MEASURE THEORY

Volume 3

Part II

D.H.Fremlin

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MEASURE THEORY

Volume 3

Measure Algebras

Part II

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Chapter 35

Riesz spaces

The next three chapters are devoted to an abstract description of the 'function spaces' described in Chapter 24, this time concentrating on their internal structure and relationships with their associated measure algebras. I find that any convincing account of these must involve a substantial amount of general theory concerning partially ordered linear spaces, and in particular various types of Riesz space or vector lattice. I therefore provide an introduction to this theory, a kind of appendix built into the middle of the volume. The relation of this chapter to the next two is very like the relation of Chapter 31 to Chapter 32. As with Chapter 31, it is not really meant to be read for its own sake; those with a particular interest in Riesz spaces might be better served by LUXEMBURG & ZAANEN 71, SCHAEFER 74, ZAANEN 83 or my own book FREMLIN 74A.

I begin with three sections in an easy gradation towards the particular class of spaces which we need to understand: partially ordered linear spaces (§351), general Riesz spaces (§352) and Archimedean Riesz spaces (§353); the last includes notes on Dedekind (σ-)complete spaces. These sections cover the fragments of the algebraic theory of Riesz spaces which I will use. In the second half of the chapter, I deal with normed Riesz spaces (in particular, L- and M-spaces)(§354), spaces of linear operators (§355) and dual Riesz spaces (§356).

351 Partially ordered linear spaces

I begin with an account of the most basic structures which involve an order relation on a linear space, partially ordered linear spaces. As often in this volume, I find myself impelled to do some of the work in very much greater generality than is strictly required, in order to show more clearly the nature of the arguments being used. I give the definition (351A) and most elementary properties (351B-351L) of partially ordered linear spaces; then I describe a general representation theorem for arbitrary partially ordered linear spaces as subspaces of reduced powers of $\mathbb{R}$ (351M-351Q). I end with a brief note on Archimedean partially ordered linear spaces (351R).

351A Definition I repeat a definition mentioned in 241E. A **partially ordered linear space** is a linear space $(U,+,\cdot)$ over $\mathbb{R}$ together with a partial order $\le$ on U such that

$$u \le v \Longrightarrow u+w \le v+w,$$

$$u \ge 0,\ \alpha \ge 0 \Longrightarrow \alpha u \ge 0$$

for u, v, $w \in U$ and $\alpha \in \mathbb{R}$.

351B Elementary facts Let U be a partially ordered linear space. We have the following elementary consequences of the definition above, corresponding to the familiar rules for manipulating inequalities among real numbers.

(a) For u, $v \in U$,

$$u \le v \Longrightarrow 0 = u+(-u) \le v+(-u) = v-u \Longrightarrow u = 0+u \le v-u+u = v,$$

$$u \le v \Longrightarrow -v = u+(-v-u) \le v+(-v-u) = -u.$$

(b) Suppose that u, $v \in U$ and $u \le v$. Then $\alpha u \le \alpha v$ for every $\alpha \ge 0$ and $\alpha v \le \alpha u$ for every $\alpha \le 0$. **P** (i) If $\alpha \ge 0$, then $\alpha(v-u) \ge 0$ so $\alpha v \ge \alpha u$. (ii) If $\alpha \le 0$ then $(-\alpha)u \le (-\alpha)v$ so

$$\alpha v = -(-\alpha)v \le -(-\alpha u) = u. \text{ **Q**}$$

(c) If $u \ge 0$ and $\alpha \le \beta$ in $\mathbb{R}$, then $(\beta-\alpha)u \ge 0$, so $\alpha u \le \beta u$. If $0 \le u \le v$ in U and $0 \le \alpha \le \beta$ in $\mathbb{R}$, then $\alpha u \le \beta u \le \beta v$.

351C Positive cones Let U be a partially ordered linear space.

(a) I will write U^+ for the **positive cone** of U, the set $\{u : u \in U,\ u \ge 0\}$.

(b) By 351Ba, the ordering is determined by the positive cone U^+, in the sense that $u \le v \iff v-u \in U^+$.

(c) It is easy to characterize positive cones. If U is a real linear space, a set $C \subseteq U$ is the positive cone for some ordering rendering U a partially ordered linear space iff

$$u+v \in C, \quad \alpha u \in C \text{ whenever } u,\, v \in C \text{ and } \alpha \ge 0,$$

$$0 \in C, \quad u \in C \,\&\, -u \in C \Longrightarrow u = 0.$$

P (i) If $C = U^+$ for some partially ordered linear space ordering $\le$ of U, then

$$u,\, v \in C \Longrightarrow 0 \le u \le u+v \Longrightarrow u+v \in C,$$

$$u \in C,\, \alpha \ge 0 \Longrightarrow \alpha u \ge 0, \text{ i.e., } \alpha u \in C,$$

$$0 \le 0 \text{ so } 0 \in C,$$

$$u,\, -u \in C \Longrightarrow u = 0 + u \le (-u) + u = 0 \le u \Longrightarrow u = 0.$$

(ii) On the other hand, if C satisfies the conditions, define the relation $\le$ by writing $u \le v \iff v - u \in C$; then

$$u - u = 0 \in C \text{ so } u \le u \text{ for every } u \in U,$$

$$\text{if } u \le v \text{ and } v \le w \text{ then } w - u = (w-v) + (v-u) \in C \text{ so } u \le w,$$

$$\text{if } u \le v \text{ and } v \le u \text{ then } u - v,\, v - u \in C \text{ so } u - v = 0 \text{ and } u = v$$

and $\le$ is a partial order; moreover,

$$\text{if } u \le v \text{ and } w \in U \text{ then } (v+w) - (u+w) = v - u \in C \text{ and } u + w \le v + w,$$

$$\text{if } u,\, \alpha \ge 0 \text{ then } \alpha u \in C \text{ and } \alpha u \ge 0,$$

$$u \ge 0 \iff u \in C.$$

So $\le$ makes U a partially ordered linear space in which C is the positive cone. **Q**

(d) An incidental useful fact. Let U be a partially ordered linear space, and $u \in U$. Then $u \ge 0$ iff $u \ge -u$. **P** If $u \ge 0$ then $0 \ge -u$ so $u \ge -u$. If $u \ge -u$ then $2u \ge 0$ so $u = \frac{1}{2} \cdot 2u \ge 0$. **Q**

(e) I have called U^+ a 'positive cone' without defining the term 'cone'. I think this is something we can pass by for the moment; but it will be useful to recognise that U^+ is always convex, for if u, $v \in U^+$ and $\alpha \in [0,1]$ then αu, $(1-\alpha)v \ge 0$ and $\alpha u + (1-\alpha)v \in U^+$, so is a 'convex cone' as defined in 3A5Ba.

351D Suprema and infima Let U be a partially ordered linear space.

(a) The definition of 'partially ordered linear space' implies that $u \mapsto u + w$ is always an order-isomorphism; on the other hand, $u \mapsto -u$ is order-reversing, by 351Ba.

(b) It follows that if $A \subseteq U$ and $v \in U$ then

$$\sup_{u\in A}(v+u) = v + \sup A \text{ if either side is defined,}$$

$$\inf_{u\in A}(v+u) = v + \inf A \text{ if either side is defined,}$$

$$\sup_{u\in A}(v-u) = v - \inf A \text{ if either side is defined,}$$

$$\inf_{u\in A}(v-u) = v - \sup A \text{ if either side is defined.}$$

(c) Moreover, we find that if A, $B \subseteq U$ and $\sup A$ and $\sup B$ are defined, then $\sup(A+B)$ is defined and equal to $\sup A + \sup B$, writing $A + B = \{u + v : u \in A,\, v \in B\}$ as usual. **P** Set $u_0 = \sup A$, $v_0 = \sup B$. Using (b), we have

$$\begin{aligned} u_0 + v_0 &= \sup_{u\in A}(u + v_0) \\ &= \sup_{u\in A}(\sup_{v\in B}(u+v)) = \sup(A+B). \text{ **Q**} \end{aligned}$$

Similarly, if A, $B \subseteq U$ and $\inf A$, $\inf B$ are defined then $\inf(A+B) = \inf A + \inf B$.

(d) If $\alpha > 0$ then $u \mapsto \alpha u$ is an order-isomorphism, so we have $\sup(\alpha A) = \alpha \sup A$ if either side is defined; similarly, $\inf(\alpha A) = \alpha \inf A$.

351E Linear subspaces If U is a partially ordered linear space, and V is any linear subspace of U, then V, with the induced linear and order structures, is a partially ordered linear space; this is obvious from the definition.

351F Positive linear operators Let U and V be partially ordered linear spaces, and write $\mathsf{L}(U;V)$ for the linear space of all linear operators from U to V. For S, $T \in \mathsf{L}(U;V)$ say that $S \le T$ iff $Su \le Tu$ for every $u \in U^+$. Under this ordering, $\mathsf{L}(U;V)$ is a partially ordered linear space; its positive cone is $\{T : Tu \ge 0$ for every $u \in U^+\}$. **P** This is an elementary verification. **Q** Note that, for $T \in \mathsf{L}(U;V)$,

$$\begin{aligned} T \ge 0 &\Longrightarrow Tu \le Tu + T(v-u) = Tv \text{ whenever } u \le v \text{ in } U \\ &\Longrightarrow 0 = T0 \le Tu \text{ for every } u \in U^+ \\ &\Longrightarrow T \ge 0, \end{aligned}$$

so that $T \ge 0$ iff T is order-preserving. In this case we say that T is a **positive** linear operator.

Clearly ST is a positive linear operator whenever U, V and W are partially ordered linear spaces and $T : U \to V$, $S : V \to W$ are positive linear operators (cf. 313Ia).

351G Order-continuous positive linear operators: Proposition Let U and V be partially ordered linear spaces and $T : U \to V$ a positive linear operator.

(a) The following are equiveridical:

(i) T is order-continuous;

(ii) $\inf T[A] = 0$ in V whenever $A \subseteq U$ is a non-empty downwards-directed set with infimum 0 in U;

(iii) $\sup T[A] = Tw$ in V whenever $A \subseteq U^+$ is a non-empty upwards-directed set with supremum w in U.

(b) The following are equiveridical:

(i) T is sequentially order-continuous;

(ii) $\inf_{n\in\mathbb{N}} Tu_n = 0$ in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U with infimum 0 in U;

(iii) $\sup_{n\in\mathbb{N}} Tu_n = Tw$ in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in U^+ with supremum w in U.

proof (a)(i)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(ii) Assuming (iii), and given that A is non-empty, downwards-directed and has infimum 0, take any $u_0 \in A$ and consider $A' = \{u : u \in A, u \le u_0\}$, $B = u_0 - A'$. Then A' is non-empty, downwards-directed and has infimum 0, so B is non-empty, upwards-directed and has supremum u_0 (using 351Db); by (iii), $\sup T[B] = Tu_0$ and (inverting again)

$$\inf T[A'] = \inf T[u_0 - B] = \inf(Tu_0 - T[B]) = Tu_0 - \sup T[B] = 0.$$

But (because T is positive) 0 is surely a lower bound for $T[A]$, so it is also the infimum of $T[A]$. As A is arbitrary, (ii) is true.

(ii)$\Rightarrow$(i) Suppose now that (ii) is true. **(α)** If $A \subseteq U$ is non-empty, downwards-directed and has infimum w, then $A - w$ is non-empty, downwards-directed and has infimum 0, so

$$\inf T[A-w] = 0, \quad \inf T[A] = \inf(T[A-w] + Tw) = Tw + \inf T[A-w] = Tw.$$

(β) If $A \subseteq U$ is non-empty, upwards-directed and has supremum w, then $-A$ is non-empty, downwards-directed and has infimum $-w$, so

$$\sup T[A] = -\inf(-T[A]) = -\inf T[-A] = -T(-w) = Tw.$$

Putting these together, T is order-continuous.

(b) The arguments are identical, replacing each directed set by an appropriate sequence.

351H Riesz homomorphisms (a) For the sake of a representation theorem below (351Q), I introduce the following definition. Let U, V be partially ordered linear spaces. A **Riesz homomorphism** from U to V is a linear operator $T : U \to V$ such that whenever $A \subseteq U$ is a finite non-empty set and $\inf A = 0$ in U, then $\inf T[A] = 0$ in V. The following facts are now nearly obvious.

(b) Any Riesz homomorphism is a positive linear operator. (For if T is a Riesz homomorphism and $u \ge 0$, then $\inf\{0,u\} = 0$ so $\inf\{0,Tu\} = 0$ and $Tu \ge 0$.)

(c) Let U and V be partially ordered linear spaces and $T : U \to V$ a Riesz homomorphism. Then

$$\inf T[A] \text{ exists} = T(\inf A), \quad \sup T[A] \text{ exists} = T(\sup A)$$

whenever $A \subseteq U$ is a finite non-empty set and $\inf A$, $\sup A$ exist. (Apply the definition in (a) to

$$A' = \{u - \inf A : u \in A\}, \quad A'' = \{\sup A - u : u \in A\}.)$$

(d) If U, V and W are partially ordered linear spaces and $T : U \to V$, $S : V \to W$ are Riesz homomorphisms then $ST : U \to W$ is a Riesz homomorphism.

351I Solid sets Let U be a partially ordered linear space. I will say that a subset A of U is **solid** if

$$A = \{v : v \in U, \ -u \le v \le u \text{ for some } u \in A\} = \bigcup_{u\in A}[-u,u]$$

in the notation of 2A1Ab. (I should perhaps remark that while this definition is well established in the case of Riesz spaces (§352), the extension to general partially ordered linear spaces is not standard. See 351Yb for a warning.)

351J Proposition Let U be a partially ordered linear space and V a solid linear subspace of U. Then the quotient linear space U/V has a partially ordered linear space structure defined by either of the rules

$u^\bullet \le w^\bullet$ iff there is some $v \in V$ such that $u \le v + w$,

$(U/V)^+ = \{u^\bullet : u \in U^+\}$,

and for this partial order on U/V the map $u \mapsto u^\bullet : U \to U/V$ is a Riesz homomorphism.

proof (a) I had better start by giving priority to one of the descriptions of the relation $\le$ on U/V; I choose the first. To see that this makes U/V a partially ordered linear space, we have to check the following.

(i) $0 \in V$ and $u \le u + 0$, so $u^\bullet \le u^\bullet$ for every $u \in U$.

(ii) If u_1, u_2, $u_3 \in U$ and $u_1^\bullet \le u_2^\bullet$, $u_2^\bullet \le u_3^\bullet$ then there are v_1, $v_2 \in V$ such that $u_1 \le u_2 + v_1$, $u_2 \le u_3 + v_2$; in which case $v_1 + v_2 \in V$ and $u_1 \le u_3 + v_1 + v_2$, so $u_1^\bullet \le u_3^\bullet$.

(iii) If u, $w \in U$ and $u^\bullet \le w^\bullet$, $w^\bullet \le u^\bullet$ then there are v, $v' \in V$ such that $u \le w + v$, $w \le u + v'$. Now there are v_0, $v_0' \in V$ such that $-v_0 \le v \le v_0$, $-v_0' \le v' \le v_0'$, and in this case v_0, $v_0' \ge 0$ (351Cd), so

$$-v_0 - v_0' \le -v' \le u - w \le v \le v_0 + v_0' \in V,$$

.

Accordingly $u - w \in V$ and $u^\bullet = w^\bullet$. Thus U/V is a partially ordered set.

(iv) If u_1, u_2, $w \in U$ and $u_1^\bullet \le u_2^\bullet$, then there is a $v \in V$ such that $u_1 \le u_2 + v$, in which case $u_1 + w \le u_2 + w + v$ and $u_1^\bullet + w^\bullet \le u_2^\bullet + w^\bullet$.

(v) If $u \in U$, $\alpha \in \mathbb{R}$, $u^\bullet \ge 0$ and $\alpha \ge 0$ then there is a $v \in V$ such that $u + v \ge 0$; now $\alpha v \in V$ and $\alpha u + \alpha v \ge 0$, so $\alpha u^\bullet = (\alpha u)^\bullet \ge 0$.

Thus U/V is a partially ordered linear space.

(b) Now $(U/V)^+ = \{u^\bullet : u \ge 0\}$. **P** If $u \ge 0$ then of course $u^\bullet \ge 0$ because $0 \in V$ and $u + 0 \ge 0$. On the other hand, if we have any element p of $(U/V)^+$, there are $u \in U$, $v \in V$ such that $u^\bullet = p$ and $u + v \ge 0$; but now $p = (u + v)^\bullet$ is of the required form. **Q**

(c) Finally, $u \mapsto u^\bullet$ is a Riesz homomorphism. **P** Suppose that $A \subseteq U$ is a non-empty finite set and that $\inf A = 0$ in U. Then $u^\bullet \ge 0$ for every $u \in A$, that is, 0 is a lower bound for $\{u^\bullet : u \in A\}$. Let $p \in U/V$ be any other lower bound for $\{u^\bullet : u \in A\}$. Express p as $w^\bullet$ where $w \in U$. For each $u \in A$, $w^\bullet \le u^\bullet$ so there is a $v_u \in V$ such that $w \le u + v_u$. Next, there is a $v_u' \in V$ such that $-v_u' \le v_u \le v_u'$. Set $v^* = \sum_{u\in A} v_u' \in V$. Then $v_u \le v_u' \le v^*$ so $w \le u + v^*$ for every $u \in A$, and $w - v^*$ is a lower bound for A in U. Accordingly $w - v^* \le 0$, $w \le 0 + v^*$ and $p = w^\bullet \le 0$. As p is arbitrary, $\inf\{u^\bullet : u \in A\} = 0$; as A is arbitrary, $u \mapsto u^\bullet$ is a Riesz homomorphism. **Q**

351K Lemma Suppose that U is a partially ordered linear space, and that W, V are solid linear subspaces of U such that $W \subseteq V$. Then $V_1 = \{v^\bullet : v \in V\}$ is a solid linear subspace of U/W.

proof (i) Because the map $u \mapsto u^\bullet$ is linear, V_1 is a linear subspace of U/W. (ii) If $p \in V_1$, there is a $v \in V$ such that $p = v^\bullet$; because V is solid in U, there is a $v_0 \in V$ such that $-v_0 \le v \le v_0$; now $v_0^\bullet \in V_1$ and $-v_0^\bullet \le p \le v_0^\bullet$. (iii)

If $p \in V_1$, $q \in U/W$ and $-p \leq q \leq p$, take $v_0 \in V$, $u \in U$ such that $v_0^\bullet = p$ and $u^\bullet = q$. Because $-v_0^\bullet \leq u^\bullet \leq v_0^\bullet$, there are w, $w' \in W$ such that $-v_0 - w \leq u \leq v_0 + w'$. Now $-v_0 - w$, $v_0 + w'$ both belong to V, which is solid, so $u \in V$ and $q = u^\bullet \in V_1$. (iv) Putting (ii) and (iii) together, V_1 is solid.

351L Products If $\langle U_i \rangle_{i \in I}$ is any family of partially ordered linear spaces, we have a product linear space $U = \prod_{i \in I} U_i$; if we set $u \leq v$ in U iff $u(i) \leq v(i)$ for every $i \in I$, U becomes a partially ordered linear space, with positive cone $\{u : u(i) \geq 0 \text{ for every } i \in I\}$. For each $i \in I$ the map $u \mapsto u(i) : U \to U_i$ is an order-continuous Riesz homomorphism (in fact, it preserves arbitrary suprema and infima).

351M Reduced powers of $\mathbb{R}$ (a) Let X be any set. Then $\mathbb{R}^X$ is a partially ordered linear space if we say that $f \leq g$ means that $f(x) \leq g(x)$ for every $x \in X$, as in 351L. If now $\mathcal{F}$ is a filter on X, we have a corresponding set

$$V = \{f : f \in \mathbb{R}^X,\ \{x : f(x) = 0\} \in \mathcal{F}\};$$

it is easy to see that V is a linear subspace of $\mathbb{R}^X$, and is solid because $f \in V$ iff $|f| \in V$. By the **reduced power** $\mathbb{R}^X|\mathcal{F}$ I shall mean the quotient partially ordered linear space $\mathbb{R}^X/V$.

(b) Note that for $f \in \mathbb{R}^X$,

$$f^\bullet \geq 0 \text{ in } \mathbb{R}^X|\mathcal{F} \iff \{x : f(x) \geq 0\} \in \mathcal{F}.$$

P (i) If $f^\bullet \geq 0$, there is a $g \in V$ such that $f + g \geq 0$; now

$$\{x : f(x) \geq 0\} \supseteq \{x : g(x) = 0\} \in \mathcal{F}.$$

(ii) If $\{x : f(x) \geq 0\} \in \mathcal{F}$, then $\{x : (|f| - f)(x) = 0\} \in \mathcal{F}$, so $f^\bullet = |f|^\bullet \geq 0$. **Q**

351N On the way to the next theorem, the main result (in terms of mathematical content) of this section, we need a string of lemmas.

Lemma Let U be a partially ordered linear space. If $u, v_0, \ldots, v_n \in U$ are such that $u \neq 0$ and $v_0, \ldots, v_n \geq 0$ then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u) \neq 0$ and $f(v_i) \geq 0$ for every i.

proof The point is that at most one of u, $-u$ can belong to the convex cone C generated by $\{v_0, \ldots, v_n\}$, because this is included in the convex cone set U^+, and since $u \neq 0$ at most one of u, $-u$ can belong to U^+.

Now however the Hahn-Banach theorem, in the form 3A5D, tells us that if $u \notin C$ there is a linear functional $f : U \to \mathbb{R}$ such that $f(u) < 0$ and $f(v_i) \geq 0$ for every i; while if $-u \notin C$ we can get $f(-u) < 0$ and $f(v_i) \geq 0$ for every i. Thus in either case we have a functional of the required type.

351O Lemma Let U be a partially ordered linear space, and u_0 a non-zero member of U. Then there is a solid linear subspace V of U such that $u_0 \notin V$ and whenever $A \subseteq U$ is finite, non-empty and has infimum 0 then $A \cap V \neq \emptyset$.

proof (a) Let $\mathcal{W}$ be the family of all solid linear subspaces of U not containing u_0. Then any non-empty totally ordered $\mathcal{V} \subseteq \mathcal{W}$ has an upper bound $\bigcup \mathcal{V}$ in $\mathcal{W}$. By Zorn's Lemma, $\mathcal{W}$ has a maximal element V say. This is surely a solid linear subspace of U not containing u_0.

(b) Now for any $w \in U^+ \setminus V$ there are $\alpha \geq 0$, $v \in V^+$ such that $-\alpha w - v \leq u_0 \leq \alpha w + v$. **P** Let V_1 be

$$\{u : u \in U, \text{ there are } \alpha \geq 0,\ v \in V^+ \text{ such that } -\alpha w - v \leq u \leq \alpha w + v\}.$$

Then it is easy to check that V_1 is a solid linear subspace of U, including V, and containing w; because $w \notin V$, $V_1 \neq V$, so $V_1 \notin \mathcal{W}$ and $u \in V_1$, as claimed. **Q**

(c) It follows that if $A \subseteq U$ is finite and non-empty and $\inf A = 0$ in U then $A \cap V \neq \emptyset$. **P?** Otherwise, for every $w \in A$ there must be $\alpha_w \geq 0$, $v_w \in V^+$ such that $-\alpha_w w - v_w \leq u_0 \leq \alpha_w w + v_w$. Set $\alpha = 1 + \sum_{w \in A} \alpha_w$, $v = \sum_{w \in A} v_w \in V$; then $-\alpha w - v \leq u_0 \leq \alpha w + v$ for every $w \in A$. Accordingly $\frac{1}{\alpha}(u_0 - v) \leq w$ for every $w \in A$ and $\frac{1}{\alpha}(u_0 - v) \leq 0$, so $u_0 \leq v$. Similarly, $-\frac{1}{\alpha}(v + u_0) \leq w$ for every $w \in A$ and $-v \leq u_0$. But (because V is solid) this means that $u_0 \in V$, which is not so. **XQ**

Accordingly V has the required properties.

351P Lemma Let U be a partially ordered linear space and u a non-zero element of U, and suppose that $A_0, \ldots, A_n$ are finite non-empty subsets of U such that $\inf A_j = 0$ for every $j \leq n$. Then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u) \neq 0$ and $\min f[A_j] = 0$ for every $j \leq n$.

proof By 351O, there is a solid linear subspace V of U such that $u \notin V$ and $A_j \cap V \neq 0$ for every $j \leq n$. Give the quotient space U/V its standard partial ordering (351J), and in U/V set $C = \{v^\bullet : v \in \bigcup_{j\leq n} A_j\}$. Then C is a finite subset of $(U/V)^+$, while $u^\bullet \neq 0$, so by 351N there is a linear functional $g : U/V \to \mathbb{R}$ such that $g(u^\bullet) \neq 0$ but $g(p) \geq 0$ for every $p \in C$. Set $f(v) = g(v^\bullet)$ for $v \in U$; then $f : U \to \mathbb{R}$ is linear, $f(u) \neq 0$ and $f(v) \geq 0$ for every $v \in \bigcup_{j\leq n} A_j$. But also, for each $j \leq n$, there is a $v_j \in A_j \cap V$, so that $f(v_j) = 0$; and this means that $\min f[A_j]$ must be 0, as required.

351Q Now we are ready for the theorem.

Theorem Let U be any partially ordered linear space. Then we can find a set X, a filter $\mathcal{F}$ on X and an injective Riesz homomorphism from U to the reduced power $\mathbb{R}^X|\mathcal{F}$ described in 351M.

proof Let $X = U'$ be the set of all linear functionals $f : U \to \mathbb{R}$; for $u \in U$ define $\hat{u} \in \mathbb{R}^X$ by setting $\hat{u}(f) = f(u)$ whenever $f \in X$ and $u \in U$. Let $\mathcal{A}$ be the family of non-empty finite sets $A \subseteq U$ such that $\inf A = 0$. For $A \in \mathcal{A}$ let F_A be the set of those $f \in X$ such that $\min f[A] = 0$. Since $0 \in F_A$ for every $A \in \mathcal{A}$, the set

$$\mathcal{F} = \{F : F \subseteq X, \text{ there are } A_0, \dots , A_n \in \mathcal{A} \text{ such that } F \supseteq \textstyle\bigcap_{j\leq n} F_{A_j}\}$$

is a filter on X. Set $\psi(u) = \hat{u}^\bullet \in \mathbb{R}^X|\mathcal{F}$ for $u \in U$. Then $\psi : U \to \mathbb{R}^X|\mathcal{F}$ is an injective Riesz homomorphism.

P (i) ψ is linear because $u \mapsto \hat{u} : U \to \mathbb{R}^X$ and $h \mapsto h^\bullet : \mathbb{R}^X \to \mathbb{R}^X|\mathcal{F}$ are linear. (ii) If $A \in \mathcal{A}$, then $F_A \in \mathcal{F}$. So, first, if $v \in A$, then $\{f : \hat{v}(f) \geq 0\} \in \mathcal{F}$, so that $\psi(v) = \hat{v}^\bullet \geq 0$ in $\mathbb{R}^X|\mathcal{F}$ (351Mb). Next, if $w \in \mathbb{R}^X|\mathcal{F}$ and $w \leq \psi(v)$ for every $v \in A$, we can express w as $h^\bullet$ where $h^\bullet \leq \hat{v}^\bullet$ for every $v \in A$, that is, $H_v = \{f : h(f) \leq \hat{v}(f)\} \in \mathcal{F}$ for every $v \in A$. But now $H = F_A \cap \bigcap_{v\in A} H_v \in \mathcal{F}$, and for $f \in H$ we have $h(f) \leq \min_{v\in A} f(v) = 0$. This means that $w = h^\bullet \leq 0$. As w is arbitrary, $\inf \psi[A] = 0$. As A is arbitrary, ψ is a Riesz homomorphism. (iii) Finally, **?** suppose, if possible, that there is a non-zero $u \in U$ such that $\psi(u) = 0$. Then $F = \{f : f(u) = 0\} \in \mathcal{F}$, and there are $A_0, \dots , A_n \in \mathcal{A}$ such that $F \supseteq \bigcap_{j\leq n} F_{A_j}$. By 351P, there is an $f \in \bigcap_{j\leq n} F_{A_j}$ such that $f(u) \neq 0$; which is impossible. **X** Accordingly ψ is injective, as claimed. **Q**

351R Archimedean spaces (a) For a partially ordered linear space U, the following are equiveridical: (i) if u, $v \in U$ are such that $nu \leq v$ for every $n \in \mathbb{N}$ then $u \leq 0$ (ii) if $u \geq 0$ in U then $\inf_{\delta>0} \delta u = 0$. **P(i)⇒(ii)** If (i) is true and $u \geq 0$, then of course $\delta u \geq 0$ for every $\delta > 0$; on the other hand, if $v \leq \delta u$ for every $\delta > 0$, then $nv \leq n \cdot \frac{1}{n} u = u$ for every $n \geq 1$, while of course $0v = 0 \leq u$, so $v \leq 0$. Thus 0 is the greatest lower bound of $\{\delta u : \delta > 0\}$. **(ii)⇒(i)** If (ii) is true and $nu \leq v$ for every $n \in \mathbb{N}$, then $0 \leq v$ and $u \leq \frac{1}{n} v$ for every $n \geq 1$. If now $\delta > 0$, then there is an $n \geq 1$ such that $\frac{1}{n} \leq \delta$, so that $u \leq \frac{1}{n} v \leq \delta v$ (351Bc). Accordingly u is a lower bound for $\{\delta v : \delta > 0\}$ and $u \leq 0$. **Q**

(b) I will say that partially ordered linear spaces satisfying the equiveridical conditions of (a) above are **Archimedean**.

(c) Any linear subspace of an Archimedean partially ordered linear space, with the induced partially ordered linear space structure, is Archimedean.

(d) Any product of Archimedean partially ordered linear spaces is Archimedean. **P** If $U = \prod_{i\in I} U_i$ is a product of Archimedean spaces, and $nu \leq v$ in U for every $n \in \mathbb{N}$, then for each $i \in I$ we must have $nu(i) \leq v(i)$ for every n, so that $u(i) \leq 0$; accordingly $u \leq 0$. **Q** In particular, $\mathbb{R}^X$ is Archimedean for any set X.

351X Basic exercises >(a) Let ζ be any ordinal. The **lexicographic ordering** of $\mathbb{R}^\zeta$ is defined by saying that $f \leq g$ iff either $f = g$ or there is a $\xi < \zeta$ such that $f(\eta) = g(\eta)$ for $\eta < \xi$ and $f(\xi) < g(\xi)$. Show that this is a total order on $\mathbb{R}^\zeta$ which renders $\mathbb{R}^\zeta$ a partially ordered linear space.

(b) Let U be a partially ordered linear space and V a linear subspace of U. Show that the formulae of 351J define a partially ordered linear space structure on the quotient U/V iff V is **order-convex**, that is, $u \in V$ whenever v_1, $v_2 \in V$ and $v_1 \leq u \leq v_2$.

(c) Let $\langle U_i\rangle_{i\in I}$ be a family of partially ordered linear spaces with product U. For $i \in I$, define $T_i : U_i \to U$ by setting $T_i x = u$ where $u(i) = x$, $u(j) = 0$ for $j \neq i$. Show that T_i is an injective order-continuous Riesz homomorphism.

>(d) Let U be a partially ordered linear space and $\langle V_i\rangle_{i\in I}$ a family of partially ordered linear spaces with product V. Show that $\mathsf{L}(U;V)$ can be identified, as partially ordered linear space, with $\prod_{i\in I}\mathsf{L}(U;V_i)$.

>(e) Show that if U, V are partially ordered linear spaces and V is Archimedean, then $\mathsf{L}(U;V)$ is Archimedean.

351Y Further exercises (a) Give an example of two partially ordered linear spaces U and V and a bijective Riesz homomorphism $T:U\to V$ such that $T^{-1}:V\to U$ is not a Riesz homomorphism.

(b)(i) Let U be a partially ordered linear space. Show that U is a solid subset of itself (on the definition 351I) iff $U=U^+-U^+$. (ii) Give an example of a partially ordered linear space U satisfying this condition with an element $u\in U$ such that the intersection of the solid sets containing u is not solid.

(c) Show that a reduced power $\mathbb{R}^X|\mathcal{F}$, as described in 351M, is totally ordered iff $\mathcal{F}$ is an ultrafilter, and in this case has a natural structure as a totally ordered field.

(d) Let U be a partially ordered linear space, and suppose that A, $B\subseteq U$ are two non-empty finite sets such that (α) $u\vee v=\sup\{u,v\}$ is defined for every $u\in A$, $v\in B$ (β) $\inf A$ and $\inf B$ and $(\inf A)\vee(\inf B)$ are defined. Show that $\inf\{u\vee v:u\in A,\,v\in B\}=(\inf A)\vee(\inf B)$. (*Hint*: show that this is true if $U=\mathbb{R}$, if $U=\mathbb{R}^X$ and if $U=\mathbb{R}^X|\mathcal{F}$, and use 351Q.)

(e) Show that a reduced power $\mathbb{R}^X|\mathcal{F}$, as described in 351M, is Archimedean iff $\bigcap_{n\in\mathbb{N}}F_n\in\mathcal{F}$ whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$.

351 Notes and comments The idea of 'partially ordered linear space' is a very natural abstraction from the elementary examples of $\mathbb{R}^X$ and its subspaces, and the only possible difficulty lies in guessing the exact boundary at which one's standard manipulations with such familiar spaces cease to be valid in the general case. (For instance, most people's favourite examples are Archimedean, in the sense of 351R, so it is prudent to check your intuitions against a non-Archimedean space like that of 351Xa.) There is really no room for any deep idea to appear in 351B-351F. When I come to what I call 'Riesz homomorphisms', however (351H), there are some more interesting possibilities in the background.

I shall not discuss the applications of Theorem 351Q to general partially ordered linear spaces; it is here for the sake of its application to Riesz spaces in the next section. But I think it is a very striking fact that not only does any partially ordered linear space U appear as a linear subspace of some reduced power of $\mathbb{R}$, but the embedding can be taken to preserve any suprema and infima of finite sets which exist in U. This is in a sense a result of the same kind as the Stone representation theorem for Boolean algebras; it gives us a chance to confirm that an intuition valid for $\mathbb{R}$ or $\mathbb{R}^X$ may in fact apply to arbitrary partially ordered linear spaces. If you like, this provides a metamathematical foundation for such results as those in 351B. I have to say that for partially ordered linear spaces it is generally quicker to find a proof directly from the definition than to trace through an argument relying on 351Q; but this is not always the case for Riesz spaces. I offer 351Yd as an example of a result where a direct proof does at least call for a moment's thought, while the argument through 351Q is straightforward.

'Reduced powers' are of course of great importance for other reasons; I mention 351Yc as a hint of what can be done.

352 Riesz spaces

In this section I sketch those fragments of the theory we need which can be expressed as theorems about general Riesz spaces or vector lattices. I begin with the definition (352A) and most elementary properties (352C-352F). In 352G-352J I discuss Riesz homomorphisms and the associated subspaces (Riesz subspaces, solid linear subspaces); I mention product spaces (352K, 352T) and quotient spaces (352Jb, 352U) and the form the representation theorem 351Q takes in the present context (352L-352M). Most of the second half of the section concerns the theory of 'bands' in Riesz spaces, with the algebras of complemented bands (352Q) and projection bands (352S) and a description of bands generated by upwards-directed sets (352V). I conclude with a description of 'f-algebras' (352W).

352A I repeat a definition from 241E.

Definition A **Riesz space** or **vector lattice** is a partially ordered linear space which is a lattice.

352B Lemma If U is a partially ordered linear space, then it is a Riesz space iff $\sup\{0,u\}$ is defined for every $u \in U$.

proof If U is a lattice, then of course $\sup\{u,0\}$ is defined for every u. If $\sup\{u,0\}$ is defined for every u, and v_1, v_2 are any two members of U, consider $w = v_1 + \sup\{0, v_2 - v_1\}$; by 351Db, $w = \sup\{v_1, v_2\}$. Next,

$$\inf\{v_1, v_2\} = -\sup\{-v_1, -v_2\}$$

must also be defined in U; as v_1 and v_2 are arbitrary, U is a lattice.

352C Notation In any Riesz space U I will write

$$u^+ = u \vee 0, \quad u^- = (-u) \vee 0 = (-u)^+, \quad |u| = u \vee (-u)$$

where (as in any lattice) $u \vee v = \sup\{u, v\}$ (and $u \wedge v = \inf\{u, v\}$).

I mention immediately a term which will be useful: a family $\langle u_i \rangle_{i \in I}$ in U is **disjoint** if $|u_i| \wedge |u_j| = 0$ for all distinct i, $j \in I$. Similarly, a set $C \subseteq U$ is **disjoint** if $|u| \wedge |v| = 0$ for all distinct u, $v \in C$.

352D Elementary identities Let U be a Riesz space. The translation-invariance of the order, and its invariance under positive scalar multiplication, reversal under negative multiplication, lead directly to the following, which are in effect special cases of 351D:

$$u + (v \vee w) = (u + v) \vee (u + w), \quad u + (v \wedge w) = (u + v) \wedge (u + w),$$

$$\alpha(u \vee v) = \alpha u \vee \alpha v, \quad \alpha(u \wedge v) = \alpha u \wedge \alpha v \text{ if } \alpha \geq 0,$$

$$-(u \vee v) = (-u) \wedge (-v).$$

Combining and elaborating on these facts, we get

$$u^+ - u^- = (u \vee 0) - ((-u) \vee 0) = u + (0 \vee (-u)) - ((-u) \vee 0) = u,$$

$$u^+ + u^- = 2u^+ - u = (2u \vee 0) - u = u \vee (-u) = |u|,$$

$$u \geq 0 \iff -u \leq 0 \iff u^- = 0 \iff u = u^+ \iff u = |u|,$$

$$|-u| = |u|, \quad |\,|u|\,| = |u|, \quad |\alpha u| = |\alpha||u|$$

(looking at the cases $\alpha \geq 0$, $\alpha \leq 0$ separately),

$$\begin{aligned} u \vee v + u \wedge v &= u + (0 \vee (v - u)) + v + ((u - v) \wedge 0) \\ &= u + (0 \vee (v - u)) + v - ((v - u) \vee 0) = u + v, \end{aligned}$$

$$u \vee v = u + (0 \vee (v - u)) = u + (v - u)^+,$$

$$u \wedge v = u + (0 \wedge (v - u)) = u - (-0 \vee (u - v)) = u - (u - v)^+,$$

$$u \vee v = \frac{1}{2}(2u \vee 2v) = \frac{1}{2}(u + v + (u - v) \vee (v - u)) = \frac{1}{2}(u + v + |u - v|),$$

$$u \wedge v = u + v - u \vee v = \frac{1}{2}(u + v - |u - v|),$$

$$u^+ \vee u^- = u \vee (-u) \vee 0 = |u|, \quad u^+ \wedge u^- = u^+ + u^- - (u^+ \vee u^-) = 0,$$

$$|u + v| = (u + v) \wedge ((-u) + (-v)) \leq (|u| + |v|) \wedge (|u| + |v|) = |u| + |v|,$$

$$||u| - |v|| = (|u| - |v|) \vee (|v| - |u|) \leq |u - v| + |v - u| = |u - v|,$$

$$|u \vee v| \leq |u| + |v|$$

(because $-|u| \leq u \vee v \leq |u| \vee |v| \leq |u| + |v|$)

for u, v, $w \in U$ and $\alpha \in \mathbb{R}$.

352E Distributive laws Let U be a Riesz space.

(a) If A, $B \subseteq U$ have suprema a, b in U, then $C = \{u \wedge v : u \in A,\ v \in B\}$ has supremum $a \wedge b$. **P** Of course $u \wedge v \le a \wedge b$ for all $u \in A$, $v \in B$, so $a \wedge b$ is an upper bound for C. Now suppose that c is any upper bound for C. If $u \in A$ and $v \in B$ then

$$u - (u-v)^+ = u \wedge v \le c, \quad u \le c + (u-v)^+ \le c + (a-v)^+$$

(because $(a-v)^+ = \sup\{a-v, 0\} \ge \sup\{u-v, 0\} = (u-v)^+$). As u is arbitrary, $a \le c + (a-v)^+$ and $a \wedge v \le c$. Now turn the argument round:

$$v = (a \wedge v) + (v-a)^+ \le c + (v-a)^+ \le c + (b-a)^+,$$

and this is true for every $v \in B$, so $b \le c + (b-a)^+$ and $a \wedge b \le c$. As c is arbitrary, $a \wedge b = \sup C$, as claimed. **Q**

(b) Similarly, or applying (a) to $-A$ and $-B$, $\inf\{u \vee v : u \in A,\ v \in B\} = \inf A \vee \inf B$ whenever A, $B \subseteq U$ and the right-hand-side is defined.

(c) In particular, U is a distributive lattice (definition: 3A1Ic).

352F Further identities and inequalities At a slightly deeper level we have the following facts.

Proposition Let U be a Riesz space.

(a) If u, v, $w \ge 0$ in U then $u \wedge (v+w) \le (u \wedge v) + (u \wedge w)$.

(b) If $u_0, \dots, u_n \in U$ are disjoint, then $|\sum_{i=0}^n \alpha_i u_i| = \sum_{i=0}^n |\alpha_i||u_i|$ for any $\alpha_0, \dots, \alpha_n \in \mathbb{R}$.

(c) If u, $v \in U$ then

$$u^+ \wedge v^+ \le (u+v)^+ \le u^+ + v^+.$$

(d) If $u_0, \dots, u_m, v_0, \dots, v_n \in U^+$ and $\sum_{i=0}^m u_i = \sum_{j=0}^n v_j$, then there is a family $\langle w_{ij} \rangle_{i \le m, j \le n}$ in U^+ such that $\sum_{i=0}^m w_{ij} = v_j$ for every $j \le n$ and $\sum_{j=0}^n w_{ij} = u_i$ for every $i \le m$.

proof (a)

$$\begin{aligned} u \wedge (v+w) &\le [(u+w) \wedge (v+w)] \wedge u \\ &\le [(u \wedge v) + w] \wedge [(u \wedge v) + u] = (u \wedge v) + (u \wedge w). \end{aligned}$$

(b)(i)($\boldsymbol{\alpha}$) A simple induction, using (a) for the inductive step, shows that if $v_0, \dots, v_m$, $w_0, \dots, w_n$ are non-negative then $\sum_{i=0}^m v_i \wedge \sum_{j=0}^n w_j \le \sum_{i=0}^m \sum_{j=0}^n v_i \wedge w_j$. **($\boldsymbol{\beta}$)** Next, if $u \wedge v = 0$ then

$$(u-v)^+ = u - (u \wedge v) = u, \quad (u-v)^- = (v-u)^+ = v - (v \wedge u) = v,$$

$$|u-v| = (u-v)^+ + (u-v)^- = u + v = |u+v|,$$

so if $|u| \wedge |v| = 0$ then

$$\begin{aligned} (u^+ + v^+) \wedge (u^- + v^-) &\le (u^+ \wedge u^-) + (u^+ \wedge v^-) + (v^+ \wedge u^-) + (v^+ \wedge v^-) \\ &\le 0 + (|u| \wedge |v|) + (|v| \wedge |u|) + 0 = 0 \end{aligned}$$

and

$$|u+v| = |(u^+ + v^+) - (u^- + v^-)| = u^+ + v^+ + u^- + v^- = |u| + |v|.$$

($\boldsymbol{\gamma}$) Finally, if $|u| \wedge |v| = 0$ and α, $\beta \in \mathbb{R}$,

$$|\alpha u| \wedge |\beta v| = |\alpha||u| \wedge |\beta||v| \le (|\alpha| + |\beta|)|u| \wedge (|\alpha| + |\beta|)|v| = (|\alpha| + |\beta|)(|u| \wedge |v|) = 0.$$

(ii) We may therefore proceed by induction. The case $n = 0$ is trivial. For the inductive step to $n+1$, setting $u_i' = \alpha_i u_i$ we have $|u_i'| \wedge |u_j'| = 0$ for all $i \ne j$, by (i-γ). By (i-α),

$$|u_{n+1}'| \wedge |\sum_{i=0}^n u_i'| \le |u_{n+1}'| \wedge \sum_{i=0}^n |u_i'| \le \sum_{i=0}^n |u_{n+1}'| \wedge |u_i'| = 0,$$

so by (i-β) and the inductive hypothesis

$$|\sum_{i=0}^{n+1} u_i'| = |u_{n+1}'| + |\sum_{i=0}^n u_i'| = \sum_{i=0}^{n+1} |u_i'|$$

as required.

(c) By 352E,

$$u^+ \wedge v^+ = (u \vee 0) \wedge (v \vee 0) = (u \wedge v) \vee 0.$$

Now

$$u \wedge v = \frac{1}{2}(u + v - |u - v|) \leq \frac{1}{2}(u + v + |u + v|) = (u + v)^+,$$

and of course $0 \leq (u+v)^+$, so $u^+ \wedge v^+ \leq (u+v)^+$.

For the other inequality we need only note that $u + v \leq u^+ + v^+$ (because $u \leq u^+$, $v \leq v^+$) and $0 \leq u^+ + v^+$.

(d) Write w for the common value of $\sum_{i=0}^m u_i$ and $\sum_{j=0}^n v_j$.

Induce on $k = \#(\{(i,j) : i \leq m,\ j \leq n,\ u_i \wedge v_j > 0\})$. If $k = 0$, that is, $u_i \wedge v_j = 0$ for all i, j, then (by (a), used repeatedly) we must have $w \wedge w = 0$, that is, $w = 0$, and we can take $w_{ij} = 0$ for all i, j. For the inductive step to $k \geq 1$, take i^*, j^* such that $w^* = u_{i^*} \wedge v_{j^*} > 0$. Set

$$\tilde{u}_{i^*} = u_{i^*} - w^*, \quad \tilde{u}_i = u_i \text{ for } i \neq i^*,$$

$$\tilde{v}_{j^*} = v_{j^*} - w^*, \quad \tilde{v}_j = v_j \text{ for } j \neq j^*.$$

Then $\sum_{i=0}^m \tilde{u}_i = \sum_{j=0}^n \tilde{v}_j = w - w^*$ and $\tilde{u}_i \wedge \tilde{v}_j \leq u_i \wedge v_j$ for all i, j, while $\tilde{u}_{i^*} \wedge \tilde{v}_{j^*} = 0$; so that

$$\#(\{(i,j) : \tilde{u}_i \wedge \tilde{v}_j > 0\}) < k.$$

By the inductive hypothesis, there are $\tilde{w}_{ij} \geq 0$, for $i \leq m$ and $j \leq n$, such that $\tilde{u}_i = \sum_{j=0}^n \tilde{w}_{ij}$ for each i and $\tilde{v}_j = \sum_{i=0}^m \tilde{w}_{ij}$ for each j. Set $w_{i^*j^*} = \tilde{w}_{i^*j^*} + w^*$, $w_{ij} = \tilde{w}_{ij}$ for $(i,j) \neq (i^*, j^*)$; then $u_i = \sum_{j=0}^n w_{ij}$ and $v_j = \sum_{i=0}^m w_{ij}$, so the induction proceeds.

352G Riesz homomorphisms: Proposition Let U be a Riesz space, V a partially ordered linear space and $T : U \to V$ a linear operator. Then the following are equiveridical:

(i) T is a Riesz homomorphism in the sense of 351H;

(ii) $(Tu)^+ = \sup\{Tu, 0\}$ is defined and equal to $T(u^+)$ for every $u \in U$;

(iii) $\sup\{Tu, -Tu\}$ is defined and equal to $T|u|$ for every $u \in U$;

(iv) $\inf\{Tu, Tv\} = 0$ in V whenever $u \wedge v = 0$ in U.

proof (i)⇒(iii) and (i)⇒(iv) are special cases of 351Hc. For (iii)⇒(ii) we have

$$\sup\{Tu, 0\} = \frac{1}{2}Tu + \sup\{\frac{1}{2}Tu, -\frac{1}{2}Tu\} = \frac{1}{2}Tu + \frac{1}{2}T|u| = T(u^+).$$

For (ii)⇒(i), argue as follows. If (ii) is true and u, $v \in U$, then

$$Tu \wedge Tv = \inf\{Tu, Tv\} = Tu + \inf\{0, Tv - Tu\} = Tu - \sup\{0, T(u - v)\}$$

is defined and equal to

$$Tu - T((u - v)^+) = T(u - (u - v)^+) = T(u \wedge v).$$

Inducing on n,

$$\inf_{i \leq n} Tu_i = T(\inf_{i \leq n} u_i)$$

for all $u_0, \dots, u_n \in U$; in particular, if $\inf_{i \leq n} u_i = 0$ then $\inf_{i \leq n} Tu_i = 0$; which is the definition I gave of Riesz homomorphism.

Finally, for (iv)⇒(ii), we know from (iv) that $0 = \inf\{T(u^+), T(u^-)\}$, so $-T(u^+) = \inf\{0, -Tu\}$ and $T(u^+) = \sup\{0, Tu\}$.

352H Proposition If U and V are Riesz spaces and $T : U \to V$ is a bijective Riesz homomorphism, then T is a partially-ordered-linear-space isomorphism, and $T^{-1} : V \to U$ is a Riesz homomorphism.

proof Use 352G(ii). If $v \in V$, set $u = T^{-1}v$; then $T(u^+) = v^+$ so $T^{-1}(v^+) = u^+ = (T^{-1}v)^+$. Thus T^{-1} is a Riesz homomorphism; in particular, it is order-preserving, so T is an isomorphism for the order structures as well as for the linear structures.

352I Riesz subspaces (a) If U is a partially ordered linear space, a **Riesz subspace** of U is a linear subspace V such that $\sup\{u,v\}$ and $\inf\{u,v\}$ are defined in U and belong to V for every u, $v \in V$. In this case they are the supremum and infimum of $\{u,v\}$ in V, so V, with the induced order and linear structure, is a Riesz space in its own right, and the embedding map $u \mapsto u : V \to U$ is a Riesz homomorphism.

(b) Generally, if U is a Riesz space, V is a partially ordered linear space and $T : U \to V$ is a Riesz homomorphism, then $T[U]$ is a Riesz subspace of V (because, by 351Hc, $Tu \vee Tu' = T(u \vee u')$, $T(u \wedge u') = Tu \wedge Tu'$ belong to $T[U]$ for all u, $u' \in U$).

(c) If U is a Riesz space and V is a linear subspace of U, then V is a Riesz subspace of U iff $|u| \in V$ for every $u \in V$. **P** In this case,

$$u \vee v = \frac{1}{2}(u + v + |u - v|), \quad u \wedge v = \frac{1}{2}(u + v - |u - v|)$$

belong to V for all u, $v \in V$. **Q**

352J Solid subsets (a) If U is a Riesz space, a subset A of U is solid (in the sense of 351I) iff $v \in A$ whenever $u \in A$ and $|v| \le |u|$. **P** (α) If A is solid, $u \in V$ and $|v| \le |u|$, then there is some $w \in A$ such that $-w \le u \le w$; in this case $|v| \le |u| \le w$ and $-w \le v \le w$ and $v \in A$. (β) Suppose that A satisfies the condition. If $u \in A$, then $|u| \in A$ and $-|u| \le u \le |u|$. If $w \in A$ and $-w \le u \le w$ then $-u \le w$, $|u| \le w = |w|$ and $u \in A$. Thus A is solid. **Q** In particular, if A is solid, then $v \in A$ iff $|v| \in A$.

For any set $A \subseteq U$, the set

$$\{u : \text{there is some } v \in A \text{ such that } |u| \le |v|\}$$

is a solid subset of U, the smallest solid set including A; we call it the **solid hull** of A in U.

Any solid linear subspace of U is a Riesz subspace (use 352Fc). If $V \subseteq U$ is a Riesz subspace, then the solid hull of V in U is

$$\{u : \text{there is some } v \in V \text{ such that } |u| \le v\}$$

and is a solid linear subspace of U.

(b) If T is a Riesz homomorphism from a Riesz space U to a partially ordered linear space V, then its kernel W is a solid linear subspace of U. **P** If $u \in W$ and $|v| \le |u|$, then $T|u| = \sup\{Tu, T(-u)\} = 0$, while $-|u| \le v \le |u|$, so that $-0 \le Tv \le 0$ and $v \in W$. **Q** Now the quotient space U/W, as defined in 351J, is a Riesz space, and is isomorphic, as partially ordered linear space, to the Riesz space $T[U]$. **P** Because U/W is the linear space quotient of V by the kernel of the linear operator T, we have an induced linear space isomorphism $S : U/W \to T[U]$ given by setting $Su^\bullet = Tu$ for every $u \in U$. If $p \ge 0$ in U/W there is a $u \in U^+$ such that $u^\bullet = p$ (351J), so that $Sp = Tu \ge 0$. On the other hand, if $p \in U/W$ and $Sp \ge 0$, take $u \in U$ such that $u^\bullet = p$. We have

$$T(u^+) = (Tu)^+ = (Sp)^+ = Sp = Tu,$$

so that $T(u^-) = Tu^+ - Tu = 0$ and $u^- \in W$, $p = (u^+)^\bullet \ge 0$. Thus $Sp \ge 0$ iff $p \ge 0$, and S is a partially-ordered-linear-space isomorphism. **Q**

(c) Because a subset of a Riesz space is a solid linear subspace iff it is the kernel of a Riesz homomorphism (see 352U below), such subspaces are sometimes called **ideals**.

352K Products If $\langle U_i \rangle_{i \in I}$ is any family of Riesz spaces, then the product partially ordered linear space $U = \prod_{i \in I} U_i$ (351L) is a Riesz space, with

$$u \vee v = \langle u(i) \vee v(i) \rangle_{i \in I}, \quad u \wedge v = \langle u(i) \wedge v(i) \rangle_{i \in I}, \quad |u| = \langle |u(i)| \rangle_{i \in I}$$

for all u, $v \in U$.

352L Theorem Let U be any Riesz space. Then there are a set X, a filter $\mathcal{F}$ on X and a Riesz subspace of the Riesz space $\mathbb{R}^X|\mathcal{F}$ (definition: 351M) which is isomorphic, as Riesz space, to U.

proof By 351Q, we can find such X and $\mathcal{F}$ and an injective Riesz homomorphism $T : U \to \mathbb{R}^X|\mathcal{F}$. By 352K, or otherwise, $\mathbb{R}^X$ is a Riesz space; by 352Jb, $\mathbb{R}^X|\mathcal{F}$ is a Riesz space (recall that it is a quotient of $\mathbb{R}^X$ by a solid linear subspace, as explained in 351M); by 352Ib, $T[U]$ is a Riesz subspace of $\mathbb{R}^X|\mathcal{F}$; and by 352H it is isomorphic to U.

352M Corollary Any identity involving the operations $+$, $-$, $\vee$, $\wedge$, $^+$, $^-$, $|\ |$ and scalar multiplication, and the relation $\le$, which is valid in $\mathbb{R}$, is valid in all Riesz spaces.

Remark I suppose some would say that a strict proof of this must begin with a formal description of what the phrase 'any identity involving the operations...' means. However I think it is clear in practice what is involved. Given a proposed identity like

$$0 \le \sum_{i=0}^{n} |\alpha_i||u_i| - |\sum_{i=0}^{n} \alpha_i u_i| \le \sum_{i \ne j}(|\alpha_i| + |\alpha_j|)(|u_i| \wedge |u_j|),$$

(compare 352Fb), then to check that it is valid in all Riesz spaces you need only check (i) that it is true in $\mathbb{R}$ (ii) that it is true in $\mathbb{R}^X$ (iii) that it is true in any $\mathbb{R}^X|\mathcal{F}$ (iv) that it is true in any Riesz subspace of $\mathbb{R}^X|\mathcal{F}$; and you can hope that the arguments for (ii)-(iv) will be nearly trivial, since (ii) is generally nothing but a coordinate-by-coordinate repetition of (i), and (iii) and (iv) involve only transformations of the formula by Riesz homomorphisms which preserve its structure.

352N Order-density and order-continuity Let U be a Riesz space.

(a) Definition A Riesz subspace V of U is **quasi-order-dense** if for every $u > 0$ in U there is a $v \in V$ such that $0 < v \le u$; it is **order-dense** if $u = \sup\{v : v \in V, 0 \le v \le u\}$ for every $u \in U^+$.

(b) If U is a Riesz space and V is a quasi-order-dense Riesz subspace of U, then the embedding $V \subseteq\!\!\!\!\!\!\!\;\to U$ is order-continuous. **P** Let $A \subseteq V$ be a non-empty set such that $\inf A = 0$ in V. **?** If 0 is not the infimum of A in U, then there is a $u > 0$ such that u is a lower bound for A in U; now there is a $v \in V$ such that $0 < v \le u$, and v is a lower bound for A in V which is strictly greater than 0. **X** Thus $0 = \inf A$ in U. As A is arbitrary, the embedding is order-continuous. **Q**

(c)(i) If $V \subseteq U$ is an order-dense Riesz subspace, it is quasi-order-dense. (ii) If V is a quasi-order-dense Riesz subspace of U and W is a quasi-order-dense Riesz subspace of V, then W is a quasi-order-dense Riesz subspace of U. (iii) If V is an order-dense Riesz subspace of U and W is an order-dense Riesz subspace of V, then W is an order-dense Riesz subspace of U. (Use (b).) (iv) If V is a quasi-order-dense solid linear subspace of U and W is a quasi-order-dense Riesz subspace of U then $V \cap W$ is quasi-order-dense in V, therefore in U.

(d) I ought somewhere to remark that a Riesz homomorphism, being a lattice homomorphism, is order-continuous iff it preserves arbitrary suprema and infima; compare 313L(b-iv) and (b-v).

(e) If V is a Riesz subspace of U, we say that it is **regularly embedded** in U if the identity map from V to U is order-continuous, that is, whenever $A \subseteq V$ is non-empty and has infimum 0 in V, then 0 is still its greatest lower bound in U. Thus quasi-order-dense Riesz subspaces and solid linear subspaces are regularly embedded.

352O Bands Let U be a Riesz space.

(a) Definition A **band** or **normal subspace** of U is an order-closed solid linear subspace.

