\mu \in Q$. We therefore have a map $\circ : G/H \times Q \to Q$ defined by setting $b^\bullet\circ\mu = b\bullet\mu$ whenever $b \in G$ and $\mu \in Q$. It is easy to check that this is an action. Moreover, it is continuous, because if $W \subseteq Q$ is relatively open then $\{(b,\mu) : b\bullet\mu \in W\}$ is open in $G \times Q$, so its image $\{(b^\bullet,\mu) : b^\bullet\circ\mu \in W\}$ is open in $(G/H) \times Q$ (using 4A2B(f-iv)). Because G/H is amenable, there is a (G/H)-invariant Radon probability measure λ on Q.

Now consider the formula $p(f) = \int_Q(\int_X f(x)\mu(dx))\lambda(d\mu)$. If $f \in C(X)$, then $\mu \mapsto \int_X f(x)\mu(dx)$ is continuous for the vague topology on Q, so $p(f)$ is well-defined. Clearly p is a linear functional, $p(f) \ge 0$ if $f \ge 0$, and $p(\chi X) = 1$; so there is a Radon probability measure ν on X such that $p(f) = \int f d\nu$ for every $f \in C(X)$ (436J/436K). If $b \in G$, then, in the language of (b) above,

$$\begin{aligned}\int f d(b\bullet\nu) &= \int fT_b d\nu = p(fT_b) = \int_Q(\int_X fT_b d\mu)\lambda(d\mu)\\ &= \int_Q\int_X f d(b\bullet\mu)\lambda(d\mu) = \int_Q\int_X f d(b^\bullet\circ\mu)\lambda(d\mu) = \int_Q\int_X f d\mu\,\lambda(d\mu)\end{aligned}$$

(because λ is G/H-invariant)

$$= \int f d\nu$$

for every $f \in C(X)$, so that $b\bullet\nu = \nu$. Thus ν is G-invariant. As X and $\bullet$ are arbitrary, G is amenable.

(d) The canonical map from H_1 to G/H_0 is a continuous surjective homomorphism. By (a), G/H_0 is amenable; by (c), G is amenable.

(e) By (c) or (d), the product of two amenable topological groups is amenable, since each can be regarded as a normal subgroup of the product. It follows that the product of finitely many amenable topological groups is amenable. Now let $\langle G_i\rangle_{i\in I}$ be any family of amenable topological groups with product G. For finite $J \subseteq I$ let H_J be the set of those $a \in G$ such that $a(i)$ is the identity in G_i for every $i \in I \setminus J$. Then H_J is isomorphic (as topological group) to $\prod_{i\in J} G_i$, so is amenable. Since $\{H_J : J \in [I]^{<\omega}\}$ is an upwards-directed family of subgroups of G with dense union, (b) tells us that G is amenable.

(f)(i) The first step is to observe that the group $\mathbb{Z}$, with its discrete topology, is amenable. **P** Let X be a compact Hausdorff space and $\bullet$ a continuous action of $\mathbb{Z}$ on X. Set $\phi(x) = 1\bullet x$ for $x \in X$. Then $\phi : X \to X$ is continuous, so by 437T there is a Radon probability measure μ on X such that μ is equal to the image measure $\mu\phi^{-1}$. Because ϕ is bijective, we see that, for $E \subseteq X$, $E \in \operatorname{dom}\mu$ iff $\phi[E] \in \operatorname{dom}(\mu\phi^{-1}) = \operatorname{dom}\mu$, and in this case $\mu\phi[E] = \mu E$; that is, ϕ^{-1}, like ϕ, is inverse-measure-preserving. Now we can induce on n to see that $\mu(\phi^n)^{-1}$ and $\mu(\phi^{-n})^{-1}$ are equal to μ for every n. Since $n\bullet x = \phi^n(x)$ for every $x \in X$ and $n \in \mathbb{Z}$, μ is $\mathbb{Z}$-invariant. As X and $\bullet$ are arbitrary, $\mathbb{Z}$ is amenable. **Q**

(ii) Now let G be any abelian topological group. For each finite set $I \subseteq G$ let $\phi_I : \mathbb{Z}^I \to G$ be the continuous homomorphism defined by setting $\phi_I(z) = \prod_{a\in I} a^{z(a)}$ for $z \in \mathbb{Z}^I$. By (i) just above and (e), we know that $\mathbb{Z}^I$ (with its discrete topology) is amenable, so (a) tells us that the subgroup $G_I = \phi_I[\mathbb{Z}^I]$ is amenable. But now $\{G_I : I \in [G]^{<\omega}\}$ is an upwards-directed family of amenable subgroups of G with union G, so from (b) we see that G is amenable.

(g) This is immediate from 443Ub. (See also 449Xe.)

449D Theorem Let G be a topological group.

(a) Write U for the set of bounded real-valued functions on G which are uniformly continuous for the right uniformity of G. Then U is an M-space, and we have an action $\bullet_l$ of G on U defined by the formula $(a\bullet_l f)(y) = f(a^{-1}y)$ for a, $y \in G$ and $f \in U$.

(b) Let $Z \subseteq \mathbb{R}^U$ be the set of Riesz homomorphisms $z : U \to \mathbb{R}$ such that $z(\chi G) = 1$. Then Z is a compact Hausdorff space, and we have a continuous action of G on Z defined by the formula $(a\bullet z)(f) = z(a^{-1}\bullet_l f)$ for $a \in G$, $z \in Z$ and $f \in U$.

(c) Setting $\hat{a}(f) = f(a)$ for $a \in G$ and $f \in U$, the map $a \mapsto \hat{a} : G \to Z$ is a continuous function from G onto a dense subset of Z. If a, $b \in G$ then $a\bullet\hat{b} = \widehat{ab}$.

(d) Now suppose that X is a compact Hausdorff space, $(a, x) \mapsto a\bullet x$ is a continuous action of G on X, and $x_0 \in X$. Then there is a unique continuous function $\phi : Z \to X$ such that $\phi(\hat{e}) = x_0$ and $\phi(a\bullet z) = a\bullet\phi(z)$ for every $a \in G$ and $z \in Z$.

(e) If G is Hausdorff then the action of G on Z is faithful and the map $a \mapsto \hat{a}$ is a homeomorphism between G and its image in Z.

proof (a) Because U is a norm-closed Riesz subspace of $C_b(G)$ containing the constant functions (4A2Jh), it is an M-space. To see that the given formula defines an action, we need to check that $a\bullet_l f$ belongs to U whenever $a \in G$ and $f \in U$. Of course $a\bullet_l f$ is continuous and $\|a\bullet_l f\|_\infty = \|f\|_\infty$ is finite. If $\epsilon > 0$ there is a neighbourhood V of the identity e in G such that $|f(b) - f(c)| \le \epsilon$ whenever b, $c \in G$ and $bc^{-1} \in V$; now $a^{-1}Va$ is a neighbourhood of e, and if $bc^{-1} \in aVa^{-1}$ then $(a^{-1}b)(a^{-1}c)^{-1} \in V$, so $|(a\bullet_l f)(b) - (a\bullet_l f)(c)| = |f(a^{-1}b) - f(a^{-1}c)| \le \epsilon$. As ϵ is arbitrary, $a\bullet_l f$ is uniformly continuous with respect to the right uniformity. Now $\bullet_l$ is an action, just as in 4A5Cc.

(b)(i) Because U is an M-space with standard order unit χG, Z is a compact Hausdorff space and U can be identified, as normed Riesz space, with $C(Z)$ (354L). For any $a \in G$, the map $f \mapsto a\bullet_l f : U \to U$ is a Riesz homomorphism leaving the constant functions fixed. So we can define $a\bullet z$, for $z \in Z$, by saying that $(a\bullet z)(f) = z(a^{-1}\bullet_l f)$ for any $f \in U$, and $a\bullet z$ will belong to Z for any $a \in G$ and $z \in Z$. As usual, it is easy to check that this formula defines an action of G on Z.

(ii) $(a, z) \mapsto a\bullet z$ is continuous. **P** Take $a_0 \in G$, $z_0 \in Z$ and any neighbourhood W of $a_0\bullet z_0$ in Z. Because U corresponds to the whole of $C(Z)$, and Z is completely regular, there is an $f \in U^+$ such that $(a_0\bullet z_0)(f) = 1$ and $z(f) = 0$ for every $z \in Z \setminus W$. Set $W_0 = \{z : z(a_0^{-1}\bullet_l f) > \frac{1}{2}\}$. Observe that $z_0(a_0^{-1}\bullet_l f) = 1$, so W_0 is an open subset of Z containing z_0. Next, set $V_0 = \{a : a \in G, \|a^{-1}\bullet_l f - a_0^{-1}\bullet_l f\|_\infty \le \frac{1}{2}\}$. There is a neighbourhood V of e such that $|f(b) - f(c)| \le \frac{1}{2}$ whenever b, $c \in G$ and $bc^{-1} \in V$. If $a \in Va_0$ then $ab(a_0b)^{-1} = aa_0^{-1} \in V$ so

$$|(a^{-1}\bullet_l f)(b) - (a_0^{-1}\bullet_l f)(b)| = |f(ab) - f(a_0b)| \le \frac{1}{2}$$

for every $b \in G$, and $a \in V_0$. Thus $V_0 \supseteq Va_0$ is a neighbourhood of a_0.

Now if $a \in V_0$ and $z \in W_0$ we shall have

$$(a\bullet z)(f) = z(a^{-1}\bullet_l f) \ge z(a_0^{-1}\bullet_l f) - \frac{1}{2} > 0$$

and $a\bullet z \in W$. As a_0, z_0 and W are arbitrary, the action of G on Z is continuous. **Q**

(c) Of course $\hat{a}$, as defined, is a Riesz homomorphism taking the correct value at χG, so belongs to Z. Because $U \subseteq C(X)$, the map $a \mapsto \hat{a}$ is continuous. **?** If $\{\hat{a} : a \in G\}$ is not dense in Z, there is a non-zero $h \in C(Z)$ such that $h(\hat{a}) = 0$ for every $a \in G$; but as U is identified with $C(Z)$, there is an $f \in U$ such that $z(f) = h(z)$ for every $z \in Z$. In this case, f cannot be the zero function, but $f(a) = \hat{a}(f) = h(\hat{a}) = 0$ for every $a \in G$. **X** Thus the image of G is dense, as claimed.

If a, $b \in G$ and $f \in U$ then

$$(a\bullet\hat{b})(f) = \hat{b}(a^{-1}\bullet_l f) = (a^{-1}\bullet_l f)(b) = f(ab) = \widehat{ab}(f),$$

so $a\bullet\hat{b} = \widehat{ab}$.

(d) We have a Riesz homomorphism $T : C(X) \to \mathbb{R}^G$ defined by setting $(Tg)(a) = g(a{\bullet}x_0)$ for every $g \in C(X)$ and $a \in G$. Now $Tg \in U$ for every $g \in C(X)$. **P** $Tg(a) = (a^{-1}{\bullet}g)(x_0)$; since the map $a \mapsto a^{-1}{\bullet}g$ is uniformly continuous (449Ba), so is Tg. $\|Tg\|_\infty \le \|g\|_\infty$ is finite, so $Tg \in U$. **Q**

Of course $T(\chi X) = \chi G$. So if $z \in Z$, $zT : C(X) \to \mathbb{R}$ is a Riesz homomorphism such that $(zT)(\chi X) = 1$. There is therefore a unique $\phi(z) \in X$ such that $(zT)(g) = g(\phi(z))$ for every $g \in C(X)$ (354L again). Since the function $z \mapsto g(\phi(z)) = z(Tg)$ is continuous for every $g \in C(X)$, ϕ is continuous.

Now suppose that $a \in G$. Then $\phi(\hat{a}) = a{\bullet}x_0$. **P** If $g \in C(X)$, then

$$g(\phi(\hat{a})) = \hat{a}(Tg) = (Tg)(a) = g(a{\bullet}x_0). \text{ **Q**}$$

So if $a, b \in G$, then

$$\phi(a{\bullet}\hat{b}) = \phi(\widehat{ab}) = (ab){\bullet}x_0 = a{\bullet}(b{\bullet}x_0) = a{\bullet}\phi(b).$$

Since $\{\hat{b} : b \in G\}$ is dense in Z, and all the functions here are continuous, $\phi(a{\bullet}z) = a{\bullet}\phi(z)$ for all $a \in G$ and $z \in Z$.

To see that ϕ is unique, observe that if $a \in G$ then $\phi(\hat{a}) = \phi(\widehat{ae}) = \phi(a{\bullet}\hat{e})$ must be $a{\bullet}\phi(\hat{e}) = a{\bullet}x_0$; since $\{\hat{a} : a \in G\}$ is dense in Z, X is Hausdorff and ϕ is declared to be continuous, ϕ is uniquely defined.

(e) Now suppose that the topology of G is Hausdorff. Then it is defined by the bounded uniformly continuous functions (4A2Ja); the map $a \mapsto \hat{a}$ is therefore injective and is a homeomorphism between G and its image in Z. If $a, b \in G$ are distinct, then $a{\bullet}\hat{e} = \hat{a} \ne \hat{b} = b{\bullet}\hat{e}$, so the action is faithful.

Remark Following Brook 70, the space Z, together with the canonical action of G on it and the map $a \mapsto \hat{a} : G \to Z$, is called the **greatest ambit** of the topological group G.

449E Corollary Let G be a topological group. Then the following are equiveridical:

(i) G is amenable;

(ii) there is a G-invariant Radon probability measure on the greatest ambit of G;

(iii) writing U for the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity, there is a positive linear functional $p : U \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a{\bullet_l}f) = p(f)$ for every $f \in U$ and $a \in G$.

proof Let Z be the greatest ambit of G.

(i)⇒(ii) As soon as we know that Z is a compact Hausdorff space and the action of G on Z is continuous (449Db), this becomes a special case of the definition of 'amenable topological group'.

(ii)⇒(i) Let μ be a G-invariant Radon probability measure on Z. Given any continuous action of G on a non-empty compact Hausdorff space X, fix $x_0 \in X$ and let $\phi : Z \to X$ be a continuous function such that $\phi(a{\bullet}z) = a{\bullet}\phi(z)$ for every $a \in G$ and $z \in Z$, as in 449Dd. Let ν be the image measure $\mu\phi^{-1}$. Then ν is a Radon probability measure on X (418I again). If $F \in \operatorname{dom}\nu$ and $a \in G$, then

$$\nu(a{\bullet}F) = \mu\phi^{-1}[a{\bullet}F] = \mu(a{\bullet}\phi^{-1}[F]) = \mu\phi^{-1}[F] = \nu F.$$

As a and F are arbitrary, ν is G-invariant; as X and $\bullet$ are arbitrary, G is amenable.

(ii)⇔(iii) The identification of U with $C(Z)$ (see (b-i) of the proof of 449D) means that we have a one-to-one correspondence between Radon probability measures μ on Z and positive linear functionals p on U such that $p(\chi G) = 1$, given by the formula $p(f) = \int z(f)\mu(dz)$ for $f \in U$ (436J/436K again). Now

$$\begin{aligned}
&\mu \text{ is } G\text{-invariant}\\
&\quad\iff \int (a{\bullet}z)(f)\mu(dz) = \int z(f)\mu(dz) \text{ for every } f \in U,\ a \in G\\
&\quad\iff \int z(a^{-1}{\bullet_l}f)\mu(dz) = \int z(f)\mu(dz) \text{ for every } f \in U,\ a \in G \qquad \text{(441L)}\\
&\quad\iff \int z(a{\bullet_l}f)\mu(dz) = \int z(f)\mu(dz) \text{ for every } f \in U,\ a \in G\\
&\quad\iff p(a{\bullet_l}f) = p(f) \text{ for every } f \in U,\ a \in G.
\end{aligned}$$

So there is a G-invariant μ, as required by (ii), iff there is a G-invariant p as required by (iii).

449F Corollary Let G be a topological group.

(a) If G is amenable, then

(i) every open subgroup of G is amenable;

(ii) every dense subgroup of G is amenable.

(b) Suppose that for every sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of the identity e of G there is a normal subgroup H of G such that $H\subseteq\bigcap_{n\in\mathbb{N}}V_n$ and G/H is amenable. Then G is amenable.

proof Write U_G for the set of bounded real-valued functions on G which are uniformly continuous for the right uniformity of G; if H is a subgroup of G, let U_H the set of bounded real-valued functions on H which are uniformly continuous for the right uniformity of H; and if $H\lhd G$, let $U_{G/H}$ the set of bounded real-valued functions on the quotient G/H which are uniformly continuous for the right uniformity of G/H.

(a)(i)($\boldsymbol{\alpha}$) Let H be an open subgroup of G. Take a set $A\subseteq G$ meeting each right coset of H in just one point, so that each member of G is uniquely expressible as ya where $y\in H$ and $a\in A$. Define $T:U_H\to\mathbb{R}^G$ by setting $(Tf)(ya)=f(y)$ whenever $f\in U_H$, $y\in H$ and $a\in A$. Then T is a positive linear operator. Also $T[U_H]\subseteq U_G$. **P** Let $f\in U_H$. Of course Tf is bounded. If $\epsilon>0$, there is a neighbourhood W of the identity in H such that $|f(x)-f(y)|\le\epsilon$ whenever x, $y\in H$ and $xy^{-1}\in W$. Because H is open, W is also a neighbourhood of the identity in G. Now suppose that x, $y\in G$ and $xy^{-1}\in W$. Express x as x_0a and y as y_0b where x_0, $y_0\in H$ and a, $b\in A$. Then

$$x_0ab^{-1}y_0^{-1}\in W\subseteq H,$$

so $ab^{-1}\in H$ and $a\in Hb$ and $a=b$ and $x_0y_0^{-1}\in W$ and

$$|(Tf)(x)-(Tf)(y)|=|f(x_0)-f(y_0)|\le\epsilon.$$

As ϵ is arbitrary, Tf is uniformly continuous and belongs to U_G. **Q**

($\boldsymbol{\beta}$) Next, $b\bullet_l(Tf)=T(b\bullet_lf)$ whenever $f\in U_H$ and $b\in H$. **P** If $x\in G$, express it as ya where $y\in H$ and $a\in A$. Then

$$(b\bullet_lTf)(x)=(Tf)(b^{-1}x)=(Tf)(b^{-1}ya)=f(b^{-1}y)=(b\bullet_lf)(y)=T(b\bullet_lf)(x).\ \textbf{Q}$$

($\boldsymbol{\gamma}$) By 449E, there is a positive linear functional $p:U_G\to\mathbb{R}$ such that $p(\chi G)=1$ and $p(a\bullet_lf)=p(f)$ whenever $f\in U_G$ and $a\in G$. Set $q(f)=p(Tf)$ for $f\in U_H$; then q is a positive linear operator, $q(\chi H)=1$ and q is H-invariant, by (ii). So by 449E in the other direction, H is amenable.

(ii) Now suppose that H is a dense subgroup of G. It is easy to see that the right uniformity of H is the subspace uniformity induced by the right uniformity of G (3A4D), so that $f{\restriction}H\in U_H$ for every $f\in U_G$. In the other direction, if $g\in U_H$, then g extends uniquely to a member of U_G, by 3A4G; write Tg for the extension. In this case, $b\bullet_lTg=T(b\bullet_lg)$ for every $g\in U_H$ and $b\in H$. **P** $b\bullet_lTg$ and $T(b\bullet_lg)$ are continuous, and for $a\in H$,

$$(b\bullet_lTg)(a)=Tg(b^{-1}a)=g(b^{-1}a)=(b\bullet_lg)(a)=T(b\bullet_lg)(a);$$

as H is dense in G, $b\bullet_lTg=T(b\bullet_lg)$. **Q**

Now we can use the same argument as in (i-γ) above to see that H is amenable.

(b)(i) Let $\mathcal{H}$ be the family of normal subgroups H of G such that G/H is amenable.

($\boldsymbol{\alpha}$) For $H\in\mathcal{H}$, let $\pi_H:G\to G/H$ be the canonical homomorphism and $p_H:U_{G/H}\to\mathbb{R}$ a positive linear functional such that $p_H(\chi(G/H))=1$ and $p_H(c\bullet_lg)=p_H(g)$ whenever $g\in U_{G/H}$ and $c\in G/H$. Let U'_H be $\{f:f\in U_G,\ f(x)=f(y)$ whenever x, $y\in G$ and $xy^{-1}\in H\}$. Then U'_H is a linear subspace of U_G containing χG.

If $f\in U'_H$ then there is a unique $g\in U_{G/H}$ such that $f=g\pi_H$. **P** Because $f(x)=f(y)$ whenever $\pi_Hx=\pi_Hy$, and π_H is surjective, there is a unique function $g:G/H\to\mathbb{R}$ such that $f=g\pi_H$; because f is bounded, so is g. Given $\epsilon>0$, there is an open neighbourhood W of e such that $|f(x)-f(y)|\le\epsilon$ whenever $xy^{-1}\in W$. In this case, $\pi_H[W]$ is a neighbourhood of the identity in G/H (4A5J(a-i)). Suppose that c_0, $c_1\in G/H$ are such that $c_0c_1^{-1}\in\pi_H[W]$. Then there are x_0, $x_1\in G$ and $x\in W$ such that $\pi_Hx_0=c_0$, $\pi_Hx_1=c_1$ and $\pi_Hx=c_0c_1^{-1}$. As $\pi_H(x_0x_1^{-1})=\pi_Hx$, there is a $y\in H$ such that $yx_0x_1^{-1}=x$ belongs to W; so that $\pi_H(yx_0)=c_0$ and

$$|g_H(c_0)-g_H(c_1)|=|f(yx_0)-f(x_1)|\le\epsilon.$$

As ϵ is arbitrary, $g\in U_{G/H}$. **Q**

We therefore have a functional $p'_H:U'_H\to\mathbb{R}$ defined by setting $p'_H(g\pi_H)=p_H(g)$ whenever $g\in U_{G/H}$. Of course $g\ge0$ whenever $g\pi_H\ge0$, so p'_H is a positive linear functional, and $p'_H(\chi G)=1$.

(β) If $f \in U'_H$ and $a \in G$ then $a\bullet_l f \in U'_H$ and $p'_H(a\bullet_l f) = p'_H(f)$. **P** Let $g \in U_{G/H}$ be such that $f = g\pi_H$. Then

$$(a\bullet_l f)(x) = f(a^{-1}x) = g\pi_H(a^{-1}x) = g(\pi_H(a)^{-1}\pi_H(x)) = (\pi_H(a)\bullet_l g)(\pi_H(x))$$

for every $x \in G$; so $a\bullet_l f = (\pi_H(a)\bullet_l g)\pi_H$ belongs to U'_H, and

$$p'_H(a\bullet_l f) = p_H(\pi_H(a)\bullet_l g) = p_H(g) = p'_H(f). \text{ **Q**}$$

(ii) For any family $\mathcal{V}$ of neighbourhoods of e, set $\mathcal{H}_\mathcal{V} = \{H : H \in \mathcal{H},\ H \subseteq \bigcap \mathcal{V}\}$. Now for any $f \in U_G$ there is a countable family $\mathcal{V}$ of neighbourhoods of e such that $f \in U'_H$ for every $H \in \mathcal{H}_\mathcal{V}$. **P** For each $n \in \mathbb{N}$ choose a neighbourhood V_n of e such that $|f(x) - f(y)| \le 2^{-n}$ whenever $xy^{-1} \in V_n$, and set $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$. **Q**

(iii) We are supposing that $\mathcal{H}_\mathcal{V}$ is non-empty for every countable family $\mathcal{V}$ of neighbourhoods of e. There is therefore an ultrafilter $\mathcal{F}$ on $\mathcal{H}$ containing $\mathcal{H}_\mathcal{V}$ for every countable $\mathcal{V}$. Now we see from (ii) that for any $f \in U_G$ the set $\{H : H \in \mathcal{H},\ f \in U'_H\}$ belongs to $\mathcal{F}$, while $|p'_H(f)| \le \|f\|_\infty$ for any H such that $f \in U'_H$; so we can set $p(f) = \lim_{H\to\mathcal{F}} p'_H(f)$ for every $f \in U_G$. In this case, of course, p is a positive linear functional and $p(\chi G) = 1$. Also, given $f \in U_G$ and $a \in G$, then $p'_H(f) = p'_H(a\bullet_l f)$ whenever $f \in U'_H$, by (i-β), so $p(f) = p(a\bullet_l f)$. Thus p satisfies (iii) of 449E and G is amenable.

449G Example Let F_2 be the free group on two generators, with its discrete topology. Then F_2 is a σ-compact unimodular locally compact Polish group. But it is not amenable. **P** Let a and b be the generators of F_2. Then every element of F_2 is uniquely expressible as a word (possibly empty) in the letters a, b, a^{-1}, b^{-1} in which the letters a, a^{-1} are never adjacent and the letters b, b^{-1} are never adjacent. Write A for the set of elements of F_2 for which the canonical word does not begin with either b or b^{-1}, and B for the set of elements of F_2 for which the canonical word does not begin with either a or a^{-1}. Then $A \cup B = F_2$ and $A \cap B = \{e\}$. **?** Suppose, if possible, that F_2 is amenable. Every member of $\ell^\infty(F_2)$ is uniformly continuous with respect to the right uniformity. So there is an F_2-invariant positive linear functional $p : \ell^\infty(F_2) \to \mathbb{R}$ such that $p(\chi F_2) = 1$. Let ν be the corresponding non-negative additive functional on $\mathcal{P}F_2$, so that $\nu C = p(\chi C)$ for every $C \subseteq F_2$. For $c \in F_2$ and $C \subseteq F_2$, $c\bullet_l \chi C = \chi(cC)$, so $\nu(cC) = \nu C$ for every $C \subseteq F_2$ and $c \in F_2$. In particular, $\nu(b^n A) = \nu A$ for every $n \in \mathbb{Z}$; but as all the $b^n A$, for $n \in \mathbb{Z}$, are disjoint, $\nu A = 0$. Similarly $\nu B = 0$ and

$$0 = \nu(A \cup B) = \nu F_2 = p(\chi F_2) = 1,$$

which is absurd. **X** Thus F_2 is not amenable, as claimed. **Q**

449H In this section so far, I have taken care to avoid assuming that groups are locally compact. Some of the most interesting amenable groups are very far from being locally compact (e.g., 449Xh). But of course a great deal of work has been done on amenable locally compact groups. In particular, there is a remarkable list of equivalent properties, some of which I will present in the next theorem. It will be useful to have the following facts to hand.

Lemma Let G be a locally compact Hausdorff topological group, and U the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity, as in 449D-449E. Let μ be a left Haar measure on G, and $*$ the corresponding convolution on $\mathcal{L}^0(\mu)$ (444O).

(a) If $h \in \mathcal{L}^1(\mu)$ and $f \in \mathcal{L}^\infty(\mu)$ then $h * f \in U$.

(b) Let $p : U \to \mathbb{R}$ be a positive linear functional such that $p(a\bullet_l f) = p(f)$ whenever $f \in U$ and $a \in G$. Then $p(h * f) = p(f)\int h\,d\mu$ for every $h \in \mathcal{L}^1(\mu)$ and $f \in U$.

proof (a) Recall that we know from 444Rc that $h * f$ is defined everywhere in G and is continuous. For any $x \in G$,

$$(h * f)(x) = \int h(xy)f(y^{-1})\mu(dy) = \int (x^{-1}\bullet_l h) \times \overset{\leftrightarrow}{f},$$

where $\overset{\leftrightarrow}{f}(y) = f(y^{-1})$ whenever $y^{-1} \in \operatorname{dom} f$ (4A5C(c-ii)). By 449Bb, applied to the left action of G on itself, $x \mapsto (x^{-1}\bullet_l h)^\bullet : G \to L^1(\mu)$ is uniformly continuous for the right uniformity of G and the norm uniformity of $L^1(\mu)$. Since $u \mapsto \int u \times v : L^1(\mu) \to \mathbb{R}$ is uniformly continuous for every $v \in L^\infty(\mu)$, $x \mapsto (h * f)(x) = \int (x^{-1}\bullet_l h)^\bullet \times \overset{\leftrightarrow}{f}{}^\bullet$ is uniformly continuous for the right uniformity (3A4Cb). Of course $\sup_{x\in G} |(h * f)(x)| \le \|h\|_1\|f\|_\infty$ is finite, so $h * f \in U$.

(b) Let $\epsilon > 0$. Then there are a compact set $K \subseteq G$ such that $\int_{G\setminus K} |h|d\mu \le \epsilon$ (412Je) and a symmetric open neighbourhood V_0 of e such that $|f(x) - f(y)| \le \epsilon$ whenever $xy^{-1} \in V_0$. Let $a_0, \dots, a_n \in G$ be such that

$K \subseteq \bigcup_{i\le n} a_iV_0$, and set $E_i = a_iV_0 \setminus \bigcup_{j<i} a_jV_0$, $\alpha_i = \int_{E_i} h\,d\mu$ for each $i \le n$ and $F = G \setminus \bigcup_{j\le n} a_jV_0$. If $x \in G$ and $y \in E_i$, then $y^{-1}x(a_i^{-1}x)^{-1} = (a_i^{-1}y)^{-1}$ belongs to V_0, so $|f(y^{-1}x) - f(a_i^{-1}x)| \le \epsilon$. So, for any $x \in G$,

$$\begin{aligned}
\bigl|(h*f)(x) - \sum_{i=0}^n \alpha_i f(a_i^{-1}x)\bigr| &\\
&= \bigl|\int h(y)f(y^{-1}x)\mu(dy) - \sum_{i=0}^n \alpha_i f(a_i^{-1}x)\bigr| \\
&= \bigl|\int_F h(y)f(y^{-1}x)\mu(dy) + \sum_{i=0}^n (\int_{E_i} h(y)f(y^{-1}x)\mu(dy) - \alpha_i f(a_i^{-1}x))\bigr| \\
&\le \|f\|_\infty \int_F |h|d\mu + \sum_{i=0}^n \bigl|\int_{E_i} h(y)(f(y^{-1}x) - f(a_i^{-1}x))\mu(dy)\bigr| \\
&\le \|f\|_\infty \int_{X\setminus K} |h|d\mu + \epsilon \sum_{i=0}^n \int_{E_i} |h|d\mu \le \epsilon(\|f\|_\infty + \|h\|_1).
\end{aligned}$$

Thus

$$\|h*f - \textstyle\sum_{i=0}^n \alpha_i a_i\bullet_l f\|_\infty \le \epsilon(\|f\|_\infty + \|h\|_1).$$

Since $p(a_i\bullet_l f) = p(f)$ for every i, it follows that

$$\begin{aligned}
|p(h*f) - p(f)\int h\,d\mu| &\le \epsilon(\|f\|_\infty + \|h\|_1)p(\chi G) + |\sum_{i=0}^n \alpha_i - \int h\,d\mu||p(f)| \\
&\le \epsilon(\|f\|_\infty + \|h\|_1)p(\chi G) + |\int_F h\,d\mu|\|f\|_\infty p(\chi G) \\
&\le \epsilon(2\|f\|_\infty + \|h\|_1)p(\chi G).
\end{aligned}$$

As ϵ is arbitrary, $p(h*f) = p(f)\int h\,d\mu$, as claimed.

449I Notation It will save repeated explanations if I say now that for the next two results, given a locally compact Hausdorff group G, Σ_G will be the algebra of Haar measurable subsets of G and $\mathcal{N}_G$ the ideal of Haar negligible subsets of G (443A), while $\mathcal{B}_G$ will be the Borel σ-algebra of G. Recall that all three are left- and right-translation-invariant and inversion-invariant, and indeed autohomeomorphism-invariant, in that if $\gamma : G \to G$ is a function of any of the types

$$x \mapsto ax, \quad x \mapsto xa, \quad x \mapsto x^{-1}$$

or is a group automorphism which is also a homeomorphism, and $E \subseteq G$, then $\gamma[E]$ belongs to Σ_G, $\mathcal{N}_G$ or $\mathcal{B}_G$ iff E does (443Aa).

449J Theorem Let G be a locally compact Hausdorff group; fix a left Haar measure μ on G. Write $\mathcal{L}^1$ for $\mathcal{L}^1(\mu)$ and L^∞ for $L^\infty(\mu)$, etc. Let C_{k1}^+ be the set of continuous functions $h : G \to [0,\infty[$ with compact supports such that $\int h\,d\mu = 1$, and suppose that $q \in [1,\infty[$. Then the following are equiveridical:

(i) G is amenable;

(ii) there is a positive linear functional $p : C_b(G) \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a\bullet_l f) = p(f)$ for every $f \in C_b(G)$ and every $a \in G$;

(iii) there is a finitely additive functional $\phi : \mathcal{B}_G \to [0,1]$ such that $\phi G = 1$, $\phi(aE) = \phi E$ for every $E \in \mathcal{B}_G$ and $a \in G$, and $\phi E = 0$ for every Haar negligible $E \in \mathcal{B}_G$;

(iv) there is a finitely additive functional $\phi : \Sigma_G \to [0,1]$ such that $\phi G = 1$, $\phi(aE) = \phi(Ea) = \phi(E^{-1}) = \phi E$ for every $E \in \Sigma_G$ and $a \in G$, and $\phi E = 0$ for every $E \in \mathcal{N}_G$;

(v) there is a positive linear functional $\tilde p : L^\infty \to \mathbb{R}$ such that $\tilde p(\chi G^\bullet) = 1$ and $\tilde p(a\bullet_l u) = \tilde p(a\bullet_r u) = \tilde p(a\bullet_c u) = \tilde p(\overset{\leftrightarrow}{u}) = \tilde p(u)$ for every $u \in L^\infty$ and every $a \in G$, where $\bullet_l$, $\bullet_r$, $\bullet_c$ and $\leftrightarrow$ are defined as in 443Af and 443Gc;

(vi) there is a positive linear functional $\tilde p : L^\infty \to \mathbb{R}$ such that $\tilde p(\chi G^\bullet) = 1$ and $\tilde p(a\bullet_l u) = \tilde p(u)$ for every $u \in L^\infty$ and every $a \in G$;

(vii) there is a positive linear functional $\tilde{p} : L^\infty \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(\nu * u) = \nu G \cdot \tilde{p}(u)$ for every $u \in L^\infty$ and every totally finite Radon measure ν on G, where $\nu * u$ is defined as in 444Ma;

(viii) there is a positive linear functional $\tilde{p} : L^\infty \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(v * u) = \tilde{p}(u) \int v$ for every $v \in L^1$ and $u \in L^\infty$;

(ix) for every finite set $J \subseteq \mathcal{L}^1$ and $\epsilon > 0$, there is an $h \in C_{k1}^+$ such that $\|g * h - (\int g\,d\mu)h\|_1 \le \epsilon$ for every $g \in J$;

(x) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is an $h \in C_{k1}^+$ such that $\|a \bullet_l h - h\|_1 \le \epsilon$ for every $a \in K$;

(xi) for any finite set $I \subseteq G$ and $\epsilon > 0$, there is a $u \in L^q$ such that $\|u\|_q = 1$ and $\|u - a \bullet_l u\|_q \le \epsilon$ for every $a \in I$;

(xii) for any finite set $I \subseteq G$ and $\epsilon > 0$, there is a compact set $L \subseteq G$ with non-zero measure such that $\mu(L \triangle aL) \le \epsilon \mu L$ for every $a \in I$;

(xiii) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is a symmetric compact neighbourhood L of the identity e in G such that $\mu(L \triangle aL) \le \epsilon \mu L$ for every $a \in K$;

(xiv) (Emerson & Greenleaf 67) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is a compact set $L \subseteq G$ with non-zero measure such that $\mu(KL) \le (1 + \epsilon)\mu L$.

proof (a)(i)⇒(vii) Write U for the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity. Then we have a positive linear functional $p : U \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a \bullet_l f) = p(f)$ for every $f \in U$ and $a \in G$ (449E). Now if $f \in \mathcal{L}^\infty$, h_1, $h_2 \in \mathcal{L}^1$ and $\int h_1 d\mu = \int h_2 d\mu$, then $p(h_1 * f) = p(h_2 * f)$. **P** By 449Ha, both $h_1 * f$ and $h_2 * f$ belong to U. Set $h = h_1 - h_2$. By 444T, there is a neighbourhood V of e such that $\|h * \nu - h\|_1 \le \epsilon$ whenever ν is a quasi-Radon measure on G such that $\nu V = \nu G = 1$, defining $h * \nu$ as in 444J. In particular, taking ν to be the indefinite-integral measure over μ defined from $g = \frac{1}{\mu V}\chi V$, $\|h * g - h\|_1 \le \epsilon$ (using 444Pb). Now

$$|p(h_1 * f) - p(h_2 * f)| = |p(h * f)| \le |p((h * g) * f)| + |p((h * g - h) * f)|$$
$$\le |p(h * (g * f))| + \|(h * g - h) * f\|_\infty$$

(because $*$ is associative, 444Oe)

$$\le |p(g * f) \int h\,d\mu| + \|h * g - h\|_1 \|f\|_\infty$$

(449Hb)

$$\le \epsilon \|f\|_\infty.$$

As ϵ is arbitrary, $p(h_1 * f) = p(h_2 * f)$, as claimed. **Q**

Of course $p(h * f) = 0$ whenever $h \in \mathcal{L}^1$, $f \in \mathcal{L}^\infty$ and $f = 0$ a.e. (444Ob). We can therefore define a functional $\tilde{p} : L^\infty \to \mathbb{R}$ by saying that $\tilde{p}(f^\bullet) = p(h * f)$ whenever $f \in \mathcal{L}^\infty$, $h \in \mathcal{L}^1$ and $\int h\,d\mu = 1$. $\tilde{p}$ is positive and linear because p is. It follows that $p(h * f) = \tilde{p}(f^\bullet) \int h\,d\mu$ whenever $h \in \mathcal{L}^1$ and $f \in \mathcal{L}^\infty$. Also $\tilde{p}(\chi G^\bullet) = p(\chi G) = 1$ because $h * \chi G = (\int h\,d\mu)\chi G$ for every $h \in \mathcal{L}^1$.

If $u \in L^\infty$ and ν is a totally finite Radon measure on G, express u as $f^\bullet$ where $f \in \mathcal{L}^\infty$, so that $\nu * u = (\nu * f)^\bullet$ (444Ma). Taking any non-negative $h \in \mathcal{L}^1$ such that $\int h\,d\mu = 1$, we have

$$h * (\nu * f) = h\mu * (\nu * f)$$

(444Pa; here $h\mu$ is the indefinite-integral measure, as in 444J)

$$= (h\mu * \nu) * f$$

(444Ic)

$$= (h * \nu)\mu * f$$

(444K)

$$= (h * \nu) * f$$

(444Pa again). So

$$\tilde{p}(\nu * u) = \tilde{p}((\nu * f)^\bullet) = p(h * (\nu * f))$$
$$= p((h * \nu) * f) = \int h * \nu\,d\mu \cdot \tilde{p}(u) = \nu G \cdot \tilde{p}(u)$$

(444K). As ν and u are arbitrary, $\tilde{p}$ has the required properties.

(b)(vii)⇒(vi) Take $\tilde{p}$ from (vii). If $a \in G$ and $u \in L^\infty$, consider the Dirac measure δ_a on G concentrated at a. Then $\delta_a * u = a\bullet_l u$. **P** Take $f \in \mathcal{L}^\infty$ such that $f^\bullet = u$. Then

$$(\delta_a * f)(x) = \int f(y^{-1}x)\delta_a(dy) = f(a^{-1}x) = (a\bullet_l f)(x)$$

whenever $a^{-1}x \in \operatorname{dom} f$, so $\delta_a * f = a\bullet_l f$ and

$$\delta_a * u = \delta_a * f^\bullet = (\delta_a * f)^\bullet = (a\bullet_l f)^\bullet = a\bullet_l u. \text{ **Q**}$$

Accordingly, using (vii),

$$\tilde{p}(a\bullet_l u) = \tilde{p}(\delta_a * u) = \delta_a(G)\tilde{p}(u) = \tilde{p}(u),$$

as required by (vi).

(c)(vi)⇒(v)(α) The first step is to note that since there is a left-invariant mean there must also be a right-invariant mean, that is, a positive linear functional $\tilde{q} : L^\infty \to \mathbb{R}$ such that $\tilde{q}(\chi G^\bullet) = 1$ and $\tilde{q}(a\bullet_r u) = \tilde{q}(u)$ for every $u \in L^\infty$ and every $a \in G$. **P** Set $\tilde{q}(u) = \tilde{p}(\overset{\leftrightarrow}{u})$ for $u \in L^\infty$. Evidently $\tilde{q}$ is a positive linear functional and $\tilde{q}(\chi G^\bullet) = 1$. By 443Gc,

$$\tilde{q}(a\bullet_r u) = \tilde{p}((a\bullet_r u)^\leftrightarrow) = \tilde{p}(a\bullet_l \overset{\leftrightarrow}{u}) = \tilde{p}(\overset{\leftrightarrow}{u}) = \tilde{q}(u)$$

whenever $u \in L^\infty$ and $a \in G$. **Q**

(β) At this point, recall that L^1 is a Banach algebra under convolution (444Sb), and that L^∞ can be identified with its normed space dual, because μ is a quasi-Radon measure, therefore localizable (415A), and we can use 243Gb. We therefore have an Arens multiplication on $(L^\infty)^* \cong (L^1)^{**}$ defined by the formulae of 4A6O. Of course $\tilde{p}$ and $\tilde{q}$ both belong to $(L^\infty)^*$; write $\tilde{r}_0 = \tilde{p}\circ\tilde{q}$ for their Arens product. To see that $\tilde{r}_0(\chi G^\bullet) = 1$, note that if u, $v \in L^1$ then, defining $\chi G^\bullet \circ u$ and $\tilde{q}\circ\chi G^\bullet$ as in 4A6O, we have

$$\int (\chi G^\bullet \circ u) \times v = \int \chi G^\bullet \times (u * v) = \int u \int v$$

as noted in 444Sb; consequently $\chi G^\bullet \circ u = (\int u)\chi G^\bullet$,

$$\int (\tilde{q}\circ\chi G^\bullet) \times u = \tilde{q}(\chi G^\bullet \circ u) = \tilde{q}((\int u)\chi G^\bullet) = \int u$$

and $\tilde{q}\circ\chi G^\bullet = \chi G^\bullet$. Now, of course,

$$\tilde{r}_0(\chi G^\bullet) = \tilde{p}(\tilde{q}\circ\chi G^\bullet) = \tilde{p}(\chi G^\bullet) = 1.$$

As noted in 4A6O, $\|\tilde{r}_0\| \le \|\tilde{p}\|\|\tilde{q}\| = 1$, so $\tilde{r}_0$ must be a positive linear functional.

(γ) We find next that $\tilde{r}_0(a\bullet_l u) = \tilde{r}_0(u)$ whenever $u \in L^\infty$ and $a \in G$. **P** By 443Ge, we have a bounded linear operator $S : L^1 \to L^1$ defined by setting $Sv = a^{-1}\bullet_l v$ for every $v \in L^1$. By 444Sa, $S(u * v) = (Su) * v$ for all u, $v \in L^1$. Identifying L^∞ with $(L^1)^*$, we have the adjoint operator $S' : L^\infty \to L^\infty$ given by saying that

$$\begin{aligned}\int S'u \times v = \int u \times Sv &= \int u \times (a^{-1}\bullet_l v) \\ &= \int a\bullet_l (u \times a^{-1}\bullet_l v) = \int (a\bullet_l u) \times v\end{aligned}$$

whenever $u \in L^\infty$ and $v \in L^1$, so that $S'u = a\bullet_l u$ for every $u \in L^\infty$. But this means that

$$(S''\tilde{p})(u) = \tilde{p}(a\bullet_l u) = \tilde{p}(u)$$

for every u, so that $S''\tilde{p} = \tilde{p}$. By 4A6O(b-i),

$$S''\tilde{r}_0 = S''(\tilde{p}\circ\tilde{q}) = (S''\tilde{p})\circ\tilde{q} = \tilde{p}\circ\tilde{q} = \tilde{r}_0,$$

that is, $\tilde{r}_0(a\bullet_l u) = \tilde{r}_0(u)$ for every $u \in L^\infty$. **Q**

(δ) In the same way, $\tilde{r}_0(a\bullet_r u) = \tilde{r}_0(u)$ whenever $u \in L^\infty$ and $a \in G$. **P** This time, define $T : L^1 \to L^1$ by setting $Tv = \Delta(a^{-1})a^{-1}\bullet_r v$ for every $v \in L^1$, where Δ is the left modular function of G; 444Sa tells us that $T(u * v) = u * Tv$ for all u, $v \in L^1$. Since $\int f d\mu = \Delta(a)\int a\bullet_r f d\mu$ for every $f \in \mathcal{L}^1$ (442Kc), $\int v = \Delta(a)\int a\bullet_r v$ for every $v \in L^1$, and

$$\int T'u \times v = \int u \times Tv = \Delta(a^{-1}) \int u \times (a^{-1}\bullet_r v)$$
$$= \int a\bullet_r (u \times a^{-1}\bullet_r v) = \int (a\bullet_r u) \times v$$

whenever $u \in L^\infty$ and $v \in L^1$. Thus $T'u = a\bullet_r u$ for every $u \in L^\infty$. But now we have

$$(T''\tilde{q})(u) = \tilde{q}(a\bullet_r u) = \tilde{q}(u)$$

for every u, so that $T''\tilde{q} = \tilde{q}$. By 4A6O(b-ii), $T''\tilde{r}_0 = \tilde{r}_0$, that is, $\tilde{r}_0(a\bullet_r u) = \tilde{r}_0(u)$ for every $u \in L^\infty$. **Q**

(ϵ) Thus $\tilde{r}_0$ is both left- and right-invariant. To get reversal-invariance, set

$$\tilde{r}(u) = \frac{1}{2}(\tilde{r}_0(u) + \tilde{r}_0(\overleftrightarrow{u}))$$

for $u \in L^\infty$. Then $\tilde{r}_0$ is a positive linear functional and $\tilde{r}_0(\chi G^\bullet) = 1$. Because

$$a\bullet_l \overleftrightarrow{u} = (a\bullet_r u)^\leftrightarrow, \quad a\bullet_r \overleftrightarrow{u} = (a\bullet_l u)^\leftrightarrow,$$

$u \mapsto \tilde{r}_0(\overleftrightarrow{u})$ and $\tilde{r}$ are also both left- and right-invariant, and of course $\tilde{r}(\overleftrightarrow{u}) = \tilde{r}(u)$ for every u. Finally,

$$\tilde{r}(a\bullet_c u) = \tilde{r}(a\bullet_l (a\bullet_r u)) = \tilde{r}(u)$$

for every $u \in L^\infty$ and $a \in G$, so $\tilde{r}$ has all the properties required by (v).

(d)(v)⇒(iv) Take $\tilde{p}$ from (v), and set $\phi E = \tilde{p}(\chi E^\bullet)$ for every $E \in \Sigma_G$. Then $\phi : \Sigma_G \to [0,1]$ is additive and $\phi G = 1$; also, if $E \in \mathcal{N}_G$, $\chi E^\bullet = 0$ in L^∞ and $\phi E = 0$. If $E \in \Sigma_G$ and $a \in G$, then $\chi(aE) = a\bullet_l(\chi E)$ (4A5C(c-ii)) and

$$\phi(aE) = \tilde{p}(\chi(aE)^\bullet) = \tilde{p}((a\bullet_l \chi E)^\bullet) = \tilde{p}(a\bullet_l(\chi E^\bullet)) = \tilde{p}(\chi E^\bullet) = \phi E.$$

Next, $\chi E^{-1} = (\chi E)^\leftrightarrow$ and $(\chi E^{-1})^\bullet = (\chi E^\bullet)^\leftrightarrow$, so

$$\phi(E^{-1}) = \tilde{p}((\chi E^\bullet)^\leftrightarrow) = \tilde{p}(\chi E^\bullet) = \phi E.$$

Consequently, for $E \in \Sigma_G$ and $a \in G$,

$$\phi(Ea) = \phi((Ea)^{-1}) = \phi(a^{-1}E^{-1}) = \phi(E^{-1}) = \phi E.$$

Thus ϕ satisfies the requirements of (iv).

(e)(iv)⇒(iii) This is trivial; we have only to take $\phi : \Sigma_G \to [0,1]$ as in (iv) and consider $\phi\restriction\mathcal{B}_G$.

(f)(iii)⇒(ii) Given $\phi : \mathcal{B}_G \to [0,1]$ as in (iii), set $p(f) = \oint f d\phi$ for $f \in C_b(G)$, where $\oint d\phi$ is as defined in 363L, that is, the unique $\| \|_\infty$-continuous linear functional on the space $L^\infty(\mathcal{B}_G)$ of bounded Borel measurable functions from G to $\mathbb{R}$ such that $\oint \chi E\, d\phi = \phi E$ for every $E \in \mathcal{B}_G$. p is positive because ϕ is non-negative (363Lc), and $p(\chi G) = \phi G = 1$. If $a \in G$, then

$$\oint a\bullet_l \chi E\, d\phi = \oint \chi(aE) d\phi = \phi(aE) = \phi E = \oint \chi E\, d\phi$$

for every $E \in \mathcal{B}_G$; because $f \mapsto \oint f d\phi$ and $f \mapsto \oint a\bullet_l f d\phi$ are both linear and $\| \|_\infty$-continuous, they agree on $L^\infty(\mathcal{B}_G) \supseteq C_b(G)$, and

$$p(a\bullet_l f) = \oint a\bullet_l f d\phi = \oint f d\phi = p(f)$$

for every $f \in C_b(G)$, as required.

(g)(ii)⇒(i) Given p as in (ii), its restriction to the space of bounded right-uniformly-continuous functions is positive, linear and G-invariant, so G is amenable, by 449E.

(h)(vii)⇒(viii) Take $\tilde{p}$ as in (vii). If $g \in \mathcal{L}^1$, $f \in \mathcal{L}^\infty$ and $g \ge 0$, then

$$\tilde{p}(g * f)^\bullet = \tilde{p}(g\mu * f)^\bullet = \tilde{p}(g\mu * f^\bullet)$$
$$= (g\mu)(G)\tilde{p}(f^\bullet) = \int g\, d\mu \cdot \tilde{p}(f^\bullet);$$

translating into terms of L^1 and L^∞ as in 444Sa, we get $\tilde{p}(v * u) = \int v \cdot \tilde{p}(u)$ for all $u \in L^\infty$ and $v \in (L^1)^+$. By linearity, the same is true for all $v \in L^1$, as required by (viii).

(i)(viii)⇒(ix) Suppose that (viii) is true.

(α) Note first that if $J \subseteq \mathcal{L}^\infty$ is finite and $\epsilon > 0$, then

$$A(J,\epsilon) = \{h : h \in C_{k1}^+,\ |\int f \times h\,d\mu - \tilde{p}(f^\bullet)| \le \epsilon \text{ for every } f \in J\}$$

is non-empty. **P** It is enough to consider the case in which $\chi G \in J$. Let $\eta \in \left]0, \frac{1}{2}\right[$ be such that $\eta + 5\eta \sup_{f\in J} \|f^\bullet\|_\infty \le \epsilon$. Because $\tilde{p} \in (L^\infty)^* \cong (L^1)^{**}$, there is a $u_0 \in L^1$ such that $\|u_0\|_1 \le 1$ and $|\tilde{p}(f^\bullet) - \int f^\bullet \times u_0| \le \eta$ for every $f \in J$ (4A4If). In particular, $\int u_0 \ge \tilde{p}(\chi G^\bullet) - \eta = 1 - \eta$. By 416I, there is a continuous $h_0 : G \to \mathbb{R}$ with compact support such that $\|u_0 - h_0^\bullet\|_1 \le \eta$. Now $\int h_0 d\mu \ge 1 - 2\eta$ and $\int |h_0| d\mu \le 1 + \eta$. So if we set $h_0^+ = h_0 \vee 0$, $\gamma = \int h_0^+ d\mu$ and $h = \frac{1}{\gamma} h_0^+$, we shall have

$$\gamma \le 1 + \eta, \quad \|h_0^+ - h_0\|_1 = \frac{1}{2}\int |h_0| - h_0 \le 2\eta, \quad \|h - h_0^+\| = |\gamma - 1| \le 2\eta,$$

so $\|u_0 - h^\bullet\|_1 \le 5\eta$, while $h \in C_{k1}^+$. This will mean that

$$|\tilde{p}(f^\bullet) - \int f \times h\,d\mu| \le \eta + 5\eta\|f^\bullet\|_\infty \le \epsilon$$

for every $f \in J$. **Q**

We therefore have a filter $\mathcal{F}$ on C_{k1}^+ containing every $A(J,\epsilon)$, and $\tilde{p}(f^\bullet) = \lim_{h\to\mathcal{F}} \int f \times h\,d\mu$ for every $f \in \mathcal{L}^\infty$.

(β) Now $0 = \lim_{h\to\mathcal{F}} (g * h)^\bullet - (\int g\,d\mu)h^\bullet$ for the weak topology of L^1, for every $g \in \mathcal{L}^1$. **P** Set $\gamma = \int g\,d\mu$. Let $f \in \mathcal{L}^\infty$. Define g' by setting $g'(x) = \Delta(x^{-1})g(x^{-1})$ whenever this is defined, where Δ is the left modular function of G, as before; then $g' \in \mathcal{L}^1$ and $\int g' d\mu = \gamma$ (442K(b-ii)). Set $v = (g')^\bullet \in L^1$. If $h \in C_{k1}^+$, then

$$\int f \times (g * h) d\mu = \iint f(xy)g(x)h(y)\mu(dx)\mu(dy)$$

(444Od)

$$= \int \Bigl(\int \Delta(x^{-1})g'(x^{-1})f(xy)\mu(dx)\Bigr)h(y)\mu(dy)$$

$$= \int (g' * f)(y)h(y)\mu(dy)$$

(444Oa)

$$= \int (g' * f) \times h\,d\mu.$$

By (viii), we have

$$\tilde{p}(g' * f)^\bullet = \tilde{p}(v * f^\bullet) = \tilde{p}(f^\bullet)\int v = \gamma\tilde{p}(f^\bullet),$$

so we get

$$\lim_{h\to\mathcal{F}} \int f \times (g * h - \gamma h) d\mu = \lim_{h\to\mathcal{F}} \int (g' * f - \gamma f) \times h\,d\mu$$
$$= \tilde{p}(g' * f)^\bullet - \gamma\tilde{p}(f^\bullet) = 0.$$

As f is arbitrary, and $(L^1)^*$ can be identified with L^∞, this is all we need. **Q**

(γ) Now take any finite set $J \subseteq \mathcal{L}^1$ and $\epsilon > 0$. On $(L^1)^J$ let $\mathfrak{T}$ be the locally convex linear space topology which is the product topology when each copy of L^1 is given the norm topology, and $\mathfrak{S}$ the corresponding weak topology. Define $T : C_k(G) \to (L^1)^J$ by setting

$$Th = \bigl\langle (g * h)^\bullet - (\int g\,d\mu)h^\bullet \bigr\rangle_{g\in J}$$

for $h \in C_k(G)$, where $C_k(G)$ is the linear space of continuous real-valued functions on G with compact support. Then T is linear. Moreover, by (β), $\lim_{h\to\mathcal{F}} Th = 0$ in $(L^1)^J$ for the product topology, if each copy of L^1 is given its weak topology. By 4A4Be, this is just $\mathfrak{S}$. In particular, 0 belongs to the $\mathfrak{S}$-closure of $T[C_{k1}^+]$. But C_{k1}^+ is convex and T is linear, so $T[C_{k1}^+]$ is convex; by 4A4Ed, 0 belongs to the $\mathfrak{T}$-closure of $T[C_{k1}^+]$. There is therefore an $h \in C_{k1}^+$ such that $\|(g * h)^\bullet - (\int g\,d\mu)h^\bullet\|_1 \le \epsilon$ for every $g \in J$. As J and ϵ are arbitrary, (ix) is true.

(j)(ix)⇒(x) Suppose that (ix) is true and that we are given a compact set $K \subseteq G$ and $\epsilon > 0$. Set $\eta = \frac{1}{3}\epsilon$. Fix any $h_0 \in C_{k1}^+$. Let V be a neighbourhood of e such that $\|c \bullet_l h_0 - h_0\|_1 \le \eta$ whenever $c \in V$ (443Gf). Let $I \subseteq G$ be a finite set such that $K \subseteq IV$. By (ix), there is an $h_1 \in C_{k1}^+$ such that

$$\|b \bullet_l h_0 * h_1 - h_1\|_1 \le \eta \text{ for every } b \in I, \quad \|h_0 * h_1 - h_1\|_1 \le \eta.$$

(I omit brackets because $(b \bullet_l h_0) * h_1 = b \bullet_l (h_0 * h_1)$, see 444Of.) Set $h = h_0 * h_1$. Then $h \in C_{k1}^+$. **P** h is continuous, by 444Rc (or otherwise). If we write M_i for the support $\operatorname{supp}(h_i)$ of h_i for both i, then $h(x) = 0$ for every $x \in G \setminus M_0 M_1$, so h has compact support. Of course $h \ge 0$, and $\int h\,d\mu = \int h_0 d\mu \int h_1 d\mu = 1$ (444Qb), so $h \in C_{k1}^+$. **Q**

If $a \in K$, there are $b \in I$, $c \in V$ such that $a = bc$, so that

$$\begin{aligned}\|a \bullet_l h - h\|_1 &= \|b \bullet_l (c \bullet_l h_0 * h_1) - h_0 * h_1\|_1 \\ &\le \|b \bullet_l (c \bullet_l h_0 - h_0) * h_1\|_1 + \|b \bullet_l h_0 * h_1 - h_1\|_1 + \|h_1 - h_0 * h_1\|_1 \\ &\le \|c \bullet_l h_0 - h_0\|_1 + \eta + \eta \le 3\eta = \epsilon.\end{aligned}$$

So this h will serve.

(k)(x)⇒(xiii) Suppose that (x) is true.

(α) I show first that for any compact set $K \subseteq G$ and $\epsilon > 0$ there is an $h \in C_{k1}^+$ such that

$$\overleftrightarrow{h} = h, \quad h(e) = \|h\|_\infty,$$

$$\|a \bullet_l h - h\|_1 \le \epsilon \text{ for every } a \in K.$$

P By (x), there is an $h_0 \in C_{k1}^+$ such that $\|a \bullet_l h_0 - h_0\|_1 \le \epsilon$ for every $a \in K$. Set $\gamma = \int \overleftrightarrow{h}_0 d\mu$; because $\overleftrightarrow{h}_0 \in C_k(X)^+ \setminus \{0\}$, γ is finite and not 0. Try $h = h_0 * \frac{1}{\gamma}\overleftrightarrow{h}_0$, so that $h(x) = \frac{1}{\gamma}\int h_0(y) h_0(x^{-1}y)\mu(dy)$ for every $x \in G$. By 444Rc, $h \in C_b(G)$ and

$$\|h\|_\infty \le \|h_0\|_2 \|\frac{1}{\gamma} h_0\|_2 = \frac{1}{\gamma}\int h_0^2 d\mu = h(e).$$

Because $h_0 \ge 0$, $h \ge 0$; by 444Qb, $\int h\,d\mu = 1$; and (as in (j) above) $\operatorname{supp}(h)$ is included in the compact set $\operatorname{supp}(h_0)\operatorname{supp}(\overleftrightarrow{h}_0)$, so $h \in C_{k1}^+$. By 444Rb, or otherwise, $h = \overleftrightarrow{h}$.

Finally, if $a \in K$, then

$$\|a \bullet_l h - h\|_1 = \frac{1}{\gamma}\|a \bullet_l (h_0 * \overleftrightarrow{h}_0) - h_0 * \overleftrightarrow{h}_0\|_1 = \frac{1}{\gamma}\|(a \bullet_l h_0 - h_0) * \overleftrightarrow{h}_0\|_1$$

(444Of once more)

$$\le \frac{1}{\gamma}\|a \bullet_l h_0 - h_0\|_1 \|\overleftrightarrow{h}_0\|_1$$

(444Qb again)

$$= \|a \bullet_l h_0 - h_0\|_1 \le \epsilon,$$

as required. **Q**

(β) Next, for any ϵ, $\delta > 0$ and any compact set $K \subseteq G$ there are an open symmetric neighbourhood V of e and a closed set F such that $\mu V < \infty$, $\mu F \le \delta$ and $\mu(aV \triangle V) \le \epsilon\mu V$ whenever $a \in K \setminus F$. **P** Of course it is enough to deal with the case in which $\mu K > 0$. Set $\eta = \epsilon\delta/\mu K$. By (α), there is an $h \in C_{k1}^+$ such that $h(e) = \|h\|_\infty$, $h = \overleftrightarrow{h}$ and $\|a \bullet_l h - h\|_1 < \eta$ for every $a \in K$.

Set $K_0 = \operatorname{supp}(h)$ and $K^* = K_0 \cup KK_0$, so that $K^* \subseteq G$ is compact. Set

$$\begin{aligned}Q = \{(a, x, t) : a \in K,\ x \in G,\ t \in \mathbb{R}, \\ \text{either } h(x) \le t < h(a^{-1}x) \text{ or } h(a^{-1}x) \le t < h(x)\}.\end{aligned}$$

Then Q is a Borel subset of $G \times G \times \mathbb{R}$ included in the compact set $K \times K^* \times [0, h(e)]$. Let μ_L be Lebesgue measure on $\mathbb{R}$, and let $\mu \times \mu \times \mu_L$ be the τ-additive product measure on $G \times G \times \mathbb{R}$ (417D). (Of course this is actually a Radon measure.) For $t \in \mathbb{R}$ let V_t be the open set $\{x : h(x) > t\}$. Now 417H tells us that

$$(\mu\times\mu\times\mu_L)(Q)=\int_{G\times G}\mu_L\{t:(a,x,t)\in Q\}(\mu\times\mu)(d(a,x))$$

(where $\mu\times\mu$ is the τ-additive product measure on $G\times G$, so that we can identify $\mu\times\mu\times\mu_L$ with $(\mu\times\mu)\times\mu_L$), as in 417Db)

$$=\int_K\int_G|h(a^{-1}x)-h(x)|\mu(dx)\mu(da)$$

(we can use 417H again because $\{x:h(a^{-1}x)\ne h(x)\}\subseteq K^*$ if $a\in K$, and μK^* is finite)

$$=\int_K\|a\bullet_l h-h\|_1\mu(da)<\eta\mu K$$

(by the choice of h)

$$=\eta\mu K\int h\,d\mu=\eta\mu K(\mu\times\mu_L)\{(x,t):0\le t<h(x)\}$$

$$=\eta\mu K\int_0^{h(e)}\mu V_t\,\mu_L(dt)$$

as in 252N. (The c.l.d. and τ-additive product measures on $G\times\mathbb{R}$ coincide, by 417T.) On the other hand,

$$(\mu\times\mu\times\mu_L)(Q)=\int_{K\times\mathbb{R}}\mu\{x:(a,x,t)\in Q\}(\mu\times\mu_L)(d(a,t))$$

(again, we can use 417H because $x\in K^*$ whenever $(a,x,t)\in Q$)

$$=\int_{K\times\mathbb{R}}\mu(V_t\triangle aV_t)(\mu\times\mu_L)(d(a,t))$$

$$=\int_0^\infty\int_K\mu(V_t\triangle aV_t)\mu(da)\mu_L(dt)$$

because $V_t=aV_t=G$ whenever $t<0$ and $a\in G$. So there must be some $t\in\,]0,h(e)[$ such that

$$\int_K\mu(V_t\triangle aV_t)\mu(da)<\eta\mu K\mu V_t=\epsilon\delta\mu V_t.$$

Set $V=V_t$ and $F=\{a:a\in K,\ \mu(V_t\triangle aV_t)\ge\epsilon\mu V_t\}$; then V is open, F is closed (443C) and $\mu(V\triangle aV)\le\epsilon\mu V$ for every $a\in K\setminus F$; also $0<\mu V<\infty$, V is symmetric (because $h=\overset{\leftrightarrow}{h}$) and $e\in V$ (because $t<h(e)$). **Q**

(γ) Now let $K\subseteq G$ be a compact set and $\epsilon>0$, as in the statement of (xiii); enlarging K and lowering ϵ if necessary, we may suppose that $\mu K>0$ and $\epsilon\le 1$. Set $K_1=K\cup KK$, so that K_1 is still compact. By (β), we have a symmetric open neighbourhood V of e, of finite measure, such that $W=\{a:a\in K_1,\ \mu(aV\triangle V)>\frac{1}{3}\epsilon\mu V\}$ has measure less than $\frac{1}{2}\mu K$. If $a\in K$, then $W\cup a^{-1}W$ cannot cover K, so there is a $b\in K\setminus W$ such that $ab\notin W$; thus b and ab both belong to $K_1\setminus W$, and

$$\mu(aV\triangle V)\le\mu(aV\triangle abV)+\mu(abV\triangle V)\le\mu(V\triangle bV)+\frac{1}{3}\epsilon\mu V\le\frac{2}{3}\epsilon\mu V.$$

We can now find a compact symmetric neighbourhood L of e, included in V, with $8\mu(V\setminus L)\le\epsilon\mu L$. In this case, we shall have $\mu L>0$ and

$$\mu(aL\triangle L)\le\mu(aV\setminus aL)+\mu(aV\triangle V)+\mu(V\setminus L)$$

$$\le 2\mu(V\setminus L)+\frac{2}{3}\epsilon(\mu L+\mu(V\setminus L))\le\epsilon\mu L$$

for every $a\in K$, as required.

(l)(xiii)$\Rightarrow$(xiv) Suppose that (xiii) is true, and that $K\subseteq G$ is compact and $\epsilon>0$. Enlarging K if necessary, we may suppose that it includes a neighbourhood of e. Of course we may also suppose that $\epsilon\le 1$.

(α) The first thing to note is that there is a set $I\subseteq G$ such that $KI=G$ and $m=\sup_{y\in G}\#(\{x:x\in I,\ y\in Kx\})$ is finite. **P** Let V be an open neighbourhood of e such that $VV^{-1}\subseteq K$. Let $I\subseteq G$ be maximal subject to

$x^{-1}V \cap y^{-1}V = \emptyset$ for all distinct x, $y \in I$. If $x \in G$, there must be a $y \in I$ such that $x^{-1}V \cap y^{-1}V \neq \emptyset$, so that $x^{-1} \in y^{-1}VV^{-1}$ and $x \in VV^{-1}y \subseteq Ky \subseteq KI$; as x is arbitrary, $G \subseteq KI$. If $y \in G$ and $I_y = \{x : x \in I, y \in Kx\}$, then $I_y \subseteq K^{-1}y$, so that $\{x^{-1}V : x \in I_y\}$ is a disjoint family of subsets of $y^{-1}KV$. But this means that $\#(I_y)\mu V \leq \mu(y^{-1}KV) = \mu(KV)$. Accordingly $\sup_{y \in G} \#(I_y) \leq \frac{\mu(KV)}{\mu V}$ is finite. **Q**

(β) Set $\gamma = \sup_{a \in K} \Delta(a)$, $K^* = KKK^{-1}$. Let $\delta > 0$ be such that

$$\delta\gamma m < \epsilon(\mu K - \delta\gamma),$$

and let $\eta > 0$ be such that

$$1 + \frac{\delta\gamma m}{\mu K - \delta\gamma} \leq (1+\epsilon)(1 - \frac{\eta}{\delta}\mu K^*).$$

By (xiii), there is a non-negligible compact set $L^* \subseteq G$ such that $\mu(aL^* \triangle L^*) \leq \eta\mu L^*$ for every $a \in K^*$. Set $L = \{x : x \in L^*, \mu(K^* \setminus L^*x^{-1}) \leq \delta\}$; note that L is closed (because $x \mapsto \mu(K^* \cap L^*x^{-1})$ is continuous, see 443C), therefore compact.

(γ) $\mu L \geq (1 - \frac{\eta}{\delta}\mu K^*)\mu L^*$. **P** Set $W = \{(x,y) : x \in K^*, y \in L^* \setminus L, xy \notin L^*\}$. Then W is a relatively compact Borel subset of $G \times G$, so we may apply Fubini's theorem (in the form 417H, as usual) to see that

$$\begin{aligned}\delta\mu(L^* \setminus L) &\leq \int_{L^* \setminus L} \mu(K^* \setminus L^*y^{-1})\mu(dy) = \int \mu W^{-1}[\{y\}]\mu(dy)\\ &= \int \mu W[\{x\}]\mu(dx) = \int_{K^*} \mu((L^* \setminus L) \setminus x^{-1}L^*)\mu(dx)\\ &\leq \int_{K^*} \mu(L^* \setminus x^{-1}L^*)\mu(dx) = \int_{K^*} \mu(xL^* \setminus L^*)\mu(dx) \leq \eta\mu K^*\mu L^*\end{aligned}$$

by the choice of L^*. Accordingly

$$\mu L = \mu L^* - \mu(L^* \setminus L) \geq (1 - \frac{\eta}{\delta}\mu K^*)\mu L^*. \ \mathbf{Q}$$

In particular, $\mu L > 0$.

(δ) Set $J = \{x : x \in I, L \cap Kx \neq \emptyset\}$. For each $x \in J$, choose $z_x \in Kx \cap L$. Then $\Delta(z_x) \leq \gamma\Delta(x)$ and $\Delta(x)(\mu K - \gamma\delta) \leq \mu(Kx \cap L^*)$ for every $x \in J$. **P** $\Delta(z_x) = \Delta(z_x x^{-1})\Delta(x) \leq \gamma\Delta(x)$ because $z_x x^{-1} \in K$. Next, because $z_x \in L$,

$$\mu(K^*z_x \setminus L^*) = \Delta(z_x)\mu(K^* \setminus L^*z_x^{-1}) \leq \delta\Delta(z_x).$$

Since $x \in K^{-1}z_x$, $Kx \subseteq KK^{-1}z_x \subseteq K^*z_x$,

$$\mu(Kx \setminus L^*) \leq \mu(K^*z_x \setminus L^*) \leq \delta\Delta(z_x) \leq \delta\gamma\Delta(x)$$

and

$$\mu(Kx \cap L^*) = \mu(Kx) - \mu(Kx \setminus L^*) \geq \Delta(x)\mu K - \delta\gamma\Delta(x) = \Delta(x)(\mu K - \delta\gamma). \ \mathbf{Q}$$

(ϵ) Now recall that $\sum_{x \in I} \chi(Kx) \leq m\chi G$, by the definition of m, so that

$$(\mu K - \delta\gamma)\textstyle\sum_{x \in J} \Delta(x) \leq \sum_{x \in J} \mu(Kx \cap L^*) \leq m\mu L^*.$$

Since $KI = G$, $L \subseteq KJ$ and

$$KL \subseteq \bigcup_{x \in J} KKx \subseteq \bigcup_{x \in J} KKK^{-1}z_x = \bigcup_{x \in J} K^*z_x,$$

$$\begin{aligned}\mu(KL \setminus L^*) &\leq \sum_{x \in J} \mu(K^*z_x \setminus L^*) = \sum_{x \in J} \Delta(z_x)\mu(K^* \setminus L^*z_x^{-1})\\ &\leq \delta\sum_{x \in J} \Delta(z_x) \leq \delta\gamma\sum_{x \in J} \Delta(x) \leq \frac{\delta\gamma m}{\mu K - \delta\gamma}\mu L^*.\end{aligned}$$

Accordingly

$$\mu(KL) \le \mu L^*(1+\frac{\delta\gamma m}{\mu K-\delta\gamma}) \le \mu L\cdot\frac{1+\frac{\delta\gamma m}{\mu K-\delta\gamma}}{1-\frac{\eta}{\delta}\mu K^*}$$

(by (γ) above)

$$\le (1+\epsilon)\mu L,$$

by the choice of δ and η. Thus we have found an appropriate set L.

(m)(xiv)⇒(xii) If $I\subseteq G$ is finite and $\epsilon>0$, $I\cup\{e\}$ is compact, so there is a compact set $L\subseteq G$, of non-zero measure, such that $\mu(IL\cup L)\le(1+\frac{1}{2}\epsilon)\mu L$. Consequently

$$\mu(L\triangle aL) = 2\mu(aL\setminus L) \le 2\mu(IL\setminus L) \le \epsilon\mu L$$

for every $a\in I$, as required by (xii).

(n)(xii)⇒(ii) Write $\mathcal{L}$ for the family of all compact subsets of G with non-zero measure. For $L\in\mathcal{L}$, define $p_L:C_b(G)\to\mathbb{R}$ by setting $p_L(f)=\frac{1}{\mu L}\int_L fd\mu$ for $f\in C_b(G)$. Of course $|p_L(f)|\le\|f\|_\infty$. For finite $I\subseteq G$, $\epsilon>0$ set

$$\mathcal{A}(I,\epsilon)=\{L: L\in\mathcal{L},\ \mu(aL\triangle L)\le\epsilon\mu L \text{ for every } a\in I\}.$$

By (xii), no $\mathcal{A}(I,\epsilon)$ is empty. So we have an ultrafilter $\mathcal{F}$ on $\mathcal{L}$ containing every $\mathcal{A}(I,\epsilon)$. Set $p(f)=\lim_{L\to\mathcal{F}}p_L(f)$ for $f\in C_b(G)$; then $p:C_b(G)\to\mathbb{R}$ is a positive linear functional and $p(\chi G)=1$.

If $a\in G$, $f\in C_b(G)$ and $L\in\mathcal{A}(\{a\},\epsilon)$, then

$$\begin{aligned}|p_L(a\bullet_l f)-p_L(f)| &= \frac{1}{\mu L}\Bigl|\int_L f(a^{-1}x)\mu(dx)-\int_L f(x)\mu(dx)\Bigr|\\ &= \frac{1}{\mu L}\Bigl|\int_{aL} fd\mu-\int_L fd\mu\Bigr| \le \frac{1}{\mu L}\|f\|_\infty\mu(aL\triangle L)\le\epsilon\|f\|_\infty.\end{aligned}$$

Since $\mathcal{F}$ contains $\mathcal{A}(\{a\},\epsilon)$ for every $\epsilon>0$,

$$|p(a\bullet_l f)-p(f)| = \lim_{L\to\mathcal{F}}|p_L(a\bullet_l f)-p_L(f)| = 0.$$

As f and a are arbitrary, p witnesses that (ii) is true.

(o)(xii)⇒(xi) Given a finite set $I\subseteq G$ and $\epsilon>0$, (xii) tells us that there is a compact set $L\subseteq G$ of non-zero measure such that $\mu(aL\triangle L)\le\epsilon^q\mu L$ for every $a\in I$. Try $u=\frac{1}{(\mu L)^{1/q}}(\chi L)^\bullet$. Then $\|u\|_q=1$. If $a\in I$, then $a\bullet_l u=\frac{1}{(\mu L)^{1/q}}\chi(aL)^\bullet$, so

$$\int|u-a\bullet_l u|^q = \int(\frac{1}{(\mu L)^{1/q}}\chi(aL\triangle L)^\bullet)^q = \frac{\mu(aL\triangle L)}{\mu L}\le\epsilon^q$$

and $\|u-a\bullet_l u\|_q\le\epsilon$, as required by (xi).

(p)(xi)⇒(xii) Now assume that (xi) is true. Let $I\subseteq G$ be a finite set, and $\epsilon>0$. Set $\delta=\frac{\epsilon}{4+\epsilon}$, and let $\eta>0$ be such that that $(1+\eta\#(I))^q\le1+\delta$.

Take $u\in L^q$ such that $\|u\|_q=1$ and $\|u-a\bullet_l u\|_q\le\eta$ for every $a\in I$. Setting $v=|u|$, we see that $a\bullet_l v=|a\bullet_l u|$, so $|v-a\bullet_l v|\le|u-a\bullet_l u|$ and $\|v-a\bullet_l v\|_q\le\eta$ for every $a\in I$, while $\|v\|_q=1$. Let $f:G\to[0,\infty[$ be a function such that $f^\bullet=v$ in L^q; then $\|f\|_q=1$ while $\|f-a\bullet_l f\|_q\le\eta$ for every $a\in I$. Set $g=\sup_{a\in I\cup\{e\}}a\bullet_l f$; then

$$f\le g\le f+\textstyle\sum_{a\in I}(a\bullet_l f-f)^+,$$

so

$$\|g\|_q\le1+\textstyle\sum_{a\in I}\|a\bullet_l f-f\|_q\le1+\eta\#(I),\qquad \int g^q d\mu\le(1+\eta\#(I))^q\le1+\delta.$$

For $t>0$, set $E_t=\{x:f(x)^q\ge t\}$, $F_t=\{t:g(x)^q\ge t\}$. Then

$$\int_0^\infty\mu E_t dt=\int f^q d\mu=1,\qquad \int_0^\infty\mu F_t dt=\int g^q d\mu\le1+\delta,$$

where the integrals here are with respect to Lebesgue measure (252O). There must therefore be a $t>0$ such that $\mu E_t>0$ and $\mu F_t\le(1+\delta)\mu E_t$.

If $a \in I$, then $a \bullet_l f \le g$, so $aE_t = \{x : (a \bullet_l f)(x) \ge t\}$ is included in F_t; also, of course, $E_t \subseteq F_t$. We therefore have

$$\mu(E_t \triangle aE_t) = 2\mu(aE_t \setminus E_t) \le 2\mu(F_t \setminus E_t) \le 2\delta\mu E_t.$$

There is no reason why E_t should be compact, so it may not itself be the L we seek. However, μE_t is certainly finite, so there must be a compact $L \subseteq E$ such that $\mu L \ge (1-\delta)\mu E_t$. In this case, $\mu L > 0$ and

$$\begin{aligned}\mu(aL \triangle L) &\le \mu(aL \triangle aE_t) + \mu(aE_t \triangle E_t) + \mu(E_t \triangle L)\\ &\le 2\mu(E_t \setminus L) + 2\delta\mu E_t \le 4\delta\mu E_t \le \tfrac{4\delta}{1-\delta}\mu L = \epsilon\mu L\end{aligned}$$

for every $a \in I$. So this L will serve.

Remark Of course there are many variations possible in the conditions listed above, some of which are in 449Xk-449Xm.

449K Proposition Let G be an amenable locally compact Hausdorff group, and H a subgroup of G. Then H is amenable.

proof (Paterson 88, 1.12) **(a)** For most of the proof (down to the end of (g) below), suppose that H is closed. Let V be a compact neighbourhood of the identity in G. Let $I \subseteq G$ be a maximal set such that $VzH \cap Vz'H = \emptyset$ for all distinct z, $z' \in I$. Then $V^{-1}VIH = G$. **P** If $x \in G$, there is a $z \in I$ such that $VxH \cap VzH \ne \emptyset$, that is,

$$x \in V^{-1}VzHH^{-1} = V^{-1}VzH \subseteq V^{-1}VIH. \text{ } \mathbf{Q}$$

(b) If $x \in G$ then

$$I \cap V^{-1}xH = \{z : z \in I,\, z \in V^{-1}xH\} = \{z : z \in I,\, x \in VzH^{-1} = VzH\}$$

has at most one element. If $K \subseteq G$ is compact, then there is a finite set $J \subseteq G$ such that $K \subseteq V^{-1}J$, and now

$$I \cap KH \subseteq \bigcup_{x \in J} I \cap V^{-1}xH$$

is finite.

(c) Let $h \in C_k(X)^+$ be such that $h \ge \chi(V^{-1}V)$; write W for the support of h. Set $g(x) = \sum_{z \in I} h(xz^{-1})$ for $x \in G$.

If $K \subseteq G$ is compact, then

$$\begin{aligned}\{z : z \in I,\, &h(xyz^{-1}) \ne 0 \text{ for some } x \in K \text{ and } y \in H\}\\ &\subseteq I \cap \{z : KHz^{-1} \cap W \ne \emptyset\}\\ &= I \cap \{z : zH^{-1}K^{-1} \cap W^{-1} \ne \emptyset\} = I \cap W^{-1}KH\end{aligned}$$

is finite. In particular, $\{z : z \in I,\, h(xz^{-1}) \ne 0\}$ and $g(x)$ are finite for every $x \in G$. Next, if $x_0 \in G$, then $J = \{z : h(xz^{-1}) \ne 0 \text{ for some } x \in x_0V\}$ is finite, and $g(x) = \sum_{z \in J} h(xz^{-1})$ for $x \in x_0V$, so g is continuous at x_0; as x_0 is arbitrary, $g \in C(G)$. Of course $g \ge 0$ because $h \ge 0$.

(d)(i) For $x \in G$, set $g_x(y) = g(x^{-1}y)$ for $y \in H$. Then $g_x \in C_k(H)$. **P** g_x is continuous because g is. Now $J = \{z : z \in I,\, h(x^{-1}yz^{-1}) \ne 0 \text{ for some } y \in H\}$ is finite, and

$$\begin{aligned}\{y : g_x(y) \ne 0\} &\subseteq \{y : \text{ there is a } z \in I \text{ such that } h(x^{-1}yz^{-1}) \ne 0\}\\ &\subseteq \{y : \text{ there is a } z \in J \text{ such that } x^{-1}yz^{-1} \in W\} = xWJ\end{aligned}$$

which is compact. **Q**

(ii) Moreover, for any $x_0 \in G$ there is a compact set $L \subseteq H$ such that $\{x : |g_x - g_{x_0}| \le \epsilon\chi L\}$ is a neighbourhood of x_0 for every $\epsilon > 0$. **P**

$$J = \{z : z \in I,\, h(x^{-1}yz^{-1}) \ne 0 \text{ for some } x \in x_0V \text{ and } y \in H\}$$

is finite, by (c) in its full strength. Let L be the compact set $H \cap x_0VWJ$.

Take any $\epsilon > 0$. If $x \in x_0V$, then $g_x(y) = \sum_{z \in J} h(x^{-1}yz^{-1})$ for every $y \in H$, and $g_x(y) = 0$ for $y \in H \setminus L$. Moreover, setting $g'_x(y) = \sum_{z \in J} h(x^{-1}yz^{-1})$ for $x \in G$ and $y \in H$, $(x, y) \mapsto g'_x(y)$ is continuous, so $x \mapsto g'_x : G \to C(H)$ is continuous if we give $C(H)$ the topology of uniform convergence on compact subsets of H (4A2G(g-i)). In particular, $x \mapsto g'_x\restriction L$ is continuous for the norm topology of $C(L)$, and $U = \{x : x \in x_0V,\ \|g'_x\restriction L - g'_{x_0}\restriction L\|_\infty \le \epsilon\}$ is a neighbourhood of x_0. But if $x \in U$, then

$$g_x(y) = g_{x_0}(y) = 0 \text{ for } y \in H \setminus L,$$

$$|g_x(y) - g_{x_0}(y)| = |g'_x(y) - g'_{x_0}(y)| \le \epsilon \text{ for } y \in L,$$

so $|g_x - g_{x_0}| \le \epsilon\chi L$. **Q**

(e) Now take a left Haar measure ν on H. (This is where it really matters whether H is closed.) Define $T : C_b(H) \to \mathbb{R}^G$ by setting

$$(Tf)(x) = \int g_x \times f d\nu = \int_H g(x^{-1}y)f(y)\nu(dy)$$

for $f \in C_b(H)$ and $x \in G$. Then $Tf \in C(G)$ for every $f \in C_b(H)$. **P** Given $x_0 \in G$ and $\epsilon > 0$, let $L \subseteq H$ be a compact set as in (d-ii). Let $\delta > 0$ be such that $\delta \int_L |f| d\nu \le \epsilon$. Then

$$\{x : |(Tf)(x) - (Tf)(x_0)| \le \epsilon\} \supseteq \{x : |g_x - g_{x_0}| \le \delta\chi L\}$$

is a neighbourhood of x_0. As x_0 and ϵ are arbitrary, Tf is continuous. **Q**

Clearly, $T : C_b(H) \to C(G)$ is a positive linear operator. Next, if $f \in C_b(H)$ and $b \in H$, $T(b\bullet_l f) = b\bullet_l(Tf)$. **P**

$$\begin{aligned} T(b\bullet_l f)(x) &= \int_H g(x^{-1}y)(b\bullet_l f)(y)\nu(dy) = \int_H g(x^{-1}y)f(b^{-1}y)\nu(dy) \\ &= \int_H g(x^{-1}by)f(y)\nu(dy) = (Tf)(b^{-1}x) = (b\bullet_l Tf)(x) \end{aligned}$$

for every $x \in G$. **Q**

We need to know that $T(\chi H)(x) > 0$ for every $x \in G$. **P** There is a $z \in I$ such that $x^{-1} \in V^{-1}VzH$, as remarked in (a). Now $x^{-1}Hz^{-1}$ meets $V^{-1}V$, so there is a $y \in H$ such that

$$1 \le h(x^{-1}yz^{-1}) \le g(x^{-1}y) = g_x(y)$$

and $T(\chi H)(x) = \int_H g_x d\nu > 0$ because g_x is continuous and non-negative and ν is strictly positive. **Q**

(f) We therefore have a positive linear operator $S : C_b(H) \to C(G)$ defined by setting $Sf = \frac{Tf}{T(\chi H)}$ for $f \in C_b(H)$. Since $S(\chi H) = \chi G$, $S[C_b(H)] \subseteq C_b(G)$; moreover, for $f \in C_b(H)$ and $b \in H$,

$$\begin{aligned} S(b\bullet_l f) &= \frac{T(b\bullet_l f)}{T(\chi H)} = \frac{T(b\bullet_l f)}{T(b\bullet_l \chi H)} \\ &= \frac{b\bullet_l (Tf)}{b\bullet_l (T\chi H)} = b\bullet_l \frac{Tf}{T\chi H} = b\bullet_l Sf. \end{aligned}$$

(g) At this point, recall that by 449J(ii) there is a positive linear functional $p : C_b(G) \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a\bullet_l f) = p(f)$ whenever $f \in C_b(G)$ and $a \in G$. Set $q = pS : C_b(H) \to \mathbb{R}$; then q is a positive linear functional, $q(\chi H) = 1$ and $q(b\bullet_l f) = q(f)$ whenever $f \in C_b(H)$ and $b \in H$. So q witnesses that 449J(ii) is true for H, and H is amenable.

(h) All this has been on the assumption that H is closed. But in general $\overline{H}$ is a closed subgroup of G, therefore amenable by (a)-(g) here, and H is dense in $\overline{H}$, therefore amenable by 449F(a-ii).

449L If we make a further step back towards the origin of this topic, and suppose that our group is discrete, then we have a striking further condition to add to the lists above. I give this as a corollary of a general result on group actions recalling the main theorems of §§395 and 448.

Tarski's theorem Let G be a group acting on a non-empty set X. Then the following are equiveridical:

(i) there is an additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu X = 1$ and $\nu(a\bullet A) = \nu A$ whenever $A \subseteq X$ and $a \in G$;

(ii) there are no $A_0, \dots, A_n$, $a_0, \dots, a_n$, $b_0, \dots, b_n$ such that $A_0, \dots, A_n$ are subsets of X covering X, $a_0, \dots, a_n$, b_0, $\dots, b_n$ belong to G, and $a_0\bullet A_0$, $b_0\bullet A_0$, $a_1\bullet A_1$, $b_1\bullet A_1, \dots, b_n\bullet A_n$ are all disjoint.

proof (a)(i)⇒(ii) This is elementary, for if $\nu : \mathcal{P}X \to [0,1]$ is a non-zero additive functional and $A_0, \dots, A_n$ cover X and $a_0, \dots, b_n \in G$, then

$$\sum_{i=0}^{n} \nu(a_i \bullet A_i) + \sum_{i=0}^{n} \nu(b_i \bullet A_i) = 2\sum_{i=0}^{n} \nu A_i \geq 2\nu X > \nu X,$$

and $a_0 \bullet A_0, \dots, b_n \bullet A_n$ cannot all be disjoint.

(b)(ii)⇒(i) Now suppose that (ii) is true.

(α) Suppose that $c_0, \dots, c_n \in G$. Then there is a finite set $I \subseteq X$ such that $\#(\{c_i \bullet x : i \leq n, x \in I\}) < 2\#(I)$. **P?** Otherwise, by the Marriage Lemma in the form 4A1H, applied to the set

$$R = \{((x, j), c_i \bullet x) : x \in X, j \in \{0, 1\}, i \leq n\} \subseteq (X \times \{0, 1\}) \times X,$$

there is an injective function $\phi : X \times \{0, 1\} \to X$ such that $\phi(x, j) \in \{c_i \bullet x : i \leq n\}$ for every $x \in X$ and $j \in \{0, 1\}$. Now set $B_{ij} = \{x : \phi(x, 0) = c_i \bullet x, \phi(x, 1) = c_j \bullet x\}$ for $i, j \leq n$, so that $X = \bigcup_{i,j \leq n} B_{ij}$. Let $A_{ij} \subseteq B_{ij}$ be such that $\langle A_{ij}\rangle_{i,j \leq n}$ is a partition of X, and set $a_{ij} = c_i$, $b_{ij} = c_j$ for $i, j \leq n$; then $a_{ij} \bullet A_{ij} \subseteq \phi[A_{ij} \times \{0\}]$, $b_{ij} \bullet A_{ij} \subseteq \phi[A_{ij} \times \{1\}]$ are all disjoint, which is supposed to be impossible. **XQ**

(β) Suppose that $J \subseteq G$ is finite and $\epsilon > 0$. Then there is a non-empty finite set $I \subseteq X$ such that $\#(I \triangle c \bullet I) \leq \epsilon\#(I)$ for every $c \in J$. **P** It is enough to consider the case in which the identity e of G belongs to J. **?** Suppose, if possible, that there is no such set I. Let $m \geq 1$ be such that $(1 + \frac{1}{2}\epsilon)^m \geq 2$. Set $K = J^m = \{c_1 c_2 \dots c_m : c_1, \dots, c_m \in J\}$. By ($\alpha$), there is a finite set $I_0 \subseteq X$ such that $\#(I_0^*) < 2\#(I_0)$, where $I_0^* = \{a \bullet x : a \in K, x \in I_0\}$. Choose $c_1, \dots, c_m$ and $I_1, \dots, I_m$ inductively such that

given that I_k is a non-empty finite subset of X, where $0 \leq k < m$, take $c_{k+1} \in J$ such that $\#(I_k \triangle c_{k+1} \bullet I_k) > \epsilon\#(I_k)$ and set $I_{k+1} = I_k \cup c_{k+1} \bullet I_k$.

Then $I_k \subseteq \{a \bullet x : a \in J^k, x \in I_0\}$ for each $k \leq m$, and in particular $I_m \subseteq I_0^*$. But also

$$\begin{aligned}\#(I_{k+1}) &= \#(I_k) + \#((c_{k+1} \bullet I_k) \setminus I_k) \\ &= \#(I_k) + \frac{1}{2}\#((c_{k+1} \bullet I_k) \triangle I_k) \geq (1 + \frac{1}{2}\epsilon)\#(I_k)\end{aligned}$$

for every $k < m$, so

$$\#(I_0^*) \geq \#(I_m) \geq (1 + \frac{1}{2}\epsilon)^m \#(I_0) \geq 2\#(I_0),$$

contrary to the choice of I_0. **XQ**

(γ) There is therefore an ultrafilter $\mathcal{F}$ on $[X]^{<\omega} \setminus \{\emptyset\}$ such that

$$\mathcal{A}_{c\epsilon} = \{I : I \in [X]^{<\omega} \setminus \{\emptyset\}, \#(I \triangle c \bullet I) \leq \epsilon\#(I)\}$$

belongs to $\mathcal{F}$ for every $c \in G$ and $\epsilon > 0$. For $I \in [X]^{<\omega} \setminus \{\emptyset\}$ and $A \subseteq X$ set $\nu_I(A) = \#(A \cap I)/\#(I)$, and set $\nu A = \lim_{I \to \mathcal{F}} \nu_I A$ for every $A \subseteq X$, so that $\nu : \mathcal{P}X \to [0, 1]$ is an additive functional and $\nu X = 1$.

Now ν is G-invariant. **P** If $A \subseteq X$ and $c \in G$ and $\epsilon > 0$, then $\mathcal{A}_{c^{-1},\epsilon} \in \mathcal{F}$. If $I \in \mathcal{A}_{c^{-1},\epsilon}$, then

$$\begin{aligned}|\nu_I(c \bullet A) - \nu_I(A)| &= \frac{1}{\#(I)}|\#(I \cap (c \bullet A)) - \#(I \cap A)| \\ &= \frac{1}{\#(I)}|\#((c^{-1} \bullet I) \cap A) - \#(I \cap A)| \\ &\leq \frac{1}{\#(I)}\#((c^{-1} \bullet I) \triangle I) \leq \epsilon.\end{aligned}$$

As ϵ is arbitrary, $\lim_{I \to \mathcal{F}} \nu_I(c \bullet A) - \nu_I(A) = 0$, and $\nu(c \bullet A) = \nu A$. As A and c are arbitrary, ν is G-invariant. **Q**

So ν witnesses that (i) is true, and the proof is complete.

449M Corollary Let G be a group with its discrete topology. Then the following are equiveridical:

(i) G is amenable;

(ii) there are no $A_0, \dots, A_n$, $a_0, \dots, a_n$, $b_0, \dots, b_n$ such that $G = \bigcup_{i \leq n} A_i$ $a_0, \dots, a_n, b_0, \dots, b_n$ belong to G, and $a_0 A_0$, $b_0 A_0$, $a_1 A_1$, $b_1 A_1, \dots, b_n A_n$ are disjoint.

proof All we have to observe is that (α) every function from G to $\mathbb{R}$ is uniformly continuous for the right uniformity of G, so that G is amenable iff there is an invariant positive linear functional $p : \ell^\infty(G) \to \mathbb{R}$ such that $p(\chi G) = 1$ (β) that a positive linear functional on $\ell^\infty(G)$ is G-invariant iff the corresponding additive functional on $\mathcal{P}G$ is G-invariant. So (i) of 449L is equivalent to amenability of G as defined in 449A.

449N Theorem Let G be a group which is amenable in its discrete topology, X a set, and $\bullet$ an action of G on X. Let $\mathcal{E}$ be a subring of $\mathcal{P}X$ and $\nu : \mathcal{E} \to [0, \infty[$ a finitely additive functional which is G-invariant in the sense that $g \bullet E \in \mathcal{E}$ and $\nu(g \bullet E) = \nu E$ whenever $E \in \mathcal{E}$ and $g \in G$. Then there is an extension of ν to a G-invariant non-negative finitely additive functional $\tilde{\nu}$ defined on the ideal $\mathcal{I}$ of subsets of X generated by $\mathcal{E}$.

proof (a) There is a non-negative finitely additive functional $\theta : \mathcal{I} \to \mathbb{R}$ extending ν. **P** Let V be the linear subspace of $\ell^\infty(X)$ generated by $\{\chi E : E \in \mathcal{E}\}$, so that V can be identified with the Riesz space $S(\mathcal{E})$ (361L). Let U be the solid linear subspace of $\ell^\infty(X)$ generated by V. For $u \in U$ set $q(u) = \inf\{⨍ v\, d\nu : v \in V, |u| \le v\}$, where $⨍ d\nu : S(\mathcal{E}) \to \mathbb{R}$ is the positive linear functional corresponding to $\nu : \mathcal{E} \to [0, \infty[$ as in 361F-361G. Then q is a seminorm, and $|⨍ v\, d\nu| \le q(v)$ for every $v \in V$. So there is a linear functional $f : U \to \mathbb{R}$ such that $f(v) = ⨍ v\, d\nu$ for every $v \in V$ and $|f(u)| \le q(u)$ for every $u \in U$ (4A4D(a-i)). Set $\theta A = f(\chi A)$ for $A \in \mathcal{I}$. Then $\theta : \mathcal{I} \to \mathbb{R}$ is additive and extends ν. If $A \in \mathcal{I}$, there is an $E \in \mathcal{E}$ including A. Now we have

$$\theta(E \setminus A) = f(\chi E - \chi A) \le q(\chi E - \chi A) \le ⨍ \chi E\, d\nu = \nu E = \theta E,$$

so $\theta A \ge 0$. So θ is non-negative. **Q**

(b) As in the proof of 449M, we have a positive G-invariant linear functional $p : \ell^\infty(G) \to \mathbb{R}$ such that $p(\chi G) = 1$. For $A \in \mathcal{I}$, set $f_A(a) = \theta(a^{-1} \bullet A)$ for $a \in G$, and $\tilde{\nu} A = p(f_A)$. Then $\tilde{\nu} : \mathcal{I} \to [0, \infty[$ is additive. If $E \in \mathcal{E}$ then $f_A(a) = \nu E$ for every a, so $\tilde{\nu}$ extends ν. If $A \in \mathcal{I}$ and $a, b \in G$, then $f_{aA}(b) = \theta(b^{-1}aA) = f_A(a^{-1}b)$, so $f_{aA} = a \bullet_l f_A$ and

$$\tilde{\nu}(aA) = p(f_{aA}) = p(f_A) = \tilde{\nu} A.$$

Thus $\tilde{\nu}$ is G-invariant, as required.

449O Corollary (Banach 1923) If $r = 1$ or $r = 2$, there is a functional $\theta : \mathcal{P}\mathbb{R}^r \to [0, \infty]$ such that (i) $\theta(A \cup B) = \theta A + \theta B$ whenever $A, B \subseteq \mathbb{R}^r$ are disjoint (ii) θE is the Lebesgue measure of E whenever $E \subseteq \mathbb{R}^r$ is Lebesgue measurable (iii) $\theta(g[A]) = \theta A$ whenever $A \subseteq \mathbb{R}^r$ and $g : \mathbb{R}^r \to \mathbb{R}^r$ is an isometry.

proof (a) The point is that the group G of all isometries of $\mathbb{R}^r$, with its discrete topology, is amenable. **P** Let $G_0 \subseteq G$ be the subgroup consisting of rotations about $\mathbf{0}$; because $r \le 2$, this is abelian, therefore amenable (449Cf). Let $G_1 \subseteq G$ be the subgroup consisting of isometries keeping $\mathbf{0}$ fixed; then G_0 is a normal subgroup of G_1, and G_1/G_0 is abelian, so G_1 is amenable (449Cc). Let $G_2 \subseteq G$ be the normal subgroup consisting of translations; then G_2 is abelian, therefore amenable. Now $G = G_1 G_2$, so G is amenable (449Cd). **Q**

(b) Let $\mathcal{E}$ be the ring of subsets of $\mathbb{R}^r$ with finite Lebesgue measure, and let ν be the restriction of Lebesgue measure to $\mathcal{E}$. Then ν is G-invariant. By 449N, there is a G-invariant extension $\tilde{\nu}$ of ν to the ideal $\mathcal{I}$ generated by $\mathcal{E}$. Setting $\theta A = \tilde{\nu} A$ for $A \in \mathcal{I}$, ∞ for $A \in \mathcal{P}\mathbb{R}^r \setminus \mathcal{I}$, we have a suitable functional θ.

449X Basic exercises >(a) Let G be a topological group. On G define a binary operation $\diamond$ by saying that $x \diamond y = yx$ for all $x, y \in G$. Show that $(G, \diamond)$ is a topological group isomorphic to G, so is amenable iff G is.

(b) Show that any finite topological group is amenable.

>(c) Show that, for any $r \in \mathbb{N}$, the isometry group of $\mathbb{R}^r$, with the topology of pointwise convergence, is amenable. (*Hint*: 443Xw, 449Cd.)

(d) Find a locally compact Polish group which is amenable but not unimodular. (*Hint*: 442Xf, 449Cd.)

(e) Prove 449Cg directly from 441C, without mentioning Haar measure.

(f) Let G be a topological group and U the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity of G. Show that $\bullet_r$, as defined in 4A5Cc, gives an action of G on U.

(g) Let $\bullet$ be an action of a group G on a set X, and U a Riesz subspace of $\ell^\infty(X)$, containing the constant functions, such that $a \bullet f \in U$ whenever $f \in U$ and $a \in G$. Show that the following are equiveridical: (i) there is a G-invariant positive linear functional $p : U \to \mathbb{R}$ such that $p(\chi X) = 1$; (ii) $\sup_{x \in X} \sum_{i=0}^n f_i(x) - f_i(a_i \bullet x) \ge 0$ whenever $f_0, \ldots, f_n \in U$ and $a_0, \ldots, a_n \in G$. (*Hint*: if (ii) is true, let V be the linear subspace generated by $\{f - a \bullet f : f \in U, a \in G\}$ and show that $\inf_{g \in V} \|g - \chi X\|_\infty = 1$.)

>(h) Let X be a set and G the group of all permutations of X. (i) Give X the zero-one metric, so that G is the isometry group of X. Show that G, with the topology of pointwise convergence (441G), is amenable. (*Hint*: for any $I \in [X]^{<\omega}$, $\{a : a \in G,\, a(x) = x$ for every $x \in X \setminus I\}$ is amenable.) (ii) Show that if X is infinite then G, with its discrete topology, is not amenable. (*Hint*: the left action of F_2 on itself can be regarded as an embedding of F_2 in G.)

(i) Let G be a Hausdorff topological group, and $\hat{G}$ its completion with respect to its bilateral uniformity (definition: 4A5Hb). Show that G is amenable iff $\hat{G}$ is.

(j)(i) Let G be the group with generators a, b and relations $a^2 = b^3 = e$ (that is, the quotient of the free group on two generators a and b by the normal subgroup generated by $\{a^2, b^3\}$). Show that G, with its discrete topology, is not amenable. (ii) Let G be the group with generators a, b and relations $a^2 = b^2 = e$. Show that G, with its discrete topology, is amenable. (*Hint*: we have a function length : $G \to \mathbb{N}$ such that $\mathrm{length}(ab) \le \mathrm{length}(a) + \mathrm{length}(b)$ for all a, $b \in G$ and $\limsup_{n\to\infty} \frac{1}{n}\#(\{a : \mathrm{length}(a) \le n\})$ is finite. See also 449Yf.)

(k) Let G be a locally compact Hausdorff group, and μ a left Haar measure on G. Show that G is amenable iff for every finite set $I \subseteq G$, finite set $J \subseteq \mathcal{L}^\infty(\mu)$ and $\epsilon > 0$, there is an $h \in C_{k1}(G)^+$ (definition: 449J) such that $|\int f(ax)h(x)\mu(dx) - \int f(x)h(x)\mu(dx)| \le \epsilon$ for every $a \in I$ and $f \in J$. (*Hint*: the image of the unit ball in L^1 is weak* dense in the unit ball of $(L^\infty)^*$.)

(l) Let G be a locally compact Hausdorff group, and μ a left Haar measure on G. Show that the following are equiveridical: (i) G is amenable; (ii) there is a positive linear functional $p^\# : L^\infty(\mu) \to \mathbb{R}$ such that $p^\#(\chi G^\bullet) = 1$ and $p^\#(a \bullet_r u) = p^\#(u)$ for every $u \in L^\infty(\mu)$ and every $a \in G$; (iii) for every finite set $I \subseteq G$, finite set $J \subseteq \mathcal{L}^\infty(\mu)$ and $\epsilon > 0$, there is an $h \in C_{k1}^+$ such that $|\int f(xa)h(x)\mu(dx) - \int f(x)h(x)\mu(dx)| \le \epsilon$ for every $a \in I$ and $f \in J$.

(m) Let G be a locally compact Hausdorff group and $\mathcal{B}\mathfrak{a}_G$ its Baire σ-algebra. Show that G is amenable iff there is a non-zero finitely additive $\phi : \mathcal{B}\mathfrak{a}_G \to [0,1]$ such that $\phi(aE) = \nu E$ for every $a \in G$ and $E \in \mathcal{B}\mathfrak{a}_G$.

(n) A **symmetric Følner sequence** in a group G is a sequence $\langle L_n\rangle_{n\in\mathbb{N}}$ of non-empty finite symmetric subsets of G such that $\lim_{n\to\infty} \frac{\#(L_n \triangle aL_n)}{\#(L_n)} = 0$ for every $a \in G$. Show that a group G has a symmetric Følner sequence iff it is countable and amenable when given its discrete topology.

>(o) Let G be a group which is amenable when given its discrete topology. Let $\phi : \mathcal{P}G \to [0,1]$ be an additive functional such that $\phi G = 1$ and $\phi(aE) = \phi E$ whenever $E \subseteq G$ and $a \in G$. For $E \subseteq G$ set $\psi E = \fint \phi(Ex)\phi(dx)$. Show that $\psi : \mathcal{P}G \to [0,1]$ is an additive functional, that $\psi G = 1$ and that $\psi(aE) = \psi(Ea) = \psi E$ for every $E \subseteq G$ and $a \in G$.

(p) Let X be a non-empty set, G a group and $\bullet$ an action of G on X. Suppose that G is an amenable group when given its discrete topology. Show that there is an additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu X = 1$ and $\nu(a \bullet A) = \nu A$ for every $A \subseteq X$ and every $a \in G$.

(q) Let G be a locally compact Hausdorff group and μ a left Haar measure on G. Show that if G, with its discrete topology, is amenable, then there is a functional $\phi : \mathcal{P}G \to [0,\infty]$, extending μ, such that $\phi(A \cup B) = \phi A + \phi B$ whenever A, $B \subseteq G$ are disjoint and $\phi(xA) = \phi A$ for every $x \in G$ and $A \subseteq G$.

(r) Let X be a compact metrizable space, $\phi : X \to X$ a continuous function and μ a Radon probability measure on X such that $\mu\phi^{-1} = \mu$. Show that for μ-almost every $x \in X$, $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$ is defined for every $f \in C(X)$. (*Hint*: 4A2Pe, 372J.)

449Y Further exercises (a) If S is a semigroup with identity e and X is a set, an **action** of S on X is a map $(s,x) \mapsto s \bullet x : S \times X \to X$ such that $s \bullet (t \bullet x) = (st) \bullet x$ and $e \bullet x = x$ for every s, $t \in S$ and $x \in X$. A topological semigroup S with identity is **amenable** if for every non-empty compact Hausdorff space X and every continuous action of S on X there is a Radon probability measure μ on X such that $\int f(s \bullet x)\mu(dx) = \int f(x)\mu(dx)$ for every $s \in S$ and $f \in C(X)$. Show that

(i) $(\mathbb{N}, +)$, with its discrete topology, is amenable;

(ii) if S is a topological semigroup and $\mathcal{S}$ is an upwards-directed family of amenable sub-semigroups of S with dense union in S, then S is amenable;

(iii) if $\langle S_i\rangle_{i\in I}$ is a family of amenable topological semigroups with product S then S is amenable;

(iv) if S is an amenable topological semigroup, S' is a topological semigroup, and there is a continuous multiplicative surjection from S onto S', then S' is amenable;

(v) if S is an abelian topological semigroup, then it is amenable.

(b) Give an example of a topological semigroup S with identity such that S is amenable in the sense of 449Ya but $(S,\diamond)$ is not, where $a\diamond b=ba$ for a, $b\in S$.

(c) Let G be a topological group and U the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity. Let M_{qR}^+ be the space of totally finite quasi-Radon measures on G. (i) Show that if $\nu\in M_{\mathrm{qR}}^+$ then $\nu*f$ (definition: 444H) belongs to U for every $f\in U$. (ii) Show that $(\nu,f)\mapsto\nu*f:M_{\mathrm{qR}}^+\times U\to U$ is continuous if M_{qR}^+ is given its narrow topology and U is given its norm topology. (iii) Show that if $p:U\to\mathbb{R}$ is a continuous linear functional such that $p(a\bullet_l f)=p(f)$ for every $f\in U$ and $a\in G$, then $p(\nu*f)=\nu G\cdot p(f)$ for every $f\in U$ and every totally finite quasi-Radon measure ν on G.

(d) Re-work 449J for general groups carrying Haar measures.

(e) Let G be a group with a symmetric Følner sequence $\langle L_n\rangle_{n\in\mathbb{N}}$ (449Xn), and $\bullet$ an action of G on a reflexive Banach space U such that $u\mapsto a\bullet u$ is a linear operator of norm at most 1 for every $a\in G$. For $n\in\mathbb{N}$ set $T_nu=\frac{1}{\#(L_n)}\sum_{a\in L_n}a\bullet u$ for $u\in U$. Show that for every $u\in U$ the sequence $\langle T_nu\rangle_{n\in\mathbb{N}}$ is norm-convergent to a $v\in U$ such that $a\bullet v=v$ for every $a\in G$. (*Hint*: 372A. See also 461Yg below.)

(f) Let G be a locally compact Hausdorff group and μ a left Haar measure on G. Suppose that G is **exponentially bounded**, that is, $\limsup_{n\to\infty}(\mu(K^n))^{1/n}\le 1$ for every compact set $K\subseteq G$. Show that G is amenable.

(g) Let G be a group and $\bullet$ an action of G on a set X. Let T be an algebra of subsets of X such that $g\bullet E\in\mathrm{T}$ for every $E\in\mathrm{T}$ and $g\in G$, and H a member of T; write T_H for $\{E:E\in\mathrm{T},\,E\subseteq H\}$. Let $\nu:\mathrm{T}_H\to[0,\infty]$ be a functional which is additive in the sense that $\nu\emptyset=0$ and $\nu(E\cup F)=\nu E+\nu F$ whenever E, $F\in\mathrm{T}_H$ are disjoint, and locally G-invariant in the sense that $g\bullet E\in\mathrm{T}$ and $\nu(g\bullet E)=\nu E$ whenever $E\in\mathrm{T}_H$, $g\in G$ and $g\bullet E\subseteq H$. Show that there is an extension of ν to a G-invariant additive functional $\tilde\nu:\mathrm{T}\to[0,\infty]$.

(h) Let X be a set, A a subset of X, and $\bullet$ an action of a group G on X. Show that the following are equiveridical: (i) there is a functional $\theta:\mathcal{P}X\to[0,\infty]$ such that $\theta A=1$, $\theta(B\cup C)=\theta B+\theta C$ and $\theta(a\bullet B)=\theta B$ for all disjoint B, $C\subseteq X$ and $a\in G$; (ii) there are no $A_0,\dots,A_n$, $a_0,\dots,a_n$, $b_0,\dots,b_n$ such that $A_0,\dots,A_n$ are subsets of G covering A, $a_0,\dots,b_n$ belong to G, and $a_0\bullet A_0$, $b_0\bullet A_0$, $a_1\bullet A_1$, $b_1\bullet A_1,\dots,b_n\bullet A_n$ are disjoint subsets of A.

(i) (Swierczkowski 58) Let G be the group of orthogonal 3×3 matrices. Set $S=\begin{pmatrix}\frac35&\frac45&0\\-\frac45&\frac35&0\\0&0&1\end{pmatrix}$ and $T=\begin{pmatrix}1&0&0\\0&\frac35&\frac45\\0&-\frac45&\frac35\end{pmatrix}$. Show that S and T are free in G (that is, no non-trivial product of the form $S^{n_0}T^{n_1}S^{n_2}T^{n_3}\dots S^{n_{2k}}$ can be the identity), so that G is not amenable in its discrete topology. (*Hint*: let R be the ring of 3×3 matrices over the field $\mathbb{Z}_5$. In R set $\sigma=\begin{pmatrix}3&4&0\\1&3&0\\0&0&0\end{pmatrix}$, $\tau=\begin{pmatrix}0&0&0\\0&3&4\\0&1&3\end{pmatrix}$. Show that $\sigma^2=\sigma$. Now suppose $\rho\in R$ is defined as a non-trivial product of the elements σ, τ and their transposes $\sigma^\top$, $\tau^\top$ in which σ and $\sigma^\top$ are never adjacent, τ and $\tau^\top$ are never adjacent, and the last term is σ. Set $\begin{pmatrix}a\\b\\c\end{pmatrix}=\rho\begin{pmatrix}1\\0\\0\end{pmatrix}$. Show that if the first term in the product is σ or $\sigma^\top$, then $c=0$ and $b\ne 0$, and otherwise $a=0$ and $b\ne 0$.)

(j) Let F_2 be the free group on two generators a, b. (i) Show that there is a partition (A,B,C,D) of F_2 such that $aA=A\cup B\cup C$ and $bB=A\cup B\cup D$. (ii) Let S_2 be the unit sphere in $\mathbb{R}^3$. Show that if S, T are the matrices of

449Yi, there is a partition (A, B, C, D, E) of S^2 such that E is countable, $S[A] = A \cup B \cup C$ and $T[B] = A \cup B \cup D$. (This is a version of the **Hausdorff paradox**.) (iii) Show that there is no non-zero rotation-invariant additive functional from $\mathcal{P}S_2$ to $[0,1]$. (iv) Show that there is no rotation-invariant additive extension of Lebesgue measure to all subsets of the unit ball $B(\mathbf{0}, 1)$. (See Wagon 85.)

449 Notes and comments The general theory of amenable groups is outside the scope of this book. Here I have tried only to indicate some of the specifically measure-theoretic arguments which are used in the theory. Primarily we have the Riesz representation theorem, enabling us to move between linear functionals and measures. Since the invariant measures considered in the definition of 'amenable group' are all Radon measures on compact Hausdorff spaces, they can equally well be thought of as linear functionals on spaces of continuous functions. What is striking is that the definition in terms of continuous actions on arbitrary compact Hausdorff spaces can be reduced to a question concerning an invariant mean on a single space easily constructed from the group (449E).

The first part of this section deals with general topological groups. It is a remarkable fact that some of the most important non-locally-compact topological groups are amenable. For most of these we shall have to wait until we have done 'concentration of measure' (§§476, 492) and can approach 'extremely amenable' groups (§493). But there is an easy example in 449Xh which already indicates one of the basic methods.

For a much fuller account of the theory of amenable locally compact groups, see Paterson 88. Theorem 449J here is mostly taken from Greenleaf 69, where you will find many references to its development. Historically the subject was dominated by the case of discrete groups, in which combinatorial rather than measure-theoretic formulations seem more appropriate. In 449J, conditions (ii)-(viii) relate to invariant means of one kind or another, strengthening that of 449E. Note that the means of 449E(iii) and 449J(ii)-(viii) are normalized by conditions $p(\chi G) = \tilde{p}(\chi G^\bullet) = \phi G = 1$, while the left Haar measure μ of 449J has a degree of freedom; so that when they come together in 449J(viii) the two sides of the equation $\tilde{p}(g * f)^\bullet = \tilde{p}(f^\bullet) \int g \, d\mu$ must move together if we change μ by a scalar factor. Of course this happens through the hidden dependence of the convolution operation on μ. (The convolutions in 449J(vii) do *not* involve μ.) Between 449J(i) and 449J(viii) there is a double step. First we note that a convolution $g * f$ is a kind of weighted average of left translates of f, so that if we have a mean which is invariant under translations we can hope that it will be invariant under convolutions (449H, 449Yc). What is more remarkable is that an invariant mean on the space U of bounded uniformly continuous functions should in the first place give rise to a left-invariant mean on the space L^∞ of (equivalence classes of) bounded Haar measurable functions (449J(vi)), and then even a two-sided-invariant mean (449J(v)).

Condition (xii) in 449J looks at a different aspect of the phenomenon. In effect, it amounts to saying that not only is there an invariant mean, but that there is an invariant mean defined by the formula $p(f) = \lim_{L \to \mathcal{F}} \frac{1}{\mu L} \int_L f$ for a suitable filter $\mathcal{F}$ on the family of sets of non-zero finite measure. This may be called a 'Følner condition', following Følner 55. (449Xk looks for an invariant mean of the form $p(f) = \lim_{h \to \mathcal{F}} \int f \times h \, d\mu$, where $\mathcal{F}$ is a suitable filter on $\mathcal{L}^1(\mu)$.) In 449J(xi), the case $q = 1$ is just a weaker version of condition (x), but the case $q = 2$ tells us something new.

The techniques developed in §444 to handle Haar measures on groups which are not locally compact can also be used in 449H-449J, using 'totally bounded for the bilateral uniformity' in place of 'compact' when appropriate (449Yd). 443L provides another route to the same generalization.

In 449K, it is natural to ask whether the hypothesis 'locally compact' is necessary. It certainly cannot be dropped completely; there is an important amenable Polish group with a closed subgroup which is not amenable (493Xf).

The words 'right' and 'left' appear repeatedly in this section, and it is not perhaps immediately clear which of the ordinary symmetries we can expect to find. The fact that the operation $x \mapsto x^{-1} : G \to G$ always gives us an isomorphism between a group and the same set with the multiplication reversed (449Xa) means that we do not have to distinguish between 'left amenable' and 'right amenable' groups, at least if we start from the definition in 449A. In 449C also there is nothing to break the symmetry between left and right. In 449B and 449D-449E, however, we must commit ourselves to the *left* action of the group on the space of functions which are uniformly continuous with respect to the *right* uniformity. If we wish to change one, we must also expect to change the other. In the list of conditions in 449J, some can be reflected straightforwardly (see 449Xl), but in groups which are not unimodular there seem to be difficulties. For semigroups we do have a difference between 'left' and 'right' amenability (449Yb).

It is not surprising that in the search for invariant means we should repeatedly use averaging and limiting processes. The 'finitely additive integrals' $⨍ d\phi$, $⨍ d\nu$ in part (f) the proof of 449J and part (a) of the proof of 449N are an effective way of using one invariant additive functional ϕ or ν to build another. Similarly, because we are looking only for finite additivity, we can be optimistic about taking cluster points of families of almost-invariant functionals, as in the proofs of 449F, 449J and 449L.

In the case of discrete groups, in which all considerations of measurability and continuity evaporate, we have a completely different technique available, as in 449L. Here we can go directly from a non-paradoxicality condition, a weaker version of conditions already introduced in 395E and 448E, to a Følner condition ((β) in part (b) of the proof of 449L) which easily implies amenability. I remind you that I still do not know how far these ideas can be applied to other algebras than $\mathcal{P}X$ (395Z). The difficulty is that the unscrupulous use of the axiom of choice in the infinitary Marriage Lemma seems to give us no control over the nature of the sets A_{ij} described in (b-α) of the proof of 449L; moreover, the structure of the proof depends on having a suitable invariant measure (counting measure on X) to begin with. For more on amenable discrete groups and their connexions with measure theory see Laczkovich 02.

Chapter 45

Perfect measures and disintegrations

One of the most remarkable features of countably additive measures is that they provide us with a framework for probability theory, as described in Chapter 27. The extraordinary achievements of probability theory since Kolmogorov are to a large extent possible because of the rich variety of probability measures which can be constructed. We have already seen image measures (234C[1]) and product measures (§254). The former are elementary, but a glance at the index will confirm that they have many surprises to offer; the latter are obviously fundamental to any idea of what probability theory means. In this chapter I will look at some further constructions. The most important are those associated with 'disintegrations' or 'regular conditional probabilities' (§§452-453) and methods for confirming the existence of measures on product spaces with given images on subproducts (§454, 455A). We find that these constructions have to be based on measure spaces of special types; the measures involved in the principal results are the Radon measures of Chapter 41 (of course), the compact and perfect measures of Chapter 34, and an intermediate class, the 'countably compact' measures of Marczewski 53 (451B). So the first section of this chapter is a systematic discussion of compact, countably compact and perfect measures.

A 'disintegration', when present, is likely to provide us with a particularly effective instrument for studying a measure, analogous to Fubini's theorem for product measures (see 452F). §§452-453 therefore concentrate on theorems guaranteeing the existence of disintegrations compatible with some pre-existing structure, typically an inverse-measure-preserving function (452I, 452O, 453K) or a product structure (452M). Both depend on the existence of suitable liftings, and for the topological version in §453 we need a 'strong' lifting, so much of that section is devoted to the study of such liftings.

One of the central concerns of probability theory is to understand 'stochastic processes', that is, models of systems evolving randomly over time. If we think of our state space as consisting of functions, so that a whole possible history is described by a random function of time, it is natural to think of our functions as members of some set $\prod_{n\in\mathbb{N}} Z_n$ (if we think of observations as being taken at discrete time intervals) or $\prod_{t\in[0,\infty[} Z_t$ (if we regard our system as evolving continuously), where Z_t represents the set of possible states of the system at time t. We are therefore led to consider measures on such product spaces, and the new idea is that we may have some definite intuition concerning the joint distribution of *finite* strings $(f(t_0),\ldots,f(t_n))$ of values of our random function, that is to say, we may think we know something about the image measures on finite products $\prod_{i\le n} Z_{t_i}$. So we come immediately to a fundamental question: given a (probability) measure μ_J on $\prod_{i\in J} Z_i$ for each finite $J\subseteq T$, when will there be a measure on $\prod_{i\in T} Z_i$ compatible with every μ_J? In §454 I give the most important generally applicable existence theorems for such measures, and in 455A-455E I show how they can be applied to a general construction for models of Markov processes. These models enable us to discuss the Markov property either in terms of disintegrations or in terms of conditional expectations (455C, 455O), and for Lévy processes, in terms of inverse-measure-preserving functions (455U).

The abstract theory of §454 yields measures on product spaces which, from the point of view of a probabilist, are unnaturally large, often much larger than intuition suggests. Some of the most powerful results in the theory of Markov processes, such as the strong Markov property (455O), depend on moving to much smaller spaces; most notably the space of càdlàg functions (455G), but the larger space of càllàl functions is also of interest. The most important example, Brownian motion, will have to wait for Chapter 47, but I give the basic general theory of Lévy processes in complete metric groups.

One of the defining characteristics of Brownian motion is the fact that all its finite-dimensional marginals are Gaussian distributions. Stochastic processes with this property form a particularly interesting class, which I examine in §456. From the point of view of this volume, one of their most striking properties is Talagrand's theorem that, regarded as measures on powers $\mathbb{R}^I$, they are τ-additive (456O).

The next two sections look again at some of the ideas of the previous sections when interpreted as answers to questions of the form 'can all the measures in such-and-such a family be simultaneously extended to a single measure?' If we seek only a *finitely* additive common extension, there is a reasonably convincing general result (457A); but countably additive measures remain puzzling even in apparently simple circumstances (457Z). In §458 I introduce 'relatively independent' families of σ-algebras, with the associated concept of 'relative product' of measures. Finally, in §459, I give some basic results on symmetric measures and exchangeable random variables, with De Finetti's theorem (459C) and corresponding theorems on representing permutation-invariant measures on products as mixtures of product measures (459E, 459H).

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[1] Formerly 112E.

451 Perfect, compact and countably compact measures

In §§342-343 I introduced 'compact' and 'perfect' measures as part of a study of the representation of homomorphisms of measure algebras by functions between measure spaces. An intermediate class of 'countably compact' measures (the 'compact' measures of Marczewski 53) has appeared in the exercises. It is now time to collect these ideas together in a more systematic way. In this section I run through the standard properties of compact, countably compact and perfect measures (451A-451J), with a couple of simple examples of their interaction with topologies (451M-451P). An example of a perfect measure space which is not countably compact is in 451U. Some new ideas, involving non-trivial set theory, show that measurable functions from compact totally finite measure spaces to metrizable spaces have 'essentially separable ranges' (451R); consequently, any measurable function from a Radon measure space to a metrizable space is almost continuous (451T).

451A Let me begin by recapitulating the principal facts already covered.

(a) A family $\mathcal{K}$ of sets is a **compact class** if $\bigcap\mathcal{K}' \ne \emptyset$ whenever $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property. If $\mathcal{K} \subseteq \mathcal{P}X$, then $\mathcal{K}$ is a compact class iff there is a compact topology on X for which every member of $\mathcal{K}$ is closed (342D). A subfamily of a compact class is compact (342Ab).

(b) A measure on a set X is **compact** if it is inner regular with respect to some compact class of sets; equivalently, if it is inner regular with respect to the closed sets for some compact topology on X (342F). All Radon measures are compact measures (416Wa). If (X, Σ, μ) is a semi-finite compact measure space with measure algebra $\mathfrak{A}$, (Y, T, ν) is a complete strictly localizable measure space with measure algebra $\mathfrak{B}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, there is a function $g : Y \to X$ such that $g^{-1}[E] \in \mathrm{T}$ and $g^{-1}[E]^\bullet = \pi(E^\bullet)$ for every $E \in \Sigma$ (343B).

(c) A family $\mathcal{K}$ of sets is a **countably compact class** if $\bigcap_{n\in\mathbb{N}} K_n \ne \emptyset$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $\bigcap_{i\le n} K_i \ne \emptyset$ for every $n \in \mathbb{N}$. Any subfamily of a countably compact class is countably compact. If $\mathcal{K}$ is a countably compact class, then there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ which is closed under finite unions and countable intersections (413R).

(d) A measure space (X, Σ, μ) is **perfect** if whenever $f : X \to \mathbb{R}$ is measurable, $E \in \Sigma$ and $\mu E > 0$, there is a compact set $K \subseteq f[E]$ such that $\mu f^{-1}[K] > 0$. A countably separated semi-finite measure space is compact iff it is perfect (343K). A measure space (X, Σ, μ) is isomorphic to the unit interval with Lebesgue measure iff it is an atomless complete countably separated perfect probability space (344Ka).

451B Now for the new class of measures.

Definition Let (X, Σ, μ) be a measure space. Then (X, Σ, μ), or μ, is **countably compact** if μ is inner regular with respect to some countably compact class of sets.

Evidently compact measures are also countably compact. A simple example of a countably compact measure which is not compact is the countable-cocountable measure on an uncountable set (342M). For an example of a perfect measure which is not countably compact, see 451U.

Note that if μ is inner regular with respect to a countably compact class $\mathcal{K}$, then it is also inner regular with respect to $\mathcal{K} \cap \Sigma$ (411B), and $\mathcal{K} \cap \Sigma$ is still countably compact.

451C Proposition (Ryll-Nardzewski 53) Any semi-finite countably compact measure is perfect.

proof The central idea is the same as in 342L, but we need to refine the second half of the argument.

(a) Let (X, Σ, μ) be a countably compact measure space, $f : X \to \mathbb{R}$ a measurable function, and $E \in \Sigma$ a set of positive measure. Let $\mathcal{K}$ be a countably compact class such that μ is inner regular with respect to $\mathcal{K}$; by 451Ac, we may suppose that $\mathcal{K}$ is closed under finite unions and countable intersections.

Because μ is semi-finite, there is a measurable set $F \subseteq E$ such that $0 < \mu F < \infty$; replacing F by a set of the form $F \cap f^{-1}[[-n, n]]$ if necessary, we may suppose that $f[F]$ is bounded; finally, we may suppose that $F \in \mathcal{K}$. Let $\langle\epsilon_q\rangle_{q\in\mathbb{Q}}$ be a family of strictly positive real numbers such that $\sum_{q\in\mathbb{Q}} \epsilon_q < \frac{1}{2}\mu F$. For each $q \in \mathbb{Q}$, set $E_q = \{x : x \in F,\ f(x) \le q\}$, $E'_q = \{x : x \in F,\ f(x) > q\}$, and choose K_q, $K'_q \in \mathcal{K} \cap \Sigma$ such that $K_q \subseteq E_q$, $K'_q \subseteq E'_q$ and $\mu(E_q \setminus K_q) \le \epsilon_q$, $\mu(E'_q \setminus K'_q) \le \epsilon_q$. Then $K = \bigcap_{q\in\mathbb{Q}}(K_q \cup K'_q) \in \mathcal{K} \cap \Sigma$, $K \subseteq F$ and

$$\mu(F \setminus K) \le \textstyle\sum_{q\in\mathbb{Q}} \mu(E_q \setminus K_q) + \mu(E'_q \setminus K'_q) < \mu F,$$

so $\mu K > 0$.

(b) Take any $t \in \overline{f[K]}$. Enumerate $\mathbb{Q}$ as $\langle q_n \rangle_{n\in\mathbb{N}}$ and define $\langle L_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ by the rule

$$\begin{aligned} L_n &= K_{q_n} \text{ if } t < q_n, \\ &= K'_{q_n} \text{ if } t > q_n, \\ &= F \text{ if } t = q_n. \end{aligned}$$

Now $\bigcap_{i\le n} L_i \ne \emptyset$ for every $n \in \mathbb{N}$. **P** Because $t \in \overline{f[K]}$, there must be some $s \in f[K]$ such that $s < q_i$ whenever $i \le n$ and $t < q_i$, while $s > q_i$ whenever $i \le n$ and $t > q_i$. Let $x \in K$ be such that $f(x) = s$. Then, for any $i \le n$,

either $t < q_i$, $f(x) < q_i$ so $x \notin K'_{q_i}$ and $x \in K_{q_i} = L_i$

or $t > q_i$, $f(x) > q_i$ so $x \notin K_{q_i}$ and $x \in K'_{q_i} = L_i$,

or $t = q_i$ and $x \in F = L_i$.

So $x \in \bigcap_{i\le n} L_i$. **Q**

As $\mathcal{K}$ is a countably compact class, there must be some $x \in \bigcap_{n\in\mathbb{N}} L_n$. But this means that, for any $n \in \mathbb{N}$,

if $t > q_n$ then $x \in K'_{q_n}$ and $f(x) > q_n$,

if $t < q_n$ then $x \in K_{q_n}$ and $f(x) < q_n$.

So in fact $f(x) = t$. Accordingly $t \in f[K]$.

(c) What this shows is that $\overline{f[K]} \subseteq f[K]$ and $f[K]$ is closed. Because (by the choice of F) it is also bounded, it is compact (2A2F). Of course we now have $f[K] \subseteq f[E]$, while $\mu f^{-1}[f[K]] \ge \mu K > 0$. As f and E are arbitrary, μ is perfect.

451D Proposition Let (X, Σ, μ) be a measure space, and $E \in \Sigma$; let μ_E be the subspace measure on E.

(a) If μ is compact, so is μ_E.

(b) If μ is countably compact, so is μ_E.

(c) If μ is perfect, so is μ_E.

proof (a)-(b) Let $\mathcal{K}$ be a (countably) compact class such that μ is inner regular with respect to $\mathcal{K}$. Then μ_E is inner regular with respect to $\mathcal{K}$ (412Oa), so is (countably) compact.

(c) Suppose that $f : E \to \mathbb{R}$ is Σ_E-measurable, where $\Sigma_E = \Sigma \cap \mathcal{P}E$ is the subspace σ-algebra, and $F \subseteq E$ is such that $\mu F > 0$. Set

$$\begin{aligned} g(x) &= \arctan f(x) \text{ if } x \in E, \\ &= 2 \text{ if } x \in X \setminus E. \end{aligned}$$

Then g is Σ-measurable, so there is a compact set $K \subseteq g[F]$ such that $\mu g^{-1}[K] > 0$. Set $L = \{\tan t : t \in K\}$; then $L \subseteq f[F]$ is compact and $f^{-1}[L] = g^{-1}[K]$ has non-zero measure. As f and F are arbitrary, μ_E is perfect.

451E Proposition Let (X, Σ, μ) be a perfect measure space.

(a) If (Y, T, ν) is another measure space and $f : X \to Y$ is an inverse-measure-preserving function, then ν is perfect.

(b) In particular, $\mu\restriction\mathrm{T}$ is perfect for any σ-subalgebra T of Σ.

proof (a) Suppose that $g : Y \to \mathbb{R}$ is T-measurable and $F \in \mathrm{T}$ is such that $\nu F > 0$. Then $gf : X \to \mathbb{R}$ is Σ-measurable and $\mu f^{-1}[F] > 0$. So there is a compact set $K \subseteq (gf)[f^{-1}[F]]$ such that $\mu(gf)^{-1}[K] > 0$. But now $K \subseteq g[F]$ and $\nu g^{-1}[K] > 0$. As g and F are arbitrary, ν is perfect.

(b) Apply (a) to $Y = X$, $\nu = \mu\restriction\mathrm{T}$ and f the identity function.

Remark We shall see in 452R that there is a similar result for countably compact measures; but for compact measures, there is not (342Xf, 451Xh).

451F Lemma (Sazonov 66) Let (X, Σ, μ) be a semi-finite measure space. Then the following are equiveridical:

(i) μ is perfect;

(ii) $\mu\restriction\mathrm{T}$ is compact for every countably generated σ-subalgebra T of Σ;

(iii) $\mu\restriction\mathrm{T}$ is perfect for every countably generated σ-subalgebra T of Σ;

(iv) for every countable set $\mathcal{E} \subseteq \Sigma$ there is a σ-algebra $\mathrm{T} \supseteq \mathcal{E}$ such that $\mu\restriction\mathrm{T}$ is perfect.

proof (a)(i)⇒(ii) Suppose that μ is perfect, and that T is a countably generated σ-subalgebra of Σ. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in T which σ-generates it, and define $f : X \to \mathbb{R}$ by setting $f(x) = \sum_{n=0}^{\infty} 3^{-n}\chi E_n(x)$ for every $x \in X$. Then f is measurable. Set $\mathcal{K} = \{f^{-1}[L] : L \subseteq f[X]$ is compact$\}$. Then $\mathcal{K}$ is a compact class. **P** If $\mathcal{K}' \subseteq \mathcal{K}$ is non-empty and has the finite intersection property, then $\mathcal{L}' = \{L : L \subseteq f[X]$ is compact, $f^{-1}[L] \in \mathcal{K}'\}$ is also a non-empty family with the finite intersection property. So there is an $\alpha \in \bigcap\mathcal{L}'$; since $\alpha \in f[X]$, there is an x such that $f(x) = \alpha$, and now $x \in \bigcap\mathcal{K}'$. As $\mathcal{K}'$ is arbitrary, $\mathcal{K}$ is a compact class. **Q**

Observe next that, for any $n \in \mathbb{N}$,

$$E_n = \{x : \exists\, I \subseteq n, \sum_{i\in I} 3^{-i} + 3^{-n} \le f(x) < \sum_{i\in I} 3^{-i} + 3^{-n+1}\}.$$

So $\mathrm{T}' = \{f^{-1}[F] : F \subseteq \mathbb{R}\}$ contains every E_n; as it is a σ-algebra of subsets of X, it includes T.

Now $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{K}$. **P** If $E \in \mathrm{T}$ and $\mu E > 0$, there is a set $F \subseteq \mathbb{R}$ such that $E = f^{-1}[F]$. Because f is Σ-measurable and μ is perfect, there is a compact set $L \subseteq f[E]$ such that $\mu f^{-1}[L] > 0$. But now $f^{-1}[L] \in \mathcal{K} \cap \mathrm{T}$, and $f^{-1}[L] \subseteq E$ because $L \subseteq F$. Because $\mathcal{K}$ is closed under finite unions, this is enough to show that $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{K}$. **Q**

Thus $\mathcal{K}$ witnesses that $\mu\restriction\mathrm{T}$ is a compact measure.

(b)(ii)⇒(i) Now suppose that $\mu\restriction\mathrm{T}$ is compact for every countably generated σ-algebra $\mathrm{T} \subseteq \Sigma$, that $f : X \to \mathbb{R}$ is a measurable function, and that $\mu E > 0$. Let $F \subseteq E$ be a measurable set of non-zero finite measure, and T the σ-algebra generated by $\{F\} \cup \{f^{-1}[\,]-\infty, q[\,] : q \in \mathbb{Q}\}$, so that T is countably generated and f is T-measurable. Because $\mu\restriction\mathrm{T}$ is compact, so is the subspace measure $(\mu\restriction\mathrm{T})_F$ (451Da); but this is now perfect (342L or 451C), while $F \in \mathrm{T}$ and $\mu F > 0$, so there is a compact set $L \subseteq f[F] \subseteq f[E]$ such that $\mu f^{-1}[L] > 0$. As f and E are arbitrary, μ is perfect.

(c)(i)⇒(iv) is trivial.

(d)(iv)⇒(iii) If (iv) is true, and T is a countably generated σ-subalgebra of Σ, let $\mathcal{E}$ be a countable set generating it. Then there is a σ-algebra $\mathrm{T}_1 \supseteq \mathcal{E}$ such that $\mu\restriction\mathrm{T}_1$ is perfect. By 451Eb, $\mu\restriction\mathrm{T} = (\mu\restriction\mathrm{T}_1)\restriction\mathrm{T}$ is compact, therefore perfect.

(e)(iii)⇒(ii) If (iii) is true, and T is a countably generated σ-subalgebra of Σ, then $\mu\restriction\mathrm{T}$ is perfect; but as (i)⇒(ii), and T is a countably generated σ-subalgebra of itself, $\mu\restriction\mathrm{T}$ is compact.

451G Proposition Let (X,Σ,μ) be a measure space. Let $(X,\hat\Sigma,\hat\mu)$ be its completion and $(X,\tilde\Sigma,\tilde\mu)$ its c.l.d. version. Then

(a)(i) if μ is compact, so are $\hat\mu$ and $\tilde\mu$;

(ii) if μ is semi-finite and either $\hat\mu$ or $\tilde\mu$ is compact, then μ is compact.

(b)(i) If μ is countably compact, so are $\hat\mu$ and $\tilde\mu$;

(ii) if μ is semi-finite and either $\hat\mu$ or $\tilde\mu$ is countably compact, then μ is countably compact.

(c)(i) If μ is perfect, so are $\hat\mu$ and $\tilde\mu$;

(ii) if $\hat\mu$ is perfect, then μ is perfect;

(iii) if μ is semi-finite and $\tilde\mu$ is perfect, then μ is perfect.

proof (a)-(b) The arguments for $\hat\mu$ and $\tilde\mu$ run very closely together. Write $\check\mu$ for either of them, and $\check\Sigma$ for its domain.

(i) If μ is inner regular with respect to $\mathcal{K}$, so is $\check\mu$ (412Ha). So if μ is (countably) compact, so is $\check\mu$.

(ii) Now suppose that μ is semi-finite. The point is that if $\mathcal{K}$ is closed under countable intersections and $\check\mu$ is inner regular with respect to $\mathcal{K}$, so is μ. **P** Suppose that $E \in \Sigma$ and that $\mu E > \gamma$. Choose sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ and $\mathcal{K}_n$ in $\mathcal{K}$ inductively, as follows. E_0 is to be such that $E_0 \subseteq E$ and $\gamma < \mu E_0 < \infty$. Given that $\gamma < \mu E_n < \infty$, let $K_n \in \mathcal{K} \cap \check\Sigma$ be such that $K_n \subseteq E_n$ and $\check\mu K_n > \gamma$; now take $E_{n+1} \in \Sigma$ such that $E_{n+1} \subseteq K_n$ and $\mu E_{n+1} = \check\mu K_n$ (212C or 213Fc), and continue. At the end of the induction, $\bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} E_n$ is a member of $\Sigma \cap \mathcal{K}$ included in E and of measure at least γ. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$. **Q**

It follows that if $\check\mu$ is compact or countably compact, so is μ. **P** Let $\mathcal{K}$ be a (countably) compact class such that $\check\mu$ is inner regular with respect to $\mathcal{K}$; by 451Aa or 451Ac, there is a (countably) compact class $\mathcal{K}^*$, including $\mathcal{K}$, which is closed under countable intersections, so that μ is inner regular with respect to $\mathcal{K}^*$, and is itself (countably) compact. **Q**

(c)(i)(α) Let $f : X \to \mathbb{R}$ be $\hat{\Sigma}$-measurable, and $E \in \hat{\Sigma}$ such that $\hat{\mu}E > 0$. Then there are a μ-conegligible set $F_0 \in \Sigma$ such that $f\restriction F_0$ is Σ-measurable (212Fa), and an $F_1 \in \Sigma$ such that $F_1 \subseteq E$ and $\hat{\mu}(E \setminus F_1) = 0$. Set $F = F_0 \cap F_1$. By 451Dc, the subspace measure μ_F is perfect, while $f\restriction F$ is Σ_F-measurable; so there is a compact set $K \subseteq f[F]$ such that $\mu(F \cap f^{-1}[K]) > 0$. But now $K \subseteq f[E]$ and $\hat{\mu}f^{-1}[K] > 0$. As f and E are arbitrary, $\hat{\mu}$ is perfect.

(β) Let $f : X \to \mathbb{R}$ be $\tilde{\Sigma}$-measurable, and $E \in \tilde{\Sigma}$ such that $\tilde{\mu}E > 0$. Then there is a set $F \in \Sigma$ such that $\mu F < \infty$ and $\hat{\mu}(F \cap E)$ is defined and greater than 0 (213D). In this case, $\hat{\mu}$ and $\tilde{\mu}$ induce the same subspace measure $\hat{\mu}_F$ on F. Accordingly $f\restriction F$ is $\hat{\Sigma}$-measurable. Because $\hat{\mu}$ is perfect (by (α) just above), so is $\hat{\mu}_F$ (451Dc), and there is a compact set $K \subseteq f[F \cap E]$ such that $\hat{\mu}_F(f\restriction F)^{-1}[K] > 0$. But now, of course, $K \subseteq f[E]$ and $\tilde{\mu}f^{-1}[K] > 0$. As f and E are arbitrary, $\tilde{\mu}$ is perfect.

(ii) Suppose that $\hat{\mu}$ is perfect. Since $\mu = \hat{\mu}\restriction\Sigma$, μ is perfect, by 451Eb.

(iii) Similarly, if $\tilde{\mu}$ is perfect and μ is semi-finite, then $\mu = \tilde{\mu}\restriction\Sigma$, by 213Hc, so μ is perfect.

451H Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of sets with product X. Suppose that $\mathcal{K}_i \subseteq \mathcal{P}X_i$ for each $i \in I$, and set $\mathcal{K} = \{\pi_i^{-1}[K] : i \in I,\, K \in \mathcal{K}_i\}$, where $\pi_i : X \to X_i$ is the coordinate map for each $i \in I$. Then

(a) if every $\mathcal{K}_i$ is a compact class, so is $\mathcal{K}$;

(b) if every $\mathcal{K}_i$ is a countably compact class, so is $\mathcal{K}$.

proof (a) For each $i \in I$, let $\mathfrak{T}_i$ be a compact topology on X_i such that every member of $\mathcal{K}_i$ is closed. Then the product topology $\mathfrak{T}$ on X is compact (3A3J), and every member of $\mathcal{K}$ is $\mathfrak{T}$-closed, so $\mathcal{K}$ is a compact class.

(b) If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $\bigcap_{k\le n} K_k \ne \emptyset$ for every $n \in \mathbb{N}$, then we must be able to express each K_n as $\pi_{j_n}^{-1}[L_n]$, where $j_n \in I$ and $L_n \in \mathcal{K}_{j_n}$ for every n. Now, for $i \in I$, $\mathcal{L}_i = \{K_{j_n} : n \in \mathbb{N},\, j_n = i\}$ is a countable subset of $\mathcal{K}_i$, and any finite subfamily of $\mathcal{L}_i$ has non-empty intersection. Since $K_0 \ne \emptyset$, $X_i \ne \emptyset$; so, whether $\mathcal{L}_i$ is empty or not, $X_i \cap \bigcap \mathcal{L}_i$ is non-empty. Accordingly

$$\bigcap_{k\in\mathbb{N}} K_k = \prod_{i\in I}(X_i \cap \bigcap \mathcal{L}_i)$$

is not empty. As $\langle K_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{K}$ is countably compact.

451I Theorem Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$.

(a) If μ and ν are compact, so is λ.

(b) If μ and ν are countably compact, so is λ.

(c) If μ and ν are perfect, so is λ.

proof (a)-(b) Let $\mathcal{K} \subseteq \mathcal{P}X$, $\mathcal{L} \subseteq \mathcal{P}Y$ be (countably) compact classes such that μ is inner regular with respect to $\mathcal{K}$ and ν is inner regular with respect to $\mathcal{L}$. Set $\mathcal{M}_0 = \{K \times Y : K \in \mathcal{K}\} \cup \{X \times L : L \in \mathcal{L}\}$. Then $\mathcal{M}_0$ is (countably) compact, by 451H. By 451Aa/451Ac, there is a (countably) compact class $\mathcal{M} \supseteq \mathcal{M}_0$ which is closed under finite unions and countable intersections. By 412R, λ is inner regular with respect to $\mathcal{M}$, so is (countably) compact.

(c)(i) Let $f : X \times Y \to \mathbb{R}$ be Λ-measurable, and $V \in \Lambda$ a set of positive measure. Then there are $G \in \Sigma$, $H \in \mathrm{T}$ such that μG, νH are both finite and $\lambda(V \cap (G \times H)) > 0$. Recall that the subspace measure $\lambda_{G\times H}$ on $G \times H$ is just the product of the subspace measures μ_G and μ_H (251P(ii-α)), and is the completion of its restriction θ to the σ-algebra $\Sigma_G\widehat{\otimes}\mathrm{T}_H$ generated by $\{E \times F : E \in \Sigma_G,\, F \in \mathrm{T}_H\}$, where Σ_G and T_H are the subspace σ-algebras on G, H respectively, the domains of μ_G and μ_H (251K). Next, for any $W \in \Sigma_G\widehat{\otimes}\mathrm{T}_H$, there are countable families $\mathcal{E} \subseteq \Sigma_G$, $\mathcal{F} \subseteq \mathrm{T}_H$ such that W belongs to the σ-algebra of subsets of $G \times H$ generated by $\{E \times F : E \in \mathcal{E},\, F \in \mathcal{F}\}$ (331Gd).

(ii) The point is that θ is perfect. **P** Let Λ' be any countably generated σ-subalgebra of $\Sigma_G\widehat{\otimes}\mathrm{T}_H$; let $\langle W_n\rangle_{n\in\mathbb{N}}$ be a sequence in Λ' generating it. Then there are countable families $\mathcal{E} \subseteq \Sigma_G$, $\mathcal{F} \subseteq \mathrm{T}_H$ such that every W_n belongs to the σ-algebra generated by $\{E \times F : E \in \mathcal{E},\, F \in \mathcal{F}\}$. Let Σ', T' be the σ-algebras of subsets of G and H generated by $\mathcal{E}$ and $\mathcal{F}$ respectively; then every W_n belongs to $\Sigma'\widehat{\otimes}\mathrm{T}'$, so $\Lambda' \subseteq \Sigma'\widehat{\otimes}\mathrm{T}'$. Let λ' be the product of the measures $\mu\restriction\Sigma' = \mu_G\restriction\Sigma'$ and $\nu\restriction\mathrm{T}'$. Then λ' is the completion of its restriction to $\Sigma'\widehat{\otimes}\mathrm{T}'$.

Now trace through the results above. μ_G and ν_H are perfect (451Dc), so $\mu_G\restriction\Sigma'$ and $\nu_H\restriction\mathrm{T}'$ are compact (451F), so λ' is compact ((a) of this theorem), so λ' is perfect (342L or 451C again). But θ must agree with λ' on Λ', by Fubini's theorem (252D), or otherwise, so $\theta\restriction\Lambda'$ is a restriction of λ', and is perfect (451Eb).

Thus $\theta\restriction\Lambda'$ is perfect for every countably generated σ-subalgebra Λ' of $\operatorname{dom}\theta$. By 451F, θ is perfect. **Q**

(iii) By 451G(c-i), $\lambda_{G\times H}$ is perfect. Now $f\restriction G \times H$ is measurable, and $\lambda_{G\times H}(V \cap (G \times H)) > 0$, so there is a compact set $K \subseteq f[V \cap (G \times H)]$ such that $\lambda_{G\times H}((G \times H) \cap f^{-1}[K]) > 0$; in which case $K \subseteq f[V]$ and $\lambda f^{-1}[K] > 0$.

As f and V are arbitary, λ is perfect.

451J Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X,Σ,μ).
(a) If every μ_i is compact, so is μ.
(b) (Marczewski 53) If every μ_i is countably compact, so is μ.
(c) If every μ_i is perfect, so is μ.

proof The same strategy as in 451I is again effective.

(a)-(b) For each $i\in I$, let $\mathcal{K}_i\subseteq\mathcal{P}X_i$ be a (countably) compact class such that μ_i is inner regular with respect to $\mathcal{K}_i$. Set $\mathcal{M}_0=\{\pi_i^{-1}[K]:i\in I,\,K\in\mathcal{K}_i\}$, so that $\mathcal{M}_0$ is (countably) compact. Let $\mathcal{M}\supseteq\mathcal{M}_0$ be a (countably) compact class which is closed under finite unions and countable intersections. By 412T, μ is inner regular with respect to $\mathcal{M}$, so is (countably) compact.

(c) Let Λ' be a countably generated σ-subalgebra of $\widehat{\bigotimes}_{i\in I}\Sigma_i$, the σ-algebra of subsets of X generated by the sets $\{x:x(i)\in E\}$ for $i\in I$ and $E\in\Sigma_i$. Then $\lambda\restriction\Lambda'$ is perfect. **P** For every $W\in\widehat{\bigotimes}_{i\in I}\Sigma_i$, we must be able to find countable subsets $\mathcal{E}_i$ of Σ_i such that W is in the σ-algebra generated by $\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$; so there are in fact countable sets $\mathcal{E}_i\subseteq\Sigma_i$ such that the σ-algebra generated by $\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$ includes Λ'. Let T_i be the σ-subalgebra of Σ_i generated by $\mathcal{E}_i$, so that $\mu_i\restriction\mathrm{T}_i$ is compact. Let λ' be the product of $\langle\mu_i\restriction\mathrm{T}_i\rangle_{i\in I}$; then λ' is compact, by (a) above, therefore perfect. Now λ is an extension of λ', by 254G or otherwise, so λ' is an extension of $\lambda\restriction\Lambda'$, and $\lambda\restriction\Lambda'$ is perfect. **Q** As Λ' is arbitrary, $\lambda\restriction\widehat{\bigotimes}_{i\in I}\Sigma_i$ is perfect, and its completion λ (254Ff) also is perfect.

Remark This theorem is generalized in 454Ab.

451K The following result is interesting because it can be reached from an unexpectedly weak hypothesis; it will be useful in §455.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets with product X, and Σ_i a σ-algebra of subsets of X_i for each i. Let λ be a perfect totally finite measure with domain $\widehat{\bigotimes}_{i\in I}\Sigma_i$. Set $\pi_J(x)=x\restriction J$ for $x\in X$ and $J\subseteq I$.

(a) Let $\mathcal{K}$ be the set $\{V:V\subseteq X,\,\pi_J[V]\in\widehat{\bigotimes}_{i\in J}\Sigma_i$ for every $J\subseteq I\}$. Then λ is inner regular with respect to $\mathcal{K}$.

(b) Let $\hat\lambda$ be the completion of λ.

(i) For any $J\subseteq I$, the completion of the image measure $\lambda\pi_J^{-1}$ on $\prod_{i\in J}X_i$ is the image measure $\hat\lambda\pi_J^{-1}$.

(ii) If W is measured by $\hat\lambda$ and W is determined by coordinates in $J\subseteq I$, then there is a $V\in\widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $V\subseteq W$, V is determined by coordinates in J and $W\setminus V$ is λ-negligible.

proof (a)(i) Take $W\in\widehat{\bigotimes}_{i\in I}\Sigma_i$. Then we can find a family $\langle\mathrm{T}_i\rangle_{i\in I}$ such that T_i is a countably generated σ-subalgebra of Σ_i for each i and $W\in\widehat{\bigotimes}_{i\in I}\mathrm{T}_i$. For each $i\in I$ and $E\in\mathrm{T}_i$ set $\lambda_iE=\lambda\{x:x\in X,\,x(i)\in E\}$; then λ_i is perfect (451Ea). Because T_i is countably generated, λ_i is compact (451F); let $\mathcal{K}_i$ be a compact class such that λ_i is inner regular with respect to $\mathcal{K}_i$. By 342D, we may suppose that $\mathcal{K}_i$ is the family of closed sets for a compact topology $\mathfrak{T}_i$ on X_i.

(ii) Let $\mathcal{V}$ be the family of all sets $V\subseteq X$ expressible in the form

$$V=\bigcap_{n\in\mathbb{N}}\bigcup_{i\in J_n}\{x:x\in X,\,x(i)\in K_{ni}\}$$

where $\langle J_n\rangle_{n\in\mathbb{N}}$ is a sequence of finite subsets of I and $K_{ni}\in\mathcal{K}_i\cap\mathrm{T}_i$ whenever $n\in\mathbb{N}$ and $i\in J_n$. Given V expressed in this form, set $V_n=\bigcap_{m\le n}\bigcup_{i\in J_m}\{x:x(i)\in K_{mi}\}$ for each n. Then $\pi_J[V]=\bigcap_{n\in\mathbb{N}}\pi_J[V_n]$ for every $J\subseteq I$. **P** The product topology $\mathfrak{T}$ on X is compact, and all the V_n are $\mathfrak{T}$-closed. If $z\in\bigcap_{n\in\mathbb{N}}\pi_J[V_n]$, then for each $n\in\mathbb{N}$ there is an $x_n\in V_n$ such that $\pi_J(x)=z$. Let x be a cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$. The topologies are not Hausdorff, so we do not know at once that $\pi_J(x)=z$; but if we define x' by saying that

$$\begin{aligned}x'(i)&=z(i)\text{ if }i\in J,\\&=x(i)\text{ if }i\in I\setminus J,\end{aligned}$$

then any neighbourhood U of x' must include a neighbourhood of the form $\{y:y(i)\in U_i\text{ for }i\in K\}$ where $K\subseteq I$ is finite and U_i is a neighbourhood of $x'(i)$ for each $i\in K$. In this case, $\{y:y\in U_i\text{ for }i\in K\setminus J\}$ is a neighbourhood of x, so

$$\{n:x_n\in U\}\supseteq\{n:x_n(i)\in U_i\text{ for }i\in K\setminus J\}$$

is infinite. Thus x' also is a cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$, while $\pi_J(x')=z$. Since $x'\in\overline{\{x_m:m\ge n\}}\subseteq V_n$ for every n, $x\in V$, and $z\in\pi_J[V]$. Thus $\bigcap_{n\in\mathbb{N}}\pi_J[V_n]\subseteq\pi_J[V]$. Since surely $\pi_J[V]\subseteq\bigcap_{n\in\mathbb{N}}\pi_J[V_n]$, we have equality. **Q**

It follows that $V \in \mathcal{K}$. **P** If $J \subseteq I$ and $n \in \mathbb{N}$, then V_n belongs to the algebra of subsets of X generated by sets of the form $\{x : x(i) \in H\}$ where $i \in I$ and $H \in \Sigma_i$, which we can identify with the free product $\bigotimes_{i\in I} \Sigma_i$ (315Ma[2]). This means that V_n can be expressed as a finite union of cylinder sets of the form $C = \prod_{i\in I} H_i$ where $H_i \in \Sigma_i$ for every i and $\{i : H_i \ne X_i\}$ is finite (315Kb[3]). But in this case $\pi_J[C]$ is either empty or $\prod_{i\in J} H_i$, and in either case belongs to $\widehat{\bigotimes}_{i\in J}\Sigma_i$. So $\pi_J[V_n]$, being a finite union of such sets, also belongs to $\widehat{\bigotimes}_{i\in J}\Sigma_i$. As this is true for every $n \in \mathbb{N}$, $\pi_J[V] = \bigcap_{n\in\mathbb{N}} \pi_J[V_n]$ belongs to $\widehat{\bigotimes}_{i\in J}\Sigma_i$. As J is arbitrary, $V \in \mathcal{K}$. **Q**

(iii) Observe next that $\mathcal{V}$ is closed under finite unions. **P** If V', $V'' \in \mathcal{V}$, express them as

$$V' = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in J'_n} \{x : x(i) \in K'_{ni}\}$$

$$V'' = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in J''_n} \{x : x(i) \in K''_{ni}\}$$

where, for each n, J'_n, $J''_n \subseteq I$ are finite, $K'_{ni} \in \mathcal{K}_{ni} \cap \Sigma_i$ for $i \in J'_n$ and $K''_{ni} \in \mathcal{K}_{ni} \cap \Sigma_i$ for $i \in J''_n$. For m, $n \in \mathbb{N}$, set $J_{mn} = J'_m \cup J''_n$ and

$$\begin{aligned} K_{mni} &= K'_{mi} \cup K''_{ni} \text{ if } i \in J'_m \cap J''_n, \\ &= K'_{mi} \text{ if } i \in J'_m \setminus J''_n, \\ &= K''_{ni} \text{ if } i \in J''_n \setminus J''_m. \end{aligned}$$

Then

$$V' \cap V'' = \bigcap_{m,n\in\mathbb{N}} \bigcup_{i\in J_{mn}} \{x : x(i) \in K_{mni}\} \in \mathcal{V}. \ \textbf{Q}$$

We see also, immediately from its definition, that $\mathcal{V}$ is closed under countable intersections.

(iv) Now consider the family $\mathcal{A}$ of sets of the form $\{x : x(i) \in E\}$ where $i \in I$ and $E \in \mathrm{T}_i$. If $A \in \mathcal{A}$ is expressed in this form, then

$$\sup\{\lambda V : V \in \mathcal{V}, V \subseteq A\} \ge \sup\{\lambda_i K : K \in \mathcal{K}_i \cap \mathrm{T}_i, K \subseteq E\} = \lambda_i E = \lambda A.$$

By 412C, $\lambda\restriction\widehat{\bigotimes}_{i\in I}\mathrm{T}_i$ is inner regular with respect to $\mathcal{V}$. In particular, returning to our original set W,

$$\mu W = \sup\{\lambda V : V \in \mathcal{V}, V \subseteq W\} = \sup\{\lambda K : K \in \mathcal{K}, K \subseteq W\}.$$

As W is arbitrary, λ is inner regular with respect to $\mathcal{K}$.

(b)(i) Write $\lambda_J = \lambda\pi_J^{-1}$ and $\hat\lambda_J$ for its completion. Since $\pi_J : X \to \prod_{i\in J} X_J$ is inverse-measure-preserving for λ and λ_J, it is inverse-measure-preserving for $\hat\lambda$ and $\hat\lambda_J$ (234Ba[4]), that is, $\hat\lambda\pi_J^{-1}$ extends $\hat\lambda_J$. Now suppose that V is measured by $\hat\lambda\pi_J^{-1}$. Since λ is inner regular with respect to $\mathcal{K}$, so is $\hat\lambda$ (412Ha again), so

$$\begin{aligned} \hat\lambda\pi_J^{-1}[V] &= \sup\{\lambda K : K \in \mathcal{K}, K \subseteq \pi_J^{-1}[V]\} \\ &\le \sup\{\lambda\pi_J^{-1}[\pi_J[K]] : K \in \mathcal{K}, K \subseteq \pi_J^{-1}[V]\} \\ &\le \sup\{\lambda\pi_J^{-1}[F] : F \in \widehat{\bigotimes}_{i\in J}\Sigma_i, F \subseteq V\}. \end{aligned}$$

As V is arbitrary, $\hat\lambda\pi_J^{-1}$ is inner regular with respect to $\widehat{\bigotimes}_{i\in J}\Sigma_i$. By 412L (or otherwise), $\hat\lambda\pi_J^{-1} = \hat\lambda_J$.

(ii) Because $\pi_J^{-1}[\pi_J[W]] = W$, $\pi_J[W]$ is measured by $\hat\lambda\pi_J^{-1} = \hat\lambda_J$. So there is a $V' \subseteq \pi_J[W]$, measured by λ_J, such that

$$0 = \hat\lambda_J(\pi_J[W] \setminus V') = \hat\lambda(W \setminus \pi_J^{-1}[V']),$$

and we can take $V = \pi_J^{-1}[V']$.

***451L** The next result is sometimes useful, as a fractionally weaker sufficient condition for compactness or countable compactness of a measure.

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[2]Formerly 315L.
[3]Formerly 315J.
[4]Formerly 235Hc.

Proposition (Borodulin-Nadzieja & Plebanek 05) Let (X,Σ,μ) be a strictly localizable measure space. Let us say that a family $\mathcal{E}\subseteq\Sigma$ is **μ-centered** if $\mu(\bigcap\mathcal{E}_0)>0$ for every non-empty finite $\mathcal{E}_0\subseteq\mathcal{E}$.

(i) Suppose that μ is inner regular with respect to some $\mathcal{K}\subseteq\Sigma$ such that every μ-centered subset of $\mathcal{K}$ has non-empty intersection. Then μ is compact.

(ii) Suppose that μ is inner regular with respect to some $\mathcal{K}\subseteq\Sigma$ such that every countable μ-centered subset of $\mathcal{K}$ has non-empty intersection. Then μ is countably compact.

proof I take the two arguments together, as follows. The case $\mu X=0$ is trivial; suppose henceforth that $\mu X>0$. Let $\hat\mu$ be the completion of μ. Then $\hat\mu$ is still strictly localizable (212Gb) so has a lifting $\phi:\Sigma\to\Sigma$ (341K). Let $\mathcal{K}_1$ be the set of all those $K\in\Sigma$ for which there is some sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that

$$K=\bigcap_{n\in\mathbb{N}}K_n\subseteq\bigcap_{n\in\mathbb{N}}\phi K_n.$$

Then μ is inner regular with respect to $\mathcal{K}_1$. **P** Suppose that $E\in\Sigma$ and $0\le\gamma<\mu E$. Because μ is semi-finite, there is an $F\in\Sigma$ such that $F\subseteq E$ and $\gamma<\mu F<\infty$. Choose $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ inductively, as follows. K_0 is to be such that $K_0\subseteq F$ and $\mu K_0>\gamma$. Given that $\mu K_n>\gamma$, then $\hat\mu(K_n\cap\phi K_n)=\hat\mu K_n>\gamma$; also $\hat\mu$ is inner regular with respect to $\mathcal{K}$ (412Ha once more), so there is a $K_{n+1}\in\mathcal{K}$ such that $K_{n+1}\subseteq K_n\cap\phi K_n$ and $\mu K_{n+1}=\hat\mu K_{n+1}>\gamma$. Continue. At the end of the induction,

$$K=\bigcap_{n\in\mathbb{N}}K_n\subseteq\bigcap_{n\in\mathbb{N}}\phi K_n$$

belongs to $\mathcal{K}_1$, is included in E and has measure at least γ. **Q**

Now $\mathcal{K}_1$ is (countably) compact. **P** Let $\mathcal{K}'\subseteq\mathcal{K}_1$ be a [countable] set with the finite intersection property. For each $K\in\mathcal{K}'$, let $\mathcal{E}_K\subseteq\mathcal{K}$ be a countable set such that $K=\bigcap\mathcal{E}_K\subseteq\bigcap\{\phi E:E\in\mathcal{E}_K\}$; set $\mathcal{E}=\bigcup_{K\in\mathcal{K}'}\mathcal{E}_K$. If $\mathcal{E}_0\subseteq\mathcal{E}$ is finite and not empty, then $\phi(\bigcap\mathcal{E}_0)=\bigcap_{E\in\mathcal{E}_0}\phi E$ includes the intersection of a finite subfamily of $\mathcal{K}'$, so is not empty, and $\mu(\bigcap\mathcal{E}_0)=\hat\mu(\bigcap\mathcal{E}_0)$ is non-zero. Thus $\mathcal{E}\subseteq\mathcal{K}$ is a [countable] μ-centered set and must have non-empty intersection. But now $\bigcap\mathcal{K}'=\bigcap\mathcal{E}$ is non-empty. As $\mathcal{K}'$ is arbitrary, $\mathcal{K}_1$ is (countably) compact. **Q**

So $\mathcal{K}_1$ witnesses that μ is (countably) compact, as claimed.

451M The following is one of the basic ways in which we can find ourselves with a compact measure.

Proposition Let (X,Σ) be a standard Borel space. Then any semi-finite measure μ with domain Σ is compact, therefore perfect.

proof If $\mathfrak{T}$ is a Polish topology on X with respect to which Σ is the Borel σ-algebra, then μ is inner regular with respect to the family $\mathcal{K}$ of $\mathfrak{T}$-compact sets (433Ca), which is a compact class.

451N Proposition Let (X,Σ,μ) be a perfect measure space and $\mathfrak{T}$ a T_0 topology on X with a countable network consisting of measurable sets. (For instance, μ might be a topological measure on a regular space with a countable network (4A2Ng), or a second-countable space. In particular, X might be a separable metrizable space.) Then μ is inner regular with respect to the compact sets.

proof This is a refinement of 343K. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ running over a network for $\mathfrak{T}$. Define $g:X\to\mathbb{R}$ by setting $g=\sum_{n=0}^\infty 3^{-n}\chi H_n$ (cf. 343E). Then g is measurable, because every χE_n is. Writing $\alpha_I=\sum_{i\in I}3^{-i}$ for $I\subseteq\mathbb{N}$, and

$$H_n=\bigcup_{I\subseteq n}\,]\alpha_I+\tfrac12 3^{-n},\alpha_I+3^{-n+1}[,$$

we see that $E_n=g^{-1}[H_n]$ for each $n\in\mathbb{N}$. This shows that g is injective, because if x, y are distinct points in X there is an open set containing one but not the other, and now there is an $n\in\mathbb{N}$ such that E_n contains that one and not the other, so that just one of $g(x)$, $g(y)$ belongs to H_n. Also $g^{-1}:g[X]\to X$ is continuous, since $(g^{-1})^{-1}[E_n]=g[E_n]=H_n\cap g[X]$ is relatively open in $g[X]$ for every $n\in\mathbb{N}$ (4A2B(a-ii)).

Now suppose that $E\in\Sigma$ and $\mu E>0$. Then there is a compact set $K\subseteq g[E]$ such that $\mu g^{-1}[K]>0$. But as g is injective, $g^{-1}[K]\subseteq E$, and as g^{-1} is continuous, $g^{-1}[K]$ is compact. By 412B, this is enough to show that μ is inner regular with respect to the compact sets.

451O Corollary Let (X,Σ,μ) be a complete perfect measure space, Y a Hausdorff space with a countable network consisting of Borel sets and $f:X\to Y$ a measurable function. If the image measure μf^{-1} is locally finite, it is a Radon measure.

proof Because f is measurable, μf^{-1} is a topological measure; by 451Ea, it is perfect; by 451N, it is tight; and it is complete because μ is. Because Y has a countable network, it is Lindelöf (4A2Nb), and μf^{-1} is σ-finite (411Ge), therefore locally determined. So it is a Radon measure.

451P Corollary Let (X, Σ, μ) be a perfect measure space, Y a separable metrizable space, and $f : X \to Y$ a measurable function.

(a) If $E \in \Sigma$ and $\gamma < \mu E$, there is a compact set $K \subseteq f[E]$ such that $\mu(E \cap f^{-1}[K]) \geq \gamma$.

(b) If $\nu = \mu f^{-1}$ is the image measure, then $\mu_* f^{-1}[B] = \nu_* B$ for every $B \subseteq Y$.

(c) If moreover μ is σ-finite, then $\mu^* f^{-1}[B] = \nu^* B$ for every $B \subseteq Y$.

proof (a) Consider the subspace measure μ_E, the measurable function $f{\restriction}E$ from E to the separable metrizable space $f[E]$, and the image measure $\nu' = \mu_E(f{\restriction}E)^{-1}$ on $f[E]$. By 451Dc, 451Ea and 451N, this is tight, while $\nu' f[E] = \mu E$; so there is a compact set $K \subseteq f[E]$ such that $\nu' K \geq \gamma$, and this serves.

(b)(i) If $F \in \operatorname{dom}\nu$ and $F \subseteq B$, then

$$\nu F = \mu f^{-1}[F] \leq \mu_* f^{-1}[B];$$

as F is arbitrary, $\mu_* f^{-1}[B] \geq \nu_* B$. (ii) If $E \in \Sigma$ and $E \subseteq f^{-1}[B]$ and $\gamma < \mu E$, then (a) tells us that there is a compact set $K \subseteq f[E]$ such that $\mu(E \cap f^{-1}[K]) \geq \gamma$, in which case

$$\nu_* B \geq \nu K \geq \gamma.$$

As E and γ are arbitrary, $\nu_* B \geq \mu_* f^{-1}[B]$.

(c)(i) If $F \in \operatorname{dom}\nu$ and $F \supseteq B$, then

$$\nu F = \mu f^{-1}[F] \geq \mu^* f^{-1}[B];$$

as F is arbitrary, $\mu^* f^{-1}[B] \leq \nu^* B$. (ii) If $\mu^* f^{-1}[B] = \infty$, then of course $\mu^* f^{-1}[B] = \nu^* B$. Otherwise, because μ is σ-finite, we can find a disjoint sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ of subsets of X of finite measure, covering X, such that $E_0 \supseteq f^{-1}[B]$ and $\mu E_0 = \mu^* f^{-1}[B]$. Let $\epsilon > 0$. For each $n \geq 1$, (a) tells us that there is a compact set $K_n \subseteq f[E_n]$ such that $\mu f^{-1}[E_n \setminus K_n] \leq 2^{-n}\epsilon$. Set $H = Y \setminus \bigcup_{n\geq 1} K_n$; then $\nu H \leq \mu E + \epsilon$, and $B \subseteq H$. So

$$\nu^* B \leq \nu H \leq \mu E + \epsilon = \mu^* f^{-1}[B] + \epsilon.$$

As ϵ is arbitrary, $\nu^* B \leq \mu^* f^{-1}[B]$.

451Q I turn now to a remarkable extension of the idea above to general metric spaces Y.

Lemma Let (X, Σ, μ) be a semi-finite compact measure space, and $\langle E_i \rangle_{i\in I}$ a disjoint family of subsets of X such that $\bigcup_{i\in J} E_i \in \Sigma$ for every $J \subseteq I$. Then $\mu(\bigcup_{i\in I} E_i) = \sum_{i\in I} \mu E_i$.

proof (a) To begin with (down to the end of part (d) of the proof) assume that μ is complete and totally finite and that every E_i is negligible. Set $X_0 = \bigcup_{i\in I} E_i$, and let μ_0 be the subspace measure on X_0. Define $f : X_0 \to I$ by setting $f(x) = i$ if $i \in I$, $x \in E_i$, and let ν be the image measure $\mu_0 f^{-1}$, so that $\nu J = \mu(\bigcup_{i\in J} E_i)$ for $J \subseteq I$; then $(I, \mathcal{P}I, \nu)$ is a totally finite measure space.

(b) ν is purely atomic. **P?** Suppose, if possible, otherwise; that there is a $K \subseteq I$ such that $\nu K > 0$ and the subspace measure $\nu{\restriction}\mathcal{P}K$ is atomless. In this case there is an inverse-measure-preserving function $g : K \to [0, \gamma]$, where $\gamma = \nu K$ and $[0, \gamma]$ is given Lebesgue measure (343Cc); write λ for Lebesgue measure on $[0, \gamma]$. Set $X_1 = f^{-1}[K] = \bigcup_{i\in K} E_i$ and let μ_1 be the subspace measure on X_1. Now $gf : X_1 \to [0, \gamma]$ is inverse-measure-preserving for μ_1 and λ. Because μ is compact, so is μ_1 (451Da), so μ_1 is perfect (342L or 451C once more). By 451O, the image measure $\lambda_1 = \mu_1(gf)^{-1}$ is a Radon measure. But λ_1 must be an extension of Lebesgue measure λ, because gf is inverse-measure-preserving for μ_1 and λ, and λ_1 and λ must agree on all compact sets. By 416E(b-ii), λ_1 and λ are identical, and, in particular, have the same domains. Now for any set $A \subseteq [0, \gamma]$, $(gf)^{-1}[A] = \bigcup_{i\in J} E_i \in \Sigma$, where $J = g^{-1}[A] \subseteq I$; so $A \in \operatorname{dom}\lambda_1 = \operatorname{dom}\lambda$. But we know from 134D or 419I that not every subset of $[0, \gamma]$ can be Lebesgue measurable. **XQ**

(c) But ν is also atomless. **P?** Suppose, if possible, that $M \subseteq I$ is an atom for ν. Set $\gamma = \nu M = \mu(\bigcup_{i\in M} E_i)$,

$$\mathcal{F} = \{F : F \subseteq M,\ \nu(M \setminus F) = 0\}.$$

Because νF is defined for every $F \subseteq M$, and M is an atom, $\mathcal{F}$ is an ultrafilter on M; and because ν is countably additive, the intersection of any sequence in $\mathcal{F}$ belongs to $\mathcal{F}$, that is, $\mathcal{F}$ is ω_1-complete (definition: 4A1Ib). Also $\mathcal{F}$ must be non-principal, because we are supposing that $\nu\{i\} = 0$ for every $i \in M$. By 4A1K, there are a regular uncountable cardinal κ and a function $h : M \to \kappa$ such that the image filter $\mathcal{H} = h[[\mathcal{F}]]$ is normal.

For each $\xi < \kappa$, $\kappa \setminus \xi \in \mathcal{H}$, so

$$G_\xi = (hf)^{-1}[\kappa \setminus \xi] = \bigcup\{E_i : h(i) \ge \xi\} \in \Sigma, \quad \mu G_\xi = \nu h^{-1}[\kappa \setminus \xi] = \gamma > 0.$$

At this point I apply the full strength of the hypothesis that μ is a compact measure. Let $\mathcal{K} \subseteq \Sigma$ be a compact class such that μ is inner regular with respect to $\mathcal{K}$, and for each $\xi < \kappa$ choose $K_\xi \in \mathcal{K}$ such that $K_\xi \subseteq G_\xi$ and $\mu K_\xi \ge \frac{1}{2}\gamma$. Let $S \subseteq [\kappa]^{<\omega}$ be the family of those finite sets $L \subseteq \kappa$ such that $\bigcap_{\xi \in L} K_\xi = \emptyset$. Because $\mathcal{H}$ is a normal ultrafilter, there is an $H \in \mathcal{H}$ such that, for every $n \in \mathbb{N}$, $[H]^n$ is either a subset of S or disjoint from S (4A1L).

If we look at $\{G_\xi : \xi \in H\}$, we see that it has empty intersection, because $h(f(x)) \ge \xi$ for every $x \in G_\xi$, and $\sup H = \kappa$. So $\bigcap_{\xi \in H} K_\xi = \emptyset$. Because all the K_ξ belong to the compact class $\mathcal{K}$, there must be a finite set $L_0 \subseteq H$ such that $\bigcap_{\xi \in L_0} K_\xi = \emptyset$, that is, $L_0 \in S$. But this means that $[H]^n \cap S \ne \emptyset$, where $n = \#(L_0)$, so that $[H]^n \subseteq S$, by the choice of H. However, H is surely infinite, so we can find distinct $\xi_0, \dots, \xi_{2n}$ in H. If we now look at $K_{\xi_0}, \dots, K_{\xi_{2n}}$, we see that $\#(\{i : i \le 2n,\, x \in K_{\xi_i}\}) < n$ for every $x \in X$, so

$$\textstyle\sum_{i=0}^{2n} \chi K_{\xi_i} \le (n-1)\chi G_0, \quad \sum_{i=0}^{2n} \int \chi K_{\xi_i} \ge \frac{1}{2}\gamma(2n+1),$$

which is impossible, because $\mu G_0 = \gamma$. **XQ**

(d) Thus ν is simultaneously atomless and purely atomic, which means that $\nu I = 0$, that is, that $\mu(\bigcup_{i \in I} E_i) = 0 = \sum_{i \in I} \mu E_i$.

(e) Now let us return to the general case. Of course

$$\textstyle\sum_{i \in I} \mu E_i = \sup_{J \subseteq I \text{ is finite}} \sum_{i \in J} \mu E_i \le \mu(\bigcup_{i \in I} E_i).$$

? Suppose, if possible, that $\sum_{i \in I} \mu E_i < \mu(\bigcup_{i \in I} E_i)$. Because μ is semi-finite, there is a set $F \subseteq \bigcup_{i \in I} E_i$ such that $\sum_{i \in I} \mu E_i < \mu F < \infty$. Set $L = \{i : i \in I,\, \mu E_i > 0\}$; then L must be countable, so $\mu(\bigcup_{i \in J} E_i) = \sum_{i \in J} \mu E_i < \mu F$, and $\mu G > 0$, where $G = F \setminus \bigcup_{i \in L} E_i$. Set $E_i' = G \cap E_i$ for every $i \in I$, and let $\hat\mu_G$ be the completion of the subspace measure μ_G on G. Then $\hat\mu_G$ is compact (451Da, 451G(a-i)) and totally finite, $\hat\mu_G E_i' = 0$ for every $i \in I$, $\bigcup_{i \in J} E_i' = G \cap \bigcup_{i \in J} E_i$ is measured by $\hat\mu_G$ for every $J \subseteq I$, every E_i' is $\hat\mu_G$-negligible, but $\hat\mu_G(\bigcup_{i \in I} E_i') = \mu G$ is not zero; which contradicts the result of (a)-(d) above. **X**

So $\sum_{i \in I} \mu E_i = \mu(\bigcup_{i \in I} E_i)$, as required.

451R Lemma Let (X, Σ, μ) be a totally finite compact measure space, Y a metrizable space, and $f : X \to Y$ a measurable function. Then there is a closed separable subspace Y_0 of Y such that $f^{-1}[Y \setminus Y_0]$ is negligible.

proof (a) (Cf. 438D.) By 4A2L(g-ii), there is a σ-disjoint base $\mathcal{U}$ for the topology of Y. Express $\mathcal{U}$ as $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ where $\mathcal{U}_n$ is disjoint for each n. Then $\langle f^{-1}[U] \rangle_{U \in \mathcal{U}_n}$ is disjoint, so $\sum_{U \in \mathcal{U}_n} \mu f^{-1}[U] \le \mu X$ is finite, and $\mathcal{V}_n = \{V : V \in \mathcal{U}_n,\, \mu f^{-1}[V] > 0\}$ is countable for each n.

If $\mathcal{W} \subseteq \mathcal{U}_n \setminus \mathcal{V}_n$, then

$$\mu(\textstyle\bigcup_{U \in \mathcal{W}} f^{-1}[U]) = f^{-1}[\bigcup \mathcal{W}]$$

is measurable. By 451Q,

$$\mu f^{-1}[\textstyle\bigcup(\mathcal{U}_n \setminus \mathcal{V}_n)] = \mu(\bigcup_{U \in \mathcal{U}_n \setminus \mathcal{V}_n} f^{-1}[U]) = \sum_{U \in \mathcal{U}_n \setminus \mathcal{V}_n} \mu f^{-1}[U] = 0.$$

Set

$$\mathcal{V} = \textstyle\bigcup_{n \in \mathbb{N}} \mathcal{V}_n, \quad Y_0 = Y \setminus \bigcup(\mathcal{U} \setminus \mathcal{V}).$$

Then Y_0 is closed, and

$$f^{-1}[Y \setminus Y_0] \subseteq \textstyle\bigcup_{n \in \mathbb{N}} f^{-1}[\bigcup(\mathcal{U}_n \setminus \mathcal{V}_n)]$$

is negligible, so $f^{-1}[Y_0]$ is conegligible. On the other hand, Y_0 is separable. **P** Because $\mathcal{U}$ is a base for the topology of X, $\{Y \cap U : U \in \mathcal{U}\}$ is a base for the topology of Y (4A2B(a-vi)). But this is included in the countable family $\{Y \cap V : V \in \mathcal{V}\} \cup \{\emptyset\}$, so Y is second-countable, therefore separable (4A2Oc). **Q**

So we have found an appropriate Y_0.

451S Proposition Let (X, Σ, μ) be a semi-finite compact measure space, Y a metrizable space and $f : X \to Y$ a measurable function.

(a) The image measure $\nu = \mu f^{-1}$ is tight.

(b) If ν is locally finite and μ is complete and locally determined, ν is a Radon measure.

proof (a) Take $F \subseteq Y$ such that $\nu F > 0$. Then $\mu f^{-1}[F] > 0$. Because μ is semi-finite, there is an $E \in \Sigma$ such that $E \subseteq f^{-1}[F]$ and $0 < \mu E < \infty$.

Consider the subspace measure μ_E and the restriction $f\restriction E$. μ_E is a totally finite compact measure and $f\restriction E$ is measurable, so 451R tells us that there is a closed separable subspace $Y_0 \subseteq Y$ such that $\mu(E \setminus f^{-1}[Y_0]) = 0$. Set $E_1 = E \cap f^{-1}[Y_0]$, so that $\mu E_1 > 0$. Again, the subspace measure μ_{E_1} is a totally finite compact measure, therefore perfect, while $f[E_1] \subseteq Y_0$. So the image measure $\mu_{E_1}(f\restriction E_1)^{-1}$ on Y_0 is perfect (451Ea), therefore tight (451N), and there is a compact set $K \subseteq Y_0 \cap F$ such that $\nu K = \mu f^{-1}[K] > 0$. By 412B, this is enough to show that ν is tight.

(b) ν is complete because μ is. Now suppose that $H \subseteq Y$ is such that $H \cap F$ belongs to the domain T of ν whenever $\mu F < \infty$. In this case μ is inner regular with respect to $\mathcal{E} = \{E : E \in \Sigma,\, E \cap f^{-1}[H] \in \Sigma\}$. **P** Suppose that $E \in \Sigma$ and that $\mu E > 0$. Applying (a) to μ_E and $f\restriction E$, there is a compact set $K \subseteq f[E]$ such that $\mu f^{-1}[K] > 0$. Now $\nu K < \infty$, because ν is locally finite, so $K \cap H \in \mathrm{T}$ and $f^{-1}[K] \cap f^{-1}[H] \in \Sigma$. Thus $f^{-1}[K]$ is a non-negligible member of $\mathcal{E}$ included in E. Since $\mathcal{E}$ is closed under finite unions, this is enough to show that μ is inner regular with respect to $\mathcal{E}$. **Q**

Accordingly $f^{-1}[H] \in \Sigma$, by 412Ja. As H is arbitrary, ν is locally determined, therefore a Radon measure.

451T Theorem (Fremlin 81, Koumoullis & Prikry 83) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space and Y a metrizable space. Then a function $f : X \to Y$ is measurable iff it is almost continuous.

proof If f is almost continuous it is surely measurable, by 418E. Now suppose that f is measurable and that $E \in \Sigma$ and $\gamma < \mu E$. Let $E_0 \subseteq E$ be such that $E_0 \in \Sigma$ and $\gamma < \mu E_0 < \infty$. Applying 451R to the subspace measure μ_{E_0} and the restricted function $f\restriction E_0$, we see that there is a closed separable subspace Y_0 of Y such that $\mu(E_0 \setminus f^{-1}[Y_0]) = 0$. Set $E_1 = E_0 \cap f^{-1}[Y_0]$; then $\mu E_1 > \gamma$. Applying 418J to μ_{E_1} and $f\restriction E_1 : E_1 \to Y_0$, we can find a measurable set $F \subseteq E_1$ such that $f\restriction F$ is continuous and $\mu F \ge \gamma$. As E and γ are arbitrary, f is almost continuous.

451U Example (Vinokurov & Makhkamov 73, Musiał 76) There is a perfect completion regular quasi-Radon probability space which is not countably compact.

proof (a) Let Ω be the set of non-zero countable limit ordinals. For each $\xi \in \Omega$, let $\langle \theta_\xi(n) \rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in ξ with supremum ξ, and set

$$Q_\xi = \{x : x \in \{0,1\}^{\omega_1},\, x(\theta_\xi(n)) = 0 \text{ for every } n \in \mathbb{N}\}.$$

Write

$$X = \{0,1\}^{\omega_1} \setminus \bigcup_{\xi\in\Omega} Q_\xi.$$

Let ν_{ω_1} be the usual measure on $\{0,1\}^{\omega_1}$, and T_{ω_1} its domain; let μ be the subspace measure on X, and $\Sigma = \operatorname{dom}\mu$.

(b) It is convenient to note immediately the following fact: for every countable set $J \subseteq \omega_1$, the set $\pi_J[X]$ is conegligible in $\{0,1\}^J$, where $\pi_J(x) = x\restriction J$ for $x \in \{0,1\}^{\omega_1}$. **P** Set

$$A = \{\xi : \xi \in \Omega,\, \theta_\xi(n) \in J \text{ for every } n \in \mathbb{N}\}.$$

Then A is countable, because $\xi \le \sup J$ for every $\xi \in A$. So

$$D = \bigcup_{\xi\in A}\{y : y \in \{0,1\}^J,\, y(\theta_\xi(n)) = 0 \text{ for every } n \in \mathbb{N}\}$$

is negligible in $\{0,1\}^J$, being a countable union of negligible sets. If $y \in \{0,1\}^J \setminus D$, define $x \in \{0,1\}^{\omega_1}$ by setting $x(\eta) = y(\eta)$ for $\eta \in J$, $x(\eta) = 1$ for $\eta \in \omega_1 \setminus J$. Then $x \notin Q_\xi$ for any $\xi \in A$, because $x\restriction J = y\restriction J$, while $x \notin Q_\xi$ for any $\xi \in \Omega \setminus A$ by the definition of A. So $x \in X$. As y is arbitrary, $\pi_J[X] \supseteq \{0,1\}^J \setminus D$ is conegligible. **Q**

(c) μ is a completion regular quasi-Radon measure because ν_{ω_1} is (415E, 415B, 412Pd). Also $\mu X = 1$. **P** Let $F \in \mathrm{T}_{\omega_1}$ be a measurable envelope for X. Then there is a countable $J \subseteq \omega_1$ such that $\nu_J \pi_J[F]$ is defined and equal to $\nu_{\omega_1}F$ (254Od), where ν_J is the usual measure on $\{0,1\}^J$. But we know that $\nu_J\pi_J[X] = 1$, so

$$\mu X = \nu^*_{\omega_1}X = \nu_{\omega_1}F = \nu_J\pi_J F = 1. \text{ **Q**}$$

(d) μ is perfect. **P** Take $E \in \Sigma$ such that $\mu E > 0$, and a measurable function $f : E \to \mathbb{R}$. Set $f_1(x) = \frac{f(x)}{1+|f(x)|}$ for $x \in E$, 1 for $x \in X \setminus E$; then $f_1 : X \to \mathbb{R}$ is measurable. Let $g : \{0,1\}^{\omega_1} \to \mathbb{R}$ be a measurable function extending f_1. By 254Pb, there are a countable set $J \subseteq \omega_1$, a conegligible set $W \subseteq \{0,1\}^J$, and a measurable $h : W \to \mathbb{R}$ such that g extends $h\pi_J$. By (b), $W' = W \cap \pi_J[X]$ is conegligible, while $W'' = \{z : z \in W',\, h_1(z) < 1\}$ is measurable and not negligible. Because W'' is a non-negligible measurable subset of the perfect measure space $\{0,1\}^J$, there is a compact set $K_1 \subseteq h[W'']$ such that $\nu_J h^{-1}[K_1] > 0$. Set $K = \{\frac{t}{1-|t|} : t \in K_1\}$; then K is compact, and we have

$$K_1 \subseteq h[W''] = h[W \cap \pi_J[X]] \cap]-\infty, 1[\subseteq g[X] \cap]-\infty, 1[= f_1[X] \cap]-\infty, 1[= f_1[E],$$

$$K \subseteq f[E],$$

while f_1, g and $h\pi_J$ all agree on the μ-conegligible set $X \cap \pi_J^{-1}[W]$, so

$$\mu f^{-1}[K] = \mu f_1^{-1}[K_1] = \mu(X \cap (h\pi_J)^{-1}[K_1])$$
$$= \nu^*_{\omega_1}(X \cap (h\pi_J)^{-1}[K_1]) = \nu_{\omega_1}(h\pi_J)^{-1}[K_1]$$

(because $\nu^*_{\omega_1} X = 1$ and $(h\pi_J)^{-1}[K_1]$ is measurable)

$$= \nu_J h^{-1}[K_1] > 0.$$

As f is arbitrary, μ is perfect. **Q**

(e) ? Suppose, if possible, that μ is countably compact. Let $\mathcal{K}$ be a countably compact class of sets such that μ is inner regular with respect to $\mathcal{K}$; we may suppose that $\mathcal{K} \subseteq \Sigma$.

(i) For $I \subseteq \omega_1$ set

$$U(I) = \{x : x \in X,\, x(\eta) = 0 \text{ for every } \eta \in I\}.$$

It will be helpful to know that if $E \in \Sigma$ and $\mu E > 0$, there is a $\gamma < \omega_1$ such that $\mu(E \cap U(I)) > 0$ for every finite $I \subseteq \omega_1 \setminus \gamma$. **P** Express E as $X \cap F$ where $F \in \mathrm{T}_{\omega_1}$. Let $J \subseteq \omega_1$ be a countable set such that $\nu_{\omega_1}(F' \setminus F) = 0$, where $F' = \pi_J^{-1}[\pi_J[F]]$ (254Od again), and $\gamma < \omega_1$ such that $J \subseteq \gamma$. If $I \subseteq \omega_1 \setminus \gamma$ is finite, then $I \cap J = \emptyset$, while $U(I)$ is determined by coordinates in I and F' is determined by coordinates in J; so

$$\mu(E \cap U(I)) = \nu^*_{\omega_1}(X \cap F \cap U(I)) = \nu_{\omega_1}(F \cap U(I))$$
$$= \nu_{\omega_1}(F' \cap U(I)) = \nu_{\omega_1} F' \cdot \nu_{\omega_1} U(I) = \mu E \cdot \nu_{\omega_1} U(I) > 0.$$

Thus this γ serves. **Q**

(ii) Let $\mathcal{M}$ be the family of countable subsets M of $\omega_1 \cup \mathcal{K}$ such that

(α) if $I \subseteq M \cap \omega_1$ is finite there is a $K \in M \cap \mathcal{K}$ such that $K \subseteq U(I)$ and $\mu K > 0$;

(β) if $K \in M \cap \mathcal{K}$, $I \subseteq M \cap \omega_1$ is finite and $\mu(K \cap U(I)) > 0$, then there is a $K' \in M \cap \mathcal{K}$ such that $K' \subseteq K \cap U(I)$ and $\mu K' > 0$;

(γ) if $\gamma \in M \cap \omega_1$ then $\gamma \subseteq M$;

(δ) if $K \in M \cap \mathcal{K}$ and $\mu K > 0$ then there is a $\gamma \in M \cap \omega_1$ such that $\mu(K \cap U(I)) > 0$ whenever $I \subseteq \omega_1 \setminus \gamma$ is finite.

Then every countable $M \subseteq \omega_1 \cup \mathcal{K}$ is included in some member M' of $\mathcal{M}$.

P Choose $\langle N_n \rangle_{n\in\mathbb{N}}$ as follows. $N_0 = M$. Given that N_n is a countable subset of $\omega_1 \cup \mathcal{K}$ then let $N_{n+1} \subseteq \omega_1 \cup \mathcal{K}$ be a countable set such that

(α) if $I \subseteq N_n \cap \omega_1$ is finite there is a $K \in N_{n+1} \cap \mathcal{K}$ such that $K \subseteq U(I)$ and $\mu K > 0$;

(β) if $K \in N_n \cap \mathcal{K}$, $I \subseteq N_n \cap \omega_1$ is finite and $\mu(K \cap U(I)) > 0$, then there is a $K' \in N_{n+1} \cap \mathcal{K}$ such that $K' \subseteq K \cap U(I)$ and $\mu K' > 0$;

(γ) if $\gamma \in N_n \cap \omega_1$ then $\gamma \subseteq N_{n+1}$;

(δ) if $K \in N_n \cap \mathcal{K}$ and $\mu K > 0$ then there is a $\gamma \in N_{n+1} \cap \omega_1$ such that $\mu(K \cap U(I)) > 0$ whenever $I \subseteq \omega_1 \setminus \gamma$ is finite;

(ϵ) $N_n \subseteq N_{n+1}$.

On completing the induction, set $M' = \bigcup_{n\in\mathbb{N}} N_n$; this serves (because every finite subset of M' is a subset of some N_n). **Q**

(iii) Choose a sequence $\langle M_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{M}$ such that, for each n, $M_n \cup \{\sup(M_n \cap \omega_1) + 1\} \subseteq M_{n+1}$. Set $\gamma_n = \sup(M_n \cap \omega_1)$ for each n. Note that $\gamma_n \subseteq M_n$, because if $\eta < \gamma_n$ then there is some $\xi \in M_n$ such that $\eta < \xi$; now $\xi \subseteq M_n$ because $M_n \in \mathcal{M}$, so $\eta \in M_n$. Also $\gamma_n + 1 \in M_{n+1}$ for each n, so $\langle \gamma_n \rangle_{n\in\mathbb{N}}$ is strictly increasing, and $\xi = \sup_{n\in\mathbb{N}} \gamma_n$ belongs to Ω.

Set $J=\{\theta_\xi(n):n\in\mathbb{N}\}$. Then $J\cap\eta$ is finite for every $\eta<\xi$, and in particular $J\cap\gamma_n$ is finite for every n. Set $I_0=J\cap\gamma_0$ and $I_n=J\cap\gamma_n\setminus\gamma_{n-1}$ for $n\ge 1$. Then $\bigcap_{n\in\mathbb{N}}U(I_n)=Q_\xi$ is disjoint from X.

Choose a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ as follows. Because I_0 is a finite subset of $M_0\cap\omega_1$, there is a $K_0\in M_0\cap\mathcal{K}$ such that $K_0\subseteq U(I_0)$ and $\mu K_0>0$. Given that $K_n\in M_n\cap\mathcal{K}$ and $\mu K_n>0$, then there is a $\beta\in M_n\cap\omega_1$ such that $\mu(K_n\cap U(I))>0$ for every finite $I\subseteq\omega_1\setminus\beta$; now $\beta\le\gamma_n$ and $I_{n+1}\cap\gamma_n=\emptyset$, so $\mu(K_n\cap U(I_{n+1}))>0$. But $K_n\in M_{n+1}\cap\mathcal{K}$ and I_{n+1} is a finite subset of $M_{n+1}\cap\omega_1$, so there is a $K_{n+1}\in M_{n+1}\cap\mathcal{K}$ such that $K_{n+1}\subseteq K_n\cap U(I_{n+1})$ and $\mu K_{n+1}>0$. Continue.

In this way we find a non-increasing sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $K_n\subseteq U(I_n)$ for every n and no K_n is empty. But in this case $\bigcap_{i\le n}K_i=K_n$ is non-empty for every n, while $\bigcap_{n\in\mathbb{N}}K_n\subseteq X\cap\bigcap_{n\in\mathbb{N}}U(I_n)$ is empty. So $\mathcal{K}$ is not a countably compact class. **X**

(f) Thus μ is not countably compact, and has all the properties claimed.

***451V Weakly α-favourable spaces** There is an interesting variation on the concept of 'countably compact' measure space, as follows. For any measure space (X,Σ,μ) we can imagine an infinite game for two players, whom I will call 'Empty' and 'Nonemepty'. Empty chooses a non-negligible measurable set E_0; Nonempty chooses a non-negligible measurable set $F_0\subseteq E_0$; Empty chooses a non-negligible measurable set $E_1\subseteq F_0$; Nonempty chooses a non-negligible measurable set $F_1\subseteq E_1$, and so on. At the end of the game, Empty wins if $\bigcap_{n\in\mathbb{N}}E_n=\bigcap_{n\in\mathbb{N}}F_n$ is empty; otherwise Nonempty wins. (If $\mu X=0$, so that Empty has no legal initial move, I declare Nonempty the winner by default.) If you have seen 'Banach-Mazur' games, you will recognise this as a similar construction, in which open sets are replaced by non-negligible measurable sets.

A **strategy** for Nonempty is a rule to determine his moves in terms of the preceding moves for Empty; that is, a function $\sigma:\bigcup_{n\in\mathbb{N}}(\Sigma\setminus\mathcal{N})^{n+1}\to\Sigma\setminus\mathcal{N}$, where $\mathcal{N}$ is the ideal of negligible sets, such that $\sigma(E_0,E_1,\dots,E_n)\subseteq E_n$, at least whenever $E_0,\dots,E_n\in\Sigma\setminus\mathcal{N}$ are such that $E_{k+1}\subseteq\sigma(E_0,\dots,E_k)$ for every $k<n$; since it never matters what Nonempty does if Empty has already broken the rules, we usually just demand that $\sigma(E_0,\dots,E_n)\subseteq E_n$ for all $E_0,\dots,E_n\in\Sigma\setminus\mathcal{N}$. σ is a **winning strategy** if $\bigcap_{n\in\mathbb{N}}E_n\ne\emptyset$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\Sigma\setminus\mathcal{N}$ such that $E_{n+1}\subseteq\sigma(E_0,\dots,E_n)$ for every $n\in\mathbb{N}$. In terms of the game, we interpret this as saying that Nonempty will win if he plays $F_n=\sigma(E_0,\dots,E_n)$ whenever faced with the position $(E_0,F_0,E_1,F_1,\dots,F_{n-1},E_n)$. (Since it is supposed that Nonempty will use the same strategy throughout the game, the moves $F_0,\dots,F_{n-1}$ are determined by $E_0,\dots,E_{n-1}$ and there is no advantage in taking them separately into account when choosing F_n.)

Now we say that the measure space (X,Σ,μ) is **weakly α-favourable** if there is such a winning strategy for Nonempty.

It turns out that the class of weakly α-favourable spaces behaves in much the same way as the class of countably compact spaces. For the moment, however, I leave the details to the exercises (451Yh-451Yr). See Fremlin 00.

451X Basic exercises (a) (i) Show that any purely atomic measure space is perfect. (ii) Show that any strictly localizable purely atomic measure space is countably compact. (iii) Show that the space of 342N is not countably compact.

>(b) Show that a compact measure space in which singleton sets are negligible is atomless.

>(c) Let (X,Σ,μ) be a measure space, and ν an indefinite-integral measure over μ (234J[5]). Show that ν is compact, or countably compact, or perfect if μ is.

(d) In 413Xo, show that μ is a countably compact measure. (*Hint*: show that the algebra Σ there is a countably compact class.)

(e) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X,Σ,μ). Show that μ is compact, or countably compact, or perfect iff every μ_i is.

(f) Let (X,Σ,μ) be a measure space and $\mathcal{K}$ a family of subsets of X such that whenever $E\in\Sigma$ and $\mu E>0$ there is a $K\in\mathcal{K}$ such that $K\subseteq E$ and $\mu_*K>0$. (i) Show that if $\mathcal{K}$ is a compact class then μ is a compact measure. (ii) Show that if $\mathcal{K}$ is a countably compact class then μ is a countably compact measure.

(g) Let (X,Σ,μ) be a measure space. For $A\subseteq X$, write μ_A for the subspace measure on A. Suppose that whenever $E\in\Sigma$ and $\mu E>0$ there is a set $A\subseteq X$ such that μ_A is perfect and $\mu^*(A\cap E)>0$. Show that μ is perfect.

[5] Formerly 234B.

(h)(i) Give an example of a compact probability space (X,Σ,μ) and a σ-subalgebra T of Σ such that $\mu\restriction\mathrm{T}$ is not compact. (ii) Give an example of a compact probability space (X,Σ,μ), a set Y and a function $f:X\to Y$ such that the image measure μf^{-1} is not compact. (*Hint*: 342M, 342Xf, 439Xa.)

(i) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, with product X. Suppose that $\mathcal{K}_i\subseteq\mathcal{P}X_i$ for each i, and set $\mathcal{K}=\{\prod_{i\in I}K_i:K_i\in\mathcal{K}_i$ for each $i\}$. (i) Show that if $\mathcal{K}_i$ is a compact class for each i, so is $\mathcal{K}$. (ii) Show that if $\mathcal{K}_i$ is a countably compact class for each i, so is $\mathcal{K}$.

(j) Let $A\subseteq[0,1]$ be a set with outer Lebesgue measure 1 and inner measure 0. Show that there is a Borel measure λ on $A\times[0,1]$ such that λ is not inner regular with respect to sets which have Borel measurable projections on the factor spaces.

(k) Let X be a Polish space and E a subset of X. Show that the following are equiveridical: (i) E is universally measurable; (ii) every Borel probability measure on E is perfect; (iii) every σ-finite Borel measure on E is compact; (iv) $f[E]$ is universally measurable in $\mathbb{R}$ for every Borel measurable function $f:X\to\mathbb{R}$.

(l) In 451N, show that μ is a compact measure.

(m) Find a Radon measure space $(X,\mathfrak{T},\Sigma,\mu)$, a continuous function $f:X\to[0,1]$ and a set $B\subseteq[0,1]$ such that $\mu^*(f^{-1}[B])<(\mu f^{-1})^*B$.

(n) Let (X,Σ,μ) be a σ-finite measure space. Show that it is perfect iff whenever $f:X\to\mathbb{R}$ is measurable there is a K_σ set $H\subseteq f[X]$ such that $f^{-1}[H]$ is conegligible.

(o) Let X be a metrizable space, and μ a semi-finite topological measure on X which (regarded as a measure) is compact. Show that μ is τ-additive.

(p) Let (X,Σ,μ) be a compact strictly localizable measure space (e.g., any Radon measure space), (Y,T,ν) a σ-finite measure space, and $f:X\to L^0(\nu)$ a function. Show that the following are equiveridical: (i) f is measurable, when $L^0(\nu)$ is given its topology of convergence in measure; (ii) there is a function $h\in\mathcal{L}^0(\lambda)$, where λ is the c.l.d. product measure on $X\times Y$, such that $f(x)=h_x^\bullet$ for almost every $x\in X$, where $h_x(y)=h(x,y)$. (*Hint*: 418R.)

>(q) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space. Show that $\Sigma=\mathcal{P}X$ iff μ is purely atomic. (*Hint*: if $\Sigma=\mathcal{P}X$, apply 451T with $Y=X$, the discrete topology on Y and the identity function from X to Y.)

(r) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space and U a normed space. Show that if f, $g:X\to U$ are measurable functions, then $f+g$ is measurable. (Cf. 418Xj.)

(s) Show that in all three of the constructions of 439A, the measure ν is countably compact. (*Hint*: for the 'third construction', consider $\{f^{-1}[F]:F\subseteq\{0,1\}^{\mathfrak{c}}$ is a zero set$\}$.)

451Y Further exercises (a) Show that for any probability space (X,Σ,μ), there is a compact probability space (Y,T,ν) with a subspace isomorphic to (X,Σ,μ).

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i a σ-algebra of subsets of X_i for each i. Suppose that for each finite $J\subseteq I$ we are given a finitely additive functional ν_J on $X_J=\prod_{i\in J}X_i$, with domain the algebra $\mathrm{T}_J=\bigotimes_{i\in J}\Sigma_i$ generated by sets of the form $\{x:x\in X_J,\,x(i)\in E\}$ for $i\in J$, $E\in\Sigma_i$, and that (α) $\nu_K\{x:x\in X_K,\,x\restriction J\in W\}=\nu_JW$ whenever $J\subseteq K\in[I]^{<\omega}$ and $W\in\mathrm{T}_J$ (β) $\mu_i=\nu_{\{i\}}$ is a countably compact probability measure for every $i\in I$. Show that there is a countably compact measure μ on $X=X_I$ such that $\mu\{x:x\in X,\,x\restriction J\in W\}=\nu_JW$ whenever $J\in[I]^{<\omega}$ and $W\in\mathrm{T}_J$. (*Hint*: 454D.) (Compare 418M.)

(c) Describe μ in the case of 451Yb in which $I=[0,1]$, $X_i=[0,1]\setminus\{i\}$, Σ_i is the algebra of Lebesgue measurable subsets of X_i, and $\nu_JE=\mu_L\{t:z_{Jt}\in E\}$ for every $E\in\bigotimes_{i\in J}\Sigma_i$, where $z_{Jt}(i)=t$ for $i\in J$, $t\in[0,1]$. Contrast this with the difficulty encountered in 418Xu.

(d) Let (X,Σ,μ) be a semi-finite compact measure space, and $\langle E_i\rangle_{i\in I}$ a point-finite family of measurable subsets of X such that $\bigcup_{i\in J}E_i\in\Sigma$ for every $J\subseteq I$. Show that $\mu(\bigcup_{i\in I}E_i)=\sup_{J\subseteq I\text{ is finite}}\mu(\bigcup_{i\in J}E_i)$. (*Hint*: 438Ya.)

(e) Let X be a hereditarily metacompact space, and μ a semi-finite topological measure on X which (regarded as a measure) is compact. Show that μ is τ-additive.

(f) Let (X,Σ,μ) be a compact measure space, V a Banach space and $f:X\to V$ a measurable function such that $\|f\|:X\to[0,\infty[$ is integrable. Show that f is Bochner integrable (253Yf).

(g) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space. Suppose that Y is a separable metrizable space and Z is a metrizable space, and that $f:X\times Y\to Z$ is a function such that $x\mapsto f(x,y)$ is measurable for every $y\in Y$ and $y\mapsto f(x,y)$ is continuous for every $x\in X$. Show that μ is inner regular with respect to $\{F:F\subseteq X,\ f\restriction F\times Y$ is continuous$\}$. (*Hint*: 418Yo.)

(h) Show that any purely atomic measure space is weakly α-favourable, so that the space of 342N is weakly α-favourable but not countably compact.

(i) Show that the direct sum of a family of weakly α-favourable measure spaces is weakly α-favourable.

(j) Show that an indefinite-integral measure over a weakly α-favourable measure is weakly α-favourable.

(k)(i) Show that a countably compact measure space is weakly α-favourable. (ii) Show that a semi-finite weakly α-favourable measure space is perfect.

(l) Show that any measurable subspace of a weakly α-favourable measure space is weakly α-favourable.

(m) Let (X,Σ,μ) be a weakly α-favourable measure space, (Y,T,ν) a semi-finite measure space, and $f:X\to Y$ a (Σ,T)-measurable function such that $f^{-1}[F]$ is negligible whenever $F\subseteq Y$ is negligible. Show that (Y,T,ν) is weakly α-favourable.

(n)(i) Show that a measure space is weakly α-favourable iff its completion is weakly α-favourable. (ii) Show that a semi-finite measure space is weakly α-favourable iff its c.l.d. version is weakly α-favourable.

(o) Show that the c.l.d. product of two weakly α-favourable measure spaces is weakly α-favourable.

(p) Show that the product of any family of weakly α-favourable probability measures is weakly α-favourable.

(q) Show that the space of 451U is not weakly α-favourable.

(r) Let (X,Σ,μ) be a complete locally determined measure space and $\underline{\phi}$ a lower density for μ such that $\underline{\phi}X=X$; let $\mathfrak{T}$ be the corresponding density topology (414P). Show that (X,Σ,μ) is weakly α-favourable iff $(X,\mathfrak{T})$ is weakly α-favourable (definition: 4A2A).

(s) Let X be a set, and $\langle\mu_i\rangle_{i\in I}$ a family of weakly α-favourable measures on X with sum μ (234G[6]). Show that if μ is semi-finite, it is weakly α-favourable.

(t) Let X and Y be locally compact Hausdorff groups and $\phi:X\to Y$ a group homomorphism which is Haar measurable in the sense of 411L, that is, $\phi^{-1}[H]$ is Haar measurable for every open $H\subseteq Y$. Show that ϕ is continuous.

451 Notes and comments For a useful survey of results on countably compact and perfect measures, with historical notes, see Ramachandran 02.

The concepts of 'compact', 'countably compact' and 'perfect' measure space can all be regarded as attempts to understand and classify the special properties of Lebesgue measure on $[0,1]$, regarded as a measure space. Because a countably separated perfect probability space is very nearly isomorphic to Lebesgue measure (451Ad), we can think of a perfect measure space as one in which the countably-generated σ-subalgebras look like Lebesgue measure (451F). The arguments of 451Ic and 451Jc already hint at the kind of results we can hope for. When we form a product measure, each measurable set in the product will depend, in effect, on sequences of measurable sets in the factors, and therefore can be studied in terms of countably generated subalgebras; so that many results about products of perfect measures will be derivable, if we wish to take that route, from results about products of copies of Lebesgue measure. Of course my normal approach in this treatise is to go straight for the general result; but like anyone else

[6] Formerly 112Ya.

I often start from a picture based on the familiar special case. In the next section we shall have some theorems for which countable compactness, rather than perfectness, seems to be the relevant property.

The first half of the section (down to 451P) is essentially a matter of tidying up the theory of compact and perfect measures, and showing that the same ideas will cover the new class of countably compact measures. (You may like to go back to 342G, in which I worked through the basic properties of compact measures, and contrast the arguments used there with the slightly more sophisticated ones above.) In 451Q-451T I enter new territory, showing that for compact measures (and therefore for Radon measures) the theory of measurable functions into metric spaces is particularly simple, without making any assumptions about measure-free cardinals.

452 Integration and disintegration of measures

A standard method of defining measures is through a formula

$$\mu E = \int \mu_y E \, \nu(dy)$$

where (Y, T, ν) is a measure space and $\langle \mu_y \rangle_{y \in Y}$ is a family of measures on another set X. In practice these constructions commonly involve technical problems concerning the domain of μ (as in 452Xi), which is why I have hardly used them so far in this treatise. There are not-quite-trivial examples in 417Yb, 434R and 436F, and the indefinite-integral measures of §234 can also be expressed in this way (452Xf); for a case in which this approach is worked out fully, see 453N. But when a formula of this kind is valid, as in Fubini's theorem, it is likely to be so useful that it dominates further investigation of the topic. In this section I give one of the two most important theorems guaranteeing the existence of appropriate families $\langle \mu_y \rangle_{y \in Y}$ when μ and ν are given (452I); the other will follow in the next section (453K). They both suppose that we are provided with a suitable function $f : X \to Y$, and rely heavily on the Lifting Theorem (§341) and on considerations of inner regularity from Chapter 41.

The formal definition of a 'disintegration' (which is nearly the same thing as a 'regular conditional probability') is in 452E. The main theorem depends, for its full generality, on the concept of 'countably compact measure' (451B). It can be strengthened when μ is actually a Radon measure (452O).

The greater part of the section is concerned with general disintegrations, in which the measures μ_y are supposed to be measures on X and are not necessarily related to any particular structure on X. However a natural, and obviously important, class of applications has $X = Y \times Z$ and each μ_y based on the section $\{y\} \times Z$, so that it can be regarded as a measure on Z. Mostly there is very little more to be said in this case (see 452B-452D); but in 452M we find that there is an interesting variation in the way that countable compactness can be used.

452A Lemma Let (Y, T, ν) be a measure space, X a set, and $\langle \mu_y \rangle_{y \in Y}$ a family of measures on X. Let $\mathcal{A}$ be the family of subsets A of X such that $\theta E = \int \mu_y E \, \nu(dy)$ is defined in $\mathbb{R}$. Suppose that $X \in \mathcal{A}$.

(a) $\mathcal{A}$ is a Dynkin class.

(b) If Σ is any σ-subalgebra of $\mathcal{A}$ then $\mu = \theta \restriction \Sigma$ is a measure on X.

(c) Suppose now that every μ_y is complete. If, in (b), $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\hat{\Sigma} \subseteq \mathcal{A}$ and $\hat{\mu} = \theta \restriction \hat{\Sigma}$.

proof For (a) and (b), we have only to look at the definitions of 'Dynkin class' and 'measure' and apply the elementary properties of the integral. For (c), if $E \in \hat{\Sigma}$, then there are E', $E'' \in \Sigma$ such that $E' \subseteq E \subseteq E''$ and $\theta E' = \theta E''$. So $\mu_y E' = \mu_y E''$ for ν-almost every y; since all the μ_y are supposed to be complete, $\mu_y E$ is defined and equal to $\mu_y E'$ for almost every y, and θE is defined and equal to $\theta E' = \mu E' = \hat{\mu} E$.

452B Theorem (a) Let X be a set, (Y, T, ν) a measure space, and $\langle \mu_y \rangle_{y \in Y}$ a family of measures on X such that $\int \mu_y X \, \nu(dy)$ is defined and finite. Let $\mathcal{E}$ be a family of subsets of X, closed under finite intersections, such that $\int \mu_y E \, \nu(dy)$ is defined in $\mathbb{R}$ for every $E \in \mathcal{E}$.

(i) If Σ is the σ-algebra of subsets of X generated by $\mathcal{E}$, we have a totally finite measure μ on X, with domain Σ, given by the formula $\mu E = \int \mu_y E \, \nu(dy)$ for every $E \in \Sigma$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E \, \nu(dy)$ is defined and equal to $\hat{\mu} E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

(b) Let Z be a set, (Y, T, ν) a measure space, and $\langle \mu_y \rangle_{y \in Y}$ a family of measures on Z such that $\int \mu_y Z \, \nu(dy)$ is defined and finite. Let $\mathcal{H}$ be a family of subsets of Z, closed under finite intersections, such that $\int \mu_y H \, \nu(dy)$ is defined in $\mathbb{R}$ for every $H \in \mathcal{H}$.

(i) If Υ is the σ-algebra of subsets of Z generated by $\mathcal{H}$, we have a totally finite measure μ on $Y \times Z$, with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, defined by setting $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E[\{y\}]\nu(dy)$ is defined and equal to $\hat{\mu}E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

proof (a) Define $\mathcal{A} \subseteq \mathcal{P}X$ as in 452A. Then $\mathcal{E} \subseteq \mathcal{A}$, so by the Monotone Class Theorem (136B) $\Sigma \subseteq \mathcal{A}$ and we have (i). Applying 452Ac to $\langle \hat{\mu}_y \rangle_{y\in Y}$ we have (ii).

(b) Set $X = Y \times Z$. For $y \in Y$, let μ'_y be the measure on X defined by setting $\mu'_y E = \mu_y E[\{y\}]$ whenever this is defined; that is, μ'_y is the image of μ_y under the function $z \mapsto (y, z) : Z \to X$. Set $\mathcal{E} = \{F \times H : F \in \mathrm{T},\ H \in \mathcal{H}\}$. Then $\mathcal{E}$ is a family of subsets of X closed under finite intersections, and

$$\int \mu'_y(F \times H)\nu(dy) = \int \chi F(y)\mu_y H\,\nu(dy)$$

is defined whenever $F \in \mathrm{T}$ and $H \in \mathcal{H}$. By (a), we have a measure μ on X, with domain the σ-algebra Σ generated by $\mathcal{E}$, defined by writing

$$\mu E = \int \mu'_y E\,\nu(dy) = \int \mu_y E[\{y\}]\nu(dy)$$

for every $E \in \Sigma$. Of course Σ includes $\mathrm{T}\widehat{\otimes}\Upsilon$ (the set $\{H : Y \times H \in \Sigma\}$ is a σ-algebra of subsets of Z including $\mathcal{H}$, so includes Υ) and is therefore equal to $\mathrm{T}\widehat{\otimes}\Upsilon$.

This proves (i). If now $E \in \hat{\Sigma}$, (a-ii) tells us that

$$\hat{\mu}E = \int \hat{\mu}'_y E\,\nu(dy) = \int \hat{\mu}_y E[\{y\}]\nu(dy).$$

452C Theorem (a) Let Y be a topological space, ν a τ-additive topological measure on Y, $(X, \mathfrak{T})$ a topological space, and $\langle \mu_y \rangle_{y\in Y}$ a family of τ-additive topological measures on X such that $\int \mu_y X\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y \mapsto \mu_y U$ is lower semi-continuous for every $U \in \mathcal{U}$.

(i) We can define a τ-additive Borel measure μ on X by writing $\mu E = \int \mu_y E\,\nu(dy)$ for every Borel set $E \subseteq X$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E\,\nu(dy)$ is defined and equal to $\hat{\mu}E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

(b) Let Y be a topological space, ν a τ-additive topological measure on Y, $(Z, \mathfrak{U})$ a topological space, and $\langle \mu_y \rangle_{y\in Y}$ a family of τ-additive topological measures on Z such that $\int \mu_y Z\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{V}$ for $\mathfrak{U}$, closed under finite unions, such that $y \mapsto \mu_y V$ is lower semi-continuous for every $V \in \mathcal{V}$.

(i) We can define a τ-additive Borel measure μ on $Y \times Z$ by writing $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every Borel set $E \subseteq Y \times Z$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E[\{y\}]\nu(dy)$ is defined and equal to $\hat{\mu}E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

proof (a) For $A \subseteq X$, set $f_A(y) = \mu_y A$ when this is defined. We may suppose that $\emptyset \in \mathcal{U}$. If $\mathcal{W} \subseteq \mathcal{U}$ is a non-empty upwards-directed set with union G, $\langle f_W \rangle_{W\in\mathcal{W}}$ is an upwards-directed family of lower semi-continuous functions with supremum f_G, because every μ_y is τ-additive. So f_G is lower semi-continuous, and also $\int f_G d\nu = \sup_{W\in\mathcal{W}} \int f_W d\nu$, by 414Ba. Taking $\mathcal{E}$ to be the family of open subsets of X in 452Ba, we see that we have a τ-additive Borel measure μ on X such that $\mu E = \int \mu_y E\,\nu(dy)$ for every Borel set $E \subseteq X$. Moreover, if $\mathcal{G}$ is a non-empty upwards-directed family of open subsets of X with union G^*, then $\mathcal{W} = \{W : W \in \mathcal{U},\ W \subseteq G \text{ for some } G \in \mathcal{G}\}$ is an upwards-directed family with union G^*, so

$$\mu G^* = \int f_{G^*} d\nu = \sup_{W\in\mathcal{W}} \int f_W d\nu \le \sup_{G\in\mathcal{G}} \mu G \le \mu G^*.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive. This proves (i); (ii) follows immediately, as in 452Ba.

(b) Let $\mathcal{U}$ be the family of sets expressible as $\bigcup_{i\le n} H_i \times V_i$ where $H_i \subseteq Y$ is open and $V_i \in \mathcal{V}$ for every $i \le n$. Because $\mathcal{V}$ is a base for $\mathfrak{U}$, $\mathcal{U}$ is a base for the topology of $X = Y \times Z$. For $y \in Y$ let μ'_y be the measure on X defined by saying that $\mu'_y E = \mu_y E[\{y\}]$ whenever this is defined. Then μ'_y is a τ-additive topological probability measure on X, by 418Ha or otherwise. If $U \in \mathcal{U}$, $y \mapsto \mu'_y U$ is lower semi-continuous. **P** Express U as $\bigcup_{i\le n} H_i \times V_i$ where $H_i \subseteq Y$ is open and $V_i \in \mathcal{V}$ for each i. Suppose that $y \in Y$ and $\gamma < \mu_y U$. Set $I = \{i : i \le n,\ y \in H_i\}$, $H = Y \cap \bigcap_{i\in I} H_i$ and $V = \bigcup_{i\in I} V_i$. Then $U[\{y\}] = V \subseteq U[\{y'\}]$ for every $y' \in H$. Also $H' = \{y' : \mu_{y'} V > \gamma\}$ is a neighbourhood of y. So $H \cap H'$ is a neighbourhood of y, and $\mu'_{y'} U > \gamma$ for every $y' \in H \cap H'$. As y and γ are arbitrary, we have the result. **Q**

Now applying (a) to $\langle \mu'_y \rangle_{y\in Y}$ we see that (b) is true.

452D Theorem (a) Let $(Y,\mathfrak{S},\mathrm{T},\nu)$ be a Radon measure space, $(X,\mathfrak{T})$ a topological space, and $\langle\mu_y\rangle_{y\in Y}$ a uniformly tight (definition: 437O) family of Radon measures on X such that $\int\mu_yX\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y\mapsto\mu_yU$ is lower semi-continuous for every $U\in\mathcal{U}$. Then we have a totally finite Radon measure $\tilde\mu$ on X defined by saying that that $\tilde\mu E=\int\mu_yE\,\nu(dy)$ whenever $\tilde\mu$ measures E.

(b) Let $(Y,\mathfrak{S},\mathrm{T},\nu)$ be a Radon measure space, $(Z,\mathfrak{U})$ a topological space, and $\langle\mu_y\rangle_{y\in Y}$ a uniformly tight family of Radon measures on Z such that $\int\mu_yZ\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{V}$ for $\mathfrak{U}$, closed under finite unions, such that $y\mapsto\mu_yV$ is lower semi-continuous for every $V\in\mathcal{V}$. Then we have a totally finite Radon measure $\tilde\mu$ on $Y\times Z$ such that $\tilde\mu E=\int\mu_yE[\{y\}]\nu(dy)$ whenever $\tilde\mu$ measures E.

proof I take the two parts together. By 452C we have a τ-additive Borel measure μ satisfying the appropriate formula. Now for any $\epsilon>0$ there is a compact set $K\subseteq X$ such that $\mu K\ge\mu X-2\epsilon$. **P** In (a), take $\eta>0$ such that $\int\min(\eta,\mu_yX)\nu(dy)\le 2\epsilon$, and K such that $\mu_y(X\setminus K)\le\eta$ for every $y\in Y$. In (b), take $\eta>0$ such that $\int\min(\eta,\mu_yZ)\nu(dy)\le\epsilon$. Now let $K_1\subseteq Y$ and $K_2\subseteq Z$ be compact sets such that

$$\int_{K_1}\mu_yZ\,\nu(dy)\ge\int_Y\mu_yZ\,\nu(dy)-\epsilon,\quad \mu_y(Z\setminus K_2)\le\eta\text{ for every }y\in Y.$$

Then $K=K_1\times K_2$ is compact and

$$\begin{aligned}\mu((Y\times Z)\setminus K)&\le\int_{Y\setminus K_1}\mu_yZ\,\nu(dy)+\int_{K_1}\mu_y(Z\setminus K_2)\nu(dy)\\&\le\epsilon+\int_{K_1}\min(\eta,\mu_yZ)\nu(dy)\le 2\epsilon.\ \mathbf{Q}\end{aligned}$$

Since μ is totally finite it is surely locally finite and effectively locally finite, so the conditions of 416F(iv) are satisfied and the c.l.d. version $\tilde\mu$ of μ is a Radon measure on X. But of course $\tilde\mu$ is just the completion of μ, so 452C(a-ii) or 452C(b-ii) tells us that the declared formula also applies to $\tilde\mu$.

452E All the constructions above can be thought of as special cases of the following.

Definition Let (X,Σ,μ) and (Y,T,ν) be measure spaces. A **disintegration** of μ over ν is a family $\langle\mu_y\rangle_{y\in Y}$ of measures on X such that $\int\mu_yE\,\nu(dy)$ is defined in $[0,\infty]$ and equal to μE for every $E\in\Sigma$. If $f:X\to Y$ is an inverse-measure-preserving function, a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν is **consistent** with f if, for each $F\in\mathrm{T}$, $\mu_yf^{-1}[F]=1$ for ν-almost every $y\in F$. $\langle\mu_y\rangle_{y\in Y}$ is **strongly consistent** with f if, for almost every $y\in Y$, μ_y is a probability measure for which $f^{-1}[\{y\}]$ is conegligible.

A trivial example of a disintegration is when ν is a probability measure and $\mu_y=\mu$ for every y. Of course this is of little interest. The archetypal disintegration is 452Bb when all the μ_y are the same, in which case Fubini's theorem tells us that we are looking at a product measure on $X=Y\times Z$. If μ is a probability measure then this disintegration is strongly consistent.

The phrase **regular conditional probability** is used for special types of disintegration; typically, when μ and ν and every μ_y are probabilities, and sometimes supposing that every μ_y has the same domain as μ. I have seen the word **decomposition** used for what I call a disintegration.

452F Proposition Let (X,Σ,μ) and (Y,T,ν) be measure spaces and $\langle\mu_y\rangle_{y\in Y}$ a disintegration of μ over ν. Then $\iint f(x)\mu_y(dx)\nu(dy)$ is defined and equal to $\int fd\mu$ for every $[-\infty,\infty]$-valued function f such that $\int fd\mu$ is defined in $[-\infty,\infty]$.

proof (a) Suppose first that f is non-negative. Let $H\in\Sigma$ be a conegligible set such that $f\restriction H$ is Σ-measurable. For $n\in\mathbb{N}$ set

$$E_{nk}=\{x:x\in H,\,2^{-n}k\le f(x)\}\text{ for }k\ge1,\quad f_n=2^{-n}\textstyle\sum_{k=1}^{4^n}\chi E_{nk}.$$

Then $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of functions with $\lim_{n\to\infty}f_n(x)=f(x)$ for every $x\in H$. Now $\int\mu_y(X\setminus H)\nu(dy)=0$, so $X\setminus H$ is μ_y-negligible for almost every y. Set

$$V=\{y:\mu_y(X\setminus H)=0,\,E_{nk}\in\operatorname{dom}\mu_y\text{ for every }n\in\mathbb{N},\,k\ge1\};$$

then V is ν-conegligible. For $y\in V$,

$$\int fd\mu_y=\lim_{n\to\infty}\int f_nd\mu_y=\lim_{n\to\infty}2^{-n}\textstyle\sum_{k=1}^{4^n}\mu_yE_{nk},$$

while each function $y\mapsto\mu_yE_{nk}$ is ν-virtually measurable, so $y\mapsto\int fd\mu_y$ is ν-virtually measurable and

$$\iint f d\mu_y \nu(dy) = \lim_{n\to\infty} \iint f_n d\mu_y \nu(dy) = \lim_{n\to\infty} 2^{-n} \sum_{k=1}^{4^n} \int \mu_y E_{nk}\, \nu(dy)$$
$$= \lim_{n\to\infty} 2^{-n} \sum_{k=1}^{4^n} \mu E_{nk} = \lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

(b) For general f we now have

$$\iint f(x)\mu_y(dx)\nu(dy) = \iint f^+(x)\mu_y(dx)\nu(dy) - \iint f^-(x)\mu_y(dx)\nu(dy)$$
$$= \int f^+ d\mu - \int f^- d\mu = \int f d\mu,$$

where f^+, f^- are the positive and negative parts of f.

Remark When $X = Y \times Z$ and our disintegration is a family $\langle \mu'_y \rangle_{y\in Y}$ of measures on X defined from a family $\langle \mu_y \rangle_{y\in Y}$ of probability measures on Z, as in 452Bb, we can more naturally write $\int f(y,z)\mu_y(dz)$ in place of $\int f(x)\mu'_y(dx)$, and we get

$$\iint f(y,z)\mu_y(dz)\nu(dy) = \int f d\mu \text{ whenever the latter is defined in } [-\infty, \infty]$$

as in 252B.

452G The most useful theorems about disintegrations of course involve some restrictions on their form, most commonly involving consistency with some kind of projection. I clear the path with statements of some elementary facts.

Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, $f : X \to Y$ an inverse-measure-preserving function, and $\langle \mu_y \rangle_{y\in Y}$ a disintegration of μ over ν.

(a) If $\langle \mu_y \rangle_{y\in Y}$ is consistent with f, and $F \in \mathrm{T}$, then $\mu_y f^{-1}[F] = \chi F(y)$ for ν-almost every $y \in Y$; in particular, almost every μ_y is a probability measure.

(b) If $\langle \mu_y \rangle_{y\in Y}$ is strongly consistent with f it is consistent with f.

(c) If ν is countably separated (definition: 343D) and $\langle \mu_y \rangle_{y\in Y}$ is consistent with f, then it is strongly consistent with f.

proof (a) We have $\mu_y f^{-1}[F] = 1$ for almost every $y \in F$. Since also

$$\mu_y(X \setminus f^{-1}[F]) = \mu_y f^{-1}[Y \setminus F] = 1, \quad \mu_y X = \mu_y f^{-1}[Y] = 1$$

for almost every $y \in Y \setminus F$, $\mu_y f^{-1}[F] = 0$ for almost every $y \in X \setminus F$.

(b) If $F \in \mathrm{T}$, then $f^{-1}[F] \supseteq f^{-1}[\{y\}]$ is μ_y-conegligible for almost every $y \in F$; since we are also told that $\mu_y X = 1$ for almost every y, $\mu_y f^{-1}[F] = 1$ for almost every $y \in F$.

(c) There is a countable $\mathcal{F} \subseteq \mathrm{T}$ separating the points of Y; we may suppose that $Y \in \mathcal{F}$ and that $Y \setminus F \in \mathcal{F}$ for every $F \in \mathcal{F}$. Now

$$H_F = F \setminus \{y : \mu_y f^{-1}[F] \text{ is defined and equal to } 1\}$$

is negligible for every $F \in \mathcal{F}$, so that

$$Z = Y \setminus \bigcup_{F\in\mathcal{F}} H_F$$

is conegligible. For $y \in Z$, set $\mathcal{F}_y = \{F : y \in F \in \mathcal{F}\}$; then

$$\{y\} = \bigcap \mathcal{F}_y, \quad f^{-1}[\{y\}] = \bigcap \{f^{-1}[F] : F \in \mathcal{F}_y\},$$

while $\mu_y f^{-1}[F] = 1$ for every $F \in \mathcal{F}_y$. Because $\mathcal{F}_y$ is countable, $\mu_y f^{-1}[\{y\}] = 1$. This is true for almost every y, so $\langle \mu_y \rangle_{y\in Y}$ is strongly consistent with f.

452H Lemma Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and $T : L^\infty(\mu) \to L^\infty(\nu)$ a positive linear operator such that $T(\chi X^\bullet) = \chi Y^\bullet$ and $\int Tu = \int u$ whenever $u \in L^\infty(\mu)^+$. Let $\mathcal{K}$ be a countably compact class of subsets of X, closed under finite unions and countable intersections, such that μ is inner regular with respect to $\mathcal{K}$. Then there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν such that

(i) μ_y is a complete probability measure on X, inner regular with respect to $\mathcal{K}$ and measuring every member of $\mathcal{K}$, for every $y \in Y$;

(ii) setting $h_g(y) = \int g\,d\mu_y$ whenever $g \in \mathcal{L}^\infty(\mu)$ and $y \in Y$ are such that the integral is defined, $h_g \in \mathcal{L}^\infty(\nu)$ and $T(g^\bullet) = h_g^\bullet$ for every $g \in \mathcal{L}^\infty(\mu)$.

proof (a) Completing ν does not change $\mathcal{L}^\infty(\nu)$ or $L^\infty(\nu)$, nor does it change the families which are disintegrations over ν; so we may assume throughout that ν is complete. It therefore has a lifting $\theta : \mathfrak{B} \to \mathrm{T}$, where $\mathfrak{B}$ is the measure algebra of ν, which gives rise to a Riesz homomorphism S from $L^\infty(\nu) \cong L^\infty(\mathfrak{B})$ to the space $L^\infty(\mathrm{T})$ of bounded T-measurable real-valued functions on Y such that $(Sv)^\bullet = v$ for every $v \in L^\infty(\nu)$ (363I, 363F, 363H).

(b) For $y \in Y$ and $E \in \Sigma$, set $\psi_y E = (ST(\chi E^\bullet))(y)$. Because $0 \le T(\chi E^\bullet) \le \chi Y^\bullet$ in $L^\infty(\nu)$, $0 \le \psi_y E \le 1$. The maps

$$E \mapsto \chi E \mapsto \chi E^\bullet \mapsto T(\chi E^\bullet) \mapsto ST(\chi E^\bullet)$$

are all additive, so $\psi_y : \Sigma \to [0,1]$ is additive for each $y \in Y$. For fixed $E \in \Sigma$,

$$\mu E = \int \chi E\,d\mu = \int (\chi E^\bullet) = \int T(\chi E^\bullet) = \int ST(\chi E^\bullet) = \int \psi_y E\,\nu(dy).$$

(c) Recall that μ is supposed to be inner regular with respect to the countably compact class $\mathcal{K}$. By 413Sa, there is for every $y \in Y$ a complete measure μ'_y on X such that $\mu'_y X \le \psi_y X \le 1$, $\mathcal{K} \subseteq \operatorname{dom}\mu'_y$, and $\mu'_y K \ge \psi_y K$ for every $K \in \mathcal{K} \cap \Sigma$.

(d) Now, for any fixed $E \in \Sigma$, $\mu'_y E$ is defined and equal to $\psi_y E$ for almost every $y \in Y$. **P** Let $\langle K_n\rangle_{n\in\mathbb{N}}$, $\langle K'_n\rangle_{n\in\mathbb{N}}$ be sequences in $\mathcal{K} \cap \Sigma$ such that $K_n \subseteq E$ and $K'_n \subseteq X \setminus E$ for every n, while $\mu E = \sup_{n\in\mathbb{N}} \mu K_n$ and $\mu(X \setminus E) = \sup_{n\in\mathbb{N}} \mu K'_n$. Set $L = \bigcup_{n\in\mathbb{N}} K_n$, $L' = \bigcap_{n\in\mathbb{N}}(X \setminus K'_n)$. Then both L and L' belong to the domain of every μ'_y, and

$$\begin{aligned}\sup_{n\in\mathbb{N}} \psi_y K_n &\le \sup_{n\in\mathbb{N}} \mu'_y K_n \le \mu'_y L \le \mu'_y L' \\ &\le \inf_{n\in\mathbb{N}} \mu'_y(X \setminus K'_n) = \mu'_y X - \sup_{n\in\mathbb{N}} \mu'_y K'_n \le 1 - \sup_{n\in\mathbb{N}} \psi_y K'_n\end{aligned}$$

for every y. On the other hand,

$$\begin{aligned}\int (1 - \sup_{n\in\mathbb{N}} \psi_y K'_n)\nu(dy) &\le \nu Y - \sup_{n\in\mathbb{N}} \int \psi_y(K'_n)\nu(dy) = \mu X - \sup_{n\in\mathbb{N}} \mu K'_n = \mu E \\ &= \sup_{n\in\mathbb{N}} \mu K_n = \sup_{n\in\mathbb{N}} \int \psi_y K_n \nu(dy) \le \int \sup_{n\in\mathbb{N}} \psi_y K_n \nu(dy).\end{aligned}$$

So

$$\sup_{n\in\mathbb{N}} \psi_y K_n = \mu'_y L = \mu'_y L' = 1 - \sup_{n\in\mathbb{N}} \psi_y K'_n$$

for almost every y. Because $L \subseteq E \subseteq L'$ and μ'_y is complete, $E \in \operatorname{dom}\mu'_y$ and

$$\mu'_y E = 1 - \sup_{n\in\mathbb{N}} \psi_y K'_n \ge 1 - \psi_y(X \setminus E) \ge \psi_y E$$

for almost every $y \in Y$. Similarly, $\mu'_y(X \setminus E) \ge \psi_y(X \setminus E)$ for almost every y. But as

$$\mu'_y E + \mu'_y(X \setminus E) = \mu'_y X \le \psi_y X \le 1$$

whenever the left-hand side is defined, we must have $\mu'_y E = \psi_y E$ for almost every y, as claimed. **Q**

It follows at once that

$$\int \mu'_y E\,\nu(dy) = \int \psi_y E\,\nu(dy) = \mu E$$

for every $E \in \Sigma$, and $\langle\mu'_y\rangle_{y\in Y}$ is a disintegration of μ over ν.

(e) At this point observe that

$$\int \mu'_y X\,\nu(dy) = \mu X = \int \chi X^\bullet = \int T(\chi X^\bullet) = \nu Y,$$

so $F_0 = \{y : \mu'_y X < 1\}$ is negligible. Taking any $y_0 \in Y \setminus F_0$ and setting

$$\begin{aligned}\mu_y &= \mu'_{y_0} \text{ for } y \in F_0 \\ &= \mu'_y \text{ for } y \in Y \setminus F_0,\end{aligned}$$

we find ourselves with a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν with the same properties as $\langle\mu'_y\rangle_{y\in Y}$, but now consisting entirely of probability measures.

(f) For $g \in \mathcal{L}^\infty(\mu)$, set $h_g(y) = \int g\,d\mu_y$ whenever $y \in Y$ is such that the integral is defined. Consider the set V of those $g \in \mathcal{L}^\infty(\mu)$ such that $h_g \in \mathcal{L}^\infty(\nu)$ and $Tg^\bullet = h_g^\bullet$ in $L^\infty(\nu)$. If $E \in \Sigma$, then $h_{\chi E}(y) = \psi_y E$ for almost every y, so

$$h_{\chi E}^\bullet = (ST(\chi E^\bullet))^\bullet = T(\chi E^\bullet);$$

accordingly $\chi E \in V$. It is easy to check that V is closed under addition and scalar multiplication, so it contains all simple functions. Next, if $\langle g_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of simple functions with limit $g \in \mathcal{L}^\infty(\nu)$, then $h_g = \sup_{n\in\mathbb{N}} h_{g_n}$ wherever the right-hand side is defined. Also T is order-continuous, because it preserves integrals, so

$$Tg^\bullet = \sup_{n\in\mathbb{N}} Tg_n^\bullet = \sup_{n\in\mathbb{N}} h_{g_n}^\bullet = h_g^\bullet$$

and $g \in V$. Finally, if $g \in \mathcal{L}^\infty(\mu)$ is zero almost everywhere, there is a negligible $E \in \Sigma$ such that $g(x) = 0$ for every $x \in X \setminus E$; $\mu_y E = 0$ for almost every y, so $h_g(y) = \int g\,d\mu_y = 0$ for almost every y and again $g \in V$. Putting these together, we see that $V = \mathcal{L}^\infty(\nu)$, as required by (ii) as stated above.

452I Theorem (Pachl 78) Let (X,Σ,μ) be a non-empty countably compact measure space, (Y,T,ν) a σ-finite measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν, consistent with f, such that μ_y is a complete probability measure on X for every $y \in Y$. Moreover,

(i) if $\mathcal{K}$ is a countably compact class of subsets of X such that μ is inner regular with respect to $\mathcal{K}$, then we can arrange that $\mathcal{K} \subseteq \operatorname{dom}\mu_y$ for every $y \in Y$;

(ii) if, in (i), $\mathcal{K}$ is closed under finite unions and countable intersections, then we can arrange that $\mathcal{K} \subseteq \operatorname{dom}\mu_y$ and μ_y is inner regular with respect to $\mathcal{K}$ for every $y \in Y$.

proof (a) Consider first the case in which ν and μ are probability measures and we are provided with a class $\mathcal{K}$ as in (ii). In this case, for each $u \in L^\infty(\mu)$, $F \mapsto \int_{f^{-1}[F]} u$ is countably additive. So we have an operator $T : L^\infty(\mu) \to L^\infty(\nu)$ defined by saying that $\int_F Tu = \int_{f^{-1}[F]} u$ whenever $u \in L^\infty(\mu)$ and $F \in \mathrm{T}$. Of course T is linear and positive and $\int Tu = \int u$ whenever $u \in L^\infty(\mu)$.

By 452H, there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν such that

(α) for every $y \in Y$, $\mu_y X = 1$, $\mathcal{K} \subseteq \operatorname{dom}\mu_y$ and μ_y is inner regular with respect to $\mathcal{K}$;

(β) $T(g^\bullet) = h_g^\bullet$ whenever $g \in \mathcal{L}^\infty(\mu)$ and $h_g(y) = \int g\,d\mu_y$ when the integral is defined.

If now $F \in \mathrm{T}$, set $g = \chi f^{-1}[F]$ in (β); then $Tg^\bullet$ is defined by saying that

$$\int_H Tg^\bullet = \int_{f^{-1}[H]} g = \mu f^{-1}[F \cap H] = \nu(F \cap H)$$

for every $H \in \mathrm{T}$, so that $Tg^\bullet = \chi F^\bullet$ and we must have $\mu_y f^{-1}[F] = 1$ for almost every $y \in F$. Thus $\langle\mu_y\rangle_{y\in Y}$ is a consistent distribution.

(b) The theorem is formulated in a way to make it quotable in parts without committing oneself to a particular class $\mathcal{K}$. But if we are given a class satisfying (i), we can extend it to one satisfying (ii), by 413R; and if we are told only that μ is countably compact, we know from the definition that we shall be able to choose a countably compact class satisfying (i).

(c) This proves the theorem on the assumption that μ and ν are probability measures. If $\mu X = \nu Y = 0$ then the result is trivial, as we can take every μ_y to be the zero measure. Otherwise, because ν is σ-finite, there is a partition $\langle Y_n\rangle_{n\in\mathbb{N}}$ of Y into measurable sets of finite measure. Let $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{n=0}^\infty \gamma_n \nu Y_n = 1$, and write

$$\nu' F = \sum_{n=0}^\infty \gamma_n \nu(F \cap Y_n) \text{ for } F \in \mathrm{T},$$

$$\mu' E = \sum_{n=0}^\infty \gamma_n \mu(E \cap X_n) \text{ for } E \in \Sigma.$$

It is easy to check (α) that ν' and μ' are probability measures (β) that f is inverse-measure-preserving for μ' and ν' (γ) that if μ is inner regular with respect to $\mathcal{K}$ so is μ'. Note that ν' and ν have the same negligible sets. By (a)-(b),

μ' has a disintegration $\langle\mu_y\rangle_{y\in Y}$ over ν' which is consistent with f, and (if appropriate) has the properties demanded in (i) or (ii). Now, if $E\in\Sigma$,

$$\mu E=\sum_{n=0}^{\infty}\gamma_n^{-1}\mu'(E\cap X_n)=\sum_{n=0}^{\infty}\gamma_n^{-1}\int\mu_y(E\cap X_n)\nu'(dy)$$
$$=\sum_{n=0}^{\infty}\gamma_n^{-1}\int_{Y_n}\mu_yE\,\nu'(dy)$$

(because $\mu_yX=1$, $\mu_yX_n=(\chi Y_n)(y)$ for ν'-almost every y, every n)

$$=\sum_{n=0}^{\infty}\int_{Y_n}\mu_yE\,\nu(dy)=\sum_{n=0}^{\infty}\int\mu_y(E\cap X_n)\nu(dy)=\int\mu_yE\,\nu(dy).$$

So $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of μ over ν. If $F\in\mathrm{T}$, then $\mu_yf^{-1}[F]=1$ for ν'-almost every y, that is, for ν-almost every y, so $\langle\mu_y\rangle_{y\in Y}$ is still consistent with f with respect to the measure ν.

452J Remarks (a) In the theorem above, I have carefully avoided making any promises about the domains of the μ_y beyond that in (i). If Σ_0 is the σ-algebra generated by $\mathcal{K}\cap\Sigma$, then whenever $E\in\Sigma$ there are E', $E''\in\Sigma_0$ such that $E'\subseteq E\subseteq E''$ and $\mu(E''\setminus E')=0$. (For μ, like ν, must be σ-finite, so we can choose E' to be a countable union of members of $\mathcal{K}\cap\Sigma$, and E'' to be the complement of such a union.) Thus we shall have a σ-algebra on which every μ_y is defined and which will be adequate to describe nearly everything about μ. The example of Lebesgue measure on the square (452E) shows that we cannot ordinarily expect the μ_y to be defined on the whole of Σ itself. In many important cases, of course, we can say more (452Xl).

(b) Necessarily (as remarked in the course of the proof) $\mu_yX=1$ for almost every y. In some applications it seems right to change μ_y for a negligible set of y's so that every μ_y is a probability measure. Of course this cannot be done if $X=\emptyset\ne Y$, but this case is trivial (we should have to have $\nu Y=0$). In other cases, we can make sure that any new μ_y is equal to some old one, so that a property required by (i) or (ii) remains true of the new disintegration. If we want to have '$\mu_yf^{-1}[\{y\}]=\mu_yX=1$ for every $y\in Y$', strengthening 'strongly consistent', we shall of course have to begin by checking that f is surjective.

(c) The question of whether 'σ-finite' can be weakened to 'strictly localizable' in the hypotheses of 452I is related to the Banach-Ulam problem (452Yb). See also 452O.

452K Example The hypothesis 'countably compact' in 452I is in fact essential (452Ye). To see at least that it cannot be omitted, we have the following elementary example. Set $Y=[0,1]$, and let ν be Lebesgue measure on Y, with domain T. Let $X\subseteq[0,1]$ have outer measure 1 and inner measure 0 (134D, 419I); let μ be the subspace measure on X. Set $f(x)=x$ for $x\in X$. Then there is no disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν which is consistent with f.

P? Suppose, if possible, that $\langle\mu_y\rangle_{y\in[0,1]}$ is such a disintegration. Then, in particular, the sets

$$H_q=[0,q]\setminus\{y:X\cap[0,q]\in\operatorname{dom}\mu_y,\ \mu_y(X\cap[0,q])=1\},$$

$$H'_q=[q,1]\setminus\{y:X\cap[q,1]\in\operatorname{dom}\mu_y,\ \mu_y(X\cap[q,1])=1\}$$

are negligible for every $q\in[0,1]$. Set $G=[0,1]\setminus\bigcup_{q\in\mathbb{Q}\cap[0,1]}(H_q\cup H'_q)$, so that G is ν-conegligible. Then there must be some $y\in G\setminus X$. Now $\mu_y(X\cap[0,q'])=\mu_y(X\cap[q,1])=1$ whenever q, $q'\in\mathbb{Q}$ and $0\le q<y<q'\le 1$, so that $\mu_y(X\cap\{y\})=1$. But $X\cap\{y\}=\emptyset$. **XQ**

452L The same ideas as in 452I can be used to prove a result on the disintegration of measures on product spaces. It will help to have a definition.

Definition Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and λ a measure on $X=\prod_{i\in I}X_i$. For each $i\in I$ set $\pi_i(x)=x(i)$ for $x\in X$. Then the image measure $\lambda\pi_i^{-1}$ is the **marginal measure** of λ on X_i.

452M I return to the context of 452B-452D.

Theorem Let Y and Z be sets and $\mathrm{T} \subseteq \mathcal{P}Y$, $\Upsilon \subseteq \mathcal{P}Z$ σ-algebras. Let μ be a non-zero totally finite measure with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, and ν the marginal measure of μ on Y. Suppose that the marginal measure λ of μ on Z is inner regular with respect to a countably compact class $\mathcal{K} \subseteq \mathcal{P}Z$ which is closed under finite unions and countable intersections. Then there is a family $\langle\mu_y\rangle_{y\in Y}$ of complete probability measures on Z, all measuring every member of $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, such that

$$\mu E = \int \mu_y E[\{y\}]\nu(dy)$$

for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$, and

$$\int f d\mu = \iint f(y,z)\mu_y(dz)\nu(dy)$$

whenever f is a $[-\infty,\infty]$-valued function such that $\int f d\mu$ is defined in $[-\infty,\infty]$.

proof (a) To begin with, assume that μ is a probability measure and that ν is complete. Let $\mathfrak{B}$ be the measure algebra of ν and $\theta : \mathfrak{B} \to \mathrm{T}$ a lifting. For $H \in \Upsilon$ and $F \in \mathrm{T}$ set $\nu_H F = \mu(F \times H)$; then $\nu_H : \mathrm{T} \to [0,1]$ is countably additive and $\nu_H F \le \nu F$ for every $F \in \mathrm{T}$, so there is a $v_H \in L^1(\nu)$ such that $\int_F v_H = \nu_H F$ for every $F \in \mathrm{T}$ and $0 \le v_H \le \chi 1$. We can therefore think of v_H as a member of $L^\infty(\nu) \cong L^\infty(\mathfrak{B})$. Let $T : L^\infty(\mathfrak{B}) \to L^\infty(\mathrm{T})$ be the Riesz homomorphism associated with θ, and set $\psi_y H = (Tv_H)(y)$ for every $y \in Y$.

Each $\psi_y : \Upsilon \to [0,\infty[$ is finitely additive. So we have a complete measure μ_y on Z such that $\mu_y Z \le \psi_y Z = 1$, $\mathcal{K} \subseteq \operatorname{dom}\mu_y$, μ_y is inner regular with respect to $\mathcal{K}$ and $\mu_y K \ge \psi_y K$ for every $K \in \mathcal{K}$ (413Sa, as before).

For $H \in \Upsilon$, $F \in \mathrm{T}$ we have

$$\int_F \psi_y H\,\nu(dy) = \int_F Tv_H = \int_F v_H = \nu_H F = \mu(F \times H).$$

So

$$\underline{\int}\mu_y K \cdot \chi F(y)\nu(dy) \ge \int_F \psi_y K\,\nu(dy) = \mu(F \times K)$$

for every $K \in \mathcal{K}$. Now note that, for any $H \in \Upsilon$ and $F \in \mathrm{T}$,

$$\begin{aligned}\mu(F \times H) - \sup_{K\in\mathcal{K},K\subseteq H} \mu(F \times K) &= \inf_{K\in\mathcal{K},K\subseteq H} \mu(F \times (H \setminus K))\\ &\le \inf_{K\in\mathcal{K},K\subseteq H} \lambda(H \setminus K) = 0\end{aligned}$$

because λ is inner regular with respect to $\mathcal{K}$ (and, like μ, is a probability measure). So

$$\begin{aligned}\underline{\int}(\mu_y)_*H \cdot \chi F(y)\nu(dy) &\ge \sup_{K\in\mathcal{K},K\subseteq H} \underline{\int}\mu_y K \cdot \chi F(y)\nu(dy)\\ &\ge \sup_{K\in\mathcal{K},K\subseteq H} \mu(F \times K) = \mu(F \times H).\end{aligned}$$

In particular,

$$\underline{\int}(\mu_y)_*H\,\nu(dy) \ge \mu(Y \times H) = \lambda H,$$

and similarly $\underline{\int}(\mu_y)_*(Z \setminus H)\nu(dy) \ge \lambda(Z \setminus H)$.

Taking ν-integrable functions g_1, g_2 such that $g_1(y) \le (\mu_y)_*H$ and $g_2(y) \le (\mu_y)_*(Z \setminus H)$ for almost every y, $\int g_1 d\nu = \underline{\int}(\mu_y)_*H\,\nu(dy)$ and $\int g_2 d\nu = \underline{\int}(\mu_y)_*(Z \setminus H)\nu(dy)$ (133Ja), we must have

$$g_1(y) + g_2(y) \le (\mu_y)_*H + (\mu_y)_*(Z \setminus H) \le \mu_y Z \le 1$$

for almost every y, while $\int g_1 + g_2\,d\nu \ge 1$; so that, for almost all y,

$$g_1(y) + g_2(y) = (\mu_y)_*H + (\mu_y)_*(Z \setminus H) = \mu_y Z = 1,$$

and (because μ_y is complete) $\mu_y H$ is defined and equal to $g_1(y)$ (413E). It now follows that

$$\int_F \mu_y H\,\nu(dy) = \int_F g_1(y)\nu(dy) = \underline{\int}(\mu_y)_*H \cdot \chi F(y)\nu(dy) \ge \mu(F \times H)$$

for every $F \in \mathrm{T}$. But since also

$$\int_F \mu_y(Z \setminus H)\nu(dy) \ge \mu(F \times (Z \setminus H)),$$

$$\int_F \mu_y H + \mu_y(Z \setminus H)\,\nu(dy) \le \nu F = \mu(F \times H) + \mu(F \times (Z \setminus H)),$$

we must actually have $\int_F \mu_y H\,\nu(dy) = \mu(F \times H)$.

All this is true whenever $F \in \mathrm{T}$ and $H \in \Upsilon$. But now, setting

$$\mathcal{E} = \{E : E \in \mathrm{T}\widehat{\otimes}\Upsilon,\ \mu E = \int \mu_y E[\{y\}]\nu(dy)\},$$

we see that $\mathcal{E}$ is a Dynkin class and includes $\mathcal{I} = \{F \times H : F \in \mathrm{T},\ H \in \Upsilon\}$, which is closed under finite intersections; so that the Monotone Class Theorem tells us that $\mathcal{E}$ includes the σ-algebra generated by $\mathcal{I}$, and is the whole of $\mathrm{T}\widehat{\otimes}\Upsilon$.

(b) The rest is just tidying up. (i) The construction in (a) allows $\mu_y Z$ to be less than 1 for a ν-negligible set of y; but of course all we have to do, if that happens, is to amend μ_y arbitrarily on that set to any of the 'ordinary' values of μ_y. (ii) If the original measure ν is not complete, let $\hat{\mu}$ and $\hat{\nu}$ be the completions of μ and ν, and $\hat{\mathrm{T}}$ the domain of $\hat{\nu}$. The projection onto Y is inverse-measure-preserving for μ and ν, so is inverse-measure-preserving for $\hat{\mu}$ and $\hat{\nu}$ (234Ba[7]), and $\hat{\mu}$ measures every member of $\hat{\mathrm{T}}\widehat{\otimes}\Upsilon$; set $\mu' = \hat{\mu}\restriction\hat{\mathrm{T}}\widehat{\otimes}\Upsilon$. Next, the marginal measure of μ' on Z is still λ (since both must have domain Υ). So we can apply (a) to μ' to get the result. (iii) If the original measure μ is not a probability measure, apply the arguments so far to suitable scalar multiples of μ and ν.

(c) Thus we have the formula

$$\mu E = \int \mu_y E[\{y\}]\nu(dy)$$

for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$. The second formula announced follows as in the remark following 452F.

452N Corollary Let Y and Z be sets and $\mathrm{T} \subseteq \mathcal{P}Y$, $\Upsilon \subseteq \mathcal{P}Z$ σ-algebras. Let μ be a probability measure with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, and ν the marginal measure of μ on Y. Suppose that

either Υ is the Baire σ-algebra with respect to a compact Hausdorff topology on Z

or Υ is the Borel σ-algebra with respect to an analytic Hausdorff topology on Z

or (Z, Υ) is a standard Borel space.

Then there is a family $\langle\mu_y\rangle_{y\in Y}$ of probability measures on Z, all with domain Υ, such that

$$\mu E = \int \mu_y E[\{y\}]\nu(dy)$$

for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$, and

$$\int f d\mu = \iint f(y,z)\mu_y(dz)\nu(dy)$$

whenever f is a $[-\infty,\infty]$-valued function such that $\int f d\mu$ is defined in $[-\infty,\infty]$.

proof In each case, the marginal measure of μ on Z is tight (that is, inner regular with respect to the closed compact sets) for a Hausdorff topology on Z. (Use 412D when Υ is the Baire σ-algebra on a compact Hausdorff space and 433Ca when it is the Borel σ-algebra on an analytic Hausdorff space; when (Z, Υ) is a standard Borel space, take any appropriate Polish topology on Z and use 423Ba.) So 452M tells us that we can achieve the formulae sought with Radon probability measures μ_y. Since (in all three cases) $\operatorname{dom}\mu_y$ will include Υ for every y, we can get the result as stated by replacing each μ_y by $\mu_y\restriction\Upsilon$.

452O Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν, consistent with f, such that every μ_y is a Radon measure on X.

proof (a) Let $\langle Y_i\rangle_{i\in I}$ be a decomposition of Y. For each $i \in I$, let ν_i be the subspace measure on Y_i and λ_i the subspace measure on $X_i = f^{-1}[Y_i]$. Then $f_i = f\restriction X_i$ is inverse-measure-preserving for λ_i and ν_i. Let $\mathcal{K}_i$ be the family of compact subsets of X_i; of course $\mathcal{K}_i$ is a (countably) compact class and λ_i is inner regular with respect to $\mathcal{K}_i$ (412Oa). By 452I, we can choose, for each $i \in I$, a disintegration $\langle\tilde{\mu}_y\rangle_{y\in Y_i}$ of λ_i over ν_i, consistent with $f\restriction X_i$, such that $\tilde{\mu}_y$ measures every compact subset of X_i and is inner regular with respect to $\mathcal{K}_i$ for every $y \in Y_i$. Adjusting any which are not probability measures, and completing them if necessary, we can suppose that every $\tilde{\mu}_y$ is a complete probability measure. By 412Ja, $\tilde{\mu}_y$ measures every relatively closed subset of X_i for every $y \in Y_i$.

For $i \in I$ and $y \in Y_i$, set

$$\mu_y E = \tilde{\mu}_y(E \cap X_i)$$

[7] Formerly 235Hc.

whenever $E \subseteq X$ and $E \cap X_i$ is measured by $\tilde{\mu}_y$. Then μ_y is a complete totally finite measure on X; it is inner regular with respect to $\mathcal{K}_i$ and measures every closed subset of X. It follows at once that it is tight and measures every Borel set, that is, is a Radon measure on X.

(b) Now $\mu E = \int \mu_y E\,\nu(dy)$ for every $E \in \Sigma$. **P** $\bigcup_{i\in J} E \cap X_i = E \cap f^{-1}[\bigcup_{i\in J} Y_i]$ belongs to Σ for every $J \subseteq I$. By 451Q, $\mu E = \sum_{i\in I} \mu(E \cap X_i)$. For $i \in I$, we have $\int_{Y_i} \tilde{\mu}_y(E \cap X_i)\nu_i(dy) = \mu(E \cap X_i)$. So

$$\mu E = \sum_{i\in I} \mu(E \cap X_i) = \sum_{i\in I} \int_{Y_i} \tilde{\mu}_y(E \cap X_i)\nu_i(dy)$$
$$= \sum_{i\in I} \int_{Y_i} \mu_y E\,\nu(dy) = \int \mu_y E\,\nu(dy)$$

by 214N. **Q**

Thus $\langle \mu_y \rangle_{y\in Y}$ is a disintegration of μ over ν.

(c) Finally, if $F \in \mathrm{T}$ and $i \in I$, then

$$Y_i \cap F \setminus \{y : \mu_y f^{-1}[F] \text{ is defined and equal to } 1\}$$
$$= (F \cap Y_i) \setminus \{y : y \in Y_i,\ \tilde{\mu}_y f^{-1}[F \cap Y_i] = 1\}$$

is negligible for every i, so $\mu_y f^{-1}[F] = 1$ for almost every y. Thus $\langle \mu_y \rangle_{y\in Y}$ is consistent with f.

452P Corollary (cf. Blackwell 56) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ an analytic Radon measure space and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y\in Y}$ of μ over ν, strongly consistent with f, such that every μ_y is a Radon measure on X.

proof By 433B, ν is countably separated; now put 452O and 452Gc together.

452Q Disintegrations and conditional expectations Fubini's theorem provides a relatively concrete description of the conditional expectation of a function on a product of probability spaces with respect to the σ-algebra defined by one of the factors, by means of the formula $g(x, y) = \int f(x, z)dz$ (253H). This generalizes straightforwardly to measures with disintegrations, as follows.

Proposition Let (X, Σ, μ) and (Y, T, ν) be probability spaces and $f : X \to Y$ an inverse-measure-preserving function. Suppose that $\langle \mu_y \rangle_{y\in Y}$ is a disintegration of μ over ν which is consistent with f, and that g is a μ-integrable real-valued function.

(a) Setting $h_0(y) = \int g\,d\mu_y$ whenever $y \in Y$ and the integral is defined in $\mathbb{R}$, h_0 is a Radon-Nikodým derivative of the functional $F \mapsto \int_{f^{-1}[F]} g\,d\mu : \mathrm{T} \to \mathbb{R}$.

(b) Now suppose that ν is complete. Setting $h_1(x) = \int g\,d\mu_{f(x)}$ whenever $x \in X$ and the integral is defined in $\mathbb{R}$, then h_1 is a conditional expectation of g on the σ-algebra $\Sigma_0 = \{f^{-1}[F] : F \in \mathrm{T}\}$.

proof (a) If $F \in \mathrm{T}$, then $f^{-1}[F]$ is μ_y-conegligible for almost every $y \in F$, and μ_y-negligible for almost every $y \in Y \setminus F$, so $\int g \times \chi f^{-1}[F]d\mu_y = h_0(y) \times \chi F(y)$ for almost every y, and

$$\int_F h_0 d\nu = \iint g \times \chi f^{-1}[F]d\mu_y \nu(dy) = \int_{f^{-1}[F]} g\,d\mu$$

(452F). As F is arbitrary, we have the result.

(b) Of course Σ_0 is a σ-algebra (111Xd), and it is included in Σ because f is inverse-measure-preserving. By 452F, $Y_0 = \{y : g \text{ is } \mu_y\text{-integrable}\}$ is conegligible, so $\operatorname{dom} h_1 = f^{-1}[Y_0]$ is conegligible. If $\alpha \in \mathbb{R}$, then

$$F = \{y : y \in Y_0,\ \int g\,d\mu_y \ge \alpha\}$$

belongs to T because $y \mapsto \int g\,d\mu_y$ is ν-virtually measurable and ν is complete. So

$$\{x : x \in \operatorname{dom} h_1,\ h_1(x) \ge \alpha\} = f^{-1}[F]$$

belongs to Σ_0, and h_1 is Σ_0-measurable. If $F \in \mathrm{T}$, then

$$\int_{f^{-1}[F]} h_1\,d\mu = \int_{f^{-1}[F]} \int g\,d\mu_{f(x)}\mu(dx) = \int_F \int g\,d\mu_y\nu(dy)$$

(235G[8])

$$= \int_F h_0 d\nu = \int_{f^{-1}[F]} g\,d\mu$$

as in (a). As F is arbitrary, h_1 is a conditional expectation of g on Σ_0, as claimed.

***452R** I take the opportunity to interpolate an interesting result about countably compact measures. It demonstrates the power of 452I to work in unexpected ways.

Theorem (Pachl 79) Let (X, Σ, μ) be a countably compact measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then ν is countably compact.

proof (a) For most of the proof (down to the end of (b) below) I suppose that μ and ν are totally finite.

Let Z be the Stone space of the Boolean algebra T. (I am *not* using the measure algebra here!) For $F \in \mathrm{T}$, let F^* be the corresponding open-and-closed subset of Z. For each $y \in Y$, the map $F \mapsto \chi F(y)$ is a Boolean homomorphism from T to $\{0, 1\}$, so belongs to Z; define $g : Y \to Z$ by saying that $g(y)(F) = \chi F(y)$ for $y \in Y$, $F \in \mathrm{T}$, that is, $g^{-1}[F^*] = F$ for every $F \in \mathrm{T}$. Let $\mathcal{Z}$ be the family of zero sets in Z, and Λ the Baire σ-algebra of Z.

The set

$$\{W : W \subseteq Z,\ g^{-1}[W] \in \mathrm{T}\}$$

is a σ-algebra of subsets of Z containing all the open-and-closed sets, so contains every zero set (4A3Oe) and includes Λ. Set $\lambda W = \nu g^{-1}[W]$ for $W \in \Lambda$. Then λ is a Baire measure on Z, so is inner regular with respect to $\mathcal{Z}$ (412D).

Set $h = gf : X \to Z$. Then h is a composition of inverse-measure-preserving functions, so is inverse-measure-preserving. By 452I, there is a disintegration $\langle \mu_z \rangle_{z \in Z}$ of μ over λ which is consistent with h.

(b) Let $\mathcal{K} \subseteq \mathcal{P}Y$ be the family of sets

$$\{g^{-1}[V] : V \in \mathcal{Z},\ \mu_z h^{-1}[V] = \mu_z X = 1 \text{ for every } z \in V\}.$$

(i) $\mathcal{K}$ is a countably compact class of sets. **P** Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{K}$ such that $\bigcap_{i \le n} K_i \neq \emptyset$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $V_n \in \mathcal{Z}$ be such that $K_n = g^{-1}[V_n]$ and $\mu_z h^{-1} V_n = \mu_z X = 1$ for every $z \in V_n$. Then

$$g^{-1}[\textstyle\bigcap_{i \le n} V_i] = \bigcap_{i \le n} K_i \neq \emptyset$$

for every $n \in \mathbb{N}$, so $\{V_n : n \in \mathbb{N}\}$ has the finite intersection property and (because Z is compact) there is a $z \in \bigcap_{n \in \mathbb{N}} V_n$. Now

$$\mu_z h^{-1}[V_n] = \mu_z X = 1$$

for every $n \in \mathbb{N}$, so

$$\emptyset \neq \textstyle\bigcap_{n \in \mathbb{N}} h^{-1}[V_n] = f^{-1}[\bigcap_{n \in \mathbb{N}} K_n].$$

Thus $\bigcap_{n \in \mathbb{N}} K_n$ is non-empty. As $\langle K_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\mathcal{K}$ is a countably compact class. **Q**

(ii) ν is inner regular with respect to $\mathcal{K}$. **P** Suppose that $F \in \mathrm{T}$ and $\gamma < \nu F$. Choose a sequence $\langle V_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{Z}$ as follows. Start with $V_0 = F^*$, so that

$$\lambda V_0 = \nu g^{-1}[V_0] = \nu F > \gamma.$$

Given that $V_n \in \mathcal{Z}$ and $\lambda V_n > \gamma$, then we know that $\mu_z h^{-1}[V_n] = \mu_z X = 1$ for λ-almost every $z \in V_n$; because λ is inner regular with respect to $\mathcal{Z}$, there is a $V_{n+1} \in \mathcal{Z}$ such that $V_{n+1} \subseteq V_n$, $\lambda V_{n+1} > \gamma$ and $\mu_z h^{-1}[V_n] = \mu_z X = 1$ for every $z \in V_{n+1}$. Continue.

At the end of the induction, set $V = \bigcap_{n \in \mathbb{N}} V_n$. Then $V \in \mathcal{Z}$. If $z \in V$, then

$$\mu_z h^{-1}[V] = \lim_{n \to \infty} \mu_z h^{-1}[V_n] = 1 = \mu_z X,$$

so $g^{-1}[V] \in \mathcal{K}$. Because $V \subseteq V_0 = F^*$, $g^{-1}[V] \subseteq F$, and

[8]Formerly 235I.

$$\nu g^{-1}[V] = \lambda V = \lim_{n\to\infty} \lambda V_n \ge \gamma.$$

As F and γ are arbitrary, ν is inner regular with respect to $\mathcal{K}$. **Q**

Thus $\mathcal{K}$ witnesses that ν is countably compact.

(c) For the general case, let $\langle Y_i\rangle_{i\in I}$ be a decomposition of Y. For each $i \in I$, set $X_i = f^{-1}[Y_i]$; let μ_i be the subspace measure on X_i and ν_i the subspace measure on Y_i. Then μ_i is countably compact (451Db) and $f\restriction X_i : X_i \to Y_i$ is inverse-measure-preserving for μ_i and ν_i, so ν_i is countably compact, by (a)-(b) above. Let $\mathcal{K}_i \subseteq \mathcal{P}Y_i$ be a countably compact class such that ν_i is inner regular with respect to $\mathcal{K}_i$. Then $\mathcal{K} = \bigcup_{i\in I} \mathcal{K}_i$ is a countably compact class (because any sequence in $\mathcal{K}$ with the finite intersection property must lie within a single $\mathcal{K}_i$). By 413R, there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ which is closed under finite unions; by 412Aa, ν is inner regular with respect to $\mathcal{K}^*$, so is countably compact. This completes the proof.

***452S Corollary** (Pachl 78) If (X, Σ, μ) is a countably compact totally finite measure space, and T is any σ-subalgebra of Σ, then $\mu\restriction\mathrm{T}$ is countably compact.

452T In 452E, I remarked in passing that Fubini's theorem on a product space $X = Y \times Z$ can be thought of as giving us a disintegration of the product measure on X over the factor measure on Y. There are other contexts in which we find that a canonical disintegration is provided for a structure (X, μ, Y, ν) without calling on the Lifting Theorem. Here I will describe an important case arising naturally in the theory of group actions.

Theorem Let X be a locally compact Hausdorff space, G a compact Hausdorff topological group and $\bullet$ a continuous action of G on X. Suppose that μ is a G-invariant Radon probability measure on X. For $x \in X$, write $f(x)$ for the corresponding orbit $\{a\bullet x : a \in G\}$ of the action. Let $Y = f[X]$ be the set of orbits, with the topology $\{W : W \subseteq Y, f^{-1}[W]$ is open in $X\}$. Write ν for the image measure μf^{-1} on Y.

(a) Y is locally compact and Hausdorff, and ν is a Radon probability measure.

(b) For each $\boldsymbol{y} \in Y$, there is a unique G-invariant Radon probability $\mu_{\boldsymbol{y}}$ on X such that $\mu_{\boldsymbol{y}}(\boldsymbol{y}) = 1$.

(c) $\langle\mu_{\boldsymbol{y}}\rangle_{\boldsymbol{y}\in Y}$ is a disintegration of μ over ν, strongly consistent with f.

proof (a) By 4A5Ja, Y is locally compact and Hausdorff, and f is an open map. By 418I, ν is a Radon measure.