(b) If $V \subseteq U$ is a solid linear subspace then it is a band iff $\sup A \in V$ whenever $A \subseteq V^+$ is a non-empty, upwards-directed subset of V with a supremum in U. **P** Of course the condition is necessary; I have to show that it is sufficient. (i) Let $A \subseteq V$ be any non-empty upwards-directed set with a supremum in V. Take any $u_0 \in A$ and set $A_1 = \{u - u_0 : u \in A, u \ge u_0\}$. Then A_1 is a non-empty upwards-directed subset of V^+, and $u_0 + A_1 = \{u : u \in A, u \ge u_0\}$ has the same upper bounds as A, so $\sup A_1 = \sup A - u_0$ is defined in U and belongs to V. Now $\sup A = u_0 + \sup A_1$ also belongs to V. (ii) If $A \subseteq V$ is non-empty, downwards-directed and has an infimum in U, then $-A \subseteq V$ is upwards-directed, so $\inf A = \sup(-A)$ belongs to V. Thus V is order-closed. **Q**

(c) For any set $A \subseteq U$ set $A^\perp = \{v : v \in U, |u| \wedge |v| = 0 \text{ for every } u \in A\}$. Then $A^\perp$ is a band. **P** (i) Of course $0 \in A^\perp$. (ii) If $v, w \in A^\perp$ and $u \in A$, then

$$|u| \wedge |v + w| \le (|u| \wedge |v|) + (|u| \wedge |w|) = 0,$$

so $v + w \in A^\perp$. (iii) If $v \in A^\perp$ and $|w| \le |v|$ then

$$0 \le |u| \wedge |w| \le |u| \wedge |v| = 0$$

for every $u \in A$, so $w \in A^\perp$. (iv) If $v \in A^\perp$ then $nv \in A^\perp$ for every n, by (ii). So if $\alpha \in \mathbb{R}$, take $n \in \mathbb{N}$ such that $|\alpha| \le n$; then

$$|\alpha v| = |\alpha||v| \le n|v| \in A^\perp$$

and $\alpha v \in A^\perp$. Thus $A^\perp$ is a solid linear subspace of U. (v) If $B \subseteq (A^\perp)^+$ is non-empty and upwards-directed and has a supremum w in U, then

$$|u| \wedge |w| = |u| \wedge w = \sup_{v\in B} |u| \wedge v = 0$$

by 352Ea, so $w \in A^\perp$. Thus $A^\perp$ is a band. **Q**

(d) For any $A \subseteq U$, $A \subseteq (A^\perp)^\perp$. Also $B^\perp \subseteq A^\perp$ whenever $A \subseteq B$. So

$$A^{\perp\perp\perp} \subseteq A^\perp \subseteq A^{\perp\perp\perp}$$

and $A^\perp = A^{\perp\perp\perp}$.

(e) If W is another Riesz space and $T : U \to W$ is an order-continuous Riesz homomorphism then its kernel is a band. (For $\{0\}$ is order-closed in W and the inverse image of an order-closed set by an order-continuous order-preserving function is order-closed.)

352P Complemented bands Let U be a Riesz space. A band $V \subseteq U$ is **complemented** if $V^{\perp\perp} = V$, that is, if V is of the form $A^\perp$ for some $A \subseteq U$ (352Od). In this case its **complement** is the complemented band $V^\perp$.

352Q Theorem In any Riesz space U, the set $\mathfrak{C}$ of complemented bands forms a Dedekind complete Boolean algebra, with

$$V \cap_{\mathfrak{C}} W = V \cap W, \quad V \cup_{\mathfrak{C}} W = (V + W)^{\perp\perp},$$

$$1_{\mathfrak{C}} = U, \quad 0_{\mathfrak{C}} = \{0\}, \quad 1_{\mathfrak{C}} \setminus_{\mathfrak{C}} V = V^\perp,$$

$$V \subseteq_{\mathfrak{C}} W \iff V \subseteq W$$

for V, $W \in \mathfrak{C}$.

proof To show that $\mathfrak{C}$ is a Boolean algebra, I use the identification of Boolean algebras with complemented distributive lattices (311L).

(a) Of course $\mathfrak{C}$ is partially ordered by $\subseteq$. If V, $W \in \mathfrak{C}$ then

$$V \cap W = V^{\perp\perp} \cap W^{\perp\perp} = (V^\perp \cup W^\perp)^\perp \in \mathfrak{C},$$

and $V \cap W$ must be $\inf\{V, W\}$ in C. The map $V \mapsto V^\perp : \mathfrak{C} \to \mathfrak{C}$ is an order-reversing permutation, so that $V \subseteq W$ iff $W^\perp \subseteq V^\perp$ and $V \vee W = \sup\{V, W\}$ will be $(V^\perp \cap W^\perp)^\perp$; thus $\mathfrak{C}$ is a lattice. Note also that $V \vee W$ must be the smallest complemented band including $V + W$, that is, it is $(V + W)^{\perp\perp}$.

(b) If V_1, V_2, $W \in \mathfrak{C}$ then $(V_1 \vee V_2) \wedge W = (V_1 \wedge W) \vee (V_2 \wedge W)$. **P** Of course $(V_1 \vee V_2) \wedge W \supseteq (V_1 \wedge W) \vee (V_2 \wedge W)$. **?** Suppose, if possible, that there is a $u \in (V_1 \vee V_2) \cap W \setminus ((V_1 \cap W) \vee (V_2 \cap W))$. Then $u \notin ((V_1 \cap W)^\perp \cap (V_2 \cap W)^\perp)^\perp$, so there is a $v \in (V_1 \cap W)^\perp \cap (V_2 \cap W)^\perp$ such that $u_1 = |u| \wedge |v| > 0$. Now $u_1 \in V_1 \vee V_2 = (V_1^\perp \cap V_2^\perp)^\perp$ so $u_1 \notin V_1^\perp \cap V_2^\perp$; say $u_1 \notin V_j^\perp$, and there is a $v_j \in V_j$ such that $u_2 = u_1 \wedge |v_j| > 0$. In this case we still have $u_2 \in (V_j \cap W)^\perp$, because $u_2 \le |v|$, but also $u_2 \in V_j$ and $u_2 \in W$ because $u_2 \le |u|$; but this means that $u_2 = u_2 \wedge u_2 = 0$, which is absurd. **X** Thus $(V_1 \vee V_2) \wedge W \subseteq (V_1 \wedge W) \vee (V_2 \wedge W)$ and the two are equal. **Q**

(c) Now if $V \in \mathfrak{C}$,

$$V \wedge V^\perp = \{0\}$$

is the least member of $\mathfrak{C}$, because if $v \in V \cap V^\perp$ then $|v| = |v| \wedge |v| = 0$. By 311L, $\mathfrak{C}$ has a Boolean algebra structure, with the Boolean relations described; by 312M, this structure is uniquely defined.

(d) Finally, if $\mathcal{V} \subseteq \mathfrak{C}$ is non-empty, then

$$\bigcap \mathcal{V} = (\bigcup_{V\in\mathcal{V}} V^\perp)^\perp \in \mathfrak{C}$$

and is $\inf \mathcal{V}$ in $\mathfrak{C}$. So $\mathfrak{C}$ is Dedekind complete.

352R Projection bands Let U be a Riesz space.

(a) A **projection band** in U is a set $V \subseteq U$ such that $V + V^\perp = U$. In this case V is a complemented band. **P** If $v \in V^{\perp\perp}$ then v is expressible as $v_1 + v_2$ where $v_1 \in V$ and $v_2 \in V^\perp$. Now $|v| = |v_1| + |v_2| \ge |v_2|$ (352Fb), so

$$|v_2| = |v_2| \wedge |v_2| \le |v_2| \wedge |v| = 0$$

and $v = v_1 \in V$. Thus $V = V^{\perp\perp}$ is a complemented band. **Q** Observe that $U = V^\perp + V^{\perp\perp}$ so $V^\perp$ also is a projection band.

(b) Because $V \cap V^\perp$ is always $\{0\}$, we must have $U = V \oplus V^\perp$ for any projection band $V \subseteq U$; accordingly there is a corresponding **band projection** $P_V : U \to U$ defined by setting $P(v + w) = v$ whenever $v \in V$, $w \in V^\perp$. In this context I will say that v is the **component** of $v + w$ in V. The kernel of P is $V^\perp$, the set of values is V, and $P^2 = P$. Moreover, P is an order-continuous Riesz homomorphism. **P** (i) P is a linear operator because V and $V^\perp$ are linear subspaces. (ii) If $v \in V$ and $w \in V^\perp$ then $|v + w| = |v| + |w|$, by 352Fb, so $P|v + w| = |v| = |P(v + w)|$; consequently P is a Riesz homomorphism (352G). (iii) If $A \subseteq U$ is downwards-directed and has infimum 0, then $Pu \le u$ for every $u \in A$, so $\inf P[A] = 0$; thus P is order-continuous. **Q**

(c) Note that for any band projection P, and any $u \in U$, we have $|Pu| \wedge |u - Pu| = 0$, so that $|u| = |Pu| + |u - Pu|$ and (in particular) $|Pu| \le |u|$; consequently $P[W] \subseteq W$ for any solid linear subspace W of U.

(d) A linear operator $P : U \to U$ is a band projection iff $Pu \wedge (u - Pu) = 0$ for every $u \in U^+$. **P** I remarked in (c) that the condition is satisfied for any band projection. Now suppose that P has the property. (i) For any $u \in U^+$, $Pu \ge 0$ and $u - Pu \ge 0$; in particular, P is a positive linear operator. (ii) If $u, v \in U^+$ then $u - Pu \le (u+v) - P(u+v)$, so

$$Pv \wedge (u - Pu) \le P(u + v) \wedge ((u + v) - P(u + v)) = 0$$

and $Pv \wedge (u - Pu) = 0$. (iii) If $u, v \in U$ then $|Pv| \le P|v|$, $|u - Pu| \le |u| - P|u|$ (because $w \mapsto w - Pw$ is a positive linear operator), so

$$|Pv| \wedge |u - Pu| \le P|v| \wedge (|u| - P|u|) = 0.$$

(iv) Setting $V = P[U]$, we see that $u - Pu \in V^\perp$ for every $u \in U$, so that

$$u = u + (u - Pu) \in V + V^\perp$$

for every u, and $U = V + V^\perp$; thus V is a projection band. (v) Since $Pu \in V$ and $u - Pu \in V^\perp$ for every $u \in U$, P is the band projection onto V. **Q**

352S Proposition Let U be any Riesz space.

(a) The family $\mathfrak{B}$ of projection bands in U is a subalgebra of the Boolean algebra $\mathfrak{C}$ of complemented bands in U.

(b) For $V \in \mathfrak{B}$ let $P_V : U \to V$ be the corresponding projection. Then for any $e \in U^+$,

$$P_{V \cap W} e = P_V e \wedge P_W e = P_V P_W e, \quad P_{V \vee W} e = P_V e \vee P_W e$$

for all V, $W \in \mathfrak{B}$. In particular, band projections commute.

(c) If $V \in \mathfrak{B}$ then the algebra of projection bands in V is just the principal ideal of $\mathfrak{B}$ generated by V.

proof (a) Of course $0_{\mathfrak{C}} = \{0\} \in \mathfrak{B}$. If $V \in \mathfrak{B}$ then $V^\perp = 1_{\mathfrak{C}} \setminus V$ belongs to $\mathfrak{B}$. If now W is another member of $\mathfrak{B}$, then

$$(V \cap W) + (V \cap W)^\perp \supseteq (V \cap W) + V^\perp + W^\perp.$$

But if $u \in U$ then we can express u as $v + v'$, where $v \in V$ and $v' \in V^\perp$, and v as $w + w'$, where $w \in W$ and $w' \in W^\perp$; and as $|w| \le |v|$, we also have $w \in V$, so that

$$u = w + v' + w' \in (V \cap W) + V^\perp + W^\perp.$$

This shows that $V \cap W \in \mathfrak{B}$. Thus $\mathfrak{B}$ is closed under intersection and complements and is a subalgebra of $\mathfrak{C}$.

(b) If V, $W \in \mathfrak{B}$ and $e \in U^+$, we have $e = e_1 + e_2 + e_3 + e_4$ where

$$e_1 = P_W P_V e \in V \cap W, \quad e_2 = P_{W^\perp} P_V e \in V \cap W^\perp,$$

$$e_3 = P_W P_{V^\perp} e \in V^\perp \cap W, \quad e_4 = P_{W^\perp} P_{V^\perp} e \in V^\perp \cap W^\perp,$$

$$e_1 + e_2 = P_V e, \quad e_1 + e_3 = P_W e.$$

Now $e_2 + e_3 + e_4$ belongs to $(V \cap W)^\perp$, so e_1 must be the component of e in $V \cap W$; similarly e_4 is the component of e in $V^\perp \cap W^\perp$, and $e_1 + e_2 + e_3$ is the component of e in $V \vee W$. But as $e_2 \wedge e_3 = 0$, we have

$$P_{V \cap W} e = e_1 = (e_1 + e_2) \wedge (e_1 + e_3) = P_V e \wedge P_W e,$$

$$P_{V \vee W} e = e_1 + e_2 + e_3 = (e_1 + e_2) \vee (e_1 + e_3) = P_V e \vee P_W e,$$

as required.

It follows that

$$P_V P_W = P_{V \cap W} = P_{W \cap V} = P_W P_V.$$

(c) If V, $W \in \mathfrak{B}$ and $W \subseteq V$, then of course W is a band in the Riesz space V (because V is order-closed in U, so that for any set $A \subseteq W$ its supremum in U will be its supremum in V). For any $v \in V$, we have an expression of it as $w + w'$, where $w \in W$ and $w' \in W^\perp$, taken in U; but as $|w| + |w'| = |w + w'| = |v| \in V$, w' belongs to V, and is in $W_V^\perp$, the band in V orthogonal to W. Thus $W + W_V^\perp = V$ and W is a projection band in V. Conversely, if W is a projection band in V, then $W^\perp$ (taken in U) includes $W_V^\perp + V^\perp$, so that

$$W + W^\perp \supseteq W + W_V^\perp + V^\perp = V + V^\perp = U$$

and $W \in \mathfrak{B}$.

Thus the algebra of projection bands in V is, as a set, equal to the principal ideal $\mathfrak{B}_V$; because their orderings agree, or otherwise, their Boolean algebra structures coincide.

352T Products again (a) If $U = \prod_{i \in I} U_i$ is a product of Riesz spaces, then for any $J \subseteq I$ we have a subspace

$$V_J = \{u : u \in U,\, u(i) = 0 \text{ for all } i \in I \setminus J\}$$

of U, canonically isomorphic to $\prod_{i \in J} U_i$. Each V_J is a projection band, its complement being $V_{I \setminus J}$; the map $J \mapsto V_J$ is a Boolean homomorphism from $\mathcal{P}I$ to the algebra $\mathfrak{B}$ of projection bands in U, and $\langle V_{\{i\}} \rangle_{i \in I}$ is a partition of unity in $\mathfrak{B}$.

(b) Conversely, if U is a Riesz space and $(V_0, \ldots, V_n)$ is a *finite* partition of unity in the algebra $\mathfrak{B}$ of projection bands in U, then every element of U is uniquely expressible as $\sum_{i=0}^n u_i$ where $u_i \in V_i$ for each i. (Induce on n.) This decomposition corresponds to a Riesz space isomorphism between U and $\prod_{i \le n} V_i$.

352U Quotient spaces (a) If U is a Riesz space and V is a solid linear subspace, then the quotient partially ordered linear space U/V (351J) is a Riesz space; if U and W are Riesz spaces and $T : U \to W$ a Riesz homomorphism, then the kernel V of T is a solid linear subspace of U and the Riesz subspace $T[U]$ of W is isomorphic to U/V (352Jb).

(b) Suppose that U is a Riesz space and V a solid linear subspace. Then the canonical map from U to U/V is order-continuous iff V is a band. **P** (i) If $u \mapsto u^\bullet$ is order-continuous, its kernel V is a band, by 352Oe. (ii) If V is a band, and $A \subseteq U$ is non-empty and downwards-directed and has infimum 0, let $p \in U/V$ be any lower bound for $\{u^\bullet : u \in A\}$. Express p as $w^\bullet$. Then $((w - u)^+)^\bullet = (w^\bullet - u^\bullet)^+ = 0$, that is, $(w - u)^+ \in V$ for every $u \in A$. But this means that

$$w^+ = \sup_{u \in A} (w - u)^+ \in V, \quad p^+ = (w^+)^\bullet = 0,$$

that is, $p \le 0$. As p is arbitrary, $\inf_{u \in A} u^\bullet = 0$; as A is arbitrary, $u \mapsto u^\bullet$ is order-continuous. **Q**

352V Principal bands Let U be a Riesz space. Evidently the intersection of any family of Riesz subspaces of U is a Riesz subspace, the intersection of any family of solid linear subspaces is a solid linear subspace, the intersection of any family of bands is a band; we may therefore speak of the band generated by a subset A of U, the intersection of all the bands including A. Now we have the following description of the band generated by a single element.

Lemma Let U be a Riesz space.

(a) If $A \subseteq U^+$ is upwards-directed and $2w \in A$ for every $w \in A$, then an element u of U belongs to the band generated by A iff $|u| = \sup_{w \in A} |u| \wedge w$.

(b) If $u \in U$ and $w \in U^+$, then u belongs to the band in U generated by w iff $|u| = \sup_{n \in \mathbb{N}} |u| \wedge nw$.

proof (a) Let W be the band generated by A and W' the set of elements of U satisfying the condition.

(i) If $u \in W'$ then $|u| \wedge w \in W$ for every $w \in A$, because W is a solid linear subspace; because W is also order-closed, $|u|$ and u belong to W. Thus $W' \subseteq W$.

(ii) Now W' is a band.

P$(\boldsymbol{\alpha})$ If $u \in W'$ and $|v| \le |u|$ then

$$\sup_{w \in A} |v| \wedge w = \sup_{w \in A} |v| \wedge |u| \wedge w = |v| \wedge \sup_{w \in A} |u| \wedge w = |v| \wedge |u| = |v|$$

by 352Ea, so $v \in W'$.

$(\boldsymbol{\beta})$ If u, $v \in W'$ then, for any w_1, $w_2 \in A$ there is a $w \in A$ such that $w \ge w_1 \vee w_2$. Now $w_1 + w_2 \le 2w \in A$, and

$$(|u| + |v|) \wedge 2w \ge (|u| \wedge w_1) + (|v| \wedge w_2).$$

So any upper bound for $\{(|u|+|v|)\wedge w : w \in A\}$ must also be an upper bound for $\{|u|\wedge w : w \in A\}+\{|v|\wedge w : w \in A\}$ and therefore greater than or equal to

$$\begin{aligned}\sup(\{|u| \wedge w : w \in A\} + \{|v| \wedge w : w \in A\}) &= \sup_{w\in A} |u| \wedge w + \sup_{w\in A} |v| \wedge w \\ &= |u| + |v|\end{aligned}$$

(351Dc). But this means that $\sup_{w\in A}(|u| + |v|) \wedge w$ must be $|u| + |v|$, and $|u| + |v|$ belongs to W'; it follows from (α) that $u + v$ belongs to W'.

$(\boldsymbol{\gamma})$ Just as in 352Oc, we now have

$$nu \in W' \text{ for every } n \in \mathbb{N},\ u \in W',$$

and therefore $\alpha u \in W'$ for every $\alpha \in \mathbb{R}$, $u \in W'$, since $|\alpha u| \le |nu|$ if $|\alpha| \le n$. Thus W' is a solid linear subspace of U.

$(\boldsymbol{\delta})$ Now suppose that $C \subseteq (W')^+$ has a supremum v in U. Then any upper bound of $\{v \wedge w : w \in A\}$ must also be an upper bound of $\{u \wedge w : u \in C,\ w \in A\}$ and greater than or equal to $u = \sup_{w\in A} u \wedge w$ for every $u \in C$, therefore greater than or equal to $v = \sup C$. Thus $v = \sup_{w\in A} v \wedge w$ and $v \in W'$. As C is arbitrary, W' is a band (352Ob). **Q**

(iii) Since A is obviously included in W', W' must include W; putting this together with (i), $W = W'$, as claimed.

(b) Apply (a) with $A = \{nw : n \in \mathbb{N}\}$.

352W f-algebras Some of the most important Riesz spaces have multiplicative structures as well as their order and linear structures. A particular class of these structures appears sufficiently often for it to be useful to develop a little of its theory. The following definition is a common approach.

(a) Definition An **f-algebra** is a Riesz space U with a multiplication $\times : U \times U \to U$ such that

$$u \times (v \times w) = (u \times v) \times w,$$

$$(u + v) \times w = (u \times w) + (v \times w), \quad u \times (v + w) = (u \times v) + (u \times w),$$

$$\alpha(u \times v) = (\alpha u) \times v = u \times (\alpha v)$$

for all u, v, $w \in U$ and $\alpha \in \mathbb{R}$, and

$$u \times v \ge 0 \text{ whenever } u,\ v \ge 0,$$

$$\text{if } u \wedge v = 0 \text{ then } (u \times w) \wedge v = (w \times u) \wedge v = 0 \text{ for every } w \ge 0.$$

An f-algebra is **commutative** if $u \times v = v \times u$ for all u, v.

(b) Let U be an f-algebra.

(i) If $u \wedge v = 0$ in U, then $u \times v = 0$. **P** $u \wedge (u \times v) = 0$ so $(u \times v) \wedge (u \times v) = 0$. **Q**

(ii) $u \times u \ge 0$ for every $u \in U$. **P**

$$\begin{aligned}(u^+-u^-)\times(u^+-u^-)&=u^+\times u^+-u^+\times u^- -u^-\times u^+ +u^-\times u^-\\&=u^+\times u^+ +u^-\times u^-\geq 0.\ \mathbf{Q}\end{aligned}$$

(iii) If u, $v\in U$ then $|u\times v|=|u|\times|v|$. **P** $u^+\times v^+$, $u^+\times v^-$, $u^-\times v^+$ and $u^+\times u^-$ are disjoint, so

$$\begin{aligned}|u\times v|&=|u^+\times v^+-u^+\times v^- -u^-\times v^+ +u^-\times v^-|\\&=u^+\times v^+ +u^+\times v^- +u^-\times v^+ +u^-\times v^-\\&=|u|\times|v|\end{aligned}$$

by 352Fb. **Q**

(iv) If $v\in U^+$ the maps $u\mapsto u\times v$, $u\mapsto v\times u:U\to U$ are Riesz homomorphisms. **P** The first four clauses of the definition in (a) ensure that they are linear operators. If $u\in U$, then

$$|u|\times v=|u\times v|,\quad v\times|u|=|v\times u|$$

by (iii), so we have Riesz homomorphisms, by 352G(iii). **Q**

(c) Let $\langle U_i\rangle_{i\in I}$ be a family of f-algebras, with Riesz space product U (352K). If we set $u\times v=\langle u(i)\times v(i)\rangle_{i\in I}$ for all u, $v\in U$, then U becomes an f-algebra.

352X Basic exercises >(a) Let U be any Riesz space. Show that $|u^+-v^+|\leq|u-v|$ for all u, $v\in U$.

>(b) Let U, V be a Riesz spaces and $T:U\to V$ a linear operator. Show that the following are equiveridical: (i) T is a Riesz homomorphism; (ii) $T(u\vee v)=Tu\vee Tv$ for all u, $v\in U$; (iii) $T(u\wedge v)=Tu\wedge Tv$ for all u, $v\in U$; (iv) $|Tu|=T|u|$ for every $u\in U$.

(c) Let U be a Riesz space and V a solid linear subspace; for $u\in U$ write $u^\bullet$ for the corresponding element of U/V. Show that if $A\subseteq U$ is solid then $\{u^\bullet:u\in A\}$ is solid in U/W.

(d) Let U be a Riesz space. Show that $\operatorname{med}(\alpha u,\alpha v,\alpha w)=\alpha\operatorname{med}(u,v,w)$ for all u, v, $w\in U$ and all $\alpha\in\mathbb{R}$. (*Hint*: 3A1Ic, 352M.)

(e) Let U and V be Riesz spaces and $T:U\to V$ a Riesz homomorphism with kernel W. Show that if W is a band in U and $T[U]$ is regularly embedded in V then T is order-continuous.

(f) Give $U=\mathbb{R}^2$ its lexicographic ordering (351Xa). Show that it has a band V which is not complemented.

(g) Let U be a Riesz space and $\mathfrak{C}$ the algebra of complemented bands in U. Show that for any $V\in\mathfrak{C}$ the algebra of complemented bands in V is just the principal ideal of $\mathfrak{C}$ generated by V.

>(h) Let $U=C([0,1])$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$, with its usual linear and order structures, so that it is a Riesz subspace of $\mathbb{R}^{[0,1]}$. Set $V=\{u:u\in U,\ u(t)=0\text{ if }t\leq\frac{1}{2}\}$. Show that V is a band in U and that $V^\perp=\{u:u(t)=0\text{ if }t\geq\frac{1}{2}\}$, so that V is complemented but is not a projection band.

(i) Show that the Boolean homomorphism $J\mapsto V_J:\mathcal{P}I\to\mathfrak{B}$ of 352Ta is order-continuous.

(j) Let U be a Riesz space and $A\subseteq U^+$ an upwards-directed set. Show that the band generated by A is $\{u:|u|=\sup_{n\in\mathbb{N},w\in A}|u|\wedge nw\}$.

>(k)(i) Let X be any set. Setting $(u\times v)(x)=u(x)v(x)$ for u, $v\in\mathbb{R}^X$, $x\in X$, show that $\mathbb{R}^X$ is a commutative f-algebra. (ii) With the same definition of $\times$, show that $\ell^\infty(X)$ is an f-algebra. (iii) If X is a topological space, show that $C(X)$, $C_b(X)$ are f-algebras. (iv) If (X,Σ,μ) is a measure space, show that $L^0(\mu)$ and $L^\infty(\mu)$ (§241, §243) are f-algebras.

(l) Let $U\subseteq\mathbb{R}^{\mathbb{Z}}$ be the set of sequences u such that $\{n:u(n)\neq 0\}$ is bounded above in $\mathbb{Z}$. For u, $v\in U$ (i) say that $u\leq v$ if either $u=v$ or there is an $n\in\mathbb{Z}$ such that $u(n)<v(n)$, $u(i)=v(i)$ for every $i>n$ (ii) say that $(u*v)(n)=\sum_{i=-\infty}^{\infty}u(i)v(n-i)$ for every $n\in\mathbb{Z}$. Show that U is an f-algebra under this ordering and multiplication.

(m) Let U be an f-algebra. (i) Show that any complemented band in U is an ideal in the ring $(U, +, \times)$. (ii) Show that if $P : U \to U$ is a band projection, then $P(u \times v) = Pu \times Pv$ for every u, $v \in U$.

(n) Let U be an f-algebra with multiplicative identity e. Show that $u - \gamma e \le \frac{1}{\gamma}u^2$ for every $u \in U$, $\gamma > 0$. (*Hint*: $(u^+ - \gamma e)^2 \ge 0$.)

352 Notes and comments In this section we begin to see a striking characteristic of the theory of Riesz spaces: repeated reflections of results in Boolean algebra. Without spelling out a complete list, I mention the distributive laws (313Bc, 352Ea) and the behaviour of order-continuous homomorphisms (313Pa, 313Qa, 352N, 352Oe, 352Ub, 352Xe). Riesz subspaces correspond to subalgebras, solid linear subspaces to ideals and Riesz homomorphisms to Boolean homomorphisms. We even have a correspondence, though a weaker one, between the representation theorems available; every Boolean algebra is isomorphic to a subalgebra of a power of $\mathbb{Z}_2$ (311D-311E), while every Riesz space is isomorphic to a Riesz subspace of a quotient of a power of $\mathbb{R}$ (352L). It would be a closer parallel if every Riesz space were embeddable in some $\mathbb{R}^X$; I must emphasize that the differences are as important as the agreements. Subspaces of $\mathbb{R}^X$ are of great importance, but are by no means adequate for our needs. And of course the details – for instance, the identities in 352D-352F, or 352V – frequently involve new techniques in the case of Riesz spaces. Elsewhere, as in 352G, I find myself arguing rather from the opposite side, when applying results from the theory of general partially ordered linear spaces, which has little to do with Boolean algebra.

In the theory of bands in Riesz spaces – corresponding to order-closed ideals in Boolean algebras – we have a new complication in the form of bands which are not complemented, which does not arise in the Boolean algebra context; but it disappears again when we come to specialize to Archimedean Riesz spaces (353B). (Similarly, order-density and quasi-order-density coincide in both Boolean algebras (313K) and Archimedean Riesz spaces (353A).) Otherwise the algebra of complemented bands in a Riesz space looks very like the algebra of order-closed ideals in a Boolean algebra (314Yh, 352Q). The algebra of projection bands in a Riesz space (352S) would correspond, in a Boolean algebra, to the algebra itself.

I draw your attention to 352H. The result is nearly trivial, but it amounts to saying that the theory of Riesz spaces will be 'algebraic', like the theories of groups or linear spaces, rather than 'analytic', like the theories of partially ordered linear spaces or topological spaces, in which we can have bijective morphisms which are not isomorphisms.

353 Archimedean and Dedekind complete Riesz spaces

I take a few pages over elementary properties of Archimedean and Dedekind (σ)-complete Riesz spaces.

353A Proposition Let U be an Archimedean Riesz space. Then every quasi-order-dense Riesz subspace of U is order-dense.

proof Let $V \subseteq U$ be a quasi-order-dense Riesz subspace, and $u \ge 0$ in U. Set $A = \{v : v \in V,\ v \le u\}$. **?** Suppose, if possible, that u is not the least upper bound of A. Then there is a $u_1 < u$ such that $v \le u_1$ for every $v \in A$. Because $0 \in A$, $u_1 \ge 0$. Because V is quasi-order-dense, there is a $v > 0$ in V such that $v \le u - u_1$. Now $nv \le u_1$ for every $n \in \mathbb{N}$. **P** Induce on n. For $n = 0$ this is trivial. For the inductive step, given $nv \le u_1$, then $(n+1)v \le u_1 + v \le u$, so $(n+1)v \in A$ and $(n+1)v \le u_1$. Thus the induction proceeds. **Q** But this is impossible, because $v > 0$ and U is supposed to be Archimedean. **X**

So $u = \sup A$. As u is arbitrary, V is order-dense.

353B Proposition Let U be an Archimedean Riesz space. Then

(a) for every $A \subseteq U$, the band generated by A is $A^{\perp\perp}$,

(b) every band in U is complemented.

proof (a) Let V be the band generated by A. Then V is surely included in $A^{\perp\perp}$, because this is a band including A (352O). **?** Suppose, if possible, that $V \ne A^{\perp\perp}$. Then there is a $w \in A^{\perp\perp} \setminus V$, so that $|w| \notin V$. Set $B = \{v : v \in V,\ v \le |w|\}$; then B is upwards-directed and non-empty. Because V is order-closed, $|w|$ cannot be the supremum of A, and there is a $u_0 > 0$ such that $|w| - u_0 \ge v$ for every $v \in B$. Now $u_0 \wedge |w| \ne 0$, so $u_0 \notin A^\perp$, and there is a $u_1 \in A$ such that $v = u_0 \wedge |u_1| > 0$. In this case $nv \in B$ for every $n \in \mathbb{N}$. **P** Induce on n. For $n = 0$ this is trivial. For the inductive step, given that $nv \in B$, then $nv \le |w| - u_0$ so $(n+1)v \le nv + u_0 \le |w|$; but also $(n+1)v \le nv + |u_1| \in V$, so $(n+1)v \in B$. **Q** But this means that $nv \le |w|$ for every n, which is impossible, because U is Archimedean. **X**

(b) Now if $V \subseteq U$ is any band, it is surely the band generated by itself, so is equal to $V^{\perp\perp}$, and is complemented (352P).

Remark We may therefore speak of the **band algebra** of an Archimedean Riesz space, rather than the 'complemented band algebra' (352Q).

353C Corollary Let U be an Archimedean Riesz space and $v \in U$. Let V be the band in U generated by v. If $u \in U$, then $u \in V$ iff there is no w such that $0 < w \le |u|$ and $w \wedge |v| = 0$.

proof By 353B, $V = \{v\}^{\perp\perp}$. Now, for $u \in U$,

$$u \notin V \iff \exists\, w \in \{v\}^{\perp},\ |u| \wedge |w| > 0 \iff \exists\, w \in \{v\}^{\perp},\ 0 < w \le |u|.$$

Turning this round, we have the condition announced.

353D Proposition Let U be an Archimedean Riesz space and V an order-dense Riesz subspace of U. Then the map $W \mapsto W \cap V$ is an isomorphism between the band algebras of U and V.

proof If $W \subseteq U$ is a band, then $W \cap V$ is surely a band in V (it is order-closed in V because it is the inverse image of the order-closed set W under the embedding $V \subsetneqq U$, which is order-continuous by 352Nc and 352Nb). If W, W' are distinct bands in U, say $W' \not\subseteq W$, then $W' \not\subseteq W^{\perp\perp}$, by 353B, so $W' \cap W^{\perp} \ne \{0\}$; because V is order-dense, $V \cap W' \cap W^{\perp} \ne \{0\}$, and $V \cap W' \ne V \cap W$. Thus $W \mapsto W \cap V$ is injective.

If $Q \subseteq V$ is a band in V, then its complementary band in V is just $Q^{\perp} \cap V$, where $Q^{\perp}$ is taken in U. So (because V, like U, is Archimedean, by 351Rc) $Q = (Q^{\perp} \cap V)^{\perp} \cap V = W \cap V$, where $W = (Q^{\perp} \cap V)^{\perp}$ is a band in U. Thus the map $W \mapsto W \cap V$ is an order-preserving bijection between the two band algebras. By 312M, it is a Boolean isomorphism, as claimed.

353E Lemma Let U be an Archimedean Riesz space and $V \subseteq U$ a band such that $\sup\{v : v \in V,\ 0 \le v \le u\}$ is defined for every $u \in U^+$. Then V is a projection band.

proof Take any $u \in U^+$ and set $v = \sup\{v' : v' \in V^+,\ v' \le u\}$, $w = u - v$. $v \in V$ because V is a band. Also $w \in V^{\perp}$. **P?** If not, there is some $v_0 \in V^+$ such that $w \wedge v_0 > 0$. Now for any $n \in \mathbb{N}$ we see that

$$nv_0 \le u \Longrightarrow nv_0 \le v \Longrightarrow (n+1)v_0 \le v + w = u,$$

so an induction on n shows that $nv_0 \le u$ for every n; which is impossible, because U is supposed to be Archimedean. **XQ** Accordingly $u = v + w \in V + V^{\perp}$. As u is arbitrary, $U^+ \subseteq V + V^{\perp}$, and V is a projection band (352R).

353F Lemma Let U be an Archimedean Riesz space. If $A \subseteq U$ is non-empty and bounded above and B is the set of its upper bounds, then $\inf(B - A) = 0$.

proof ? If not, let $w > 0$ be a lower bound for $B - A$. If $u \in A$ and $v \in B$, then $v - u \ge w$, that is, $u \le v - w$; as u is arbitrary, $v - w \in B$. Take any $u_0 \in A$, $v_0 \in B$. Inducing on n, we see that $v_0 - nw \in B$ for every $n \in \mathbb{N}$, so that $v_0 - nw \ge u_0$, $nw \le v_0 - u_0$ for every n; but this is impossible, because U is supposed to be Archimedean. **X**

353G Dedekind completeness Recall that a partially ordered set P is Dedekind (σ)-complete if (countable) non-empty sets with upper and lower bounds have suprema and infima in P (314A). For a Riesz space U, U is Dedekind complete iff every non-empty upwards-directed subset of U^+ with an upper bound has a least upper bound, and is Dedekind σ-complete iff every non-decreasing sequence in U^+ with an upper bound has a least upper bound. **P** (Compare 314Bc.) (i) Suppose that any non-empty upwards-directed order-bounded subset of U^+ has an upper bound, and that $A \subseteq U$ is any non-empty set with an upper bound. Take $u_0 \in A$ and set

$$B = \{u_0 \vee u_1 \vee \ldots \vee u_n - u_0 : u_1, \ldots, u_n \in A\}.$$

Then B is an upwards-directed subset of U^+, and if w is an upper bound of A then $w - u_0$ is an upper bound of B. So $\sup B$ is defined in U, and in this case $u_0 + \sup B = \sup A$. As A is arbitrary, U is Dedekind complete. (ii) Suppose that order-bounded non-decreasing sequences in U^+ have suprema, and that $A \subseteq U$ is any countable non-empty set with an upper bound. Let $\langle u_n \rangle_{n\in\mathbb{N}}$ be a sequence running over A, and set $v_n = \sup_{i\le n} u_i - u_0$ for each n. Then $\langle v_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing order-bounded sequence in U^+, and $u_0 + \sup_{n\in\mathbb{N}} v_n = \sup A$. (iii) Finally, still supposing that order-bounded non-decreasing sequences in U^+ have suprema, if $A \subseteq U$ is non-empty, countable and bounded below, $\inf A$ will be defined and equal to $-\sup(-A)$. **Q**

353H Proposition Let U be a Dedekind σ-complete Riesz space.

(a) U is Archimedean.

(b) For any $v \in U$ the band generated by v is a projection band.

(c) If $u, v \in U$, then u is uniquely expressible as $u_1 + u_2$, where u_1 belongs to the band generated by v and $|u_2| \wedge |v| = 0$.

proof (a) Suppose that $u, v \in U$ are such that $nu \le v$ for every $n \in \mathbb{N}$. Then $nu^+ \le v^+$ for every n, and $A = \{nu^+ : n \in \mathbb{N}\}$ is a countable non-empty upwards-directed set with an upper bound; say $w = \sup A$. Since $A + u^+ \subseteq A$, $w + u^+ = \sup(A + u^+) \le w$, and $u \le u^+ \le 0$. As u, v are arbitrary, U is Archimedean.

(b) Let V be the band generated by v. Take any $u \in U^+$ and set $A = \{v' : v' \in V, 0 \le v' \le u\}$. Then $\{u \wedge n|v| : n \in \mathbb{N}\}$ is a countable set with an upper bound, so has a supremum u_1 say in U. Now u_1 is an upper bound for A. **P** If $v' \in A$, then

$$v' = \sup_{n\in\mathbb{N}} v' \wedge n|v| \le u_1$$

by 352Vb. **Q** Since $u \wedge n|v| \in A \subseteq V$ for every n, $u_1 \in V$ and $u_1 = \sup A$.

As u is arbitrary, 353E tells us that V is a projection band.

(c) Again let V be the band generated by v. Then $\{v\}^{\perp\perp}$ is a band containing v, so

$$\{v\} \subseteq V \subseteq \{v\}^{\perp\perp}, \quad \{v\}^{\perp} \supseteq V^{\perp} \supseteq \{v\}^{\perp\perp\perp} = \{v\}^{\perp}$$

(352Od), and $V^{\perp} = \{v\}^{\perp}$.

Now, if $u \in U$, u is uniquely expressible in the form $u_1 + u_2$ where $u_1 \in V$ and $u_2 \in V^{\perp}$, by (b). But

$$u_2 \in V^{\perp} \iff u_2 \in \{v\}^{\perp} \iff |u_2| \wedge |v| = 0.$$

So we have the result.

353I Proposition In a Dedekind complete Riesz space, all bands are projection bands.

proof Use 353E, noting that the sets $\{v : v \in V, 0 \le v \le u\}$ there are always non-empty, upwards-directed and bounded above, so always have suprema.

353J Proposition (a) Let U be a Dedekind σ-complete Riesz space.

(i) If V is a solid linear subspace of U, then V is (in itself) Dedekind σ-complete.

(ii) If V is a sequentially order-closed Riesz subspace of U then V is Dedekind σ-complete.

(iii) If V is a sequentially order-closed solid linear subspace of U, the canonical map from U to V is sequentially order-continuous, and the quotient Riesz space U/V also is Dedekind σ-complete.

(b) Let U be a Dedekind complete Riesz space.

(i) If V is a solid linear subspace of U, then V is (in itself) Dedekind complete.

(ii) If $V \subseteq U$ is an order-closed Riesz subspace then V is Dedekind complete.

proof (a)(i) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in V^+ with an upper bound $v \in V$, then $w = \sup_{n\in\mathbb{N}} u_n$ is defined in U; but as $0 \le w \le v$, $w \in V$ and $w = \sup_{n\in\mathbb{N}} u_n$ in V. Thus V is Dedekind σ-complete.

(ii) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing order-bounded sequence in W, then $u = \sup_{n\in\mathbb{N}} u_n$ is defined in U; but because V is sequentially order-closed, $u \in V$ and $u = \sup_{n\in\mathbb{N}} u_n$ in V.

(iii) Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in U with supremum u. Then of course $u^\bullet$ is an upper bound for $A = \{u_n^\bullet : n \in \mathbb{N}\}$ in U/V. Now let p be any other upper bound for A. Express p as $v^\bullet$. Then for each $n \in \mathbb{N}$ we have $u_n^\bullet \le p$, so that $(u_n - v)^+ \in V$. Because V is sequentially order-closed, $(u - v)^+ = \sup_{n\in\mathbb{N}}(u_n - v)^+ \in V$ and $u^\bullet \le p$. Thus $u^\bullet$ is the least upper bound of A. By 351Gb, $u \mapsto u^\bullet$ is sequentially order-continuous.

Now suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $(U/V)^+$ with an upper bound $p \in (U/V)^+$. Let $u \in U^+$ be such that $u^\bullet = p$, and for each $n \in \mathbb{N}$ let $u_n \in U^+$ be such that $u_n^\bullet = p_n$. Set $v_n = u \wedge \sup_{i\le n} u_i$ for each n; then $v_n^\bullet = p_n$ for each n, and $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing order-bounded sequence in U. Set $v = \sup_{n\in\mathbb{N}} v_n$; by the last paragraph, $v^\bullet = \sup_{n\in\mathbb{N}} p_n$ in U/V. As $\langle p_n\rangle_{n\in\mathbb{N}}$ is arbitrary, U/V is Dedekind σ-complete, as claimed.

(b) The argument is the same as parts (i) and (ii) of the proof of (a).

353K Proposition Let U be a Riesz space and V a quasi-order-dense Riesz subspace of U which is (in itself) Dedekind complete. Then V is a solid linear subspace of U.

proof Suppose that $v \in V$, $u \in U$ and that $|u| \le |v|$. Consider $A = \{w : w \in V, 0 \le w \le u^+\}$. Then A is a non-empty subset of V with an upper bound in V (viz., $|v|$). So A has a supremum v_0 in V. Because the embedding $V \subsetneqq U$ is order-continuous (352Nb), v_0 is the supremum of A in U. But as V is order-dense (353A), $v_0 = u^+$ and $u^+ \in V$. Similarly, $u^- \in V$ and $u \in V$. As u, v are arbitrary, V is solid.

353L Order units Let U be a Riesz space.

(a) An element e of U^+ is an **order unit** in U if U is the solid linear subspace of itself generated by e; that is, if for every $u \in U$ there is an $n \in \mathbb{N}$ such that $|u| \le ne$. (For the solid linear subspace generated by $v \in U^+$ is $\bigcup_{n\in\mathbb{N}}[-nv, nv]$.)

(b) An element e of U^+ is a **weak order unit** in U if U is the principal band generated by e; that is, if $u = \sup_{n\in\mathbb{N}} u \wedge ne$ for every $u \in U^+$ (352Vb).

Of course an order unit is a weak order unit.

(c) If U is Archimedean, then an element e of U^+ is a weak order unit iff $\{e\}^{\perp\perp} = U$ (353B), that is, iff $\{e\}^\perp = \{0\}$ (because

$$\{e\}^\perp = \{0\} \Longrightarrow \{e\}^{\perp\perp} = \{0\}^\perp = U \Longrightarrow \{e\}^\perp = \{e\}^{\perp\perp\perp} = U^\perp = \{0\},)$$

that is, iff $u \wedge e > 0$ whenever $u > 0$.

353M Theorem Let U be an Archimedean Riesz space with order unit e. Then it can be embedded as an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, in such a way that e corresponds to χX; moreover, this embedding is essentially unique.

Remark Here $C(X)$ is the space of all continuous functions from X to $\mathbb{R}$; because X is compact, they are all bounded, so that χX is an order unit in $C(X) = C_b(X)$.

proof (a) Let X be the set of Riesz homomorphisms x from U to $\mathbb{R}$ such that $x(e) = 1$. Define $T : U \to \mathbb{R}^X$ by setting $(Tu)(x) = x(u)$ for $x \in X$, $u \in U$; then it is easy to check that T is a Riesz homomorphism, just because every member of X is a Riesz homomorphism, and of course $Te = \chi X$.

(b) The key to the proof is the fact that X separates the points of U, that is, that T is injective. I choose the following method to show this. Suppose that $w \in U$ and $w > 0$. Because U is Archimedean, there is a $\delta > 0$ such that $(w - \delta e)^+ \ne 0$. Now there is an $x \in X$ such that $x(w) \ge \delta$. **P** (i) By 351O, there is a solid linear subspace V of U such that $(w - \delta e)^+ \notin V$ and whenever $u \wedge v = 0$ in U then one of u, v belongs to V. (ii) Because $V \ne U$, $e \notin V$, so no non-zero multiple of e can belong to V. Also observe that if u, $v \in U \setminus V$, then one of $(u - v)^+$, $(v - u)^+$ must belong to V, while neither $u = u \wedge v + (u - v)^+$ nor $v = u \wedge v + (v - u)^+$ does; so $u \wedge v \notin V$. (iii) For each $u \in U$ set $A_u = \{\alpha : \alpha \in \mathbb{R}, (u - \alpha e)^+ \in V\}$. Then

$$\alpha \ge \beta \in A_u \Longrightarrow 0 \le (u - \alpha e)^+ \le (u - \beta e)^+ \in V \Longrightarrow \alpha \in A_u.$$

Also A_u is non-empty and bounded below, because if $\alpha \ge 0$ is such that $-\alpha e \le u \le \alpha e$ then $\alpha \in A_u$ and $-\alpha - 1 \notin A_u$ (since $(u - (-\alpha - 1)e)^+ \ge e \notin V$). (iv) Set $x(u) = \inf A_u$ for every $u \in U$; then $\alpha \in A_u$ for every $\alpha > x(u)$, $\alpha \notin A_u$ for every $\alpha < x(u)$. (v) If u, $v \in U$, $\alpha > x(u)$ and $\beta > x(v)$ then

$$((u + v) - (\alpha + \beta)e)^+ \le (u - \alpha e)^+ + (v - \beta e)^+ \in V$$

(352Fc), so $\alpha + \beta \in A_{u+v}$; as α and β are arbitrary, $x(u + v) \le x(u) + x(v)$. (vi) If u, $v \in U$ and $\alpha < x(u)$, $\beta < x(v)$ then

$$((u + v) - (\alpha + \beta)e)^+ \ge (u - \alpha e)^+ \wedge (v - \beta e)^+ \notin V,$$

using (ii) of this argument and 352Fc, so $\alpha + \beta \notin A_{u+v}$. As α and β are arbitrary, $x(u + v) \ge x(u) + x(v)$. (vii) Thus $x : U \to \mathbb{R}$ is additive. (viii) If $u \in U$, $\gamma > 0$ then

$$\alpha \in A_u \Longrightarrow (\gamma u - \alpha\gamma e)^+ = \gamma(u - \alpha e)^+ \in V \Longrightarrow \gamma\alpha \in A_{\gamma u};$$

thus $A_{\gamma u} \supseteq \gamma A_u$; similarly, $A_u \supseteq \gamma^{-1} A_{\gamma u}$ so $A_{\gamma u} = \gamma A_u$ and $x(\gamma u) = \gamma x(u)$. (ix) Consequently x is linear, since we know already from (vii) that $x(0u) = 0.x(u)$, $x(-u) = -x(u)$. (x) If $u \ge 0$ then $u + \alpha e \ge \alpha e \notin V$ for every $\alpha > 0$, that is, $-\alpha \notin A_u$ for every $\alpha > 0$, and $x(u) \ge 0$; thus x is a positive linear functional. (xi) If $u \wedge v = 0$, then one of u, v belongs to V, so $\min(x(u), x(v)) \le 0$ and (using (x)) $\min(x(u), x(v)) = 0$; thus x is a Riesz homomorphism (352G(iv)). (xii) $A_e = [1, \infty[$ so $x(e) = 1$. Thus $x \in X$. (xiii) $\delta \notin A_w$ so $x(w) \ge \delta$. **Q**

(c) Thus $Tw \neq 0$ whenever $w > 0$; consequently $|Tw| = T|w| \neq 0$ whenever $w \neq 0$, and T is injective. I now have to define the topology of X. This is just the subspace topology on X if we regard X as a subset of $\mathbb{R}^U$ with its product topology. To see that X is compact, observe that if for each $u \in U$ we choose an α_u such that $|u| \leq \alpha_u e$, then X is a subspace of $Q = \prod_{u \in U}[-\alpha_u, \alpha_u]$. Because Q is a product of compact spaces, it is compact, by Tychonoff's theorem (3A3J). Now X is a closed subset of Q. **P** X is just the intersection of the sets

$$\{x : x(u+v) = x(u) + x(v)\}, \quad \{x : x(\alpha u) = \alpha x(u)\},$$

$$\{x : x(u^+) = \max(x(u), 0)\}, \quad \{x : x(e) = 1\}$$

as u, v run over U and α over $\mathbb{R}$; and each of these is closed, so X is an intersection of closed sets and therefore itself closed. **Q** Consequently X also is compact. Moreover, the coordinate functionals $x \mapsto x(u)$ are continuous on Q, therefore on X also, that is, $Tu : X \to \mathbb{R}$ is a continuous function for every $u \in U$.

Note also that because Q is a product of Hausdorff spaces, Q and X are Hausdorff (3A3Id).

(d) So T is a Riesz homomorphism from U to $C(X)$. Now $T[U]$ is a Riesz subspace of $C(X)$, containing χX, and such that if x, $y \in X$ are distinct there is an $f \in T[U]$ such that $f(x) \neq f(y)$ (because there is surely a $u \in U$ such that $x(u) \neq y(u)$). By the Stone-Weierstrass theorem (281A), $T[U]$ is $\| \|_\infty$-dense in $C(X)$.

Consequently it is also order-dense. **P** If $f > 0$ in $C(X)$, set $\epsilon = \frac{1}{3}\|f\|_\infty$, and let $u \in U$ be such that $\|f - Tu\|_\infty \leq \epsilon$; set $v = (u - \epsilon e)^+$. Since

$$0 < (f - 2\epsilon\chi X)^+ \leq (Tu - \epsilon\chi X)^+ \leq f^+ = f,$$

$0 < Tv \leq f$. As f is arbitrary, $T[U]$ is quasi-order-dense, therefore order-dense (353A). **Q**

(e) I have still to show that the representation is (essentially) unique. Suppose, then, that we have another representation of U as a norm-dense Riesz subspace of $C(Z)$, with e this time corresponding to χZ; to simplify the notation, let us suppose that U is actually a subspace of $C(Z)$. Then for each $z \in Z$, we have a functional $\hat{z} : U \to \mathbb{R}$ defined by setting $\hat{z}(u) = u(z)$ for every $u \in U$; of course $\hat{z}$ is a Riesz homomorphism such that $\hat{z}(e) = 1$, that is, $\hat{z} \in X$. Thus we have a function $z \mapsto \hat{z} : Z \to X$. For any $u \in U$, the function $z \mapsto \hat{z}(u) = u(z)$ is continuous, so the function $z \mapsto \hat{z}$ is continuous (3A3Ib). If z_1, z_2 are distinct members of Z, there is an $f \in C(Z)$ such that $f(z_1) \neq f(z_2)$ (3A3Bf); now there is a $u \in U$ such that $\|f - u\|_\infty \leq \frac{1}{3}|f(z_1) - f(z_2)|$, so that $u(z_1) \neq u(z_2)$ and $\hat{z}_1 \neq \hat{z}_2$. Thus $z \mapsto \hat{z}$ is injective. Finally, it is also surjective. **P** Suppose that $x \in X$. Set $V = \{u : u \in U, x(u) = 0\}$; then V is a solid linear subspace of U (352Jb), not containing e. For $z \in V^+$ set $G_v = \{z : v(z) > 1\}$. Because $e \notin V$, $G_v \neq Z$. $\mathcal{G} = \{G_v : v \in V^+\}$ is an upwards-directed family of open sets in Z, not containing Z; consequently, because Z is compact, $\mathcal{G}$ cannot be an open cover of Z. Take $z \in Z \setminus \bigcup \mathcal{G}$. Then $v(z) \leq 1$ for every $v \in V^+$; because $\alpha|v| \in V^+$ whenever $v \in V$, $\alpha \geq 0$, we must have $v(z) = 0$ for every $v \in V$. Now, given any $u \in U$, consider $v = u - x(u)e$. Then $x(v) = 0$ so $v \in V$ and $v(z) = 0$, that is,

$$u(z) = (v + x(u)e)(z) = v(z) + x(u)e(z) = x(u).$$

As u is arbitrary, $\hat{z} = x$; as x is arbitrary, we have the result. **Q**

Thus $z \mapsto \hat{z}$ is a continuous bijection from the compact Hausdorff space Z to the compact Hausdorff space X; it must therefore be a homeomorphism (3A3Dd).

This argument shows that if U is embedded as a norm-dense Riesz subspace of $C(Z)$, where Z is compact and Hausdorff, then Z must be homeomorphic to X. But it shows also that a homeomorphism is canonically defined by the embedding; $z \in Z$ corresponds to the Riesz homomorphism $u \mapsto u(z)$ in X.

353N Lemma Let U be a Riesz space, V an Archimedean Riesz space and $S, T : U \to V$ Riesz homomorphisms such that $Su \wedge Tu' = 0$ in V whenever $u \wedge u' = 0$ in U. Set $W = \{u : Su = Tu\}$. Then W is a solid linear subspace of U; if S and T are order-continuous, W is a band.

proof (a) It is easy to check that, because S and T are Riesz homomorphisms, W is a Riesz subspace of U.

(b) If $w \in W$ and $0 \leq u \leq w$ in U, then $Su \leq Tu$. **P?** Otherwise, set $e = Sw = Tw$, and let V_e be the solid linear subspace of V generated by e, so that V_e is an Archimedean Riesz space with order unit, containing both Su and Tu. By 353M (or its proof), there is a Riesz homomorphism $x : V_e \to \mathbb{R}$ such that $x(e) = 1$ and $x(Su) > x(Tu)$. Take α such that $x(Su) > \alpha > x(Tu)$, and consider $u' = (u - \alpha w)^+$, $u'' = (\alpha w - u)^+$. Then

$$x(Su') = \max(0, x(Su) - \alpha x(Sw)) = \max(0, x(Su) - \alpha) > 0,$$

$$x(Tu'') = \max(0, \alpha x(Tw) - x(Tu)) = \max(0, \alpha - x(Tu)) > 0,$$

so

$$x(Su' \wedge Tu'') = \min(x(Su'), x(Tu'')) > 0$$

and $Su' \wedge Tu'' > 0$, while $u' \wedge u'' = 0$. **XQ**

Similarly, $Tu \leq Su$ and $u \in W$. As u and w are arbitrary, W is a solid linear subspace.

(c) Finally, suppose that S and T are order-continuous, and that $A \subseteq W$ is a non-empty upwards-directed set with supremum u in U. Then

$$Su = \sup S[A] = \sup T[A] = Tu$$

and $u \in W$. As u and A are arbitrary, W is a band (352Ob).

353O *f*-algebras I give two results on f-algebras, intended to clarify the connexions between the multiplicative and lattice structures of the Riesz spaces in Chapter 36.

Proposition Let U be an Archimedean f-algebra (352W). Then

(a) the multiplication is separately order-continuous in the sense that the maps $u \mapsto u \times w$, $u \mapsto w \times u$ are order-continuous for every $w \in U^+$;

(b) the multiplication is commutative.

proof (a) Let $A \subseteq U$ be a non-empty set with infimum 0, and $v_0 \in U^+$ a lower bound for $\{u \times w : u \in A\}$. Fix $u_0 \in A$. If $u \in A$ and $\delta > 0$, then $v_0 \wedge (u_0 - \frac{1}{\delta}u)^+ \leq \delta u_0 \times w$. **P** Set $v = v_0 \wedge (u_0 - \frac{1}{\delta}u)^+$. Then

$$\delta v \wedge (u - \delta u_0)^+ \leq (\delta u_0 - u)^+ \wedge (u - \delta u_0)^+ = 0,$$

so $v \wedge (u - \delta u_0)^+ = 0$ and $v \wedge ((u - \delta u_0)^+ \times w) = 0$. But

$$v \leq v_0 \leq u \times w \leq (u - \delta u_0)^+ \times w + \delta u_0 \times w,$$

so

$$v \leq ((u - \delta u_0)^+ \times w) \wedge v + (\delta u_0 \times w) \wedge v \leq \delta u_0 \times w,$$

by 352Fa. **Q**

Taking the infimum over u, and using the distributive laws (352E), we get

$$v_0 \wedge u_0 \leq \delta u_0 \times w.$$

Taking the infimum over δ, and using the hypothesis that U is Archimedean,

$$v_0 \wedge u_0 = 0.$$

But this means that $v_0 \wedge (u_0 \times w) = 0$, while $v_0 \leq u_0 \times w$, so $v_0 = 0$. As v_0 is arbitrary, $\inf_{u \in A} u \times w = 0$; as A is arbitrary, $u \mapsto u \times w$ is order-continuous. Similarly, $u \mapsto w \times u$ is order-continuous.

(b)(i) Fix $v \in U^+$, and for $u \in U$ set

$$Su = u \times v, \quad Tu = v \times u.$$

Then S and T are both order-continuous Riesz homomorphisms from U to itself (352W(b-iv) and (a) above). Also, $Su \wedge Tu' = 0$ whenever $u \wedge u' = 0$. **P**

$$0 = (u \times v) \wedge u' = (u \times v) \wedge (v \times u'). \text{ } \mathbf{Q}$$

So $W = \{u : u \times v = v \times u\}$ is a band in U (353N). Of course $v \in W$ (because $Sv = Tv = v^2$). If $u \in W^\perp$, then $v \wedge |u| = 0$ so $Su = Tu = 0$ (352W(b-i)), and $u \in W$; but this means that $W^\perp = \{0\}$ and $W = W^{\perp\perp} = U$ (353Bb). Thus $v \times u = u \times v$ for every $u \in U$.

(ii) This is true for every $v \in U^+$. Of course it follows that $v \times u = u \times v$ for every u, $v \in U$, so that multiplication is commutative.

353P Proposition Let U be an Archimedean f-algebra with multiplicative identity e.

(a) e is a weak order unit in U.

(b) If u, $v \in U$ then $u \times v = 0$ iff $|u| \wedge |v| = 0$.

(c) If $u \in U$ has a multiplicative inverse u^{-1} then $|u|$ also has a multiplicative inverse; if $u \geq 0$ then $u^{-1} \geq 0$ and u is a weak order unit.