(b) Let λ be the unique Haar probability measure on G (442Id). By 443Ub-443Ud, applied to the action $\bullet\restriction G\times\boldsymbol{y}$ of G on $\boldsymbol{y}$, we have a unique G-invariant Radon probability measure $\mu'_{\boldsymbol{y}}$ on $\boldsymbol{y}$ defined by saying that $\mu'_{\boldsymbol{y}}E = \lambda\{g : g\bullet x \in E\}$ for every $x \in \boldsymbol{y}$ and Borel set $E \subseteq \boldsymbol{y}$. Now $\mu_{\boldsymbol{y}}$ must be the unique extension of $\mu'_{\boldsymbol{y}}$ to X. Of course we still have $\mu_{\boldsymbol{y}}E = \lambda\{g : g\bullet x \in E\}$ for every $x \in \boldsymbol{y}$ and Borel set $E \subseteq X$.

(c)(i) Let $V \subseteq X$ be an open set, and set $h_V(\boldsymbol{y}) = \mu_{\boldsymbol{y}}V$ for $\boldsymbol{y} \in Y$. Then h_V is lower semi-continuous. **P** Suppose that $\boldsymbol{y} \in Y$ and $\alpha \in \mathbb{R}$ are such that $h_V(\boldsymbol{y}) > \alpha$. Then there is a compact set $K \subseteq V$ such that $\mu_{\boldsymbol{y}}K > \alpha$. Fix $x \in \boldsymbol{y}$, and set $L = \{g : g\bullet x \in K\}$, so that L is a compact subset of G and $\lambda L > \alpha$. The set $\{(g, x') : g \in L, x' \in X, g\bullet x' \notin V\}$ is closed in $L \times X$, so its projection $\{x' : \exists\, g \in L, g\bullet x' \notin V\}$ is closed (4A2Gm) and $U = \{x' : g\bullet x' \in V$ for every $g \in L\}$ is open in X. Now $f[U]$ is open in Y, because f is an open map. Of course $x \in U$ and $\boldsymbol{y} \in f[U]$. But if $\boldsymbol{y}' \in f[U]$, there is an $x' \in U$ such that $f(x') = \boldsymbol{y}'$, and now

$$h_V(\boldsymbol{y}') = \mu_{\boldsymbol{y}'}V = \lambda\{g : g\bullet x' \in V\} \ge \lambda L > \alpha.$$

As $\boldsymbol{y}$ and α are arbitrary, h_V is lower semi-continuous. **Q**

(ii) In particular, h_V is Borel measurable; because f is inverse-measure-preserving for μ and ν,

$$\int h_V\, d\nu = \int h_V(f(x))\mu(dx)$$

(235G again)

$$= \int \lambda\{g : g\bullet x \in V\}\mu(dx) = \int \mu\{x : g\bullet x \in V\}\lambda(dg)$$

(by 417H, because μ and λ are totally finite Radon measures and $\{(g, x) : g\bullet x \in V\}$ is an open set in $G \times X$)

$$= \int \mu(g^{-1}\bullet V)\lambda(dg) = \int \mu V\lambda(dg)$$

(because μ is G-invariant)

$$= \mu V.$$

By the Monotone Class Theorem, as usual, it follows that $\int \mu_{\boldsymbol{y}} E\,\nu(d\boldsymbol{y}) = \mu E$ for every Borel set $E \subseteq X$ (apply 136C to μ and $E \mapsto \int \mu_{\boldsymbol{y}} E\,\nu(d\boldsymbol{y})$), and therefore (because every $\mu_{\boldsymbol{y}}$ is complete and μ is the completion of a Borel measure) for every $E \in \operatorname{dom}\mu$. So $\langle \mu_{\boldsymbol{y}} \rangle_{\boldsymbol{y} \in Y}$ is a disintegration of μ over ν. Since

$$\mu_{\boldsymbol{y}} f^{-1}[\{\boldsymbol{y}\}] = \mu_{\boldsymbol{y}}(\boldsymbol{y}) = 1$$

for every $\boldsymbol{y} \in Y$, the disintegration is strongly consistent with f.

452X Basic exercises (a) Let Y be a first-countable topological space, ν a topological probability measure on Y, Z a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a family of topological probability measures on Z such that $y \mapsto \mu_y V$ is lower semi-continuous for every open set $V \subseteq Z$. Show that there is a Borel probability measure μ on $Y \times Z$ such that $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every Borel set $E \subseteq Y \times Z$. (*Hint*: 434R.)

(b) Let (Y, T, ν) be a probability space, Z a topological space and P the set of topological probability measures on Z with its narrow topology (437Jd). Let $y \mapsto \mu_y : Y \to P$ be a function which is measurable in the sense of 411L. Show that, writing $\mathcal{B}(Z)$ for the Borel σ-algebra of Z, we have a probability measure μ defined on $\mathrm{T}\widehat{\otimes}\mathcal{B}(Z)$ such that $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every $E \in \mathrm{T}\widehat{\otimes}\mathcal{B}(Z)$.

(c) Let (Y, T, ν) be a probability space, Z a topological space and $P_{\mathcal{B}\mathfrak{a}}$ the set of Baire probability measures on Z with its vague topology (437Jc). Let $y \mapsto \mu_y : Y \to P_{\mathcal{B}\mathfrak{a}}$ be a measurable function. Show that, writing $\mathcal{B}\mathfrak{a}(Z)$ for the Baire σ-algebra of Z, we have a probability measure μ defined on $\mathrm{T}\widehat{\otimes}\mathcal{B}\mathfrak{a}(Z)$ such that $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every $E \in \mathrm{T}\widehat{\otimes}\mathcal{B}\mathfrak{a}(Z)$.

(d) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon probability space, $(X, \mathfrak{T})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a family of Radon probability measures on X. Suppose that (i) there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y \mapsto \mu_y U$ is lower semi-continuous for every $U \in \mathcal{U}$ (ii) ν is inner regular with respect to the family $\{K : K \subseteq Y,\ \{\mu_y : y \in K\}$ is uniformly tight$\}$. Show that we have a Radon probability measure $\tilde{\mu}$ on X such that $\tilde{\mu} E = \int \mu_y E\nu(dy)$ whenever $\tilde{\mu}$ measures E.

(e) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon probability space, $(Z, \mathfrak{U})$ a Prokhorov Hausdorff space (437U), and P the space of Radon probability measures on Z with its narrow topology. Suppose that $y \mapsto \mu_y : Y \to P$ is almost continuous. Show that we have a Radon probability measure $\tilde{\mu}$ on $Y \times Z$ such that $\tilde{\mu} E = \int \mu_y E[\{y\}]\nu(dy)$ whenever $\tilde{\mu}$ measures E.

(f) Let (X, T, ν) be a measure space, and μ an indefinite-integral measure over ν (234J[9]). Show that there is a disintegration $\langle \mu_x \rangle_{x \in X}$ of μ over ν such that $\mu_x\{x\} = \mu_x X$ for every $x \in X$.

>(g) Let (X, Σ, μ) and (Y, T, ν) be measure spaces and $\langle \mu_y \rangle_{y \in Y}$ a disintegration of μ over ν. Show that $\langle \hat{\mu}_y \rangle_{y \in Y}$ is a disintegration of $\hat{\mu}$ over ν, where $\hat{\mu}_y$ and $\hat{\mu}$ are the completions of μ_y and μ respectively.

>(h) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and ν' an indefinite-integral measure over ν, defined from a ν-virtually measurable function $g : Y \to [0, \infty[$. Suppose that $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν'. Show that $\langle g(y)\mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν.

(i) Let (Y, T, ν) be a probability space, X a set and $\langle \mu_y \rangle_{y \in Y}$ a family of probability measures on X. Set $\theta A = \overline{\int} \mu_y^*(A)\,\nu(dy)$ for every $A \subseteq X$. (i) Show that θ is an outer measure on X. (ii) Let μ be the measure on X defined from θ by Carathéodory's construction. Show that $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν. (iii) Suppose that $X = [0,1]^2$, ν is Lebesgue measure on $[0,1] = Y$ and $\mu_y E = \nu\{x : (x,y) \in E\}$ whenever this is defined. Show that, for any E measured by μ, $\mu_y E \in \{0,1\}$ for ν-almost every y.

(j) Explore connexions between 452F and the formula $\int f d\mu = \iint f d\nu_z \lambda(dz)$ of 443Qe.

(k) Let (X, Σ, μ) be a countably compact σ-finite measure space, (Y, T, ν) a σ-finite measure space, and $f : X \to Y$ a (Σ, T)-measurable function such that $f^{-1}[F]$ is μ-negligible whenever $F \subseteq Y$ is ν-negligible. Show that there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν such that, for each $F \in \mathrm{T}$, $\mu_y(X \setminus f^{-1}[F]) = 0$ for almost every $y \in F$. (*Hint*: Reduce to the case in which μ is totally finite, and disintegrate μ over $\nu' = (\mu f^{-1}) \restriction \mathrm{T}$.)

[9] Formerly 234B.

>(l) Let (X,Σ,μ) be a non-empty countably compact measure space such that Σ is countably generated (as σ-algebra), (Y,T,ν) a σ-finite measure space, and $f:X\to Y$ an inverse-measure-preserving function. (i) Show that there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν, consistent with f, such that every μ_y is a probability measure with domain Σ. (ii) Show that if $\langle\mu'_y\rangle_{y\in Y}$ is any other disintegration of μ over ν which is consistent with f, then $\mu_y=\mu'_y\restriction\Sigma$ for almost every y.

(m) Let (X,Σ) be a non-empty standard Borel space, μ a measure with domain Σ, (Y,T,ν) a σ-finite measure space, and $f:X\to Y$ an inverse-measure-preserving function. (i) Show that there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν, consistent with f, such that every μ_y is a probability measure with domain Σ. (ii) Show that if $\langle\mu'_y\rangle_{y\in Y}$ is any other disintegration of μ over ν which is consistent with f, then $\mu_y=\mu'_y\restriction\Sigma$ for almost every y.

(n) Let (X,Σ,μ) be a totally finite countably compact measure space and $\mathrm{T}\subseteq\Sigma$ a countably-generated σ-algebra; set $\nu=\mu\restriction\mathrm{T}$. Show that there is a disintegration $\langle\mu_x\rangle_{x\in X}$ of μ over ν such that $\mu_xH_x=\mu_xX=1$ for every $x\in X$, where $H_x=\bigcap\{F:x\in F\in\mathrm{T}\}$ for every x. (*Hint*: apply 452I with $Y=\{H_x:x\in X\}$.)

(o) Show that 452I can be deduced from 452M. (*Hint*: start with the case $\nu Y=1$; set $\lambda W=\mu\{x:(x,f(x))\in W\}$ for $W\in\Sigma\widehat{\otimes}\mathrm{T}$.)

(p) Show that, in 452M, we shall have $\hat\mu E=\int\mu_yE[\{y\}]\nu(dy)$ whenever the completion $\hat\mu$ of μ measures E.

>(q) Let T be the Borel σ-algebra of $[0,1]$, ν the restriction of Lebesgue measure to T, $Z\subseteq[0,1]$ a set with inner measure 0 and outer measure 1, and Υ the Borel σ-algebra of Z. Show that there is a probability measure μ on $[0,1]\times Z$ defined by setting $\mu E=\nu^*\{y:(y,y)\in E\}$ for $E\in\mathrm{T}\widehat{\otimes}\Upsilon$. Show that there is no disintegration of μ over ν which is consistent with the projection $(y,z)\mapsto y$.

>(r) Let (X,Σ,μ) be a complete totally finite countably compact measure space and T a σ-subalgebra of Σ containing all negligible sets. Show that there is a family $\langle\mu_x\rangle_{x\in X}$ of probability measures on X such that (i) $x\mapsto\mu_xE$ is T-measurable and $\int\mu_xE\,\mu(dx)=\mu E$ for every $E\in\Sigma$ (ii) if $F\in\mathrm{T}$, then $\mu_xF=1$ for almost every $x\in F$. Show that if g is any μ-integrable real-valued function, then g is μ_x-integrable for almost every x, and $x\mapsto\int g\,d\mu_x$ is a conditional expectation of g on T.

(s) Let (X_0,Σ_0,μ_0) and (X_1,Σ_1,μ_1) be σ-finite measure spaces. For each i, let (Y_i,T_i,ν_i) be a measure space and $\langle\mu^{(i)}_y\rangle_{y\in Y_i}$ a disintegration of μ_i over ν_i. Show that $\langle\mu^{(0)}_{y_0}\times\mu^{(1)}_{y_1}\rangle_{(y_0,y_1)\in Y_0\times Y_1}$ is a disintegration of $\mu_0\times\mu_1$ over $\nu_0\times\nu_1$, where each product here is a c.l.d. product measure.

(t) In 452M, suppose that Z is a metrizable space and $\mathcal{K}$ is the family of compact subsets of Z, and let $(Y,\hat{\mathrm{T}},\hat\nu)$ be the completion of (Y,T,ν). Show that $y\mapsto\mu_y$ is a $\hat{\mathrm{T}}$-measurable function from Y to the set of Radon probability measures on Z with its narrow topology. (*Hint*: 437Rh.)

(u) $SU(r)$, for $r\ge2$, is the set of $r\times r$ matrices T with complex coefficients such that $\det T=1$ and $TT^*=I$, where T^* is the complex conjugate of the transpose of T. (i) Show that under the natural action $(T,u)\mapsto Tu:SU(r)\times\mathbb{C}^r\to\mathbb{C}^r$ the orbits are the spheres $\{u:u\,\textbf{.}\,\bar u=\gamma\}$, for $\gamma>0$, together with $\{0\}$. (ii) Show that if a Borel set $C\subseteq\mathbb{C}^r$ is such that $\gamma C\subseteq C$ for every $\gamma>0$, and μ_0, μ_1 are two $SU(r)$-invariant Radon probability measures on $\mathbb{C}^r$ such that $\mu_0\{0\}=\mu_1\{0\}$, than $\mu_0C=\mu_1C$.[10]

452Y Further exercises (a) Let Z be a set, (Y,T,ν) a measure space, and $\langle\mu_y\rangle_{y\in Y}$ a family of measures on Z. Let Υ be a σ-algebra of subsets of Z such that, for every $H\in\Upsilon$, $y\mapsto\mu_yH:Y\to[0,\infty]$ is defined ν-a.e. and is ν-virtually measurable. For $F\in\mathrm{T}$, set $\mathcal{H}_F=\{H:H\in\Upsilon,\ \mu_yH$ is defined for every $y\in F$ and $\sup_{y\in F}\mu_yH<\infty\}$. Show that there is a measure μ on $Y\times Z$, with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, defined by setting

$$\mu E=\sup\{\sum_{i=0}^n\int_{F_i}\mu_y(E[\{y\}]\cap H_i)\nu(dy):F_0,\dots,F_n\in\mathrm{T}\text{ are disjoint},$$
$$\nu F_i<\infty\text{ and }H_i\in\mathcal{H}_{F_i}\text{ for every }i\le n\}$$

for $E\in\mathrm{T}\widehat{\otimes}\Upsilon$.

[10] I am grateful to G.Vitillaro for bringing this to my attention.

(b) Let (X,Σ,μ) be a semi-finite countably compact measure space, (Y,T,ν) a strictly localizable measure space, and $f:X\to Y$ an inverse-measure-preserving function. Suppose that the magnitude of ν (definition: 332Ga) is finite or a measure-free cardinal (definition: 438A). Show that there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν which is consistent with f.

(c) Give an example to show that the phrase 'strictly localizable' in the statements of 452O and 452Yb cannot be dispensed with.

(d) Give an example to show that, in 452M, we cannot always arrange that $\Upsilon\subseteq\operatorname{dom}\mu_y$ for ν-almost every $y\in Y$.

(e) Let (X,Σ,μ) be a probability space such that whenever (Y,T,ν) is a probability space and $f:X\to Y$ is an inverse-measure-preserving function, there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν which is consistent with f. Show that μ is countably compact. (*Hint*: 452R, or Pachl 78.)

(f) Let X be a K-analytic Hausdorff space and μ a totally finite measure on X which is inner regular with respect to the closed sets. Show that μ is countably compact. (*Hint*: 432D.)

(g) Let X be a set, and $\langle\mu_i\rangle_{i\in I}$ a family of countably compact measures on X with sum μ (234G[11]). Show that if μ is semi-finite, it is countably compact.

(h) Let X be a locally compact Hausdorff space, G a compact Hausdorff group, and $\bullet$ a continuous action of G on X. Let H be another group and $\circ$ a continuous action of H on X which commutes with $\bullet$ in the sense that $g\bullet(h\circ x)=h\circ(g\bullet x)$ for all $g\in G$, $h\in H$ and $x\in X$. (i) Show that $((g,h),x)\to g\bullet(h\circ x):(G\times H)\times X\to X$ is a continuous action of the product group $G\times H$ on X. (ii) Suppose that the action in (i) is transitive. Show that if μ, μ' are G-invariant Radon probability measures on X and $E\subseteq X$ is a Borel set such that $h\circ E=E$ for every $h\in H$, then $\mu E=\mu'E$.

452 Notes and comments 452B and 452C correspond respectively to the ordinary and τ-additive product measures of §§251 and 417. I have not attempted to find a suitable general formulation for the constructions when the measures involved are not totally finite. In 452Ya I set out a possible version which at least agrees with the c.l.d. product measure when all the μ_y are the same. Any product measure which has an associated Fubini theorem can be expected to be generalizable in the same way; for instance, 434R becomes 452Xa.

The hypotheses in 452B are closely matched with the conclusion, and clearly cannot be relaxed substantially if the theorem is to remain true. 452C and 452D are a rather different matter. While the condition '$y\mapsto\mu_yV$ is lower semi-continuous' is a natural one, and plainly necessary for the argument given, the integrated measure μ can be τ-additive or Radon for other reasons. In particular, the most interesting specific example in this book of a Radon measure constructed through these formulae (453N below) does not satisfy the lower semi-continuity condition for the section measures.

Early theorems on disintegrations concentrated on cases in which all the measure spaces involved were 'standard' in that the measures were defined on standard Borel algebras, or were the completions of such measures. Theorem 452I here is the end (so far) of a long search for ways to escape from topological considerations. As usual, of course, the most important applications (in probability theory) are still rooted in the standard case. Being countably separated, such spaces automatically yield disintegrations which are concentrated on fibers, in the sense that $\mu_yf^{-1}[\{y\}]=\mu_yX=1$ for almost every y (452P). The general question of when we can expect to find disintegrations of this type is an important one to which I will return in the next section.

452I and 452O, as stated, assume that the functions $f:X\to Y$ controlling the disintegrations are inverse-measure-preserving. In fact it is easy to weaken this assumption (452Xk). Note the constructions for conditional expectations in 452Q and 452Xr.

Obviously 452I and 452M are nearly the same theorem; but I write out formally independent proofs because the constructions needed to move between them are not quite trivial. In fact I think it is easier to deduce 452I from 452M than the other way about (452Xo). The point of 452N is that the spaces (X,Σ) there have the 'countably compact measure property', that is, any totally finite measure with domain Σ is countably compact. I will return to this in the exercises to §454 (454Xf *et seq.*).

The method of 452R, due to J.Pachl, may have inspired the proof of (vi)$\Rightarrow$(i) in 343B. In the general introduction to this work I wrote 'I have very little confidence in anything I have ever read concerning the history of ideas'. We

[11] Formerly 112Ya.

have here a case indicating the difficulties a historian faces. I proved 343B in the winter of 1996-97, while a guest of the University of Wisconsin at Madison. Around that time I was renewing my acquaintance with Pachl 78. I know I ran my eye over the proof of 452R, without, I may say, understanding it, as became plain when I came to write the first draft of the present section in the summer of 1997; whether I had understood it twenty years earlier I do not know. It is entirely possible that a subterranean percolation of Pachl's idea was what dislodged an obstacle to my attempts to prove 343B, but I was not at the time conscious of any connexion.

453 Strong liftings

The next step involves the concept of 'strong' lifting on a topological measure space (453A); I devote a few pages to describing the principal cases in which strong liftings are known to exist (453B-453J). When we have *Radon* measures μ and ν, with an *almost continuous* inverse-measure-preserving function between them, and a *strong* lifting for ν, we can hope for a disintegration $\langle\mu_y\rangle_{y\in Y}$ such that (almost) every μ_y lives on the appropriate fiber. This is the content of 453K. I end the section with a note on the relation between strong liftings and Stone spaces (453M) and with V.Losert's example of a space with no strong lifting (453N).

Much of the work here is based on ideas in Ionescu Tulcea & Ionescu Tulcea 69.

453A The proof of the first disintegration theorem I presented, 452H, depended on two essential steps: the use of a lifting for (Y,T,ν) to define the finitely additive functionals ψ_y, and the use of a countably compact class to convert these into countably additive functionals. In 452O I observed that if our countably compact class is the family of compact sets in a Hausdorff space, we can get Radon measures in our disintegration. Similarly, if we have a lifting of a special type, we can hope for special properties of the disintegration. A particularly important kind of lifting, in this context, is the following.

Definition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a topological measure space. A lifting $\phi:\Sigma\to\Sigma$ is **strong** or **of local type** if $\phi G\supseteq G$ for every open set $G\subseteq X$, that is, if $\phi F\subseteq F$ for every closed set $F\subseteq X$. I will say that ϕ is **almost strong** if $\bigcup_{G\in\mathfrak{T}}G\setminus\phi G$ is negligible.

Similarly, if $\mathfrak{A}$ is the measure algebra of μ, a lifting $\theta:\mathfrak{A}\to\Sigma$ is **strong** if $\theta G^\bullet\supseteq G$ for every open set $G\subseteq X$, and **almost strong** if $\bigcup_{G\in\mathfrak{T}}G\setminus\theta G^\bullet$ is negligible.

Obviously a strong lifting is almost strong.

453B We already have the machinery to describe a particularly striking class of strong liftings.

Theorem Let X be a topological group with a Haar measure μ, and Σ its algebra of Haar measurable sets.

(a) If $\phi:\Sigma\to\Sigma$ is a left-translation-invariant lifting, in the sense of 447A, then ϕ is strong.

(b) μ has a strong lifting.

proof (a) Apply 447B with $Y=\{e\}$ and $\underline{\phi}=\phi$.

(b) For there is a left-translation-invariant lifting (447J).

Remark In particular, translation-invariant liftings on $\mathbb{R}^r$ or $\{0,1\}^I$ (§345) are strong.

453C Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a topological measure space and $\phi:\Sigma\to\Sigma$ a lifting. Write $\mathcal{L}^\infty$ for the space of bounded Σ-measurable real-valued functions on X, so that $\mathcal{L}^\infty$ can be identified with $L^\infty(\Sigma)$ (363H) and the Boolean homomorphism $\phi:\Sigma\to\Sigma$ gives rise to a Riesz homomorphism $T:\mathcal{L}^\infty\to\mathcal{L}^\infty$ (363F).

(a) If ϕ is a strong lifting, then $Tf=f$ for every bounded continuous function $f:X\to\mathbb{R}$.

(b) If $(X,\mathfrak{T})$ is completely regular and $Tf=f$ for every $f\in C_b(X)$, then ϕ is strong.

proof (a) Suppose first that $f\ge0$. For $\alpha\in\mathbb{R}$, set $G_\alpha=\{x:x\in X,\,f(x)>\alpha\}$; then G_α is open, so $\phi G_\alpha\supseteq G_\alpha$. We have $f\ge\alpha\chi G_\alpha$, so

$$Tf\ge\alpha T(\chi G_\alpha)=\alpha\chi(\phi G_\alpha)\ge\alpha\chi G_\alpha,$$

that is, $(Tf)(x)\ge\alpha$ whenever $f(x)>\alpha$. As α is arbitrary, $Tf\ge f$. At the same time, setting $\gamma=\|f\|_\infty$, we have

$$T(\gamma\chi X-f)\ge\gamma\chi X-f,\quad T(\gamma\chi X)=\gamma\chi(\phi X)=\gamma\chi X,$$

so $Tf\le f$ and $Tf=f$.

For general $f\in C_b(X)$,

$$Tf = T(f^+ - f^-) = Tf^+ - Tf^- = f^+ - f^- = f,$$

where f^+ and f^- are the positive and negative parts of f.

(b) Let $G \subseteq X$ be open and x any point of G. Then there is an $f \in C_b(X)$ such that $f \leq \chi G$ and $f(x) = 1$. In this case

$$f = Tf \leq T(\chi G) = \chi(\phi G),$$

so $x \in \phi G$. As x is arbitrary, $G \subseteq \phi G$; as G is arbitrary, ϕ is strong.

453D Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space.

(a) If μ has a strong lifting it is strictly positive (definition: 411Nf).

(b) If μ is strictly positive and complete, and has an almost strong lifting, it has a strong lifting.

(c) If μ has an almost strong lifting it is τ-additive, so has a support.

(d) If μ is complete and $\mu X > 0$ and the subspace measure μ_E has an almost strong lifting for some conegligible set $E \subseteq X$, then μ has an almost strong lifting.

proof (a) If $\phi : \Sigma \to \Sigma$ is a strong lifting, then $G \subseteq \phi G = \emptyset$ whenever G is a negligible open set, so μ is strictly positive.

(b) If μ is strictly positive and complete and $\phi : \Sigma \to \Sigma$ is an almost strong lifting, set $A = \bigcup_{G \in \mathfrak{T}} G \setminus \phi G$. For each $x \in A$, let $\mathcal{I}_x$ be the ideal of subsets of X generated by

$$\{F : F \subseteq X \text{ is closed, } x \notin F\} \cup \{B : B \subseteq X \text{ is negligible}\}.$$

Then $X \notin \mathcal{I}_x$, because μ is strictly positive, so a closed set not containing x cannot be conegligible. There is therefore a Boolean homomorphism $\psi_x : \mathcal{P}X \to \{0, 1\}$ such that $\psi_x F = 0$ for every $F \in \mathcal{I}_x$ (311D). Set

$$\tilde{\phi} E = (\phi E \setminus A) \cup \{x : x \in A,\ \psi_x E = 1\}$$

for $E \in \Sigma$. It is easy to check that $\tilde{\phi} : \Sigma \to \mathcal{P}X$ is a Boolean homomorphism. (Compare the proof of 341J.) If $E \in \Sigma$, then

$$E \triangle \tilde{\phi} E \subseteq (E \triangle \phi E) \cup A$$

is negligible, so (because μ is complete) $\tilde{\phi} E \in \Sigma$. If E is negligible, then $E \in \mathcal{I}_x$ and $\psi_x E = 0$ for every $x \in A$, so $\tilde{\phi} E = \phi E = \emptyset$. Thus $\tilde{\phi}$ is a lifting. Now suppose that $x \in G \in \mathfrak{T}$. If $x \in A$, then $X \setminus G \in \mathcal{I}_x$, so $\psi_x(X \setminus G) = 0$, $\psi_x G = 1$ and $x \in \tilde{\phi} G$. If $x \notin A$, then $x \in \phi G$ and again $x \in \tilde{\phi} G$. As x and G are arbitrary, $\tilde{\phi}$ is a strong lifting.

(c) Suppose that $\phi : \Sigma \to \Sigma$ is an almost strong lifting. Let $\mathcal{G}$ be a non-empty upwards-directed family of open sets with union H. If $\sup_{G \in \mathcal{G}} \mu G = \infty$, this is surely equal to μH. Otherwise, there is a non-decreasing sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{G}$ such that $G \setminus G^*$ is negligible for every $G \in \mathcal{G}$, where $G^* = \bigcup_{n \in \mathbb{N}} G_n$ (215Ab). Then $\phi G \subseteq \phi G^*$ for every $G \in \mathcal{G}$. This means that

$$H \setminus \phi G^* \subseteq \bigcup_{G \in \mathcal{G}} G \setminus \phi G$$

is negligible, because ϕ is almost strong, and

$$\mu H \leq \mu(\phi G^*) = \mu G^* = \lim_{n \to \infty} \mu G_n = \sup_{G \in \mathcal{G}} \mu G.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive. By 411Nd, it has a support.

(d) Now suppose that μ is complete, that $\mu X > 0$ and that there is a conegligible $E \subseteq X$ such that μ_E has an almost strong lifting ϕ. Let $\psi : \mathcal{P}X \to \{\emptyset, X\}$ be any Boolean homomorphism such that $\psi A = \emptyset$ whenever A is negligible. (This is where I use the hypothesis that X is not negligible.) Define $\tilde{\phi} : \Sigma \to \mathcal{P}X$ by setting

$$\tilde{\phi} F = \phi(E \cap F) \cup (\psi F \setminus E).$$

Then $\tilde{\phi}$ is a Boolean homomorphism because ϕ and ψ are;

$$F \triangle \tilde{\phi} F \subseteq ((E \cap F) \triangle \phi(E \cap F)) \cup (X \setminus E)$$

is negligible, so $\tilde{\phi} F \in \Sigma$, for every $F \in \Sigma$, because μ is complete; and if F is negligible, then $\phi(E \cap F) = \psi F = \emptyset$ so $\tilde{\phi} F = \emptyset$. Thus $\tilde{\phi}$ is a lifting. Finally,

$$\bigcup_{G \in \mathfrak{T}} G \setminus \tilde{\phi} G \subseteq (X \setminus E) \cup \bigcup_{G \in \mathfrak{T}} ((G \cap E) \setminus \phi(G \cap E))$$

is negligible because ϕ is almost strong and E is conegligible.

453E Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete strictly localizable topological measure space with an almost strong lifting, and $A\subseteq X$ a non-negligible set. Then the subspace measure μ_A has an almost strong lifting.

proof Let $\phi:\Sigma\to\Sigma$ be an almost strong lifting. Because μ is strictly localizable, A has a measurable envelope W say (put 213J and 213L together). Write Σ_A for the subspace σ-algebra on A. Let $\psi:\Sigma_A\to\{\emptyset,A\}$ be any Boolean homomorphism such that $\psi H=\emptyset$ for every negligible set $H\subseteq A$.

If E, $F\in\Sigma$ and $E\cap A=F\cap A$, then $\phi E\cap\phi W=\phi F\cap\phi W$. **P**

$$\mu((E\triangle F)\cap W)=\mu^*((E\triangle F)\cap A)=0,$$

so

$$(\phi E\cap\phi W)\triangle(\phi F\cap\phi W)=\phi((E\triangle F)\cap W)=\emptyset. \text{ **Q**}$$

We can therefore define a function $\tilde\phi:\Sigma_A\to\mathcal{P}A$ by setting

$$\tilde\phi H=(\phi E\cap\phi W\cap A)\cup(\psi H\setminus\phi W)$$

whenever $E\in\Sigma$ and $H=E\cap A$. It is easy to check that $\tilde\phi$ is a Boolean homomorphism. If $E\in\Sigma$ then

$$(E\cap A)\triangle\tilde\phi(E\cap A)\subseteq(E\triangle\phi E)\cup(A\setminus\phi W)\subseteq(E\triangle\phi E)\cup(W\setminus\phi W)$$

is negligible, so $\tilde\phi(E\cap A)\in\Sigma_A$ (because μ and μ_A are complete). If $H\in\Sigma_A$ is negligible, then

$$\tilde\phi H\subseteq\phi H\cup\psi H=\emptyset,$$

so $\tilde\phi$ is a lifting for μ_A.

Now set $B=(A\setminus\phi W)\cup\bigcup_{G\in\mathfrak{T}}G\setminus\phi G$. Because ϕ is almost strong, B is negligible. If $H\subseteq A$ is relatively open, then $H\setminus\tilde\phi H\subseteq B$. **P** Take $x\in H\setminus\tilde\phi H$. Express H as $G\cap A$ where $G\subseteq X$ is open. If $x\in\phi W$, then $x\notin\phi G$ so $x\in B$; if $x\notin\phi W$, then of course $x\in B$. **Q** Thus

$$\bigcup\{H\setminus\tilde\phi H:H\subseteq A\text{ is relatively open}\}\subseteq B$$

is negligible and $\tilde\phi$ is almost strong.

453F Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete strictly localizable topological measure space.

(a) If $\mathfrak{T}$ has a countable network, any lifting for μ is almost strong.

(b) Suppose that $\mu X>0$ and μ is inner regular with respect to

$$\mathcal{K}=\{K:K\in\Sigma,\ \mu_K\text{ has an almost strong lifting}\},$$

where μ_K is the subspace measure on K. Then μ has an almost strong lifting.

proof (a) Let $\mathcal{E}$ be a countable network for $\mathfrak{T}$, and $\phi:\Sigma\to\Sigma$ a lifting. For each $E\in\mathcal{E}$, let $\hat E$ be a measurable envelope of E (213J/213L again). Then

$$\bigcup_{G\in\mathfrak{T}}G\setminus\phi G=\bigcup_{G\in\mathfrak{T},E\in\mathcal{E},E\subseteq G}E\setminus\phi G\subseteq\bigcup_{G\in\mathfrak{T},E\in\mathcal{E},E\subseteq G}\hat E\setminus\phi\hat E$$

(because if $E\subseteq G\in\mathfrak{T}$, then $G\in\Sigma$, so $\mu(\hat E\setminus G)=0$ and $\phi\hat E\subseteq\phi G$)

$$\subseteq\bigcup_{E\in\mathcal{E}}\hat E\setminus\phi\hat E$$

is negligible, so ϕ is almost strong.

(b) Let $\mathcal{L}\subseteq\mathcal{K}$ be a disjoint family such that $\mu^*A=\sum_{L\in\mathcal{L}}\mu^*(A\cap L)$ for every $A\subseteq X$ (412Ib). For each $L\in\mathcal{L}$, let Σ_L be the corresponding subspace σ-algebra and $\phi_L:\Sigma_L\to\Sigma_L$ an almost strong lifting. Set $E=\bigcup\mathcal{L}$; then

$$\mu^*(X\setminus E)=\textstyle\sum_{L\in\mathcal{L}}\mu(L\setminus E)=0,$$

so E is conegligible. For $F\in\Sigma_E$ set $\phi F=\bigcup_{L\in\mathcal{L}}\phi_L(F\cap L)$; then ϕ is a Boolean homomorphism from Σ_E to $\mathcal{P}E$. If $F\in\Sigma_E$, then

$$\mu^*(F\triangle\phi F)=\textstyle\sum_{L\in\mathcal{L}}\mu^*(L\cap(F\triangle\phi F))=\sum_{L\in\mathcal{L}}\mu^*((F\cap L)\triangle\phi_L(F\cap L))=0,$$

while if $\mu F=0$ then $\phi_L(F\cap L)=\emptyset$ for every L, so $\phi F=\emptyset$. Thus ϕ is a lifting. Now set

$$A = \bigcup\{H \setminus \phi H : H \subseteq E \text{ is relatively open}\}.$$

If $L \in \mathcal{L}$, then

$$A \cap L = \bigcup\{(H \cap L) \setminus \phi_L(H \cap L) : H \subseteq E \text{ is relatively open}\}$$

is negligible, because ϕ_L is almost strong; thus ϕ is an almost strong lifting for μ_E. By 453Dd, μ also has an almost strong lifting.

453G Corollary (a) A non-zero quasi-Radon measure on a separable metrizable space has an almost strong lifting.

(b) A non-zero Radon measure μ on an analytic Hausdorff space X has an almost strong lifting.

proof (a) A quasi-Radon measure is complete and strictly localizable (415A), so, if non-zero, has a lifting (341K). A separable metrizable space has a countable network (4A2P(a-iii)), so this lifting must be almost strong.

(b) If $K \subseteq X$ is compact and non-negligible, it is metrizable (423Dc), so that the subspace measure μ_K has an almost strong lifting, by (a); as μ is tight (that is, inner regular with respect to the closed compact sets), it has an almost strong lifting, by 453Fb.

Remark In particular, Lebesgue measure on $\mathbb{R}^r$ has an almost strong lifting and therefore, by 453Db, a strong lifting, as already noted in 453B.

453H Lemma Let (X, Σ, μ) be a complete locally determined measure space and $\mathfrak{T}$ a topology on X generated by a family $\mathcal{U} \subseteq \Sigma$. Suppose that $\phi : \Sigma \to \Sigma$ is a lifting such that $\phi U \supseteq U$ for every $U \in \mathcal{U}$. Then μ is a τ-additive topological measure, and ϕ is a strong lifting.

proof Of course ϕ is a lower density, and $\phi X = X$, so by 414P we have a density topology

$$\mathfrak{T}_d = \{E : E \in \Sigma,\ E \subseteq \phi E\}$$

with respect to which μ is a τ-additive topological measure. But our hypothesis is that $\mathcal{U} \subseteq \mathfrak{T}_d$, so $\mathfrak{T} \subseteq \mathfrak{T}_d$ and μ is a τ-additive topological measure with respect to $\mathfrak{T}$. Also, of course, $\phi G \supseteq G$ for every $G \in \mathfrak{T}$, so ϕ is a strong lifting.

453I Proposition Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of topological probability spaces such that every $\mathfrak{T}_i$ has a countable network and every μ_i is strictly positive. Let λ be the (ordinary) complete product measure on $X = \prod_{i\in I} X_i$. Then λ is a τ-additive topological measure and has a strong lifting.

proof (a) The strategy of the proof is as follows. We may suppose that $I = \kappa$ is a cardinal. Write Λ for the domain of λ, and for each $\xi \le \kappa$ let Λ_ξ be the σ-algebra of members of Λ determined by coordinates less than ξ; write $\pi_\xi : X \to X_\xi$ for the canonical map. I seek to define a lifting $\phi : \Lambda \to \Lambda$ such that $\phi W \supseteq W$ for every open set $W \in \Lambda$. This will be the last in a family $\langle\phi_\xi\rangle_{\xi\le\kappa}$ of partial liftings, constructed inductively as in the proof of 341H, with $\operatorname{dom}\phi_\xi = \Lambda_\xi$ for each ξ. The inductive hypothesis will be that ϕ_ξ extends ϕ_η whenever $\eta \le \xi$, and $\phi_\xi\pi_\eta^{-1}[G] \supseteq \pi_\eta^{-1}[G]$ for every $\eta < \xi$ and every open $G \subseteq X_\eta$.

The induction starts with $\Lambda_0 = \{\emptyset, X\}$, $\phi_0\emptyset = \emptyset$, $\phi_0 X = X$. For $\xi \le \kappa$, set $\mathfrak{B}_\xi = \{W^\bullet : W \in \Lambda_\xi\}$.

(b) *Inductive step to a successor ordinal* $\xi + 1$ Suppose that ϕ_ξ has been defined, where $\xi < \kappa$.

(i) By 341Nb, there is a lifting $\phi'_\xi : \Lambda \to \Lambda$ extending ϕ_ξ. Let $\mathcal{E}_\xi$ be a countable network for $\mathfrak{T}_\xi$. For each $E \in \mathcal{E}_\xi$ let $\hat{E}$ be a measurable envelope of E. Set

$$Q = \bigcup\{\pi_\xi^{-1}[\hat{E}] \setminus \phi'_\xi(\pi_\xi^{-1}[\hat{E}]) : E \in \mathcal{E}_\xi\};$$

then Q is negligible.

(ii) For $x \in Q$, let $\mathcal{I}_x \subseteq \Lambda$ be the ideal generated by

$$\{W : W \in \Lambda_\xi,\ x \notin \phi_\xi W\} \cup \{\pi_\xi^{-1}[F] : F \subseteq X_\xi \text{ is closed},\ \pi_\xi(x) \notin F\} \cup \{W : \lambda W = 0\}.$$

Then $X \notin \mathcal{I}_x$. **P?** Otherwise, there are a $W \in \Lambda_\xi$, a closed $F \subseteq X_\xi$ and a negligible $W' \in \Lambda$ such that $W \cup W' \cup \pi_\xi^{-1}[F] = X$ while $x \notin \phi_\xi W \cup \pi_\xi^{-1}[F]$. But in this case

$$\begin{aligned}0 = \lambda W' &\ge \lambda((X \setminus W) \cap (X \setminus \pi_\xi^{-1}[F]))\\ &= \lambda(X \setminus W) \cdot \lambda(X \setminus \pi_\xi^{-1}[F]) = \lambda(X \setminus W) \cdot \mu_\xi(X_\xi \setminus F) > 0\end{aligned}$$

because μ_ξ is strictly positive and $\phi_\xi W \ne X$. **XQ**

There is therefore a Boolean homomorphism $\psi_x : \Lambda \to \{0,1\}$ which is zero on $\mathcal{I}_x$.

(iii) Set

$$\phi_{\xi+1}W = (\phi'_\xi W \setminus Q) \cup \{x : x \in Q,\, \psi_x W = 1\}$$

for every $W \in \Lambda_{\xi+1}$. Then $\phi_{\xi+1}$ is a Boolean homomorphism from $\Lambda_{\xi+1}$ to $\mathcal{P}X$. Because $\phi_{\xi+1}W \triangle \phi'_\xi W \subseteq Q$ is negligible, $\phi_{\xi+1}W \in \Lambda$ and $W \triangle \phi_{\xi+1}W$ is negligible for every $W \in \Lambda_{\xi+1}$. If $\lambda W = 0$ then $\phi'_\xi W = \emptyset$ and $\psi_x W = 0$ for every $x \in Q$, so $\phi_{\xi+1}W = \emptyset$; thus $\phi_{\xi+1} : \Lambda_{\xi+1} \to \Lambda$ is a partial lifting. If $W \in \Lambda_\xi$, then, for $x \in Q$,

$$\begin{aligned} x \in \phi_\xi W &\Longrightarrow x \notin \phi_\xi(X \setminus W) \Longrightarrow X \setminus W \in \mathcal{I}_x \\ &\Longrightarrow \psi_x(X \setminus W) = 0 \Longrightarrow \psi_x W = 1 \iff x \in \phi_{\xi+1}W \\ &\Longrightarrow W \notin \mathcal{I}_x \Longrightarrow x \in \phi_\xi W, \end{aligned}$$

so $\phi_{\xi+1}W = \phi_\xi W$. Thus $\phi_{\xi+1}$ extends ϕ_ξ.

(iv) Suppose that $\eta \le \xi$ and $G \subseteq X_\eta$ is open. If $\eta < \xi$ then

$$\phi_{\xi+1}(\pi_\eta^{-1}[G]) = \phi_\xi(\pi_\eta^{-1}[G]) \supseteq \pi_\eta^{-1}[G]$$

by the inductive hypothesis. If $\eta = \xi$, take any $x \in \pi_\xi^{-1}[G]$. If $x \in Q$, then $X \setminus \pi_\xi^{-1}[G] \in \mathcal{I}_x$, so $\psi_x(\pi_\xi^{-1}[G]) = 1$ and $x \in \phi_{\xi+1}(\pi_\xi^{-1}[G])$. If $x \notin Q$, there is an $E \in \mathcal{E}_\xi$ such that $x(\xi) \in E \subseteq G$. In this case, $x(\xi) \in \hat{E}$, so

$$x \in \pi_\xi^{-1}[\hat{E}] \setminus Q \subseteq \phi'_\xi(\pi_\xi^{-1}[\hat{E}]) \setminus Q \subseteq \phi_{\xi+1}(\pi_\xi^{-1}[\hat{E}]) \subseteq \phi_{\xi+1}(\pi_\xi^{-1}[G])$$

because $\hat{E} \setminus G$ and $\pi_\xi^{-1}[\hat{E}] \setminus \pi_\xi^{-1}[G]$ are negligible. As x is arbitary, $\pi_\eta^{-1}[G] \subseteq \phi_{\xi+1}(\pi_\eta^{-1}[G])$ in this case also.

Thus the induction continues.

(c) *Inductive step to a non-zero limit ordinal ξ of countable cofinality* Suppose that $0 < \xi \le \kappa$, that $\operatorname{cf}\xi = \omega$ and that ϕ_η has been defined for every $\eta < \xi$. Let $\langle \zeta_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in ξ with limit ξ. Then $\mathfrak{B}_\xi$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_{\zeta_n}$ (using 254N and 254Fe, or otherwise). By 341G, there is a partial lower density $\underline{\phi} : \Lambda_\xi \to \Lambda$ extending every ϕ_{ζ_n}, and therefore extending ϕ_η for every $\eta < \xi$. By 341Jb (applied to $\lambda\restriction\widehat{\Lambda}_\xi$, where $\widehat{\Lambda}_\xi$ is the σ-subalgebra of Λ generated by $\Lambda_\xi \cup \{W : \lambda W = 0\}$), there is a partial lifting $\phi_\xi : \Lambda_\xi \to \Lambda$ such that $\underline{\phi}W \subseteq \phi_\xi W$ for every $W \in \Lambda_\xi$.

If $\eta < \xi$ and $W \in \Lambda_\eta$, then

$$\phi_\eta W = \underline{\phi}W \subseteq \phi_\xi W, \quad X \setminus \phi_\eta W = \phi_\eta(X \setminus W) \subseteq \phi_\xi(X \setminus W) = X \setminus \phi_\xi W,$$

so ϕ_ξ extends ϕ_η. If $\eta < \xi$ and $G \subseteq X_\eta$ is open,

$$\phi_\xi(\pi_\eta^{-1}[G]) = \phi_{\eta+1}(\pi_\eta^{-1}[G]) \supseteq \pi_\eta^{-1}[G].$$

So again the induction continues.

(d) *Inductive step to a limit ordinal ξ of uncountable cofinality* In this case, $\mathfrak{B}_\xi = \bigcup_{\eta<\xi} \mathfrak{B}_\eta$, as in the proof of 341H; so there will be a unique partial lifting $\phi_\xi : \Lambda_\xi \to \Lambda$ extending ϕ_η for every $\eta < \xi$ (set $\phi_\xi W = \phi_\eta W'$ whenever $W \in \Lambda_\xi$, $\eta < \xi$, $W' \in \Lambda_\eta$ and $W \triangle W'$ is negligible). As in (c), we again have

$$\phi_\xi(\pi_\eta^{-1}[G]) = \phi_{\eta+1}(\pi_\eta^{-1}[G]) \supseteq \pi_\eta^{-1}[G]$$

whenever $\eta < \xi$ and $G \subseteq X_\eta$ is open.

(e) At the end of the induction, we have a lifting $\phi = \phi_\kappa$ of Λ such that $\phi U \supseteq U$ for every $U \in \mathcal{U}$, where $\mathcal{U} = \{\pi_\xi^{-1}[G] : \xi < \kappa,\, G \in \mathfrak{T}_\xi\}$. By 453H, λ is a τ-additive topological measure and ϕ is a strong lifting.

453J Corollary Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i\in I}$ be a family of quasi-Radon probability spaces such that every $\mathfrak{T}_i$ has a countable network consisting of measurable sets and every μ_i is strictly positive. Then the ordinary product measure λ on $X = \prod_{i\in I} X_i$ is quasi-Radon and has a strong lifting. If every X_i is compact and Hausdorff, then λ is a Radon measure.

proof We have just seen that λ is a τ-additive topological measure with a strong lifting; but also it is inner regular with respect to the closed sets, by 412Ua, so it is a quasi-Radon measure. If all the X_i are compact and Hausdorff, so is X, so λ is a Radon measure (416G).

453K We come now to the construction of disintegrations from strong liftings.

Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces and $f:X\to Y$ an almost continuous inverse-measure-preserving function. Suppose that ν has an almost strong lifting. Then there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν such that every μ_y is a Radon measure and $\mu_yX=\mu_yf^{-1}[\{y\}]=1$ for almost every $y\in Y$.

proof (a)(i) Suppose first that X is compact, μ is a probability measure and that f is continuous.

Turn back to the proofs of 452H-452I. In part (a) of the proof of 452H, suppose that the lifting $\theta:\mathfrak{B}\to\mathrm{T}$ corresponds to an almost strong lifting $\phi:\mathrm{T}\to\mathrm{T}$ (see 341Ba). Set $B=\bigcup_{H\in\mathfrak{S}}H\setminus\phi H$, so that B is negligible. In part (c) of the proof of 452H, take $\mathcal{K}$ to be the family of compact subsets of X. Then all the μ_y, as constructed in 452H, will be Radon probability measures. For every y, $f^{-1}[\{y\}]$ is a closed set, so is necessarily measured by μ_y. But also it is μ_y-conegligible for every $y\in Y\setminus B$. **P** Let $K\subseteq X\setminus f^{-1}[\{y\}]$ be a compact set. Then $f[K]$ is a compact set not containing y. Because Y is Hausdorff, there is an open set H containing y such that $\overline{H}\cap f[K]=\emptyset$ (4A2F(h-i)). Now

$$y\in H\setminus B\subseteq\phi H\subseteq\phi\overline{H}.$$

Let E be the compact set $f^{-1}[\overline{H}]$. Taking $T:L^\infty(\mu)\to L^\infty(\nu)$ as in part (a) of the proof of 452I, $T(\chi E^\bullet)=\chi\overline{H}^\bullet$, so

$$\psi_yE=(ST(\chi E^\bullet))(y)=(S(\chi\overline{H}^\bullet))(y)=(\chi(\phi\overline{H}))(y)=1.$$

Because $E\in\mathcal{K}$, $\mu_yE\ge\psi_yE$; since we always have $\mu_yX=1$, E is μ_y-conegligible. But $K\cap E=\emptyset$, so $\mu_yK=0$. As K is arbitrary, $\mu_y(X\setminus f^{-1}[\{y\}])=0$. **Q**

Thus $\mu_yf^{-1}[\{y\}]=1$ for almost every $y\in Y$, while $\mu_yX\le1$ for every y.

(ii) The result for totally finite μ and ν and continuous f follows at once.

(b) Now suppose that μ and ν are totally finite, and that f is almost continuous.

(i) Let $\mathcal{K}$ be the family of subsets $K\subseteq X$ such that

K is compact and $f\restriction K$ is continuous,

whenever $F\in\mathrm{T}$ and $\nu(F\cap f[K])>0$ then $\mu(K\cap f^{-1}[F])>0$,

either $K=\emptyset$ or $\mu K>0$.

Take any $E\in\Sigma$ such that $\mu E>0$. Then there is a $K\in\mathcal{K}$ such that $K\subseteq E$ and $\mu K>0$. **P** Let $K_0\subseteq E$ be a compact set such that $f\restriction K_0$ is continuous and $\mu K_0>0$. Let $\delta>0$ be such that $\mu K_0-\delta\nu Y>0$. For compact sets $K\subseteq K_0$ set $q(K)=\mu K-\delta\nu f[K]$. Choose $\langle\alpha_n\rangle_{n\in\mathbb{N}}$, $\langle K_n\rangle_{n\ge1}$ as follows. Given that K_n is a compact subset of K_0, where $n\in\mathbb{N}$, set

$$\alpha_n=\sup\{q(K):K\subseteq K_n\text{ is compact}\},$$

and choose a compact subset K_{n+1} of K_n such that $q(K_{n+1})\ge\max(q(K_n),\alpha_n-2^{-n})$. Continue.

Set $K=\bigcap_{n\in\mathbb{N}}K_n$. We have

$$\begin{aligned}q(K)&=\mu K-\delta\nu f[K]\\&\ge\lim_{n\to\infty}\mu K_n-\delta\inf_{n\in\mathbb{N}}\nu f[K_n]=\lim_{n\to\infty}q(K_n)=\sup_{n\in\mathbb{N}}q(K_n)\end{aligned}$$

because $\langle q(K_n)\rangle_{n\in\mathbb{N}}$ is non-decreasing. Of course $K\subseteq E$,

$$\mu K\ge q(K)\ge q(K_0)>0,$$

and $f\restriction K$ is continuous because $K\subseteq K_0$.

? If there is an $F\in\mathrm{T}$ such that $\nu(F\cap f[K])>0$ but $\mu(K\cap f^{-1}[F])=0$, take a compact set $K'\subseteq K\setminus f^{-1}[F]$ such that $\mu K'>\mu K-\delta\nu(F\cap f[K])$. Then $f[K']\subseteq f[K]\setminus F$, so

$$q(K')=\mu K'-\delta\nu f[K']\ge\mu K'-\delta(\nu f[K]-\nu(F\cap f[K]))>\mu K-\delta\nu f[K]=q(K).$$

Let $n\in\mathbb{N}$ be such that $q(K')>q(K)+2^{-n}$; then K' is a compact subset of K_n, so

$$\alpha_n\ge q(K')>q(K)+2^{-n}\ge q(K_{n+1})+2^{-n}\ge\alpha_n,$$

which is impossible. **X** Thus K belongs to $\mathcal{K}$ and will serve. **Q**

(ii) By 342B, there is a countable disjoint set $\mathcal{K}_0 \subseteq \mathcal{K}$ such that $\mu(X \setminus \bigcup \mathcal{K}_0) = 0$. Enumerate $\mathcal{K}_0$ as $\langle K_n \rangle_{n<\#(\mathcal{K}_0)}$; for convenience of notation, if $\mathcal{K}_0$ is finite, set $K_n = \emptyset$ for $n \ge \#(\mathcal{K}_0)$, so that every K_n belongs to $\mathcal{K}$ and $\mu E = \sum_{n=0}^{\infty} \mu(E \cap K_n)$ for every $E \in \Sigma$.

(iii) For each $n \in \mathbb{N}$, define $\lambda_n : \mathrm{T} \to \mathbb{R}$ by setting $\lambda_n F = \mu(K_n \cap f^{-1}[F])$ for every $F \in \mathrm{T}$. Then λ_n is a measure dominated by ν, so there is a T-measurable $g_n : Y \to [0,1]$ such that $\lambda_n F = \int_F g_n$ for every $F \in \mathrm{T}$, by the Radon-Nikodým theorem. Because $\lambda_n(Y \setminus f[K_n]) = 0$, we may suppose that $g_n(y) = 0$ for $y \notin f[K_n]$. We have

$$\int_F \sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty} \int_F g_n = \sum_{n=0}^{\infty} \mu(K_n \cap f^{-1}[F]) = \mu f^{-1}[F] = \nu F$$

for every $F \in \mathrm{T}$, so $\sum_{n=0}^{\infty} g_n(y) = 1$ for ν-almost every y. Reducing the g_n further on a set of measure zero, if need be, we may suppose that $\sum_{n=0}^{\infty} g_n(y) \le 1$ for every y.

(iv) For each $n \in \mathbb{N}$, let $\tilde{\lambda}_n$ be the subspace measure on $f[K_n]$ induced by λ_n, and $\tilde{\mu}_n$ the subspace measure on K_n induced by μ. Then $f \restriction K_n$ is inverse-measure-preserving for $\tilde{\mu}_n$ and $\tilde{\lambda}_n$. Also, $\tilde{\lambda}_n$ has an almost strong lifting. **P** If $K_n = \emptyset$, this is trivial. Otherwise, $\nu f[K_n] \ge \mu K_n > 0$, so the subspace measure $\tilde{\nu}_n$ induced by ν on $f[K_n]$ has an almost strong lifting, by 453E. But $\tilde{\nu}_n$ and $\tilde{\lambda}_n$ have the same domain $\mathrm{T} \cap \mathcal{P}(f[K_n])$ and the same null ideal, because $K_n \in \mathcal{K}$; so an almost strong lifting for $\tilde{\nu}_n$ is an almost strong lifting for $\tilde{\lambda}_n$. **Q**

By (a) above, we can find a disintegration $\langle \mu_{ny} \rangle_{y \in f[K_n]}$ of $\tilde{\mu}_n$ over $\tilde{\lambda}_n$ such that every μ_{ny} is a Radon measure on K_n, $\mu_{ny} K_n \le 1$ for every y and

$$\mu_{ny}\{x : x \in K_n,\ f(x) = y\} = 1$$

for $\tilde{\lambda}_n$-almost every $y \in f[K_n]$, that is, for ν-almost every $y \in f[K_n]$. For $y \in Y \setminus f[K_n]$, let μ_{ny} be the zero measure on K_n.

(v) Now, for $y \in Y$, set

$$\mu_y E = \sum_{n=0}^{\infty} g_n(y) \mu_{ny}(E \cap K_n)$$

for all those $E \subseteq X$ such that the sum is defined. Then μ_y is a Radon measure and $\mu_y X \le 1$. **P** Because every μ_{ny} is a complete measure, so is μ_y. We have

$$\mu_y X = \sum_{n=0}^{\infty} g_n(y) \mu_{ny} K_n \le \sum_{n=0}^{\infty} g_n(y) \le 1$$

by the choice of the g_n. If $G \subseteq X$ is open then μ_{ny} measures $G \cap K_n$ for every n, so μ_y measures G; accordingly μ_y measures every compact set. If $\mu_y E > 0$, there is some $n \in \mathbb{N}$ such that $g_n(y) > 0$ and $\mu_{ny}(E \cap K_n) > 0$; now there is a compact set $K \subseteq E \cap K_n$ such that $\mu_{ny} K > 0$, in which case $\mu_y K > 0$. By 412B, μ_y is tight, and is a Radon measure. **Q**

(vi) $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν. **P** If $E \in \Sigma$ then

$$\mu E = \sum_{n=0}^{\infty} \mu(E \cap K_n)$$

(by the choice of the K_n in (ii) above)

$$= \sum_{n=0}^{\infty} \tilde{\mu}_n(E \cap K_n) = \sum_{n=0}^{\infty} \int_{f[K_n]} \mu_{ny}(E \cap K_n) \tilde{\lambda}_n(dy)$$

(because $\langle \mu_{ny} \rangle_{y \in f[K_n]}$ is a disintegration of $\tilde{\mu}_n$ over $\tilde{\lambda}_n$)

$$= \sum_{n=0}^{\infty} \int_{f[K_n]} \mu_{ny}(E \cap K_n) \lambda_n(dy) = \sum_{n=0}^{\infty} \int \mu_{ny}(E \cap K_n) \lambda_n(dy)$$

(because $\lambda_n(Y \setminus f[K_n]) = 0$)

$$= \sum_{n=0}^{\infty} \int g_n(y) \mu_{ny}(E \cap K_n) \nu(dy)$$

(235A)

$$= \int \sum_{n=0}^{\infty} g_n(y) \mu_{ny}(E \cap K_n) \nu(dy) = \int \mu_y E \, \nu(dy). \textbf{ Q}$$

(vii) It follows that $\mu_y f^{-1}[\{y\}]=1$ for almost every y. **P**

$$\begin{aligned}\{y:\mu_y f^{-1}[\{y\}]\ne 1\}&\subseteq\{y:\mu_y X\ne 1\}\cup\{y:\mu_y^*(X\setminus f^{-1}[\{y\}])>0\}\\&\subseteq\{y:\mu_y X\ne 1\}\cup\bigcup_{n\in\mathbb{N}}\{y:y\in f[K_n],\,\mu_{ny}(K_n\setminus f^{-1}[\{y\}])>0\}\end{aligned}$$

is negligible. **Q**

(c) Now let us turn to the general case. This proceeds just as in 452O. Let $\langle Y_i\rangle_{i\in I}$ be a decomposition of Y. For each $i\in I$, take X_i, λ_i and ν_i as in the proof of 452O. Note that λ_i and ν_i are Radon measures, so that we can apply (b) above to find a disintegration $\langle\tilde\mu_y\rangle_{y\in Y_i}$ of λ_i over ν_i such that every $\tilde\mu_y$ is a Radon measure and $\tilde\mu_y X_i=\tilde\mu_y f^{-1}[\{y\}]=1$ for ν_i-almost every $y\in Y_i$. Just as in 452O, we can set

$$\mu_y E=\tilde\mu_y(E\cap X_i)$$

whenever $y\in Y_i$ and μ_y measures $E\cap X_i$, to obtain a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν in which every μ_y is a Radon measure and $\mu_y X=1$ for almost every y; and this time

$$\{y:y\in Y,\,\mu_y f^{-1}[\{y\}]\ne 1\}=\bigcup_{i\in I}\{y:y\in Y_i,\,\tilde\mu_y f^{-1}[\{y\}]\ne 1\}$$

is negligible. So we have a disintegration of the required type.

453L Remark If f is surjective, we can arrange that every μ_y is a Radon probability measure for which $X_y=f^{-1}[\{y\}]$ is μ_y-conegligible, just by changing some of the μ_y to Dirac measures. If f is not surjective, then we can still (if X itself is not empty) arrange that every μ_y is a Radon probability measure; but it might be more appropriate to make some of the μ_y the zero measure, so that X_y is always μ_y-conegligible.

I have continued to express this theorem in terms of measures μ_y on the whole space X. Of course, if we take it that X_y is to be μ_y-conegligible for every y, it will sometimes be easier to think of μ_y as a measure on X_y; this is very much what we do in the case of Fubini's theorem, where all the X_y are, in effect, the same.

453M Strong liftings and Stone spaces Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space, and $(Z,\mathfrak{S},\mathrm{T},\nu)$ the Stone space of the measure algebra $(\mathfrak{A},\bar\mu)$ of μ. For $E\in\Sigma$ let $E^*\subseteq Z$ be the open-and-closed set corresponding to the equivalence class $E^\bullet\in\mathfrak{A}$. Let R be the relation

$$\bigcap_{F\subseteq X\text{ is closed}}\{(z,x):z\in Z\setminus F^*\text{ or }x\in F\}\subseteq Z\times X$$

(415Q). For every lifting $\phi:\Sigma\to\Sigma$ we have a unique function $g_\phi:X\to Z$ such that $\phi E=g_\phi^{-1}[E^*]$ for every $E\in\Sigma$ (see 341P). Now we have the following easy facts.

(a) ϕ is strong iff $(g_\phi(x),x)\in R$ for every $x\in X$. **P**

$$\begin{aligned}&(g_\phi(x),x)\in R\text{ for every }x\in X\\&\qquad\iff x\in F\text{ whenever }F\text{ is closed and }g_\phi(x)\in F^*\\&\qquad\iff F\subseteq g_\phi^{-1}[F^*]\text{ for every closed set }F\subseteq X\\&\qquad\iff F\subseteq\phi F\text{ for every closed set }F\subseteq X\\&\qquad\iff \phi\text{ is strong. }\mathbf{Q}\end{aligned}$$

(b) If $\mathfrak{T}$ is Hausdorff, so that R is the graph of a function f (415Ra), then ϕ is strong iff $fg_\phi(x)=x$ for every $x\in X$. (For $(g_\phi(x),x)\in R$ iff $fg_\phi(x)=x$.)

453N Losert's example (Losert 79) There is a compact Hausdorff space with a strictly positive completion regular Radon probability measure which has no strong lifting.

proof (a) Let ν be the usual measure on $\{0,1\}^{\mathbb{N}}=Y$. Let $M\subseteq Y$ be a closed nowhere dense set such that $\nu M>0$ (cf. 419B), and ν_1 a Radon probability measure on Y such that $\nu_1 M=1$ (e.g., a Dirac measure concentrated at some point of M).

Let I be any set of cardinal at least ω_2 such that $I\cap(I\times I)=\emptyset$. Let λ be the product measure on Y^I, giving each factor the measure ν; of course λ can be identified with the usual measure on $\{0,1\}^{\mathbb{N}\times I}$ (254N). Note that λ and ν are both strictly positive. For $i\in I$ write $M_i=\{z:z\in Y^I,\,z(i)\in M\}$; then M_i is closed in Y^I.

Set $A = \{(i,j) : i, j \in I, i \neq j\}$. For $z \in Y^I$ and $(i,j) \in A$ let $\nu_{ij}^{(z)}$ be the Radon probability measure on Y given by setting

$$\nu_{ij}^{(z)} = \nu_1 \text{ if } z \in M_i \cap M_j,$$
$$= \nu \text{ otherwise.}$$

Now, for $z \in Y^I$, let λ_z be the Radon product measure of $\langle \nu_{ij}^{(z)} \rangle_{(i,j)\in A}$ on Y^A.

(b) Let $\mathcal{U}$ be the family of sets $U \subseteq Y^A$ of the form $\{u : u(i,j) \in U_{ij}$ for $(i,j) \in B\}$, where $B \subseteq A$ is finite and $U_{ij} \subseteq Y$ is open-and-closed for every $(i,j) \in B$. Then the function $z \mapsto \lambda_z U : Y^I \to [0,1]$ is Borel measurable for every $U \in \mathcal{U}$. **P** Express U in the given form. For $C \subseteq B$ set

$$E_C = \{z : z \in Y^I, C = \{(i,j) : (i,j) \in B, z \in M_i \cap M_j\}\},$$

so that $\langle E_C \rangle_{C \subseteq B}$ is a partition of Y^I into Borel sets. For any $C \subseteq B$,

$$\lambda_z U = \prod_{(i,j)\in B} \nu_{ij}^{(z)}(U_{ij}) = \prod_{(i,j)\in C} \nu_1 U_{ij} \cdot \prod_{(i,j)\in B\setminus C} \nu U_{ij}$$

is constant for $z \in E_C$. **Q**

(c) There is a Radon measure μ on $X = Y^I \times Y^A$ specified by the formula

$$\mu E = \int \lambda_z E[\{z\}]\lambda(dz)$$

for every Baire set $E \subseteq X$. **P** Let $\mathcal{E}$ be the class of those sets $E \subseteq X$ such that $\int \lambda_z E[\{z\}]\lambda(dz)$ is defined. Then $\mathcal{E}$ is closed under monotone limits of sequences, and $E \setminus E' \in \mathcal{E}$ whenever $E, E' \in \mathcal{E}$ and $E' \subseteq E$; also $\mathcal{E}$ contains all the basic open-and-closed sets in X of the form $V \times U$, where $V \subseteq Y^I$ is open-and-closed and $U \in \mathcal{U}$. By the Monotone Class Theorem (136B), $\mathcal{E}$ includes the σ-algebra generated by such sets, which is the Baire σ-algebra $\mathcal{B}\mathfrak{a}$ of X (4A3Of). Of course $E \mapsto \int \lambda_z E[\{z\}]\lambda(dz)$ is countably additive on $\mathcal{B}\mathfrak{a}$, so is a Baire measure on X, and has a unique extension to a Radon measure, by 432F. **Q**

μ is strictly positive. **P** Let $W \subseteq X$ be any non-empty open set. Then it includes an open set of the form $V \times U$ where $V = \{z : z \in Y^I, z(i) \in V_i$ for every $i \in J\}$, $U = \{u : u \in Y^A, u(j,k) \in U_{jk}$ for every $(j,k) \in B\}$, $J \subseteq I$ and $B \subseteq A$ are finite sets, and V_i, $U_{jk} \subseteq Y$ are non-empty open sets for every $i \in J$ and $(j,k) \in B$. Now ν is strictly positive, so $\lambda V' > 0$, where

$$V' = \{z : z \in V, z \notin M_j \text{ whenever } (j,k) \in B\}.$$

(This is where we need to know that the M_j are nowhere dense.) But if $z \in V'$ then $\nu_{jk}^{(z)} = \nu$ for every $(j,k) \in B$, so

$$\lambda_z U = \prod_{(j,k)\in B} \nu U_{jk} > 0.$$

Accordingly

$$\mu W \geq \int_{V'} \lambda_z U \lambda(dz) > 0.$$

As W is arbitrary, μ is strictly positive. **Q**

Write Σ for the domain of μ.

(d) Fix on a self-supporting compact set $K \subseteq X$. I seek to show that, regarded as a subset of $Y^{I\cup A}$, K is determined by coordinates in some countable set.

(i) There is a zero set $L \supseteq K$ such that $\mu L = \mu K$. **P** Let $\langle K_n \rangle_{n\in\mathbb{N}}$ be a sequence of compact subsets of $X \setminus K$ such that $\lim_{n\to\infty} \mu K_n = \mu(X \setminus K)$. For each $n \in \mathbb{N}$ there is a continuous function $f_n : X \to [0,1]$ which is zero on K and 1 on K_n; now $L = \{x : f_n(x) = 0$ for every $n \in \mathbb{N}\}$ is a zero set including K and of the same measure as K. **Q**

(ii) By 4A3Nc, L is determined by coordinates in a countable subset of $I \cup A$, that is, there are countable sets $J_0 \subseteq I$, $B_0 \subseteq A$ such that whenever $(z,u) \in L$, $(z',u') \in X$, $z\restriction J_0 = z'\restriction J_0$ and $u\restriction B_0 = u'\restriction B_0$ we shall have $(z',u') \in L$. Set

$$J = J_0 \cup \{i : (i,j) \in B_0\} \cup \{j : (i,j) \in B_0\}, \quad B = A \cap (J \times J);$$

then $J \supseteq J_0$ and $B \supseteq B_0$ are still countable, and L is determined by coordinates in $J \cup B$.

(iii) Take any $(z_0,u_0)\in X\setminus K$. Because K is closed, we can find finite sets $J_1\subseteq I$ and $B_1\subseteq A$, open-and-closed sets $V_i\subseteq Y$ for $i\in J_1$, and open-and-closed sets $U_{ij}\subseteq Y$ for $(i,j)\in B_1$, such that

$$W=\{(z,u):z(i)\in V_i \text{ for every } i\in J_1,\ u(i,j)\in U_{ij} \text{ for every } (i,j)\in B_1\}$$

contains (z_0,u_0) and is disjoint from K. Set

$$W_1=\{(z,u):(z,u)\in X, z(i)\in V_i \text{ for every } i\in J_1\cap J,\\ u(i,j)\in U_{ij} \text{ for every } (i,j)\in B_1\cap B\},$$

$$Q=\{z:z\in Y^I,\ \lambda_z((L\cap W_1)[\{z\}])>0\},$$

so that W_1 is an open-and-closed set in X and Q is a Borel set in Y^I ((b) above). Now Q is determined by coordinates in J. **P** Suppose that $z\in Q$, $z'\in Y^I$ and $z\restriction J=z'\restriction J$. Because both L and W_1 are determined by coordinates in $J\cup B$, $(L\cap W_1)[\{z\}]=(L\cap W_1)[\{z'\}]=H$ say, and H is determined by coordinates in B. At the same time, for any $(i,j)\in B$, $M_i\cap M_j$ is determined by coordinates in J, so contains z iff it contains z', and $\nu_{ij}^{(z)}=\nu_{ij}^{(z')}$. This means that, writing λ'_z and $\lambda'_{z'}$ for the products of $\langle\nu_{ij}^{(z)}\rangle_{(i,j)\in B}$ and $\langle\nu_{ij}^{(z')}\rangle_{(i,j)\in B}$ on Y^B, $\lambda'_z=\lambda'_{z'}$. So

$$\lambda_{z'}((L\cap W_1)[\{z'\}])=\lambda_{z'}H=\lambda'_{z'}H'=\lambda'_zH'=\lambda_zH=\lambda_z((L\cap W_1)[\{z\}])>0,$$

where $H'=\{u\restriction B:u\in H\}$ (254Ob), and $z'\in Q$. **Q**

(iv) Set

$$J_2=(\{i:(i,j)\in B_1\setminus B\}\cup\{j:(i,j)\in B_1\setminus B\})\setminus J.$$

Then J_2 is a finite subset of $I\setminus J$, and $B_1\subseteq(J\cup J_2)\times(J\cup J_2)$. Set

$$G=\{z:z\in Y^I,\ z(i)\notin M \text{ for every } i\in J_2\},$$

so that G is a dense open subset of Y^I. Set

$$G_1=\{z:z\in Y^I,\ z(i)\in V_i \text{ for every } i\in J_1\setminus J\}.$$

Then G_1 is a non-empty open set, so $G\cap G_1\ne\emptyset$ and $\lambda(G\cap G_1)>0$.

(v) Set

$$U=\{u:u\in Y^A,\ u(i,j)\in U_{ij} \text{ for every } (i,j)\in B_1\setminus B\}.$$

If $z\in G$, then $z\notin M_i\cap M_j$ whenever $(i,j)\in B_1\setminus B$, so $\nu_{ij}^{(z)}=\nu$ for every $(i,j)\in B_1\setminus B$, and

$$\lambda_zU=\prod_{(i,j)\in B_1\setminus B}\nu U_{ij}>0.$$

(vi) ? Suppose, if possible, that $\lambda Q>0$. Because Q is determined by coordinates in J and $G\cap G_1$ is determined by coordinates in $J_2\cup(J_1\setminus J)$,

$$\lambda(Q\cap G\cap G_1)=\lambda Q\cdot\lambda(G\cap G_1)>0.$$

If $z\in Q\cap G\cap G_1$,

$$\lambda_z((L\cap W)[\{z\}])=\lambda_z(U\cap(L\cap W_1)[\{z\}])$$

(because $W=W_1\cap(Y^I\times U)\cap(G_1\times Y^A)$, and $z\in G_1$)

$$=\lambda_zU\cdot\lambda_z((L\cap W_1)[\{z\}])$$

(because $(L\cap W_1)[\{z\}]$ is determined by coordinates in B, while U is determined by coordinates in $B_1\setminus B$, and λ_z is a product measure)

$$>0$$

because $z\in G\cap Q$. But this means that

$$0<\int\lambda_z((L\cap W)[\{z\}])\lambda(dz)=\mu(L\cap W)=\mu(K\cap W)=\mu\emptyset,$$

which is absurd. **X**

Thus λQ must be zero.

(vii) Consequently

$$\mu(K\cap W_1)=\mu(L\cap W_1)=\int\lambda_z((L\cap W_1)[\{z\}])\lambda(dz)=0;$$

because K is self-supporting, $K\cap W_1=\emptyset$. And W_1 contains (z_0,u_0) and is determined by coordinates in $J\cup B$.

(viii) What this means is that there can be no $(z,u)\in K$ such that $z\restriction J=z_0\restriction J$ and $u\restriction B=u_0\restriction B$. At this point, recall that (z_0,u_0) was an arbitrary point of $X\setminus K$. So what must be happening is that K is determined by coordinates in the countable set $J\cup B$. By 4A3Nc again, in the other direction, K is a zero set.

(e) Part (d) shows that every self-supporting compact subset of X is a zero set. Since μ is certainly inner regular with respect to the self-supporting compact sets, it is inner regular with respect to the zero sets, that is, is completion regular.

It follows that whenever $E\in\Sigma$ there is an $E'\subseteq E$, determined by coordinates in a countable subset of $I\cup A$, such that $E\setminus E'$ is negligible. (Take E' to be a countable union of self-supporting compact sets.)

(f) **?** Now suppose, if possible, that we could find a strong lifting ϕ for μ. For each $i\in I$, take a set $E_i\subseteq\phi(M_i\times Y^A)$ such that $\mu E_i=\mu\phi(M_i\times Y^A)$ and E_i is determined by coordinates in $J_i\cup B_i$, where $J_i\subseteq I$ and $B_i\subseteq A$ are countable. Set

$$J_i^*=\{j:(j,k)\in B_i\}\cup\{k:(j,k)\in B_i\},$$

so that J_i^* also is countable. Because $\#(I)\ge\omega_2$, there are distinct i, $j\in I$ such that $i\notin J_j^*$ and $j\notin J_i^*$ (4A1Ea). So $(i,j)\notin B_i\cup B_j$.

Set

$$F=\{u:u\in Y^A,\ u(i,j)\in M\}.$$

Then $\mu((M_i\cap M_j)\times(Y^A\setminus F))=0$. **P** If $z\in M_i\cap M_j$, then

$$\lambda_z(Y^A\setminus F)=\nu_{ij}^{(z)}(Y\setminus M)=0.$$

But $(M_i\cap M_j)\times(Y^A\setminus F)$ is a Baire set, so

$$\begin{aligned}\mu((M_i\cap M_j)\times(Y^A\setminus F))&=\int\lambda_z((M_i\cap M_j)\times(Y^A\setminus F))[\{z\}]\lambda(dz)\\&=\int_{M_i\cap M_j}\lambda_z(Y^A\setminus F)\lambda(dz)=0.\ \mathbf{Q}\end{aligned}$$

Accordingly

$$E_i\cap E_j\subseteq\phi(M_i\times Y^A)\cap\phi(M_j\times Y^A)=\phi((M_i\cap M_j)\times Y^A)\subseteq\phi(Y^I\times F)$$

(because $((M_i\cap M_j)\times Y^A)\setminus(Y^I\times F)$ is negligible)

$$\subseteq Y^I\times F$$

because $Y^I\times F$ is closed and ϕ is supposed to be strong. However, $E_i\cap E_j$ is determined by coordinates in $J_i\cup J_j\cup B_i\cup B_j$, while $Y^I\times F$ is determined by coordinates in $\{(i,j)\}$, which does not meet $B_i\cup B_j$. So either $E_i\cap E_j$ is empty or $F=Y^A$. But $F\ne Y^A$ because $M\ne Y$, while

$$\begin{aligned}\mu(E_i\cap E_j)=\mu(\phi(M_i\times Y^A)\cap\phi(M_j\times Y^A))&=\mu((M_i\cap M_j)\times Y^A)\\&=\lambda(M_i\cap M_j)=(\nu M)^2>0,\end{aligned}$$

so $E_i\cap E_j\ne\emptyset$. **X**

(g) Thus μ has no strong lifting, as claimed.

453X Basic exercises >(a) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(Z,\mathfrak{T},\Sigma,\mu)$ its Stone space. Show that the canonical lifting for μ (341O) is strong.

(b) Let $\mathfrak{S}$ be the right-facing Sorgenfrey topology on $\mathbb{R}$ (415Xc). Show that there is a lifting for Lebesgue measure on $\mathbb{R}$ which is strong with respect to $\mathfrak{S}$. (*Hint*: set $\underline{\phi}E=\{x:\lim_{\delta\downarrow 0}\frac{1}{\delta}\mu(E\cap[x,x+\delta])=1\}$, and use 341Jb.)

>(c) Let μ be the usual measure on the split interval (343J, 419L). Show that μ has a strong lifting.

(d) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete locally determined topological measure space such that μ is inner regular with respect to the closed sets, and $\phi:\Sigma\to\Sigma$ a strong lifting. Show that μ is a quasi-Radon measure with respect to the lifting topology $\mathfrak{T}_l$ (414Q). Show that if $\mathfrak{T}$ is regular then $\mathfrak{T}_l\supseteq\mathfrak{T}$.

(e) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a topological measure space which has an almost strong lifting. Show that any non-zero indefinite-integral measure over μ (234J[12]) has an almost strong lifting.

(f) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space such that (X,Σ,μ) is countably separated and $\mu X>0$; for example, $(X,\mathfrak{T})$ could be an analytic space (433B). Show that μ is inner regular with respect to the compact metrizable subsets of X, so has an almost strong lifting. (*Hint*: there is an injective measurable $f:X\to\mathbb{R}$, which must be almost continuous.)

(g) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete locally determined topological measure space such that μ is effectively locally finite and inner regular with respect to the closed sets, and $\underline{\phi}:\Sigma\to\Sigma$ a lower density such that $\underline{\phi}G\supseteq G$ for every open $G\subseteq X$. Show that μ is a quasi-Radon measure with respect to both $\mathfrak{T}$ and the density topology associated with $\underline{\phi}$.

(h) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space and $\underline{\phi}:\Sigma\to\Sigma$ a lower density such that $\underline{\phi}G\supseteq G$ for every open $G\subseteq X$. Let $\langle G_x\rangle_{x\in X}$ be a family of open sets in X such that $x\notin\underline{\phi}(X\setminus G_x)$ for every $x\in X$. (i) Show that $A\setminus\bigcup_{x\in A}(G_x\cap U_x)$ is negligible whenever $A\subseteq X$ and U_x is a neighbourhood of x for every $x\in A$. (ii) Let $\mathfrak{S}$ be the topology on X generated by $\mathfrak{T}\cup\{\{x\}\cup G_x:x\in X\}$. Show that μ is quasi-Radon with respect to $\mathfrak{S}$.

(i) Let X and Y be Hausdorff spaces, and μ a Radon probability measure on $X\times Y$; set $\pi(x,y)=y$ for $x\in X$, $y\in Y$, and let ν be the image measure $\mu\pi^{-1}$. Suppose that ν has an almost strong lifting. Show that there is a family $\langle\mu_y\rangle_{y\in Y}$ of Radon probability measures on X such that $\mu E=\int\mu_y(E^{-1}[\{y\}])\nu(dy)$ for every $E\in\operatorname{dom}\mu$.

(j) Use 453Xe to simplify part (b) of the proof of 453K.

(k) In 453N, show that $\int\lambda_z E[\{z\}]\lambda(dz)$ is defined and equal to μE whenever μ measures E.

453Y Further exercises (a) Let $(Y,\mathfrak{S},\mathrm{T},\nu)$ be a Radon measure space such that $\nu Y>0$ and whenever $(X,\mathfrak{T},\Sigma,\mu)$ is a Radon measure space and $f:X\to Y$ is an almost continuous inverse-measure-preserving function, then there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν such that $\mu_y f^{-1}[\{y\}]=1$ for almost every y. Show that ν has an almost strong lifting. (*Hint*: Start with the case in which Y is compact. Take $f:X\to Y$ to be the function described in 415R, 416V and 453Mb. Set $\underline{\phi}E=\{y:\mu_y E^*=\mu_y X=1\}$.)

453Z Problems (a) If $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ are compact Radon measure spaces with strong liftings, does their product necessarily have a strong lifting? What if they are both Stone spaces?

(b) If $(X,\mathfrak{T},\Sigma,\mu)$ is a Radon probability space with countable Maharam type, must it have an almost strong lifting?

453 Notes and comments As I noted in §452, early theorems on disintegrations concentrated on cases in which all the measure spaces involved were 'standard' in that the measures were defined on standard Borel spaces (§424), or were the completions of such measures. Under these conditions the distinction between 452I and 453K becomes blurred; measures (when completed) have to be Radon measures (433Cb), liftings have to be almost strong (453F) and disintegrations have to be concentrated on fibers (452Gc). Theorem 453K provides disintegrations concentrated on fibers without any limitation on the size of the spaces involved, though making strong topological assumptions.

The strength of 453K derives from the remarkable variety of the (Radon) measure spaces which have strong liftings, as in 453F, 453G, 453I and 453J. For some ten years there were hopes that every strictly positive Radon measure had a strong lifting, which were finally dashed by Losert 79; I give a version of the example in 453N. This is a special construction, and it remains unclear whether some much more direct approach might yield another example (453Za). I should perhaps remark straight away that if the continuum hypothesis is true, then any strictly positive Radon measure with Maharam type at most ω_1 has a strong lifting (see 535I in Volume 5). In particular, subject to the continuum hypothesis, $Z\times Z$ has a strong lifting, where Z is the Stone space of the Lebesgue measure algebra, and we have a positive answer to 453Zb.

[12] Formerly 234B.

454 Measures on product spaces

A central concern of probability theory is the study of 'processes', that is, families $\langle X_t\rangle_{t\in T}$ of random variables thought of as representing the evolution of a system in time. The representation of such processes as random variables in the modern sense, that is, measurable functions on an abstract probability space, was one of the first challenges faced by Kolmogorov. In this section I give a version of Kolmogorov's theorem on the extension of consistent families of measures on subproducts to a measure on the whole product (454D). It turns out that some restriction on the marginal measures is necessary, and 'perfectness' seems to be an appropriate hypothesis, necessarily satisfied if the factor spaces are standard Borel spaces or the marginal measures are Radon measures. If we have marginal measures with stronger properties then we shall be able to infer corresponding properties of the measure on the product space (454A, generalizing 451J).

The apparatus here makes it easy to describe joint distributions of arbitrary families of real-valued random variables (454J-454P), extending the ideas of §271. For the sake of the theorem that almost all Brownian paths are continuous (477B) I briefly investigate measures on $C(T)$, where T is a Polish space (454Q-454S).

454A Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a non-empty family of totally finite measure spaces. Set $X=\prod_{i\in I}X_i$ and let μ be a measure on X which is inner regular with respect to the σ-algebra $\widehat{\bigotimes}_{i\in I}\Sigma_i$ generated by $\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$, where $\pi_i:X\to X_i$ is the coordinate map for each $i\in I$. Suppose that every π_i is inverse-measure-preserving.

(a) If $\mathcal{K}\subseteq\mathcal{P}X$ is a family of sets which is closed under finite unions and countable intersections, and μ_i is inner regular with respect to $\mathcal{K}_i=\{K:K\subseteq X_i,\,\pi_i^{-1}[K]\in\mathcal{K}\}$ for every $i\in I$, then μ is inner regular with respect to $\mathcal{K}$.

(b)(i) If every μ_i is a compact measure, so is μ;

(ii) if every μ_i is a countably compact measure, so is μ;

(iii) if every μ_i is a perfect measure, so is μ.

proof If X is empty this is all trivial, so we may suppose that $X\ne\emptyset$.

(a) Set $\mathcal{A}=\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$. If $A\in\mathcal{A}$, $V\in\Sigma$ and $\mu(A\cap V)>0$, there is a $K\in\mathcal{K}\cap\mathcal{A}$ such that $K\subseteq A$ and $\mu(K\cap V)>0$. **P** Express A as $\pi_i^{-1}[E]$, where $E\in\Sigma_i$; take $L\in\mathcal{K}_i$ such that $L\subseteq E$ and $\mu_iL>\mu_iE-\mu(A\cap V)$, and set $K=\pi_i^{-1}[L]$. **Q**

By 412C, $\mu\restriction\widehat{\bigotimes}_{i\in I}\Sigma_i$ is inner regular with respect to $\mathcal{K}$; by 412Ab, so is μ.

(b)(i)-(ii) Suppose that every μ_i is (countably) compact. Then for each $i\in I$ we can find a (countably) compact class $\mathcal{K}_i\subseteq\mathcal{P}X_i$ such that μ_i is inner regular with respect to $\mathcal{K}_i$. Set $\mathcal{L}=\{\pi_i^{-1}[K]:i\in I,\,K\in\mathcal{K}_i\}$. Then $\mathcal{L}$ is (countably) compact (451H). So there is a (countably) compact $\mathcal{K}\supseteq\mathcal{L}$ which is closed under finite unions and countable intersections (342D, 413R). Now μ is inner regular with respect to $\mathcal{K}$, by (a), and therefore (countably) compact.

(iii) Let T_0 be a countably generated σ-subalgebra of $\widehat{\bigotimes}_{i\in I}\Sigma_i$. Then there must be some countable subfamily $\mathcal{E}$ of $\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$ such that T_0 is included in the σ-algebra generated by $\mathcal{E}$ (use 331Gd). Set $\mathcal{E}_i=\{E:E\in\Sigma_i,\,\pi_i^{-1}[E]\in\mathcal{E}\}$ for each i, so that $\mathcal{E}_i$ is countable, and let Σ_i' be the σ-algebra generated by $\mathcal{E}_i$. Then $\mu_i\restriction\Sigma_i'$ is compact (451F). Applying (i), we see that $\mu\restriction\widehat{\bigotimes}_{i\in I}\Sigma_i'$ is compact, therefore perfect; while $\mathrm{T}_0\subseteq\widehat{\bigotimes}_{i\in I}\Sigma_i'$. As T_0 is arbitrary, $\mu\restriction\widehat{\bigotimes}_{i\in I}\Sigma_i$ is perfect (451F). But as the completion of μ is exactly the completion of $\mu\restriction\widehat{\bigotimes}_{i\in I}\Sigma_i$, μ also is perfect, by 451Gc.

454B Corollary Let $\langle X_i\rangle_{i\in I}$ be a family of Polish spaces with product X. Then any totally finite Baire measure on X is a compact measure.

proof If μ is a Baire measure on X, then its domain $\mathcal{B}\mathfrak{a}(X)$ is $\widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i)$, where $\mathcal{B}(X_i)$ is the Borel σ-algebra of X_i for each $i\in I$ (4A3Na). So each image measure μ_i on X_i is a Borel measure, therefore tight (that is, inner regular with respect to the closed compact sets, 433Ca), and by 454A(b-i) μ is compact.

454C Theorem (Marczewski & Ryll-Nardzewski 53) Let (X,Σ,μ) be a perfect totally finite measure space and (Y,T,ν) any totally finite measure space. Let $\Sigma\otimes\mathrm{T}$ be the algebra of subsets of $X\times Y$ generated by $\{E\times F:E\in\Sigma,\,F\in\mathrm{T}\}$. If $\lambda_0:\Sigma\otimes\mathrm{T}\to[0,\infty[$ is a non-negative finitely additive functional such that $\lambda_0(E\times Y)=\mu E$ and $\lambda_0(X\times F)=\nu F$ whenever $E\in\Sigma$ and $F\in\mathrm{T}$, then λ_0 has a unique extension to a measure defined on the σ-algebra $\Sigma\widehat{\otimes}\mathrm{T}$ generated by $\Sigma\otimes\mathrm{T}$.

proof (a) By 413Kb, it will be enough to show that $\lim_{n\to\infty}\lambda_0 W_n = 0$ for every non-increasing sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\Sigma\otimes\mathrm{T}$ with empty intersection. Take such a sequence. Each W_n must belong to the algebra generated by some finite subset of $\{E\times F : E\in\Sigma,\, F\in\mathrm{T}\}$, so there must be a countable set $\mathcal{E}\subseteq\Sigma$ such that every W_n belongs to the algebra generated by $\{E\times F : E\in\mathcal{E},\, F\in\mathrm{T}\}$; let Σ_0 be the σ-subalgebra of Σ generated by $\mathcal{E}$, so that every W_n belongs to $\Sigma_0\widehat{\otimes}\mathrm{T}$.

(b) By 451F, $\mu\restriction\Sigma_0$ is a compact measure; let $\mathcal{K}\subseteq\mathcal{P}X$ be a compact class such that $\mu\restriction\Sigma_0$ is inner regular with respect to $\mathcal{K}$. We may suppose that $\mathcal{K}$ is the family of closed sets for a compact topology on X (342Da). Let $\mathcal{W}$ be the family of those elements W of $\Sigma_0\otimes\mathrm{T}$ such that every horizontal section $W^{-1}[\{y\}]$ belongs to $\mathcal{K}$. Then $\mathcal{W}$ is closed under finite unions and intersections.

(c) If $W\in\Sigma_0\otimes\mathrm{T}$ and $\epsilon>0$, then there is a $W'\in\mathcal{W}$ such that $W'\subseteq W$ and $\lambda_0(W\setminus W')\le\epsilon$. **P** Express W as $\bigcup_{i\le n}E_i\times F_i$, where $E_i\in\Sigma_0$ and $F_i\in\mathrm{T}$ for each $i\le n$. (Cf. 315Kb[13].) For each $i\le n$, take $K_i\in\mathcal{K}\cap\Sigma_0$ such that $\mu(E_i\setminus K_i)\le\frac{1}{n+1}\epsilon$, and set $W'=\bigcup_{i\le n}K_i\times F_i$. Then $W'\in\mathcal{W}$, $W'\subseteq W$ and

$$\begin{aligned}\lambda_0(W\setminus W') &\le \sum_{i=0}^{n}\lambda_0((E_i\times F_i)\setminus(K_i\times F_i)) \le \sum_{i=0}^{n}\lambda_0((E_i\setminus K_i)\times Y)\\ &= \sum_{i=0}^{n}\mu_0(E_i\setminus K_i)\le\epsilon. \text{ } \mathbf{Q}\end{aligned}$$

(d) Take any $\epsilon>0$. Then for each $n\in\mathbb{N}$ we can find $W_n'\in\mathcal{W}$ such that $W_n'\subseteq W_n$ and $\lambda_0(W_n\setminus W_n')\le 2^{-n}\epsilon$. Set $V_n=\bigcap_{i\le n}W_i'$, so that $V_n\in\mathcal{W}$ and

$$\lambda_0(W_n\setminus V_n)\le\textstyle\sum_{i=0}^{n}\lambda_0(W_i\setminus W_i')\le 2\epsilon$$

for each n, and $\langle V_n\rangle_{n\in\mathbb{N}}$ is non-increasing, with empty intersection.

Because $V_n\in\Sigma_0\otimes\mathrm{T}$, its projection $H_n=V_n[X]$ belongs to T, for each n. Of course $\langle H_n\rangle_{n\in\mathbb{N}}$ is non-increasing; also $\bigcap_{n\in\mathbb{N}}H_n=\emptyset$. **P** If $y\in Y$, then $\langle V_n^{-1}[\{y\}]\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, because $\bigcap_{n\in\mathbb{N}}V_n\subseteq\bigcap_{n\in\mathbb{N}}W_n$ is empty. But $\mathcal{K}$ is a compact class, so there must be some n such that $V_n^{-1}[\{y\}]$ is empty, that is, $y\notin H_n$. **Q** Accordingly $\lim_{n\to\infty}\nu H_n=0$. But as $V_n\subseteq X\times H_n$, $\lim_{n\to\infty}\lambda_0 V_n=0$.

This means that $\lim_{n\to\infty}\lambda_0 W_n\le 2\epsilon$. But as ϵ is arbitrary, $\lim_{n\to\infty}\lambda_0 W_n=0$, as required.

454D Theorem (Kolmogorov 33, §III.4) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of totally finite perfect measure spaces. Set $X=\prod_{i\in I}X_i$, and write $\bigotimes_{i\in I}\Sigma_i$ for the algebra of subsets of X generated by $\{\pi_i^{-1}[E] : i\in I,\, E\in\Sigma_i\}$, where $\pi_i : X\to X_i$ is the coordinate map for each $i\in I$. Suppose that $\lambda_0 : \bigotimes_{i\in I}\Sigma_i\to[0,\infty[$ is a non-negative finitely additive functional such that $\lambda_0\pi_i^{-1}[E]=\mu_i E$ whenever $i\in I$ and $E\in\Sigma_i$. Then λ_0 has a unique extension to a measure λ with domain $\widehat{\bigotimes}_{i\in I}\Sigma_i$, and λ is perfect.

proof (a) The argument follows the same pattern as that of 454C. This time, take a non-increasing sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\bigotimes_{i\in I}\Sigma_i$ with empty intersection. Each W_n belongs to the algebra generated by some finite subset of $\{\pi_i^{-1}[E] : i\in I,\, E\in\Sigma_i\}$, so we can find countable sets $\mathcal{E}_i\subseteq\Sigma_i$ such that every W_n belongs to the subalgebra generated by $\{\pi_i^{-1}[E] : i\in I,\, E\in\mathcal{E}_i\}$. Let T_i be the σ-subalgebra of Σ_i generated by $\mathcal{E}_i$, so that every W_n belongs to $\bigotimes_{i\in I}\mathrm{T}_i$.

(b) For each $i\in I$, $\mu_i\restriction\mathrm{T}_i$ is compact (451F); let $\mathfrak{T}_i$ be a compact topology on X_i such that $\mu_i\restriction\mathrm{T}_i$ is inner regular with respect to the closed sets (342F). Let $\mathfrak{T}$ be the product topology on X, so that $\mathfrak{T}$ is compact (3A3J). Let $\mathcal{W}$ be the family of closed sets in X belonging to $\bigotimes_{i\in I}\mathrm{T}_i$.

(c) If $W\in\bigotimes_{i\in I}\mathrm{T}_i$ and $\epsilon>0$, there is a $W'\in\mathcal{W}$ such that $W'\subseteq W$ and $\lambda_0(W\setminus W')\le\epsilon$. **P** We can express W as $\bigcup_{k\le n}\bigcap_{i\in J_k}\pi_i^{-1}[E_{ki}]$ where each J_k is a finite subset of I and $E_{ki}\in\Sigma_i$ for $k\le n$, $i\in J_k$ (cf. 315Kb). Let $\langle\epsilon_{ki}\rangle_{k\le n,i\in J_k}$ be a family of strictly positive numbers with sum at most ϵ. For each $k\le n$, $i\in J_k$ take a closed set $K_{ki}\in\mathrm{T}_i$ such that $K_{ki}\subseteq E_{ki}$ and $\mu_i(E_{ki}\setminus K_{ki})\le\epsilon_{ki}$, and set $W'=\bigcup_{k\le n}\bigcap_{i\in J_k}\pi_i^{-1}[K_{ki}]$. **Q**

(d) Take any $\epsilon>0$. Then for each $n\in\mathbb{N}$ we can find $W_n'\in\mathcal{W}$ such that $W_n'\subseteq W_n$ and $\lambda_0(W_n\setminus W_n')\le 2^{-n}\epsilon$. Set $V_n=\bigcap_{i\le n}W_i'$. Then $\langle V_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed sets in the compact space X, and has empty intersection, so there is some n such that V_n is empty, and

[13] Formerly 315J.

$$\lambda_0 W_n \le \sum_{i=0}^n \lambda_0(W_i \setminus W_i') \le 2\epsilon.$$

As ϵ is arbitrary, $\lim_{n\to\infty} \lambda_0 W_n = 0$.

(e) As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, λ_0 has a unique countably additive extension to $\widehat{\bigotimes}_{i\in I}\Sigma_i$, by 413Kb, as before. Of course the extension is perfect, by 454A(b-iii).

454E Corollary Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of perfect measure spaces. Let $\mathcal{C}$ be the family of subsets of $X=\prod_{i\in I} X_i$ expressible in the form $X\cap\bigcap_{i\in J}\pi_i^{-1}[E_i]$ where $J\subseteq I$ is finite and $E_i\in\Sigma_i$ for every $i\in I$, writing $\pi_i(x)=x(i)$ for $x\in X$, $i\in I$. Suppose that $\lambda_0:\mathcal{C}\to\mathbb{R}$ is a functional such that (i) $\lambda_0\pi_i^{-1}[E]=\mu_i E$ whenever $i\in I$ and $E\in\Sigma_i$ (ii) $\lambda_0 C=\lambda_0(C\cap\pi_i^{-1}[E])+\lambda_0(C\setminus\pi_i^{-1}[E])$ whenever $C\in\mathcal{C}$, $i\in I$ and $E\in\Sigma_i$. Then λ_0 has a unique extension to a measure on $\widehat{\bigotimes}_{i\in I}\Sigma_i$, which is necessarily perfect.

proof By 326E[14], λ_0 has an extension to an additive functional on $\bigotimes_{i\in I}\Sigma_i$, so we can apply 454D.

454F Corollary Let $\langle(X_i,\Sigma_i)\rangle_{i\in I}$ be a family of standard Borel spaces. Set $X=\prod_{i\in I}X_i$, and let $\bigotimes_{i\in I}\Sigma_i$ be the algebra of subsets of X generated by $\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$, where $\pi_i:X\to X_i$ is the coordinate map for each i. Let $\lambda_0:\bigotimes_{i\in I}\Sigma_i\to[0,\infty[$ be a non-negative finitely additive functional such that all the marginal functionals $E\mapsto\lambda_0\pi_i^{-1}[E]:\Sigma_i\to[0,\infty[$ are countably additive. Then λ_0 has a unique extension to a measure defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$, which is a compact measure.

proof This follows immediately from 454D and 454A if we note that all the measures $\lambda_0\pi_i^{-1}$ are necessarily compact, therefore perfect (451M).

454G Corollary Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i a σ-algebra of subsets of X_i for each $i\in I$. Suppose that for each finite set $J\subseteq I$ we are given a totally finite measure μ_J on $Z_J=\prod_{i\in J}X_i$ with domain $\widehat{\bigotimes}_{i\in J}\Sigma_i$ such that (i) whenever J, K are finite subsets of I and $J\subseteq K$, then the canonical projection from Z_K to Z_J is inverse-measure-preserving (ii) every marginal measure $\mu_{\{i\}}$ on $Z_{\{i\}}\cong X_i$ is perfect. Then there is a unique measure μ defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$ such that the canonical projection $\tilde\pi_J:\prod_{i\in I}X_i\to Z_J$ is inverse-measure-preserving for every finite $J\subseteq I$.

proof All we need to observe is that

$$\bigotimes_{i\in I}\Sigma_i=\{\tilde\pi_J^{-1}[V]:J\in[I]^{<\omega},\,V\in\bigotimes_{i\in J}\Sigma_i\}.$$

Because all the canonical projections from X_K onto X_J are inverse-measure-preserving, we have $\mu_J V=\mu_K V'$ whenever J, K are finite subsets of I, $V\in\bigotimes_{i\in J}\Sigma_i$, $V'\in\bigotimes_{i\in K}\Sigma_i$ and $\tilde\pi_J^{-1}[V]=\tilde\pi_K^{-1}[V']$. So we have a functional $\lambda_0:\bigotimes_{i\in I}\Sigma_i\to[0,\infty[$ such that $\lambda_0\tilde\pi_J^{-1}[V]=\mu_J V$ whenever $J\subseteq I$ is finite and $V\in\bigotimes_{i\in J}\Sigma_i$. It is easy to check that λ_0 is finitely additive and satisfies the conditions of 454D. So λ_0 can be extended to a measure μ defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$.

If $J\subseteq I$ is finite, then μ_J and $\mu\tilde\pi_J^{-1}$ agree on $\bigotimes_{i\in J}\Sigma_J$ and therefore (by the Monotone Class Theorem, 136C) on $\widehat{\bigotimes}_{i\in J}\Sigma_i$; that is, $\tilde\pi_J$ is inverse-measure-preserving. To see that μ itself is unique, observe that the conditions define its values on $\bigotimes_{i\in I}\Sigma_i$ and therefore on $\widehat{\bigotimes}_{i\in I}\Sigma_i$, by the Monotone Class Theorem once more.

454H Corollary Let $\langle(X_n,\Sigma_n)\rangle_{n\in\mathbb{N}}$ be a sequence of standard Borel spaces. For each $n\in\mathbb{N}$ set $Z_n=\prod_{i<n}X_i$ and $\mathrm{T}_n=\widehat{\bigotimes}_{i<n}\Sigma_i$. (For $n=0$, we have $Z_0=\{\emptyset\}$, $\mathrm{T}_0=\{\emptyset,Z_0\}$.) For $n\in\mathbb{N}$, $W\in\mathrm{T}_{n+1}$, $z\in Z_n$ write $W[\{z\}]=\{\xi:\xi\in X_n,\,(z,\xi)\in W\}$; set $X=\prod_{n\in\mathbb{N}}X_n$ and write $\tilde\pi_n$ for the canonical projection of X onto Z_n. Suppose that for each $n\in\mathbb{N}$ and $z\in Z_n$ we are given a probability measure ν_z on X_n with domain Σ_n such that $z\mapsto\nu_z(E)$ is T_n-measurable for every $E\in\Sigma_n$. Then there is a unique probability measure μ on $X=\prod_{n\in\mathbb{N}}X_n$, with domain $\Sigma=\widehat{\bigotimes}_{n\in\mathbb{N}}\Sigma_n$, such that, writing $\tilde\mu_n$ for the image measure $\mu\tilde\pi_n^{-1}$ on Z_n,

$$\tilde\mu_{n+1}(W)=\int\nu_z W[\{z\}]\tilde\mu_n(dz)$$

for every $n\in\mathbb{N}$ and $W\in\mathrm{T}_{n+1}$, and

[14] Formerly 326Q.

$$\int f d\tilde{\mu}_{n+1} = \iint \ldots \iint f(\xi_0, \ldots, \xi_n)\nu_{(\xi_0,\ldots,\xi_{n-1})}(d\xi_n)$$
$$\nu_{(\xi_0,\ldots,\xi_{n-2})}(d\xi_{n-1}) \ldots \nu_{\xi_0}(d\xi_1)\nu_{\emptyset}(d\xi_0)$$

for every $n \in \mathbb{N}$ and $\tilde{\mu}_{n+1}$-integrable real-valued function f.

proof (a) We must check that the first formula given actually defines $\tilde{\mu}_{n+1}(W)$ for every $W \in \mathrm{T}_{n+1}$. Of course this is an induction on n. $\tilde{\mu}_0$ will be the unique probability measure on the singleton set Z_0. Given that $\mathrm{dom}\,\tilde{\mu}_n = \mathrm{T}_n$, the class $\mathcal{W}$ of sets $W \subseteq Z_{n+1}$ for which $\int \nu_z W[\{z\}]\tilde{\mu}_n(dz)$ is defined will contain all sets of the form $\prod_{i\le n} E_i$ where $E_i \in \Sigma_i$ for every $i \le n$, just because the function $z \mapsto \nu_z E_n$ is T_n-measurable. Since $\mathcal{W}$ is closed under increasing sequential unions and differences of comparable sets, the Monotone Class Theorem (136B) tells us that it includes the σ-algebra generated by the cylinder sets, which is T_{n+1}.

(b) Accordingly $\langle \nu'_z \rangle_{z\in Z_n}$ is a disintegration of $\tilde{\mu}_{n+1}$ over $\tilde{\mu}_n$, where ν'_z is the measure with domain T_{n+1} defined by writing $\nu'_z(W) = \nu_z W[\{z\}]$ for $W \in \mathrm{T}_{n+1}$, $z \in Z_n$. By 452F,

$$\begin{aligned}\int_{Z_{n+1}} f d\tilde{\mu}_{n+1} &= \int_{Z_n} \int_{Z_{n+1}} f(w)\nu'_z(dw)\tilde{\mu}_n(dz) \\ &= \int_{Z_n} \int_{X_n} f(z, \xi_{n+1})\nu_z(d\xi_{n+1})\tilde{\mu}_n(dz)\end{aligned}$$

for every $\tilde{\mu}_{n+1}$-integrable function f. (The second equality can be regarded as an application of the change-of-variable formula 235Gb applied to the (ν_z, ν'_z)-inverse-measure-preserving function $\xi \mapsto (z, \xi) : X_n \to Z_{n+1}$.) Now a direct induction yields the general formula for $\int f d\tilde{\mu}_{n+1}$ in the statement of this corollary.

(c) The canonical maps from Z_{n+1} to Z_n are all inverse-measure-preserving, just because every ν_z is a probability measure. We therefore have a well-defined functional $\lambda_0 : \bigotimes_{n\in\mathbb{N}} \Sigma_n \to [0,1]$ defined by setting $\lambda_0\tilde{\pi}_n^{-1}[W] = \tilde{\mu}_n W$ whenever $n \in \mathbb{N}$ and $W \in \bigotimes_{i<n} \Sigma_i$, and this λ_0 is finitely additive; moreover, each marginal measure $\lambda_0\pi_n^{-1}$, where $\pi_n : X \to X_n$ is the coordinate map, is countably additive, because it is expressible as an image measure of $\tilde{\mu}_{n+1}$ on X_n.

(d) Everything so far has been valid for any sequence $\langle (X_n, \Sigma_n) \rangle_{n\in\mathbb{N}}$ of sets with attached σ-algebras. But at this point we note that the marginal measures $\lambda_0\pi_n^{-1}$ must be perfect, because (X_n, Σ_n) is a standard Borel space. So Theorem 454D gives the result.

454I Remarks In 454F and 454H the hypotheses call for 'standard Borel spaces' (X_i, Σ_i). As the proofs make clear, what is needed in each case is that 'every totally finite measure with domain Σ_i must be perfect'. We have already seen other ways in which this can be true: for instance, if X is any Radon Hausdorff space (434C), and Σ its Borel σ-algebra. Further examples are in 454Xd, 454Xh-454Xi and 454Yb-454Yc. Indeed, even weaker hypotheses can be fully adequate. In 454H, for instance, it will be quite enough if all the marginal measures on the factors X_n are perfect; in view of 454A and 451E, this will be so iff all the measures $\tilde{\mu}_n$ on the partial products Z_n are perfect. It may be difficult to be sure of this unless either we have some argument from the nature of the factor spaces (X_n, Σ_n), as suggested above, or a clear understanding of the marginal measures. In applications such as 455A below, however, there may be other approaches available.

454J Distributions of random processes For the next few paragraphs I shift to probabilists' notation.

Proposition Let (Ω, Σ, μ) be a probability space and $\langle X_i \rangle_{i\in I}$ a family of real-valued random variables on Ω (see §271).

(i) There is a unique complete probability measure ν on $\mathbb{R}^I$, measuring every Baire set and inner regular with respect to the zero sets, such that

$$\nu\{x : x \in \mathbb{R}^I,\ x(i_r) \le \alpha_r \text{ for every } r \le n) = \Pr(X_{i_r} \le \alpha_r \text{ for every } r \le n)$$

whenever $i_0, \ldots, i_n \in I$ and $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$.

(ii) If $i_0, \ldots, i_n \in I$ and $\tilde{\pi}(x) = (x(i_0), \ldots, x(i_n))$ for $x \in \mathbb{R}^I$, then the image measure $\nu\tilde{\pi}^{-1}$ on $\mathbb{R}^{n+1}$ is the joint distribution of $X_{i_0}, \ldots, X_{i_n}$ as defined in 271C.

(iii) ν is a compact measure. If I is countable then ν is a Radon measure.

(iv) If every X_i is defined everywhere on Ω, then the function $\omega \mapsto \langle X_i(\omega)\rangle_{i\in I} : \Omega \to \mathbb{R}^I$ is inverse-measure-preserving for $\hat{\mu}$ and ν, where $\hat{\mu}$ is the completion of μ.

proof (a) Completing μ, and adjusting the X_i on negligible sets, does not change any of the joint distributions of families $X_{i_0},\dots,X_{i_n}$ (271Ad), so we may suppose henceforth that μ is complete and that every X_i is defined on the whole of Ω. Set $\phi(\omega) = \langle X_i(\omega)\rangle_{i\in I}$ for $\omega \in \Omega$. Then $\{F : F \subseteq \mathbb{R}^I,\ \phi^{-1}[F] \in \Sigma\}$ is a σ-algebra of subsets of $\mathbb{R}^I$ containing $\{x : x(i) \le \alpha\}$ whenever $i \in I$ and $\alpha \in \mathbb{R}$, so includes the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$ of $\mathbb{R}^I$ (4A3Na). If we define $\nu_0 F = \mu\phi^{-1}[F]$ for $F \in \mathcal{B}\mathfrak{a}(\mathbb{R}^I)$, ν_0 is a Baire measure on $\mathbb{R}^I$ for which ϕ is inverse-measure-preserving. We are supposing that μ is complete, so ϕ is still inverse-measure-preserving for μ and the completion ν of ν_0 (234Ba[15]). Since ν_0 is inner regular with respect to the zero sets (412D), so is ν (412Ha), and of course ν measures every Baire set. By 454B, ν_0 is compact, so ν also is (451Ga).

(b) If $i_0,\dots,i_n \in I$ and $\alpha_0,\dots,\alpha_n \in \mathbb{R}$, then

$$\begin{aligned}\Pr(X_{i_r} \le \alpha_r \text{ for every } r \le n) &= \mu\{\omega : X_{i_r}(\omega) \le \alpha_r \text{ for } r \le n\}\\ &= \nu\{x : \phi(x)(i_r) \le \alpha_r \text{ for } r \le n\}\\ &= \nu\{x : x(i_r) \le \alpha_r \text{ for } r \le n\}.\end{aligned}$$

(c) If $i_0,\dots,i_n \in I$ and we set $\tilde{\pi}(x) = (x(i_0),\dots,x(i_n))$ for $x \in \mathbb{R}^I$, then $\nu\tilde{\pi}^{-1}$ is a Radon measure (451O). Since it agrees with the distribution of $X_{i_0},\dots,X_{i_n}$ on all sets of the form $\{z : z(r) \le \alpha_r \text{ for } r \le n\}$, it must be exactly the distribution of $X_{i_0},\dots,X_{i_n}$ (271Ba).

(d) If I is countable, then $\mathbb{R}^I$ is Polish, so ν is a Radon measure (433Cb).

(e) The only point I have not covered is the uniqueness of ν. But suppose that ν' is another measure on $\mathbb{R}^I$ with the properties described in (i). If $i_0,\dots,i_n \in I$ and $\tilde{\pi}(x) = (x(i_0),\dots,x(i_n))$ for $x \in \mathbb{R}^I$, then the image measures $\nu\tilde{\pi}^{-1}$ and $\nu'\tilde{\pi}^{-1}$ on $\mathbb{R}^{n+1}$ are both the distribution of $X_{i_0},\dots,X_{i_n}$, by the argument of (c) above. This means that ν and ν' agree on the algebra of subsets of $\mathbb{R}^I$ generated by sets of the form $\{x : x(i) \in E\}$ where $i \in I$ and $E \subseteq \mathbb{R}$ is Borel. By 454D, they agree on all zero sets, and must be equal (412L).

454K Definition In the context of 454J, I will call ν the **(joint) distribution** of the process $\langle X_i\rangle_{i\in I}$.

Note that if $I = n \in \mathbb{N}\setminus\{0\}$, then ν is a Radon measure on $\mathbb{R}^n$, so is the distribution of $\langle X_i\rangle_{i<n}$ in the sense of 271C.

454L Independence With this extension of the notion of 'distribution' we have a straightforward reformulation of the characterization of independence in 272G.

Theorem Let (Ω,Σ,μ) be a probability space and $\langle X_i\rangle_{i\in I}$ a family of real-valued random variables on Ω, with distribution ν on $\mathbb{R}^I$. Then $\langle X_i\rangle_{i\in I}$ is independent iff ν is the c.l.d. product of the marginal measures on $\mathbb{R}$.

proof (a) For $i \in I$, write ν_i for the marginal measure $\mu\pi_i^{-1}$ on $\mathbb{R}$, taking $\pi_i(x) = x(i)$ as usual. If $J \subseteq I$ is finite, and $\tilde{\pi}_J(x) = x{\restriction}J$, then $\nu\tilde{\pi}_J^{-1}$ is the distribution (in the sense of Chapter 27) of $\langle X_i\rangle_{i\in J}$, by 454J(iii). In particular, ν_i is the distribution of X_i for each i.

(b) If ν is the product measure $\prod_{i\in I}\nu_i$, and $J \subseteq I$ is finite, then $\nu\tilde{\pi}_J^{-1}$ is the product measure $\prod_{i\in J}\nu_i$ (254Oa), so $\langle X_i\rangle_{i\in J}$ is independent (272G). As J is arbitrary, $\langle X_i\rangle_{i\in I}$ is independent (272Bb).

(c) Conversely, if $\langle X_i\rangle_{i\in I}$ is independent, then ν agrees with $\lambda = \prod_{i\in I}\nu_i$ on all sets of the form $\{x : x(i) \le \alpha_i \text{ for } i \in J\}$ where $J \subseteq I$ is finite and $\langle\alpha_i\rangle_{i\in J} \in \mathbb{R}^J$. By the uniqueness assertion in 454J(i), $\nu = \lambda$.

454M The fundamental existence theorem 454G takes a more direct form in this context.

Proposition Let I be a set, and suppose that for each finite $J \subseteq I$ we are given a Radon probability measure ν_J on $\mathbb{R}^J$ such that whenever K is a finite subset of I and $J \subseteq K$, then the canonical projection from $\mathbb{R}^K$ to $\mathbb{R}^J$ is inverse-measure-preserving. Then there is a unique complete probability measure ν on $\mathbb{R}^I$, measuring every Baire set and inner regular with respect to the zero sets, such that the canonical projection from $\mathbb{R}^I$ to $\mathbb{R}^J$ is inverse-measure-preserving for every finite $J \subseteq I$.

[15] Formerly 235Hc.

proof For finite $J \subseteq I$, let μ_J be the restriction of ν_J to the Borel σ-algebra $\mathcal{B}(\mathbb{R}^J)$. Then the canonical projection from $\mathbb{R}^K$ to $\mathbb{R}^J$ is inverse-measure-preserving for μ_K and μ_J whenever $J \subseteq K$ are finite subsets of I. Moreover, $\mu_{\{i\}}$ is a Borel measure on $\mathbb{R}$, therefore perfect, for every $i \in I$. By 454G, we have a unique Baire probability measure μ on $\mathbb{R}^I$ such that the projections $\mathbb{R}^I \to \mathbb{R}^J$ are (μ, μ_J)-inverse-measure-preserving for all finite $J \subseteq I$. Let ν be the completion of μ; then the projections are (ν, ν_J)-inverse-measure-preserving because ν_J is always the completion of μ_J. Finally, ν is unique because $\nu\restriction\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$ must have the defining property for μ.

454N We know that Radon measures are often determined by the integrals they give to continuous functions (415I). If we look at distributions we get a stronger result for probability measures.

Proposition Let Ω be a Hausdorff space, μ and ν two Radon probability measures on Ω, and $\langle X_i\rangle_{i\in I}$ a family of continuous functions separating the points of Ω. If μ and ν give $\langle X_i\rangle_{i\in I}$ the same distribution, they are equal.

proof (a) If K and L are disjoint compact subsets of Ω, there is an open set G such that $K \subseteq G \subseteq X \setminus L$ and $\mu G = \nu G$. **P** $W_i = \{(\omega, \omega') : X_i(\omega) \neq X_i(\omega')\}$ is an open subset of $\Omega \times \Omega$, and $\bigcup_{i\in I} W_i$ includes the compact set $K \times L$. So there is a finite set $J \subseteq I$ such that $K \times L \subseteq \bigcup_{i\in J} W_i$. Define $f : \Omega \to \mathbb{R}^J$ by setting $f(\omega)(i) = X_i(\omega)$ for $\omega \in \Omega$ and $i \in J$; then f is continuous, and $f[K] \cap f[L] = \emptyset$. Also the image measures μf^{-1} and νf^{-1} must be the same, because they are both the common distribution of $\langle X_i\rangle_{i\in J}$. Set $G = \Omega \setminus f^{-1}[L]$; this works. **Q**

(b) Now if $E \subseteq \Omega$ is a Borel set, and $\epsilon > 0$, there are compact sets $K \subseteq E$, $L \subseteq \Omega \setminus E$ such that $\mu K \geq \mu E - \epsilon$ and $\nu L \geq \nu(\Omega \setminus E) - \epsilon$. Let G be an open set such that $\mu G = \nu G$ and $K \subseteq G \subseteq \Omega \setminus L$. Then

$$\begin{aligned}\mu E &\leq \epsilon + \mu K \leq \epsilon + \mu G = \epsilon + \nu G \leq \epsilon + \nu(\Omega \setminus L)\\ &= \epsilon + 1 - \nu L \leq 2\epsilon + 1 - \nu(\Omega \setminus E) = 2\epsilon + \nu E.\end{aligned}$$

As ϵ is arbitrary, $\mu E \leq \nu E$; similarly, $\nu E \leq \mu E$. As E is arbitrary, μ and ν agree on the Borel sets and must coincide.

454O Proposition Let (Ω, Σ, μ) be a probability space and $\langle X_i\rangle_{i\in I}$ a family of random variables on Ω with distribution ν. If $\langle i_n\rangle_{n\in\mathbb{N}}$ is a sequence in I and $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a Borel measurable function, then the random variables

$$\omega \mapsto f(\langle X_{i_n}(\omega)\rangle_{n\in\mathbb{N}}) : \textstyle\bigcap_{n\in\mathbb{N}} \operatorname{dom} X_{i_n} \to \mathbb{R}, \qquad x \mapsto f(\langle x(i_n)\rangle_{n\in\mathbb{N}}) : \mathbb{R}^I \to \mathbb{R}$$

have the same distribution.

proof Completing μ and extending each X_i to the whole of Ω, we may suppose that $\phi : \Omega \to \mathbb{R}^I$ is inverse-measure-preserving for μ and ν, where $\phi(\omega) = \langle X_i(\omega)\rangle_{i\in I}$ for $\omega \in \Omega$. Now, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}\nu\{x : f(\langle x(i_n)\rangle_{n\in\mathbb{N}}) \leq \alpha\} &= \mu\{\omega : f(\langle \phi(\omega)(i_n)\rangle_{n\in\mathbb{N}}) \leq \alpha\}\\ &= \mu\{\omega : f(\langle X_{i_n}(\omega)\rangle_{n\in\mathbb{N}}) \leq \alpha\},\end{aligned}$$

so the distributions are the same.

454P Theorem Let I be a set.

(a) Let ν and ν' be Baire probability measures on $\mathbb{R}^I$ such that $\int e^{if(x)}\nu(dx) = \int e^{if(x)}\nu'(dx)$ for every continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$. Then $\nu = \nu'$.

(b) Let $\langle X_j\rangle_{j\in I}$ and $\langle Y_j\rangle_{j\in I}$ be two families of random variables such that

$$\mathbb{E}(\exp(i\textstyle\sum_{r=0}^n \alpha_r X_{j_r})) = \mathbb{E}(\exp(i\textstyle\sum_{r=0}^n \alpha_r Y_{j_r}))$$

whenever $j_0, \dots, j_n \in I$ and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$. Then $\langle X_j\rangle_{j\in I}$ and $\langle Y_j\rangle_{j\in I}$ have the same distribution.

proof (a) For each finite set $J \subseteq I$, write $\tilde\pi_J(x) = x\restriction J$ for $x \in X$. Then we have Radon measures μ_J and μ'_J on $\mathbb{R}^J$ defined by saying that $\mu_J F = \nu\tilde\pi_J^{-1}[F]$, $\mu'_J F = \nu'\tilde\pi_J^{-1}[F]$ for Borel sets $F \subseteq \mathbb{R}^J$. If $\langle\alpha_j\rangle_{j\in J} \in \mathbb{R}^J$, then

$$\begin{aligned}\int \exp(i\sum_{j\in J}\alpha_j z(j))\mu_J(dz) &= \int \exp(i\sum_{j\in J}\alpha_j x(j))\nu(dx)\\ &= \int \exp(i\sum_{j\in J}\alpha_j x(j))\nu'(dx) = \int \exp(i\sum_{j\in J}\alpha_j z(j))\mu'_J(dz),\end{aligned}$$

so μ_J and μ'_J have the same characteristic function, therefore are equal (285M). This is true for every J, so ν and ν' are equal, by 454D.

(b) Taking ν and ν' to be the two distributions, (a) (with 454O) tells us that their restrictions to the Baire σ-algebra of $\mathbb{R}^I$ are the same, so they must be identical.

454Q Continuous processes The original, and still by far the most important, context for 454D is when every (X_i, Σ_i) is $\mathbb{R}$ with its Borel σ-algebra, so that $X = \prod_{i\in I} X_i$ can be identified with $\mathbb{R}^I$. In the discussion so far, the set I has been an abstract set, except in the very special case of 454H. But some of the most important applications (to which I shall come in §455) involve index sets carrying a topological structure; for instance, I could be the unit interval $[0,1]$ or the half-line $[0,\infty[$. In such a case, we have a wide variety of subspaces of $\mathbb{R}^I$ (for instance, the space of continuous functions) marked out as special, and it is important to know when, and in what sense, our measures on the product space $\mathbb{R}^I$ can be regarded as, or replaced by, measures on the subspace of interest. In the next few paragraphs I look briefly at spaces of continuous functions on Polish spaces.

Lemma Let T be a separable metrizable space and (X, Σ, μ) a semi-finite measure space. Let $\mathfrak{T}$ be a topology on X such that μ is inner regular with respect to the closed sets.

(a) Let $\phi : X \times T \to \mathbb{R}$ be a function such that (i) for each $x \in X$, $t \mapsto \phi(x,t)$ is continuous (ii) for each $t \in T$, $x \mapsto \phi(x,t)$ is Σ-measurable. Then μ is inner regular with respect to $\mathcal{K} = \{K : K \subseteq X,\ \phi\restriction K \times T \text{ is continuous}\}$.

(b) Let $\theta : X \to C(T)$ be a function such that $x \mapsto \theta(x)(t)$ is Σ-measurable for every $t \in T$. Give $C(T)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T. Then θ is almost continuous.

proof The result is trivial if T is empty, so we may suppose that $T \ne \emptyset$.

(a) Take $E \in \Sigma$ and $\gamma < \mu E$; take $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$. Let $\mathcal{U}$ be a countable base for the topology of T consisting of non-empty sets, D a countable dense subset of T and $\mathcal{V}$ a countable base for the topology of $\mathbb{R}$. For $U \in \mathcal{U}$, $V \in \mathcal{V}$ set

$$E_{UV} = \{x : \phi(x,t) \in V \text{ for every } t \in U \cap D\};$$

then $E_{UV} \in \Sigma$. Let $\langle \epsilon_{UV} \rangle_{U\in\mathcal{U}, V\in\mathcal{V}}$ be a family of strictly positive numbers with sum at most $\mu F - \gamma$. For each $U \in \mathcal{U}$, $V \in \mathcal{V}$ take a closed set $F_{UV} \subseteq F \setminus E_{UV}$ such that $\mu F_{UV} \ge \mu(F \setminus E_{UV}) - \epsilon_{UV}$. Consider

$$K = \bigcap_{U\in\mathcal{U}, V\in\mathcal{V}} F_{UV} \cup (F \cap E_{UV}).$$

Then $K \subseteq E$ and $\mu K \ge \gamma$.

If $x \in K$, $t \in T$ and $\phi(x,t) \in V_0 \in \mathcal{V}$, let $V \in \mathcal{V}$ be such that $\phi(x,t) \in V$ and $\overline{V} \subseteq V_0$. Then $\{t' : \phi(x,t') \in V\}$ is an open set containing t, so there is some $U \in \mathcal{U}$ such that $t \in U$ and $\phi(x,t') \in V$ for every $t' \in U$. This means that $x \in E_{UV}$, so that $(K \setminus F_{UV}) \times U$ contains (x,t), and is a relatively open set in $K \times T$. If $(x',t') \in (K \setminus F_{UV}) \times U$, then $x' \in E_{UV}$, so $\phi(x',t'') \in V$ whenever $t'' \in U \cap D$; as D is dense, $\phi(x',t'') \in \overline{V}$ whenever $t'' \in U$; in particular, $\phi(x',t') \in \overline{V} \subseteq V_0$. This shows that $(K \times T) \cap \phi^{-1}[V_0]$ is relatively open in $K \times T$; as V_0 is arbitrary, $\phi\restriction K \times T$ is continuous.

So $K \in \mathcal{K}$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$.

(b) Set $\phi(x,t) = \theta(x)(t)$ for $x \in X$, $t \in T$. Because $\theta(x) \in C(T)$ for every x, ϕ is continuous in the second variable; and the hypothesis on θ is just that ϕ is measurable in the first variable. So μ is inner regular with respect to $\mathcal{K}$ as described in (a). But $\theta\restriction K$ is continuous for every $K \in \mathcal{K}$, by 4A2G(g-ii). So θ is almost continuous.

454R Proposition Let T be an analytic metrizable space (e.g., a Polish space, or any Souslin-F subset of a Polish space), and μ a probability measure on $C(T)$ with domain the σ-algebra Σ generated by the evaluation functionals $f \mapsto f(t) : C(T) \to \mathbb{R}$ for $t \in T$. Give $C(T)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T. Then the completion of μ is a $\mathfrak{T}_c$-Radon measure.

proof If T is empty this is trivial, so let us suppose henceforth that $T \ne \emptyset$.

(a) Let D be a countable dense subset of T. Let $\pi : C(T) \to \mathbb{R}^D$ be the restriction map. Set $X = \pi[C(T)] \subseteq \mathbb{R}^D$; then X, with the topology it inherits from $\mathbb{R}^D$, is a separable metrizable space. Note that, because D is dense, π is injective.

We need to know that π is an isomorphism between $(C(T), \Sigma)$ and $(X, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of X. **P** Since the Borel σ-algebra of $\mathbb{R}^D$ is just the σ-algebra generated by the functionals $g \mapsto g(t) : \mathbb{R}^D \to \mathbb{R}$ as t runs over D (4A3Dc/4A3E), $\mathcal{B}$ is the σ-algebra of subsets of X generated by the functionals $g \mapsto g(t) : X \to \mathbb{R}$ for $t \in D$.

So π is surely $(\Sigma,\mathcal{B})$-measurable. On the other hand, if $t\in X$, there is a sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ in D converging to t, so that $\pi^{-1}(g)(t)=\lim_{n\to\infty}g(t_n)$ for every $g\in X$, and $g\mapsto\pi^{-1}(g)(t):X\to\mathbb{R}$ is $\mathcal{B}$-measurable. Accordingly π^{-1} is $(\mathcal{B},\Sigma)$-measurable. **Q**

(b) Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a base for the topology of T, with no U_n empty. For each $n\in\mathbb{N}$, $g\in\mathbb{R}^D$ set

$$\omega_n(g)=\sup_{t,u\in U_n\cap D}\min(1,g(t)-g(u)),$$

so that $\omega_n:\mathbb{R}^D\to[0,1]$ is T-measurable, where T is the Borel (or Baire) algebra of $\mathbb{R}^D$. For $g\in\mathbb{R}^D$, $g\in X$ iff g has an extension to a continuous function on T, that is,

for every $t\in T$, $k\in\mathbb{N}$ there is an $n\in\mathbb{N}$ such that $t\in U_n$ and $\omega_n(g)\le 2^{-k}$.

Turning this round, $\mathbb{R}^D\setminus X$ is the projection onto the first coordinate of the set

$$Q=\bigcup_{k\in\mathbb{N}}\bigcap_{n\in\mathbb{N}}\{(g,t)\text{: either } t\notin U_n \text{ or } \omega_n(g)>2^{-k}\}\subseteq\mathbb{R}^D\times T.$$

But (because every U_n is an open set and every ω_n is Borel measurable) Q is a Borel set in the analytic space $\mathbb{R}^D\times T$. So Q is analytic (423Eb) and $\mathbb{R}^D\setminus X$ is analytic (423Bb). Since $\mathbb{R}^D$, being Polish (4A2Qc), is a Radon space (434Kb), X is a Radon space (434Fd).

(c) The image measure $\nu=\mu\pi^{-1}$ on X is a Borel probability measure. Because X is a Radon space, ν is tight, and its completion $\hat\nu$ is a Radon measure.

By 454Qb, $\pi^{-1}:X\to C(T)$ is almost continuous if we give $C(T)$ the topology $\mathfrak{T}_c$. So the image measure $\lambda=\hat\nu(\pi^{-1})^{-1}$ is a Radon measure for $\mathfrak{T}_c$ (418I). But of course λ is the completion of μ, just because π is a bijection and $\hat\nu$ is the completion of ν.

454S Corollary Let T be an analytic metrizable space.

(a) $C(T)$, with either the topology $\mathfrak{T}_p$ of uniform convergence on finite subsets of T or the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T, is a measure-compact Radon space.

(b) Let μ be a Baire probability measure on $\mathbb{R}^T$ such that $\mu^*C(T)=1$. Then the subspace measure $\hat\mu_C$ on $C(T)$ induced by the completion of μ is a Radon measure on $C(T)$ if $C(T)$ is given either $\mathfrak{T}_p$ or $\mathfrak{T}_c$. μ itself is τ-additive and has a unique extension $\tilde\mu$ which is a Radon measure on $\mathbb{R}^T$; $\hat\mu_C$ is the subspace measure on $C(T)$ induced by $\tilde\mu$.

proof (a) Let μ be a probability measure on $C(T)$ which is either a Baire measure or a Borel measure with respect to either $\mathfrak{T}_p$ or $\mathfrak{T}_c$. Let $\tilde\mu$ be the completion of $\mu\restriction\Sigma$, where Σ is the σ-algebra generated by the functionals $f\mapsto f(t)$; because all these are $\mathfrak{T}_p$-continuous, Σ is certainly included in the Baire σ-algebra for $\mathfrak{T}_p$, so that $\Sigma\subseteq\operatorname{dom}\mu$. 454R tells us that $\tilde\mu$ is a Radon measure for $\mathfrak{T}_c$. Because $\mathfrak{T}_p$ is a coarser Hausdorff topology, $\tilde\mu$ is also a Radon measure for $\mathfrak{T}_p$. Also $\tilde\mu$ must extend μ, because its domain includes that of μ and the completion of μ must extend $\tilde\mu$ (in fact, of course, this means that $\tilde\mu$ is actually the completion of μ). Now $\tilde\mu$ is τ-additive (for either topology), so μ also is; as μ is arbitrary, $C(T)$ is measure-compact (for either topology). On the other hand, if μ is a Borel measure (for either topology), it must be tight for that topology; so that $C(T)$ is a Radon space.

(b) Write μ_C for the subspace measure on $C(T)$. Recall that the domain Σ of μ is just the σ-algebra generated by the functionals $f\mapsto f(t):\mathbb{R}^T\to\mathbb{R}$, as t runs over T (4A3Na), so that the domain Σ_C of μ_C is the σ-algebra of subsets of $C(T)$ generated by the functionals $f\mapsto f(t):C(T)\to\mathbb{R}$. By 454R, the completion of μ_C is a Radon measure on $C(T)$ if we give $C(T)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T, and therefore also for the weaker Hausdorff topology $\mathfrak{T}_p$. Because the μ_C-negligible sets for μ_C are just the intersections of $C(T)$ with μ-negligible sets (214Cb), the completion of μ_C is the subspace measure $\hat\mu_C$ induced by the completion of μ (214Ib[16]).

The embedding $C(T)\subsetneqq\mathbb{R}^T$ is of course continuous for $\mathfrak{T}_c$ and the product topology on $\mathbb{R}^T$, so we have a Radon image measure $\tilde\mu$ on $\mathbb{R}^T$ defined by saying that $\tilde\mu E=\hat\mu_C(E\cap C(T))$ whenever $E\cap C(T)$ is measured by $\hat\mu_C$. If $E\in\Sigma$, then

$$\tilde\mu E=\hat\mu_C(E\cap C(T))=\mu_C(E\cap C(T))=\mu^*(E\cap C(T))=\mu E$$

because $\mu^*C(T)=1$, so $\tilde\mu$ extends μ. Of course $\tilde\mu C(T)=1$ and the subspace measure on $C(T)$ induced by $\tilde\mu$ is just $\hat\mu_C$.

Finally, because μ has an extension to a Radon measure, it must itself be τ-additive. Because Σ includes a base for the topology of $\mathbb{R}^T$, μ can have only one extension to a Radon measure on $\mathbb{R}^T$ (415H(iv)).

[16] Formerly 214Xb.

454T A bit out of order, I give an elementary remark on completion regular measures on products of compact spaces.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of compact Hausdorff spaces, and μ a completion regular topological measure on $X=\prod_{i\in I}X_i$. Then the marginal measure μ_i of μ on X_i is completion regular for each $i\in I$.

proof Let $\pi_i:X\to X_i$ be the coordinate map. If $E\subseteq X_i$ and $\gamma<\mu_i E$, there is a zero set $Z\subseteq\pi_i^{-1}[E]$ such that $\mu Z\ge\gamma$. Now $\pi_i[Z]\subseteq E$ is a zero set in X_i (4A2G(c-ii), using 4A2B(f-i)), and

$$\mu_i\pi_i[Z]=\mu\pi_i^{-1}[\pi_i[Z]]\ge\mu Z\ge\gamma.$$

As E and γ are arbitrary, μ_i is inner regular with respect to the zero sets, so is completion regular.