(d) If V is another Archimedean f-algebra with multiplicative identity e', and $T : U \to V$ is a positive linear operator such that $Te = e'$, then T is a Riesz homomorphism iff $T(u \times v) = Tu \times Tv$ for all u, $v \in U$.

proof (a) $e = e^2 \geq 0$ by 352W(b-ii). If $u \in U$ and $e \wedge |u| = 0$ then $|u| = (e \times |u|) \wedge |u| = 0$; by 353Lc, e is a weak order unit.

(b) If $|u| \wedge |v| = 0$ then $u \times v = 0$, by 352W(b-i). If $w = |u| \wedge |v| > 0$, then $w^2 \leq |u| \times |v|$. Let $n \in \mathbb{N}$ be such that $nw \not\leq e$, and set $w_1 = (nw - e)^+$, $w_2 = (e - nw)^+$. Then

$$\begin{aligned} 0 \neq w_1 = w_1 \times e &= w_1 \times w_2 + w_1 \times (e \wedge nw) \\ &= w_1 \times (e \wedge nw) \leq (nw)^2 \leq n^2 |u| \times |v| = n^2 |u \times v|, \end{aligned}$$

so $u \times v \neq 0$.

(c) $u \times u^{-1} = e$ so $|u| \times |u^{-1}| = |e| = e$ (352W(b-iii)), and $|u^{-1}| = |u|^{-1}$. (Recall that inverses in any semigroup with identity are unique, so that we need have no inhibitions in using the formulae u^{-1}, $|u|^{-1}$.)

Now suppose that $u \geq 0$. Then $u^{-1} = |u^{-1}| \geq 0$. If $u \wedge |v| = 0$ then

$$e \wedge |v| = (u \times u^{-1}) \wedge |v| = 0,$$

so $v = 0$; accordingly u is a weak order unit.

(d)(i) If T is multiplicative, and $u \wedge v = 0$ in U, then $Tu \times Tv = T(u \times v) = 0$ and $Tu \wedge Tv = 0$, by (b). So T is a Riesz homomorphism, by 352G.

(ii) Accordingly I shall henceforth assume that T is a Riesz homomorphism and seek to show that it is multiplicative.

If u, $v \in U^+$, then $T(u \times v)$ and $Tu \times Tv$ both belong to the band generated by Tu. **P** Write W for this band. (α) For any $n \geq 1$ we have $(v - ne)^2 \geq 0$, that is, $2nv \leq v^2 + n^2 e$, so

$$n(v - ne) \leq 2nv - n^2 e \leq v^2.$$

Consequently

$$T(u \times v) - nTu = T(u \times v) - nT(u \times e) = T(u \times (v - ne)) \leq \frac{1}{n} T(u \times v^2)$$

because $v' \mapsto T(u \times v')$ is a positive linear operator; as V is Archimedean, $\inf_{n\in\mathbb{N}}(T(u \times v) - nTu)^+ = 0$ and $T(u \times v) = \sup_{n\in\mathbb{N}} T(u \times v) \wedge nTu$ belongs to W. (β) If $w \wedge |Tu| = 0$ then

$$w \wedge |Tu \times Tv| = w \wedge (|Tu| \times |Tv|) = 0;$$

so $Tu \times Tv \in W^{\perp\perp} = W$. **Q**

(iii) Fix $v \in U^+$. For $u \in U$, set $S_1 u = Tu \times Tv$ and $S_2 u = T(u \times v)$. Then S_1 and S_2 are both Riesz homomorphisms from U to V. If $u \wedge u' = 0$ in U, then $S_1 u \wedge S_2 u' = 0$ in V, because (by (ii) just above) $S_1 u$ belongs to the band generated by Tu, while $S_2 u'$ belongs to the band generated by Tu', and $Tu \wedge Tu' = T(u \wedge u') = 0$. By 353N, $W = \{u : S_1 u = S_2 u\}$ is a solid linear subspace of U. Of course it contains e, since

$$S_1 e = Te \times Tv = e' \times Tv = Tv = T(e \times v) = S_2 e.$$

In fact $u \in W$ for every $u \in U^+$. **P** As noted in (ii) just above, $u - ne \leq \frac{1}{n} u^2$ for every $n \geq 1$. So

$$\begin{aligned} |S_1 u - S_2 u| &= |S_1(u - ne)^+ + S_1(u \wedge ne) - S_2(u - ne)^+ - S_2(u \wedge ne)| \\ &\leq S_1(u - ne)^+ + S_2(u - ne)^+ \leq \frac{1}{n}(S_1 u^2 + S_2 u^2) \end{aligned}$$

for every $n \geq 1$, and $|S_1 u - S_2 u| = 0$, that is, $S_1 u = S_2 u$. **Q**

So $W = U$, that is, $Tu \times Tv = T(u \times v)$ for every $u \in U$. And this is true for every $v \in U^+$. It follows at once that it is true for every $v \in U$, so that T is multiplicative, as claimed.

353Q Proposition Let U be a Riesz space and V an order-dense Riesz subspace of U. If V is Archimedean, so is U.

proof ? Otherwise, let u', $u \in U$ be such that $u' > 0$ and $nu' \leq u$ for every $n \in \mathbb{N}$. Let $v' \in V$ be such that $0 < v' \leq u'$; set $\tilde{u} = u - v'$; let $v \in V$ be such that $v \leq u$ but $v \not\leq \tilde{u}$. (This is where we need V to be order-dense rather than just quasi-order-dense.) Let $w \in V$ be such that $w > 0$ and $w \leq (v - \tilde{u})^+$; note that $w \leq u - \tilde{u} = v'$.

Because V is Archimedean, there is an $n \ge 1$ such that $nw \not\le v$. In this case,

$$0 < (nw - v)^+ \le ((n+1)v' - (v + v'))^+ \le (u - (v + v'))^+ = (\tilde{u} - v)^+$$

but

$$(nw - v)^+ \wedge (\tilde{u} - v)^+ \le nw \wedge n(\tilde{u} - v)^+ = n((v - \tilde{u})^+ \wedge (\tilde{u} - v)^+) = 0,$$

which is impossible. **X**

353X Basic exercises >(a) Let U be a Riesz space in which every band is complemented. Show that U is Archimedean.

(b) A Riesz space U has the **principal projection property** iff the band generated by any single member of U is a projection band. Show that any Dedekind σ-complete Riesz space has the principal projection property, and that any Riesz space with the principal projection property is Archimedean.

>(c) Fill in the missing part (b-iii) of 353J.

(d) Let U be an Archimedean f-algebra with an order-unit which is a multiplicative identity. Show that U can be identified, as f-algebra, with a subspace of $C(X)$ for some compact Hausdorff space X.

353Y Further exercises (a) Let U be a Riesz space in which every quasi-order-dense solid linear subspace is order-dense. Show that U is Archimedean.

(b) Let X be a completely regular Hausdorff space. Show that $C(X)$ is Dedekind complete iff $C_b(X)$ is Dedekind complete iff X is extremally disconnected.

(c) Let X be a compact Hausdorff space. Show that $C(X)$ is Dedekind σ-complete iff $\overline{G}$ is open for every cozero set $G \subseteq X$. (Cf. 314Yf.) Show that in this case X is zero-dimensional.

(d) Let U be an Archimedean Riesz space such that $\{u_n : n \in \mathbb{N}\}$ has a supremum in U whenever $\langle u_n \rangle_{n\in\mathbb{N}}$ is a sequence in U such that $u_m \wedge u_n = 0$ whenever $m \ne n$. Show that U has the principal projection property, but need not be Dedekind σ-complete.

(e) Let U be an Archimedean Riesz space. Show that the following are equiveridical: (i) U has the countable sup property (241Ye) (ii) for every $A \subseteq U$ there is a countable $B \subseteq A$ such that A and B have the same upper bounds; (iii) every order-bounded disjoint subset of U^+ is countable.

(f) Let U be an Archimedean f-algebra. Show that an element e of U is a multiplicative identity iff $e^2 = e$ and e is a weak order unit. (*Hint*: start by showing that under these conditions, $e \times u = 0 \Rightarrow u = 0$.)

(g) Let U be an Archimedean f-algebra with a multiplicative identity. Show that if $u \in U$ then u is invertible iff $|u|$ is invertible.

353 Notes and comments As in the last section, many of the results above have parallels in the theory of Boolean algebras; thus 353A corresponds to 313K, 353G corresponds in part to remarks in 314Bc and 314Xa, and 353J corresponds to 314C-314E. Riesz spaces are more complicated; for instance, principal ideals in Boolean algebras are straightforward, while in Riesz spaces we have to distinguish between the solid linear subspace generated by an element and the band generated by the same element. Thus an 'order unit' in a Boolean ring would just be an identity, while in a Riesz space we must distinguish between 'order unit' and 'weak order unit'. As this remark may suggest to you, (Archimedean) Riesz spaces are actually closer in spirit to arbitrary Boolean rings than to the Boolean algebras we have been concentrating on so far; to the point that in §361 below I will return briefly to general Boolean rings.

Note that the standard definition of 'order-dense' in Boolean algebras, as given in 313J, corresponds to the definition of 'quasi-order-dense' in Riesz spaces (352Na); the point here being that Boolean algebras behave like Archimedean Riesz spaces, in which there is no need to make a distinction.

I give the representation theorem 353M more for completeness than because we need it in any formal sense. In 351Q and 352L I have given representation theorems for general partially ordered linear spaces, and general Riesz spaces, as quotients of spaces of functions; in 368F below I give a theorem for Archimedean Riesz spaces corresponding rather more closely to the expressions of the L^p spaces as quotients of spaces of measurable functions. In 353M, by contrast, we have a theorem expressing Archimedean Riesz spaces with order units as true spaces of functions, rather than as spaces of equivalence classes of functions. All these theorems are important in forming an appropriate mental picture of ordered linear spaces, as in 352M.

I give a bare-handed proof of 353M, using only the Riesz space structure of $C(X)$; if you know a little about extreme points of dual unit balls you can approach from that direction instead, using 354Yj. The point is that (as part (d) of the proof of 353M makes clear) the space X can be regarded as a subset of the normed space dual U^* of U with its weak* topology. In this treatise generally, and in the present chapter in particular, I allow myself to be slightly prejudiced against normed-space methods; you can find them in any book on functional analysis, and I prefer here to develop techniques like those in part (b) of the proof of 353M, which will be a useful preparation for such theorems as 368E.

There is a very close analogy between 353M and the Stone representation of Boolean algebras (311E, 311I-311K). Just as the proof of 311E looked at the set of ring homomorphisms from $\mathfrak{A}$ to the elementary Boolean algebra $\mathbb{Z}_2$, so the proof of 353M looks at Riesz homomorphisms from U to the elementary M-space $\mathbb{R}$. Later on, the most important M-spaces, from the point of view of this treatise, will be the L^∞ spaces of §363, explicitly defined in terms of Stone representations (363A).

Of the two parts of 353O, it is (a) which is most important for the purposes of this book. The f-algebras we shall encounter in Chapter 36 can be seen to be commutative for different, and more elementary, reasons. The (separate) order-continuity of multiplication, however, is not always immediately obvious. Similarly, the uniferent Riesz homomorphisms we shall encounter can generally be seen to be multiplicative without relying on the arguments of 353Pd.

354 Banach lattices

The next step is a brief discussion of norms on Riesz spaces. I start with the essential definitions (354A, 354D) with the principal properties of general Riesz norms (354B-354C) and order-continuous norms (354E). I then describe two of the most important classes of Banach lattice: M-spaces (354F-354L) and L-spaces (354M-354R), with their elementary properties. For M-spaces I give the basic representation theorem (354K-354L), and for L-spaces I give a note on uniform integrability (354P-354R).

354A Definitions (a) If U is a Riesz space, a **Riesz norm** or **lattice norm** on U is a norm $\| \|$ such that $\|u\| \le \|v\|$ whenever $|u| \le |v|$; that is, a norm such that $\||u|\| = \|u\|$ for every u and $\|u\| \le \|v\|$ whenever $0 \le u \le v$.

(b) A **Banach lattice** is a Riesz space with a Riesz norm under which it is complete.

Remark We have already seen many examples of Banach lattices; I list some in 354Xa below.

354B Lemma Let U be a Riesz space with a Riesz norm $\| \|$.

(a) U is Archimedean.

(b) The maps $u \mapsto |u|$ and $u \mapsto u^+$ are uniformly continuous.

(c) For any $u \in U$, the sets $\{v : v \le u\}$ and $\{v : v \ge u\}$ are closed; in particular, the positive cone of U is closed.

(d) Any band in U is closed.

(e) If V is a norm-dense Riesz subspace of U, then $V^+ = \{v : v \in V, v \ge 0\}$ is norm-dense in the positive cone U^+ of U.

proof (a) If u, $v \in U$ are such that $nu \le v$ for every $n \in \mathbb{N}$, then $nu^+ \le v^+$ so $n\|u^+\| \le \|v^+\|$ for every n, and $\|u^+\| = 0$, that is, $u^+ = 0$ and $u \le 0$. As u, v are arbitrary, U is Archimedean.

(b) For any u, $v \in U$, $||u| - |v|| \le |u - v|$ (352D), so $\||u| - |v|\| \le \|u - v\|$; thus $u \mapsto |u|$ is uniformly continuous. Consequently $u \mapsto \frac{1}{2}(u + |u|) = u^+$ is uniformly continuous.

(c) Now $\{v : v \le u\} = \{v : (v - u)^+ = 0\}$ is closed because the function $v \mapsto (v - u)^+$ is continuous and $\{0\}$ is closed. Similarly $\{v : v \ge u\} = \{v : (u - v)^+ = 0\}$ is closed.

(d) If $V \subseteq U$ is a band, then $V = V^{\perp\perp}$ (353Bb), that is, $V = \{v : |v| \wedge |w| = 0 \text{ for every } w \in V^\perp\}$. Because the function $v \mapsto |v| \wedge |w| = \frac{1}{2}(|v| + |w| - ||v| - |w||)$ is continuous, all the sets $\{v : |v| \wedge |w| = 0\}$ are closed, and so is their intersection V.

(e) Observe that $V^+ = \{v^+ : v \in V\}$ and $U^+ = \{u^+ : u \in U\}$; recall that $u \mapsto u^+$ is continuous, and apply 3A3Eb.

354C Lemma If U is a Banach lattice and $\langle u_n \rangle_{n\in\mathbb{N}}$ is a sequence in U such that $\sum_{n=0}^\infty \|u_n\| < \infty$, then $\sup_{n\in\mathbb{N}} u_n$ is defined in U, with $\|\sup_{n\in\mathbb{N}} u_n\| \le \sum_{n=0}^\infty \|u_n\|$.

proof Set $v_n = \sup_{i\le n} u_i$ for each n. Then

$$0 \le v_{n+1} - v_n \le (u_{n+1} - u_n)^+ \le |u_{n+1} - u_n|$$

for each $n \in \mathbb{N}$, so

$$\textstyle\sum_{n=0}^\infty \|v_{n+1} - v_n\| \le \sum_{n=0}^\infty \|u_{n+1} - u_n\| \le \sum_{n=0}^\infty \|u_{n+1}\| + \|u_n\|$$

is finite, and $\langle v_n \rangle_{n\in\mathbb{N}}$ is Cauchy. Let u be its limit; because $\langle v_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, and the sets $\{v : v \ge v_n\}$ are all closed, $u \ge v_n$ for each $n \in \mathbb{N}$. On the other hand, if $v \ge v_n$ for every n, then

$$(u - v)^+ = \lim_{n\to\infty} (v_n - v)^+ = 0,$$

and $u \le v$. So

$$u = \sup_{n\in\mathbb{N}} v_n = \sup_{n\in\mathbb{N}} u_n$$

is the required supremum.

To estimate its norm, observe that $|v_n| \le \sum_{i=0}^n |u_i|$ for each n (induce on n, using the last item in 352D for the inductive step), so that

$$\|u\| = \lim_{n\to\infty} \|v_n\| \le \textstyle\sum_{i=0}^\infty \||u_i|\| = \sum_{i=0}^\infty \|u_i\|.$$

354D I come now to the basic properties according to which we classify Riesz norms.

Definitions (a) A **Fatou norm** on a Riesz space U is a Riesz norm on U such that whenever $A \subseteq U^+$ is non-empty and upwards-directed and has a least upper bound in U, then $\|\sup A\| = \sup_{u\in A} \|u\|$. (Observe that, once we know that $\|\,\|$ is a Riesz norm, we can be sure that $\|u\| \le \|\sup A\|$ for every $u \in A$, so that all we shall need to check is that $\|\sup A\| \le \sup_{u\in A} \|u\|$.)

(b) A Riesz norm on a Riesz space U has the **Levi property** if every upwards-directed norm-bounded set is bounded above.

(c) A Riesz norm on a Riesz space U is **order-continuous** if $\inf_{u\in A} \|u\| = 0$ whenever $A \subseteq U$ is a non-empty downwards-directed set with infimum 0.

354E Proposition Let U be a Riesz space with an order-continuous Riesz norm $\|\,\|$.

(a) If $A \subseteq U$ is non-empty and upwards-directed and has a supremum, then $\sup A \in \overline{A}$.

(b) $\|\,\|$ is Fatou.

(c) If $A \subseteq U$ is non-empty and upwards-directed and bounded above, then for every $\epsilon > 0$ there is a $u \in A$ such that $\|(v - u)^+\| \le \epsilon$ for every $v \in A$; that is, the filter $\mathcal{F}(A\uparrow)$ on U generated by $\{\{v : v \in A,\, u \le v\} : u \in A\}$ is a Cauchy filter.

(d) Any non-decreasing order-bounded sequence in U is Cauchy.

(e) If U is a Banach lattice it is Dedekind complete.

(f) Every order-dense Riesz subspace of U is norm-dense.

(g) Every norm-closed solid linear subspace of U is a band.

proof (a) Suppose that $A \subseteq U$ is non-empty and upwards-directed and has a least upper bound u_0. Then $B = \{u_0 - u : u \in A\}$ is downwards-directed and has infimum 0. So $\inf_{u\in A} \|u_0 - u\| = 0$, and $u_0 \in \overline{A}$.

(b) If, in (a), $A \subseteq U^+$, then we must have

$$\|u_0\| \le \inf_{u\in A} \|u\| + \|u - u_0\| \le \sup_{u\in A} \|u\|.$$

As A is arbitary, $\|\,\|$ is a Fatou norm.

(c) Let B be the set of upper bounds for A. Then B is downwards-directed; because A is upwards-directed, $B - A = \{v - u : v \in B,\, u \in A\}$ is downwards-directed. By 353F, $\inf(B - A) = 0$. So there are $w \in B$, $u \in A$ such that $\|w - u\| \le \epsilon$. Now if $v \in A$,

$$(v-u)^+ = (v \vee u) - u \le w - u,$$

so $\|(v-u)^+\| \le \epsilon$.

In terms of the filter $\mathcal{F}(A\uparrow)$, this tells us that if v_0, v_1 belong to $F_u = \{v : v \in A,\, v \ge u\}$ then $|v_0 - v_1| \le w - u$ so $\|v_0 - v_1\| \le \epsilon$ and the diameter of F_u is at most ϵ. As ϵ is arbitrary, $\mathcal{F}(A\uparrow)$ is a Cauchy filter.

(d) If $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing order-bounded sequence, and $\epsilon > 0$, then, applying (c) to $\{u_n : n \in \mathbb{N}\}$, we find that there is an $m \in \mathbb{N}$ such that $\|u_m - u_n\| \le \epsilon$ whenever $m \ge n$.

(e) Now suppose that U is a Banach lattice. Let $A \subseteq U$ be any non-empty set with an upper bound. Set $B = \{u_0 \vee \ldots \vee u_n : u_0, \ldots , u_n \in A\}$, so that B is upwards-directed and has the same upper bounds as A. Let $\mathcal{F}(B\uparrow)$ be the filter on U generated by $\{B \cap [v, \infty[\,: v \in B\}$. By (c), this is a Cauchy filter with a limit u^* say. For every $u \in A$, $[u, \infty[$ is a closed set belonging to $\mathcal{F}(B\uparrow)$, so contains u^*; thus u^* is an upper bound for A. If w is any upper bound for A, $]-\infty, w]$ is a closed set belonging to $\mathcal{F}(B\uparrow)$, so contains u^*; thus $u^* = \sup A$ and A has a supremum.

(f) If V is an order-dense Riesz subspace of U and $u \in U^+$, set $A = \{v : v \in V,\, v \le u\}$. Then A is upwards-directed and has supremum u, so $u \in \overline{A} \subseteq \overline{V}$, by (a). Thus $U^+ \subseteq \overline{V}$; it follows at once that $U = U^+ - U^+ \subseteq \overline{V}$.

(g) If V is a norm-closed solid linear subspace of U, and $A \subseteq V^+$ is a non-empty, upwards-directed subset of V with a supremum in U, then $\sup A \in \overline{A} \subseteq V$, by (a); by 352Ob, V is a band.

354F Lemma If U is an Archimedean Riesz space with an order unit e (definition: 353L), there is a Riesz norm $\|\,\|_e$ defined on U by the formula

$$\|u\|_e = \min\{\alpha : \alpha \ge 0,\, |u| \le \alpha e\}$$

for every $u \in U$.

proof This is a routine verification. Because e is an order-unit, $\{\alpha : \alpha \ge 0,\, |u| \le \alpha e\}$ is always non-empty, so always has an infimum α_0 say; now $|u| - \alpha_0 e \le \delta e$ for every $\delta > 0$, so (because U is Archimedean) $|u| - \alpha_0 e \le 0$ and $|u| \le \alpha_0 e$, so that the minimum is attained. In particular, $\|u\|_e = 0$ iff $u = 0$. The subadditivity and homogeneity of $\|\,\|_e$ are immediate from the facts that $|u + v| \le |u| + |v|$, $|\alpha u| = |\alpha||u|$.

354G Definitions (a) If U is an Archimedean Riesz space and e an order unit in U, the norm $\|\,\|_e$ as defined in 354F is the **order-unit norm** on U associated with e.

(b) An M**-space** is a Banach lattice in which the norm is an order-unit norm.

(c) If U is an M-space, its **standard order unit** is the order unit e such that $\|\,\|_e$ is the norm of U. (To see that e is uniquely defined, observe that it is $\sup\{u : u \in U,\, \|u\| \le 1\}$.)

354H Examples (a) For any set X, $\ell^\infty(X)$ is an M-space with standard order unit χX. (As remarked in 243Xl, the completeness of $\ell^\infty(X)$ can be regarded as the special case of 243E in which X is given counting measure.)

(b) For any topological space X, the space $C_b(X)$ of bounded continuous real-valued functions on X is an M-space with standard order unit χX. (It is a Riesz subspace of $\ell^\infty(X)$ containing the order unit of $\ell^\infty(X)$, therefore in its own right an Archimedean Riesz space with order unit. To see that it is complete, it is enough to observe that it is closed in $\ell^\infty(X)$ because a uniform limit of continuous functions is continuous (3A3Nb).)

(c) For any measure space (X, Σ, μ), the space $L^\infty(\mu)$ is an M-space with standard order unit $\chi X^\bullet$.

354I Lemma Let U be an Archimedean Riesz space with order unit e, and V a subset of U which is dense for the order-unit norm $\|\,\|_e$. Then for any $u \in U$ there are sequences $\langle v_n \rangle_{n\in\mathbb{N}}$, $\langle w_n \rangle_{n\in\mathbb{N}}$ in V such that $v_n \le v_{n+1} \le u \le w_{n+1} \le w_n$ and $\|w_n - v_n\|_e \le 2^{-n}$ for every n; so that $u = \sup_{n\in\mathbb{N}} v_n = \inf_{n\in\mathbb{N}} w_n$ in U.

If V is a Riesz subspace of U, and $u \ge 0$, we may suppose that $v_n \ge 0$ for every n. Consequently V is order-dense in U.

proof For each $n \in \mathbb{N}$, take v_n, $w_n \in V$ such that

$$\|u - \tfrac{3}{2^{n+3}}e - v_n\|_e \le \tfrac{1}{2^{n+3}}, \quad \|u + \tfrac{3}{2^{n+3}}e - w_n\|_e \le \tfrac{1}{2^{n+3}}.$$

Then

$$u - \frac{1}{2^{n+1}}e \le v_n \le u - \frac{1}{2^{n+2}}e \le u \le u + \frac{1}{2^{n+2}}e \le w_n \le u + \frac{1}{2^{n+1}}e.$$

Accordingly $\langle v_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, $\langle w_n \rangle_{n\in\mathbb{N}}$ is non-increasing and $\|w_n - v_n\|_e \le 2^{-n}$ for every n. Because U is Archimedean, $\sup_{n\in\mathbb{N}} v_n = \inf_{n\in\mathbb{N}} w_n = u$.

If V is a Riesz subspace of U, then replacing v_n by v_n^+ if necessary we may suppose that every v_n is non-negative; and V is order-dense by the definition in 352Na.

354J Proposition Let U be an Archimedean Riesz space with an order unit e. Then $\| \|_e$ is Fatou and has the Levi property.

proof This is elementary. If $A \subseteq U^+$ is non-empty, upwards-directed and norm-bounded, then it is bounded above by αe, where $\alpha = \sup_{u\in A} \|u\|_e$. This is all that is called for in the Levi property. If moreover $\sup A$ is defined, then $\sup A \le \alpha e$ so $\|\sup A\| \le \alpha$, as required in the Fatou property.

354K Theorem Let U be an Archimedean Riesz space with order unit e. Then it can be embedded as an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, in such a way that e corresponds to χX and $\| \|_e$ corresponds to $\| \|_\infty$; moreover, this embedding is essentially unique.

proof This is nearly word-for-word a repetition of 353M. The only addition is the mention of the norms. Let X and $T : U \to C(X)$ be as in 353M. Then, for any $u \in U$, $|u| \le \|u\|_e e$, so that

$$|Tu| = T|u| \le \|u\|_e Te = \|u\|_e \chi X,$$

and $\|Tu\|_\infty \le \|u\|_e$. On the other hand, if $0 < \delta < \|u\|_e$ then $u_1 = (|u| - \delta e)^+ > 0$, so that $Tu_1 = (|Tu| - \delta\chi X)^+ > 0$ and $\|Tu\|_\infty \ge \delta$; as δ is arbitrary, $\|Tu\|_\infty \ge \|u\|_e$.

354L Corollary Any M-space U is isomorphic, as Banach lattice, to $C(X)$ for some compact Hausdorff X, and the isomorphism is essentially unique. X can be identified with the set of Riesz homomorphisms $x : U \to \mathbb{R}$ such that $x(e) = 1$, where e is the standard order unit of U, with the topology induced by the product topology on $\mathbb{R}^U$.

proof By 354K, there are a compact Hausdorff space X and an embedding of U as a norm-dense Riesz subspace of $C(X)$ matching $\| \|_e$ to $\| \|_\infty$. Since U is complete under $\| \|_e$, its image is closed in $C(X)$ (3A4Ff), and must be the whole of $C(X)$. The expression is unique just in so far as the expression of 353M/354K is unique. In particular, we may, if we wish, take X to be the set of normalized Riesz homomorphisms from U to $\mathbb{R}$, as in the proof of 353M.

Remark The set of uniferent Riesz homomorphisms from U to $\mathbb{R}$ is sometimes called the **spectrum** of U.

354M I come now to a second fundamental class of Banach lattices, in a strong sense 'dual' to the class of M-spaces, as will appear in §356.

Definition An ***L*-space** is a Banach lattice U such that $\|u + v\| = \|u\| + \|v\|$ whenever $u, v \in U^+$.

Example If (X, Σ, μ) is any measure space, then $L^1(\mu)$, with its norm $\| \|_1$, is an L-space (242D, 242F). In particular, taking μ to be counting measure on $\mathbb{N}$, ℓ^1 is an L-space (242Xa).

354N Theorem If U is an L-space, then its norm is order-continuous and has the Levi property.

proof (a) Both of these are consequences of the following fact: if $A \subseteq U$ is norm-bounded and non-empty and upwards-directed, then $\sup A$ is defined in U and belongs to the norm-closure of A in U. **P** Fix $u_0 \in A$; set $B = \{u - u_0 : u \in A,\, u \ge u_0\}$. Then $B \subseteq U^+$ is norm-bounded, non-empty and upwards-directed. Set $\gamma = \sup_{u\in B} \|u\|$. Consider the filter $\mathcal{F}(B\uparrow)$ on U generated by sets of the form $\{v : v \in B,\, v \ge u\}$ for $u \in B$. If $\epsilon > 0$ there is a $u \in B$ such that $\|u\| \ge \gamma - \epsilon$; now if $v, v' \in B \cap [u, \infty[$, there is a $w \in B$ such that $v \vee v' \le w$, so that

$$\|v - v'\| \le \|w - u\| = \|w\| - \|u\| \le \epsilon.$$

As ϵ is arbitrary, $\mathcal{F}(B\uparrow)$ is Cauchy and has a limit u^* say. If $u \in B$, $[u, \infty[$ is a closed set belonging to $\mathcal{F}(B\uparrow)$, so contains u^*; thus u^* is an upper bound for B. If w is an upper bound for B, then $]-\infty, w]$ is a closed set belonging to $\mathcal{F}(B\uparrow)$, so contains u^*; thus u^* is the least upper bound of B. And $B \in \mathcal{F}(B\uparrow)$, so $u^* \in \overline{B}$.

Because $u \mapsto u_0$ is an order-preserving homeomorphism,

$$u^* + u_0 = \sup\{u : u_0 \le u \in A\} = \sup A$$

and $u^* + u_0 \in \overline{A}$, as required. **Q**

(b) This shows immediately that the norm has the Levi property. But also it must be order-continuous. **P** If $A \subseteq U$ is non-empty and downwards-directed and has infimum 0, take any $u_0 \in A$ and consider $B = \{u_0 - u : u \in A,\ u \le u_0\}$. Then B is upwards-directed and has supremum u_0, so $u_0 \in \overline{B}$ and

$$\inf_{u\in A} \|u\| \le \inf_{v\in B} \|u_0 - v\| = 0. \ \mathbf{Q}$$

354O Proposition If U is an L-space and V is a norm-closed Riesz subspace of U, then V is an L-space in its own right. In particular, any band in U is an L-space.

proof For any Riesz subspace V of U, we surely have $\|u+v\| = \|u\|+\|v\|$ whenever $u, v \in V^+$; so if V is norm-closed, therefore a Banach lattice, it must be an L-space. But in any Banach lattice, a band is norm-closed (354Bd), so a band in an L-space is again an L-space.

354P Uniform integrability in L-spaces Some of the ideas of §246 can be readily expressed in this abstract context.

Definition Let U be an L-space. A set $A \subseteq U$ is **uniformly integrable** if for every $\epsilon > 0$ there is a $w \in U^+$ such that $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$.

354Q Since I have already used the phrase 'uniformly integrable' based on a different formula, I had better check instantly that the two definitions are consistent.

Proposition If (X, Σ, μ) is any measure space, then a subset of $L^1 = L^1(\mu)$ is uniformly integrable in the sense of 354P iff it is uniformly integrable in the sense of 246A.

proof (a) If $A \subseteq L^1$ is uniformly integrable in the sense of 246A, then for any $\epsilon > 0$ there are $M \ge 0$, $E \in \Sigma$ such that $\mu E < \infty$ and $\int (|u| - M\chi E^\bullet)^+ \le \epsilon$ for every $u \in A$; now $w = M\chi E^\bullet$ belongs to $(L^1)^+$ and $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$. As ϵ is arbitrary, A is uniformly integrable in the sense of 354P.

(b) Now suppose that A is uniformly integrable in the sense of 354P. Let $\epsilon > 0$. Then there is a $w \in (L^1)^+$ such that $\|(|u| - w)^+\| \le \frac{1}{2}\epsilon$ for every $u \in A$. There is a simple function $h : X \to \mathbb{R}$ such that $\|w - h^\bullet\| \le \frac{1}{2}\epsilon$ (242Mb); now take $E = \{x : h(x) \ne 0\}$, $M = \sup_{x\in X} |h(x)|$ (I pass over the trivial case $X = \emptyset$), so that $h \le M\chi E$ and

$$(|u| - M\chi E^\bullet)^+ \le (|u| - w)^+ + (w - M\chi E^\bullet)^+ \le (|u| - w)^+ + (w - h^\bullet)^+,$$

$$\int (|u| - M\chi E^\bullet)^+ \le \|(|u| - w)^+\| + \|w - h^\bullet\| \le \epsilon$$

for every $u \in A$. As ϵ is arbitrary, A is uniformly integrable in the sense of 354P.

354R I give abstract versions of the easiest results from §246.

Theorem Let U be an L-space.

(a) If $A \subseteq U$ is uniformly integrable, then

(i) A is norm-bounded;

(ii) every subset of A is uniformly integrable;

(iii) for any $\alpha \in \mathbb{R}$, αA is uniformly integrable;

(iv) there is a uniformly integrable, solid, convex, norm-closed set $C \supseteq A$;

(v) for any other uniformly integrable set $B \subseteq U$, $A \cup B$ and $A + B$ are uniformly integrable.

(b) For any set $A \subseteq U$, the following are equiveridical:

(i) A is uniformly integrable;

(ii) $\lim_{n\to\infty}(|u_n| - \sup_{i<n} |u_i|)^+ = 0$ for every sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in A;

(iii) *either* A is empty *or* for every $\epsilon > 0$ there are $u_0, \dots , u_n \in A$ such that $\|(|u| - \sup_{i\le n} |u_i|)^+\| \le \epsilon$ for every $u \in A$;

(iv) A is norm-bounded and any disjoint sequence in the solid hull of A is norm-convergent to 0.

(c) If $V \subseteq U$ is a closed Riesz subspace then a subset of V is uniformly integrable when regarded as a subset of V iff it is uniformly integrable when regarded as a subset of U.

proof (a)(i) There must be a $w \in U^+$ such that $\int (|u| - w)^+ \le 1$ for every $u \in A$; now

$$|u| \le |u| - w + |w| \le (|u| - w)^+ + |w|, \quad \|u\| \le \|(|u| - w)^+\| + \|w\| \le 1 + \|w\|$$

for every $u \in A$, so A is norm-bounded.

(ii) This is immediate from the definition.

(iii) Given $\epsilon > 0$, we can find $w \in U^+$ such that $|\alpha|\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$; now $\|(|v| - |\alpha|w)^+\| \le \epsilon$ for every $v \in \alpha A$.

(iv) If A is empty, take $C = A$. Otherwise, try

$$C = \{v : v \in U, \ \|(|v| - w)^+\| \le \sup_{u \in A} \|(|u| - w)^+\| \text{ for every } w \in U^+\}.$$

Evidently $A \subseteq C$, and C satisfies the definition 354M because A does. The functionals

$$v \mapsto \|(|v| - w)^+\| : U \to \mathbb{R}$$

are all continuous for $\| \ \|$ (because the operators $v \mapsto |v|$, $v \mapsto v - w$, $v \mapsto v^+$, $v \mapsto \|v\|$ are continuous), so C is closed. If $|v'| \le |v|$ and $v \in C$, then

$$\|(|v'| - w)^+\| \le \|(|v| - w)^+\| \le \sup_{u \in A} \|(|u| - w)^+\|$$

for every w, and $v' \in C$. If $v = \alpha v_1 + \beta v_2$ where v_1, $v_2 \in C$, $\alpha \in [0, 1]$ and $\beta = 1 - \alpha$, then $|v| \le \alpha|v_1| + \beta|v_2|$, so

$$|v| - w \le (\alpha|v_1| - \alpha w) + (\beta|v_2| - \beta w) \le (\alpha|v_1| - \alpha w)^+ + (\beta|v_2| - \beta w)^+$$

and

$$(|v| - w)^+ \le \alpha(|v_1| - w)^+ + \beta(|v_2| - w)^+$$

for every w; accordingly

$$\begin{aligned}\|(|v| - w)^+\| &\le \alpha\|(|v_1| - w)^+\| + \beta\|(|v_2| - w)^+\| \\ &\le (\alpha + \beta) \sup_{u \in A} \|(|u| - w)^+\| = \sup_{u \in A} \|(|u| - w)^+\|\end{aligned}$$

for every w, and $v \in C$.

Thus C has all the required properties.

(v) I show first that $A \cup B$ is uniformly integrable. **P** Given $\epsilon > 0$, let w_1, $w_2 \in U^+$ be such that

$$\|(|u| - w_1)^+\| \le \epsilon \text{ for every } u \in A, \quad \|(|u| - w_2)^+\| \le \epsilon \text{ for every } u \in B.$$

Set $w = w_1 \vee w_2$; then $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A \cup B$. As ϵ is arbitrary, $A \cup B$ is uniformly integrable. **Q**

Now (iv) tells us that there is a convex uniformly integrable set C including $A \cup B$, and in this case $A + B \subseteq 2C$, so $A + B$ is also uniformly integrable, using (ii) and (iii).

(b)(i)⇒(ii)&(iv) Suppose that A is uniformly integrable and that $\langle u_n \rangle_{n \in \mathbb{N}}$ is any sequence in the solid hull of A. Set $v_n = \sup_{i \le n} |u_i|$ for $n \in \mathbb{N}$ and

$$v'_0 = v_0 = |u_0|, \quad v'_n = v_n - v_{n-1} = (|u_n| - \sup_{i<n} |u_i|)^+$$

for $n \ge 1$. Given $\epsilon > 0$, there is a $w \in U^+$ such that $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$, and therefore for every u in the solid hull of A. Of course $\sup_{n \in \mathbb{N}} \|v_n \wedge w\| \le \|w\|$ is finite, so there is an $n \in \mathbb{N}$ such that $\|v_i \wedge w\| \le \epsilon + \|v_n \wedge w\|$ for every $i \in \mathbb{N}$. But now, for $m > n$,

$$\begin{aligned}v'_m \le (|u_m| - v_n)^+ &\le (|u_m| - |u_m| \wedge w)^+ + ((|u_m| \wedge w) - v_n)^+ \\ &\le (|u_m| - w)^+ + (v_m \wedge w) - (v_n \wedge w),\end{aligned}$$

so that

$$\begin{aligned}\|v'_m\| &\le \|(|u_m| - w)^+\| + \|(v_m \wedge w) - (v_n \wedge w)\| \\ &= \|(|u_m| - w)^+\| + \|v_m \wedge w\| - \|v_n \wedge w\| \le 2\epsilon,\end{aligned}$$

using the L-space property of the norm for the equality in the middle. As ϵ is arbitrary, $\lim_{n \to \infty} v'_n = 0$. As $\langle u_n \rangle_{n \in \mathbb{N}}$ is arbitrary, condition (ii) is satisfied; but so is condition (iv), because we know from (a-i) that A is norm-bounded, and if $\langle u_n \rangle_{n \in \mathbb{N}}$ is disjoint then $v'_n = |u_n|$ for every n, so that in this case $\lim_{n \to \infty} u_n = 0$.

(ii)⇒(iii)⇒(i) are elementary.

not-(i)⇒not-(iv) Now suppose that A is not uniformly integrable. If it is not norm-bounded, we can stop. Otherwise, there is some $\epsilon > 0$ such that $\sup_{u \in A} \|(|u| - w)^+\| > \epsilon$ for every $w \in U^+$. Consequently we shall be able to choose inductively a sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in A such that $\|(|u_n| - 2^n \sup_{i<n} |u_i|)^+\| > \epsilon$ for every $n \geq 1$. Because A is norm-bounded, $\sum_{i=0}^{\infty} 2^{-i} \|u_i\|$ is finite, and we can set

$$v_n = (|u_n| - 2^n \sup_{i<n} |u_i| - \sum_{i=n+1}^{\infty} 2^{-i} |u_i|)^+$$

for each n. (The sum $\sum_{i=n+1}^{\infty} 2^{-i}|u_i|$ is defined because $\langle \sum_{i=n+1}^{m} 2^{-i}|u_i| \rangle_{m \geq n+1}$ is a Cauchy sequence.) We have $v_m \leq |u_m|$,

$$\begin{aligned} v_m \wedge v_n &\leq (|u_m| - 2^{-n}|u_n|)^+ \wedge (|u_n| - 2^n|u_m|)^+ \\ &\leq (2^n|u_m| - |u_n|)^+ \wedge (|u_n| - 2^n|u_m|)^+ = 0 \end{aligned}$$

whenever $m < n$, so $\langle v_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in the solid hull of A; while

$$\|v_n\| \geq \|(|u_n| - 2^n \sup_{i<n} |u_i|)^+\| - \sum_{i=n+1}^{\infty} 2^{-i}\|u_i\| \geq \epsilon - 2^{-n} \sup_{u \in A} \|u\| \to \epsilon$$

as $n \to \infty$, so condition (iv) is not satisfied.

(c) Now this follows at once, because conditions (b-ii) and (b-iv) are satisfied in V iff they are satsified in U.

354X Basic exercises >(a) Work through the proofs that the following are all Banach lattices. (i) $\mathbb{R}^r$ with (α) $\|x\|_1 = \sum_{i=1}^{r} |\xi_i|$ (β) $\|x\|_2 = \sqrt{\sum_{i=1}^{r} |\xi_i|^2}$ (γ) $\|x\|_\infty = \max_{i \leq r} |\xi_i|$, where $x = (\xi_1, \dots, \xi_r)$. (ii) $\ell^p(X)$, for any set X and any $p \in [1, \infty]$ (242Xa, 243Xl, 244Xn). (iii) $L^p(\mu)$, for any measure space (X, Σ, μ) and any $p \in [1, \infty]$ (242F, 243E, 244G). (iv) $\boldsymbol{c}_0$, the space of sequences convergent to 0, with the norm $\| \, \|_\infty$ inherited from ℓ^∞.

(b) Let $\langle U_i \rangle_{i \in I}$ be any family of Banach lattices. Write U for their Riesz space product (352K), and in U set

$$\|u\|_1 = \sum_{i \in I} \|u(i)\|, \quad V_1 = \{u : \|u\|_1 < \infty\},$$

$$\|u\|_\infty = \sup_{i \in I} \|u(i)\| \text{ (counting } \sup \emptyset \text{ as 0)}, \quad V_\infty = \{u : \|u\|_\infty < \infty\}.$$

Show that V_1, V_∞ are solid linear subspaces of U and are Banach lattices under their norms $\| \, \|_1$, $\| \, \|_\infty$.

(c) Let U be a Riesz space with a Riesz norm. Show that the maps $(u, v) \mapsto u \wedge v$, $(u, v) \mapsto u \vee v : U \times U \to U$ are uniformly continuous.

>(d) Let U be a Riesz space with a Riesz norm. (i) Show that any order-bounded set in U is norm-bounded. (ii) Show that in $\mathbb{R}^r$, with any of the standard Riesz norms (354Xa(i)), norm-bounded sets are order-bounded. (iii) Show that in $\ell^1(\mathbb{N})$ there is a sequence converging to 0 (for the norm) which is not order-bounded. (iv) Show that in $\boldsymbol{c}_0$ any sequence converging to 0 is order-bounded, but there is a norm-bounded set which is not order-bounded.

(e) Let U be a Riesz space with a Riesz norm. Show that it is a Banach lattice iff non-decreasing Cauchy sequences are convergent. (*Hint*: if $\|u_{n+1} - u_n\| \leq 2^{-n}$ for every n, show that $\langle \sup_{i \leq n} u_i \rangle_{n \in \mathbb{N}}$ is Cauchy, and that $\langle u_n \rangle_{n \in \mathbb{N}}$ converges to $\inf_{n \in \mathbb{N}} \sup_{m \geq n} u_m$.)

(f) Let U be a Riesz space with a Riesz norm. Show that U is a Banach lattice iff every non-decreasing Cauchy sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in U^+ has a least upper bound u with $\|u\| = \lim_{n \to \infty} \|u_n\|$.

(g) Let U be a Banach lattice. Suppose that $B \subseteq U$ is solid and $\sup_{n \in \mathbb{N}} u_n \in B$ whenever $\langle u_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in B with a supremum in U. Show that B is closed. (*Hint*: show first that $u \in B$ whenever there is a sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in $B \cap U^+$ such that $\|u - u_n\| \leq 2^{-n}$ for every n; do this by considering $v_m = \inf_{n \geq m} u_n$.)

(h) Let U be any Riesz space with a Riesz norm. Show that the Banach space completion of U (3A5Jb) has a unique partial ordering under which it is a Banach lattice.

>(i) Show that $\boldsymbol{c}_0$ is a Banach lattice with an order-continuous norm which does not have the Levi property.

>(j) Show that ℓ^∞, with $\| \, \|_\infty$, is a Banach lattice with a Fatou norm which has the Levi property but is not order-continuous.

(k) Let U be a Riesz space with a Fatou norm. Show that if $V \subseteq U$ is a regularly embedded Riesz subspace then the induced norm on V is a Fatou norm.

(l) Let U be a Riesz space and $\| \ \|$ a Riesz norm on U which is order-continuous in the sense of 354Dc. Show that its restriction to U^+ is order-continuous in the sense of 313H.

(m) Let U be a Riesz space with an order-continuous norm. Show that if $V \subseteq U$ is a regularly embedded Riesz subspace then the induced norm on V is order-continuous.

(n) Let U be a Dedekind σ-complete Riesz space with a Fatou norm which has the Levi property. Show that it is a Banach lattice. (*Hint*: 354Xf.)

(o) Let $\langle U_i \rangle_{i \in I}$ be any family of Banach lattices and let V_1, V_∞ be the subspaces of $U = \prod_{i \in I} U_i$ as described in 354Xb. (i) Show that V_1, V_∞ have norms which are Fatou, or have the Levi property, iff every U_i has. (ii) Show that the norm of V_1 is order-continuous iff the norm of every U_i is. (iii) Show that V_∞ is an M-space iff every U_i is. (iv) Show that V_1 is an L-space iff every U_i is.

(p) Let U be a Banach lattice with an order-continuous norm. (i) Show that a sublattice of U is norm-closed iff it is order-closed in the sense of 313Da. (ii) Show that a norm-closed Riesz subspace of U is itself a Banach lattice with an order-continuous norm.

>(q) Let U be an M-space and V a norm-closed Riesz subspace of U containing the standard order unit of U. (i) Show that V, with the induced norm, is an M-space. (ii) Deduce that the space $\boldsymbol{c}$ of convergent sequences is an M-space if given the norm $\| \ \|_\infty$ inherited from ℓ^∞.

(r) Show that a Banach lattice U is an M-space iff (α) its norm is a Fatou norm with the Levi property (β) $\|u \vee v\| = \max(\|u\|, \|v\|)$ for all u, $v \in U^+$.

>(s) Describe a topological space X such that the space $\boldsymbol{c}$ of convergent sequences (354Xq) can be identified with $C(X)$.

(t) Let $D \subseteq \mathbb{R}$ be any non-empty set, and V the space of functions $f : D \to \mathbb{R}$ of bounded variation (§224). For $f \in V$ set $\|f\| = \sup\{|f(t_0)| + \sum_{i=1}^n |f(t_i) - f(t_{i-1})| : t_0 \le t_1 \le \ldots \le t_n \text{ in } D\}$ (224Yb). Let C be the set of bounded non-decreasing functions from D to $[0, \infty[$. Show that C is the positive cone of V for a Riesz space ordering under which V is an L-space.

354Y Further exercises (a) Let U be a Riesz space with a Riesz norm, and V a norm-dense Riesz subspace of U. Suppose that the induced norm on V is Fatou, when regarded as a norm on the Riesz space V. Show (i) that V is order-dense in U (ii) that the norm of U is Fatou. (*Hint*: for (i), show that if $u \in U^+$, $v_n \in V^+$ and $\|u - v_n\| \le 2^{-n-2}\|u\|$ for every n, then $\|v_0 - \inf_{i \le n} v_i\| \le \frac{1}{2}\|u\|$ for every n, so that 0 cannot be $\inf_{n \in \mathbb{N}} v_n$ in V.)

(b) Let U be a Riesz space with a Riesz norm. Show that the following are equiveridical: (i) $\lim_{n\to\infty} u_n = 0$ whenever $\langle u_n \rangle_{n \in \mathbb{N}}$ is a disjoint order-bounded sequence in U^+ (ii) $\lim_{n\to\infty} u_{n+1} - u_n = 0$ for every order-bounded non-decreasing sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in U (iii) whenever $A \subseteq U^+$ is a non-empty downwards-directed set in U^+ with infimum 0, $\inf_{u \in A} \sup_{v \in A, v \le u} \|u - v\| = 0$. (*Hint*: for (i)$\Rightarrow$(ii), show by induction that $\lim_{n\to\infty} u_n = 0$ whenever $\langle u_n \rangle_{n \in \mathbb{N}}$ is an order-bounded sequence such that, for some fixed $k \ge 1$, $\inf_{i \in K} u_i = 0$ for every $K \subseteq \mathbb{N}$ of size k; now show that if $\langle u_n \rangle_{n \in \mathbb{N}}$ is non-decreasing and $0 \le u_n \le u$ for every n, then $\inf_{i \in K}(u_{i+1} - u_i - \frac{1}{k}u)^+ = 0$ whenever $K \subseteq \mathbb{N}$ and $\#(K) = k \ge 1$. For (iii)$\Rightarrow$(i), set $A = \{u : \exists\, n,\ u \ge u_i\ \forall\, i \ge n\}$. See Fremlin 74a, 24H.)

(c) Show that any Riesz space with an order-continuous norm has the countable sup property (definition: 241Ye).

(d) Let U be a Banach lattice. Show that the following are equiveridical: (i) the norm on U is order-continuous; (ii) U satisfies the conditions of 354Yb; (iii) every order-bounded monotonic sequence in U is Cauchy.

(e) Let U be a Riesz space with a Fatou norm. Show that the norm on U is order-continuous iff it satisfies the conditions of 354Yb.

(f) For $f \in C([0,1])$, set $\|f\|_1 = \int |f(x)|dx$. Show that $\| \|_1$ is a Riesz norm on $C([0,1])$ satisfying the conditions of 354Yb, but is not order-continuous.

(g) Let U be a Riesz space with a Riesz norm $\| \|$. Show that $(U, \| \|)$ satisfies the conditions of 354Yb iff the norm of its completion is order-continuous.

(h) Let U be a Riesz space with a Riesz norm, and $V \subseteq U$ a norm-dense Riesz subspace such that the induced norm on V is order-continuous. Show that the norm of U is order-continuous. (*Hint*: use 354Ya.)

(i) Let U be an Archimedean Riesz space. For any $e \in U^+$, let U_e be the solid linear subspace of U generated by e, so that e is an order unit in U_e, and let $\| \|_e$ be the corresponding order-unit norm on U_e. We say that U is **uniformly complete** if U_e is complete under $\| \|_e$ for every $e \in U^+$. (i) Show that any Banach lattice is uniformly complete. (ii) Show that any Dedekind σ-complete Riesz space is uniformly complete (cf. 354Xn). (iii) Show that if U is a uniformly complete Riesz space with a Riesz norm which has the Levi property, then U is a Banach lattice. (iv) Show that if U is a Banach lattice then a set $A \subseteq U$ is closed, for the norm topology, iff $A \cap U_e$ is $\| \|_e$-closed for every $e \in U^+$. (v) Let V be a solid linear subspace of U. Show that the quotient Riesz space U/V is Archimedean iff $V \cap U_e$ is $\| \|_e$-closed for every $e \in U^+$. (vi) Show that if U is uniformly complete and $V \subseteq U$ is a solid linear subspace such that U/V is Archimedean, then U/V is uniformly complete. (vii) Show that U is Dedekind σ-complete iff it is uniformly complete and has the principal projection property (353Xb). (*Hint*: for (vii), use 353Yc.)

(j) Let U be an Archimedean Riesz space with an order unit, endowed with its order-unit norm. Let Z be the unit ball of U^*. Show that for a linear functional $f : U \to \mathbb{R}$ the following are equiveridical: (i) f is an **extreme point** of Z, that is, $f \in Z$ and $Z \setminus \{f\}$ is convex (ii) $|f(e)| = 1$ and one of f, $-f$ is a Riesz homomorphism.

(k) Let U be a Banach lattice such that $\|u + v\| = \|u\| + \|v\|$ whenever $u \wedge v = 0$. Show that U is an L-space. (*Hint*: by 354Yd, the norm is order-continuous, so U is Dedekind complete. If u, $v \geq 0$, set $e = u + v$, and represent U_e as $C(X)$ where X is extremally disconnected (353Yb); now approximate u and v by functions taking only finitely many values to show that $\|u + v\| = \|u\| + \|v\|$.)

(l) Let U be a uniformly complete Archimedean Riesz space (354Yi). Set $U_{\mathbb{C}} = U \times U$ with the complex linear structure defined by identifying $(u, v) \in U \times U$ as $u + iv \in U_{\mathbb{C}}$, so that $u = \mathcal{R}\mathrm{e}(u + iv)$, $v = \mathcal{I}\mathrm{m}(u + iv)$ and $(\alpha + i\beta)(u + iv) = (\alpha u - \beta v) + i(\alpha v + \beta u)$. (i) Show that for $w \in U_{\mathbb{C}}$ we can define $|w| \in U$ by setting $|w| = \sup_{|\zeta|=1} \mathcal{R}\mathrm{e}(\zeta w)$. (ii) Show that if U is a uniformly complete Riesz subspace of $\mathbb{R}^X$ for some set X, then we can identify $U_{\mathbb{C}}$ with the linear subspace of $\mathbb{C}^X$ generated by U. (iii) Show that $|w + w'| \leq |w| + |w'|$, $|\gamma w| = |\gamma||w|$ for all $w \in U_{\mathbb{C}}$, $\gamma \in \mathbb{C}$. (iv) Show that if $w \in U_{\mathbb{C}}$ and $|w| \leq u_1 + u_2$, where u_1, $u_2 \in U^+$, then w is expressible as $w_1 + w_2$ where $|w_j| \leq u_j$ for both j. (*Hint*: set $e = u_1 + u_2$ and represent U_e as $C(X)$.) (v) Show that if U_0 is a solid linear subspace of U, then, for $w \in U_{\mathbb{C}}$, $|w| \in U_0$ iff $\mathcal{R}\mathrm{e}\, w$, $\mathcal{I}\mathrm{m}\, w$ both belong to U_0. (vi) Show that if U has a Riesz norm then we have a norm on $U_{\mathbb{C}}$ defined by setting $\|w\| = \||w|\|$, and that if U is a Banach lattice then $U_{\mathbb{C}}$ is a (complex) Banach space. (vii) Show that if $U = L^p(\mu)$, where (X, Σ, μ) is a measure space and $p \in [1, \infty]$, then $U_{\mathbb{C}}$ can be identified with $L^p_{\mathbb{C}}(\mu)$ as defined in 242P, 243K and 244P. (We may call $U_{\mathbb{C}}$ the **complexification** of the Riesz space U.)

(m) Let (X, Σ, μ) be a measure space and V a Banach lattice. Write $\mathcal{L}^1_V$ for the space of Bochner integrable functions from conegligible subsets of X to V, and L^1_V for the corresponding set of equivalence classes (253Yf). (i) Show that L^1_V is a Banach lattice under the ordering defined by saying that $f^\bullet \leq g^\bullet$ iff $f(x) \leq g(x)$ in V for μ-almost every $x \in X$. (ii) Show that when $V = L^1(\nu)$, for some other measure space (Y, T, ν), then this ordering of L^1_V agrees with the ordering of $L^1(\lambda)$ where λ is the (c.l.d.) product measure on $X \times Y$ and we identify L^1_V with $L^1(\lambda)$, as in 253Yi. (iii) Show that if V has an order-continuous norm, so has L^1_V. (*Hint*: 354Yd.) (iv) Show that if μ is Lebesgue measure on $[0,1]$ and $V = \ell^\infty$, then L^1_V is not Dedekind σ-complete.

354 Notes and comments Apart from some of the exercises, the material of this section is pretty strictly confined to ideas which will be useful later in this volume. The basic Banach lattices of measure theory are the L^p spaces of Chapter 24; these all have Fatou norms with the Levi property (244Yf-244Yg), and for $p < \infty$ their norms are order-continuous (244Ye). In Chapter 36 I will return to these spaces in a more abstract context. Here I am mostly concerned to establish a vocabulary in which their various properties, and the relationships between these properties, can be expressed.

In normed Riesz spaces we have a very rich mixture of structures, and must take particular care over the concepts of 'boundedness', 'convergence' and 'density', which have more than one possible interpretation. In particular, we must scrupulously distinguish between 'order-bounded' and 'norm-bounded' sets. I have not yet formally introduced any of the various concepts of order-convergence (see §367), but I think that even so it is best to get into the habit of reminding oneself, when a convergent sequence appears, that it is convergent for the norm topology, rather than in any sense related directly to the order structure.

I should perhaps warn you that for the study of M-spaces 354L is not as helpful as it may look. The trouble is that apart from a few special cases (as in 354Xs) the topological space used in the representation is actually more complicated and mysterious than the M-space it is representing.

After the introduction of M-spaces, this section becomes a natural place for 'uniformly complete' spaces (354Yi). For the moment I leave these in the exercises. But I mention them now because they offer a straightforward route towards a theory of 'complex Riesz spaces' (354Yl). In large parts of functional analysis it is natural, and in some parts it is necessary, to work with normed spaces over $\mathbb{C}$ rather than over $\mathbb{R}$, and for L^2 spaces in particular it is useful to have a proper grasp of the complex case. And while the insights offered by the theory of Riesz spaces are not especially important in such areas, I think we should always seek connexions between different topics. So it is worth remembering that uniformly complete Riesz spaces have complexifications.

I shall have a great deal more to say about L-spaces when I come to spaces of additive functionals (§362) and to L^1 spaces again (§365) and to linear operators on them (§371); and before that, there will be something in the next section on their duals, and on L-spaces which are themselves dual spaces. For the moment I just give some easy results, direct translations of the corresponding facts in §246, which have natural expressions in the language of this section, holding deeper ideas over. In particular, the characterization of uniformly integrable sets as relatively weakly compact sets (247C) is valid in general L-spaces (356Q).

For an extensive treatment of Banach lattices, going very much deeper than I have space for in this volume, see Lindenstrauss & Tzafriri 79. For a careful exposition of a great deal of useful information, see Schaefer 74.

355 Spaces of linear operators

We come now to a discussion of linear operators between Riesz spaces. Linear operators are central to any kind of functional analysis, and a feature of the theory of Riesz spaces is the way the order structure picks out certain classes of operators for special consideration. Here I concentrate on positive and order-continuous operators, with a brief mention of sequential order-continuity. It turns out, in fact, that we need to work with operators which are differences of positive operators or of order-continuous positive operators. I define the basic spaces $\mathsf{L}^\sim$, $\mathsf{L}^\times$ and $\mathsf{L}^\sim_c$ (355A, 355G), with their most important properties (355B, 355E, 355H-355I) and some remarks on the special case of Banach lattices (355C, 355K). At the same time I give an important theorem on extension of operators (355F) and a corollary (355J).

The most important case is of course that in which the codomain is $\mathbb{R}$, so that our operators become real-valued functionals; I shall come to these in the next section.

355A Definition Let U and V be Riesz spaces. A linear operator $T : U \to V$ is **order-bounded** if $T[A]$ is order-bounded in V for every order-bounded $A \subseteq U$.

I will write $\mathsf{L}^\sim(U;V)$ for the set of order-bounded linear operators from U to V.

355B Lemma If U and V are Riesz spaces,

(a) a linear operator $T : U \to V$ is order-bounded iff $\{Tu : 0 \le u \le w\}$ is bounded above in V for every $w \in U^+$;

(b) in particular, any positive linear operator from U to V belongs to $\mathsf{L}^\sim = \mathsf{L}^\sim(U;V)$;

(c) $\mathsf{L}^\sim$ is a linear space;

(d) if W is another Riesz space and $T : U \to V$ and $S : V \to W$ are order-bounded linear operators, then $ST : U \to W$ is order-bounded.

proof (a) This is elementary. If $T \in \mathsf{L}^\sim$ and $w \in U^+$, $[0,w]$ is order-bounded, so its image must be order-bounded in V, and in particular bounded above. On the other hand, if T satisfies the condition, and A is order-bounded, then $A \subseteq [u_1,u_2]$ for some $u_1 \le u_2$, and

$$T[A] \subseteq T[u_1 + [0,u_2 - u_1]] = Tu_1 + T[[0,u_2 - u_1]]$$

is bounded above; similarly, $T[-A]$ is bounded above, so $T[A]$ is bounded below; as A is arbitrary, T is order-bounded.

(b) If T is positive then $\{Tu : 0 \le u \le w\}$ is bounded above by Tw for every $w \ge 0$, so $T \in \mathsf{L}^\sim$.

(c) If T_1, $T_2 \in \mathsf{L}^\sim$, $\alpha \in \mathbb{R}$ and $A \subseteq U$ is order-bounded, then there are v_1, $v_2 \in V$ such that $T_i[A] \subseteq [-v_i, v_i]$ for both i. Setting $v = (1+|\alpha|)v_1 + v_2$, $(\alpha T_1 + T_2)[A] \subseteq [-v, v]$; as A is arbitrary, $\alpha T_1 + T_2$ belongs to $\mathsf{L}^\sim$; as α, T_1, T_2 are arbitrary, and since the zero operator surely belongs to $\mathsf{L}^\sim$, $\mathsf{L}^\sim$ is a linear subspace of the space of all linear operators from U to V.

(d) This is immediate from the definition; if $A \subseteq U$ is order-bounded, then $T[A] \subseteq V$ and $(ST)[A] = S[T[A]] \subseteq W$ are order-bounded.

355C Theorem If U and V are Banach lattices then every order-bounded linear operator (in particular, every positive linear operator) from U to V is continuous.

proof ? Suppose, if possible, that $T : U \to V$ is an order-bounded linear operator which is not continuous. Then for each $n \in \mathbb{N}$ we can find a $u_n \in U$ such that $\|u_n\| \le 2^{-n}$ but $\|Tu_n\| \ge n$. Now $u = \sup_{n\in\mathbb{N}} |u_n|$ is defined in U (354C), and there is a $v \in V$ such that $-v \le Tw \le v$ whenever $-u \le w \le u$; but this means that $\|v\| \ge \|Tu_n\| \ge n$ for every n, which is impossible. **X**

355D Lemma Let U be a Riesz space and V any linear space over $\mathbb{R}$. Then a function $T : U^+ \to V$ extends to a linear operator from U to V iff

$$T(u + u') = Tu + Tu', \quad T(\alpha u) = \alpha Tu$$

for all u, $u' \in U^+$ and every $\alpha > 0$, and in this case the extension is unique.

proof For in this case we can, and must, set

$$T_1 u = Tu_1 - Tu_2 \text{ whenever } u_1,\, u_2 \in U^+ \text{ and } u = u_1 - u_2;$$

it is elementary to check that this defines $T_1 u$ uniquely for every $u \in U$, and that T_1 is a linear operator extending T.

355E Theorem Let U be a Riesz space and V a Dedekind complete Riesz space.

(a) The space $\mathsf{L}^\sim$ of order-bounded linear operators from U to V is a Dedekind complete Riesz space; its positive cone is the set of positive linear operators from U to V. In particular, every order-bounded linear operator from U to V is expressible as the difference of positive linear operators.

(b) For $T \in \mathsf{L}^\sim$, T^+ and $|T|$ are defined in the Riesz space $\mathsf{L}^\sim$ by the formulae

$$T^+(w) = \sup\{Tu : 0 \le u \le w\},$$

$$|T|(w) = \sup\{Tu : |u| \le w\} = \sup\{\textstyle\sum_{i=0}^n |Tu_i| : \sum_{i=0}^n |u_i| = w\}$$

for every $w \in U^+$.

(c) If S, $T \in \mathsf{L}^\sim$ then

$$(S \vee T)(w) = \sup_{0\le u\le w} Su + T(w-u), \quad (S \wedge T)(w) = \inf_{0\le u\le w} Su + T(w-u)$$

for every $w \in U^+$.

(d) Suppose that $A \subseteq \mathsf{L}^\sim$ is non-empty and upwards-directed. Then A is bounded above in $\mathsf{L}^\sim$ iff $\{Tu : T \in A\}$ is bounded above in V for every $u \in U^+$, and in this case $(\sup A)(u) = \sup_{T\in A} Tu$ for every $u \ge 0$.

(e) Suppose that $A \subseteq (\mathsf{L}^\sim)^+$ is non-empty and downwards-directed. Then $\inf A = 0$ in $\mathsf{L}^\sim$ iff $\inf_{T\in A} Tu = 0$ in V for every $u \in U^+$.

proof (a)(i) Suppose that $T \in \mathsf{L}^\sim$. For $w \in U^+$ set $R_T(w) = \sup\{Tu : 0 \le u \le w\}$; this is defined because V is Dedekind complete and $\{Tu : 0 \le u \le w\}$ is bounded above in V. Then $R_T(w_1 + w_2) = R_T w_1 + R_T w_2$ for all w_1, $w_2 \in U^+$. **P** Setting $A_i = [0, w_i]$ for each i, $w = w_1 + w_2$ and $A = [0, w]$, then of course $A_1 + A_2 \subseteq A$; but also $A \subseteq A_1 + A_2$, because if $u \in A$ then $u = (u \wedge w_1) + (u - w_1)^+$, and $0 \le (u - w_1)^+ \le (w - w_1)^+ = w_2$, so $u \in A_1 + A_2$. Consequently

$$\begin{aligned} R_T w &= \sup T[A] = \sup T[A_1 + A_2] = \sup(T[A_1] + T[A_2]) \\ &= \sup T[A_1] + \sup T[A_2] = R_T w_1 + R_T w_2 \end{aligned}$$

by 351Dc. **Q** Next, it is easy to see that $R_T(\alpha w) = \alpha R_T w$ for $w \in U^+$ and $\alpha > 0$, just because $u \mapsto \alpha u$, $v \mapsto \alpha v$ are isomorphisms of the partially ordered linear spaces U and V. It follows from 355D that we can extend R_T to a linear operator from U to V.

Because $R_T u \geq T0 = 0$ for every $u \in U^+$, R_T is a positive linear operator. But also $R_T u \geq Tu$ for every $u \in U^+$, so $R_T - T$ is also positive, and $T = R_T - (R_T - T)$ is the difference of two positive linear operators.

(ii) This shows that every order-bounded operator is a difference of positive operators. But of course if T_1 and T_2 are positive, then $(T_1 - T_2)u \leq T_1 w$ whenever $0 \leq u \leq w$ in U, so that $T_1 - T_2$ is order-bounded, by the criterion in 355Ba. Thus $\mathsf{L}^\sim$ is precisely the set of differences of positive operators.

(iii) Just as in 351F, $\mathsf{L}^\sim$ is a partially ordered linear space if we say that $S \leq T$ iff $Su \leq Tu$ for every $u \in U^+$. Now it is a Riesz space. **P** Take any $T \in \mathsf{L}^\sim$. Then R_T, as defined in (i), is an upper bound for $\{0, T\}$ in $\mathsf{L}^\sim$. If $S \in \mathsf{L}^\sim$ is any other upper bound for $\{0, T\}$, then for any $w \in U^+$ we must have $Sw \geq Su \geq Tu$ whenever $u \in [0, w]$, so that $Sw \geq R_T w$; as w is arbitrary, $S \geq R_T$; as S is arbitrary, $R_T = \sup\{0, T\}$ in $\mathsf{L}^\sim$. Thus $\sup\{0, T\}$ is defined in $\mathsf{L}^\sim$ for every $T \in \mathsf{L}^\sim$; by 352B, $\mathsf{L}^\sim$ is a Riesz space. **Q**

(I defer the proof that it is Dedekind complete to (d-ii) below.)

(b) As remarked in (a-iii), $R_T = T^+$ for each $T \in \mathsf{L}^\sim$; but this is just the formula given for T^+. Now, if $T \in \mathsf{L}^\sim$ and $w \in U^+$,

$$\begin{aligned} |T|(w) &= 2T^+ w - Tw = 2 \sup_{u \in [0,w]} Tu - Tw \\ &= \sup_{u \in [0,w]} T(2u - w) = \sup_{u \in [-w,w]} Tu, \end{aligned}$$

which is the first formula offered for $|T|$. In particular, if $|u| \leq w$ then Tu, $-Tu = T(-u)$ are both less than or equal to $|T|(w)$, so that $|Tu| \leq |T|(w)$. So if $u_0, \dots, u_n$ are such that $\sum_{i=0}^n |u_i| = w$, then

$$\sum_{i=0}^n |Tu_i| \leq \sum_{i=0}^n |T|(|u_i|) = |T|(w).$$

Thus $B = \{\sum_{i=0}^n |Tu_i| : \sum_{i=0}^n |u_i| = w\}$ is bounded above by $|T|(w)$. On the other hand, if v is an upper bound for B and $|u| \leq w$, then

$$Tu \leq |Tu| + |T(w - |u|)| \leq v;$$

as u is arbitrary, $|T|(w) \leq v$; thus $|T|(w)$ is the least upper bound for B. This completes the proof of part (b) of the theorem.

(c) We know that $S \vee T = T + (S - T)^+$ (352D), so that

$$\begin{aligned} (S \vee T)(w) &= Tw + (S - T)^+(w) = Tw + \sup_{0 \leq u \leq w} (S - T)(u) \\ &= \sup_{0 \leq u \leq w} Tw + (S - T)(u) = \sup_{0 \leq u \leq w} Su + T(w - u) \end{aligned}$$

for every $w \in U^+$, by the formula in (b). Also from 352D we have $S \wedge T = S + T - T \vee S$, so that

$$\begin{aligned} (S \wedge T)(w) &= Sw + Tw - \sup_{0 \leq u \leq w} Tu + S(w - u) \\ &= \inf_{0 \leq u \leq w} Sw + Tw - Tu - S(w - u) \end{aligned}$$

(351Db)

$$= \inf_{0 \leq u \leq w} Su + T(w - u)$$

for $w \in U^+$.

(d)(i) Now suppose that $A \subseteq \mathsf{L}^\sim$ is non-empty and upwards-directed and that $\{Tu : T \in A\}$ is bounded above in V for every $u \in U^+$. In this case, because V is Dedekind complete, we may set $Ru = \sup_{T \in A} Tu$ for every $u \in U^+$. Now $R(u_1 + u_2) = Ru_1 + Ru_2$ for all u_1, $u_2 \in U^+$. **P** Set $B_i = \{Tu_i : T \in A\}$ for each i, $B = \{T(u_1 + u_2) : T \in A\}$. Then $B \subseteq B_1 + B_2$, so

$$R(u_1 + u_2) = \sup B \leq \sup(B_1 + B_2) = \sup B_1 + \sup B_2 = Ru_1 + Ru_2.$$

On the other hand, if $v_i \in B_i$ for both i, there are $T_i \in A$ such that $v_i = T_i u_i$ for each i; because A is upwards-directed, there is a $T \in A$ such that $T \ge T_i$ for both i, and now

$$R(u_1 + u_2) \ge T(u_1 + u_2) = Tu_1 + Tu_2 \ge T_1 u_1 + T_2 u_2 = v_1 + v_2.$$

As v_1, v_2 are arbitrary,

$$R(u_1 + u_2) \ge \sup(B_1 + B_2) = \sup B_1 + \sup B_2 = Ru_1 + Ru_2. \text{ **Q**}$$

It is also easy to see that $R(\alpha u) = \alpha Ru$ for every $u \in U^+$ and $\alpha > 0$. So, using 355D again, R has an extension to a linear operator from U to V.

If we fix any $T_0 \in A$, we have $T_0 u \le Ru$ for every $u \in U^+$, so $R - T_0$ is a positive linear operator, and $R = (R - T_0) + T_0$ belongs to $\mathsf{L}^\sim$. Again, $Tu \le Ru$ for every $T \in A$ and $u \in U^+$, so R is an upper bound for A in $\mathsf{L}^\sim$; and, finally, if S is any upper bound for A in $\mathsf{L}^\sim$, then Su is an upper bound for $\{Tu : T \in A\}$, and must be greater than or equal to Ru, for every $u \in U^+$; so that $R \le S$ and $R = \sup A$ in $\mathsf{L}^\sim$.

(ii) Consequently $\mathsf{L}^\sim$ is Dedekind complete. **P** If $A \subseteq \mathsf{L}^\sim$ is non-empty and bounded above by S say, then $A' = \{T_0 \vee T_1 \vee \ldots \vee T_n : T_0, \ldots , T_n \in A\}$ is upwards-directed and bounded above by S, so $\{Tu : T \in A'\}$ is bounded above by Su for every $u \in U^+$; by (i) just above, A' has a supremum in $\mathsf{L}^\sim$, which will also be the supremum of A. **Q**

(e) Suppose that $A \subseteq (\mathsf{L}^\sim)^+$ is non-empty and downwards-directed. Then $-A = \{-T : T \in A\}$ is non-empty and upwards-directed, so

$$\begin{aligned}\inf A = 0 &\iff \sup(-A) = 0\\ &\iff \sup_{T\in A}(-Tu) = 0 \text{ for every } u \in U^+\\ &\iff \inf_{T\in A} Tu = 0 \text{ for every } u \in U^+.\end{aligned}$$

355F Theorem Let U and V be Riesz spaces, $U_0 \subseteq U$ a Riesz subspace and $T_0 : U_0 \to V$ a positive linear operator such that $Su = \sup\{T_0 w : w \in U_0,\ 0 \le w \le u\}$ is defined in V for every $u \in U^+$. Suppose *either* that U_0 is order-dense and that T_0 is order-continuous *or* that U_0 is solid.

(a) There is a unique positive linear operator $T : U \to V$, extending T_0, which agrees with S on U^+.

(b) If T_0 is a Riesz homomorphism so is T.

(c) If T_0 is order-continuous so is T.

(d) If U_0 is order-dense and T_0 is an injective Riesz homomorphism, then T is injective.

(e) If U_0 is order-dense and T_0 is order-continuous then T is the only order-continuous positive linear operator from U to V extending T_0.

proof (a)(i) (The key.) If u, $u' \in U^+$ then $S(u + u') = Su + Su'$. **P** If w, $w' \in U_0^+$, $w \le u$ and $w' \le u'$, then $w + w' \le u + u'$, so

$$T_0 w + T_0 w' = T_0(w + w') \le S(u + u');$$

as w and w' are arbitrary, $Su+Su' \le S(u+u')$ (351Dc). In the other direction, suppose that $w \in U_0^+$ and $w \le u+u'$.

case 1 Suppose that U_0 is solid. Then $w \wedge u$ and $(w - u)^+$ belong to U_0, while $w \wedge u \le u$ and $(w - u)^+ \le (u + u' - u)^+ = u'$; so

$$T_0 w = T_0(w \wedge u + (w - u)^+) = T_0(w \wedge u) + T_0(w - u)^+ \le Su + Su';$$

as w is arbitrary, $S(u + u') \le Su + Su'$ and we must have equality.

case 2 Suppose that U_0 is order-dense and T_0 is order-continuous. Set $A = \{v : v \in U_0^+,\ v \le w \wedge u\}$ and $B = \{v : v \in U_0^+,\ v \le (w - u)^+\}$. Then (taking the suprema in U) $w \wedge u = \sup A$ and $(w - u)^+ = \sup B$, because U_0 is order-dense; by 351Dc again, $w = \sup(A + B)$ in U and therefore $w = \sup(A + B)$ in U_0. Also both A and B are upwards-directed, so $A + B$ also is. Because T_0 is order-continuous,

$$T_0 w = \sup T_0[A + B] = \sup(T_0[A] + T_0[B]) \le Su + Su'.$$

So once again we must have $S(u + u') \le Su + Su'$ and therefore $S(u + u') = Su + Su'$. **Q**

(ii) Of course $S(\alpha u) = \alpha Su$ whenever $u \in U^+$ and $\alpha \ge 0$. By 355D, S has a unique extension to a linear operator $T : U \to V$. As $Tu = Su \ge 0$ whenever $u \ge 0$, T is positive. If $u \in U_0^+$ then $Su = T_0 u$, so T extends T_0.

(b) Suppose that T_0 is a Riesz homomorphism. If $u \wedge u' = 0$ in U, then $w \wedge w' = 0$ and $T_0 w \wedge T_0 w' = 0$ whenever $w \in U_0 \cap [0, u]$ and $w' \in U_0 \cap [0, u']$. By 352Ea, $Tu \wedge Tu' = Su \wedge Su' = 0$ in V. By 352G(iv), T is a Riesz homomorphism.

(c) Now suppose that T_0 is order-continuous. Suppose that $B \subseteq U^+$ is non-empty and upwards-directed and has a supremum $u_0 \in U$. Of course $Tu \le Tu_0$ for every $u \in B$, so Tu_0 is an upper bound for $T[B]$. On the other hand, suppose that v is an upper bound for $T[B]$. If $w \in U_0^+$ and $u \in U^+$, $w \wedge u = \sup\{w' : w' \in U_0,\ 0 \le w' \le w \wedge u\}$. **P** If U_0 is solid, $w \wedge u \in U_0$; and otherwise U_0 is order-dense. **Q** So if $w \in U_0$ and $0 \le w \le u_0$,

$$w = w \wedge \sup B = \sup_{u \in B} w \wedge u = \sup_{u \in B} \sup(U_0 \cap [0, w \wedge u]) = \sup C,$$

where

$$C = \{w' : w' \in U_0^+,\ w' \le w \wedge u \text{ for some } u \in B\}.$$

Since C is upwards-directed,

$$T_0 w = \sup T_0[C] \le v.$$

As w is arbitrary, $Tu_0 \le v$; as v is arbitrary, $Tu_0 = \sup T[B]$; as B is arbitrary, T is order-continuous (351Ga).

(d) If U_0 is order-dense and T_0 is an injective Riesz homomorphism, then for any non-zero $u \in U$ there is a non-zero $w \in U_0$ such that $|w| \le |u|$; so that

$$|Tu| = T|u| \ge T_0|w| > 0$$

because T is a Riesz homomorphism, by (b). As u is arbitrary, T is injective.

(e) Finally, if U_0 is order-dense then any order-continuous positive linear operator extending T_0 must agree with S on U^+ and is therefore equal to T.

355G Definition Let U be a Riesz space and V a Dedekind complete Riesz space. Then $\mathsf{L}^\times(U;V)$ will be the set of those $T \in \mathsf{L}^\sim(U;V)$ expressible as the difference of order-continuous positive linear operators, and $\mathsf{L}^\sim_c(U;V)$ will be the set of those $T \in \mathsf{L}^\sim(U;V)$ expressible as the difference of sequentially order-continuous positive linear operators.

Because a composition of (sequentially) order-continuous functions is (sequentially) order-continuous, we shall have

$$ST \in \mathsf{L}^\times(U;W) \text{ whenever } S \in \mathsf{L}^\times(V;W),\ T \in \mathsf{L}^\times(U;V),$$

$$ST \in \mathsf{L}^\sim_c(U;W) \text{ whenever } S \in \mathsf{L}^\sim_c(V;W),\ T \in \mathsf{L}^\sim_c(U;V),$$

for all Riesz spaces U and all Dedekind complete Riesz spaces V, W.

355H Theorem Let U be a Riesz space and V a Dedekind complete Riesz space. Then
(i) $\mathsf{L}^\times = \mathsf{L}^\times(U;V)$ is a band in $\mathsf{L}^\sim = \mathsf{L}^\sim(U;V)$, therefore a Dedekind complete Riesz space in its own right;
(ii) a member T of $\mathsf{L}^\sim$ belongs to $\mathsf{L}^\times$ iff $|T|$ is order-continuous.

proof There is a fair bit to check, but each individual step is easy enough.

(a) Suppose that S, T are order-continuous positive linear operators from U to V. Then $S+T$ is order-continuous. **P** If $A \subseteq U$ is non-empty, downwards-directed and has infimum 0, then for any u_1, $u_2 \in A$ there is a $u \in A$ such that $u \le u_1$, $u \le u_2$, and now $(S+T)(u) \le Su_1 + Tu_2$. Consequently any lower bound for $(S+T)[A]$ must also be a lower bound for $S[A] + T[A]$. But since

$$\inf(S[A] + T[A]) = \inf S[A] + \inf T[A] = 0$$

(351Dc), $\inf(S+T)[A]$ must also be 0; as A is arbitrary, $S+T$ is order-continuous, by 351Ga. **Q**

(b) Consequently $S+T \in \mathsf{L}^\times$ for all S, $T \in \mathsf{L}^\times$. Since $-T$ and αT belong to $\mathsf{L}^\times$ for every $T \in \mathsf{L}^\times$ and $\alpha \ge 0$, we see that $\mathsf{L}^\times$ is a linear subspace of $\mathsf{L}^\sim$.

(c) If $T : U \to V$ is an order-continuous linear operator, $S : U \to V$ is linear and $0 \le S \le T$, then S is order-continuous. **P** If $A \subseteq U$ is non-empty, downwards-directed and has infimum 0, then any lower bound of $S[A]$ must also be a lower bound of $T[A]$, so $\inf S[A] = 0$; as A is arbitrary, S is order-continuous. **Q**

It follows that $\mathsf{L}^\times$ is a solid linear subspace of $\mathsf{L}^\sim$. **P** If $T \in \mathsf{L}^\times$ and $|S| \le |T|$ in $\mathsf{L}^\sim$, express T as $T_1 - T_2$ where T_1, T_2 are order-continuous positive linear operators. Then

$$S^+,\, S^- \le |S| \le |T| \le T_1 + T_2,$$

so S^+ and S^- are order-continuous and $S = S^+ - S^- \in \mathsf{L}^\times$. **Q**

Accordingly $\mathsf{L}^\times$ is a Dedekind complete Riesz space in its own right (353J(b-i)).

(d) The argument of (c) also shows that if $T \in \mathsf{L}^\times$ then $|T|$ is order-continuous; so that for $T \in \mathsf{L}^\sim$,

$$T \in \mathsf{L}^\times \iff |T| \in \mathsf{L}^\times \iff |T| \text{ is order-continuous.}$$

(e) If $C \subseteq (\mathsf{L}^\times)^+$ is non-empty, upwards-directed and has a supremum $T \in L^\sim$, then T is order-continuous, so belongs to $\mathsf{L}^\times$. **P** Suppose that $A \subseteq U^+$ is non-empty, upwards-directed and has supremum w. Then

$$Tw = \sup_{S\in C} Sw = \sup_{S\in C} \sup_{u\in A} Su = \sup_{u\in A} Tu,$$

putting 355Ed and 351G(a-iii) together. So (using 351Ga again) T is order-continuous. **Q** Consequently $\mathsf{L}^\times$ is a band in $\mathsf{L}^\sim$ (352Ob).

This completes the proof.

355I Theorem Let U be a Riesz space and V a Dedekind complete Riesz space. Then $\mathsf{L}^\sim_c(U;V)$ is a band in $\mathsf{L}^\sim(U;V)$, and a member T of $\mathsf{L}^\sim(U;V)$ belongs to $\mathsf{L}^\sim_c(U;V)$ iff $|T|$ is sequentially order-continuous.

proof Copy the arguments of 355H.

355J Proposition Let U be a Riesz space and V a Dedekind complete Riesz space. Let $U_0 \subseteq U$ be an order-dense Riesz subspace; then $T \mapsto T\restriction U_0$ is an embedding of $\mathsf{L}^\times(U;V)$ as a solid linear subspace of $\mathsf{L}^\times(U_0;V)$. In particular, any operator in $\mathsf{L}^\times(U_0;V)$ has at most one extension in $\mathsf{L}^\times(U;V)$.

proof (a) Because the embedding $U_0 \subsetneqq U$ is positive and order-continuous (352Nb), $T\restriction U_0$ is positive and order-continuous whenever T is; so $T\restriction U_0 \in \mathsf{L}^\times(U_0;V)$ whenever $T \in \mathsf{L}^\times(U;V)$. Because the map $T \mapsto T\restriction U_0$ is linear, the image W of $\mathsf{L}^\times(U;V)$ is a linear subspace of $\mathsf{L}^\times(U_0;V)$.

(b) If $T \in \mathsf{L}^\times(U;V)$ and $T\restriction U_0 \ge 0$, then $T \ge 0$. **P?** Suppose, if possible, that there is a $u \in U^+$ such that $Tu \not\ge 0$. Because $|T| \in \mathsf{L}^\times(U;V)$ is order-continuous and $A = \{v : v \in U_0,\, v \le u\}$ is an upwards-directed set with supremum u, $\inf\{|T|(u-v) : v \in A\} = 0$ and there is a $v \in A$ such that $Tu + |T|(u-v) \not\ge 0$. But $Tv = Tu + T(v-u) \le Tu + |T|(u-v)$ so $Tv \not\ge 0$ and $T\restriction U_0 \not\ge 0$. **XQ**

This shows that the map $T \mapsto T\restriction U_0$ is an order-isomorphism between $\mathsf{L}^\times(U;V)$ and W, and in particular is injective.

(c) Now suppose that $S_0 \in W$ and that $|S| \le |S_0|$ in $\mathsf{L}^\times(U_0;V)$. Then $S \in W$. **P** Take $T_0 \in \mathsf{L}^\times(U;V)$ such that $T_0\restriction U_0 = S_0$. Then $S_1 = |T_0|\restriction U_0$ is a positive member of W such that $S_0 \le S_1$ and $-S_0 \le S_1$, so $S^+ \le S_1$. Consequently, for any $u \in U^+$,

$$\sup\{S^+v : v \in U_0,\, 0 \le v \le u\} \le \sup\{S_1 v : v \in U_0,\, 0 \le v \le u\} \le |T_0|(u)$$

is defined in V (recall that we are assuming that V is Dedekind complete). But this means that S^+ has an extension to an order-continuous positive linear operator from U to V (355F), and belongs to W. Similarly, $S^- \in W$, so $S \in W$. **Q**

This shows that W is a solid linear subspace of $\mathsf{L}^\times(U_0;V)$, as claimed.

355K Proposition Let U be a Banach lattice with an order-continuous norm.

(a) If V is any Archimedean Riesz space and $T : U \to V$ is a positive linear operator, then T is order-continuous.

(b) If V is a Dedekind complete Riesz space then $\mathsf{L}^\times(U;V) = \mathsf{L}^\sim(U;V)$.

proof (a) Suppose that $A \subseteq U^+$ is non-empty and downwards-directed and has infimum 0. Then for each $n \in \mathbb{N}$ there is a $u_n \in A$ such that $\|u_n\| \le 4^{-n}$. By 354C, $u = \sup_{n\in\mathbb{N}} 2^n u_n$ is defined in U. Now $Tu_n \le 2^{-n}Tu$ for every n, so any lower bound for $T[A]$ must also be a lower bound for $\{2^{-n}Tu : n \in \mathbb{N}\}$ and therefore (because V is Archimedean) less than or equal to 0. Thus $\inf T[A] = 0$; as A is arbitrary, T is order-continuous.

(b) This is now immediate from 355Ea and the definition of $\mathsf{L}^\times$.

355X Basic exercises >(a) Let U and V be arbitrary Riesz spaces. (i) Show that the set $\mathsf{L}(U;V)$ of all linear operators from U to V is a partially ordered linear space if we say that $S \le T$ whenever $Su \le Tu$ for every $u \in U^+$. (ii) Show that if U and V are Banach lattices then the set of positive operators is closed in the normed space $\mathsf{B}(U;V)$ of bounded linear operators from U to V.

>(b) If U is a Riesz space and $\| \, \|$, $\| \, \|'$ are two norms on U both rendering it a Banach lattice, show that they are equivalent, that is, give rise to the same topology.

(c) Let U be a Riesz space with a Riesz norm, V an Archimedean Riesz space with an order unit, and $T : U \to V$ a linear operator which is continuous for the given norm on U and the order-unit norm on V. Show that T is order-bounded.

(d) Let U be a Riesz space, V an Archimedean Riesz space, and $T : U^+ \to V^+$ a map such that $T(u_1 + u_2) = Tu_1 + Tu_2$ for all u_1, $u_2 \in U^+$. Show that T has an extension to a linear operator from U to V.

>(e) Show that if r, $s \ge 1$ are integers then the Riesz space $\mathsf{L}^\sim(\mathbb{R}^r;\mathbb{R}^s)$ can be identified with the space of real $s \times r$ matrices, saying that a matrix is positive iff every coefficient is positive, so that if $T = \langle \tau_{ij} \rangle_{1 \le i \le s, 1 \le j \le r}$ then $|T|$, taken in $\mathsf{L}^\sim(\mathbb{R}^r;\mathbb{R}^s)$, is $\langle |\tau_{ij}| \rangle_{1 \le i \le s, 1 \le j \le r}$. Show that a positive matrix represents a Riesz homomorphism iff each row has at most one non-zero coefficient.

>(f) Let U be a Riesz space and V a Dedekind complete Riesz space. Show that if $T_0, \ldots, T_n \in \mathsf{L}^\sim(U;V)$ then

$$(T_0 \vee \ldots \vee T_n)(w) = \sup\{\sum_{i=0}^n T_i u_i : u_i \ge 0 \ \forall \ i \le n, \ \sum_{i=0}^n u_i = w\}$$

for every $w \in U^+$.

>(g) Let U be a Riesz space, V a Dedekind complete Riesz space, and $A \subseteq \mathsf{L}^\sim(U;V)$ a non-empty set. Show that A is bounded above in $\mathsf{L}^\sim(U;V)$ iff $C_w = \{\sum_{i=0}^n T_i u_i : T_0, \ldots, T_n \in A, \ u_0, \ldots, u_n \in U^+, \ \sum_{i=0}^n u_i = w\}$ is bounded above in V for every $w \in U^+$, and in this case $(\sup A)(w) = \sup C_w$ for every $w \in U^+$.

355Y Further exercises (a) Let U and V be Banach lattices. For $T \in \mathsf{L}^\sim = \mathsf{L}^\sim(U;V)$, set

$$\|T\|_\sim = \sup_{w \in U^+, \|w\| \le 1} \inf\{\|v\| : |Tu| \le v \text{ whenever } |u| \le w\}.$$

Show that $\| \, \|_\sim$ is a norm on $\mathsf{L}^\sim$ under which $\mathsf{L}^\sim$ is a Banach space, and that the set of positive linear operators is closed in $\mathsf{L}^\sim$.

(b) Give an example of a continuous linear operator from ℓ^2 to itself which is not order-bounded.

(c) Let U and V be Riesz spaces and $T : U \to V$ a linear operator. (i) Show that for any $w \in U^+$, $C_w = \{\sum_{i=0}^n |Tu_i| : u_0, \ldots, u_n \in U^+, \ \sum_{i=0}^n u_i = w\}$ is upwards-directed, and has the same upper bounds as $\{Tu : |u| \le w\}$. (*Hint*: 352Fd.) (ii) Show that if $\sup C_w$ is defined for every $w \in U^+$, then $S = T \vee (-T)$ is defined in the partially ordered linear space $\mathsf{L}^\sim(U;V)$ and $Sw = \sup C_w$ for every $w \in U^+$.

(d) Let U, V and W be Riesz spaces, of which V and W are Dedekind complete. (i) Show that for any $S \in \mathsf{L}^\times(V;W)$, the map $T \mapsto ST : L^\sim(U;V) \to \mathsf{L}^\sim(U;W)$ belongs to $\mathsf{L}^\times(\mathsf{L}^\sim(U;V);\mathsf{L}^\sim(U;W))$, and is a Riesz homomorphism if S is. (*Hint*: 355Yc.) (ii) Show that for any $T \in \mathsf{L}^\sim(U;V)$, the map $S \mapsto ST : L^\sim(V;W) \to \mathsf{L}^\sim(U;W)$ belongs to $\mathsf{L}^\times(L^\sim(V;W);\mathsf{L}^\sim(U;W))$.

(e) Let $\nu_{\mathbb{N}}$ be the usual measure on $\{0,1\}^{\mathbb{N}}$ and $\boldsymbol{c}$ the Banach lattice of convergent sequences. Find a linear operator $T : L^2(\nu_{\mathbb{N}}) \to \boldsymbol{c}$ which is norm-continuous, therefore order-bounded, such that 0 and T have no common upper bound in the partially ordered linear space of all linear operators from $L^2(\nu_{\mathbb{N}})$ to $\boldsymbol{c}$.

(f) Let U and V be Banach lattices. Let $\mathsf{L}^{\mathrm{reg}}$ be the linear space of operators from U to V expressible as the difference of positive operators. For $T \in \mathsf{L}^{\mathrm{reg}}$ let $\|T\|_{\mathrm{reg}}$ be

$$\inf\{\|T_1 + T_2\| : T_1, \ T_2 : U \to V \text{ are positive}, \ T = T_1 - T_2\}.$$

Show that $\| \, \|_{\mathrm{reg}}$ is a norm under which $\mathsf{L}^{\mathrm{reg}}$ is complete.

(g) Let U and V be Riesz spaces. For this exercise only, say that $\mathsf{L}^\times(U;V)$ is to be the set of linear operators $T:U\to V$ such that whenever $A\subseteq U$ is non-empty, downwards-directed and has infimum 0 then $\{v:v\in V^+,\ \exists\, w\in A,\ |Tu|\le v$ whenever $|u|\le w\}$ has infimum 0 in V. (i) Show that $\mathsf{L}^\times(U;V)$ is a linear space. (ii) Show that if U is Archimedean then $\mathsf{L}^\times(U;V)\subseteq\mathsf{L}^\sim(U;V)$. (iii) Show that if U is Archimedean and V is Dedekind complete then this definition agrees with that of 355G. (iv) Show that for any Riesz spaces U, V and W, $ST\in\mathsf{L}^\times(U;W)$ for every $S\in\mathsf{L}^\times(V;W)$ and $T\in\mathsf{L}^\times(U;V)$. (v) Show that if U and V are Banach lattices, then $\mathsf{L}^\times(U;V)$ is closed in $\mathsf{L}^\sim(U;V)$ for the norm $\|\,\|_\sim$ of 355Ya. (vi) Show that if V is Archimedean and U is a Banach lattice with an order-continuous norm, then $\mathsf{L}^\times(U;V)=\mathsf{L}^\sim(U;V)$.

(h) Let U be a Riesz space and V a Dedekind complete Riesz space. Show that the band projection $P:\mathsf{L}^\sim(U;V)\to\mathsf{L}^\times(U;V)$ is given by the formula

$$(PT)(w)=\inf\{\sup_{u\in A}Tu : A\subseteq U^+\text{ is non-empty, upwards-directed and has supremum } w\}$$

for every $w\in U^+$, $T\in(\mathsf{L}^\sim(U;V))^+$. (Cf. 362Bd.)

(i) Show that if U is a Riesz space with the countable sup property (241Ye), then $\mathsf{L}^\sim_c(U;V)=\mathsf{L}^\times(U;V)$ for every Dedekind complete Riesz space V.

(j) Let U and V be Riesz spaces, of which V is Dedekind complete, and U_0 a solid linear subspace of U. Show that the map $T\mapsto T\restriction U_0$ is an order-continuous Riesz homomorphism from $\mathsf{L}^\times(U;V)$ onto a solid linear subspace of $\mathsf{L}^\times(U_0;V)$.

(k) Let U be a uniformly complete Riesz space (354Yi) and V a Dedekind complete Riesz space. Let $U_{\mathbb C}$, $V_{\mathbb C}$ be their complexifications (354Yl). Show that the complexification of $\mathsf{L}^\sim(U;V)$ can be identified with the complex linear space of linear operators $T:U_{\mathbb C}\to V_{\mathbb C}$ such that $B_T(w)=\{|Tu|:|u|\le w\}$ is bounded above in V for every $w\in U^+$, and that now $|T|(w)=\sup B_T(w)$ for every $T\in\mathsf{L}^\sim(U;V)_{\mathbb C}$ and $w\in U^+$. (*Hint*: if u, $v\in U$ and $|u+iv|=w$, then u and v can be simultaneously approximated for the order-unit norm $\|\,\|_w$ on the solid linear subspace generated by w by finite sums $\sum_{j=0}^n(\cos\theta_j)w_j$, $\sum_{j=0}^n(\sin\theta_j)w_j$ where $w_j\in U^+$, $\sum_{j=0}^n w_j=w$. Consequently $|T(u+iv)|\le|T|(w)$ for every $T\in\mathsf{L}^\sim_{\mathbb C}$.)

355 Notes and comments I have had to make some choices in the basic definitions of this chapter (355A, 355G). For Dedekind complete codomains V, there is no doubt what $\mathsf{L}^\sim(U;V)$ should be, since the order-bounded operators (in the sense of 355A) are just the differences of positive operators (355Ea). (These are sometimes called '**regular**' operators.) When V is not Dedekind complete, we have to choose between the two notions, as not every order-bounded operator need be regular (355Ye). In my previous book (Fremlin 74a) I chose the regular operators; I have still not encountered any really persuasive reason to settle definitively on either class. In 355G the technical complications in dealing with any natural equivalent of the larger space (see 355Yg) are such that I have settled for the narrower class, but explicitly restricting the definition to the case in which V is Dedekind complete. In the applications in this book, the codomains are nearly always Dedekind complete, so we can pass these questions by.

The elementary extension technique in 355D may recall the definition of the Lebesgue integral (122L-122M). In the same way, 351G may remind you of the theorem that a linear operator between normed spaces is continuous everywhere if it is continuous anywhere, or of the corresponding results about Boolean homomorphisms and additive functionals on Boolean algebras (313L, 326Ka, 326R).

Of course 355Ea is the central fact about the space $\mathsf{L}^\sim(U;V)$ for Dedekind complete V; because we get a new Riesz space from old ones, the prospect of indefinite recursion immediately presents itself. For Banach lattices, $\mathsf{L}^\sim(U;V)$ is a linear subspace of the space $\mathsf{B}(U;V)$ of bounded linear operators (355C); the question of when the two are equal will be of great importance to us. I give only the vaguest hints on how to show that they can be different (355Yb, 355Ye), but these should be enough to make it plain that equality is the exception rather than the rule. It is also very useful that we have effective formulae to describe the Riesz space operations on $\mathsf{L}^\sim(U;V)$ (355E, 355Xf-355Xg, 355Yc). You may wish to compare these with the corresponding formulae for additive functionals on Boolean algebras in 326Yd and 362B.

If we think of $\mathsf{L}^\sim$ as somehow corresponding to the space of bounded additive functionals on a Boolean algebra, the bands $\mathsf{L}^\sim_c$ and $\mathsf{L}^\times$ correspond to the spaces of countably additive and completely additive functionals. In fact (as will appear in §362) this correspondence is very close indeed. For the moment, all I have sought to establish is that $\mathsf{L}^\sim_c$ and $\mathsf{L}^\times$ are indeed bands. Of course any case in which $\mathsf{L}^\sim(U;V) = \mathsf{L}^\sim_c(U;V)$ or $\mathsf{L}^\sim_c(U;V) = \mathsf{L}^\times(U;V)$ is of interest (355Kb, 355Yi).

Between Banach lattices, positive linear operators are continuous (355C); it follows at once that the Riesz space structure determines the topology (355Xb), so that it is not to be wondered at that there are further connexions between the norm and the spaces $\mathsf{L}^\sim$ and $\mathsf{L}^\times$, as in 355K.

355F will be a basic tool in the theory of representations of Riesz spaces; if we can represent an order-dense Riesz subspace of U as a subspace of a Dedekind complete space V, we have at least some chance of expressing U also as a subspace of V. Of course it has other applications, starting with analysis of the dual spaces.

356 Dual spaces

As always in functional analysis, large parts of the theory of Riesz spaces are based on the study of linear functionals. Following the scheme of the last section, I define spaces $U^\sim$, $U^\sim_c$ and $U^\times$, the 'order-bounded', 'sequentially order-continuous' and 'order-continuous' duals of a Riesz space U (356A). These are Dedekind complete Riesz spaces (356B). If U carries a Riesz norm they are closely connected with the normed space dual U^*, which is itself a Banach lattice (356D). For each of them, we have a canonical Riesz homomorphism from U to the corresponding bidual. The map from U to $U^{\times\times}$ is particularly important (356I); when this map is an isomorphism we call U 'perfect' (356J). The last third of the section deals with L- and M-spaces and the duality between them (356N, 356P), with two important theorems on uniform integrability (356O, 356Q).

356A Definition Let U be a Riesz space.

(a) I write $U^\sim$ for the space $\mathcal{L}^\sim(U;\mathbb{R})$ of order-bounded real-valued linear functionals on U, the **order-bounded dual** of U.

(b) $U^\sim_c$ will be the space $\mathcal{L}^\sim_c(U;\mathbb{R})$ of differences of sequentially order-continuous positive real-valued linear functionals on U, the **sequentially order-continuous dual** of U.

(c) $U^\times$ will be the space $\mathcal{L}^\times(U;\mathbb{R})$ of differences of order-continuous positive real-valued linear functionals on U, the **order-continuous dual** of U.

Remark It is easy to check that the three spaces $U^\sim$, $U^\sim_c$ and $U^\times$ are in general different (356Xa-356Xc). But the examples there leave open the question: can we find a Riesz space U, for which $U^\sim_c \ne U^\times$, and which is actually Dedekind complete, rather than just Dedekind σ-complete, as in 356Xc? This leads to unexpectedly deep water; it is yet another form of the Banach-Ulam problem. Really this is a question for Volume 5, but in 363S below I collect the relevant ideas which are within the scope of the present volume.

356B Theorem For any Riesz space U, its order-bounded dual $U^\sim$ is a Dedekind complete Riesz space in which $U^\sim_c$ and $U^\times$ are bands, therefore Dedekind complete Riesz spaces in their own right. For $f \in U^\sim$, f^+ and $|f| \in U^\sim$ are defined by the formulae

$$f^+(w) = \sup\{f(u) : 0 \le u \le w\}, \quad |f|(w) = \sup\{f(u) : |u| \le w\}$$

for every $w \in U^+$. A non-empty upwards-directed set $A \subseteq U^\sim$ is bounded above iff $\sup_{f\in A} f(u)$ is finite for every $u \in U$, and in this case $(\sup A)(u) = \sup_{f\in A} f(u)$ for every $u \in U^+$.

proof 355E, 355H, 355I.

356C Proposition Let U be any Riesz space and P a band projection on U. Then its adjoint $P' : U^\sim \to U^\sim$, defined by setting $P'(f) = fP$ for every $f \in U^\sim$, is a band projection on $U^\sim$.

proof Because $P : U \to U$ is a positive linear operator, $P'f \in U^\sim$ for every $f \in U^\sim$ (355Bd), and P' is a positive linear operator from $U^\sim$ to itself. Set $Q = I - P$, the complementary band projection on U; then Q' is another positive linear operator on $U^\sim$, and $P'f + Q'f = f$ for every f. Now $P'f \wedge Q'f = 0$ for every $f \ge 0$. **P** For any $w \in U^+$,

$$(P'f - Q'f)^+(w) = \sup_{0 \le u \le w} (P'f - Q'f)(u)$$
$$= \sup_{0 \le u \le w} f(Pu - Qu) = f(Pw)$$

(because $Pu - Qu \le Pu \le Pw = P(Pw) - Q(Pw)$ whenever $0 \le u \le w$)

$$= (P'f)(w),$$

so $(P'f - Q'f)^+ = P'f$, that is, $P'f \wedge Q'f = 0$. **Q** By 352Rd, P' is a band projection.

356D Proposition Let U be a Riesz space with a Riesz norm.

(a) The normed space dual U^* of U is a solid linear subspace of $U^\sim$, and in itself is a Banach lattice with a Fatou norm and has the Levi property.

(b) The norm of U is order-continuous iff $U^* \subseteq U^\times$.

(c) If U is a Banach lattice, then $U^* = U^\sim$, so that $U^\sim$, $U^\times$ and $U_c^\sim$ are all Banach lattices.

(d) If U is a Banach lattice with order-continuous norm then $U^* = U^\times = U^\sim$.

proof (a)(i) If $f \in U^*$ then

$$\sup_{|u| \le w} f(u) \le \sup_{|u| \le w} \|f\|\|u\| = \|f\|\|w\| < \infty$$

for every $w \in U^+$, so $f \in U^\sim$ (355Ba). Thus $U^* \subseteq U^\sim$.

(ii) If $f \in U^\sim$, $g \in U^*$ and $|f| \le |g|$, then for any $w \in U$

$$|f(w)| \le |f|(|w|) \le |g|(|w|) = \sup_{|u| \le |w|} g(u) \le \sup_{|u| \le |w|} \|g\|\|u\| \le \|g\|\|w\|.$$

As w is arbitrary, $f \in U^*$ and $\|f\| \le \|g\|$; as f and g are arbitrary, U^* is a solid linear subspace of $U^\sim$ and the norm of U^* is a Riesz norm. Because U^* is a Banach space it is also a Banach lattice.

(iii) If $A \subseteq (U^*)^+$ is non-empty and upwards-directed and $M = \sup_{f \in A} \|f\|$ is finite, then $\sup_{f \in A} f(u) \le M\|u\|$ is finite for every $u \in U^+$, so $g = \sup A$ is defined in $U^\sim$ (355Ed). Now $g(u) = \sup_{f \in A} f(u)$ for every $u \in U^+$, as also noted in 355Ed, so

$$|g(u)| \le g(|u|) \le M\||u|\| = M\|u\|$$

for every $u \in U$, and $\|g\| \le M$. But as A is arbitrary, this proves simultaneously that the norm of $U^\sim$ is Fatou and has the Levi property.

(b)(i) Suppose that the norm of U is order-continuous. If $f \in U^*$ and $A \subseteq U$ is a non-empty downwards-directed set with infimum 0, then

$$\inf_{u \in A} |f|(u) \le \inf_{u \in A} \|f\|\|u\| = 0,$$

so $|f| \in U^\times$ and $f \in U^\times$. Thus $U^* \subseteq U^\times$.

(ii) Now suppose that the norm is not order-continuous. Then there is a non-empty downwards-directed set $A \subseteq U$, with infimum 0, such that $\inf_{u \in A} \|u\| = \delta > 0$. Set

$$B = \{v : v \ge u \text{ for some } u \in A\}.$$

Then B is convex. **P** If v_1, $v_2 \in B$ and $\alpha \in [0,1]$, there are u_1, $u_2 \in A$ such that $v_i \ge u_i$ for both i; now there is a $u \in A$ such that $u \le u_1 \wedge u_2$, so that

$$u = \alpha u + (1-\alpha)u \le \alpha v_1 + (1-\alpha)v_2,$$

and $\alpha v_1 + (1-\alpha)v_2 \in B$. **Q** Also $\inf_{v \in B} \|v\| = \delta > 0$. By the Hahn-Banach theorem (3A5Cb), there is an $f \in U^*$ such that $\inf_{v \in B} f(v) > 0$. But now

$$\inf_{u \in A} |f|(u) \ge \inf_{u \in A} f(u) > 0$$

and $|f|$ is not order-continuous; so $U^* \not\subseteq U^\times$.

(c) By 355C, $U^\sim \subseteq U^*$, so $U^\sim = U^*$. Now $U^\times$ and $U_c^\sim$, being bands, are closed linear subspaces (354Bd), so are Banach lattices in their own right.

(d) Put (b) and (c) together.

356E Biduals If you have studied any functional analysis at all, it will come as no surprise that duals-of-duals are important in the theory of Riesz spaces. I start with a simple lemma.

Lemma Let U be a Riesz space and $f : U \to \mathbb{R}$ a positive linear functional. Then for any $u \in U^+$ there is a positive linear functional $g : U \to \mathbb{R}$ such that $0 \le g \le f$, $g(u) = f(u)$ and $g(v) = 0$ whenever $u \wedge v = 0$.

proof Set $g(v) = \sup_{\alpha \ge 0} f(v \wedge \alpha u)$ for every $v \in U^+$. Then it is easy to see that $g(\beta v) = \beta g(v)$ for every $v \in U^+$, $\beta \in [0, \infty[$. If $v, w \in U^+$ then

$$(v \wedge \alpha u) + (w \wedge \alpha u) \le (v + w) \wedge 2\alpha u \le (v \wedge 2\alpha u) + (w \wedge 2\alpha u)$$

for every $\alpha \ge 0$ (352Fa), so $g(v + w) = g(v) + g(w)$. Accordingly g has an extension to a linear functional from U to $\mathbb{R}$ (355D). Of course $0 \le g(v) \le f(v)$ for $v \ge 0$, so $0 \le g \le f$ in $U^\sim$. We have $g(u) = f(u)$, while if $u \wedge v = 0$ then $\alpha u \wedge v = 0$ for every $\alpha \ge 0$, so $g(v) = 0$.

356F Theorem Let U be a Riesz space and V a solid linear subspace of $U^\sim$. For $u \in U$ define $\hat{u} : V \to \mathbb{R}$ by setting $\hat{u}(f) = f(u)$ for every $f \in V$. Then $u \mapsto \hat{u}$ is a Riesz homomorphism from U to $V^\times$.

proof (a) By the definition of addition and scalar multiplication in V, $\hat{u}$ is linear for every u; also $\widehat{\alpha u} = \alpha \hat{u}$ and $(u_1 + u_2)\hat{} = \hat{u}_1 + \hat{u}_2$ for all u, u_1, $u_2 \in U$ and $\alpha \in \mathbb{R}$. If $u \ge 0$ then $\hat{u}(f) = f(u) \ge 0$ for every $f \in V^+$, so $\hat{u} \ge 0$; accordingly every $\hat{u}$ is the difference of two positive functionals, and $u \mapsto \hat{u}$ is a linear operator from U to $V^\sim$.

(b) If $B \subseteq V$ is a non-empty downwards-directed set with infimum 0, then $\inf_{f \in B} f(u) = 0$ for every $u \in U^+$, by 355Ee. But this means that $\hat{u}$ is order-continuous for every $u \in U^+$, so that $\hat{u} \in V^\times$ for every $u \in U$.

(c) If $u \wedge v = 0$ in U, then for any $f \in V^+$ there is a $g \in [0, f]$ such that $g(u) = f(u)$ and $g(v) = 0$ (356E). So

$$(\hat{u} \wedge \hat{v})(f) \le \hat{u}(f - g) + \hat{v}(g) = f(u) - g(u) + g(v) = 0.$$

As f is arbitrary, $\hat{u} \wedge \hat{v} = 0$. As u and v are arbitrary, $u \mapsto \hat{u}$ is a Riesz homomorphism (352G).

356G Lemma Suppose that U is a Riesz space such that $U^\sim$ separates the points of U. Then U is Archimedean.

proof ? Otherwise, there are u, $v \in U$ such that $v > 0$ and $nv \le u$ for every $n \in \mathbb{N}$. Now there is an $f \in U^\sim$ such that $f(v) \ne 0$; but $|f(v)| \le |f|(v) \le \frac{1}{n}|f|(u)$ for every n, so this is impossible. **X**

356H Lemma Let U be an Archimedean Riesz space and $f > 0$ in $U^\times$. Then there is a $u \in U$ such that (i) $u > 0$ (ii) $f(v) > 0$ whenever $0 < v \le u$ (iii) $g(u) = 0$ whenever $g \wedge f = 0$ in $U^\times$. Moreover, if $u_0 \in U^+$ is such that $f(u_0) > 0$, we can arrange that $u \le u_0$.

proof (a) Because $f > 0$ there is certainly some $u_0 \in U$ such that $f(u_0) > 0$. Set $A = \{v : 0 \le v \le u_0,\ f(v) = 0\}$. Then $(v_1 + v_2) \wedge u_0 \in A$ for all v_1, $v_2 \in A$, so A is upwards-directed. Because $f(u_0) > 0 = \sup f[A]$ and f is order-continuous, u_0 cannot be the least upper bound of A, and there is another upper bound u_1 of A strictly less than u_0.

Set $u = u_0 - u_1 > 0$. If $0 \le v \le u$ and $f(v) = 0$, then

$$w \in A \implies w \le u_1 \implies w + v \le u_0 \implies w + v \in A;$$

consequently $nv \in A$ and $nv \le u_0$ for every $n \in \mathbb{N}$, so $v = 0$. Thus u has properties (i) and (ii).

(b) Now suppose that $g \wedge f = 0$ in $U^\times$. Let $\epsilon > 0$. Then for each $n \in \mathbb{N}$ there is a $v_n \in [0, u]$ such that $f(v_n) + g(u - v_n) \le 2^{-n}\epsilon$ (355Ec). If $v \le v_n$ for every $n \in \mathbb{N}$ then $f(v) = 0$ so $v = 0$; thus $\inf_{n \in \mathbb{N}} v_n = 0$. Set $w_n = \inf_{i \le n} v_i$ for each $n \in \mathbb{N}$; then $\langle w_n \rangle_{n \in \mathbb{N}}$ is non-increasing and has infimum 0 so (because g is order-continuous) $\inf_{n \in \mathbb{N}} g(w_n) = 0$. But

$$u - w_n = \sup_{i \le n} u - v_i \le \sum_{i=0}^n u - v_i,$$

so

$$g(u - w_n) \le \sum_{i=0}^n g(u - v_i) \le 2\epsilon$$

for every n, and

$$g(u) \le 2\epsilon + \inf_{n \in \mathbb{N}} g(w_n) = 2\epsilon.$$

As ϵ is arbitrary, $g(u) = 0$; as g is arbitrary, u has the third required property.

356I Theorem Let U be any Archimedean Riesz space. Then the canonical map from U to $U^{\times\times}$ (356F) is an order-continuous Riesz homomorphism from U onto an order-dense Riesz subspace of $U^{\times\times}$. If U is Dedekind complete, its image in $U^{\times\times}$ is solid.

proof (a) By 356F, $u \mapsto \hat{u} : U \to U^{\times\times}$ is a Riesz homomorphism.

To see that it is order-continuous, take any non-empty downwards-directed set $A \subseteq U$ with infimum 0. Then $C = \{\hat{u} : u \in A\}$ is downwards-directed, and for any $f \in (U^\times)^+$

$$\inf_{\phi\in C} \phi(f) = \inf_{u\in A} f(u) = 0$$

because f is order-continuous. As f is arbitrary, $\inf C = 0$ (355Ee); as A is arbitrary, $u \mapsto \hat{u}$ is order-continuous (351Ga).

(b) Now suppose that $\phi > 0$ in $U^{\times\times}$. By 356H, there is an $f > 0$ in $U^\times$ such that $\phi(f) > 0$ and $\phi(g) = 0$ whenever $g \wedge f = 0$. Next, there is a $u > 0$ in U such that $f(u) > 0$. Since $u \ge 0$, $\hat{u} \ge 0$; since $\hat{u}(f) > 0$, $\hat{u} \wedge \phi > 0$.

Because $U^{\times\times}$ (being Dedekind complete) is Archimedean, $\inf_{\alpha>0} \alpha\hat{u} = 0$, and there is an $\alpha > 0$ such that

$$\psi = (\hat{u} \wedge \phi - \alpha\hat{u})^+ > 0.$$

Let $g \in (U^\times)^+$ be such that $\psi(g) > 0$ and $\theta(g) = 0$ whenever $\theta \wedge \psi = 0$ in $U^{\times\times}$. Let $v \in U^+$ be such that $g(v) > 0$ and $h(v) = 0$ whenever $h \wedge g = 0$ in $U^\times$.

Because $\hat{v}(g) = g(v) > 0$, $\hat{v} \wedge \psi > 0$. As $\psi \le \hat{u}$, $\hat{v} \wedge \hat{u} > 0$ and $\hat{v} \wedge \alpha\hat{u} > 0$. Set $w = v \wedge \alpha u$; then $\hat{w} = \hat{v} \wedge \alpha\hat{u}$, by 356F, so $\hat{w} > 0$.

? Suppose, if possible, that $\hat{w} \not\le \phi$. Then $\theta = (\hat{w} - \phi)^+ > 0$, so there is an $h \in (U^\times)^+$ such that $\theta(h) > 0$ and $\theta(h') > 0$ whenever $0 < h' \le h$ (356H, for the fourth and last time). Now examine

$$\theta(h \wedge g) \le (\alpha\hat{u} - \phi \wedge \hat{u})^+(g)$$

(because $\hat{w} \le \alpha\hat{u}$, $\phi \wedge \hat{u} \le \phi$, $h \wedge g \le g$)

$$= 0$$

because $(\alpha\hat{u} - \phi \wedge \hat{u})^+ \wedge \psi = 0$. So $h \wedge g = 0$ and $h(v) = 0$. But this means that

$$\theta(h) \le \hat{w}(h) \le \hat{v}(h) = 0,$$

which is impossible. **X**

Thus $0 < \hat{w} \le \phi$. As ϕ is arbitrary, the image $\hat{U}$ of U is quasi-order-dense in $U^{\times\times}$, therefore order-dense (353A).

(c) Now suppose that U is Dedekind complete and that $0 \le \phi \le \psi \in \hat{U}$. Express ψ as $\hat{u}$ where $u \in U$, and set $A = \{v : v \in U, v \le u^+, \hat{v} \le \phi\}$. If $v \in U$ and $0 \le \hat{v} \le \phi$, then $w = v^+ \wedge u^+ \in A$ and $\hat{w} = \hat{v}$; thus $\phi = \sup\{\hat{v} : v \in A\} = \hat{v}_0$, where $v_0 = \sup A$. So $\phi \in \hat{U}$. As ϕ and ψ are arbitrary, $\hat{U}$ is solid in $U^{\times\times}$.