454X Basic exercises >(a) Let μ be Lebesgue measure on $[0,1]$, and Σ its domain. Let X_0, $X_1\subseteq[0,1]$ be disjoint sets of full outer measure. For each i, let Σ_i be the relative σ-algebra on X_i. Show that we have a finitely additive functional λ defined on $\Sigma_0\otimes\Sigma_1$ by the formula

$$\lambda((E\cap X_0)\times(F\cap X_1))=\mu(E\cap F)\text{ for all }E,\,F\in\Sigma,$$

and that λ has no extension to a measure on $X_0\times X_1$.

(b) Adapt the example of 419K to provide a counter-example for 454G if we omit the hypothesis that the marginal measures $\mu_{\{i\}}$ must be perfect.

(c) Adapt the example of 419K/454Xb to provide a counter-example for 454H if we omit the hypothesis that the (X_n,Σ_n) must be standard Borel spaces. (*Hint*: if $z\in\prod_{i\le n}X_i$, try $\nu_z(E)=1$ if $z(n)\in E$, 0 otherwise.)

>(d) Let X be a set and Σ a σ-algebra of subsets of X. Let us say that (X,Σ) has the **perfect measure property** if every totally finite measure with domain Σ is perfect. Show that (i) if (X,Σ) has the perfect measure property, so does (E,Σ_E) for any $E\in\Sigma$, where Σ_E is the subspace σ-algebra on E (ii) if $\langle(X_i,\Sigma_i)\rangle_{i\in I}$ is a family of spaces with the perfect measure property, then $(\prod_{i\in I}X_i,\widehat{\bigotimes}_{i\in I}\Sigma_i)$ has the perfect measure property.

(e) Let (X,Σ) be a space with the perfect measure property, and T the smallest σ-algebra including Σ and closed under Souslin's operation. Show that (X,T) has the perfect measure property.

(f) Let X be a set and Σ a σ-algebra of subsets of X. Let us say that (X,Σ) has the **countably compact measure property** if every totally finite measure with domain Σ is countably compact. Show that (i) if (X,Σ) has the countably compact measure property it has the perfect measure property (ii) if (X,Σ) has the countably compact measure property so does (E,Σ_E) for every $E\in\Sigma$, where Σ_E is the subspace σ-algebra on E (iii) if $\langle(X_i,\Sigma_i)\rangle_{i\in I}$ is a family of spaces with the countably compact measure property, then $(\prod_{i\in I}X_i,\widehat{\bigotimes}_{i\in I}\Sigma_i)$ has the countably compact measure property.

(g) Suppose that (X,Σ) has the countably compact measure property. (i) Let μ be a totally finite measure with domain Σ, (Y,T,ν) a measure space, and $f:X\to Y$ an inverse-measure-preserving function. Show that μ has a disintegration $\langle\mu_y\rangle_{y\in Y}$ over ν which is consistent with f. (ii) Let Y be any set, T a σ-algebra of subsets of Y, and λ a probability measure with domain $\Sigma\widehat{\otimes}\mathrm{T}$. Show that there is a family $\langle\mu_y\rangle_{y\in Y}$ of probability measures on X such that $\lambda W=\int\mu_y W^{-1}[\{y\}]\nu(dy)$ for every $W\in\Sigma\widehat{\otimes}\mathrm{T}$, where ν is the marginal measure of λ on Y. (*Hint*: 452M.)

(h)(i) Let X be any set, and Σ the countable-cocountable algebra on X. Show that (X,Σ) has the countably compact measure property. (ii) Show that any standard Borel space has the countably compact measure property.

(i) Let X be a Radon Hausdorff space, and Σ_{um} the algebra of universally measurable sets in X (434D). Show that (X,Σ_{um}) has the countably compact measure property.

>(j) Let $\langle X_i\rangle_{i\in I}$ be an independent family of real-valued random variables. Show that its distribution is a quasi-Radon measure on $\mathbb{R}^I$. (*Hint*: 415E.)

(k) Let I be a set and ν, ν' two quasi-Radon measures on $\mathbb{R}^I$ such that $\int e^{if(x)}\nu(dx)=\int e^{if(x)}\nu'(dx)$ for every continuous linear functional $f:\mathbb{R}^I\to\mathbb{R}$. Show that $\nu=\nu'$.

>(l) Let Σ be the σ-algebra of subsets of $C([0,\infty[)$ generated by the functionals $f \mapsto f(t)$ for $t \ge 0$. Give $C([0,\infty[)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact sets. (i) Show that $\mathfrak{T}_c$ is Polish, and that $\Sigma \cap \mathfrak{T}_c$ is a base for $\mathfrak{T}_c$ which generates Σ as σ-algebra. (ii) Use this to give a quick proof of 454R in this case.

(m) Let T be a Polish space, and $\mathfrak{T}_c$ the topology on $C(T)$ of uniform convergence on compact sets. Show that if $\mathfrak{T}$ is any Hausdorff topology on $C(T)$, coarser than $\mathfrak{T}_c$, such that all the functionals $f \mapsto f(t)$, for $t \in T$, are Baire measurable for $\mathfrak{T}$, then $(C(T), \mathfrak{T})$ is a measure-compact Radon space.

(n) Give an example of a metrizable space Ω with a continuous injective function $X : \Omega \to [0,1]$ and two different quasi-Radon probability measures μ, ν on Ω giving the same distribution to the random variable X.

454Y Further exercises (a) In 454Ab, show that μ is weakly α-favourable (definition: 451V) if every μ_i is.

(b) Let Σ be the algebra of Lebesgue measurable subsets of $\mathbb{R}$. Show that $(\mathbb{R}, \Sigma)$ has the perfect measure property (454Xd) iff $\mathfrak{c}$ is measure-free.

(c) Let $\mathcal{B}$ be the Borel σ-algebra of ω_1 with its order topology. Show that $(\omega_1, \mathcal{B})$ has the perfect measure property. (*Hint*: 439Yf.)

(d) Let (X, Σ, μ) be a semi-finite measure space with a topology such that μ is inner regular with respect to the closed sets, T a second-countable space and Y a separable metrizable space. Suppose that $\phi : X \times T \to Y$ is continuous in the second variable and measurable in the first, as in 454Q. Show that μ is inner regular with respect to $\mathcal{K} = \{K : K \subseteq X,\ \phi \restriction K \times T \text{ is continuous}\}$.

454 Notes and comments 454A generalizes Theorem 451J, which gave the same result (with essentially the same proof) for product measures. One of the themes of this section is the idea that we can deduce properties of measures on product spaces from properties of their marginal measures, that is, the image measures on the factors. The essence of 'compactness', 'countable compactness' and 'perfectness' is that we can find enough points in the measure space to do what we want. (See, for instance, the characterization of compactness in 343B, or Pachl's characterization of countable compactness in 452Ye.) Since the canonical feature of a product space is that we put in every point the Axiom of Choice provides us with, it's perhaps not surprising that such properties can be inherited by measures on product spaces.

Theorems 454C and 454D can be regarded as further variations on the same theme. A finitely additive non-negative functional on an algebra of sets will have an extension to a measure if, and only if, it is sequentially smooth in the sense that the measures of a decreasing sequence of sets with empty intersection converge to zero (413K). If we have a decreasing sequence of sets, with measures bounded away from zero, but with empty intersection, one interpretation of the phenomenon is that some points which ought to have been present got left out of the sets. What 454D tells us is that perfectness (and countable additivity) of the marginal measures is enough to ensure that there are enough points in the product to stop this happening. In effect, 454C tells us that it will be enough if every marginal but one is perfect.

These results are of course associated with the projective limit constructions in 418M-418Q. In the theorems there we had Radon measures, so that they were actually compact rather than perfect; in return for the stronger hypothesis on the measures, we could handle projective limits corresponding to rather small subsets of the product spaces (see the formulae in 418O-418Q). Just as in §418, the patterns change when we have countable rather than uncountable families to deal with (418P-418Q, 454H).

In 454J-454P, I insist rather arbitrarily that 'the' joint distribution of a family $\langle X_i \rangle_{i \in I}$ of real-valued random variables is the completion of a Baire measure on $\mathbb{R}^I$. Of course all the ideas can also be expressed in terms of the Baire measure itself, but I have sought a formulation which is consistent with the rules set out in §271. When I is countable, we get a Radon measure (454J(iii)), as in the finite-dimensional case. There are other cases in which the distribution is a quasi-Radon measure (454Xj). As always, we can ask whether the distribution is τ-additive; in this case it will have a canonical extension to a quasi-Radon measure (415N). Important examples of this phenomenon are described in 455H and 456O. Because $\mathbb{R}^I$ has a linear topological space structure, we have a notion of 'characteristic function' for any probability measure on $\mathbb{R}^I$ measuring the zero sets, and the characteristic function of a Baire measure determines that measure (454P, 454Xk).

In 454R, $C(T)$, with $\mathfrak{T}_c$, has a countable network (4A2Oe), so the subspace measure μ_C induced by μ on $C(T)$ must be a τ-additive topological measure with respect to $\mathfrak{T}_c$ (414O) and has a unique extension to a quasi-Radon measure on $C(T)$ (415M). The hard bit is the next step, showing that $C(T)$, under $\mathfrak{T}_c$, is a Radon space; this is the real point of 454Q-454R. For the most important case, in which $T=[0,\infty[$, we have a useful simplification, because $\mathfrak{T}_c$ is actually Polish (454Xl). Even in this case, however, we need to observe that the measure we are seeking is a little more complicated than a simple completion of a measure on $\mathbb{R}^T$. We must complete the *subspace* measure on $C(T)$, and $C(T)$ is far from being a measurable set. The measure $\tilde{\mu}$ of 454S will not as a rule be completion regular, for instance. Spaces of continuous functions are so important that it is worth noticing that the results here will be valid for various topologies on $C(T)$ (454Xm).

455 Markov and Lévy processes

For a 'Markov' process, in which the evolution of the system after a time t depends only on the state at time t, the general theory of §454 leads to a straightforward existence theorem (at least for random variables taking values in standard Borel spaces) dependent only on a natural consistency condition on the transitional probabilities (455A, 455E). The formulation leads naturally to descriptions of the 'Markov property' (for stopping times taking only countably many values) in terms of disintegrations and conditional expectations (455C, 455Ec). With appropriate continuity conditions, we find that the process can be represented either by a Radon measure (455H) or by a measure on the set of càdlàg paths (455Gc) for which we have a formulation of the strong Markov property (for general stopping times) in terms of disintegrations (455O). These conditions are satisfied by Lévy processes (455P-455R). For these, we have an alternative expression of the strong Markov property in terms of inverse-measure-preserving functions (455U). By far the most important example of a continuous-time Markov process is Brownian motion, but I defer discussion of this to §477.

455A Theorem Let T be a totally ordered set with least element t^*, and for each $t\in T$ let Ω_t be a non-empty set and T_t a σ-algebra of subsets of Ω_t containing all singleton subsets of Ω_t. Set $\Omega=\prod_{t\in T}\Omega_t$ and for $t\in T$, $\omega\in\Omega$ set $X_t(\omega)=\omega(t)$. Fix $x^*\in\Omega_{t^*}$. Suppose that we are given, for each pair $s<t$ in T, a family $\langle\nu_x^{(s,t)}\rangle_{x\in\Omega_s}$ of perfect probability measures on Ω_t, all with domain T_t, and suppose that

(†) whenever $s<t<u$ in T and $x\in\Omega_s$, then $\langle\nu_y^{(t,u)}\rangle_{y\in\Omega_t}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$.

For $J\subseteq T$ write π_J for the canonical map from Ω onto $Z_J=\prod_{t\in J}\Omega_t$. Then there is a unique probability measure μ on Ω, with domain $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$, such that, writing λ_J for the image measure $\mu\pi_J^{-1}$,

$$\begin{aligned}\int f d\lambda_J &= \int f(\omega(t^*),\omega(t_1),\dots,\omega(t_n))\mu(d\omega)\\ &= \int\dots\iint f(x^*,x_1,\dots,x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\\ &\qquad\qquad \nu_{x_{n-2}}^{(t_{n-2},t_{n-1})}(dx_{n-1})\dots\nu_{x^*}^{(t^*,t_1)}(dx_1)\end{aligned}$$

whenever $t^*<t_1<\dots<t_n$, $J=\{t^*,t_1,\dots,t_n\}$ and f is λ_J-integrable. μ is perfect, and the marginal measure $\mu_t=\mu X_t^{-1}$ is equal to $\nu_{x^*}^{(t^*,t)}$, if $t>t^*$, while $\mu_{t^*}\{x^*\}=1$.

proof (a) For $I\subseteq T$, write $\mathrm{T}_I=\widehat{\bigotimes}_{t\in I}\mathrm{T}_t$. If $I=\{t_0,t_1,\dots,t_n\}$ is a finite subset of T with $t^*=t_0<t_1<\dots<t_n$, then we have a probability measure λ_I on Z_I with domain T_I such that

$$\begin{aligned}\int f d\lambda_I &= \int\dots\iint f(x^*,x_1,\dots,x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\\ &\qquad\qquad \nu_{x_{n-2}}^{(t_{n-2},t_{n-1})}(dx_{n-1})\dots\nu_{x^*}^{(t^*,t_1)}(dx_1)\end{aligned}$$

for every λ_I-integrable function f. **P** Use 454H on the finite sequence $(\Omega_{t_0},\dots,\Omega_{t_n})$. The measures ν_z required by 454H must be constructed by the rule

$$\nu_z=\nu_{z(t_m)}^{(t_m,t_{m+1})}$$

for $m<n$, $z\in\prod_{i\le m}\Omega_{t_i}$, while of course $\nu_\emptyset\{x^*\}=1$. (Having a finite sequence rather than an infinite one clearly makes things easier; we can stop the argument at the end of part (b) of the proof of 454H.) **Q**

When $I=\{t^*\}$, so that Z_I can be identified with Ω_{t^*}, I mean to interpret these formulae in such a way that $\lambda_I\{x^*\}=1$. When $J=\{t^*,t\}$, with $t^*<t$, and $E\in\mathrm{T}_t$, then we can apply the formula above to the function $z\mapsto\chi E(z(t))$ to get $\lambda_J\{z:z(t)\in E\}=\nu^{(t^*,t)}_{x^*}(E)$.

(b) Of course the point of this is that these measures λ_I form a consistent family; if $t^*\in I\subseteq J\in[T]^{<\omega}$, then the canonical projection $\pi_{IJ}:Z_J\to Z_I$ is inverse-measure-preserving. **P** It is enough to consider the case in which J has just one more point than I, since then we can induce on $\#(J\setminus I)$. In this case, express J as $\{t_0,\ldots,t_n\}$ where $t^*=t_0<\ldots<t_n$, and suppose that $I=J\setminus\{t_m\}$. If $W\in\mathrm{T}_I$, then

$$\begin{aligned}\lambda_J\pi_{IJ}^{-1}[W]=\int\ldots\iint\ldots\int\chi W(x^*,x_1,\ldots,x_{m-1},x_{m+1},\ldots,x_n)\nu^{(t_{n-1},t_n)}_{x_{n-1}}(dx_n)\\ \ldots\nu^{(t_m,t_{m+1})}_{x_m}(dx_{m+1})\nu^{(t_{m-1},t_m)}_{x_{m-1}}(dx_m)\ldots\nu^{(t^*,t_1)}_{x^*}(dx_1)\\ =\int\ldots\iint g_{(x_1,\ldots,x_{m-1})}(x_{m+1})\nu^{(t_m,t_{m+1})}_{x_m}(dx_{m+1})\\ \nu^{(t_{m-1},t_m)}_{x_{m-1}}(dx_m)\ldots\nu^{(t^*,t_1)}_{x^*}(dx_1)\end{aligned}\qquad (*)$$

where

$$\begin{aligned}g_{(x_1,\ldots,x_{m-1})}(x_{m+1})=\int\ldots\int\chi W(x^*,x_1\ldots,x_{m-1},x_{m+1},\ldots,x_n)\\ \nu^{(t_{n-1},t_n)}_{x_{n-1}}(dx_n)\ldots\nu^{(t_{m+1},t_{m+2})}_{x_{m+1}}(dx_{m+2}).\end{aligned}$$

Here, of course, we use the hypothesis (†); since $\langle\nu^{(t_m,t_{m+1})}_y\rangle_{y\in\Omega_{t_m}}$ is a disintegration of $\nu^{(t_{m-1},t_{m+1})}_{x_{m-1}}$ over $\nu^{(t_{m-1},t_m)}_{x_{m-1}}$, and $g_{(x_1,\ldots,x_{m-1})}$ is bounded and $\nu^{(t_{m-1},t_{m+1})}_{x_{m-1}}$-integrable (by 454H),

$$\begin{aligned}\int g_{(x_1,\ldots,x_{m-1})}(x_{m+1})\nu^{(t_{m-1},t_{m+1})}_{x_{m-1}}(dx_{m+1})\\ =\iint g_{(x_1,\ldots,x_{m-1})}(x_{m+1})\nu^{(t_m,t_{m+1})}_{x_m}(dx_{m+1})\nu^{(t_{m-1},t_m)}_{x_{m-1}}(dx_m)\end{aligned}$$

(452F). Substituting this into (*) above,

$$\begin{aligned}\lambda_J\pi_{IJ}^{-1}[W]&=\int\ldots\iint g_{(x_1,\ldots,x_{m-1})}(x_{m+1})\\ &\qquad\nu^{(t_m,t_{m+1})}_{x_m}(dx_{m+1})\nu^{(t_{m-1},t_m)}_{x_{m-1}}(dx_m)\ldots\nu^{(t^*,t_1)}_{x^*}(dx_1)\\ &=\int\ldots\int g_{(x_1,\ldots,x_{m-1})}(x_{m+1})\nu^{(t_{m-1},t_{m+1})}_{x_{m-1}}(dx_{m+1})\ldots\nu^{(t^*,t_1)}_{x^*}(dx_1)\\ &=\int\ldots\int\ldots\int\chi W(x^*,x_1,\ldots,x_{m-1},x_{m+1},\ldots,x_n)\\ &\qquad\nu^{(t_{n-1},t_n)}_{x_{n-1}}(dx_n)\ldots\nu^{(t_{m-1},t_{m+1})}_{x_{m-1}}(dx_{m+1})\ldots\nu^{(t^*,t_1)}_{x^*}(dx_1)\\ &=\lambda_IW,\end{aligned}$$

applying the formula in (a) again. **Q**

(Some of the formulae here are inappropriate if $m=n>1$. In this case, of course,

$$\begin{aligned}\lambda_J\pi_{IJ}^{-1}[W]&=\int\ldots\int\chi W(x^*,x_1,\ldots,x_{n-1})\nu^{(t_{n-1},t_n)}_{x_{n-1}}(dx_n)\ldots\nu^{(t^*,t_1)}_{x^*}(dx_1)\\ &=\int\ldots\int\chi W(x^*,x_1,\ldots,x_{n-1})\nu^{(t_{n-2},t_{n-1})}_{x_{n-2}}(dx_{n-1})\ldots\nu^{(t^*,t_1)}_{x^*}(dx_1)=\lambda_IW.\end{aligned}$$

If $m=1<n$, there is a collapse of a different kind; we must look at

$$\lambda_J\pi_{IJ}^{-1}[W] = \int\ldots\iint \chi W(x^*,x_2,\ldots,x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\ldots\nu_{x_1}^{(t_1,t_2)}(dx_2)\nu_{x^*}^{(t^*,t_1)}(dx_1)$$
$$= \int\ldots\iint \chi W(x^*,x_2,\ldots,x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\ldots\nu_{x^*}^{(t^*,t_2)}(dx_2) = \lambda_I W.$$

If $m=n=1$ then

$$\lambda_J\pi_{IJ}^{-1}[W] = \int\chi W(x^*)\nu_{x^*}^{(t^*,t_1)}(dx_1) = \chi W(x^*) = \lambda_I W).$$

(c) Part (b) tells us that we have a consistent family of measures on the finite products Z_J, and therefore have a functional λ on $\bigotimes_{t\in T}\mathrm{T}_t$ defined by setting $\lambda\pi_J^{-1}[W] = \lambda_J W$ for every finite $J\subseteq T$ containing t^* and $W\in\bigotimes_{t\in J}\mathrm{T}_t$. λ is finitely additive, and its images $\mu_t = \lambda X_t^{-1}$ are all countably additive and perfect because $\mu_t = \nu_{x^*}^{(t^*,t)}$ for $t>t^*$, while μ_{t^*} is concentrated at $\{x^*\}$.

By 454D, we have a perfect measure μ extending λ. We have to check that each λ_J is the image measure $\mu\pi_J^{-1}$; but this is true because they agree on $\bigotimes_{t\in J}\mathrm{T}_t$ (using the Monotone Class Theorem in the form 136C, as always). So the integral formula sought for λ_J is just that obtained in part (a). By the last remark in (a), we have the declared formulae for the marginal measures μ_t.

455B Lemma Suppose that T, t^*, $\langle(\Omega_t,\mathrm{T}_t)\rangle_{t\in T}$, Ω, x^* and $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in\Omega_s}$ are as in 455A.

(a) Suppose that μ is constructed from x^* and $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in\Omega_s}$ as in 455A. If $F\in\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ is determined by coordinates in $[t^*,t_0]$ and $H^* = \{\omega:\omega(t_i)\in E_i \text{ for } 1\le i\le n\}$ where $t_0<t_1\ldots<t_n$ and $E_i\in\mathrm{T}_{t_i}$ for $1\le i\le n$, then

$$\mu(H^*\cap F) = \int_F\int\ldots\int\chi H(y_1,\ldots,y_n)\nu_{y_{n-1}}^{(t_{n-1},t_n)}(dy_n)\ldots\nu_{\omega(t_0)}^{(t_0,t_1)}(dy_1)\mu(d\omega) \qquad (*)$$

where $H=\prod_{1\le i\le n}E_i$.

(b) Suppose that $\omega\in\Omega$ and $a\in T\cup\{\infty\}$, where ∞ is taken to be greater than every element of T. For $s<t$ in T and $x\in\Omega_s$ set

$$\nu_{\omega ax}^{(s,t)} = \nu_x^{(s,t)} \text{ if } a<s,$$
$$= \nu_{\omega(a)}^{(a,t)} \text{ if } s\le a<t,$$
$$= \delta_{\omega(t)}^{(t)} \text{ if } t\le a,$$

here writing $\delta_x^{(t)}$ for the probability measure with domain T_t such that $\delta_x^{(t)}(\{x\})=1$.

(i) $\nu_{\omega ax}^{(s,t)}$ is always a perfect probability measure with domain T_t, and $\langle\nu_{\omega ay}^{(t,u)}\rangle_{y\in\Omega_t}$ is a disintegration of $\nu_{\omega ax}^{(s,u)}$ over $\nu_{\omega ax}^{(s,t)}$ whenever $s<t<u$ in T and $x\in\Omega_s$.

(ii) Taking $\mu_{\omega a}$ to be the measure on Ω defined from $\omega(t^*)$ and $\langle\nu_{\omega ax}^{(s,t)}\rangle_{s<t,x\in\Omega_t}$ by the method of 455A, then $\{\omega':\omega'\in\Omega,\ \omega'\restriction D=\omega\restriction D\}$ is $\mu_{\omega a}$-conegligible for every countable $D\subseteq T\cap[t^*,a]$.

(iii) If ω, $\omega'\in\Omega$ and $\omega\restriction[t^*,a]=\omega'\restriction[t^*,a]$ then $\mu_{\omega a}=\mu_{\omega' a}$.

proof (a)(i) Suppose first that F is of the form $\{\omega:\omega(s_i)\in F_i \text{ for } i\le m\}$ where $t^*=s_0<\ldots<s_m=t_0$. For $x\in\Omega_{t_0}$ set

$$f(x) = \int\ldots\int\chi H(y_1,\ldots,y_n)\nu_{y_{n-1}}^{(t_{n-1},t_n)}(dy_n)\ldots\nu_x^{(t_0,t_1)}(dy_1).$$

Writing $G=\prod_{i\le m}F_i$, we have

$$\mu(H^*\cap F) = \int\ldots\iint\ldots\int\chi G(x^*,x_1,\ldots,x_m)\chi H(y_1,\ldots,y_n)\nu_{y_{n-1}}^{(t_{n-1},t_n)}(dy_n)$$
$$\ldots\nu_{x_m}^{(s_m,t_1)}(dy_1)\nu_{x_{m-1}}^{(s_{m-1},s_m)}(dx_m)\ldots\nu_{x^*}^{(t^*,s_1)}(dx_1)$$
$$= \int\ldots\int\chi G(x^*,x_1,\ldots,x_m)f(x_m)\nu_{x_{m-1}}^{(s_{m-1},s_m)}(dx_m)\ldots\nu_{x^*}^{(t^*,s_1)}(dx_1)$$
$$= \int g\,d\lambda_J$$

(where $J=\{t^*,s_1,\ldots,s_m\}$, $g(z)=\chi G(z(t^*),\ldots,z(s_m))f(z(s_m))$ for $z\in\prod_{s\in J}\Omega_s$, and λ_J is defined as in 455A)

$$= \int g\pi_J d\mu = \int_F f(\omega(t_0))\mu(d\omega)$$

(because $g\pi_J(\omega) = f(\omega(s_m)) = f(\omega(t_0))$ if $\omega \in F$, 0 otherwise)

$$= \int_F \int \ldots \int \chi H(y_1, \ldots , y_n)\nu^{(t_{n-1},t_n)}_{y_{n-1}}(dy_n) \ldots \nu^{(t_0,t_1)}_{\omega(t_0)}(dy_1)\mu(d\omega).$$

(ii) Let $\mathcal{I}$ be the family of sets F of the type dealt with in (a). Since the intersection of two members of $\mathcal{I}$ belongs to $\mathcal{I}$, the Monotone Class Theorem tells us that (*) is true for all sets in the σ-algebra T generated by $\mathcal{I}$. But any member of $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ determined by coordinates in $[t^*, t_0]$ belongs to T. **P** Fix $v \in \prod_{s\in T\setminus[t^*,t_0]} \Omega_s$. For $\omega \in \Omega$ define $f(\omega) \in \Omega$ by setting

$$f(\omega)(s) = \omega(s) \text{ if } s \le t_0,$$
$$= v(s) \text{ if } s > t_0.$$

Then $\mathrm{T}' = \{F : F \subseteq \Omega,\ f^{-1}[F] \in \mathrm{T}\}$ is a σ-algebra of subsets of Ω containing $\{\omega : \omega(t) \in E\}$ whenever $t \in T$ and $E \in \mathrm{T}_t$, so includes $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$. If $F \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ and F is determined by coordinates in $[t^*, t_0]$, then $F = f^{-1}[F] \in \mathrm{T}$. **Q**

So (*) is true of every $F \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$, as claimed.

(b)(i) Of course every $\nu^{(s,t)}_{\omega ax}$ is a perfect probability measure with domain T_t. If $s < t < u$ and $E \in \mathrm{T}_u$, then

$$\int_{\Omega_t} \nu^{(t,u)}_{\omega ay}(E)\nu^{(s,t)}_{\omega ax}(dy) = \int_{\Omega_t} \nu^{(t,u)}_{y}(E)\nu^{(s,t)}_{x}(dy) = \nu^{(s,u)}_{x}(E) = \nu^{(s,u)}_{\omega ax}(E)$$
$$\text{if } a < s,$$
$$= \int_{\Omega_t} \nu^{(t,u)}_{y}(E)\nu^{(a,t)}_{\omega(a)}(dy) = \nu^{(a,u)}_{\omega(a)}(E) = \nu^{(s,u)}_{\omega ax}(E)$$
$$\text{if } s \le a < t,$$
$$= \int_{\Omega_t} \nu^{(t,u)}_{\omega(t)}(E)\delta^{(t)}_{\omega(t)}(dy) = \nu^{(t,u)}_{\omega(t)}(E) = \nu^{(s,u)}_{\omega ax}(E)$$
$$\text{if } a = t,$$
$$= \int_{\Omega_t} \nu^{(a,u)}_{\omega(a)}(E)\delta^{(t)}_{\omega(t)}(dy) = \nu^{(a,u)}_{\omega(a)}(E) = \nu^{(s,u)}_{\omega ax}(E)$$
$$\text{if } t < a < u,$$
$$= \int_{\Omega_t} \delta^{(u)}_{\omega(u)}(E)\delta^{(t)}_{\omega(t)}(dy) = \delta^{(u)}_{\omega(u)}(E) = \nu^{(s,u)}_{\omega ax}(E)$$
$$\text{if } u \le a.$$

(ii) Consider first the case $D = \{t\}$, where $t^* < t \le a$. Then

$$\mu_{\omega a}\{\omega' : \omega'(t) = \omega(t)\} = \nu^{(t^*,t)}_{\omega,a,\omega(t^*)}\{\omega(t)\} = \delta^{(t)}_{\omega(t)}\{\omega(t)\} = 1.$$

As for $D = \{t^*\}$, $\mu_{\omega a}$ starts at $\omega(t^*)$, so (as noted in the last clause of the statement of 455A) $\mu_{\omega a}\{\omega' : \omega'(t^*) = \omega(t^*)\} = 1$.

For general D, we have an intersection of countably many sets of these types, which will be $\mu_{\omega a}$-conegligible.

(iii) Looking at the definition, we see that $\nu^{(s,t)}_{\omega' ax} = \nu^{(s,t)}_{\omega ax}$ for all s, t and x, and of course $\omega'(t^*) = \omega(t^*)$, so $\mu_{\omega' a} = \mu_{\omega a}$.

455C Theorem Suppose that T, t^*, $\langle(\Omega_t, \mathrm{T}_t)\rangle_{t\in T}$, Ω, x^*, $\langle\nu^{(s,t)}_x\rangle_{s<t,x\in\Omega_s}$ and μ are as in 455A. Adjoin a point ∞ to T above any point of T, and let $\tau : \Omega \to T \cup \{\infty\}$ be a function taking countably many values and such that $\{\omega : \tau(\omega) \le s\}$ belongs to $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ and is determined by coordinates in $[t^*, s]$ for every $s \in T$.

(a) For $\omega \in \Omega$ define $\nu^{(s,t)}_{\omega,\tau(\omega),x}$, for $s < t$ and $x \in \Omega_s$, as in 455Bb, and let $\mu_{\omega,\tau(\omega)}$ be the corresponding measure on Ω. Then $\langle\mu_{\omega,\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of μ over itself.

(b) Let Σ_τ be the set of those $E \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ such that $E \cap \{\omega : \tau(\omega) \le t\}$ is determined by coordinates in $[t^*, t]$ for every $t \in T$. Then Σ_τ is a σ-subalgebra of $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$. If f is any μ-integrable real-valued function, and we set $g_f(\omega) = \int f d\mu_{\omega,\tau(\omega)}$ when this is defined in $\mathbb{R}$, then g_f is a conditional expectation of f on Σ_τ.

proof (a)(i) Set $F_t = \{\omega : \omega \in \Omega,\, \tau(\omega) = t\}$ for $t \in T \cup \{\infty\}$; note that $F_t \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ for every $t \in T \cup \{\infty\}$, and that F_t is determined by coordinates in $[t^*, t]$ for $t \in T$.

(ii) Consider first the case in which τ takes only finitely many values. Suppose that $J \subseteq T$ is a finite set including $\{t^*\} \cup (T \cap \tau[\Omega])$. Enumerate J as $\langle t_i\rangle_{i\le n}$. Suppose that $E_i \in \mathrm{T}_{t_i}$ for $i \le n$ and set $H^* = \{\omega : \omega(t_i) \in E_i$ for every $i \le n\}$. We need to calculate $\int_\Omega \mu_{\omega,\tau(\omega)}(H^*)\mu(d\omega)$.

Set $H = \prod_{i\le n} E_i$,

$$H_j = \prod_{j<i\le n} E_j, \quad H_j^* = \{\omega : \omega \in \Omega,\, \omega(t_i) \in E_i \text{ for } j < i \le n\},$$

$$G_j^* = \{\omega : \omega(t_i) \in E_i \text{ for } i \le j\}$$

for $j \le n$. If $i < n$, $j \le n$, $\omega \in F_{t_j}$ and $x \in \Omega_{t_i}$, then

$$\begin{aligned}\nu^{(t_i,t_{i+1})}_{\omega,\tau(\omega),x} &= \nu^{(t_i,t_{i+1})}_x \text{ if } i > j,\\ &= \nu^{(t_j,t_{j+1})}_{\omega(t_j)} \text{ if } i = j,\\ &= \delta^{(t_{i+1})}_{\omega(t_{j+1})} \text{ if } i < j.\end{aligned}$$

So if $j \le n$ and $\omega \in F_{t_j}$,

$$\begin{aligned}\mu_{\omega,\tau(\omega)}(H^*) &= \int \ldots \int \chi H(\omega(t^*), x_1, \ldots, x_n)\nu^{(t_{n-1},t_n)}_{\omega,\tau(\omega),x_{n-1}}(dx_n)\ldots\nu^{(t^*,t_1)}_{\omega,\tau(\omega),\omega(t^*)}(dx_1)\\ &= \iint \ldots \int \chi H(x_0, \ldots, x_n)\nu^{(t_{n-1},t_n)}_{\omega,\tau(\omega),x_{n-1}}(dx_n)\ldots\nu^{(t^*,t_1)}_{\omega,\tau(\omega),x_0}(dx_1)\delta^{(t^*)}_{\omega(t^*)}(dx_0)\\ &= \iint \ldots \iiint \ldots \int \chi H(x_0, \ldots, x_n)\nu^{(t_{n-1},t_n)}_{\omega,\tau(\omega),x_{n-1}}(dx_n)\\ &\qquad \ldots\nu^{(t_{j+1},t_{j+2})}_{\omega,\tau(\omega),x_{j+1}}(dx_{j+2})\nu^{(t_j,t_{j+1})}_{\omega,\tau(\omega),x_j}(dx_{j+1})\nu^{(t_{j-1},t_j)}_{\omega,\tau(\omega),x_{j-1}}(dx_j)\\ &\qquad \ldots\delta^{(t_1)}_{\omega(t_1)}(dx_1)\delta^{(t^*)}_{\omega(t^*)}(dx_0)\\ &= \int \ldots \iiint \ldots \int \chi H(x_0, \ldots, x_n)\nu^{(t_{n-1},t_n)}_{x_{n-1}}(dx_n)\\ &\qquad \ldots\nu^{(t_j,t_{j+1})}_{x_j}(dx_{j+1})\nu^{(t_j,t_{j+1})}_{\omega(t_j)}(dx_{j+1})\delta^{(t_j)}_{\omega(t_j)}(dx_j)\ldots\delta^{(t^*)}_{\omega(t^*)}(dx_0)\\ &= \int \ldots \int \chi H(\omega(t^*), \ldots, \omega(t_j), x_{j+1}, \ldots, x_n)\nu^{(t_{n-1},t_n)}_{x_{n-1}}(dx_n)\\ &\qquad \ldots\nu^{(t_j,t_{j+1})}_{\omega(t_j)}(dx_{j+1})\\ &= \int \ldots \int \chi H_j(x_{j+1}, \ldots, x_n)\nu^{(t_{n-1},t_n)}_{x_{n-1}}(dx_n)\ldots\nu^{(t_j,t_{j+1})}_{\omega(t_j)}(dx_{j+1})\\ &\qquad\qquad \text{if } \omega \in G_j^*,\\ &= 0 \text{ otherwise.}\end{aligned}$$

As noted in 455B(b-ii), $\mu_{\omega,\tau(\omega)}(H^*) = \chi H^*(\omega)$ if $\tau(\omega) = \infty$.

Now

$$\int_\Omega \mu_{\omega,\tau(\omega)}(H^*)\mu(d\omega) = \sum_{j=0}^n \int_{F_{t_j}} \mu_{\omega,\tau(\omega)}(H^*)\mu(d\omega) + \int_{F_\infty} \mu_{\omega,\tau(\omega)}(H^*)\mu(d\omega)$$

$$= \sum_{j=0}^n \int_{F_{t_j}\cap G_j^*} \int \ldots \int \chi H_j(x_{j+1},\ldots,x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)$$

$$\ldots \nu_{\omega(t_j)}^{(t_j,t_{j+1})}(dx_{j+1})\mu(d\omega) + \mu(F_\infty \cap H^*)$$

$$= \sum_{j=0}^n \mu(F_{t_j} \cap G_j^* \cap H_j^*) + \mu(H^* \cap F_\infty)$$

(by 455Ba)

$$= \sum_{j=0}^n \mu(F_{t_j} \cap H^*) + \mu(F_\infty \cap H^*) = \mu H^*.$$

Thus we have the formula we need when E is of the special form $\{\omega : \omega(t) \in E_t$ for every $t \in J\}$, $J \subseteq T$ being a finite set and E_t being a member of T_t for every $t \in J$. By the Monotone Class Theorem (136B), we shall have $\int \mu_{\omega,\tau(\omega)}(E)\mu(d\omega) = \mu E$ for every $E \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$, so that $\langle \mu_{\omega,\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of μ over itself.

(iii) If τ takes infinitely many values, enumerate them as $\langle t_n\rangle_{n\in\mathbb{N}}$, and for $n \in \mathbb{N}$ define $\tau_n : T \to T \cup \{\infty\}$ by setting

$$\tau_n(\omega) = t_i \text{ if } i \le n \text{ and } \tau(\omega) = t_i,$$
$$= \infty \text{ if } \tau(\omega) \notin \{t_i : i \le n\}.$$

Then τ_n takes only finitely many values, and $\{\omega : \tau_n(\omega) \le t\} \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ is determined by coordinates in $[0,t]$ for every $t \in T$. So we shall have

$$\int \mu_{\omega,\tau_n(\omega)}(E)\mu(d\omega) = \mu E$$

for every $E \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$. Now observe that $\mu_{\omega,\tau_n(\omega)} = \mu_{\omega,\tau(\omega)}$ whenever $\tau(\omega) = \tau_n(\omega)$. So, for each ω, $\mu_{\omega,\tau_n(\omega)} = \mu_{\omega,\tau(\omega)}$ for all but finitely many n. This means that, for every $E \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$,

$$\mu_{\omega,\tau(\omega)}(E) = \lim_{n\to\infty} \mu_{\omega,\tau_n(\omega)}(E)$$

for every $\omega \in \Omega$, and

$$\int \mu_{\omega,\tau(\omega)}(E)\mu(d\omega) = \lim_{n\to\infty} \int \mu_{\omega,\tau_n(\omega)}(E)\mu(d\omega) = \mu E,$$

as required.

(b)(i) Since $\{\omega : \tau(\omega) \le t\}$ is determined by coordinates in $[t^*, t]$ for every $t \in T$, $\Omega \in \Sigma_\tau$, and it is now elementary to confirm that Σ_τ is a σ-algebra.

(ii) I had better note that g_f is defined almost everywhere; this is because, by (a) above and 452F,

$$\int g_f d\mu = \iint f d\mu_{\omega,\tau(\omega)}\mu(d\omega) = \int f d\mu.$$

(iii) If ω, $\omega' \in \Omega$ and $\omega'\restriction[t^*,\tau(\omega)] = \omega\restriction[t^*,\tau(\omega)]$, then $g_f(\omega) = g_f(\omega')$ if either is defined. **P** Since $F_{\tau(\omega)}$ is determined by coordinates in $[t^*,\tau(\omega)]$, $\tau(\omega') = \tau(\omega)$. By 455B(b-ii), $\mu_{\omega',\tau(\omega')} = \mu_{\omega,\tau(\omega)}$, so $g_f(\omega) = g_f(\omega')$ if either is defined. **Q**

(iv) If $F \in \Sigma_\tau$ and $\omega \in \Omega$, then $\mu_{\omega,\tau(\omega)}F = 1$ if $\omega \in F$, 0 otherwise. **P** Setting $b = \tau(\omega)$, $F \cap F_b$ and $F_b \setminus F$ are determined by coordinates in a countable subset of $T \cap [t^*, b]$, so by 455B(b-ii) we have $\mu_{\omega b}F = 1$ if $\omega \in F_b \cap F$ and $\mu_{\omega b}(F_b \setminus F) = 1$ if $\omega \in F_b \setminus F$. **Q**

It follows that if f is μ-integrable and $F \in \Sigma_\tau$, then $g_{f\times\chi F} = g_f \times \chi F$. **P** If $\omega \in F$, then $\mu_{\omega,\tau(\omega)}F = 1$ and

$$g_{f\times\chi F}(\omega) = \int_F f d\mu_{\omega,\tau(\omega)} = \int f d\mu_{\omega,\tau(\omega)};$$

if $\omega \notin F$ then $\mu_{\omega,\tau(\omega)}F = 0$ and

$$g_{f\times\chi F}(\omega)=\int_F f d\mu_{\omega,\tau(\omega)}=0.\ \mathbf{Q}$$

(v) Now let f be any μ-integrable real-valued function. Then there is a Σ_τ-measurable function $g'_f:\Omega\to\,]-\infty,\infty]$ such that $g'_f=_{\text{a.e.}}g_f$, $g'_f(\omega)\le g_f(\omega)$ for every $\omega\in\operatorname{dom}g_f$, and $g'_f(\omega)=-\infty$ for every $\omega\in\Omega\setminus\operatorname{dom}g_f$. **P** For $q\in\mathbb{Q}$, set $W_q=\{\omega:\omega\in\operatorname{dom}g_f,\ g_f(\omega)\ge q\}$. For $q\in\mathbb{Q}$ and $b\in\tau[\Omega]$, consider $W_{bq}=W_q\cap F_b$. W_{bq} is measured by the completion $\hat\mu$ of μ, and is determined by coordinates in $T\cap[t^*,b]$, by (iii). By 451K(b-ii) there is a $W'_{bq}\in\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ such that $W'_{bq}\subseteq W_{bq}$, $W_{bq}\setminus W'_{bq}$ is negligible and W'_{bq} is determined by coordinates in $T\cap[t^*,b]$.

Having defined the family $\langle W'_{bq}\rangle_{b\in\tau[\Omega],q\in\mathbb{Q}}$, set $W'_q=\bigcup_{b\in\tau[\Omega]}W'_{bq}$ for $q\in\mathbb{Q}$. Then $W'_q\in\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ and $W'_q\cap F_b=W'_{bq}$ is determined by coordinates in $T\cap[t^*,b]$ for every $b\in\tau[\Omega]$, so $W'_q\in\Sigma_\tau$. Also $W'_q\subseteq W_q$ and $W_q\setminus W'_q$ is negligible.

Set

$$g'_f(\omega)=\sup\{q:q\in\mathbb{Q},\ \omega\in W'_q\}$$

for $\omega\in\Omega$, counting $\sup\emptyset$ as $-\infty$. Then g'_f is Σ_τ-measurable, $g'_f(\omega)=-\infty$ for $\omega\notin\operatorname{dom}g_f$, $g'_f(\omega)\le g_f(\omega)$ for $\omega\in\operatorname{dom}g_f$, and $g'_f=g_f$ on $\operatorname{dom}g_f\setminus\bigcup_{q\in\mathbb{Q}}W_q\setminus W'_q$, so $g'_f=_{\text{a.e.}}g_f$. **Q**

(vi) Continuing from (v), we find that g'_f is a conditional expectation of f on Σ_τ.[17] **P** I have already shown that g'_f is Σ_τ-measurable. If $F\in\Sigma_\tau$ then

$$\int_F g'_f d\mu=\int g'_f\times\chi F\,d\mu=\int g_f\times\chi F\,d\mu$$

(because $g_f=_{\text{a.e.}}g'_f$)

$$=\int g_{f\times\chi F}d\mu$$

(by (iv))

$$=\iint f\times\chi F\,d\mu_{\omega,\tau(\omega)}\mu(d\omega)=\int f\times\chi F\,d\mu$$

(452F once again)

$$=\int_F f d\mu.\ \mathbf{Q}$$

(vii) Similarly, or applying the arguments of (v)-(vi) to $-f$, we see that for any μ-integrable function f there is a conditional expectation g''_f of f on Σ_τ such that $g''_f(\omega)\ge g_f(\omega)$ when $\omega\in\operatorname{dom}g_f$ and $g''_f(\omega)=\infty$ when $g_f(\omega)$ is undefined. Now $g'_f=_{\text{a.e.}}g''_f$ and both are Σ_τ-measurable. It follows that g_f is defined, and equal to both g'_f and g''_f, $(\mu\restriction\Sigma_\tau)$-a.e.; so that g_f itself is also a conditional expectation of f on Σ_τ.

455D Remarks (a) The idea of the construction in 455A is that $\langle X_t\rangle_{t\in T}$ is a family of random variables, and that we start from the assurance that 'history is irrelevant'; if, at time b, we wish to make guesses about the behaviour of X_t, the state of the system at a future time t, then we expect that it will be useful to look at the current state X_b, but once we know the value of X_b then any further information about X_s for $s<b$ will tell us nothing more about X_t. We are given the **transitional probabilities** $\nu_x^{(s,t)}$, which can be thought of as the conditional distributions of X_t given that $X_s=x$. The condition (†) of 455A is plainly necessary if the system is going to make sense at all; the content of the theorem is that it is also sufficient, at least when all the conditional expectations are perfect measures, to ensure that the system as a whole can indeed be represented as a family of random variables, in Kolmogorov's sense, on a suitable probability space.

(b) The statement '$\langle\mu_{\omega,\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of μ over itself' in 455Ca is not obviously a target worth working very hard for. But the point of this particular family is that not only does $\mu_{\omega,\tau(\omega)}$ follow ω up to and including time $\tau(\omega)$ (455B(b-ii)), but also $\mu_{\omega,\tau(\omega)}=\mu_{\omega',\tau(\omega')}$ whenever $\omega'\restriction[t^*,\tau(\omega)]=\omega\restriction[t^*,\tau(\omega)]$, as noted in (b-iii) of the proof of 455B.

[17] The definition of 'conditional expectation' in 233D was directed towards real-valued functions, and g'_f is permitted to take the values $\pm\infty$. So what I really mean here is that the restriction of g'_f to the set on which it is finite is a conditional expectation of f.

If we take τ in 455C to be constant, with value $b \in T$, then we get a precise description of what it means for 'history to be irrelevant'. In this case, we can take the measures $\mu_{\omega b}$, and project them onto $\prod_{t \geq b} \Omega_t$; let $\lambda^{(\omega)}_{[b,\infty[}$ be the image measure. Then it is easy to check that $\lambda^{(\omega)}_{[b,\infty[}$ is the measure defined from the point $\omega(b)$ and the family $\langle \nu^{(s,t)}_x \rangle_{b \leq s < t, x \in \Omega_s}$ by the method of 455A; so that $\lambda^{(\omega)}_{[b,\infty[} = \lambda^{(\omega')}_{[b,\infty[}$ whenever $\omega(b) = \omega'(b)$.

(c) I have called 455C a 'theorem', and there are certainly enough ideas in it to warrant the title. But the restriction to stopping times taking only countably many values means that we are a large step away from a result which is really useful in continuous time. The calculations with sets $\{\omega : \tau(\omega) = b\}$ in the proofs of 455C and 455E are a clear sign that we are not yet ready for continuous stopping times, in which $\{\omega : \tau(\omega) = b\}$ will usually be negligible for every b, except perhaps $b = \infty$. Of course we can use 455C with $T = \mathbb{N}$; but it must be obvious that there are better and cleaner expressions of the result in this case. In the work below, 455C is going to function as a lemma, the first stage in much stronger results (starting with 455O) which depend on special properties of the measures $\nu^{(s,t)}_x$.

(d) In the context of 455A, it seemed to involve fewer explanations to take a fixed σ-algebra T_t for each t and to define μ on $\widehat{\bigotimes}_{t \in T} \mathrm{T}_t$. As you know, I ordinarily have a strong prejudice in favour of completing measures. In the situations most important to us, this is perfectly straightforward, if a touch laborious; I present a version in the next theorem.

455E Theorem Let T be a totally ordered set with least element t^*. Let $\langle \Omega_t \rangle_{t \in T}$ be a family of Hausdorff spaces; suppose that we are given an $x^* \in \Omega_{t^*}$ and, for each pair $s < t$ in T, a family $\langle \nu^{(s,t)}_x \rangle_{x \in \Omega_s}$ of Radon probability measures on Ω_t such that

$\langle \nu^{(t,u)}_y \rangle_{y \in \Omega_s}$ is a disintegration of $\nu^{(s,u)}_x$ over $\nu^{(s,t)}_x$ whenever $s < t < u$ in T and $x \in \Omega_s$.

Write $\Omega = \prod_{t \in T} \Omega_t$; for $t \in T$ let $\mathcal{B}(\Omega_t)$ be the Borel σ-algebra of Ω_t, and $X_t : \Omega \to \Omega_t$ the canonical map; for $J \subseteq T$ write π_J for the canonical map from Ω onto $\prod_{t \in J} \Omega_t$. For $t \in T$ and $x \in \Omega_t$ let $\delta^{(t)}_x$ be the Dirac measure on Ω_t concentrated at x.

(a) There is a unique complete probability measure $\hat{\mu}$ on Ω, inner regular with respect to $\widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t)$, such that, writing $\hat{\lambda}_J$ for the image measure $\hat{\mu}\pi_J^{-1}$,

$$\begin{aligned} \int f d\hat{\lambda}_J &= \int f(\omega(t^*), \omega(t_1), \dots, \omega(t_n)) \hat{\mu}(d\omega) \\ &= \int \dots \iint f(x^*, x_1, \dots, x_n) \nu^{(t_{n-1}, t_n)}_{x_{n-1}}(dx_n) \\ &\qquad\qquad \nu^{(t_{n-2}, t_{n-1})}_{x_{n-2}}(dx_{n-1}) \dots \nu^{(t^*, t_1)}_{x^*}(dx_1) \end{aligned}$$

whenever $t^* < t_1 < \dots < t_n$ in T, $J = \{t^*, t_1, \dots, t_n\}$ and f is $\hat{\lambda}_J$-integrable. In particular, the image measure $\hat{\mu}X_t^{-1}$ is equal to $\nu^{(t^*,t)}_{x^*}$ if $t > t^*$, and to $\delta^{(t^*)}_{x^*}$ if $t = t^*$.

(b)(i) For $\omega \in \Omega$ and $a \in T \cup \{\infty\}$ define $\langle \nu^{(s,t)}_{\omega a x} \rangle_{s < t, x \in \Omega_s}$ by setting

$$\begin{aligned} \nu^{(s,t)}_{\omega a x} &= \nu^{(s,t)}_x \text{ if } a < s, \\ &= \nu^{(a,t)}_{\omega(a)} \text{ if } s \leq a < t, \\ &= \delta^{(t)}_{\omega(t)} \text{ if } t \leq a. \end{aligned}$$

The family $\langle \nu^{(s,t)}_{\omega a x} \rangle_{s < t, x \in \Omega_s}$, together with the point $\omega(t^*) \in \Omega_{t^*}$, satisfy the conditions of (a), so can be used to define a complete measure $\hat{\mu}_{\omega a}$ on Ω.

(ii) If $\omega \in \Omega$ and $D \subseteq T \cap [t^*, a]$ is countable, then $\hat{\mu}_{\omega a}\{\omega' : \omega' \restriction D = \omega \restriction D\} = 1$.

(iii) If ω, $\omega' \in \Omega$ and $\omega' \restriction [t^*, a] = \omega \restriction [t^*, a]$, then $\hat{\mu}_{\omega' a} = \hat{\mu}_{\omega a}$.

(c) Let Σ be the domain of $\hat{\mu}$. Suppose that $\tau : \Omega \to T \cup \{\infty\}$ is a function taking countably many values and such that $\{\omega : \tau(\omega) \leq t\}$ belongs to Σ and is determined by coordinates in $[t^*, t]$ for every $t \in T$.

(i) $\langle \hat{\mu}_{\omega, \tau(\omega)} \rangle_{\omega \in \Omega}$ is a disintegration of $\hat{\mu}$ over itself.

(ii) Let Σ_τ be the set

$$\{E : E \in \Sigma,\ E \cap \{\omega : \tau(\omega) \leq t\} \text{ is determined by coordinates in } [t^*, t] \text{ for every } t \in T\}.$$

Then Σ_τ is a σ-subalgebra of Σ. If f is any $\hat{\mu}$-integrable real-valued function, and we set $g_f(\omega) = \int f d\hat{\mu}_{\omega,\tau(\omega)}$ when this is defined in $\mathbb{R}$, then g_f is a conditional expectation of f on Σ_τ.

proof My aim is to apply 455A-455C to the Borel measures $\grave{\nu}_x^{(s,t)} = \nu_x^{(s,t)} \restriction \mathcal{B}(\Omega_t)$, and take $\hat{\mu}$ to be the completion of the Baire measure μ produced by the method of 455A. The essential discipline is to check carefully that almost every measure ζ is the completion of an appropriate measure $\grave{\zeta}$.

(a) At the start, every Radon probability measure is the completion of the corresponding Borel measure, so that the $\nu_x^{(s,t)}$ are indeed the completions of the $\grave{\nu}_x^{(s,t)}$ defined from them. Since completing a measure does not affect the associated integration (212Fb), the condition

whenever $s < t < u$ in T, $x \in \Omega_s$ and $E \subseteq \Omega_u$ is a Borel set, then $\grave{\nu}_s^{(s,u)}(E) = \int \grave{\nu}_y^{(t,u)}(E) \grave{\nu}_x^{(s,t)}(dy)$

follows at once from

whenever $s < t < u$ in T and $x \in \Omega_s$, then $\langle \nu_y^{(t,u)} \rangle_{y \in \Omega_s}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$.

Also the $\grave{\nu}_x^{(s,t)}$, being tight Borel measures, are all perfect (342L/451C). So we can indeed form a measure μ on Ω with domain $\widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t)$ by the process in 455A, and complete it.

The next step has a little more content in it: I need to show that for any $J \subseteq T$, the image measure $\hat{\mu}\pi_J^{-1}$ on $\prod_{t \in J} \Omega_t$ is the completion of the image measure $\mu\pi_J^{-1}$. But here we just have to recall that μ is perfect (454D), so that we can use 451Kb. For finite $J \subseteq T$ we can therefore write $\hat{\lambda}_J$ indifferently for the completion of $\lambda_J = \mu\pi_J^{-1}$ and for $\hat{\mu}\pi_J^{-1}$, and the formula for $\int f d\hat{\lambda}_J$ can be read off from 455A, since it deals only with integrals, which are unaffected by completions.

(b) This follows 455Bb. This time we must start by noting that every $\nu_{\omega a x}^{(s,t)}$ is a Radon probability measure.

(i) The formulae of part (i) of the proof of 455Bb can still be applied to show that

$$\int_{\Omega_t} \nu_{\omega a y}^{(t,u)}(E) \nu_{\omega a x}^{(s,t)}(dy) = \nu_{\omega a x}^{(s,u)}(E)$$

whenever $s < t < u$, $x \in \Omega_s$ and $E \in \mathcal{B}(\Omega_u)$. Since any set measured by $\nu_{\omega a x}^{(s,u)}$ can be approximated internally and externally by Borel sets, we see that $\langle \nu_{\omega a y}^{(t,u)} \rangle_{y \in \Omega_t}$ is a disintegration of $\nu_{\omega a x}^{(s,u)}$ over $\nu_{\omega a x}^{(s,t)}$. (Cf. 452Xg.)

(ii) Similarly, the argument of part (ii) of the proof of 455Bb can still be used to show that whenever $\omega \in \Omega$ and $D \subseteq T \cap [t^*, a]$ is countable, then $\omega' \restriction D = \omega \restriction D$ for $\hat{\mu}_{\omega a}$-almost every $\omega' \in \Omega$.

(iii) Once again, we can use the argument from 455B; if $\omega' \restriction [t^*, a] = \omega \restriction [t^*, a]$, then $\nu_{\omega' a x}^{(s,t)} = \nu_{\omega a x}^{(s,t)}$ for all x, s and t, and $\hat{\mu}_{\omega' a} = \hat{\mu}_{\omega a}$.

(c)(i)(α) The key step here is to observe that there is a function $\grave{\tau} : \Omega \to T \cup \{\infty\}$ which satisfies the properties required in 455C and is equal $\hat{\mu}$-almost everywhere to τ. **P** For each $a \in T \cap \tau[\Omega]$, $F_a = \tau^{-1}[\{a\}]$ belongs to Σ and is determined by coordinates in $[t^*, a]$. By 451K(b-ii) again, there is an $F'_a \in \widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t)$ such that $F'_a \subseteq F_a$, F'_a is determined by coordinates in $[t^*, a]$ and $\hat{\mu}(F_a \setminus F'_a) = 0$. Define $\grave{\tau}$ by setting

$$\begin{aligned} \grave{\tau}(\omega) &= a \text{ if } a \in T \cap \tau[\Omega] \text{ and } \omega \in F'_a, \\ &= \infty \text{ if } \omega \in \Omega \setminus \bigcup_{a \in T \cap \tau[\Omega]} F'_a. \end{aligned}$$

It is easy to check that this $\grave{\tau}$ will serve. **Q**

(β) For $\omega \in \Omega$ and $a \in T$, define $\langle \grave{\nu}_{\omega a x}^{(s,t)} \rangle_{s<t, x \in \Omega_t}$ and $\langle \mu_{\omega a} \rangle_{\omega \in \Omega}$ from $\langle \grave{\nu}_x^{(s,t)} \rangle_{s<t, x \in \Omega_t}$ and $\grave{\tau}$ as in 455Bb. If $\tau(\omega) = \grave{\tau}(\omega)$ then $\grave{\nu}_{\omega,\grave{\tau}(\omega),x}^{(s,t)} = \nu_{\omega,\tau(\omega),x}^{(s,t)} \restriction \mathcal{B}(\Omega_t)$ for all s, t and x, so that $\hat{\mu}_{\omega,\tau(\omega)}$ is the completion of $\mu_{\omega,\grave{\tau}(\omega)}$. This is true for almost all ω. Now we know from 455Ca that $\langle \mu_{\omega,\grave{\tau}(\omega)} \rangle_{\omega \in \Omega}$ is a disintegration of μ over itself, and therefore also over $\hat{\mu}$. It follows that $\langle \hat{\mu}_{\omega,\grave{\tau}(\omega)} \rangle_{\omega \in \Omega}$ is a disintegration of $\hat{\mu}$ over $\hat{\mu}$, by 452B(ii). But $\hat{\mu}_{\omega,\grave{\tau}(\omega)} = \hat{\mu}_{\omega,\tau(\omega)}$ for $\hat{\mu}$-almost every ω, so $\langle \hat{\mu}_{\omega,\tau(\omega)} \rangle_{\omega \in \Omega}$ also is a disintegration of $\hat{\mu}$ over itself.

(ii)(α) Just as in part (b-i) of the proof of 455C, Σ_τ is a σ-algebra because it contains Ω.

(β) Recall the F_a, F'_a in (i-α) above. Set $F_\infty = \tau^{-1}[\{\infty\}]$, and take $F'_\infty \in \widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t)$ such that $F'_\infty \subseteq F_\infty$ and $F_\infty \setminus F'_\infty$ is negligible. Then $F^* = \bigcup_{a \in \tau[\Omega]} F'_a$ is conegligible in Ω. Write $\grave{\Sigma}_{\grave{\tau}}$ for the set of those $F \in \widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t)$ such that $F \cap F'_a$ is determined by coordinates in $[t^*, a]$ for every $a \in T \cap \grave{\tau}[\Omega]$. Then $F^* \in \grave{\Sigma}_{\grave{\tau}} \cap \Sigma_\tau$ because

$F^* \cap F'_a = F^* \cap F_a = F'_a$ for every $a \in \tau[\Omega]$. In fact we have more. First, $\tau \restriction F^* = \grave{\tau} \restriction F^*$. Next, if $F \subseteq F^*$ and $F \in \grave{\Sigma}_{\grave{\tau}}$, then $F \in \Sigma_\tau$. **P** For any $a \in T \cap \tau[\Omega]$, $F \cap F_a = F \cap F'_a$ belongs to $\widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t) \subseteq \Sigma$ and is determined by coordinates in $[t^*, a]$. **Q** And thirdly, if $F \in \Sigma_\tau$, there is a $G \in \grave{\Sigma}_{\grave{\tau}}$ such that $G \subseteq F$ and $F \setminus G$ is negligible. **P** As in (iv-α), we can find for each $a \in \tau[\Omega]$ a set $G_a \in \widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t)$, determined by coordinates in $T \cap [t^*, a]$, such that $G_a \subseteq F \cap F_a$ and $(F \cap F_a) \setminus G_a$ is negligible. Set $G = \bigcup_{a \in \tau[\Omega]} G_a$. **Q**

(γ) Now take a $\hat{\mu}$-integrable function f. Then it is μ-integrable. By 455Cb, $\grave{g}_f$ is a conditional expectation of f on $\grave{\Sigma}_{\grave{\tau}}$, where

$$\grave{g}_f(\omega) = \int f d\mu_{\omega, \grave{\tau}(\omega)} = \int f d\hat{\mu}_{\omega, \grave{\tau}(\omega)}$$

whenever the integral is defined in $\mathbb{R}$. We know that there is a $\grave{\Sigma}_{\grave{\tau}}$-measurable function $g' : \Omega \to \mathbb{R}$ equal to $\grave{g}_f$ except perhaps on a negligible set H belonging to $\grave{\Sigma}_{\grave{\tau}}$. Replacing g' by $g' \times \chi F^*$ and H by $H \cup (\Omega \setminus F^*)$ if necessary, we can suppose that g' is zero outside F^* and that $\Omega \setminus H \subseteq F^*$. In this case, g' is Σ_τ-measurable. **P** For any $\alpha \in \mathbb{R}$,

$$\begin{aligned}\{\omega : g'(\omega) \geq \alpha\} &= \{\omega : \omega \in F^*,\, g'(\omega) \geq \alpha\} \cup (\Omega \setminus F^*) \text{ if } \alpha \leq 0,\\ &= \{\omega : \omega \in F^*,\, g'(\omega) \geq \alpha\} \text{ if } \alpha > 0,\end{aligned}$$

and in either case belongs to Σ_τ, by (β). **Q** At the same time, we note that $H \in \Sigma_\tau$.

If $\omega \in \Omega \setminus H$, then $\omega \in F^*$, $\tau(\omega) = \grave{\tau}(\omega)$, $\hat{\mu}_{\omega, \tau(\omega)} = \hat{\mu}_{\omega, \grave{\tau}(\omega)}$ and

$$g_f(\omega) = \int f d\hat{\mu}_{\omega, \tau(\omega)} = \int f d\hat{\mu}_{\omega, \grave{\tau}(\omega)} = \int f d\mu_{\omega, \grave{\tau}(\omega)} = \grave{g}_f(\omega) = g'(\omega).$$

So g_f is defined and equal to g' and $\grave{g}_f$ except perhaps on the negligible set H belonging to Σ_τ; consequently g_f is defined $(\hat{\mu} \restriction \Sigma_\tau)$-a.e. and is $(\hat{\mu} \restriction \Sigma_\tau)$-virtually measurable.

If $F \in \Sigma_\tau$, there is a $G \in \grave{\Sigma}_{\grave{\tau}}$ such that $G \subseteq F$ and $F \setminus G$ is negligible, by the last remark in (β). So

$$\int_F f d\hat{\mu} = \int_G f d\mu = \int_G \grave{g}_f d\mu$$

(because $\grave{g}_f$ is a conditional expectation of f on $\grave{\Sigma}_{\grave{\tau}}$)

$$= \int_F \grave{g}_f d\hat{\mu} = \int_F g_f d\hat{\mu} = \int_F g_f d(\hat{\mu} \restriction \Sigma_\tau).$$

As F is arbitrary, g_f is a conditional expectation of f on Σ_τ, and the proof is complete.

455F Of course the leading example for the work above is the case in which $T = [0, \infty[$ and $\Omega_t = \mathbb{R}$ for every $t \geq 0$. Moving towards this, a natural intermediate stage is when $T = [0, \infty[$ and all the Ω_t are the same, so that we can regard an element of $\prod_{t \in T} \Omega_t$ as the path of a moving point. In this case we can begin to think about paths which are more or less continuous. The next theorem gives a widely applicable condition for existence of many paths which are one-sidedly continuous. It depends on a fairly strong continuity property for the transitional probabilities.

Definitions (a) Let U be a Hausdorff space and $\langle \nu_x^{(s,t)} \rangle_{0 \leq s < t, x \in U}$ a family of Radon probability measures on U. I will say that $\langle \nu_x^{(s,t)} \rangle_{0 \leq s < t, x \in U}$ is **narrowly continuous** if it is continuous, as a function from $\{(s,t) : 0 \leq s < t\} \times U$ to the set of Radon probability measures on U, when the latter is given its narrow topology (437Jd).

Remark I speak of the 'narrow' topology here partly because, in the present treatise, this has become the standard topology on spaces of Radon measures, and partly because the phrase 'vaguely continuous' seems inappropriate. But, as will appear, all the results below will rely on the fact that the vague topology (437Jc) is coarser than the narrow topology. In the present context, in which we have Radon measures on a completely regular Hausdorff space, the two topologies actually coincide (437L). So $\langle \nu_x^{(s,t)} \rangle_{0 \leq s < t, x \in U}$ is narrowly continuous iff $(s,t,x) \mapsto \int f d\nu_x^{(s,t)}$ is continuous for every bounded continuous $f : \Omega \to \mathbb{R}$.

(b) Let (U, ρ) be a metric space, and $\langle \nu_x^{(s,t)} \rangle_{0 \leq s < t, x \in U}$ a family of Radon probability measures on U. I will say that $\langle \nu_x^{(s,t)} \rangle_{0 \leq s < t, x \in U}$ is **uniformly time-continuous on the right** if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\nu_x^{(s,t)} B(x, \epsilon) \geq 1 - \epsilon$ whenever $x \in U$ and $0 \leq s < t \leq s + \delta$.

455G Theorem Let (U,ρ) be a complete metric space and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0\le s<t<u$ and $x\in U$. Take a point $\tilde\omega$ in $\Omega=U^{[0,\infty[}$, and $a\in[0,\infty]$. Let $\hat\mu_{\tilde\omega a}$ be the completed probability measure on Ω defined from $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$, $\tilde\omega$ and a as in 455Eb.

(a) For $\hat\mu_{\tilde\omega a}$-almost every $\omega\in\Omega$, $\lim_{q\in\mathbb{Q},q\downarrow t}\omega(q)$ and $\lim_{q\in\mathbb{Q},q\uparrow t}\omega(q)$ are defined in U for every $t>a$.

(b)(i) If $a\le t<\infty$, then $\omega(t)=\lim_{q\in\mathbb{Q},q\downarrow t}\omega(q)$ for $\hat\mu_{\tilde\omega a}$-almost every $\omega\in\Omega$.

(ii) If $a<t<\infty$, then $\omega(t)=\lim_{q\in\mathbb{Q},q\uparrow t}\omega(q)$ for $\hat\mu_{\tilde\omega a}$-almost every $\omega\in\Omega$.

(c)(i) Let $C^{\Downarrow}$ be the set of càllàl functions from $[0,\infty[$ to U (438S). If $\tilde\omega\in C^{\Downarrow}$, $C^{\Downarrow}$ has full outer measure for $\hat\mu_{\tilde\omega a}$.

(ii) Let C_{dlg} be the set of càdlàg functions from $[0,\infty[$ to U. If $\tilde\omega\in C_{\mathrm{dlg}}$, C_{dlg} has full outer measure for $\hat\mu_{\tilde\omega a}$.

Remark In this result and the ones to follow, I have not spelt out separately what it means if $a=0$; but of course this is the case in which we are starting the process at time $t^*=0$ and value $x^*=\tilde\omega(0)$, just as in the original construction 455A.

proof (a) Of course we can assume in this part of the proof that a is finite.

(i) Suppose that $\eta\in\,]0,1[$ and ϵ, $\delta>0$ are such that $\nu_x^{(s,t)}B(x,\epsilon)\ge1-\eta$ whenever $x\in U$ and $0\le s<t\le s+\delta$. Then

$$\hat\mu_{\tilde\omega a}\{\omega:\omega\in\Omega,\ \operatorname{diam}\omega[D]\le4\epsilon\}\ge\tfrac{1-2\eta}{1-\eta}$$

whenever $D\subseteq[a,\infty[$ is a countable set of diameter at most δ.

P $(\boldsymbol{\alpha})$ For finite D, I seek to induce on $\#(D)$. If $\#(D)\le1$ then of course $\operatorname{diam}\omega[D]\le4\epsilon$ for every ω and we can stop. So suppose that $D=\{t_0,\dots,t_n\}$ where $n\ge1$ and $a\le t_0<\dots<t_n$. To begin with, I go through the formulae when $t_0>0$.

For $k\le n$ set

$$E_k=\{\omega:\rho(\omega(t_k),\omega(t_0))>2\epsilon,\ \rho(\omega(t_i),\omega(t_0))\le2\epsilon\text{ for }i<k\},$$

$$F_k=\{\omega:\omega\in E_k,\ \rho(\omega(t_n),\omega(t_k))\le\epsilon\},$$

$$G_k=\{(x_0,\dots,x_k):\rho(x_k,x_0)>2\epsilon,\ \rho(x_i,x_0)\le2\epsilon\text{ for }i<k\}\subseteq U^{k+1}.$$

If $1\le k<n$ then

$$\begin{aligned}\hat\mu_{\tilde\omega a}F_k=\lambda_{\{0,t_0,\dots,t_k,t_n\}}\{(x,x_0,\dots,x_k,x_n):\rho(x_i,x_0)\le2\epsilon\text{ for }i<k,\\ \rho(x_0,x_k)>2\epsilon,\ \rho(x_k,x_n)\le\epsilon\}\end{aligned}$$

(defining λ_J as the image measure of $\hat\mu_{\tilde\omega a}$ on U^J, as in 455E)

$$=\int\dots\int\chi G_k(x_0,\dots,x_k)\chi B(x_k,\epsilon)(x_n)\nu_{\tilde\omega a x_k}^{(t_k,t_n)}(dx_n)\dots\nu_{\tilde\omega(0)}^{(0,t_0)}(dx_0)$$

(were $\nu_{\tilde\omega a x}^{(s,t)}$ is defined as in 455Eb)

$$=\int\dots\int\chi G_k(x_0,\dots,x_k)\chi B(x_k,\epsilon)(x_n)\nu_{x_k}^{(t_k,t_n)}(dx_n)\dots\nu_{\tilde\omega(0)}^{(0,t_0)}(dx_0)$$

(because $a\le t_0<\dots<t_n$)

$$=\int\dots\int\chi G_k(x_0,\dots,x_k)\nu_{x_k}^{(t_k,t_n)}(B(x_k,\epsilon))\nu_{x_{k-1}}^{(t_{k-1},t_k)}(dx_k)\dots\nu_{\tilde\omega(0)}^{(0,t_0)}(dx_0)$$

$$\ge\int\dots\int(1-\eta)\chi G_k(x_0,\dots,x_k)\nu_{x_{k-1}}^{(t_{k-1},t_k)}(dx_k)\dots\nu_{\tilde\omega(0)}^{(0,t_0)}(dx_0)$$

(because $t_k<t_n\le t_k+\delta$, so $\nu_x^{(t_k,t_n)}B(x,\epsilon)\ge1-\eta$ for every x)

$$\begin{aligned}&=(1-\eta)\lambda_{\{0,t_0,\dots,t_k\}}\{(x,x_0,\dots,x_k,x_n):\rho(x_i,x_0)\le2\epsilon\text{ for }i<k,\\&\qquad\rho(x_0,x_k)>2\epsilon\}\\&=(1-\eta)\hat\mu_{\tilde\omega a}E_k.\end{aligned}$$

If $k=n$, then of course $F_k=E_k$, so again $\hat\mu_{\tilde\omega a}F_k\ge(1-\eta)\hat\mu_{\tilde\omega a}E_k$. Accordingly

$$\begin{aligned}(1-\eta)\sum_{k=1}^{n}\hat\mu_{\tilde\omega a}E_k\le\sum_{k=1}^{n}\hat\mu_{\tilde\omega a}F_k&\le\hat\mu_{\tilde\omega a}\{\omega:\rho(\omega(t_n),\omega(t_0))>\epsilon\}\\&=\lambda_{\{0,t_0,t_n\}}\{(x,x_0,x_n):\rho(x_0,x_n)>\epsilon\}\\&=\int\nu_{x_0}^{(t_0,t_n)}(U\setminus B(x_0,\epsilon))\nu_{\tilde\omega(0)}^{(0,t_0)}(dx_0)\le\eta\end{aligned}$$

because $t_n-t_0\le\delta$ so $\nu_x^{(t_0,t_n)}(U\setminus B(x,\epsilon))\le\eta$ for every x.

But now we have

$$\begin{aligned}\hat\mu_{\tilde\omega a}\{\omega:\omega\in\Omega,\ \operatorname{diam}\omega[D]\le 4\epsilon\}&\ge\hat\mu_{\tilde\omega a}\{\omega:\rho(\omega(t_k),\omega(t_0))\le 2\epsilon\text{ for }1\le k\le n\}\\&=\hat\mu_{\tilde\omega a}(\Omega\setminus\bigcup_{1\le k\le n}E_k)\ge 1-\tfrac{\eta}{1-\eta}=\tfrac{1-2\eta}{1-\eta}\end{aligned}$$

as required.

(β) If $t_0=a=0$ the formulae simplify slightly, but the ideas are the same. We have $\omega(0)=\tilde\omega(0)$ for $\hat\mu_{\tilde\omega 0}$-almost every ω, so

$$\begin{aligned}\hat\mu_{\tilde\omega 0}F_k&=\lambda_{0,t_1,\dots,t_k,t_n}\{(\tilde\omega(0),x_1,\dots,x_k,x_n):\rho(x_i,\tilde\omega(0))\le 2\epsilon\text{ for }i<k,\\&\qquad\qquad\rho(\tilde\omega(0),x_k)>2\epsilon,\ \rho(x_k,x_n)\le\epsilon\}\\&=\int\dots\int\chi G_k(\tilde\omega(0),x_1,\dots,x_k)\chi B(x_k,\epsilon)(x_n)\nu_{\tilde\omega a x_k}^{(t_k,t_n)}(dx_n)\dots\nu_{\tilde\omega(0)}^{(0,t_1)}(dx_1)\\&\ge(1-\eta)\int\dots\int\chi G_k(\tilde\omega(0),x_1,\dots,x_k)\nu_{\tilde\omega a x_{k-1}}^{(t_{k-1},t_k)}(dx_k)\dots\nu_{\tilde\omega(0)}^{(0,t_1)}(dx_1)\\&=(1-\eta)\hat\mu_{\tilde\omega 0}E_k\end{aligned}$$

for $1\le k<n$,

$$\begin{aligned}(1-\eta)\sum_{k=1}^{n}\hat\mu_{\tilde\omega 0}E_k&\le\lambda_{\{0,t_n\}}\{(\tilde\omega(0),x_n):\rho(\tilde\omega(0),x_n)>\epsilon\}\\&=\int\nu_{\tilde\omega(0)}^{(t_0,t_n)}(U\setminus B(\tilde\omega(0),\epsilon))\le\eta,\end{aligned}$$

and the final calculation is unchanged.

(γ) For countably infinite D, let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of finite sets with union D; then $\langle\{\omega:\omega\in\Omega,\ \operatorname{diam}\omega[I_n]\le 4\epsilon\}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with intersection $\{\omega:\omega\in\Omega,\ \operatorname{diam}\omega[D]\le 4\epsilon\}$, so the measure of the limit is the limit of the measures, and is at most $\frac{1-2\eta}{1-\eta}$. **Q**

(ii) For $m\in\mathbb{N}$, $\epsilon>0$ and $A\subseteq[0,\infty[$ let $G(A,\epsilon,m)$ be

$$\begin{aligned}\{\omega:\omega\in\Omega,\ &\text{there are }s_0<s_0'\le s_1<s_1'\le\dots\le s_m\le s_m'\text{ in }A\\&\text{such that }\rho(\omega(s_i'),\omega(s_i))>4\epsilon\text{ for every }i\le m\}.\end{aligned}$$

Let $\delta>0$ be such that $\nu_x^{(s,t)}B(x,\epsilon)\ge\frac{4}{5}$ whenever $x\in U$ and $s<t\le s+\delta$. Then $\hat\mu_{\tilde\omega a}G(D,\epsilon,m)\le 2^{-m}$ whenever $m\in\mathbb{N}$ and $D\subseteq[a,\infty[$ is a countable set of diameter at most δ.

P **(α)** As in (i), first consider finite D. For these, we can induce on m. If $m=0$ then $G(D,\epsilon,0)=\{\omega:\operatorname{diam}\omega[D]>4\epsilon\}$ so (i), with $\eta=\frac{1}{5}$, tells us that $\hat\mu_{\tilde\omega a}G(D,\epsilon,0)\le\frac{2\eta}{1-\eta}=\frac{1}{2}$. For the inductive step to $m+1$, define $\tau:\Omega\to[0,\infty]$ by setting

$$\begin{aligned}\tau(\omega)&=\min\{t:t\in D,\ \omega\in G(D\cap[a,t],\epsilon,m)\}\text{ if }\omega\in G(D,\epsilon,m),\\&=\infty\text{ otherwise.}\end{aligned}$$

Then τ takes only finitely many values, all strictly greater than a, and $\{\omega : \tau(\omega) = t\}$ belongs to $\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U) = \widehat{\bigotimes}_{t\in[0,\infty[}\mathcal{B}(U)$ and is determined by coordinates in $[0,t]$ for every $t \ge 0$. We can therefore apply 455E(b)-(c).

For each $\omega \in \Omega$, define $\langle \nu^{(s,t)}_{\omega,\tau(\omega),x}\rangle_{s<t,x\in U}$ from $\langle \nu^{(s,t)}_x\rangle_{s<t,x\in U}$ as in 455Eb; let $\langle \tilde\nu^{(s,t)}_{\omega,\tau(\omega),x}\rangle_{s<t,x\in U}$ be the family defined in the same way from $\langle \nu^{(s,t)}_{\tilde\omega ax}\rangle_{s<t,x\in U}$. Let $\hat\mu_{\omega,\tau(\omega)}$ be defined from $\omega(0)$ and $\langle \nu^{(s,t)}_{\omega,\tau(\omega),x}\rangle_{s<t,x\in U}$, and $\hat\mu'_{\omega,\tau(\omega)}$ from $\omega(0)$ and $\langle \tilde\nu^{(s,t)}_{\omega,\tau(\omega),x}\rangle_{s<t,x\in U}$, again as in 455Eb. Then 455E(c-i) tells us that $\langle \hat\mu'_{\omega,\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of $\hat\mu_{\tilde\omega a}$ over itself. But now observe that, for any $\omega \in \Omega$ and $x \in U$,

$$\nu^{(s,t)}_{\omega,\tau(\omega),x} = \nu^{(s,t)}_x = \nu^{(s,t)}_{\tilde\omega ax} = \tilde\nu^{(s,t)}_{\omega,\tau(\omega),x} \text{ if } \tau(\omega) < s < t,$$

(because $a < \tau(\omega)$)

$$= \nu^{(\tau(\omega),t)}_{\omega(\tau(\omega))} = \nu^{(\tau(\omega),t)}_{\tilde\omega,a,\omega(\tau(\omega))} = \tilde\nu^{(s,t)}_{\omega,\tau(\omega),x} \text{ if } s \le \tau(\omega) < t,$$
$$= \delta^{(t)}_{\omega(t)} = \tilde\nu^{(s,t)}_{\omega,\tau(\omega),x} \text{ if } s < t \le \tau(\omega),$$

so $\hat\mu_{\omega,\tau(\omega)} = \hat\mu'_{\omega,\tau(\omega)}$. Accordingly $\langle \hat\mu_{\omega,\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of $\hat\mu_{\tilde\omega a}$ over itself.

Now, for $\omega \in \Omega$, consider

$$H_\omega = \{\omega' : \omega' \in G(D,\epsilon,m+1),\ \omega'\restriction D\cap[0,\tau(\omega)] = \omega\restriction D\cap[0,\tau(\omega)]\}.$$

If $\omega \notin G(D,\epsilon,m)$ then $\tau(\omega) = \infty$ and $H_\omega = \emptyset$, because $G(D,\epsilon,m+1) \subseteq G(D,\epsilon,m)$ are determined by coordinates in D. If $\omega \in G(D,\epsilon,m)$ and $\tau(\omega) = b$, then

$$H_\omega = \{\omega' : \omega'\restriction D\cap[0,b] = \omega\restriction D\cap[0,b] \text{ and } \operatorname{diam}(\omega'[D\cap[b,\infty[\,]) > 4\epsilon\},$$

so that $\hat\mu_{\omega,\tau(\omega)}H_\omega \le \frac{1}{2}$ by (i), again with $\eta = \frac{1}{5}$.

So

$$\hat\mu_{\tilde\omega a}G(D,\epsilon,m+1) = \int \hat\mu_{\omega,\tau(\omega)}G(D,\epsilon,m+1)\hat\mu_{\tilde\omega a}(d\omega) = \int \hat\mu_{\omega,\tau(\omega)}H_\omega\hat\mu_{\tilde\omega a}(d\omega)$$

(using 455E(b-ii))

$$= \int_{G(D,\epsilon,m)} \hat\mu_{\omega,\tau(\omega)}H_\omega\hat\mu_{\tilde\omega a}(d\omega) \le \frac{1}{2}\hat\mu_{\tilde\omega a}G(D,\epsilon,m) \le 2^{-m-1}$$

by the inductive hypothesis. Thus the induction proceeds.

(β) Now, for countably infinite D, again express D as the union of a non-decreasing sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ of finite sets, and observe that $\langle G(I_n,\epsilon,m)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with union $G(D,\epsilon,m)$; so

$$\hat\mu_{\tilde\omega a}G(D,\epsilon,m) = \lim_{n\to\infty} \hat\mu_{\tilde\omega a}G(I_n,\epsilon,m) \le 2^{-m}$$

for every $m \in \mathbb{N}$. **Q**

(iii) For $n \in \mathbb{N}$, let $\delta_n > 0$ be such that $\nu^{(s,t)}_x B(x,2^{-n}) \ge \frac{4}{5}$ whenever $x \in U$ and $s < t \le s+\delta_n$. Consider the set

$$E = \bigcup_{n,k\in\mathbb{N}}\bigcap_{m\in\mathbb{N}} G(\mathbb{Q}\cap[a+k\delta_n, a+(k+1)\delta_n], 2^{-n+2}, m).$$

Then $\hat\mu_{\tilde\omega a}E = 0$. Suppose that $\omega \in \Omega\setminus E$ and $t > a$. **?** If $\lim_{q\in\mathbb{Q},q\uparrow t}\omega(q)$ is undefined, then (because U is complete under ρ) there must be an $n \in \mathbb{N}$ and a strictly increasing sequence $\langle q_i\rangle_{i\in\mathbb{N}}$ in $\mathbb{Q}$, with supremum t, such that $\rho(\omega(q_{i+1}),\omega(q_i)) \ge 2^{-n+2}$ for every $i \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $t \in \,]a+k\delta_n, a+(k+1)\delta_n]$; let $l \in \mathbb{N}$ be such that $q_l \ge a+k\delta_n$. Then, for every $m \in \mathbb{N}$, $(q_l, q_{l+1}, q_{l+1}, q_{l+2}, \dots, q_{m-1}, q_m)$ witnesses that $\omega \in G(\mathbb{Q}\cap[a+k\delta_n, a+(k+1)\delta_n], 2^{-n+2}, m)$; which is impossible. **X** So $\lim_{q\in\mathbb{Q},q\uparrow t}\omega(q)$ is defined; similarly, $\lim_{q\in\mathbb{Q},q\downarrow t}\omega(q)$ is defined.

As E is $\hat\mu_{\tilde\omega a}$-negligible, this proves (a).

(b)(i) This is actually easier. Consider part (a-i) of the proof above. Given $n \in \mathbb{N}$, we see that there is a $\delta_n > 0$ such that $\hat\mu_{\tilde\omega a}\{\omega : \operatorname{diam}\omega[D] \le 2^{-n}\} \ge 1-2^{-n}$ whenever $D \subseteq [a,\infty[$ is a countable set of diameter at most δ_n. Set

$$D_n = \{t\}\cup(\mathbb{Q}\cap[t,t+\delta_n]), \quad E_n = \{\omega : \operatorname{diam}\omega[D_n] \le 2^{-n}\}$$

for each $n \in \mathbb{N}$, and $E = \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} E_m$. Then $\hat\mu_{\tilde\omega a} E = 1$, and for $\omega \in E$ we have an $n \in \mathbb{N}$ such that $\rho(\omega(t), \omega(q)) \le 2^{-m}$ whenever $m \ge n$ and $q \in \mathbb{Q} \cap [t, t+\delta_m]$, so that $\omega(t) = \lim_{q\in\mathbb{Q}, q\downarrow t} \omega(q)$.

(ii) If $t > a$, the same argument applies on the other side of t, taking $D_n = \{t\} \cup (\mathbb{Q} \cap [\max(a, t-\delta_n), t])$, to see that $\omega(t) = \lim_{q\in\mathbb{Q}, q\uparrow t} \omega(q)$ for $\hat\mu_{\tilde\omega a}$-almost every ω.

(c)($\boldsymbol{\alpha}$) Suppose that $E \subseteq \Omega$ and $\hat\mu_{\tilde\omega a} E > 0$. Then there is an $\omega^* \in E$ such that

$$\omega^*(t) = \tilde\omega(t) \text{ for every } t \le a,$$

$$\omega^*(t) = \lim_{s\downarrow t} \omega^*(s) \text{ for every } t \ge a,$$

$$\lim_{s\uparrow t} \omega^*(t) \text{ is defined for every } t > a.$$

P Let $E' \in \widehat{\bigotimes}_{[0,\infty[} \mathcal{B}(U)$ be such that $E' \subseteq E$ and $\hat\mu_{\tilde\omega a} E' > 0$. Let $D \subseteq [0, \infty[$ be a countable set such that E' is determined by coordinates in D; we can suppose that $a \in D$ if a is finite. Let F be the set of those $\omega \in \Omega$ such that

$$\lim_{q\in\mathbb{Q}, q\downarrow t} \omega(q) \text{ and } \lim_{q\in\mathbb{Q}, q\uparrow t} \omega(q) \text{ are defined in } U \text{ for every } t > a,$$

$$\omega(t) = \lim_{q\in\mathbb{Q}, q\downarrow t} \omega(t) \text{ for every } t \in D \cap [a, \infty[,$$

$$\omega(t) = \tilde\omega(t) \text{ for every } t \in D \cap [0, a].$$

Then (a) and (b), with 455E(b-ii), tell us that F is $\hat\mu_{\tilde\omega a}$-conegligible. So there is an $\omega \in E \cap F$. Define $\omega^* \in \Omega$ by setting

$$\begin{aligned}\omega^*(t) &= \tilde\omega(t) \text{ if } t \le a,\\ &= \lim_{q\in\mathbb{Q}, q\downarrow t} \omega(t) \text{ if } t \ge a;\end{aligned}$$

note that the definitions of $\omega^*(a)$ are consistent if a is finite, and that $\omega^* \restriction D = \omega \restriction D$, so that $\omega^* \in E' \subseteq E$.

If $t \le a$, then of course $\omega^*(t) = \tilde\omega(t)$. If $t \ge a$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\rho(\omega(q), \omega^*(t)) \le \epsilon$ whenever $q \in \mathbb{Q} \cap]t, t+\delta]$; in which case $\rho(\omega^*(s), \omega^*(t)) \le \epsilon$ whenever $s \in [t, t+\delta[$; as ϵ is arbitrary, $\omega^*(t) = \lim_{s\downarrow t} \omega^*(s)$. If $t > a$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\rho(\omega(q), \omega(q')) \le \epsilon$ whenever $q \in \mathbb{Q} \cap [t-\delta, t[$; in which case $\rho(\omega^*(s), \omega^*(s')) \le \epsilon$ whenever $s \in [t-\delta, t[$; as ϵ is arbitrary and U is complete, $\lim_{s\uparrow t} \omega^*(s)$ is defined in U. So we have an appropriate ω^*. **Q**

($\boldsymbol{\beta}$) Suppose, in (α), that $\tilde\omega \in C^{\Vert}$. Then $\omega^* \in C^{\Vert}$. **P**

$$\lim_{s\uparrow t} \omega^*(s) = \lim_{s\uparrow t} \tilde\omega(s) \text{ is defined whenever } 0 < t \le a,$$

$$\lim_{s\downarrow t} \omega^*(s) = \lim_{s\downarrow t} \tilde\omega(s) \text{ is defined whenever } 0 \le t < a,$$

$$\text{if } a > 0,\ \lim_{s\downarrow 0} \omega^*(s) = \lim_{s\downarrow 0} \tilde\omega(s) = \tilde\omega(0) = \omega^*(0),$$

if $0 < t < a$, then $\omega^*(t) = \tilde\omega(t)$ is equal to at least one of $\lim_{s\uparrow t} \omega^*(s) = \lim_{s\uparrow t} \tilde\omega(s)$, $\lim_{s\downarrow t} \omega^*(s) = \lim_{s\downarrow t} \tilde\omega(s)$.

Since we already know that

$$\omega^*(t) = \lim_{s\downarrow t} \omega^*(s) \text{ for every } t \ge a,$$

$$\lim_{s\uparrow t} \omega^*(t) \text{ is defined for every } t > a,$$

ω^* is càllàl. **Q**

As E is arbitrary, it follows that if $\tilde\omega \in C^{\Vert}$ then $C^{\Vert}$ meets every non-negligible $\hat\mu_{\tilde\omega a}$-measurable set, so that $\hat\mu^*_{\tilde\omega a} C^{\Vert} = 1$, as required by (i).

($\boldsymbol{\gamma}$) Similarly, if $\tilde\omega \in C_{\mathrm{dlg}}$, then any ω^* with the properties described in (α) also belongs to C_{dlg}. **P** This time, we have

$$\text{if } 0 \le t < a, \text{ then } \omega^*(t) = \tilde\omega(t) = \lim_{s\downarrow t} \omega^*(s) = \lim_{s\downarrow t} \tilde\omega(s),$$

which with the other properties listed is enough to ensure that $\omega^* \in C_{\mathrm{dlg}}$. **Q** Since E is arbitrary, $\hat\mu^*_{\tilde\omega a} C_{\mathrm{dlg}} = 1$.

This completes the proof of part (c).

455H Corollary Let (U,ρ) be a complete metric space and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0\le s<t<u$ and $x\in U$. Let $C^{\Vdash}(U)$ be the set of càllàl functions from $[0,\infty[$ to U. Suppose that $\tilde\omega\in C^{\Vdash}(U)$, and $a\in[0,\infty]$; let $\hat\mu_{\tilde\omega a}$ be the completed probability measure on $\Omega=U^{[0,\infty[}$ defined from $\tilde\omega$, a and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ as in 455Eb. Then $\hat\mu_{\tilde\omega a}$ has a unique extension to a Radon measure $\tilde\mu_{\tilde\omega a}$ on Ω, and $\tilde\mu_{\tilde\omega a}C^{\Vdash}(U)=1$.

proof (a) In the language of 455E(b-i), $\nu_{\tilde\omega ax}^{(0,t)}$ is a Radon measure whenever $t>0$ and $x\in U$, so the image measure defined from $\hat\mu_{\tilde\omega a}$ and the map $\omega\mapsto\omega(t)$ is always a Radon measure on U, and there there is a σ-compact set $H_t\subseteq U$ such that $\omega(t)\in H_t$ for $\hat\mu_{\tilde\omega a}$-almost every ω. Set $U_0=\overline{\bigcup_{q\in\mathbb{Q}\cap[0,\infty[}H_q}$; then U_0 is separable and $\hat\mu_{\tilde\omega a}E=1$, where $E=\{\omega:\omega(q)\in U_0\text{ for every }q\in\mathbb{Q}\cap[0,\infty[\}$. By 455G(c-i), $E\cap C^{\Vdash}(U)$ has full outer measure; and if $\omega\in E\cap C^{\Vdash}(U)$, then $\omega(t)\in U_0$ for every $t\ge 0$.

(b) Thus $E\cap C^{\Vdash}(U)$ is included in $C^{\Vdash}(U_0)$, the set of càllàl functions from $[0,\infty[$ to the Polish space U_0. So $\hat\mu^*_{\tilde\omega a}C^{\Vdash}(U_0)=1$. Let $\hat\mu_C$ be the subspace probability measure on $C^{\Vdash}(U_0)$.

Since $\hat\mu_{\tilde\omega a}$ is inner regular with respect to $\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$, $\hat\mu_C$ is inner regular with respect to the σ-algebra $\Sigma=\{E\cap C^{\Vdash}(U_0):E\in\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)\}$ (412Ob). But Σ is just the σ-algebra generated by the maps $\omega\mapsto\omega(t):C^{\Vdash}(U_0)\to U_0$ for $t\ge 0$, which is the Baire σ-algebra of $C^{\Vdash}(U_0)$ (4A3Nd). Accordingly $\hat\mu_C\restriction\Sigma$ is a Baire measure and is inner regular with respect to the closed sets (412D); it follows that its completion $\hat\mu_C$ is inner regular with respect to the closed sets (412Ab).

At this point, recall that $C^{\Vdash}(U_0)$ is K-analytic (438Sc). So $\hat\mu_C$ has an extension to a Radon measure $\tilde\mu_C$ on $C^{\Vdash}(U_0)$ (432D). Now $\tilde\mu_C$ has an extension to a Radon probability measure $\tilde\mu_{\tilde\omega a}$ on Ω such that $\tilde\mu_{\tilde\omega a}C^{\Vdash}(U)=\tilde\mu_{\tilde\omega a}C^{\Vdash}(U_0)=1$. And if $\hat\mu_{\tilde\omega a}$ measures E, then

$$\tilde\mu_{\tilde\omega a}E=\tilde\mu_C(E\cap C^{\Vdash}(U_0))=\hat\mu_C(E\cap C^{\Vdash}(U_0))=\hat\mu^*_{\tilde\omega a}(E\cap C^{\Vdash}(U_0))=\hat\mu_{\tilde\omega a}E,$$

so $\tilde\mu_{\tilde\omega a}$ extends $\hat\mu_{\tilde\omega a}$.

(c) As for uniqueness, observe that $\operatorname{dom}\hat\mu_{\tilde\omega a}$ includes a base for the topology of Ω, so by 415H there can be at most one Radon measure extending $\hat\mu_{\tilde\omega a}$.

455I In fact we can go farther; the Radon measure $\tilde\mu$ is much more closely related to the completed Baire measure it extends than one might expect.

Lemma Let (U,ρ) be a complete separable metric space and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0\le s<t<u$ and $x\in U$. Suppose that $\tilde\omega\in\Omega$, and $a\in[0,\infty]$; let $\hat\mu_{\tilde\omega a}$ be the completed probability measure on $\Omega=U^{[0,\infty[}$ defined from $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$, $\tilde\omega$ and a as in 455Eb.

(a) Suppose that $0\le q_0<q_1$ and $\epsilon>0$. For $\omega\in\Omega$, I will say that $]q_0,q_1[$ is an **ϵ-shift interval** of ω with **(q_0,q_1,ϵ)-shift point** t if $\rho(\omega(q_0),\omega(q_1))>2\epsilon$ and

$$\begin{aligned}t&=\sup\{q:q\in\mathbb{Q}\cap\,]q_0,q_1[\,,\ \rho(\omega(q),\omega(q_0))\le\epsilon\}\\&=\inf\{q:q\in\mathbb{Q}\cap\,]q_0,q_1[\,,\ \rho(\omega(q),\omega(q_1))\le\epsilon\}.\end{aligned}$$

Let E be the set of such ω.

(i) $E\in\mathcal{B}\mathfrak{a}(\Omega)=\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$.

(ii) The function $f:E\to\,]q_0,q_1[$ which takes each $\omega\in E$ to its (q_0,q_1,ϵ)-shift point is $\mathcal{B}\mathfrak{a}(\Omega)$-measurable.

(iii) If $q_0\ge a$, the set $\{\omega:\omega\in E,\ f(\omega)=t\}$ is $\hat\mu_{\tilde\omega a}$-negligible for every $t\in\,]q_0,q_1[$.

(iv) If q_0, $q_1\in\mathbb{Q}$, $\omega\in E$, $\omega'\in\Omega$ and $\omega'\restriction\mathbb{Q}=\omega\restriction\mathbb{Q}$, then $\omega'\in E$ and $f(\omega')=f(\omega)$.

(b) Suppose that $\langle q_i\rangle_{i\le n}$, $\langle q'_i\rangle_{i\le n}$, $\langle\le_i\rangle_{i\le n}$, $\epsilon>0$, $E\in\mathcal{B}\mathfrak{a}(\Omega)$ and $\langle f_i\rangle_{i\le n}$ are such that, for every $i\le n$,

$$q_i,\,q'_i\in\mathbb{Q},\quad q_i<q'_i,\quad \le_i\text{ is either }\le\text{ or }\ge,$$

$$]q_i,q'_i[\text{ is an }\epsilon\text{-shift interval of }\omega\text{ with }(q_i,q'_i,\epsilon)\text{-shift point }f_i(\omega),\text{ for every }\omega\in E,$$

and also

$$a\le q_0,\quad q'_i\le q_{i+1}\text{ for every }i<n,$$

$$\text{whenever } \omega,\, \omega' \in E \text{ there is an } i \le n \text{ such that } f_i(\omega') \le_i f_i(\omega).$$

Then E is $\hat\mu_{\tilde\omega a}$-negligible.

(c) Suppose that $\langle q_i\rangle_{i\le n}$, $\langle q'_i\rangle_{i\le n}$, $\langle \le_i\rangle_{i\le n}$, $\epsilon > 0$, $E \in \mathcal{B}\mathfrak{a}(\Omega)$ and $\langle f_i\rangle_{i\le n}$ are such that, for every $i \le n$,

$$q_i,\, q'_i \in \mathbb{Q}, \quad q_i < q'_i, \quad \le_i \text{ is either } \le \text{ or } \ge,$$

$$]q_i, q'_i[\text{ is an } \epsilon\text{-shift interval of } \omega \text{ with } (q_i, q'_i, \epsilon)\text{-shift point } f_i(\omega), \text{ for every } \omega \in E,$$

and also

$$a \le q_0, \quad q'_i \le q_{i+1} \text{ for every } i < n.$$

Then for $\hat\mu_{\tilde\omega a}$-almost every $\omega \in E$ there is an $\omega' \in E$ such that $f_i(\omega') <_i f_i(\omega)$ for every $i \le n$.

proof (a)(i) Note that by 4A3Na we can identify $\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$ with the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\Omega)$ of Ω. If s, $t \ge 0$, then $\omega \mapsto (\omega(s), \omega(t)) : \Omega \to U^2$ is $\mathcal{B}\mathfrak{a}(\Omega)$-measurable, by 418Bb; so $\omega \mapsto \rho(\omega(s), \omega(t))$ is $\mathcal{B}\mathfrak{a}(\Omega)$-measurable. For $\omega \in \Omega$, $\omega \in E$ iff (α) $\rho(\omega(q_0), \omega(q_1)) > 2\epsilon$ (β) whenever q, $q' \in \mathbb{Q} \cap\,]q_0, q_1[$, $\rho(\omega(q), \omega(q_0)) \le \epsilon$ and $\rho(\omega(q'), \omega(q_1)) \le \epsilon$ then $q \le q'$ (γ) for every $n \in \mathbb{N}$ there are q, $q' \in \mathbb{Q} \cap\,]q_0, q_1[$ such that $\rho(\omega(q), \omega(q_0)) \le \epsilon$, $\rho(\omega(q'), \omega(q'_0)) \le \epsilon$ and $q' \le q + 2^{-n}$. So $E \in \mathcal{B}\mathfrak{a}(\Omega)$.

(ii) Now, for any t,

$$\{\omega : \omega \in E,\, f(\omega) > t\} = \bigcup_{q\in\mathbb{Q}\cap]t,q_1[}\{\omega : \omega \in E,\, \rho(\omega(q), \omega(q_0)) \le \epsilon\}$$

belongs to $\mathcal{B}\mathfrak{a}(\Omega)$, so f is $\mathcal{B}\mathfrak{a}(\Omega)$-measurable.

(iii) Consider the set E' of those $\omega \in \Omega$ such that

$$\lim_{q\in\mathbb{Q},q\uparrow t} \omega(q) = \omega(t) = \lim_{q\in\mathbb{Q},q\downarrow t} \omega(q).$$

If $\omega \in E \cap E'$, at least one of $\rho(\omega(t), \omega(q_0))$, $\rho(\omega(t), \omega(q_1))$ must be greater than ϵ; in the first case, t cannot be $\sup\{q : q \in \mathbb{Q}\cap]q_0, q_1[,\, \rho(\omega(q), \omega(q_0)) \le \epsilon\}$; in the second case, t cannot be $\inf\{q : q \in \mathbb{Q}\cap]q_0, q_1[,\, \rho(\omega(q), \omega(q_1)) \le \epsilon\}$; so in either case $f(\omega)$ cannot be equal to t. Now $\hat\mu_{\tilde\omega a}E' = 1$, by 455Gb, so $\{\omega : \omega \in E,\, f(\omega) = t\} \subseteq \Omega \setminus E'$ is $\hat\mu_{\tilde\omega a}$-negligible.

(iv) Immediate from the definitions.

(b) Induce on n. Of course we need consider only the case $E \ne \emptyset$.

(i) If $n = 0$, f_0 must be constant on E, so E must be negligible, by (a-iii).

(ii) For the inductive step to $n \ge 1$, set $E_t = \{\omega : \omega \in E,\, f_0(\omega) = t\}$ for $t \in\,]q_0, q'_0[$; by (a-ii), $E_t \in \mathcal{B}\mathfrak{a}(\Omega)$.

(α) There is a countable set $J \subseteq\,]q_0, q'_0[$ such that whenever $t \in [q_0, q'_0] \setminus J$ and ω, $\omega' \in E_t$ then there is an i such that $1 \le i \le n$ and $f_i(\omega) \le_i f_i(\omega')$. **P** Let $\mathcal{W}$ be a countable base for the topology of $\prod_{1\le i\le n}]q_i, q'_i[$. For $\omega \in E$ set $g(\omega) = \langle f_i(\omega)\rangle_{1\le i\le n}$; note that $g : E \to \prod_{1\le i\le n}]q_i, q'_i[$ is $\mathcal{B}\mathfrak{a}(\Omega)$-measurable. For $W \in \mathcal{W}$, set

$$A_W = \{t : t \in\,]q_0, q'_0[\text{ and there is an } \omega \in E_t \text{ such that } g(\omega) \in W\}.$$

Set

$$J = \{t : t \in\,]q_0, q'_0[,\, t \text{ is either } \inf A_W \text{ or } \sup A_W \text{ for some } W \in \mathcal{W}\}.$$

Then J is a countable subset of $]q_0, q'_0[$. **?** Suppose that $t \in\,]q_0, q'_0[\setminus J$ and ω, $\omega' \in E_t$ are such that $f_i(\omega') <_i f_i(\omega)$ for $1 \le i \le n$. Let $W \in \mathcal{W}$ be such that $g(\omega') \in W$ and $z(i) <_i f_i(\omega)$ whenever $1 \le i \le n$ and $z \in W$. Then ω' witnesses that $t \in A_W$; since t is neither the greatest nor the least element of A_W, there is a $t' \in A_W$ such that $t' <_0 t$; take $\omega'' \in E_{t'}$ such that $g(\omega'') \in W$. Then

$$f_0(\omega'') = t' <_0 t = f_0(\omega),$$

$$f_i(\omega'') = g(\omega'')(i) <_i f_i(\omega) \text{ for } 1 \le i \le n,$$

which is impossible. **X** Thus J has the required property. **Q**

(β) Now consider the family $\langle\hat\mu_{\omega q_1}\rangle_{\omega\in\Omega}$. Because $q_1 > a$, this is a disintegration of $\hat\mu_{\tilde\omega a}$ over itself. **P** As in part (a-ii-α) of the proof of 455G, we can think of each $\hat\mu_{\omega q_1}$ as defined either from $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ or from $\langle\nu_{\tilde\omega a x}^{(s,t)}\rangle_{a\le s<t,x\in U}$; and in the latter form we can apply 455E(c-i). **Q**

Consider $\mu_{\omega q_1}(E)$ for $\omega \in \Omega$. This time, note that $\{\omega' : \omega'\restriction[0,q_1]\cap\mathbb{Q} = \omega\restriction[0,q_1]\cap\mathbb{Q}\}$ is $\mu_{\omega q_1}$-conegligible. In particular, $\mu_{\omega q_1}(E) = 0$ unless $]q_0, q_0'[$ is an ϵ-shift interval of ω. Next,

$$\{\omega :]q_0, q_0'[\text{ is an } \epsilon\text{-shift interval of } \omega \text{ with } (q_0, q_0', \epsilon)\text{-shift point in } J\}$$

is $\hat\mu_{\tilde\omega a}$-negligible, by (a-iii) again. Finally, suppose that $\omega \in \Omega$ is such that $]q_0, q_0'[$ is an ϵ-shift interval of ω with (q_0, q_0', ϵ)-shift point $t \in]q_0, q_0'[\setminus J$. Then

$$\mu_{\omega q_1}E = \mu_{\omega q_1}\{\omega' : \omega' \in E,\ \omega'\restriction\mathbb{Q}\cap[0,q_1] = \omega\restriction\mathbb{Q}\cap[0,q_1]\}$$
$$= \mu_{\omega q_1}\{\omega' : \omega' \in E_t\}.$$

But the choice of J in (α) ensured that E_t would be a set of the same type as E, one level down, determined by intervals starting from q_1, so that $\hat\mu_{\omega q_1}E_t = 0$, by the inductive hypothesis applied to ω and q_1 in place of $\tilde\omega$ and a.

(γ) So we see that $\hat\mu_{\omega q_1}E = 0$ for $\hat\mu_{\tilde\omega a}$-almost every ω, and $\hat\mu_{\tilde\omega a}E = 0$. Thus the induction proceeds.

(c) Let F be the set of those $\omega \in E$ for which there is no $\omega' \in E$ such that $f_i(\omega') <_i f_i(\omega)$ for every $i \le n$. Then $F \in \mathcal{B}\mathfrak{a}(\Omega)$. **P** For each $\omega \in E$ set $f(\omega) = \langle f_i(\omega)\rangle_{i\le n}$ and

$$W_\omega = \{z : z \in \textstyle\prod_{i\le n}]q_i, q_i'[,\ f_i(\omega) <_i z(i) \text{ for every } i \le n\},$$

so that W_ω is open in $\prod_{i\le n}]q_i, q_i'[$. Set $W = \bigcup_{\omega\in E} W_\omega$. Then W is open and $F = \{\omega : \omega \in E,\ f(\omega) \notin W\}$ belongs to $\mathcal{B}\mathfrak{a}(\Omega)$. **Q**

If ω, $\omega' \in F$ then there is surely some $i \le n$ such that $f_i(\omega') \le_i f_i(\omega)$. By (b), $\hat\mu_{\tilde\omega a}F = 0$.

455J Theorem Let (U,ρ) be a complete separable metric space and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0 \le s < t < u$ and $x \in U$. Write $C^{\Downarrow}$ for the set of càllàl functions from $[0,\infty[$ to U. Suppose that $\tilde\omega \in C^{\Downarrow}$, and $a \in [0,\infty]$; let $\hat\mu_{\tilde\omega a}$ be the completed probability measure on $\Omega = U^{[0,\infty[}$ defined from $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$, $\tilde\omega$ and a as in 455Eb, and $\tilde\mu_{\tilde\omega a}$ its extension to a Radon measure on Ω, as in 455H. Then $\tilde\mu_{\tilde\omega a}$ is inner regular with respect to sets of the form $F \cap C^{\Downarrow}$ where $F \subseteq \Omega$ is a zero set.

proof (a) As in 455I, $\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$ is the Baire σ-algebra of Ω. Let D be $((\{a\}\cup\mathbb{Q})\cap[0,\infty[)\cup\{t : t \ge 0,\ \tilde\omega$ is not continuous at $t\}$; then D is countable (438S(a-i)). Let E^* be $\{\omega : \omega \in \Omega,\ \omega\restriction D\cap[0,a] = \tilde\omega\restriction D\cap[0,a]\}$; then E^* is $\hat\mu_{\tilde\omega a}$-conegligible, by 455E(b-ii) once again.

(b) Let $\mathcal{G}$ be a countable base for the topology of U. Let $\mathcal{W}$ be the family of open subsets of Ω of the form $\{\omega : \omega(q) \in G_q$ for every $q \in J\}$ where $J \subseteq D$ is finite and $G_q \in \mathcal{G}$ for every $q \in J$. Let Θ be the set of all strings

$$\theta = (q_0, q_0', \dots, q_n, q_n', \le_0, \dots, \le_n, k, W)$$

such that

$$q_0, \dots, q_n' \in \mathbb{Q}, \quad a \le q_0 < q_0' \le q_1 < q_1' \le \dots \le q_n < q_n',$$

for each $i \le n$, $\le_i$ is either $\le$ or $\ge$,

$$k \in \mathbb{N}, \quad W \in \mathcal{W};$$

then Θ is countable.

(c) Let $K \subseteq E^* \cap C^{\Downarrow}$ be compact. Set $L = \pi_D^{-1}[\pi_D[K]]$, where $\pi_D(\omega) = \omega\restriction D$ for $\omega \in \Omega$; then L is a Baire subset of Ω, because $\pi_D[K]$ is a compact subset of the metrizable space U^D.

(i) For

$$\theta = (q_0, q_0', \dots, q_n, q_n', \le_0, \dots, \le_n, k, W) \in \Theta$$

let E_θ be the set of those $\omega \in L \cap W$ such that, for each $i \le n$, $]q_i, q_i'[$ is a 2^{-k}-shift interval of ω (definition: 455Ia). For $\omega \in E_\theta$ and $i \le n$ let $f_i(\theta,\omega)$ be the $(q_i, q_i', 2^{-k})$-shift point of ω. By 455Ia, E_θ is a Baire subset of Ω and $\omega \mapsto f_i(\theta,\omega)$ is Baire measurable. Let F_θ be the set of those $\omega \in E_\theta$ such that there is no $\omega' \in E_\theta$ with $f_i(\theta,\omega') <_i f_i(\theta,\omega)$ for every $i \le n$; by 455Ic, F_θ is $\hat\mu_{\tilde\omega a}$-negligible. So $F^* = \bigcup_{\theta\in\Theta} F_\theta$ is $\hat\mu_{\tilde\omega a}$-negligible.

(ii) Suppose that $\omega \in K \setminus F^*$. Let A be the set of points in $]a,\infty[$ at which ω is discontinuous. If $J \subseteq A$ is finite and $\epsilon_t \in \{-1,1\}$ for each $t \in J$, there is an $\omega' \in K$ such that $\omega'\restriction D = \omega\restriction D$ and ω' is continuous on the right at every point t of J such that $\epsilon_t = 1$, while ω' is continuous on the left at every point t of J such that $\epsilon_t = -1$. **P** This is trivial if J is empty. Otherwise, enumerate J in ascending order as $t_0 < t_1 < \ldots < t_n$. Set $x_i = \lim_{t\uparrow t_i} \omega(t)$, $y_i = \lim_{t\downarrow t_i} \omega(t)$; because $\omega \in C^{\upharpoonleft\downharpoonright}$ these are defined, and because ω is not continuous at t they are different.

Let $k \in \mathbb{N}$ be such that $\rho(x_i, y_i) > 2^{-k+1}$ for each $i \le n$. For $i \le n$, let let q_i, $q'_i \in \mathbb{Q}$ be such that $q_i < t_i < q'_i$, $\rho(\omega(t), x_i) \le 2^{-k-1}$ for $t \in [q_i, t_i[$, and $\rho(\omega(t), y_i) \le 2^{-k-1}$ for $t \in]t_i, q'_i]$. Of course we can suppose that $a \le q_0$ and that $q'_i \le q_{i+1}$ for $i < n$. Observe that this will ensure that every $]q_i, q'_i[$ is a 2^{-k}-shift interval of ω with $(q_i, q'_i, 2^{-k})$-shift point t_i.

Let $\le_i$ be $\le$ if $\epsilon_{t_i} = 1$, $\ge$ if $\epsilon_{t_i} = -1$. For each $W \in \mathcal{W}$ containing ω, let $\theta_W \in \Theta$ be $(q_0, \ldots, q'_n, \le_0, \ldots, \le_n, k, W)$. Then $\omega \in E_{\theta_W}$, and $f_i(\theta_W, \omega) = t_i$ for each $i \le n$. Because $\omega \notin F_{\theta_W}$, there is an $\omega_W \in E_{\theta_W}$ such that $f_i(\theta_W, \omega_W) <_i f_i(\theta_W, \omega) = t_i$ for every $i \le n$. Let $\omega'_W \in K$ be such that $\omega'_W \restriction D = \omega_W \restriction D$; then $\omega'_W \in E_{\theta_W}$ and

$$f_i(\theta_W, \omega'_W) = f_i(\theta_W, \omega_W) <_i t_i$$

for every $i \le n$ (455I(a-iv)).

If $i \le n$ and $\epsilon_{t_i} = 1$,

$$\rho(\omega'_W(q), \omega'_W(q'_i)) \le 2^{-k}$$

for every rational $q \in]f_i(\theta_W, \omega'_W), q'_i]$; because $\omega'_W \in C^{\upharpoonleft\downharpoonright}$ and $t_i \in]f_i(\theta_W, \omega'_W), q'_i]$ $\rho(\omega'_W(t_i), \omega'_W(q'_i)) \le 2^{-k}$. Similarly, if $\epsilon_{t_i} = -1$, $\rho(\omega'_W(t_i), \omega'_W(q_i)) \le 2^{-k}$.

Let $\mathcal{F}$ be an ultrafilter on $\mathcal{W}$ containing all sets of the form $\{W : \omega \in W \subseteq W_0\}$ where $\omega \in W_0 \in \mathcal{W}$, and set $\omega' = \lim_{W\to\mathcal{F}} \omega'_W \in K$. Then $\omega' \restriction D = \omega \restriction D$, because ω_W and ω'_W belong to W whenever $\omega \in W \in \mathcal{W}$. If $i \le n$ and $\epsilon_{t_i} = 1$, then

$$\rho(\omega'(t_i), \omega'(q'_i)) = \lim_{W\to\mathcal{F}} \rho(\omega'_W(t_i), \omega'_W(q'_i)) \le 2^{-k},$$

$$\rho(\omega'(q_i), \omega'(q'_i)) = \rho(\omega(q_i), \omega(q'_i)) > 2^{-k+1},$$

so $\rho(\omega'(q_i), \omega'(t_i)) > 2^{-k}$. On the other hand,

$$\rho(\omega'(q_i), \omega'(q)) = \rho(\omega(q_i), \omega(q)) \le 2^{-k}$$

for every rational $q \in [q_i, t_i[$. So ω' cannot be continuous on the left at t_i; because $\omega' \in C^{\upharpoonleft\downharpoonright}$, it must be continuous on the right at t_i. Similarly, if $i \le n$ and $\epsilon_{t_i} = -1$, ω' cannot be continuous on the right at t_i and must be continuous on the left at t_i. But this is what we need to know. **Q**

(iii) Suppose that $\omega \in K \setminus F^*$, $\omega' \in C^{\upharpoonleft\downharpoonright}$ and $\omega \restriction D = \omega' \restriction D$. Then $\omega' \in K$. **P** Let A be the set of points in $]a, \infty[$ where ω is not continuous, and for $t \in A$ let ϵ_t be 1 if ω' is continuous on the right at t, -1 if ω' is continuous on the left at t. For each finite $J \subseteq A$, (ii) tells us that there is an $\omega_J \in K$ such that $\omega_J \restriction D = \omega \restriction D = \omega' \restriction D$ and, for $t \in J$, ω_J is continuous on the right at t if $\epsilon_t = 1$, and continuous on the left at t if $\epsilon_t = -1$. As both ω_J and ω' are càllàl, this means that $\omega_J(t) = \omega'(t)$ for $t \in J$. Taking a cluster point $\omega^* \in K$ of ω_J as J increases through the finite subsets of A, we see that $\omega^* \restriction (A \cup D) = \omega' \restriction (A \cup D)$.

Now recall that $\omega \in E^*$, so that

$$\omega' \restriction D \cap [0, a] = \omega \restriction D \cap [0, a] = \tilde{\omega} \restriction D \cap [0, a].$$

Since both ω' and $\tilde{\omega}$ are càllàl, $\tilde{\omega}$ is discontinuous at any point of $[0, a[$ at which ω' is discontinuous. Since I arranged that a (if finite) would be in D, $D \cup A$ contains every point at which ω' is discontinuous. But this means that $\omega^* = \omega'$ (438S(a-ii)). So $\omega' \in K$. **Q**

(iv) Suppose that $\tilde{\mu}_{\tilde{\omega}a} K > \gamma \ge 0$. Then there is a zero set $F \subseteq \Omega$ such that $F \cap C^{\upharpoonleft\downharpoonright} \subseteq K$ and $\tilde{\mu}_{\tilde{\omega}a}(F \cap C^{\upharpoonleft\downharpoonright}) \ge \gamma$. **P** Because $\tilde{\mu}_{\tilde{\omega}a} F^* = \hat{\mu}_{\tilde{\omega}a} F^* = 0$, there is a compact $K' \subseteq K \setminus F^*$ such that $\tilde{\mu}_{\tilde{\omega}a} K' \ge \gamma$. Set $F = \pi_D^{-1}[\pi_D[K']]$; F is a zero set in Ω because $\pi_D[K']$ is a zero set in U^D. By (iii), $F \cap C^{\upharpoonleft\downharpoonright} \subseteq K$; and

$$\tilde{\mu}_{\tilde{\omega}a}(F \cap C^{\upharpoonleft\downharpoonright}) \ge \tilde{\mu}_{\tilde{\omega}a} K' \ge \gamma. \textbf{ Q}$$

(c) Since E^* and $C^{\upharpoonleft\downharpoonright}$ are $\tilde{\mu}_{\tilde{\omega}a}$-conegligible, the Radon measure $\tilde{\mu}_{\tilde{\omega}a}$ is certainly inner regular with respect to the compact subsets of $E^* \cap C^{\upharpoonleft\downharpoonright}$; by (b-iv), $\tilde{\mu}_{\tilde{\omega}a}$ is inner regular with respect to the intersections of $C^{\upharpoonleft\downharpoonright}$ with zero sets.

455K Corollary Suppose, in 455J, that $\tilde{\omega} \in C_{\mathrm{dlg}}$, the space of càdlàg functions from $[0, \infty[$ to U. Then the subspace measure $\ddot{\mu}_{\tilde{\omega}a}$ on C_{dlg} induced by $\hat{\mu}_{\tilde{\omega}a}$ is a completion regular quasi-Radon measure.

proof The point is that the outer measures $\tilde{\mu}^*_{\tilde{\omega}a}$ and $\hat{\mu}^*_{\tilde{\omega}a}$ agree on subsets of C_{dlg}. **P** Since $\tilde{\mu}_{\tilde{\omega}a}$ extends $\hat{\mu}_{\tilde{\omega}a}$, $\tilde{\mu}^*_{\tilde{\omega}a}A \le \hat{\mu}^*_{\tilde{\omega}a}A$ for every $A \subseteq \Omega$. On the other hand, if $A \subseteq C_{\text{dlg}}$ and $\tilde{\mu}^*_{\tilde{\omega}a}A < \gamma$, there is an $E \supseteq A$ such that $\tilde{\mu}_{\tilde{\omega}a}E < \gamma$. By 455J, there is a zero set $F \subseteq \Omega$ such that $E \cap F \cap C^{\Downarrow} = \emptyset$ and $\tilde{\mu}_{\tilde{\omega}a}(F \cap C^{\Downarrow}) \ge 1 - \gamma$. Now

$$\hat{\mu}^*_{\tilde{\omega}a}A \le \hat{\mu}^*_{\tilde{\omega}a}(C_{\text{dlg}} \setminus F) = \hat{\mu}_{\tilde{\omega}a}(\Omega \setminus F)$$

(because $\hat{\mu}^*_{\tilde{\omega}a}C_{\text{dlg}} = 1$, by 455G(c-ii))

$$= \tilde{\mu}_{\tilde{\omega}a}(\Omega \setminus F) = \tilde{\mu}_{\tilde{\omega}a}(C^{\Downarrow} \setminus F) \le \gamma.$$

As γ is arbitrary, $\hat{\mu}^*_{\tilde{\omega}a}A \le \tilde{\mu}^*_{\tilde{\omega}a}A$. **Q**

Write $\ddot{\tilde{\mu}}_{\tilde{\omega}a}$ for the subspace measure on C_{dlg} induced by $\tilde{\mu}_{\tilde{\omega}a}$. By 214Cd, the outer measures $\ddot{\tilde{\mu}}^*_{\tilde{\omega}a} = \tilde{\mu}^*_{\tilde{\omega}a}\restriction\mathcal{P}C_{\text{dlg}}$ and $\ddot{\mu}^*_{\tilde{\omega}a}$ are the same. Because $\ddot{\tilde{\mu}}_{\tilde{\omega}a}$ and $\ddot{\mu}_{\tilde{\omega}a}$ are both complete probability measures, they must be identical (213C). Because $\tilde{\mu}_{\tilde{\omega}a}$ is a Radon measure, $\ddot{\tilde{\mu}}_{\tilde{\omega}a} = \ddot{\mu}_{\tilde{\omega}a}$ is quasi-Radon (415B). Because $\hat{\mu}_{\tilde{\omega}a}$ is the completion of a Baire measure, therefore inner regular with respect to the zero sets in Ω (412D, 412Ha), $\ddot{\mu}_{\tilde{\omega}a}$ is inner regular with respect to the zero sets in C_{dlg}, by 412Pd, and is completion regular.

455L Stopping times We need the continuous-time version of the concept of 'stopping time' introduced in §275. Let Ω be a set, Σ a σ-algebra of subsets of Ω and $\langle\Sigma_t\rangle_{t\ge0}$ a non-decreasing family of σ-subalgebras of Σ. (Such a family is called a **filtration**.) For $t \ge 0$, set $\Sigma_t^+ = \bigcap_{s>t}\Sigma_s$, so that $\langle\Sigma_t^+\rangle_{t\ge0}$ also is a non-decreasing family of σ-algebras. Of course $\Sigma_t^+ = \bigcap_{s>t}\Sigma_s^+$ for every $t \ge 0$.

(a) A function $\tau : \Omega \to [0,\infty]$ is a **stopping time adapted to** $\langle\Sigma_t\rangle_{t\ge0}$ if $\{\omega : \omega \in \Omega,\ \tau(\omega) \le t\}$ belongs to Σ_t for every $t \ge 0$.

Note that in this case τ will be Σ-measurable.

(b) A function $\tau : \Omega \to [0,\infty]$ is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$ iff $\{\omega : \tau(\omega) < t\} \in \Sigma_t$ for every $t \ge 0$. **P** (i) If τ is adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$ and $t \ge 0$, then $\{\omega : \tau(\omega) \le q\} \in \Sigma_q^+ \subseteq \Sigma_t$ whenever $0 \le q < t$, so

$$\{\omega : \tau(\omega) < t\} = \bigcup_{q\in\mathbb{Q}\cap[0,t[}\{\omega : \tau(\omega) \le q\} \in \Sigma_t.$$

Thus τ satisfies the condition. (ii) If τ satisfies the condition and $t \ge 0$, set $t_n = t + 2^{-n}$ for each n. Then

$$\{\omega : \tau(\omega) < t_n\} \in \Sigma_{t_n} \subseteq \Sigma_{t_m}$$

whenever $m \le n$, so

$$\{\omega : \tau(\omega) \le t\} = \bigcap_{n\ge m}\{\omega : \tau(\omega) < t_n\} \in \Sigma_{t_m}$$

for every m, and

$$\{\omega : \tau(\omega) \le t\} \in \bigcap_{m\in\mathbb{N}}\Sigma_{t_m} = \Sigma_t^+.$$

As t is arbitrary, τ is adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$. **Q**

(c)(i) Constant functions on Ω are of course stopping times.

(ii) If τ and τ' are stopping times adapted to $\langle\Sigma_t\rangle_{t\ge0}$, so is $\tau + \tau'$. **P**

$$\{\omega : \tau(\omega) + \tau'(\omega) \le t\} = \bigcap_{q\in\mathbb{Q}\cap[0,t]}\{\omega : \tau(\omega) \le q\} \cup \{\omega : \tau'(\omega) \le t - q\} \in \Sigma_t$$

for every $t \ge 0$. **Q**

(iii) (Compare 455Cb and 455E(c-ii)). If τ is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge0}$, then

$$\Sigma_\tau = \{E : E \in \Sigma,\ E \cap \{\omega : \tau(\omega) \le t\} \in \Sigma_t \text{ for every } t \ge 0\}$$

is a σ-subalgebra of Σ. (The check is elementary.)

(iv) If $\langle\tau_i\rangle_{i\in I}$ is a countable family of stopping times adapted to $\langle\Sigma_t\rangle_{t\ge0}$, then $\tau = \sup_{i\in I}\tau_i$ is adapted to $\langle\Sigma_t\rangle_{t\ge0}$. **P** For any $t \ge 0$,

$$\{\omega : \tau(\omega) \le t\} = \bigcap_{i\in I}\{\omega : \tau_i(\omega) \le t\} \in \Sigma_t. \textbf{ Q}$$

(v) If $\langle\tau_i\rangle_{i\in I}$ is a countable family of stopping times adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$, then $\tau = \inf_{i\in I}\tau_i$ is adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$, because

$$\{\omega : \tau(\omega) < t\} = \bigcup_{i\in I}\{\omega : \tau_i(\omega) < t\} \in \Sigma_t$$

for every $\tau \ge 0$.

(d) Now suppose that Y is a topological space and we have a family $\langle X_t\rangle_{t\ge 0}$ of functions from Ω to Y, and that $\tau : \Omega \to [0,\infty]$ is any Σ-measurable function. Set $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ when $\tau(\omega) < \infty$. If $(t,\omega) \mapsto X_t(\omega) : [0,\infty[\times\Omega \to Y$ is $\mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma$-measurable, where $\mathcal{B}([0,\infty[)$ is the Borel σ-algebra of $[0,\infty[$, then $X_\tau : \{\omega : \tau(\omega) < \infty\} \to Y$ is Σ-measurable. **P** Setting $\Omega_0 = \{\omega : \tau(\omega) < \infty\}$, the map $\omega \mapsto (\tau(\omega),\omega) : \Omega_0 \to [0,\infty[\times\Omega$ is $(\Sigma,\mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma)$-measurable (4A3Bc), so X_τ is the composition of a $(\Sigma,\mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma)$-measurable function with a $\mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma$-measurable function and is Σ-measurable, by 4A3Bb. **Q**

***(e)** Again take a topological space Y, a family $\langle X_t\rangle_{t\ge 0}$ of functions from Ω to Y, and a stopping time $\tau : \Omega \to [0,\infty]$ adapted to $\langle\Sigma_t\rangle_{t\ge 0}$. This time, suppose that $\langle X_t\rangle_{t\ge 0}$ is **progressively measurable**, that is, that $(s,\omega) \mapsto X_s(\omega) : [0,t]\times\Omega \to Y$ is $\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t$-measurable for every $t \ge 0$, and moreover that Σ_t is closed under Souslin's operation (421B) for every t. Then X_τ, as defined in (d), will be Σ_τ-measurable. **P** Suppose that $H \subseteq Y$ is open, and set $E = \{\omega : \omega \in \operatorname{dom} X_\tau,\ X_\tau(\omega) \in H\}$. Of course $\langle X_t\rangle_{t\ge 0}$ satisfies the condition of (d), so $E \in \Sigma$. Take any $t \ge 0$. Then

$$\begin{aligned}\{(s,\omega) : 0 \le s \le t,\ \tau(\omega) = s\} &= \bigcap_{n\in\mathbb{N}}\bigcup_{i\in\mathbb{N}}\{s : 2^{-n}(i-1) < s \le \min(t,2^{-n}i)\}\\ &\qquad\qquad \times\{\omega : 2^{-n}(i-1) < \tau(\omega) \le \min(t,2^{-n}i)\}\\ &\in \mathcal{B}([0,t])\widehat{\otimes}\Sigma_t,\end{aligned}$$

$$\{(s,\omega) : s \le t,\ X_s(\omega) \in H\} \in \mathcal{B}([0,t])\widehat{\otimes}\Sigma_t,$$

so

$$W = \{(s,\omega) : s \le t,\ \tau(\omega) = s,\ X_s(\omega) \in H\}$$

also belongs to $\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t$. Consequently the projection of W onto Ω belongs to $\mathcal{S}(\Sigma_t) = \Sigma_t$ (423M). But this is just

$$\{\omega : \tau(\omega) \le t,\ X_{\tau(\omega)} \in H\} = E \cap \{\omega : \tau(\omega) \le t\}.$$

As t is arbitrary, $E \in \Sigma_\tau$; as H is arbitrary, X_τ is Σ_τ-measurable. **Q**

***(f)** There are some technical points concerning stopping times which are perhaps worth noting here.

(i) Suppose that μ is a probability measure with domain Σ and null ideal $\mathcal{N}(\mu)$. Then we can form the completion $\hat\mu$ with domain $\hat\Sigma$. If we now set $\hat\Sigma_t = \{E\triangle A : E \in \Sigma_t,\ A \in \mathcal{N}(\mu)\}$, $\langle\hat\Sigma_t\rangle_{t\ge 0}$ and $\langle\hat\Sigma_t^+\rangle_{t\ge 0}$ are filtrations, where $\hat\Sigma_t^+ = \bigcap_{s>t}\hat\Sigma_s$ for $t \ge 0$.

(ii) We find that $\hat\Sigma_t^+ = \{E\triangle A : E \in \Sigma_t^+,\ A \in \mathcal{N}(\mu)\}$ for every $t \ge 0$. **P** Of course

$$\{E\triangle A : E \in \Sigma_t^+,\ A \in \mathcal{N}(\mu)\} \subseteq \bigcap_{s>t}\{E\triangle A : E \in \Sigma_s,\ A \in \mathcal{N}(\mu)\} = \hat\Sigma_t^+.$$

If $F \in \hat\Sigma_t^+$, then for every $q \in \mathbb{Q}$ such that $q > t$ there is an $E_q \in \Sigma_q$ such that $F\triangle E_q$ is negligible. Set

$$E = \bigcup_{q\in\mathbb{Q},q>t}\bigcap_{q'\in\mathbb{Q},t<q'\le q} E_{q'}, \qquad A = F\triangle E;$$

then $E \in \Sigma_t^+$, $A \in \mathcal{N}(\mu)$ and $F = E\triangle A$. **Q**

(iii) Of course every stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$ is adapted to $\langle\hat\Sigma_t^+\rangle_{t\ge 0}$. Conversely, if $\tau : \Omega \to [0,\infty]$ is a stopping time adapted to $\langle\hat\Sigma_t^+\rangle_{t\ge 0}$, there is a stopping time τ', adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$, such that $\tau =_{\text{a.e.}} \tau'$. **P** For each $q \in \mathbb{Q}\cap[0,\infty[$, set $F_q = \{\omega : \tau(\omega) < q\}$; by (b), $F_q \in \hat\Sigma_q$ and there is an $E_q \in \Sigma_q$ such that $F_q\triangle E_q$ is negligible. For $\omega \in \Omega$, set $\tau'(\omega) = \inf\{q : q \in \mathbb{Q}\cap[0,\infty[,\ \omega \in E_q\}$, counting $\inf\emptyset$ as ∞. Then $\{\omega : \tau'(\omega) < t\} = \bigcup_{q\in\mathbb{Q}\cap[0,t[} E_q$ belongs to Σ_t for every t, so τ' is adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$. And $\{\omega : \tau'(\omega) \ne \tau(\omega)\} \subseteq \bigcup_{q\in\mathbb{Q}\cap[0,\infty[} E_q\triangle F_q$ is negligible. **Q**

(iv) Continuing from (iii) just above, we find that, defining $\hat\Sigma_\tau^+$ from $\langle\hat\Sigma_t^+\rangle_{t\ge 0}$ and τ and $\Sigma_{\tau'}^+$ from $\langle\Sigma_t^+\rangle_{t\ge 0}$ and τ' by the formula in (c-iii), then $\hat\Sigma_\tau^+ = \{F\triangle A : F \in \Sigma_{\tau'}^+,\ A \in \mathcal{N}(\mu)\}$. **P** Let A_0 be the negligible set $\{\omega : \tau(\omega) \ne \tau'(\omega)\}$. (α) If $E \in \Sigma_{\tau'}^+$, then for every $t \ge 0$ we have

$$E\cap\{\omega:\tau'(\omega)\le t\}\in\Sigma_t^+,$$

$$(E\cap\{\omega:\tau(\omega)\le t\})\triangle(E\cap\{\omega:\tau'(\omega)\le t\})\subseteq A_0\in\mathcal{N}(\mu),$$

so (using (ii)) $E\cap\{\omega:\tau(\omega)\le t\}\in\hat\Sigma_t^+$; as t is arbitrary, $E\in\hat\Sigma_\tau^+$. (β) If $F\in\hat\Sigma_\tau^+$, then for every $q\in\mathbb{Q}\cap[0,\infty[$ the sets $F\cap\{\omega:\tau(\omega)\le q\}$ and $F\cap\{\omega:\tau'(\omega)\le q\}$ belong to $\hat\Sigma_q^+$, so there is an $E_q\in\Sigma_q^+$ such that $E_q\triangle(F\cap\{\omega:\tau'(\omega)\le q\})$ is negligible. Set $E'_q=\bigcup_{r\in\mathbb{Q}\cap[0,q]}E_r$ for $q\in\mathbb{Q}\cap[0,\infty[$; then $E'_q\in\Sigma_q^+$ and $E'_q\triangle(F\cap\{\omega:\tau'(\omega)\le q\})$ is negligible for each q, while $E'_q\subseteq E'_r$ if $q\le r$ in $\mathbb{Q}\cap[0,\infty[$. It follows that

$$\bigcap_{q\in\mathbb{Q}\cap[t,\infty[}E'_q=\bigcap_{q\in\mathbb{Q}\cap[t,s]}E'_q\in\Sigma_s^+$$

whenever $t<s$ in $[0,\infty[$, so that $\bigcap_{q\in\mathbb{Q}\cap[t,\infty[}E'_q\in\Sigma_t^+$ for every t. Set

$$E=\bigcap_{q\in\mathbb{Q}\cap[0,\infty[}E'_q\cup\{\omega:\tau'(\omega)>q\}.$$

Then

$$\begin{aligned}
&E\cap\{\omega:\tau'(\omega)\le t\}\\
&\quad=\{\omega:\tau'(\omega)\le t\}\cap\bigcap_{q\in\mathbb{Q}\cap[0,\infty[}(E'_q\cup\{\omega:\tau'(\omega)>q\})\\
&\quad=\{\omega:\tau'(\omega)\le t\}\cap\bigcap_{q\in\mathbb{Q}\cap[0,t[}(E'_q\cup\{\omega:\tau'(\omega)>q\})\cap\bigcap_{q\in\mathbb{Q}\cap[t,\infty[}E'_q\\
&\quad\in\Sigma_t^+
\end{aligned}$$

for any $t\ge0$, so $E\in\Sigma_{\tau'}^+$. If we look at $(E\triangle F)\cap\{\omega:\tau'(\omega)<\infty\}$, we see that this is included in the negligible set

$$\bigcup_{q\in\mathbb{Q}\cap[0,\infty[}E'_q\triangle(F\cap\{\omega:\tau'(\omega)\le q\})$$

because $\{\omega:\tau'(\omega)<\infty\}\cap F$ is just

$$\{\omega:\tau'(\omega)<\infty\}\cap\bigcap_{q\in\mathbb{Q}\cap[0,\infty[}(F\cap\{\omega:\tau'(\omega)\le q\})\cup\{\omega:\tau'(\omega)>q\}.$$

As for the set $H=\{\omega:\tau'(\omega)=\infty\}$, this belongs to Σ, and every subset of H belonging to Σ also belongs to $\Sigma_{\tau'}^+$. Let $H'\in\Sigma$ be such that $H'\subseteq H$ and $H'\triangle(F\cap H)$ is negligible; then $E'=H'\cup(E\setminus H)$ belongs to $\Sigma_{\tau'}^+$ and differs from F by a negligible set. **Q**

455M Hitting times I mention a class of stopping times which is particularly important in applications, and also very helpful in giving an idea of the concept.

Proposition Let U be a Polish space and C_{dlg} the set of càdlàg functions from $[0,\infty[$ to U. Let $A\subseteq U$ be an analytic set, and define $\tau:C_{\mathrm{dlg}}\to[0,\infty]$ by setting

$$\tau(\omega)=\inf\{t:\omega(t)\in A\}$$

for $\omega\in C_{\mathrm{dlg}}$, counting $\inf\emptyset$ as ∞.

(a) Let Σ be a σ-algebra of subsets of C_{dlg} closed under Souslin's operation and including the algebra generated by the functionals $\omega\mapsto\omega(t)$ for $t\ge0$. Then τ is Σ-measurable.

(b) For $t\ge0$ let Σ_t be

$$\{F:F\in\Sigma,\ \omega'\in F\text{ whenever }\omega,\,\omega'\in C_{\mathrm{dlg}},\ \omega\in F\text{ and }\omega\restriction[0,t]=\omega'\restriction[0,t]\},$$

and $\Sigma_t^+=\bigcap_{s>t}\Sigma_t$. Then τ is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$.

(c) If A is closed, then τ is adapted to $\langle\Sigma_t\rangle_{t\ge0}$.

proof (a)(i) It will help to recall from 4A3W that there is a Polish topology $\mathfrak{S}$ on C_{dlg} such that the Borel σ-algebra $\mathcal{B}(C_{\mathrm{dlg}})$ is just the σ-algebra generated by the coordinate functions $\omega\mapsto\omega(t)$. In this case, every $\mathfrak{S}$-analytic set, being $\mathfrak{S}$-Souslin-F (423Eb), belongs to Σ.

(ii) The set

$$W=\{(\omega,t,x):\omega\in C_{\mathrm{dlg}},\ t\ge0,\ x\in U,\ \omega(t)=x\}$$

is a Borel subset of $C_{\mathrm{dlg}}\times[0,\infty[\times U$. **P** If ρ is a metric on U inducing its topology and D is a countable dense subset of U,

$$W=\bigcap_{n\in\mathbb{N}}\bigcup_{q\in\mathbb{Q},q\ge 0}\bigcup_{y\in D}\{(\omega,t,x):t\in[q-2^{-n},q],$$
$$\rho(\omega(q),y)\le 2^{-n},\ \rho(x,y)\le 2^{-n}\}.\ \mathbf{Q}$$

(iii) Since C_{dlg}, $[0,\infty[$ and U are all Polish, W is an analytic set. Now, for any $t\ge 0$,

$$W'=\{(\omega,s,x):s\in[0,t[,\ x\in A,\ (\omega,s,x)\in W\}$$

is analytic and its projection

$$\{\omega:\tau(\omega)<t\}=\{\omega:\text{there are } s,\ x \text{ such that } (\omega,s,x)\in W'\}$$

is analytic and belongs to Σ. As t is arbitrary, τ is Σ-measurable.

(b) Now, given $t\ge 0$, $F=\{\omega:\tau(\omega)<t\}$ belongs to Σ, and if $\omega\in F$, $\omega'\in C_{\mathrm{dlg}}$ are such that $\omega'\restriction[0,t]=\omega\restriction[0,t]$, there is an $s<t$ such that $\omega'(s)=\omega(s)\in A$, so $\tau(\omega')<t$ and $\omega'\in F$. Thus $F\in\Sigma_t$. As t is arbitrary, τ is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$, by 455Lb.

(c) As A is closed and every member of C_{dlg} is continuous on the right, $\omega(\tau(\omega))\in A$ whenever $\tau(\omega)<\infty$. So if ω, $\omega'\in C_{\mathrm{dlg}}$, $\tau(\omega)\le t$ and $\omega'\restriction[0,t]=\omega\restriction[0,t]$, then $\omega'(\tau(\omega))\in A$ and $\tau(\omega')\le t$. Thus $\{\omega:\tau(\omega)\le t\}\in\Sigma_t$ for every t, and τ is adapted to $\langle\Sigma_t\rangle_{t\ge 0}$.

455N We need an elementary fact about narrow (more properly, vague) convergence.

Lemma Let (U,ρ) be a metric space, $n\in\mathbb{N}$ and $f:U^{n+1}\to\mathbb{R}$ a bounded uniformly continuous function. Let $\langle\nu_x^{(k)}\rangle_{k<n,x\in U}$ be a family of topological probability measures on U such that $x\mapsto\nu_x^{(k)}$ is continuous for the narrow topology for every $k<n$. Then

$$y\mapsto\iint\ldots\int f(y,x_1,\ldots,x_n)\nu_{x_{n-1}}^{(n-1)}(dx_n)\ldots\nu_{x_1}^{(1)}(dx_2)\nu_y^{(0)}(dx_1)$$

is defined everywhere on U and continuous.

proof Induce on n. If $n=0$ the formula is just $y\mapsto f(y)$, so the result is trivial. For the inductive step to $n\ge 1$, set

$$g(y,x_1)=\int\ldots\int f(y,x_1,\ldots,x_n)\nu_{x_{n-1}}^{(n-1)}(dx_n)\ldots\nu_{x_1}^{(1)}(dx_2)$$

for y, $x_1\in U$; by the inductive hypothesis this is well-defined and $x_1\mapsto g(y,x_1)$ is continuous. Note that g is bounded because f is. It follows that $h(y)=\int g(y,x_1)\nu_y^{(0)}(dx_1)$ is defined for every y. I need to show that h is continuous. Take any $y\in U$ and $\epsilon>0$. Then there is a $\delta_0>0$ such that $|f(y',x_1,\ldots,x_n)-f(y,x_1,\ldots,x_n)|\le\epsilon$ whenever $\rho(y',y)\le\delta_0$ and $x_1,\ldots,x_n\in U$; so that $|g(y',x_1)-g(y,x_1)|\le\epsilon$ whenever $\rho(y',y)\le\delta_0$ and $x_1\in U$. Next, because $x\mapsto\nu_x^{(0)}$ is narrowly continuous, $x\mapsto\int g(y,x_1)\nu_x^{(0)}(dx_1)$ is continuous (437Jf/437Kb), and there is a $\delta\in\,]0,\delta_0]$ such that $|\int g(y,x_1)\nu_{y'}^{(0)}(dx_1)-\int g(y,x_1)\nu_y^{(0)}(dx_1)|\le\epsilon$ whenever $\rho(y',y)\le\delta$. So if $\rho(y',y)\le\delta$,

$$\begin{aligned}|h(y')-h(y)|&\le|\int g(y',x_1)\nu_{y'}^{(0)}(dx_1)-\int g(y,x_1)\nu_{y'}^{(0)}(dx_1)|\\&\qquad+|\int g(y,x_1)\nu_{y'}^{(0)}(dx_1)-\int g(y,x_1)\nu_y^{(0)}(dx_1)|\\&\le\int|g(y',x_1)-g(y,x_1)|\nu_{y'}^{(0)}(dx_1)+\epsilon\le 2\epsilon.\end{aligned}$$

As y and ϵ are arbitrary, h is continuous and the induction proceeds.

455O If both the continuity conditions in 455F are satisfied, we have a version of 455C/455Eb which is much more to the point.

Theorem Suppose that (U,ρ) is a complete metric space, x^* is a point of U, $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ is a family of Radon probability measures on U which is both narrowly continuous and uniformly time-continuous on the right, and that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $x\in U$ and $s<t<u$. Let $\hat\mu$ be the corresponding completed measure on $\Omega=U^{[0,\infty[}$, as in 455E. Let C_{dlg} be the set of càdlàg functions from $[0,\infty[$ to U, $\ddot\mu$ the subspace measure on C_{dlg}, and $\ddot\Sigma$ its domain. For $t\ge 0$, let $\ddot\Sigma_t$ be

$$\{F : F \in \ddot{\Sigma},\ \omega' \in F \text{ whenever } \omega \in F,\ \omega' \in C_{\text{dlg}} \text{ and } \omega' \restriction [0,t] = \omega \restriction [0,t]\},$$

and $\ddot{\Sigma}_t^+ = \bigcap_{s>t} \ddot{\Sigma}_t$.

For $\omega \in \Omega$ and $a \geq 0$ let $\hat{\mu}_{\omega a}$ be the completed measure on Ω built from $\omega(0)$ and $\langle \nu_{\omega ax}^{(s,t)} \rangle_{0 \leq s < t, x \in U}$ as in 455Eb; let $\ddot{\mu}_{\omega a}$ be the subspace measure on C_{dlg}. Let $\tau : C_{\text{dlg}} \to [0,\infty]$ be a stopping time adapted to $\langle \ddot{\Sigma}_t^+ \rangle_{t \geq 0}$.

(a) $\langle \ddot{\mu}_{\omega,\tau(\omega)} \rangle_{\omega \in C_{\text{dlg}}}$ is a disintegration of $\ddot{\mu}$ over itself.

(b) Set

$$\ddot{\Sigma}_\tau^+ = \{F : F \in \ddot{\Sigma},\ F \cap \{\omega : \tau(\omega) \leq t\} \in \ddot{\Sigma}_t^+ \text{ for every } t \geq 0\}.$$

Then $\ddot{\Sigma}_\tau^+$ is a σ-algebra of subsets of C_{dlg}. For a $\ddot{\mu}$-integrable function f on C_{dlg}, write $\ddot{g}_f(\omega) = \int_{C_{\text{dlg}}} f d\ddot{\mu}_{\omega,\tau(\omega)}$ when this is defined in $\mathbb{R}$. Then $\ddot{g}_f$ is a conditional expectation of f on $\ddot{\Sigma}_\tau^+$.

(c) If τ is adapted to $\langle \ddot{\Sigma}_t \rangle_{t \geq 0}$, set

$$\ddot{\Sigma}_\tau = \{F : F \in \ddot{\Sigma},\ F \cap \{\omega : \tau(\omega) \leq t\} \in \ddot{\Sigma}_t \text{ for every } t \geq 0\}.$$

Then $\ddot{\Sigma}_\tau$ is a σ-algebra of subsets of C_{dlg}, and $\ddot{g}_f$ is a conditional expectation of f on $\ddot{\Sigma}_\tau$, for every $f \in \mathcal{L}^1(\ddot{\mu})$.

proof (a)(i) I had better begin by checking that the ground is clear. By 455G, $\hat{\mu}^* C_{\text{dlg}} = \hat{\mu}_{\omega a}^* C_{\text{dlg}} = 1$ for every $\omega \in C_{\text{dlg}}$ and $a \geq 0$, so that $\ddot{\mu}$ and $\ddot{\mu}_{\omega a}$ (for $\omega \in C_{\text{dlg}}$) are all probability measures.

Of course $\langle \ddot{\Sigma}_t \rangle_{t \geq 0}$ is a non-decreasing family of σ-subalgebras of $\ddot{\Sigma}$, so that $\langle \ddot{\Sigma}_t^+ \rangle_{t \geq 0}$ is another such family, and we are in the territory explored in 455L.

(ii) Write Σ for the domain of $\hat{\mu}$, and for $t \geq 0$ set

$$\Sigma_t = \{E : E \in \Sigma,\ E \text{ is determined by coordinates in } [0,t]\}.$$

Then $\ddot{\Sigma}_t = \{E \cap C_{\text{dlg}} : E \in \Sigma_t\}$. **P** If $E \in \Sigma_t$, then $E \cap C_{\text{dlg}} \in \ddot{\Sigma}$ and clearly $E \cap C_{\text{dlg}} \in \ddot{\Sigma}_t$. If $F \in \ddot{\Sigma}_t$, let $E \in \Sigma$ be such that $E \cap C_{\text{dlg}} = F$. Applying 455Ec to the stopping time with constant value t, we have

$$\hat{\mu}E = \int_\Omega \hat{\mu}_{\omega t}(E) \hat{\mu}(d\omega).$$

Set

$$E^* = \{\omega : \omega \in \Omega,\ \hat{\mu}_{\omega t}(E) \text{ is defined}\},$$

$$E_0 = \{\omega : \omega \in E^*,\ \hat{\mu}_{\omega t}(E) = 0\}, \quad E_1 = \{\omega : \omega \in E^*,\ \hat{\mu}_{\omega t}(E) = 1\}.$$

Then E^*, E_0 and E_1 are measured by $\hat{\mu}$ and are determined by coordinates in $[0,t]$ (by 455E(b-iii)), and $\hat{\mu}E^* = 1$.

If $\omega \in E^* \cap C_{\text{dlg}}$, then $\hat{\mu}_{\omega t}^* C_{\text{dlg}} = 1$, so

$$\ddot{\mu}_{\omega t}(F) = \hat{\mu}_{\omega t}^*(E \cap C_{\text{dlg}}) = \hat{\mu}_{\omega t}(E).$$

If $\omega \in C_{\text{dlg}}$, let D be a countable dense subset of $[0,t]$ containing t; then

$$\begin{aligned} 1 &= \hat{\mu}_{\omega t}\{\omega' : \omega' \in \Omega,\ \omega' \restriction D = \omega \restriction D\} = \ddot{\mu}_{\omega t}\{\omega' : \omega' \in C_{\text{dlg}},\ \omega' \restriction D = \omega \restriction D\} \\ &= \ddot{\mu}_{\omega t}\{\omega' : \omega' \in C_{\text{dlg}},\ \omega' \restriction [0,t] = \omega \restriction [0,t]\}. \end{aligned}$$

So if $\omega \in E^* \cap C_{\text{dlg}}$,

$$\hat{\mu}_{\omega t}(E) = \ddot{\mu}_{\omega t}(F) = \ddot{\mu}_{\omega t}\{\omega' : \omega' \in F,\ \omega' \restriction [0,t] = \omega \restriction [0,t]\} = \chi F(\omega) \in \{0,1\}$$

because F is determined (relative to C_{dlg}) by coordinates in $[0,t]$. This means that $E_1 \cap C_{\text{dlg}} \subseteq F$ and $E_0 \cap F = \emptyset$, while $E_0 \cup E_1$ is $\hat{\mu}$-conegligible. So if we take

$$E' = E_1 \cup \{\omega : \omega\Omega \setminus E_1 \text{ and there is an } \omega' \in F \text{ such that } \omega' \restriction [0,t] = \omega \restriction [0,t]\},$$

$E' \cap C_{\text{dlg}} = F$, E' is determined by coordinates in $[0,t]$, $E_1 \subseteq E' \subseteq \Omega \setminus E_0$, $\hat{\mu}$ measures E' and $E' \in \Sigma_t$. Thus $\ddot{\Sigma}_t = \{E \cap C_{\text{dlg}} : E \in \Sigma_t\}$, as claimed. **Q**

(iii) Take $n \in \mathbb{N}$, and set $D_n = \{2^{-n}i : i \in \mathbb{N}\}$. Suppose that $\tau : C_{\text{dlg}} \to D_n \cup \{\infty\}$ is a stopping time adapted to $\langle \ddot{\Sigma}_t \rangle_{t \geq 0}$. Then $\langle \hat{\mu}_{\omega,\tau(\omega)} \rangle_{\omega \in C_{\text{dlg}}}$ is a disintegration of $\hat{\mu}$ over $\ddot{\mu}$. **P** For each $i \in \mathbb{N}$, $F_i = \tau^{-1}[\{2^{-n}i\}]$ belongs to $\ddot{\Sigma}_{2^{-n}i}$, so there is an $E_i \in \Sigma$, determined by coordinates in $[0, 2^{-n}i]$, such that $F_i = E_i \cap C_{\text{dlg}}$. For $\omega \in \Omega$, set

$$\dot{\tau}(\omega) = \inf\{2^{-n}i : i \in \mathbb{N},\ \omega \in E_i\},$$

counting $\inf\emptyset$ as ∞. Then $\dot\tau\restriction C_{\mathrm{dlg}} = \tau$. Also $\dot\tau[\Omega] \subseteq D_n \cup \{\infty\}$ is countable, and $\dot\tau^{-1}[\{b\}] \in \Sigma$ is determined by coordinates in $[0, b]$ for every $b \in D_n$. By 455Ec, $\langle\hat\mu_{\omega,\dot\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of $\hat\mu$ over itself.

Now take any $E \in \Sigma$. Then

$$\hat\mu E = \int_\Omega \hat\mu_{\omega,\dot\tau(\omega)}(E)\hat\mu(d\omega) = \int_{C_{\mathrm{dlg}}} \hat\mu_{\omega,\dot\tau(\omega)}(E)\ddot\mu(d\omega)$$

(214F)

$$= \int_{C_{\mathrm{dlg}}} \hat\mu_{\omega,\tau(\omega)}(E)\ddot\mu(d\omega). \ \mathbf{Q}$$

It follows that $\langle\ddot\mu_{\omega,\tau(\omega)}\rangle_{\omega\in C_{\mathrm{dlg}}}$ is a disintegration of $\ddot\mu$ over itself. **P** If $F \in \ddot\Sigma$, there is an $E \in \Sigma$ such that $F = E \cap C_{\mathrm{dlg}}$. Now

$$\ddot\mu F = \hat\mu E = \int_{C_{\mathrm{dlg}}} \hat\mu_{\omega,\tau(\omega)}(E)\ddot\mu(d\omega)$$

$$= \int_{C_{\mathrm{dlg}}} \ddot\mu_{\omega,\tau(\omega)}(E \cap C_{\mathrm{dlg}})\ddot\mu(d\omega) = \int_{C_{\mathrm{dlg}}} \ddot\mu_{\omega,\tau(\omega)}(F)\ddot\mu(d\omega). \ \mathbf{Q}$$

(iv) Now let $\tau : C_{\mathrm{dlg}} \to [0, \infty]$ be any stopping time adapted to $\langle\ddot\Sigma_t^+\rangle_{t\ge0}$. For each $n \in \mathbb{N}$, define $\tau_n : C_{\mathrm{dlg}} \to D_n \cup \{\infty\}$ by setting

$$\begin{aligned}\tau_n(\omega) &= 2^{-n}(i+1) \text{ if } i \in \mathbb{N} \text{ and } 2^{-n}i \le \tau(\omega) < 2^{-n}(i+1),\\ &= \infty \text{ if } \tau(\omega) = \infty.\end{aligned}$$

By 455Lb, $\{\omega : \tau_n(\omega) = t\} \in \ddot\Sigma_t$ for every $t \in D_n$. So (iii) tells us that $\langle\ddot\mu_{\omega,\tau_n(\omega)}\rangle_{\omega\in C_{\mathrm{dlg}}}$ is a disintegration of $\ddot\mu$ over itself.

(v) Suppose that $k \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_k$, $h : U^{k+1} \to \mathbb{R}$ is bounded and uniformly continuous, and $\omega \in C_{\mathrm{dlg}}$. Then

$$\textstyle\int_\Omega h(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega') = \lim_{n\to\infty} \int_\Omega h(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau_n(\omega)}(d\omega').$$

P Recall from 455E that

$$\int_\Omega h(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega')$$

$$= \int_U \ldots \int_U h(\omega(0), x_1, \ldots, x_k)\nu^{(t_{k-1},t_k)}_{\omega,\tau(\omega),x_{k-1}}(dx_k) \ldots \nu^{(0,t_1)}_{\omega(0)}(dx_1),$$

and similarly for each τ_n. If $\tau(\omega) \ge t_k$, then

$$\nu^{(t_{i-1},t_i)}_{\omega,\tau_n(\omega),x} = \delta_{\omega(t_i)} = \nu^{(t_{i-1},t_i)}_{\omega,\tau(\omega),x}$$

for $1 \le i \le k$, $n \in \mathbb{N}$ and $x \in U$, so the result is trivial. If $j \le k$ is such that $t_{j-1} \le \tau(\omega) < t_j$, then

$$\begin{aligned}\nu^{(t_{i-1},t_i)}_{\omega,\tau(\omega),x} &= \delta_{\omega(t_i)} \text{ if } i < j,\\ &= \nu^{(\tau(\omega),t_j)}_{\omega(\tau(\omega))} \text{ if } i = j,\\ &= \nu^{(t_{i-1},t_i)}_x \text{ if } j < i < k.\end{aligned}$$

So

$$\int_\Omega h(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega')$$

$$= \int_U \int_U \ldots \int_U h(\omega(0), \ldots, \omega(j-1), x_j, \ldots, x_k)$$

$$\nu^{(t_{k-1},t_k)}_{x_{k-1}}(dx_k) \ldots \nu^{(t_j,t_{j+1})}_{x_j}(dx_{j+1})\nu^{(\tau(\omega),t_j)}_{\omega(\tau(\omega))}(dx_j).$$

Moreover, there is some n_0 such that $\tau_n(\omega) < t_j$ for every $n \ge n_0$, so that we can use this formula for all such n. Setting

$$g(x) = \int_U \ldots \int_U h(\omega(0), \ldots, \omega(j-1), x, x_{j+1}, \ldots, x_k)\nu_{x_{k-1}}^{(t_{k-1},t_k)}(dx_k) \ldots \nu_x^{(t_j,t_{j+1})}(dx_{j+1})$$

for $x \in U$, we see from 455N that g is continuous, while of course it is also bounded, because h is bounded. At this point, recall that ω is supposed to be continuous on the right, while the system of transitional probabilities is jointly continuous, so that

$$\nu_{\omega(\tau(\omega))}^{(\tau(\omega),t_j)} = \lim_{n\to\infty} \nu_{\omega(\tau_n(\omega))}^{(\tau_n(\omega),t_j)}$$

for the narrow topology, and

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega & h(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau_n(\omega)}(d\omega') \\
&= \lim_{n\to\infty} \int_U \int_U \ldots \int_U h(\omega(0), \ldots, \omega(j-1), x_j, \ldots, x_k) \\
&\qquad \nu_{x_{k-1}}^{(t_{k-1},t_k)}(dx_k) \ldots \nu_{x_j}^{(t_j,t_{j+1})}(dx_{j+1})\nu_{\omega(\tau_n(\omega))}^{(\tau_n(\omega),t_j)}(dx_j) \\
&= \lim_{n\to\infty} \int_U g(x_j)\nu_{\omega(\tau_n(\omega))}^{(\tau_n(\omega),t_j)}(dx_j) \\
&= \int_U g(x_j)\nu_{\omega(\tau(\omega))}^{(\tau(\omega),t_j)}(dx_j) \\
&= \int_\Omega h(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega'),
\end{aligned}$$

as claimed. **Q**

(vi) Again suppose that $0 = t_0 < t_1 < \ldots < t_k$. If $h : U^{k+1} \to \mathbb{R}$ is bounded and uniformly continuous, then

$$\int_{C_{\mathrm{dlg}}} \int_\Omega h(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega')\ddot\mu(d\omega) = \int_\Omega h(\omega(t_0), \ldots, \omega(t_k))\hat\mu(d\omega).$$

P The point here is that

$$\int_{C_{\mathrm{dlg}}} \int_\Omega h(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau_n(\omega)}(d\omega')\ddot\mu(d\omega) = \int_\Omega h(\omega(t_0), \ldots, \omega(t_k))\hat\mu(d\omega)$$

is defined for every $n \in \mathbb{N}$, by (iii) and 452F, as usual. Now the integrands

$$\omega \mapsto \int_\Omega h(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau_n(\omega)}(d\omega')$$

converge at every point of C_{dlg}, by (v), and are uniformly bounded, because h is, so that

$$\begin{aligned}
\int_{C_{\mathrm{dlg}}} \int_\Omega & h(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega')\ddot\mu(d\omega) \\
&= \lim_{n\to\infty} \int_{C_{\mathrm{dlg}}} \int_\Omega h(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau_n(\omega)}(d\omega')\ddot\mu(d\omega) \\
&= \int_\Omega h(\omega(t_0), \ldots, \omega(t_k))(d\omega).
\end{aligned}$$ **Q**

If $G \subseteq U^{k+1}$ is open, there is a non-decreasing sequence $\langle h_m\rangle_{m\in\mathbb{N}}$ of uniformly continuous functions from U^{k+1} to $[0,1]$ such that $\chi G = \sup_{m\in\mathbb{N}} h_m$, in which case

$$\begin{aligned}
\int_{C_{\mathrm{dlg}}} & \hat\mu_{\omega,\tau(\omega)}\{\omega' : (\omega'(t_0), \ldots, \omega'(t_k)) \in G\}\ddot\mu(d\omega) \\
&= \lim_{m\to\infty} \int_{C_{\mathrm{dlg}}} \int_\Omega h_m(\omega'(t_0), \ldots, \omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega')\ddot\mu(d\omega) \\
&= \lim_{m\to\infty} \int_\Omega h_m(\omega(t_0), \ldots, \omega(t_k))\hat\mu(d\omega) \\
&= \hat\mu\{\omega : (\omega(t_0), \ldots, \omega(t_k)) \in G\}.
\end{aligned}$$

By the Monotone Class Theorem, we get

$$\int_{C_{\mathrm{dlg}}} \hat\mu_{\omega,\tau(\omega)}\{\omega' : (\omega'(t_0),\dots,\omega'(t_k)) \in E\}\ddot\mu(d\omega)$$
$$= \hat\mu\{\omega : (\omega(t_0),\dots,\omega(t_k)) \in E\}$$

for every Borel set $E \subseteq U^{k+1}$. Now recall that $t_0,\dots,t_k$ were any strictly increasing sequence starting at 0, so we can use the Monotone Class Theorem yet again to see that

$$\int_{C_{\mathrm{dlg}}} \hat\mu_{\omega,\tau(\omega)}(E)\ddot\mu(d\omega) = \hat\mu(E)$$

for every $E \in \widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$ and therefore for every $E \in \Sigma$.

(vii) Finally, if $F \in \ddot\Sigma$, there is an $E \in \Sigma$ such that $F = E \cap C_{\mathrm{dlg}}$, so that

$$\ddot\mu(F) = \hat\mu(E) = \int_{C_{\mathrm{dlg}}} \hat\mu_{\omega,\tau(\omega)}(E)\ddot\mu(d\omega) = \int_{C_{\mathrm{dlg}}} \ddot\mu_{\omega,\tau(\omega)}(F)\ddot\mu(d\omega);$$

which is what we set out to prove.

(b)(i) By 455L(c-iii), $\ddot\Sigma^+_\tau$ is a σ-algebra. If f is a $\ddot\mu$-integrable real-valued function, then $\int_{C_{\mathrm{dlg}}} \ddot g_f \ddot\mu = \int_{C_{\mathrm{dlg}}} f\ddot\mu$, by (a) and 452F. For $\alpha \in \mathbb{R}$ set

$$E(f,\alpha) = \{\omega : \omega \in C_{\mathrm{dlg}},\ \ddot g_f(\omega) \text{ is defined in } \mathbb{R} \text{ and } \ddot g_f(\omega) \le \alpha\},$$

so that $E(f,\alpha) \in \ddot\Sigma$. For $t \ge 0$, set

$$H_t = \{\omega : \omega \in C_{\mathrm{dlg}},\ \tau(\omega) \le t\}, \quad H'_t = \{\omega : \omega \in C_{\mathrm{dlg}},\ \tau(\omega) < t\},$$

so that $H_t \in \ddot\Sigma^+_t$ and $H'_t \in \ddot\Sigma_t$ (455Lb).

(ii) If ω, $\omega' \in C_{\mathrm{dlg}}$ and $s > \tau(\omega)$ are such that $\omega'\restriction[0,s] = \omega\restriction[0,s]$, then $\tau(\omega') = \tau(\omega)$. **P** $H_{\tau(\omega)}$, $H'_{\tau(\omega)}$ and their difference belong to $\ddot\Sigma_s$, so are determined (relative to C_{dlg}) by coordinates in $[0,s]$; since $H_{\tau(\omega)} \setminus H'_{\tau(\omega)}$ contains ω, it also contains ω', and $\tau(\omega') = \tau(\omega)$. **Q**

(iii) If f is $\ddot\mu$-integrable, $\alpha \in \mathbb{R}$ and $s > 0$, then $E(f,\alpha) \cap H'_s \in \ddot\Sigma_s$. **P** Certainly $E(f,\alpha) \cap H'_s \in \ddot\Sigma$. If ω, $\omega' \in C_{\mathrm{dlg}}$ and $\omega\restriction[0,s] = \omega'\restriction[0,s]$, then

$$\omega \in E(f,\alpha) \cap H'_s \implies \tau(\omega) < s \text{ and } \int_{C_{\mathrm{dlg}}} f d\ddot\mu_{\omega,\tau(\omega)} \le \alpha$$
$$\implies \tau(\omega') = \tau(\omega) < s \text{ and } \int_{C_{\mathrm{dlg}}} f d\ddot\mu_{\omega,\tau(\omega')} \le \alpha$$

(by (ii))

$$\implies \omega' \in E(f,\alpha) \cap H'_s.$$

So $E(f,\alpha) \cap H'_s$ is determined (relative to C_{dlg}) by coordinates in $[0,s]$ and belongs to $\ddot\Sigma_s$. **Q**

Consequently $E(f,\alpha) \cap H_t \in \ddot\Sigma^+_t$ for every $t \ge 0$. **P** $H_t = \bigcap_{n\in\mathbb{N}} H'_{t_n}$ where $t_n = t + 2^{-n}$ for each n, so

$$E(f,\alpha) \cap H_t = \bigcap_{n\ge m} E(f,\alpha) \cap H'_{t_n}$$

belongs to $\ddot\Sigma_{t_m}$ for every $m \in \mathbb{N}$, and $E(f,\alpha) \cap H_t \in \ddot\Sigma^+_t$. **Q**

Thus $E(f,\alpha) \in \ddot\Sigma^+_\tau$ for every α. As α is arbitrary, $\operatorname{dom}\ddot g_f \in \ddot\Sigma^+_\tau$ and $\ddot g_f$ is $\ddot\Sigma^+_\tau$-measurable.

(iv) Define $\langle\tau_n\rangle_{n\in\mathbb{N}}$ as in (a-iv) above, so that each τ_n is a stopping time adapted to $\langle\ddot\Sigma_t\rangle_{t\ge0}$ and $\langle\tau_n(\omega)\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with limit $\tau(\omega)$ for every ω. For a $\ddot\mu$-integrable real-valued function f on C_{dlg}, $\omega \in C_{\mathrm{dlg}}$ and $n \in \mathbb{N}$, set

$$\ddot g_f^{(n)}(\omega) = \int_{C_{\mathrm{dlg}}} f d\ddot\mu_{\omega,\tau_n(\omega)}$$

whenever the right-hand side is defined in $\mathbb{R}$. By (a), $\int_{C_{\mathrm{dlg}}} \ddot g_f^{(n)} d\ddot\mu = \int_{C_{\mathrm{dlg}}} f d\ddot\mu$. We have seen also, in (a-iii), that each τ_n has an extension $\grave\tau_n$ which is a stopping time on Ω of the type considered in 455Ec. So if we take a $\hat\mu$-integrable function $\tilde f$ extending f, and set

$$g^{(n)}_{\tilde f}(\omega)=\int_\Omega \tilde f d\hat\mu_{\omega,\dot\tau_n(\omega)}$$

whenever $\omega\in\Omega$ is such that the integral is defined in $\mathbb{R}$, $g^{(n)}_{\tilde f}$ will be a conditional expectation of $\tilde f$ on $\Sigma_{\dot\tau_n}$, the algebra of sets $E\in\Sigma$ such that $E\cap\{\omega:\dot\tau_n(\omega)\le t\}$ is determined by coordinates in $[0,t]$ for every $t\ge 0$.

If $\omega\in C_{\rm dlg}$, then $C_{\rm dlg}$ has full outer measure for $\hat\mu_{\omega,\tau_n(\omega)}=\hat\mu_{\omega,\dot\tau_n(\omega)}$, so

$$g^{(n)}_{\tilde f}(\omega)=\int_\Omega \tilde f d\hat\mu_{\omega,\dot\tau_n(\omega)}=\int_{C_{\rm dlg}} fd\ddot\mu_{\omega,\tau_n(\omega)}=\ddot g^{(n)}_f(\omega)$$

whenever either is defined.

(v) Set

$$\ddot\Sigma_{\tau_n}=\{F:F\in\ddot\Sigma,\ F\cap\{\omega:\tau_n(\omega)\le t\}\in\ddot\Sigma_t\text{ for every }t\ge 0\}.$$

Then every $F\in\ddot\Sigma_{\tau_n}$ is of the form $\tilde F\cap C_{\rm dlg}$ where $\tilde F\in\Sigma_{\dot\tau_n}$. **P** Recall that τ_n and $\dot\tau_n$ take values in $D_n\cup\{\infty\}$, where $D_n=\{2^{-n}i:i\in\mathbb{N}\}$. For each $i\in\mathbb{N}$, set $F_i=\{\omega:\omega\in F,\ \tau_n(\omega)=2^{-n}i\}$; then $F_i\in\ddot\Sigma_{2^{-n}i}$, so there is an $E_i\in\Sigma_{2^{-n}i}$ such that $F_i=E_i\cap C_{\rm dlg}$ (a-ii). Let $E_\infty\in\Sigma$ be such that $E_\infty\cap C_{\rm dlg}=\{\omega:\tau_n(\omega)=\infty\}$, and try

$$\tilde F=\bigcup_{i\in\mathbb{N}}(E_i\cap\dot\tau_n^{-1}[\{2^{-n}i\}])\cup(E_\infty\cap\dot\tau_n^{-1}[\{\infty\}]).$$

Then $\tilde F\cap C_{\rm dlg}=F$ (because $\dot\tau_n$ extends τ_n) and $\tilde F\in\Sigma_{\dot\tau_n}$ (because

$$\tilde F\cap\dot\tau_n^{-1}[\{2^{-n}i\}]=E_i\cap\dot\tau_n^{-1}[\{2^{-n}i\}]\in\Sigma_{2^{-n}i}$$

for every i). **Q**

(vi) If f is $\ddot\mu$-integrable, then $\ddot g^{(n)}_f$ is a conditional expectation of f on $\ddot\Sigma_{\tau_n}$ for every n. **P** Take $F\in\ddot\Sigma_{\tau_n}$. Then there are an $\tilde F\in\Sigma_{\dot\tau_n}$ such that $F=\tilde F\cap C_{\rm dlg}$, and a $\hat\mu$-integrable $\tilde f$ such that $f=\tilde f\restriction C_{\rm dlg}$. So

$$\int_F fd\ddot\mu=\int_{\tilde F}\tilde fd\hat\mu=\int_{\tilde F}g^{(n)}_{\tilde f}d\hat\mu$$

(455E(c-ii))

$$=\int_{\tilde F}\int_\Omega\tilde fd\hat\mu_{\omega,\dot\tau_n(\omega)}\hat\mu(d\omega)=\int_F\int_\Omega\tilde fd\hat\mu_{\omega,\tau_n(\omega)}\ddot\mu(d\omega)$$

(because $\hat\mu^*C_{\rm dlg}=1$, $F=\tilde F\cap C_{\rm dlg}$ and $\tau_n=\dot\tau_n\restriction C_{\rm dlg}$)

$$=\int_F\int_{C_{\rm dlg}}fd\ddot\mu_{\omega,\tau_n(\omega)}\ddot\mu(d\omega)=\int_F\ddot g^{(n)}_fd\ddot\mu.\ \textbf{Q}$$

(vii) Let Φ be the set of those $\ddot\mu$-integrable real-valued functions f such that $\lim_{n\to\infty}\int_{C_{\rm dlg}}|\ddot g_f-\ddot g^{(n)}_f|d\ddot\mu=0$. For $J\subseteq[0,\infty[$ let $\pi_J:\Omega\to U^J$ be the restriction map. By (a-v), $f=h\pi_J\restriction C_{\rm dlg}$ belongs to Φ whenever $J\subseteq[0,\infty[$ is finite and $h:U^J\to\mathbb{R}$ is bounded and uniformly continuous, since in this case

$$\ddot g_f(\omega)=\int_{C_{\rm dlg}}h\pi_Jd\ddot\mu_{\omega,\tau(\omega)}=\int_\Omega h\pi_Jd\hat\mu_{\omega,\tau(\omega)}=\lim_{n\to\infty}\int_\Omega h\pi_Jd\hat\mu_{\omega,\tau_n(\omega)}$$
$$=\lim_{n\to\infty}\int_{C_{\rm dlg}}h\pi_Jd\ddot\mu_{\omega,\tau_n(\omega)}=\lim_{n\to\infty}\ddot g^{(n)}_f(\omega)$$

for every $\omega\in C_{\rm dlg}$. Next, $\ddot g_{\alpha f}=_{\text{a.e.}}\alpha\ddot g_f$, $\ddot g_{f+f'}=_{\text{a.e.}}\ddot g_f+\ddot g_{f'}$ and

$$\int_{C_{\rm dlg}}|\ddot g_f-\ddot g_{f'}|d\ddot\mu=\int_{C_{\rm dlg}}|\ddot g_{f-f'}|d\ddot\mu\le\int_{C_{\rm dlg}}\ddot g_{|f-f'|}d\ddot\mu=\int_{C_{\rm dlg}}|f-f'|d\ddot\mu$$

for all f, $f'\in\mathcal{L}^1(\ddot\mu)$ and $\alpha\in\mathbb{R}$; and we have similar expressions for every $\ddot g^{(n)}_f$. So $f+f'\in\Phi$ and $\alpha f\in\Phi$ whenever f, $f'\in\Phi$, and moreover $f\in\Phi$ whenever $f\in\mathcal{L}^1(\ddot\mu)$ and there is a sequence $\langle f_k\rangle_{f\in\mathbb{N}}$ in Φ such that $\lim_{k\to\infty}\int_{C_{\rm dlg}}|f-f_k|d\ddot\mu=0$.

If $J\subseteq[0,\infty[$ is finite and $G\subseteq U^J$, then $(\chi G)\pi_J\restriction C_{\rm dlg}\in\Phi$. **P** There is a non-decreasing sequence $\langle h_k\rangle_{k\in\mathbb{N}}$ of bounded uniformly continuous functions on U^J with limit χG; now $h_k\pi_J\restriction C_{\rm dlg}\in\Phi$ and $(\chi G)\pi_J\restriction C_{\rm dlg}=\lim_{k\to\infty}h_k\pi_J\restriction C_{\rm dlg}$. **Q** By the Monotone Class Theorem, $(\chi E)\pi_J\restriction C_{\rm dlg}\in\Phi$ whenever $J\subseteq[0,\infty[$ is finite and $E\in\mathcal{B}(U^J)$. By the Monotone Class Theorem again, $\chi(E\cap C_{\rm dlg})\in\Phi$ whenever $E\in\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$. Since we surely have $f'\in\Phi$ whenever

$f \in \Phi$ and $f' = f$ $\ddot{\mu}$-a.e., $\chi(E \cap C_{\mathrm{dlg}}) \in \Phi$ whenever $E \in \Sigma$, that is, $\chi E \in \Phi$ for every $E \in \ddot{\Sigma}$. It follows at once that $\Phi = \mathcal{L}^1(\ddot{\mu})$.

(viii) We are nearly home. Suppose that $f \in \mathcal{L}^1(\ddot{\mu})$ and $F \in \ddot{\Sigma}^+_\tau$. If $n \in \mathbb{N}$, then $F \in \ddot{\Sigma}_{\tau_n}$. **P** For any $t > 0$,

$$F \cap \{\omega : \tau(\omega) < t\} = \bigcup_{q \in \mathbb{Q}, q < t} F \cap \{\omega : \tau(\omega) \le q\} \in \ddot{\Sigma}_t.$$

So, for any $i \in \mathbb{N}$,

$$F \cap \{\omega : \tau_n(\omega) \le 2^{-n} i\} = F \cap \{\omega : \tau(\omega) < 2^{-n} i\} \in \ddot{\Sigma}_{2^{-n} i}. \text{ } \mathbf{Q}$$

So $\int_F \ddot{g}^{(n)}_f d\ddot{\mu} = \int_F f d\ddot{\mu}$. But $f \in \Phi$, so

$$\int_F \ddot{g}_f d\ddot{\mu} = \lim_{n\to\infty} \int_F \ddot{g}^{(n)}_f d\ddot{\mu} = \int_F f d\ddot{\mu}.$$

Since we already know, from (iii) above, that $\operatorname{dom} \ddot{g}_f \in \ddot{\Sigma}^+_\tau$ and $\ddot{g}_f$ is $\ddot{\Sigma}^+_\tau$-measurable, $\ddot{g}_f$ is a conditional expectation of f on $\ddot{\Sigma}^+_\tau$, as claimed.

(c)(i) By 455L(c-iii) again, $\ddot{\Sigma}_\tau$ is a σ-algebra.

(ii) If ω, $\omega' \in C_{\mathrm{dlg}}$ and $\omega' \restriction [0, \tau(\omega)] = \omega \restriction [0, \tau(\omega)]$ then $\ddot{\mu}_{\omega', \tau(\omega')} = \ddot{\mu}_{\omega, \tau(\omega)}$. **P** Set $t = \tau(\omega)$. This time, H_t and H'_t, defined as in (b-i), belong to $\ddot{\Sigma}_t$, so their difference belongs to $\ddot{\Sigma}_t$ and is determined (relative to C_{dlg}) by coordinates in $[0, t]$; so $\omega' \in H_t \setminus H'_t$ and $\tau(\omega') = t$. Now, reading off the definition in 455Eb, $\nu^{(s,u)}_{\omega' t x} = \nu^{(s,u)}_{\omega t x}$ for all s, u and x, so $\ddot{\mu}_{\omega' t} = \ddot{\mu}_{\omega t}$. **Q**

(iii) It follows that if $f \in \mathcal{L}^1(\ddot{\mu})$ and $\alpha \in \mathbb{R}$ then $F = \{\omega : \omega \in C_{\mathrm{dlg}},\ \ddot{g}_f(\omega)$ is defined and at most $\alpha\}$ belongs to $\ddot{\Sigma}_\tau$. **P** We know from (b-iii) that $F \in \ddot{\Sigma}$. If $t \ge 0$, $\omega \in F$, $\omega' \in C_{\mathrm{dlg}}$, $\tau(\omega) \le t$ and $\omega' \restriction [0, t] = \omega \restriction [0, t]$ then $\ddot{\mu}_{\omega', \tau(\omega')} = \ddot{\mu}_{\omega, \tau(\omega)}$, so $\ddot{g}_f(\omega') = \ddot{g}_f(\omega)$ and $\omega' \in F$. Thus $F \cap \{\omega : \tau(\omega) \le t\}$ is determined (relative to C_{dlg}) by coordinates in $[0, t]$ and belongs to $\ddot{\Sigma}_t$. **Q**

(iv) Thus $\operatorname{dom} \ddot{g}_f \in \ddot{\Sigma}_\tau$ and $\ddot{g}_f$ is $\ddot{\Sigma}_\tau$-measurable. As we already know that it is a conditional expectation of f on $\ddot{\Sigma}^+_\tau \supseteq \ddot{\Sigma}_\tau$, it is a conditional expectation of f on $\ddot{\Sigma}_\tau$.

455P The eventual objective of this section is to provide a foundation for study of the original, and still by far the most important, example of a continuous-time Markov process, Brownian motion. In the language developed above, we shall have $U = \mathbb{R}$ (or, when we come to the applications in §§477-479, $U = \mathbb{R}^r$), and all the transitional probabilities $\nu^{(s,t)}_x$ will be Gaussian. But the techniques so far developed can tell us a great deal about much more general processes with some of the same features.

Theorem Let U be a metrizable topological group which is complete under a right-translation-invariant metric ρ inducing its topology. Let $\langle \lambda_t \rangle_{t>0}$ be a family of Radon probability measures on U such that $\lambda_s * \lambda_t = \lambda_{s+t}$ for all s, $t > 0$. Suppose that $\lim_{t \downarrow 0} \lambda_t G = 1$ for every open neighbourhood G of the identity in U. For $x \in U$ and $0 \le s < t$, let $\nu^{(s,t)}_x$ be the Radon probability measure on U defined by saying that $\nu^{(s,t)}_x(E) = \lambda_{t-s}(Ex^{-1})$ whenever λ_{t-s} measures Ex^{-1}.

(a) $\langle \nu^{(t,u)}_y \rangle_{y \in U}$ is a disintegration of $\nu^{(s,u)}_x$ over $\nu^{(s,t)}_x$ whenever $x \in U$ and $0 \le s < t < u$.

(b) $\langle \nu^{(s,t)}_x \rangle_{0 \le s < t, x \in U}$ is narrowly continuous and uniformly time-continuous on the right.

(c)(i) For any $x^* \in U$, we can define a complete measure $\hat{\mu}$ on $U^{[0,\infty[}$ by the method of 455E applied to x^* and $\langle \nu^{(s,t)}_x \rangle_{0 \le s < t, x \in U}$.

(ii) If C_{dlg} is the space of càdlàg functions from $[0, \infty[$ to U, then $\hat{\mu}^* C_{\mathrm{dlg}} = 1$, and the subspace measure $\ddot{\mu}$ on C_{dlg} will have the properties described in 455O.

(iii) $\hat{\mu}$ has a unique extension to a Radon measure $\tilde{\mu}$ on $U^{[0,\infty[}$.

proof (a) Note first that $y \mapsto yx$ is inverse-measure-preserving for λ_{t-s} and $\nu^{(s,t)}_x$, so that $\int f(y) \nu^{(s,t)}_x(dy) = \int f(yx) \lambda_{t-s}(dy)$ for any real-valued function on U for which either is defined (235Gb). If $E \subseteq U$ is measured by $\nu^{(s,u)}_x$, then

$$\nu^{(s,u)}_x(E) = \lambda_{u-s}(Ex^{-1}) = (\lambda_{u-t} * \lambda_{t-s})(Ex^{-1}) = \int \lambda_{u-t}(Ex^{-1}y^{-1}) \lambda_{t-s}(dy)$$

(444A)

$$= \int \nu_{yx}^{(t,u)}(E)\lambda_{t-s}(dy) = \int \nu_y^{(t,u)}(E)\nu_x^{(s,t)}(dy);$$

as E is arbitrary, $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$.

(b)(i)(α) Suppose that $x\in U$ and $0\le s<t$; set $u=t-s$. Let $f:U\to\mathbb{R}$ be a bounded continuous function and set $M=\|f\|_\infty$. Take $\epsilon\in\,]0,1[$. Let $K\subseteq U$ be a compact set such that $\lambda_u K\ge 1-\epsilon$. Then there is a symmetric open neighbourhood V of the identity e of U such that $|f(wyx^{-1})-f(wx^{-1})|\le 2\epsilon$ whenever $w\in K$ and $y\in V^2$. **P** For each $w\in K$ there is a neighbourhood W_w of e such that $|f(wyx^{-1})-f(wx^{-1})|\le\epsilon$ whenever $y\in W_w^2$. Because K is compact, there are $w_0,\dots,w_n\in K$ such that $K\subseteq\bigcup_{i\le n}w_iW_{w_i}$; set $W=\bigcap_{i\le n}W_{w_i}$. If $w\in K$ and $y\in W$, there is an $i\le n$ such that $w\in w_iW_{w_i}$, in which case

$$\begin{aligned}|f(wyx^{-1})-f(wx^{-1})| &\le |f(w_i(w_i^{-1}wy)x^{-1})-f(w_ix^{-1})|\\ &\qquad\qquad + |f(w_ix^{-1})-f(w_i(w_i^{-1}w)x^{-1})|\\ &\le 2\epsilon\end{aligned}$$

because both $w_i^{-1}wy$ and $w_i^{-1}w$ belong to $W_{w_i}^2$. So if we take a symmetric open neighbourhood V of e such that $V^2\subseteq W$, this will serve. **Q**

(β) Let $\delta>0$ be such that $\lambda_vV\ge 1-\epsilon$ whenever $0<v\le 2\delta$. It will be worth noting that $\lambda_v(KV)\ge 1-2\epsilon$ whenever $0<v<u$ and $u-v\le 2\delta$. **P** In this case, $\lambda_u=\lambda_v*\lambda_{u-v}$. Now $U\setminus K\supseteq(U\setminus KV)V^{-1}$. So

$$\epsilon\ge\lambda_u(U\setminus K)\ge\lambda_v(U\setminus KV)\lambda_{u-v}(V^{-1})\ge(1-\epsilon)\lambda_v(U\setminus KV)$$

and

$$\lambda_v(KV)\ge 1-\tfrac{\epsilon}{1-\epsilon}\ge 1-2\epsilon.\ \mathbf{Q}$$

(γ) Suppose that $0\le s'<t'$ and $y\in U$ are such that $y^{-1}x\in W$, $|s'-s|\le\delta$ and $|t'-t|\le\delta$. Then $|\int fd\nu_y^{(s',t')}-\int fd\nu_x^{(s,t)}|\le(6M+4)\epsilon$. **P** Set $u'=t'-s'$, so that $|u-u'|\le 2\delta$. We have

$$|\int fd\nu_y^{(s',t')}-\int fd\nu_x^{(s,t)}|=|\int f(wy^{-1})\lambda_{u'}(dw)-\int f(wx^{-1})\lambda_u(dw)|.$$

case 1 Suppose that $u'<u$. Then $\lambda_u=\lambda_{u'}*\lambda_{u-u'}$, so

$$\begin{aligned}|\int fd\nu_y^{(s',t')}-\int fd\nu_x^{(s,t)}| &= |\int f(wy^{-1})\lambda_{u'}(dw)-\int f(wx^{-1})(\lambda_{u'}*\lambda_{u-u'})(dw)|\\ &= |\int f(wy^{-1})\lambda_{u'}(dw)-\iint f(wzx^{-1})\lambda_{u-u'}(dz)\lambda_{u'}(dw)|\end{aligned}$$

(444C)

$$\begin{aligned}&\le\int|f(wy^{-1})-\int f(wzx^{-1})\lambda_{u-u'}(dz)|\lambda_{u'}(dw)\\ &\le 4M\epsilon+\sup_{w\in KV}|f(wy^{-1})-\int f(wzx^{-1})\lambda_{u-u'}(dz)|\end{aligned}$$

(because $\lambda_{u'}(U\setminus KV)\le 2\epsilon$, by ($\beta$), and $|f(wy^{-1})-\int f(wzx^{-1})\lambda_{u-u'}(dz)|\le 2M$ for every w)

$$\begin{aligned}&\le 4M\epsilon+\sup_{w\in KV}\int|f(wy^{-1})-\int f(wzx^{-1})|\lambda_{u-u'}(dz)\\ &\le 6M\epsilon+\sup_{w\in KV,z\in V}|f(wy^{-1})-\int f(wzx^{-1})|\end{aligned}$$

(because $\lambda_{u-u'}V\ge 1-\epsilon$)

$$\begin{aligned}&\le 6M\epsilon+\sup_{w\in K,v\in V,z\in V}|f(wvy^{-1}xx^{-1})-f(wvzx^{-1})|\\ &\le 6M\epsilon+\sup_{w\in K,v\in V,z\in V}\bigl(|f(wvy^{-1}xx^{-1})-f(wx^{-1})|\\ &\qquad\qquad\qquad + |f(wvzx^{-1})-f(wx^{-1})|\bigr)\\ &\le 6M\epsilon+4\epsilon\end{aligned}$$

by the choice of V, because $y^{-1}x \in V$.

case 2 Suppose that $u' = u$. Then

$$\begin{aligned}|\int f d\nu_y^{(s',t')} - \int f d\nu_x^{(s,t)}| &\le \int |f(wy^{-1}) - f(wx^{-1})|\lambda_u(dw) \\ &\le 2M\epsilon + \sup_{w\in K} |f(wy^{-1}xx^{-1}) - f(wx^{-1})| \\ &\le 2M\epsilon + \epsilon.\end{aligned}$$

case 3 Suppose that $u' > u$. Then $\lambda_{u'} = \lambda_u * \lambda_{u'-u}$, so

$$\begin{aligned}|\int f d\nu_y^{(s',t')} - \int f d\nu_x^{(s,t)}| &= |\iint f(wzy^{-1})\lambda_{u'-u}(dz)\lambda_u(dw) - \int f(wx^{-1})\lambda_u(dw)| \\ &\le \int |\int f(wzy^{-1})\lambda_{u'-u}(dz) - f(wx^{-1})|\lambda_u(dw) \\ &\le 2M\epsilon + \sup_{w\in K} |\int f(wzy^{-1})\lambda_{u'-u}(dz) - f(wx^{-1})| \\ &\le 2M\epsilon + 2M\epsilon + \sup_{w\in K, z\in V} |f(wzy^{-1}) - f(wx^{-1})| \\ &\le 4M\epsilon + 2\epsilon.\end{aligned}$$

So we have the result in all cases. **Q**

(δ) As s, t, x, ϵ and f are arbitrary, $\langle \nu_x^{(s,t)}\rangle_{0\le s<t, x\in U}$ is narrowly (= vaguely) continuous.

(ii) Given $\epsilon > 0$, there is a $\delta > 0$ such that $\lambda_t\{x : \rho(x,e) < \epsilon\} \ge 1 - \epsilon$ whenever $0 < t \le \delta$. Now suppose that $x \in U$ and $0 \le s < t \le s + \delta$. Then

$$\nu_x^{(s,t)} B(x,\epsilon) = \lambda_{t-s}(B(x,\epsilon)x^{-1}) = \lambda_{t-s} B(e,\epsilon)$$

(because ρ is right-translation-invariant)

$$\ge 1 - \epsilon.$$

As ϵ is arbitrary, $\langle \nu_x^{(s,t)}\rangle_{0\le s<t, x\in U}$ is uniformly time-continuous on the right.

(c) This is now just a matter of putting 455O and 455H together.

455Q Lévy processes If we approach as probabilists, without prejudices in favour of any particular realization, the processes in 455P manifest themselves as follows. Let U be a separable metrizable topological group with identity e, and consider the following list of properties of a family $\langle X_t\rangle_{t\ge 0}$ of U-valued random variables:

$X_0 = e$ almost everywhere,

$\Pr(X_t X_s^{-1} \in F) = \Pr(X_{t-s} \in F)$ whenever $0 \le s < t$ and $F \subseteq U$ is Borel

(the process is **stationary**),

whenever $0 \le t_0 < t_1 < \ldots < t_n$, then $X_{t_1}X_{t_0}^{-1}, X_{t_2}X_{t_1}^{-1}, \ldots, X_{t_n}X_{t_{n-1}}^{-1}$ are independent

in the sense of 418U (the process has **independent increments**),

$X_t \to e$ in measure as $t \downarrow 0$

(that is, $\lim_{t\downarrow 0} \Pr(X_t \in G) = 1$ for every neighbourhood G of the identity). I say here that U should be separable and metrizable in order to ensure that all the functions $X_t X_s^{-1}$ should be measurable (of course it will be enough if U is metrizable and of measure-free weight, as in 438E). Such a family I will call a **Lévy process**.

455R Theorem Let U be a Polish group with identity e which is complete under a right-translation-invariant metric inducing its topology. A family $\langle X_t\rangle_{t\ge 0}$ of U-valued random variables is a Lévy process iff there is a family $\langle \lambda_t\rangle_{t>0}$ of Radon probability measures on U, satisfying the conditions of 455P, such that if we start from $x^* = e$ and build the measure $\hat\mu$ on $U^{[0,\infty[}$ as in 455P, then

$$\Pr(X_{t_i} \in F_i \text{ for every } i \le n) = \hat\mu\{\omega : \omega(t_i) \in F_i \text{ for every } i \le n\}$$

whenever $t_0, \dots, t_n \in [0, \infty[$ and $F_i \subseteq U$ is a Borel set for every $i \le n$.

proof (a) Suppose we have a family $\langle \lambda_t \rangle_{t>0}$ of Radon probability measures on U such that $\lambda_s * \lambda_t = \lambda_{s+t}$ for all s, $t > 0$ and $\lim_{t\downarrow 0} \lambda_t G = 1$ for every open neighbourhood G of e in U. Define $\langle \nu_x^{(s,t)} \rangle_{0 \le s < t, x \in U}$ as in 455P, and let $\hat\mu$ be the corresponding completed measure on $\Omega = U^{[0,\infty[}$ as in 455Pc, starting from $x^* = e$. Set $X_t(\omega) = \omega(t)$ for $t \ge 0$ and $\omega \in \Omega$. Then $X_0 = e$ a.e. (455Ea) and

$$\Pr(X_t \in F) = \hat\mu X_t^{-1}[F] = \nu_0^{(0,t)} F = \lambda_t F$$

for $t > 0$ and $F \in \mathcal{B}(U)$ (455Ea again). In particular,

$$\lim_{t\downarrow 0} \Pr(X_t \in G) = \lim_{t\downarrow 0} \lambda_t G = 1$$

for every neighbourhood G of the identity. If $0 < s < t$ and $F \in \mathcal{B}(U)$, set $H = \{(e, x, y) : yx^{-1} \in F\} \subseteq U^3$. Then

$$\begin{aligned}
\Pr(X_t X_s^{-1} \in F) &= \hat\mu\{\omega : (\omega(0), \omega(s), \omega(t)) \in H\} \\
&= \iint \chi H(e, x, y) \nu_x^{(s,t)}(dy) \nu_e^{(0,s)}(dx)
\end{aligned}$$

(455E)

$$\begin{aligned}
&= \iint \chi H(e, x, yx) \lambda_{t-s}(dy) \lambda_s(dx) \\
&= \iint \chi F(y) \lambda_{t-s}(dy) \lambda_s(dx) = \lambda_{t-s}(F) = \Pr(X_{t-s} \in F).
\end{aligned}$$

If $0 = s < t$ then $X_t X_s^{-1} =_{\text{a.e.}} X_t = X_{t-s}$, of course. If $0 = t_0 < t_1 < \ldots < t_n$ and $F_0, \dots, F_{n-1} \in \mathcal{B}(U)$, set

$$E_k = \{\omega : \omega \in \Omega,\, \omega(t_{i+1})\omega(t_i)^{-1} \in F_i \text{ for every } i < k\},$$

$$H_k = \{(x_0, \dots, x_k) : x_{i+1}x_i^{-1} \in F_i \text{ for every } i < k\} \subseteq U^{k+1}$$

for $k \le n$. Then

$$\hat\mu E_1 = \hat\mu\{\omega : \omega(t_1)\omega(0)^{-1} \in F_0\} = \hat\mu\{\omega : \omega(t_1) \in F_0\} = \nu_e^{(0,t_1)} F_0 = \lambda_{t_1} F_0,$$

and for $k \ge 2$

$$\begin{aligned}
&\Pr(X_{t_{i+1}} X_{t_i}^{-1} \in F_i \text{ for every } i < k) \\
&\quad = \hat\mu E_k = \int \dots \int \chi H_k(e, x_1, \dots, x_k) \nu_{x_{k-1}}^{(t_{k-1},t_k)}(dx_k) \dots \nu_e^{(0,t_1)}(dx_1)
\end{aligned}$$

(455E)

$$\begin{aligned}
&= \int \dots \iint \chi H_k(e, x_1, \dots, x_{k-1}, x_k x_{k-1}) \lambda_{t_k - t_{k-1}}(dx_k) \\
&\qquad \nu_{x_{k-2}}^{(t_{k-2},t_{k-1})}(dx_{k-1}) \dots \nu_e^{(0,t_1)}(dx_1) \\
&= \int \dots \iint \chi H_{k-1}(e, x_1, \dots, x_{k-1}) \chi F_{k-1}(x_k) \lambda_{t_k - t_{k-1}}(dx_k) \\
&\qquad \nu_{x_{k-2}}^{(t_{k-2},t_{k-1})}(dx_{k-1}) \dots \nu_e^{(0,t_1)}(dx_1) \\
&= \lambda_{t_k - t_{k-1}}(F_{k-1}) \int \dots \int \chi H_{k-1}(e, x_1, \dots, x_{k-1}) \\
&\qquad \nu_{x_{k-2}}^{(t_{k-2},t_{k-1})}(dx_{k-1}) \dots \nu_e^{(0,t_1)}(dx_1) \\
&= \lambda_{t_k - t_{k-1}}(F_{k-1}) \cdot \hat\mu E_{k-1}.
\end{aligned}$$

So

$$\Pr(X_{t_{i+1}}X_{t_i}^{-1} \in F_i \text{ for every } i < n) = \hat{\mu}E_n = \prod_{i=0}^{n-1} \lambda_{t_i - t_{i-1}} F_i$$
$$= \prod_{i=0}^{n-1} \Pr(X_{t_i - t_{i-1}} \in F_i) = \prod_{i=0}^{n-1} \Pr(X_{t_i}X_{t_{i-1}}^{-1} \in F_i).$$

As $F_0, \dots, F_{n-1}$ are arbitrary, $X_{t_1}X_{t_0}^{-1}, X_{t_2}X_{t_1}^{-1}, \dots, X_{t_n}X_{t_{n-1}}^{-1}$ are independent. Thus all the conditions of 455Q are satisfied.

(b)(i) In the other direction, given a family $\langle X_t \rangle_{t \ge 0}$ with the properties listed in 455Q, then for each $t > 0$ there is a Radon measure λ_t on U such that $\lambda_t F = \Pr(X_t \in F)$ for every $F \in \mathcal{B}(U)$, for each $t > 0$. **P** U is Polish, therefore analytic, and we can apply 433Cb to the Borel measure $F \mapsto \Pr(X_t \in F)$. **Q** If s, $t > 0$, then the distribution of $X_{s+t}X_s^{-1}$ is the same as the distribution of X_t, so is λ_t.

If s, $t > 0$ then $\lambda_{s+t} = \lambda_s * \lambda_t$. **P** If F_1, $F_2 \in \mathcal{B}(U)$ then

$$\Pr((X_t, X_{s+t}X_t^{-1}) \in F_1 \times F_2) = \Pr(X_t \in F_1, X_{s+t}X_t^{-1} \in F_2)$$
$$= \Pr(X_t \in F_1) \Pr(X_{s+t}X_t^{-1} \in F_2)$$

(because X_t and $X_{s+t}X_t^{-1}$ are independent)

$$= \lambda_t F_1 \cdot \lambda_s F_2 = (\lambda_t \times \lambda_s)(F_1 \times F_2).$$

By the Monotone Class Theorem, or otherwise,

$$\Pr((X_t, X_{s+t}X_t^{-1}) \in H) = (\lambda_t \times \lambda_s)H$$

for every $H \in \mathcal{B}(U)\widehat{\otimes}\mathcal{B}(U) = \mathcal{B}(U^2)$. So if $F \in \mathcal{B}(U)$ we shall have

$$(\lambda_s * \lambda_t)(F) = (\lambda_s \times \lambda_t)\{(x,y) : xy \in F\} = (\lambda_t \times \lambda_s)\{(y,x) : xy \in F\}$$
$$= \Pr(X_{s+t}X_t^{-1}X_t \in F) = \lambda_{s+t}F,$$

and $\lambda_s * \lambda_t = \lambda_{s+t}$. (Cf. 272T[18].) **Q**

Next, for any neighbourhood G of e,

$$\lim_{t \downarrow 0} \lambda_t G = \lim_{t \downarrow 0} \Pr(X_t \in G) = 1.$$

So $\langle \lambda_t \rangle_{t > 0}$ satisfies the conditions of 455P. Let $\hat{\mu}$ be the corresponding completed measure on $U^{[0,\infty[}$ as in 455Pc.

(ii) $\Pr(X_{t_i} \in F_i \text{ for every } i \le k) = \hat{\mu}\{\omega : \omega(t_i) \in F_i \text{ for every } i \le k\}$ whenever $t_0, \dots, t_n \in [0, \infty[$ and $F_i \in \mathcal{B}(U)$ for every $i \le k$. **P** It is enough to consider the case $0 = t_0 < t_1 < \dots < t_n$. In this case, whenever $E_0, \dots, E_n \in \mathcal{B}(U)$,

$$\Pr((X_{t_0}, X_{t_1}X_{t_0}^{-1}, \dots, X_{t_n}X_{t_{n-1}}^{-1}) \in E_0 \times \dots \times E_n)$$
$$= \Pr(X_{t_0} \in E_0) \Pr(X_{t_1}X_{t_0}^{-1} \in E_1) \dots \Pr(X_{t_n}X_{t_{n-1}}^{-1}) \in E_n)$$
$$= \delta_e(E_0)\lambda_{t_1 - t_0}(E_1) \dots \lambda_{t_n - t_{n-1}}(E_n)$$

(where δ_e is the Dirac measure concentrated at e)

$$= \hat{\mu}\{\omega : \omega(t_0) \in E_0\}\hat{\mu}\{\omega : \omega(t_1)\omega(t_0)^{-1} \in E_1\} \dots$$
$$\hat{\mu}\{\omega : \omega(t_n)\omega(t_{n-1})^{-1} \in E_n\}$$
$$= \hat{\mu}\{\omega : \omega(t_0) \in E_0,\, \omega(t_1)\omega(t_0)^{-1} \in E_1,\, \dots \omega(t_n)\omega(t_{n-1})^{-1} \in E_n\}$$

(by (a) above)

$$= \hat{\mu}\{\omega : (\omega(t_0), \omega(t_1)\omega(t_0)^{-1}, \dots, \omega(t_n)\omega(t_{n-1})^{-1}) \in E_0 \times \dots \times E_n\}.$$

So in fact

[18] Formerly 272S.

$$\Pr((X_{t_0},X_{t_1}X_{t_0}^{-1},\dots,X_{t_n}X_{t_{n-1}}^{-1})\in H)$$
$$=\hat\mu\{\omega:(\omega(t_0),\omega(t_1)\omega(t_0)^{-1},\dots,\omega(t_n)\omega(t_{n-1})^{-1})\in H\}$$

for every Borel set $H\subseteq U^{n+1}$.

Set

$$\phi(x_0,\dots,x_n)=(x_0,x_1x_0,x_2x_1x_0,\dots,x_nx_{n-1}\dots x_1x_0)$$

for $x_0,\dots,x_n\in U$, so that $\phi:U^{n+1}\to U^{n+1}$ is continuous. If $H\in\mathcal{B}(U^{n+1})$, then

$$\begin{aligned}\Pr(X_{t_0},\dots,X_{t_n}\in H)&=\Pr(\phi(X_{t_0},X_{t_1}X_{t_0}^{-1},\dots,X_{t_n}X_{t_{n-1}}^{-1})\in H)\\&=\Pr((X_{t_0},X_{t_1}X_{t_0}^{-1},\dots,X_{t_n}X_{t_{n-1}}^{-1})\in\phi^{-1}[H])\\&=\hat\mu\{\omega:((\omega(t_0),\omega(t_1)\omega(t_0)^{-1},\dots,\omega(t_n)\omega(t_{n-1})^{-1})\in\phi^{-1}[H]\}\\&=\hat\mu\{\omega:(\omega(t_0),\dots,\omega(t_n))\in H\}.\end{aligned}$$

Taking $H=F_0\times\dots\times F_n$ we have the result. **Q**

455S Lemma Let U be a metrizable topological group which is complete under a right-translation-invariant metric inducing its topology. Let $\langle\lambda_t\rangle_{t>0}$ be a family of Radon probability measures on U such that $\lambda_s*\lambda_t=\lambda_{s+t}$ for all s, $t>0$ and $\lim_{t\downarrow 0}\lambda_tG=1$ for every open neighbourhood G of the identity e in U. For $x\in U$ and $0\le s<t$, let $\nu_x^{(s,t)}$ be the Radon probability measure on U defined by saying that $\nu_x^{(s,t)}(E)=\lambda_{t-s}(Ex^{-1})$ whenever λ_{t-s} measures Ex^{-1}.

(a) If $0\le t_0<t_1<\dots<t_n$, $z\in U$ and $f:\mathbb{R}^J\to\mathbb{R}$ is a bounded Borel measurable function, where $J=\{t_0,\dots,t_n\}$, then

$$\begin{aligned}&\iint\dots\int f(z,x_1,\dots,x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\dots\nu_{x_1}^{(t_1,t_2)}(dx_2)\nu_z^{(t_0,t_1)}(dx_1)\\&\quad=\iint\dots\int f(z,x_1z,\dots,x_nz)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\\&\qquad\qquad\dots\nu_{x_1}^{(t_1,t_2)}(dx_2)\nu_e^{(t_0,t_1)}(dx_1).\end{aligned}$$

(b) Take $\omega\in U^{[0,\infty[}$ and $a\ge 0$. Let $\hat\mu$ and $\hat\mu_{\omega a}$ be the measures on $U^{[0,\infty[}$ defined from $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in U}$ by the method of 455E, starting from $x^*=e$. Define $\phi_{\omega a}:U^{[0,\infty[}\to U^{[0,\infty[}$ by setting

$$\begin{aligned}\phi_{\omega a}(\omega')(t)&=\omega(t)\text{ if }t<a,\\&=\omega'(t-a)\omega(a)\text{ if }t\ge a.\end{aligned}$$

Then $\hat\mu_{\omega a}$ is the image measure $\hat\mu\phi_{\omega a}^{-1}$.

(c) In (b), suppose that ω belongs to the set C_{dlg} of càdlàg functions from $[0,\infty[$ to U. Then $\phi_{\omega a}(\omega')\in C_{\text{dlg}}$ for every $\omega'\in C_{\text{dlg}}$, and $\phi_{\omega a}:C_{\text{dlg}}\to C_{\text{dlg}}$ is inverse-measure-preserving for the subspace measures $\ddot\mu$ and $\ddot\mu_{\omega a}$ on C_{dlg}.

proof (a)(i) If $x\in U$ and $0\le s<t$, then $\nu_x^{(s,t)}(E)=\nu_e^{(s,t)}(Ex^{-1})$ for any $E\subseteq U$ such that either is defined; so $\int f(y)\nu_x^{(s,t)}(dy)=\int f(yx)\nu_e^{(s,t)}(dy)$ for any function $f:U\to\mathbb{R}$ for which either is defined. More generally,

$$\int f(y)\nu_{xz}^{(s,t)}(dy)=\int f(yxz)\nu_e^{(s,t)}(dy)=\int f(yz)\nu_x^{(s,t)}(dy)$$

whenever f is such that any of the three integrals is defined.

(ii) Now induce on n. For the case $n=0$, the natural interpretation of both sides of the formula presented is $f(z)$. For the inductive step to $n+1$, we have

$$\begin{aligned}&\int\dots\int f(z,x_1,x_2,\dots,x_{n+1})\nu_{x_n}^{(t_n,t_{n+1})}(dx_{n+1})\dots\nu_z^{(t_0,t_1)}(dx_1)\\&\quad=\int\dots\int f(z,x_1z,\dots,x_nz,x_{n+1})\nu_{x_nz}^{(t_n,t_{n+1})}(dx_{n+1})\dots\nu_e^{(t_0,t_1)}(dx_1)\end{aligned}$$

(by the inductive hypothesis applied to $(x_0,x_1,\dots,x_n)\mapsto\int f(x_0,\dots,x_n,x_{n+1})\nu_{x_n}^{(t_n,t_{n+1})}(dx_{n+1})$)

$$= \int \ldots \int f(z, x_1 z, \ldots, x_n z, x_{n+1} z) \nu_{x_n}^{(t_n, t_{n+1})}(dx_{n+1}) \ldots \nu_e^{(t_0, t_1)}(dx_1)$$

by (i) applied to the functions $y \mapsto f(z, x_1 z, \ldots, x_n z, y)$ for each $x_1, \ldots, x_n$.

(b)(i) Suppose that $J \subseteq [0, \infty[$ is a finite set containing both 0 and a, enumerated in increasing order as $(t_0, \ldots, t_n)$ with $a = t_j$. Set $z = \omega(a)$. Let $f : \mathbb{R}^J \to \mathbb{R}$ be a function. Then $f\pi_J \phi_{\omega a} = g\pi_K$ where $K = \{0, t_{j+1} - a, \ldots, t_n - a\}$ and $g(x_j, \ldots, x_n) = f(\omega(0), \omega(t_1), \ldots, \omega(t_{j-1}), x_j z, \ldots, x_n z)$ for $x_j, \ldots, x_n \in U$. **P** For $\omega' \in U^{[0,\infty[}$,

$$\begin{aligned} f\pi_J \phi_{\omega a}(\omega') &= (f(\phi_{\omega a}(\omega')(t_0)), \ldots, f(\phi_{\omega a}(\omega')(t_n)) \\ &= (f(\omega(0)), \ldots, f(\omega(t_{j-1})), f(\omega'(0)z), \ldots f(\omega'(t_n - a)z)) \\ &= g(\omega'(0), \ldots, \omega'(t_n - a)) = g\pi_K(\omega'). \mathbf{Q} \end{aligned}$$

(ii) Again suppose that $J \subseteq [0, \infty[$ is a finite set containing both 0 and a, enumerated in increasing order as $(t_0, \ldots, t_n)$ with $a = t_j$, and set $z = \omega(a)$. This time, let $f : \mathbb{R}^J \to \mathbb{R}$ be a bounded Borel measurable function. Then

$$\begin{aligned} \int f\pi_J d\hat{\mu}_{\omega a} &= \int \ldots \int f(e, x_1, \ldots, x_n) \nu_{\omega a x_{n-1}}^{(t_{n-1}, t_n)}(dx_n) \ldots \nu_{\omega a 0}^{(0, t_1)}(dx_1) \\ &= \int \ldots \iint \ldots \int f(e, x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \\ &\qquad \nu_{x_{n-1}}^{(t_{n-1}, t_n)}(dx_n) \ldots \nu_z^{(a, t_{j+1})}(dx_{j+1}) \delta_z(dx_j) \ldots \delta_{\omega(t_1)}(dx_1) \end{aligned}$$

(reading from the formulae in 455E; here each δ_x is a Dirac measure on U)

$$\begin{aligned} &= \int \ldots \int f(e, \omega(t_1), \ldots, z, x_{j+1}, \ldots, x_n) \\ &\qquad \nu_{x_{n-1}}^{(t_{n-1}, t_n)}(dx_n) \ldots \nu_z^{(a, t_{j+1})}(dx_{j+1}) \\ &= \int \ldots \int f(e, \omega(t_1), \ldots, z, x_{j+1} z, \ldots, x_n z) \\ &\qquad \nu_{x_{n-1}}^{(t_{n-1}, t_n)}(dx_n) \ldots \nu_e^{(a, t_{j+1})}(dx_{j+1}) \end{aligned}$$

(applying (a) to the function $(y_0, \ldots, y_{n-j}) \mapsto f(e, \ldots, \omega(t_{j-1}), y_0, \ldots, y_{n-j})$)

$$= \int \ldots \int g(e, x_{j+1}, \ldots, x_n) \nu_{x_{n-1}}^{(t_{n-1}, t_n)}(dx_n) \ldots \nu_e^{(a, t_{j+1})}(dx_{j+1})$$

(where $g(x_j, \ldots, x_n) = f(\omega(0), \omega(t_1), \ldots, \omega(t_{j-1}), x_j z, \ldots, x_n z)$ for $x_j, \ldots, x_n \in U$)

$$= \int \ldots \int g(e, x_{j+1}, \ldots, x_n) \nu_{x_{n-1}}^{(t_{n-1} - a, t_n - a)}(dx_n) \ldots \nu_e^{(0, t_{j+1} - a)}(dx_{j+1})$$

(because $\nu_x^{(s-a, t-a)} E = \lambda_{t-s}(Ex^{-1}) = \nu_x^{(s,t)} E$ whenever $E \subseteq \mathbb{R}^K$ is Borel, $x \in U$ and $a \le s < t$)

$$= \int g\pi_K d\hat{\mu}$$

(where $K = \{0, t_{j+1} - a, \ldots, t_n - a\}$)

$$= \int f\pi_J \phi_{\omega a} d\hat{\mu} = \int f\pi_J d(\hat{\mu}\phi_{\omega a}^{-1}).$$

As f and J are arbitrary, $\hat{\mu}_{\omega a}$ and $\hat{\mu}\phi_{\omega a}^{-1}$ agree on the algebra $\bigotimes_{[0,\infty[} \mathcal{B}(U)$ generated by sets of the form $\{\omega : \omega(t) \in E\}$ for $t \ge 0$ and Borel sets $E \subseteq U$. By the Monotone Class Theorem, the measures agree on the σ-algebra $\widehat{\bigotimes}_{[0,\infty[} \mathcal{B}(U)$ generated by $\bigotimes_{[0,\infty[} \mathcal{B}(U)$; because they are both defined as complete measures inner regular with respect to this σ-algebra, they are identical.

(c) The defining formula for $\phi_{\omega a}$ makes it plain that $\phi_{\omega a}(\omega')$ is càdlàg whenever ω, ω' are càdlàg; and in fact that, if ω is càdlàg, then $\phi_{\omega a}(\omega')$ is càdlàg iff ω' is càdlàg. If W is measured by $\ddot{\mu}_{\omega a}$, there is a $W' \in \operatorname{dom} \hat{\mu}_{\omega a}$ such that $W = W' \cap C_{\mathrm{dlg}}$. In this case, $\phi_{\omega a}^{-1}[W] = \phi_{\omega a}^{-1}[W'] \cap C_{\mathrm{dlg}}$ while $\phi_{\omega a}^{-1}[W'] \in \operatorname{dom} \hat{\mu}$; so $\ddot{\mu}\phi_{\omega a}^{-1}[W]$ is defined and equal to

$$\hat{\mu}\phi_{\omega a}^{-1}[W'] = \hat{\mu}_{\omega a} W' = \ddot{\mu}_{\omega a} W.$$

455T Corollary Let U be a metrizable topological group which is complete under a right-translation-invariant metric inducing its topology. Let $\langle\lambda_t\rangle_{t>0}$ be a family of Radon probability measures on U such that $\lambda_s*\lambda_t=\lambda_{s+t}$ for all s, $t>0$ and $\lim_{t\downarrow 0}\lambda_tG=1$ for every open neighbourhood G of the identity e in U; let $\hat{\mu}$ be the measure on $U^{[0,\infty[}$ defined from $\langle\lambda_t\rangle_{t>0}$ by the method of 455P, starting from $x^*=e$. Let $C_{\rm dlg}$ be the set of càdlàg functions from $[0,\infty[$ to U, $\ddot{\mu}$ the subspace measure on $C_{\rm dlg}$ and $\ddot{\Sigma}$ its domain. For $t\ge 0$, let $\ddot{\Sigma}_t$ be

$$\{F:F\in\ddot{\Sigma},\,\omega'\in F\text{ whenever }\omega\in F,\,\omega'\in C_{\rm dlg}\text{ and }\omega'\restriction[0,t]=\omega\restriction[0,t]\}$$

and $\hat{\ddot{\Sigma}}_t=\{F\triangle A:F\in\ddot{\Sigma}_t,\,\ddot{\mu}A=0\}$. Then $\hat{\ddot{\Sigma}}_t=\bigcap_{s>t}\hat{\ddot{\Sigma}}_s$ includes $\ddot{\Sigma}_t^+=\bigcap_{s>t}\ddot{\Sigma}_s$.

proof (a) I show first that $\ddot{\Sigma}_t^+\subseteq\hat{\ddot{\Sigma}}_t$. **P** Take $E\in\ddot{\Sigma}_t^+$. Let $\tau:C_{\rm dlg}\to[0,\infty]$ be the constant stopping time with value t, and f the characteristic function χE. Set $g(\omega)=\int_{C_{\rm dlg}}fd\ddot{\mu}_{\omega t}$ when this is defined in $\mathbb{R}$, where $\ddot{\mu}_{\omega t}$ is defined as in 455O; then g is a conditional expectation of f on $\ddot{\Sigma}_\tau$ (455Ob). Since

$$\begin{aligned}\ddot{\Sigma}_\tau^+&=\{H:H\in\ddot{\Sigma},\,H\cap\{\omega:\tau(\omega)\le s\}\in\ddot{\Sigma}_s^+\text{ for every }s\ge 0\}\\&=\{H:H\in\ddot{\Sigma},\,H\in\ddot{\Sigma}_s^+\text{ for every }s\ge t\}=\ddot{\Sigma}_t^+\end{aligned}$$

contains E, $g=_{\rm a.e.}\chi E$. Setting $F=\{\omega:\omega\in\operatorname{dom}g,\,g(\omega)=1\}$, $F\in\ddot{\Sigma}$ (remember that $\ddot{\mu}$ is complete), and $E\triangle F$ is negligible.

Now 455Sc, with 235Gb, tells us that

$$g(\omega)=\int_{C_{\rm dlg}}fd\ddot{\mu}_{\omega t}=\int_{C_{\rm dlg}}f\phi_{\omega t}d\ddot{\mu}$$

whenever either integral is defined in $\mathbb{R}$, where

$$\begin{aligned}\phi_{\omega t}(\omega')(s)&=\omega(s)\text{ if }s<t,\\&=\omega'(s-t)\omega(t)\text{ if }s\ge t.\end{aligned}$$

If ω_0, $\omega_1\in C_{\rm dlg}$ and $\omega_0\restriction[0,t]=\omega_1\restriction[0,t]$, then $\phi_{\omega_0 t}=\phi_{\omega_1 t}$ so $g(\omega_0)=g(\omega_1)$ if either is defined. It follows that $\omega_0\in F$ iff $\omega_1\in F$. As ω_0 and ω_1 are arbitrary, $F\in\ddot{\Sigma}_t$ and $E\in\hat{\ddot{\Sigma}}_t$.

(b) Of course $\hat{\ddot{\Sigma}}_t\subseteq\hat{\ddot{\Sigma}}_s$ whenever $s>t$. Putting (a) and 455L(f-ii) together,

$$\bigcap_{s>t}\hat{\ddot{\Sigma}}_s=\{E\triangle A:E\in\ddot{\Sigma}_t^+,\,\ddot{\mu}A=0\}\subseteq\{E\triangle A:E\in\hat{\ddot{\Sigma}}_t,\,\ddot{\mu}A=0\}=\hat{\ddot{\Sigma}}_t$$

and we have equality.

455U Theorem Let U be a metrizable topological group which is complete under a right-translation-invariant metric inducing its topology. Let $\langle\lambda_t\rangle_{t>0}$ be a family of Radon probability measures on U such that $\lambda_s*\lambda_t=\lambda_{s+t}$ for all s, $t>0$ and $\lim_{t\downarrow 0}\lambda_tG=1$ for every open neighbourhood G of the identity e in U; let $\hat{\mu}$ be the measure on $U^{[0,\infty[}$ defined from $\langle\lambda_t\rangle_{t>0}$ by the method of 455P, starting from $x^*=e$. Let $C_{\rm dlg}$ be the set of càdlàg functions from $[0,\infty[$ to U, $\ddot{\mu}$ the subspace measure on $C_{\rm dlg}$ and $\ddot{\Sigma}$ its domain. For $t\ge 0$, let $\ddot{\Sigma}_t$ be

$$\{F:F\in\ddot{\Sigma},\,\omega'\in F\text{ whenever }\omega\in F,\,\omega'\in C_{\rm dlg}\text{ and }\omega'\restriction[0,t]=\omega\restriction[0,t]\},$$

and $\ddot{\Sigma}_t^+=\bigcap_{s>t}\ddot{\Sigma}_s$; let $\tau:C_{\rm dlg}\to[0,\infty]$ be a stopping time adapted to $\langle\ddot{\Sigma}_t^+\rangle_{t\ge0}$. Define $\phi_\tau:C_{\rm dlg}\times C_{\rm dlg}\to C_{\rm dlg}$ by setting

$$\begin{aligned}\phi_\tau(\omega,\omega')(t)&=\omega'(t-\tau(\omega))\omega(\tau(\omega))\text{ if }t\ge\tau(\omega),\\&=\omega(t)\text{ otherwise.}\end{aligned}$$

Then ϕ_τ is inverse-measure-preserving for the product measure $\ddot{\mu}\times\ddot{\mu}$ on $C_{\rm dlg}\times C_{\rm dlg}$ and $\ddot{\mu}$ on $C_{\rm dlg}$.

proof (a) First, I ought to remark that we know from 455P that the conditions of 455O are satisfied. I aim to apply 455Oa, using 455S to give a description of the measures $\ddot{\mu}_{\omega,\tau(\omega)}$. Now if f is $\ddot{\mu}$-integrable, we have, in the notation of 455O and 455S,

$$\int_{C_{\rm dlg}}fd\ddot{\mu}=\int_{C_{\rm dlg}}\int_{C_{\rm dlg}}fd\ddot{\mu}_{\omega,\tau(\omega)}\ddot{\mu}(d\omega)$$

(455Oa)

$$= \int_{C_{\mathrm{dlg}}} \int_{C_{\mathrm{dlg}}} f\phi_{\omega,\tau(\omega)}(\omega')\ddot{\mu}(d\omega')\ddot{\mu}(d\omega)$$

(455Sc)

$$= \int_{C_{\mathrm{dlg}}} \int_{C_{\mathrm{dlg}}} f\phi_{\tau}(\omega,\omega')\ddot{\mu}(d\omega')\ddot{\mu}(d\omega)$$

.

(b) To convert the repeated integral into the product measure, we have still to check for measurability. The point is that, writing Λ for the domain of the product measure $\ddot{\mu}\times\ddot{\mu}$, ϕ_τ is $(\Lambda, \widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U))$-measurable.

P(i) Consider first the case in which U is separable. Take $t \ge 0$. Then $E_t = \{\omega : \omega \in C_{\mathrm{dlg}},\ \tau(\omega) \le t\}$ belongs to $\ddot{\Sigma}$. The function

$$\omega \mapsto t - \tau(\omega) : E_t \to [0,\infty[$$

is $(\ddot{\Sigma}, \mathcal{B}([0,\infty[))$-measurable; the function

$$(\omega', s) \mapsto \omega'(s) : C_{\mathrm{dlg}} \times [0,\infty[\to U$$

is $(\ddot{\Sigma}\widehat{\otimes}\mathcal{B}([0,\infty[), \mathcal{B}(U))$-measurable, by 4A3Wc, because U is Polish; so the function

$$(\omega,\omega') \mapsto \omega'(t-\tau(\omega)) : E_t \times C_{\mathrm{dlg}} \to U$$

is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma}, \mathcal{B}(U))$-measurable. Next, similarly,

$$\omega \mapsto \omega(\tau(\omega)) : E_t \to U$$

is $(\ddot{\Sigma}, \mathcal{B}(U))$-measurable, while

$$(y,z) \mapsto yz : U \times U \to U$$

is $(\mathcal{B}(U)\widehat{\otimes}\mathcal{B}(U), \mathcal{B}(U))$-measurable, because U is a second-countable topological group. But this means that

$$(\omega,\omega') \mapsto \omega'(t-\tau(\omega))\omega(\tau(\omega)) : E_t \times C_{\mathrm{dlg}} \to U$$

is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma}, \mathcal{B}(U))$-measurable. On the other hand, of course,

$$(\omega,\omega') \mapsto \omega(t) : (C_{\mathrm{dlg}} \setminus E_t) \times C_{\mathrm{dlg}} \to U$$

is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma}, \mathcal{B}(U))$-measurable. Putting these together,

$$(\omega,\omega') \mapsto \phi_\tau(\omega,\omega')(t) : C_{\mathrm{dlg}} \times C_{\mathrm{dlg}} \to U$$

is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma}, \mathcal{B}(U))$-measurable. This is true for every $t \ge 0$, so $\phi_\tau \restriction C_{\mathrm{dlg}} \times C_{\mathrm{dlg}}$ is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma}, \widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U))$-measurable.

(ii) For the general case, we can use the trick in 455H. There is a separable subset U' of U such that $\nu_e^{(0,q)}U' = 1$ for every rational $q \ge 0$. We can suppose that U' is a closed subgroup of U. Because U' is closed,

$$\begin{aligned} C'_{\mathrm{dlg}} &= \{\omega : \omega \in C_{\mathrm{dlg}},\ \omega(t) \in U' \text{ for every } t \ge 0\} \\ &= \{\omega : \omega \in C_{\mathrm{dlg}},\ \omega(q) \in U' \text{ for every } q \in \mathbb{Q} \cap [0,\infty[\} \end{aligned}$$

is $\ddot{\mu}$-conegligible in C_{dlg}, and because U' is a subgroup, $\phi_\tau(\omega,\omega') \in C'_{\mathrm{dlg}}$ for all ω, $\omega' \in C'_{\mathrm{dlg}}$. Now the argument of (i) shows that $\phi_\tau \restriction C'_{\mathrm{dlg}} \times C'_{\mathrm{dlg}}$ is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma}, \widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U'))$-measurable, therefore $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma}, \widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U))$-measurable. Since $C'_{\mathrm{dlg}} \times C'_{\mathrm{dlg}}$ is $(\ddot{\mu}\times\ddot{\mu})$-conegligible, ϕ_τ is $(\Lambda, \widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U))$-measurable. **Q**

(c) It follows that if $E \in \widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$ then, setting $f = \chi(E \cap C_{\mathrm{dlg}})$ in (a),

$$(\ddot{\mu}\times\ddot{\mu})\phi_\tau^{-1}[E \cap C_{\mathrm{dlg}}] = \int f(\phi_\tau(\omega,\omega'))\ddot{\mu}(d\omega')\ddot{\mu}(d\omega) = \int f d\ddot{\mu} = \ddot{\mu}(E \cap C_{\mathrm{dlg}})$$

by Fubini's theorem. But $\ddot{\mu}$ is the subspace measure generated by the completion of a measure with domain $\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$, so is inner regular with respect to sets of the form $E \cap C_{\mathrm{dlg}}$ with $E \in \widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$; by 412K, ϕ_τ is inverse-measure-preserving.

455X Basic exercises (a) Let $\langle A_n\rangle_{n\ge1}$ be a non-increasing sequence of subsets of $[0,1]$, all with Lebesgue outer measure 1, and with empty intersection. Set $T = \{0\} \cup \{\frac{1}{n} : n \ge 1\}$, $\Omega_0 = \{0\}$, $\Omega_{1/n} = A_n$ for $n \ge 1$; for $t \in T$ let T_t be the Borel σ-algebra of Ω_t. For $s < t$ in T and $x \in \Omega_s$ define a Borel measure $\nu_x^{(s,t)}$ on Ω_t by saying that if $n \ge 1$, then $\nu_0^{(0,1/n)}(E \cap A_n)$ is the Lebesgue measure of E for every Borel set $E \subseteq [0,1]$,

if $0 < s < t$, then $\nu_x^{(s,t)}\{x\} = 1$.

Show that $\langle \nu_y^{(t,u)} \rangle_{y \in \Omega_t}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $s < t < u$ in T and $x \in \Omega_s$. Taking $x^* = 0$, show that there is no measure μ on $\prod_{t\in T}\Omega_t$ with the properties listed in 455A.

(b) Let T, t^*, $\langle(\Omega_t, \mathrm{T}_t)\rangle_{t\in T}$, x^*, $\langle \nu_x^{(s,t)} \rangle_{s<t,x\in\Omega_s}$ and μ be as in 455A. Suppose that we are given a family $\langle(\Omega'_t, \mathrm{T}'_t, \pi_t)\rangle_{t\in T}$ such that (α) Ω'_t is a set, T'_t is a σ-algebra of subsets of Ω'_t and $\pi_t : \Omega_t \to \Omega'_t$ is a surjective $(\mathrm{T}_t, \mathrm{T}'_t)$-measurable function for every $t \in T$ (β) whenever $s < t$ in T and x, $x' \in \Omega_s$ are such that $\pi_s(x) = \pi_s(x')$, then $\nu_x^{(s,t)}$ and $\nu_{x'}^{(s,t)}$ agree on $\{\pi_t^{-1}[F] : F \in \mathrm{T}'_t\}$. (i) Show that if we set $\acute\nu_w^{(s,t)}(F) = \nu_x^{(s,t)}\pi_t^{-1}[F]$ whenever $s < t$ in T, $x \in \Omega_s$, $w = \pi_s(x)$ and $F \in \mathrm{T}'_t$, then every $\acute\nu_w^{(s,t)}$ is a perfect probability measure, and $\langle \acute\nu_z^{(t,u)} \rangle_{z\in\Omega'_t}$ is a disintegration of $\acute\nu_w^{(s,u)}$ over $\acute\nu_w^{(s,t)}$ whenever $s < t < u$ in T and $w \in \Omega'_s$. (ii) Let μ' be the measure on $\Omega' = \prod_{t\in T}\Omega'_t$ defined by the method of 455A from $\pi_{t^*}(x^*)$ and $\langle \acute\nu_w^{(s,t)} \rangle_{s<t,w\in\Omega'_s}$. Show that $\pi : \Omega \to \Omega$ is inverse-measure-preserving for μ and μ', where $\pi(\omega)(t) = \pi_t(\omega(t))$ for $\omega \in \Omega$ and $t \in T$.

(c) In 455E, set $T = \{-1\} \cup [0,\infty[$, let each Ω_t be $\mathbb{R}$, and for $x \in \mathbb{R}$, $0 \le s < t$ let $\nu_x^{(s,t)}$ be the Dirac measure on $\mathbb{R}$ concentrated at $\psi(x, t-s)$ on $\mathbb{R}$, where

$$\begin{aligned}\psi(x,t) &= \tfrac{x}{1-xt} \text{ if } xt \ne 1 \text{ and } x \ne 0,\\ &= 0 \text{ if } xt = 1,\\ &= -\frac{1}{t} \text{ if } x = 0.\end{aligned}$$

Let ν be any atomless Radon probability measure on $\mathbb{R}$, and complete the definition by setting $\nu_x^{(-1,t)}(E) = \nu\{y : \psi(y,t) \in E\}$ whenever $t \ge 0$ and this is defined; set $x^* = 0$. Show that the conditions of 455E are satisfied, that the measure $\hat\mu$ constructed in 455E is a distribution on $\mathbb{R}^T$, and that $\hat\mu$ is not τ-additive. (*Hint*: setting $\phi(y)(-1) = 0$, $\phi(y)(t) = \psi(y,t)$ for $t \ge 0$ and $y \in \mathbb{R}$, show that $\phi : \mathbb{R} \to \mathbb{R}^T$ is inverse-measure-preserving for ν_0 and $\hat\mu$, and that every point of $\mathbb{R}^T$ has a neighbourhood of zero measure.)

(d) Let T, t^*, $\langle \Omega_t \rangle_{t\in T}$, x^*, $\langle \nu_x^{(s,t)} \rangle_{s<t,x\in\Omega_s}$ and $\hat\mu$ be as in 455E. Suppose that we are given a family $\langle(\Omega'_t, \pi_t)\rangle_{t\in T}$ such that (α) Ω'_t is a Hausdorff space and $\pi_t : \Omega_t \to \Omega'_t$ is a continuous surjective function for every $t \in T$ (β) whenever $s < t$ in T and x, $x' \in \Omega_s$ are such that $\pi_s(x) = \pi_s(x')$, then the image measures $\nu_x^{(s,t)}\pi_t^{-1}$ and $\nu_{x'}^{(s,t)}\pi_t^{-1}$ on Ω'_t are the same. (i) Show that if we set $\acute\nu_w^{(s,t)} = \nu_x^{(s,t)}\pi_t^{-1}$ whenever $s < t$ in T, $x \in \Omega_s$ and $w = \pi_s(x)$, then every $\acute\nu_w^{(s,t)}$ is a Radon probability measure, and $\langle \acute\nu_z^{(t,u)} \rangle_{z\in\Omega'_t}$ is a disintegration of $\acute\nu_w^{(s,u)}$ over $\acute\nu_w^{(s,t)}$ whenever $s < t < u$ in T and $w \in \Omega'_s$. (ii) Let $\hat\mu'$ be the measure on $\Omega' = \prod_{t\in T}\Omega'_t$ defined by the method of 455E from $\pi_{t^*}(x^*)$ and $\langle \acute\nu_w^{(s,t)} \rangle_{s<t,w\in\Omega'_s}$. Show that $\pi : \Omega \to \Omega$ is inverse-measure-preserving for $\hat\mu$ and $\hat\mu'$, where $\pi(\omega)(t) = \pi_t(\omega(t))$ for $\omega \in \Omega$ and $t \in T$.

(e) Let U be a locally compact metrizable group and ν any Radon probability measure on U. For $t > 0$ let λ_t be the Radon probability measure

$$e^{-t}(\delta_e + t\nu + \frac{t^2}{2!}\nu * \nu + \frac{t^3}{3!}\nu * \nu * \nu + \ldots),$$

where δ_e is the Dirac measure on U concentrated at the identity e of U, and the sum is defined as in 234G[19]. Show that $\langle \lambda_t \rangle_{t>0}$ satisfies the conditions of 455P (with respect to an appropriate metric on U). (*Hint*: 4A5Mb, 4A5Q(iv).)

(f) Let U, $\langle \lambda_t \rangle_{t>0}$ and $\hat\mu$ be as in 455P. Let V be a Hausdorff space, z^* a point of V and $\bullet$ a continuous action of U on V; set $\pi(x) = x{\bullet}z^*$ for $x \in U$. Define $\langle \nu_x^{(s,t)} \rangle_{0\le s<t,x\in U}$ from $\langle \lambda_t \rangle_{t>0}$ as in 455P. (i) Show that if x, $x' \in U$ and $\pi(x) = \pi(x')$, then the image measures $\nu_x^{(s,t)}\pi^{-1}$, $\nu_{x'}^{(s,t)}\pi^{-1}$ on V are equal whenever $s < t$. (ii) Let $\hat\mu'$ be the measure on $V^{[0,\infty[}$ defined as in 455Xd. Show that if we define $\tilde\pi : U^{[0,\infty[} \to V^{[0,\infty[}$ by setting $\tilde\pi(\omega)(t) = \omega(t){\bullet}z^*$ for every $\omega \in U^{[0,\infty[}$ and $t \ge 0$, $\tilde\pi$ is inverse-measure-preserving for $\hat\mu$ and $\hat\mu'$ and also for the Radon measures extending them; moreover, that the restriction of $\tilde\pi$ to $C_{\mathrm{dlg}}(U)$, the space of càdlàg functions from $[0,\infty[$ to U, is inverse-measure-preserving for the subspace measures on $C_{\mathrm{dlg}}(U)$ and $C_{\mathrm{dlg}}(V)$.

[19] Formerly 112Ya.

>(g) For $t>0$, let λ_t be the normal distribution on $\mathbb{R}$ with expectation 0 and variance t. Show that $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455Q.

>(h) For $t>0$, let λ_t be the Poisson distribution with expectation t, that is, $\lambda_t(E)=e^{-t}\sum_{m\in E\cap\mathbb{N}}t^m/m!$ for $E\subseteq\mathbb{R}$ (cf. 285Q, 285Xo). (i) Show that $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455Q. (ii) Show that if $\tilde\mu$ is the Radon measure defined from $\langle\lambda_t\rangle_{t>0}$ as in 455Pc, then ω is non-decreasing and $\omega[\,[0,\infty[\,]=\mathbb{N}$ for $\tilde\mu$-almost every $\omega\in\mathbb{R}^{[0,\infty[}$.

>(i) For $t>0$, let λ_t be the Cauchy distribution with centre 0 and scale parameter t, that is, the distribution with probability density function $x\mapsto\frac{t}{\pi(x^2+t^2)}$ (285Xm). (i) Show that $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455Q. (ii) Show that if $\hat\mu$ is the corresponding distribution on $\mathbb{R}^{[0,\infty[}$, then $C([0,\infty[)$ is $\hat\mu$-negligible. (*Hint*: estimate $\Pr(|X_{(i+1)/n}-X_{i/n}|\le\epsilon$ for every $i<n$).) (iii) Suppose that $\alpha>0$. Define $T_\alpha:\mathbb{R}^{[0,\infty[}\to\mathbb{R}^{[0,\infty[}$ by setting $(T_\alpha\omega)(t)=\frac{1}{\alpha}\omega(\alpha t)$ for $t\ge 0$ and $\omega\in\mathbb{R}^{[0,\infty[}$. Show that T_α is inverse-measure-preserving for $\hat\mu$. (iv) For $\omega\in\mathbb{R}^{[0,\infty[}$ set $(R\omega)(t)=t^2\omega(\frac{1}{t})$ if $t>0$, $\omega(0)$ if $t=0$. Show that $R:\mathbb{R}^{[0,\infty[}\to\mathbb{R}^{[0,\infty[}$ is inverse-measure-preserving for $\hat\mu$. (Compare 477E below.)

>(j)(i) The **standard gamma distribution** with parameter t is the probability distribution λ_t on $\mathbb{R}$ with probability density function $x\mapsto\frac{1}{\Gamma(t)}x^{t-1}e^{-x}$ for $x>0$. Show that its expectation is t. (*Hint*: 225Xj(iv).) Show that its variance is t. (ii) Show that $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455Q. (*Hint*: 272U[20], 252Yf.) (iii) Show that $\lim_{t\downarrow 0}t\Gamma(t)=1$, so that $\lim_{t\downarrow 0}\frac{1}{t}\lambda_t[1,\infty[\,=\int_1^\infty\frac{1}{x}te^{-x}dx>0$. (iv) Show that if $\tilde\mu$ is the Radon measure on $\mathbb{R}^{[0,\infty[}$ defined from $\langle\lambda_t\rangle_{t>0}$ as in 455Pc, then $\{\omega:\omega$ is strictly increasing and not continuous$\}$ is $\tilde\mu$-conegligible.

(k) Let U be an abelian Hausdorff topological group. Let $\langle\lambda'_t\rangle_{t>0}$, $\langle\lambda''_t\rangle_{t>0}$ be two families of Radon probability measures on U and set $\lambda_t=\lambda'_t*\lambda''_t$ for $t>0$. (i) Show that if $\lambda'_{s+t}=\lambda'_s*\lambda'_t$ and $\lambda''_{s+t}=\lambda''_s*\lambda''_t$ for all s, $t>0$, then $\lambda_{s+t}=\lambda_s*\lambda_t$ for all s, $t>0$. (ii) Show that if $\lim_{t\downarrow 0}\lambda'_tG=\lim_{t\downarrow 0}\lambda''_tG=1$ for every open set containing the identity e of U, then $\lim_{t\downarrow 0}\lambda_tG=1$ for every open set G containing e. (iii) Now suppose that U is metrizable and complete under a right-translation-invariant metric inducing its topology. Let $\hat\mu'$, $\hat\mu''$ and $\hat\mu$ be the measures on $U^{[0,\infty[}$ defined from $\langle\lambda'_t\rangle_{t>0}$, $\langle\lambda''_t\rangle_{t>0}$ and $\langle\lambda_t\rangle_{t>0}$ as in 455P. Set $\theta(\omega,\omega')(t)=\omega(t)\omega'(t)$ for ω, $\omega'\in U^{[0,\infty[}$ and $t\ge 0$. Show that $\theta:U^{[0,\infty[}\times U^{[0,\infty[}\to U^{[0,\infty[}$ is inverse-measure-preserving for $\hat\mu'\times\hat\mu''$ and $\hat\mu$. (iv) Repeat (iii) for the subspace measures on the space of càdlàg functions from $[0,\infty[$ to U.

455Y Further exercises (a) Let $\langle X_n\rangle_{n\in\mathbb{Z}}$ be a double-ended sequence of real-valued random variables such that (i) for each $n\in\mathbb{Z}$, $Y_n=X_{n+1}-X_n$ is independent of $\{X_i:i\le n\}$ (ii) $\langle Y_n\rangle_{n\in\mathbb{Z}}$ is identically distributed. Show that the Y_n are essentially constant. (*Hint*: 285Yc.)

(b) For $0\le s<t$ and $x\in\mathbb{R}$ define a Radon probability measure $\nu_x^{(s,t)}$ on $\mathbb{R}$ by saying that

$$\begin{aligned}\nu_x^{(s,t)}&=\tfrac{1-t}{1-s}\delta_0+\tfrac{t-s}{1-s}\lambda_{[s,t]}\text{ if }x=0\text{ and }t\le 1,\\&=\tfrac{t-1}{1-s}\delta_0+\tfrac{2-t-s}{1-s}\lambda_{[s,2-t]}\text{ if }x=0\text{ and }1\le t<2-s,\\&=\delta_0\text{ if }0<x<1\text{ and }2-x\le t,\\&=\delta_x\text{ otherwise,}\end{aligned}$$

writing δ_x for the Dirac measure on $\mathbb{R}$ concentrated at x, and $\lambda_{[s,t]}$ for the uniform distribution based on the interval $[s,t]$. (i) Show that $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in\mathbb{R}}$ satisfies the conditions of 455E. (ii) Starting from $x^*=0$, let $\hat\mu$ be the corresponding measure on $\mathbb{R}^{[0,\infty[}$. Show that $\hat\mu^*C_{\mathrm{dlg}}=1$, where C_{dlg} is the space of càdlàg functions from $[0,\infty[$ to $\mathbb{R}$. (iii) Show that $\hat\mu$ has a unique extension to a Radon measure $\tilde\mu$ on $\mathbb{R}^{[0,\infty[}$. (iv) Show that $\tilde\mu C_{\mathrm{dlg}}=0$. (v) Show that the subspace measure on C_{dlg} induced by $\hat\mu$ is not τ-additive.

(c) A probability distribution λ on $\mathbb{R}$ is **infinitely divisible** if for every $n\ge 1$ it is expressible as a convolution $\nu*\ldots*\nu$ of n copies of a probability distribution. Let ϕ be the characteristic function (§285) of an infinitely divisible distribution λ. (i) Show that for each $n\ge 1$ there is a characteristic function ϕ_n such that $\phi_n^n=\phi$. (ii) Show that

[20]Formerly 272T.

if $\delta>0$ is such that $\phi(y)\ne 0$ for $|y|\le\delta$, then $\lim_{n\to\infty}\phi_n(y)=1$ for $|y|\le\delta$. (iii) Show that $\lim_{n\to\infty}\phi_n(y)=1$ for every $y\in\mathbb{R}$. (*Hint*: for any characteristic function ψ, $4\,\mathcal{R}\mathrm{e}\,\psi(y)\le 3+\mathcal{R}\mathrm{e}\,\psi(2y)$ for every y.) (iv) Show that ϕ is never zero, and that there is a unique family $\langle\lambda_t\rangle_{t>0}$ of distributions satisfying the conditions in 455P and such that $\lambda_1=\lambda$. (v) Show that if λ has finite expectation, then $\mathbb{E}(\lambda_t)$ is defined and equal to $t\mathbb{E}(\lambda)$ for every $t>0$. (vi) Show that if λ has finite variance, then $\mathrm{Var}(\lambda_t)$ is defined and equal to $t\,\mathrm{Var}(\lambda)$ for every $t>0$.

(d) Let U be a Hausdorff topological group and $\langle\lambda_t\rangle_{t>0}$ a family of Radon probability measures on U such that $\lambda_{s+t}=\lambda_s*\lambda_t$ whenever $s,\ t>0$. (i) Show that we can define a family $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ as in 455P, and that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $x\in U$ and $s<t<u$, so that, starting from $x^*=e$ the identity of U, we can apply 455E to obtain a measure $\hat\mu$ on $U^{[0,\infty[}$. (ii) Now suppose that $U=\mathbb{R}^r$ where $r\ge 1$. For $t>0$, $E\subseteq U$ set $\overset{\leftrightarrow}{\lambda}_tE=\lambda_t(-E)$ whenever λ_t measures $-E$; now set $\lambda_t^\#=\lambda_t*\overset{\leftrightarrow}{\lambda}_t$. Show that $\lambda_{s+t}^\#=\lambda_s^\#*\lambda_t^\#$ for all $s,\ t>0$, and that $\lim_{t\downarrow 0}\lambda_t^\# G=1$ for every open neighbourhood G of 0. Show that $\hat\mu$ has an extension to a Radon measure on $(\mathbb{R}^r)^{[0,\infty[}$.

(e) Let Y be a metrizable space and C_{dlg} the set of càdlàg functions from $[0,\infty[$ to Y. For $\omega\in C_{\mathrm{dlg}}$ and $t\ge 0$ set $X_t(\omega)=\omega(t)$. Let Σ be a σ-algebra of subsets of C_{dlg} such that $X_t:C_{\mathrm{dlg}}\to Y$ is measurable for every $t\ge 0$. For $t\ge 0$ let Σ_t be

$$\{F:F\in\Sigma,\ \omega'\in F\text{ whenever }\omega,\ \omega'\in C_{\mathrm{dlg}},\ \omega\in F\text{ and }\omega\restriction[0,t]=\omega'\restriction[0,t]\}.$$

Show that $\langle X_t\rangle_{t\ge 0}$ is progressively measurable with respect to $\langle\Sigma_t\rangle_{t\ge 0}$.

455 Notes and comments This section has grown into the longest in this treatise. There are some big theorems here. I am trying to do two rather different things: sketch the fundamental properties of Markov processes, and work through the details of particular realizations of them. I remarked in the introduction to Chapter 27 that probability theory is not really about measure spaces and measurable functions. It is much more about distributions, and by 'distribution' here I do not really mean a Radon probability measure on $\mathbb{R}^r$, let alone a completed Baire measure on $\mathbb{R}^I$, as in 454K. I mean rather the family of probabilities of the type $\Pr(X_i\le\alpha_i\ \forall\ i\le n)$; everything else is formal structure, offering proofs and (I hope) some kinds of deeper understanding, but essentially secondary. The appalling formulae above ($\nu_{\omega,\tau(\omega),x_j}^{(t_j,t_{j+1})}(dx_{j+1})$, $\ddot\Sigma\widehat\otimes\ddot\Sigma$ and so on) arise from my attempts to distinguish clearly among the host of probability spaces which present themselves to us as relevant.

However one of the messages of this section is that for many stochastic processes it is possible to identify semi-canonical realizations. We already have a crude one in 454J; starting from any family $\langle X_i\rangle_{i\in I}$ of real-valued random variables on any probability space, we can move to a measure on $\mathbb{R}^I$ which is in some sense unique and carries the probabilistic content of the original family. I noted in §454 that when this measure is τ-additive we have a canonical extension to a quasi-Radon measure, just as good regarded as a realization of the abstract process, and possibly with useful further properties. In 455H we find that many of the most important processes can be represented by Radon measures; I do not think these Radon measures have been much studied, except, of course, in the case of Brownian motion. But 455O and 455U show that for some purposes we are better off with quasi-Radon measures on the set of càdlàg functions. The most important stopping times are the hitting times of 455M, which are adapted to families of the form $\langle\Sigma_t^+\rangle_{t\ge 0}$; and for such a stopping time to be approximated by discrete stopping times, as in parts (a-vi) and (b-vii) of the proof of 455O, we need to know that our paths are continuous on the right.

It is of course true that when the complete metric space U, in 455O or later, is separable, then we have a standard Borel structure on the space C_{dlg} of càdlàg functions (4A3Wb), so that the measures $\ddot\mu$ are Radon measures for appropriate Polish topologies on C_{dlg}.

Returning to the detailed exposition, 455A is an attempt at a continuous-time version of 454H. I use the letters t, T to suggest the probabilistic intuitions behind these results; we think of the spaces Ω_t in 455A as being the sets of possible states of a system at 'time' t, so that the measures $\nu_x^{(s,t)}$ are descriptions of how we believe the system is likely to evolve between times s and t, having observed that it is in state x at time s. In the case of 'discrete time', when we observe the system only at clearly separated moments, it is easy to handle non-Markov processes, in which evolution between times $n-1$ and n can depend on the whole history up to time $n-1$; thus in 454H the measures $\nu_z=\nu_z^{(n-1,n)}$ are defined for every $z\in\prod_{i<n}X_i$, but we make no attempt to describe measures $\nu_z^{(n-1,m)}$ for any $m>n$. In 'continuous time' we do need to say something about arbitrary time steps, and it is hard to formulate a consistency condition to fill the place of (†) in 455A without limiting the kind of process being examined. At the cost of an appalling increase in complexity, of course, the formulae of 455A can sometimes be adapted to general processes, if we replace the 'current' state space Ω_t by the 'historical' state space $\prod_{t^*\le s\le t}\Omega_s$. (For we can hope that

$(\prod_{t^* \le s \le t} \Omega_s, \widehat{\bigotimes}_{t^* \le s \le t} \mathrm{T}_s)$ will have the 'perfect measure property' of 454Xd.) We should finish up with a measure on $\prod_{t \in T}(\prod_{t^* \le s \le t} \Omega_s)$. But the important applications, even when not Markov, are open to more economical and more enlightening approaches. We really do need a least element t^* of T; see 455Ya.

I have not yet come to the reason why this section is such hard work. This is in its attempt to analyze the 'Markov property' of the distributions being examined here. The point about the families $\langle \nu_x^{(s,t)} \rangle_{s<t, x \in \Omega_s}$ of transitional probabilities is that they not only give us stochastic processes, as in 455A, but also recipes for conditional expectations, derived from the truncated families $\langle \nu_x^{(s,t)} \rangle_{a \le s < t, x \in \Omega_s}$. These lead to measures μ'_{ax} on $\prod_{t \ge a} \Omega_t$ which can be thought of as distributions of future paths given that we have reached the point x at time a. It is no surprise that these should provide straightforward descriptions of conditional expectations on algebras of the form

$$\{F : F \text{ is determined by coordinates in } [t^*, t]\}.$$

Without much more trouble, we can extend this to suitable algebras defined from simple 'stopping times', as in 455C. The arguments there have some technical features which you may find annoying (and I invite you to find your own way past the complications), but are essentially elementary, as they have to be in such a general context. It is interesting that we can move to stopping times taking countably many values without further difficulty.

However, we are still only seven pages into the section, and not everything to come is as straightforward as the completion processes described in 455E. An essential aspect of continuous-time Markov processes is the possibility of stopping times which take a continuum of values, as is typically the case in the examples provided by 455M. These are much harder to deal with, and we have to restrict sharply the class of processes we examine. The particular restriction I have chosen is described by the definitions in 455F. I should of course say that these, particularly 455Fb ('uniformly time-continuous on the right') are more limiting than is strictly necessary; in 'Feller processes' (Rogers & Williams 94, III.6) we have a slightly different approach to the same intuitive target. The aim is to find sufficient conditions for the 'strong Markov property', in which we can find disintegrations and conditional expectations associated with general stopping times, as in 455O. To do this, we have to abandon the set $\Omega = U^{[0,\infty[}$ and move to the correct set of full outer measure, the set C_{dlg} of 'càdlàg' functions, which dominates the central part of this section. The first thing the definition 455Fb must do is to ensure that C_{dlg} has full outer measure not only for the distribution on Ω but also for the conditional distributions we shall be using (455G). If U is a Polish space, C_{dlg} has a standard Borel structure (4A3Wb), which is comforting.

I hope that you are becoming resigned to the view that the notational complexities of this section are not solely due to an inconsiderate disregard for the reader's eyesight. The original probability measures $\nu_x^{(s,t)}$ of 455A really do form a three-parameter family, the conversion of these into finite-dimensional distributions λ_J really is a multiple repeated integral, the derived probabilities $\nu_{\omega a x}^{(s,t)}$ in 455B are a five-parameter system. Without wishing to insist on my use of grave accents in the proof of 455E, it is surely safer to have a way of distinguishing between completed and uncompleted measures, and while the result may be 'obvious', I think there are some twists on the way which not everyone would foresee. Again, if you wish to dispense with the double-dotted symbols from 455O on, you will have to find some other way of reminding yourself that we are looking at a new representation of the process on a new probability space.

This treatise as a whole is theory-heavy and example-light. I assure you that all the theory here is in fact example-driven. You should start with the four examples of Lévy processes in 455Xg-455Xj. Of these, 455Xg is **Brownian motion**, the starting point of the whole theory; I will return to this in §477. A problem with the formalization in 455A is that we have to start with an exact description of the transitional probabilities $\nu_x^{(s,t)}$. It does not help at all in establishing the existence of such families matching some probabilistic intuition. Only in rather special cases do we have elegant formulae for these systems. In 455Xb, 455Xd and 455Xf I try to show how the general theory gives us methods of using one system to build others.

I suppose that 455O is the summit; from here on the going is easier. In 455P I introduce 'Lévy processes', a particularly interesting class intermediate in generality between the continuous processes of 455O and Brownian motion. These have of course mostly been considered in the case $U = \mathbb{R}$, but the extension to Banach spaces U is an obvious one, and we can even manage non-abelian groups if we are careful. (For an elementary example of a process which can really exploit a non-abelian group, see 455Xe.) The 'Poisson process' in 455Xh is by some way the most important example after Brownian motion itself. Lévy processes on $\mathbb{R}$ are well understood; the family $\langle \lambda_t \rangle_{t>0}$ is determined by λ_1, any 'infinitely divisible' distribution can be taken for λ_1 (455Yc), and a complete description of infinitely divisible distributions is provided by the Lévy-Khintchine representation theorem (Fristedt & Gray 97, 16.3). As a final result in the general theory, I give an alternative version of the strong Markov property in 455U. For Lévy processes, we can re-start, following any of the usual stopping times, with an exact copy of the process, and this corresponds to a true inverse-measure-preserving function from C_{dlg}^2 to C_{dlg}.

A comment on 455T. The idea behind the σ-algebras Σ_t, $\ddot{\Sigma}_t$ of 455M, 455O and later is that they consist of events 'observable at time t', that is, determined by the path taken up to and including time t. We quickly find ourselves forced to consider augmented algebras $\Sigma_t^+ = \bigcap_{s>t} \Sigma_s$, where somehow we are allowed infinitesimal intuitions into the immediate future. (A typical situation is that of 455Mb when the set A is open, so that if $\omega(t) \in \overline{A}$ we can expect that there will be paths which continue immediately into A, and others which do not, and it may not be obvious which, if either, should be regarded as typical.) The question is, whether Σ_t is really different from Σ_t^+. The claim of 455T is that $\ddot{\Sigma}_t^+$ is included in a kind of completion $\hat{\ddot{\Sigma}}_t$ of $\ddot{\Sigma}_t$. Of course the completion is in terms of the measure $\ddot{\mu}$ on the whole space C_{dlg} of càdlàg paths; we need advance knowledge of which subsets of C_{dlg} are negligible. But if we are interested in the measure algebra $\mathfrak{A}$ of $\ddot{\mu}$ and its closed subalgebras $\mathfrak{A}_t = \{E^\bullet : E \in \ddot{\Sigma}_t\}$, 455T tells us that (in the context of Lévy processes) we can expect to have $\mathfrak{A}_t = \bigcap_{s>t} \mathfrak{A}_s$. Turning to the definition of $\ddot{\mu}$ in 455O as a subspace measure, we see that $\mathfrak{A}$ can be regarded as the measure algebra of the measure $\hat{\mu}$ on $U^{[0,\infty[}$ defined by the formulae of 455E; and even that $\mathfrak{A}_t$ can be identified with

$$\{E^\bullet : E \in \operatorname{dom} \hat{\mu},\ E \text{ is determined by coordinates in } [0,t]\}$$

(see part (a-ii) of the proof of 455O). But I think that this last step will not usually be helpful, because (as noted above) $\ddot{\mu}$ will commonly be a Radon measure for an appropriate topology, while $\hat{\mu}$ is likely at best to be the completion of a Baire measure.

I have cast the second half of the section in terms of measures on C_{dlg}, because it is reasonably well adapted to Lévy processes in general. When we come to look at particular processes, we often find that there is a smaller class of functions (e.g., continuous functions in the case of Brownian motion, or non-decreasing $\mathbb{N}$-valued functions in the case of the Poisson process) which is fully adequate and easier to focus on. For the detailed study of such processes, as in §477 below, I think it will usually be helpful to make the shift. But there may be rival conegligible subsets of C_{dlg} with different virtues, as in 477Ef.

456 Gaussian distributions

Uncountable powers of $\mathbb{R}$ are not as a rule measure-compact (439P, 455Xc; see also 533J in Volume 5). Accordingly distributions, in the sense of 454K, need not be τ-additive. But some, at least, of the distributions most important to us are indeed τ-additive, and therefore have interesting canonical extensions. This section is devoted to a remarkable result, taken from Talagrand 81, concerning a class of distributions which are of great importance in probability theory. It demands a combination of techniques from classical probability theory and from the topological measure theory of this volume. I begin with the definition and fundamental properties of what I call 'centered Gaussian distributions' (456A-456I). These are fairly straightforward adaptations of the classical finite-dimensional theory, and will be useful in §477 when we come to study Brownian motion. Another relatively easy idea is that of 'universal' Gaussian distribution (456J-456L). In 456M we come to a much deeper result, a step towards classifying the ways in which a Gaussian family of n-dimensional random variables can accumulate at 0. The ideas are combined in 456N-456O to complete the proof of Talagrand's theorem that Gaussian distributions on powers of $\mathbb{R}$ are τ-additive.

456A Definitions (a) Write μ_G for the Radon probability measure on $\mathbb{R}$ which is the distribution of a standard normal random variable, that is, the probability distribution with density function $x \mapsto \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ (274A). For any set I, write $\mu_G^{(I)}$ for the measure on $\mathbb{R}^I$ which is the product of copies of μ_G; this is always quasi-Radon (415E/453J); if I is countable, it is Radon (417Q); if $I = n \in \mathbb{N} \setminus \{0\}$, it is the probability distribution with density function $x \mapsto (2\pi)^{-n/2} e^{-x \cdot x/2}$ (272I); if $I = \emptyset$, it is the unique probability measure on the singleton set $\mathbb{R}^\emptyset$.

(b) I will use the phrase **centered Gaussian distribution** to mean a measure μ on a power $\mathbb{R}^I$ of $\mathbb{R}$ such that μ is the completion of a Baire measure (that is, is a distribution in the sense of 454K) and every continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$ is either zero almost everywhere or is a normal random variable with zero expectation. (Note that I call the distribution concentrated at the point 0 in $\mathbb{R}^I$ a 'Gaussian distribution'.)

(c) If I is a set and μ is a centered Gaussian distribution on $\mathbb{R}^I$, its **covariance matrix** is the family $\langle \sigma_{ij} \rangle_{i,j \in I}$ where $\sigma_{ij} = \int x(i)x(j)\mu(dx)$ for i, $j \in I$. (The integral is always defined and finite because each function $x \mapsto x(i)$ is either essentially constant or normally distributed, and in either case is square-integrable.)

456B I start with some fundamental facts about Gaussian distributions.

Proposition (a) Suppose that I and J are sets, μ is a centered Gaussian distribution on $\mathbb{R}^I$, and $T:\mathbb{R}^I\to\mathbb{R}^J$ is a continuous linear operator. Then there is a unique centered Gaussian distribution on $\mathbb{R}^J$ for which T is inverse-measure-preserving; if J is countable, this is the image measure μT^{-1}.

(b) Let I be a set, and μ, ν two centered Gaussian distributions on $\mathbb{R}^I$. If they have the same covariance matrices they are equal.

(c) For any set I, $\mu_G^{(I)}$ is the centered Gaussian distribution on $\mathbb{R}^I$ with the identity matrix for its covariance matrix.

(d) Suppose that I is a countable set. Then a measure μ on $\mathbb{R}^I$ is a centered Gaussian distribution iff it is of the form $\mu_G^{(\mathbb{N})}T^{-1}$ where $T:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^I$ is a continuous linear operator.

(e) Suppose $\langle I_j\rangle_{j\in J}$ is a disjoint family of sets with union I, and that for each $j\in J$ we have a centered Gaussian distribution ν_j on $\mathbb{R}^{I_j}$. Then the product ν of the measures ν_j, regarded as a measure on $\mathbb{R}^I$, is a centered Gaussian distribution.

(f) Let I be any set, μ a centered Gaussian distribution on $\mathbb{R}^I$ and $E\subseteq\mathbb{R}^I$ a set such that μ measures E. Writing $-E=\{-x:x\in E\}$, $\mu(-E)=\mu E$.

proof (a)(i) For Baire sets $F\subseteq\mathbb{R}^J$, set $\nu F=\mu T^{-1}[F]$; this is always defined because T is continuous (4A3Kc). This makes ν a Baire probability measure on $\mathbb{R}^J$ for which T is inverse-measure-preserving. Because μ is complete, T is still inverse-measure-preserving for the completion $\hat\nu$ of ν (234Ba[21]). If $g:\mathbb{R}^J\to\mathbb{R}$ is a continuous linear functional, so is $gT:\mathbb{R}^I\to\mathbb{R}$; now $\nu\{y:g(y)\le\alpha\}=\mu\{x:gT(x)\le\alpha\}$ for every α, so g and gT have the same distribution, and are both either zero a.e. or normal random variables. As g is arbitrary, $\hat\nu$ is a centered Gaussian distribution as defined in 456Ab. Of course it is the only such distribution on $\mathbb{R}^J$ for which T is inverse-measure-preserving.

(ii) Now suppose that J is countable. Then $\mathbb{R}^J$ is Polish (4A2Qc), so ν is a Borel measure and $\hat\nu$ is a Radon measure (433Cb). $\mathbb{R}^J$ has a countable network consisting of Borel sets, μ is perfect (454A(b-iii)) and totally finite, and T is measurable (418Bd), so μT^{-1} is a Radon measure (451O). Thus $\hat\nu$ and μT^{-1} are Radon measures agreeing on the Borel sets and must be equal.

(b) The point is that $\mu f^{-1}=\nu f^{-1}$ for every continuous linear functional $f:\mathbb{R}^I\to\mathbb{R}$. **P** By (a), μf^{-1} and νf^{-1} are Radon measures on $\mathbb{R}$, and by the definition of 'Gaussian distribution' each is either a normal distribution with expectation zero, or is concentrated at 0. By 4A4Be, we can express f in the form $f(x)=\sum_{i\in I}\beta_i x(i)$ for every $x\in\mathbb{R}^I$, where $\{i:\beta_i\ne 0\}$ is finite. In this case

$$\int t^2(\mu f^{-1})(dt)=\int f(x)^2\mu(dx)$$

(235G[22])

$$=\sum_{i,j\in I}\beta_i\beta_j\int x(i)x(j)\mu(dx)$$

$$=\sum_{i,j\in I}\beta_i\beta_j\int x(i)x(j)\nu(dx)=\int t^2(\nu f^{-1})(dt)$$

because μ and ν have the same covariance matrices. But this means that μf^{-1} and νf^{-1} have the same variance; if this is zero, they both give measure 1 to $\{0\}$; otherwise, they are normal distributions with the same expectation and the same variance, so again are equal. **Q**

By 454P, $\mu=\nu$.

(c) Being a completion regular quasi-Radon probability measure (415E), $\mu_G^{(I)}$ is the completion of a Baire probability measure on $\mathbb{R}^I$. If $f:\mathbb{R}^I\to\mathbb{R}$ is a continuous linear functional, then it is expressible in the form $f(z)=\sum_{i\in I}\beta_i z(i)$, where $J=\{i:\beta_i\ne 0\}$ is finite. I need to show that f is either zero a.e. or a normal random variable with expectation 0. If $J=\emptyset$ then $f=0$ everywhere and we can stop. Otherwise, $f=\sum_{i\in J}\beta_i\pi_i$, where $\pi_i(x)=x(i)$ for $i\in I$ and $x\in\mathbb{R}^I$. Now, with respect to the measure $\mu_G^{(I)}$, $\langle\pi_i\rangle_{i\in J}$ is an independent family of normal random variables with zero expectation (272G). So $\langle\beta_i\pi_i\rangle_{i\in J}$ is independent (272E), and $\beta_i\pi_i$ is normal for

[21] Formerly 235Hc.

[22] Formerly 235I.

$i \in J$ (274Ae). By 274B, $f = \sum_{i \in J} \beta_i \pi_i$ is normal, and of course it has zero expectation. As f is arbitrary, $\mu_G^{(I)}$ is a centered Gaussian distribution.

We have

$$\int x(i)x(i)\mu_G^{(I)}(dx) = \int t^2 \mu_G(dt) = 1$$

for $i \in I$, and

$$\int x(i)x(j)\mu_G^{(I)}(dx) = \int t\mu_G(dt) \cdot \int t\mu_G(dt) = 0$$

if i, $j \in I$ are distinct. So the covariance matrix of $\mu_G^{(J)}$ is the identity matrix. By (b), it is the only centered Gaussian distribution with this covariance matrix.

(d)(i) It follows from (c) and (a) that if $\mu = \mu_G^{(\mathbb{N})} T^{-1}$, where $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^I$ is a continuous linear operator, then μ is a centered Gaussian distribution.

(ii) Now suppose that μ is a centered Gaussian distribution on $\mathbb{R}^I$. Set $\pi_i(x) = x(i)$ for $i \in I$ and $x \in \mathbb{R}^I$; for $i \in I$, set $u_i = \pi_i^\bullet$ in $L^2 = L^2(\mu)$. By 4A4Jh, there is a countable orthonormal family $\langle v_j \rangle_{j \in J}$ in L^2 such that every v_j is a linear combination of the u_i, and every u_i is a linear combination of the v_j. We may suppose that $J \subseteq \mathbb{N}$. For $i \in I$, express u_i as $\sum_{j \in J} \alpha_{ij} v_j$, where $\{j : \alpha_{ij} \neq 0\}$ is finite. Define $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^I$ by setting $(Tz)(i) = \sum_{j \in J} \alpha_{ij} z(j)$ for every $z \in \mathbb{R}^{\mathbb{N}}$ and $i \in I$. Then T is a continuous linear functional. Set $\nu = \mu_G^{(\mathbb{N})} T^{-1}$, so that ν is a centered Gaussian distribution on $\mathbb{R}^I$, by (a). Because $\mathbb{R}^I$ is Polish, both μ and ν must be Radon measures.

(iii) Now μ and ν have the same covariance matrices. **P** If i, $i' \in I$ then

$$\begin{aligned}\int x(i)x(i')\mu(dx) &= (u_i|u_{i'}) = \sum_{j,j' \in J} \alpha_{ij}\alpha_{i'j'}(v_j|v_{j'}) \\ &= \sum_{j \in J} \alpha_{ij}\alpha_{i'j} = \sum_{j,j' \in J} \alpha_{ij}\alpha_{i'j'} \int z(j)z(j')\mu_G^{(\mathbb{N})}(dz) \\ &= \int (Tz)(i)(Tz)(i')\mu_G^{(\mathbb{N})}(dz) = \int x(i)x(i')\nu(dx). \text{ } \mathbf{Q}\end{aligned}$$

By (b), $\mu = \nu$ is of the required form.