356J Definition A Riesz space U is **perfect** if the canonical map from U to $U^{\times\times}$ is an isomorphism.

356K Proposition A Riesz space U is perfect iff (i) it is Dedekind complete (ii) $U^\times$ separates the points of U (iii) whenever $A \subseteq U$ is non-empty and upwards-directed and $\{f(u) : u \in A\}$ is bounded for every $f \in U^\times$, then A is bounded above in U.

proof (a) Suppose that U is perfect. Because it is isomorphic to $U^{\times\times}$, which is surely Dedekind complete, U also is Dedekind complete. Because the map $u \mapsto \hat{u} : U \to U^{\times\times}$ is injective, $U^\times$ separates the points of U. If $A \subseteq U$ is non-empty and upwards-directed ad $\{f(u) : u \in A\}$ is bounded above for every $f \in U^\times$, then $B = \{\hat{u} : u \in A\}$ is non-empty and upwards-directed and $\sup_{\phi\in B} \phi(f) < \infty$ for every $f \in U^\times$, so $\sup B$ is defined in $U^{\times\sim}$ (355Ed); but $U^{\times\times}$ is a band in $U^{\times\sim}$, so $\sup B$ belongs to $U^{\times\times}$ and is of the form $\hat{w}$ for some $w \in U$. Because $u \mapsto \hat{u}$ is a Riesz space isomorphism, $w = \sup A$ in U. Thus U satisfies the three conditions.

(b) Suppose that U satisfies the three conditions. We know that $u \mapsto \hat{u}$ is an order-continuous Riesz homomorphism onto an order-dense Riesz subspace of $U^{\times\times}$ (356I). It is injective because $U^\times$ separates the points of U. If $\phi \ge 0$ in $U^{\times\times}$, set $A = \{u : u \in U^+, \hat{u} \le \phi\}$. Then A is non-empty and upwards-directed and for any $f \in U^\times$

$$\sup_{u\in A} f(u) \le \sup_{u\in A} |f|(u) \le \sup_{u\in A} \hat{u}(|f|) \le \phi(|f|) < \infty,$$

so by condition (iii) A has an upper bound in U. Since U is Dedekind complete, $w = \sup A$ is defined in U. Now

$$\hat{w} = \sup_{u \in A} \hat{u} = \phi.$$

As ϕ is arbitrary, the image of U includes $(U^{\times\times})^+$, therefore is the whole of $U^{\times\times}$, and $u \mapsto \hat{u}$ is a bijective Riesz homomorphism, that is, a Riesz space isomorphism.

356L Proposition (a) Any band in a perfect Riesz space is a perfect Riesz space in its own right.

(b) For any Riesz space U, $U^\sim$ is perfect; consequently $U^\sim_c$ and $U^\times$ are perfect.

proof (a) I use the criterion of 356K. Let U be a perfect Riesz space and V a band in U. Then V is Dedekind complete because U is (353Jb). If $v \in V \setminus \{0\}$ there is an $f \in U^\times$ such that $f(v) \neq 0$; but the embedding $V \subsetneq U$ is order-continuous (352N), so $g = f \restriction V$ belongs to $V^\times$, and $g(v) \neq 0$. Thus $V^\times$ separates the points of V. If $A \subseteq V$ is non-empty and upwards-directed and $\sup_{v \in A} g(v)$ is finite for every $g \in V^\times$, then $\sup_{v \in A} f(v) < \infty$ for every $f \in U^\times$ (again because $f \restriction V \in V^\times$), so A has an upper bound in U; because U is Dedekind complete, $\sup A$ is defined in U; because V is a band, $\sup A \in V$ and is an upper bound for A in V. Thus V satisfies the conditions of 356K and is perfect.

(b) $U^\sim$ is Dedekind complete, by 355Ea. If $f \in U^\sim \setminus \{0\}$, there is a $u \in U$ such that $f(u) \neq 0$; now $\hat{u}(f) \neq 0$, where $\hat{u} \in U^{\sim\times}$ (356F). Thus $U^{\sim\times}$ separates the points of $U^\sim$. If $A \subseteq U^\sim$ is non-empty and upwards-directed and $\sup_{f \in A} \phi(f)$ is finite for every $\phi \in U^{\sim\times}$, then, in particular,

$$\sup_{f \in A} f(u) = \sup_{f \in A} \hat{u}(f) < \infty$$

for every $u \in U$, so A is bounded above in $U^\sim$, by 355Ed. Thus $U^\sim$ satisfies the conditions of 356K and is perfect.

By (a), it follows at once that $U^\times$ and $U^\sim_c$ are perfect.

356M Proposition If U is a Banach lattice in which the norm is order-continuous and has the Levi property, then U is perfect.

proof By 356Db, $U^* = U^\times$; since U^* surely separates the points of U, so does $U^\times$. By 354Ee, U is Dedekind complete. If $A \subseteq U$ is non-empty and upwards-directed and $f[A]$ is bounded for every $f \in U^\times$, then A is norm-bounded, by the Uniform Boundedness Theorem (3A5Hb). Because the norm is supposed to have the Levi property, A is bounded above in U. Thus U satisfies all the conditions of 356K and is perfect.

356N L- and M-spaces I come now to the duality between L-spaces and M-spaces which I hinted at in §354.

Proposition Let U be an Archimedean Riesz space with an order-unit norm.

(a) $U^* = U^\sim$ is an L-space.

(b) If e is the standard order unit of U, then $\|f\| = |f|(e)$ for every $f \in U^*$.

(c) A linear functional $f : U \to \mathbb{R}$ is positive iff it belongs to U^* and $\|f\| = f(e)$.

(d) If $e \neq 0$ there is a positive linear functional f on U such that $f(e) = 1$.

proof (a)-(b) We know already that $U^* \subseteq U^\sim$ is a Banach lattice (356Da). If $f \in U^\sim$ then

$$\sup\{|f(u)| : \|u\| \le 1\} = \sup\{|f(u)| : |u| \le e\} = |f|(e),$$

so $f \in U^*$ and $\|f\| = |f|(e)$; thus $U^\sim = U^*$. If $f, g \ge 0$ in U^*, then

$$\|f + g\| = (f + g)(e) = f(e) + g(e) = \|f\| + \|g\|;$$

thus U^* is an L-space.

(c) As already remarked, if f is positive then $f \in U^*$ and $\|f\| = f(e)$. On the other hand, if $f \in U^*$ and $\|f\| = f(e)$, take any $u \ge 0$. Set $v = (1 + \|u\|)^{-1}u$. Then $0 \le v \le e$ and $\|e - v\| \le 1$ and

$$f(e - v) \le |f(e - v)| \le \|f\| = f(e).$$

But this means that $f(v) \ge 0$ so $f(u) \ge 0$. As u is arbitrary, $f \ge 0$.

(d) By the Hahn-Banach theorem (3A5Ac), there is an $f \in U^*$ such that $f(e) = \|f\| = 1$; by (c), f is positive.

356O Theorem Let U be an Archimedean Riesz space with order-unit norm. Then a set $A \subseteq U^* = U^\sim$ is uniformly integrable iff it is norm-bounded and $\lim_{n \to \infty} \sup_{f \in A} |f(u_n)| = 0$ for every order-bounded disjoint sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in U^+.

proof (a) Suppose that A is uniformly integrable. Then it is surely norm-bounded (354Ra). If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in U^+ bounded above by w, then for any $\epsilon>0$ we can find an $h\ge 0$ in U^* such that $\|(|f|-h)^+\|\le\epsilon$ for every $f\in A$. Now $\sum_{i=0}^n h(u_i)\le h(w)$ for every n, and $\lim_{n\to 0}h(u_n)=0$; since at the same time

$$|f(u_n)|\le|f|(u_n)\le h(u_n)+(|f|-h)^+(u_n)\le h(u_n)+\epsilon\|u_n\|\le h(u_n)+\epsilon\|w\|$$

for every $f\in A$ and $n\in\mathbb{N}$, $\limsup_{n\to\infty}\sup_{f\in A}|f|(u_n)\le\epsilon\|w\|$. As ϵ is arbitrary,

$$\lim_{n\to\infty}\sup_{f\in A}|f|(u_n)=0,$$

and the conditions are satisfied.

(b)(i) Now suppose that A is norm-bounded but not uniformly integrable. Write B for the solid hull of A, M for $\sup_{f\in A}\|f\|=\sup_{f\in B}\|f\|$; then there is a disjoint sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in $B\cap(U^*)^+$ which is not norm-convergent to 0 (354R(b-iv)), that is,

$$\delta=\tfrac{1}{2}\limsup_{n\to\infty}g_n(e)=\tfrac{1}{2}\limsup_{n\to\infty}\|g_n\|>0,$$

where e is the standard order unit of U.

(ii) Set

$$C=\{v:0\le v\le e,\ \sup_{g\in B}g(v)\ge\delta\},$$

$$D=\{w:0\le w\le e,\ \limsup_{n\to\infty}g_n(w)>\delta\}.$$

Then for any $u\in D$ we can find $v\in C$ and $w\in D$ such that $v\wedge w=0$. **P** Set $\delta'=\limsup_{n\to\infty}g_n(u)$, $\eta=(\delta'-\delta)/(3+M)>0$; take $k\in\mathbb{N}$ so large that $k\eta\ge M$.

Because $g_n(u)\ge\delta'-\eta$ for infinitely many n, we can find a set $K\subseteq\mathbb{N}$, of size k, such that $g_i(u)\ge\delta'-\eta$ for every $i\in K$. Now we know that, for each $i\in K$, $g_i\wedge k\sum_{j\in K,j\ne i}g_j=0$, so there is a $v_i\le u$ such that $g_i(u-v_i)+k\sum_{j\in K,j\ne i}g_j(v_i)\le\eta$ (355Ec). Now

$$g_i(v_i)\ge g_i(u)-\eta\ge\delta'-2\eta,\quad g_i(v_j)\le\frac{\eta}{k}\text{ for }i,\,j\in K,\,i\ne j.$$

Set $v'_i=(v_i-\sum_{j\in K,j\ne i}v_j)^+$ for each $i\in K$; then

$$g_i(v'_i)\ge g_i(v_i)-\sum_{j\in K,j\ne i}g_i(v_j)\ge\delta'-3\eta$$

for every $i\in K$, while $v'_j\wedge v'_i=0$ for distinct $i,\,j\in K$.

For each $n\in\mathbb{N}$,

$$\sum_{i\in K}g_n(u\wedge\frac{1}{\eta}v'_i)\le g_n(u)\le\|g_n\|\le\eta k,$$

so there is some $i(n)\in K$ such that

$$g_n(u\wedge\frac{1}{\eta}v'_{i(n)})\le\eta,\quad g_n(u-\frac{1}{\eta}v'_{i(n)})^+\ge g_n(u)-\eta.$$

Since $\{n:g_n(u)\ge\delta+2\eta\}$ is infinite, there is some $m\in K$ such that $J=\{n:g_n(u)\ge\delta+2\eta,\ i(n)=m\}$ is infinite. Try

$$v=(v'_m-\eta u)^+,\quad w=(u-\frac{1}{\eta}v'_m)^+.$$

Then v, $w\in[0,u]$ and $v\wedge w=0$. Next,

$$g_m(v)\ge g_m(v'_m)-\eta M\ge\delta'-3\eta-\eta M=\delta,$$

so $v\in C$, while for any $n\in J$

$$g_n(w)=g_n(u-\frac{1}{\eta}v'_{i(n)})^+\ge g_n(u)-\eta\ge\delta+\eta;$$

since J is infinite,

$$\limsup_{n\to\infty}g_n(w)\ge\delta+\eta>\delta$$

and $w\in D$. **Q**

(iii) Since $e\in D$, we can choose inductively sequences $\langle w_n\rangle_{n\in\mathbb{N}}$ in D, $\langle v_n\rangle_{n\in\mathbb{N}}$ in C such that $w_0=e$, $v_n\wedge w_{n+1}=$

0, $v_n \vee w_{n+1} \le w_n$ for every $n \in \mathbb{N}$. But in this case $\langle v_n\rangle_{n\in\mathbb{N}}$ is a disjoint order-bounded sequence in $[0,u]$, while for each $n \in \mathbb{N}$, we can find $f_n \in A$ such that $|f_n|(v_n) > \frac{2}{3}\delta$. Now there is a $u_n \in [0, v_n]$ such that $|f_n(u_n)| \ge \frac{1}{3}\delta$. **P** Set $\gamma = \sup_{0\le v\le v_n} |f_n(v)|$. Then $f_n^+(v_n)$, $f_n^-(v_n)$ are both less than or equal to γ, so $|f_n|(v_n) \le 2\gamma$ and $\gamma > \frac{1}{3}\delta$; so there is a $u_n \in [0, v_n]$ such that $|f_n(u_n)| \ge \frac{1}{3}\delta$. **Q**

Accordingly we have a disjoint sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $[0,e]$ such that $\sup_{f\in A} |f(u_n)| \ge \frac{1}{3}\delta$ for every $n \in \mathbb{N}$.

(iv) All this is on the assumption that A is norm-bounded and not uniformly integrable. So, turning it round, we see that if A is norm-bounded and $\lim_{n\to\infty} \sup_{f\in A} |f(u_n)| = 0$ for every order-bounded disjoint sequence $\langle u_n\rangle_{n\in\mathbb{N}}$, A must be uniformly integrable.

This completes the proof.

356P Proposition Let U be an L-space.

(a) U is perfect.

(b) $U^* = U^\sim = U^\times$ is an M-space; its standard order unit is the functional $\int$ defined by setting $\int u = \|u^+\| - \|u^-\|$ for every $u \in U$.

(c) If $A \subseteq U$ is non-empty and upwards-directed and $\sup_{u\in A} \int u$ is finite, then $\sup A$ is defined in U and $\int \sup A = \sup_{u\in A} \int u$.

proof (a) By 354N we know that the norm on U is order-continuous and has the Levi property, so 356M tells us that U is perfect.

(b) 356Dd tells us that $U^* = U^\sim = U^\times$.

The L-space property tells us that the functional $u \mapsto \|u\| : U^+ \to \mathbb{R}$ is additive; of course it is also homogeneous, so by 355D it has an extension to a linear functional $\int : U \to \mathbb{R}$ satisfying the given formula. Because $\int u = \|u\| \ge 0$ for $u \ge 0$, $\int \in (U^\sim)^+$. For $f \in U^\sim$,

$$\begin{aligned} |f| \le \int &\iff |f|(u) \le \int u \text{ for every } u \in U^+ \\ &\iff |f(v)| \le \|u\| \text{ whenever } |v| \le u \in U \\ &\iff |f(v)| \le \|v\| \text{ for every } v \in U \\ &\iff \|f\| \le 1, \end{aligned}$$

so the norm on $U^* = U^\sim$ is the order-unit norm defined from $\int$, and $U^\sim$ is an M-space, as claimed.

(c) Fix $u_0 \in A$, and set $B = \{u^+ : u \in A,\ u \ge u_0\}$. Then $B \subseteq U^+$ is upwards-directed, and

$$\begin{aligned} \sup_{v\in B} \|v\| &= \sup_{u\in A, u\ge u_0} \int u^+ = \sup_{u\in A, u\ge u_0} \int u + \int u^- \\ &\le \sup_{u\in A, u\ge u_0} \int u + \int u_0^- < \infty. \end{aligned}$$

Because $\| \; \|$ has the Levi property, B is bounded above. But (because A is upwards-directed) every member of A is dominated by some member of B, so A also is bounded above. Because U is Dedekind complete, $\sup A$ is defined in U. Finally, $\int \sup A = \sup_{u\in A} \int u$ because $\int$, being a positive member of $U^\times$, is order-continuous.

356Q Theorem Let U be any L-space. Then a subset of U is uniformly integrable iff it is relatively weakly compact.

proof (a) Let $A \subseteq U$ be a uniformly integrable set.

(i) Suppose that $\mathcal{F}$ is an ultrafilter on X containing A. Then $A \ne \emptyset$. Because A is norm-bounded, $\sup_{u\in A} |f(u)| < \infty$ and $\phi(f) = \lim_{u\to\mathcal{F}} f(u)$ is defined in $\mathbb{R}$ for every $f \in U^*$ (2A3Se).

If f, $g \in U^*$ then

$$\phi(f + g) = \lim_{u\to\mathcal{F}} f(u) + g(u) = \lim_{u\to\mathcal{F}} f(u) + \lim_{u\to\mathcal{F}} g(u) = \phi(f) + \phi(g)$$

(2A3Sf). Similarly,

$$\phi(\alpha f) = \lim_{u\to\mathcal{F}} \alpha f(u) = \alpha\phi(f)$$

whenever $f \in U^*$ and $\alpha \in \mathbb{R}$. Thus $\phi : U^* \to \mathbb{R}$ is linear. Also

$$|\phi(f)| \le \sup_{u\in A} |f(u)| \le \|f\| \sup_{u\in A} \|u\|,$$

so $\phi \in U^{**} = U^{*\sim}$.

(ii) Now the point of this argument is that $\phi \in U^{*\times}$. **P** Suppose that $B \subseteq U^*$ is non-empty and downwards-directed and has infimum 0. Fix $f_0 \in B$. Let $\epsilon > 0$. Then there is a $w \in U^+$ such that $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$, which means that

$$|f(u)| \le |f|(|u|) \le |f|(w) + |f|(|u| - w)^+ \le |f|(w) + \epsilon\|f\|$$

for every $f \in U^*$ and every $u \in A$. Accordingly $|\phi(f)| \le |f|(w) + \epsilon\|f\|$ for every $f \in U^*$. Now $\inf_{f\in B} f(w) = 0$ (using 355Ee, as usual), so there is an $f_1 \in B$ such that $f_1 \le f_0$ and $f_1(w) \le \epsilon$. In this case

$$|\phi|(f_1) = \sup_{|f|\le f_1} |\phi(f)| \le \sup_{|f|\le f_1} |f|(w) + \epsilon\|f\| \le f_1(w) + \epsilon\|f_1\| \le \epsilon(1 + \|f_0\|).$$

As ϵ is arbitrary, $\inf_{f\in B} |\phi|(f) = 0$; as B is arbitrary, $|\phi|$ is order-continuous and $\phi \in U^{*\times}$. **Q**

(iii) At this point, we recall that $U^* = U^\times$ and that the canonical map from U to $U^{\times\times}$ is surjective (356P). So there is a $u_0 \in U$ such that $\hat{u}_0 = \phi$. But now we see that

$$f(u_0) = \phi(f) = \lim_{u\to\mathcal{F}} f(u)$$

for every $f \in U^*$; which is just what is meant by saying that $\mathcal{F} \to u_0$ for the weak topology on U (2A3Sd).

Accordingly every ultrafilter on U containing A has a limit in U. But because the weak topology on U is regular (3A3Be), it follows that the closure of A for the weak topology is compact (3A3De), so that A is relatively weakly compact.

(b) For the converse I use the criterion of 354R(b-iv). Suppose that $A \subseteq U$ is relatively weakly compact. Then A is norm-bounded, by the Uniform Boundedness Theorem. Now let $\langle u_n\rangle_{n\in\mathbb{N}}$ be any disjoint sequence in the solid hull of A. For each n, let U_n be the band in U generated by u_n. Let P_n be the band projection from U onto U_n (353Hb). Let $v_n \in A$ be such that $|u_n| \le |v_n|$; then

$$|u_n| = P_n|u_n| \le P_n|v_n| = |P_n v_n|,$$

so $\|u_n\| \le \|P_n v_n\|$ for each n. Let $g_n \in U^*$ be such that $\|g_n\| = 1$ and $g_n(P_n v_n) = \|P_n v_n\|$.

Define $T : U \to \mathbb{R}^{\mathbb{N}}$ by setting $Tu = \langle g_n(P_n u)\rangle_{n\in\mathbb{N}}$ for each $u \in U$. Then T is a continuous linear operator from U to ℓ^1. **P** For $m \ne n$, $U_m \cap U_n = \{0\}$, because $|u_m| \wedge |u_n| = 0$. So, for any $u \in U$, $\langle P_n u\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in U, and

$$\sum_{i=0}^n \|P_i u\| = \|\sum_{i=0}^n |P_i u|\| = \|\sup_{i\le n} |P_i u|\| \le \|u\|$$

for every n; accordingly

$$\|Tu\|_1 = \sum_{i=0}^\infty |g_i P_i u| \le \sum_{i=0}^\infty \|P_i u\| \le \|u\|.$$

Since T is certainly a linear operator (because every coordinate functional $g_i P_i$ is linear), we have the result. **Q**

Consequently $T[A]$ is relatively weakly compact in ℓ^1, because T is continuous for the weak topologies (2A5If). But ℓ^1 can be identified with $L^1(\mu)$, where μ is counting measure on $\mathbb{N}$. So $T[A]$ is uniformly integrable in ℓ^1, by 247C, and in particular $\lim_{n\to\infty} \sup_{w\in T[A]} |w(n)| = 0$. But this means that

$$\lim_{n\to\infty} \|u_n\| \le \lim_{n\to\infty} |g_n(P_n v_n)| = \lim_{n\to\infty} |(Tv_n)(n)| = 0.$$

As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, A satisfies the conditions of 354R(b-iv) and is uniformly integrable.

356X Basic exercises (a) Show that if $U = \ell^\infty$ then $U^\times = U_c^\sim$ can be identified with ℓ^1, and is properly included in $U^\sim$. (*Hint*: show that if $f \in U_c^\sim$ then $f(u) = \sum_{n=0}^\infty u(n) f(e_n)$, where $e_n(n) = 1$, $e_n(i) = 0$ for $i \ne n$.)

(b) Show that if $U = C([0, 1])$ then $U^\times = U_c^\sim = \{0\}$. (*Hint*: show that if $f \in (U_c^\sim)^+$ and $\langle q_n\rangle_{n\in\mathbb{N}}$ enumerates $\mathbb{Q} \cap [0, 1]$, then for each $n \in \mathbb{N}$ there is a $u_n \in U^+$ such that $u_n(q_n) = 1$ and $f(u_n) \le 2^{-n}$.)

(c) Let X be an uncountable set, μ the countable-cocountable measure on X and Σ its domain (211R). Let U be the space of bounded Σ-measurable real-valued functions on X. Show that U is a Dedekind σ-complete Banach lattice if given the supremum norm $\| \|_\infty$. Show that $U^\times$ can be identified with $\ell^1(X)$ (cf. 356Xa), and that $u \mapsto \int u\,d\mu$ belongs to $U_c^\sim \setminus U^\times$.

(d) Let U be a Dedekind σ-complete Riesz space and $f \in U_c^\sim$. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be an order-bounded sequence in U which is order-convergent to $u \in U$ in the sense that $u = \inf_{n\in\mathbb{N}} \sup_{m\ge n} u_m = \sup_{n\in\mathbb{N}} \inf_{m\ge n} u_m$. Show that $\lim_{n\to\infty} f(u_n)$ exists and is equal to $f(u)$.

(e) Let U be any Riesz space. Show that the band projection $P : U^\sim \to U^\times$ is defined by the formula

$$(Pf)(u) = \inf\{\sup_{v\in A} f(v) : A \subseteq U \text{ is non-empty, upwards-directed and has supremum } u\}$$

for every $f \in (U^\sim)^+$, $u \in U^+$. (*Hint*: show that the formula for Pf always defines an order-continuous linear functional. Compare 355Yh, 356Yb and 362Bd.)

(f) Let U be any Riesz space. Show that the band projection $P : U^\sim \to U^\sim_c$ is defined by the formula

$$(Pf)(u) = \inf\{\sup_{n\in\mathbb{N}} f(v_n) : \langle v_n\rangle_{n\in\mathbb{N}} \text{ is a non-decreasing sequence with supremum } u\}$$

for every $f \in (U^\sim)^+$, $u \in U^+$.

(g) Let U be a Riesz space with a Riesz norm. Show that U^* is perfect.

(h) Let U be a Riesz space with a Riesz norm. Show that the canonical map from U to U^{**} is a Riesz homomorphism.

(i) Let V be a perfect Riesz space and U any Riesz space. Show that $\mathcal{L}^\sim(U;V)$ is perfect. (*Hint*: show that if $u \in U$ and $g \in V^\times$ then $T \mapsto g(Tu)$ belongs to $\mathcal{L}^\sim(U;V)^\times$.)

(j) Let U be an M-space. Show that it is perfect iff it is Dedekind complete and $U^\times$ separates the points of U.

(k) Let U be a Banach lattice which, as a Riesz space, is perfect. Show that its norm has the Levi property.

(l) Write out a proof from first principles that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in ℓ^1 such that $|u_n(n)| \ge \delta > 0$ for every $n \in \mathbb{N}$, then $\{u_n : n \in \mathbb{N}\}$ is not relatively weakly compact.

(m) Let U be an L-space and $A \subseteq U$ a non-empty set. Show that the following are equiveridical: (i) A is uniformly integrable (ii) $\inf_{f\in B} \sup_{u\in A} |f(u)|$ for every non-empty downwards-directed set $B \subseteq U^\times$ with infimum 0 (iii) $\inf_{n\in\mathbb{N}} \sup_{u\in A} |f_n(u)| = 0$ for every non-increasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $U^\times$ with infimum 0 (iv) A is norm-bounded and $\lim_{n\to\infty} \sup_{u\in A} |f_n(u)| = 0$ for every disjoint order-bounded sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $U^\times$.

356Y Further exercises (a) Let U be a Riesz space with the countable sup property. Show that $U^\times = U^\sim_c$.

(b) Let U be a Riesz space, and $\mathcal{A}$ a family of non-empty downwards-directed subsets of U^+ all with infimum 0. (i) Show that $U^\sim_{\mathcal{A}} = \{f : f \in U^\sim, \inf_{u\in A} |f|(u) = 0 \text{ for every } A \in \mathcal{A}\}$ is a band in $U^\sim$. (ii) Set $\mathcal{A}^* = \{A_0 + \ldots + A_n : A_0, \ldots, A_n \in \mathcal{A}\}$. Show that $U^\sim_{\mathcal{A}} = U^\sim_{\mathcal{A}^*}$. (iii) Take any $f \in (U^\sim)^+$, and let g, h be the components of f in $U^\sim_{\mathcal{A}}$, $(U^\sim_{\mathcal{A}})^\perp$ respectively. Show that

$$g(u) = \inf_{A\in\mathcal{A}^*} \sup_{v\in A} f(u-v)^+, \quad h(u) = \sup_{A\in\mathcal{A}^*} \inf_{v\in A} f(u \wedge v)$$

for every $u \in U^+$. (Cf. 362Xi.)

(c) Let U be a Riesz space. For any band $V \subseteq U$ write V° for $\{f : f \in U^\times, f(v) \le 1 \text{ for every } v \in V\} = \{f : f \in U^\times, f(v) = 0 \text{ for every } v \in V\}$. Show that $V \mapsto (V^\perp)^\circ$ is a surjective order-continuous Boolean homomorphism from the algebra of complemented bands of U onto the band algebra of $U^\times$, and that it is injective iff $U^\times$ separates the points of U.

(d) Let U be a Dedekind complete Riesz space such that $U^\times$ separates the points of U and U is the solid linear subspace of itself generated by a countable set. Show that U is perfect.

(e) Let U be an L-space and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in U such that $\langle f(u_n)\rangle_{n\in\mathbb{N}}$ is Cauchy for every $f \in U^*$. Show that $\langle u_n\rangle_{n\in\mathbb{N}}$ is convergent for the weak topology of U. (*Hint*: use 356Xm(iv) to show that $\{u_n : n \in \mathbb{N}\}$ is relatively weakly compact.)

(f) Let U be a perfect Banach lattice with order-continuous norm and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in U such that $\langle f(u_n)\rangle_{n\in\mathbb{N}}$ is Cauchy for every $f \in U^*$. Show that $\langle u_n\rangle_{n\in\mathbb{N}}$ is convergent for the weak topology of U. (*Hint*: set $\phi(f) = \lim_{n\to\infty} f_n(u)$. For any $g \in (U^*)^+$ let V_g be the solid linear subspace of U^* generated by g, $W_g = \{u : g(|u|) = 0\}^\perp$, $\|u\|_g = g(|u|)$ for $u \in W_g$. Show that the completion of W_g under $\| \|_g$ is an L-space with dual isomorphic to V_g, and hence (using 356Ye) that $\phi\restriction V_g$ belongs to $V_g^\times$; as g is arbitrary, $\phi \in V^\times$ and may be identified with an element of U.)

(g) Let U be a uniformly complete Archimedean Riesz space with complexification V (354Yl). (i) Show that the complexification of $U^\sim$ can be identified with the space of linear functionals $f : V \to \mathbb{C}$ such that $\sup_{|v|\le u} |f(v)|$ is finite for every $u \in U^+$. (ii) Show that if U is a Banach lattice, then the complexification of $U^\sim = U^*$ can be identified (as normed space) with V^*. (See 355Yk.)

(h) Let U be a perfect Banach lattice. Show that the family of closed balls in U is a compact class. (*Hint*: 342Ya.)

356 Notes and comments The section starts easily enough, with special cases of results in §355 (356B). When U has a Riesz norm, the identification of U^* as a subspace of $U^\sim$, and the characterization of order-continuous norms (356D) are pleasingly comprehensive and straightforward. Coming to biduals, we need to think a little (356F), but there is still no real difficulty at first. In 356H-356I, however, something more substantial is happening. I have written these arguments out in what seems to be the shortest route to the main theorem, at the cost perhaps of neglecting any intuitive foundation. What I think we are really doing is matching bands in U, $U^\times$ and $U^{\times\times}$, as in 356Yc.

From now on, almost the first thing we shall ask of any new Riesz space will be whether it is perfect, and if not, which of the three conditions of 356K it fails to satisfy. For reasons which will I hope appear in the next chapter, perfect Riesz spaces are especially important in measure theory; in particular, all L^p spaces for $p \in [1, \infty[$ are perfect (366Dd), as are the L^∞ spaces of localizable measure spaces (365N). Further examples will be discussed in §369 and §374. Of course we have to remember that there are also important Riesz spaces which are not perfect, of which $C([0,1])$ and $\boldsymbol{c}_0$ are two of the simplest examples.

The duality between L- and M-spaces (356N, 356P) is natural and satisfying. We are now in a position to make a determined attempt to tidy up the notion of 'uniform integrability'. I give two major theorems. The first is yet another 'disjoint-sequence' characterization of uniformly integrable sets, to go with 246G and 354R. The essential difference here is that we are looking at disjoint sequences in a predual; in a sense, this means that the result is a sharper one, because the M-space U need not be Dedekind complete (for instance, it could be $C([0,1])$ – this indeed is the archetype for applications of the theorem) and therefore need not have as many disjoint sequences as its dual. (For instance, in the dual of $C([0,1])$ we have all the point masses δ_t, where $\delta_t(u) = u(t)$; these form a disjoint family in $C([0,1])^\sim$ not corresponding to any disjoint family in $C([0,1])$.) The essence of the proof is a device to extract a disjoint sequence in U to match approximately a subsequence of a given disjoint sequence in $U^\sim$. In the example just suggested, this would correspond, given a sequence $\langle t_n \rangle_{n\in\mathbb{N}}$ of distinct points in $[0,1]$, to finding a subsequence $\langle t_{n(i)} \rangle_{i\in\mathbb{N}}$ which is discrete, so that we can find disjoint $u_i \in C([0,1])$ with $u_i(t_{n(i)}) = 1$ for each i.

The second theorem, 356Q, is a new version of a result already given in §247: in any L-space, uniform integrability is the same as relative weak compactness. I hope you are not exasperated by having been asked, in Volume 2, to master a complex argument (one of the more difficult sections of that volume) which was going to be superseded. Actually it is worse than that. A theorem of Kakutani (369E) tells us that every L-space is isomorphic to an L^1 space. So 356Q is itself a consequence of 247C. I do at least owe you an explanation for writing out two proofs. The first point is that the result is sufficiently important for it to be well worth while spending time in its neighbourhood, and the contrasts and similarities between the two arguments are instructive. The second is that the proof I have just given was not really accessible at the level of Volume 2. It does not rely on every single page of this chapter, but the key idea (that U is isomorphic to $U^{\times\times}$, so it will be enough if we can show that A is relatively compact in $U^{\times\times}$) depends essentially on 356I, which lies pretty deep in the abstract theory of Riesz spaces. The third is an aesthetic one: a theorem about L-spaces ought to be proved in the category of normed Riesz spaces, without calling on a large body of theory outside. Of course this is a book on measure theory, so I did the measure theory first, but if you look at everything that went into it, the proof in §247 is I believe longer, in the formal sense, than the one here, even setting aside the labour of proving Kakutani's theorem.

Let us examine the ideas in the two proofs. First, concerning the proof that uniformly integrable sets are relatively compact, the method here is very smooth and natural; the definition I chose of 'uniform integrability' is exactly adapted to showing that uniformly integrable sets are relatively compact in the order-continuous bidual; all the effort goes into the proof that L-spaces are perfect. The previous argument depended on identifying the dual of L^1 as L^∞ – and was disagreeably complicated by the fact that the identification is not always valid, so that I needed to reduce the problem to the σ-finite case (part (b-ii) of the proof of 247C). After that, the Radon-Nikodým theorem did the trick. Actually Kakutani's theorem shows that the side-step to σ-finite spaces is irrelevant. It directly represents an abstract L-space as $L^1(\mu)$ for a localizable measure μ, in which case $(L^1)^* \cong L^\infty$ exactly.

In the other direction, both arguments depend on a disjoint-sequence criterion for uniform integrability (246G(iii) or 354R(b-iv)). These criteria belong to the 'easy' side of the topic; straightforward Riesz space arguments do the job, whether written out in that language or not. (Of course the new one in this section, 356O, lies a little deeper.) I go a bit faster this time because I feel that you ought by now to be happy with the Hahn-Banach theorem and the Uniform Boundedness Theorem, which I was avoiding in Volume 2. And then of course I quote the result for ℓ^1. This looks like cheating. But ℓ^1 really is easier, as you will find if you just write out part (a) of the proof of 247C for this case. It is not exactly that you can dispense with any particular element of the argument; rather it is that the formulae become much more direct when you can write $u(i)$ in place of $\int_{F_i} u$, and 'cluster points for the weak topology' become pointwise limits of subsequences, so that the key step (the 'sliding hump', in which $u_{k(j)}(n(k(j)))$ is the only significant coordinate of $u_{k(j)}$), is easier to find.

We now have a wide enough variety of conditions equivalent to uniform integrability for it to be easy to find others; I give a couple in 356Xm, corresponding in a way to those in 246G. You may have noticed, in the proof of 247C, that in fact the full strength of the hypothesis 'relatively weakly compact' is never used; all that is demanded is that a couple of sequences should have cluster points for the weak topology. So we see that a set A is uniformly integrable iff every sequence in A has a weak cluster point. But this extra refinement is nothing to do with L-spaces; it is generally true, in any normed space U, that a set $A \subseteq U$ is relatively weakly compact iff every sequence in A has a cluster point in U for the weak topology ('Eberlein's theorem'; see 462D in Volume 4, Köthe 69, 24.2.1, or Dunford & Schwartz 57, V.6.1).

There is a very rich theory concerning weak compactness in perfect Riesz spaces, based on the ideas here; some of it is explored in Fremlin 74a. As a sample, I give one of the basic properties of perfect Banach lattices with order-continuous norms: they are 'weakly sequentially complete' (356Yf).

Chapter 36

Function Spaces

Chapter 24 of Volume 2 was devoted to the elementary theory of the 'function spaces' L^0, L^1, L^2 and L^∞ associated with a given measure space. In this chapter I return to these spaces to show how they can be related to the more abstract themes of the present volume. In particular, I develop constructions to demonstrate, as clearly as I can, the way in which all the function spaces associated with a measure space in fact depend only on its measure algebra; and how many of their features can (in my view) best be understood in terms of constructions involving measure algebras.

The chapter is very long, not because there are many essentially new ideas, but because the intuitions I seek to develop depend, for their logical foundations, on technically complex arguments. This is perhaps best exemplified by §364. If two measure spaces (X,Σ,μ) and (Y,T,ν) have isomorphic measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ then the spaces $L^0(\mu)$, $L^0(\nu)$ are isomorphic as topological f-algebras; and more: for any isomorphism between $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ there is a unique corresponding isomorphism between the L^0 spaces. The intuition involved is in a way very simple. If f, g are measurable real-valued functions on X and Y respectively, then $f^\bullet\in L^0(\mu)$ will correspond to $g^\bullet\in L^0(\nu)$ if and only if $[\![f^\bullet>\alpha]\!]=\{x:f(x)>\alpha\}^\bullet\in\mathfrak{A}$ corresponds to $[\![g^\bullet>\alpha]\!]=\{y:g(y)>\alpha\}^\bullet\in\mathfrak{B}$ for every α. But the check that this formula is consistent, and defines an isomorphism of the required kind, involves a good deal of detailed work. It turns out, in fact, that the *measures* μ and ν do not enter this part of the argument at all, except through their ideals of negligible sets (used in the construction of $\mathfrak{A}$ and $\mathfrak{B}$). This is already evident, if you look for it, in the theory of $L^0(\mu)$; in §241, as written out, you will find that the measure of an individual set is not once mentioned, except in the exercises. Consequently there is an invitation to develop the theory with algebras $\mathfrak{A}$ which are not necessarily measure algebras. Here is another reason for the length of the chapter; substantial parts of the work are being done in greater generality than the corresponding sections of Chapter 24, necessitating a degree of repetition. Of course this is not 'measure theory' in the strict sense; but for thirty years now measure theory has been coloured by the existence of these generalizations, and I think it is useful to understand which parts of the theory apply only to measure algebras, and which can be extended to other σ-complete Boolean algebras, like the algebraic theory of L^0, or even to all Boolean algebras, like the theory of L^∞.

Here, then, are two of the objectives of this chapter: first, to express the ideas of Chapter 24 in ways making explicit their independence of particular measure spaces, by setting up constructions based exclusively on the measure algebras involved; second, to set out some natural generalizations to other algebras. But to justify the effort needed I ought to point to some mathematically significant idea which demands these constructions for its expression, and here I mention the categorical nature of the constructions. Between Boolean algebras we have a variety of natural and important classes of 'morphism'; for instance, the Boolean homomorphisms and the order-continuous Boolean homomorphisms; while between measure algebras we have in addition the measure-preserving Boolean homomorphisms. Now it turns out that if we construct the L^p spaces in the natural ways then morphisms between the underlying algebras give rise to morphisms between their L^p spaces. For instance, any Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces a multiplicative norm-contractive Riesz homomorphism from $L^\infty(\mathfrak{A})$ to $L^\infty(\mathfrak{B})$; if $\mathfrak{A}$ and $\mathfrak{B}$ are Dedekind σ-complete, then any sequentially order-continuous Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces a sequentially order-continuous multiplicative Riesz homomorphism from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$; and if $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are measure algebras, then any measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces norm-preserving Riesz homomorphisms from $L^p(\mathfrak{A},\bar\mu)$ to $L^p(\mathfrak{B},\bar\nu)$ for every $p\in[1,\infty]$. All of these are 'functors', that is, a composition of homomorphisms between algebras gives rise to a composition of the corresponding operators between their function spaces, and are 'covariant', that is, a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ leads to an operator from $L^p(\mathfrak{A})$ to $L^p(\mathfrak{B})$. But the same constructions lead us to a functor which is 'contravariant': starting from an order-continuous Boolean homomorphism from a semi-finite measure algebra $(\mathfrak{A},\bar\mu)$ to a measure algebra $(\mathfrak{B},\bar\nu)$, we have an operator from $L^1(\mathfrak{B},\bar\nu)$ to $L^1(\mathfrak{A},\bar\mu)$. This last is in fact a kind of conditional expectation operator. In my view it is not possible to make sense of the theory of measure-preserving transformations without at least an intuitive grasp of these ideas.

Another theme is the characterization of each construction in terms of universal mapping theorems: for instance, each L^p space, for $1\le p\le\infty$, can be characterized as Banach lattice in terms of factorizations of functions of an appropriate class from the underlying algebra to Banach lattices.

Now let me try to sketch a route-map for the journey ahead. I begin with two sections on the space $S(\mathfrak{A})$; this construction applies to any Boolean algebra (indeed, any Boolean ring), and corresponds to the space of 'simple functions' on a measure space. Just because it is especially close to the algebra (or ring) $\mathfrak{A}$, there is a particularly large number of universal mapping theorems corresponding to different aspects of its structure (§361). In §362 I seek to relate ideas on additive functionals on Boolean algebras from Chapter 23 and §§326-327 to the theory of Riesz space duals in §356. I then turn to the systematic discussion of the function spaces of Chapter 24: L^∞ (§363), L^0 (§364),

L^1 (§365) and other L^p (§366), followed by an account of convergence in measure (§367). While all these sections are dominated by the objectives sketched in the paragraphs above, I do include a few major theorems not covered by the ideas of Volume 2, such as the Kelley-Nachbin characterization of the Banach spaces $L^\infty(\mathfrak{A})$ for Dedekind complete $\mathfrak{A}$ (363R). In the last two sections of the chapter I turn to the use of L^0 spaces in the representation of Archimedean Riesz spaces (§368) and of Banach lattices separated by their order-continuous duals (§369).

361 *S*

This is the fundamental Riesz space associated with a Boolean ring $\mathfrak{A}$. When $\mathfrak{A}$ is a ring of sets, $S(\mathfrak{A})$ can be regarded as the linear space of 'simple functions' generated by the indicator functions of members of $\mathfrak{A}$ (361L). Its most important property is the universal mapping theorem 361F, which establishes a one-to-one correspondence between (finitely) additive functions on $\mathfrak{A}$ (361B-361C) and linear operators on $S(\mathfrak{A})$. Simple universal mapping theorems of this type can be interesting, but do not by themselves lead to new insights; what makes this one important is the fact that $S(\mathfrak{A})$ has a canonical Riesz space structure, norm and multiplication (361E). From this we can deduce universal mapping theorems for many other classes of function (361G, 361H, 361I, 361Xb). (Particularly important are countably additive and completely additive real-valued functionals, which will be dealt with in the next section.) While the exact construction of $S(\mathfrak{A})$ (and the associated map from $\mathfrak{A}$ to $S(\mathfrak{A})$) can be varied (361D, 361L, 361M, 361Ya), its structure is uniquely defined, so homomorphisms between Boolean rings correspond to maps between their $S(\,)$-spaces (361J), and (when $\mathfrak{A}$ is a Boolean algebra) $\mathfrak{A}$ can be recovered from the Riesz space $S(\mathfrak{A})$ as the algebra of its projection bands (361K).

361A Boolean rings In this section I speak of Boolean *rings* rather than *algebras*; there are ideas in §365 below which are more naturally expressed in terms of the ring of elements of finite measure in a measure algebra than in terms of the whole algebra. I should perhaps therefore recall some of the ideas of §311, which is the last time when Boolean rings without identity were mentioned, and set out some simple facts.

(a) Any Boolean ring $\mathfrak{A}$ can be represented as the ring of compact open subsets of its Stone space Z, which is a zero-dimensional locally compact Hausdorff space (311I); Z is just the set of surjective ring homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$ (311E).

(b) If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a function, then the following are equiveridical: (i) π is a ring homomorphism; (ii) $\pi(a \setminus b) = \pi a \setminus \pi b$ for all a, $b \in \mathfrak{A}$; (iii) π is a lattice homomorphism and $\pi 0 = 0$. **P** See 312H. To prove (ii)⇒(iii), observe that if a, $b \in \mathfrak{A}$ then

$$\pi(a \cap b) = \pi a \setminus \pi(a \setminus b) = \pi a \setminus (\pi a \setminus \pi b) = \pi a \cap \pi b,$$

$$\pi a = \pi((a \cup b) \cap a) = \pi(a \cup b) \cap \pi a \subseteq \pi(a \cup b),$$

$$\pi(b \setminus a) = \pi((a \cup b) \setminus a) = \pi(a \cup b) \setminus \pi a,$$

$$\pi(a \cup b) = \pi a \cup \pi(b \setminus a) = \pi a \cup (\pi b \setminus \pi a) = \pi a \cup \pi b. \textbf{ Q}$$

(c) If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a ring homomorphism, then π is order-continuous iff $\inf \pi[A] = 0$ whenever $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and $\inf A = 0$ in $\mathfrak{A}$; while π is sequentially order-continuous iff $\inf_{n\in\mathbb{N}} \pi a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0. (See 313L.)

(d) The following will be a particularly important type of Boolean ring for us. If $(\mathfrak{A}, \bar\mu)$ is a measure algebra, then the ideal $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar\mu a < \infty\}$ is a Boolean ring in its own right. Now suppose that $(\mathfrak{B}, \bar\nu)$ is another measure algebra and $\mathfrak{B}^f \subseteq \mathfrak{B}$ the corresponding ring of elements of finite measure. We can say that a ring homomorphism $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ is **measure-preserving** if $\bar\nu\pi a = \bar\mu a$ for every $a \in \mathfrak{A}^f$. In this case π is order-continuous. **P** If $A \subseteq \mathfrak{A}^f$ is non-empty, downwards-directed and has infimum 0, then $\inf_{a\in A} \bar\mu a = 0$, by 321F; but this means that $\inf_{a\in A} \bar\nu\pi a = 0$, and $\inf \pi[A] = 0$ in $\mathfrak{B}^f$. **Q**

361B Definition Let $\mathfrak{A}$ be a Boolean ring and U a linear space. A function $\nu : \mathfrak{A} \to U$ is **finitely additive**, or just **additive**, if $\nu(a \cup b) = \nu a + \nu b$ whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0$.

361C Elementary facts We have the following immediate consequences of this definition, corresponding to 326B and 313L. Let $\mathfrak{A}$ be a Boolean ring, U a linear space and $\nu : \mathfrak{A} \to U$ an additive function.

(a) $\nu 0 = 0$ (because $\nu 0 = \nu 0 + \nu 0$).

(b) If $a_0, \dots , a_m$ are disjoint in $\mathfrak{A}$, then $\nu(\sup_{j\le m} a_j) = \sum_{j=0}^{m} \nu a_j$. (Induce on m.)

(c) If $\mathfrak{B}$ is another Boolean ring and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a ring homomorphism, then $\nu\pi : \mathfrak{B} \to U$ is additive. In particular, if $\mathfrak{B}$ is a subring of $\mathfrak{A}$, then $\nu\restriction\mathfrak{B} : \mathfrak{B} \to U$ is additive.

(d) If V is another linear space and $T : U \to V$ is a linear operator, then $T\nu : \mathfrak{A} \to V$ is additive.

(e) If U is a partially ordered linear space, then ν is order-preserving iff it is non-negative, that is, $\nu a \ge 0$ for every $a \in \mathfrak{A}$. **P** (α) If ν is order-preserving, then of course $0 = \nu 0 \le \nu a$ for every $a \in \mathfrak{A}$. (β) If ν is non-negative, and $a \subseteq b$ in $\mathfrak{A}$, then

$$\nu a \le \nu a + \nu(b \setminus a) = \nu b. \text{ } \mathbf{Q}$$

(f) If U is a partially ordered linear space and ν is non-negative, then (i) ν is order-continuous iff $\inf \nu[A] = 0$ whenever $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum 0 (ii) ν is sequentially order-continuous iff $\inf_{n\in\mathbb{N}} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

P(i) If ν is order-continuous, then of course $\inf \nu[A] = \nu 0 = 0$ whenever $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum 0. If ν satisfies the condition, and $A \subseteq \mathfrak{A}$ is a non-empty upwards-directed set with supremum c, then $\{c \setminus a : a \in \mathfrak{A}\}$ is downwards-directed with infimum 0 (313Aa), so that

$$\sup_{a\in A} \nu a = \sup_{a\in A} \nu c - \nu(c \setminus a) = \nu c - \inf_{a\in A} \nu(c \setminus a)$$

(by 351Db)

$$= \nu c.$$

Similarly, if $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum c, then

$$\inf_{a\in A} \nu a = \inf_{a\in A} \nu c + \nu(a \setminus c) = \nu c + \inf_{a\in A} \nu(a \setminus c) = \nu c.$$

Putting these together, ν is order-continuous.

(ii) If ν is sequentially order-continuous, then of course $\inf_{n\in\mathbb{N}} \nu a_n = \nu 0 = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0. If ν satisfies the condition, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum c, then $\langle c \setminus a_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0, so that

$$\sup_{n\in\mathbb{N}} \nu a_n = \sup_{n\in\mathbb{N}} \nu c - \nu(c \setminus a_n) = \nu c - \inf_{n\in\mathbb{N}} \nu(c \setminus a_n) = \nu c.$$

Similarly, if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum c, then $\langle a_n \setminus c\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0, so that

$$\inf_{n\in\mathbb{N}} \nu a_n = \inf_{n\in\mathbb{N}} \nu c + \nu(c \setminus a_n) = \nu c + \inf_{n\in\mathbb{N}} \nu(c \setminus a_n) = \nu c.$$

Thus ν is sequentially order-continuous. **Q**

361D Construction Let $\mathfrak{A}$ be a Boolean ring, and Z its Stone space. For $a \in \mathfrak{A}$ write χa for the indicator function of the open-and-compact subset $\widehat{a}$ of Z corresponding to a. Note that $\chi a = 0$ iff $a = 0$. Let $S(\mathfrak{A})$ be the linear subspace of $\mathbb{R}^Z$ generated by $\{\chi a : a \in \mathfrak{A}\}$. Because χa is a bounded function for every a, $S(\mathfrak{A})$ is a subspace of the M-space $\ell^\infty(Z)$ of all bounded real-valued functions on Z (354Ha), and $\| \|_\infty$ is a norm on $S(\mathfrak{A})$. Because $\chi a \times \chi b = \chi(a \cap b)$ for all a, $b \in \mathfrak{A}$ (writing $\times$ for pointwise multiplication of functions, as in 281B), $S(\mathfrak{A})$ is closed under $\times$.

361E I give a portmanteau proposition running through the elementary, mostly algebraic, properties of $S(\mathfrak{A})$.

Proposition Let $\mathfrak{A}$ be a Boolean ring, with Stone space Z. Write S for $S(\mathfrak{A})$.

(a) If $a_0,\dots,a_n\in\mathfrak{A}$, there are disjoint $b_0,\dots,b_m$ such that each a_i is expressible as the supremum of some of the b_j.

(b) If $u\in S$, it is expressible in the form $\sum_{j=0}^m\beta_j\chi b_j$ where $b_0,\dots,b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j\in\mathbb{R}$ for each j. If all the b_j are non-zero then $\|u\|_\infty=\sup_{j\le m}|\beta_j|$.

(c) If $u\in S$ is non-negative, it is expressible in the form $\sum_{j=0}^m\beta_j\chi b_j$ where $b_0,\dots,b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j\ge 0$ for each j, and simultaneously in the form $\sum_{j=0}^m\gamma_j\chi c_j$ where $c_0\supseteq c_1\supseteq\dots\supseteq c_m$ and $\gamma_j\ge 0$ for every j.

(d) If $u=\sum_{j=0}^m\beta_j\chi b_j$ where $b_0,\dots,b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j\in\mathbb{R}$ for each j, then $|u|=\sum_{j=0}^m|\beta_j|\chi b_j\in S$.

(e) S is a Riesz subspace of $\mathbb{R}^Z$; in its own right, it is an Archimedean Riesz space. If $\mathfrak{A}$ is a Boolean algebra, then S has an order unit $\chi 1$ and $\|u\|_\infty=\min\{\alpha:\alpha\ge 0,\,|u|\le\alpha\chi 1\}$ for every $u\in S$.

(f) The map $\chi:\mathfrak{A}\to S$ is injective, additive, non-negative, a lattice homomorphism and order-continuous.

(g) Suppose that $u\ge 0$ in S and $\delta\ge 0$ in $\mathbb{R}$. Then

$$[\![u>\delta]\!]=\max\{a:a\in\mathfrak{A},\,(\delta+\eta)\chi a\le u\text{ for some }\eta>0\}$$

is defined in $\mathfrak{A}$, and

$$\delta\chi[\![u>\delta]\!]\le u\le\delta\chi[\![u>0]\!]\vee\|u\|_\infty[\![u>\delta]\!].$$

In particular, $u\le\|u\|_\infty\chi[\![u>0]\!]$ and there is an $\eta>0$ such that $\eta\chi[\![u>0]\!]\le u$. If u, $v\ge 0$ in S then $u\wedge v=0$ iff $[\![u>0]\!]\cap[\![v>0]\!]=0$.

(h) Under $\times$, S is an f-algebra (352W) and a commutative normed algebra (2A4J).

(i) For any $u\in S$, $u\ge 0$ iff $u=v\times v$ for some $v\in S$.

proof Write $\widehat{a}$ for the open-and-compact subset of Z corresponding to $a\in\mathfrak{A}$.

(a) Induce on n. If $n=0$ take $m=0$, $b_0=a_0$. For the inductive step to $n\ge 1$, take disjoint $b_0,\dots,b_m$ such that a_i is the supremum of some of the b_j for each $i<n$; now replace $b_0,\dots,b_m$ with $b_0\cap a_n,\dots,b_m\cap a_n,b_0\setminus a_n,\dots,b_m\setminus a_n$, $a_n\setminus\sup_{j\le m}b_j$ to obtain a suitable string for $a_0,\dots,a_n$.

(b) If $u=0$ set $m=0$, $b_0=0$, $\beta_0=0$. Otherwise, express u as $\sum_{i=0}^n\alpha_i\chi a_i$ where $a_0,\dots,a_n\in\mathfrak{A}$ and $\alpha_0,\dots,\alpha_n$ are real numbers. Let $b_0,\dots,b_m$ be disjoint and such that every a_i is expressible as the supremum of some of the b_j. Set $\gamma_{ij}=1$ if $b_j\subseteq a_i$, 0 otherwise, so that, because the b_j are disjoint, $\chi a_i=\sum_{j=0}^m\gamma_{ij}\chi b_j$ for each i. Then

$$u=\sum_{i=0}^n\alpha_i\chi a_i=\sum_{i=0}^n\sum_{j=0}^m\alpha_i\gamma_{ij}\chi b_j=\sum_{j=0}^m\beta_j\chi b_j,$$

setting $\beta_j=\sum_{i=0}^n\alpha_i\gamma_{ij}$ for each $j\le m$.

The expression for $\|u\|_\infty$ is now obvious.

(c)(i) If $u\ge 0$ in (b), we must have $\beta_j=u(z)\ge 0$ whenever $z\in\widehat{b}_j$, so that $\beta_j\ge 0$ whenever $b_j\ne 0$; consequently $u=\sum_{j=0}^m|\beta_j|\chi b_j$ is in the required form.

(ii) If we suppose that every β_j is non-negative, and rearrange the terms of the sum so that $\beta_0\le\dots\le\beta_m$, then we may set $\gamma_0=\beta_0$, $\gamma_j=\beta_j-\beta_{j-1}$ for $1\le j\le m$, $c_j=\sup_{i\ge j}b_i$ to get

$$\sum_{j=0}^m\gamma_j\chi c_j=\sum_{j=0}^m\sum_{i=j}^m\gamma_j\chi b_i=\sum_{i=0}^m\sum_{j=0}^i\gamma_j\chi b_i=\sum_{i=0}^m\beta_i\chi b_i=u.$$

(d) is trivial, because $\widehat{b}_0,\dots,\widehat{b}_n$ are disjoint.

(e) By (d), $|u|\in S$ for every $u\in S$, so S is a Riesz subspace of $\mathbb{R}^Z$, and in itself is an Archimedean Riesz space. If $\mathfrak{A}$ is a Boolean algebra, then $\chi 1$, the constant function with value 1, belongs to S, and is an order unit of S; while

$$\|u\|_\infty=\min\{\alpha:\alpha\ge 0,\,|u(z)|\le\alpha\ \forall\,z\in Z\}=\min\{\alpha:\alpha\ge 0,\,|u|\le\alpha\chi 1\}$$

for every $u\in S$.

(f) χ is injective because $\widehat{a}\ne\widehat{b}$ whenever $a\ne b$. χ is additive because $\widehat{a}\cap\widehat{b}=\emptyset$ whenever $a\cap b=0$. Of course χ is non-negative. It is a lattice homomorphism because $a\mapsto\widehat{a}:\mathfrak{A}\to\mathcal{P}Z$ and $E\mapsto\chi E:\mathcal{P}Z\to\mathbb{R}^Z$ are. To see that ν is order-continuous, take a non-empty downwards-directed $A\subseteq\mathfrak{A}$ with infimum 0. **?** Suppose, if possible, that $\{\chi a:a\in A\}$ does not have infimum 0 in S. Then there is a $u>0$ in S such that $u\le\chi a$ for every $a\in\mathfrak{A}$. Now u can be expressed as $\sum_{j=0}^m\beta_j\chi b_j$ where $b_0,\dots,b_m$ are disjoint. There must be some $z_0\in Z$ such that $u(z_0)>0$; take j such that $z_0\in\widehat{b}_j$, so that $b_j\ne 0$ and $\beta_j=u(z_0)>0$. But now, for any $z\in\widehat{b}_j$, $a\in A$,

$$(\chi a)(z)\ge u(z)=\beta_j>0$$

and $z \in \widehat{a}$. As z is arbitrary, $\widehat{b}_j \subseteq \widehat{a}$ and $b_j \subseteq a$; as a is arbitrary, b_j is a non-zero lower bound for A in $\mathfrak{A}$. **X** So $\inf \chi[A] = 0$ in S. As A is arbitrary, χ is order-continuous, by the criterion of 361C(f-i).

(g) Express u as $\sum_{j=0}^m \beta_j \chi b_j$ where $b_0, \dots, b_m$ are disjoint and every $\beta_j \ge 0$. Then given $\delta \ge 0$, $\eta > 0$ and $a \in \mathfrak{A}$ we have $(\delta+\eta)\chi a \le u$ iff $a \subseteq \sup\{b_j : j \le m,\, \beta_j \ge \delta+\eta\}$. So $[\![u > \delta]\!] = \sup\{b_j : j \le m,\, \beta_j > \delta\}$. Writing $c = [\![u > \delta]\!]$, $d = [\![u > 0]\!] = \sup\{b_j : \beta_j > 0\}$, we have

$$\begin{aligned} u(z) &\le \|u\|_\infty \text{ if } z \in \widehat{c}, \\ &\le \delta \text{ if } z \in \widehat{d} \setminus \widehat{c}, \\ &= 0 \text{ if } z \in Z \setminus \widehat{d}. \end{aligned}$$

So

$$\delta\chi c \le u \le \|u\|_\infty \chi c \vee \delta\chi d,$$

as claimed. Taking $\delta = 0$ we get $u \le \|u\|_\infty \chi d$. Set

$$\eta = \min(\{1\} \cup \{\beta_j : j \le m,\, \beta_j > 0\});$$

then $\eta > 0$ and $\eta\chi d \le u$.

If u, $v \in S^+$ take η, $\eta' > 0$ such that

$$\eta\chi[\![u > 0]\!] \le u, \quad \eta'\chi[\![v > 0]\!] \le v.$$

Then

$$\min(\eta, \eta')\chi([\![u > 0]\!] \cap [\![v > 0]\!]) \le u \wedge v \le \max(\|u\|_\infty, \|v\|_\infty)\chi([\![u > 0]\!] \cap [\![v > 0]\!]).$$

So

$$u \wedge v = 0 \Longrightarrow [\![u > 0]\!] \cap [\![v > 0]\!] = 0 \Longrightarrow u \wedge v = 0.$$

(h) S is a commutative f-algebra and normed algebra just because it is a Riesz subspace of the f-algebra and commutative normed algebra $\ell^\infty(Z)$ and is closed under multiplication.

(i) If $u = \sum_{j=0}^m \beta_j \chi b_j$ where $b_0, \dots, b_m$ are disjoint and $\beta_j \ge 0$ for every j, then $u = v \times v$ where $v = \sum_{j=0}^m \sqrt{\beta_j} \chi b_j$.

361F I now turn to the universal mapping theorems which really define the construction.

Theorem Let $\mathfrak{A}$ be a Boolean ring, and U any linear space. Then there is a one-to-one correspondence between additive functions $\nu : \mathfrak{A} \to U$ and linear operators $T : S(\mathfrak{A}) \to U$, given by the formula $\nu = T\chi$.

proof (a) The core of the proof is the following observation. Let $\nu : \mathfrak{A} \to U$ be additive. If $a_0, \dots, a_n \in \mathfrak{A}$ and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ are such that $\sum_{i=0}^n \alpha_i \chi a_i = 0$ in $S = S(\mathfrak{A})$, then $\sum_{i=0}^n \alpha_i \nu a_i = 0$ in U. **P** By 361Ea, we can find disjoint $b_0, \dots, b_m$ such that each a_i is the supremum of some of the b_j; set $\gamma_{ij} = 1$ if $b_j \subseteq a_i$, 0 otherwise, so that $\chi a_i = \sum_{j=0}^m \gamma_{ij} \chi b_j$ and $\nu a_i = \sum_{j=0}^m \gamma_{ij} \nu b_j$ for each i. Set $\beta_j = \sum_{i=0}^n \alpha_i \gamma_{ij}$ for each j, so that

$$0 = \sum_{i=0}^n \alpha_i \chi a_i = \sum_{j=0}^m \beta_j \chi b_j.$$

Now $\beta_j \nu b_j = 0$ in U for each j, because either $b_j = 0$ and $\nu b_j = 0$, or there is some $z \in \widehat{b}_j$ so that β_j must be 0. Accordingly

$$0 = \sum_{j=0}^m \beta_j \nu b_j = \sum_{j=0}^m \sum_{i=0}^n \alpha_i \gamma_{ij} \nu b_j = \sum_{i=0}^n \alpha_i \nu a_i. \quad \mathbf{Q}$$

(b) It follows that if $u \in S$ is expressible simultaneously as $\sum_{i=0}^n \alpha_i \chi a_i = \sum_{j=0}^m \beta_j \chi b_j$, then

$$\sum_{i=0}^n \alpha_i \chi a_i + \sum_{j=0}^m (-\beta_j) \chi b_j = 0 \text{ in } S,$$

so that

$$\sum_{i=0}^n \alpha_i \nu a_i + \sum_{j=0}^m (-\beta_j) \nu b_j = 0 \text{ in } U,$$

and

$$\sum_{i=0}^n \alpha_i \nu a_i = \sum_{j=0}^m \beta_j \nu b_j.$$

We can therefore define $T : S \to U$ by setting

$$T(\sum_{i=0}^n \alpha_i \chi a_i) = \sum_{i=0}^n \alpha_i \nu a_i$$

whenever $a_0, \dots , a_n \in \mathfrak{A}$ and $\alpha_0, \dots , \alpha_n \in \mathbb{R}$.

(c) It is now elementary to check that T is linear, and that $T\chi a = \nu a$ for every $a \in \mathfrak{A}$. Of course this last condition uniquely defines T, because $\{\chi a : a \in \mathfrak{A}\}$ spans the linear space S.

361G Theorem Let $\mathfrak{A}$ be a Boolean ring, and U a partially ordered linear space. Let $\nu : \mathfrak{A} \to U$ be an additive function, and $T : S(\mathfrak{A}) \to U$ the corresponding linear operator.

(a) ν is non-negative iff T is positive.

(b) In this case,

(i) if T is order-continuous or sequentially order-continuous, so is ν;

(ii) if U is Archimedean and ν is order-continuous or sequentially order-continuous, so is T.

(c) If U is a Riesz space, then the following are equiveridical:

(i) T is a Riesz homomorphism;

(ii) $\nu a \wedge \nu b = 0$ in U whenever $a \cap b = 0$ in $\mathfrak{A}$;

(iii) ν is a lattice homomorphism.

proof Write S for $S(\mathfrak{A})$.

(a) If T is positive, then surely $\nu a = T\chi a \ge 0$ for every $a \in \mathfrak{A}$, so $\nu = T\chi$ is non-negative. If ν is non-negative, and $u \ge 0$ in S, then u is expressible as $\sum_{j=0}^m \beta_j \chi b_j$ where $b_0, \dots , b_m \in \mathfrak{A}$ and $\beta_j \ge 0$ for every j (361Ec), so that

$$Tu = \sum_{j=0}^m \beta_j \nu b_j \ge 0.$$

Thus T is positive.

(b)(i) If T is order-continuous (resp. sequentially order-continuous) then $\nu = T\chi$ is the composition of two order-continuous (resp. sequentially order-continuous) functions (361Ef), so must be order-continuous (resp. sequentially order-continuous).

(ii) Assume now that U is Archimedean.

(α) Suppose that ν is order-continuous and that $A \subseteq S$ is non-empty, downwards-directed and has infimum 0. Fix $u_0 \in A$, set $\alpha = \|u\|_\infty$ and $a_0 = [\![u > 0]\!]$ (in the language of 361Eg). If $\alpha = 0$ then of course $\inf_{u\in A} Tu = Tu_0 = 0$. Otherwise, take any $w \in U$ such that $w \not\le 0$. Then there is some $\delta > 0$ such that $w \not\le \delta\nu a_0$, because U is Archimedean. Set $A' = \{u : u \in A,\ u \le u_0\}$; because A is downwards-directed, A' has the same lower bounds as A, and $\inf A' = 0$, while A' is still downwards-directed. For $u \in A'$ set $c_u = [\![u > \delta]\!]$, so that

$$\delta\chi c_u \le u \le \alpha\chi c_u + \delta\chi[\![u > 0]\!] \le \alpha\chi c_u + \delta\chi a_0$$

(361Eg). If $u, v \in A'$ and $u \le v$, then $c_u \subseteq c_v$, so $C = \{c_u : u \in A'\}$ is downwards-directed; but if c is any lower bound for C in $\mathfrak{A}$, $\delta\chi c$ is a lower bound for A' in S, so is zero, and $c = 0$ in $\mathfrak{A}$. Thus $\inf C = 0$ in $\mathfrak{A}$, and $\inf_{u\in A'} \nu c_u = 0$ in U. But this means, in particular, that $\frac{1}{\alpha}(w - \delta\nu a_0)$ is not a lower bound for $\nu[C]$, and there is some $u \in A'$ such that $\frac{1}{\alpha}(w - \delta\nu a_0) \not\le \nu c_u$, that is, $w - \delta\nu a_0 \not\le \alpha\nu c_u$, that is, $w \not\le \delta\nu a_0 + \alpha\nu c_u$. As $u \le \alpha\chi c_u + \delta\chi a_0$,

$$Tu \le T(\alpha\chi c_u + \delta\chi a_0) = \alpha\nu c_u + \delta\nu a_0,$$

and $w \not\le Tu$. Since w is arbitrary, this means that $0 = \inf T[A]$; as A is arbitrary, T is order-continuous.

(β) The argument for sequential order-continuity is essentially the same. Suppose that ν is sequentially order-continuous and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in S with infimum 0. Again set $\alpha = \|u_0\|$, $a_0 = [\![u_0 > 0]\!]$; again we may suppose that $\alpha > 0$; again take any $w \in U$ such that $w \not\le 0$. As before, there is some $\delta > 0$ such that $w \not\le \delta\nu a_0$. For $n \in \mathbb{N}$ set $c_n = [\![u_n > \delta]\!]$, so that

$$\delta\chi c_n \le u_n \le \alpha\chi c_n + \delta\chi a_0.$$

The sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ is non-increasing because $\langle u_n\rangle_{n\in\mathbb{N}}$ is, and if $c \subseteq c_n$ for every n, then $\delta\chi c \le u_n$ for every n, so is zero, and $c = 0$ in $\mathfrak{A}$. Thus $\inf_{n\in\mathbb{N}} c_n = 0$ in $\mathfrak{A}$, and $\inf_{n\in\mathbb{N}} \nu c_n = 0$ in U, because ν is sequentially order-continuous. Replacing A', C in the argument above by $\{u_n : n \in \mathbb{N}\}$, $\{c_n : n \in \mathbb{N}\}$ we find an n such that $w \not\le Tu_n$. Since w is arbitrary, this means that $0 = \inf_{n\in\mathbb{N}} Tu_n$; as $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, T is sequentially order-continuous.

(c)(i)$\Rightarrow$(iii) If $T : S(\mathfrak{A}) \to U$ is a Riesz homomorphism, and $\nu = T\chi$, then surely ν is a lattice homomorphism because T and χ are.

(iii)⇒(ii) is trivial.

(ii)⇒(i) If $\nu a \wedge \nu b = 0$ whenever $a \cap b = 0$, then for any $u \in S(\mathfrak{A})$ we have an expression of u as $\sum_{j=0}^{m} \beta_j \chi b_j$, where $b_0, \dots, b_m \in \mathfrak{A}$ are disjoint. Now

$$|Tu| = |\sum_{j=0}^{m} \beta_j \nu b_j| = \sum_{j=0}^{m} |\beta_j| \nu b_j = T(\sum_{j=0}^{m} |\beta_j| \chi b_j) = T(|u|)$$

by 352Fb and 361Ed. As u is arbitrary, T is a Riesz homomorphism (352G).

361H Theorem Let $\mathfrak{A}$ be a Boolean ring and U a Dedekind complete Riesz space. Suppose that $\nu : \mathfrak{A} \to U$ is an additive function and $T : S = S(\mathfrak{A}) \to U$ is the corresponding linear operator. Then $T \in \mathsf{L}^{\sim} = \mathsf{L}^{\sim}(S; U)$ iff $\{\nu b : b \subseteq a\}$ is order-bounded in U for every $a \in \mathfrak{A}$, and in this case $|T| \in \mathsf{L}^{\sim}$ corresponds to $|\nu| : \mathfrak{A} \to U$, defined by setting

$$\begin{aligned}|\nu|(a) &= \sup\{\sum_{j=0}^{n} |\nu a_i| : a_0, \dots, a_n \subseteq a \text{ are disjoint}\}\\ &= \sup\{\nu b - \nu(a \setminus b) : b \subseteq a\}\end{aligned}$$

for every $a \in \mathfrak{A}$.

proof (a) Suppose that $T \in \mathsf{L}^{\sim}$ and $a \in \mathfrak{A}$. Then for any $b \subseteq a$, we have $\chi b \le \chi a$ so

$$|\nu b| = |T\chi b| \le |T|(\chi a).$$

Accordingly $\{\nu b : b \subseteq a\}$ is order-bounded in U.

(b) Now suppose that $\{\nu b : b \subseteq a\}$ is order-bounded in U for every $a \in \mathfrak{A}$. Then for any $a \in \mathfrak{A}$ we can define $w_a = \sup\{|\nu b| : b \subseteq a\}$; in this case, $\nu b - \nu(a \setminus b) \le 2w_a$ whenever $b \subseteq a$, so $\theta a = \sup_{b \subseteq a} \nu b - \nu(a \setminus b)$ is defined in U. Considering $b = a$, $b = 0$ we see that $\theta a \ge |\nu a|$. Next, $\theta : \mathfrak{A} \to U$ is additive. **P** Take a_1, $a_2 \in \mathfrak{A}$ such that $a_1 \cap a_2 = 0$; set $a_0 = a_1 \cup a_2$. For each $j \le 2$ set

$$A_j = \{\nu(a_j \cap b) - \nu(a_j \setminus b) : b \in \mathfrak{A}\} \subseteq U.$$

Then $A_0 \subseteq A_1 + A_2$, because

$$\nu(a_0 \cap b) - \nu(a_0 \setminus b) = \nu(a_1 \cap b) - \nu(a_1 \setminus b) + \nu(a_2 \cap b) - \nu(a_2 \setminus b)$$

for every $b \in \mathfrak{A}$. But also $A_1 + A_2 \subseteq A_0$, because if b_1, $b_2 \in \mathfrak{A}$ then

$$\nu(a_1 \cap b_1) - \nu(a_1 \setminus b_1) + \nu(a_2 \cap b_2) - \nu(a_2 \setminus b_2) = \nu(a_0 \cap b) - \nu(a_0 \setminus b)$$

where $b = (a_1 \cap b_1) \cup (a_2 \cap b_2)$. So $A_0 = A_1 + A_2$, and

$$\theta a_0 = \sup A_0 = \sup A_1 + \sup A_2 = \theta a_1 + \theta a_2$$

(351Dc). **Q**

We therefore have a corresponding positive operator $T_1 : S \to U$ such that $\theta = T_1\chi$. But we also see that $\theta a = \sup\{\sum_{i=0}^{n} |\nu a_i| : a_0, \dots, a_n \subseteq a \text{ are disjoint}\}$ for every $a \in \mathfrak{A}$. **P** If $a_0, \dots, a_n$ are disjoint and included in a, then

$$\sum_{i=0}^{n} |\nu a_i| \le \sum_{i=0}^{n} \theta a_i = \theta(\sup_{i \le n} a_i) \le \theta a.$$

On the other hand,

$$\theta a \le \sup_{b \subseteq a} |\nu b| + |\nu(a \setminus b)| \le \sup\{\sum_{i=0}^{n} |\nu a_i| : a_0, \dots, a_n \subseteq a \text{ are disjoint}\}. \textbf{ Q}$$

It follows that $T \in \mathsf{L}^{\sim}$. **P** Take any $u \ge 0$ in S. Set $a = [\![u > 0]\!]$ (361Eg) and $\alpha = \|u\|_\infty$. If $0 < |v| \le u$, then v is expressible as $\sum_{i=0}^{n} \alpha_i \chi a_i$ where $a_0, \dots, a_n$ are disjoint and no α_i nor a_i is zero. Since $|v| \le \alpha\chi a$, we must have $|\alpha_i| \le \alpha$, $a_i \subseteq a$ for each i. So

$$|Tv| = |\sum_{i=0}^{n} \alpha_i \nu a_i| \le \sum_{i=0}^{n} |\alpha_i||\nu a_i| \le \alpha \sum_{i=0}^{n} |\nu a_i| \le \alpha\theta a.$$

Thus $\{|Tv| : |v| \le u\}$ is bounded above by $\alpha\theta a$. As u is arbitrary, $T \in \mathsf{L}^{\sim}$. **Q**

(c) Thus $T \in \mathsf{L}^{\sim}$ iff ν is order-bounded on the sets $\{b : b \subseteq a\}$, and in this case the two formulae offered for $|\nu|$ are consistent and make $|\nu| = \theta$. Finally, $\theta = |T|\chi$. **P** Take $a \in \mathfrak{A}$. If $a_0, \dots, a_n \subseteq a$ are disjoint, then

$$\sum_{i=0}^{n} |\nu a_i| = \sum_{i=0}^{n} |T\chi a_i| \le \sum_{i=0}^{n} |T|(\chi a_i) \le |T|(\chi a);$$

so $\theta a \le |T|(\chi a)$. On the other hand, the argument at the end of (b) above shows that $|T|(\chi a) \le \theta a$ for every a. Thus $|T|(\chi a) = \theta a$ for every $a \in \mathfrak{A}$, as required. **Q**

361I Theorem Let $\mathfrak{A}$ be a Boolean ring, U a normed space and $\nu : \mathfrak{A} \to U$ an additive function. Give $S = S(\mathfrak{A})$ its norm $\| \|_\infty$, and let $T : S \to U$ be the linear operator corresponding to ν. Then T is a bounded linear operator iff $\{\nu a : a \in \mathfrak{A}\}$ is bounded, and in this case $\|T\| = \sup_{a,b\in\mathfrak{A}} \|\nu a - \nu b\|$.

proof (a) If T is bounded, then

$$\|\nu a - \nu b\| = \|T(\chi a - \chi b)\| \le \|T\|\|\chi a - \chi b\|_\infty \le \|T\|$$

for every $a \in \mathfrak{A}$, so ν is bounded and $\sup_{a,b\in\mathfrak{A}} \|\nu a - \nu b\| \le \|T\|$.

(b)(i) For the converse, we need a refinement of an idea in 361Ec. If $u \in S$ and $u \ge 0$ and $\|u\|_\infty \le 1$, then u is expressible as $\sum_{i=0}^m \gamma_i \chi c_i$ where $\gamma_i \ge 0$ and $\sum_{i=0}^m \gamma_i = 1$. **P** If $u = 0$, take $n = 0$, $c_0 = 0$, $\gamma_0 = 1$. Otherwise, start from an expression $u = \sum_{j=0}^n \gamma_j \chi c_j$ where $c_0 \supseteq \ldots \supseteq c_n$ and every γ_j is non-negative, as in 361Ec. We may suppose that $c_n \ne 0$, in which case

$$\sum_{j=0}^n \gamma_j = u(z) \le 1$$

for every $z \in \widehat{c}_n \subseteq Z$, the Stone space of $\mathfrak{A}$. Set $m = n + 1$, $c_m = 0$ and $\gamma_m = 1 - \sum_{j=0}^n \gamma_j$ to get the required form. **Q**

(ii) The next fact we need is an elementary property of real numbers: if $\gamma_0, \ldots , \gamma_m$, $\gamma'_0, \ldots , \gamma'_n \ge 0$ and $\sum_{i=0}^m \gamma_i = \sum_{j=0}^n \gamma'_j$, then there are $\delta_{ij} \ge 0$ such that $\gamma_i = \sum_{j=0}^n \delta_{ij}$ for every $i \le m$ and $\gamma'_j = \sum_{i=0}^m \delta_{ij}$ for every $j \le n$. **P** This is just the case $U = \mathbb{R}$ of 352Fd. **Q**

(iii) Now suppose that ν is bounded; set $\alpha_0 = \sup_{a\in\mathfrak{A}} \|\nu a\| < \infty$. Then

$$\alpha = \sup_{a,b\in\mathfrak{A}} \|\nu a - \nu b\| \le 2\alpha_0$$

is also finite. If $u \in S$ and $\|u\|_\infty \le 1$, then we can express u as $u^+ - u^-$ where u^+, u^- are non-negative and also of norm at most 1. By (i), we can express these as

$$u^+ = \sum_{i=0}^m \gamma_i \chi c_i, \quad u^- = \sum_{j=0}^n \gamma'_j \chi c'_j$$

where all the γ_i, γ'_j are non-negative and $\sum_{i=0}^m \gamma_i = \sum_{j=0}^n \gamma'_j = 1$. Take $\langle \delta_{ij} \rangle_{i\le m,j\le n}$ from (ii). Set $c_{ij} = c_i$, $c'_{ij} = c'_j$ for all i, j, so that

$$u^+ = \sum_{i=0}^m \sum_{j=0}^n \delta_{ij} \chi c_{ij}, \quad u^- = \sum_{i=0}^m \sum_{j=0}^n \delta_{ij} \chi c'_{ij},$$

$$u = \sum_{i=0}^m \sum_{j=0}^n \delta_{ij} (\chi c_{ij} - \chi c'_{ij}),$$

$$Tu = \sum_{i=0}^m \sum_{j=0}^n \delta_{ij} (\nu c_{ij} - \nu c'_{ij}),$$

$$\|Tu\| \le \sum_{i=0}^m \sum_{j=0}^n \delta_{ij} \|\nu c_{ij} - \nu c'_{ij}\| \le \sum_{i=0}^m \sum_{j=0}^n \delta_{ij} \alpha = \alpha.$$

As u is arbitrary, T is a bounded linear operator and $\|T\| \le \alpha$, as required.

361J The last few paragraphs describe the properties of $S(\mathfrak{A})$ in terms of universal mapping theorems. The next theorem looks at the construction as a functor which converts Boolean algebras into Riesz spaces and ring homomorphisms into Riesz homomorphisms.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ a ring homomorphism.

(a) We have a Riesz homomorphism $T_\pi : S(\mathfrak{A}) \to S(\mathfrak{B})$ given by the formula

$$T_\pi(\chi a) = \chi(\pi a) \text{ for every } a \in \mathfrak{A}.$$

For any $u \in S(\mathfrak{A})$, $\|T_\pi u\|_\infty = \min\{\|u'\|_\infty : u' \in S(\mathfrak{A}), T_\pi u' = T_\pi u\}$; in particular, $\|T_\pi u\|_\infty \le \|u\|_\infty$. Moreover, $T_\pi(u \times u') = T_\pi u \times T_\pi u'$ for all u, $u' \in S(\mathfrak{A})$.

(b) T_π is surjective iff π is surjective, and in this case $\|v\|_\infty = \min\{\|u\|_\infty : u \in S(\mathfrak{A}), T_\pi u = v\}$ for every $v \in S(\mathfrak{B})$.

(c) The kernel of T_π is just the set of those $u \in S(\mathfrak{A})$ such that $\pi[\![|u| > 0]\!] = 0$, defining $[\![\ldots > \ldots]\!]$ as in 361Eg.

(d) T_π is injective iff π is injective, and in this case $\|T_\pi u\|_\infty = \|u\|_\infty$ for every $u \in S(\mathfrak{A})$.

(e) T_π is order-continuous iff π is order-continuous.

(f) T_π is sequentially order-continuous iff π is sequentially order-continuous.

(g) If $\mathfrak{C}$ is another Boolean ring and $\phi : \mathfrak{B} \to \mathfrak{C}$ is another ring homomorphism, then $T_{\phi\pi} = T_\phi T_\pi : S(\mathfrak{A}) \to S(\mathfrak{C})$.

proof (a) The map $\chi\pi : \mathfrak{A} \to S(\mathfrak{B})$ is additive (361Cc), so corresponds to a linear operator $T = T_\pi : S(\mathfrak{A}) \to S(\mathfrak{B})$, by 361F. χ and π are both lattice homomorphisms, so $\chi\pi$ also is, and T is a Riesz homomorphism (361Gc). If $u = \sum_{i=0}^n \alpha_i \chi a_i$, where $a_0, \dots , a_n$ are disjoint, then look at $I = \{i : i \le n,\ \pi a_i \ne 0\}$. We have

$$Tu = \sum_{i=0}^n \alpha_i \chi(\pi a_i) = \sum_{i \in I} \alpha_i \chi(\pi a_i)$$

and $\pi a_0, \dots , \pi a_n$ are disjoint, so that

$$\|Tu\|_\infty = \sup_{i\in I} |\alpha_i| = \|u'\|_\infty \le \sup_{a_i \ne 0} |\alpha_i| \le \|u\|_\infty,$$

where $u' = \sum_{i\in I} \alpha_i \chi a_i$, so that $Tu' = Tu$. If a, $a' \in \mathfrak{A}$, then

$$T(\chi a \times \chi a') = T\chi(a \cap a') = \chi\pi(a \cap a') = \chi\pi a \times \chi\pi a' = T\chi a \times T\chi a',$$

so T is multiplicative.

(b) If π is surjective, then $T[S(\mathfrak{A})]$ must be the linear span of

$$\{T(\chi a) : a \in \mathfrak{A}\} = \{\chi(\pi a) : a \in \mathfrak{A}\} = \{\chi b : b \in \mathfrak{B}\},$$

so is the whole of $S(\mathfrak{B})$. If T is surjective, and $b \in \mathfrak{B}$, then there must be a $u \in \mathfrak{A}$ such that $Tu = \chi b$. We can express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots , a_n$ are disjoint; now

$$\chi b = Tu = \sum_{i=0}^n \alpha_i \chi(\pi a_i),$$

and $\pi a_0, \dots , \pi a_n$ are disjoint in $\mathfrak{B}$, so we must have

$$b = \sup_{i\in I} \pi a_i = \pi(\sup_{i\in I} a_i) \in \pi[\mathfrak{A}],$$

where $I = \{i : \alpha_i = 1\}$. As b is arbitrary, π is surjective. Of course the formula for $\|v\|_\infty$ is a consequence of the formula for $\|Tu\|_\infty$ in (a).

(c)(i) If $\pi[\![|u| > 0]\!] = 0$ then $|u| \le \alpha\chi a$, where $\alpha = \|u\|_\infty$, and $a = [\![|u| > 0]\!]$, so

$$|Tu| = T|u| \le \alpha T(\chi a) = \alpha\chi(\pi a) = 0,$$

and $Tu = 0$.

(ii) If $u \in S(\mathfrak{A})$ and $Tu = 0$, express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots , a_n$ are disjoint and every α_i is non-zero (361Eb). In this case

$$0 = |Tu| = T|u| = \sum_{i=0}^n |\alpha_i| \chi(\pi a_i),$$

so $\pi a_i = 0$ for every i, and

$$\pi[\![|u| > 0]\!] = \pi(\sup_{i\le n} a_i) = \sup_{i\le n} \pi a_i = 0.$$

(d) If T is injective and $a \in \mathfrak{A} \setminus \{0\}$, then $\chi(\pi a) = T(\chi a) \ne 0$, so $\pi a \ne 0$; as a is arbitrary, π is injective. If π is injective then $\pi[\![|u| > 0]\!] \ne 0$ for every non-zero $u \in S(\mathfrak{A})$, so T is injective, by (c). In this case the formula in (a) shows that T is norm-preserving.

(e)(i) If T is order-continuous and $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum 0 in $\mathfrak{A}$, let b be any lower bound for $\pi[A]$ in $\mathfrak{B}$. Then

$$\chi b \le \chi(\pi a) = T(\chi a)$$

for any $a \in A$. But $T\chi$ is order-continuous, by 361Ef, so $\inf_{a\in A} T(\chi a) = 0$, and b must be 0. As b is arbitrary, $\inf_{a\in A} \pi a = 0$; as A is arbitrary, π is order-continuous.

(ii) If π is order-continuous, so is $\chi\pi : \mathfrak{A} \to S(\mathfrak{B})$, using 361Ef again; but now by 361G(b-ii) T must be order-continuous.

(f)(i) If T is sequentially order-continuous, and $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, let b be any lower bound for $\{\pi a_n : n \in \mathbb{N}\}$ in $\mathfrak{B}$. Then

$$\chi b \le \chi(\pi a_n) = T(\chi a_n)$$

for any $a \in A$. But $T\chi$ is sequentially order-continuous so $\inf_{n\in\mathbb{N}} T(\chi a_n) = 0$, and b must be 0. As b is arbitrary, $\inf_{n\in\mathbb{N}} \pi a_n = 0$; as A is arbitrary, π is sequentially order-continuous.

(ii) If π is sequentially order-continuous, so is $\chi\pi : \mathfrak{A} \to S(\mathfrak{B})$; but now T must be sequentially order-continuous.

(g) We need only check that

$$T_{\phi\pi}(\chi a) = \chi(\phi(\pi a)) = T_\phi(\chi(\pi a)) = T_\phi T_\pi(\chi a)$$

for every $a \in \mathfrak{A}$.

361K Proposition Let $\mathfrak{A}$ be a Boolean algebra. For $a \in \mathfrak{A}$ write V_a for the solid linear subspace of $S(\mathfrak{A})$ generated by χa. Then $a \mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands in $S(\mathfrak{A})$.

proof Write S for $S(\mathfrak{A})$.

(a) The point is that, for any $a \in \mathfrak{A}$,

(i) $|u| \wedge |v| = 0$ whenever $u \in V_a$, $v \in V_{1\setminus a}$,

(ii) $V_a + V_{1\setminus a} = S$.

P (i) is just because $\chi a \wedge \chi(1 \setminus a) = 0$. As for (ii), if $w \in S$ then

$$w = (w \times \chi a) + (w \times \chi(1 \setminus a)) \in V_a + V_{1\setminus a}. \text{ } \mathbf{Q}$$

(b) Accordingly $V_a + V_a^\perp \supseteq V_a + V_{1\setminus a} = S$ and V_a is a projection band (352R). Next, any projection band $U \subseteq S$ is of the form V_a. **P** We know that $\chi 1 = u + v$ where $u \in U$, $v \in U^\perp$. Since $|u| \wedge |v| = 0$, u and v must be the indicator functions of complementary subsets of Z, the Stone space of $\mathfrak{A}$. But $\{z : u(z) \neq 0\} = \{z : u(z) \geq 1\}$ must be of the form $\widehat{a}$, where $a = [\![u > 0]\!]$, in which case $u = \chi a$ and $v = \chi(1 \setminus a)$. Accordingly $U \supseteq V_a$ and $U^\perp \supseteq V_{1\setminus a}$. But this means that U must be V_a precisely. **Q**

(c) Thus $a \mapsto V_a$ is a surjective function from $\mathfrak{A}$ onto the algebra of projection bands in S. Now

$$a \subseteq b \iff \chi a \in V_b \iff V_a \subseteq V_b,$$

so $a \mapsto V_a$ is order-preserving and bijective. By 312M it is a Boolean isomorphism.

361L Proposition Let X be a set, and Σ a ring of subsets of X, that is, a subring of the Boolean ring $\mathcal{P}X$. Then $S(\Sigma)$ can be identified, as ordered linear space, with the linear subspace of $\ell^\infty(X)$ generated by the indicator functions of members of Σ, which is a Riesz subspace of $\ell^\infty(X)$. The norm of $S(\Sigma)$ corresponds to the uniform norm on $\ell^\infty(X)$, and its multiplication to pointwise multiplication of functions.

proof Let Z be the Stone space of Σ, and for $E \in \Sigma$ write χE for the indicator function of E as a subset of X, $\hat{\chi}E$ for the indicator function of the open-and-compact subset of Z corresponding to E. Of course $\chi : \Sigma \to \ell^\infty(X)$ is additive, so by 361F there is a linear operator $T : S \to \ell^\infty(X)$, writing S for $S(\Sigma)$, such that $T(\hat{\chi}E) = \chi E$ for every $E \in \Sigma$.

If $u \in S$, $Tu \geq 0$ iff $u \geq 0$. **P** Express u as $\sum_{j=0}^m \beta_j \hat{\chi} E_j$ where $E_0, \dots, E_m$ are disjoint. Then $Tu = \sum_{j=0}^m \beta_j \chi E_j$, so

$$u \geq 0 \iff \beta_j \geq 0 \text{ whenever } E_j \neq \emptyset \iff Tu \geq 0. \text{ } \mathbf{Q}$$

But this means (α) that

$$Tu = 0 \iff Tu \geq 0 \;\&\; T(-u) \geq 0 \iff u \geq 0 \;\&\; -u \geq 0 \iff u = 0,$$

so that T is injective and is a linear space isomorphism between S and its image $\mathcal{S}$, which must be the linear space spanned by $\{\chi E : E \in \Sigma\}$ (β) that T is an order-isomorphism between S and $\mathcal{S}$.

Because $\chi E \wedge \chi F = 0$ whenever E, $F \in \Sigma$ and $E \cap F = \emptyset$, T is a Riesz homomorphism and $\mathcal{S}$ is a Riesz subspace of $\ell^\infty(X)$ (361Gc). Now

$$\|u\|_\infty = \inf\{\alpha : |u| \leq \alpha\hat{\chi}X\} = \inf\{\alpha : |Tu| \leq \alpha\chi X\} = \|Tu\|_\infty$$

for every $u \in S$. Finally,

$$T(\hat{\chi}E \times \hat{\chi}F) = T(\hat{\chi}(E \cap F)) = \chi(E \cap F) = T(\hat{\chi}E) \times T(\hat{\chi}F)$$

for all E, $F \in \Sigma$, so $\mathcal{S}$ is closed under pointwise multiplication and the multiplications of S, $\mathcal{S}$ are identified by T.

361M Proposition Let X be a set, Σ a ring of subsets of X, and $\mathcal{I}$ an ideal of Σ; write $\mathfrak{A}$ for the quotient ring $\Sigma/\mathcal{I}$. Let $\mathcal{S}$ be the linear span of $\{\chi E : E \in \Sigma\}$ in $\mathbb{R}^X$, and write

$$V=\{f:f\in\mathcal{S},\,\{x:f(x)\ne 0\}\in\mathcal{I}\}.$$

Then V is a solid linear subspace of $\mathcal{S}$. Now $S(\mathfrak{A})$ becomes identified with the quotient Riesz space $\mathcal{S}/V$, if for every $E\in\Sigma$ we identify $\chi(E^\bullet)\in S(\mathfrak{A})$ with $(\chi E)^\bullet\in\mathcal{S}/V$. If we give $\mathcal{S}$ its uniform norm inherited from $\ell^\infty(X)$, V is a closed linear subspace of $\mathcal{S}$, and the quotient norm on $\mathcal{S}/V$ corresponds to the norm of $S(\mathfrak{A})$:

$$\|f^\bullet\|=\min\{\alpha:\{x:|f(x)|>\alpha\}\in\mathcal{I}\}.$$

If we write $\times$ for pointwise multiplication on $\mathcal{S}$, then V is an ideal of the ring $(\mathcal{S},+,\times)$, and the multiplication induced on $\mathcal{S}/V$ corresponds to the multiplication of $S(\mathfrak{A})$.

proof Use 361J and 361L. We can identify $\mathcal{S}$ with $S(\Sigma)$. Now the canonical ring homomorphism $E\mapsto E^\bullet$ corresponds to a surjective Riesz homomorphism T from $S(\Sigma)$ to $S(\mathfrak{A})$ which takes χE to $\chi(E^\bullet)$. For $f\in\mathcal{S}$, $[\![|f|>0]\!]$ is just $\{x:f(x)\ne 0\}$, so the kernel of T is just the set of those $f\in\mathcal{S}$ such that $\{x:f(x)\ne 0\}\in\mathcal{I}$, which is V. So

$$S(\mathfrak{A})=T[\mathcal{S}]\cong\mathcal{S}/V.$$

As noted in 361Ja, $T(f\times g)=Tf\times Tg$ for all f, $g\in\mathcal{S}$, so the multiplications of $\mathcal{S}/V$ and $S(\mathfrak{A})$ match. As for the norms, the norm of $S(\mathfrak{A})$ corresponds to the norm of $\mathcal{S}/V$ by the formulae in 361Ja or 361Jb. To see that V is closed in $\mathcal{S}$, we need note only that if $f\in\overline{V}$ then

$$\|Tf\|_\infty=\inf_{g\in V}\|f+g\|_\infty=\inf_{g\in V}\|f-g\|_\infty=0,$$

so that $Tf=0$ and $f\in V$. To check the formula for $\|f^\bullet\|$, take any $f\in\mathcal{S}$. Express it as $\sum_{i=0}^n\alpha_i\chi E_i$ where $E_0,\dots,E_n\in\Sigma$ are disjoint. Set $I=\{i:E_i\notin\mathcal{I}\}$; then

$$\|Tf\|_\infty=\max_{i\in I}|\alpha_i|=\min\{\alpha:\{x:|f(x)|>\alpha\}\in\mathcal{I}\}.$$

361X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean ring and U a linear space. Show that a function $\nu:\mathfrak{A}\to U$ is additive iff $\nu 0=0$ and $\nu(a\cup b)+\nu(a\cap b)=\nu a+\nu b$ for all a, $b\in\mathfrak{A}$.

>(b) Let U be an **algebra over** $\mathbb{R}$, that is, a real linear space endowed with a multiplication $\times$ such that $(U,+,\times)$ is a ring and $\alpha(w\times z)=(\alpha w)\times z=w\times(\alpha z)$ for all w, $z\in U$ and all $\alpha\in\mathbb{R}$. Let $\mathfrak{A}$ be a Boolean ring, $\nu:\mathfrak{A}\to U$ an additive function and $T:S(\mathfrak{A})\to U$ the corresponding linear operator. Show that T is multiplicative iff $\nu(a\cap b)=\nu a\times\nu b$ for all a, $b\in\mathfrak{A}$.

>(c) Let $\mathfrak{A}$ be a Boolean ring, and U a Dedekind complete Riesz space. Suppose that $\nu:\mathfrak{A}\to U$ is an additive function such that the corresponding linear operator $T:S(\mathfrak{A})\to U$ belongs to $\mathsf{L}^\sim=\mathsf{L}^\sim(S(\mathfrak{A});U)$. Show that $T^+\in\mathsf{L}^\sim$ corresponds to $\nu^+:\mathfrak{A}\to U$, where $\nu^+a=\sup_{b\subseteq a}\nu b$ for every $a\in\mathfrak{A}$.

(d) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. Show that there is a natural one-to-one correspondence between Boolean homomorphisms $\pi:\mathfrak{A}\to\mathfrak{B}$ and Riesz homomorphisms $T:S(\mathfrak{A})\to S(\mathfrak{B})$ such that $T(\chi 1_{\mathfrak{A}})=\chi 1_{\mathfrak{B}}$, given by setting $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$.

(e) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean rings and $T:S(\mathfrak{A})\to S(\mathfrak{B})$ a linear operator such that $T(u\times v)=Tu\times Tv$ for all u, $v\in S(\mathfrak{A})$. Show that there is a ring homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ such that $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$.

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings. Show that any isomorphism of the algebras $S(\mathfrak{A})$ and $S(\mathfrak{B})$ (using the word 'algebra' in the sense of 361Xb) must be a Riesz space isomorphism, and therefore corresponds to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

(g) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and $T:S(\mathfrak{A})\to S(\mathfrak{B})$ a Riesz homomorphism. Show that there are a ring homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ and a non-negative $v\in S(\mathfrak{B})$ such that $T(\chi a)=v\times\chi(\pi a)$ for every $a\in\mathfrak{A}$.

(h) Let $\mathfrak{A}$ be a Boolean algebra, $\pi:\mathfrak{A}\to\mathfrak{A}$ a Boolean homomorphism and $T_\pi:S(\mathfrak{A})\to S(\mathfrak{A})$ the associated Riesz homomorphism. Let $\mathfrak{C}$ be the fixed-point subalgebra of π (312K). Show that $S(\mathfrak{C})$ may be identified with the linear subspace of $S(\mathfrak{A})$ generated by $\{\chi c:c\in\mathfrak{C}\}$, and that this is $\{u:u\in S(\mathfrak{A}),\,T_\pi u=u\}$.

(i) Let $\mathfrak{A}$ be a Boolean ring. Show that for any $u\in S(\mathfrak{A})$ the solid linear subspace of $S(\mathfrak{A})$ generated by u is a projection band in $S(\mathfrak{A})$. Show that the set of such bands is an ideal in the algebra of all projection bands, and is isomorphic to $\mathfrak{A}$.

>(j) Let X be a set and Σ a σ-algebra of subsets of X. Show that the linear span $\mathcal{S}$ in $\mathbb{R}^X$ of $\{\chi E : E \in \Sigma\}$ is just the set of Σ-measurable functions $f : X \to \mathbb{R}$ which take only finitely many values.

(k) For any Boolean ring $\mathfrak{A}$, we may define its 'complex S-space' $S_{\mathbb{C}}(\mathfrak{A})$ as the linear span in $\mathbb{C}^Z$ of the indicator functions of open-and-compact subsets of the Stone space Z of $\mathfrak{A}$. State and prove results corresponding to 361Eb, 361Ed, 361Eh, 361F, 361L and 361M.

361Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean ring. For $a \in \mathfrak{A}$ let $e_a \in \mathbb{R}^{\mathfrak{A}}$ be the function such that $e_a(a) = 1$, $e_a(b) = 0$ for $b \in \mathfrak{A} \setminus \{a\}$; let V be the linear subspace of $\mathbb{R}^{\mathfrak{A}}$ generated by $\{e_a : a \in \mathfrak{A}\}$. Let $W \subseteq V$ be the linear subspace spanned by members of V of the form $e_{a\cup b} - e_a - e_b$ where a, $b \in \mathfrak{A}$ are disjoint. Define $\chi' : \mathfrak{A} \to V/W$ by taking $\chi' a = e_a^\bullet$ to be the image in V/W of $e_a \in V$. Show, without using the axiom of choice, that the pair $(V/W, \chi')$ has the universal mapping property of $(S(\mathfrak{A}), \chi)$ as described in 361F and that V/W has a Riesz space structure, a norm and a multiplicative structure as described in 361D-361E. Prove results corresponding to 361E-361M.

(b) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a non-empty family of Boolean algebras, with free product $\mathfrak{A}$; write $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ for the canonical maps, and

$$C = \{\inf_{j \in J} \varepsilon_j(a_j) : J \subseteq I \text{ is finite, } a_j \in \mathfrak{A}_j \text{ for every } j \in J\}.$$

Suppose that U is a linear space and $\theta : C \to U$ is such that

$$\theta c = \theta(c \cap \varepsilon_i(a)) + \theta(c \cap \varepsilon_i(1 \setminus a))$$

whenever $c \in C$, $i \in I$ and $a \in \mathfrak{A}_i$. Show that there is a unique additive function $\nu : \mathfrak{A} \to U$ extending θ. (*Hint*: 326E.)

(c) Let $\mathfrak{A}$ be a Boolean ring and U a Dedekind complete Riesz space. Let $A \subseteq \mathsf{L}^\sim = \mathsf{L}^\sim(S(\mathfrak{A}); U)$ be a non-empty set. Suppose that $\tilde{T} = \sup A$ is defined in $\mathsf{L}^\sim$, and that $\tilde{\nu} = \tilde{T}\chi$. Show that for any $a \in \mathfrak{A}$,

$$\tilde{\nu} a = \sup\{\sum_{i=0}^n T_i(\chi a_i) : T_0, \dots, T_n \in A,\ a_0, \dots, a_n \subseteq a \text{ are disjoint, } \sup_{i \le n} a_i = a\}.$$

(d) Let $\mathfrak{A}$ be a Boolean algebra. Show that the algebra of all bands of $S(\mathfrak{A})$ can be identified with the Dedekind completion of $\mathfrak{A}$ (314U).

(e) Let $\mathfrak{A}$ be a Boolean ring, and U a complex normed space. Let $\nu : \mathfrak{A} \to U$ be an additive function and $T : S_{\mathbb{C}}(\mathfrak{A}) \to U$ the corresponding linear operator (cf. 361Xk). Show that (giving $S_{\mathbb{C}}(\mathfrak{A})$ its usual norm $\| \|_\infty$)

$$\|T\| = \sup\{\|\sum_{j=0}^n \zeta_j \nu a_j\| : a_0, \dots, a_n \in \mathfrak{A} \text{ are disjoint, } |\zeta_j| = 1 \text{ for every } j\}$$

if either is finite.

(f) Let U be a Riesz space. Show that it is isomorphic to $S(\mathfrak{A})$, for some Boolean algebra $\mathfrak{A}$, iff it has an order unit and every solid linear subspace of U is a projection band.

361 Notes and comments The space $S(\mathfrak{A})$ corresponds of course to the idea of 'simple function' which belongs to the very beginnings of the theory of integration (122A). All that 361D is trying to do is to set up a logically sound description of this obvious concept which can be derived from the Boolean ring $\mathfrak{A}$ itself. To my eye, there is a defect in the construction there. It relies on the axiom of choice, since it uses the Stone space; but none of the elementary properties of $S(\mathfrak{A})$ have anything to do with the axiom of choice. In 361Ya I offer an alternative construction which is in a formal sense more 'elementary'. If you work through the suggestion there you will find, however, that the technical details become significantly more complicated, and would be intolerable were it not for the intuition provided by the Stone space construction. Of course this intuition is chiefly valuable in the finitistic arguments used in 361E, 361F and 361I; and for these arguments we really need the Stone representation only for finite Boolean rings, which does not depend on the axiom of choice.

It is quite true that in most of this volume (and in most of this chapter) I use the axiom of choice without scruple and without comment. I mention it here only because I find myself using arguments dependent on choice to prove theorems of a type to which the axiom cannot be relevant.

The linear space structure of $S(\mathfrak{A})$, together with the map χ, are uniquely determined by the first universal mapping theorem here, 361F. This result says nothing about the order structure, which needs the further refinement in 361Ga. What is striking is that the partial order defined by 361Ga is actually a lattice ordering, so that we can have a universal mapping theorem for functions to Riesz spaces, as in 361Gc and 361Ja. Moreover, the same ordering provides a happy abundance of results concerning order-continuous functions (361Gb, 361Je-361Jf). When the codomain is a Dedekind complete Riesz space, so that we have a Riesz space $\mathsf{L}^\sim(S;U)$, and a corresponding modulus function $T \mapsto |T|$ for linear operators, there are reasonably natural formulae for $|T|\chi$ in terms of $T\chi$ (361H); see also 361Xc and 361Yc. The multiplicative structure of $S(\mathfrak{A})$ is defined by 361Xb, and its norm by 361I.

The Boolean ring $\mathfrak{A}$ cannot be recovered from the linear space structure of $S(\mathfrak{A})$ alone (since this tells us only the cardinality of $\mathfrak{A}$), but if we add either the ordering or the multiplication of $S(\mathfrak{A})$ then $\mathfrak{A}$ is easy to identify (361K, 361Xf).

The most important Boolean algebras of measure theory arise either as algebras of sets or as their quotients, so it is a welcome fact that in such cases the spaces $S(\mathfrak{A})$ have straightforward representations in terms of the construction of $\mathfrak{A}$ (361L-361M).

In Chapter 24 I offered a paragraph in each section to sketch a version of the theory based on the field of complex numbers rather than the field of real numbers. This was because so many of the most important applications of these ideas involve complex numbers, even though (in my view) the ideas themselves are most clearly and characteristically expressed in terms of real numbers. In the present chapter we are one step farther away from these applications, and I therefore relegate complex numbers to the exercises, as in 361Xk and 361Ye.

362 $S^\sim$

The next stage in our journey is the systematic investigation of linear functionals on spaces $S = S(\mathfrak{A})$. We already know that these correspond to additive real-valued functionals on the algebra $\mathfrak{A}$ (361F). My purpose here is to show how the structure of the Riesz space dual $S^\sim$ and its bands is related to the classes of additive functionals introduced in §§326-327. The first step is just to check the identification of the linear and order structures of $S^\sim$ and the space M of bounded finitely additive functionals (362A); all the ideas needed for this have already been set out, and the basic properties of $S^\sim$ are covered by the general results in §356. Next, we need to be able to describe the operations on M corresponding to the Riesz space operations $|\ |$, $\vee$, $\wedge$ on $S^\sim$, and the band projections from $S^\sim$ onto $S^\sim_c$ and $S^\times$; these are dealt with in 362B, with a supplementary remark in 362D. In the case of measure algebras, we have some further important bands which present themselves in M, rather than in $S^\sim$, and which are treated in 362C. Since all these spaces are L-spaces, it is worth taking a moment to identify their uniformly integrable subsets; I do this in 362E.

While some of the ideas here have interesting extensions to the case in which $\mathfrak{A}$ is a Boolean ring without identity, these can I think be left to one side; the work of this section will be done on the assumption that every $\mathfrak{A}$ is a Boolean algebra.

362A Theorem Let $\mathfrak{A}$ be a Boolean algebra. Write S for $S(\mathfrak{A})$.

(a) The partially ordered linear space of all finitely additive real-valued functionals on $\mathfrak{A}$ may be identified with the partially ordered linear space of all real-valued linear functionals on S.

(b) The linear space of bounded finitely additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S^\sim$ of order-bounded linear functionals on S. If $f \in S^\sim$ corresponds to $\nu : \mathfrak{A} \to \mathbb{R}$, then $f^+ \in S^\sim$ corresponds to ν^+, where

$$\nu^+ a = \sup_{b \subseteq a} \nu b$$

for every $a \in \mathfrak{A}$, and

$$\|f\| = \sup_{a \in \mathfrak{A}} \nu a - \nu(1 \setminus a).$$

(c) The linear space of bounded countably additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S^\sim_c$.

(d) The linear space of completely additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S^\times$.

proof By 361F, we have a canonical one-to-one correspondence between linear functionals $f : S \to \mathbb{R}$ and additive functionals $\nu_f : \mathfrak{A} \to \mathbb{R}$, given by setting $\nu_f = f\chi$.

(a) Now it is clear that $\nu_{f+g} = \nu_f + \nu_g$, $\nu_{\alpha f} = \alpha\nu_f$ for all f, g and α, so this one-to-one correspondence is a linear space isomorphism. To see that it is also an order-isomorphism, we need note only that ν_f is non-negative iff f is, by 361Ga.

(b) Recall from 356N that, because S is a Riesz space with order unit (361Ee), $S^\sim$ has a corresponding norm under which it is an L-space.

(i) If $f \in S^\sim$, then

$$\sup_{b\in\mathfrak{A}} |\nu_f b| = \sup_{b\in\mathfrak{A}} |f(\chi b)| \le \sup\{|f(u)| : u \in S,\ |u| \le \chi 1\}$$

is finite, and ν_f is bounded.

(ii) Now suppose that ν_f is bounded and that $v \in S^+$. Then there is an $\alpha \ge 0$ such that $v \le \alpha\chi 1$ (361Ee). If $u \in S$ and $|u| \le v$, then we can express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n$ are disjoint (361Eb); now $|\alpha_i| \le \alpha$ whenever $a_i \ne 0$, so

$$|f(u)| = |\sum_{i=0}^n \alpha_i \nu_f a_i| \le \alpha \sum_{i=0}^n |\nu_f a_i| = \alpha(\nu_f c_1 - \nu_f c_2) \le 2\alpha \sup_{b\in\mathfrak{A}} |\nu_f b|,$$

setting $c_1 = \sup\{a_i : i \le n,\ \nu_f a_i \ge 0\}$, $c_2 = \sup\{a_i : i \le n,\ \nu_f a_i < 0\}$. This shows that $\{f(u) : |u| \le v\}$ is bounded. As v is arbitrary, $f \in S^\sim$ (356Aa).

(iii) To check the correspondence between f^+ and ν_f^+, refine the arguments of (i) and (ii) as follows. Take any $f \in S^\sim$. If $a \in \mathfrak{A}$,

$$\nu_f^+ a = \sup_{b\subseteq a} \nu_f b = \sup_{b\subseteq a} f(\chi b) \le \sup\{f(u) : u \in S,\ 0 \le u \le \chi a\} = f^+(\chi a).$$

On the other hand, if $u \in S$ and $0 \le u \le \chi a$, then we can express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n$ are disjoint; now $0 \le \alpha_i \le 1$ whenever $a_i \ne 0$, so

$$f(u) = \sum_{i=0}^n \alpha_i \nu_f a_i \le \nu_f c \le \nu_f^+ a,$$

where $c = \sup\{a_i : i \le n,\ \nu_f a_i \ge 0\}$. As u is arbitrary, $f^+(\chi a) \le \nu_f^+ a$. This shows that $\nu_f^+ = f^+\chi$ is finitely additive, and that $\nu_f^+ = \nu_{f^+}$, as claimed.

(iv) Now, for any $f \in S^\sim$,

$$\|f\| = |f|(\chi 1)$$

(356N)

$$= (2f^+ - f)(\chi 1) = 2\nu_f^+ 1 - \nu_f 1$$

(by (iii) just above)

$$= \sup_{a\in\mathfrak{A}} 2\nu_f a - \nu_f 1 = \sup_{a\in\mathfrak{A}} \nu_f a - \nu_f(1 \setminus a).$$

(c) If $f \ge 0$ in $S^\sim$, then f is sequentially order-continuous iff ν_f is sequentially order-continuous (361Gb), that is, iff ν_f is countably additive (326Kc). Generally, an order-bounded linear functional belongs to $S_c^\sim$ iff it is expressible as the difference of two sequentially order-continuous positive linear functionals (356Ab), while a bounded finitely additive functional is countably additive iff it is expressible as the difference of two non-negative countably additive functionals (326L); so in the present context $f \in S_c^\sim$ iff ν_f is bounded and countably additive.

(d) If $f \ge 0$ in $S^\sim$, then f is order-continuous iff ν_f is order-continuous (361Gb), that is, iff ν_f is completely additive (326Oc). Generally, an order-bounded linear functional belongs to $S^\times$ iff it is expressible as the difference of two order-continuous positive linear functionals (356Ac), while a finitely additive functional is completely additive iff it is expressible as the difference of two non-negative completely additive functionals (326Q); so in the present context $f \in S^\times$ iff ν_f is completely additive.

362B Spaces of finitely additive functionals The identifications in the last theorem mean that we can relate the Riesz space structure of $S(\mathfrak{A})^\sim$ to constructions involving finitely additive functionals. I have already set out the most useful facts as exercises (326Yd, 326Ym, 326Yn, 326Yp, 326Yq); it is now time to repeat them more formally.

Theorem Let $\mathfrak{A}$ be a Boolean algebra. Let M be the Riesz space of bounded finitely additive real-valued functionals on $\mathfrak{A}$, $M_\sigma \subseteq M$ the space of bounded countably additive functionals, and $M_\tau \subseteq M_\sigma$ the space of completely additive functionals.

(a) For any $\mu, \nu \in M$, $\mu \vee \nu$, $\mu \wedge \nu$ and $|\nu|$ are defined by the formulae

$$(\mu \vee \nu)(a) = \sup_{b\subseteq a} \mu b + \nu(a \setminus b),$$

$$(\mu \wedge \nu)(a) = \inf_{b \subseteq a} \mu b + \nu(a \setminus b),$$

$$|\nu|(a) = \sup_{b \subseteq a} \nu b - \nu(a \setminus b) = \sup_{b,c \subseteq a} \nu b - \nu c$$

for every $a \in \mathfrak{A}$. Setting

$$\|\nu\| = |\nu|(1) = \sup_{a \in \mathfrak{A}} \nu a - \nu(1 \setminus a),$$

M becomes an L-space.

(b) M_σ and M_τ are projection bands in M, therefore L-spaces in their own right. In particular, $|\nu| \in M_\sigma$ for every $\nu \in M_\sigma$, and $|\nu| \in M_\tau$ for every $\nu \in M_\tau$.

(c) The band projection $P_\sigma : M \to M_\sigma$ is defined by the formula

$$(P_\sigma \nu)(c) = \inf\{\sup_{n \in \mathbb{N}} \nu a_n : \langle a_n \rangle_{n \in \mathbb{N}} \text{ is a non-decreasing sequence with supremum } c\}$$

whenever $c \in \mathfrak{A}$ and $\nu \geq 0$ in M.

(d) The band projection $P_\tau : M \to M_\tau$ is defined by the formula

$$(P_\tau \nu)(c) = \inf\{\sup_{a \in A} \nu a : A \text{ is a non-empty upwards-directed set with supremum } c\}$$

whenever $c \in \mathfrak{A}$ and $\nu \geq 0$ in M.

(e) If $A \subseteq M$ is upwards-directed, then A is bounded above in M iff $\{\nu 1 : \nu \in A\}$ is bounded above in $\mathbb{R}$, and in this case (if $A \neq \emptyset$) $\sup A$ is defined by the formula

$$(\sup A)(a) = \sup_{\nu \in A} \nu a \text{ for every } a \in \mathfrak{A}.$$

(f) Suppose that μ, $\nu \in M$.

(i) The following are equiveridical:

(α) ν belongs to the band in M generated by μ;

(β) for every $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu a| \leq \epsilon$ whenever $|\mu|a \leq \delta$;

(γ) $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ such that $\lim_{n\to\infty} |\mu|(a_n) = 0$.

(ii) Now suppose that μ, $\nu \geq 0$, and let ν_1, ν_2 be the components of ν in the band generated by μ and its complement. Then

$$\nu_1 c = \sup_{\delta > 0} \inf_{\mu a \leq \delta} \nu(c \setminus a), \quad \nu_2 c = \inf_{\delta > 0} \sup_{a \subseteq c, \mu a \leq \delta} \nu a$$

for every $c \in \mathfrak{A}$.

proof (a) Of course $\mu \vee \nu = \nu + (\mu - \nu)^+$, $\mu \wedge \nu = \nu - (\nu - \mu)^+$, $|\nu| = \nu \vee (-\nu)$ (352D), so the formula of 362Ab gives

$$\begin{aligned}
(\mu \vee \nu)(a) &= \nu a + \sup_{b \subseteq a} \mu b - \nu b = \sup_{b \subseteq a} \mu b + \nu(a \setminus b), \\
(\mu \wedge \nu)(a) &= \nu a - \sup_{b \subseteq a} \nu b - \mu b = \inf_{b \subseteq a} \mu b + \nu(a \setminus b), \\
|\nu|(a) &= \sup_{b \subseteq a} \nu b - \nu(a \setminus b) \leq \sup_{b,c \subseteq a} \nu b - \nu c = \sup_{b,c \subseteq a} \nu(b \setminus c) - \nu(c \setminus b) \\
&\leq \sup_{b,c \subseteq a} |\nu|(b \setminus c) + |\nu|(c \setminus b) = \sup_{b,c \subseteq a} |\nu|(b \triangle c) \leq |\nu|(a).
\end{aligned}$$

The formula offered for $\|\nu\|$ corresponds exactly to the formula in 362Ab for the norm of the associated member of $S(\mathfrak{A})^\sim$; because $S(\mathfrak{A})^\sim$ is an L-space under its norm, so is M.

(b) By 362Ac-362Ad, M_σ and M_τ may be identified with $S(\mathfrak{A})^\sim_c$ and $S(\mathfrak{A})^\times$, which are bands in $S(\mathfrak{A})^\sim$ (356B), therefore projection bands (353I); so that M_σ and M_τ are projection bands in M, and are L-spaces in their own right (354O).