(e) We must first confirm that ν is the completion of a Baire measure. **P** If we write $\mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j})$ for the Baire σ-algebra of $\mathbb{R}^{I_j}$, then each ν_j is the completion of its restriction $\nu_j \restriction \mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j})$, so ν is also the product of the measures $\nu_j \restriction \mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j})$ (254I), and is therefore the completion of its restriction to $\widehat{\bigotimes}_{j \in J} \mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j})$ (254Ff). But as $\mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j}) = \widehat{\bigotimes}_{i \in I_j} \mathcal{B}\mathfrak{a}(\mathbb{R})$ for every j (4A3Na), $\widehat{\bigotimes}_{j \in J} \mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j})$ can be identified with $\widehat{\bigotimes}_{i \in I} \mathcal{B}\mathfrak{a}(\mathbb{R}) = \mathcal{B}\mathfrak{a}(\mathbb{R}^I)$, so that ν is indeed the completion of $\nu \restriction \mathcal{B}\mathfrak{a}(\mathbb{R}^I)$. **Q**

Now suppose that $f : \mathbb{R}^I \to \mathbb{R}$ is a continuous linear functional. Then we can express f in the form $f(x) = \sum_{i \in K} \alpha_i x(i)$ for every $x \in \mathbb{R}^I$, where $K \subseteq I$ is finite. Set $L = \{j : K \cap I_j \neq \emptyset\}$ and $K_j = K \cap I_j$ for $j \in L$, so that L and every K_j are finite; for $j \in L$ and $x \in \mathbb{R}^I$ set $f_j(x) = \sum_{i \in K_j} \alpha_i x(i)$. Now $f = \sum_{j \in L} f_j$.

If we set $g_j(y) = \sum_{i \in K_j} \alpha_i y(i)$ for $y \in \mathbb{R}^{I_j}$, then g_j is either zero a.e. or a normal random variable with respect to the probability measure ν_j. Since

$$\nu\{x : f_j(x) \leq \alpha\} = \nu\{x : g_j(x \restriction I_j) \leq \alpha\} = \nu_j\{y : g_j(y) \leq \alpha\}$$

for every $\alpha \in \mathbb{R}$, f_j (regarded as a random variable on $(\mathbb{R}^I, \nu)$) has the same distribution as g_j (regarded as a random variable on $(\mathbb{R}^{I_j}, \nu_j)$). This is true for every $j \in L$. Moreover, the different f_j, as j runs over L, are independent. So $f = \sum_{j \in L} f_j$ is the sum of independent random variables which are all either normal or essentially constant. By 274B again, f also is either normal or essentially constant. And of course its expectation is zero. As f is arbitrary, this shows that ν is a centered Gaussian distribution.

(f) Set $Tx = -x$ for $x \in \mathbb{R}^I$, so that T is a continuous linear operator and we have a unique centered Gaussian distribution ν on $\mathbb{R}^I$ such that T is inverse-measure-preserving for μ and ν, by (a). For any i, $j \in I$,

$$\int x(i)x(j)\nu(dx) = \int (Tx)(i)(Tx)(j)\mu(dx) = \int x(i)x(j)\mu(dx),$$

so μ and ν have the same covariance matrices and are equal, by (b). Accordingly

$$\mu(-E) = \mu T^{-1}[E] = \nu E = \mu E$$

whenever μ measures E.

456C Since a Gaussian distribution is determined by its covariance matrix (456Bb), we naturally seek descriptions of which matrices can arise.

Theorem Let I be a set and $\langle\sigma_{ij}\rangle_{i,j\in I}$ a family of real numbers. Then the following are equiveridical:

(i) $\langle\sigma_{ij}\rangle_{i,j\in I}$ is the covariance matrix of a centered Gaussian distribution on $\mathbb{R}^I$;

(ii) there are a (real) Hilbert space U and a family $\langle u_i\rangle_{i\in I}$ in U such that $(u_i|u_j)=\sigma_{ij}$ for all i, $j\in I$;

(iii) for every finite $J\subseteq I$, $\langle\sigma_{ij}\rangle_{i,j\in J}$ is the covariance matrix of a centered Gaussian distribution on $\mathbb{R}^J$;

(iv) $\langle\sigma_{ij}\rangle_{i,j\in I}$ is symmetric and positive semi-definite in the sense that $\sigma_{ij}=\sigma_{ji}$ for all i, $j\in I$ and $\sum_{i,j\in J}\alpha_i\alpha_j\sigma_{ij}\ge 0$ whenever $J\subseteq I$ is finite and $\langle\alpha_i\rangle_{i\in J}\in\mathbb{R}^J$.

proof (i)⇒(ii) If μ is a centered Gaussian distribution on $\mathbb{R}^I$ with covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in I}$, then $L^2(\mu)$ is a Hilbert space. Setting $X_i(x)=x(i)$ for $x\in\mathbb{R}^I$, $u_i=X_i^\bullet$ belongs to the Hilbert space $L^2(\mu)$ for every $i\in I$, and

$$(u_i|u_j)=\int X_i\times X_jd\mu=\int x(i)x(j)\mu(dx)=\sigma_{ij}$$

for all i, $j\in I$.

(ii)⇒(iv) In this context,

$$\sigma_{ij}=(u_i|u_j)=(u_j|u_i)=\sigma_{ji},$$

$$\textstyle\sum_{i,j\in J}\alpha_i\alpha_j\sigma_{ij}=\sum_{i,j\in J}\alpha_i\alpha_j(u_i|u_j)=\|\sum_{i\in J}\alpha_iu_i\|^2\ge 0.$$

(iv)⇒(iii) Here we have to know something about symmetric matrices. Given a family $\langle\sigma_{ij}\rangle_{i,j\in I}$ satisfying the conditions of (iv), and a finite set $J\subseteq I$, we have a linear operator $T:\mathbb{R}^J\to\mathbb{R}^J$ defined by saying that $(Tz)(i)=\sum_{j\in J}\sigma_{ij}z(j)$ for $z\in\mathbb{R}^J$ and $i\in J$. Give $\mathbb{R}^J=\ell^2(J)$ its usual inner product, so that $w\,.\,z=\sum_{j\in J}w(j)z(j)$ for w, $z\in\mathbb{R}^J$; then $\mathbb{R}^J$ is a Hilbert space and

$$\textstyle Tw\,.\,z=\sum_{i\in J}\sum_{j\in J}\sigma_{ij}w(j)z(i)=\sum_{j\in J}\sum_{i\in J}\sigma_{ji}z(i)w(j)=w\,.\,Tz$$

for all w, $z\in\mathbb{R}^J$, so that T is self-adjoint. Moreover, if $z\in\mathbb{R}^J$,

$$\textstyle Tz\,.\,z=\sum_{i,j\in J}\sigma_{ij}z(i)z(j)\ge 0$$

by the other condition on $\langle\sigma_{ij}\rangle_{i,j\in I}$.

By 4A4M[23], $\mathbb{R}^J$ has an orthonormal basis consisting of eigenvectors for T; if $\#(J)=n$, we have a basis $\langle u_k\rangle_{k<n}$ and a family $\langle\gamma_k\rangle_{k<n}$ of real numbers such that $Tu_k=\gamma_ku_k$ for each $k<n$. We need to know that $\sum_{k<n}u_k(i)u_k(j)=1$ if $i=j$, 0 otherwise. **P** Let $\langle v_i\rangle_{i\in J}$ be the standard basis of $\mathbb{R}^J$, so that $v_i(j)=1$ if $i=j$, 0 if $i\ne j$. Then

$$\textstyle v_i=\sum_{k<n}(v_i\,.\,u_k)u_k=\sum_{k<n}u_k(i)u_k$$

for $i\in J$, so

$$\begin{aligned}\sum_{k<n}u_k(i)u_k(j)=\sum_{k,l<n}u_k(i)u_l(j)u_k\,.\,u_l=v_i\,.\,v_j&=1\text{ if }i=j,\\ &=0\text{ otherwise. }\mathbf{Q}\end{aligned}$$

Now $\gamma_k=Tu_k\,.\,u_k\ge 0$, so $\sqrt{\gamma_k}$ is defined for each k, and we have a linear operator $S:\mathbb{R}^n\to\mathbb{R}^J$ defined by setting $Se_k=\sqrt{\gamma_k}u_k$ for each k, where $\langle e_k\rangle_{k<n}$ is the standard basis of $\mathbb{R}^n$, defined by saying that $e_k(l)=1$ if $k=l$, 0 otherwise.

Taking $\mu_G^{(n)}$ to be the standard Gaussian distribution on $\mathbb{R}^n$, $\mu=\mu_G^{(n)}S^{-1}$ is a centered Gaussian distribution on $\mathbb{R}^J$, by 456Ba. For i, $j\in J$,

[23] Or, rather, its finite-dimensional special case, which is easier; you may know it under the slogan 'symmetric matrices are diagonalisable'.

$$\begin{aligned}\int w(i)w(j)\mu(dw) &= \int (Sz)(i)(Sz)(j)\mu_G^{(n)}(dz)\\ &= \int \sum_{k<n}\sqrt{\gamma_k}z(k)u_k(i)\cdot\sum_{l<n}\sqrt{\gamma_l}z(l)u_l(j)\mu_G^{(n)}(dz)\\ &= \sum_{k<n}\sum_{l<n}\sqrt{\gamma_k\gamma_l}u_k(i)u_l(j)\int z(k)z(l)\mu_G^{(n)}(dz)\\ &= \sum_{k<n}\gamma_k u_k(i)u_k(j) = \sum_{k<n}Tu_k(i)u_k(j)\\ &= \sum_{k<n,l\in J}\sigma_{il}u_k(l)u_k(j) = \sum_{l\in J}\sigma_{il}\sum_{k<n}u_k(l)u_k(j) = \sigma_{ij}.\end{aligned}$$

So μ is the distribution we are looking for.

(iii)⇒(i) I seek to apply 454M. For each finite $J \subseteq I$, let μ_J be a centered Gaussian distribution on $\mathbb{R}^J$ with covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in J}$; by 456Bb, it is unique. If $K \subseteq I$ is finite and $J \subseteq K$, set $T_{KJ}z = z\restriction J$ for $z \in \mathbb{R}^K$; then $\mu_K T_{KJ}^{-1}$ is a centered Gaussian distribution on $\mathbb{R}^J$, by 456Ba, and its covariance matrix is that of μ_J, so $\mu_J = \mu_K T_{KJ}^{-1}$. By 454M, we have a distribution μ on $\mathbb{R}^I$, the completion of a Baire probability measure, such that $\mu_J = \mu T_J^{-1}$ for every finite $J \subseteq I$, setting $T_J x = x\restriction J$ for $x \in \mathbb{R}^I$.

Applying this with $J = \{i, j\}$, we see that $\int x(i)x(j)\mu(dx) = \sigma_{ij}$ for all i, $j \in I$. To see that μ is a centered Gaussian distribution in the sense of 456Ab, take a continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$. Then there is a finite family $\langle\beta_i\rangle_{i\in J}$ in $\mathbb{R}$ such that $f(x) = \sum_{i\in J}\beta_i x(i)$ for each $x \in \mathbb{R}^I$. Setting $g(z) = \sum_{i\in J}\beta_i z(i)$ for $z \in \mathbb{R}^J$, we have $f = gT_J$, so that the image distribution μf^{-1} on $\mathbb{R}$ is just $\mu_J g^{-1}$, and (because μ_J is a centered Gaussian distribution) is either normal or concentrated at 0. As f is arbitrary, μ itself is a centered Gaussian distribution.

456D Gaussian processes I take a page to spell out the connexion between centered Gaussian distributions, and the processes considered in 454J-454K.

Definition A family $\langle X_i\rangle_{i\in I}$ of real-valued random variables on a probability space is a **centered Gaussian process** if its distribution (454J) is a centered Gaussian distribution.

456E Independence and correlation We have an important characterization of independence of families forming a Gaussian process. The essential idea is in (a) below. I give the more elaborate version (b) for the sake of an application in §477.

Proposition (a) Let $\langle X_i\rangle_{i\in I}$ be a centered Gaussian process. Then $\langle X_i\rangle_{i\in I}$ is independent iff $\mathbb{E}(X_i \times X_j) = 0$ for all distinct i, $j \in I$.

(b) Let $\langle X_i\rangle_{i\in I}$ be a centered Gaussian process on a complete probability space (Ω, Σ, μ), and $\mathcal{J}$ a disjoint family of subsets of I; for $J \in \mathcal{J}$ let Σ_J be the σ-algebra of subsets of Ω generated by $\{X_i^{-1}[F] : i \in J, F \subseteq \mathbb{R}$ is Borel$\}$. Suppose that $\mathbb{E}(X_i \times X_j) = 0$ whenever J, J' are distinct members of $\mathcal{J}$, $i \in J$ and $j \in J'$. Then $\langle\Sigma_J\rangle_{J\in\mathcal{J}}$ is independent.

proof (a)(i) If $\langle X_i\rangle_{i\in I}$ is independent, and i, $j \in I$ are distinct, then $\mathbb{E}(X_i \times X_j) = \mathbb{E}(X_i)\mathbb{E}(X_j) = 0$, by 272R[24].

(ii) If $\mathbb{E}(X_i \times X_j) = 0$ for all distinct i, $j \in I$, let μ be the distribution of $\langle X_i\rangle_{i\in I}$ and $\langle\sigma_{ij}\rangle_{i,j\in I}$ its covariance matrix. Then $\sigma_{ij} = 0$ whenever $i \ne j$. So if we take ν_i to be the normal distribution on $\mathbb{R}$ with expectation 0 and variance σ_{ii} (or the distribution concentrated at 0 if $\sigma_{ii} = 0$), the product $\nu = \prod_{i\in I}\nu_i$ will be a centered Gaussian distribution on $\mathbb{R}^I$ (456Be) also with covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in I}$, and is equal to μ, by 456Bb. Thus μ is a product measure and $\langle X_i\rangle_{i\in I}$ is independent (454L).

(b) Set $K = \bigcup\mathcal{J}$. For each $J \in \mathcal{J}$, let ν_J be the distribution of $\langle X_i\rangle_{i\in J}$, and let $\nu = \prod_{J\in\mathcal{J}}\nu_J$ be the product measure on $\prod_{J\in\mathcal{J}}\mathbb{R}^J$, which we can identify with $\mathbb{R}^K$. Then ν is a centered Gaussian distribution, and its covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in K}$ is such that

$$\begin{aligned}\sigma_{ij} &= \mathbb{E}(X_i \times X_j) \text{ if } i,\ j \text{ belong to the same member of } \mathcal{J},\\ &= \mathbb{E}(X_i)\mathbb{E}(X_j) = 0 \text{ otherwise;}\end{aligned}$$

[24] Formerly 272Q.

that is, it is the covariance matrix of the process $\langle X_i\rangle_{i\in K}$. Let $f:\Omega\to\mathbb{R}^K$ be a function such that $f(\omega)(i)=X_i(\omega)$ whenever $i\in K$ and $\omega\in\operatorname{dom}X_i$; because μ is complete, f is $(\Sigma,\mathcal{B}\mathfrak{a}(\mathbb{R}^K))$-measurable. For $J\in\mathcal{J}$ and $\omega\in\Omega$ set $f_J(\omega)=f(\omega)\restriction J$.

Now suppose that $\mathcal{J}_0\subseteq\mathcal{J}$ is non-empty and finite and that $E_J\in\Sigma_J$ for each $J\in\mathcal{J}_0$. Then for each $J\in\mathcal{J}_0$ there is a Baire set $F_J\subseteq\mathbb{R}^J$ such that $E_J\triangle f_J^{-1}[F_J]$ is μ-negligible, and $\mu E_J=\nu_JF_J$. Next, the distribution of $\langle X_i\rangle_{i\in K}$ is a centered Gaussian distribution on $\mathbb{R}^K$, and has covariance matrix $\langle\mathbb{E}(X_i\times X_j)\rangle_{i,j\in K}=\langle\sigma_{ij}\rangle_{i,j\in K}$, so it must be ν. But this means that, setting $F=\{x:x\in\mathbb{R}^K,\ x\restriction J\in F_J\text{ for every }J\in\mathcal{J}_0\}\in\mathcal{B}\mathfrak{a}(\mathbb{R}^K)$,

$$\begin{aligned}\mu(\bigcap_{J\in\mathcal{J}}E_J)&=\mu(\bigcap_{J\in\mathcal{J}}f_J^{-1}[F_J])=\mu f^{-1}[F]\\&=\nu F=\prod_{J\in\mathcal{J}}\nu_JF_J=\prod_{J\in\mathcal{J}}\mu E_J.\end{aligned}$$

As $\langle E_J\rangle_{J\in\mathcal{J}_0}$ is arbitrary, $\langle\Sigma_J\rangle_{J\in\mathcal{J}}$ is independent.

456F Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of random variables on a probability space (Ω,Σ,μ). Then the following are equiveridical:

(i) the distribution of $\langle X_i\rangle_{i\in I}$, in the sense of 454K, is a centered Gaussian distribution;

(ii) whenever $i_0,\dots,i_n\in I$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$ then $\sum_{r=0}^n\alpha_rX_{i_r}$ is either zero a.e. or a normal random variable with zero expectation;

(iii) whenever $i_0,\dots,i_n\in I$ then the joint distribution of $X_{i_0},\dots,X_{i_n}$, in the sense of 271C, is a centered Gaussian distribution;

(iv) whenever $J\subseteq I$ is finite then there is an independent family $\langle Y_k\rangle_{k\in K}$ of standard normal random variables on Ω such that each X_i, for $i\in J$, is almost everywhere equal to a linear combination of the Y_k.

proof (ii)$\Leftrightarrow$(i) is immediate from the definition in 456Ab and 454O.

(i)$\Leftrightarrow$(iii) is also direct from 456Ab and the identification of the two concepts of 'distribution' (454K).

(iv)$\Rightarrow$(ii) is direct from 274A-274B.

(i)$\Rightarrow$(iv) For $i\in J$ set $u_i=X_i^\bullet$ in $L^2(\mu)$. By 4A4Jh again there is an orthonormal family $\langle v_k\rangle_{k\in K}$ in $L^2(\mu)$ such that each v_k is a linear combination of the u_i and each u_i is a linear combination of the v_k. Take Y_k such that $Y_k^\bullet=v_k$ for each k; then each X_i is equal almost everywhere to a linear combination of the Y_k, while each Y_k is equal almost everywhere to a linear combination of the X_i. As $\#(K)$ must be the dimension of the linear span of $\{u_i:i\in J\}$, K is finite. Any linear combination of the Y_k is equal almost everywhere to a linear combination of the X_k, so is either zero a.e. or a normal random variable with zero expectation. Because (ii)$\Rightarrow$(iii), $\langle Y_k\rangle_{k\in K}$ has a centered Gaussian distribution ν say. Each Y_k has variance $(v_k|v_k)=1$, so is a standard normal random variable.

The covariance matrix of ν is given by

$$\begin{aligned}\int y(j)y(k)\nu(dy)=\mathbb{E}(Y_j\times Y_k)=(v_j|v_k)&=1\text{ if }j=k,\\&=0\text{ otherwise.}\end{aligned}$$

By 456E, $\langle Y_k\rangle_{k\in K}$ is independent, so we have found a suitable family.

456G Now I start work on material for the main theorem of this section.

Lemma Let I be a finite set and μ a centered Gaussian distribution on $\mathbb{R}^I$. Suppose that $\gamma\ge0$ and $\alpha=\mu\{x:\sup_{i\in I}|x(i)|\ge\gamma\}$. Then $\mu\{x:\sup_{i\in I}|x(i)|\ge\frac{1}{2}\gamma\}\ge2\alpha(1-\alpha)^3$.

proof (a) The case $\gamma=0$, $\alpha=1$ is trivial; suppose that $\gamma>0$. We may suppose that I has a total order $\le$. Give $\mathbb{R}^{I\times4}\cong(\mathbb{R}^I)^4$ the product λ of four copies of μ; then λ is a centered Gaussian distribution (456Be). Define $T:\mathbb{R}^{I\times4}\to\mathbb{R}^I$ by setting $(Ty)(i)=\frac{1}{2}\sum_{r=0}^3y(i,r)$ for $i\in I$, $y\in\mathbb{R}^{I\times4}$; then λT^{-1} is a centered Gaussian distribution on $\mathbb{R}^I$ (456Ba). Now λT^{-1} has the same covariance matrix as μ. **P** If i, $j\in I$ then

$$\int x(i)x(j)(\lambda T^{-1})(dx) = \int (Ty)(i)(Ty)(j)\lambda(dy) = \frac{1}{4}\sum_{r=0}^{3}\sum_{s=0}^{3}\int y(i,r)y(j,s)\lambda(dy)$$
$$= \frac{1}{4}\sum_{r=0}^{3}\int y(i,r)y(j,r)\lambda(dy) = \frac{1}{4}\sum_{r=0}^{3}\int x(i)x(j)\mu(dx)$$

(because the map $y \mapsto \langle y(i,r)\rangle_{i\in I} : \mathbb{R}^{I\times 4} \to \mathbb{R}^I$ is inverse-measure-preserving for each r)

$$= \int x(i)x(j)\mu(dx). \text{ } \mathbf{Q}$$

So $\lambda T^{-1} = \mu$, by 456Bb.

(b) Define

$$E_{ir} = \{y : y \in \mathbb{R}^{I\times 4},\ |y(i,r)| \ge \gamma\}$$

for $r < 4$ and $i \in I$, and

$$E_r = \bigcup_{i\in I} E_{ir}$$

for $r < 4$, so that

$$\lambda E_r = \lambda\{y : \sup_{i\in I}|y(i,r)| \ge \gamma\} = \mu\{x : \sup_{i\in I}|x(i)| \ge \gamma\} = \alpha.$$

Set $E'_r = E_r \setminus \bigcup_{s\ne r} E_s$, so that

$$\lambda E'_r = \lambda\{y : \sup_{i\in I}|y(i,r)| \ge \gamma,\ \sup_{i\in I}|y(i,s)| < \gamma \text{ for } s \ne r)$$
$$= \lambda\{y : \sup_{i\in I}|y(i,r)| \ge \gamma\} \cdot \prod_{s\ne r}\lambda\{y : \sup_{i\in I}|y(i,s)| < \gamma\}$$

(because these are independent events)

$$= \alpha(1-\alpha)^3.$$

(c) Next, for $i \in I$ and $r < 4$, set $E'_{ir} = E_{ir} \setminus (\bigcup_{j<i} E_{jr} \cup \bigcup_{s\ne r} E_s)$, so that $E'_r = \bigcup_{i\in I} E'_{ir}$. Observe that $\langle E'_{ir}\rangle_{i\in I,r<4}$ is disjoint. Set

$$F_{ir} = \{y : y \in E'_{ir},\ y(i,r)\textstyle\sum_{s\ne r} y(i,s) \ge 0\}.$$

Then $\nu F_{ir} \ge \frac{1}{2}\nu E'_{ir}$. **P** We can think of $\mathbb{R}^{I\times 4}$ as a product $\mathbb{R}^J \times \mathbb{R}^K$, where $J = I \times \{r\}$ and $K = I \times (4 \setminus \{r\})$. In this case, λ becomes identified with a product $\lambda_J \times \lambda_K$, where λ_J and λ_K are centered Gaussian distributions on $\mathbb{R}^J$ and $\mathbb{R}^K$ respectively, and E'_{ir} is of the form $V \times W$, where

$$V = \{v : v \in \mathbb{R}^J,\ |v(i,r)| \ge \gamma,\ |v(j,r)| < \gamma \text{ for } j < i\},$$

$$W = \{w : w \in \mathbb{R}^K,\ |w(j,s)| < \gamma \text{ for every } j \in I,\ s \ne r\}.$$

In the same representation, F_{ir} becomes $(V^+ \times W^+) \cup (V^- \times W^-)$, where

$$V^+ = \{v : v \in V,\ v(i,r) \ge \gamma\}, \quad V^- = \{v : v \in V,\ v(i,r) \le -\gamma\},$$

$$W^+ = \{w : w \in W,\ \textstyle\sum_{r\ne s} w(i,s) \ge 0\}, \quad W^- = \{w : w \in W,\ \textstyle\sum_{r\ne s} w(i,s) \le 0\}.$$

By 456Bf, $\lambda_J V^+ = \lambda_J V^-$ and $\lambda_K W^+ = \lambda_K W^-$; since $V^- = V \setminus V^+$, while $W^+ \cup W^- = W$, we have

$$\lambda_J V^+ = \lambda_J V^- = \frac{1}{2}\lambda_J V, \quad \lambda_K W^+ = \lambda_K W^- \ge \frac{1}{2}\lambda_K W.$$

But this means that

$$\lambda F_{ir} = \lambda(V^+ \times W^+) + \lambda(V^- \times W^-)$$
$$= \lambda_J V^+ \cdot \lambda_K W^+ + \lambda_J V^- \cdot \lambda_K W^- \ge \frac{1}{2}\lambda_J V \cdot \lambda_K W = \frac{1}{2}\lambda E'_{ir},$$

as claimed. **Q**

(d) At this point, observe that if $y \in F_{ir}$ then $|\sum_{s=0}^{3} y(i,s)| \ge |y(i,r)| \ge \gamma$. So

$$\begin{aligned}\mu\{x : \sup_{i\in I} |x(i)| \ge \frac{1}{2}\gamma\}) &= \lambda T^{-1}[\{x : \sup_{i\in I} |x(i)| \ge \frac{1}{2}\gamma\})] \\ &= \lambda\{y : \sup_{i\in I} |\frac{1}{2}\sum_{r=0}^{3} y(i,r)| \ge \frac{1}{2}\gamma\}) \\ &= \lambda\{y : \sup_{i\in I} |\sum_{r=0}^{3} y(i,r)| \ge \gamma) \\ &\ge \lambda(\bigcup_{i\in I, r<4} F_{ir}) = \sum_{i\in I, r<4} \lambda F_{ir} \\ &\ge \frac{1}{2}\sum_{r<4}\sum_{i\in I} \lambda E'_{ir} \ge \frac{1}{2}\sum_{r<4} \lambda E'_r = 2\alpha(1-\alpha)^3,\end{aligned}$$

which is what we set out to prove.

456H The support of a Gaussian distribution: Proposition Let I be a set and μ a centered Gaussian distribution on $\mathbb{R}^I$. Write Z for the set of those $x \in \mathbb{R}^I$ such that $f(x) = 0$ whenever $f : \mathbb{R}^I \to \mathbb{R}$ is a continuous linear functional and $f = 0$ a.e. Then Z is a self-supporting closed linear subspace of $\mathbb{R}^I$ with full outer measure. If I is countable Z is the support of μ.

proof (a) Being the intersection of a family of closed linear subspaces, of course Z is a closed linear subspace.

(b) Z has full outer measure. **P** Let $F \subseteq \mathbb{R}^I$ be a non-negligible zero set. Let $J \subseteq I$ be a countable set such that F is determined by coordinates in J. For $i \in I$ and $x \in \mathbb{R}^I$ set $\pi_i(x) = x(i)$; then each π_i is either normally distributed or zero almost everywhere, so is square-integrable; set $u_i = \pi_i^\bullet$ in $L^2 = L^2(\mu)$. Let $\langle v_k\rangle_{k\in K}$ be a countable orthonormal family in L^2 such that every v_k is a linear combination of the u_i, for $i \in J$, and every u_i, for $i \in J$, is a linear combination of the v_k (4A4Jh once more). Extend $\langle v_k\rangle_{k\in K}$ to a Hamel basis $\langle v_l\rangle_{l\in L}$ of L^2. For every $i \in I$, we can express u_i as $\sum_{l\in L} \alpha_{il} v_l$, where $\{l : \alpha_{il} \ne 0\}$ is finite; and the construction ensures that $\alpha_{il} = 0$ if $i \in J$ and $l \in L \setminus K$.

Consider the linear operator $T_0 : \mathbb{R}^K \to \mathbb{R}^J$ defined by setting $(T_0 z)(i) = \sum_{k\in K} \alpha_{ik} z(k)$ for $z \in \mathbb{R}^K$ and $i \in J$. If we give $\mathbb{R}^K$ the product measure $\mu_G^{(K)}$, then the image measure $\mu_G^{(K)} T_0^{-1}$ is a Gaussian distribution (456Ba), with covariance matrix

$$\begin{aligned}\sigma_{ii'} &= \int x(i)x(i')(\mu_G^{(K)} T_0^{-1})dx = \int (T_0 z)(i)(T_0 z)(i')\mu_G^{(K)}(dz) \\ &= \sum_{k,k'\in K} \alpha_{ik}\alpha_{i'k'} z(k)z(k')\mu_G^{(K)}(dz) = \sum_{k\in K} \alpha_{ik}\alpha_{i'k} \\ &= \sum_{k,k'\in K} \alpha_{ik}\alpha_{i'k}(v_k|v_{k'}) = (u_i|u_{i'}) = \int x(i)x(i')\mu(dx).\end{aligned}$$

But this means that $\mu_G^{(K)} T_0^{-1}$ has the same covariance matrix as $\mu\tilde\pi_J^{-1}$, where $\tilde\pi_J x = x\restriction J$ for $x \in \mathbb{R}^I$. Since this also is a centered Gaussian distribution, the two measures must be equal (456Bb). We know that $\tilde\pi_J[F]$ has non-zero measure, so there is a $z_0 \in \mathbb{R}^K$ such that $T_0 z_0 \in \tilde\pi_J[F]$. Extend z_0 arbitrarily to $z_1 \in \mathbb{R}^L$.

Set $x_1(i) = \sum_{l\in L} \alpha_{il} z_1(l)$ for $i \in I$. Then $\tilde\pi_J x_1 = T_0 z_0 \in \tilde\pi_J[F]$, so $x_1 \in F$, because F is determined by coordinates in J. If a continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$ is zero a.e., it can be expressed in the form $f(x) = \sum_{i\in I} \beta_i x(i)$ where $\{i : \beta_i \ne 0\}$ is finite (4A4Be again). In this case,

$$0 = f^\bullet = \textstyle\sum_{i\in I} \beta_i u_i = \sum_{l\in L}\sum_{i\in I} \beta_i \alpha_{il} v_l$$

in L^2. Since $\langle v_l\rangle_{l\in L}$ is linearly independent, $\sum_{i\in I} \beta_i\alpha_{il} = 0$ for every $l \in L$. But this means that

$$f(x_1) = \textstyle\sum_{i\in I, l\in L} \beta_i \alpha_{il} z_1(l) = 0.$$

As f is arbitrary, $x_1 \in Z$ and $Z \cap F \ne \emptyset$. As F is arbitrary, and μ is inner regular with respect to the zero sets, Z has full outer measure. **Q**

(c) Z is self-supporting. **P** If $W \subseteq \mathbb{R}^I$ is an open set meeting Z, there is an open set V, depending on coordinates in a finite set $J \subseteq I$, such that $V \subseteq W$ and $V \cap Z \neq \emptyset$. Write $\tilde{\pi}_J(x) = x\restriction J$ for $x \in \mathbb{R}^I$, and ν_J for the image measure $\mu\tilde{\pi}_J^{-1}$ on $\mathbb{R}^J$; by 456Ba, this is a centered Gaussian distribution. By 456Bd, there is a continuous linear operator $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^J$ such that $\nu_J = \mu_G^{(\mathbb{N})}T^{-1}$. Since the support of μ_G is $\mathbb{R}$, the support of $\mu_G^{(\mathbb{N})}$ is $\mathbb{R}^{\mathbb{N}}$ (417E(iv), or otherwise), and the support of ν_J is $Z_1 = \overline{T[\mathbb{R}^{\mathbb{N}}]}$ (411Ne).

Write Q for the set of linear functionals $g : \mathbb{R}^J \to \mathbb{R}$ (necessarily continuous, because J is finite) which are zero on Z_1. If $g \in Q$, then $gT = 0$, so $g = 0$ ν_J-a.e. and $g\tilde{\pi}_J = 0$ μ-a.e. This means that $g\tilde{\pi}_J(x) = 0$ for every $x \in Z$, that is, $g(y) = 0$ for every $y \in \tilde{\pi}_J[Z]$. Because Z_1 is a linear subspace of $\mathbb{R}^J$, this is enough to show that $\tilde{\pi}_J[Z] \subseteq Z_1$.

Now recall that $V \cap Z \neq \emptyset$ so $Z_1 \cap \tilde{\pi}_J[V] \neq \emptyset$, while $V = \tilde{\pi}_J^{-1}[\tilde{\pi}_J[V]]$. Since $\tilde{\pi}_J$ is an open map (4A2B(f-i)), $\tilde{\pi}_J[V]$ is open and

$$\mu^*(W \cap Z) = \mu W \ge \mu V = \nu_J\tilde{\pi}_J[V] > 0,$$

because Z_1 is the support of ν_J. **Q**

(d) If I is countable, μ is a topological measure so measures Z, and Z is the support of μ.

456I Remarks (a) In the context of 456H, I will call Z the **support** of the centered Gaussian distribution μ, even though μ need not be a topological measure, so the definition 411Nb is not immediately applicable. In 456P we shall see that Z really is the support of a canonical extension of μ.

(b) It is worth making one elementary point at once. If I and J are sets, μ and ν are centered Gaussian distributions on $\mathbb{R}^I$ and $\mathbb{R}^J$ respectively with supports Z and Z', and $T : \mathbb{R}^I \to \mathbb{R}^J$ is an inverse-measure-preserving continuous linear operator, then $Tz \in Z'$ for every $z \in Z$. **P** If $g : \mathbb{R}^J \to \mathbb{R}$ is a continuous linear functional which is zero ν-a.e., then $gT : \mathbb{R}^I \to \mathbb{R}$ is a continuous linear functional which is zero μ-a.e., so $g(Tz) = (gT)(z) = 0$. **Q**

456J Universal Gaussian distributions: Definition A centered Gaussian distribution on $\mathbb{R}^I$ is **universal** if its covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in I}$ is the inner product for a Hilbert space structure on I. (See 456Xe.)

456K Proposition Let I be any set, and μ a centered Gaussian distribution on I. Then there are a set J, a universal centered Gaussian distribution ν on $\mathbb{R}^J$, and a continuous inverse-measure-preserving linear operator $T : \mathbb{R}^J \to \mathbb{R}^I$.

proof (a) Set $J = L^2(\mu)$. Then for any finite $K \subseteq J$ there is a centered Gaussian distribution μ_K on $\mathbb{R}^K$ such that $\int x(u)x(v)\mu(dx) = (u|v)$ for all u, $v \in K$. **P** If $K = \emptyset$ or $K = \{0\}$ this is trivial, as we take μ to be the trivial distribution concentrated at 0. Otherwise, let $\langle w_i\rangle_{i<n}$ be an orthonormal basis for the linear subspace of J generated by K. For each $u \in K$, express it as $\sum_{i=0}^{n-1}\alpha_{ui}w_i$. Define $T : \mathbb{R}^n \to \mathbb{R}^K$ by setting $(Tz)(u) = \sum_{i=0}^{n-1}\alpha_{ui}z(i)$ for $z \in \mathbb{R}^n$ and $u \in K$. Set $\mu_K = \mu_G^{(n)}T^{-1}$. Then μ_K is a centered Gaussian distribution, by 456Ba, and its covariance matrix is given by

$$\begin{aligned}\int x(u)x(v)\mu_K(dx) &= \int(Ty)(u)(Ty)(v)\mu_G^{(n)}(dy) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\alpha_{ui}\alpha_{vj}\int y(i)y(j)\mu_G^{(n)}(dy)\\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\alpha_{ui}\alpha_{vj}(w_i|w_j) = (u|v)\end{aligned}$$

for all u, $v \in K$. So μ_K is the distribution we seek. **Q**

(b) If $K \subseteq J$ is finite and $L \subseteq K$, then $\mu_L = \mu_K\pi_{KL}^{-1}$, where $\pi_{KL}(x) = x\restriction L$ for $x \in \mathbb{R}^K$. **P** Since $\mu_K\pi_{KL}^{-1}$ is a centered Gaussian distribution, all we have to do is to check its covariance matrix. But if u, $v \in L$ then

$$\begin{aligned}\int y(u)y(v)(\mu_K\pi_{KL}^{-1})(dy) &= \int(\pi_{KL}x)(u)(\pi_{KL}x)(v)\mu_K(dx)\\ &= \int x(u)x(v)\mu_K(dx) = (u|v) = \int y(u)y(v)\mu_L(dy).\end{aligned}$$

By 456Bb, $\mu_L = \mu_K\pi_{KL}^{-1}$. **Q**

(c) By 454G, there is a Baire measure ν' on $\mathbb{R}^J$ such that $\nu'\pi_{JK}^{-1}[E] = \mu_K E$ for every finite $K \subseteq J$ and every Borel set $E \subseteq \mathbb{R}^K$. Take ν to be the completion of ν'. Then π_{JK} is inverse-measure-preserving for ν and μ_K, for every finite $K \subseteq J$. If $f : \mathbb{R}^J \to \mathbb{R}$ is a continuous linear functional, there are a finite $K \subseteq J$ and a linear functional $g : \mathbb{R}^K \to \mathbb{R}$ such that $f = \pi_{JK} g$, so that

$$\nu\{x : f(x) \le \alpha\} = \mu_K\{x : g(x) \le \alpha\}$$

for every α, and f and g have the same distribution; as g is either normal with zero expectation or zero a.e., so is f. As f is arbitrary, ν is a centered Gaussian distribution.

(d) ν is universal. **P** If u, $v \in J$ set $K = \{u, v\}$. Then

$$\begin{aligned}\int x(u)x(v)\nu(dx) &= \int (\pi_{JK}x)(u)(\pi_{JK}x)(v)\nu(dx)\\ &= \int y(u)y(v)\mu_K(dy) = (u|v).\end{aligned}$$

Thus the covariance matrix of ν is just the inner product of the standard Hilbert space structure of J. **Q**

(e) For $i \in I$, let $u_i \in J$ be the equivalence class of the square-integrable function $x \mapsto x(i) : \mathbb{R}^I \to \mathbb{R}$. Define $T : \mathbb{R}^J \to \mathbb{R}^I$ by setting $(Ty)(i) = y(u_i)$ for every $i \in I$ and $y \in \mathbb{R}^J$. Then there is a centered Gaussian distribution μ' on $\mathbb{R}^I$ such that T is inverse-measure-preserving for ν and μ'. Now the covariance matrix of μ' is defined by

$$\begin{aligned}\int x(i)x(j)\mu'(dx) = \int (Ty)(i)(Ty)(j)\nu(dy) &= \int y(u_i)y(u_j)\nu(dy)\\ = (u_i|u_j) &= \int x(i)x(j)\mu(dx)\end{aligned}$$

for all i, $j \in I$. So μ and μ' are equal and T is inverse-measure-preserving for ν and μ.

456L Lemma Let μ be a universal centered Gaussian distribution on $\mathbb{R}^I$; give I a corresponding Hilbert space structure such that $\int x(i)x(j)\mu(dx) = (i|j)$ for all i, $j \in I$. Let $F \in \operatorname{dom}\mu$ be a set determined by coordinates in J, where $J \subseteq I$ is a closed linear subspace for the Hilbert space structure of I. Let W be the union of all the open subsets of $\mathbb{R}^I$ which meet F in a negligible set, and W' the union of the open subsets of $\mathbb{R}^I$ which meet F in a negligible set and are determined by coordinates in J. If $F \subseteq W$ then $F \subseteq W'$.

proof (a) Let Z be the support of μ in the sense of 456H. We need to know that Z is just the set of all linear functionals from I to $\mathbb{R}$. **P** If $K \subseteq I$ is finite and $\langle\alpha_i\rangle_{i\in K} \in \mathbb{R}^K$ and $f(x) = \sum_{i\in I}\alpha_i x(i)$ for $x \in \mathbb{R}^I$, then

$$\|f\|_2^2 = \textstyle\sum_{i,j\in K}\alpha_i\alpha_j(i|j) = \|\sum_{i\in K}\alpha_i i\|^2.$$

So, for $x \in \mathbb{R}^I$,

$$\begin{aligned}x \in Z &\iff f(x) = 0 \text{ whenever } f \in (\mathbb{R}^I)^* \text{ and } \|f\|_2 = 0\\ &\iff \sum_{i\in K}\alpha_i x(i) = 0 \text{ whenever } K \subseteq I \text{ is finite and } \sum_{i\in K}\alpha_i i = 0 \text{ in } I\\ &\iff x : I \to \mathbb{R} \text{ is linear. } \mathbf{Q}\end{aligned}$$

(b) Let $J^\perp$ be the orthogonal complement of J in I, so that $I = J \oplus J^\perp$ (4A4Jf). Give $\mathbb{R}^J$ and $\mathbb{R}^{J^\perp}$ the centered Gaussian distributions μ_J, $\mu_{J^\perp}$ induced by μ and the projections $x \mapsto x\restriction J$, $x \mapsto x\restriction J^\perp$. Then the product measure λ on $\mathbb{R}^J \times \mathbb{R}^{J^\perp}$ is also a centered Gaussian distribution (456Be). Define $T : \mathbb{R}^J \times \mathbb{R}^{J^\perp} \to \mathbb{R}^I$ by setting $T(u, v)(j + k) = u(j) + v(k)$ whenever $j \in J$, $k \in J^\perp$, $u \in \mathbb{R}^J$ and $v \in \mathbb{R}^{J^\perp}$. Then T is inverse-measure-preserving for λ and μ. **P** T is a continuous linear operator so we have a centered Gaussian distribution μ' on $\mathbb{R}^I$ such that T is inverse-measure-preserving for λ and μ'. If j, $j' \in J$ and k, $k' \in J^\perp$,

$$\begin{aligned}\int x(j+k)x(j'+k')\mu'(dx) &= \int T(u,v)(j+k)T(u,v)(j'+k')\lambda(d(u,v))\\ &= \int (u(j)+v(k))(u(j')+v(k'))\lambda(d(u,v))\\ &= \int u(j)u(j')\mu_J(du) + \int v(k)v(k')\mu_{J^\perp}(dv)\\ &= \int x(j)x(j')\mu(dx) + \int x(k)x(k')\mu(dx)\\ &= (j|j') + (k|k') = (j+k|j'+k')\\ &= \int x(j+k)x(j'+k')\mu(dx).\end{aligned}$$

Thus μ and μ' have the same covariance matrix and are equal, and T is inverse-measure-preserving for λ and μ. **Q**

(c) Take any $z \in W \cap Z$. Then there is an open set V, determined by coordinates in a finite set $K_0 \subseteq I$, such that $z \in V$ and $\mu(V \cap F) = 0$. Let $\epsilon > 0$ be such that $y \in V$ whenever $x \in \mathbb{R}^I$ and $|x(i) - z(i)| < 2\epsilon$ for every $i \in K_0$. Express each $k \in K_0$ as $k' + k''$ where $k' \in J$ and $k'' \in J^\perp$. Set

$$V' = \{x : x \in \mathbb{R}^I,\ |x(k') - z(k')| < \epsilon \text{ for every } k \in K_0\}.$$

Then V' is an open set, determined by coordinates in J, and contains z. Also $\mu(V' \cap F) = 0$. **P** Set $V'' = \{x : x \in \mathbb{R}^I$, $|x(k'') - z(k'')| < \epsilon$ for every $k \in K_0\}$. Then $V \supseteq V' \cap V'' \cap Z$. Since Z has full outer measure (456H),

$$\mu(V' \cap V'' \cap F) = \mu^*(V' \cap V'' \cap F \cap Z) \le \mu(V \cap F) = 0.$$

Now

$$\begin{aligned}0 = \mu(V' \cap V'' \cap F) &= \lambda T^{-1}[V' \cap F \cap V'']\\ &= \lambda\{(x\restriction J, x\restriction J^\perp) : x \in V' \cap F \cap V''\}\end{aligned}$$

(because $V' \cap F \cap V''$ is determined by coordinates in $J \cup J^\perp$)

$$= \mu_J\{x\restriction J : x \in V' \cap F\} \cdot \mu_{J^\perp}\{x\restriction J^\perp : x \in V''\}$$

because $V' \cap F$ is determined by coordinates in J, while V'' is determined by coordinates in $J^\perp$. However, $z \in V''$, and $z\restriction J^\perp$ belongs to the support Z' of $\mu_{J^\perp}$, by 456Ib; since Z' is self-supporting, and $\{x\restriction J^\perp : x \in V''\}$ is open, $\mu_{J^\perp}\{x\restriction J^\perp : x \in V''\} > 0$. We conclude that

$$0 = \mu_J\{x\restriction J : x \in V' \cap F\} = \mu(V' \cap F). \text{ **Q**}$$

(d) This shows that $z \in W'$. As z is arbitrary, $W \cap Z \subseteq W'$.

? Suppose, if possible, that there is a point $z_0 \in F \setminus W'$. If i, $j \in J$ and $\alpha \in \mathbb{R}$, then

$$\{x : x(i+j) \ne x(i) + x(j)\}, \quad \{x : x(\alpha i) \ne \alpha x(i)\}$$

are negligible open sets determined by coordinates in J, so are included in W' and do not contain z_0. Thus $z_0\restriction J : J \to \mathbb{R}$ is linear. Let $z : I \to \mathbb{R}$ be a linear functional extending $z_0\restriction J$. Then $z \in Z$ and $z\restriction J = z_0\restriction J$; as both F and W' are determined by coordinates in J, $z \in F \setminus W'$. But this means that $z \in Z \cap W \setminus W'$, which is impossible. **X**

So $F \subseteq W'$, as claimed.

456M Cluster sets: Lemma Let I be a countable set, $n \ge 1$ an integer and μ a centered Gaussian distribution on $\mathbb{R}^{I\times n}$. For $\epsilon > 0$ set

$$I_\epsilon = \{i : i \in I,\ \int |x(i,r)|^2 \mu(dx) \le \epsilon^2 \text{ for every } r < n\};$$

suppose that no I_ϵ is empty.

(a) There is a closed set $F \subseteq \mathbb{R}^n$ such that

$$F = \bigcap_{\epsilon>0} \overline{\{\langle x(i,r)\rangle_{r<n} : i \in I_\epsilon\}}$$

for almost every $x \in \mathbb{R}^{I\times n}$.

(b) If $z \in F$ and $-1 \le \alpha \le 1$, then $\alpha z \in F$.

(c) If F is bounded, then there is some $\epsilon > 0$ such that $\sup_{i \in I_\epsilon, r<n} |x(i,r)| < \infty$ for almost every $x \in \mathbb{R}^{I\times n}$.

proof (a)(i) For $x \in \mathbb{R}^{I\times n}$ and $i \in I$ set $S_i(x) = \langle x(i,r)\rangle_{r<n} \in \mathbb{R}^n$. For $x \in \mathbb{R}^{I\times n}$ set

$$F_x = \bigcap_{\epsilon>0} \overline{\{S_i x : i \in I_\epsilon\}},$$

so that F_x is a closed subset of $\mathbb{R}^n$. For $A \subseteq \mathbb{R}^n$ set $E_A = \{x : x \in \mathbb{R}^{I\times n},\ A \cap F_x \ne \emptyset\}$.

(ii) By 456Bd, there is a continuous linear operator $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{I\times n}$ such that $\mu = \mu_G^{(\mathbb{N})} T^{-1}$. Set $T_i = S_i T$ for $i \in I$; then $T_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$ is a continuous linear operator. For $y \in \mathbb{R}^{\mathbb{N}}$, set

$$\tilde F_y = F_{T(y)} = \bigcap_{\epsilon>0} \overline{\{T_i(y) : i \in I_\epsilon\}}.$$

For $A \subseteq \mathbb{R}^n$ set

$$\tilde E_A = T^{-1}[E_A] = \{y : y \in \mathbb{R}^{\mathbb{N}},\ A \cap \tilde F_y \ne \emptyset\}.$$

(iii) If $K \subseteq \mathbb{R}^n$ is compact, then $\tilde E_K$ is a Borel subset of $\mathbb{R}^{\mathbb{N}}$. **P** Let $\mathcal{V}$ be a countable base for the topology of $\mathbb{R}^n$, and for $k \ge 1$ let $\mathcal{V}_k$ be the set of members of $\mathcal{V}$ with diameter at most $1/k$ which meet K. Then $T_i^{-1}[V]$ is a Borel set for every $V \in \mathcal{V}$ and $i \in I$, so

$$E' = \bigcap_{k\ge1} \bigcup_{V\in\mathcal{V}_k} \bigcup_{i\in I_{1/k}} T_i^{-1}[V]$$

is a Borel set.

If $y \in \tilde E_K$, take $k \ge 1$. There is a $z \in \tilde F_y \cap K$. Let $V \in \mathcal{V}$ be such that $z \in V$ and $\operatorname{diam} V \le 1/k$; in this case $V \in \mathcal{V}_k$. Because $z \in \overline{\{T_i(y) : i \in I_{1/k}\}}$, there is an $i \in I_{1/k}$ such that $T_i(y) \in V$. As k is arbitrary, this shows that $y \in E'$; thus $\tilde E_K \subseteq E'$.

If $y \notin \tilde E_K$, then K is a compact set disjoint from the closed set $\tilde F_y$. There is therefore some $\epsilon > 0$ such that $K \cap \overline{\{T_i(y) : i \in I_\epsilon\}} = \emptyset$ (since these form a downwards-directed family of compact sets with empty intersection). Next, there is a $\delta > 0$ such that $B(z,\delta) \cap \overline{\{T_i(y) : i \in I_\epsilon\}} = \emptyset$ for every $z \in K$ (2A2Ed). Let $k \ge 1$ be such that $1/k \le \min(\epsilon,\delta)$. If $V \in \mathcal{V}_k$ and $i \in I_{1/k}$, there is some $z \in K \cap V$ so $V \subseteq B(z,\delta)$ and $T_i(y) \notin V$. This shows that $y \notin E'$. As y is arbitrary, $E' \subseteq \tilde E_K$.

So $\tilde E_K = E'$ is a Borel set. **Q**

It follows at once that $\tilde E_H$ is a Borel set for every K_σ set H, in particular, for any open or closed set H.

(iv) We need a simple estimate on the coefficients of the linear operators T_i. Let α_{irj} be such that $T_i(y) = \langle \sum_{j=0}^\infty \alpha_{irj} y(j)\rangle_{r<n}$ for $i \in I$, $r < n$ and $y \in \mathbb{R}^{\mathbb{N}}$. (Of course $\{j : \alpha_{irj} \ne 0\}$ is finite for each i and r.) Then

$$\int x(i,r)^2 \mu(dx) = \int T_i(y)(r)^2 \mu_G^{(\mathbb{N})}(dy) = \sum_{j=0}^\infty \alpha_{irj}^2,$$

so $|\alpha_{irj}| \le \epsilon$ whenever $i \in I_\epsilon$, $r < n$ and $j \in \mathbb{N}$.

(v) Now suppose that $H \subseteq \mathbb{R}^n$ is open, that $K \subseteq H$ is compact, and that $\mu_G^{(\mathbb{N})} \tilde E_K > 0$. Then $\mu_G^{(\mathbb{N})} \tilde E_H = 1$. **P** Let $\epsilon > 0$. Let $\langle \epsilon_j\rangle_{j\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{j=0}^\infty \epsilon_j \le \frac12 \min(\epsilon, \mu_G^{(\mathbb{N})} \tilde E_K)$, and for each $j \in \mathbb{N}$ let $\gamma_j \ge 0$ be such that $\mu_G[-\gamma_j,\gamma_j] \ge 1 - \epsilon_j$. Then

$$\tilde E = \{y : y \in \tilde E_K,\ |y(j)| \le \gamma_j \text{ for every } j \in \mathbb{N}\}$$

has measure at least $\mu_G^{(\mathbb{N})} \tilde E_K - \sum_{j=0}^\infty \epsilon_j > 0$. By 254Sb, there are an $m \in \mathbb{N}$ and a set $\tilde E'$, of measure at least $1 - \frac12\epsilon$, such that for every $y' \in \tilde E'$ there is a $y \in \tilde E$ such that $y(j) = y'(j)$ whenever $j > m$. Set $\tilde E'' = \{y : y \in \tilde E',\ |y(j)| \le \gamma_j$ for every $j \in \mathbb{N}\}$, so that $\mu_G^{(\mathbb{N})} \tilde E'' \ge 1 - \epsilon$.

Let $\delta > 0$ be such that $z' \in H$ whenever $z \in K$ and $\|z - z'\| \le 2\delta$. Let $\eta > 0$ be such that $2\eta\sqrt{n} \sum_{j=0}^m \gamma_j \le \delta$. If $y' \in \tilde E''$ and $i \in I_\eta$, there is a $y \in \tilde E$ such that $y(j) = y'(j)$ for $j > m$. Also $|y(j) - y'(j)| \le 2\gamma_j$ for $j \le m$, so

$$|T_i(y)(r) - T_i(y')(r)| \le \sum_{j=0}^\infty |\alpha_{irj}||y(j) - y'(j)| \le \sum_{j=0}^m 2\eta\gamma_j \le \frac{\delta}{\sqrt{n}}$$

for every $r < n$, and $\|T_i(y) - T_i(y')\| \le \delta$.

Now $\tilde F_y \cap K \ne \emptyset$; take $z \in \tilde F_y \cap K$. For every $\zeta > 0$, there is an $i \in I_{\min(\eta,\zeta)}$ such that $\|z - T_i(y)\| \le \delta$, so that $\|z - T_i(y')\| \le 2\delta$. This means that $B(z,2\delta) \cap \{T_i(y') : i \in I_\zeta\}$ is not empty. As $B(z,2\delta)$ is compact, it must meet $\tilde F_{y'}$. But this means that $H \cap \tilde F_{y'} \ne \emptyset$, by the choice of δ. As y' is arbitrary, $\tilde E'' \subseteq \tilde E_H$, while $\mu_G^{(\mathbb{N})} \tilde E'' \ge 1 - \epsilon$.

This works for every $\epsilon > 0$. So $\tilde{E}_H$ is conegligible, as claimed. **Q**

(vi) If $H \subseteq \mathbb{R}^n$ is open and $\mu_G^{(\mathbb{N})}\tilde{E}_H > 0$, then (because H is σ-compact) there is a compact set $K \subseteq H$ such that $\mu_G^{(\mathbb{N})}\tilde{E}_K > 0$, and (e) tells us that $\mu_G^{(\mathbb{N})}\tilde{E}_H = 1$.

Set

$$\mathcal{V}_0 = \{V : V \in \mathcal{V},\, \mu_G^{(\mathbb{N})}\tilde{E}_V = 0\} = \{V : V \in \mathcal{V},\, \mu_G^{(\mathbb{N})}\tilde{E}_V < 1\}.$$

Then we see that

$$\mathcal{V}_0 = \{V : V \in \mathcal{V},\, \tilde{F}_y \cap V = \emptyset\}$$

for almost every $y \in \mathbb{R}^{\mathbb{N}}$, that is,

$$\mathcal{V}_0 = \{V : V \in \mathcal{V},\, F_x \cap V = \emptyset\}$$

for almost every $x \in \mathbb{R}^{I\times n}$. But as every F_x is closed, we have $F_x = \mathbb{R}^n \setminus \bigcup\mathcal{V}_0$ for almost every x. So we can set $F = \mathbb{R}^n \setminus \bigcup\mathcal{V}_0$.

(b)(i) Give $\mathbb{R}^{I\times 2n} \cong (\mathbb{R}^{I\times n})^2$ the measure λ corresponding to the product measure $\mu\times\mu$; by 456Be, this is a centered Gaussian distribution. For $(x_1, x_2) \in (\mathbb{R}^{I\times n})^2$, set

$$F'_{x_1x_2} = \bigcap_{\epsilon>0}\overline{\{(S_ix_1, S_ix_2) : i \in I_\epsilon\}}.$$

By (a), we have a closed set $F' \subseteq \mathbb{R}^{2n}$ such that $F' = F'_{x_1x_2}$ for almost all x_1, x_2. (Of course $I_\epsilon = \{i : \int|x_j(i,r)|^2\lambda(dx) \le \epsilon^2$ for every $j \in \{1,2\}$, $r < n\}$ whenever $\epsilon > 0$.)

(ii) Now $(z, 0) \in F'$. **P** Take $x_1 \in \mathbb{R}^{I\times n}$ such that $F = F_{x_1}$ and $E = \{x_2 : F' = F'_{x_1x_2}\}$ is conegligible. (Almost every point of $\mathbb{R}^{I\times n}$ has these properties.) For $k \in \mathbb{N}$, $z \in \overline{\{S_ix_1 : i \in I_{2^{-k}}\}}$; let $i_k \in I_{2^{-k}}$ be such that $\sum_{r<n}|z(r) - x_1(i_k, r)| \le 2^{-k}$. Next,

$$\sum_{k=0}^{\infty}\sum_{r=0}^{n-1}\int|x(i_k,r)|^2\mu(dx) \le \sum_{k=0}^{\infty}2^{-2k}n < \infty,$$

so $\sum_{k=0}^{\infty}\sum_{r=0}^{n-1}|x(i_k,r)|^2$ is finite for almost every $x \in \mathbb{R}^I$, and there must be an $x_2 \in E$ such that $\sum_{k=0}^{\infty}\sum_{r=0}^{n-1}|x_2(i_k,r)|^2$ is finite. But in this case $\lim_{k\to\infty}x_2(i_k,r) = 0$ for every r, while $\lim_{k\to\infty}x_1(i_k,r) = z(r)$. Accordingly $(z,0) \in F'_{x_1x_2} = F'$. **Q**

(iii) Set $\beta = \sqrt{1-\alpha^2}$ and define $\tilde{T} : (\mathbb{R}^{I\times n})^2 \to \mathbb{R}^{I\times n}$ by setting $\tilde{T}(x_1,x_2) = \alpha x_1 + \beta x_2$ for x_1, $x_2 \in \mathbb{R}^{I\times n}$. Then $\tilde{T}$ is a continuous linear operator, so the image measure $\lambda\tilde{T}^{-1}$ is a centered Gaussian distribution on $\mathbb{R}^{I\times n}$ (456Ba). Moreover, it has the same covariance matrix as μ. **P** If i, $j \in I$ then

$$\begin{aligned}\int x(i)x(j)(\lambda\tilde{T}^{-1})(dx) &= \int\tilde{T}(x_1,x_2)(i)\tilde{T}(x_1,x_2)(j)\lambda(d(x_1,x_2))\\ &= \int(\alpha x_1(i) + \beta x_2(i))(\alpha x_1(j) + \beta x_2(j))\lambda(d(x_1,x_2))\\ &= \alpha^2\int x_1(i)x_1(j)\lambda(d(x_1,x_2)) + \beta^2\int x_2(i)x_2(j)\lambda(d(x_1,x_2))\\ &= (\alpha^2+\beta^2)\int x(i)x(j)\mu(dx) = \int x(i)x(j)\mu(dx). \text{ **Q**}\end{aligned}$$

So $\lambda\tilde{T}^{-1} = \mu$ (456Bb).

(iv) If x_1, $x_2 \in \mathbb{R}^I$ are such that $(z,0) \in F'_{x_1x_2}$, then $\alpha z \in F_{\tilde{T}(x_1,x_2)}$. **P** For every $\epsilon > 0$ there is an $i \in I_\epsilon$ such that $|z(r) - x_1(i,r)| \le \epsilon$ and $|x_2(i,r)| \le \epsilon$ for every $r < n$. But now $|\alpha z(r) - \tilde{T}(x_1,x_2)(r)| \le 2\epsilon$ for every $r < n$. **Q**

So

$$\tilde{T}^{-1}[\{x : \alpha z \in F_x\}] = \{(x_1,x_2) : \alpha z \in F_{\tilde{T}(x_1,x_2)}\} \supseteq \{(x_1,x_2) : (z,0) \in F'_{x_1x_2}\}$$

is λ-conegligible, and $\alpha z \in F_x$ for μ-almost every x, that is, $\alpha z \in F$, as claimed.

(c) Suppose now that F is bounded.

(i) For $L \subseteq I$, $\alpha \ge 0$ set

$$Q(L,\alpha) = \bigcup_{i\in L, r<n}\{x : |x(i,r)| \ge \alpha\}.$$

By 456G, applied to the image of μ under the map $x \mapsto x\restriction L \times n : \mathbb{R}^{I\times n} \to \mathbb{R}^{L\times n}$,

$$\mu Q(L, \tfrac{1}{2}\alpha) \ge 2\mu Q(L,\alpha)(1-\mu Q(L,\alpha))^3$$

for every finite $L \subseteq I$ and every $\alpha \ge 0$.

Let $\beta > 0$ be such that $\delta = 2\beta(1-(n+1)\beta)^3 - \beta > 0$, and let $\alpha_0 > 0$ be such that $\alpha_0^2\beta \ge 1$ and $\|z\| < \frac{1}{2}\alpha_0$ for every $z \in F$, so that $\mu\{x : |x(i,r)| \ge \alpha_0\} \le \beta$ whenever $i \in I_1$ and $r < n$, and $\mu Q(\{i\},\alpha_0) \le n\beta$ for every $i \in I_1$. Set $K = \{z : z \in \mathbb{R}^n,\ \frac{1}{2}\alpha_0 \le \max_{r<n}|z(r)| \le \alpha_0\}$, so that K is a compact set disjoint from F. For almost every x,

$$\emptyset = K\cap F = K \cap F_x = \bigcap_{k\ge 1} K \cap \overline{\{S_i x : i \in I_{1/k}\}},$$

so there is a $k \ge 1$ such that $S_i x \notin K$ for every $i \in I_{1/k}$. Since the sets

$$\{x : S_i(x) \in K \text{ for some } i \in I_{1/k}\}$$

form a non-increasing sequence of measurable sets with negligible intersection, there is a $k \ge 1$ such that

$$\mu\{x : S_i(x) \in K \text{ for some } i \in I_{1/k}\} < \delta.$$

(ii) ? Suppose, if possible, that

$$\mu Q(I_{1/k}, \alpha_0) > \beta.$$

Let $L \subseteq I_{1/k}$ be a finite set of minimal size such that $\gamma = \mu Q(L,\alpha_0) \ge \beta$. Since $\mu Q(\{i\},\alpha_0) \le n\beta$ for any $i \in L$, and L is minimal, we must have

$$\beta \le \gamma \le (n+1)\beta.$$

Now this means that

$$\mu Q(L, \frac{1}{2}\alpha_0) \ge 2\gamma(1-\gamma)^3$$

(see (i) above)

$$\ge 2\gamma(1-(n+1)\beta)^3 = \gamma(1+\frac{\delta}{\beta}) \ge \gamma + \delta,$$

so that $\mu(Q(L,\frac{1}{2}\alpha_0)\setminus Q(L,\alpha_0)) \ge \delta$. But if $x \in Q(L,\frac{1}{2}\alpha_0)\setminus Q(L,\alpha_0)$ there is some $i \in L$ such that $\max_{r<n}|x(i,r)| \ge \frac{1}{2}\alpha_0$ while $\max_{r<n}|x(i,r)| < \alpha_0$, in which case $S_i(x) \in K$. So we get

$$\delta \le \mu(Q(L,\frac{1}{2}\alpha_0)\setminus Q(L,\alpha_0)) \le \mu\{x : S_i(x) \in K \text{ for some } i \in L\}$$

$$\le \mu\{x : S_i(x) \in K \text{ for some } i \in I_{1/k}\} < \delta$$

which is absurd. **X**

(iii) Thus $\mu Q(I_{1/k},\alpha_0) \le \beta$. For $\alpha \ge 0$, set $f(\alpha) = \mu Q(I_{1/k},\alpha)$; then f is non-increasing. Also $f(\frac{1}{2}\alpha) \ge 2f(\alpha)(1-f(\alpha))^3$ for every α. **P?** Otherwise, because

$$f(\alpha) = \sup\{\mu Q(L,\alpha) : L \subseteq I_{1/k} \text{ is finite}\},$$

there is a finite $L \subseteq I_{1/k}$ such that $f(\frac{1}{2}\alpha) < 2\gamma(1-\gamma)^3$, where $\gamma = \mu Q(L,\alpha)$. But in this case $\mu Q(L,\frac{1}{2}\alpha) \le f(\frac{1}{2}\alpha) < 2\gamma(1-\gamma)^3$, which is impossible, as remarked in (i). **XQ**

Set $\zeta = \lim_{\alpha\to\infty} f(\alpha)$. Then

$$\zeta = \lim_{\alpha\to\infty} f(\tfrac{1}{2}\alpha) \ge 2\zeta(1-\zeta)^3.$$

But we also know, from (ii), that $\zeta \le f(\alpha_0) \le \beta$. So $(1-\zeta)^3 \ge (1-\beta)^3 > \frac{1}{2}$ and ζ must be 0.

What this means is that if we set $\epsilon = \frac{1}{k}$ then

$$\lim_{\alpha\to\infty}\mu\{x : \sup_{i\in I_\epsilon, r<n}|x(i,r)| > \alpha\} = 0,$$

that is, $\sup_{i\in I_\epsilon, r<n}|x(i,r)|$ is finite for almost every $x \in \mathbb{R}^{I\times n}$, as claimed.

456N Lemma Let J be a set and μ a centered Gaussian distribution on $\mathbb{R}^J$. Let M be the linear subspace of $L^2(\mu)$ generated by $\{\pi_j^\bullet : j \in J\}$, where $\pi_j(x) = x(j)$ for $x \in \mathbb{R}^J$ and $j \in J$. If M is separable (for the norm topology) then μ is τ-additive.

proof Suppose, if possible, otherwise.

(a) There is an upwards-directed family $\mathcal{G}$ of open Baire sets in $\mathbb{R}^J$ such that $W_0 = \bigcup \mathcal{G}$ is a Baire set and $\mu W_0 > \sup_{G \in \mathcal{G}} \mu G$. Let $\mathcal{G}_0 \subseteq \mathcal{G}$ be a countable upwards-directed set such that $\sup_{G \in \mathcal{G}_0} \mu G = \sup_{G \in \mathcal{G}} \mu G$, and set $W_1 = W_0 \setminus \bigcup \mathcal{G}_0$; then $\mu W_1 > 0$ and $\mu(W_1 \cap G) = 0$ for every $G \in \mathcal{G}$. Let W be a non-negligible zero set included in W_1.

For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be a countable base for the topology of $\mathbb{R}^n$ consisting of open balls. Let $\mathcal{G}^*_n$ be the family of open sets of $\mathbb{R}^J$ of the form $T^{-1}[V]$, where $T : \mathbb{R}^J \to \mathbb{R}^n$ is a continuous linear operator, $V \subseteq \mathbb{R}^n$ is open and $\mu(W \cap T^{-1}[V]) = 0$. Of course $\mathcal{G}^*_0 = \emptyset$.

(b) For $n \ge 1$ and $V \in \mathcal{V}_n$, let $\mathcal{T}_{nV}$ be the family of continuous linear operators $T : \mathbb{R}^J \to \mathbb{R}^n$ such that $W \cap T^{-1}[V]$ is negligible, but not included in $\bigcup \mathcal{G}^*_{n-1}$. Index $\mathcal{T}_{nV}$ as $\langle T_i \rangle_{i \in I(n,V)}$; it will be convenient to do this in such a way that all the $I(n,V)$ are disjoint. Define f_{ir}, for $i \in I(n,V)$ and $r < n$, by saying that $T_i(x) = \langle f_{ir}(x) \rangle_{r<n}$ for $x \in \mathbb{R}^J$. Define $\phi_n : \bigcup_{V \in \mathcal{V}_n} I(n,V) \to M^n$ by setting $\phi_n(i) = \langle f^\bullet_{ir} \rangle_{r<n}$ for each $i \in \bigcup_{V \in \mathcal{V}_n} I(n,V)$. Because M is separable (in its norm topology), M^n is separable in its product topology (4A2P(a-v)). Fix a countable set $I'(n,V) \subseteq I(n,V)$ such that $\{\phi_n(i) : i \in I'(n,V)\}$ is dense in $\{\phi_n(i) : i \in I(n,V)\}$. Set $\rho_n(\langle u_r \rangle_{r<n}, \langle v_r \rangle_{r<n}) = \max_{r<n} \|u_r - v_r\|_2$ for $\langle u_r \rangle_{r<n}, \langle v_r \rangle_{r<n} \in M^n$, so that ρ_n is a metric defining the product topology of M^n.

(c) If $j \in I(n,V)$, then there is a $\delta > 0$ such that

$$\{T_i(x) : i \in I'(n,V),\ \rho_n(\phi_n(i), \phi_n(j)) \le \delta\}$$

is bounded for almost every $x \in \mathbb{R}^J$. **P** Define $S : \mathbb{R}^J \to \mathbb{R}^{I'(n,V)\times n}$ by setting $(Sx)(i,r) = T_i(x,r) - T_j(x,r)$ for $x \in \mathbb{R}^J$, $i \in I'(n,V)$ and $r < n$. By 456Ba, the image measure $\lambda = \nu S^{-1}$ is a centered Gaussian distribution on $\mathbb{R}^{I'(n,V)\times n}$. For $\delta > 0$, set

$$\begin{aligned} I'_\delta &= \{i : i \in I'(n,V),\ \int |y(i,r)|^2 \lambda(dy) \le \delta^2 \text{ for every } r < n\} \\ &= \{i : i \in I'(n,V),\ \int |Sx(i,r)|^2 \nu(dx) \le \delta^2 \text{ for every } r < n\} \\ &= \{i : i \in I'(n,V),\ \int |T_i(x)(r) - T_j(x)(r)|^2 \nu(dx) \le \delta^2 \text{ for every } r < n\} \\ &= \{i : i \in I'(n,V),\ \rho_n(\phi_n(i), \phi_n(j)) \le \delta\}. \end{aligned}$$

By 456Ma, there is a closed set $F \subseteq \mathbb{R}^n$ such that $F = \bigcap_{\delta>0} \overline{\{y(i,r) : i \in I'_\delta\}}$ for λ-almost every $y \in \mathbb{R}^{I'(n,V)\times n}$, so that $F = \bigcap_{\delta>0} \overline{\{Sx(i,r) : i \in I'_\delta\}}$ for ν-almost every $x \in \mathbb{R}^J$.

By 456Mb, $\alpha z \in F$ whenever $z \in F$ and $|\alpha| \le 1$. **?** If F is not bounded, then it must include a line L through 0. (The sets $\{\frac{1}{n} z : z \in F,\ \|z\| = n\}$, for $n \ge 1$, form a non-increasing sequence of non-empty compact sets, so there is a point z_0 belonging to them all; take L to be the set of multiples of z_0.) Let $D \subseteq L$ be a countable dense set. For $z \in D$ and $k \in \mathbb{N}$ we know that

for every $i \in I(n,V)$, $T_i(x) \notin V$ for almost every $x \in W$,

for almost every $x \in \mathbb{R}^J$ there is an $i \in I'(n,V)$ such that $\|T_i(x) - T_j(x) - z\| \le 2^{-k}$

and therefore

for almost every $x \in W$, $T_i(x) \notin V$ for every $i \in I'(n,V)$, but there is an $i \in I'(n,V)$ such that $\|T_i(x) - T_j(x) - z\| \le 2^{-k}$

so that

for almost every $x \in W$, the distance from $T_j(x) + z$ to the closed set $\mathbb{R}^n \setminus V$ is at most 2^{-k}.

This is true for every $k \in \mathbb{N}$, so we get

$T_j(x) + z \notin V$ for almost every $x \in W$.

And *this* is true for every $z \in D$, so we get

for almost every $x \in W$, $T_j(x) + z \notin V$ for every $z \in D$, so $T_j(x) \notin V + L$.

Let $S_0 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a linear operator with kernel L, and set $V' = S_0[V]$. Then $V' \subseteq \mathbb{R}^{n-1}$ is open, and $W \cap (S_0 T_j)^{-1}[V'] = W \cap T_j^{-1}[V + L]$ is negligible. But this means that $T_j^{-1}[V] \subseteq (S_0 T_j)^{-1}[V'] \in \mathcal{G}^*_{n-1}$ and $T_j^{-1}[V]$ is included in $\bigcup \mathcal{G}^*_{n-1}$; which contradicts the definition of $\mathcal{T}_{nV}$. **X**

So F is bounded. By 456Mc, there is some $\delta > 0$ such that $\sup_{i\in I'_\delta, r<n} |y(i,r)| < \infty$ for λ-almost every $y \in \mathbb{R}^{I'(n,V)\times n}$, in which case $\sup_{i\in I'_\delta, r<n} |Sx(i,r)| < \infty$ for ν-almost every $x \in \mathbb{R}^J$, that is, $\{T_i(x) - T_j(x) : i \in I'_\delta\}$ is bounded for ν-almost every x. Of course this means that $\{T_i(x) : i \in I(n,V),\ \rho_n(\phi_n(i),\phi_n(j)) \le \delta\}$ is bounded for almost every $x \in \mathbb{R}^J$. **Q**

(d) Accordingly $\phi_n[I(n,V)]$ is covered by the family $\mathcal{U}_{nV}$ of open sets $U \subseteq M^n$ such that $\{T_i(x) : i \in I'(n,V),\ \phi_n(i) \in U\}$ is bounded for almost every x. Because M^n is separable and metrizable, it is hereditarily Lindelöf (4A2P(a-iii)), so there is a sequence $\langle U_{nVk}\rangle_{k\in\mathbb{N}}$ in $\mathcal{U}_{nV}$ covering $\phi_n[I(n,V)]$. For each k, set $I_{nVk} = I'(n,V) \cap \phi_n^{-1}[U_{nVk}]$. Then $\{T_i(x) : i \in I_{nVk}\}$ is bounded for almost every x. Because $\phi_n[I'(n,V)]$ is dense in $\phi_n[I(n,V)]$, $\phi_n[I_{nVk}] = \phi_n[I'(n,V)] \cap U_{nVk}$ is dense in $\phi_n[I(n,V)] \cap U_{nVk}$. So for every $i \in I(n,V)$ there is a $k \in \mathbb{N}$ such that $\phi_n(i) \in \overline{\phi_n[I_{nVk}]}$.

(e) Recall that $W \subseteq \mathbb{R}^J$ is a non-negligible zero set included in

$$\bigcup_{n\ge1} \bigcup \mathcal{G}_n^* = \bigcup_{n\ge1} \bigcup_{V\in\mathcal{V}_n} \bigcup_{i\in I(n,V)} T_i^{-1}[V].$$

Let $J_0 \subseteq J$ be a countable set such that W is determined by coordinates in J_0.

Let $\langle\epsilon_j\rangle_{j\in J_0}$ and $\langle\epsilon'_{nVk}\rangle_{n\ge1,V\in\mathcal{V}_n,k\in\mathbb{N}}$ be families of strictly positive real numbers such that $\sum_{n=1}^\infty \sum_{V\in\mathcal{V}_n} \sum_{k=0}^\infty \epsilon'_{nVk}$ and $\sum_{j\in J_0} \epsilon_j$ are both at most $\frac{1}{3}\mu W$. Let $\langle\gamma_j\rangle_{j\in J_0}$ and $\langle\gamma'_{nVk}\rangle_{n\ge1,V\in\mathcal{V}_n,k\in\mathbb{N}}$ be such that

$$\mu\{x : x \in \mathbb{R}^J,\ |x(j)| \ge \gamma_j\} \le \epsilon_j \text{ for every } j \in J_0,$$

$$\mu\{x : x \in \mathbb{R}^J,\ \sup_{i\in I_{nVk}} \|T_i(x)\| \ge \gamma'_{nVk}\} \le \epsilon'_{nVk} \text{ for every } n \ge 1,\ V \in \mathcal{V}_n,\ k \in \mathbb{N}.$$

Set $W' = \{x : x \in W,\ |x(j)| \le \gamma_j \text{ for every } j \in J_0\}$; then $\mu W' \ge \frac{2}{3}\mu W$ and W' is of the form $C \times \mathbb{R}^{J\setminus J_0}$, where $C \subseteq \mathbb{R}^{J_0}$ is compact. Set

$$W'' = \{x : x \in W',\ \|T_i(x)\| \le \gamma'_{nVk} \text{ whenever } n \ge 1,\ V \in \mathcal{V}_n,\ k \in \mathbb{N} \text{ and } i \in I_{nVk}\};$$

then $\mu W'' \ge \frac{1}{3}\mu W$.

(f) Set $I = \bigcup_{n\ge1,V\in\mathcal{V}_n} I(n,V)\times n$. If $K \subseteq I$ is finite, then

$$W'_K = \{x : x \in W',\ \|f_{ir}(x)\| \le \gamma'_{nVk} \text{ whenever } n \ge 1,\ V \in \mathcal{V}_n,\ k \in \mathbb{N},\ (i,r) \in K \text{ and } \phi_n(i) \in \overline{\phi_n[I_{nVk}]}\}$$

has measure at least $\frac{1}{3}\mu W$. **P** For each quintuple (i,r,n,V,k) with $n \ge 1$, $V \in \mathcal{V}_n$, $k \in \mathbb{N}$, $(i,r) \in K$ and $\phi_n(i) \in \overline{\phi_n[I_{nVk}]}$, there is a sequence $\langle i_m\rangle_{m\in\mathbb{N}}$ in I_{nVk} such that $\rho_n(\phi_n(i),\phi_n(i_m)) \le 2^{-m}$ for every m; so that $\|f_{ir} - f_{i_m r}\|_2 \le 2^{-m}$ for every m. But this means that $f_{ir}(x) = \lim_{m\to\infty} f_{i_m r}(x)$ for almost every $x \in \mathbb{R}^J$. Accordingly $|f_{ir}(x)| \le \gamma'_{nVk}$ for almost every $x \in W''$. Since there are only countably many such quintuples (i,r,n,V,k), we see that $W'' \setminus W'_K$ is negligible, so $\mu W'_K \ge \mu W'' \ge \frac{1}{3}\mu W$. **Q**

(g) For $x \in \mathbb{R}^J$, define $Tx \in \mathbb{R}^I$ by setting $(Tx)(i,r) = f_{ir}(x)$ for $i \in I(n,V)$ and $r < n$. Then $T : \mathbb{R}^J \to \mathbb{R}^I$ is a continuous linear operator. By 4A4H, $T[W']$ is closed.

For finite $K \subseteq I$, let $\mathcal{H}_K$ be the family of open subsets H of $\mathbb{R}^K$ such that $\mu\{x : x \in W,\ Tx{\restriction}K \in H\} = 0$. Then $\mathcal{H}_K$ is closed under countable unions so has a largest member H_K. Now there is a $K \in [I]^{<\omega}$ such that $Tx{\restriction}K \in H_K$ for every $x \in W'_K$. **P?** Otherwise, choose for each $K \in [I]^{<\omega}$ an $x_K \in W'_K$ such that $Tx_K{\restriction}K \notin H_K$. Let $\mathcal{F}$ be an ultrafilter on $[I]^{<\omega}$ containing $\{K : L \subseteq K \in [I]^{<\omega}\}$ for every finite $L \subseteq I$. If $(i,r) \in I$, there are $n \ge 1$, $V \in \mathcal{V}_n$ and $k \in \mathbb{N}$ such that $r < n$ and $\phi_n(i) \in \overline{\phi_n[I_{nVk}]}$, in which case $|f_{ir}(x_K)| \le \gamma'_{nVk}$ whenever $K \in [I]^{<\omega}$ contains (i,r). This means that $\lim_{K\to\mathcal{F}} f_{ir}(x_K)$ must be defined in $[-\gamma'_{nVk},\gamma'_{nVk}]$; consequently $y^* = \lim_{K\to\mathcal{F}} Tx_K$ is defined in $\mathbb{R}^I$. Since $x_K \in W'$ for every K, $y^* \in \overline{T[W']} = T[W']$.

Let $x^* \in W'$ be such that $Tx^* = y^*$. Since $x^* \in W$, there are $n \ge 1$, $V \in \mathcal{V}_n$ and $i \in I(n,V)$ such that $T_i(x^*) \in V$. Set $L = \{(i,r) : r < n\}$, $H = \{z : z \in \mathbb{R}^L,\ \langle z(i,r)\rangle_{r<n} \in V\}$; then $\{x : Tx{\restriction}L \in H\} = T_i^{-1}[V]$. Since $y^*{\restriction}L = Tx^*{\restriction}L$ belongs to H, and H is open, there must be a $K \supseteq L$ such that $Tx_K{\restriction}L \in H$. But in this case $H' = \{z : z \in \mathbb{R}^K,\ z{\restriction}L \in H\}$ is an open subset of $\mathbb{R}^K$ and

$$\{x : Tx{\restriction}K \in H'\} = \{x : Tx{\restriction}L \in H\} = \{x : T_i(x) \in V\}$$

meets W in a negligible set, and $H' \subseteq H_K$. But this means that $Tx_K{\restriction}K \in H_K$, contrary to the choice of x_K. **XQ**

(h) Putting (f) and (g) together, we find ourselves trying to believe simultaneously that $\mu W'_K > 0$ and that $Tx{\restriction}K \in H_K$ for every $x \in W'_K$ and that $W'_K \subseteq W$ and that $\{x : x \in W,\ Tx{\restriction}K \in H_K\}$ is negligible. Faced with this we have to abandon the original supposition that μ is not τ-additive.

456O We now have all the ideas needed for the main theorem of this section.

Theorem (TALAGRAND 81) Every centered Gaussian distribution is τ-additive.

proof ? Suppose, if possible, that μ is a centered Gaussian distribution on a set $\mathbb{R}^I$ which is not τ-additive.

(a) By 456K, there are a set J and a universal centered Gaussian distribution ν on $\mathbb{R}^J$ and a continuous linear operator $T:\mathbb{R}^J\to\mathbb{R}^I$ which is inverse-measure-preserving for ν and μ. By 418Ha, ν is not τ-additive.

(b) As in part (a) of the proof of 456N, there are a non-negligible zero set $W\subseteq\mathbb{R}^J$ and a family $\mathcal{G}$ of open sets, covering W, such that $\nu(W\cap G)=0$ for every $G\in\mathcal{G}$. Give J a Hilbert space structure such that $\int x(i)x(j)\nu(dx)=(i|j)$ for all i, $j\in J$. Let $K_0\subseteq J$ be a countable set such that W is determined by coordinates in K_0, and let K be the closed linear subspace of J generated by K_0. Let $\mathcal{G}'$ be the family of open sets determined by coordinates in K which meet W in negligible sets. Then $W\subseteq\bigcup\mathcal{G}'$, by 456L.

Let λ be the centered Gaussian distribution on $\mathbb{R}^K$ for which the map $\tilde\pi_K=x\mapsto x\restriction K:\mathbb{R}^J\to\mathbb{R}^K$ is inverse-measure-preserving. Then $\tilde\pi_K[W]$ is a zero set in $\mathbb{R}^K$, $\lambda\tilde\pi_K[W]=\nu W>0$, $\{\tilde\pi_K[G]:G\in\mathcal{G}'\}$ is a family of open sets in $\mathbb{R}^K$ covering $\tilde\pi_K[W]$, and $\lambda(\tilde\pi_K[W]\cap\tilde\pi_K[G])=\nu(W\cap G)=0$ for every $G\in\mathcal{G}'$; so λ is not τ-additive. However, K, regarded as a normed space, is separable (see 4A4Bg); and if we set $\pi_j(y)=y(j)$ for $y\in\mathbb{R}^K$ and $j\in K$, then $\|\pi_i^\bullet-\pi_j^\bullet\|_2=\|i-j\|$ for all i, $j\in K$. So $\{\pi_j^\bullet:j\in K\}$ is separable in $L^2(\lambda)$. And this is impossible, by 456N. **X**

Thus every centered Gaussian distribution must be τ-additive.

456P Corollary If μ is a centered Gaussian distribution on $\mathbb{R}^I$, there is a unique quasi-Radon measure $\tilde\mu$ on $\mathbb{R}^I$ extending μ. The support of μ as defined in 456H is the support of $\tilde\mu$ as defined in 411N.

proof By 415L, μ has a unique extension to a quasi-Radon measure $\tilde\mu$. Now the support Z of μ is a closed set, so $\tilde\mu Z=\mu^*Z$ (415L(i)). Also Z is self-supporting for μ. If $G\subseteq\mathbb{R}^I$ is an open set meeting Z, then there is a cozero set $H\subseteq G$ which also meets Z, and $\mu^*(Z\cap H)>0$. It follows that $\mu^*(Z\setminus H)<1$; as $\tilde\mu$ extends μ, $\tilde\mu(Z\setminus H)<1$ and $\tilde\mu(Z\cap G)>0$. This shows that Z is self-supporting for $\tilde\mu$, so must be the support of $\tilde\mu$ in the standard sense.

456Q Proposition Let I be a set and R the set of functions $\sigma:I\times I\to\mathbb{R}$ which are symmetric and positive semi-definite in the sense of 456C; give R the subspace topology induced by the usual topology of $\mathbb{R}^{I\times I}$. Let $P_{\mathrm{qR}}(\mathbb{R}^I)$ be the space of quasi-Radon probability measures on $\mathbb{R}^I$ with its narrow topology (437Jd). For $\sigma\in R$, let μ_σ be the centered Gaussian distribution on $\mathbb{R}^I$ with covariance matrix σ (456C), and $\tilde\mu_\sigma$ the quasi-Radon measure extending μ_σ (456P). Then R is a closed subset of $\mathbb{R}^{I\times I}$ and the function $\sigma\mapsto\tilde\mu_\sigma:R\to P_{\mathrm{qR}}(\mathbb{R}^I)$ is continuous.

proof (a) From 456C(iv) we see at once that R is closed. So the rest of this proof will be devoted to showing that $\sigma\mapsto\tilde\mu_\sigma$ is continuous.

(b) I had better begin with the one-dimensional case. If $I=\{j\}$ is a singleton, and we identify $\mathbb{R}^I$ with $\mathbb{R}$, then $\tilde\mu_\sigma$ is the ordinary normal distribution with mean 0 and variance $\sigma(j,j)$, counting the Dirac measure centered at 0 as a normal distribution with zero variance. If $H\subseteq\mathbb{R}$ is open and $\gamma\in\mathbb{R}$, set

$$G=\{\alpha:\alpha>0,\ \frac{1}{\sqrt{2\pi\alpha}}\int_H e^{-t^2/\alpha}dt>\gamma\};$$

then G is open. If $0\notin H$, then

$$\{\sigma:\tilde\mu_\sigma H>\gamma\}=\{\sigma:\sigma(i,i)\in G\}$$

is open. If $0\in H$ and $\gamma\ge1$, then $\{\sigma:\tilde\mu_\sigma H>\gamma\}$ is empty; if $0\in H$ and $\gamma<1$, then

$$\{\sigma:\tilde\mu_\sigma H>\gamma\}=\{\sigma:\sigma(i,i)\in G\}\cup\{0\}$$

is open because there is an $\eta>0$ such that $[-\eta,\eta]\subseteq H$ and $\alpha\in G$ whenever $\alpha>0$ and

$$\frac{1}{\sqrt{2\pi}}\int_{-\eta/\sqrt{\alpha}}^{\eta/\sqrt{\alpha}}e^{-t^2/2}dt>\gamma.$$

As H is arbitrary, $\sigma\mapsto\tilde\mu_\sigma$ is continuous.

(c) Now suppose that I is finite. Let $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ be a sequence in R with limit $\sigma\in R$. Let ϕ_n, ϕ be the characteristic functions of $\tilde\mu_{\sigma_n}$, $\tilde\mu_\sigma$ respectively (§285). If $y\in\mathbb{R}^I$, set $f(x)=x\,.\,y$ for $x\in\mathbb{R}^I$; then

$$\phi(y)=\int e^{if(x)}\tilde\mu_\sigma(dx)=\int e^{it}(\tilde\mu_\sigma f^{-1})(dt),$$

writing $\tilde{\mu}_\sigma f^{-1}$ for the image Radon measure on $\mathbb{R}$. Now $\tilde{\mu}_\sigma f^{-1}$ is the one-dimensional Gaussian distribution with variance $\sum_{j,k\in I}\sigma(j,k)y(j)y(k)$ (see part (b) of the proof of 456B). But since

$$\sum_{j,k\in I}\sigma(j,k)y(j)y(k) = \lim_{n\to\infty}\sum_{j,k\in I}\sigma_n(j,k)y(j)y(k,$$

(a) tells us that $\tilde{\mu}_\sigma f^{-1} = \lim_{n\to\infty}\tilde{\mu}_{\sigma_n}f^{-1}$ for the narrow topology on $P_{\mathrm{qR}}(\mathbb{R})$, therefore also for the vague topology (437L), and $\phi(y) = \lim_{n\to\infty}\phi_n(y)$. By 285L, $\tilde{\mu}_\sigma = \lim_{n\to\infty}\tilde{\mu}_{\sigma_n}$ for the vague topology, therefore also for the narrow topology.

Thus $\sigma\mapsto\tilde{\mu}_\sigma$ is sequentially continuous. As I is countable, R is metrizable, and $\sigma\mapsto\tilde{\mu}_\sigma$ is continuous.

(d) For the general case, suppose that $H\subseteq\mathbb{R}^I$ is an open set and that $\gamma\in\mathbb{R}$. Set $G_{H\gamma} = \{\sigma:\sigma\in R,\ \tilde{\mu}_\sigma H>\gamma\}$.

(i) If H is determined by coordinates in a finite set $J\subseteq I$ then $G_{H\gamma}$ is open in R. **P** Let R_J be the set of symmetric positive semi-definite functions on $\mathbb{R}^{J\times J}$; write $h(\sigma) = \sigma\restriction J\times J$ for $\sigma\in R$, and $\tilde{\pi}_J(x) = x\restriction J$ for $x\in\mathbb{R}^I$. Of course $h(\sigma)\in R_J$ for $\sigma\in R$, and $h:R\to R_J$ is continuous. For $\sigma\in R$, we know that there is a centered Gaussian distribution ν on $\mathbb{R}^J$ such that $\tilde{\pi}_J$ is inverse-measure-preserving for μ_σ and ν, by 456Ba; the covariance matrix of ν is of course $h(\sigma)$, so we can call it $\mu_{h(\sigma)}$. Next, there is a quasi-Radon measure $\tilde{\nu}$ on $\mathbb{R}^J$ such that $\tilde{\pi}_J$ is inverse-measure-preserving for $\tilde{\mu}_\sigma$ and $\tilde{\nu}$ (418Hb); as $\tilde{\nu}$ must extend the Baire measure ν, it is the unique quasi-Radon measure extending ν, and we can call it $\tilde{\mu}_{h(\sigma)}$.

Because H is determined by coordinates in J, $H = \tilde{\pi}_J^{-1}[H']$ where $H' = \tilde{\pi}_J[H]$ is open in $\mathbb{R}^J$ (4A2B(f-i) again). So $G' = \{\tau:\tau\in R_J,\ \tilde{\mu}_\tau H'>\gamma\}$ is open in R_J, by (b), and

$$G_{H\gamma} = \{\sigma:(\tilde{\mu}_\sigma\tilde{\pi}_J^{-1})(H')>\gamma\} = \{\sigma:\tilde{\mu}_{h(\sigma)}H'>\gamma\} = h^{-1}[G']$$

is open in R. **Q**

(ii) In fact $G_{H\gamma}$ is open in R for any open set $H\subseteq\mathbb{R}^I$ and $\gamma\in\mathbb{R}$. **P** Take any $\sigma\in G_{H\gamma}$. Because $\tilde{\mu}_\sigma$ is τ-additive, and the family

$$\mathcal{V} = \{V: V\subseteq\mathbb{R}^I \text{ is open and determined by coordinates in a finite set}\}$$

is a base for the topology of $\mathbb{R}^I$ closed under finite unions, there is a $V\in\mathcal{V}$ such that $V\subseteq H$ and $\tilde{\mu}_\sigma V>\gamma$. Now $\sigma\in G_{V\gamma}\subseteq G_{H\gamma}$; by (i), $G_{V\gamma}$ is open, so $\sigma\in\operatorname{int}G_{H\gamma}$; as σ is arbitrary, $G_{H\gamma}$ is open. **Q** But this is just what we need to know to see that $\sigma\mapsto\tilde{\mu}_\sigma$ is continuous for the narrow topology on $P_{\mathrm{qR}}(\mathbb{R}^I)$, and the proof is complete.

456X Basic exercises (a) Let I be any set. (i) Show that if $y\in\ell^1(I)$ then $\int\sum_{i\in I}|y(i)x(i)|\mu_G^{(I)}(dx) = \frac{2}{\sqrt{2\pi}}\|y\|_1$. (*Hint*: start by evaluating $\mathbb{E}(|Z|)$ where Z is a standard normal random variable.) (ii) Show that if $y\in\ell^2(I)$ then $\int\sum_{i\in I}|y(i)x(i)|^2\mu_G^{(I)}(dx) = \|y\|_2^2$.

(b) Let $n\ge 1$ be an integer. (i) Show that $\mu_G^{(n)}T^{-1} = \mu_G^{(n)}$ for any orthogonal linear operator $T:\mathbb{R}^n\to\mathbb{R}^n$. (ii) Set $p(x) = \frac{1}{\|x\|}x$ for $x\in\mathbb{R}^n\setminus\{0\}$; take $p(0)$ to be any point of S^{n-1}. Show that $\mu_G^{(n)}p^{-1}$ is a multiple of $(n-1)$-dimensional Hausdorff measure on S^{n-1}. (*Hint*: 443U.)

(c) Let G be a group, and $h:G\to\mathbb{R}$ a real positive definite function (definition: 445L). (i) Show that we have a centered Gaussian distribution μ on $\mathbb{R}^G$ with covariance matrix $\langle h(a^{-1}b)\rangle_{a,b\in G}$. (ii) Show that μ is invariant under the left shift action $\bullet_l$ of G on $\mathbb{R}^G$ (4A5Cc).

(d) Let I be a countable set, μ a centered Gaussian distribution on $\mathbb{R}^I$, and $\gamma\ge 0$. Set $\alpha = \mu\{x:\sup_{i\in I}|x(i)|\ge\gamma\}$. Show that $\mu\{x:\sup_{i\in I}|x(i)|\ge\frac{1}{2}\gamma\}\ge 2\alpha(1-\alpha)^3$.

(e) Let I be a set and $\langle\sigma_{ij}\rangle_{i,j\in I}$ a family of real numbers. Show that there is at most one inner product space structure on I for which $\sigma_{ij} = (i|j)$ for all i, $j\in I$.

(f) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be an independent sequence of standard normal random variables, and $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ a square-summable real sequence. (i) Show that for any permutation $\pi:\mathbb{N}\to\mathbb{N}$, $X = \sum_{n=0}^\infty\alpha_nX_n$ and $\sum_{n=0}^\infty\alpha_{\pi(n)}X_{\pi(n)}$ are finite and equal a.e. (*Hint*: 273B.) (ii) Show that X is normal, with mean 0 and variance $\sum_{n=0}^\infty\alpha_n^2$.

>(g) For any set I, I will say that a **centered Gaussian quasi-Radon measure** on $\mathbb{R}^I$ is a quasi-Radon measure μ on $\mathbb{R}^I$ such that every continuous linear functional $f:\mathbb{R}^I\to\mathbb{R}$ is either zero a.e. or is normally distributed with zero expectation. Show that

(i) there is a one-to-one correspondence between centered Gaussian quasi-Radon measures μ on $\mathbb{R}^I$ and centered Gaussian distributions ν on $\mathbb{R}^I$ obtained by matching μ with ν iff they agree on the zero sets of $\mathbb{R}^I$;

(ii) if μ, ν are centered Gaussian quasi-Radon measures on $\mathbb{R}^I$ and $\int x(i)x(j)\mu(dx) = \int x(i)x(j)\nu(dx)$ for all i, $j \in I$, then $\mu = \nu$;

(iii) the support of a centered Gaussian quasi-Radon measure on $\mathbb{R}^I$ is a linear subspace of $\mathbb{R}^I$;

(iv) if $\langle I_j \rangle_{j\in J}$ is a disjoint family of sets with union I, and μ_j is a centered Gaussian quasi-Radon measure on $\mathbb{R}^{I_j}$ for each $j \in J$, then the quasi-Radon product of $\langle \mu_j \rangle_{j\in J}$, regarded as a measure on $\mathbb{R}^I$, is a centered Gaussian quasi-Radon measure.

(h) Let I be a set, and let H be a Hilbert space with orthonormal basis $\langle e_i \rangle_{i\in I}$. For $i \in I$, $x \in \mathbb{R}^I$ set $f_i(x) = x(i)$. Show that there is a bounded linear operator $T : H \to L^1(\mu_G^{(I)})$ such that $Te_i = f_i^\bullet$ for every $i \in I$, and that $\|Tu\|_1 = \frac{2}{\sqrt{2\pi}}\|u\|_2$ for every $u \in H$.

456Y Further exercises (a) Let (Ω, Σ, μ) be a probability space with measure algebra $(\mathfrak{A}, \bar\mu)$, and $\langle u_i \rangle_{i\in I}$ a family in $L^2(\mu) \cong L^2(\mathfrak{A}, \bar\mu)$ which is a centered Gaussian process in the sense that whenever $X_i \in \mathcal{L}^2(\mu)$ is such that $X_i^\bullet = u_i$ for every i, then $\langle X_i \rangle_{i\in I}$ is a centered Gaussian process. Suppose that $\gamma \ge 0$ and that $\alpha = \bar\mu(\sup_{i\in I} [\![|u_i| \ge \gamma]\!])$. Show that $\bar\mu(\sup_{i\in I} [\![|u_i| \ge \frac{1}{2}\gamma]\!]) \ge 2\alpha(1-\alpha)^3$.

(b) Let U be a Hilbert space with an orthonormal basis $\langle u_j \rangle_{j\in J}$, and μ the universal centered Gaussian distribution on $\mathbb{R}^U$ with covariance matrix defined by the inner product of U. Show that there is a function $T : \mathbb{R}^J \to \mathbb{R}^U$, inverse-measure-preserving for $\mu_G^{(J)}$ and μ, such that whenever $\langle j_n \rangle_{n\in\mathbb{N}}$ is a sequence of distinct elements of J and $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ is a square-summable sequence in $\mathbb{R}$, then $(Tx)(\sum_{n=0}^\infty \alpha_n u_{j_n}) = \sum_{n=0}^\infty \alpha_n x(j_n)$ for almost every $x \in \mathbb{R}^J$.

(c) Let U be an infinite-dimensional Hilbert space and μ the universal centered Gaussian distribution on $\mathbb{R}^U$ with covariance matrix defined by the inner product of U. Show that $\mu C = 0$ for every compact set $C \subseteq \mathbb{R}^U$.

(d) Let I be a set and μ be a centered Gaussian distribution on $\mathbb{R}^I$. Show that the following are equiveridical: (i) μ has countable Maharam type; (ii) $L^2(\mu)$ is separable; (iii) I is separable under the pseudometric $(i,j) \mapsto \sqrt{\int (x(i)-x(j))^2\mu(dx)}$.

456 Notes and comments This section has aimed for a direct route to Talagrand's theorem 456O, leaving most of the real reasons for studying Gaussian processes (see FERNIQUE 97) to one side. It should nevertheless be clear from such fragments as 252Xi, 456Bb, 456G and the exercises here that they are one of the many concepts of probability theory which are both significant and delightful. Very much the most important Gaussian processes are those associated with Brownian motion, which will be treated in §477 *et seq.*

You will of course have observed that the methods used here are entirely different from those in §455, even though one of the concerns of that section was a check for τ-additive distributions and corresponding quasi-Radon versions, as in 455K. However the results of §455 were based on the fact that in the most important cases the distributions there have extensions to Radon measures (455H). Gaussian distributions need not be like this at all, even when they have countable Maharam type; see 456Yc.

457 Simultaneous extension of measures

The questions addressed in §§451, 454 and 455 can all be regarded as special cases of a general class of problems: given a set X and a family $\langle \nu_i \rangle_{i\in I}$ of (probability) measures on X, when can we expect to find a measure on X extending every ν_i? An alternative formulation, superficially more general, is to ask: given a set X, a family $\langle (Y_i, \mathrm{T}_i, \nu_i) \rangle_{i\in I}$ of probability spaces, and functions $\phi_i : X \to Y_i$ for each i, when can we find a measure on X for which every ϕ_i is inverse-measure-preserving? Even the simplest non-trivial case, when $X = \prod_{i\in I} Y_i$ and every ϕ_i is the coordinate map, demands a significant construction (the product measures of Chapter 25). In this section I bring together a handful of important further cases which are accessible by the methods of this chapter. I begin with a discussion of extensions of finitely additive measures (457A-457D), which are much easier, before considering the problems associated with countably additive measures (457E-457G), with examples (457H-457J). In 457K-457M I look at a pair of optimisation problems.

457A It is helpful to start with a widely applicable result on common extensions of finitely additive measures.

Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\langle\mathfrak{B}_i\rangle_{i\in I}$ a non-empty family of subalgebras of $\mathfrak{A}$. For each $i \in I$, we may identify $L^\infty(\mathfrak{B}_i)$ with the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi b : b \in \mathfrak{B}_i\}$ (363Ga). Suppose that for each $i \in I$ we are given a finitely additive functional $\nu_i : \mathfrak{B}_i \to [0,1]$ such that $\nu_i 1 = 1$; write $⨍\ldots d\nu_i$ for the corresponding positive linear functional on $L^\infty(\mathfrak{B}_i)$ (363L). Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathfrak{A} \to [0,1]$ extending every ν_i;

(ii) whenever $i_0, \dots, i_n \in I$, $a_k \in \mathfrak{B}_{i_k}$ for $k \le n$, and $\sum_{k=0}^n \chi a_k \ge m\chi 1$ in $S(\mathfrak{A})$, where $m \in \mathbb{N}$, then $\sum_{k=0}^n \nu_{i_k} a_k \ge m$;

(iii) whenever $i_0, \dots, i_n \in I$, $a_k \in \mathfrak{B}_{i_k}$ for $k \le n$, and $\sum_{k=0}^n \chi a_k \le m\chi 1$, where $m \in \mathbb{N}$, then $\sum_{k=0}^n \nu_{i_k} a_k \le m$;

(iv) whenever $i_0, \dots, i_n \in I$ are distinct, $u_k \in L^\infty(\mathfrak{B}_{i_k})$ for every $k \le n$, and $\sum_{k=0}^n u_k \ge \chi 1$, then $\sum_{i=0}^n ⨍ u_k d\nu_{i_k} \ge 1$;

(v) whenever $i_0, \dots, i_n \in I$ are distinct, $u_k \in L^\infty(\mathfrak{B}_{i_k})$ for every $k \le n$, and $\sum_{k=0}^n u_k \le \chi 1$, then $\sum_{i=0}^n ⨍ u_k d\nu_{i_k} \le 1$.

proof (a) It is elementary to check that if (i) is true then (ii)-(v) are all true, simply because we have a positive linear functional $⨍\, d\mu$ extending all the functionals $⨍\, d\nu_i$.

(b)(ii)⇒(iii) Given that $a_k \in \mathfrak{B}_{i_k}$ and $\sum_{k=0}^n \chi a_k \le m\chi 1$, then

$$\sum_{k=0}^n \chi(1 \setminus a_k) = (n+1)\chi 1 - \sum_{k=0}^n \chi a_k \ge (n+1-m)\chi 1,$$

so

$$\sum_{k=0}^n \nu_{i_k} a_k = n+1 - \sum_{k=0}^n \nu_{i_k}(1 \setminus a_k) \le n+1-(n+1-m) = m,$$

as required by (iii).

(c)(iii)⇒(i) Assume (iii). Set $\psi a = \sup\{\nu_i a : i \in I,\, a \in \mathfrak{B}_i\}$ for $a \in \mathfrak{A}$ (interpreting $\sup\emptyset$ as 0, as usual in such contexts). Then ψ satisfies the condition (ii) of 391F. **P?** Otherwise, there is a finite indexed family $\langle a_k\rangle_{k\in K}$ in $\mathfrak{A}$ such that $\inf_{k\in J} a_k = 0$ whenever $J \subseteq K$ and $\#(J) \ge \sum_{k\in K} \psi a_i$. The general hypothesis of the lemma implies that $\mathfrak{A} \ne \{0\}$, so $\inf\emptyset = 1 \ne 0$ and K is non-empty. Taking K to be of minimal size, we get an example in which $\psi a_k > 0$ for every $k \in K$. Set $m = \|\sum_{k\in K} \chi a_k\|_\infty$; then $m \in \mathbb{N}$ and $m < \sum_{k\in K} \psi a_k$, so we can find for each $k \in K$ an $i_k \in I$ such that $a_k \in \mathfrak{B}_{i_k}$ and $m < \sum_{k\in K} \nu_{i_k} a_k$. But this contradicts our hypothesis (iii). **XQ**

By 391F, there is a non-negative finitely additive functional μ such that $\mu 1 = 1$ and $\mu a \ge \psi a$ for every $a \in \mathfrak{A}$, that is, $\mu b \ge \nu_i b$ whenever $i \in I$ and $b \in \mathfrak{B}_i$. But observe now that, because $\mu 1 = \nu_i 1$ and $\mu(1 \setminus b) \ge \nu_i(1 \setminus b)$, we actually have $\mu b = \nu_i b$ for every $b \in \mathfrak{B}_i$, so that μ extends ν_i, for every $i \in I$.

(d)(iv)⇒(ii) Suppose that (iv) is true, and that $i_0, \dots, i_n \in I$, $a_k \in \mathfrak{B}_{i_k}$ for $k \le n$, and $\sum_{k=0}^n \chi a_k \ge m\chi 1$ in $S(\mathfrak{A})$, where $m \in \mathbb{N}$. If $m = 0$ then of course $\sum_{k=0}^n \nu_{i_k} a_k \ge m$. Otherwise, set $J = \{i_k : k \le n\}$ and enumerate J as $\langle j_l\rangle_{l\le r}$. For $l \le r$ set $u_l = \frac{1}{m}\sum_{k\le n, i_k = j_l} \chi a_k$. Then $u_l \in S(\mathfrak{B}_{j_l})$ for each l, and

$$\sum_{l=0}^r u_l = \frac{1}{m}\sum_{l=0}^r \sum_{k\le n, i_k=j_l} \chi a_k = \frac{1}{m}\sum_{k=0}^n \chi a_k \ge \chi 1.$$

As $j_0, \dots, j_l$ are distinct,

$$\sum_{l=0}^r ⨍ u_l d\nu_{j_l} = \frac{1}{m}\sum_{k=0}^n \nu_{i_k} a_k \ge 1.$$

So (ii) is true.

(e)(v)⇒(iii) Use the same argument as in (d) above.

457B Corollary Let X be a set and $\langle Y_i\rangle_{i\in I}$ a family of sets. Suppose that for each $i \in I$ we have an algebra $\mathcal{E}_i$ of subsets of Y_i, an additive functional $\nu_i : \mathcal{E}_i \to [0,1]$ such that $\nu_i Y_i = 1$, and a function $f_i : X \to Y_i$. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathcal{P}X \to [0,1]$ such that $\mu f_i^{-1}[E] = \nu_i E$ whenever $i \in I$ and $E \in \mathcal{E}_i$;

(ii) whenever $i_0, \dots, i_n \in I$ and $E_k \in \mathcal{E}_{i_k}$ for $k \le n$, then there is an $x \in X$ such that $\sum_{k=0}^n \nu_{i_k} E_k \le \#(\{k : k \le n,\, f_{i_k}(x) \in E_k\})$.

proof (i)⇒(ii) is elementary; if $m = \lceil\sum_{k=0}^n \nu_{i_k} E_k\rceil - 1$, then $\sum_{k=0}^n \mu f_{i_k}^{-1}[E_k] > m\mu X$, so $\sum_{k=0}^n \chi f_{i_k}^{-1} E_k \not\le m\chi X$, that is, there is an $x \in X$ such that

$$\#(\{k : f_{i_k}(x) \in E_k\}) = \sum_{k=0}^n (\chi f_{i_k}^{-1}[E_k])(x) \ge m+1 \ge \sum_{k=0}^n \nu_{i_k} E_k.$$

(ii)⇒(i) Now suppose that (ii) is true. For $i \in I$ set $\mathfrak{B}_i = \{f_i^{-1}[E] : E \in \mathcal{E}_i\}$. Note that if $E \in \mathcal{E}_i$ and $\nu_i E > 0$, then (applying (ii) with $n = 0$, $i_0 = i$ and $E_0 = E$) $f_i^{-1}[E]$ cannot be empty; accordingly we have an additive functional $\nu_i' : \mathfrak{B}_i \to [0,1]$ defined by setting $\nu_i' f^{-1}[E] = \nu_i E$ for every $E \in \mathcal{E}_i$, and $\nu_i' X = 1$. If $i_0, \dots, i_n \in I$, $H_0 \in \mathfrak{B}_{i_0}, \dots, H_n \in \mathfrak{B}_{i_n}$ and $m \in \mathbb{N}$ are such that $\sum_{k=0}^n \chi H_k \le m\chi X$, express each H_k as $f_{i_k}^{-1}[E_k]$, where $E_k \in \mathcal{E}_{i_k}$; then there is an $x \in X$ such that

$$\textstyle\sum_{k=0}^n \nu_{i_k}' H_k = \sum_{k=0}^n \nu_{i_k} E_k \le \#(\{k : f_k(x) \in E_k\}) = \sum_{k=0}^m \chi H_k(x) \le m.$$

But this means that the condition of 457A(iii) is satisfied, with $\mathfrak{A} = \mathcal{P}X$, so 457A(i) and (i) here are also true.

457C Corollary (a) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}_1$, $\mathfrak{B}_2$ two subalgebras of $\mathfrak{A}$ with finitely additive functionals $\nu_i : \mathfrak{B}_i \to [0,1]$ such that $\nu_1 1 = \nu_2 1 = 1$. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathfrak{A} \to [0,1]$ extending both the ν_i;

(ii) whenever $b_1 \in \mathfrak{B}_1$, $b_2 \in \mathfrak{B}_2$ and $b_1 \cup b_2 = 1$, then $\nu_1 b_1 + \nu_2 b_2 \ge 1$;

(iii) whenever $b_1 \in \mathfrak{B}_1$, $b_2 \in \mathfrak{B}_2$ and $b_1 \cap b_2 = 0$, then $\nu_1 b_1 + \nu_2 b_2 \le 1$.

(b) Let X, Y_1, Y_2 be sets, and for $i \in \{1,2\}$ let $\mathcal{E}_i$ be an algebra of subsets of Y_i, $\nu_i : \mathcal{E}_i \to [0,1]$ an additive functional such that $\nu_i Y_i = 1$, and $f_i : X \to Y_i$ a function. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathcal{P}X \to [0,1]$ such that $\mu f_i^{-1}[E] = \nu_i E$ whenever $i \in \{1,2\}$ and $E \in \mathcal{E}_i$;

(ii) $f_1^{-1}[E_1] \cap f_2^{-1}[E_2] \ne \emptyset$ whenever $E_1 \in \mathcal{E}_1$, $E_2 \in \mathcal{E}_2$ and $\nu_1 E_1 + \nu_2 E_2 > 1$;

(iii) $\nu_1 E_1 \le \nu_2 E_2$ whenever $E_1 \in \mathcal{E}_1$, $E_2 \in \mathcal{E}_2$ and $f_1^{-1}[E_1] \subseteq f_2^{-1}[E_2]$.

proof (a)(i)⇒(iii) is elementary (and is a special case of 457A(i)⇒457A(iii)).

(iii)⇒(ii) If (iii) is true, and $b_1 \in \mathfrak{B}_1$, $b_2 \in \mathfrak{B}_2$ are such that $b_1 \cup b_2 = 1$, then $(1 \setminus b_1) \cap (1 \setminus b_2) = 0$, so

$$\nu_1 b_1 + \nu_2 b_2 = 2 - \nu_1(1 \setminus b_1) - \nu_2(1 \setminus b_2) \ge 1.$$

(ii)⇒(i) The point is that (ii) here implies (ii) of 457A. **P** Suppose that $i_0, \dots, i_n \in \{1,2\}$, $a_k \in \mathfrak{B}_{i_k}$ for $k \le n$ and $\sum_{k=0}^n \chi a_k \ge m\chi 1$ in $S(\mathfrak{A})$, where $m \in \mathbb{N}$. Set $K_j = \{k : k \le n,\ i_k = j\}$ for each j, $u = \sum_{k \in K_1} \chi a_k \in S(\mathfrak{B}_1)$, $v = \sum_{k \in K_2} \chi a_k \in S(\mathfrak{B}_2)$. Then we can express u as $\sum_{j=0}^{m_1} \chi c_j$ where $c_j \in \mathfrak{B}_1$ for each $j \le m_1$ and $c_0 \supseteq c_1 \supseteq \dots \supseteq c_{m_1}$ (see the proof of 361Ec). Taking $c_j = 0$ for $m_1 < j \le m$ if necessary, we may suppose that $m_1 \ge m$. Similarly, $v = \sum_{j=0}^{m_2} \chi d_j$ where $m_2 \ge m$, $d_j \in \mathfrak{B}_2$ for each $j \le m_2$ and $d_0 \supseteq \dots \supseteq d_{m_2}$.

For $j < m$, set $b_j = 1 \setminus (c_j \cup d_{m-j-1})$. Then, because $b_j \cap c_j = 0$,

$$u \times \chi b_j = \textstyle\sum_{r=0}^{m_1} \chi(c_r \cap b_j) = \sum_{r=0}^{j-1} \chi(c_r \cap b_j) \le j\chi b_j,$$

and similarly $v \times \chi b_j \le (m - j - 1)\chi b_j$, so

$$m\chi b_j \le (u + v) \times \chi b_j = u \times \chi b_j + v \times \chi b_j \le (m-1)\chi b_j,$$

and b_j must be 0.

Thus $c_j \cup d_{m-j-1} = 1$ for every $j < m$. But this means that $\nu_1 c_j + \nu_2 d_{m-j-1} \ge 1$ for every $j < m$, so that

$$\begin{aligned}\sum_{k=0}^n \nu_{i_k} a_k &= \sum_{k \in K_1} \nu_1 a_k + \sum_{k \in K_2} \nu_2 a_k = \int u\,d\nu_1 + \int v\,d\nu_2 \\ &= \sum_{j=0}^{m_1} \nu_1 c_j + \sum_{j=0}^{m_2} \nu_2 d_j \ge \sum_{j=0}^{m-1} \nu_1 c_j + \nu_2 d_{m-1-j} \ge m,\end{aligned}$$

as required. **Q**

Because 457A(ii) implies 457A(i), we have the result.

(b) We can convert (i) and (ii) here into (a-i) and (a-iii) just above by the same translation as in 457B. So (i) and (ii) are equiveridical. As for (iii), this corresponds exactly to replacing E_2 by $Y_2 \setminus E_2$ in (ii).

***457D** The proof of 457A is based, at some remove, on the Hahn-Banach theorem, as applied in the proof of 391E-391F. An alternative proof uses the max-flow min-cut theorem of graph theory. To show the power of this method I apply it to an elaboration of 457C, as follows.

Proposition (Strassen 65) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}_1$, $\mathfrak{B}_2$ two subalgebras of $\mathfrak{A}$. Suppose that $\nu_i : \mathfrak{B}_i \to [0,1]$ are finitely additive functionals such that $\nu_1 1 = \nu_2 1 = 1$, and $\theta : \mathfrak{A} \to [0,\infty[$ another additive functional. Then the following are equiveridical:

(i) there is an additive functional $\mu:\mathfrak{A}\to[0,\infty[$ extending both the ν_i, and such that $\mu a\le\theta a$ for every $a\in\mathfrak{A}$;
(ii) $\nu_1b_1+\nu_2b_2\le 1+\theta(b_1\cap b_2)$ whenever $b_1\in\mathfrak{B}_1$ and $b_2\in\mathfrak{B}_2$.

proof (a) As usual in this context, (i)$\Rightarrow$(ii) is elementary; if $\mu\le\theta$ extends both ν_j, and $b_j\in\mathfrak{B}_j$ for both j, then

$$\nu_1b_1+\nu_2b_2=\mu b_1+\mu b_2=\mu(b_1\cup b_2)+\mu(b_1\cap b_2)\le 1+\theta(b_1\cap b_2).$$

(b) For the reverse implication, suppose to begin with (down to the end of (d) below) that $\mathfrak{A}$ is finite. Let I, J and K be the sets of atoms of $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $\mathfrak{A}$ respectively. Consider the transportation network (V,E,γ) where

$$V=\{(0,0)\}\cup\{(b,1):b\in I\}\cup\{(d,2):d\in K\}\cup\{(c,3):c\in J\}\cup\{(1,4)\},$$

$$E=\{e^0_b:b\in I\}\cup\{e^1_d:d\in K\}\cup\{e^2_d:d\in K\}\cup\{e^3_c:c\in J\},$$

where

for $b\in I$, e^0_b runs from $(0,0)$ to $(b,1)$,
for $d\in K$, e^1_d runs from $(b,1)$ to $(d,2)$, where b is the member of I including d,
for $d\in K$, e^2_d runs from $(d,2)$ to $(c,3)$, where c is the member of J including d,
for $c\in J$, e^3_c runs from $(c,3)$ to $(1,4)$.

Define the capacity $\gamma(e)$ of each link by setting

$$\gamma(e^0_b)=\nu_1b\text{ for }b\in I,$$

$$\gamma(e^1_d)=\gamma(e^2_d)=\theta d\text{ for }d\in K,$$

$$\gamma(e^3_c)=\nu_2c\text{ for }c\in J.$$

By the max-flow min-cut theorem (4A4N), there are a flow ϕ and a cut X of the same value; that is, we have a function $\phi:E\to[0,\infty[$ and a set $X\subseteq E$ such that

$$\sum_{e\text{ starts from }v}\phi(e)=\sum_{e\text{ ends at }v}\phi(e)$$

for every $v\in V\setminus\{(0,0),(1,4)\}$,

$$\phi(e)\le\gamma(e)$$

for every $e\in E$,

$$\sum_{e\text{ starts from }(0,0)}\phi e=\sum_{e\text{ ends at }(1,4)}\phi e=\sum_{e\in X}\gamma(e),$$

and there is no path from $(0,0)$ to $(1,4)$ using only links in $E\setminus X$.

Now, for any $d\in K$, there is exactly one link e^1_d ending at d and exactly one link e^2_d starting from d. So $\phi(e^1_d)=\phi(e^2_d)$, and we may define an additive functional μ on $\mathfrak{A}$ by setting

$$\mu a=\sum_{d\in K,d\subseteq a}\phi(e^1_d)=\sum_{d\in K,d\subseteq a}\phi(e^2_d)$$

for every $a\in\mathfrak{A}$.

(c)(i) $\mu b\le\nu_1b$ for every $b\in\mathfrak{B}_1$. **P** Because I is the set of atoms of the finite Boolean algebra $\mathfrak{B}_1$, it is enough to show that $\mu b\le\nu_1b$ for every $b\in I$. Now, for such b,

$$\begin{aligned}\mu b=\sum_{d\in K,d\subseteq b}\phi(e^1_d)&=\sum_{e\text{ starts from }(b,1)}\phi(e)\\&=\sum_{e\text{ ends at }(b,1)}\phi(e)=\phi(e^0_b)\le\gamma(e^0_b)=\nu_1b,\end{aligned}$$

because the only link ending at $(b,1)$ is e^0_b. **Q**

(ii) Similarly, because the only link starting at $(c,3)$ has capacity ν_2c, $\mu c\le\nu_2c$ for every $c\in J$. But this means that $\mu c\le\nu_2c$ for every $c\in\mathfrak{B}_2$.

(iii) In third place, because

$$\mu d=\phi(e^1_d)\le\gamma(e^1_d)=\theta d$$

for every $d\in K$, $\mu a\le\theta a$ for every $a\in\mathfrak{A}$.

(d) (The key.) $\mu 1 \ge 1$. **P** We have

$$\begin{aligned}\mu 1 &= \sum_{d\in K} \mu d = \sum_{d\in K} \phi(e^1_d)\\ &= \sum_{b\in I} \sum_{d\in K, d\subseteq b} \phi(e^1_d) = \sum_{b\in I} \sum_{e \text{ starts from } (b,1)} \phi(e)\\ &= \sum_{b\in I} \sum_{e \text{ ends at } (b,1)} \phi(e) = \sum_{e \text{ starts from } (0,0)} \phi(e) = \sum_{e\in X} \gamma(e).\end{aligned}$$

Set

$$\begin{aligned}b^* &= \sup\{b : b\in I,\, e^0_b \in X\} \in \mathfrak{B}_1,\\ a^*_1 &= \sup\{d : d\in K,\, e^1_d \in X\},\\ a^*_2 &= \sup\{d : d\in K,\, e^2_d \in X\},\\ c^* &= \sup\{c : c\in J,\, e^3_c \in X\} \in \mathfrak{B}_2.\end{aligned}$$

For any $d\in K$, we have a four-link path $e^0_b, e^1_d, e^2_d, e^3_c$ from $(0,0)$ to $(1,4)$, where $b\in I$, $c\in J$ are the atoms of $\mathfrak{B}_1$, $\mathfrak{B}_2$ including d. At least one of the links in this path must belong to X, so that d is included in $b^*\cup a^*_1\cup a^*_2\cup c^*$. Thus, writing $a = (1\setminus b^*)\cap(1\setminus c^*)$, $a\subseteq a^*_1\cup a^*_2$ and $\theta a \le \theta a^*_1 + \theta a^*_2$. But this means that

$$\begin{aligned}\mu 1 &= \sum_{e\in X} \gamma(e)\\ &= \sum_{b\in I, e^0_b\in X} \gamma(e^0_b) + \sum_{d\in K, e^1_d\in X} \gamma(e^1_d) + \sum_{d\in K, e^2_d\in X} \gamma(e^2_d) + \sum_{c\in J, e^3_c\in X} \gamma(e^3_c)\\ &= \sum_{b\in I, e^0_b\in X} \nu_1 b + \sum_{d\in K, e^1_d\in X} \theta d + \sum_{d\in K, e^2_d\in X} \theta d + \sum_{c\in J, e^3_c\in X} \nu_2 c\\ &= \nu_1 b^* + \theta a^*_1 + \theta a^*_2 + \nu_2 c^*\end{aligned}$$

(remember that θ is additive)

$$\ge \nu_1 b^* + \theta((1\setminus b^*)\cap(1\setminus c^*)) + \nu_2 c^* \ge \nu_1 b^* + \nu_1(1\setminus b^*) + \nu_2(1\setminus c^*) - 1 + \nu_2 c^*$$

(applying the hypothesis (ii))

$$= 1,$$

as claimed. **Q**

Since we already know that $\nu_1 1 = 1$ and that $\mu b \le \nu_1 b$ for every $b\in\mathfrak{B}_1$, we must have $\mu 1 = 1$ and $\mu b = \nu_1 b$ for every $b\in\mathfrak{B}$, so that μ extends ν_1. Similarly, μ extends ν_2.

(e) Thus the proposition is proved in the case in which $\mathfrak{A}$ is finite. In the general case, for each finite subset K of $\mathfrak{A}$ write $\mathfrak{A}_K$ for the subalgebra of $\mathfrak{A}$ generated by K. Then (b)-(d) tell us that there is a non-negative additive functional μ_K on $\mathfrak{A}_K$, dominated by θ on $\mathfrak{A}_K$, agreeing with ν_1 on $\mathfrak{A}_K\cap\mathfrak{B}_1$ and agreeing with ν_2 on $\mathfrak{A}_K\cap\mathfrak{B}_2$. Let μ be any cluster point of the μ_K in $[0,1]^{\mathfrak{A}}$ as K increases through the finite subsets of $\mathfrak{A}$; then μ will be a non-negative additive functional on $\mathfrak{A}$, dominated by θ, and extending ν_1 and ν_2.

This proves the result.

457E Proposition Let X be a non-empty set and $\langle\nu_i\rangle_{i\in I}$ a family of probability measures on X satisfying the conditions of Lemma 457A, taking $\mathfrak{A} = \mathcal{P}X$ and $\mathfrak{B}_i = \operatorname{dom}\nu_i$ for each i. Suppose that there is a countably compact class $\mathcal{K}\subseteq\mathcal{P}X$ such that every ν_i is inner regular with respect to $\mathcal{K}$. Then there is a probability measure μ on X extending every ν_i.

proof If $I=\emptyset$ this is trivial. Otherwise, by 457A, there is a finitely additive functional ν on $\mathcal{P}X$ extending every ν_i. Now 413Sa tells us that there is a complete measure μ on X such that $\mu X \le \nu X$ and $\mu K \ge \nu K$ for every $K\in\mathcal{K}$. In this case, for any $i\in I$ and $E\in\mathrm{T}_i = \operatorname{dom}\nu_i$, we must have

$$\mu_*E \ge \sup_{K\in\mathcal{K},K\subseteq E} \mu K \ge \sup_{K\in\mathcal{K}\cap\operatorname{dom}\nu_i,K\subseteq E} \mu K$$
$$\ge \sup_{K\in\mathcal{K}\cap\operatorname{dom}\nu_i,K\subseteq E} \nu K = \sup_{K\in\mathcal{K}\cap\operatorname{dom}\nu_i,K\subseteq E} \nu_i K = \nu_i E.$$

In particular, $\mu X \ge \nu_i X = 1$. Also $\mu X \le \nu X = 1$, so

$$\mu^* E = 1 - \mu_*(X \setminus E) \le 1 - \nu_i(X \setminus E) = \nu_i E$$

for any $E \in \mathrm{T}_i$; as μ is complete, μE is defined and equal to $\nu_i E$ for every $E \in \mathrm{T}_i$, and μ extends ν_i, as required.

457F Proposition (a) Let (X,Σ,μ) be a perfect probability space and (Y,T,ν) any probability space. Write $\Sigma\otimes\mathrm{T}$ for the algebra of subsets of $X\times Y$ generated by $\{E\times F : E\in\Sigma,\ F\in\mathrm{T}\}$. Suppose that $Z\subseteq X\times Y$ is such that

(i) Z is expressible as the intersection of a sequence in $\Sigma\otimes\mathrm{T}$,

(ii) $Z\cap(E\times F)\ne\emptyset$ whenever $E\in\Sigma$, $F\in\mathrm{T}$ are such that $\mu E+\nu F>1$.

Then there is a probability measure λ on Z such that the maps $(x,y)\mapsto x : Z\to X$ and $(x,y)\mapsto y : Z\to Y$ are both inverse-measure-preserving.

(b) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of perfect probability spaces. Write $\bigotimes_{i\in I}\Sigma_i$ for the algebra of subsets of $X=\prod_{i\in I}X_i$ generated by $\{\{x : x\in X,\ x(i)\in E\} : i\in I,\ E\in\Sigma_i\}$. Suppose that $Z\subseteq X$ is such that

(i) Z is expressible as the intersection of a sequence in $\bigotimes_{i\in I}\Sigma_i$,

(ii) whenever $i_0,\dots,i_n\in I$ and $E_k\in\Sigma_{i_k}$ for $k\le n$, there is a $z\in Z$ such that $\#(\{k : k\le n, z(i_k)\in E_k\})\ge\sum_{k=0}^n\mu_{i_k}E_k$.

Then there is a perfect probability measure λ on Z such that $z\mapsto z(i) : Z\to X_i$ is inverse-measure-preserving for every $i\in I$.

proof (a) Apply 457Cb to the coordinate maps $f_1 : Z\to X$ and $f_2 : Z\to Y$. The condition (ii) here shows that 457C(b-ii) is satisfied, so there is an additive functional $\theta : \mathcal{P}Z\to[0,1]$ such that $\theta f_1^{-1}[E]=\mu E$ for every $E\in\Sigma$ and $\theta f_2^{-1}[F]=\nu F$ for every $F\in\mathrm{T}$.

Define $\theta' : \Sigma\otimes\mathrm{T}\to[0,1]$ by setting $\theta' W=\theta(Z\cap W)$ for every $W\in\Sigma\otimes\mathrm{T}$. Then $\theta'(E\times Y)=\mu E$ for every $E\in\Sigma$ and $\theta'(X\times F)=\nu F$ for every $F\in\mathrm{T}$. Because μ is perfect, θ' has an extension to a measure $\tilde\lambda$ defined on $\Sigma\widehat{\otimes}\mathrm{T}$ (454C). Now Z is supposed to be expressible as $\bigcap_{n\in\mathbb{N}}W_n$ where $W_n\in\Sigma\otimes\mathrm{T}$ for every n; since

$$\tilde\lambda W_n=\theta' W_n=\theta(Z\cap W_n)=\theta Z=1$$

for every n, $\tilde\lambda Z=1$. So if we take λ to be the subspace measure on Z induced by $\tilde\lambda$, λ will be a probability measure on Z. If $E\in\Sigma$, then

$$\lambda(Z\cap(E\times Y))=\tilde\lambda(Z\cap(E\times Y))=\tilde\lambda(E\times Y)$$
$$=\theta'(E\times Y)=\theta(Z\cap(E\times Y))=\mu E.$$

So $f_1 : Z\to X$ is inverse-measure-preserving for λ and μ. Similarly, $f_2 : Z\to Y$ is inverse-measure-preserving for λ and ν.

(b) We use the same ideas, but appealing to 457B and 454D instead of 457Cb and 454C. Taking $f_i : X\to X_i$ to be the coordinate map for each $i\in I$, (ii) here, with 457B, tells us that there is an additive functional $\theta : \mathcal{P}Z\to[0,1]$ such that $\theta f_i^{-1}[E]=\mu_i E$ whenever $i\in I$ and $E\in\Sigma_i$.

Define $\theta' : \bigotimes_{i\in I}\Sigma_i\to[0,1]$ by setting $\theta' W=\theta(Z\cap W)$ for every $W\in\bigotimes_{i\in I}\Sigma_i$. Then

$$\theta'\{x : x\in X,\ x(i)\in E\}=\theta\{z : z\in Z,\ z(i)\in E\}=\mu_i E$$

whenever $i\in I$ and $E\in\Sigma_i$. Because every μ_i is perfect, θ' has an extension to a perfect measure $\tilde\lambda$ defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$ (454D). Now Z is supposed to be expressible as $\bigcap_{n\in\mathbb{N}}W_n$ where $W_n\in\bigotimes_{i\in I}\Sigma_i$ for every n; since

$$\tilde\lambda W_n=\theta' W_n=\theta(Z\cap W_n)=\theta Z=1$$

for every n, $\tilde\lambda Z=1$. So if we take λ to be the subspace measure on Z induced by $\tilde\lambda$, λ will be a probability measure on Z; by 451Dc, λ is perfect. If $i\in I$ and $E\in\Sigma_i$, then

$$\lambda\{z : z\in Z,\ z(i)\in E\}=\tilde\lambda\{x : x\in X,\ x(i)\in E\}=\theta'\{x : x\in X,\ x(i)\in E\}$$
$$=\theta\{z : z\in Z,\ z(i)\in E\}=\mu_i E.$$

So $z \mapsto z(i) : Z \to X_i$ is inverse-measure-preserving for λ and μ_i for every $i \in I$, as required.

457G Theorem Let X be a set and $\langle \mu_i \rangle_{i \in I}$ a family of probability measures on X which is upwards-directed in the sense that for any i, $j \in I$ there is a $k \in I$ such that μ_k extends both μ_i and μ_j. Suppose that for any countable $J \subseteq I$ there is a measure on X extending μ_i for every $i \in J$. Then there is a measure on X extending μ_i for every $i \in I$.

proof Set $\Sigma_i = \operatorname{dom} \mu_i$ for each $i \in I$. Because $\langle \mu_i \rangle_{i \in I}$ is upwards-directed, $\mathrm{T} = \bigcup_{i \in I} \Sigma_i$ is an algebra of subsets of X, and we have a finitely additive functional $\nu : \mathrm{T} \to [0,1]$ defined by saying that $\nu E = \mu_i E$ whenever $i \in I$ and $E \in \Sigma_i$. Now if $\langle E_n \rangle_{n \in \mathbb{N}}$ is any non-increasing sequence in T with empty intersection, there is a countable set $J \subseteq I$ such that $E_n \in \bigcup_{i \in J} \Sigma_i$ for every $n \in \mathbb{N}$. We are told that there is a measure λ on X extending μ_i for every $i \in J$; now $\nu E_n = \lambda E_n$ for every $n \in \mathbb{N}$, so $\lim_{n \to \infty} \nu E_n = 0$. By 413Kb, ν has an extension to a measure on X, which of course extends every μ_i.

457H Example Set $X = \{(x,y) : 0 \le x < y \le 1\} \subseteq [0,1]^2$. Write π_1, $\pi_2 : X \to \mathbb{R}$ for the coordinate maps, and μ_L for Lebesgue measure on $[0,1]$, with Σ_L its domain.

(a) There is a finitely additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu \pi_i^{-1}[E] = \mu_L E$ whenever $i \in \{1,2\}$ and $E \in \Sigma_L$. **P** If E_1, $E_2 \in \Sigma_L$ and $\mu_L E_1 + \mu_L E_2 > 1$, then neither is empty and $\inf E_1 < \sup E_2$, so there are $x \in E_1$, $y \in E_2$ such that $x < y$, and $(x,y) \in \pi_1^{-1}[E_1] \cap \pi_2^{-1}[E_2]$. So the result follows by 457Cb. **Q**

(b) However, there is no measure μ on X for which both π_1 and π_2 are inverse-measure-preserving. **P?** If there were,

$$\int \pi_1(x,y)\mu(d(x,y)) = \int x\mu_L(dx) = \int y\mu_L(dy) = \int \pi_2(x,y)\mu(d(x,y))$$

by 235G; but $\pi_1(x,y) < \pi_2(x,y)$ for every $(x,y) \in X$, so this is impossible. **XQ**

(c) If we write $\mathrm{T}_i = \{\pi_i^{-1}[E] : E \subseteq [0,1] \text{ is Borel}\}$ for each i, then we have a measure ν_i with domain T_i defined by setting $\nu_i \pi_i^{-1}[E] = \mu_L E$ for each Borel set $E \subseteq [0,1]$. Now ν_1 and ν_2 have no common extension to a Borel measure on X, even though X is a Polish space and each ν_i is a compact measure, being inner regular with respect to the compact class $\mathcal{K}_i = \{\pi_i^{-1}[K] : K \subseteq \,]0,1[\text{ is compact}\}$. (The trouble is that $\mathcal{K}_1 \cup \mathcal{K}_2$ is *not* compact, so we cannot apply 457E.)