(c) Take any $\nu \geq 0$ in M. Set

$$\nu_\sigma c = \inf\{\sup_{n \in \mathbb{N}} \nu a_n : \langle a_n \rangle_{n \in \mathbb{N}} \text{ is a non-decreasing sequence with supremum } c\}$$

for every $c \in \mathfrak{A}$. Then of course $0 \leq \nu_\sigma c \leq \nu c$ for every c. The point is that ν_σ is countably additive. **P** Let $\langle c_i \rangle_{i \in \mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$, with supremum c. Then for any $\epsilon > 0$ we have non-decreasing sequences $\langle a_n \rangle_{n \in \mathbb{N}}$, $\langle a_{in} \rangle_{n \in \mathbb{N}}$, for $i \in \mathbb{N}$, such that

$$\sup_{n \in \mathbb{N}} a_n = c, \quad \sup_{n \in \mathbb{N}} a_{in} = c_i \text{ for } i \in \mathbb{N},$$

$$\sup_{n\in\mathbb{N}} \nu a_n \le \nu_\sigma c + \epsilon,$$

$$\sup_{n\in\mathbb{N}} \nu a_{in} \le \nu_\sigma c_i + 2^{-i}\epsilon \text{ for every } i \in \mathbb{N}.$$

Set $b_n = \sup_{i\le n} a_{in}$ for each n; then $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, and

$$\sup_{n\in\mathbb{N}} b_n = \sup_{i,n\in\mathbb{N}} a_{in} = \sup_{i\in\mathbb{N}} c_i = c,$$

so

$$\begin{aligned}\nu_\sigma c &\le \sup_{n\in\mathbb{N}} \nu b_n = \sup_{n\in\mathbb{N}} \sum_{i=0}^{n} \nu a_{in}\\ &= \sum_{i=0}^{\infty} \sup_{n\in\mathbb{N}} \nu a_{in} \le \sum_{i=0}^{\infty} \nu_\sigma c_i + 2^{-i}\epsilon = \sum_{i=0}^{\infty} \nu_\sigma c_i + 2\epsilon.\end{aligned}$$

On the other hand, $\langle a_n \cap c_i\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum $c\cap c_i = c_i$ for each i, so $\nu_\sigma c_i \le \sup_{n\in\mathbb{N}} \nu(a_n\cap c_i)$, and

$$\sum_{i=0}^{\infty} \nu_\sigma c_i \le \sum_{i=0}^{\infty} \sup_{n\in\mathbb{N}} \nu(a_n\cap c_i) = \sup_{n\in\mathbb{N}} \sum_{i=0}^{\infty} \nu(a_n\cap c_i)$$

(because $\langle a_n\rangle_{n\in\mathbb{N}}$ is non-decreasing)

$$\le \sup_{n\in\mathbb{N}} \nu a_n$$

(because $\langle c_i\rangle_{i\in\mathbb{N}}$ is disjoint)

$$\le \nu_\sigma c + \epsilon.$$

As ϵ is arbitrary, $\nu_\sigma c = \sum_{i=0}^{\infty} \nu_\sigma c_i$; as $\langle c_i\rangle_{i\in\mathbb{N}}$ is arbitrary, ν_σ is countably additive. **Q**

Thus $\nu_\sigma \in M_\sigma$. On the other hand, if $\nu' \in M_\sigma$ and $0 \le \nu' \le \nu$, then whenever $c \in \mathfrak{A}$ and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum c,

$$\nu' c = \sup_{n\in\mathbb{N}} \nu' a_n \le \sup_{n\in\mathbb{N}} \nu a_n.$$

So we must have $\nu' c \le \nu_\sigma c$. This means that

$$\nu_\sigma = \sup\{\nu' : \nu' \in M_\sigma,\ \nu' \le \nu\} = P_\sigma \nu,$$

as claimed.

(d) The same ideas, with essentially elementary modifications, deal with the completely additive part. Take any $\nu \ge 0$ in M. Set

$$\nu_\tau c = \inf\{\sup_{a\in A} \nu a : A \text{ is a non-empty upwards-directed set with supremum } c\}$$

for every $c \in \mathfrak{A}$. Then of course $0 \le \nu_\tau c \le \nu c$ for every c. The point is that ν_τ is completely additive. **P** Note first that if $c \in \mathfrak{A}$, $\epsilon > 0$ there is a non-empty upwards-directed A, with supremum c, such that $\sup_{a\in A} \nu a \le \nu_\tau c + \epsilon \nu c$; for if $\nu c = 0$ we can take $A = \{c\}$. Now let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$. Then for any $\epsilon > 0$ we have non-empty upwards-directed sets A, A_i, for $i \in I$, such that

$$\sup A = 1, \quad \sup A_i = c_i \text{ for } i \in I, \quad \sup_{a\in A} \nu a \le \nu_\tau 1 + \epsilon\nu 1,$$

$$\sup_{a\in A_i} \nu a \le \nu_\tau c_i + \epsilon \nu c_i \text{ for every } i \in I.$$

Set

$$B = \{\sup_{i\in J} a_i : J \subseteq I \text{ is finite, } a_i \in A_i \text{ for every } i \in J\};$$

then B is non-empty and upwards-directed, and

$$\sup B = \sup(\bigcup_{i\in I} A_i) = 1,$$

so

$$\nu_\tau 1 \le \sup_{b\in B} \nu b = \sup\{\sum_{i\in J} \nu a_i : J \subseteq I \text{ is finite, } a_i \in A_i \ \forall\, i \in J\}$$
$$\le \sum_{i\in I} \nu_\tau c_i + \epsilon \nu c_i \le \epsilon\nu 1 + \sum_{i\in I} \nu_\tau c_i.$$

On the other hand, $A'_i = \{a \cap c_i : a \in A\}$ is a non-empty upwards-directed set with supremum c_i for each i, so $\nu_\tau c_i \le \sup_{a\in A'_i} \nu a$, and

$$\sum_{i\in I} \nu_\tau c_i \le \sum_{i\in I} \sup_{a\in A} \nu(a \cap c_i) = \sup_{a\in A} \sum_{i\in I} \nu(a \cap c_i)$$
$$\le \sup_{a\in A} \nu a \le \nu_\tau 1 + \epsilon\nu 1.$$

As ϵ is arbitrary, $\nu_\tau c = \sum_{i\in I} \nu_\tau c_i$; as $\langle c_i\rangle_{i\in I}$ is arbitrary, ν_τ is completely additive, by 326R. **Q**

Thus $\nu_\tau \in M_\tau$. On the other hand, if $\nu' \in M_\tau$ and $0 \le \nu' \le \nu$, then whenever $c \in \mathfrak{A}$ and A is a non-empty upwards-directed set with supremum c,

$$\nu' c = \sup_{a\in A} \nu' a \le \sup_{a\in A} \nu a$$

(using 326Oc). So we must have $\nu' c \le \nu_\tau c$. This means that

$$\nu_\tau = \sup\{\nu' : \nu' \in M_\tau,\, \nu' \le \nu\} = P_\tau \nu,$$

as claimed.

(e) If A is empty, of course it is bounded above in M, and $\{\nu 1 : \nu \in A\} = \emptyset$ is bounded above in $\mathbb{R}$; so let us suppose that A is not empty. In this case, if $\lambda_0 \in M$ is an upper bound for A, then $\lambda_0 1$ is an upper bound for $\{\nu 1 : \nu \in A\}$. On the other hand, if $\sup_{\nu\in A} \nu 1 = \gamma$ is finite, $\gamma^* = \sup\{\nu a : \nu \in A,\, a \in \mathfrak{A}\}$ is finite. **P** Fix $\nu_0 \in A$. Set $\gamma_1 = \sup_{a\in\mathfrak{A}} |\nu_0 a| < \infty$. Then for any $\nu \in A$ and $a \in \mathfrak{A}$ there is a $\nu' \in A$ such that $\nu_0 \vee \nu \le \nu'$, so that

$$\nu a \le \nu' a = \nu' 1 - \nu'(1 \setminus a) \le \gamma - \nu_0(1 \setminus a) \le \gamma + \gamma_1.$$

So

$$\gamma^* \le \gamma + \gamma_1 < \infty. \ \mathbf{Q}$$

Set $\lambda a = \sup_{\nu\in A} \nu a$ for every $a \in \mathfrak{A}$. Then $\lambda : \mathfrak{A} \to \mathbb{R}$ is additive. **P** If a, $b \in \mathfrak{A}$ are disjoint, then

$$\lambda(a \cup b) = \sup_{\nu\in A} \nu(a \cup b) = \sup_{\nu\in A} \nu a + \nu b = \sup_{\nu\in A} \nu a + \sup_{\nu\in A} \nu b$$

(because A is upwards-directed)

$$= \lambda a + \lambda b. \ \mathbf{Q}$$

Also $\lambda a \le \gamma^*$ for every a, so

$$|\lambda a| = \max(\lambda a, -\lambda a) = \max(\lambda a, \lambda(1 \setminus a) - \lambda 1) \le \gamma^* + |\lambda 1|$$

for every $a \in \mathfrak{A}$, and λ is bounded.

This shows that $\lambda \in M$, so that A is bounded above in M. Of course λ must be actually the least upper bound of A in M.

(f)(i)(α)$\Rightarrow$(β) Suppose that ν belongs to the band in M generated by μ, that is, $|\nu| = \sup_{n\in\mathbb{N}} |\nu| \wedge n|\mu|$ (352Vb). Let $\epsilon > 0$. Then there is an $n \in \mathbb{N}$ such that $|\nu|(1) \le \frac{1}{2}\epsilon + (|\nu| \wedge n|\mu|)(1)$ ((e) above). Set $\delta = \frac{1}{2n+1}\epsilon > 0$. If $|\mu|(a) \le \delta$, then

$$|\nu a| \le |\nu|(a) = (|\nu| \wedge n|\mu|)(a) + (|\nu| - |\nu| \wedge n|\mu|)(a)$$
$$\le n|\mu|(a) + (|\nu| - |\nu| \wedge n|\mu|)(1) \le n\delta + \frac{1}{2}\epsilon \le \epsilon.$$

So (β) is satisfied.

not-(α)$\Rightarrow$not-(β) Suppose that ν does not belong to the band in M generated by $|\mu|$. Then there is a $\nu_1 > 0$ such that $\nu_1 \le |\nu|$ and $\nu_1 \wedge |\mu| = 0$ (353C). For any $\delta > 0$, there is an $a \in \mathfrak{A}$ such that $\nu_1(1 \setminus a) + |\mu|(a) \le \min(\delta, \frac{1}{2}\nu_1 1)$ ((a) above); now $|\mu|(a) \le \delta$ but

$$|\nu|(a) \ge \nu_1 a = \nu_1 1 - \nu_1(1 \setminus a) \ge \nu_1 1 - \tfrac{1}{2}\nu_1 1 = \tfrac{1}{2}\nu_1 1.$$

Thus μ, ν do not satisfy (β) (with $\epsilon = \frac{1}{2}\nu_1 1$).

(β)$\Rightarrow$(γ) is trivial.

(γ)$\Rightarrow$(α) Observe first that if $\langle c_k\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ such that $\lim_{k\to\infty} |\mu| c_k = 0$, then $\lim_{k\to\infty} \nu^+ c_k = 0$. **P** Let $\epsilon > 0$. Because $\nu^+ \wedge \nu^- = 0$, there is a $b \in \mathfrak{A}$ such that $\nu^+ b + \nu^-(1 \setminus b) \le \epsilon$, by part (a). Now $\langle c_k \setminus b\rangle_{k\in\mathbb{N}}$ is non-increasing and $\lim_{k\to\infty} |\mu|(c_k \setminus b) = 0$, so $\lim_{k\to\infty} \nu(c_k \setminus b) = 0$ and

$$\begin{aligned}\limsup_{k\to\infty} \nu^+ c_k &= \limsup_{k\to\infty} \nu^+(c_k \cap b) + \nu(c_k \setminus b) + \nu^-(c_k \setminus b)\\ &\le \nu^+ b + \nu^-(1 \setminus b) \le \epsilon.\end{aligned}$$

As ϵ is arbitrary, $\lim_{k\to\infty} \nu^+ c_k = 0$. **Q**

? Now suppose, if possible, that ν^+ does not belong to the band generated by μ. Then there is a $\nu_1 > 0$ such that $\nu_1 \le \nu^+$ and $\nu_1 \wedge |\mu| = 0$. Set $\epsilon = \frac{1}{4}\nu_1 1 > 0$. For each $n \in \mathbb{N}$, we can choose $a_n \in \mathfrak{A}$ such that $|\mu| a_n + \nu_1(1 \setminus a_n) \le 2^{-n}\epsilon$, by part (a) again. For $n \ge k$, set $b_{kn} = \sup_{k\le i\le n} a_i$; then

$$|\mu| b_{kn} \le \textstyle\sum_{i=k}^n |\mu| a_i \le 2^{-k+1}\epsilon,$$

and $\langle b_{kn}\rangle_{n\ge k}$ is non-decreasing. Set $\gamma_k = \sup_{n\ge k} \nu_1 b_{kn}$ and choose $m(k) \ge k$ such that $\nu_1 b_{k,m(k)} \ge \gamma_k - 2^{-k}\epsilon$. Setting $b_k = b_{k,m(k)}$, we see that $b_k \cup b_{k+1} = b_{kn}$ where $n = \max(m(k), m(k+1))$, so that

$$\nu_1(b_k \cup b_{k+1}) \le \gamma_k \le \nu_1 b_k + 2^{-k}\epsilon$$

and $\nu_1(b_{k+1} \setminus b_k) \le 2^{-k}\epsilon$. Set $c_k = \inf_{i\le k} b_i$ for each k; then

$$\nu_1(b_{k+1} \setminus c_{k+1}) = \nu_1(b_{k+1} \setminus c_k) \le \nu_1(b_{k+1} \setminus b_k) + \nu_1(b_k \setminus c_k) \le 2^{-k}\epsilon + \nu_1(b_k \setminus c_k)$$

for each k; inducing on k, we see that

$$\nu_1(b_k \setminus c_k) \le \textstyle\sum_{i=0}^{k-1} 2^{-i}\epsilon \le 2\epsilon$$

for every k. This means that

$$\nu^+ c_k \ge \nu_1 c_k \ge \nu_1 b_k - 2\epsilon \ge \nu_1 a_k - 2\epsilon = \nu_1 1 - \nu_1(1 \setminus a_k) - 2\epsilon \ge 4\epsilon - \epsilon - 2\epsilon = \epsilon$$

for every $k \in \mathbb{N}$. On the other hand, $\langle c_k\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence and

$$|\mu| c_k \le |\mu| b_k \le 2^{-k+1}\epsilon$$

for every k, which contradicts the paragraph just above. **X**

This means that ν^+ must belong to the band generated by μ. Similarly $\nu^- = (-\nu)^+$ belongs to the band generated by μ and $\nu = \nu^+ + \nu^-$ also does.

(ii) Take $c \in \mathfrak{A}$. Set

$$\beta_1 = \sup_{\delta>0} \inf_{\mu a\le\delta} \nu(c \setminus a), \quad \beta_2 = \inf_{\delta>0} \sup_{a\subseteq c, \mu a\le\delta} \nu a.$$

Then

$$\beta_1 = \sup_{\delta>0} \inf_{a\subseteq c, \mu a\le\delta} \nu(c \setminus a) = \nu c - \beta_2.$$

Take any $\epsilon > 0$. Because ν_1 belongs to the band generated by μ, part (i) tells us that there is a $\delta > 0$ such that $\nu_1 a \le \epsilon$ whenever $\mu a \le \delta$. In this case, if $\mu a \le \delta$,

$$\nu(c \setminus a) = \nu c - \nu(c \cap a) \ge \nu c - \epsilon \ge \nu_1 c - \epsilon;$$

thus

$$\beta_1 \ge \inf_{\mu a\le\delta} \nu(c \setminus a) \ge \nu_1 c - \epsilon.$$

As ϵ is arbitrary, $\beta_1 \ge \nu_1 c$. On the other hand, given ϵ, $\delta > 0$, there is an $a \subseteq c$ such that $\mu a + \nu_2(c \setminus a) \le \min(\delta, \epsilon)$, because $\mu \wedge \nu_2 = 0$ (using (a) again). In this case, of course, $\mu a \le \delta$, while

$$\nu a \ge \nu_2 a = \nu_2 c - \nu_2(c \setminus a) \ge \nu_2 c - \epsilon.$$

Thus $\sup_{a\subseteq c,\mu a\le\delta}\nu a\ge\nu_2c-\epsilon$. As δ is arbitrary, $\beta_2\ge\nu_2c-\epsilon$. As ϵ is arbitrary, $\beta_2\ge\nu_2c$; but as

$$\beta_1+\beta_2=\nu c=\nu_1c+\nu_2c,$$

$\beta_i=\nu_ic$ for both i, as claimed.

Remark The L-space norm $\|\;\|$ on M, described in (a) above, is the **total variation norm**.

362C The formula in 362B(f-i) has, I hope, already reminded you of the concept of 'absolutely continuous' additive functional from the Radon-Nikodým theorem (Chapter 23, §327). The expressions in 362Bf are limited by the assumption that μ, like ν, is finite-valued. If we relax this we get an alternative version of some of the same ideas.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and M be the Riesz space of bounded finitely additive real-valued functionals on $\mathfrak{A}$. Write

$$M_{ac}=\{\nu:\nu\in M\text{ is absolutely continuous with respect to }\bar\mu\}$$

(see 327A),

$$M_{tc}=\{\nu:\nu\in M\text{ is continuous with respect to the measure-algebra topology on }\mathfrak{A}\},$$

$$M_t=\{\nu:\nu\in M,\,|\nu|1=\sup_{\bar\mu a<\infty}|\nu|a\}.$$

Then M_{ac}, M_{tc} and M_t are bands in M.

proof (a)(i) It is easy to check that M_{ac} is a linear subspace of M.

(ii) If $\nu\in M_{ac}$, $\nu'\in M$ and $|\nu'|\le|\nu|$ then $\nu'\in M_{ac}$. **P** Given $\epsilon>0$ there is a $\delta>0$ such that $|\nu a|\le\frac12\epsilon$ whenever $\bar\mu a\le\delta$. Now

$$|\nu'a|\le|\nu'|(a)\le|\nu|(a)\le 2\sup_{c\subseteq a}|\nu c|\le\epsilon$$

(using the formula for $|\nu|$ in 362Ba) whenever $\bar\mu a\le\delta$. As ϵ is arbitrary, ν' is absolutely continuous. **Q**

(iii) If $A\subseteq M_{ac}$ is non-empty and upwards-directed and $\nu=\sup A$ in M, then $\nu\in M_{ac}$. **P** Let $\epsilon>0$. Then there is a $\nu'\in A$ such that $\nu1\le\nu'1+\frac12\epsilon$ (362Be). Now there is a $\delta>0$ such that $|\nu a|\le\frac12\epsilon$ whenever $\bar\mu a\le\delta$. If now $\bar\mu a\le\delta$,

$$|\nu a|\le|\nu'a|+(\nu-\nu')(a)\le\tfrac12\epsilon+(\nu-\nu')(1)\le\epsilon.$$

As ϵ is arbitrary, ν is absolutely continuous with respect to $\bar\mu$. **Q**

Putting these together, we see that M_{ac} is a band.

(b)(i) We know that M_{tc} consists just of those $\nu\in M$ which are continuous at 0 (327Bc). Of course this is a linear subspace of M.

(ii) If $\nu\in M_{tc}$, $\nu'\in M$ and $|\nu'|\le|\nu|$ then $|\nu|\in M_{tc}$. **P** Write $\mathfrak{A}^f=\{d:d\in\mathfrak{A},\,\bar\mu d<\infty\}$. Given $\epsilon>0$ there are $d\in\mathfrak{A}^f$, $\delta>0$ such that $|\nu a|\le\frac12\epsilon$ whenever $\bar\mu(a\cap d)\le\delta$. Now

$$|\nu'a|\le|\nu'|(a)\le|\nu|(a)\le 2\sup_{c\subseteq a}|\nu c|\le\epsilon$$

whenever $\bar\mu(a\cap d)\le\delta$. As ϵ is arbitrary, ν' is continuous at 0 and belongs to M_{tc}. **Q**

(iii) If $A\subseteq M_{tc}$ is non-empty and upwards-directed and $\nu=\sup A$ in M, then $\nu\in M_{tc}$. **P** Let $\epsilon>0$. Then there is a $\nu'\in A$ such that $\nu1\le\nu'1+\frac12\epsilon$. There are $d\in\mathfrak{A}^f$, $\delta>0$ such that $|\nu a|\le\frac12\epsilon$ whenever $\bar\mu(a\cap d)\le\delta$. If now $\bar\mu(a\cap d)\le\delta$,

$$|\nu a|\le|\nu'a|+(\nu-\nu')(a)\le\tfrac12\epsilon+(\nu-\nu')(1)\le\epsilon.$$

As ϵ is arbitrary, ν is continuous at 0, therefore belongs to M_{tc}. **Q**

Putting these together, we see that M_{tc} is a band.

(c)(i) M_t is a linear subspace of M. **P** Suppose that ν_1, $\nu_2\in M_t$ and $\alpha\in\mathbb{R}$. Given $\epsilon>0$, there are a_1, $a_2\in\mathfrak{A}^f$ such that $|\nu_1|(1\setminus a_1)\le\frac{\epsilon}{1+|\alpha|}$ and $|\nu_2|(1\setminus a_2)\le\epsilon$. Set $a=a_1\cup a_2$; then $\bar\mu a<\infty$ and

$$|\nu_1+\nu_2|(1\setminus a)\le 2\epsilon,\quad |\alpha\nu_1|(1\setminus a)\le\epsilon.$$

As ϵ is arbitrary, $\nu_1+\nu_2$ and $\alpha\nu_1$ belong to M_t; as ν_1, ν_2 and α are arbitrary, M_t is a linear subspace of M. **Q**

(ii) If $\nu \in M_t$, $\nu' \in M$ and $|\nu'| \le |\nu|$ then

$$\inf_{\bar\mu a<\infty} |\nu'|(1 \setminus a) \le \inf_{\bar\mu a<\infty} |\nu|(1 \setminus a) = 0,$$

so $\nu' \in M_t$. Thus M_t is a solid linear subspace of M.

(iii) If $A \subseteq M_t^+$ is non-empty and upwards-directed and $\nu = \sup A$ is defined in M, then $\nu \in M_t$. **P**

$$|\nu|1 = \nu 1 = \sup_{\nu' \in A} \nu' 1 = \sup_{\nu' \in A, \bar\mu a<\infty} \nu' a = \sup_{\bar\mu a<\infty} \nu a.$$

As A is arbitrary, $\nu \in M_t$. **Q** Thus M_t is a band in M.

362D For semi-finite measure algebras, among others, the formula of 362Bd takes a special form.

Proposition Let $\mathfrak{A}$ be a weakly (σ,∞)-distributive Boolean algebra. Let M be the space of bounded finitely additive functionals on $\mathfrak{A}$, $M_\tau \subseteq M$ the space of completely additive functionals, and $P_\tau : M \to M_\tau$ the band projection, as in 362B. Then for any $\nu \in M^+$ and $c \in \mathfrak{A}$ there is a non-empty upwards-directed set $A \subseteq \mathfrak{A}$ with supremum c such that $(P_\tau\nu)(c) = \sup_{a\in A} \nu a$; that is, the 'inf' in 362Bd can be read as 'min'.

proof By 362Bd, we can find for each n a non-empty upwards-directed A_n, with supremum c, such that $\sup_{a\in A_n} \nu a \le (P_\tau\nu)(c) + 2^{-n}$. Set $B_n = \{c \setminus a : a \in A_n\}$ for each n, so that B_n is downwards-directed and has infimum 0. Because $\mathfrak{A}$ is weakly (σ,∞)-distributive,

$$B = \{b : \text{for every } n \in \mathbb{N} \text{ there is a } b' \in B_n \text{ such that } b \supseteq b'\}$$

is also a downwards-directed set with infimum 0. Consequently $A = \{c \setminus b : b \in B\}$ is upwards-directed and has supremum c. Moreover, for any $n \in \mathbb{N}$ and $a \in A$, there is an $a' \in A_n$ such that $a \subseteq a'$; so, using 362Bd again and referring to the choice of A_n,

$$(P_\tau\nu)(c) \le \sup_{a\in A} \nu a \le \sup_{a'\in A_n} \nu a' \le (P_\tau\nu)(c) + 2^{-n}.$$

As n is arbitrary, A has the required property.

362E Uniformly integrable sets The spaces $S^\sim$, $S_c^\sim$ and $S^\times$ of 362A, or, if you prefer, the spaces M, M_σ, M_τ, M_{ac}, M_{tc}, M_t of 362B-362C, are all L-spaces, and any serious study of them must involve a discussion of their uniformly integrable (= relatively weakly compact) subsets. The basic work has been done in 356O; I spell out its application in this context.

Theorem Let $\mathfrak{A}$ be a Boolean algebra and M the L-space of bounded finitely additive functionals on $\mathfrak{A}$. Then a norm-bounded set $C \subseteq M$ is uniformly integrable iff $\lim_{n\to\infty} \sup_{\nu\in C} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$.

proof Write S for $S(\mathfrak{A})$ and $\tilde C$ for the set $\{f : f \in S^\sim, f\chi \in C\}$. Because the map $f \mapsto f\chi$ is a normed Riesz space isomorphism between $S^\sim$ and M, $\tilde C$ is uniformly integrable in M iff C is uniformly integrable in $S^\sim$.

(a) Suppose that C is uniformly integrable and that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$. Then $\langle \chi a_n\rangle_{n\in\mathbb{N}}$ is a disjoint order-bounded sequence in $S^\sim$, while $\tilde C$ is uniformly integrable, so $\lim_{n\to\infty} \sup_{f\in\tilde C} |f(\chi a_n)| = 0$, by 356O; but this means that $\lim_{n\to\infty} \sup_{\nu\in C} |\nu a_n| = 0$. Thus the condition is satisfied.

(b) Now suppose that C is not uniformly integrable. By 356O, in the other direction, there is a disjoint sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in S such that $0 \le u_n \le \chi 1$ for each n and $\limsup_{n\to\infty} \sup_{f\in\tilde C} |f(u_n)| > 0$. For each n, take $c_n = [\![u_n > 0]\!]$ (361Eg); then $0 \le u_n \le \chi c_n$ and $\langle c_n\rangle_{n\in\mathbb{N}}$ is disjoint. Now

$$\begin{aligned}\limsup_{n\to\infty} \sup_{\nu\in C} |\nu|(c_n) &= \limsup_{n\to\infty} \sup_{f\in\tilde C} |f|(\chi c_n) \\ &\ge \limsup_{n\to\infty} \sup_{f\in\tilde C} |f(u_n)| > 0.\end{aligned}$$

So if we choose $\nu_n \in C$ such that $|\nu_n|(c_n) \ge \frac{1}{2} \sup_{\nu\in C} |\nu|(c_n)$, we shall have $\limsup_{n\to\infty} |\nu_n|(c_n) > 0$. Next, for each n, we can find $a_n \subseteq c_n$ such that $|\nu_n a_n| \ge \frac{1}{2}|\nu_n|(c_n)$, so that

$$\limsup_{n\in\mathbb{N}} \sup_{\nu\in C} |\nu a_n| \ge \limsup_{n\to\infty} |\nu_n a_n| > 0.$$

Since $\langle a_n\rangle_{n\in\mathbb{N}}$, like $\langle c_n\rangle_{n\in\mathbb{N}}$, is disjoint, the condition is not satisfied. This completes the proof.

362X Basic exercises (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν_1, ν_2 two countably additive functionals on $\mathfrak{A}$. Show that $|\nu_1| \wedge |\nu_2| = 0$ in the Riesz space of bounded finitely additive functionals on $\mathfrak{A}$ iff there is a $c \in \mathfrak{A}$ such that $\nu_1 a = \nu_1(a \cap c)$ and $\nu_2 a = \nu_2(a \setminus c)$ for every $a \in \mathfrak{A}$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and take M, M_{ac} as in 362C. Show that for any non-negative $\nu \in M$, the component ν_{ac} of ν in M_{ac} is given by the formula

$$\nu_{ac}c = \sup_{\delta>0} \inf_{\bar{\mu}a \le \delta} \nu(c \setminus a).$$

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and take M, M_t as in 362C. (i) Show that M_t is just the set of those $\nu \in M$ such that $\nu a = \lim_{b \to \mathcal{F}} \nu(a \cap b)$ for every $a \in \mathfrak{A}$, where $\mathcal{F}$ is the filter on $\mathfrak{A}$ generated by the sets $\{b : b \in \mathfrak{A}^f, b \supseteq b_0\}$ as b_0 runs over the set $\mathfrak{A}^f$ of elements of $\mathfrak{A}$ of finite measure. (ii) Show that the complementary band $M_t^\perp$ of M_t in M is just the set of those $\nu \in M$ such that $\nu a = 0$ for every $a \in \mathfrak{A}^f$. (iii) Show that for any $\nu \in M$, its component ν_t in M_t is given by the formula $\nu_t a = \lim_{b \to \mathcal{F}} \nu(a \cap b)$ for every $a \in \mathfrak{A}$.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Write M, M_σ, M_τ, M_{ac}, M_{tc} and M_t as in 362B-362C. Show that (i) $M_\sigma \subseteq M_{ac}$ (ii) $M_{ac} \cap M_t = M_{tc} \subseteq M_\tau$ (iii) if $(\mathfrak{A}, \bar{\mu})$ is σ-finite, then $M_\sigma = M_{tc}$.

(e) Let $\mathfrak{A}$ be a Boolean algebra, and M the space of bounded additive functionals on $\mathfrak{A}$. Let us say that a non-zero finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **atomic** if whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0$ then at least one of νa, νb is zero. (i) Show that for a non-zero finitely additive functional ν on $\mathfrak{A}$ the following are equiveridical: (α) ν is atomic; (β) $\nu \in M$ and $|\nu|$ is atomic; (γ) $\nu \in M$ and the corresponding linear functional $f_{|\nu|} = |f_\nu| \in S(\mathfrak{A})^\sim$ is a Riesz homomorphism; (δ) there are a multiplicative linear functional $f : S(\mathfrak{A}) \to \mathbb{R}$ and an $\alpha \in \mathbb{R}$ such that $\nu a = \alpha f(\chi a)$ for every $a \in \mathfrak{A}$; (ϵ) $\nu \in M$ and the band in M generated by ν is the set of multiples of ν. (ii) Show that a completely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is atomic iff there are $a \in \mathfrak{A}$ and $\alpha \in \mathbb{R} \setminus \{0\}$ such that a is an atom in $\mathfrak{A}$ and $\nu b = \alpha$ when $a \subseteq b$, 0 when $a \cap b = 0$.

(f) Let $\mathfrak{A}$ be a Boolean algebra. (i) Show that the properly atomless functionals (definition: 326F) form a band M_c in the Riesz space M of all bounded finitely additive functionals on $\mathfrak{A}$. (ii) Show that the complementary band $M_c^\perp$ consists of just those $\nu \in M$ expressible as a sum $\sum_{i \in I} \nu_i$ of countably many atomic functionals $\nu_i \in M$. (iii) Show that if $\mathfrak{A}$ is purely atomic then a properly atomless completely additive functional on $\mathfrak{A}$ must be 0.

(g) Let X be a set and Σ an algebra of subsets of X. Let M be the Riesz space of bounded finitely additive functionals on Σ, M_τ the space of completely additive functionals and M_p the space of functionals expressible in the form $\nu E = \sum_{x \in E} \alpha_x$ for some absolutely summable family $\langle \alpha_x \rangle_{x \in X}$ of real numbers. (i) Show that M_p is a band in M. (ii) Show that if all singleton subsets of X belong to Σ then $M_p = M_\tau$. (iii) Show that if Σ is a σ-algebra then every member of M_p is countably additive. (iv) Show that if X is a compact zero-dimensional Hausdorff space and Σ is the algebra of open-and-closed subsets of X then the complementary band $M_p^\perp$ of M_p in M is the band M_c of properly atomless functionals described in 362Xf.

(h) Let (X, Σ, μ) be a measure space. Let M be the Riesz space of bounded finitely additive functionals on Σ and M_σ the space of bounded countably additive functionals. Let M_{tc}, M_{ac} be the spaces of truly continuous and bounded absolutely continuous additive functionals as defined in 232A. Show that M_{tc} and M_{ac} are bands in M and that $M_{tc} \subseteq M_\sigma \cap M_{ac}$. Show that if μ is σ-finite then $M_{tc} = M_\sigma \cap M_{ac}$.

(i) Let $\mathfrak{A}$ be a Boolean algebra and M the Riesz space of bounded finitely additive functionals on $\mathfrak{A}$. (i) For any non-empty downwards-directed set $A \subseteq \mathfrak{A}$ set $N_A = \{\nu : \nu \in M, \inf_{a \in A} |\nu|a = 0\}$. Show that N_A is a band in M. (ii) For any non-empty set $\mathcal{A}$ of non-empty downwards-directed sets in $\mathfrak{A}$ set $M_{\mathcal{A}} = \{\nu : \nu \in M, \inf_{a \in A} |\nu|a = 0 \; \forall \, A \in \mathcal{A}\}$. Show that $M_{\mathcal{A}}$ is a band in M. (iii) Explain how to represent as such $M_{\mathcal{A}}$ the bands M_σ, M_τ, M_t, M_{ac}, M_{tc} described in 362B-362C, and also any band generated by a single element of M. (iv) Suppose, in (ii), that $\mathcal{A}$ has the property that for any A, $A' \in \mathcal{A}$ there is a $B \in \mathcal{A}$ such that for every $b \in B$ there are $a \in A$, $a' \in A'$ such that $a \cup a' \subseteq b$. Show that for any non-negative $\nu \in M$, the component ν_1 of ν in $M_{\mathcal{A}}$ is given by the formula $\nu_1 c = \inf_{A \in \mathcal{A}} \sup_{a \in A} \nu(c \setminus a)$, so that the component ν_2 of ν in $M_{\mathcal{A}}^\perp$ is given by the formula $\nu_2 c = \sup_{A \in \mathcal{A}} \inf_{a \in A} \nu(c \cap a)$. (Cf. 356Yb.)

362Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. Let $\mathfrak{C}$ be the band algebra of the Riesz space M of bounded finitely additive functionals on $\mathfrak{A}$ (353B). Show that the bands M_σ, M_τ, M_c (362B, 362Xf) generate a subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$ with at most six atoms. Give an example in which $\mathfrak{C}_0$ has six atoms. How many atoms can it have if (i) $\mathfrak{A}$ is atomless (ii) $\mathfrak{A}$ is purely atomic (iii) $\mathfrak{A}$ is Dedekind σ-complete?

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Let $\mathfrak{C}$ be the band algebra of the Riesz space M of bounded finitely additive functionals on $\mathfrak{A}$. Show that the bands M_σ, M_τ, M_c, M_{ac}, M_{tc}, M_t (362B, 362C, 362Xf) generate a subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$ with at most twelve atoms. Give an example in which $\mathfrak{C}_0$ has twelve atoms. How many atoms can it have if (i) $\mathfrak{A}$ is atomless (ii) $\mathfrak{A}$ is purely atomic (iii) $(\mathfrak{A}, \bar{\mu})$ is semi-finite (iv) $(\mathfrak{A}, \bar{\mu})$ is localizable (v) $(\mathfrak{A}, \bar{\mu})$ is σ-finite (vi) $(\mathfrak{A}, \bar{\mu})$ is totally finite?

(c) Give an example of a set X, a σ-algebra Σ of subsets of X, and a functional in M_p (as defined in 362Xg) which is not completely additive.

(d) Let U be a Riesz space and f, $g \in U^\sim$. Show that the following are equiveridical: (α) g is in the band in $U^\sim$ generated by f; (β) for every $u \in U^+$, $\epsilon > 0$ there is a $\delta > 0$ such that $|g(v)| \le \epsilon$ whenever $0 \le v \le u$ and $|f|(v) \le \delta$; (γ) $\lim_{n\to\infty} g(u_n) = 0$ whenever $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U^+ and $\lim_{n\to\infty} |f|(u_n) = 0$.

(e) Let $\mathfrak{A}$ be a weakly σ-distributive Boolean algebra (316Ye). Show that the 'inf' in the formula for $P_\sigma\nu$ in 362Bc can be replaced by 'min'.

(f) Let $\mathfrak{A}$ be any Boolean algebra and M the space of bounded finitely additive functionals on $\mathfrak{A}$. Let $C \subseteq M$ be such that $\sup_{\nu\in C} |\nu a| < \infty$ for every $a \in \mathfrak{A}$. (i) Suppose that $\sup_{n\in\mathbb{N}} \sup_{\nu\in C} |\nu a_n|$ is finite for every disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is norm-bounded. (ii) Suppose that $\lim_{n\to\infty} \sup_{\nu\in C} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is uniformly integrable.

(g) Let $\mathfrak{A}$ be a Boolean algebra and M_τ the space of completely additive functionals on $\mathfrak{A}$. Let $C \subseteq M_\tau$ be such that $\sup_{\nu\in C} |\nu a| < \infty$ for every atom $a \in \mathfrak{A}$. (i) Suppose that $\sup_{n\in\mathbb{N}} \sup_{\nu\in C} |\nu a_n|$ is finite for every disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is norm-bounded. (ii) Suppose that $\lim_{n\to\infty} \sup_{\nu\in C} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is uniformly integrable.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle \nu_n \rangle_{n\in\mathbb{N}}$ a sequence of countably additive real-valued functionals on $\mathfrak{A}$. Suppose that $\nu a = \lim_{n\to\infty} \nu_n a$ is defined in $\mathbb{R}$ for every $a \in \mathfrak{A}$. Show that ν is countably additive and that $\{\nu_n : n \in \mathbb{N}\}$ is uniformly integrable. (*Hint*: 246Yg.) Show that if every ν_n is completely additive, so is ν.

(i) Let $\mathfrak{A}$ be a Boolean algebra, M the Riesz space of bounded finitely additive functionals on $\mathfrak{A}$, and $M_c \subseteq M$ the band of properly atomless functionals (362Xf). Show that for a non-negative $\nu \in M$ the component ν_c of ν in M_c is given by the formula

$$\nu_c a = \inf_{\delta>0} \sup\{\sum_{i=0}^n \nu a_i : a_0, \dots, a_n \subseteq a \text{ are disjoint, } \nu a_i \le \delta \text{ for every } i\}$$

for each $a \in \mathfrak{A}$.

(j) Let $\mathfrak{A}$ be a Boolean algebra and M the L-space of bounded additive real-valued functionals on $\mathfrak{A}$. Show that the complexification of M, as defined in 354Yl, can be identified with the Banach space of bounded additive functionals $\nu : \mathfrak{A} \to \mathbb{C}$, writing

$$\|\nu\| = \sup\{\sum_{i=0}^n |\nu a_i| : a_0, \dots, a_n \text{ are disjoint elements of } \mathfrak{A}\}$$

for such ν.

(k) Let $\mathfrak{A}$ be a Boolean algebra and M the L-space of bounded additive real-valued functionals on $\mathfrak{A}$. Suppose that M_0 is a norm-closed linear subspace of M and that $a \mapsto \nu(a \cap c) : \mathfrak{A} \to \mathbb{R}$ belongs to M_0 whenever $\nu \in M_0$ and $c \in \mathfrak{A}$. Show that M_0 is a band in M. (*Hint*: 436L.)

362 Notes and comments The Boolean algebras most immediately important in measure theory are of course σ-algebras of measurable sets and their quotient measure algebras. It is therefore natural to begin any investigation by concentrating on Dedekind σ-complete algebras. Nevertheless, in this section and the last (and in §326), I have gone to some trouble not to specialize to σ-complete algebras except when necessary. Partly this is just force of habit, but partly it is because I wish to lay a foundation for a further step forward: the investigation of the ways in which additive functionals on general Boolean algebras reflect the concepts of measure theory, and indeed can generate them. Some of the results in this direction can be surprising. I do not think it obvious that the condition (γ) in 362B(f-i), for instance, is sufficient in the absence of any hypothesis of Dedekind σ-completeness or countable additivity.

Given a Boolean algebra $\mathfrak{A}$ with the associated Riesz space $M \cong S(\mathfrak{A})^\sim$ of bounded additive functionals on $\mathfrak{A}$, we now have a substantial list of bands in M: M_σ, M_τ, M_c (362Xf), and for a measure algebra the further bands M_{ac}, M_{tc} and M_t; for an algebra of sets we also have M_p (362Xg). These bands can be used to generate finite subalgebras of the band algebra of M (362Ya-362Yb), and for any such finite subalgebra we have a corresponding decomposition of M as a direct sum of the bands which are the atoms of the subalgebra (352Tb). This decomposition of M can be regarded as a recipe for decomposing its members into finite sums of functionals with special properties. What I called the 'Lebesgue decomposition' in 232I is just such a recipe. In that context I had a measure space (X, Σ, μ) and was looking at the countably additive functionals from Σ to $\mathbb{R}$, that is, at M_σ in the language of this section, and the bands involved in the decomposition were M_p, M_{ac} and M_{tc}. But I hope that it will be plain that these ideas can be refined indefinitely, as we refine the classification of additive functionals. At each stage, of course, the exact enumeration of the subalgebra of bands generated by the classification (as in 362Ya-362Yb) is a necessary check that we have understood the relationships between the classes we have described.

These decompositions are of such importance that it is worth examining the corresponding band projections. I give formulae for the action of band projections on (non-negative) functionals in 362Bc, 362Bd, 362B(f-ii), 362Xb, 362Xc(iii), 362Xi(iv) and 362Yi. Of course these are readily adapted to give formulae for the projections onto the complementary bands, as in 362Bf and 362Xi.

If we have an algebra of sets, the completely additive functionals are (usually) of relatively minor importance; in the standard examples, they correspond to functionals defined as weighted sums of point masses (362Xg(ii)). The point is that measure algebras $\mathfrak{A}$ appear as quotients of σ-algebras Σ of sets by σ-ideals $\mathcal{I}$; consequently the countably additive functionals on $\mathfrak{A}$ correspond exactly to the countably additive functionals on Σ which are zero on $\mathcal{I}$; but the canonical homomorphism from Σ to $\mathfrak{A}$ is hardly ever order-continuous, so completely additive functionals on $\mathfrak{A}$ rarely correspond to completely additive functionals on Σ. On the other hand, when we are looking at countably additive functionals on Σ, we have to consider the possibility that they are singular in the sense that they are carried on some member of $\mathcal{I}$; in the measure algebra context this possibility disappears, and we can often be sure that every countably additive functional is absolutely continuous, as in 327Bb.

For any Boolean algebra $\mathfrak{A}$, we can regard it as the algebra of open-and-closed subsets of its Stone space Z; the points of Z correspond to Boolean homomorphisms from $\mathfrak{A}$ to $\{0, 1\}$, which are the normalised 'atomic elements' in the space of additive functionals on $\mathfrak{A}$ (362Xe, 362Xg(iv)). It is the case that all non-negative additive functionals on a Boolean algebra $\mathfrak{A}$ can be represented by appropriate measures on its Stone space (see 416Q in Volume 4), but I prefer to hold this result back until it can take its place among other theorems on representing functionals by measures and integrals.

It is one of the leitmotivs of this chapter, that Boolean algebras and Riesz spaces are Siamese twins; again and again, matching results are proved by the application of identical ideas. A typical example is the pair 362B(f-i) and 362Yd. Many of us have been tempted to try to describe something which would provide a common generalization of Boolean algebras and Riesz spaces (and lattice-ordered groups). I have not yet seen any such structure which was worth the trouble. Most of the time, in this chapter, I shall be using ideas from the general theory of Riesz spaces to suggest and illuminate questions in measure theory; but if you pursue this subject you will surely find that intuitions often come to you first in the context of Boolean algebras, and the applications to Riesz spaces are secondary.

In 362E I give a condition for uniform integrability in terms of disjoint sequences, following the pattern established in 246G and repeated in 354R and 356O. The condition of 362E assumes that the set is norm-bounded; but if you have 246G to hand, you will see that it can be done with weaker assumptions involving atoms, as in 362Yf-362Yg.

I mention once again the Banach-Ulam problem: if $\mathfrak{A}$ is Dedekind complete, can $S(\mathfrak{A})_c^\sim$ be different from $S(\mathfrak{A})^\times$? This is obviously equivalent to the form given in the notes to §326 above. See 363S below.

363 L^∞

In this section I set out to describe an abstract construction for L^∞ spaces on arbitrary Boolean algebras, corresponding to the $L^\infty(\mu)$ spaces of §243. I begin with the definition of $L^\infty(\mathfrak{A})$ (363A) and elementary facts concerning its own structure and the embedding $S(\mathfrak{A}) \subsetneq L^\infty(\mathfrak{A})$ (363B-363D). I give the basic universal mapping theorems which define the Banach lattice structure of L^∞ (363E) and a description of the action of Boolean homomorphisms on L^∞ spaces (363F-363G) before discussing the representation of $L^\infty(\Sigma)$ and $L^\infty(\Sigma/\mathcal{I})$ for σ-algebras Σ and ideals $\mathcal{I}$ of sets (363H). This leads at once to the identification of $L^\infty(\mu)$, as defined in Volume 2, with $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the measure algebra of μ (363I). Like $S(\mathfrak{A})$, $L^\infty(\mathfrak{A})$ determines the algebra $\mathfrak{A}$ (363J). I briefly discuss the dual spaces of L^∞; they correspond exactly to the duals of S described in §362 (363K). Linear functionals on L^∞ can for some purposes be treated as 'integrals' (363L).

In the second half of the section I present some of the theory of Dedekind complete and σ-complete algebras. First, $L^\infty(\mathfrak{A})$ is Dedekind (σ-)complete iff $\mathfrak{A}$ is (363M). The spaces $L^\infty(\mathfrak{A})$, for Dedekind σ-complete $\mathfrak{A}$, are precisely the Dedekind σ-complete Riesz spaces with order unit (363N-363P). The spaces $L^\infty(\mathfrak{A})$, for Dedekind complete $\mathfrak{A}$, are precisely the normed spaces which may be put in place of $\mathbb{R}$ in the Hahn-Banach theorem (363R). Finally, I mention some equivalent forms of the Banach-Ulam problem (363S).

363A Definition Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z. I will write $L^\infty(\mathfrak{A})$ for the space $C(Z) = C_b(Z)$ of continuous real-valued functions from Z to $\mathbb{R}$, endowed with the linear structure, order structure, norm and multiplication of $C(Z) = C_b(Z)$. (Recall that because Z is compact (311I), $\{u(z) : z \in Z\}$ is bounded for every $u \in L^\infty(\mathfrak{A}) = C(Z)$ (2A3N(b-iii)), that is, $C(Z) = C_b(Z)$. Of course if $\mathfrak{A} = \{0\}$, so that $Z = \emptyset$, then $C(Z)$ has just one member, the empty function.)

363B Theorem Let $\mathfrak{A}$ be any Boolean algebra; write L^∞ for $L^\infty(\mathfrak{A})$.

(a) L^∞ is an M-space; its standard order unit is the constant function taking the value 1 at each point; in particular, L^∞ is a Banach lattice with a Fatou norm and the Levi property.

(b) L^∞ is a commutative Banach algebra and an f-algebra.

(c) If $u \in L^\infty$ then $u \ge 0$ iff there is a $v \in L^\infty$ such that $u = v \times v$.

proof (a) See 354Hb and 354J.

(b)-(c) are obvious from the definitions of Banach algebra (2A4J) and f-algebra (352W) and the ordering of $L^\infty = C(Z)$.

363C Proposition Let $\mathfrak{A}$ be any Boolean algebra. Then $S(\mathfrak{A})$ is a norm-dense, order-dense Riesz subspace of $L^\infty(\mathfrak{A})$, closed under multiplication.

proof Let Z be the Stone space of $\mathfrak{A}$. Using the definition of $S = S(\mathfrak{A})$ set out in 361D, it is obvious that S is a linear subspace of $L^\infty = L^\infty(\mathfrak{A}) = C(Z)$ closed under multiplication. Because S, like L^∞, is a Riesz subspace of $\mathbb{R}^Z$ (361Ee), S is a Riesz subspace of L^∞. By the Stone-Weierstrass theorem (in either of the forms given in 281A and 281E), S is norm-dense in L^∞. Consequently it is order-dense (354I).

363D Proposition Let $\mathfrak{A}$ be a Boolean algebra. If we regard $\chi a \in S(\mathfrak{A})$ (361D) as a member of $L^\infty(\mathfrak{A})$ for each $a \in \mathfrak{A}$, then $\chi : \mathfrak{A} \to L^\infty(\mathfrak{A})$ is additive, order-preserving, order-continuous and a lattice homomorphism.

proof Because the embedding $S = S(\mathfrak{A}) \subsetneqq L^\infty(\mathfrak{A}) = L^\infty$ is a Riesz homomorphism, $\chi : \mathfrak{A} \to L^\infty$ is additive and a lattice homomorphism (361F-361G). Because S is order-dense in L^∞ (363C), the embedding $S \subsetneqq L^\infty$ is order-continuous (352Nb), so $\chi : \mathfrak{A} \to L^\infty$ is order-continuous (361Gb).

363E Theorem Let $\mathfrak{A}$ be a Boolean algebra, and U a Banach space. Let $\nu : \mathfrak{A} \to U$ be a bounded additive function.

(a) There is a unique bounded linear operator $T : L^\infty(\mathfrak{A}) \to U$ such that $T\chi = \nu$; in this case $\|T\| = \sup_{a,b\in\mathfrak{A}} \|\nu a - \nu b\|$.

(b) If U is a Banach lattice then T is positive iff ν is non-negative; and in this case T is order-continuous iff ν is order-continuous, and sequentially order-continuous iff ν is sequentially order-continuous.

(c) If U is a Banach lattice then T is a Riesz homomorphism iff ν is a lattice homomorphism iff $\nu a \wedge \nu b = 0$ whenever $a \cap b = 0$.

proof Write $S = S(\mathfrak{A})$, $L^\infty = L^\infty(\mathfrak{A})$.

(a) By 361I there is a unique bounded linear operator $T_0 : S \to U$ such that $T_0\chi = \nu$, and $\|T_0\| = \sup\{\|\nu a - \nu b\| : a, b \in \mathfrak{A}\}$. But because U is a Banach space and S is dense in L^∞, T_0 has a unique extension to a bounded linear operator $T : L^\infty \to U$ with the same norm (2A4I).

(b)(i) If T is positive then T_0 is positive so ν is non-negative, by 361Ga.

(ii) If ν is non-negative then T_0 is positive, by 361Ga in the other direction. But if $u \in L^{\infty+}$ and $\epsilon > 0$, then by 354I there is a $v \in S^+$ such that $\|u - v\|_\infty \le \epsilon$; now $\|Tu - Tv\| \le \epsilon\|T\|$. But $Tv = T_0v$ belongs to the positive cone U^+ of U. As ϵ is arbitrary, Tu belongs to the closure of U^+, which is U^+ (354Bc). As u is arbitrary, T is positive.

(iii) Now suppose that ν is order-continuous as well as non-negative, and that $A \subseteq L^\infty$ is a non-empty downwards-directed set with infimum 0. Set

$$B = \{v : v \in S, \text{ there is some } u \in A \text{ such that } v \geq u\}.$$

Then B is downwards-directed (indeed, $v_1 \wedge v_2 \in B$ for every v_1, $v_2 \in B$), and $u = \inf\{v : v \in B, u \leq v\}$ for every $u \in A$ (354I again), so B has the same lower bounds as A and $\inf B = 0$ in L^∞ and in S. But we know from 361Gb that T_0 is order-continuous, while any lower bound for $\{Tu : u \in A\}$ in U must also be a lower bound for $\{Tv : v \in B\} = \{T_0 v : v \in B\}$, so $\inf_{u \in A} Tu = \inf_{v \in B} T_0 v = 0$ in U. As A is arbitrary, T is order-continuous (351Ga).

(iv) Suppose next that ν is only sequentially order-continuous, and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in L^∞ with infimum 0. For each n, k choose $w_{nk} \in S$ such that $u_n \leq w_{nk}$ and $\|w_{nk} - u_n\|_\infty \leq 2^{-k}$ (354I once more), and set $w'_n = \inf_{j,k\leq n} w_{jk}$ for each n. Then $\langle w'_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in S, and any lower bound of $\{w'_n : n \in \mathbb{N}\}$ is also a lower bound of $\{u_n : n \in \mathbb{N}\}$, so $0 = \inf_{n\in\mathbb{N}} w'_n$ in S and L^∞. Since $T_0 : S \to U$ is sequentially order-continuous (361Gb),

$$\inf_{n\in\mathbb{N}} Tu_n \leq \inf_{n\in\mathbb{N}} Tw'_n = \inf_{n\in\mathbb{N}} T_0 w'_n = 0$$

in U. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, T is sequentially order-continuous.

(v) On the other hand, if T is order-continuous or sequentially order-continuous, so is $\nu = T\chi$, because χ is order-continuous (363D).

(c) We know that $T_0 : S \to U$ is a Riesz homomorphism iff ν is a lattice homomorphism iff $\nu a \wedge \nu b = 0$ whenever $a \cap b = 0$, by 361Gc. But T_0 is a Riesz homomorphism iff T is. **P** If T is a Riesz homomorphism so is T_0, because the embedding $S \subsetneq L^\infty$ is a Riesz homomorphism. On the other hand, if T_0 is a Riesz homomorphism, then the functions $u \mapsto u^+ \mapsto T(u^+)$, $u \mapsto Tu \mapsto (Tu)^+$ are continuous (by 354Bb) and agree on S, so agree on L^∞, and T is a Riesz homomorphism, by 352G. **Q**

363F Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a) There is an associated multiplicative Riesz homomorphism $T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$, of norm at most 1, defined by saying that $T_\pi(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$.

(b) For any $u \in L^\infty(\mathfrak{A})$, there is a $u' \in L^\infty(\mathfrak{A})$ such that $T_\pi u = T_\pi u'$ and $\|u'\|_\infty = \|T_\pi u\|_\infty \leq \|u\|_\infty$.

(c)(i) The kernel of T_π is the norm-closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi a : a \in \mathfrak{A}, \pi a = 0\}$.

(ii) The set of values of T_π is the norm-closed linear subspace of $L^\infty(\mathfrak{B})$ generated by $\{\chi(\pi a) : a \in \mathfrak{A}\}$.

(d) T_π is surjective iff π is surjective, and in this case $\|v\|_\infty = \min\{\|u\|_\infty : T_\pi u = v\}$ for every $v \in L^\infty(\mathfrak{B})$.

(e) T_π is injective iff π is injective, and in this case $\|T_\pi u\|_\infty = \|u\|_\infty$ for every $u \in L^\infty(\mathfrak{A})$.

(f) T_π is order-continuous, or sequentially order-continuous, iff π is.

(g) If $\mathfrak{C}$ is another Boolean algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ is another Boolean homomorphism, then $T_{\theta\pi} = T_\theta T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{C})$.

proof Let Z and W be the Stone spaces of $\mathfrak{A}$ and $\mathfrak{B}$. By 312Q there is a continuous function $\phi : W \to Z$ such that $\widehat{\pi a} = \phi^{-1}[\widehat{a}]$ for every $a \in \mathfrak{A}$, where $\widehat{a}$ is the open-and-closed subset of Z corresponding to $a \in \mathfrak{A}$. Write T for T_π.

(a) For $u \in L^\infty(\mathfrak{A}) = C(Z)$, set $Tu = u\phi : W \to \mathbb{R}$. Then $Tu \in C(W) = L^\infty(\mathfrak{B})$. It is obvious, or at any rate very easy to check, that $T : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$ is linear, multiplicative, a Riesz homomorphism and of norm 1 unless $\mathfrak{B} = \{0\}$, $W = \emptyset$. If $a \in \mathfrak{A}$, then

$$T(\chi a) = (\chi a)\phi = (\chi\widehat{a})\phi = \chi(\phi^{-1}[\widehat{a}]) = \chi(\pi a),$$

identifying $\chi a \in L^\infty(\mathfrak{A})$ with the indicator function $\chi\widehat{a} : Z \to \{0,1\}$ of the set $\widehat{a}$. Of course $T_\pi = T$ is the only continuous linear operator with these properties, by 363Ea.

(b) Set $\alpha = \|Tu\|_\infty$, $u'(z) = \mathrm{med}(-\alpha, u(z), \alpha)$ for $z \in Z$; that is, $u' = \mathrm{med}(-\alpha e, u, \alpha e)$ in $L^\infty(\mathfrak{A})$, where e is the standard order unit of $L^\infty(\mathfrak{A})$. Then Te is the standard order unit of $L^\infty(\mathfrak{B})$, so

$$Tu' = \mathrm{med}(-\alpha Te, Tu, \alpha Te) = Tu$$

(because T is a lattice homomorphism, see 3A1Ic), while

$$\|u'\|_\infty \leq \alpha = \|Tu\|_\infty = \|Tu'\|_\infty \leq \|u'\|_\infty \leq \|u\|_\infty.$$

(c)(i) Let U be the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi a : \pi a = 0\}$, and U_0 the kernel of T. Because T is continuous and linear, U_0 is a closed linear subspace, and $T(\chi a) = \chi 0 = 0$ whenever $\pi a = 0$; so $U \subseteq U_0$. Now take any $u \in U_0$ and $\epsilon > 0$. Then $T(u^+) = (Tu)^+ = 0$, so $u^+ \in U_0$. By 354I there is a $u' \in S(\mathfrak{A})$ such that $0 \le u' \le u^+$ and $\|u^+ - u'\|_\infty \le \epsilon$. Now $0 \le Tu' \le Tu^+ = 0$, so $Tu' = 0$. Express u' as $\sum_{i=0}^n \alpha_i \chi a_i$ where $\alpha_i \ge 0$ for each i. For each i, $\alpha_i \chi(\pi a_i) = T(\alpha_i \chi a_i) = 0$, so $\pi a_i = 0$ or $\alpha_i = 0$; in either case $\alpha_i \chi a_i \in U$. Consequently $u' \in U$. As ϵ is arbitrary and U is closed, $u^+ \in U$. Similarly, $u^- = (-u)^+ \in U$ and $u = u^+ - u^- \in U$. As u is arbitrary, $U_0 \subseteq U$ and $U_0 = U$.