457I Example Let μ_L be Lebesgue measure on $[0,1]$ and Σ_L its domain. Set

$$X = \{(\xi_1,\xi_2,\xi_3) : 0 \le \xi_i \le 1 \text{ for each } i,\ \sum_{i=1}^3 \xi_i \le \tfrac{3}{2},\ \sum_{i=1}^3 \xi_i^2 \le 1\}.$$

For $1 \le i \le 3$ set $\pi_i(x) = \xi_i$ for $x = (\xi_1,\xi_2,\xi_3) \in X$.

(a) If $E_i \in \Sigma_L$ for $i \le 3$, then there is an $x \in X$ such that $\#(\{i : \pi_i(x) \in E_i\}) \ge \sum_{i=1}^3 \mu_L E_i$. **P** Set $\alpha_i = \inf(E_i \cup \{1\})$ for each i, and set

$$m = \lceil \sum_{i=1}^3 \mu_L E_i \rceil \le \lceil \sum_{i=1}^3 1 - \alpha_i \rceil = 3 - \lfloor \sum_{i=1}^3 \alpha_i \rfloor,$$

so that $\sum_{i=1}^3 \alpha_i < 4 - m$. Take $\xi_i \in E_i \cup \{1\}$ such that $\sum_{i=1}^3 \xi_i < 4 - m$. It will be enough to consider the case in which $\xi_1 \le \xi_2 \le \xi_3$.

(i) If $m = 1$, then $\sum_{i=1}^3 \xi_i < 3$ so $\xi_1 < 1$ and $\xi_1 \in E_1$. Set $x = (\xi_1,0,0)$; then $x \in X$ and

$$\#(\{i : \pi_i(x) \in E_i\}) \ge 1 \ge \sum_{i=1}^3 \mu_L E_i.$$

(ii) If $m = 2$, then $\sum_{i=1}^3 \xi_i < 2$ so $\xi_2 < 1$ and $\xi_1 \in E_1$, $\xi_2 \in E_2$. Set $x = (\xi_1,\xi_2,0)$. We have $\xi_1 + \xi_2 \le \frac{4}{3} \le \frac{3}{2}$. Also

$$\xi_2 \le \frac{1}{2}(\xi_2 + \xi_3) \le 1 - \frac{1}{2}\xi_1,$$

so

$$\xi_1^2 + \xi_2^2 \le \xi_1^2 + (1 - \tfrac{1}{2}\xi_1)^2 = 1 - \xi_1 + \tfrac{5}{4}\xi_1^2 \le 1$$

because $\xi_1 \le \frac{2}{3} \le \frac{4}{5}$. So $x \in X$ and

$$\#(\{i : \pi_i(x) \in E_i\}) \ge 2 \ge \sum_{i=1}^{3} \mu_L E_i.$$

(iii) If $m = 3$ then $\sum_{i=1}^{3} \xi_i < 1$ so $\xi_i \in E_i$ for every i; set $x = (\xi_1, \xi_2, \xi_3)$. Since $\sum_{i=1}^{3} \xi_i^2 \le \sum_{i=1}^{3} \xi_i \le 1$, $x \in X$ and

$$\#(\{i : \pi_i(x) \in E_i\}) = 3 \ge \sum_{i=1}^{3} \mu_L E_i.$$

Putting these together, we have the result. **Q**

(b) There is no finitely additive functional ν on X such that $\nu\pi_i^{-1}[E] = \mu_L E$ for each i and every $E \in \Sigma_L$. **P?** Suppose there were. Set $\mathrm{T}_i = \{\pi_i^{-1}[E] : E \in \Sigma_L\}$ and $\nu_i = \nu\restriction\mathrm{T}_i$ for each i. Then ν_i is a probability measure on X; moreover, because X is compact, $\pi_i^{-1}[K]$ is compact for every compact $K \subseteq [0,1]$, so ν_i is inner regular with respect to the compact subsets of X. By 457E, the ν_i have a common extension to a countably additive measure μ. Now

$$\int_X \xi_1 + \xi_2 + \xi_3 \, \mu(dx) = 3\int_0^1 t\,dt = \frac{3}{2},$$

so we must have $\xi_1 + \xi_2 + \xi_3 = \dfrac{3}{2}$ for μ-almost every x; similarly,

$$\int_X \xi_1^2 + \xi_2^2 + \xi_3^2 \, \mu(dx) = 3\int_0^1 t^2\,dt = 1,$$

so we must have $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$ for μ-almost every x. Since

$$(\frac{3}{2} - \xi_3)^2 = (\xi_1 + \xi_2)^2 \le 2(\xi_1^2 + \xi_2^2) \le 2(1 - \xi_3^2)$$

for almost every x, $\xi_3 - \xi_3^2 \ge \frac{1}{12}$ for almost every x, which is impossible, since $\mu\{x : \xi_3 \le \dfrac{1}{2} - \dfrac{1}{\sqrt{6}}\} > 0$. **XQ**

457J Example There are a set X and a family $\langle\mu_i\rangle_{i\in I}$ of probability measures on X such that (i) for every countable set $J \subseteq I$ there is a measure on X extending μ_i for every $i \in J$ (ii) there is no measure on X extending μ_i for every $i \in I$.

proof By 439Fc, there is an uncountable universally negligible subset of $[0,1]$. Because $[0,1]$ and $\mathcal{P}\mathbb{N}$ are uncountable Polish spaces, they have isomorphic Borel structures (424Cb), so there is an uncountable universally negligible set $X_0 \subseteq \mathcal{P}\mathbb{N}$. The map $a \mapsto \mathbb{N} \setminus a$ is an autohomeomorphism of $\mathcal{P}\mathbb{N}$, so $X_1 = \{\mathbb{N} \setminus a : a \in X_0\}$ is universally negligible, and $X = X_0 \cup X_1$ is universally negligible (439Cb).

For $n \in \mathbb{N}$, set $E_n = \{a : n \in a \in X\}$ and $\Sigma_n = \{\emptyset, E_n, X \setminus E_n, X\}$; note that, because X is closed under complementation, neither E_n nor $X \setminus E_n$ is empty, and we have a probability measure μ_n with domain Σ_n defined by setting $\mu_n E_n = \mu_n(X \setminus E_n) = \frac{1}{2}$. Next, for $a \in X$, set $\Sigma'_a = \{\emptyset, \{a\}, X \setminus \{a\}, X)$, and let μ'_a be the probability measure with domain Σ'_a defined by setting $\mu'_a\{a\} = 0$.

If $J \subseteq X$ is countable, then there is a probability measure on X extending μ_n for every $n \in \mathbb{N}$ and μ'_a for every $a \in J$. **P** Because X_0 is uncountable, there is a $b \in X_0$ such that neither b nor $b' = \mathcal{P}\mathbb{N} \setminus b$ belongs to J. Let μ be the probability measure with domain $\mathcal{P}X$ defined by setting $\mu\{b\} = \mu\{b'\} = \frac{1}{2}$; this extends all the μ_n and all the μ'_a for $a \in J$. **Q**

? Suppose, if possible, that μ is a measure on X extending every μ_n and every μ'_a. In this case, because μ extends every μ_n, its domain includes the Borel σ-algebra $\mathcal{B}$ of X, and $\mu\restriction\mathcal{B}$ is a Borel probability measure on X. Since X is universally negligible, there is a point $a \in X$ such that $\mu\{a\} > 0$; in which case μ cannot extend μ'_a. **X**

Thus the μ_n, μ'_a constitute a family of the kind required.

457K In addition to existence, we can ask for solutions to simultaneous-extension problems which are optimal in some sense; some transportation problems can be interpreted as questions of this kind. In this direction I give just one result, which is also connected to the ideas of §437.[25]

Definition (Bogachev 07, §8.10(viii)) Let (X, ρ) be a metric space. For quasi-Radon probability measures μ, ν on X, set

$$\rho_{\mathrm{W}}(\mu, \nu) = \sup\{|\int u\,d\mu - \int u\,d\nu| : u : X \to \mathbb{R} \text{ is bounded and 1-Lipschitz}\}.$$

(Compare the metric ρ_{KR} of 437Qb. ρ_{W} is sometimes called the 'Wasserstein metric'.)

[25] I am indebted to J.Pachl for leading me to this material.

457L Theorem Let (X,ρ) be a metric space and P_{qR} the set of quasi-Radon probability measures on X; define ρ_{W} as in 457K.

(a) For all μ, ν and λ in P_{qR},

$$\rho_{\mathrm{W}}(\mu,\nu)=\rho_{\mathrm{W}}(\nu,\mu),\quad \rho_{\mathrm{W}}(\mu,\lambda)\le\rho_{\mathrm{W}}(\mu,\nu)+\rho_{\mathrm{W}}(\nu,\lambda),$$

$$\rho_{\mathrm{W}}(\mu,\nu)=0 \text{ iff } \mu=\nu.$$

(b) (cf. VASERSHTEIN 69) If μ, $\nu\in P_{\mathrm{qR}}$, then $\rho_{\mathrm{W}}(\mu,\nu)=\inf_{\lambda\in Q(\mu,\nu)}\int\rho(x,y)\lambda(d(x,y))$, where $Q(\mu,\nu)$ is the set of quasi-Radon probability measures on $X\times X$ with marginal measures μ and ν.

(c) In (b), if μ and ν are Radon measures, $Q(\mu,\nu)$ is included in $P_{\mathrm{R}}(X\times X)$, the space of Radon probability measures on $X\times X$, and is compact for the narrow topology on $P_{\mathrm{R}}(X\times X)$; and there is a $\lambda\in Q(\mu,\nu)$ such that $\rho_{\mathrm{W}}(\mu,\nu)=\int\rho(x,y)\lambda(d(x,y))$.

(d) If ρ is bounded, then ρ_{W} is a metric on P_{qR} inducing the narrow topology (definition: 437Jd).

proof (a) The first two clauses are immediate from the definition. For the third, observe that if $\mu\ne\nu$ then $\rho_{\mathrm{W}}(\mu,\nu)\ge\rho_{\mathrm{KR}}(\mu,\nu)>0$ by 437R.

(b) Write $\zeta\in[0,\infty]$ for $\rho_{\mathrm{W}}(\mu,\nu)$, $\mathcal{L}^\infty_{\mathrm{dom}\,\mu}$ for the space of bounded $\mathrm{dom}\,\mu$-measurable functions from X to $\mathbb{R}$ and $\mathcal{L}^\infty_{\mathrm{dom}\,\nu}$ for the space of bounded $\mathrm{dom}\,\nu$-measurable functions from X to $\mathbb{R}$.

(i) We have

$$\zeta=\sup\{\int u\,d\mu+\int v\,d\nu:u\in\mathcal{L}^\infty_{\mathrm{dom}\,\mu},\,v\in\mathcal{L}^\infty_{\mathrm{dom}\,\nu},$$
$$u(x)+v(y)\le\rho(x,y)\text{ for all }x,\,y\in X\}.$$

$\mathbf{P}(\boldsymbol{\alpha})$ Suppose that $u\in\mathcal{L}^\infty_{\mathrm{dom}\,\mu}$, $v\in\mathcal{L}^\infty_{\mathrm{dom}\,\nu}$ and $u(x)+v(y)\le\rho(x,y)$ for all x, $y\in X$. Set

$$w(x)=\inf_{y\in X}\rho(x,y)-v(y)$$

for $x\in X$. Then $u(x)\le w(x)$ and $w(x)+v(x)\le 0$ for every x, so $u\le w\le -v$ and w is bounded; also w is 1-Lipschitz, because if x, $x'\in X$ then

$$w(x)-\rho(x,x')=\inf_{y\in X}\rho(x,y)-v(y)-\rho(x,x')\le\inf_{y\in X}\rho(x',y)-v(y)=w(x').$$

Accordingly

$$\int u\,d\mu+\int v\,d\nu\le\int w\,d\mu-\int w\,d\nu\le\zeta.$$

$(\boldsymbol{\beta})$ In the other direction, given $\gamma<\zeta$, there is a bounded 1-Lipschitz function $u:X\to\mathbb{R}$ such that $|\int u\,d\mu-\int u\,d\nu|\ge\gamma$. Replacing u by $-u$ if necessary, we can arrange that $\int u\,d\mu-\int u\,d\nu\ge\gamma$. Now set $v=-u$; then $u(x)+v(y)\le\rho(x,y)$ for all x, y, and $\int u\,d\mu+\int v\,d\nu\ge\gamma$. $\mathbf{Q}$

It follows that if $u\in\mathcal{L}^\infty_{\mathrm{dom}\,\mu}$, $v\in\mathcal{L}^\infty_{\mathrm{dom}\,\nu}$ and $u(x)+v(y)\le\beta\rho(x,y)$ for all x, $y\in X$, where $\beta>0$, then

$$\int u\,d\mu+\int v\,d\nu=\beta(\int\frac{1}{\beta}u\,d\mu+\int\frac{1}{\beta}v\,d\nu)\le\beta\zeta.$$

(ii) $\int\rho\,d\lambda\ge\zeta$ for every $\lambda\in Q(\mu,\nu)$. $\mathbf{P}$ If $u\in\mathcal{L}^\infty_{\mathrm{dom}\,\mu}$, $v\in\mathcal{L}^\infty_{\mathrm{dom}\,\nu}$ and $u(x)+v(y)\le\rho(x,y)$ for all x, $y\in X$, then

$$\int u\,d\mu+\int v\,d\nu=\int u(x)\lambda(d(x,y))+\int v(y)\lambda(d(x,y))$$

(235G)

$$\le\int\rho\,d\lambda$$

so (i) gives us the result. $\mathbf{Q}$

If $\zeta=\infty$, we can stop; so henceforth suppose that ζ is finite.

(iii) Define $p:\ell^\infty(X\times X)\to[0,\infty[$ by setting

$$p(w)=\inf\{\alpha+\beta\zeta:\alpha,\,\beta>0,\,w(x,y)\le\alpha+\beta\rho(x,y)\text{ for all }x,\,y\in X\}.$$

Then $p(w+w') \le p(w)+p(w')$ and $p(\alpha w)=\alpha p(w)$ whenever w, $w' \in \ell^\infty(X\times X)$ and $\alpha \in [0,\infty[$. For u, $v \in \mathbb{R}^X$ define $u\otimes v \in \mathbb{R}^{X\times X}$ by setting $(u\otimes v)(x,y)=u(x)v(y)$ for all x, $y\in X$ (cf. 253B); set

$$V=\{(u\otimes\chi X)+(\chi X\otimes v): u\in\mathcal{L}^\infty_{\operatorname{dom}\mu},\, v\in\mathcal{L}^\infty_{\operatorname{dom}\nu}\}.$$

Let $\mu\times\nu$ be the quasi-Radon product measure on $X\times X$ (417R). Then we have a linear functional $h_0 : V\to\mathbb{R}$ defined by saying that $h_0(w)=\int w\,d(\mu\times\nu)$ for $w\in V$. The point is that $h_0(w)\le p(w)$ for every $w\in V$. **P** We have $u\in\mathcal{L}^\infty_{\operatorname{dom}\mu}$, $v\in\mathcal{L}^\infty_{\operatorname{dom}\nu}$ such that $w(x,y)=u(x)+v(y)$ for all x, $y\in X$. If α, $\beta>0$ are such that $w(x,y)\le\alpha+\beta\rho(x,y)$ for all x, $y\in X$, set $u_0(x)=u(x)-\alpha$ for every x; then $u_0(x)+v(y)\le\beta\rho(x,y)$ for all x and y, so

$$\begin{aligned} h_0(w) &= \int u\otimes\chi X\,d(\mu\times\nu)+\int\chi X\otimes v\,d(\mu\times\nu)=\int u\,d\mu+\int v\,d\nu\\ &=\alpha+\int u_0\,d\mu+\int v\,d\nu\le\alpha+\beta\zeta \end{aligned}$$

by the last remark in (i). As α and β are arbitrary, $h_0(w)\le p(w)$. **Q**

(iv) By the Hahn-Banach theorem (3A5Aa), there is a linear functional $h:\ell^\infty(X\times X)\to\mathbb{R}$, extending h_0, such that $h(w)\le p(w)$ for every $w\in\ell^\infty(X\times X)$. In this case, h must be a positive linear functional, because if $w\ge 0$ then $p(-w)=0$, so $h(-w)\le 0$. Since also

$$h(\chi(X\times X))=h_0(\chi(X\times X))=(\mu\times\nu)(X\times X)=1,$$

$\|h\|=1$ in $\ell^\infty(X\times X)^*$. If u, $v\in C_b(X)$ then

$$h(u\otimes\chi X)=h_0(u\otimes\chi X)=\int u\,d\mu,\quad h(\chi X\otimes v)=h_0(\chi X\otimes v)=\int v\,d\nu.$$

Let $\theta:\mathcal{P}(X\times X)\to[0,1]$ be the additive functional defined by setting $\theta W=h(\chi W)$ for $W\subseteq X\times X$. Observe that $\theta(E\times X)=\mu E$ for every $E\in\operatorname{dom}\mu$ and $\theta(X\times E)=\nu E$ for every $E\in\operatorname{dom}\nu$.

(v) Because both μ and ν are inner regular with respect to the totally bounded sets (434L), there is a separable subset Y of X such that $\mu Y=\nu Y=1$, and we can take Y to be a Borel set. Now let $\epsilon>0$. Then we have a countable partition $\langle E_i\rangle_{i\in I}$ of Y into non-empty Borel sets of diameter at most ϵ. For i, $j\in I$, set

$$\begin{aligned} \alpha_{ij}&=\tfrac{\theta(E_i\times E_j)}{\mu E_i\,\nu E_j}\text{ if }\mu E_i\cdot\nu E_j>0,\\ &=0\text{ otherwise.} \end{aligned}$$

Since $\theta(E_i\times E_j)\le\min(\mu E_i,\nu E_j)$, $\theta(E_i\times E_j)=\alpha_{ij}\mu E_i\nu E_j$. If $i\in I$ is such that $\mu E_i>0$, then $\sum_{j\in I}\alpha_{ij}\nu E_j=1$. **P** For any $\eta>0$ there is a finite $K_0\subseteq I$ such that $\nu(X\setminus\bigcup_{j\in K_0}E_j)\le\eta$. Now

$$\begin{aligned} |1-\sum_{j\in K}\alpha_{ij}\nu E_j|\mu E_i &= |\mu E_i-\sum_{j\in K}\theta(E_i\times E_j)|=|\theta(E_i\times X)-\theta(E_i\times\bigcup_{j\in K}E_j)|\\ &=\theta(E_i\times(X\setminus\bigcup_{j\in K}E_j))\le\theta(X\times(X\setminus\bigcup_{j\in K}E_j))\\ &=\nu(X\setminus\bigcup_{j\in K}E_j)\le\eta \end{aligned}$$

whenever K is a finite subset of I including K_0; as η is arbitrary, $\mu E_i\cdot\sum_{j\in I}\alpha_{ij}\nu E_j=\mu E_i$ and $\sum_{j\in I}\alpha_{ij}\nu E_j=1$. **Q** Similarly, $\sum_{i\in I}\alpha_{ij}\mu E_i=1$ whenever $\nu E_j>0$.

(vi) Define a Borel measurable function $w_0:X\times X\to[0,\infty[$ by setting

$$\begin{aligned} w_0(x,y)&=\alpha_{ij}\text{ if }i,\,j\in I,\,x\in E_i\text{ and }y\in E_j,\\ &=0\text{ if }(x,y)\in(X\times X)\setminus(Y\times Y). \end{aligned}$$

Let λ be the indefinite-integral measure over $\mu\times\nu$ defined by w_0; then λ is a quasi-Radon probability measure with marginals μ, ν. **P** If $E\in\operatorname{dom}\mu$, then

$$\lambda(E \times X) = \int_{E\times X} w_0 d(\mu\times\nu) = \sum_{i,j\in I}\int_{(E\cap E_i)\times E_j} w_0 d(\mu\times\nu)$$
$$= \sum_{i,j\in I}\alpha_{ij}\mu(E\cap E_i)\cdot\nu E_j = \sum_{i\in I}\mu(E\cap E_i)$$

(because $\sum_{j\in I}\alpha_{ij}\nu E_j = 1$ whenever $\mu E_i > 0$)

$$= \mu E.$$

In particular, $\lambda(X\times X) = 1$, so λ is a probability measure, and is quasi-Radon by 415Ob; and the coordinate projection $(x, y) \mapsto x$ is inverse-measure-preserving for λ and μ. To see that μ is exactly the image measure, observe that if $E \subseteq X$ is such that $\lambda(E \times X)$ is defined, then $(E \cap E_i) \times E_j$ must be measured by $\mu\times\nu$ whenever $\alpha_{ij} > 0$. For any $i \in I$ such that $\mu E_i > 0$, there is surely some j such that $\alpha_{ij} > 0$, in which case $E \cap E_i \in \operatorname{dom}\mu$; since $\bigcup_{i\in I} E_i$ is μ-conegligible (and μ is complete and I is countable), $E \in \operatorname{dom}\mu$. Thus μ is the marginal of λ on the first coordinate. Similarly, ν is the marginal of λ on the second coordinate. **Q**

For i, $j \in I$ we have

$$\lambda(E_i \times E_j) = \alpha_{ij}\mu E_i \cdot \nu E_j = \theta(E_i \times E_j).$$

(vii) $\int \rho\, d\lambda \le \zeta + 2\epsilon$. **P** For i, $j \in I$, set

$$\beta_{ij} = \inf_{x\in E_i, y\in E_j} \rho(x, y);$$

set

$$w(x, y) = \beta_{ij} \text{ if } i,\ j \in I,\ x \in E_i \text{ and } y \in E_j,$$
$$= 0 \text{ if } (x, y) \in (X \times X) \setminus (Y \times Y).$$

Then

$$w \le \rho \times \chi(Y \times Y) \le w + 2\epsilon\chi(X \times X),$$

so

$$\int \rho\, d\lambda = \int_{Y\times Y} \rho\, d\lambda \le 2\epsilon + \int w\, d\lambda$$
$$= 2\epsilon + \sum_{i,j\in I}\beta_{ij}\lambda(E_i\times E_j) = 2\epsilon + \sum_{i,j\in I}\beta_{ij}\theta(E_i\times E_j).$$

Now, for any finite $K \subseteq I$,

$$\textstyle\sum_{i,j\in K}\beta_{ij}\theta(E_i\times E_j) = h(w\times\chi(\bigcup_{i,j\in K}E_i\times E_j)) \le h(\rho) \le p(\rho) \le \zeta$$

by the definition of p. So $\int \rho\, d\lambda \le 2\epsilon + \zeta$, as claimed. **Q**

(viii) As ϵ is arbitrary,

$$\inf_{\lambda\in Q(\mu,\nu)} \textstyle\int \rho\, d\lambda \le \zeta.$$

With (ii), this completes the proof of (b).

(c) For every $\epsilon > 0$, there is a compact set $K \subseteq X$ such that $\mu(X \setminus K) + \nu(X \setminus K) \le \epsilon$. In this case $\lambda((X \times X) \setminus (K \times K)) \le \epsilon$ for every $\lambda \in Q(\mu, \nu)$. In the first place, this shows that if $\lambda \in Q(\mu, \nu)$, then $\lambda W = \sup_{L\subseteq X\times X \text{ is compact}} \lambda(W \cap L)$ for every $W \in \operatorname{dom}\lambda$; by 416F, λ is a Radon measure. Thus $Q(\mu, \nu) \subseteq P_{\mathrm{R}}(X \times X)$. Next, we see also that $Q(\mu, \nu)$ is uniformly tight (437O), therefore relatively compact in the space $M^+_{\mathrm{R}}(X \times X)$ of totally finite Radon measures on $X \times X$ (437P).

Writing π_1, π_2 for the coordinate projections from $X \times X$ to X, we see that

$$Q(\mu, \nu) = \{\lambda : \lambda \in M^+_{\mathrm{R}}(X \times X),\ \lambda\pi_1^{-1} = \mu \text{ and } \lambda\pi_2^{-1} = \nu\}.$$

Since the functions $\lambda \mapsto \lambda\pi_1^{-1}$ and $\lambda \mapsto \lambda\pi_2^{-1}$ from $M^+_{\mathrm{R}}(X \times X)$ to $M^+_{\mathrm{R}}(X)$ are continuous (437N), and $M^+_{\mathrm{R}}(X)$ is Hausdorff in its narrow topology (437R(a-ii)), $Q(\mu, \nu)$ is closed in $M^+_{\mathrm{R}}(X \times X)$, therefore compact.

Finally, the function $\lambda \mapsto \int \rho\,d\lambda$ from $M_{\mathrm{R}}^{+}(X\times X)$ to $[0,\infty]$ is lower semi-continuous (437Jg), and must attain its infimum on the compact set $Q(\mu,\nu)$ (4A2B(d-viii)). But (b) tells us that this infimum is just $\rho_{\mathrm{W}}(\mu,\nu)$.

(d)(i) Suppose first that $\rho(x,y)\le 2$ for all x, $y\in X$. Then $\rho_{\mathrm{W}}=\rho_{\mathrm{KR}}\restriction P_{\mathrm{qR}}\times P_{\mathrm{qR}}$. **P** As already noted in (a), $\rho_{\mathrm{W}}(\mu,\nu)\ge\rho_{\mathrm{KR}}(\mu,\nu)$ for all μ, $\nu\in P_{\mathrm{qR}}$. In the other direction, if μ, $\nu\in P_{\mathrm{qR}}$ and $u:X\to\mathbb{R}$ is 1-Lipschitz, then $|u(x)-u(y)|\le 2$ for all x, $y\in X$, so there is an $\alpha\in\mathbb{R}$ such that $|u(x)-\alpha|\le 1$ for all $x\in X$. Set $v(x)=u(x)-\alpha$ for every x; then $v:X\to[-1,1]$ is 1-Lipschitz, so

$$|\int u\,d\mu-\int u\,d\nu|=|\int v\,d\mu-\int v\,d\nu|$$

(because $\mu X=\nu X$)

$$\le\rho_{\mathrm{KR}}(\mu,\nu).$$

As u is arbitrary, $\rho_{\mathrm{W}}(\mu,\nu)\le\rho_{\mathrm{KR}}(\mu,\nu)$ and the two metrics are equal. **Q**

(ii) In general, take $\gamma>0$ such that $\rho(x,y)\le 2\gamma$ for all x, $y\in X$. Set $\sigma=\dfrac{1}{\gamma}\rho$, so that σ is a metric on X equivalent to ρ. Now σ_{KR} defines the narrow topology on P_{qR}, by 437R(g-i), so $\rho_{\mathrm{W}}=\gamma\sigma_{\mathrm{W}}=\gamma\sigma_{\mathrm{KR}}\restriction P_{\mathrm{qR}}\times P_{\mathrm{qR}}$ also does.

457M If we relax our demands, and look for measures dominated by each measure in a family rather than extending them, similar methods give further results.

Theorem (see KELLERER 84) Let X be a Hausdorff space and $\langle\nu_i\rangle_{i\in I}$ a non-empty finite family of locally finite measures on X all inner regular with respect to the closed sets.

(a) For $A\subseteq X\times[0,\infty[$, set

$$c(A)=\inf\{\sum_{i\in I}\int h_i d\nu_i: h_i:X\to[0,\infty]\text{ is }\operatorname{dom}\nu_i\text{-measurable for each }i\in I,$$
$$\alpha\le\sum_{i\in I}h_i(x)\text{ whenever }(x,\alpha)\in A\}.$$

(i) c is a Choquet capacity (definition: 432J).

(ii) For every $A\subseteq X\times[0,\infty[$, the infimum in the definition of $c(A)$ is attained.

(b) Let $f:X\to[0,\infty[$ be a function such that $\{x:f(x)\ge\alpha\}$ is K-analytic for every $\alpha>0$. Then

$$\inf\{\sum_{i\in I}\int h_i d\nu_i: h_i:X\to[0,\infty]\text{ is }\operatorname{dom}\nu_i\text{-measurable for each }i\in I,\ f\le\sum_{i\in I}h_i\}$$
$$=\sup\{\int f\,d\mu:\mu\text{ is a Radon measure on }X\text{ and }\mu\le\nu_i\text{ for every }i\in I\},$$

where '$\mu\le\nu_i$' here is to be interpreted in the sense of 234P.

proof (a)(i)($\boldsymbol{\alpha}$) For $f:X\to[0,\infty]$ set

$$\Omega_f=\{(x,\alpha):x\in X,\ \alpha\le f(x)\},\quad \Omega'_f=\{(x,\alpha):x\in X,\ \alpha<f(x)\}$$

as in 252N.

It will be convenient to amalgamate the ν_i into a single measure, as follows. Let (Y,T,ν) be the direct sum of the family $\langle(X_i,\nu_i)\rangle_{i\in I}$ in the sense of 214L, so that $Y=X\times I$ and $\nu E=\sum_{i\in I}\nu_i\{x:(x,i)\in E\}$ for those $E\subseteq Y$ for which the sum is defined. Give Y its disjoint-union topology, that is, the product topology if I is given the discrete topology; then it is easy to check that ν is locally finite (see 411Xh) and inner regular with respect to the closed sets (see 412Xm). For $h\in[0,\infty]^Y$ and $x\in X$ set $(Th)(x)=\sum_{i\in I}h(x,i)$; observe that $T(h+h')=Th+Th'$ and $T(\alpha h)=\alpha Th$ for all h, $h':Y\to[0,\infty]$ and $\alpha\ge0$. Now, for any $A\subseteq X\times[0,\infty[$, we have

$$c(A)=\inf\{\int h\,d\nu:h:Y\to[0,\infty]\text{ is T-measurable},$$
$$\alpha\le Th(x)\text{ whenever }(x,\alpha)\in A\}$$

(because $\int h\,d\nu=\sum_{i\in I}\int h(x,i)\nu_i(dx)$ for non-negative h, by 214M)

$$= \inf\{\int h\,d\nu : h : Y \to [0,\infty] \text{ is T-measurable, } A \subseteq \Omega_{Th}\}.$$

(β) Of course $c : \mathcal{P}(X \times [0,\infty[) \to [0,\infty]$ is non-decreasing. To see that it is sequentially order-continuous on the left, I show in fact that if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of $X \times [0,\infty]$ with union A, and $\gamma = \sup_{n\in\mathbb{N}} c(A_n)$ is finite, then there is a T-measurable $h : Y \to [0,\infty]$ such that $\alpha \le Th(x)$ whenever $(x,\alpha) \in A$ and $\int h\,d\nu = \gamma$. **P** Surely $c(A) \ge \gamma$. For each $n \in \mathbb{N}$ we have a T-measurable $h_n : Y \to [0,\infty]$ such that $\int h_n d\nu \le \gamma + 2^{-n}$ and $A_n \subseteq \Omega_{Th_n}$. By Komlós' theorem (276H), there is a strictly increasing sequence $\langle n(k)\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\lim_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^m h_{n(k)}$ is defined ν-a.e.; set $h = \limsup_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^m h_{n(k)}$. Then $h : Y \to [0,\infty]$ is T-measurable, and $h =_{\text{a.e.}} \liminf_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^m h_{n(k)}$. By Fatou's Lemma,

$$\int h\,d\nu \le \liminf_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^m \int h_{n(k)}d\nu \le \gamma,$$

while if $j \in \mathbb{N}$ and $(x,\alpha) \in A_j$. $\alpha \le Th_{n(k)}(x)$ for every $k \ge j$, so

$$\alpha \le \liminf_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^m Th_{n(k)}(x) \le \limsup_{m\to\infty} \sum_{i\in I} \frac{1}{m+1}\sum_{k=0}^m h_{n(k)}(x,i)$$

$$\le \sum_{i\in I} \limsup_{m\to\infty} \frac{1}{m+1}\sum_{k=0}^m h_{n(k)}(x,i)$$

(because I is finite)

$$= \sum_{i\in I} h(x,i) = Th(x).$$

Thus $A \subseteq \Omega_{Th}$, so

$$c(A) \le \int h\,d\nu \le \gamma \le c(A)$$

and we have equality. **Q**

(γ) Now suppose that $K \subseteq X \times [0,\infty[$ is compact, and $\epsilon > 0$. Set $L = \pi_1[K]$, where $\pi_1 : X \times [0,\infty[\to X$ is the canonical map; then $L \subseteq X$ and $L \times I \subseteq Y$ are compact. Because ν is locally finite, there is an open set $H \subseteq Y$ such that $L \times I \subseteq H \in \mathrm{T}$ and νH is finite (see 411Ga). Let ν_H be the subspace measure induced by ν on H, and T_H its domain; then ν_H is totally finite and inner regular with respect to the closed sets (412Pc), therefore outer regular with respect to the open sets (411D). Let $h : Y \to [0,\infty]$ be a T-measurable function such that $A \subseteq \Omega_{Th}$ and $\int h\,d\nu \le c(K) + \epsilon$. Set $h_1(y) = h(y) + \frac{\epsilon}{\nu H}$ for $y \in H$; then $\int_H h_1 d\nu_H \le c(K) + 2\epsilon$. By 412Wa, there is a lower semi-continuous T_H-measurable $g_1 : H \to [0,\infty]$ such that $h_1 \le g_1$ and $\int_H g_1 d\nu_H \le c(K) + 3\epsilon$. Extend g_1 to a function $g : Y \to [0,\infty]$ by setting $g(y) = 0$ for $y \in Y \setminus H$; then g is T-measurable and lower semi-continuous and $\int g\,d\nu \le c(K) + 3\epsilon$. Moreover, if $(x,\alpha) \in K$, then

$$Tg(x) > T(h \times \chi H)(x) = Th(x)$$

(because $\{x\} \times I \subseteq H$)

$$\ge \alpha,$$

so $K \subseteq \Omega'_{Tg}$.

The point is that Ω'_{Tg} is open in $X \times [0,\infty[$. **P** If $x \in X$ and $0 \le \alpha < Tg(x) = \sum_{i\in I} g(x,i)$, let $\langle\alpha_i\rangle_{i\in I}$ be such that $0 \le \alpha_i < g(x,i)$ for each $i \in I$ and $\sum_{i\in I} \alpha_i = \alpha' > \alpha$. Set $G = \bigcap_{i\in I}\{z : z \in X,\ g(z,i) > \alpha_i\}$; then G is an open subset of X, and $(x,\alpha) \in G \times [0,\alpha'[\subseteq \Omega'_{Tg}$. Thus $(x,\alpha) \in \operatorname{int}\Omega'_g$; as (x,α) is arbitrary, Ω'_{Tg} is open. **Q**

Since $c(\Omega'_{Tg})$ is surely less than or equal to $\int g\,d\nu$, and ϵ is arbitrary, we have

$$c(K) = \inf\{c(U) : U \subseteq X \times [0,\infty[\text{ is open and } K \subseteq U\}.$$

Thus all the conditions of 432Ja are satisfied, and c is a Choquet capacity.

(ii) We need consider only the case $c(A) < \infty$, which is dealt with in (i-β) above, if we take $A_n = A$ for every n.

(b)(i) For $g : X \to [-\infty, \infty]$, set

$$p(g) = \inf\{\sum_{i\in I}\int h_i d\nu_i : h_i \in [0,\infty]^X \text{ is } \operatorname{dom}\nu_i\text{-measurable for each } i \in I,$$
$$|g| \le \sum_{i\in I} h_i\}$$
$$= \inf\{\int h\,d\nu : h : Y \to [0,\infty] \text{ is T-measurable, } \Omega_{|g|} \subseteq \Omega_{Th}\} = c(\Omega_{|g|}).$$

Then $p(\alpha g) = |\alpha| p(g)$ whenever $g \in [-\infty,\infty]^X$ and $\alpha \in \mathbb{R}$, $p(g_1) \le p(g_2)$ whenever $|g_1| \le |g_2|$, and $p(g_1 + g_2) \le p(g_1) + p(g_2)$ for all g_1, $g_2 : X \to [-\infty,\infty]$; so if we set $V = \{g : g \in \mathbb{R}^X,\ p(g) < \infty\}$, V is a solid linear subspace of $\mathbb{R}^X$ and $p\restriction V$ is a seminorm.

Suppose that μ is a Radon measure on X and $\mu \le \nu_i$ for every $i \in I$. Then $\int f d\mu \le p(f)$. **P** Because μ measures every K-analytic set (432A), $\int f d\mu$ is defined. If $p(f) = \infty$ then of course $\int f d\mu \le p(f)$. Otherwise, for any $\gamma > p(f)$, we have $\operatorname{dom}\nu_i$-measurable functions $h_i : X \to [0,\infty]$ such that $f \le \sum_{i\in I} h_i$ and $\sum_{i\in I}\int h_i d\nu_i \le \gamma$. But now $\int h_i d\mu$ is defined and less than or equal to $\int h_i d\nu_i$ for each i (234Qc), so $\int f d\mu \le \sum_{i\in I}\int h_i d\mu \le \gamma$. As γ is arbitrary, $\int f d\mu \le p(f)$. **Q**

(ii)(α) In the other direction, suppose that $\gamma < p(f)$, and set $A = \{(x,\alpha) : 0 < \alpha < f(x)\}$; then

$$A = \bigcup_{q\in\mathbb{Q}}\{(x,\alpha) : f(x) \ge q > \alpha > 0\}$$

is K-analytic (422Ge, 422Hc, 423Ba, 423C). On the other hand, for any $h : Y \to [0,\infty]$, $A \subseteq \Omega_{Th}$ iff $\Omega_f \subseteq \Omega_{Th}$. So

$$c(A) = c(\Omega_f) = p(f) > \gamma.$$

By Choquet's theorem 432K, there is a compact set $K \subseteq A$ such that $c(K) > \gamma$. Set

$$f_1(x) = \sup(\{0\} \cup K[\{x\}])$$

for $x \in X$. As in (a-i-γ) above, we have for any $i \in I$ an open set G including $L_0 = \pi_1[K]$ such that $\nu_i G$ is defined and finite, so χL_0 and f_1 belong to V. By the Hahn-Banach theorem (4A4Da), there is a linear functional $\theta : V \to \mathbb{R}$ such that $|\theta(g)| \le p(g)$ for every $g \in V$ and $\theta(f_1) = p(f_1)$. Since $|\theta(g)| \le p(g_0)$ whenever $|g| \le g_0$, θ is order-bounded, and if θ^+ is its positive part (355Eb), we shall still have $\theta^+(g) \le p(g)$ for every $g \in V$ and $\theta^+(f_1) = p(f_1)$.

(β) Set $\mu_0 C = \theta^+(\chi(C \cap L_0))$ for $C \subseteq X$. Then $\mu_0 : \mathcal{P}X \to [0,\infty[$ is additive. By 416K, there is a Radon measure μ on X such that $\mu L \ge \mu_0 L$ for every compact $L \subseteq X$ and $\mu G \le \mu_0 G$ for every open $G \subseteq X$. Now $\operatorname{dom}\nu_i \subseteq \operatorname{dom}\mu$ for every $i \in I$. **P** Suppose that $E \in \operatorname{dom}\nu_i$. Let $L \subseteq X$ be compact. Then there is an open set $G_0 \supseteq L$ such that $\nu_i G_0$ is defined and finite. Take any $\delta > 0$. Because the subspace measure induced by ν_i on G_0 is totally finite and inner regular with respect to the closed sets, there are a closed set F and an open set G, both measured by ν_i, such that $F \subseteq E \cap G_0 \subseteq G$ and $\nu_i(G \setminus F) \le \delta$. In this case

$$\mu(G \setminus F) \le \mu_0(G \setminus F) = \theta^+(\chi(L_0 \cap G \setminus F)) \le p(\chi(G \setminus F)) \le \int \chi(G \setminus F) d\nu_i \le \delta.$$

So

$$\mu^*(E \cap G_0) \le \mu G \le \mu F + \delta \le \mu_*(E \cap G_0) + \delta;$$

as δ is arbitrary, $\mu^*(E \cap G_0) = \mu_*(E \cap G_0)$ and μ measures $E \cap G_0$ (413Ef), and therefore also measures $E \cap L = E \cap G_0 \cap L$. As L is arbitrary, μ measures E (412Ja). **Q**

In fact, $\mu \le \nu_i$. **P** If ν_i measures E and $L \subseteq X$ is compact, the arguments just above show that for any $\delta > 0$ there is an open set $G \supseteq E \cap L$ such that $\nu_i G \le \nu_i E + \delta$, so that

$$\mu(E \cap L) \le \mu G \le \mu_0 G = \theta^+(\chi(G \cap L_0)) \le p(\chi(G \cap L_0)) \le \nu_i G \le \nu_i E + \delta.$$

As L and δ are arbitrary, $\mu E \le \nu_i E$. **Q**

(γ) To estimate $\int f d\mu$, recall that $\theta^+(f_1) > \gamma$, while $\theta^+(\chi L_0)$ is finite. There is therefore an $\eta > 0$ such that $\eta\theta^+(\chi L_0) \le \theta^+(f_1) - \gamma$ and $\theta^+(f_2) \ge \gamma$, where $f_2 = (f_1 - \eta\chi L_0)^+ = (f_1 - \eta\chi X)^+$. For $k \in \mathbb{N}$ set $F_k =$

$\pi_1[K \cap [(k+1)\eta, \infty[]$, so that each F_k is a compact subset of L_0 and $f_2 \le \sum_{k=0}^m \eta\chi F_k \le f$, where $m \in \mathbb{N}$ is such that $K \subseteq X \times [0, m\eta]$. Now

$$\gamma \le \theta^+(f_2) \le \theta^+(\sum_{k=0}^m \eta\chi F_k) = \eta\sum_{k=0}^m \theta^+(\chi F_k)$$
$$= \eta\sum_{k=0}^m \mu_0 F_k \le \eta\sum_{k=0}^m \mu F_k \le \int f d\mu.$$

(δ) As γ is arbitrary,

$$\sup\{\int f d\mu : \mu \text{ is a Radon measure on } X,\ \mu \le \nu_i \text{ for every } i \in I\} \ge p(f)$$

and we must have equality. This completes the proof.

457N Remarks It may not be quite obvious how close the domination requirement '$\mu \le \nu_i$ for every $i \in I$' is to the marginal requirement '$\nu_i = \mu\pi_i^{-1}$ for every $i \in I$', so I spell out the correspondence. Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, $X = \prod_{i\in I} X_i$, and $\pi_i : X \to X_i$ the canonical map for each i.

(a) For each $i \in I$ we have a (unique) pull-back probability measure ν_i on X with domain $\{\pi_i^{-1}[E] : E \in \Sigma_i\}$ such that the image measure $\nu_i\pi_i^{-1}$ is μ_i (see 234F). Now it is elementary to check that, for a measure μ on X, $\mu \le \nu_i$ iff $\mu\pi_i^{-1} \le \mu_i$; and if μ is required to be a probability measure, then $\mu \le \nu_i$ iff μ extends ν_i iff $\mu\pi_i^{-1}$ extends μ_i.

(b) We find also that if $\mu \le \nu_i$ for every i, then there is a probability measure μ' on X such that $\mu \le \mu'$ and μ' extends ν_i for every i. **P** Set $\gamma = \mu X$. If $\gamma = 1$, set $\mu' = \mu$. Otherwise, for each $i \in I$, set $\lambda_i E = \frac{1}{1-\gamma}(\mu_i E - \mu\pi_i^{-1}[E])$ for $E \in \Sigma_i$. Then λ_i is a probability measure on X_i; let $\lambda = \prod_{i\in I} \lambda_i$ be the product measure, and set $\mu' = \mu + (1-\gamma)\lambda$. Then

$$\mu'\pi_i^{-1} = \mu\pi_i^{-1} + (1-\gamma)\lambda\pi_i^{-1} = \mu\pi_i^{-1} + (1-\gamma)\lambda_i = \mu_i$$

and μ' extends ν_i for each i. **Q**

(c) In the simplest intended applications, therefore, in which we have two Radon probability spaces (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) and a profit function $f : X \to [0, \infty[$, and we are looking for a Radon probability measure μ on $X = X_1 \times X_2$, with marginals μ_1 and μ_2, maximising $\int f d\mu$, then we can seek to apply 457Mb with the pull-back measures ν_1 and ν_2 of (a) here to see that the optimum is

$$\inf\{\int h_1 d\mu_1 + \int h_2 d\mu_2 : f(x_1, x_2) \le h_1(x_1) + h_2(x_2)\ \forall\, x_1 \in X_1,\ x_2 \in X_2\}.$$

If the process of part (b-ii) of the proof of 457M leads to a more or less optimal measure μ which is not itself a probability measure, we can increase it to μ' with $\mu'\pi_i^{-1}$ extending μ_i for each i; and in this case we shall have $\mu'\pi_i^{-1} = \mu_i$ for each i, by 418I and 416E, as usual. Of course we shall need to confirm that $\int f d\mu'$ is defined, but in the context of 457Mb, this will automatically be so.

(d) There is an obvious parallel between the formulae of 457M and that in part (b-i) of the proof of 457L. Allowing for the change of direction, where an infimum in 457L corresponds to a supremum in 457M, the pattern of the duality is the same in both cases, and there is some overlap (457Xq). But the arguments of the two theorems – in particular, the proofs that we can get countably additive measures from the finitely additive measures provided by the Hahn-Banach theorem – are rather different.

457X Basic exercises (a) Let X be a non-empty set and $\langle\nu_i\rangle_{i\in I}$ a family of probability measures on X satisfying the conditions of Lemma 457A, taking $\mathfrak{A} = \mathcal{P}X$ and $\mathfrak{B}_i = \operatorname{dom}\nu_i$ for each i. Suppose that there is a totally finite measure θ on X such that θE is defined and greater than or equal to $\nu_i E$ whenever $i \in I$ and ν_i measures E. Show that there is a measure on X extending every ν_i. (*Hint*: 391E.)

(b) Find a set X and non-negative additive functionals μ_1, μ_2 defined on subalgebras of $\mathcal{P}X$ which agree on $\operatorname{dom}\mu_1 \cap \operatorname{dom}\mu_2$ but have no common extension to a non-negative additive functional. (*Hint*: take $\#(X) = 3$.)

(c) Let $\mathfrak{A}$ be a Boolean algebra and $\langle\nu_i\rangle_{i\in I}$ a family of non-negative finitely additive functionals, each ν_i being defined on a subalgebra $\mathfrak{B}_i$ of $\mathfrak{A}$. Show that if any finite number of the ν_i have a common extension to an additive functional on a subalgebra of $\mathfrak{A}$, then the whole family has a common extension to an additive functional on the whole algebra $\mathfrak{A}$.

(d) Set $X = \{0,1,2\}$ and in the algebra $\mathcal{P}X$ let $\mathfrak{B}_i$ be the subalgebra $\{\emptyset,\{i\},X\setminus\{i\},X\}$ for each i. Let $\nu_i:\mathfrak{B}_i\to[0,1]$ be the additive functional such that $\nu_i\{i\}=\frac{1}{2}$, $\nu_iX=1$. Show that any pair of ν_0, ν_1, ν_2 have a common extension to an additive functional on $\mathcal{P}X$, but that the three together have no such extension.

(e) Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, and $\nu:\mathfrak{B}\to[0,\infty[$, $\theta:\mathfrak{A}\to[0,\infty[$ additive functionals such that $\nu b\le\theta b$ for every $b\in\mathfrak{B}$. Show directly, without using either 457D or 391F, that there is an additive functional $\mu:\mathfrak{A}\to[0,\infty[$, extending ν, such that $\mu a\le\theta a$ for every $a\in\mathfrak{A}$. (*Hint*: first consider the case in which $\mathfrak{A}$ is the algebra generated by $\mathfrak{B}\cup\{c\}$.)

>(f) Let $(Y_1,\mathfrak{S}_1,\mathrm{T}_1,\nu_1)$ and $(Y_2,\mathfrak{S}_2,\mathrm{T}_2,\nu_2)$ be Radon probability spaces and $X\subseteq Y_1\times Y_2$ a closed set. Show that the following are equiveridical: (i) there is a measure on X such that the coordinate map from X to Y_i is inverse-measure-preserving for both i; (ii) there is a Radon measure on X such that the coordinate map from X to Y_i is inverse-measure-preserving for both i; (iii) for every compact $K\subseteq Y_1$, $\nu_1K\le\nu_2^*(X[K])$. (*Hint*: for (iii)⇒(ii), use 457C to show that there is a finitely additive functional ν on $\mathcal{P}X$ of the required type; now observe that ν must give large mass to compact subsets of X, and apply 413S.)

>(g) Suppose that $\mathfrak{A}$ is a Boolean algebra, $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and $I\subseteq\mathfrak{A}$ a finite set; let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by $I\cup\mathfrak{B}$ and $\nu:\mathfrak{C}\to[0,\infty[$ a finitely additive functional. (i) Show that if $\nu{\restriction}\mathfrak{B}$ is completely additive then ν is completely additive. (ii) Show that if $\mathfrak{A}$ is Dedekind σ-complete, $\mathfrak{B}$ is a σ-subalgebra and $\nu{\restriction}\mathfrak{B}$ is countably additive then ν is countably additive.

(h) Let (X,Σ,μ) be a probability space, $\mathcal{A}$ a finite family of subsets of X and T the subalgebra of $\mathcal{P}X$ generated by $\Sigma\cup\mathcal{A}$. Show that if $\nu:\mathrm{T}\to[0,1]$ is a finitely additive functional extending μ, then ν is countably additive.

(i) Let (X,Σ,μ) be a probability space, $\langle A_i\rangle_{i\in I}$ a partition of X and $\langle\alpha_i\rangle_{i\in I}$ a family in $[0,1]$ summing to 1. Show that the following are equiveridical: (i) there is a measure ν on X, extending μ, such that $\nu A_i=\alpha_i$ for every $i\in I$; (ii) there is a finitely additive functional $\nu:\mathcal{P}X\to[0,1]$, extending μ, such that $\nu A_i=\alpha_i$ for every $i\in I$; (iii) $\mu_*(\bigcup_{i\in J}A_i)\le\sum_{i\in J}\alpha_i$ for every $J\subseteq I$; (iv) $\mu^*(\bigcup_{i\in J}A_i)\ge\sum_{i\in J}\alpha_i$ for every finite $J\subseteq I$. (*Hint*: for (ii)⇒(i) use 457Xh.)

(j) Let $X\subseteq[0,1]^2$ be a Lebesgue measurable set such that $X\cap(E\times F)$ is not negligible for any non-negligible sets E, $F\subseteq[0,1]$. (For the construction of such sets, see the notes to §325.) Show that there is a Radon measure on X such that both the coordinate projections from X to $[0,1]$ are inverse-measure-preserving, where $[0,1]$ is given Lebesgue measure. (*Hint*: show that there is a measure-preserving bijection ϕ between conegligible subsets of $[0,1]$ which is covered by X; ϕ can be taken to be of the form $\phi(x)=x-\alpha_n$ for $x\in E_n$.)

(k) Set $X=\{(t,2t):0\le t\le\frac{1}{2}\}\cup\{(t,2t-1):\frac{1}{2}\le t\le 1\}$. Show that there is a Radon measure on X for which both the coordinate maps onto $[0,1]$ are inverse-measure-preserving, but that X does not include the graph of any measure-preserving bijection between conegligible subsets of $[0,1]$.

(l) Let X be the eighth-sphere $\{x:x\in[0,1]^3,\ \|x\|=1\}$. Show that there is a measure on X such that all three coordinate maps from X onto $[0,1]$ are inverse-measure-preserving. (*Hint*: 265Xe.)

(m) Set $X=\{x:x\in[0,1]^3,\ \xi_1+\xi_2+\xi_3=\frac{3}{2}\}$. Show that there is a measure on X such that all the coordinate maps from X onto $[0,1]$ are inverse-measure-preserving. (*Hint*: note that X is a regular hexagon; try one-dimensional Hausdorff measure on its boundary.)

(n) Explain how to adapt the example in 457J to provide a family $\langle\mu_i\rangle_{i\in I}$ of probability measures on a set X such that (i) $\langle\mu_i\rangle_{i\in I}$ is upwards-directed, in the sense of 457G (iii) there is no measure on X extending μ_i for every $i\in I$.

(o) Let X be a topological space and P_{qR} the set of quasi-Radon probability measures on X. For $\mu,\nu\in P_{\mathrm{qR}}$, write $Q(\mu,\nu)$ for the set of quasi-Radon probability measures on $X\times X$ which have marginal measures μ on the first copy of X, ν on the second. (i) For a bounded continuous pseudometric ρ on X, set $\rho_{\mathrm{W}}(\mu,\nu)=\inf\{\int\rho(x,y)\lambda(d(x,y)):\lambda\in Q(\mu,\nu)\}$. Show that ρ_{W} is a pseudometric on P_{qR}. (ii) Show that if X is completely regular and P is a family of bounded pseudometrics defining the topology of X, then $\{\rho_{\mathrm{W}}:\rho\in\mathrm{P}\}$ defines the narrow topology of P_{qR}.

(p) Suppose that X, $\langle\nu_i\rangle_{i\in I}$ and $c:\mathcal{P}(X\times[0,\infty[)\to[0,\infty]$ are as in 457M. (i) Show that c is a submeasure. (ii) Show that if every ν_i is outer regular with respect to the open sets, then c is an outer regular Choquet capacity.

(q) Show that if the metric ρ is bounded, then 457Lc can be deduced from 457Mb and part (b-i) of the proof of 457L.

(r) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\le n}$ be a finite family of Radon probability spaces, $X=\prod_{i\in I}X_i$, and $f:X\to\mathbb{R}$ a bounded Baire measurable function. Show that

$$\begin{aligned}\inf\{\int fd\mu:\mu &\text{ is a Radon measure on } X \text{ with marginal measure } \mu_i \text{ on each } X_i\}\\ &=\sup\{\sum_{i=0}^n\int h_id\mu_i:h_i\in\ell^\infty(X_i)\text{ is }\Sigma_i\text{-measurable for each } i,\\ &\qquad\qquad\sum_{i=0}^n h_i(\xi_i)\le f(x)\text{ whenever } x=(\xi_0,\dots,\xi_n)\in X\}.\end{aligned}$$

(*Hint*: reduce to the case in which every X_i is K_σ.)

457Y Further exercises (a) Show that for any $n\ge2$ there are a finite set X and a family $\langle\mu_i\rangle_{i\le n}$ of measures on X such that $\{\mu_i:i\le n,\,i\ne j\}$ have a common extension to a measure on X for every $j\le n$, but the whole family $\{\mu_i:i\le n\}$ has no such extension.

(b) Show that the example in 457H has the property: if f_i is a ν_i-integrable real-valued function for each i, and $\int f_1d\nu_1+\int f_2d\nu_2<1$, then there is an $(x,y)\in\operatorname{dom}f_1\cap\operatorname{dom}f_2$ such that $f_1(x,y)+f_2(x,y)<1$.

(c) Suppose we replace the set X in 457H with $X'=X\cup\{(x,x):x\in[0,\frac12]\}$, and write ν_i' for the measures on X' defined by the coordinate projections. Show that (i) if f_i is a ν_i'-integrable real-valued function on X' for each i, and $\int f_1d\nu_1'+\int f_2d\nu_2'\le1$, then there is an $(x,y)\in\operatorname{dom}f_1\cap\operatorname{dom}f_2$ such that $f_1(x,y)+f_2(x,y)\le1$ (ii) there is no measure on X' extending both ν_i'.

(d) In 457Xm, show that there are many Radon measures on X such that all the coordinate maps from X onto $[0,1]$ are inverse-measure-preserving.

(e) Give an example of a compact Hausdorff space X, a sequence $\langle\nu_n\rangle_{n\in\mathbb{N}}$ of tight probability measures on X, and a K_σ set $E\subseteq X$ such that

$$\inf\{\textstyle\sum_{n=0}^\infty\int h_nd\nu_n:\chi E\le\sum_{n=0}^\infty h_n\}=1,$$

$$\sup\{\mu E:\mu\text{ is a Radon measure on } X \text{ and } \mu\le\nu_n\text{ for every } n\in\mathbb{N}\}\le\frac12.$$

457Z Problems Give $[0,1]$ Lebesgue measure.

(a) Characterize the sets $X\subseteq[0,1]^2$ for which there is a measure on X such that both the projections from X to $[0,1]$ are inverse-measure-preserving.

(b) Set $X=\{x:x\in[0,1]^3,\ \|x\|=1\}$. Is there more than one Radon measure on X for which all three coordinate maps from X onto $[0,1]$ are inverse-measure-preserving? (See 457Xl, 457Yd.)

457 Notes and comments In the context of this section, as elsewhere (compare 391E-391G and 391J), finitely additive extensions, as in 457A-457D, generally present easier problems than countably additive extensions. So techniques for turning additive functionals into measures (391D, 413K, 413S, 416K, 454C, 454D, 457E, 457G, 457Lb, 457Mb, 457Xi) are very valuable. Note that 457D offers possibilities in this direction: if θ there is countably additive, μ also will be (457Xa).

457H and 457J demonstrate obstacles which can arise when seeking countably additive extensions even when finitely additive extensions give no difficulty. For finitely additive extensions a problem can arise at any finite number of measures (see 457Ya), but there is no further obstruction with infinite families (457Xc). For countably additive measures we have a positive result (457G) only under very restricted circumstances; relaxing any of the hypotheses can lead to failure (457J, 457Xn). Even in the apparently concrete case in which we have an open or closed set $X \subseteq [0,1]^2$ and we are seeking a measure on X with prescribed image measures on each coordinate, there can be surprises (457H, 457Xj, 457Xk), and I know of no useful description of the sets for which such a measure can be found (457Za).

The two-dimensional case has a special feature: when verifying the conditions (ii) or (iii) in 457A, or the condition (ii) of 457B, it is enough to consider only one set associated with each coordinate (457C). Put another way, in conditions (iv) and (v) of 457A it is enough to examine indicator functions. This is not the case as soon as we have three coordinates (457I). Compare 457A(ii)-(iii) with the definition of 'intersection number' of an indexed family in a Boolean algebra (391H), where we had to allow repetitions for essentially the same reason.

In 457K-457L, we can of course work with τ-additive Borel measures in place of quasi-Radon measures, as in 437M. The essential content of 457L is already displayed in the case of separable X, in which case all Borel measures are τ-additive, and we can fractionally simplify our hypotheses; indeed this is true whenever X has measure-free weight (438J).

The functional ρ_{W} of 457K-457L is a kind of $[0,\infty]$-valued metric; see 4A2T for another occasion on which it would have saved explanation if the definition of 'metric' allowed infinite distances. In 457Lb we think of the metric ρ as representing a cost to be minimised, and in 457Mb we think of f as a profit to be maximised; since both arguments rely on the functions being non-negative, they cannot be simply inverted unless ρ or f is bounded above (as in 457Xq), and there is a further complication from the asymmetric nature of the condition '$\{x : f(x) \ge \alpha\}$ is K-analytic' in 457M. However, for the primary applications, as in 457Xr, this is not a problem. Observe that the same pattern has already appeared in 457A(iv)-(v).

458 Relative independence and relative products

Stochastic independence is one of the central concepts of probability theory, and pervades measure theory. We come now to a generalization of great importance. If X_1, X_2 and Y are random variables, we may find that X_1 and X_2 are 'relatively independent over Y', or 'independent when conditioned on Y', in the sense that if we know the value of Y, then we learn nothing further about one of the X_i if we are told the value of the other. For any stochastic process, where information comes to us piecemeal, this idea is likely to be fundamental. In this section I set out a general framework for discussion of relative independence (458A), introducing relative distributions (458I) and relative independence in measure algebras (458L-458M). In the second half of the section I look at 'relative product measures' (458N, 458Q), giving the basic existence theorems (458O, 458S, 458T).

458A Relative independence Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ.

(a) I say that a family $\langle E_i\rangle_{i\in I}$ in Σ is **relatively (stochastically) independent** over T if whenever $J \subseteq I$ is finite and not empty, and g_i is a conditional expectation of χE_i on T for each $i \in J$, then $\mu(F \cap \bigcap_{i\in J} E_i) = \int_F \prod_{i\in J} g_i d\mu$ for every $F \in \mathrm{T}$; that is, $\prod_{i\in J} g_i$ is a conditional expectation of $\chi(\bigcap_{i\in J} E_i)$ on T. (Note that this does not depend on which conditional expectations g_i we take, since any two conditional expectations of χE_i must be equal almost everywhere.) A family $\langle\Sigma_i\rangle_{i\in I}$ of subalgebras of Σ is **relatively independent** over T if $\langle E_i\rangle_{i\in I}$ is relatively independent over T whenever $E_i \in \Sigma_i$ for every $i \in I$.

(b) I say that a family $\langle f_i\rangle_{i\in I}$ in $\mathcal{L}^0(\mu)$ (the space of almost-everywhere-defined virtually measurable real-valued functions, or 'random variables') is **relatively independent** over T if $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T (with respect to the completion of μ), where Σ_i is the σ-algebra defined by f_i in the sense of 272C, that is, the σ-algebra generated by $\{f_i^{-1}[F] : F \subseteq \mathbb{R} \text{ is a Borel set}\}$.

(c) I remark at once that a family of σ-algebras or random variables is relatively independent iff every finite subfamily is (cf. 272Bb).

(d) It will be convenient to have a shorthand referring to lattices of σ-algebras of sets. If Σ, T are algebras of subsets of a set X, I will write $\Sigma \vee \mathrm{T}$ for the σ-algebra of subsets of X generated by $\Sigma \cup \mathrm{T}$; similarly, if $\langle\Sigma_i\rangle_{i\in I}$ is a family of algebras of subsets of X, then $\bigvee_{i\in I} \Sigma_i$ will be the σ-algebra generated by $\bigcup_{i\in I} \Sigma_i$. Note that the functions $\vee$, $\bigvee$ here are always supposed to yield σ-algebras, even if we start with algebras which are not closed under countable unions, so that $\Sigma \vee \Sigma$ could in principle be strictly larger than Σ.

458B There are some surprising results at the very beginning of the theory of relative independence; see 458Xa, for instance. On the positive side, we have the following facts.

Lemma Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of subalgebras of Σ such that $\mathrm{T}\subseteq\bigcup_{i\in I}\Sigma_i$. Suppose that whenever $J\subseteq I$ is finite and not empty, $E_i\in\Sigma_i$ and g_i is a conditional expectation of χE_i on T for each $i\in J$, then $\mu(\bigcap_{i\in J}E_i)=\int\prod_{i\in J}g_i d\mu$. Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

proof Take $F\in\mathrm{T}$, a finite non-empty $J\subseteq I$ and $E_i\in\Sigma_i$ for $i\in J$. Let $j\in I$ be such that $F\in\Sigma_j$. Set $K=J\cup\{j\}$; if $j\notin J$, set $E_j=X$. Now set $E'_j=E_j\cap F$ and $E'_i=E_i$ for $i\in K\setminus\{j\}$.

For $i\in K$, let g_i be a conditional expectation of χE_i on T. Set $g'_j=g_j\times\chi F$ and $g'_i=g_i$ for $i\in K\setminus\{j\}$; then g'_i is a conditional expectation of $\chi E'_i$ for each $i\in K$. So we have

$$\mu(F\cap\bigcap_{i\in J}E_i)=\mu(\bigcap_{i\in K}E'_i)=\int\prod_{i\in K}g'_i d\mu=\int_F\prod_{i\in J}g_i d\mu.$$

As F and $\langle E_i\rangle_{i\in J}$ are arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

458C Proposition Let (X,Σ,μ) be a probability space, $\mathbb{T}$ a non-empty upwards-directed family of σ-subalgebras of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ which is relatively independent over T for every $\mathrm{T}\in\mathbb{T}$. Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over $\bigvee\mathbb{T}$.

proof (a) Suppose first that $\mathbb{T}$ is countable; because it is upwards-directed, there is a non-decreasing sequence $\langle\mathrm{T}_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{T}$ such that $\bigcup\mathbb{T}=\bigcup_{n\in\mathbb{N}}\mathrm{T}_n$ and $\bigvee\mathbb{T}=\bigvee_{n\in\mathbb{N}}\mathrm{T}_n$. Take a non-empty finite set $J\subseteq I$ and $E_i\in\Sigma_i$ for $i\in J$; set $E=\bigcap_{i\in J}E_i$. For $i\in J$, let g_{ni} be a conditional expectation of χE_i on T_n for each n; then $g_i=\lim_{n\to\infty}g_{ni}$ is a conditional expectation of χE_i on $\bigvee\mathbb{T}$ (275I). Similarly, if h_n is a conditional expectation of χE on T_n for each n, $h=\lim_{n\to\infty}h_n$ is a conditional expectation of χE on $\bigvee\mathbb{T}$. Since $\langle E_i\rangle_{i\in J}$ is relatively independent over T_n, $h_n=_{\text{a.e.}}\prod_{i\in J}g_{ni}$ for each n; accordingly $h=_{\text{a.e.}}\prod_{i\in J}h_i$, and $\prod_{i\in J}h_i$ is a conditional expectation of χE on $\bigvee\mathbb{T}$. As $\langle E_i\rangle_{i\in J}$ is arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over $\bigvee\mathbb{T}$.

(b) For the general case, take a non-empty finite $J\subseteq I$ and $E_i\in\Sigma_i$ for $i\in J$; set $E=\bigcap_{i\in I}E_i$. For each $i\in J$, let $g_i:X\to[0,1]$ be a $\bigvee\mathbb{T}$-measurable conditional expectation of χE_i on $\bigvee\mathbb{T}$, and $g:X\to[0,1]$ a $\bigvee\mathbb{T}$-measurable conditional expectation of χE on $\bigvee\mathbb{T}$. Then for every $i\in J$ and $q\in\mathbb{Q}$ there is a countable set $\mathbb{T}_{iq}\subseteq\mathbb{T}$ such that $\{x:g_i(x)\ge q\}\in\bigvee\mathbb{T}_{iq}$; similarly, there is for each $q\in\mathbb{Q}$ a countable set $\mathbb{T}'_q\subseteq\mathbb{T}$ such that $\{x:g(x)\ge q\}\in\bigvee\mathbb{T}'_q$. Let $\tilde{\mathbb{T}}$ be a countable upwards-directed subset of $\mathbb{T}$ including $\bigcup_{i\in J,q\in\mathbb{Q}}\mathbb{T}_{iq}\cup\bigcup_{q\in\mathbb{Q}}\mathbb{T}'_q$. Then every g_i is $\bigvee\tilde{\mathbb{T}}$-measurable, so is a conditional expectation of χE_i on $\bigvee\tilde{\mathbb{T}}$; similarly, g is a conditional expectation of χE on $\bigvee\tilde{\mathbb{T}}$. By (i), $g=_{\text{a.e.}}\prod_{i\in J}g_i$, so $\prod_{i\in J}g_i$ is a conditional expectation of χE on $\bigvee\mathbb{T}$. As $\langle E_i\rangle_{i\in J}$ is arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over $\bigvee\mathbb{T}$, as claimed.

458D Proposition Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ and $\langle\Sigma_i\rangle_{i\in I}$ a family of subalgebras of Σ which is relatively independent over T.

(a) If $J\subseteq I$ and Σ'_i is a subalgebra of Σ_i for $i\in J$, then $\langle\Sigma'_j\rangle_{i\in J}$ is relatively independent over T.

(b) Set $\Sigma^*_i=\Sigma_i\vee\mathrm{T}$ for $i\in I$. Then $\langle\Sigma^*_i\rangle_{i\in I}$ is relatively independent over T.

(c) If $\mathcal{E}\subseteq\bigcup_{i\in I}\Sigma_i$, then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over the σ-algebra generated by $\mathrm{T}\cup\mathcal{E}$.

proof (a) Immediate from the definition in 458Aa.

(b)(i) Suppose that $F_0\in\mathrm{T}$ and that Σ'_i is the algebra generated by $\Sigma_i\cup\{F_0\}$ for each $i\in I$. Then $\langle\Sigma'_i\rangle_{i\in I}$ is relatively independent over T. **P** Suppose that $J\subseteq I$ is finite and not empty, and that $E_i\in\Sigma'_i$ for each $i\in J$. For $i\in I$, we can express E_i as $(G_i\cap F_0)\cup(H_i\setminus F_0)$, where G_i, $H_i\in\Sigma_i$. Let g_i, h_i be conditional expectations of χG_i, χH_i on T; then $f_i=g_i\times\chi F_0+h_i\times\chi(X\setminus F_0)$ is a conditional expectation of χE on T. Now, for any $F\in\mathrm{T}$, we have

$$\begin{aligned}\int_F\prod_{i\in J}f_i&=\int_F\prod_{i\in J}g_i\times\chi F_0+\prod_{i\in J}h_i\times\chi(X\setminus F_0)\\&=\int_{F\cap F_0}\prod_{i\in J}g_i+\int_{F\setminus F_0}\prod_{i\in J}h_i=\mu(F\cap\bigcap_{i\in J}G_i\cap F_0)+\mu(F\cap\bigcap_{i\in J}H_i\setminus F_0)\end{aligned}$$

(because the families $\langle G_i\rangle_{i\in J}$ and $\langle H_i\rangle_{i\in J}$ are both relatively independent over T)

$$= \mu(F \cap \bigcap_{i \in J} E_i).$$

As $\langle E_i \rangle_{i \in J}$ is arbitrary, $\langle \Sigma'_i \rangle_{i \in I}$ is relatively independent over T. **Q**

(ii) Suppose that $\mathcal{E} \subseteq \mathrm{T}$ is finite, and that Σ'_i is the algebra generated by $\Sigma_i \cup \mathcal{E}$ for each i. Then $\langle \Sigma'_i \rangle_{i \in I}$ is relatively independent over T. **P** Induce on $\#(\mathcal{E})$, using (i) for the inductive step. **Q**

(iii) Suppose that Σ'_i is the algebra generated by $\Sigma_i \cup \mathrm{T}$ for each $i \in I$. Then $\langle \Sigma'_i \rangle_{i \in I}$ is relatively independent over T. **P** If $J \subseteq I$ is finite and not empty, and $E_i \in \Sigma'_i$ for each $i \in J$, then there is a finite set $\mathcal{E} \subseteq \mathrm{T}$ such that E_i belongs to the algebra Σ''_i generated by $E_i \cup \mathcal{E}$ for every $i \in J$. By (ii), $\langle \Sigma''_i \rangle_{i \in I}$ is relatively independent over T, so $\langle E_i \rangle_{i \in J}$ is relatively independent over T; as $\langle E_i \rangle_{i \in J}$ is arbitrary, $\langle \Sigma'_i \rangle_{i \in I}$ is relatively independent over T. **Q**

(iv) Finally, suppose that $J \subseteq I$ is finite and not empty, that $E_i \in \Sigma^*_i$ for each $i \in J$, that $F \in \mathrm{T}$ and that $\epsilon > 0$. For $i \in J$, let Σ'_i be the algebra generated by $\Sigma_i \cup \mathrm{T}$; then there is an $E'_i \in \Sigma'_i$ such that $\mu(E'_i \triangle E_i) \le \epsilon$ (136H). Let g_i, g'_i be conditional expectations of χE_i, $\chi E'_i$ on T; we can arrange that they are all defined on the whole of X and take values in $[0, 1]$. Then

$$|\mu(F \cap \bigcap_{i \in J} E_i) - \int_F \prod_{i \in J} g_i| \le \sum_{i \in J} \mu(E_i \triangle E'_i) + |\mu(F \cap \bigcap_{i \in J} E'_i) - \int_F \prod_{i \in J} g'_i| + \int_F |\prod_{i \in J} g'_i - \prod_{i \in J} g_i|$$

$$\le \epsilon\#(J) + 0 + \int_F \sum_{i \in J} |g'_i - g_i|$$

((iii) above and 285O)

$$\le \epsilon\#(J) + \sum_{i \in J} \int |g'_i - g_i|$$

$$\le \epsilon\#(J) + \sum_{i \in J} \int |\chi E'_i - \chi E_i|$$

(233J or 242Je)

$$= \epsilon\#(J) + \sum_{i \in J} \mu(E'_i \triangle E_i) \le 2\epsilon\#(J).$$

As ϵ is arbitrary,

$$\mu(F \cap \textstyle\bigcap_{i \in J} E_i) = \int_F \prod_{i \in J} g_i.$$

As $\langle E_i \rangle_{i \in J}$ and F are arbitrary, $\langle \Sigma^*_i \rangle_{i \in I}$ is relatively independent.

(c) For any $\mathcal{E} \subseteq \bigcup_{i \in I} \Sigma_i$, write $\mathrm{T}_{\mathcal{E}}$ for the σ-algebra generated by $\mathrm{T} \cup \mathcal{E}$.

(i) Suppose that i, $j \in I$ are distinct, $E \in \Sigma_i$, g is a conditional expectation of χE on T, and $H \in \Sigma_j$. Then g is a conditional expectation of χE on $\mathrm{T}_{\{H\}}$. **P** Let h be a conditional expectation of χH on T. If $F \in \mathrm{T}$, then

$$\mu(F \cap H \cap E_i) = \int_F g \times h$$

(because Σ_j and Σ_i are relatively independent over T)

$$= \int_F g \times \chi H$$

(because $g \times h$ is a conditional expectation of $g \times \chi H$ on T, see 233Eg)

$$= \int_{F \cap H} g.$$

Similarly, $\mu(F \cap E_i \setminus H) = \int_{F \setminus H} g$. Now any $G \in \mathrm{T}_{\{H\}}$ is expressible as $(F_1 \cap H) \cup (F_2 \setminus H)$ where F_1, $F_2 \in \mathrm{T}$, so that

$$\mu(G\cap E)=\mu(F_1\cap E\cap H)+\mu(F_2\cap E\setminus H)=\int_{F_1\cap H}g+\int_{F_2\setminus H}g=\int_G g,$$

as required. **Q**

(ii) If $j\in I$ and $H\in\Sigma_j$, $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over $\mathrm{T}_{\{H\}}$.

P **(α)** Let $J\subseteq I$ be a non-empty finite set containing j, and $\langle E_i\rangle_{i\in J}$ a family such that $E_i\in\Sigma_i$ for $i\in J$. Set $K=J\setminus\{j\}$. For $i\in K$, let $g_i:X\to[0,1]$ be a T-measurable conditional expectation of χE_i on T. Then g_i is a conditional expectation of χE_i on $\mathrm{T}_{\{H\}}$, by (i). Let g_j be a conditional expectation of χE_j on $\mathrm{T}_{\{H\}}$, and g_j' a conditional expectation of $\chi(E_j\cap H)$ on T. Then, for any $F\in\mathrm{T}$,

$$\mu(F\cap H\cap\bigcap_{i\in J}E_i)=\mu(F\cap(E_j\cap H)\cap\bigcap_{i\in K}E_i)=\int_F g_j'\times\prod_{i\in K}g_i$$

(because $\langle\Sigma_i\rangle_{i\in J}$ is relatively independent over T)

$$=\int_F\chi(E_j\cap H)\times\prod_{i\in K}g_i$$

(233Eg again, because $\prod_{i\in K}g_i$ is bounded and T-measurable)

$$=\int_{F\cap H}\chi E_j\times\prod_{i\in K}g_i=\int_{F\cap H}g_j\times\prod_{i\in K}g_i$$

(because $\prod_{i\in K}g_i$ is bounded and $\mathrm{T}_{\{H\}}$-measurable). Similarly,

$$\mu(F\cap\bigcap_{i\in J}E_i\setminus H)=\int_{F\setminus H}g_j\times\prod_{i\in K}g_i=\int_{F\setminus H}\prod_{i\in J}g_i$$

for every $F\in\mathrm{T}$; putting these together, as in (i),

$$\mu(G\cap\bigcap_{i\in J}E_i)=\int_G\prod_{i\in J}g_i$$

for every $G\in\mathrm{T}_{\{H\}}$, and $\prod_{i\in J}g_i$ is a conditional expectation of $\chi(\bigcap_{i\in J}E_i)$ on $\mathrm{T}_{\{H\}}$.

(β) This is not exactly the formula demanded by the definition in 458Aa, because I supposed that $j\in J$; but if we have a non-empty finite $J\subseteq I\setminus\{j\}$ and $\langle E_j\rangle_{i\in J}\in\prod_{i\in J}\Sigma_j$, set $J'=J\cup\{j\}$ and $E_j=X$ to see that there is a family $\langle g_i\rangle_{i\in J'}$ such that g_i is a conditional expectation of χE_i on $\mathrm{T}_{\{H\}}$ for every i, and

$$\mu(G\cap\bigcap_{i\in J}E_i)=\mu(G\cap\bigcap_{i\in J'}E_i)=\int_G\prod_{i\in J'}g_i=\int_G\prod_{i\in J}g_i$$

for every $G\in\mathrm{T}_{\{H\}}$. So $\langle\Sigma_i\rangle_{i\in I}$ really is relatively independent over $\mathrm{T}_{\{H\}}$. **Q**

(iii) Inducing on $\#(\mathcal{E})$, we see that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over $\mathrm{T}_{\mathcal{E}}$ whenever $\mathcal{E}\subseteq\bigcup_{i\in I}\Sigma_i$ is finite. By 458C, $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over $\mathrm{T}_{\mathcal{E}}$ for every $\mathcal{E}\subseteq\bigcup_{i\in I}\Sigma_i$.

458E Example The simplest examples of relatively independent σ-algebras arise as follows. Let (X,Σ,μ) be a probability space, $\langle\mathrm{T}_i\rangle_{i\in I}$ an independent family of σ-subalgebras of Σ, as in 272Ab, and T a σ-subalgebra of Σ which is independent of $\bigvee_{i\in I}\mathrm{T}_i$. For each $i\in I$, let Σ_i be $\mathrm{T}\vee\mathrm{T}_i$. Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

proof In view of 458Db, it is enough to show that $\langle\mathrm{T}_i\rangle_{i\in I}$ is relatively independent over T. But if we have a non-empty finite $J\subseteq I$ and $E_i\in\mathrm{T}_i$ for $i\in I$, then $\mu(E_i\cap F)=\mu E_i\cdot\mu F$ for $F\in\mathrm{T}$, so $f_i=\mu E_i\cdot\chi X$ is a conditional expectation of χE_i on T, for each i. Similarly, setting $E=\bigcap_{i\in J}E_i$, $\mu E\cdot\chi X$ is a conditional expectation of χE on T. Since $\mu E=\prod_{i\in J}\mu E_j$, $\prod_{i\in J}f_i$ is a conditional expectation of χE on T, which is what we need to know.

458F The following facts are elementary but occasionally useful.

Proposition Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ.

(a) Let $\langle f_i\rangle_{i\in I}$ be a family of non-negative μ-integrable functions on X which is relatively independent over T. For each $i\in I$ let g_i be a conditional expectation of f_i on T. Then for any $F\in\mathrm{T}$ and $i_0,\dots,i_n\in I$,

$$\int_F\prod_{j=0}^n g_{i_j}\le\int_F\prod_{j=0}^n f_{i_j}$$

with equality if all the i_j are distinct.

(b) Suppose that Σ_1, Σ_2 are σ-subalgebras of Σ which are relatively independent over T, and that $f\in\mathcal{L}^1(\mu\restriction\Sigma_1)$. If g is a conditional expectation of f on T, then it is a conditional expectation of f on $\mathrm{T}\vee\Sigma_2$.

proof (a) Let Σ_i be the σ-algebra generated by f_i for each i, so that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

(i) To begin with, suppose that $i_0,\dots,i_n$ are all different.

(α) If $f_i=\chi E_i$ for each $i\in I$, where $E_i\in\Sigma_i$, the result is just the definition of 'relative independence' in 458Aa.

(β) Because both sides of the desired equality are multilinear expressions of the inputs, and conditional expectation is an essentially linear operation, the same is true if all the f_i are simple functions.

(γ) For general non-negative integrable random variables f_i, let $\langle f_{ik}\rangle_{k\in\mathbb{N}}$ be a non-decreasing sequence of non-negative Σ_i-simple functions converging almost everywhere to f_i for each i, and g_{ik} a conditional expectation of f_{ik} for all i and k. Then $\langle g_{ik}\rangle_{k\in\mathbb{N}}$ is non-decreasing almost everywhere and converges a.e. to the given conditional expectation g_i of f_i. So

$$\int_F\prod_{j=0}^n g_{i_j}=\lim_{k\to\infty}\int_F\prod_{j=0}^n g_{i_jk}=\lim_{k\to\infty}\int_F\prod_{j=0}^n f_{i_jk}=\int_F\prod_{j=0}^n f_{i_j},$$

as required.

(ii)(α) Now suppose that the $i_0,\dots,i_n$ are not all distinct, but that all the f_{i_j} are bounded. Let $l_0,\dots,l_m$ enumerate $\{i_0,\dots,i_n\}$ and for $j\le m$ set $k_j=\#(\{r:i_r=l_j\})$. For each $j\le m$, let h_j be a conditional expectation of $f_{l_j}^{k_j}=|f_{l_j}|^{k_j}$. Because $t\mapsto|t|^{k_j}$ is convex, $g_{l_j}^{k_j}\le_{\text{a.e.}}h_j$ (233J). So

$$\int_F\prod_{j=0}^n g_{i_j}=\int_F\prod_{j=0}^m g_{l_j}^{k_j}\le\int_F\prod_{j=0}^m h_j=\int_F\prod_{j=0}^m f_{l_j}^{k_j}$$

(by part (i), because each $f_{l_j}^{k_j}$ is Σ_{l_j}-measurable)

$$=\int_F\prod_{j=0}^n f_{i_j},$$

as required.

(β) Finally, for the general case, take simple functions f_{ik} and conditional expectations g_{ik} as in (a-iii) above. Then

$$\int_F\prod_{j=0}^n g_{i_j}=\lim_{k\to\infty}\int_F\prod_{j=0}^n g_{i_jk}\le\lim_{k\to\infty}\prod_{j=0}^n\int_F f_{i_jk}=\int_F\prod_{j=0}^n f_{i_j},$$

and the proof is complete.

(b) Adjusting f on a negligible set if necessary, we may suppose that f is Σ_1-measurable. Take any $F\in\Sigma_2\vee\mathrm{T}$, and let $h\ge 0$ be a conditional expectation of χF on T. By 458Db and 458Da, Σ_1 and $\Sigma_2\vee\mathrm{T}$ are relatively independent over T, so f and χF are relatively independent over T. Accordingly

$$\int_F f=\int f\times\chi F=\int g\times h$$

(applying (a) to the positive and negative parts of f)

$$=\int g\times\chi F$$

(233K)

$$=\int_F g.$$

As F is arbitrary and

$$g\in\mathcal{L}^1(\mu\restriction\mathrm{T})\subseteq\mathcal{L}^1(\mu\restriction\Sigma_2\vee\mathrm{T}),$$

g is a conditional expectation of f on $\Sigma_2\vee\mathrm{T}$.

Remark In (a), I have avoided speaking of conditional expectations of products $\prod_{j=0}^n f_{i_j}$ because these need not be integrable functions. But when $\prod_{j=0}^n f_{i_j}$ is integrable and has a conditional expectation g, then we must have $\prod_{j=0}^n g_{i_j}\le_{\text{a.e.}} g$, with equality almost everywhere when the i_j are distinct.

***458G** It is sometimes useful to know that 'relative independence' can be defined without using the apparatus of conditional expectations; indeed, we have a formulation which can be used with finitely additive functionals rather than measures.

Lemma Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ. Let $\mathbb{T}$ be the family of finite subalgebras of T. For $\Lambda\in\mathbb{T}$ write $\mathcal{A}_\Lambda$ for the set of non-negligible atoms in Λ. For non-empty finite $J\subseteq I$, $\langle E_i\rangle_{i\in J}\in\prod_{i\in J}\Sigma_i$ and $F\in\mathrm{T}$, set

$$\phi_\Lambda(F,\langle E_i\rangle_{i\in J})=\sum_{H\in\mathcal{A}_\Lambda}\mu(H\cap F)\cdot\prod_{i\in J}\frac{\mu(E_i\cap H)}{\mu H}.$$

Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T iff $\lim_{\Lambda\in\mathbb{T},\Lambda\uparrow}\phi_\Lambda(F,\langle E_i\rangle_{i\in J})=\mu(F\cap\bigcap_{i\in J}E_i)$ whenever $J\subseteq I$ is finite and not empty, $E_i\in\Sigma_i$ for every $i\in J$ and $F\in\mathrm{T}$.

proof (a) The point is just that if $J\subseteq I$ is finite and not empty, $E_i\in\Sigma_i$ for $i\in J$, g_i is a conditional expectation of χE_i on T for each i, and $F\in\mathrm{T}$, then $\int_F\prod_{i\in J}g_i d\mu=\lim_{\Lambda\uparrow}\phi_\Lambda(F,\langle E_i\rangle_{i\in J})$. **P** Adjusting each g_i on a negligible set if necessary, we may suppose that it is T-measurable, defined everywhere on X and takes values between 0 and 1.

Fix $n\in\mathbb{N}$ for the moment. Let Λ_n be the finite subalgebra of T generated by sets of the form $\{x:g_i(x)\le 2^{-n}k\}$ for $i\in J$ and $k\le 2^n$, and Λ any finite subalgebra of Σ_0 including Λ_n. If H is an atom of Λ and $\mu H>0$, then there are integers k_i, for $i\in J$, such that $2^{-n}k_i\le g_i(x)<2^{-n}(k_i+1)$ for every $i\in J$ and $x\in H$. So

$$2^{-n}k_i\le\frac{\mu_i(E\cap H)}{\mu H}<2^{-n}(k_i+1)$$

for each i. Accordingly

$$\begin{aligned}\sum_{H\in\mathcal{A}_\Lambda}\mu(H\cap F)\cdot\prod_{i\in J}2^{-n}k_i&\le\phi_\Lambda(F,\langle E_i\rangle_{i\in J})\\&\le\sum_{H\in\mathcal{A}_\Lambda}\mu(H\cap F)\cdot\prod_{i\in J}\min(1,2^{-n}(k_i+1)),\end{aligned}$$

that is,

$$\int_F\prod_{i\in J}g'_{in}d\mu\le\phi_\Lambda(F,\langle E_i\rangle_{i\in J})\le\int_F\prod_{i\in J}g''_{in}d\mu,$$

where $g'_{in}(x)=2^{-n}k$, $g''_{in}(x)=\min(1,2^{-n}(k+1))$ when $2^{-n}k\le g_i(x)<2^{-n}(k+1)$. But this means that

$$\begin{aligned}|\phi_\Lambda(F,\langle E_i\rangle_{i\in J})-\int_F\prod_{i\in J}g_i d\mu|&\le\max(\int|\prod_{i\in J}g'_{in}-\prod_{i\in J}g_i|d\mu,\int|\prod_{i\in J}g''_{in}-\prod_{i\in J}g_i|d\mu)\\&\le\max(\int\sum_{i\in J}|g'_{in}-g_i|d\mu,\int\sum_{i\in J}|g''_{in}-g_i|d\mu)\end{aligned}$$

(because all the g_i, g'_{in}, g''_{in} take values in $[0,1]$)

$$\le 2^{-n}\#(J).$$

Since this is true for every $\Lambda\supseteq\Lambda_n$ and every $n\in\mathbb{N}$, $\lim_{\Lambda\uparrow}\phi_\Lambda(F,\langle E_i\rangle_{i\in J})=\int_F\prod_{i\in J}g_i d\mu$. **Q**

(b) Accordingly the condition given exactly matches the definition in 458A.

458H All the fundamental theorems concerning stochastic independence have relativized forms. A simple one is the following.

Proposition (Compare 272K.) Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let $\langle\Sigma_i\rangle_{i\in I}$ be a family of σ-subalgebras of Σ which is relatively independent over T. Let $\langle I_j\rangle_{j\in J}$ be a partition of I, and for each $j\in J$ let $\tilde\Sigma_j$ be $\bigvee_{i\in I_j}\Sigma_i$.

(a) If $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T, then $\langle\tilde\Sigma_j\rangle_{j\in J}$ is relatively independent over T.

(b) Suppose that $\langle\tilde\Sigma_j\rangle_{j\in J}$ is relatively independent over T and that $\langle\Sigma_i\rangle_{i\in I_j}$ is relatively independent over T for every $j\in J$. Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

proof For each $E\in\Sigma$ let f_E be a conditional expectation of χE on T.

(a) Take any finite $K \subseteq J$, and let $\boldsymbol{W}$ be the set of families $\langle W_j \rangle_{j \in K}$ such that $W_j \in \tilde{\Sigma}_j$ for each $j \in K$ and $\mu(F \cap \bigcap_{j \in K} W_j) = \int_F \prod_{j \in K} f_{W_j} d\mu$ for every $F \in \mathrm{T}$. For each $j \in K$, let $\mathcal{C}_j$ be the family of measurable cylinders expressible as $W = X \cap \bigcap_{i \in L} E_i$ where $L \subseteq I_j$ is finite and $E_i \in \Sigma_i$ for $i \in L$. Note that in this case

$$\mu(F \cap W) = \mu(F \cap \textstyle\bigcap_{i \in L} E_i) = \int_F \prod_{i \in L} f_{E_i} d\mu$$

for every $F \in \mathrm{T}$, so $f_W =_{\text{a.e.}} \prod_{i \in L} f_{E_i}$, taking the product to be χX if L is empty.

If $W_j \in \mathcal{C}_j$ for each $j \in K$, then $\langle W_j \rangle_{j \in K} \in \boldsymbol{W}$. **P** Express W_j as $X \cap \bigcap_{i \in L_j} E_i$ where $L_j \subseteq I_j$ is finite and $E_i \in \Sigma_i$ whenever $j \in K$ and $i \in L_j$. Then, setting $L = \bigcup_{j \in K} L_j$,

$$\mu(F \cap \bigcap_{j \in K} W_j) = \mu(F \cap \bigcap_{i \in L} E_i) = \int_F \prod_{i \in L} f_{E_i} d\mu$$

(because $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent)

$$= \int_F \prod_{j \in K} \prod_{i \in L_j} f_{E_i} d\mu = \int_F \prod_{j \in K} f_{W_i} d\mu$$

for every $F \in \mathrm{T}$. **Q**

Observe next that if we fix $k \in K$, and a family $\langle W_j \rangle_{j \in K \setminus \{k\}}$, then the set of those $W_k \in \tilde{\Sigma}_k$ such that $\langle W_j \rangle_{j \in K} \in \boldsymbol{W}$ is a Dynkin class, so if it includes $\mathcal{C}_k$ it must include the σ-algebra generated by $\mathcal{C}_k$, viz., $\tilde{\Sigma}_k$. Now an easy induction on n shows that if $\langle W_j \rangle_{j \in K} \in \prod_{j \in K} \tilde{\Sigma}_j$ and $\#(\{j : W_j \notin \mathcal{C}_j\}) = n$, then $\langle W_j \rangle_{j \in K} \in \boldsymbol{W}$. Taking $n = \#(K)$ we see that $\prod_{j \in K} \tilde{\Sigma}_j \subseteq \boldsymbol{W}$.

As this is true for every finite $K \subseteq J$, $\langle \tilde{\Sigma}_j \rangle_{j \in J}$ is relatively independent over T, as claimed.

(b) This time, let $K \subseteq I$ be a non-empty finite set, and $E_i \in \Sigma_i$ for $i \in K$. Set $L = \{j : j \in J,\ K \cap I_j \neq \emptyset\}$, and for $j \in L$ set $G_j = \bigcap_{i \in K \cap I_j} E_i$; set $E = \bigcap_{i \in J} E_i = \bigcap_{j \in L} G_j$. Because $\langle \Sigma_i \rangle_{i \in I_j}$ is relatively independent over T, $f_{G_j} =_{\text{a.e.}} \prod_{i \in K \cap I_j} f_{E_i}$. Because $\langle \tilde{\Sigma}_j \rangle_{j \in J}$ is relatively independent over T,

$$f_E =_{\text{a.e.}} \textstyle\prod_{j \in L} f_{G_j} =_{\text{a.e.}} \prod_{i \in K} f_{E_i}.$$

As $\langle E_i \rangle_{i \in K}$ is arbitrary, we have the result.

458I For the next, we need a concept of 'relative probability distribution', as follows.

Definition Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $f \in \mathcal{L}^0(\mu)$. Then a **relative distribution** of f over T will be a family $\langle \nu_x \rangle_{x \in X}$ of Radon probability measures on $\mathbb{R}$ such that $x \mapsto \nu_x(H) : X \to [0, 1]$ is T-measurable and $\int_F \nu_x(H) \mu(dx) = \mu(F \cap f^{-1}[H])$ for every Borel set $H \subseteq \mathbb{R}$ and every $F \in \mathrm{T}$, that is, $x \mapsto \nu_x H$ is a conditional expectation of $\chi f^{-1}[H]$ on T.

458J Theorem Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $f \in \mathcal{L}^0(\mu)$. Then there is a relative distribution of f over T, which is essentially unique in the sense that if $\langle \nu_x \rangle_{x \in X}$ and $\langle \nu'_x \rangle_{x \in X}$ are two such families, then $\nu_x = \nu'_x$ for $\mu \restriction \mathrm{T}$-almost every x.

proof (a) Write μ_0 for the restriction of μ to T, $\hat{\mu}$ for the completion of μ, $\hat{\Sigma}$ for the domain of $\hat{\mu}$, and $\mathcal{B}$ for the Borel σ-algebra of $\mathbb{R}$. Then the function $x \mapsto (x, f(x)) : \operatorname{dom} f \to X \times \mathbb{R}$ is $(\hat{\Sigma}, \mathrm{T} \widehat{\otimes} \mathcal{B})$-measurable, just because $F \cap f^{-1}[H] \in \hat{\Sigma}$ for every $F \in \mathrm{T}$ and $H \in \mathcal{B}$. So we have a probability measure λ on $X \times \mathbb{R}$ defined by setting $\lambda W = \mu\{x : (x, f(x)) \in W\}$ for every $W \in \mathrm{T} \widehat{\otimes} \mathcal{B}$. The marginal measure on $\mathbb{R}$ is tight just because it is a Borel probability measure (433Ca). By 452M, we have a family $\langle \lambda_x \rangle_{x \in X}$ of Radon probability measures on $\mathbb{R}$ such that $\lambda W = \int \lambda_x W[\{x\}] \mu_0(dx)$ for every $W \in \mathrm{T} \widehat{\otimes} \mathcal{B}$.

(b) The functions $x \mapsto \lambda_x H$ need not be T-measurable. However, if we set

$$g_q(x) = \lambda_x(]-\infty, q])$$

for $q \in \mathbb{Q}$, then every g_q is μ_0-virtually measurable, so there is a μ_0-conegligible set G such that every $g_q \restriction G$ is T-measurable. By the Monotone Class Theorem (136B), the family

$$\{H : H \in \mathcal{B},\ x \mapsto \lambda_x H : G \to [0, 1] \text{ is T-measurable}\}$$

is the whole of $\mathcal{B}$. So if we set $\nu_x = \lambda_x$ for $x \in G$, and take λ_x to be the Dirac measure concentrated at 0 (for instance) for $x \in X \setminus G$, we shall have a relative distribution of f over T as defined in 458I.

(c) Now suppose that $\langle \nu'_x \rangle_{x \in X}$ is another relative distribution of f over T. Then for each $H \in \mathcal{B}$ we have $\int_F \nu_x H \mu(dx) = \int_F \nu'_x H \mu(dx)$ for every $F \in \mathrm{T}$, so that $\nu_x H = \nu'_x H$ for μ_0-almost every x. But this means that for μ_0-almost every x, we have $\nu_x H = \nu'_x H$ for every interval H with rational endpoints; and for such x we must have $\nu_x = \nu'_x$ (415H(v)).

458K Now we can state and prove a result corresponding to 272G.

Theorem Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_i \rangle_{i \in I}$ a family in $\mathcal{L}^0(\mu)$. For each $i \in I$, let $\langle \nu_{ix} \rangle_{x \in X}$ be a relative distribution of f_i over T, and $\tilde{f}_i : X \to \mathbb{R}$ an arbitrary extension of f_i to the whole of X. Then the following are equiveridical:

(i) $\langle f_i \rangle_{i \in I}$ is relatively independent over T;

(ii) for any Baire set $W \subseteq \mathbb{R}^I$ and any $F \in \mathrm{T}$,

$$\hat{\mu}(F \cap \boldsymbol{f}^{-1}[W]) = \int_F \lambda_x W \mu(dx),$$

where $\hat{\mu}$ is the completion of μ, $\boldsymbol{f}(x) = \langle \tilde{f}_i(x) \rangle_{i \in I}$ for $x \in X$, and λ_x is the product of $\langle \nu_{ix} \rangle_{i \in I}$ for each x;

(iii) for any non-negative Baire measurable function $h : \mathbb{R}^I \to \mathbb{R}$ and any $F \in \mathrm{T}$,

$$\int_F h\boldsymbol{f} d\mu = \int_F \int h \, d\lambda_x \mu(dx).$$

proof (a) Note first that if $i \in I$ and $H \subseteq \mathbb{R}$ is a Borel set, then $\int_F \nu_{ix} H \mu(dx) = \hat{\mu}(F \cap f_i^{-1}[H])$ for every $F \in \mathrm{T}$, so $x \mapsto \nu_{ix} H$ is a conditional expectation of $\chi f_i^{-1}[H]$ on T.

Suppose that $\langle f_i \rangle_{i \in I}$ is relatively independent, and $F \in \mathrm{T}$. Let $\mathcal{C}$ be the family of Baire measurable cylinders of $\mathbb{R}^I$ expressible in the form $C = \{z : z \in \mathbb{R}^I,\ z(i) \in H_i \text{ for every } i \in J\}$ where $J \subseteq I$ is finite and $H_i \subseteq \mathbb{R}$ is a Borel set for each $i \in J$. For such a set C,

$$\hat{\mu}(F \cap \boldsymbol{f}^{-1}[C]) = \hat{\mu}(F \cap \bigcap_{i \in J} \tilde{f}_i^{-1}[H_i]) = \int_F \prod_{i=0}^{n} \nu_{ix} H_i \mu(dx)$$

(interpreting an empty product as χX)

$$= \int_F \lambda_x C \mu(dx).$$

So

$$\mathcal{W} = \{W : W \subseteq \mathbb{R}^I,\ \mu(F \cap \boldsymbol{f}^{-1}[W]) = \int \lambda_x W \mu(dx)\}$$

includes $\mathcal{C}$; since it is a Dynkin class, it contains every Baire subset of $\mathbb{R}^I$ (by the Monotone Class Theorem, 136B), and (ii) is true.

(b) Now suppose that (ii) is true. Let Σ_i be the σ-algebra defined by f_i for each i. If $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for each $i \in J$, then there are Borel sets $H_i \subseteq \mathbb{R}$ such that $E_i \triangle f_i^{-1}[H_i]$ is negligible for each i, so that $x \mapsto \nu_{ix} H_i$ is a conditional expectation of χE_i on T. Now by the same equations as before, in the opposite direction,

$$\int_F \prod_{i \in J} \nu_{ix} H_i \mu(dx) = \int_F \lambda_x C \mu(dx)$$

(where $C = \{z : z(i) \in H_i \text{ for } i \in J\}$)

$$= \hat{\mu}(F \cap \boldsymbol{f}^{-1}[C]) = \hat{\mu}(F \cap \bigcap_{i \in J} f_i^{-1}[H_i]) = \hat{\mu}(F \cap \bigcap_{i \in J} E_i)$$

for every $F \in \mathrm{T}$. As $\langle E_i \rangle_{i \in J}$ is arbitrary, $\langle \Sigma_i \rangle_{i \in I}$ and $\langle f_i \rangle_{i \in I}$ are relatively independent.

(c) Thus (i)$\Leftrightarrow$(ii). For (ii)$\Rightarrow$(iii), observe that (ii) covers the case in which h is an indicator function χW; for the general case, express h as the supremum of a non-decreasing sequence of linear combinations of indicator functions, as usual. And (iii)$\Rightarrow$(ii) is trivial.

Remarks Of course the ungainly shift to $\tilde{f}_i$ is unnecessary if I is countable; but for uncountable I the intersection $\bigcap_{i\in I}\operatorname{dom} f_i$, which is the only suitable domain for $\boldsymbol{f}$, may not be conegligible.

I said that λ_x should be 'the product of $\langle\nu_{ix}\rangle_{i\in I}$'. Since the ν_{ix} are Radon probability measures, we have two possible interpretations of this: either the 'ordinary' product measure of §254 or the 'quasi-Radon' product measure of §417. But as we are interested only in the values of $\lambda_x W$ for Baire sets W, it makes no difference which we use.

458L Measure algebras We can look at the same ideas in the context of measure algebras. Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$.

(a) If $a\in\mathfrak{A}$, then we can say that $u\in L^\infty(\mathfrak{C})$ is the conditional expectation of χa on $\mathfrak{C}$ if $\int_c u=\bar\mu(c\cap a)$ for every $c\in\mathfrak{C}$ (365R). Now we can say that a family $\langle b_i\rangle_{i\in I}$ in $\mathfrak{A}$ is **relatively (stochastically) independent** over $\mathfrak{C}$ if $\bar\mu(c\cap\inf_{i\in J}b_i)=\int_c\prod_{i\in J}u_i$ whenever $J\subseteq I$ is a non-empty finite set and u_i is the conditional expectation of χb_i on $\mathfrak{C}$ for every $i\in J$; while a family $\langle\mathfrak{B}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is **relatively (stochastically) independent over** $\mathfrak{C}$ if $\langle b_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ whenever $b_i\in\mathfrak{B}_i$ for every $i\in I$.

Corresponding to 458Ab, we can say that a family $\langle w_i\rangle_{i\in I}$ in $L^0(\mathfrak{A})$ is **relatively (stochastically) independent over** $\mathfrak{C}$ if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively stochastically independent, where $\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{[\![w_i>\alpha]\!]:\alpha\in\mathbb{R}\}$ for each i.

Returning to the original form of these ideas, we say that a family $\langle b_i\rangle_{i\in I}$ in $\mathfrak{A}$ is **(stochastically) independent** if it is relatively independent over $\{0,1\}$, that is, if $\bar\mu(\inf_{i\in J}b_i)=\prod_{i\in J}\bar\mu b_i$ whenever $J\subseteq I$ is finite. Similarly, a family $\langle\mathfrak{B}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is (stochastically) independent if $\bar\mu(\inf_{i\in J}b_i)=\prod_{i\in J}\bar\mu b_i$ whenever $J\subseteq I$ is finite and $b_i\in\mathfrak{B}_i$ for every i.

(b) Let (X,Σ,μ) be a probability space and $(\mathfrak{A},\bar\mu)$ its measure algebra. Let $\langle E_i\rangle_{i\in I}$, $\langle\Sigma_i\rangle_{i\in I}$ and $\langle f_i\rangle_{i\in I}$ be, respectively, a family in Σ, a family of subalgebras of Σ, and a family of μ-virtually measurable real-valued functions defined almost everywhere on X; let T be a σ-subalgebra of Σ. For $i\in I$, set $a_i=E_i^\bullet\in\mathfrak{A}$, $\mathfrak{B}_i=\{E^\bullet:E\in\Sigma_i\}$, and $w_i=f_i^\bullet\in L^0(\mathfrak{A})$, identified with $L^0(\mu)$ (364Ic[26]). Let T be a σ-subalgebra of $\mathfrak{A}$ and $\mathfrak{C}=\{F^\bullet:F\in\mathrm{T}\}$. Then

$\langle a_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ iff $\langle E_i\rangle_{i\in I}$ is relatively independent over T,

$\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ iff $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T,

$\langle w_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ iff $\langle f_i\rangle_{i\in I}$ is relatively independent over T.

P The point is that if $f\in\mathcal{L}^1(\mu)$ (in particular, if $f=\chi E$ for some $E\in\Sigma$), and $g\in\mathcal{L}^1(\mu\restriction\mathrm{T})\subseteq\mathcal{L}^1(\mu)$ is a conditional expectation of f on T, then $g^\bullet$ is a conditional expectation of $f^\bullet$ on $\mathfrak{C}$; see 242J and 365R. **Q**

(c) Corresponding to 458B, we see that if $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ such that $\mathfrak{C}\subseteq\bigcup_{i\in I}\mathfrak{A}_i$, and $\int\prod_{i\in J}u_i d\bar\mu=\bar\mu(\inf_{i\in J}a_i)$ whenever $J\subseteq I$ is finite and not empty and $a_i\in\mathfrak{A}_i$, $u_i\in L^\infty(\mathfrak{C})$ is a conditional expectation of χa_i on $\mathfrak{C}$ for each i, then $\langle\mathfrak{A}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$.

(d) Corresponding to 458Db, we see that if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$, and $\mathfrak{B}_i^*$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}_i\cup\mathfrak{C}$ for each i, then $\langle\mathfrak{B}_i^*\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$. The most natural proof, from where we are now standing, is to express $(\mathfrak{A},\bar\mu)$ as the measure algebra of a probability space (X,Σ,μ), set $\mathrm{T}=\{F:F^\bullet\in\mathfrak{C}\}$ and $\Sigma_i=\{E:E^\bullet\in\mathfrak{B}_i\}$ for each $i\in I$, and use 458D.

Corresponding to 458Dc, we see that if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$, $D_i\subseteq\mathfrak{B}_i$ for every $i\in I$, and $\mathfrak{D}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\bigcup_{i\in I}D_i$, then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$.

(e) Following 458H, we have the result that if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$, and $\langle I_j\rangle_{j\in J}$ is a partition of I, and $\tilde{\mathfrak{B}}_j$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I_j}\mathfrak{B}_i$ for every $j\in J$, then $\langle\tilde{\mathfrak{B}}_j\rangle_{j\in J}$ is relatively independent over $\mathfrak{C}$.

(f) Note that if $a\in\mathfrak{A}$ and u is the conditional expectation of χa on $\mathfrak{C}$, then $[\![u>0]\!]=\operatorname{upr}(a,\mathfrak{C})$, by 365Rc. So if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$, and $J\subseteq I$ is finite, and $b_i\in\mathfrak{B}_i$ for each $i\in J$, then $\inf_{i\in J}b_i=0$ iff $\inf_{i\in J}\operatorname{upr}(b_i,\mathfrak{C})=0$. (If u_i is a conditional expectation of χb_i on $\mathfrak{C}$ for each i, then

$$\inf_{i\in J}\operatorname{upr}(b_i,\mathfrak{C})=\inf_{i\in J}[\![u_i>0]\!]=[\![\textstyle\prod_{i\in J}u_i>0]\!]$$

is zero iff $\bar\mu(\inf_{i\in J}b_i)=\int\prod_{i\in J}u_i=0$.)

[26]Formerly 364Jc.

(g) We have a straightforward version of 458E, as follows. If $\langle\mathfrak{C}_i\rangle_{i\in I}$ is a stochastically independent family of closed subalgebras of $\mathfrak{A}$, $\mathfrak{C}$ is independent of the algebra generated by $\bigcup_{i\in I}\mathfrak{C}_i$, and $\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\mathfrak{C}_i$ for each i, then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$. (Either repeat the proof of 458E, looking at $\mathfrak{B}_{i0}=\mathfrak{C}_i$ and $\mathfrak{B}_{i1}=\mathfrak{C}$ for each i, or move to a measure space representing $\mathfrak{A}$ and quote 458E.)

(h) Similarly, we can translate 458F into this language. Let $P:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{C},\bar\mu\restriction\mathfrak{C})\subseteq L^1(\mathfrak{A},\bar\mu)$ be the conditional expectation operator associated with $\mathfrak{C}$ (365R). Suppose that $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of closed subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$. Then

$$\int_c\prod_{j=0}^n Pu_j\le\int_c\prod_{j=0}^n u_j$$

whenever $c\in\mathfrak{C}$, $i_0,\dots,i_n\in I$ and $u_j\in L^1(\mathfrak{B}_{i_j},\bar\mu\restriction\mathfrak{B}_{i_j})^+$ for each $j\le n$, with equality if $i_0,\dots,i_n$ are distinct.

458M Proposition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\mathfrak{B}$, $\mathfrak{C}$ closed subalgebras of $\mathfrak{A}$. Write $P_{\mathfrak{B}}$, $P_{\mathfrak{C}}$ and $P_{\mathfrak{B}\cap\mathfrak{C}}$ for the conditional expectation operators associated with $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{B}\cap\mathfrak{C}$. Then the following are equiveridical:

(i) $\mathfrak{B}$ and $\mathfrak{C}$ are relatively independent over $\mathfrak{B}\cap\mathfrak{C}$;

(ii) $P_{\mathfrak{B}\cap\mathfrak{C}}(v\times w)=P_{\mathfrak{B}\cap\mathfrak{C}}v\times P_{\mathfrak{B}\cap\mathfrak{C}}w$ whenever $v\in L^\infty(\mathfrak{B})$ and $w\in L^\infty(\mathfrak{C})$;

(iii) $P_{\mathfrak{B}}P_{\mathfrak{C}}=P_{\mathfrak{B}\cap\mathfrak{C}}$;

(iv) $P_{\mathfrak{B}}P_{\mathfrak{C}}=P_{\mathfrak{C}}P_{\mathfrak{B}}$;

(v) $P_{\mathfrak{B}}u\in L^0(\mathfrak{C})$ for every $u\in L^1(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$.

proof Write P for $P_{\mathfrak{B}\cap\mathfrak{C}}$.

(i)⇒(ii) If (i) is true, $v\in L^\infty(\mathfrak{B})$ and $w\in L^\infty(\mathfrak{C})$, then $Pv\times Pw$ certainly belongs to $L^\infty(\mathfrak{B}\cap\mathfrak{C})$, and if $d\in\mathfrak{B}\cap\mathfrak{C}$, $\int_d Pv\times Pw=\int_d v\times w$ by 458Lh. So $Pv\times Pw=P(v\times w)$.

(ii)⇒(i) If (ii) is true, $b\in\mathfrak{B}$, $c\in\mathfrak{C}$ and $d\in\mathfrak{B}\cap\mathfrak{C}$, then

$$\bar\mu(d\cap b\cap c)=\int_d\chi b\times\chi c=\int_d P(\chi b\times\chi c)=\int_d P(\chi b)\times P(\chi c)$$

as required by the definition in 458La.

(ii)⇒(iii) Suppose that (ii) is true. First note that if $w\in L^\infty(\mathfrak{C})$ then $Pw=P_{\mathfrak{B}}w$. **P** Of course $Pw\in L^\infty(\mathfrak{B}\cap\mathfrak{C})\subseteq L^\infty(\mathfrak{B})$. If $b\in\mathfrak{B}$, then

$$\int_b w=\int\chi b\times w=\int P(\chi b\times w)=\int P\chi b\times Pw=\int\chi b\times Pw$$

(365Ra)

$$=\int_b Pw,$$

so that Pw possesses the defining properties of $P_{\mathfrak{B}}w$. **Q**

But this means that if $u\in L^\infty(\mathfrak{A})$, $P_{\mathfrak{B}}P_{\mathfrak{C}}u=PP_{\mathfrak{C}}u$, which in turn is equal to Pu just because $\mathfrak{B}\cap\mathfrak{C}\subseteq\mathfrak{C}$ (see 233Eh). As u is arbitrary, $P_{\mathfrak{B}}P_{\mathfrak{C}}$ agrees with P on $L^\infty(\mathfrak{A})$; but $L^\infty(\mathfrak{A})$ is $\|\,\|_1$-dense in $L^1(\mathfrak{A},\bar\mu)$, and $P_{\mathfrak{B}}P_{\mathfrak{C}}$ and P are both $\|\,\|_1$-continuous, so they agree everywhere on $L^1(\mathfrak{A},\bar\mu)$ and are equal, as required by (iii).

(iii)⇒(ii) Suppose that (iii) is true, and that $v\in L^\infty(\mathfrak{B})$, $w\in L^\infty(\mathfrak{C})$ and $d\in\mathfrak{B}\cap\mathfrak{C}$. Then

$$\int_d Pv\times Pw=\int\chi d\times Pv\times Pw=\int\chi d\times v\times Pw$$

(because $\chi d\times Pw\in L^\infty(\mathfrak{B}\cap\mathfrak{C})$)

$$=\int\chi d\times v\times P_{\mathfrak{B}}w$$

(because $P_{\mathfrak{C}}w=w$)

$$=\int\chi d\times v\times w$$

(because $\chi d\times v\in L^\infty(\mathfrak{B})$)

$$= \int_d v \times w.$$

As d is arbitrary and $Pv \times Pw \in L^\infty(\mathfrak{B} \cap \mathfrak{C})$, $Pv \times Pw = P(v \times w)$.

(i)⇒(iv) follows immediately from (i)⇒(iii) and the symmetry of the relation '$\mathfrak{B}$ and $\mathfrak{C}$ are relatively independent over $\mathfrak{B} \cap \mathfrak{C}$'.

(iv)⇒(v) If (iv) is true and $u \in L^1(\mathfrak{C}, \bar\mu\restriction\mathfrak{C})$, then

$$P_{\mathfrak{B}}u = P_{\mathfrak{B}}P_{\mathfrak{C}}u = P_{\mathfrak{C}}P_{\mathfrak{B}}u \in L^0(\mathfrak{C}),$$

so (v) is true.

(v)⇒(iii) If (v) is true, and $u \in L^1_{\bar\mu}$, then $P_{\mathfrak{C}}u \in L^1(\mathfrak{C}, \bar\mu\restriction\mathfrak{C})$, so $P_{\mathfrak{B}}P_{\mathfrak{C}}u$ belongs to $L^0(\mathfrak{C}) \cap L^0(\mathfrak{B}) = L^0(\mathfrak{B} \cap \mathfrak{C})$, and of course

$$\int_d P_{\mathfrak{B}}P_{\mathfrak{C}}u = \int_d P_{\mathfrak{C}}u = \int_d u$$

for every $d \in \mathfrak{B} \cap \mathfrak{C}$. So $P_{\mathfrak{B}}P_{\mathfrak{C}}u = Pu$. As u is arbitrary, (iii) is true.

458N Relative free products of probability algebras: Definition Let $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras and $(\mathfrak{C}, \bar\nu)$ a probability algebra, and suppose that we are given a measure-preserving Boolean homomomorphism $\pi_i : \mathfrak{C} \to \mathfrak{A}_i$ for each $i \in I$. A **relative free product** of $\langle(\mathfrak{A}_i, \bar\mu_i, \pi_i)\rangle_{i\in I}$ over $(\mathfrak{C}, \bar\nu)$ is a probability algebra $(\mathfrak{A}, \bar\mu)$, together with a measure-preserving Boolean homomorphism $\phi_i : \mathfrak{A}_i \to \mathfrak{A}$ for each $i \in I$, such that

$\mathfrak{A}$ is the closed subalgebra of itself generated by $\bigcup_{i\in I} \phi_i[\mathfrak{A}_i]$,

$\phi_i\pi_i = \phi_j\pi_j : \mathfrak{C} \to \mathfrak{A}$ for all i, $j \in I$,

writing $\mathfrak{D}$ for the common value of the $\phi_i[\pi_i[\mathfrak{C}]]$, $\langle\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$.

Remark The homomorphisms π_i and ϕ_i are essential for the formal content of this definition, and will necessarily appear in the basic result 458O; but conceptually they are a nuisance; we should much prefer to think of every $\mathfrak{A}_i$ as a subalgebra of $\mathfrak{A}$, and of $\mathfrak{C}$ as actually equal to $\mathfrak{D}$. It may help if I spell out the key condition '$\langle\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$' in terms of $\mathfrak{C}$ and the π_i.

The common value π of the $\phi_i\pi_i$ is a measure-preserving isomorphism between $\mathfrak{C}$ and $\mathfrak{D}$, so gives rise to an f-algebra isomorphism $S : L^0(\mathfrak{C}) \to L^0(\mathfrak{D})$ such that $S(\chi c) = \chi(\pi c)$ for every $c \in \mathfrak{C}$ (364P[27]); note that $S[L^\infty(\mathfrak{C})] = L^\infty(\mathfrak{D})$ and $\int Su\,d\bar\mu = \int u\,d\bar\nu$ for every $u \in L^1(\mathfrak{C})$ (365O). If $u \in L^\infty(\mathfrak{C})$ and $d \in \mathfrak{D}$, then

$$\begin{aligned}\int_d Su\,d\bar\mu &= \int Su \times \chi d\,d\bar\mu = \int Su \times S(\chi(\pi^{-1}d))d\bar\mu \\ &= \int S(u \times \chi(\pi^{-1}d))d\bar\mu = \int u \times \chi(\pi^{-1}d)\,d\bar\nu = \int_{\pi^{-1}d} u\,d\bar\nu.\end{aligned}$$

Next, for $i \in I$ and $a \in \mathfrak{A}_i$, we have a completely additive functional $c \mapsto \bar\mu_i(a \cap \pi_i c) : \mathfrak{C} \to [0,1]$; let $u_{ia} \in L^\infty(\mathfrak{C})$ be a corresponding Radon-Nikodým derivative, so that $\int_c u_{ia}d\bar\nu = \bar\mu_i(a \cap \pi_i c)$ for every $c \in \mathfrak{C}$ (365E). (Thus $u_{ia} \in L^\infty(\mathfrak{C})$ corresponds to the conditional expectation of χa on the algebra $\pi_i[\mathfrak{C}] \subseteq \mathfrak{A}_i$.) The image Su_{ia} in $L^\infty(\mathfrak{D})$ is defined by the property

$$\int_d Su_{ia}d\bar\mu = \int_{\pi^{-1}d} u_{ia}d\bar\nu = \bar\mu_i(a \cap \pi_i(\phi_i\pi_i)^{-1}d) = \bar\mu(\phi_i a \cap d)$$

for every $d \in \mathfrak{D}$; that is, Su_{ia} is the conditional expectation of $\chi(\phi_i a)$ on $\mathfrak{D}$.

Note also that $\mathfrak{D} \subseteq \phi_i[\mathfrak{A}_i]$ for every $i \in I$. So we can use the criterion of 458B/458Lc to see that

$\langle\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$

iff $\bar\mu(\inf_{i\in J} \phi_i a_i) = \int \prod_{i\in J} Su_{i,a_i}d\bar\mu$

whenever $J \subseteq I$ is finite and not empty and $a_i \in \mathfrak{A}_i$ for $i \in J$

iff $\bar\mu(\inf_{i\in J} \phi_i a_i) = \int \prod_{i\in J} u_{i,a_i}d\bar\nu$

whenever $J \subseteq I$ is finite and not empty and $a_i \in \mathfrak{A}_i$ for $i \in J$

[27]Formerly 364R.

because S is multiplicative, so we always have

$$\int \prod_{i\in J} Su_{i,a_i} d\bar{\mu} = \int S(\prod_{i\in J} u_{i,a_i}) d\bar{\mu} = \int \prod_{i\in J} u_{i,a_i} d\bar{\nu}.$$

458O Theorem Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, $(\mathfrak{C},\bar{\nu})$ a probability algebra and $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ a measure-preserving Boolean homomomorphism for each $i\in I$. Then $\langle(\mathfrak{A}_i,\bar{\mu}_i,\pi_i)\rangle_{i\in I}$ has an essentially unique relative free product over $(\mathfrak{C},\bar{\nu})$.

proof (a)(i) Let $\mathfrak{B}$ be the free product of $\langle\mathfrak{A}_i\rangle_{i\in I}$ (315I[28]); write $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{B}$ for the canonical embedding of $\mathfrak{A}_i$ in $\mathfrak{B}$. For each $i\in I$, $a\in\mathfrak{A}_i$ let $u_{ia}\in L^\infty(\mathfrak{C})$ be such that $\int_c u_{ia}d\bar{\nu}=\bar{\mu}_i(a\cap\pi_i c)$ for every $c\in\mathfrak{C}$ (458N). Of course $u_{i1}=\chi 1$ in $L^\infty(\mathfrak{C})$ for every $i\in I$, interpreting the '1' in u_{i1} in the Boolean algebra $\mathfrak{A}_i$, and the '1' in $\chi 1$ in the Boolean algebra $\mathfrak{C}$; so (this time interpreting '1' in $\mathfrak{B}$) $\lambda 1=1$ (the final '1' being a real number, of course).

Because the map $a\mapsto u_{ia}:\mathfrak{A}_i\to L^\infty(\mathfrak{C})$ is additive for each i, 326E[29] tells us that there is a unique additive functional $\lambda:\mathfrak{B}\to[0,1]$ such that

$$\lambda(\inf_{i\in J}\varepsilon_i a_i)=\int\prod_{i\in J}u_{i,a_i}d\bar{\nu}$$

whenever $J\subseteq I$ is a non-empty finite set and $a_i\in\mathfrak{A}_i$ for every $i\in J$.

(ii) By 392I, there are a probability algebra $(\mathfrak{A},\bar{\mu})$ and a Boolean homomorphism $\phi:\mathfrak{B}\to\mathfrak{A}$ such that $\lambda=\bar{\mu}\phi$. We can of course suppose that $\mathfrak{A}$ is the order-closed subalgebra of itself generated by $\phi[\mathfrak{B}]$ (which is in fact automatically the case if we use the construction in the proof of 392I).

For each $i\in I$, set $\phi_i=\phi\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$. It is a Boolean homomorphism because ϕ and ε_i are. If $a\in\mathfrak{A}_i$, then

$$\bar{\mu}\phi_i a=\bar{\mu}(\phi\varepsilon_i a)=\lambda\varepsilon_i a=\int u_{ia}d\bar{\nu}=\bar{\mu}_i(a\cap\pi_i 1)=\bar{\mu}_i a,$$

so ϕ_i is measure-preserving.

If i, $j\in I$ and $c\in\mathfrak{C}$ then

$$\begin{aligned}\bar{\mu}(\phi_i\pi_i c\,\triangle\,\phi_j\pi_j c)&=\lambda(\varepsilon_i\pi_i c\,\triangle\,\varepsilon_j\pi_j c)\\&=\lambda(\varepsilon_i\pi_i c)+\lambda(\varepsilon_j\pi_j c)-2\lambda(\varepsilon_i\pi_i c\cap\varepsilon_j\pi_j c)\\&=\int u_{i,\pi_i c}d\bar{\nu}+\int u_{j,\pi_j c}d\bar{\nu}-2\int u_{i,\pi_i c}\times u_{j,\pi_j c}d\bar{\nu}\\&=\int\chi c\,d\bar{\nu}+\int\chi c\,d\bar{\nu}-2\int\chi c\times\chi c\,d\bar{\nu}=0.\end{aligned}$$

So $\phi_i\pi_i=\phi_j\pi_j$. Let $\mathfrak{D}$ be the common value of $\phi_i[\pi_i[\mathfrak{C}]]$. (In the trivial case $I=\emptyset$, take $\mathfrak{D}=\mathfrak{A}=\{0,1\}$.)

(iii) Suppose that $J\subseteq I$ is finite and not empty and that $a_i\in\mathfrak{A}_i$ for each $i\in J$. Then

$$\begin{aligned}\bar{\mu}(\inf_{i\in J}\phi_i a_i)=\lambda(\inf_{i\in J}\varepsilon_i a_i)&=\int\prod_{i\in J}u_{i,a_i}d\bar{\nu}\\&=\int\prod_{i\in J}u_{i,a_i}d\bar{\nu}=\int\prod_{i\in J}u_{i,a_i}d\bar{\nu}.\end{aligned}$$

But this is precisely the condition described in 458N, so $\langle\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$, and $(\mathfrak{A},\bar{\mu},\langle\phi_i\rangle_{i\in I})$ is a relative free product of $\langle(\mathfrak{A}_i,\bar{\mu}_i,\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar{\nu})$.

(b) Now suppose that $(\mathfrak{A}',\bar{\mu}',\langle\phi'_i\rangle_{i\in I})$ is another relative free product of $\langle(\mathfrak{A}_i,\bar{\mu}_i,\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar{\nu})$. Then we have a Boolean homomorphism $\psi:\mathfrak{B}\to\mathfrak{A}'$ such that $\phi'_i=\psi\varepsilon_i$ for every $i\in I$ (315J[30]). In this case, $\bar{\mu}'\psi=\lambda$. **P** Let π' be the common value of $\phi'_i\pi_i$ for $i\in I$, set $\mathfrak{D}'=\pi'[\mathfrak{C}]$, and let $S':L^0(\mathfrak{C})\to L^0(\mathfrak{D}')$ be the isomorphism corresponding to $\pi':\mathfrak{C}\to\mathfrak{D}$. If $a\in\mathfrak{A}$ and $c\in\mathfrak{C}$, then

$$\bar{\mu}'(\phi'_i a\cap\pi' c)=\bar{\mu}_i(a\cap\pi_i c)=\bar{\mu}(\phi_i a\cap\pi c)=\int_c u_{ia}d\bar{\nu}=\int_{\pi' c}S'u_{ia}d\bar{\mu}'$$

by 365H (compare 458N above); it follows that $S'u_{ia}$ is the conditional expectation of $\chi(\phi'_i a)$ on $\mathfrak{D}'$.

If $J\subseteq I$ is finite and not empty, and $a_i\in\mathfrak{A}_i$ for $i\in J$, then

[28] Formerly 315H.
[29] Formerly 326Q.
[30] Formerly 315I.

$$\bar{\mu}'(\psi(\inf_{i\in J}\varepsilon_i a_i)) = \bar{\mu}'(\inf_{i\in J}\phi'_i a_i) = \int\prod_{i\in J}S'u_{i,a_i}d\bar{\mu}'$$
$$= \int S'(\prod_{i\in J}u_{i,a_i})d\bar{\mu}' = \int\prod_{i\in J}u_{i,a_i}d\bar{\nu} = \lambda(\inf_{i\in J}\varepsilon_i a_i).$$

Because λ is the only additive functional on $\mathfrak{B}$ taking the right values on elements of this form, $\bar{\mu}'\psi = \lambda$. **Q**

In particular, $\psi b = 0$ whenever $b \in \mathfrak{B}$ and $\lambda b = 0$. It follows that $\psi b = \psi b'$ whenever b, $b' \in \mathfrak{B}$ and $\phi b = \phi b'$, since in this case $\lambda(b \triangle b') = \bar{\mu}(\phi b \triangle \phi b')$ is zero. So we have a function $\theta : \phi[\mathfrak{B}] \to \mathfrak{A}'$ defined by setting $\theta(\phi b) = \psi b$ for every $b \in \mathfrak{B}$, and of course θ is a Boolean homomorphism; moreover,

$$\bar{\mu}'\theta(\phi b) = \bar{\mu}'\psi b = \lambda b = \bar{\mu}\phi b$$

for every b, so θ is measure-preserving and an isometry for the measure metrics of $\mathfrak{A}$ and $\mathfrak{A}'$. If $i \in I$ and $a \in \mathfrak{A}$, then

$$\theta\phi_i a = \theta\phi\varepsilon_i a = \psi\varepsilon_i a = \phi'_i a,$$

so $\theta\phi_i = \phi'_i$ for every i. Because $\mathfrak{A}$ and $\mathfrak{A}'$ are the closed subalgebras generated by $\bigcup_{i\in I}\phi_i[\mathfrak{A}_i]$ and $\bigcup_{i\in I}\phi'_i[\mathfrak{A}_i]$ respectively, $\phi[\mathfrak{B}]$ and $\psi[\mathfrak{B}]$ are dense (323J). The isometry θ therefore extends uniquely to a measure algebra isomorphism $\hat{\theta} : \mathfrak{A} \to \mathfrak{A}'$ which must be the unique isomorphism such that $\hat{\theta}\phi_i = \phi'_i$ for every i. Thus $(\mathfrak{A}, \bar{\mu}, \langle\phi_i\rangle_{i\in I})$ and $(\mathfrak{A}', \bar{\mu}', \langle\phi'_i\rangle_{i\in I})$ are isomorphic, and the relative free product is essentially unique.

458P Developing the argument of the last part of the proof of 458O, we have the following.

Theorem Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$, $\langle(\mathfrak{A}'_i, \bar{\mu}'_i)\rangle_{i\in I}$ be two families of probability algebras, and $\psi_i : \mathfrak{A}_i \to \mathfrak{A}'_i$ a measure-preserving Boolean homomorphism for each i. Let $(\mathfrak{C}, \bar{\nu})$, $(\mathfrak{C}', \bar{\nu}')$ be probability algebras and $\pi_i : \mathfrak{C} \to \mathfrak{A}_i$, $\pi'_i : \mathfrak{C}' \to \mathfrak{A}'_i$ measure-preserving Boolean homomomorphisms for each $i \in I$; suppose that we have a measure-preserving isomorphism $\psi : \mathfrak{C} \to \mathfrak{C}'$ such that $\pi'_i\psi = \psi_i\pi_i : \mathfrak{C} \to \mathfrak{A}'_i$ for each i. Let $(\mathfrak{A}, \bar{\mu}, \langle\phi_i\rangle_{i\in I})$ and $(\mathfrak{A}', \bar{\mu}', \langle\phi'_i\rangle_{i\in I})$ be relative free products of $\langle(\mathfrak{A}_i, \bar{\mu}_i, \pi_i)\rangle_{i\in I}$, $\langle(\mathfrak{A}'_i, \bar{\mu}'_i, \pi'_i)\rangle_{i\in I}$ over $(\mathfrak{C}, \bar{\nu})$, $(\mathfrak{C}', \bar{\nu}')$ respectively. Then there is a unique measure-preserving Boolean homomorphism $\hat{\psi} : \mathfrak{A} \to \mathfrak{A}'$ such that $\hat{\psi}\phi_i = \phi'_i\pi_i : \mathfrak{A}_i \to \mathfrak{A}'$ for every $i \in I$.

proof By the uniqueness assertion of 458O, we may suppose that $(\mathfrak{A}, \bar{\mu}, \langle\phi_i\rangle_{i\in I})$ has been constructed by the method in the proof of 458O.

(a) For $i \in A$, $a \in \mathfrak{A}$, $a' \in \mathfrak{A}'$ let $u_{ia} \in L^\infty(\mathfrak{C})$, $u'_{ia'} \in L^\infty(\mathfrak{C}')$ be defined as in the proof of 458O, so that

$$\int_c u_{ia}d\bar{\nu} = \bar{\mu}_i(a \cap \pi_i c), \quad \int_{c'} u'_{ia'}d\bar{\nu}' = \bar{\mu}'_i(a' \cap \pi'_i c')$$

whenever $c \in \mathfrak{C}$ and $c' \in \mathfrak{C}'$. Let $T : L^0(\mathfrak{C}) \to L^0(\mathfrak{C}')$ be the f-algebra isomorphism such that $T(\chi c) = \chi(\psi c)$ for every $c \in \mathfrak{C}$. Now $u'_{i,\psi_i a} = Tu_{ia}$ whenever $i \in I$ and $a \in \mathfrak{A}_i$. **P** If $c \in \mathfrak{C}$, then

$$\int_{\psi c} Tu_{ia}d\bar{\nu}' = \int_c u_{ia}d\bar{\nu} = \bar{\mu}_i(a \cap \pi_i c)$$
$$= \bar{\mu}'_i\psi_i(a \cap \pi_i c) = \bar{\mu}'_i(\psi_i a \cap \pi'_i\psi c) = \int_{\psi c} u'_{i,\psi_i a}d\bar{\nu}'.$$

Because ψ is surjective, it follows that $Tu_{ia} = u'_{i,\psi_i a}$. **Q**

(b) Let $\mathfrak{B}$ be the free product of $\langle\mathfrak{A}_i\rangle_{i\in I}$ and $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{B}$ the canonical embedding for each i; let λ be the functional on $\mathfrak{B}$ defined by the process of (a-i) in the proof of 458O. By 315J, there is a Boolean homomorphism $\theta : \mathfrak{B} \to \mathfrak{A}'$ such that $\theta\varepsilon_i = \phi'_i\psi_i : \mathfrak{A}_i \to \mathfrak{A}'$ for every i. Now $\bar{\mu}'\theta = \lambda$. **P** If $J \subseteq I$ is finite and $a_i \in \mathfrak{A}_i$ for every $i \in J$, then

$$\bar{\mu}'\theta(\inf_{i\in J}\varepsilon_i a_i) = \bar{\mu}'(\inf_{i\in J}\phi'_i\psi_i a_i) = \int\prod_{i\in J}u'_{i,\psi_i a_i}d\bar{\nu}'$$
$$= \int\prod_{i\in J}Tu_{i,a_i}d\bar{\nu}' = \int T(\prod_{i\in J}u_{i,a_i})d\bar{\nu}'$$

(because $T{\restriction}L^\infty(\mathfrak{C})$ is multiplicative)

$$= \int \prod_{i\in J} u_{i,a_i} d\bar{\nu} = \lambda(\inf_{i\in J} \varepsilon_i a_i).$$

As λ, θ and $\bar{\nu}$ are all additive, $\lambda = \bar{\nu}\theta'$ (using 315Kb[31]). **Q**

(c) Let $\phi : \mathfrak{B} \to \mathfrak{A}$ be the map described in (a-ii) of the proof of 458O. Then $\bar{\mu}(\phi b) = \lambda b = \bar{\mu}'(\theta b)$ for every $b \in \mathfrak{B}$; in particular,

$$\phi b = 0 \implies \bar{\mu}(\phi b) = 0 \implies \bar{\mu}'(\theta b) = 0 \implies \theta b = 0.$$

There is therefore a Boolean homomorphism $\tilde{\theta} : \phi[\mathfrak{B}] \to \mathfrak{A}'$ such that $\tilde{\theta}\phi = \theta$, and $\tilde{\theta}$ is measure-preserving on $\phi[\mathfrak{B}]$. Since $\phi[\mathfrak{B}]$ is topologically dense in $\mathfrak{A}$ (use 323J), $\tilde{\theta}$ has an extension to a measure-preserving Boolean homomorphism $\hat{\psi} : \mathfrak{A} \to \mathfrak{A}'$ (324O). Now, for $i \in I$ and $a \in \mathfrak{A}_i$,

$$\hat{\psi}\phi_i a = \hat{\psi}\phi\varepsilon_i a = \tilde{\theta}\phi\varepsilon_i a = \theta\varepsilon_i a = \phi'_i\psi_i a,$$

as required.

(d) To see that $\hat{\psi}$ is unique, we need observe only that the given formula defines it on the subalgebra $\phi[\mathfrak{B}]$ and that this is topologically dense in $\mathfrak{A}$, while $\hat{\psi}$, being measure-preserving, must be continuous.

458Q Relative product measures: Definitions (a) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, Y a set, and $\pi_i : X_i \to Y$ a function for each $i \in I$. The **fiber product** of $\langle(X_i, \pi_i)\rangle_{i\in I}$ is the set $\Delta = \{x : x \in \prod_{i\in I} X_i,\ \pi_i x(i) = \pi_j x(j)$ for all i, $j \in I\}$.

(b) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces and (Y, T, ν) a probability space, and suppose that we are given an inverse-measure-preserving function $\pi_i : X_i \to Y$ for each $i \in I$; let Δ be the fiber product of $\langle(X_i, \pi_i)\rangle_{i\in I}$. A **relative product measure** on Δ is a probability measure μ on Δ such that

(†) whenever $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for $i \in J$, and g_i is a Radon-Nikodým derivative with respect to ν of the functional $F \mapsto \mu_i(E \cap \pi_i^{-1}[F]) : \mathrm{T} \to [0, 1]$ for each $i \in J$, then $\mu\{x : x \in \Delta,$ $x(i) \in E_i$ for every $i \in J\}$ is defined and equal to $\int \prod_{i\in J} g_i d\nu$;

(‡) for every $W \in \Sigma$ there is a W' in the σ-algebra generated by $\{\{x : x \in \Delta,\ x(i) \in E\} : i \in I,$ $E \in \Sigma_i\}$ such that $\mu(W \triangle W') = 0$.

Remark If μ is a relative product measure of $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ over ν, then all the functions $x \mapsto x(i) : \Delta \to X_i$ are inverse-measure-preserving. **P** The condition (†) tells us that if $E \in \Sigma_i$ and g is any Radon-Nikodým derivative of $F \mapsto \mu_i(E \cap \pi_i^{-1}[F])$, then

$$\mu\{x : x(i) \in E\} = \int g\, d\nu = \mu_i E. \ \mathbf{Q}$$

It follows that if I is not empty then we have an inverse-measure-preserving function $\pi : \Delta \to Y$ defined by setting $\pi x = \pi_i x(i)$ whenever $x \in \Delta$ and $i \in I$.

Note that when verifying (†) we need check the equality $\mu\{x : x \in \Delta,\ x(i) \in E_i$ for every $i \in J\} = \int \prod_{i\in J} g_i d\nu$ for only one representative family $\langle g_i\rangle_{i\in J}$ of Radon-Nikodým derivatives for any given $\langle E_i\rangle_{i\in J}$.

458R Proposition Suppose that $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ is a family of probability spaces, (Y, T, ν) a probability space, $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each $i \in I$, Δ the fiber product of $\langle(X_i, \pi_i)\rangle_{i\in I}$ and μ a relative product measure of $\langle(\mu_i, \pi_i)\rangle_{i\in I}$. Let $(\mathfrak{A}_i, \bar{\mu}_i)$, $(\mathfrak{C}, \bar{\nu})$ and $(\mathfrak{A}, \bar{\mu})$ be the measure algebras of μ_i, ν and μ respectively, and for $i \in I$ define $\bar{\pi}_i : \mathfrak{C} \to \mathfrak{A}_i$ and $\bar{\phi}_i : \mathfrak{A}_i \to \mathfrak{A}$ by setting $\bar{\pi}_i F^\bullet = \pi_i^{-1}[F]^\bullet$, $\bar{\phi}_i E^\bullet = \{x : x \in \Delta,$ $x(i) \in E\}^\bullet$ whenever $F \in \mathrm{T}$ and $E \in \Sigma_i$. Then $(\mathfrak{A}, \bar{\mu}, \langle\bar{\phi}_i\rangle_{i\in I})$ is a relative free product of $\langle(\mathfrak{A}_i, \bar{\mu}_i, \bar{\pi}_i)\rangle_{i\in I}$ over $(\mathfrak{C}, \bar{\nu})$.

proof The case $I = \emptyset$ is trivial (if you care to follow through the definitions to the letter, $\Delta = \prod_{i\in I} X_i = \{\emptyset\}$ and $\mathfrak{A}$ is the two-point algebra). So I will take it that I is not empty. For $i \in I$ define $\phi_i : \Delta \to X_i$ by setting $\phi_i(x) = x(i)$ for $x \in \Delta$.

(a) Of course we have to check that all the $\bar{\pi}_i$ and $\bar{\phi}_i$ are measure-preserving Boolean homomorphisms between the appropriate algebras, but in view of the remark following the definition 458Q, this is elementary. The condition that $\mathfrak{A}$ should be the closed subalgebra generated by $\bigcup_{i\in I} \bar{\phi}_i[\mathfrak{A}_i]$ is just a translation of the condition (‡).

[31] Formerly 315J.

(b) As I is not empty, we have a well-defined inverse-measure-preserving map $\pi : \Delta \to Y$ given by the formula $\pi(x) = \pi_i x(i)$ whenever $x \in \Delta$ and $i \in I$. Let $\bar{\pi} : \mathfrak{C} \to \mathfrak{A}$ be the corresponding measure-preserving homomorphism, so that $\bar{\pi} = \bar{\phi}_i \bar{\pi}_i$ for every i. Set $\mathfrak{D} = \bar{\pi}[\mathfrak{C}] \subseteq \mathfrak{A}$, and let $T : L^\infty(\mathfrak{C}) \to L^\infty(\mathfrak{D})$ be the f-algebra isomorphism corresponding to $\bar{\pi}$ (363F). For $i \in I$ and $E \in \Sigma_i$, let g_{iE} be a Radon-Nikodým derivative with respect to ν of the functional $F \mapsto \mu_i(E \cap \pi_i^{-1}[F])$, and set $u_{iE} = Tg_{iE}^\bullet \in L^\infty(\mathfrak{D})$. Then u_{iE} is the conditional expectation of $\chi\{x : x(i) \in E\}^\bullet$ on $\mathfrak{D}$. **P** If $d \in \mathfrak{D}$, it is of the form $\bar{\pi} F^\bullet = \bar{\phi}_i \bar{\pi}_i F^\bullet$ where $F \in \mathrm{T}$, so that $\chi d = T(\chi F)^\bullet$ and

$$\begin{aligned}\int_d u_{iE} d\bar{\mu} &= \int u_{iE} \times \chi d \, d\bar{\mu} = \int T(g_{iE}^\bullet \times \chi F^\bullet) d\bar{\mu} \\ &= \int g_{iE}^\bullet \times \chi F^\bullet d\bar{\nu} = \int_F g_{iE} d\nu \\ &= \mu_i(E \cap \pi_i^{-1}[F]) = \mu(\phi_i^{-1}[E] \cap \phi_i^{-1}[\pi_i^{-1}[F]]) = \bar{\mu}(d \cap \phi_i^{-1}[E]^\bullet).\end{aligned}$$

As d is arbitrary, we have the result. **Q**

(c) It follows that if $J \subseteq I$ is finite and not empty, and $a_i \in \bar{\phi}_i[\mathfrak{A}_i]$ and v_i is the conditional expectation of χa_i on $\mathfrak{D}$ for each $i \in J$, then $\bar{\mu}(\inf_{i \in J} a_i) = \int \prod_{i \in J} v_i d\bar{\mu}$. **P** Express a_i as $\phi_i^{-1}[E_i]^\bullet$, where $E_i \in \Sigma_i$, so that $v_i = u_{i,E_i}$ for each i. Then

$$\begin{aligned}\int \prod_{i \in J} v_i d\bar{\mu} &= \int \prod_{i \in J} T g_{i,E_i}^\bullet d\bar{\mu} = \int T(\prod_{i \in J} g_{i,E_i}^\bullet) d\bar{\mu} = \int \prod_{i \in J} g_{i,E_i}^\bullet d\bar{\nu} \\ &= \int \prod_{i \in J} g_{i,E_i} d\nu = \mu(\bigcap_{i \in J} \phi_i^{-1}[E_i]) = \bar{\mu}(\inf_{i \in J} a_i). \ \mathbf{Q}\end{aligned}$$

But this is exactly what we need to know to see that $\langle \bar{\phi}_i[\mathfrak{A}_i] \rangle_{i \in I}$ is relatively independent over $\mathfrak{D}$, completing the proof that $(\mathfrak{A}, \bar{\mu}, \langle \bar{\phi}_i \rangle_{i \in I})$ is a relative free product of $\langle (\mathfrak{A}_i, \bar{\mu}_i, \bar{\pi}_i) \rangle_{i \in I}$ over $(\mathfrak{C}, \bar{\nu})$.

458S There is no general result on relative product measures to match 458O (see 458Xj-458Xm). The general question of when we can expect relative product measures to exist seems interesting (458Ye, 458Yf). Here I give a couple of sample results dealing with important special cases.

Proposition Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of probability spaces, (Y, T, ν) a probability space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each i. Suppose that for each i we have a disintegration $\langle \mu_{iy} \rangle_{y \in Y}$ of μ_i such that $\mu_{iy}^* \pi_i^{-1}[\{y\}] = \mu_{iy} X_i = 1$ for every $y \in Y$. Let $\Delta \subseteq \prod_{i \in I} X_i$ be the fiber product of $\langle (X_i, \pi_i) \rangle_{i \in I}$, and Υ the subspace σ-algebra on Δ induced by $\widehat{\bigotimes}_{i \in I} \Sigma_i$. For $y \in Y$, let λ_y be the product of $\langle \mu_{iy} \rangle_{i \in I}$, $(\lambda_y)_\Delta$ the subspace measure on Δ and λ'_y its restriction to Υ. Then $\mu W = \int \lambda'_y W \nu(dy)$ is defined for every $W \in \Upsilon$, and μ is a relative product measure of $\langle (\mu_i, \pi_i) \rangle_{i \in I}$ over ν.

proof If $y \in Y$, then

$$\lambda'_y \Delta = \lambda_y^* \Delta = (\lambda_y)^*(\prod_{i \in I} \pi_i^{-1}[\{y\}]) = 1$$

(254Lb). For $i \in I$ and $E \in \Sigma_i$ set $g_{iE}(y) = \mu_{iy} E$ when this is defined; then g_{iE} is a Radon-Nikodým derivative of $F \mapsto \mu_i(E \cap \pi_i^{-1}[F]) : \mathrm{T} \to [0,1]$ (452Qa). Write X for $\prod_{i \in I} X_i$; for $i \in I$ and $x \in X$ set $\phi_i(x) = x(i)$. If $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for each $i \in J$, then

$$\begin{aligned}\int \lambda'_y(\Delta \cap \bigcap_{i \in J} \phi_i^{-1}[E_i]) \nu(dy) &= \int (\lambda_y)_\Delta(\Delta \cap \bigcap_{i \in J} \phi_i^{-1}[E_i]) \nu(dy) \\ &= \int \lambda_y(X \cap \bigcap_{i \in J} \phi_i^{-1}[E_i]) \nu(dy)\end{aligned}$$

(because $\lambda_y^* \Delta = 1$ and λ_y measures every $\phi_i^{-1}[E_i]$ for almost every y)

$$= \int \prod_{i \in J} \mu_{iy} E_i \nu(dy) = \int \prod_{i \in J} g_{i,E_i} d\nu.$$

In particular, $\int \lambda'_y(\Delta \cap \bigcap_{i \in J} \phi_i^{-1}[E_i]) \nu(dy)$ is defined.

The set $\{W : W \subseteq X, \int \lambda'_y(W \cap \Delta)\nu(dy)$ is defined$\}$ is a Dynkin class of subsets of X containing $\bigcap_{i\in J} \phi_i^{-1}[E_i]$ whenever $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for each $i \in J$; by the Monotone Class Theorem, it includes $\widehat{\bigotimes}_{i\in I}\Sigma_i$. So $\mu W = \int \lambda'_y W\nu(dy)$ is defined for every $W \in \Upsilon$. Moreover, the formula displayed above tells us that $\mu(\Delta \cap \bigcap_{i\in I} \phi_i^{-1}[E_i]) = \int \prod_{i\in J} g_{i,E_i} d\nu$ whenever $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for each $i \in I$. Thus (†) of 458Q is satisfied. And (‡) is true by the choice of Υ.

458T The latitude I have permitted in the definition of 'relative product' makes it possible to look for relative product measures with further properties, as in the following.

Proposition Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of compact Radon probability spaces, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ a Radon probability space, and $\pi_i : X_i \to Y$ a continuous inverse-measure-preserving function for each i. Then $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure μ over ν which is a Radon measure for the topology on the fiber product of $\langle(X_i, \pi_i)\rangle_{i\in I}$ induced by the product topology on $\prod_{i\in I} X_i$.

proof (a) For each $i \in I$, $E \in \Sigma_i$ let g_{iE} be a Radon-Nikodým derivative of $F \mapsto \mu_i(E \cap \pi_i^{-1}[F]) : \mathrm{T} \to [0, 1]$. Let $\mathcal{C}$ be the family of measurable cylinders in $X = \prod_{i\in I} X_i$; write $\phi_i(x) = x(i)$ for $x \in X$ and $i \in I$. We have a functional $\lambda_0 : \mathcal{C} \to [0, 1]$ defined by setting

$$\lambda_0(\bigcap_{i\in J} \phi_i^{-1}[E_i]) = \int \prod_{i\in J} g_{i,E_i} d\nu$$

whenever $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for every $i \in I$. It is easy to check that λ_0 is additive in the sense of 454E so (because every μ_i is perfect, by 416Wa and 343K) it has an extension to a measure λ on X with domain $\widehat{\bigotimes}_{i\in I}\Sigma_i$. By 454Aa, with $\mathcal{K}$ the family of compact subsets of X, λ is inner regular with respect to the compact sets. By 416O, there is a Radon measure $\tilde\lambda$ on X extending λ.

(b) Let Δ be the fiber product of $\langle(X_i, \pi_i)\rangle_{i\in I}$. Now the point is that Δ is $\tilde\lambda$-conegligible. **P** Because every π_i is continuous, Δ is closed. **?** If it is not conegligible, then, because $\tilde\lambda$ is τ-additive, there must be a basic open set of non-zero measure disjoint from Δ; express such a set as $W = \prod_{i\in J} \phi_i^{-1}[G_i]$ where $J \subseteq I$ is finite and $G_i \subseteq X_i$ is open for each $i \in J$. Because $\tilde\lambda$ is inner regular with respect to the compact sets, there is a compact set $K \subseteq W$ such that $\tilde\lambda K > 0$; setting $K_i = \phi_i[K]$, $K_i \subseteq G_i$ is compact for each i and $W' = \bigcap_{i\in J} \phi_i^{-1}[K_i]$ is non-negligible. Now we have

$$0 < \tilde\lambda(\prod_{i\in J} \phi_i^{-1}[K_i]) = \lambda_0(\prod_{i\in J} \phi_i^{-1}[K_i]) = \int \prod_{i\in J} g_{i,K_i} d\nu,$$

so $F = \{y : y \in Y, g_{i,K_i}(y) > 0$ for every $i \in J\}$ is non-negligible. On the other hand, for each $i \in J$ we have

$$\int_{Y\setminus\pi_i[K_i]} g_{i,K_i} d\nu = \mu_i(K_i \cap \pi_i^{-1}[Y \setminus K_i]) = 0$$

so that $F \setminus \pi_i[K_i]$ is negligible. Accordingly $\bigcap_{i\in J} \pi_i[K_i]$ is non-negligible, and must meet the support Y_0 of Y; let y be any point of the intersection. For $i \in J$, choose $x(i) \in K_i$ such that $\pi_i x(i) = y$. For $i \in I \setminus J$, $\pi_i[X_i]$ is a compact subset of Y, and $\nu\pi_i[X_i] = \mu_i\pi_i^{-1}[\pi_i[X_i]] = 1$, so $Y_0 \subseteq \pi_i[X_i]$ and we can therefore choose $x(i) \in X_i$ with $\pi_i x(i) = y$. This defines $x \in \Delta$. But as $x(i) \in K_i$ for $i \in J$, we also have

$$x \in \bigcap_{i\in J} \phi_i^{-1}[K_i] \subseteq \bigcap_{i\in J} \phi_i^{-1}[G_i] \subseteq X \setminus \Delta,$$

which is impossible. **X**

Thus Δ is $\tilde\lambda$-conegligible, as claimed. **Q**

(c) Let μ be the subspace measure on Δ induced by $\tilde\lambda$, and Σ its domain, so that μ is a Radon probability measure on Δ with its subspace topology (416Rb). Concerning (†) of 458Q, if $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for $i \in J$, then

$$\mu(\Delta \cap \bigcap_{i\in J} \phi_i^{-1}[E_i]) = \tilde\lambda(\bigcap_{i\in J} \phi_i^{-1}[E_i]) = \lambda_0(\bigcap_{i\in J} \phi_i^{-1}[E_i]) = \int \prod_{i\in J} g_{i,E_i} d\nu,$$

as required. Finally, for (‡), the σ-algebra Υ of subsets of Δ generated by $\{\Delta \cap \phi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$ is just the subspace σ-algebra induced by $\widehat{\bigotimes}_{i\in I}\Sigma_i$. Let $\mathfrak{A}$ be the measure algebra of $\tilde\lambda$ and $\mathfrak{B} \subseteq \mathfrak{A}$ the set $\{W^\bullet : W \in \widehat{\bigotimes}_{i\in I}\Sigma_i\}$. Then $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$. If $W \subseteq X$ is open, then for every $\epsilon > 0$ there is a $W_0 \in \widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $W_0 \subseteq W$ and $\tilde\lambda(W \setminus W_0) \le \epsilon$, so $W^\bullet \in \mathfrak{B}$; accordingly $\{W : W^\bullet \in \mathfrak{B}\}$ contains every open set and every Borel set and must be the whole of $\operatorname{dom}\lambda$. Returning to the measure μ, we see that if $W \in \Sigma$ there must be a $W_0 \in \widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $\tilde\lambda(W \triangle W_0) = 0$; now $W_0 \cap \Delta \in \Upsilon$ and $\mu(W \triangle (W_0 \cap \Delta)) = 0$. So (‡) also is true, and we have a relative product measure of the declared type.

458U We can of course make a general search through theorems about product measures, looking for ways of re-presenting them as theorems about relative product measures. There is an associative law, for instance (458Xr). To give an idea of what is to be expected, I offer a result corresponding to 253D.

Proposition Let (X_1,Σ_1,μ_1), (X_2,Σ_2,μ_2) and (Y,T,ν) be probability spaces, and $\pi_1 : X_1 \to Y$, $\pi_2 : X_2 \to Y$ inverse-measure-preserving functions. Let Δ be the fiber product of (X_1,π_1) and (X_2,π_2), and suppose that μ is a relative product measure of (μ_1,π_1) and (μ_2,π_2) over ν; set $\pi x = \pi_1 x(1) = \pi_2 x(2)$ for $x \in \Delta$. Take $f_1 \in \mathcal{L}^1(\mu_1)$ and $f_2 \in \mathcal{L}^2(\mu_2)$, and set $(f_1 \otimes f_2)(x) = f_1(x(1))f_2(x(2))$ when $x \in \Delta \cap (\operatorname{dom} f_1 \times \operatorname{dom} f_2)$. For $i = 1$, 2 let $g_i \in \mathcal{L}^1(\nu)$ be a Radon-Nikodým derivative of $F \mapsto \int_{\pi_i^{-1}[F]} f_i d\mu_i : \mathrm{T} \to \mathbb{R}$. Then $\int_F g_1 \times g_2 d\nu = \int_{\pi^{-1}[F]} f_1 \otimes f_2 d\mu$ for every $F \in \mathrm{T}$.

proof When f_1 and f_2 are indicator functions of measurable sets, this is just the definition of 'relative product measure'. The formula for g_i corresponds to a linear operator from $L^1(\mu_i)$ to $L^1(\nu)$, so the result is true for simple functions f_1 and f_2. If f_1 and f_2 are almost everywhere limits of non-decreasing sequences $\langle f_{1n}\rangle_{n\in\mathbb{N}}$, $\langle f_{2n}\rangle_{n\in\mathbb{N}}$ of non-negative simple functions, then the corresponding sequences $\langle g_{1n}\rangle_{n\in\mathbb{N}}$, $\langle g_{2n}\rangle_{n\in\mathbb{N}}$ will also be non-decreasing and non-negative and convergent to g_1, g_2 ν-a.e.; moreover, because $x \mapsto x(1)$ and $x \mapsto x(2)$ are inverse-measure-preserving, $f_1 \otimes f_2 = \lim_{n\to\infty} f_{1n} \otimes f_{2n}$ μ-a.e. So in this case we shall have

$$\begin{aligned}\int_{\pi^{-1}[F]} f_1 \otimes f_2 d\mu &= \lim_{n\to\infty} \int_{\pi^{-1}[F]} f_{1n} \otimes f_{2n} d\mu \\ &= \lim_{n\to\infty} \int_F g_{1n} \times g_{2n} d\nu = \int_F g_1 \times g_2 d\nu\end{aligned}$$

for every $F \in \mathrm{T}$. Finally, considering positive and negative parts, we can extend the result to general integrable f_1 and f_2.

458X Basic exercises >(a) Find an example of a probability space (X,Σ,μ) with σ-subalgebras Σ_1, Σ_2 and T of Σ such that Σ_1 and Σ_2 are independent but are not relatively independent over T.

(b) Let (X,Σ,μ) be a probability space and T, Σ_1 and Σ_2 σ-subalgebras of Σ. Show that if $\Sigma_1 \subseteq \mathrm{T}$ then Σ_1 and Σ_2 are relatively independent over T.

>(c) Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let $\langle \mathcal{E}_i\rangle_{i\in I}$ be a family of subsets of Σ such that (i) each $\mathcal{E}_i$ is closed under finite intersections (ii) $\langle E_i\rangle_{i\in I}$ is relatively independent over T whenever $E_i \in \mathcal{E}_i$ for every i. For each $i \in I$, let Σ_i be the σ-subalgebra of Σ generated by $\mathcal{E}_i$. Show that $\langle \Sigma_i\rangle_{i\in I}$ is relatively independent over T.

>(d) Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let f_1, f_2 be μ-integrable real-valued functions which are relatively independent over T, and suppose that $f_1 \times f_2$ is integrable. Let g_1, g_2 be conditional expectations of f_1, f_2 on T. Show that $g_1 \times g_2$ is a conditional expectation of $f_1 \times f_2$ on T.

(e) In 458I, show that (writing $\hat\mu$ for the completion of μ) $\hat\mu(F \cap f^{-1}[H]) = \int_F \nu_x H \mu(dx)$ for every $F \in \mathrm{T}$ and every universally measurable $H \subseteq \mathbb{R}$.

(f) Let (X,Σ,μ) be a probability space and Σ_1, Σ_2 and T σ-subalgebras. Show that the following are equiveridical: (i) Σ_1 and Σ_2 are relatively independent over T; (ii) whenever $f \in \mathcal{L}^1(\mu\restriction\Sigma_1)$ and g is a conditional expectation of f on T, then g is a conditional expectation of f on $\Sigma_2 \vee \mathrm{T}$.

(g) Prove 458Ld directly from 313G, without appealing to 458H.

(h) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras. Show that their probability algebra free product (325K) can be identified with their relative free product over $(\mathfrak{C},\bar\nu)$ if $\mathfrak{C}$ is the two-element Boolean algebra, $\bar\nu$ its unique probability measure, and $\pi_i : \mathfrak{C} \to \mathfrak{A}_i$ the trivial Boolean homomorphism for every i.

>(i) Let Y be a set, $\langle Z_i\rangle_{i\in I}$ a family of sets, and $\pi_i : Y \times Z_i \to Y$ the canonical map for each i. Show that the fiber product of $\langle(Y \times Z_i,\pi_i)\rangle_{i\in I}$ can be identified with $Y \times \prod_{i\in I} Z_i$.

(j) Let ν be Lebesgue measure on $[0,1]$, and X_1, $X_2 \subseteq [0,1]$ disjoint sets with outer measure 1. For each $i \in \{1,2\}$ let μ_i be the subspace measure on X_i and $\pi_i : X_i \to [0,1]$ the identity map. Show that (μ_1,π_1) and (μ_2,π_2) have no relative product measure over ν.

(k) Let ν be the usual measure on the split interval $I^{\parallel}$ (343J), and μ Lebesgue measure on $[0,1]$. Set $\pi_1(t) = t^+$, $\pi_2(t) = t^-$ for $t \in [0,1]$. Show that (μ, π_1) and (μ, π_2) have no relative product measure over ν.

(l) Let ν be Lebesgue measure on $[0,1]$. For each $t \in [0,1]$, set $X_t = [0,1] \setminus \{t\}$; let μ_t be the subspace measure on X_t and $\pi_t : X_t \to [0,1]$ the identity map. Show that $\langle(\mu_t, \pi_t)\rangle_{t\in[0,1]}$ has no relative product measure over ν.

(m)(i) Show that there is a set $X \subseteq [0,1]^2$ with outer planar Lebesgue measure 1 and just one point in each vertical section. (*Hint*: 419H-419I.) (ii) Set $X_1 = X_2 = X$ and $\mu_1 = \mu_2$ the subspace measure on X; let (Y, T, ν) be $[0,1]$ with Lebesgue measure, and $\pi_1 = \pi_2$ the first-coordinate projection from X to Y. Show that (μ_1, π_1) and (μ_2, π_2) have no relative product measure over ν.

(n) Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ, all including T. Set $\pi_i(x) = x$ for every $x \in X$. Show that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T iff $\mu\restriction\Sigma^*$ is a relative product measure of $\langle(\mu\restriction\Sigma_i, \pi_i)\rangle_{i\in I}$ over $\mu\restriction\mathrm{T}$, where $\Sigma^* = \bigvee_{i\in I}\Sigma_i$.

(o)(i) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces and (X, Σ, μ) their ordinary probability space product. Show that μ is a relative product measure of $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ over ν where Y is a singleton set, ν its unique probability measure, and $\pi_i : X_i \to Y$ the unique function for each i. (ii) Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces and $(X, \mathfrak{T}, \Sigma, \mu)$ their quasi-Radon probability space product (417R). Show that μ is a relative product measure of $\langle\mu_i\rangle_{i\in I}$ in the same sense as in (i).

(p) Suppose that $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ is a family of probability spaces and (Y, T, ν) is a probability space, and that for each $i \in I$ we are given an inverse-measure-preserving function $\pi_i : X_i \to Y$. Write $\hat\mu_i$ and $\hat\nu$ for the completions of μ_i, ν respectively. Show that $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure over ν iff $\langle(\hat\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure over $\hat\nu$.

(q) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y, T, ν) a probability space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each $i \in I$. Show that if $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure over ν, so does $\langle(\mu_i, \pi_i)\rangle_{i\in J}$ for any $J \subseteq I$.

(r) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y, T, ν) a probability space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each $i \in I$. Let $\langle J_k\rangle_{k\in K}$ be a partition of I into non-empty sets. For each $k \in K$, let Δ_k be the fiber product of $\langle(X_i, \pi_i)\rangle_{i\in J_k}$; suppose that $\tilde\mu_k$ is a relative product measure of $\langle(\mu_i, \pi_i)\rangle_{i\in J_k}$. Define $\tilde\pi_k : \Delta_k \to Y$ by setting $\tilde\pi_k(x) = \pi_i x(i)$ whenever $x \in \Delta_k$ and $i \in J_k$, so that $\tilde\pi_k$ is inverse-measure-preserving. Suppose that μ is a relative product measure of $\langle(\tilde\mu_k, \tilde\pi_k)\rangle_{k\in K}$ over ν. Show that μ can be regarded as a relative product measure of $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ over ν.

(s) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ and $\langle(X'_i, \Sigma'_i, \mu'_i)\rangle_{i\in I}$ be two families of probability spaces, (Y, T, ν) and (Y', T', ν') probability spaces, and $\pi_i : X_i \to Y$, $\pi'_i : X'_i \to Y'$ inverse-measure-preserving functions for each i. Suppose that we have a measure space isomorphism $g : Y \to Y'$ and inverse-measure-preserving functions $f_i : X_i \to X'_i$, for $i \in I$, such that $g\pi_i = \pi'_i f_i$ for every i. Show that if there is a relative product measure of $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ over ν, then there is a relative product measure of $\langle(\mu'_i, \pi'_i)\rangle_{i\in I}$ over ν'.

(t) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a countable family of probability spaces, (Y, T, ν) a probability space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each i. Suppose that for each i we have a disintegration of μ_i over ν which is strongly consistent with π_i. Show that $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure over ν.

(u) Let (X, Σ, μ), (X', Σ', μ') and (Y, T, ν) be probability spaces. Suppose that $\pi : X \to Y$ and $\pi' : X' \to Y$ are inverse-measure-preserving functions, and that μ' has a disintegration $\langle\mu'_y\rangle_{y\in Y}$ over (Y, T, ν) which is strongly consistent with π'. Show that (μ, π) and (μ', π') have a relative product measure over ν. (*Hint*: set $\lambda W = \int \mu'_{\pi(x)} W[\{x\}]\mu(dx)$ for every $W \in \Sigma\widehat{\otimes}\Sigma'$.)

>(v) Let Y be a Hausdorff space, $\langle Z_i\rangle_{i\in I}$ a family of Hausdorff spaces, μ_i a Radon probability measure on $Z_i \times Y$ and $\pi_i : Y \times Z_i \to Y$ the canonical map for each i. Suppose that all the image measures $\mu_i\pi_i^{-1}$ on Y are the same, and that all but countably many of the Z_i are compact. Show that there is a Radon probability measure μ on $Y \times \prod_{i\in I} Z_i$ such that $\mu_i = \mu\phi_i^{-1}$ for each i, where $\phi_i(y, z) = (y, z(i))$ for $y \in Y$, $z \in \prod_{j\in I} Z_j$.

(w) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of Radon probability spaces, $(Y,\mathfrak{S},\mathrm{T},\nu)$ a Radon probability space, and $\pi_i:X_i\to Y$ an almost continuous inverse-measure-preserving function for each i. Show that $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over ν which is a Radon measure for the topology on the fiber product of $\langle(X_i,\pi_i)\rangle_{i\in I}$ induced by the product topology on $\prod_{i\in I}X_i$. Discuss the relation of this result to 418Q.

458Y Further exercises (a)(i) Let (X,Σ,μ) be a probability space, $\langle\mathrm{T}_n\rangle_{n\in\mathbb{N}}$ a non-increasing sequence of σ-subalgebras of Σ with intersection T, and $\langle\Sigma_i\rangle_{i\in I}$ a family of subalgebras of Σ. Suppose that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T_n for every n. Show that it is relatively independent over T. (*Hint*: 275K.) (ii) Give an example of a probability space (X,Σ,μ), a downwards-directed family $\mathbb{T}$ of σ-subalgebras of Σ, and a family $\langle E_i\rangle_{i\in I}$ in Σ which is relatively independent over T for every $\mathrm{T}\in\mathbb{T}$, but not over $\bigcap\mathbb{T}$.

(b) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_n\rangle_{n\in\mathbb{N}}$ a sequence of σ-subalgebras of Σ which is relatively independent over T. For each $n\in\mathbb{N}$ let Σ_n^* be $\bigvee_{m\ge n}\Sigma_m$, and set $\Sigma_\infty=\bigcap_{n\in\mathbb{N}}\Sigma_n^*$. Show that for every $E\in\Sigma_\infty$ there is an $F\in\mathrm{T}$ such that $E\triangle F$ is negligible. (Compare 272O.)

(c) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and f, $g\in\mathcal{L}^0(\mu)$ relatively independent over T; suppose that $\langle\nu_x\rangle_{x\in X}$ and $\langle\nu'_x\rangle_{x\in X}$ are relative distributions of f and g over T. Show that $\langle\nu_x*\nu'_x\rangle_{x\in X}$ is a relative distribution of $f+g$ over T. (Compare 272T.)

(d) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}^2(\mu)$ such that $\langle f_n\rangle_{n\in\mathbb{N}}$ is relatively independent over T and $\int_F f_n d\mu=0$ for every $n\in\mathbb{N}$ and every $F\in\mathrm{T}$. (i) Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^\infty\frac{1}{\beta_n^2}\|f_n\|_2^2<\infty$. Show that $\lim_{n\to\infty}\frac{1}{\beta_n}\sum_{i=0}^n f_i=0$ a.e. (ii) Suppose that $\sup_{n\in\mathbb{N}}\|f_n\|_2<\infty$. Show that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f_i=0$ a.e. (Compare 273D.)

(e) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, (Y,T,ν) a measure space, and $\pi_i:X_i\to Y$ an inverse-measure-preserving function for each $i\in I$. For $i\in I$ and $E\in\Sigma_i$ let g_{iE} be a Radon-Nikodým derivative of the functional $F\mapsto\mu_i(E\cap\pi_i^{-1}[F])$. Let $\mathcal{C}$ be the family of measurable cylinders in $X=\prod_{i\in I}X_i$. If $C=\{x:x\in X,\ x(i)\in E_i$ for every $i\in J\}$ where $J\subseteq I$ is finite and not empty and $E_i\in\Sigma_i$ for $i\in J$, set $\lambda_0C=\int\prod_{i\in J}g_{i,E_i}d\nu$. Let $\Delta\subseteq X$ be the fiber product of $\langle(X_i,\pi_i)\rangle_{i\in I}$. Show that the following are equiveridical: (i) $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over ν (ii) whenever $\langle C_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}$ covering Δ, $\sum_{n=0}^\infty\lambda_0C_n\ge1$.

(f) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y,T,ν) a probability space, and $\pi_i:X_i\to Y$ a surjective inverse-measure-preserving function for each $i\in I$. Suppose that $\langle(\mu_i,\pi_i)\rangle_{i\in J}$ has a relative product measure over ν for every countable $J\subseteq I$. Show that $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over ν.

(g) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of perfect probability spaces, (Y,T,ν) a countably separated probability space, and $\pi_i:X_i\to Y$ an inverse-measure-preserving function for each $i\in I$. Show that $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over ν.

(h) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Let $\mathfrak{C}_0\subseteq\mathfrak{C}$ be the core subalgebra of countable Maharam type described in the canonical form of such structures given in 333N. Show that there is a closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, including $\mathfrak{C}_0$, such that $\mathfrak{B}$ and $\mathfrak{C}$ are relatively independent over $\mathfrak{C}_0$, and $\mathfrak{A}$ is the closed subalgebra of itself generated by $\mathfrak{B}\cup\mathfrak{C}$.

(i) Let X be a set, Σ a σ-algebra of subsets of X, and T a σ-subalgebra of Σ. Let $\langle\mathcal{E}_i\rangle_{i\in I}$ be a family of subsets of Σ such that (i) $E\cap F\in\mathcal{E}_i$ whenever $i\in I$ and E, $F\in\mathcal{E}_i$ (ii) $\langle E_i\rangle_{i\in I}$ is relatively independent over T whenever $E_i\in\mathcal{E}_i$ for every $i\in I$. For each $i\in I$, let Σ_i be the σ-algebra generated by $\mathcal{E}_i$. Show that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

458 Notes and comments The elementary theory of relative independence has two aspects. First, there is the matter of systematically formulating and verifying appropriate variations on standard results on stochastic independence; 458F, 458H, 458J, 458K, 458Xd, 458Yb-458Yd come under this heading. More interestingly, we study the new phenomena associated with changes in the core σ-algebras, as in 458C, 458D and 458Xa.

At a couple of points in Volume 3 (Dye's theorem, in §388, and Kawada's theorem, in §395) I took the trouble to generalize standard theorems to 'non-ergodic' forms. In both 388L and 395P the results are complicated by potentially non-trivial closed subalgebras of the probability algebra we are studying. I remarked on both occasions that the generalization is only a matter of technique, but I do not suppose that it was obvious just why this must be so. It is however a fundamental theorem of the topic of 'random reals' in the theory of forcing that *any* theorem about probability algebras must have a relativized form as a theorem about probability algebras with arbitrary closed subalgebras. The concept of 'relative Maharam type' from §333, for instance, is what matches 'Maharam type' for simple algebras; the concept of 'exchangeable' sequence (definition: 459C) is what matches 'independent identically distributed' sequence. (In probability theory, the keyword is 'mixture'.) In this section I present another example in the idea of 'relatively independent' closed subalgebras (458L-458M). I should emphasize that the forcing method, when we eventually come to it in §556 in Volume 5, will not as a rule apply directly to measure spaces; it deals with measure algebras. But of course the ideas generated by this theory can often be profitably applied to constructions in measure spaces, and this is what I am seeking to do with relatively independent σ-algebras and relative product measures.

Just as independent σ-algebras are associated with product spaces (272J), relatively independent algebras are associated with relative products (458Xn). The archetype of a relative product measure is 458S; it is a kind of disintegrated product. It is frequently profitable to express the 'relative' concepts of measure theory in terms of disintegrations.

I introduce 'relative free products' of probability algebras before proceeding to measure spaces because the uniqueness property proved in 458O shows that we have an unambiguous definition. For measure spaces it seems for the moment better to leave ourselves a bit of freedom, not (for instance) favouring one product construction over another (458Xo). The requirement that a relative product measure be carried by the fiber product is seriously limiting (458Xj-458Xl, 458Ye), and forces us to seek strongly consistent disintegrations (458S), at least for uncountable products (see 458Xt). However, as we might hope, the special case of compact spaces with Radon measures and continuous functions is amenable to a different approach (458T); and we have a one-sided method for the product of two spaces (458Xu) which is reminiscent of 454C and 457F.

There are corresponding complications when we come to look at maps between different relative products. For measure algebras, we have a natural theorem (458P), based on the same algebraic considerations as the corresponding theorems in §§315 and 325; the only possibly surprising feature is the need to assume that $\psi : \mathfrak{C} \to \mathfrak{C}'$ is actually an isomorphism. For measure spaces there is a similar result (458Xs).

459 Symmetric measures and exchangeable random variables

Among the relatively independent families of random variables discussed in 458K, it is natural to give extra attention to those which are 'relatively identically distributed'. It turns out that these have a particularly appealing characterization as the 'exchangeable' families (459C). In the same way, among the measures on a product space X^I there is a special place for those which are invariant under permutations of coordinates (459E, 459H). A more abstract kind of permutation-invariance is examined in 495I-495J.

459A The following elementary fact seems to have gone unmentioned so far.

Lemma Let (X, Σ, μ) and (Y, T, ν) be probability spaces and $\phi : X \to Y$ an inverse-measure-preserving function; set $\Sigma_0 = \{\phi^{-1}[F] : F \in \mathrm{T}\}$. Let T_1 be a σ-subalgebra of T and $\Sigma_1 = \{f^{-1}[F] : F \in \mathrm{T}_1\}$. If $g \in \mathcal{L}^1(\nu)$ and h is a conditional expectation of g on T_1, then $h\phi$ is a conditional expectation of $g\phi$ on Σ_1.

proof h is $\nu{\restriction}\mathrm{T}_1$-integrable and ϕ is inverse-measure-preserving for $\mu{\restriction}\Sigma_1$ and $\nu{\restriction}\mathrm{T}_1$, so $h\phi$ is $\mu{\restriction}\Sigma_1$-integrable (235G). If $E \in \Sigma_1$ then there is an $F \in \mathrm{T}_1$ such that $E = \phi^{-1}[F]$, and now

$$\int_E g\phi\,d\mu = \int_{f^{-1}[F]} g\phi\,d\mu = \int_F g\,d\nu = \int_F h\,d\nu = \int_E h\phi\,d\mu.$$

As E is arbitrary, $h\phi$ is a conditional expectation of $g\phi$ on Σ_1.

459B Theorem Let (X, Σ, μ) be a probability space, Z a set, Υ a σ-algebra of subsets of Z and $\langle f_i \rangle_{i \in I}$ an infinite family of (Σ, Υ)-measurable functions from X to Z. For each $i \in I$, set $\Sigma_i = \{f_i^{-1}[H] : H \in \Upsilon\}$. Then the following are equiveridical:

(i) whenever $i_0, \dots, i_r \in I$ are distinct, $j_0, \dots, j_r \in I$ are distinct, and $H_k \in \Upsilon$ for each $k \le r$, then $\mu(\bigcap_{k \le r} f_{i_k}^{-1}[H_k]) = \mu(\bigcap_{k \le r} f_{j_k}^{-1}[H_k])$;

(ii) there is a σ-subalgebra T of Σ such that

(α) $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over T,

(β) whenever i, $j \in I$, $H \in \Upsilon$ and $F \in \mathrm{T}$, then $\mu(F \cap f_i^{-1}[H]) = \mu(F \cap f_j^{-1}[H])$.

Moreover, if I is totally ordered by $\le$, we can add

(iii) whenever $i_0 < \ldots < i_r \in I$, $j_0 < \ldots < j_r \in I$ and $H_k \in \Upsilon$ for each $k \le r$, then $\mu(\bigcap_{k\le r} f_{i_k}^{-1}[H_k]) = \mu(\bigcap_{k\le r} f_{j_k}^{-1}[H_k])$.

proof (a) Since there is always some total order on I, we may assume that we have one from the start. Of course (i)$\Rightarrow$(iii). Also (ii)$\Rightarrow$(i). **P** Suppose that (ii) is true. Then (ii-β) tells us that for each $H \in \Upsilon$ there is a T-measurable function $g_H : X \to [0,1]$ which is a conditional expectation of $\chi(f_i^{-1}[H])$ on T for every $i \in I$. Now suppose that $i_0, \ldots, i_r \in I$ are distinct, $j_0, \ldots, j_r \in I$ are distinct, and $H_k \in \Upsilon$ for each $k \le r$. Then

$$\mu(\bigcap_{k\le r} f_{i_k}^{-1}[H_k]) = \int(\prod_{k=0}^{r} g_{H_k})d\mu = \mu(\bigcap_{k\le r} f_{j_k}^{-1}[H_k])$$

because $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T. So (i) is true. **Q**

So henceforth I will suppose that (iii) is true and seek to prove (ii).

(b) Suppose first that $I = \mathbb{N}$ with its usual ordering.

(α) For each n, $r \in \mathbb{N}$, let Σ_{nr} be the σ-subalgebra of Σ generated by $\bigcup_{n\le i\le n+r} \Sigma_i$; let T_n be the σ-algebra generated by $\bigcup_{r\in\mathbb{N}} \Sigma_{nr}$, and $\mathrm{T} = \bigcap_{n\in\mathbb{N}} \mathrm{T}_n$. For $n \in \mathbb{N}$ and $H \in \Upsilon$, let $g_{nH} : X \to \mathbb{R}$ be a T-measurable function which is a conditional expectation of $\chi f_n^{-1}[H]$ on T.

(β) (The key.) For any $n \in \mathbb{N}$ and Borel set $H \subseteq \mathbb{R}$, g_{nH} is a conditional expectation of $\chi f_n^{-1}[H]$ on T_{n+1}. **P** For m, $r \in \mathbb{N}$, let $h_{mr} : X \to [0,1]$ be a Σ_{mr}-measurable function which is a conditional expectation of $\chi f_n^{-1}[H]$ on Σ_{mr}; for $m \in \mathbb{N}$, set $h_m = \lim_{r\to\infty} h_{mr}$ where this is defined. By Lévy's martingale theorem (275I) h_m is defined almost everywhere and is a conditional expectation of $\chi f_n^{-1}[H]$ on T_m.

For m, $r \in \mathbb{N}$, define $F_{mr} : X \to Z^{r+2}$ by setting $F_{mr}(x) = (f_n(x), f_m(x), f_{m+1}(x), \ldots, f_{m+r}(x))$ for $x \in X$. At this point, examine the hypothesis (iii). This implies that if $m > n$ and $r \in \mathbb{N}$ then

$$\begin{aligned}\mu(F_{mr}^{-1}[H' \times H_0 \times \ldots \times H_r]) &= \mu(f_n^{-1}[H'] \cap \bigcap_{k\le r} f_{m+k}^{-1}[H_k]) \\ &= \mu(f_n^{-1}[H'] \cap \bigcap_{k\le r} f_{m+1+k}^{-1}[H_k]) \\ &= \mu(F_{m+1,r}^{-1}[H' \times H_0 \times \ldots \times H_r])\end{aligned}$$

for all H', $H_0, \ldots, H_r \in \Upsilon$. By the Monotone Class Theorem (136C), the image measures μF_{mr}^{-1} and $\mu F_{m+1,r}^{-1}$ agree on the σ-algebra $\widehat{\bigotimes}_{r+2}\Upsilon$ of subsets of Z^{r+2} generated by measurable cylinders; set

$$\lambda = \mu F_{mr}^{-1}\restriction\widehat{\bigotimes}_{r+2}\Upsilon = \mu F_{m+1,r}^{-1}\restriction\widehat{\bigotimes}_{r+2}\Upsilon.$$

Let Λ be the σ-subalgebra of $\widehat{\bigotimes}_{r+2}\Upsilon$ generated by sets of the form $Z \times H_0 \times \ldots \times H_r$ where $H_0, \ldots, H_r \in \Upsilon$, and let h be a conditional expectation of $\chi(H \times Z^{r+1})$ on Λ with respect to λ. Then 459A tells us that hF_{mr} is a conditional expectation of $\chi(f_n^{-1}[H])$ on Σ_{mr}, and is therefore equal almost everywhere to h_{mr}. Similarly, $hF_{m+1,r} =_{\text{a.e.}} h_{m+1,r}$, and this is true for every $r \in \mathbb{N}$. But as F_{mr} and $F_{m+1,r}$ are both inverse-measure-preserving for μ and λ, this means that h_{mr}, h and $h_{m+1,r}$ all have the same distribution. In particular, $\int h_{mr}^2 d\mu = \int h_{m+1,r}^2 d\mu$. Now $\langle h_{mr}\rangle_{r\in\mathbb{N}}$ and $\langle h_{m+1,r}\rangle_{r\in\mathbb{N}}$ converge almost everywhere to h_m and h_{m+1} respectively, so

$$\int h_m^2 d\mu = \lim_{r\to\infty}\int h_{mr}^2 d\mu = \lim_{r\to\infty}\int h_{m+1,r}^2 d\mu = \int h_{m+1}^2 d\mu.$$

On the other hand, $\mathrm{T}_{m+1} \subseteq \mathrm{T}_m$, so h_{m+1} is a conditional expectation of h_m on T_{m+1} (233Eh). This means that

$$\int h_m \times h_{m+1} d\mu = \int h_{m+1} \times h_{m+1} d\mu$$

(233Eg). A direct calculation tells us that $\int(h_m - h_{m+1})^2 d\mu = 0$, so that $h_m =_{\text{a.e.}} h_{m+1}$. Inducing on r, we see that $h_m =_{\text{a.e.}} h_r$ whenever $n < m \le r$.

Now the reverse martingale theorem (275K) tells us that $\lim_{m\to\infty} h_m$ is defined almost everywhere and is a conditional expectation of $\chi f_n^{-1}[H]$ on T, that is, is equal almost everywhere to g_{nH}. Since the h_m, for $m > n$, are equal almost everywhere, they are all equal to g_{nH} a.e. In particular, g_{nH} is equal a.e. to h_{n+1}, and is a conditional expectation of $\chi f_n^{-1}[H]$ on T_{n+1}. **Q**

(γ) If $n \in \mathbb{N}$ and $H_0, \ldots, H_r \in \Upsilon$, then $\prod_{i=0}^{r} g_{n+i,H_i}$ is a conditional expectation of $\chi(\bigcap_{i\le r} f_{n+i}^{-1}[H_i])$ on T. **P** Induce on r. For $r = 0$ this is just the definition of g_{nH_0}. For the inductive step to $r \ge 1$, observe that $g_{nH_0} \times \prod_{i=1}^{r} \chi f_{n+i}^{-1}[H_i]$ is a conditional expectation of $\prod_{i=0}^{r} \chi f_{n+i}^{-1}[H_i]$ on T_{n+1}, by 233Eg or 233K, because

g_{nH_0} is a conditional expectation of $\chi f_n^{-1}[H_0]$ on T_{n+1} and $\prod_{i=1}^r \chi f_{n+i}^{-1}[H_i]$ is T_{n+1}-measurable. But as (by the inductive hypothesis) $\prod_{i=1}^r g_{n+i,H_i}$ is a conditional expectation of $\prod_{i=1}^r \chi f_{n+i}^{-1}[H_i]$ on T, while g_{nH_0} is T-measurable, $\prod_{i=0}^r g_{n+i,H_i}$ is a conditional expectation of $\prod_{i=0}^r \chi f_{n+i}^{-1}[H_i]$ on T, by 233Eg/233K again. **Q**

(δ) In particular, $\prod_{i=0}^r g_{iH_i}$ is a conditional expectation of $\chi(\bigcap_{i\le r} f_i^{-1}[H_i])$ on T for every $r \in \mathbb{N}$ and $H_0, \dots, H_r \in \Upsilon$. This shows that $\langle \Sigma_n \rangle_{n\in\mathbb{N}}$ is relatively independent over T.

(ϵ) Now consider part (β) of the condition (ii). For this, observe that if $m > 0$, $H \in \Upsilon$ and $H_i \in \Upsilon$ for $i \le r$, then

$$\mu(f_0^{-1}[H] \cap \bigcap_{i\le r} f_{m+i+1}^{-1}[H_i]) = \mu(f_m^{-1}[H] \cap \bigcap_{i\le r} f_{m+i+1}^{-1}[H_i]).$$

By the Monotone Class Theorem,

$$\mu(F \cap f_0^{-1}[H]) = \mu(F \cap f_m^{-1}[H])$$

for any $F \in \mathrm{T}_{m+1}$ and in particular for any $F \in \mathrm{T}$. Thus (ii) is true.

(c) Now suppose that there is a strictly increasing sequence $\langle j_k \rangle_{k\in\mathbb{N}}$ in I. For each n, let T_n be the σ-algebra generated by $\bigcup_{k\ge n} \Sigma_{j_k}$, and set $\mathrm{T} = \bigcap_{n\in\mathbb{N}} \mathrm{T}_n$. Then (b), applied to $\langle f_{j_k} \rangle_{k\in\mathbb{N}}$, tells us that $\langle \Sigma_{j_k} \rangle_{k\in\mathbb{N}}$ is relatively independent over T and that for each $H \in \Upsilon$ there is a function g_H which is a conditional expectation of $\chi(f_{j_k}^{-1}[H])$ on T for every $k \in \mathbb{N}$.

(α) If $i_0, \dots, i_r \in I$ are distinct and $H_0, \dots, H_r \in \Upsilon$, then $\mu(\bigcap_{k\le r} f_{i_k}^{-1}[H_k]) = \mu(\bigcap_{k\le r} f_{j_k}^{-1}[H_k])$. **P** Let ρ be the permutation of $\{0, \dots, r\}$ such that $i_{\rho(0)} < i_{\rho(1)} < \dots < i_{\rho(r)}$. Then

$$\begin{aligned}\mu(\bigcap_{k\le r} f_{i_k}^{-1}[H_k]) &= \mu(\bigcap_{k\le r} f_{i_{\rho(k)}}^{-1}[H_{\rho(k)}]) = \mu(\bigcap_{k\le r} f_{j_k}^{-1}[H_{\rho(k)}]) \\ &= \int (\prod_{k=0}^r g_{H_{\rho(k)}}) d\mu = \int (\prod_{k=0}^r g_{H_k}) d\mu = \mu(\bigcap_{k\le r} f_{j_k}^{-1}[H_k]). \text{ } \mathbf{Q}\end{aligned}$$

(β) Now suppose that $i_0, \dots, i_r \in I$ are distinct. Then there is some $m \in \mathbb{N}$ such that $j_k \ne i_l$ for any $l \le r$ and $k \ge m$. In this case, consider the sequence $f_{i_0}, \dots, f_{i_r}, f_{j_m}, f_{j_{m+1}}, \dots$. By ($\alpha$) here, this sequence satisfies the condition (iii). We can therefore apply the construction of (b). But observe that the tail σ-algebra obtained from $f_{i_0}, \dots, f_{i_r}, f_{j_m}, \dots$ is precisely T, as defined from $\langle f_{j_k} \rangle_{k\in\mathbb{N}}$ just above. So $\langle \Sigma_{i_k} \rangle_{k\le r}$ is relatively independent over T. As $i_0, \dots, i_r$ are arbitrary, $\langle \Sigma_i \rangle_{i\in I}$ is relatively independent over T. At the same time we see that if $H \in \Upsilon$ then all the $\chi(f_{i_k}^{-1}[H])$ have the same conditional expectations over T as $\chi(f_{j_m}^{-1}[H])$. So (ii-β) is satisfied.

(d) Finally, if there is no strictly increasing sequence in I, then $(I, \ge)$ is well-ordered; since I is infinite, the well-ordering starts with an initial segment of order type ω, that is, a sequence $\langle j_k \rangle_{k\in\mathbb{N}}$ such that $j_0 > j_1 > \dots$. But note now that (iii) tells us that $\mu(\bigcap_{k\le r} f_{i_k}^{-1}[H_k]) = \mu(\bigcap_{k\le r} f_{j_k}^{-1}[H_k])$ whenever $i_0 > \dots > i_r$ and $j_0 > \dots > j_r$ and $H_k \in \Upsilon$ for every k. So we can apply (c) to $(I, \ge)$ to get the result in this case also.

459C Exchangeable random variables I spell out the leading special case of this theorem.

De Finetti's theorem Let (X, Σ, μ) be a probability space, and $\langle f_i \rangle_{i\in I}$ an infinite family in $\mathcal{L}^0(\mu)$. Then the following are equiveridical:

(i) the joint distribution of $(f_{i_0}, f_{i_1}, \dots, f_{i_r})$ is the same as the joint distribution of $(f_{j_0}, f_{j_1}, \dots, f_{j_r})$ whenever $i_0, \dots, i_r \in I$ are distinct and $j_0, \dots, j_r \in I$ are distinct;

(ii) there is a σ-subalgebra T of Σ such that $\langle f_i \rangle_{i\in I}$ is relatively independent over T and all the f_i have the same relative distributions over T.

Moreover, if I is totally ordered by $\le$, we can add

(iii) the joint distribution of $(f_{i_0}, f_{i_1}, \dots, f_{i_r})$ is the same as the joint distribution of $(f_{j_0}, f_{j_1}, \dots, f_{j_r})$ whenever $i_0 < \dots < i_r$ and $j_0 < \dots < j_r$ in I.

Remark Families of random variables satisfying the condition in (i) are called **exchangeable**. The equivalence of (i) and (ii) can be expressed by saying that 'an exchangeable family of random variables is a mixture of independent identically distributed families'.

proof Changing each f_i on a negligible set will not change either their joint distributions (271De) or their relative distributions over T or their relative independence; so we may suppose that every f_i is a Σ-measurable function from X to $\mathbb{R}$. Now look at 459B, taking (Z,Υ) to be $\mathbb{R}$ with its Borel σ-algebra. The condition 459B(i) reads

whenever $i_0,\dots,i_r \in I$ are distinct, $j_0,\dots,j_r \in I$ are distinct, and $H_k \in \Upsilon$ for each $k \le r$, then $\mu(\bigcap_{k\le r} f_{i_k}^{-1}[H_k]) = \mu(\bigcap_{k\le r} f_{j_k}^{-1}[H_k])$,

matching (i) here, by 271B; similarly, (iii) of 459B matches (iii) here. Equally, condition (ii) here is just a re-phrasing of 459B(ii) in the language of 458A and 458I. So 459B gives the result.

459D Specializing 459B in another direction, we have the case in which X is actually the product Z^I. In this case, the condition 459B(i) corresponds to a strong kind of symmetry in the measure μ. It now makes sense to look for subsets of $X = Z^I$ which are essentially invariant under permutations, and we have the following result.

Proposition Let Z be a set, Υ a σ-algebra of subsets of Z, I an infinite set and μ a measure on Z^I with domain the σ-algebra $\widehat{\bigotimes}_I\Upsilon$ generated by $\{\pi_i^{-1}[H] : i\in I,\ H\in\Upsilon\}$, taking $\pi_i(x) = x(i)$ for $x\in Z^I$ and $i\in I$. For each permutation ρ of I, define $\hat\rho : Z^I \to Z^I$ by setting $\hat\rho(x) = x\rho$ for $x \in Z^I$. Suppose that $\mu = \mu\hat\rho^{-1}$ for every ρ. Let $\mathcal{E}$ be the family of those sets $E \in \widehat{\bigotimes}_I\Upsilon$ such that $\mu(E\triangle\hat\rho^{-1}[E]) = 0$ for every permutation ρ of I, and $\mathcal{V}$ the family of those sets $V\in\widehat{\bigotimes}_I\Upsilon$ such that V is determined by coordinates in $I\setminus\{i\}$ for every $i \in I$.

(a) $\mathcal{E}$ is a σ-subalgebra of $\widehat{\bigotimes}_I\Upsilon$.

(b) $\mathcal{V}$ is a σ-subalgebra of $\mathcal{E}$.

(c) If $E\in\mathcal{E}$ and $J\subseteq I$ is infinite, then there is a $V\in\mathcal{V}$, determined by coordinates in J, such that $\mu(E\triangle V) = 0$.

(d) Setting $\Sigma_i = \{\pi_i^{-1}[H] : H\in\Upsilon\}$ for each $i \in I$,

(α) $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over $\mathcal{E}$,

(β) for every $H\in\Upsilon$ there is an $\mathcal{E}$-measurable function $g_H : Z^I\to[0,1]$ which is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on $\mathcal{E}$ for every $i\in I$.

proof (a) is elementary.

(b) Let $V\in\mathcal{V}$. Suppose that $\rho : I\to I$ is a permutation, $J\subseteq I$ is finite and $H_j\in\Upsilon$ for every $j\in J$. Then there is a permutation $\sigma : I\to I$ such that $\sigma(j) = \rho(j)$ for every $j\in J$ and $J' = \{i : \sigma(i)\ne i\}$ is finite. By 254Ta, V is determined by coordinates in $I\setminus J'$, so $\hat\sigma^{-1}[V] = V$. Now

$$\mu(\hat\rho^{-1}[V]\cap\bigcap_{j\in J}\pi_j^{-1}[H_j]) = \mu(V\cap\bigcap_{j\in J}\pi_{\rho(j)}^{-1}[H_j]) = \mu(V\cap\bigcap_{j\in J}\pi_{\sigma(j)}^{-1}[H_j])$$
$$= \mu(\hat\sigma^{-1}[V]\cap\bigcap_{j\in J}\pi_j^{-1}[H_j]) = \mu(V\cap\bigcap_{j\in J}\pi_j^{-1}[H_j]).$$

By the Monotone Class Theorem, as usual, $\mu(E\cap\hat\rho^{-1}[V]) = \mu(E\cap V)$ for every $E\in\widehat{\bigotimes}_I\Upsilon$. In particular, taking $E = V$ and $E = Z^I\setminus V$, we see that $V\triangle\hat\rho^{-1}[V]$ is negligible. As ρ is arbitrary, $V\in\mathcal{E}$.

This shows that $\mathcal{V}\subseteq\mathcal{E}$. Of course $\mathcal{V}$ is a σ-algebra, since it is just the intersection of the σ-algebras $\{V : V\in\widehat{\bigotimes}_I\Sigma$, V is determined by coordinates in $I\setminus\{i\}\}$.

(c) For each $n\in\mathbb{N}$, there is a set $E_n\in\widehat{\bigotimes}_I\Sigma$, determined by a finite set J_n of coordinates, such that $\mu(E\triangle E_n)\le 2^{-n}$. Choose permutations ρ_n of I such that $\langle\rho_n[J_n]\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of subsets of J. Set $F_n = \hat\rho_n^{-1}[E_n]$; then F_n is determined by coordinates in $\rho_n[J_n]$ for each $n\in\mathbb{N}$, so $V = \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}F_m$ belongs to $\mathcal{V}$ and is determined by coordinates in J. Also

$$\mu(E\triangle F_n) = \mu(\hat\rho[E]\triangle E_n) = \mu(E\triangle E_n)\le 2^{-n}$$

for each n, so $\mu(E\triangle V) = 0$, as required.

(d) Let $\langle j_n\rangle_{n\in\mathbb{N}}$ be any sequence of distinct points of I. Set $J = \{j_n : n\in\mathbb{N}\}$. For $n\in\mathbb{N}$ let T_n be the σ-algebra generated by $\bigcup_{k\ge n}\Sigma_{j_k}$, and set $\mathrm{T} = \bigcap_{n\in\mathbb{N}}\mathrm{T}_n$, so that $\mathrm{T} = \{V : V\in\mathcal{V},\ V$ is determined by coordinates in $J\}$. **P** Of course $\mathrm{T}\subseteq\mathcal{V}$ and every member of T is determined by coordinates in J, because every member of T_0 is. On the other hand, if $V\in\mathcal{V}$ is determined by coordinates in J, then fix some $w\in Z^{I\setminus J}$. In this case, identifying Z^I with $Z^J\times Z^{I\setminus J}$, the set $V_1 = \{z : z\in Z^J,\ (z,w)\in V\}$ must belong to $\widehat{\bigotimes}_J\Upsilon$, so $V = V_1\times Z^{I\setminus J}$ belongs to T_0. Applying the same idea to $J\setminus\{j_k : k<n\}$, we see that $V\in\mathrm{T}_n$ for every n, so that $V\in\mathrm{T}$. **Q**

Part (c) of the proof of 459B tells us that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T and that for every $H\in\Upsilon$ there is a T-measurable g_H which is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on T for every $i\in I$. Now (c) here tells us that g_H is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on $\mathcal{E}$; and examining the definition in 458Aa, we see that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over $\mathcal{E}$, as claimed.

459E If μ is countably compact, we have a strong disintegration theorem, as follows.

Theorem Let Z be a set, Υ a σ-algebra of subsets of Z, I an infinite set, and μ a countably compact probability measure on Z^I with domain the σ-algebra $\widehat{\bigotimes}_I\Upsilon$ generated by $\{\pi_i^{-1}[H] : i\in I,\ H\in\Upsilon\}$, taking $\pi_i(x)=x(i)$ for $x\in Z^I$ and $i\in I$. Then the following are equiveridical:

(i) for every permutation ρ of I, $x\mapsto x\rho : Z^I\to Z^I$ is inverse-measure-preserving for μ;

(ii) for every transposition ρ of two elements of I, $x\mapsto x\rho : Z^I\to Z^I$ is inverse-measure-preserving for μ;

(iii) for each $n\in\mathbb{N}$ and any two injective functions p, $q : n\to I$ the maps $x\mapsto xp : Z^I\to Z^n$, $x\mapsto xq : Z^I\to Z^n$ induce the same measure on Z^n;

(iv) there are a probability space (Y,T,ν) and a family $\langle\lambda_y\rangle_{y\in Y}$ of probability measures on Z such that $\langle\lambda_y^I\rangle_{y\in Y}$ is a disintegration of μ over ν, writing λ_y^I for the product of copies of λ_y.

Moreover, if I is totally ordered, we can add

(v) for each $n\in\mathbb{N}$ and any two strictly increasing functions p, $q : n\to I$ the maps $x\mapsto xp : Z^I\to Z^n$, $x\mapsto xq : Z^I\to Z^n$ induce the same measure on Z^n.

If the conditions (i)-(v) are satisfied, then there is a countably compact measure λ, with domain Υ, which is the common marginal measure of μ on every coordinate; and if $\mathcal{K}$ is a countably compact class of subsets of Z, closed under finite unions and countable intersections, such that λ is inner regular with respect to $\mathcal{K}$, then

(iv)′ there are a probability space (Y,T,ν) and a family $\langle\lambda_y\rangle_{y\in Y}$ of complete probability measures on Z, all with domains including $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, such that $\langle\lambda_y^I\rangle_{y\in Y}$ is a disintegration of μ over ν.

proof (a) Since any set I can be totally ordered, we may suppose from the outset that we have been given a total ordering $\le$ of I. I start with the easy bits.

(iv)′⇒(iv) is trivial, at least if there is a common countably compact marginal measure on Z.

(iv)⇒(i) If (iv) is true and $\rho : I\to I$ is a permutation, take any $E\in\widehat{\bigotimes}_I\Sigma$ and set $E'=\{x : x\in Z^I,\ x\rho\in E\}$. For any $y\in Y$, $x\mapsto x\rho$ is an isomorphism of the measure space (Z^I,λ_y^I), so

$$\mu E'=\int\lambda_y^I E'\,\nu(dy)=\int\lambda_y^I E\,\nu(dy)=\mu E.$$

As E is arbitrary, (i) is true.

(i)⇒(ii) is trivial.

(ii)⇒(iii) There is a permutation ρ of I such that $q=\rho p$ and ρ moves only finitely many points of I, that is, ρ is a product of transpositions. By (ii), $x\mapsto x\rho$ and $x\mapsto x\rho^{-1}$ are inverse-measure-preserving for μ, that is, are isomorphisms of (Z^I,μ). But this means that $x\mapsto xp$ and $x\mapsto x\rho p=xq$ must induce the same measure on Z^n.

(iii)⇒(v) is trivial.

(b) So for the rest of the proof I assume that (v) is true. Taking $n=1$ in the statement of (v), we see that there is a common image measure $\lambda=\mu\pi_i^{-1}$ for every $i\in I$. By 452R, λ is countably compact. Let $\mathcal{K}\subseteq\mathcal{P}Z$ be a countably compact class, closed under finite unions and countable intersections, such that λ is inner regular with respect to $\mathcal{K}$.

In 459B, set $X=Z^I$ and $\Sigma=\widehat{\bigotimes}_I\Upsilon$ and $f_i=\pi_i : X\to Z$ for $i\in I$. Then (v) here corresponds to (iii) of 459B, so (translating (ii) of 459B) we have a σ-subalgebra T of $\widehat{\bigotimes}_I\Upsilon$ and a family $\langle g_H\rangle_{H\in\Upsilon}$ of T-measurable functions from Z^I to $[0,1]$ such that

$$\mu(\bigcap_{i\in J}\pi_i^{-1}[H_i])=\int(\prod_{i\in J}g_{H_i})d\mu$$

whenever $J\subseteq I$ is finite and not empty and $H_i\in\Upsilon$ for $i\in J$. In particular, g_H is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on T whenever $H\in\Upsilon$ and $i\in I$.

Fix $i^*\in I$ for the moment. Set $\nu=\mu\restriction\mathrm{T}$. The inverse-measure-preserving function π_{i^*} from (X,μ) to (Z,λ) gives us an integral-preserving Riesz homomorphism $T_0 : L^\infty(\lambda)\to L^\infty(\mu)$ defined by setting $T_0h^\bullet=(h\pi_{i^*})^\bullet$ for every $h\in\mathcal{L}^\infty(\lambda)$. Let $P : L^1(\mu)\to L^1(\nu)$ be the conditional expectation operator; then $T=PT_0 : L^\infty(\lambda)\to L^\infty(\nu)$ is an integral-preserving positive linear operator, and $T(\chi Z^\bullet)=\chi X^\bullet$.

By 452H, we have a family $\langle\lambda_x\rangle_{x\in X}$ of complete probability measures on Z, all with domains including $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, such that $\int_F h\pi_{i^*}d\mu = \int_F\int_Z h\,d\lambda_x\nu(dx)$ for every $h\in\mathcal{L}^\infty(\lambda)$ and $F\in\mathrm{T}$. In particular, setting $g'_H(x)=\lambda_xH$ whenever $H\in\Upsilon$ and $x\in X$ are such that $H\in\operatorname{dom}\lambda_x$, then g'_H will be a conditional expectation of $\chi\pi_{i^*}^{-1}[H]$ on T, and will be equal ν-almost everywhere to g_H.

This means that if $J\subseteq I$ is finite and not empty and $H_i\in\Upsilon$ for $i\in J$,

$$\begin{aligned}\int_X\lambda_x^I(\bigcap_{i\in J}\pi_i^{-1}[H_i])\nu(dx) &= \int_X\prod_{i\in J}\lambda_xH_i\,\nu(dx) = \int_X\prod_{i\in J}g'_{H_i}d\nu\\ &= \int_X\prod_{i\in J}g_{H_i}d\nu = \int_X\prod_{i\in J}g_{H_i}d\mu = \mu(\bigcap_{i\in J}\pi_i^{-1}[H_i]).\end{aligned}$$

Thus the family $\mathcal{W}$ of sets $E\subseteq X$ such that $\int\lambda_x^IE\,\nu(dx)$ and μE are defined and equal contains all measurable cylinders. As $\mathcal{W}$ is a Dynkin class it includes $\widehat{\bigotimes}_I\Upsilon$. But this says exactly that $\langle\lambda_x^I\rangle_{x\in X}$ is a disintegration of μ over ν, as required by (iv)$'$.

Thus (v)$\Rightarrow$(iv)$'$ and the proof is complete.

459F Lemma Let X be a Hausdorff space and $P_{\mathrm{R}}(X)$ the space of Radon probability measures on X with its narrow topology (definition: 437Jd). If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of compact subsets of X, then $A=\{\mu:\mu\in P_R(X),\ \mu(\bigcup_{n\in\mathbb{N}}K_n)=1\}$ is a K-analytic subset of $P_R(X)$.

proof (Recall that $P_{\mathrm{R}}(X)$ is Hausdorff, by 437R(a-ii).) For each $n\in\mathbb{N}$, let C_n be the set of Radon measures on K_n with magnitude at most 1; by 437R(f-ii), C_n is compact in its narrow topology. Let C be the compact space $\prod_{n\in\mathbb{N}}C_n$; for the rest of this proof, I will use the formula $\boldsymbol{\mu}=\langle\mu_n\rangle_{n\in\mathbb{N}}$ to describe the coordinates of members of C. Define $\psi:C\to[0,1]^{\mathbb{N}}$ by setting $\psi(\boldsymbol{\mu})(n)=\mu_nK_n$ for $n\in\mathbb{N}$ and $\boldsymbol{\mu}\in C$. Then ψ is continuous. Since $B=\{\langle\alpha_n\rangle_{n\in\mathbb{N}}:\sum_{n=0}^\infty\alpha_n=1\}$ is a Borel subset of $[0,1]^{\mathbb{N}}$, therefore a Baire set (4A3Kb), $D=\psi^{-1}[B]$ is a Baire subset of C (4A3Kc), therefore Souslin-F (421L) and K-analytic (422Hb).

For $\boldsymbol{\mu}\in D$, define a function $\phi(\boldsymbol{\mu})$ by saying that

$$\phi(\boldsymbol{\mu})(E)=\textstyle\sum_{n=0}^\infty\mu_n(E\cap K_n)\text{ if }E\subseteq X\text{ and }\mu_n\text{ measures }E\cap K_n\text{ for every }n$$

and is undefined otherwise. It is easy to check that $\phi(\boldsymbol{\mu})\in P_{\mathrm{R}}(X)$. Also $\phi:D\to P_{\mathrm{R}}(X)$ is continuous. **P** If $G\subseteq X$ is open, then $\nu\mapsto\nu(G\cap K_n):C_n\to[0,1]$ and therefore $\boldsymbol{\mu}\mapsto\mu_n(G\cap K_n):D\to[0,1]$ are lower semi-continuous for each n (4A2B(d-ii)), so $\boldsymbol{\mu}\mapsto\phi(\boldsymbol{\mu})(G)$ is lower semi-continuous (4A2B(d-iii), 4A2B(d-v)), and $\{\boldsymbol{\mu}:\phi(\boldsymbol{\mu})(G)>\alpha\}$ is open for every α; by 4A2B(a-iii), ϕ is continuous. **Q**

Consequently $A=\phi[D]$ is K-analytic (422Gd).

459G Lemma Let X be a topological space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ a totally finite quasi-Radon measure space, $y\mapsto\mu_y$ a continuous function from Y to the space $M_{\mathrm{qR}}^+(X)$ of totally finite quasi-Radon measures on X with its narrow topology, and $\mathcal{U}$ a base for the topology of X, containing X and closed under finite intersections. If $\mu\in M_{\mathrm{qR}}^+(X)$ is such that $\mu U=\int\mu_yU\,\nu(dy)$ for every $U\in\mathcal{U}$, then $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of μ over ν.

proof (a) Let $\mathcal{E}$ be the family of subsets E of X such that μE and $\int\mu_yE\,\nu(dy)$ are defined and equal. Because $X\in\mathcal{E}$, $\mathcal{E}$ is a Dynkin class; as $\mathcal{U}$ is included in $\mathcal{E}$ and is closed under finite intersections, the σ-algebra of sets generated by $\mathcal{U}$ is included in $\mathcal{E}$, and in particular any finite union of members of $\mathcal{U}$ belongs to $\mathcal{E}$.

(b) In fact every open subset of X belongs to $\mathcal{E}$. **P** If $G\subseteq X$ is open, set $\mathcal{H}=\{H:H\subseteq G$ is a finite union of members of $\mathcal{U}\}$. Then $\mathcal{H}$ is upwards-directed and has union G. Set $f_H(y)=\mu_yH$ for $y\in Y$ and $H\in\mathcal{H}$. Since $\lambda\mapsto\lambda H:M_{\mathrm{qR}}^+(X)\to\mathbb{R}$ is lower semi-continuous (by the definition of the narrow topology) and $y\mapsto\mu_y$ is continuous, $f_H:Y\to\mathbb{R}$ is lower semi-continuous (4A2B(d-ii) again). Moreover, $\{f_H:H\in\mathcal{H}\}$ is an upwards-directed family of functions with supremum f_G, where $f_G(y)=\mu_yG$ for each y, because every μ_y is τ-additive. Now

$$\mu G=\sup_{H\in\mathcal{H}}\mu H=\sup_{H\in\mathcal{H}}\int f_Hd\nu=\int f_Gd\nu$$

(414Ba)

$$=\int\mu_yG\,\nu(dy)$$

and $G \in \mathcal{E}$. **Q**

(c) It follows that every Borel subset of X belongs to $\mathcal{E}$, that is, that $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of the restriction $\mu_{\mathcal{B}}$ to the Borel σ-algebra of X. Since every μ_y is complete, $\langle \mu_y \rangle_{y \in Y}$ is also a disintegration over ν of the completion of $\mu_{\mathcal{B}}$ (452B(a-ii)), which is μ.

459H Theorem Let Z be a Hausdorff space, I an infinite set, and $\tilde{\mu}$ a quasi-Radon probability measure on Z^I such that the marginal measures on each copy of Z are Radon measures. Write $P_{\mathrm{R}}(Z)$ for the set of Radon probability measures on Z with its narrow topology. Then the following are equiveridical:

(i) for every permutation ρ of I, $w \mapsto w\rho : Z^I \to Z^I$ is inverse-measure-preserving for $\tilde{\mu}$;

(ii) for every transposition ρ of two elements of I, $w \mapsto w\rho : Z^I \to Z^I$ is inverse-measure-preserving for $\tilde{\mu}$;

(iii) for each $n \in \mathbb{N}$ and any two injective functions p, $q : n \to I$ the maps $w \mapsto wp : Z^I \to Z^n$ and $w \mapsto wq : Z^I \to Z^n$ induce the same measure on Z^n;

(iv) there are a probability space (Y, T, ν) and a family $\langle \mu_y \rangle_{y \in Y}$ of τ-additive Borel probability measures on Z such that $\langle \tilde{\mu}_y^I \rangle_{y \in Y}$ is a disintegration of $\tilde{\mu}$ over ν, writing $\tilde{\mu}_y^I$ for the τ-additive product of copies of μ_y;

(v) there is a Radon probability measure $\tilde{\nu}$ on $P_{\mathrm{R}}(Z)$ such that $\langle \tilde{\theta}^I \rangle_{\theta \in P_{\mathrm{R}}(Z)}$ is disintegration of $\tilde{\mu}$ over $\tilde{\nu}$, writing $\tilde{\theta}^I$ for the quasi-Radon product of copies of θ.

Moreover, if I is totally ordered, we can add

(vi) for each $n \in \mathbb{N}$ and any two strictly increasing functions $p, q : n \to I$ the maps $w \mapsto wp : Z^I \to Z^n$ and $w \mapsto wq : Z^I \to Z^n$ induce the same measure on Z^n.

proof (a) As in 459E, we need consider only the case in which I is totally ordered, and the implications

$$\text{(v)}\Rightarrow\text{(iv)}\Rightarrow\text{(i)}\Rightarrow\text{(ii)}\Rightarrow\text{(iii)}\Rightarrow\text{(vi)}$$

are elementary. So henceforth I will suppose that (vi) is true and seek to prove (v).

(b) We are going to need a second topology on the set Z, so I will call the original topology $\mathfrak{T}$, and for the rest of this proof I will declare the topology on which each topological concept or construction is based. Write μ for $\tilde{\mu} \restriction \widehat{\bigotimes}_I \mathcal{B}(Z, \mathfrak{T})$, where $\mathcal{B}(Z, \mathfrak{T})$ is the Borel σ-algebra of Z for the topology $\mathfrak{T}$. Then (vi) is also true of μ. (Strictly speaking, we ought to check that the different images of μ all have the same domain. But this is true, because the image of μ corresponding to a strictly increasing function $p : r \to I$ has domain $\widehat{\bigotimes}_r \mathcal{B}(Z, \mathfrak{T})$.) The (unique) marginal measure λ of μ is the restriction to $\mathcal{B}(Z, \mathfrak{T})$ of the $\mathfrak{T}$-Radon measure $\tilde{\lambda}$ which is the marginal of $\tilde{\mu}$, so is a $\mathfrak{T}$-tight $\mathfrak{T}$-Borel measure, therefore countably compact. By 454A(b-ii), μ is countably compact. So 459E, with $\mathcal{K}$ the family of $\mathfrak{T}$-compact subsets of Z, tells us that there are a probability space $(Y_0, \mathrm{T}_0, \nu_0)$ and a family $\langle \mu_y \rangle_{y \in Y_0}$ in $P_{\mathrm{R}}(Z, \mathfrak{T})$ such that $\langle \mu_y^I \rangle_{y \in Y_0}$ is a disintegration of μ over ν_0, writing μ_y^I for the ordinary product of copies of μ_y. We can of course suppose that ν_0 is complete. Note also that $\langle \mu_y \rangle_{y \in Y_0}$ is a disintegration of λ; this is clearly achieved by the proof of 459E, and it is necessarily true if $\langle \mu_y^I \rangle_{y \in Y_0}$ is to be a disintegration of μ. Because every μ_y is complete, $\langle \mu_y \rangle_{y \in Y_0}$ is also a disintegration of the completion $\tilde{\lambda}$ of λ.

(c) Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be a disjoint sequence of $\mathfrak{T}$-compact subsets of Z such that $\sum_{n=0}^{\infty} \tilde{\lambda} K_n = 1$ (412Aa). Let $\mathfrak{S}$ be

$$\{H : H \subseteq Z,\ Z \setminus (H \cap K_n) \in \mathfrak{T} \text{ for every } n \in \mathbb{N}\}.$$

Then $\mathfrak{S}$ is a locally compact topology on Z finer than $\mathfrak{T}$. (If you like, $\mathfrak{S}$ is the disjoint union topology corresponding to the partition $\{K_n : n \in \mathbb{N}\} \cup \{\{z\} : z \in Z \setminus \bigcup_{n \in \mathbb{N}} K_n\}$.) Note that the subspace topologies on any K_n induced by $\mathfrak{S}$ and $\mathfrak{T}$ are the same, so that a $\mathfrak{T}$-compact subset of K_n is $\mathfrak{S}$-compact. Because $\mathfrak{S}$ is finer than $\mathfrak{T}$, $P_{\mathrm{R}}(Z, \mathfrak{S}) \subseteq P_{\mathrm{R}}(Z, \mathfrak{T})$ (use 418I). If $\theta \in P_{\mathrm{R}}(Z, \mathfrak{T})$ and $\theta(\bigcup_{n \in \mathbb{N}} K_n) = 1$, then, from the standpoint of the topology $\mathfrak{S}$, θ is a complete topological probability measure inner regular with respect to the compact sets, so belongs to $P_{\mathrm{R}}(Z, \mathfrak{S})$. In particular, $\tilde{\lambda} \in P_{\mathrm{R}}(Z, \mathfrak{S})$.

We shall need to know that the family $\mathcal{V}$ of $\mathfrak{T}$-Borel $\mathfrak{S}$-cozero subsets of Z is a base for $\mathfrak{S}$. **P** If $z \in H \in \mathfrak{S}$, then if $z \notin \bigcup_{n \in \mathbb{N}} K_n$ the singleton $\{z\}$ belongs to $\mathcal{V}$. If $n \in \mathbb{N}$ and $z \in K_n$, then $H \cap K_n \in \mathfrak{S}$; as $\mathfrak{S}$ is locally compact, there is an $\mathfrak{S}$-cozero set G such that $z \in G \subseteq H \cap K_n$, and now G is $\mathfrak{T}$-relatively open in the $\mathfrak{T}$-compact set K_n, so G is $\mathfrak{T}$-Borel. **Q**

(d) We know that

$$\int \mu_y(\bigcup_{n \in \mathbb{N}} K_n) \nu_0(dy) = \tilde{\lambda}(\bigcup_{n \in \mathbb{N}} K_n) = 1;$$

since $\mu_y Z = 1$ for every y, the set $Y = \{y : y \in Y_0,\, \mu_y(\bigcup_{n\in\mathbb{N}} K_n) = 1\}$ must be ν_0-conegligible. Let ν be the subspace measure induced by ν_0 on Y. Then $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of $\tilde\lambda$ over ν, and $\mu_y \in P_{\mathrm{R}}(Z,\mathfrak{S})$ for every $y \in Y$, by (c).

(e) By 459F, the set

$$A = \{\theta : \theta \in P_{\mathrm{R}}(Z,\mathfrak{S}),\, \theta(\bigcup_{n\in\mathbb{N}} K_n = 1\}$$

is K-analytic in its narrow topology, while $\mu_y \in A$ for every $y \in Y$. If $G \in \mathcal{V}$ and $\alpha > 0$, $\{y : y \in Y_0,\, \mu_y G > \alpha\} \in \mathrm{T}_0$, so $\{y : y \in Y,\, \mu_y G > \alpha\}$ is measured by ν. By 432I, applied to the map $y \mapsto \mu_y : Y \to A$, there is a Radon probability measure $\tilde\nu_A$ on A such that

$$\int h\,d\tilde\nu_A = \int h(\mu_y)\nu(dy)$$

for every bounded continuous $h : A \to \mathbb{R}$.

(f) Now suppose that $f : Z \to \mathbb{R}$ is bounded and $\mathfrak{S}$-continuous. Then $\theta \mapsto \int f d\theta : P_{\mathrm{R}}(Z,\mathfrak{S}) \to \mathbb{R}$ is continuous (437K), so that

$$\iint f d\theta\,\tilde\nu_A(d\theta) = \iint f d\mu_y \nu(dy).$$

If $G \in \mathcal{V}$, there is a non-decreasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of non-negative $\mathfrak{S}$-continuous functions with supremum χG, so

$$\begin{aligned}\int \theta G\,\tilde\nu_A(d\theta) &= \sup_{n\in\mathbb{N}} \iint f_n d\theta\,\tilde\nu_A(d\theta) = \sup_{n\in\mathbb{N}} \iint f_n d\mu_y \nu(dy)\\ &= \int \mu_y G\,\nu(dy) = \lambda G = \tilde\lambda G.\end{aligned}$$

So we can apply 459G to the identity map from A to itself and the family $\langle\theta\rangle_{\theta\in A}$ to see that $\langle\theta\rangle_{\theta\in A}$ is a disintegration of $\tilde\lambda$ over $\tilde\nu_A$.

It follows that if $E \subseteq Z$ is $\tilde\lambda$-negligible, then $\theta E = 0$ for $\tilde\nu_A$-almost every θ. Moreover, since $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of $\tilde\lambda$ over ν, $\mu_y E = 0$ for ν-almost every y.

(g) If $J \subseteq I$ is finite, $G_j \in \mathfrak{T}$ for $j \in J$, and $W = \{w : w \in Z^I,\, w(j) \in G_j \text{ for } j \in J\}$, then

$$\tilde\mu W = \int \theta^I W \tilde\nu_A(d\theta).$$

P Because $\mathcal{V}$ is a base for $\mathfrak{S}$ closed under countable unions, and $\tilde\lambda$ is $\mathfrak{S}$-Radon, there is for each $j \in J$ a $G'_j \in \mathcal{V}$, included in G_j, such that $\tilde\lambda G'_J = \tilde\lambda G_j$. Set $W' = \{w : w \in Z^I,\, w(j) \in G'_j \text{ for } j \in J\}$. We have

$$W \setminus W' \subseteq \bigcup_{j\in J}\{w : w(j) \in G_j \setminus G'_j\},$$

while

$$\tilde\mu\{w : w(j) \in G_j \setminus G'_j\} = \tilde\lambda(G_j \setminus G'_j) = 0$$

for each j, so $\tilde\mu W'$ is defined and equal to $\tilde\mu W = \mu W$. Note that the same calculation shows that $\theta^I W = \theta^I W'$ whenever $\theta \in A$ is such that $\theta G'_j = \theta G_j$ for every j, that is, for $\tilde\nu_A$-almost every θ. Now, for each $j \in J$, we have a non-decreasing sequence $\langle f_{jn}\rangle_{n\in\mathbb{N}}$ of non-negative $\mathfrak{S}$-continuous real-valued functions with supremum $\chi G'_j$. Set $g_n(w) = \prod_{j\in J} f_{jn}(w(j))$ for $w \in Z^I$ and $n \in \mathbb{N}$. (I suppose you should take $g_n(w) = 1$ if J is empty.) Then each g_n is $\mathfrak{S}^I$-continuous, so if we set $h_n(\theta) = \int g_n d\theta^I$ for $\theta \in A$, h_n is continuous (put 437Mb and 437Kb together). Also $\langle g_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum $\chi W'$, so

$$\theta^I W' = \sup_{n\in\mathbb{N}} \int g_n d\theta^I = \sup_{n\in\mathbb{N}} h_n(\theta)$$

for $\theta \in A$. Accordingly

$$\begin{aligned}\tilde\mu W = \tilde\mu W' &= \int \mu_y^I W' \nu(dy) = \sup_{n\in\mathbb{N}} \int h_n(\mu_y)\nu(dy)\\ &= \sup_{n\in\mathbb{N}} \int h_n d\tilde\nu_A = \sup_{n\in\mathbb{N}} \iint g_n d\theta^I\,\tilde\nu_A(d\theta)\\ &= \int \theta^I W'\,\tilde\nu_A(d\theta) = \int \theta^I W\,\tilde\nu_A(d\theta),\end{aligned}$$

as required. **Q**

(h) We are nearly ready to dispense with the topology $\mathfrak{S}$. Since the embeddings $A \subsetneqq P_{\mathrm{R}}(Z,\mathfrak{S}) \subsetneqq P_{\mathrm{R}}(Z,\mathfrak{T})$ are continuous (437Jh), we have an image Radon probability measure $\tilde\nu$ on $P_{\mathrm{R}}(Z,\mathfrak{T})$, and

$$\int_{P_{\mathrm{R}}(Z,\mathfrak{T})} h\,d\tilde\nu = \int_A h\,d\tilde\nu_A$$

for every $h : P_{\mathrm{R}}(Z,\mathfrak{T}) \to \mathbb{R}$ such that $\int_A h\,d\tilde\nu_A$ is defined.

In particular, if we take $\mathcal{W}$ to be the family of $\mathfrak{T}^I$-open cylinder sets expressible as $\{w : w \in Z^I,\ w(j) \in G_j$ for $j \in J\}$ where $J \subseteq I$ is finite and $G_j \in \mathfrak{T}$ for each j, (g) tells us that

$$\tilde\mu W = \int \theta^I W\,\tilde\nu_A(d\theta) = \int \theta^I W\,\tilde\nu(d\theta) = \int \tilde\theta^I W\,\tilde\nu(d\theta)$$

for every $W \in \mathcal{W}$, where I now write $\mathfrak{T}^I$ for the product topology on Z^I corresponding to the topology $\mathfrak{T}$ on Z, and $\tilde\theta^I$ for the $\mathfrak{T}^I$-quasi-Radon product measure on Z^I corresponding to the $\mathfrak{T}$-Radon measure θ (417R). Now turn again to 437Mb and 459G; $\theta \mapsto \tilde\theta^I$ is a continuous function from $P_{\mathrm{R}}(Z,\mathfrak{T})$ to the space $P_{\mathrm{qR}}(Z^I,\mathfrak{T}^I)$ of $\mathfrak{T}^I$-quasi-Radon probability measures on Z^I, and $\mathcal{W}$ is a base for the topology $\mathfrak{T}^I$, so $\langle\tilde\theta^I\rangle_{\theta\in P_{\mathrm{R}}(Z,\mathfrak{T})}$ is a disintegration of $\tilde\mu$ over $\tilde\nu$, which is what I set out to prove.

459I I come now to a lemma based on ideas in TAO 07. It is in a form more elaborate than is required for the elementary application here (459J), but which will be needed in §497.

Lemma Let (X,Σ,μ) be a probability space and I a set. For a family $\mathbb{T}$ of subalgebras of $\mathcal{P}X$, write $\bigvee\mathbb{T}$ for the σ-algebra generated by $\bigcup\mathbb{T}$, as in 458Ad. Let G be the group of permutations ϕ of I such that $\{i : \phi(i) \ne i\}$ is finite. Suppose that $\bullet$ is an action of G on X such that $x \mapsto \phi\bullet x$ is inverse-measure-preserving for each $\phi \in G$; set $\phi\bullet A = \{\phi\bullet x : x \in A\}$ for $\phi \in G$ and $A \subseteq X$, as in 441Aa. Let $\langle\Sigma_J\rangle_{J\subseteq I}$ be a family of σ-subalgebras of Σ such that

(i) for every $J \subseteq I$, Σ_J is the σ-algebra generated by $\bigcup_{K\subseteq J \text{ is finite}}\Sigma_K$;

(ii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$, then $\phi\bullet E \in \Sigma_{\phi[J]}$;

(iii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$ is such that $\phi(i) = i$ for every $i \in J$, then $\phi\bullet E = E$.

Suppose that $\mathcal{J}^*$ is a filter on I not containing any infinite set, and that $K \subseteq I$, $\mathcal{K} \subseteq \mathcal{P}I$ and $\mathcal{J} \subseteq \mathcal{J}^*$ are such that for every $K' \in \mathcal{K}$ there is a $J \in \mathcal{J}$ such that $K \cap K' \subseteq J$. Then Σ_K and $\bigvee_{K'\in\mathcal{K}}\Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}}\Sigma_J$.

proof (a)(i) Let us note straight away that condition (i) above implies that $\Sigma_K \subseteq \Sigma_J$ whenever $K \subseteq J \subseteq I$.

(ii) For any σ-subalgebra T of Σ, I will (slightly abusing notation, as in 242Jh) write $L^2(\mu\restriction\mathrm{T})$ for the $\|\,\|_2$-closed linear subspace of $L^2(\mu)$ consisting of equivalence classes of μ-square-integrable T-measurable real-valued functions defined on X, and $P_{\mathrm{T}} : L^2(\mu) \to L^2(\mu\restriction\mathrm{T})$ for the corresponding conditional-expectation operator (244M). Note that P_{T} is an orthogonal projection (244Nb).

(iii) We have an action of G on $L^2(\mu)$, defined by saying that

$$(\phi\bullet f)(x) = f(\phi^{-1}\bullet x) \text{ for } \phi \in G,\ x \in X \text{ and } f \in \mathbb{R}^X$$

(4A5C(c-i)),

$$\phi\bullet f^\bullet = (\phi\bullet f)^\bullet \text{ for } \phi \in G \text{ and } f \in \mathcal{L}^2(\mu) \cap \mathbb{R}^X$$

(441Kc).

(iv) If $\mathbb{T}$ is the family of σ-algebras of subsets of X, we have an action of G on $\mathbb{T}$ defined by setting

$$\phi\bullet\mathrm{T} = \{\phi\bullet E : E \in \mathrm{T}\}$$

for $\mathrm{T} \in \mathbb{T}$ and $\phi \in G$. If $\langle\mathrm{T}_\gamma\rangle_{\gamma\in\Gamma}$ is a family in $\mathbb{T}$, then $\phi\bullet\bigvee_{\gamma\in\Gamma}\mathrm{T}_\gamma = \bigvee_{\gamma\in\Gamma}\phi\bullet\mathrm{T}_\gamma$ for every $\phi \in G$, just because $E \mapsto \phi\bullet E$ is an automorphism of the Boolean algebra $\mathcal{P}X$.

(v) If $\phi \in G$ and $L \subseteq I$, then $\phi\bullet\Sigma_L = \Sigma_{\phi[L]}$. **P** Condition (ii) of this lemma says just that $\phi\bullet\Sigma_L = \{\phi\bullet E : E \in \Sigma_L\}$ is included in $\Sigma_{\phi[L]}$; and now of course

$$\Sigma_{\phi[L]} = \phi\bullet\phi^{-1}\bullet\Sigma_L \subseteq \phi\bullet\Sigma_{\phi^{-1}[\phi[L]]} = \phi\bullet\Sigma_L. \mathbf{Q}$$

(vi) If $\phi \in G$ and T is a σ-subalgebra of Σ, then $\phi\bullet(P_{\mathrm{T}}u) = P_{\phi\bullet\mathrm{T}}(\phi\bullet u)$ for every $u \in L^2(\mu)$. **P** I should of course note that $\phi\bullet\Sigma = \Sigma$ because $x \mapsto \phi\bullet x$ is an automorphism of (X,Σ,μ), so $\phi\bullet\mathrm{T} \subseteq \Sigma$ and we can speak of $P_{\phi\bullet\mathrm{T}}$. Let $f : X \to \mathbb{R}$ be a Σ-measurable function such that $f^\bullet = u$, and $g : X \to \mathbb{R}$ a T-measurable function which is a conditional expectation of f on T. In this case, for any $\alpha \in \mathbb{R}$,

$$\{x:(\phi\bullet g)(x)>\alpha\}=\{x:g(\phi^{-1}\bullet x)>\alpha\}=\{\phi\bullet x:g(x)>\alpha\}$$
$$=\phi\bullet\{x:g(x)>\alpha\}\in\phi\bullet\mathrm{T},$$

so $\phi\bullet g$ is $(\phi\bullet\mathrm{T})$-measurable. Next, for any $F\in\phi\bullet\mathrm{T}$,

$$\int_F\phi\bullet g\,d\mu=\int_F g(\phi^{-1}\bullet x)\mu(dx)=\int_{\phi^{-1}\bullet F}g(x)\mu(dx)$$

(applying 235G to the inverse-measure-preserving function $x\mapsto\phi\bullet x:X\to X$ and the integrable function $x\mapsto g(\phi^{-1}\bullet x)$)

$$=\int_{\phi^{-1}\bullet F}f\,d\mu$$

(because $\phi^{-1}\bullet F\in\mathrm{T}$)

$$=\int_F\phi\bullet f d\mu.$$

As F is arbitrary, $\phi\bullet g$ is a conditional expectation of $\phi\bullet f$ on $\phi\bullet\mathrm{T}$, and

$$\phi\bullet(P_{\mathrm{T}}u)=\phi\bullet g^\bullet=(\phi\bullet g)^\bullet=P_{\phi\bullet\mathrm{T}}(\phi\bullet f)^\bullet=P_{\phi\bullet\mathrm{T}}(\phi\bullet u).\ \mathbf{Q}$$

(b)(i) Let $\langle J_\gamma\rangle_{\gamma\in\Gamma}$ be a non-empty finite family of subsets of I with infinite intersection, and set $\Lambda=\bigvee_{\gamma\in\Gamma}\Sigma_{J_\gamma}$. Suppose that K, $\langle K_\gamma\rangle_{\gamma\in\Gamma}$ are such that

$$K\in[I]^{<\omega},\quad K_\gamma\in[I]^{<\omega}\text{ and }K\cap K_\gamma\subseteq J_\gamma\text{ for every }\gamma\in\Gamma.$$

Take $E\in\Sigma_K$ and $F_\gamma\in\Sigma_{K_\gamma}$ for every $\gamma\in\Gamma$, and set $F=\bigcap_{\gamma\in\Gamma}F_\gamma$. Let g, $h:X\to[0,1]$ be Λ-measurable functions which are conditional expectations of χE, χF respectively on Λ. Let $\epsilon>0$.

(ii) For $L\subseteq I$ set $\Lambda_L=\bigvee_{\gamma\in\Gamma}\Sigma_{J_\gamma\cap L}\subseteq\Lambda$. For any $u\in L^2(\mu)$ there is a finite $L\subseteq I$ such that $\|P_{\mathrm{T}}u-P_\Lambda u\|_2\le\epsilon$ whenever T is a σ-subalgebra of Λ including Λ_L. $\mathbf{P}$ By condition (i) of this lemma, Λ is the σ-algebra generated by

$$\bigcup_{L\subseteq I\text{ is finite}}\bigcup_{\gamma\in\Gamma}\Sigma_{J_\gamma\cap L},$$

so $\{\Lambda_L:L\in[I]^{<\omega}\}$ is an upwards-directed family of σ-algebras whose union σ-generates Λ, and $\bigcup_{L\subseteq I\text{ is finite}}L^2(\mu\restriction\Lambda_L)$ is norm-dense in $L^2(\mu\restriction\Lambda)$. There are therefore a finite $L\subseteq I$ and a $v\in L^2(\mu\restriction\Lambda_L)$ such that $\|v-P_\Lambda u\|_2\le\epsilon$. If now $\Lambda_L\subseteq\mathrm{T}\subseteq\Lambda$, $v\in L^2(\mu\restriction\mathrm{T})$, while P_{T} is the orthogonal projection onto $L^2(\mu\restriction\mathrm{T})$, so

$$\|P_{\mathrm{T}}u-P_\Lambda u\|_2=\|P_{\mathrm{T}}P_\Lambda u-P_\Lambda u\|_2\le\|v-P_\Lambda u\|_2\le\epsilon.\ \mathbf{Q}$$

(iii Set $u=\chi E^\bullet$ and $v=\chi F^\bullet$, so that $g^\bullet=P_\Lambda u$ and $h^\bullet=P_\Lambda v$. By (b), there is an $L_0\in[I]^{<\omega}$ such that

$$\|P_{\mathrm{T}}u-P_\Lambda u\|_2\le\epsilon,\quad\|P_{\mathrm{T}}v-P_\Lambda v\|_2\le\epsilon,\quad\|P_{\mathrm{T}}(u\times v)-P_\Lambda(u\times v)\|_2\le\epsilon$$

whenever T is a σ-subalgebra of Λ including Λ_{L_0}. We can suppose that $L_0\supseteq K\cup\bigcup_{\gamma\in\Gamma}K_\gamma$. Write T_0 for Λ_{L_0}. We have

$$\|P_\Lambda u\times P_\Lambda v-P_{\mathrm{T}_0}u\times P_{\mathrm{T}_0}v\|_2$$
$$\le\|P_\Lambda u\times(P_\Lambda v-P_{\mathrm{T}_0}v)\|_2+\|(P_\Lambda u-P_{\mathrm{T}_0}u)\times P_{\mathrm{T}_0}v\|_2$$
$$\le\|P_\Lambda v-P_{\mathrm{T}_0}v\|_2+\|P_\Lambda u-P_{\mathrm{T}_0}u\|_2$$

(because $\|P_\Lambda u\|_\infty$ and $\|P_{\mathrm{T}_0}v\|_\infty$ are both at most 1)

$$\le 2\epsilon.$$

(iv) Let $L_1\subseteq\bigcap_{\gamma\in\Gamma}J_\gamma\setminus L_0$ be a set of size $\#(L_0\setminus K)$; let $\phi\in G$ be such that $\phi[L_0\setminus K]=L_1$, ϕ^2 is the identity and $\phi(i)=i$ for $i\in I\setminus(L_1\cup(L_0\setminus K))$. In this case, $\phi(i)=i$ for $i\in K$, so $\phi[L]\subseteq(L\cap K)\cup\bigcap_{\gamma\in\Gamma}J_\gamma$ for every $L\subseteq L_0$. Setting $M_\gamma=(L_0\cap J_\gamma)\cup\phi[L_0\cap J_\gamma]$, we have

$$L_0\cap J_\gamma\subseteq M_\gamma=\phi[M_\gamma]\subseteq J_\gamma,\quad\phi[K_\gamma]\subseteq J_\gamma$$

for each $\gamma\in\Gamma$. (This is where we need to know that $K\cap K_\gamma\subseteq J_\gamma$.)

Now

$$\phi\bullet u=\phi\bullet(\chi E^\bullet)=\chi(\phi\bullet E)^\bullet=\chi E^\bullet=u$$

by condition (iii) of this lemma; also

$$\|\phi\bullet(P_{\mathrm{T}_0}u)-P_\Lambda u\|_2\le 3\epsilon.$$

P By (a-iv) and (a-v),

$$\begin{aligned}\phi\bullet\mathrm{T}_0=\phi\bullet\bigvee_{\gamma\in\Gamma}\Sigma_{L_0\cap J_\gamma}&=\bigvee_{\gamma\in\Gamma}\phi\bullet\Sigma_{L_0\cap J_\gamma}\\&=\bigvee_{\gamma\in\Gamma}\Sigma_{\phi[L_0\cap J_\gamma]}\subseteq\bigvee_{\gamma\in\Gamma}\Sigma_{M_\gamma}\subseteq\bigvee_{\gamma\in\Gamma}\Sigma_{J_\gamma}=\Lambda.\end{aligned}$$

Set $\mathrm{T}=\mathrm{T}_0\vee\phi\bullet\mathrm{T}_0$; then $\mathrm{T}_0\subseteq\mathrm{T}=\phi[\mathrm{T}]\subseteq\Lambda$. But now

$$\phi\bullet(P_{\mathrm{T}}u)=P_{\phi\bullet\mathrm{T}}(\phi\bullet u)=P_{\mathrm{T}}u$$

(see (a-vi)), so

$$\begin{aligned}\|\phi\bullet(P_{\mathrm{T}_0}u)-P_\Lambda u\|_2&\le\|\phi\bullet(P_{\mathrm{T}_0}u)-\phi\bullet(P_{\mathrm{T}}u)\|_2+\|P_{\mathrm{T}}u-P_\Lambda u\|_2\\&=\|P_{\mathrm{T}}u-P_{\mathrm{T}_0}u\|_2+\|P_{\mathrm{T}}u-P_\Lambda u\|_2\\&\le\|P_\Lambda u-P_{\mathrm{T}_0}u\|_2+2\|P_{\mathrm{T}}u-P_\Lambda u\|_2\le 3\epsilon.\ \mathbf{Q}\end{aligned}$$

(v) Set

$$\mathrm{T}^*=\bigvee\nolimits_{\gamma\in\Gamma}\Sigma_{K_\gamma\cup M_\gamma}.$$

Because $L_0\cap J_\gamma\subseteq M_\gamma$ for every γ, T^* and

$$\phi\bullet\mathrm{T}^*=\bigvee\nolimits_{\gamma\in\Gamma}\Sigma_{\phi[K_\gamma]\cup M_\gamma}$$

include $\Lambda_{L_0}=\mathrm{T}_0$, while $\phi\bullet\mathrm{T}^*\subseteq\Lambda$ because $\phi[K_\gamma]\cup M_\gamma\subseteq J_\gamma$ for every γ. Also $F\in\mathrm{T}^*$, because $F_\gamma\in\Sigma_{K_\gamma}\subseteq\mathrm{T}^*$ for every γ. Now
and

$$\|P_{\mathrm{T}_0}(u\times v)-P_{\mathrm{T}_0}u\times P_{\mathrm{T}_0}v\|_2=\|P_{\mathrm{T}_0}P_{\mathrm{T}^*}(u\times v)-P_{\mathrm{T}_0}u\times P_{\mathrm{T}_0}v\|_2$$

(because $T_0\subseteq T^*$)

$$=\|P_{\mathrm{T}_0}(v\times P_{\mathrm{T}^*}u)-P_{\mathrm{T}_0}(v\times P_{\mathrm{T}_0}u)\|_2$$

(because $v\in L^2(\mu\restriction\mathrm{T}^*)$ and $P_{\mathrm{T}_0}u\in L^2(\mu\restriction\mathrm{T}_0)$, see 242L)

$$\le\|v\times P_{\mathrm{T}^*}u-v\times P_{\mathrm{T}_0}u\|_2\le\|P_{\mathrm{T}^*}u-P_{\mathrm{T}_0}u\|_2$$

(because $\|v\|_\infty\le 1$)

$$\begin{aligned}&=\|\phi\bullet(P_{\mathrm{T}^*}u)-\phi\bullet(P_{\mathrm{T}_0}u)\|_2\\&=\|P_{\phi\bullet\mathrm{T}^*}(\phi\bullet u)-\phi\bullet(P_{\mathrm{T}_0}u)\|_2\\&\le\|P_{\phi\bullet\mathrm{T}^*}u-P_\Lambda u\|_2+\|P_\Lambda u-\phi\bullet(P_{\mathrm{T}_0}u)\|_2\\&\le\epsilon+3\epsilon=4\epsilon.\end{aligned}$$

(vi) Putting these together,

$$\begin{aligned}\|P_\Lambda(u\times v)-P_\Lambda u\times P_\Lambda v\|_2&\le\|P_\Lambda(u\times v)-P_{\mathrm{T}_0}(u\times v)\|_2\\&\qquad+\|P_{\mathrm{T}_0}(u\times v)-P_{\mathrm{T}_0}u\times P_{\mathrm{T}_0}v\|_2\\&\qquad+\|P_\Lambda u\times P_\Lambda v-P_{\mathrm{T}_0}u\times P_{\mathrm{T}_0}v\|_2\\&\le\epsilon+4\epsilon+2\epsilon=7\epsilon.\end{aligned}$$

(vii) As ϵ is arbitrary, $P_\Lambda(u\times v)=P_\Lambda u\times P_\Lambda v$, that is, $g\times h$ is a conditional expectation of $\chi(E\cap F)$ on Λ, and E and F are relatively independent over Λ.

(c) It follows that Σ_K and $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$ are relatively independent over Λ. **P** Suppose that $E\in\Sigma_K$, and consider the set

$$\mathcal{E}=\{F:F\in\Sigma,\,P_\Lambda\chi(E\cap F)^\bullet=P_\Lambda(\chi E^\bullet)\times P_\Lambda(\chi F^\bullet)\}.$$

Then $\mathcal{E}$ is a Dynkin class, and by (b) above it contains

$$\mathcal{E}_0=\{\bigcap_{\gamma\in\Gamma}F_\gamma:F_\gamma\in\Sigma_{K_\gamma}\text{ for every }\gamma\in\Gamma\},$$

which is closed under $\cap$. Accordingly $\mathcal{E}$ includes the σ-algebra generated by $\mathcal{E}_0$, which is $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$. Thus

$$P_\Lambda\chi(E\cap F)^\bullet=P_\Lambda(\chi E^\bullet)\times P_\Lambda(\chi F^\bullet)$$

for every $E\in\Sigma_K$ and $F\in\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$, and Σ_K and $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$ are relatively independent over Λ. **Q**

(d) Now suppose that $\langle J_\gamma\rangle_{\gamma\in\Gamma}$ is a non-empty finite family of subsets of I with infinite intersection. As before, write Λ for $\bigvee_{\gamma\in\Gamma}\Sigma_{J_\gamma}$. Suppose that $K\subseteq I$ and that $\langle K_\gamma\rangle_{\gamma\in\Gamma}$ is a family of subsets of I such that $K\cap K_\gamma\subseteq J_\gamma$ for every $\gamma\in\Gamma$. Then Σ_K and $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$ are relatively independent over Λ. **P** Set $\mathrm{T}=\bigcup\{\Sigma_L:L\in[K]^{<\omega}\}$ and for $\gamma\in\Gamma$ set $\mathrm{T}_\gamma=\bigcup\{\Sigma_L:L\in[K_\gamma]^{<\omega}\}$. Then (b)-(c) tell us that T and the algebra T' σ-generated by $\bigcup_{\gamma\in\Gamma}\mathrm{T}_\gamma$ are relatively independent over Λ. Since Σ_K is the σ-algebra generated by T, while $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$ is the σ-algebra generated by T', 458Da-458Db tell us that Σ_K and $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$ are relatively independent over Λ. **Q**

(e) At last we are ready to approach the sets K, $\mathcal{K}$ and $\mathcal{J}$ of the statement of this lemma. The case $\mathcal{J}=\emptyset$ is trivial (as then $\mathcal{K}$ must also be empty), so suppose that $\mathcal{J}$ is non-empty.

(i) To begin with, suppose that $\mathcal{J}$ and $\mathcal{K}$ are finite. In this case, we can find finite families $\langle J_\gamma\rangle_{\gamma\in\Gamma}$ and $\langle K_\gamma\rangle_{\gamma\in\Gamma}$ running over $\mathcal{J}$, $\mathcal{K}\cup\{\emptyset\}$ respectively such that $K\cap K_\gamma\subseteq J_\gamma$ for every γ. So (d) tells us that Σ_K and $\bigvee_{K'\in\mathcal{K}}\Sigma_{K'}\vee\Sigma_\emptyset$ are relatively independent over $\bigvee_{J\in\mathcal{J}}\Sigma_J$.

(ii) If $\mathcal{K}$ is finite but $\mathcal{J}$ is infinite, then let $\mathcal{J}_0\subseteq\mathcal{J}$ be a finite set such that for every $K'\in\mathcal{K}$ there is a $J\in\mathcal{J}_0$ including $K\cap K'$. Then for any finite $\mathcal{J}'\subseteq\mathcal{J}$ including $\mathcal{J}_0$, Σ_K and $\bigvee_{K'\in\mathcal{K}}\Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}'}\Sigma_J$. Since

$$\{\bigvee_{J\in\mathcal{J}'}\Sigma_J:\mathcal{J}_0\subseteq\mathcal{J}'\in[\mathcal{J}]^{<\omega}\}$$

is an upwards-directed family of σ-algebras whose union σ-generates $\bigvee_{J\in\mathcal{J}}\Sigma_J$, 458C tells us that Σ_K and $\bigvee_{K'\in\mathcal{K}}\Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}}\Sigma_J$.

(iii) Finally, for the general case, (ii) tells us that Σ_K and $\bigvee_{K'\in\mathcal{K}'}\Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}}\Sigma_J$ for every finite $\mathcal{K}'\subseteq\mathcal{K}$, so Σ_K and $\bigvee_{K'\in\mathcal{K}}\Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}}\Sigma_J$, by 458D again.

459J Corollary Let (X,Σ,μ) be a probability space and I a set. Let G be the group of permutations ϕ of I such that $\{i:\phi(i)\ne i\}$ is finite. Suppose that $\bullet$ is an action of G on X such that $x\mapsto\phi{\bullet}x$ is inverse-measure-preserving for each $\phi\in G$. Let $\langle\Sigma_J\rangle_{J\subseteq I}$ be a family of σ-subalgebras of Σ such that

(i) for every $J\subseteq I$, Σ_J is the σ-algebra generated by $\bigcup_{K\subseteq J\text{ is finite}}\Sigma_K$;

(ii) if $J\subseteq I$, $E\in\Sigma_J$ and $\phi\in G$, then $\phi{\bullet}E\in\Sigma_{\phi[J]}$;

(iii) if $J\subseteq I$, $E\in\Sigma_J$ and $\phi\in G$ is such that $\phi(i)=i$ for every $i\in J$, then $\phi{\bullet}E=E$.

Then if $J\subseteq I$ is infinite and $\langle K_\gamma\rangle_{\gamma\in\Gamma}$ is a family of subsets of I such that $K_\gamma\cap K_\delta\subseteq J$ for all distinct γ, $\delta\in\Gamma$, $\langle\Sigma_{K_\gamma}\rangle_{\gamma\in\Gamma}$ is relatively independent over Σ_J.

proof By 459I, Σ_{K_γ} and $\bigvee_{\delta\in\Delta}\Sigma_{K_\delta}$ are relatively independent over Σ_J whenever $\Delta\subseteq\Gamma$ and $\gamma\in\Gamma\setminus\Delta$. Now 458Hb tells us that we can induce on $\#(\Delta)$ to see that $\langle\Sigma_{K_\gamma}\rangle_{\gamma\in\Delta}$ is relatively independent over Σ_J for every finite $\Delta\subseteq\Gamma$, and it follows at once that $\langle\Sigma_{K_\gamma}\rangle_{\gamma\in\Gamma}$ is relatively independent over Σ_J, as remarked in 458Ac.

proof Note first that if G is the group of permutations ϕ of I such that $\{i:\phi(i)\ne i\}$ is finite, then any $\phi\in G$ is expressible as the product of finitely many transpositions, so $w\mapsto w\phi$ is an automorphism of (X^I,μ). Let $\bullet$ be the action of G on X^I defined by saying that $\phi{\bullet}w=w\phi^{-1}$ for $x\in X^I$ and $\phi\in G$. Then $w\mapsto\phi{\bullet}w$ is inverse-measure-preserving for every ϕ.

If $L\subseteq I$ then Σ_L is the σ-algebra of subsets of X^I generated by sets of the form $\{x:x(i)\in E\}$ where $i\in L$ and $E\in\Sigma$. So Σ_L is the σ-algebra generated by $\bigcup\{\Sigma_K:K\in[L]^{<\omega}\}$.

If $i\in I$, $E\in\Sigma$ and $\phi\in G$, then

$$\phi{\bullet}\{x:x(i)\in E\}=\{\phi{\bullet}x:x(i)\in E\}=\{x:(\phi^{-1}{\bullet}x)(i)\in E\}=\{x:x(\phi(i))\in E\}.$$

So if $L \subseteq I$ and $\phi \in G$, $\{W : \phi\bullet W \in \Sigma_{\phi[L]}\}$ is a σ-algebra of subsets of X^I containing $\{x : x(i) \in E\}$ whenever $i \in L$ and $E \in \Sigma$, therefore including Σ_L; that is, $\phi\bullet W \in \Sigma_{\phi[L]}$ whenever $W \in \Sigma_L$.

If $L \subseteq I$ and $\phi \in G$ is such that $\phi(i) = i$ for every $i \in L$, then $\{W : \phi\bullet W = W\}$ is a σ-algebra of subsets of X^I containing $\{x : x(i) \in E\}$ whenever $i \in L$ and $E \in \Sigma$, so $\phi\bullet W = W$ for every $W \in \Sigma_L$.

Thus the conditions of 459I are satisfied, and the result follows at once.

459K Following the results of §452 (especially 452Ye), we do not generally expect to find disintegrations of measures which are not countably compact. It may however illuminate the constructions here if I give a specific example related to the contexts of 459E and 459H.

Example (Dubins & Freedman 79) There are a separable metrizable space Z and a quasi-Radon measure on $Z^{\mathbb{N}}$, invariant under permutations of coordinates, which cannot be disintegrated into powers of measures on Z.

proof (a) Let λ be Lebesgue measure on $[0,1]$. $Q = [0,1] \times [0,1]^{\mathbb{N}}$, with its usual topology, is a compact metrizable space, so has just $\mathfrak{c}$ Borel sets (4A3F). Let $\langle W_\xi\rangle_{\xi<\mathfrak{c}}$ enumerate the Borel subsets of Q with non-zero measure for the product measure $\lambda \times \lambda^{\mathbb{N}}$. (Remember that $\lambda \times \lambda^{\mathbb{N}}$ is a Radon measure, by 416U.) For each ξ, we have $0 < (\lambda \times \lambda^{\mathbb{N}})(W_\xi) = \int \lambda^{\mathbb{N}}(W_\xi[\{t\}])\lambda(dt)$, so $A_\xi = \{t : W_\xi[\{t\}] \ne \emptyset\}$ has cardinal $\mathfrak{c}$ (419H); we can therefore choose $\langle t_\xi\rangle_{\xi<\mathfrak{c}}$ in $[0,1]$ such that $t_\xi \in A_\xi \setminus \{t_\eta : \eta < \xi\}$ for every $\xi < \mathfrak{c}$. Now choose $t_{\xi n}$, for $\xi < \mathfrak{c}$ and $n \in \mathbb{N}$, such that $(t_\xi, \langle t_{\xi n}\rangle_{n\in\mathbb{N}}) \in W_\xi$. Set $Z = \{(t_\xi, t_{\xi n}) : \xi < \mathfrak{c},\ n \in \mathbb{N}\} \subseteq [0,1]^2$.

(b) Set $X = ([0,1]^2)^{\mathbb{N}}$ and define $\phi : Q \to X$ by setting $\phi(t, \langle t_n\rangle_{n\in\mathbb{N}}) = \langle (t, t_n)\rangle_{n\in\mathbb{N}}$ for t, $t_n \in [0,1]$. Then ϕ is a homeomorphism between Q and $\phi[Q]$, so there is a unique Radon measure $\mu^\#$ on X such that ϕ is inverse-measure-preserving for $\lambda \times \lambda^{\mathbb{N}}$ and $\mu^\#$. Now $\mu^\#$ is invariant under permutations of coordinates, because if $\rho : \mathbb{N} \to \mathbb{N}$ is a permutation and $\hat\rho(x) = x\rho$ for $x \in X$, then $\hat\rho\phi = \phi\bar\rho$, where $\bar\rho(t, \langle t_n\rangle_{n\in\mathbb{N}}) = (t, \langle t_{\rho(n)}\rangle_{n\in\mathbb{N}})$; and as $\bar\rho : Q \to Q$ is inverse-measure-preserving, so is $\hat\rho : X \to X$.

Also $Z^{\mathbb{N}}$ has full outer measure for $\mu^\#$. **P** If $\mu^\# W > 0$, then $(\lambda \times \lambda^{\mathbb{N}})\phi^{-1}[W] > 0$, so there is some $\xi < \mathfrak{c}$ such that $W_\xi \subseteq \phi^{-1}[W]$. Now $\langle (t_\xi, t_{\xi n})\rangle_{n\in\mathbb{N}} \in Z^{\mathbb{N}} \cap W$. **Q** Accordingly the subspace measure $\tilde\mu$ on Z is a probability measure. Because $\mu^\#$ is invariant under permutations of coordinates, so is $\tilde\mu$; because $\mu^\#$ is a Radon measure, $\tilde\mu$ is a quasi-Radon measure (416Ra).

(c) ? Suppose, if possible, that there are a probability space (Y, T, ν) and a family $\langle \mu_y\rangle_{y\in Y}$ of probability measures on Z such that $\tilde\mu E = \int \mu_y^{\mathbb{N}} E\,\nu(dy)$ for every Borel set $E \subseteq Z^{\mathbb{N}}$. (The argument to follow will not depend on which product measure is used in forming the $\mu_y^{\mathbb{N}}$.) Looking at sets of the form $(Z \cap H) \times Z \times Z \times \ldots$, where $H \subseteq [0,1]^2$ is a Borel set, we see that $\mu_y(Z \cap H)$ must be defined for almost every y; as Z is second-countable, μ_y must be a topological measure for almost every y. Looking at sets of the form $(Z \cap (G_0 \times [0,1])) \times (Z \cap (G_1 \times [0,1])) \times Z \times \ldots$, where G_0 and G_1 are disjoint Borel subsets of $[0,1]$, we see that $\mu_y(Z \cap (G_0 \times [0,1])) \cdot \mu_y(Z \cap (G_1 \times [0,1])) = 0$ for almost every y; as $[0,1]$ is second-countable and Hausdorff, there must be, for almost every $y \in Y$, an $s_y \in [0,1]$ such that $\mu_y(Z \cap (\{s_y\} \times [0,1])) = 1$.

Next, if $G \subseteq [0,1]$ is a Borel set, then $\mu_y(Z \cap ([0,1] \times G)) = \lambda G$ for almost every y. **P**

$$h(y) = \mu_y^{\mathbb{N}}((Z \cap ([0,1] \times G)) \times Z \times \ldots) = \mu_y(Z \cap ([0,1] \times G))$$

is defined for almost every y, and h is ν-integrable, with

$$\int h\,d\nu = \tilde\mu((Z \cap ([0,1] \times G)) \times Z \times \ldots) = \mu^\#(([0,1] \times G) \times [0,1]^2 \times \ldots) = \lambda G.$$

At the same time,

$$\begin{aligned}\int h(y)(1 - h(y))\nu(dy) &= \tilde\mu((Z \cap ([0,1] \times G)) \times (Z \cap ([0,1] \times ([0,1] \setminus G))) \times Z \times \ldots)\\ &= \mu^\#(([0,1] \times G) \times ([0,1] \times ([0,1] \setminus G)) \times [0,1]^2 \times \ldots)\\ &= \lambda G(1 - \lambda G).\end{aligned}$$

Rearranging, we see that $\int h^2 d\nu = (\int h)^2$. But this means that $\int (h(y) - \int h)^2 \nu(dy) = 0$ and $h(y) = \lambda G$ for almost every y. **Q**

It follows that, for at least some y, $\mu_y(Z \cap (\{s_y\} \times G)) = \lambda G$ for every interval $G \subseteq [0,1]$ with rational endpoints. But this is impossible, because all the vertical sections of Z are countable. **X**

Thus there is no such disintegration, as claimed.

459X Basic exercises >(a) Let (X,Σ,μ) be a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ an exchangeable sequence of real-valued random variables on X all with finite expectation. Use 459C and 273I to show that $\langle\frac{1}{n+1}\sum_{i=0}^n f_i\rangle_{n\in\mathbb{N}}$ converges a.e. (Compare 276Xg[32].)

(b) Let (X,Σ,μ) be a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ an exchangeable sequence of real-valued random variables on X all with finite variance, such that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f_i=0$ a.e. Show that $\langle\Pr(\sum_{i=0}^n f_i\ge\alpha\sqrt{n+1})\rangle_{n\in\mathbb{N}}$ is convergent for every $\alpha\in\mathbb{R}$. (*Hint*: 274I.)

(c) Let X be a completely regular topological space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ a totally finite quasi-Radon measure space, and $y\mapsto\mu_y$ a continuous function from Y to the space $M_{\mathrm{qR}}^+(X)$ of totally finite quasi-Radon measures on X with its narrow topology. Show that if $\mu\in M_{\mathrm{qR}}^+(X)$ is such that $\int fd\mu=\iint fd\mu_y\,\nu(dy)$ for every $f\in C_b(X)$, then $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of μ over ν.

>(d) (Diaconis & Freedman 80) Let Z be a non-empty compact Hausdorff space and I an infinite set including $\mathbb{N}$. Let $\tilde\mu$ be a Radon probability measure on Z^I invariant under permutations of I. For $k\le n$ let $D_{nk}\subseteq n^k$ be the set of injective functions from k to n and Ω_{nk} the set $Z^I\times n^k\times D_{nk}$, endowed with the product λ_{nk} of $\tilde\mu$ and the uniform probability measures on the finite sets n^k and D_{nk}. Define $\phi_{nk}:\Omega_{nk}\to Z^k$ and $\psi_{nk}:\Omega_{nk}\to Z^k$ by setting

$$\begin{aligned}\phi_{nk}(w,p,q)&=wp,\\ \psi_{nk}(w,p,q)&=wp\text{ if }p\in D_{nk},\\ &=wq\text{ otherwise.}\end{aligned}$$

(i) Show that there is a disintegration $\langle\mu_{nw}^k\rangle_{w\in Z^I}$ of the image measure $\lambda_{nk}\phi_{nk}^{-1}$ over $\tilde\mu$ where each μ_{nw} is a suitable point-supported measure on Z^k. (ii) Show that the image measure $\lambda_{nk}\psi_{nk}^{-1}$ is the image measure $\tilde\mu_k=\tilde\mu\tilde\pi_k^{-1}$, where $\tilde\pi_k(w)=w{\restriction}k$ for $w\in Z^I$. (iii) Show that if $n>0$ then $|\tilde\mu_kW-\int\mu_{nw}^kW\,\tilde\mu(dw)|\le\frac{k(k-1)}{2n}$ for every Baire set $W\subseteq Z^k$. (iv) Show that there is a Radon probability measure $\tilde\nu_n$ on $P_{\mathrm{R}}(Z)$ for which $w\mapsto\mu_{nw}$ is inverse-measure-preserving. (v) Show that if $\tilde\nu$ is any cluster point of $\langle\tilde\nu_n\rangle_{n\in\mathbb{N}}$ in $P_{\mathrm{R}}(Z)$ then $\langle\tilde\theta^I\rangle_{\theta\in P_{\mathrm{R}}(Z)}$ is a disintegration of $\tilde\mu$ over $\tilde\nu$, writing $\tilde\theta^I$ for the Radon product of copies of any $\theta\in P_{\mathrm{R}}(Z)$.

>(e) (Hewitt & Savage 55) Let X be a non-empty compact Hausdorff space and I an infinite set. Let Q be the set of Radon probability measures on X^I which are invariant under permutations of I. Show that (i) Q is a closed convex subset of the set $P_{\mathrm{R}}(X^I)$ of all Radon probability measures on X^I with its narrow topology; (ii) Q is isomorphic, as topological convex structure, to $P_{\mathrm{R}}(P_{\mathrm{R}}(X))$; (iii) the extreme points of Q are just the powers of Radon probability measures on X.

(f) Let X, I be sets, Σ a σ-algebra of subsets of X and μ a probability measure with domain $\widehat{\bigotimes}_I\Sigma$ which is transposition-invariant in the sense that for every transposition $\tau:I\to I$ the function $x\mapsto x\tau:X^I\to X^I$ is inverse-measure-preserving. For $J\subseteq I$, let Σ_J be the σ-algebra

$$W:W\in\widehat{\bigotimes}_I\Sigma,\ W\text{ is determined by coordinates in }J\}.$$

Show that if $J\subseteq I$ is infinite and $\langle K_\gamma\rangle_{\gamma\in\Gamma}$ is a family of subsets of I such that $K_\gamma\cap K_\delta\subseteq J$ for all distinct γ, $\delta\in\Gamma$, $\langle\Sigma_{K_\gamma}\rangle_{\gamma\in\Gamma}$ is relatively independent over Σ_J (i) using 459D (ii) using 459J.

459Y Further exercises (a) Let X be a topological space and I an infinite set. Write $P_\tau(X)$, $P_\tau(X^I)$ and $P_\tau(P_\tau(X))$ for the spaces of τ-additive Borel probability measures in X, X^I and $P_\tau(X)$ respectively, with their narrow topologies. (i) For $\theta\in P_\tau(X)$ write $\tilde\theta^I$ for the τ-additive Borel measure on X^I corresponding to θ, that is, the restriction to the Borel σ-algebra of X^I of the τ-additive product measure described in 417G. Show that $\theta\mapsto\tilde\theta^I:P_\tau(X)\mapsto P_\tau(X^I)$ is continuous. (ii) Show that if $\nu\in P_\tau(P_\tau(X))$ there is a unique $\mu_\nu\in P_\tau(X^I)$ such that $\langle\tilde\theta^I\rangle_{\theta\in P_\tau(X)}$ is a disintegration of μ_ν over ν, where $\tilde\theta^I$ is the τ-additive Borel product measure on X^I corresponding to $\theta\in P_\tau(X)$. (iii) Show that $\nu\mapsto\mu_\nu$ is a homeomorphism between $P_\tau(P_\tau(X))$ and its image in $P_\tau(X^I)$.

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[32]Formerly 276Xe.

(b) Discuss the problems which arise in 459B, 459C, 459E and 459H if the index set I is finite.

459 Notes and comments As I have presented this material, the centre of the argument of 459A-459H lies in the martingales in part (b-β) of the proof of 459B. We are trying to resolve the functions f_i into 'common' and 'independent' parts. The 'common' part is given by the conditional expectations of the f_i over an appropriate σ-algebra T, and we approach these by looking at the conditional expectations of each f_i on σ-algebras T_n generated by 'distant' f_j. All the most important ideas are already exhibited when the index set I is equal to $\mathbb{N}$. Note in particular that in the basic hypothesis that all finite strings $(f_{i_0}, \dots, f_{i_r})$ have the same joint distribution, it is enough to look at increasing strings. But there is a striking phenomenon which appears in sharper relief with uncountable sets I: any sequence $\langle j_k \rangle_{k\in\mathbb{N}}$ of distinct elements of I can be used to generate an adequate σ-algebra, because while the tail σ-algebra of sets depends on the choice of the j_k, they all lead to the same closed subalgebra of the measure algebra (459D).

Perhaps I should emphasize at this point that I really does have to be infinite, though for large finite I there are approximations to the results here.

The proof of 459B is one of the standard proofs of De Finetti's theorem, with trifling modifications. In the case of real-valued random variables we have a notion of relative distribution (458I) which gives a quick way of saying that all the f_i have the same conditional expectations over T, as in 459C(ii). For variables taking values in other spaces the situation may be different (459K), unless (as in §452) we have a countably compact measure (459E).

Specializing to the case $X = Z^I$ in 459B, we find ourselves examining symmetric measures on infinite product spaces, which are of great interest in themselves. Note that while in the hypothesis of 459E I have asked for the measure μ on the product space Z^I to be countably compact, what is actually necessary is that the marginal measure on Z should be countably compact. By 454Ab, this comes to the same thing.

As in 452O, we can look for a disintegration consisting of Radon measures, provided of course that the marginal measure is a Radon measure. What we have to work harder for is a direct expression in terms of an integral $\int \tilde{\theta}^I \tilde{\nu}(d\theta)$ where $\tilde{\nu}$ is itself a Radon probability measure on the space of Radon probability measures θ (459H). But most of the extra work consists of finding the correct reduction to the case of locally compact spaces. For compact spaces we can approach by a completely different route (459Xd). I will not go farther with this idea here, but I note that the method can be used in a wide variety of problems involving symmetric structures.

Lemma 459I is entirely different. I include it here because it gives another approach to relative independence and looks at permutation-invariant measures, though in a more abstract setting which does not bind us to the product spaces which are their most natural expressions. Its power lies precisely in the fact that in its hypotheses we do *not* suppose that $\Sigma_{J\cup K} = \Sigma_J \vee \Sigma_K$ for J, $K \subseteq I$, so the σ-algebras $\bigvee_{K'\in\mathcal{K}} \Sigma_{K'}$ and $\bigvee_{J\in\mathcal{J}} \Sigma_J$ have to be handled with special care.

Concordance to chapters 41-45

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to these chapters, and which have since been changed.

411I Completion regular measures The definition of 'completion regular' measure, referred to in Fremlin 00, has been moved to 411J.

414N Density topologies The note on 'density topologies', referred to in the 2001 edition of Volume 2, has been moved to 414P.

415Yd Sorgenfrey line This exercise, referred to in the 2001 edition of Volume 2, has been moved to 415Ye.

416M Henry's theorem, mentioned in König 04, has been re-named 416N.

416P The algebra of open-and-closed subsets This paragraph, referred to in the 2002 edition of Volume 3, has been moved to 416Q.

416T Kakutani's theorem The description of the usual measure on $\{0,1\}^\kappa$, referred to in Fremlin & Plebanek 03, has been moved to 416U.

417E τ-additive product measures The reference in Fremlin 00 to Kakutani's theorem that the product measure on $\{0,1\}^I$ is completion regular should be directed to 415E or 416U rather than 417E.

419H The example of a measure inner regular with respect to the Borel sets but with no extension to a topological measure, mentioned in König p09, is now 419J.

419J Partitions into sets of full outer measure Lemma 419J, mentioned in the 2004 printing of Volume 1, has been moved to 419I.

432I Capacitability Definition 432I, referred to in the 2008 edition of Volume 5, is now 432J.

434S-434T Vague topologies The material on vague topologies, referred to in the 2001 edition of Volume 2, has been moved to §437.

439H τ-smooth functionals The example of a τ-smooth functional which is not representable as an integral, referred to in Bogachev 07, is now 439I.

439J A non-Radon space The example of a first-countable compact Hausdorff space which is not Radon, referred to in Bogachev 07, is now 439K.

439N Baire measure The example of a Baire probability measure with no extension to a Borel measure, referred to in Bogachev 07, is now 439M.

441A Shift actions The definition of shift actions in 441Ac of the 2003 and 2006 printings, called on in the 2008 edition of Volume 5, has been moved to 4A5Cc.

444Xn Orthonormal bases The sketch of a construction of an orthonormal basis in L^2 consisting of equivalence classes of continuous functions, referred to in Bogachev 07, is now 444Ym.

§**445 Convolutions** The material mentioned in the notes to §257 in the 2001 edition of Volume 2 has been moved to §444. The particular result referred to as '445K' is now 444R.

445Xq The exercise 445Xq, referred to in the 2002 edition of Volume 3, has been moved to 445Xp.

449I, 449J Tarski's theorem The proof of Tarski's theorem, referred to in the 2002 and 2004 printings of Volume 3, is now in 449L.

452I In Fremlin 00 I quote Pachl's result that if (X, Σ, μ) is countably compact, (Y, T, ν) is strictly localizable and $f : X \to Y$ is inverse-measure-preserving, then ν is countably compact; this is now in 452R.

455D The material on Brownian motion in §455, mentioned in König 04 and König 06, has been moved to §477.

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