(ii) Let V be the closed linear subspace of $L^\infty(\mathfrak{B})$ generated by $\{\chi(\pi a) : a \in \mathfrak{A}\}$, and $V_0 = T[L^\infty(\mathfrak{A})]$. Then $T[S(\mathfrak{A})] \subseteq V$, so

$$V_0 = T[\overline{S(\mathfrak{A})}] \subseteq \overline{T[S(\mathfrak{A})]} \subseteq \overline{V} = V.$$

On the other hand, V_0 is a closed linear subspace in $L^\infty(\mathfrak{B})$. **P** It is a linear subspace because T is a linear operator. To see that it is closed, take any $v \in \overline{V}_0$. Then there is a sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ in V_0 such that $\|v - v_n\|_\infty \le 2^{-n}$ for every $n \in \mathbb{N}$. Choose $u_n \in L^\infty(\mathfrak{A})$ such that $Tu_0 = v_0$, while $Tu_n = v_n - v_{n-1}$ and $\|u_n\|_\infty = \|v_n - v_{n-1}\|_\infty$ for $n \ge 1$ (using (b) above). Then

$$\sum_{n=1}^\infty \|u_n\|_\infty \le \sum_{n=1}^\infty \|v - v_n\|_\infty + \|v - v_{n-1}\|_\infty$$

is finite, so $u = \lim_{n\to\infty} \sum_{i=0}^n u_i$ is defined in the Banach space $L^\infty(\mathfrak{A})$, and

$$Tu = \lim_{n\to\infty} \sum_{i=0}^n Tu_i = \lim_{n\to\infty} v_n = v.$$

As v is arbitrary, V_0 is closed. **Q** Since $\chi(\pi a) = T(\chi a) \in V_0$ for every $a \in \mathfrak{A}$, $V \subseteq V_0$ and $V = V_0$, as required.

(d) If π is surjective, then T is surjective, by (c-ii). If T is surjective and $b \in \mathfrak{B}$, then there is a $u \in L^\infty(\mathfrak{A})$ such that $Tu = \chi b$. Now there is a $u' \in S(\mathfrak{A})$ such that $\|u - u'\|_\infty \le \frac{1}{3}$, so that $\|Tu' - \chi b\|_\infty \le \frac{1}{3}$. Taking $a \in \mathfrak{A}$ such that $\{z : u'(z) \ge \frac{1}{2}\} = \widehat{a}$, we must have $\pi a = b$, since

$$\widehat{b} = \{w : (Tu')(w) \ge \tfrac{1}{2}\} = \phi^{-1}[\widehat{a}] = \widehat{\pi a}.$$

As b is arbitrary, π is surjective.

Now (b) tells us that in this case $\|v\|_\infty = \min\{\|u\|_\infty : Tu = v\}$ for every $v \in L^\infty(\mathfrak{B})$.

(e) By (c-i), T is injective iff π is injective. In this case, for any $u \in L^\infty(\mathfrak{A})$,

$$\|Tu\|_\infty = \|T|u|\|_\infty$$

(because T is a Riesz homomorphism)

$$\ge \sup\{\|Tu'\|_\infty : u' \in S(\mathfrak{A}),\ u' \le |u|\}$$
$$= \sup\{\|u'\|_\infty : u' \in S(\mathfrak{A}),\ u' \le |u|\}$$

(by 361Jd)

$$= \|u\|_\infty$$

(by 354I)

$$\ge \|Tu\|_\infty,$$

and $\|Tu\|_\infty = \|u\|_\infty$.

(f) If T is (sequentially) order-continuous then $\pi = T\chi$ is (sequentially) order-continuous, by 363D. If π is (sequentially) order-continuous then $\chi\pi : \mathfrak{A} \to L^\infty(\mathfrak{B})$ is (sequentially) order-continuous, so T is (sequentially) order-continuous, by 363Eb.

(g) This is elementary, in view of the uniqueness of $T_{\theta\pi}$.

363G Corollary Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, then $L^\infty(\mathfrak{C})$ can be identified, as Banach lattice and as Banach algebra, with the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi c : c \in \mathfrak{C}\}$.

(b) If $\mathcal{I}$ is an ideal of $\mathfrak{A}$, then $L^\infty(\mathfrak{A}/\mathcal{I})$ can be identified, as Banach lattice and as Banach algebra, with the quotient space $L^\infty(\mathfrak{A})/V$, where V is the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi a : a \in \mathcal{I}\}$.

proof Apply 363Fc-363Fd to the identity map from $\mathfrak{C}$ to $\mathfrak{A}$ and the canonical map from $\mathfrak{A}$ onto $\mathfrak{A}/\mathcal{I}$.

363H Representations of $L^\infty(\mathfrak{A})$ Much of the importance of the concept of $L^\infty(\mathfrak{A})$ arises from the way it is naturally represented in the contexts in which the most familiar Boolean algebras appear.

Proposition Let X be a set and Σ an algebra of subsets of X.

(a) Write $S(\Sigma)$ for the linear subspace of $\ell^\infty(X)$ generated by the indicator functions of members of Σ, and $\mathcal{L}^\infty$ for its $\| \|_\infty$-closure in $\ell^\infty(X)$.

(i) $L^\infty(\Sigma)$ can be identified, as Banach lattice and Banach algebra, with $\mathcal{L}^\infty$; if $E \in \Sigma$, then χE, defined in $L^\infty(\Sigma)$ as in 361D, can be identified with the indicator function of E regarded as a subset of X.

(ii) A bounded function $f : X \to \mathbb{R}$ belongs to $\mathcal{L}^\infty$ iff whenever $\alpha < \beta$ in $\mathbb{R}$ there is an $E \in \Sigma$ such that $\{x : f(x) > \beta\} \subseteq E \subseteq \{x : f(x) > \alpha\}$.

(iii) In particular, $L^\infty(\mathcal{P}X)$ can be identified with $\ell^\infty(X)$.

(b) Now suppose that Σ is a σ-algebra of subsets of X.

(i) $\mathcal{L}^\infty$ is just the set of bounded Σ-measurable real-valued functions on X.

(ii) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\pi : \Sigma \to \mathfrak{A}$ is a surjective sequentially order-continuous Boolean homomorphism with kernel $\mathcal{I}$, then $L^\infty(\mathfrak{A})$ can be identified, as Banach lattice and Banach algebra, with $\mathcal{L}^\infty/\mathcal{W}$, where $\mathcal{W} = \{f : f \in \mathcal{L}^\infty, \{x : f(x) \neq 0\} \in \mathcal{I}\}$ is a solid linear subspace and closed ideal of $\mathcal{L}^\infty$. For $f \in \mathcal{L}^\infty$,

$$\|f^\bullet\|_\infty = \min\{\alpha : \alpha \geq 0, \{x : |f(x)| > \alpha\} \in \mathcal{I}\}.$$

(iii) In particular, if $\mathcal{I}$ is any σ-ideal of Σ and $E \mapsto E^\bullet$ is the canonical homomorphism from Σ onto $\mathfrak{A} = \Sigma/\mathcal{I}$, then we have an identification of $L^\infty(\mathfrak{A})$ with a quotient of $\mathcal{L}^\infty$, and for any $E \in \Sigma$ we can identify $\chi(E^\bullet) \in L^\infty(\mathfrak{A})$ with the equivalence class $(\chi E)^\bullet \in \mathcal{L}^\infty/\mathcal{W}$ of the indicator function χE.

proof (a)(i) By 361L, $S(\Sigma)$, as described here, can be identified with $S(\Sigma)$ as defined in 361D. Because the normed space $\ell^\infty(X)$ is complete, $\mathcal{L}^\infty$ can be identified with the normed space completion of $S(\Sigma)$ for $\| \|_\infty$; but 363C shows that the same is true of $L^\infty(\Sigma)$. Thus we have a canonical Banach space isomorphism between $\mathcal{L}^\infty$ and $L^\infty(\Sigma)$. Because multiplication and the lattice operations are $\| \|_\infty$-continuous, both in $\mathcal{L}^\infty$ and in $L^\infty(\Sigma)$, this isomorphism is multiplicative and order-preserving, that is, identifies $\mathcal{L}^\infty$ with $L^\infty(\Sigma)$ as Banach algebra and Banach lattice. In the language of 363E, $\mathcal{L}^\infty$ is the image of $L^\infty(\Sigma)$ in $\ell^\infty(X)$ under the operator associated with the additive function $E \mapsto \chi E : \Sigma \to \ell^\infty(X)$.

(ii)(α) If $f \in \mathcal{L}^\infty$ and $\alpha < \beta$ in $\mathbb{R}$, let $g \in S(\Sigma)$ be such that $\|f - g\|_\infty \leq \frac{1}{2}(\beta - \alpha)$. Set $E = \{x : g(x) > \frac{1}{2}(\alpha + \beta)\}$; by 361G or otherwise, $E \in \Sigma$, and $\{x : f(x) > \beta\} \subseteq E \subseteq \{x : f(x) > \alpha\}$.

(β) If f satisfies the condition, take any $\epsilon > 0$. Let $n \in \mathbb{N}$ be such that $\|f\|_\infty < n\epsilon$. For $-n \leq i \leq n$, let $E_i \in \Sigma$ be such that $\{x : f(x) > (i+1)\epsilon\} \subseteq E_i \subseteq \{x : f(x) > i\epsilon\}$. Set $g(x) = \epsilon \sum_{i=-n}^{n} \chi E_i - \epsilon n$ for $x \in X$; then $g \in S(\Sigma)$ and $\|f - g\|_\infty \leq \epsilon$. As ϵ is arbitrary, $f \in \mathcal{L}^\infty$.

(iii) Now (ii) shows that if $\Sigma = \mathcal{P}X$ we shall have $\mathcal{L}^\infty = \ell^\infty(X)$ and $L^\infty(\mathcal{P}X)$ becomes identified with $\ell^\infty(X)$.

(b)(i) If Σ is a σ-algebra and $f : X \to \mathbb{R}$ is bounded then

$$\begin{aligned} f \text{ is } \Sigma\text{-measurable} &\iff \{x : f(x) > \alpha\} \in \Sigma \text{ for every } \alpha \in \mathbb{R} \\ &\iff \text{ whenever } \alpha \in \mathbb{R},\ n \in \mathbb{N} \text{ there is an } E \in \Sigma \\ &\qquad \text{such that } \{x : f(x) > \alpha + 2^{-n}\} \subseteq E \subseteq \{x : f(x) > \alpha\} \\ &\iff \text{ whenever } \beta > \alpha \text{ there is an } E \in \Sigma \\ &\qquad \text{such that } \{x : f(x) > \beta\} \subseteq E \subseteq \{x : f(x) > \alpha\} \\ &\iff f \in \mathcal{L}^\infty \end{aligned}$$

by (a-ii) above.

(ii)(α) By 363F, we have a multiplicative Riesz homomorphism $T = T_\pi$ from $L^\infty(\Sigma)$ to $L^\infty(\mathfrak{A})$ which is surjective (363Fd) and has kernel the closed linear subspace W of $L^\infty(\Sigma)$ generated by $\{\chi E : E \in \mathcal{I}\}$. Now under the identification described in (a), W corresponds to $\mathcal{W}$. **P** $\mathcal{W}$ is a linear subspace of $\mathcal{L}^\infty$ because

$$\{x : (f+g)(x) \neq 0\} \subseteq \{x : f(x) \neq 0\} \cup \{x : g(x) \neq 0\} \in \mathcal{I},$$

$$\{x : (\alpha f)(x) \neq 0\} \subseteq \{x : f(x) \neq 0\} \in \mathcal{I}$$

whenever f, $g \in \mathcal{W}$ and $\alpha \in \mathbb{R}$. If $\langle f_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathcal{W}$ converging to $f \in \mathcal{L}^\infty$, then

$$\{x : f(x) \neq 0\} \subseteq \bigcup_{n\in\mathbb{N}}\{x : f_n(x) \neq 0\} \in \mathcal{I},$$

so $f \in \mathcal{W}$. Thus $\mathcal{W}$ is a closed linear subspace of $\mathcal{L}^\infty$. If $E \in \mathcal{I}$, then χE, taken in $S(\Sigma)$ or $L^\infty(\Sigma)$, corresponds to the function $\chi E : X \to \{0,1\}$, which belongs to $\mathcal{W}$; so that W must correspond to the closed linear span in $\mathcal{L}^\infty$ of such indicator functions, which is a subspace of $\mathcal{W}$. On the other hand, if $f \in \mathcal{W}$ and $\epsilon > 0$, set

$$E_n = \{x : n\epsilon < f(x) \leq (n+1)\epsilon\}, \quad E'_n = \{x : -(n+1)\epsilon \leq f(x) < -n\epsilon\}$$

for $n \in \mathbb{N}$; all these belong to $\mathcal{I}$, so $g = \epsilon\sum_{n=0}^\infty(\chi E_n - \chi E'_n) \in \mathcal{W}$ corresponds to a member of W, while $\|f-g\|_\infty \leq \epsilon$. As W is closed, f also must correspond to some member of W. As f is arbitrary, W and $\mathcal{W}$ match exactly. **Q**

(β) Because T is a multiplicative Riesz homomorphism, $L^\infty(\mathfrak{A}) \cong L^\infty(\Sigma)/W$ is matched canonically, in its linear, order and multiplicative structures, with $\mathcal{L}^\infty/\mathcal{W}$. We know also that

$$\|v\|_\infty = \min\{\|u\|_\infty : u \in L^\infty(\Sigma),\, Tu = v\}$$

for every $v \in L^\infty(\mathfrak{A})$ (363Fd), that is, that the norm of $L^\infty(\mathfrak{A})$ corresponds to the quotient norm on $L^\infty(\Sigma)/W$.

As for the given formula for the norm, take any $f \in \mathcal{L}^\infty$. There is a $g \in \mathcal{L}^\infty$ such that $Tf = Tg$ and $\|Tf\|_\infty = \|g\|_\infty$. (Here I am treating T as an operator from $\mathcal{L}^\infty$ onto $L^\infty(\mathfrak{A})$.) In this case

$$\{x : |f(x)| > \|Tf\|_\infty\} \subseteq \{x : f(x) \neq g(x)\} \in \mathcal{I}.$$

On the other hand, if $\alpha \geq 0$ and $\{x : |f(x)| > \alpha\} \in \mathcal{I}$, and we set $h = \mathrm{med}(-\alpha\chi X, f, \alpha\chi X)$, then $Th = Tf$, so $\|Tf\|_\infty \leq \|h\|_\infty \leq \alpha$.

(iii) Put (a-i) and (ii) just above together.

363I Corollary Let (X,Σ,μ) be a measure space, with measure algebra $\mathfrak{A}$. Then $L^\infty(\mu)$ can be identified, as Banach lattice and Banach algebra, with $L^\infty(\mathfrak{A})$; the identification matches $(\chi E)^\bullet \in L^\infty(\mu)$ with $\chi(E^\bullet) \in L^\infty(\mathfrak{A})$, for every $E \in \Sigma$.

Remark The space I called $\mathcal{L}^\infty(\mu)$ in Chapter 24 is not strictly speaking the space $\mathcal{L}^\infty \cong L^\infty(\Sigma)$ of 363H; I took $\mathcal{L}^\infty(\mu) \subseteq \mathcal{L}^0(\mu)$ to be the set of essentially bounded, virtually measurable functions defined almost everywhere in X, and in general this is larger. But, as remarked in the notes to §243, $L^\infty(\mu)$ can equally well be regarded as a quotient of what I there called $\mathcal{L}^\infty_\Sigma$, which is the $\mathcal{L}^\infty$ above, because every function in $\mathcal{L}^\infty(\mu)$ is equal almost everywhere to some member of $\mathcal{L}^\infty_\Sigma$.

363J Recovering the algebra $\mathfrak{A}$: Proposition Let $\mathfrak{A}$ be a Boolean algebra. For $a \in \mathfrak{A}$ write V_a for the solid linear subspace of $L^\infty(\mathfrak{A})$ generated by χa. Then $a \mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands in $L^\infty(\mathfrak{A})$.

proof The proof is nearly identical to that of 361K. If $a \in \mathfrak{A}$, $u \in V_a$ and $v \in V_{1\setminus a}$, then $|u| \wedge |v| = 0$ because $\chi a \wedge \chi(1\setminus a) = 0$; and if $w \in L^\infty(\mathfrak{A})$ then

$$w = (w \times \chi a) + (w \times \chi(1\setminus a)) \in V_a + V_{1\setminus a}$$

because $|w \times \chi a| \leq \|w\|_\infty\chi a$ and $|w \times \chi(1\setminus a)| \leq \|w\|_\infty\chi(1\setminus a)$. So V_a and $V_{1\setminus a}$ are complementary projection bands in $L^\infty = L^\infty(\mathfrak{A})$. Next, if $U \subseteq L^\infty$ is a projection band, then $\chi 1$ is expressible as $u+v$ where $u \in U$, $v \in U^\perp$; thinking of L^∞ as the space of continuous real-valued functions on the Stone space Z of $\mathfrak{A}$, u and v must be the indicator functions of complementary subsets E, F of Z, which must be open-and-closed, so that $E = \widehat{a}$, $F = \widehat{1\setminus a}$. In this case $V_a \subseteq U$ and $V_{1\setminus a} \subseteq U^\perp$, so U must be V_a precisely. Thus $a \mapsto V_a$ is surjective. Finally, just as in 361K, $a \subseteq b \iff V_a \subseteq V_b$, so we have a Boolean isomorphism.

363K Dual spaces of L^∞ The questions treated in §362 yield nothing new in the present context. I spell out the details.

Proposition Let $\mathfrak{A}$ be a Boolean algebra. Let M, M_σ and M_τ be the L-spaces of bounded finitely additive functionals, bounded countably additive functionals and completely additive functionals on $\mathfrak{A}$. Then the embedding $S(\mathfrak{A}) \subsetneqq L^\infty(\mathfrak{A})$ induces Riesz space isomorphisms between $S(\mathfrak{A})^\sim \cong M$ and $L^\infty(\mathfrak{A})^\sim = L^\infty(\mathfrak{A})^*$, $S(\mathfrak{A})^\sim_c \cong M_\sigma$ and $L^\infty(\mathfrak{A})^\sim_c$, and $S(\mathfrak{A})^\times \cong M_\tau$ and $L^\infty(\mathfrak{A})^\times$.

proof Write $S = S(\mathfrak{A})$, $L^\infty = L^\infty(\mathfrak{A})$.

(a) For the identifications $S^\sim \cong M$, $S^\sim_c \cong M_\sigma$ and $S^\times \cong M_\tau$ see 362A.

(b) $L^{\infty*} = L^{\infty\sim}$ either because L^∞ is a Banach lattice (356Dc) or because L^∞ has an order-unit norm, so that a linear functional on L^∞ is order-bounded iff it is bounded on the unit ball.

(c) If f is a positive linear functional on L^∞, then $f{\restriction}S$ is a positive linear functional. Because S is order-dense in L^∞ (363C), the embedding is order-continuous (352Nb); so if f is (sequentially) order-continuous, so is $f{\restriction}S$. Accordingly the restriction operator $f \mapsto f{\restriction}S$ gives maps from $L^{\infty\sim}$ to $S^\sim$, $(L^\infty)^\sim_c$ to $S^\sim_c$ and $L^{\infty\times}$ to $S^\times$. If $f \in L^{\infty\sim}$ and $f{\restriction}S \ge 0$, then $f(u^+) \ge 0$ for every $u \in S$ and therefore for every $u \in L^\infty$, and $f \ge 0$; so all these restriction maps are injective positive linear operators.

(d) I need to show that they are surjective.

(i) If $g \in S^\sim$, then g is bounded on the unit ball $\{u : u \in S, \|u\|_\infty \le 1\}$, so has an extension to a continuous linear $f : L^\infty \to \mathbb{R}$ (2A4I); thus $S^\sim = \{f{\restriction}S : f \in L^{\infty\sim}\}$. This means that $f \mapsto f{\restriction}S$ is actually a Riesz space isomorphism between $L^{\infty\sim}$ and $S^\sim$. In particular, $|f|{\restriction}S = |f{\restriction}S|$ for any $f \in L^{\infty\sim}$.

(ii) If $f : L^\infty \to \mathbb{R}$ is a positive linear operator and $f{\restriction}S \in S^\sim_c$, let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence in L^∞ with infimum 0. For each n, $k \in \mathbb{N}$ there is a $v_{nk} \in S$ such that $u_n \le v_{nk} \le u_n + 2^{-k}e$, where e is the standard order unit of L^∞ (354I, as usual); set $w_n = \inf_{i,k\le n} v_{ik}$; then $\langle w_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in S with infimum 0, so

$$0 \le \inf_{n\in\mathbb{N}} f(u_n) \le \inf_{n\in\mathbb{N}} f(w_n) = 0.$$

As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $f \in (L^\infty)^\sim_c$. Consequently, for general $f \in L^{\infty\sim}$,

$$f \in (L^\infty)^\sim_c \iff |f| \in (L^\infty)^\sim_c \iff |f{\restriction}S| \in S^\sim_c \iff f{\restriction}S \in S^\sim_c,$$

and the map $f \mapsto f{\restriction}S : (L^\infty)^\sim_c \to S^\sim_c$ is a Riesz space isomorphism.

(iii) Similarly, if $f \in L^{\infty\sim}$ is non-negative and $f{\restriction}S \in S^\times$, then whenever $A \subseteq L^\infty$ is non-empty, downwards-directed and has infimum 0, $B = \{w : w \in S, \exists\, u \in A, w \ge u\}$ has infimum 0, so $\inf_{u\in A} f(u) \le \inf_{w\in B} f(w) \le 0$ and $f \in L^{\infty\times}$. As in (ii), it follows that $f \mapsto f{\restriction}S$ is a surjection from $L^{\infty\times}$ onto $S^\times$.

***363L Integration with respect to a finitely additive functional (a)** If $\mathfrak{A}$ is a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ is a bounded additive functional, then by 363K we have a corresponding functional $f_\nu \in L^\infty(\mathfrak{A})^*$ defined by saying that $f_\nu(\chi a) = \nu a$ for every $a \in \mathfrak{A}$. There are contexts in which it is convenient, and even helpful, to use the formula $\fint u\,d\nu$ in place of $f_\nu(u)$ for $u \in L^\infty = L^\infty(\mathfrak{A})$. When doing so, we must of course remember that we may have lost some of the standard properties of 'integration'. But enough of our intuitions (including, for instance, the idea of stochastic independence) remain valid to make the formula a guide to interesting ideas.

(b) Let M be the L-space of bounded finitely additive functionals on $\mathfrak{A}$ (362B). Then we have a function $(u,\nu) \mapsto \fint u\,d\nu : L^\infty \times M \to \mathbb{R}$. Now this map is bilinear. **P** For μ, $\nu \in M$, u, $v \in L^\infty$ and $\alpha \in \mathbb{R}$,

$$\fint u+v\,d\nu = \fint u\,d\nu + \fint v\,d\nu, \quad \fint \alpha u\,d\nu = \alpha\fint u\,d\nu$$

just because f_ν is linear. On the other side, we have

$$(f_\mu + f_\nu)(\chi a) = f_\mu(\chi a) + f_\nu(\chi a) = \mu a + \nu a = (\mu+\nu)(a) = f_{\mu+\nu}(\chi a)$$

for every $a \in \mathfrak{A}$, so that $f_\mu + f_\nu$ and $f_{\mu+\nu}$ must agree on $S(\mathfrak{A})$ and therefore on L^∞. But this means that $\fint u\,d(\mu+\nu) = \fint u\,d\mu + \fint u\,d\nu$. Similarly, $\fint u\,d(\alpha\mu) = \alpha\fint u\,d\mu$. **Q**

(c) If ν is non-negative, we have $\fint u\,d\nu \ge 0$ whenever $u \ge 0$, as in part (c) of the proof of 363K. Consequently, for any $\nu \in M$ and $u \in L^\infty$,

$$\begin{aligned} |\fint u\,d\nu| &= |\fint u^+\,d\nu^+ - \fint u^-d\nu^+ - \fint u^+d\nu^- + \fint u^-\,d\nu^-| \\ &\le \fint u^+\,d\nu^+ + \fint u^-d\nu^+ + \fint u^+d\nu^- + \fint u^-\,d\nu^- \\ &= \fint |u| d|\nu| \le \fint \|u\|_\infty \chi 1\,d|\nu| = \|u\|_\infty|\nu|(1) = \|u\|_\infty\|\nu\|. \end{aligned}$$

So $(u,\nu) \mapsto \fint u\,d\nu$ has norm (as defined in 253Ab) at most 1. If $\mathfrak{A} \ne 0$, the norm is exactly 1. (For this we need to know that there is a $\nu \in M^+$ such that $\nu 1 = 1$. Take any z in the Stone space of $\mathfrak{A}$ and set $\nu a = 1$ if $z \in \widehat{a}$, 0 otherwise.)

(d) We do not have any result corresponding to B.Levi's theorem in this language, because (even if ν is non-negative and countably additive) there is no reason to suppose that $\sup_{n\in\mathbb{N}} u_n$ is defined in L^∞ just because $\sup_{n\in\mathbb{N}}\oint u_n d\nu$ is finite. But if ν is countably additive and $\mathfrak{A}$ is Dedekind σ-complete, we have something corresponding to Lebesgue's Dominated Convergence Theorem (363Yg).

(e) One formula which we can imitate in the present context is that of 252O, where the ordinary integral is represented in the form

$$\oint f d\mu = \int_0^\infty \mu\{x : f(x) \geq t\}dt$$

for non-negative f. In the context of general Boolean algebras, we cannot directly represent the set $[\![f \geq t]\!] = \{x : f(x) \geq t\}$ (though in the next section I will show that in Dedekind σ-complete Boolean algebras there is an effective expression of this idea, and I will use it in the principal definition of §365). But what we can say is the following. If $\mathfrak{A}$ is any Boolean algebra, and $\nu : \mathfrak{A} \to [0, \infty[$ is a non-negative additive functional, and $u \in L^\infty(\mathfrak{A})^+$, then

$$\oint u\, d\nu = \int_0^\infty \sup\{\nu a : t\chi a \leq u\}dt,$$

where the right-hand integral is taken with respect to Lebesgue measure. **P** (i) For $t \geq 0$ set $h(t) = \sup\{\nu a : t\chi a \leq u\}$. Then h is non-increasing and zero for $t > \|u\|_\infty$, so $\int_0^\infty h(t)dt$ is defined in $\mathbb{R}$. If we set $h_n(t) = h(2^{-n}(k+1))$ whenever $k, n \in \mathbb{N}$ and $2^{-n}k \leq t < 2^{-n}(k+1)$, then $\langle h_n(t)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence which converges to $h(t)$ whenever h is continuous at t, which is almost everywhere (222A, or otherwise); so $\int_0^\infty h(t)dt = \lim_{n\to\infty} \int_0^\infty h_n(t)dt$. Next, given $n \in \mathbb{N}$ and $\epsilon > 0$, we can choose for each $k \leq k^* = \lfloor 2^n\|u\|_\infty \rfloor$ an a_k such that $2^{-n}(k+1)\chi a_k \leq u$ and $\nu a_k \geq h(2^{-n}(k+1)) - \epsilon$. In this case $\sum_{k=0}^{k^*} 2^{-n}\chi a_k \leq u$, so

$$\begin{aligned}\int_0^\infty h_n(t)dt = 2^{-n}\sum_{k=0}^{k^*} h(2^{-n}(k+1)) &\leq \|u\|_\infty\epsilon + 2^{-n}\sum_{k=0}^{k^*}\nu a_k \\ &= \|u\|_\infty\epsilon + \oint \sum_{k=0}^{k^*} 2^{-n}\chi a_k d\nu \leq \|u\|_\infty\epsilon + \oint u\, d\nu.\end{aligned}$$

As n and ϵ are arbitrary, $\int_0^\infty h(t)dt \leq \oint u\, d\nu$. (ii) In the other direction, there is for any $\epsilon > 0$ a $v \in S(\mathfrak{A})$ such that $v \leq u \leq v + \epsilon\chi 1$. If we express v as $\sum_{j=0}^m \gamma_j\chi c_j$ where $c_0 \supseteq \ldots \supseteq c_m$ and $\gamma_j \geq 0$ for every j (361Ec), then we shall have $h(t) \geq \nu c_k$ whenever $t \leq \sum_{j=0}^k \gamma_j$, so

$$\int_0^\infty h(t)dt \geq \sum_{k=0}^m \gamma_k\nu c_k = \oint v\, d\nu \geq \oint u\, d\nu - \epsilon\nu 1.$$

As ϵ is arbitrary, $\int_0^\infty h(t)dt \geq \oint u\, d\nu$ and the two 'integrals' are equal. **Q**

(f) The formula $\oint d\nu$ is especially natural when $\mathfrak{A}$ is an algebra of sets, so that L^∞ can be directly interpreted as a space of functions (363Ha); better still, when $\mathfrak{A}$ is actually a σ-algebra of subsets of a set X, L^∞ can be identified with the space of bounded $\mathfrak{A}$-measurable functions on X, as in 363Hb. So in such contexts I may write $\oint g\, d\nu$ or even $\oint g(x)\nu(dx)$ when $g : X \to \mathbb{R}$ is bounded and $\mathfrak{A}$-measurable, and $\nu : \mathfrak{A} \to \mathbb{R}$ is a bounded additive functional. But I will try to take care to signal any such deviation from the normal principle that the symbol $\int$ refers to the sequentially order-continuous integral defined in §122 with the minor modifications introduced in §§133 and 135.

363M Now I come to a fundamental fact underlying a number of theorems in both this volume and the last.

Theorem Let $\mathfrak{A}$ be a Boolean algebra.

(a) $\mathfrak{A}$ is Dedekind σ-complete iff $L^\infty(\mathfrak{A})$ is Dedekind σ-complete.

(b) $\mathfrak{A}$ is Dedekind complete iff $L^\infty(\mathfrak{A})$ is Dedekind complete.

proof (a)(i) Suppose that $\mathfrak{A}$ is Dedekind σ-complete. By 314M, we may identify $\mathfrak{A}$ with a quotient $\Sigma/\mathcal{M}$, where $\mathcal{M}$ is the ideal of meager subsets of the Stone space Z of $\mathfrak{A}$, and $\Sigma = \{E\triangle A : E \in \mathcal{E},\ A \in \mathcal{M}\}$, writing $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$ for the algebra of open-and-closed subsets of Z. By 363Hb, $L^\infty = L^\infty(\mathfrak{A})$ can be identified with $\mathcal{L}^\infty/\mathcal{V}$, where $\mathcal{L}^\infty$ is the space of bounded Σ-measurable functions from Z to $\mathbb{R}$, and $\mathcal{V}$ is the space of functions zero except on a member of $\mathcal{M}$.

Now suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^∞ with an upper bound $u \in L^\infty$. Express u_n, u as $f_n^\bullet$, $f^\bullet$ where f_n, $f \in \mathcal{L}^\infty$. Set $g(z) = \sup_{n\in\mathbb{N}} \min(f_n(z), f(z))$ for every $z \in Z$; then $g \in \mathcal{L}^\infty$ (121F), so we have a corresponding member $v = g^\bullet$ of L^∞. For each $n \in \mathbb{N}$, $u \geq u_n$ so $(f_n - f)^+ \in \mathcal{V}$,

$$\{z : f_n(z) > g(z)\} \subseteq \{z : f_n(z) > f(z)\} \in \mathcal{M}$$

and $v \ge u_n$. If $w \in L^\infty$ and $w \ge u_n$ for every n, then express w as $h^\bullet$ where $h \in \mathcal{L}^\infty$; we have $(f_n - h)^+ \in \mathcal{V}$ for every n, so

$$\{z : g(z) > h(z)\} \subseteq \bigcup_{n\in\mathbb{N}}\{z : f_n(z) > h(z)\} \in \mathcal{M}$$

because $\mathcal{M}$ is a σ-ideal, and $(g-h)^+ \in \mathcal{V}$, so $w \ge v$. Thus $v = \sup_{n\in\mathbb{N}} u_n$ in L^∞. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, L^∞ is Dedekind σ-complete (using 353G).

(ii) Now suppose that L^∞ is Dedekind σ-complete, and that A is a countable non-empty set in $\mathfrak{A}$. In this case $\{\chi a : a \in A\}$ has a least upper bound u in L^∞. Take $v \in S(\mathfrak{A})$ such that $0 \le v \le u$ and $\|u - v\|_\infty \le \frac{1}{3}$; set $b = [\![v > \frac{1}{3}]\!]$, as defined in 361Eg. If $a \in A$, then $\|(\chi a - v)^+\|_\infty \le \|u - v\|_\infty \le \frac{1}{3}$, so $\frac{2}{3}\chi a \le v$ and $a \subseteq b$. If $c \in \mathfrak{A}$ is any upper bound for A, then $v \le u \le \chi c$ so $b \subseteq c$. Thus $b = \sup A$ in $\mathfrak{A}$. As A is arbitrary, $\mathfrak{A}$ is Dedekind σ-complete.

(b)(i) For the second half of this theorem I use an argument which depends on joining the representation described in (a-i) above with the original definition of L^∞ in 363A. The point is that $C(Z) \subseteq \mathcal{L}^\infty$, and for any $f \in C(Z) = L^\infty(\mathfrak{A})$ its equivalence class $f^\bullet$ in $\mathcal{L}^\infty/\mathcal{V}$ corresponds to f itself. **P** Perhaps it will help to give a name T to the canonical isomorphism from $\mathcal{L}^\infty/\mathcal{V}$ to L^∞. Then $V = \{f : Tf^\bullet = f\}$ is a closed linear subspace of $C(Z)$, because $f \mapsto f^\bullet$ and T are continuous linear operators. But if $a \in \mathfrak{A}$, then $(\widehat{a})^\bullet$, the equivalence class of $\widehat{a} \in \Sigma$ in $\Sigma/\mathcal{M}$, corresponds to a (see the proof of 314M), so $(\chi\widehat{a})^\bullet \in \mathcal{L}^\infty/\mathcal{V}$ corresponds to χa; that is, $T(\chi\widehat{a})^\bullet = \chi\widehat{a}$, if we identify $\chi a \in L^\infty$ with $\chi\widehat{a} : Z \to \{0,1\}$. So V contains $\chi\widehat{a}$ for every $a \in \mathfrak{A}$; because V is a linear subspace, $S(\mathfrak{A}) \subseteq V$; because V is closed, $L^\infty \subseteq V$. **Q**

For a general $f \in \mathcal{L}^\infty$, $g = Tf^\bullet$ must be the unique member of $C(Z)$ such that $g^\bullet = f^\bullet$, that is, such that $\{z : g(z) \ne f(z)\}$ is meager.

(ii) Suppose now that $\mathfrak{A}$ is actually Dedekind complete. In this case Z is extremally disconnected (314S). Consequently every open set belongs to Σ. **P** If G is open, then $\overline{G}$ is open-and-closed; but $A = \overline{G} \setminus G$ is a closed set with empty interior, so is meager, and $G = \overline{G} \triangle A \in \Sigma$. **Q**

Let $A \subseteq L^\infty = C(Z)$ be any non-empty set with an upper bound in $C(Z)$. For each $z \in Z$ set $g(z) = \sup_{u\in A} u(z)$. Then

$$G_\alpha = \{z : g(z) > \alpha\} = \bigcup_{u\in A}\{z : u(z) > \alpha\}$$

is open for every $\alpha \in \mathbb{R}$ (that is, g is lower semi-continuous). Thus $G_\alpha \in \Sigma$ for every α, so $g \in \mathcal{L}^\infty$, and $v = Tg^\bullet$ is defined in $C(Z)$. For any $u \in A$, $g \ge u$ in $\mathcal{L}^\infty$, so

$$v = Tg^\bullet \ge Tu^\bullet = u$$

in L^∞; thus v is an upper bound for A in L^∞. On the other hand, if w is any upper bound for A in $L^\infty = C(Z)$, then surely $w(z) \ge u(z)$ for every $z \in Z$ and $u \in A$, so $w \ge g$ and

$$w = Tw^\bullet \ge Tg^\bullet = v.$$

This means that v is the least upper bound of A. As A is arbitrary, L^∞ is Dedekind complete.

(iii) Finally, if L^∞ is Dedekind complete, then the argument of (a-ii), applied to arbitrary non-empty subsets A of $\mathfrak{A}$, shows that $\mathfrak{A}$ also is Dedekind complete.

363N Much of the importance of L^∞ spaces in the theory of Riesz spaces arises from the next result.

Proposition Let U be a Dedekind σ-complete Riesz space with an order unit. Then U is isomorphic, as Riesz space, to $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the algebra of projection bands in U.

proof (a) By 353M, U is isomorphic to a norm-dense Riesz subspace of $C(X)$ for some compact Hausdorff space X; for the rest of this argument, therefore, we may suppose that U actually is such a subspace.

(b) $U = C(X)$. **P** If $g \in C(X)$ then by 354I there are sequences $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle f'_n\rangle_{n\in\mathbb{N}}$ in U such that $f_n \le g \le g_n$ and $\|g_n - f_n\|_\infty \le 2^{-n}$ for every n. Now $\{f_n : n \in \mathbb{N}\}$ has a least upper bound f in U; since we must have $f_n \le f \le g_n$ for every n, $f = g$ and $g \in U$. **Q**

(c) Next, X is zero-dimensional. **P** Suppose that $G \subseteq X$ is open and $x \in G$. Then there is an open set G_1 such that $x \in G_1 \subseteq \overline{G}_1 \subseteq G$ (3A3Bb). There is an $f \in C(X)$ such that $0 \le f \le \chi G_1$ and $f(x) > 0$ (also by 3A3Bb); write H for $\{y : f(y) > 0\}$. Set $g = \sup_{n\in\mathbb{N}}(nf \wedge \chi X)$, the supremum being taken in $U = C(X)$. For each $y \in H$, we must have $g(y) \ge \min(1, nf(y))$ for every n, so that $g(y) = 1$. On the other hand, if $y \in X \setminus \overline{H}$, there is an $h \in C(X)$ such that $h(y) > 0$ and $0 \le h \le \chi(X \setminus \overline{H})$; now $h \wedge f = 0$ so $h \wedge g = 0$ and $g(y) = 0$. Thus $\chi H \le g \le \chi\overline{H}$. The set $\{y : g(y) \in \{0,1\}\}$ is closed and includes $H \cup (X \setminus \overline{H})$ so must be the whole of X; thus $G_2 = \{y : g(y) > \frac{1}{2}\} = \{y : g(y) \ge \frac{1}{2}\}$ is open-and-closed, and we have

$$x \in H \subseteq G_2 \subseteq \overline{H} \subseteq \overline{G}_1 \subseteq G.$$

As x, G are arbitrary, the set of open-and-closed subsets of X is a base for the topology of X, and X is zero-dimensional. **Q**

(d) We can therefore identify X with the Stone space of its algebra $\mathcal{E}$ of open-and-closed sets (311J). But in this case 363A immediately identifies $U = C(X)$ with $L^\infty(\mathcal{E})$. By 363J, $\mathcal{E}$ is isomorphic to $\mathfrak{A}$, so $U \cong L^\infty(\mathfrak{A})$.

Remark Note that in part (c) of the argument above, we have to take care over the interpretation of 'sup'. In the space of all real-valued functions on X, the supremum of $\{nf \wedge \chi X : n \in \mathbb{N}\}$ is just χH. But g is supposed to be the least *continuous* function greater than or equal to $nf \wedge \chi X$ for every n, and is therefore likely to be strictly greater than χH, even though sandwiched between χH and $\chi\overline{H}$.

363O Corollary Let U be a Dedekind σ-complete M-space. Then U is isomorphic, as Banach lattice, to $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the algebra of projection bands of U.

proof This is merely the special case of 363N in which U is known from the start to be complete under an order-unit norm.

363P Corollary Let U be any Dedekind σ-complete Riesz space and $e \in U^+$. Then the solid linear subspace U_e of U generated by e is isomorphic, as Riesz space, to $L^\infty(\mathfrak{A})$ for some Dedekind σ-complete Boolean algebra $\mathfrak{A}$; and if U is Dedekind complete, so is $\mathfrak{A}$.

proof Because U is Dedekind σ-complete, so is U_e (353J(a-i)). Apply 363N to U_e to see that $U_e \cong L^\infty(\mathfrak{A})$ for some $\mathfrak{A}$. Because U_e is Dedekind σ-complete, so is $\mathfrak{A}$, by 363Ma; while if U is Dedekind complete, so are U_e and $\mathfrak{A}$, by 353J(b-i) and 363Mb.

363Q The next theorem will be a striking characterization of the Dedekind complete L^∞ spaces as normed spaces. As a warming-up exercise I give a much simpler result concerning their nature as Banach lattices.

Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. Then for any Banach lattice U, a linear operator $T : U \to L^\infty = L^\infty(\mathfrak{A})$ is continuous iff it is order-bounded, and in this case $\|T\| = \||T|\|$, where the modulus $|T|$ is taken in $\mathsf{L}^\sim(U; L^\infty)$.

proof It is generally true that order-bounded operators between Banach lattices are continuous (355C). If $T : U \to L^\infty$ is continuous, then for any $w \in U^+$

$$|u| \le w \Longrightarrow \|u\| \le \|w\| \Longrightarrow \|Tu\|_\infty \le \|T\|\|w\| \Longrightarrow |Tu| \le \|T\|\|w\|ee,$$

where e is the standard order unit of L^∞. So T is order-bounded. As L^∞ is Dedekind complete (363Mb), $|T|$ is defined in $\mathsf{L}^\sim(U; L^\infty)$ (355Ea). For any $w \in U$,

$$|T||w| = \sup\{|Tu| : |u| \le |w|\} \le \|T\|\|w\|e,$$

so $\||T|(w)\| \le \|T\|\|w\|$; accordingly $\||T|\| \le \|T\|$. On the other hand, of course,

$$|Tw| \le |T||w| \le \||T|\|\|w\|e$$

for every $w \in U$, so $\|T\| \le \||T|\|$ and the two norms are equal.

Remark Of course what is happening here is that the spaces $L^\infty(\mathfrak{A})$, for Dedekind complete $\mathfrak{A}$, are just the Dedekind complete M-spaces; this is an elementary consequence of 363N and 363M.

363R Now for something much deeper.

Theorem Let U be a normed space over $\mathbb{R}$. Then the following are equiveridical:

(i) there is a Dedekind complete Boolean algebra $\mathfrak{A}$ such that U is isomorphic, as normed space, to $L^\infty(\mathfrak{A})$;

(ii) whenever V is a normed space, V_0 a linear subspace of V, and $T_0 : V_0 \to U$ is a bounded linear operator, there is an extension of T_0 to a bounded linear operator $T : V \to U$ with $\|T\| = \|T_0\|$.

proof For the purposes of the argument below, let us say that a normed space U satisfying the condition (ii) has the 'Hahn-Banach property'.

Part A: (i)⇒(ii) I have to show that $L^\infty(\mathfrak{A})$ has the Hahn-Banach property for every Dedekind complete Boolean algebra $\mathfrak{A}$. Let V be a normed space, V_0 a linear subspace of V, and $T_0 : V_0 \to L^\infty = L^\infty(\mathfrak{A})$ a bounded linear operator. Set $\gamma = \|T_0\|$.

Let $\mathfrak{P}$ be the set of all functions T such that $\operatorname{dom} T$ is a linear subspace of V including V_0 and $T : \operatorname{dom} T \to U$ is a bounded linear operator extending T_0 and with norm at most γ. Order $\mathfrak{P}$ by saying that $T_1 \le T_2$ if T_2 extends T_1. Then any non-empty totally ordered subset $\mathfrak{Q}$ of $\mathfrak{P}$ has an upper bound in $\mathfrak{P}$. **P** Set $\operatorname{dom} T = \bigcup\{\operatorname{dom} T_1 : T_1 \in \mathfrak{Q}\}$, $Tv = T_1 v$ whenever $T_1 \in \mathfrak{Q}$ and $v \in \operatorname{dom} T_1$; it is elementary to check that $T \in \mathfrak{P}$, so that T is an upper bound for $\mathfrak{Q}$ in $\mathfrak{P}$. **Q**

By Zorn's Lemma, $\mathfrak{P}$ has a maximal element $\tilde T$. Now $\operatorname{dom}\tilde T = V$. **P?** Suppose, if possible, otherwise. Write $\tilde V = \operatorname{dom}\tilde T$ and take any $\tilde v \in V \setminus \tilde V$; let V_1 be the linear span of $\tilde V \cup \{\tilde v\}$, that is, $\{v + \alpha\tilde v : v \in \tilde V, \alpha \in \mathbb{R}\}$.

If v_1, $v_2 \in \tilde V$ then, writing e for the standard order unit of L^∞,

$$\begin{aligned}\tilde T v_1 + \tilde T v_2 = \tilde T(v_1 + v_2) &\le \|\tilde T(v_1 + v_2)\|_\infty e \\ &\le \gamma\|v_1 + v_2\|e \le \gamma\|v_1 - \tilde v\|e + \gamma\|v_2 + \tilde v\|e,\end{aligned}$$

so

$$\tilde T v_1 - \gamma\|v_1 - \tilde v\|e \le \gamma\|v_2 + \tilde v\|e - \tilde T v_2.$$

Because L^∞ is Dedekind complete (363Mb),

$$\tilde u = \sup_{v_1 \in \tilde V} \tilde T v_1 - \gamma\|v_1 - \tilde v\|e$$

is defined in L^∞ and $\tilde u \le \gamma\|v_2 + \tilde v\|e - Tv_2$ for every $v_2 \in \tilde V$. Putting these together, we have

$$\tilde T v + \tilde u \le \gamma\|v + \tilde v\|e, \quad \tilde T v - \tilde u \le \gamma\|v - \tilde v\|e$$

for all $v \in \tilde V$. Consequently, if $v \in \tilde V$, then for $\alpha > 0$

$$\tilde T v + \alpha\tilde u = \alpha(\tilde T(\tfrac{1}{\alpha}v) + \tilde u) \le \alpha\gamma\|\tfrac{1}{\alpha}v + \tilde v\|e = \gamma\|v + \alpha\tilde v\|e,$$

while for $\alpha < 0$

$$\tilde T v + \alpha\tilde u = |\alpha|(\tilde T(-\tfrac{1}{\alpha}v) - \tilde u) \le |\alpha|\gamma\| - \tfrac{1}{\alpha}v - \tilde v\|e = \gamma\|v + \alpha\tilde v\|e,$$

and of course

$$\tilde T v \le \|\tilde T v\|_\infty e \le \gamma\|v\|e.$$

So we have

$$\tilde T v + \alpha\tilde u \le \gamma\|v + \alpha\tilde v\|e$$

for every $v \in \tilde V$, $\alpha \in \mathbb{R}$.

Define $T_1 : V_1 \to L^\infty$ by setting $T_1(v + \alpha\tilde v) = \tilde T v + \alpha\tilde u$ for every $v \in \tilde V$, $\alpha \in \mathbb{R}$. (This is well-defined because $\tilde v \notin \tilde V$, so any member of V_1 is uniquely expressible as $v + \alpha\tilde v$ where $v \in \tilde V$ and $\alpha \in \mathbb{R}$.) Then T_1 is a linear operator, extending T_0, from a linear subspace of V to L^∞. But from the calculations above we know that $T_1 v \le \gamma\|v\|e$ for every $v \in V_1$; since we also have

$$T_1 v = -T_1(-v) \ge -\gamma\| - v\|e = -\gamma\|v\|e,$$

$\|T_1 v\|_\infty \le \gamma\|v\|$ for every $v \in V_1$, and $T_1 \in \mathfrak{P}$. But now T_1 is a member of $\mathfrak{P}$ properly extending $\tilde T$, which is supposed to be impossible. **XQ**

Accordingly $\tilde T : V \to L^\infty$ is an extension of T_0 to the whole of V, with the same norm as T_0. As V and T_0 are arbitrary, L^∞ has the Hahn-Banach property.

Part B: (ii)⇒(i) Now let U be a normed space with the Hahn-Banach property. If $U = \{0\}$ then of course it is isomorphic to $L^\infty(\mathfrak{A})$, where $\mathfrak{A} = \{0\}$, so henceforth I will take it for granted that $U \ne \{0\}$.

(a) Let Z be the unit ball of the dual U^* of U, with the weak* topology. Then Z is a compact Hausdorff space (3A5F). For $u \in U$ set $Z_u = \{z : z \in Z, |z(u)| = \|u\|\}$; then Z_u is a closed subset of Z (because $f \mapsto f(u)$ is continuous), and is non-empty, by the Hahn-Banach theorem (3A5Ab, or Part A above!) Now let $\mathfrak{P}$ be the set of those closed sets $X \subseteq Z$ such that $X \cap Z_u \ne \emptyset$ for every $u \in U$. If $\mathfrak{Q} \subseteq \mathfrak{P}$ is non-empty and totally ordered, then $\bigcap \mathfrak{Q} \in \mathfrak{P}$, because for any $u \in U$

$$\{X \cap Z_u : X \in \mathfrak{Q}\}$$

is a downwards-directed family of non-empty compact sets, so must have non-empty intersection. By Zorn's Lemma, upside down, $\mathfrak{P}$ has a minimal element X; with its relative topology, X is a compact Hausdorff space.

(b) We have a linear operator $R : U \to C(X)$ given by setting $(Ru)(x) = x(u)$ for every $u \in U$, $x \in X$; because $X \subseteq Z$, $\|Ru\|_\infty \le \|u\|$, and because $X \in \mathfrak{P}$, $\|Ru\|_\infty = \|u\|$, for every $u \in U$. Moreover, if $G \subseteq X$ is a non-empty open set (in the relative topology of X) then $X \setminus G$ cannot belong to $\mathfrak{P}$, because X is minimal, so there is a (non-zero) $u \in U$ such that $|x(u)| < \|u\|$ for every $x \in X \setminus G$. Replacing u by $\|u\|^{-1}u$ if need be, we may suppose that $\|u\| = 1$.

What this means is that $W = R[U]$ is a linear subspace of $C(X)$ which is isomorphic, as normed space, to U, and has the property that whenever $G \subseteq X$ is a non-empty relatively open set there is an $f \in W$ such that $\|f\|_\infty = 1$ and $|f(x)| < 1$ for every $x \in X \setminus G$. Observe that, because $X \setminus G$ is compact, there is now some $\alpha < 1$ such that $|f(u)| \le \alpha$ for every $f \in X \setminus G$.

Because W is isomorphic to U, it has the Hahn-Banach property.

(c) Now consider $V = \ell^\infty(X)$, $V_0 = W$, $T_0 : V_0 \to W$ the identity map. Because W has the Hahn-Banach property, there is a linear operator $T : \ell^\infty(X) \to W$, extending T_0, and of norm $\|T_0\| = 1$.

(d) If $h \in \ell^\infty(X)$ and $x_0 \in X \setminus \overline{\{x : h(x) \ne 0\}}$, then $(Th)(x_0) = 0$. **P?** Otherwise, set $G = \{y : y \in X \setminus \overline{\{x : h(x) \ne 0\}}, (Th)(y) \ne 0\}$. This is a non-empty open set in X, so there are $f \in W$, $\alpha < 1$ such that $\|f\|_\infty = 1$ and $|f(x)| \le \alpha$ for every $x \in X \setminus G$.

Because $\|f\|_\infty = 1$, there must be some $x_1 \in X$ such that $|f(x_1)| = 1$, and of course $x_1 \in G$, so that $(Th)(x_1) \ne 0$. But let $\delta > 0$ be such that $\delta\|h\|_\infty \le 1 - \alpha$. Then, because $h(x) = 0$ for $x \in G$, $|f(x)| + |\delta h(x)| \le 1$ for every $x \in X$, and $\|f + \delta h\|_\infty$, $\|f - \delta h\|_\infty$ are both less than or equal to 1. As $Tf = f$ and $\|T\| = 1$, this means that

$$\|f + \delta Th\|_\infty \le 1, \quad \|f - \delta Th\|_\infty \le 1;$$

consequently

$$|f(x_1)| + \delta|(Th)(x_1)| = \max(|(f + \delta Th)(x_1)|, |(f - \delta Th)(x_1)|) \le 1.$$

But $|f(x_1)| = 1$ and $\delta(Th)(x_1) \ne 0$, so this is impossible. **XQ**

(e) It follows that $Th = h$ for every $h \in C(X)$. **P?** Suppose, if possible, otherwise. Then there is a $\delta > 0$ such that $G = \{x : |(Th)(x) - h(x)| > \delta\}$ is not empty. Let $f \in W$ be such that $\|f\| = 1$ but $|f(x)| < 1$ for every $x \in X \setminus G$. Then there is an $x_0 \in X$ such that $|f(x_0)| = 1$; of course x_0 must belong to G. Set $f_1 = \frac{h(x_0)}{f(x_0)}f$, so that $f_1 \in W$ and $f_1(x_0) = h(x_0)$. Set

$$h_1(x) = \operatorname{med}(h(x) - \delta, f_1(x), h(x) + \delta)$$

for $x \in X$. Then $h_1 \in C(X)$. Setting

$$H = \{x : |h(x) - h(x_0)| + |f_1(x) - f_1(x_0)| < \delta\},$$

H is an open set containing x_0 and

$$|f_1(x) - h(x)| \le |f_1(x_0) - h(x_0)| + \delta = \delta, \quad h_1(x) = f_1(x)$$

for every $x \in H$. Consequently $x_0 \notin \overline{\{x : (f_1 - h_1)(x) \ne 0\}}$, and $T(f_1 - h_1)(x_0) = 0$, by (d). But this means that

$$(Th_1)(x_0) = (Tf_1)(x_0) = f_1(x_0) = h(x_0),$$

so that

$$|h(x_0) - (Th)(x_0)| = |T(h_1 - h)(x_0)| \le \|T(h_1 - h)\|_\infty \le \|h_1 - h\|_\infty \le \delta,$$

which is impossible, because $x_0 \in G$. **XQ**

(f) This tells us at once that $W = C(X)$. But (d) also tells us that X is extremally disconnected. **P** Let $G \subseteq X$ be any open set. Then $\chi X = \chi G + \chi(X \setminus G)$, so

$$\chi X = T(\chi X) = h_1 + h_2,$$

where $h_1 = T(\chi G)$, $h_2 = T(\chi(X \setminus G))$. Now from (d) we see that h_1 must be zero on $X \setminus \overline{G}$ while h_2 must be zero on G. Thus we have $h_1(x) = 1$ for $x \in G$; as h_1 is continuous, $h_1(x) = 1$ for $x \in \overline{G}$, and $h_1 = \chi\overline{G}$. Of course it follows that $\overline{G}$ is open. As G is arbitrary, X is extremally disconnected. **Q**

(g) Being also compact and Hausdorff, therefore regular (3A3Bb), X is zero-dimensional (3A3Bd). We may therefore identify X with the Stone space of its regular open algebra $\mathrm{RO}(X)$ (314S), and $W = C(X)$ with $L^\infty(\mathrm{RO}(X))$. Thus $R : U \to C(X)$ is a Banach space isomorphism between U and $C(X) \cong L^\infty(\mathrm{RO}(X))$; so U is of the type declared.

363S The Banach-Ulam problem At a couple of points already (232Hc, the notes to §326) I have remarked on a problem which was early recognised as a fundamental question in abstract measure theory. I now set out some formulations of the problem which arise naturally from the work done so far. I will do this by writing down a list of equiveridical statements; the 'Banach-Ulam problem' asks whether they are true.

I should remark that this is not generally counted as an 'open' problem. It is in fact believed by most of us that these statements are independent of the usual axioms of Zermelo-Fraenkel set theory, including the axiom of choice and even the continuum hypothesis. As such, this problem belongs to Volume 5 rather than anywhere earlier, but its manifestations will become steadily more obtrusive as we continue through this volume and the next, and I think it will be helpful to begin collecting them now. The ideas needed to show that the statements here imply each other are already accessible; in particular, they involve no set theory beyond Zorn's Lemma. These implications constitute the following theorem, derived from Luxemburg 67a.

Theorem The following statements are equiveridical.

(i) There are a set X and a probability measure ν, with domain $\mathcal{P}X$, such that $\nu\{x\} = 0$ for every $x \in X$.

(ii) There are a localizable measure space (X, Σ, μ) and an absolutely continuous countably additive functional $\nu : \Sigma \to \mathbb{R}$ which is not truly continuous, so has no Radon-Nikodým derivative (definitions: 232Ab, 232Hf).

(iii) There are a Dedekind complete Boolean algebra $\mathfrak{A}$ and a countably additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ which is not completely additive.

(iv) There is a Dedekind complete Riesz space U such that $U_c^\sim \neq U^\times$.

proof (a)(i)⇒(ii) Let X be a set with a probability measure ν, defined on $\mathcal{P}X$, such that $\nu\{x\} = 0$ for every $x \in X$. Let μ be counting measure on X. Then $(X, \mathcal{P}X, \mu)$ is strictly localizable, and $\nu : \mathcal{P}X \to \mathbb{R}$ is countably additive; also $\nu E = 0$ whenever μE is finite, so ν is absolutely continuous with respect to μ. But if $\mu E < \infty$ then E is finite and $\nu(X \setminus E) = 1$, so ν is not truly continuous, and has no Radon-Nikodým derivative (232D).

(b)(ii)⇒(iii) Let (X, Σ, μ) be a localizable measure space and $\nu : \Sigma \to \mathbb{R}$ an absolutely continuous countably additive functional which is not truly continuous. Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ; then we have an absolutely continuous countably additive functional $\bar\nu : \mathfrak{A} \to \mathbb{R}$ defined by setting $\bar\nu E^\bullet = \nu E$ for every $E \in \Sigma$ (327C). Since ν is not truly continuous, $\bar\nu$ is not completely additive (327Ce). Also $\mathfrak{A}$ is Dedekind complete, because μ is localizable, so $\mathfrak{A}$ and $\bar\nu$ witness (iii).

(c)(iii)⇒(i) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional which is not completely additive. Because ν is bounded (326M), therefore expressible as the difference of non-negative countably additive functionals (326L), there must be a non-negative countably additive functional ν' on $\mathfrak{A}$ which is not completely additive.

By 326R, there is a partition of unity $\langle a_i \rangle_{i\in I}$ in $\mathfrak{A}$ such that $\sum_{i\in I} \nu' a_i < \nu' 1$. Set $K = \{i : i \in I,\ \nu' a_i > 0\}$; then K must be countable, so

$$\nu'(\sup_{i\in I\setminus K} a_i) = \nu' 1 - \nu'(\sup_{i\in K} a_i) = \nu' 1 - \sum_{i\in K} \nu' a_i > 0.$$

For $J \subseteq I$ set $\mu J = \nu'(\sup_{i\in J\setminus K} a_i)$; the supremum is always defined because $\mathfrak{A}$ is Dedekind complete. Because ν' is countably additive and non-negative, so is μ; because $\nu' a_i = 0$ for $i \in J \setminus K$, $\mu\{i\} = 0$ for every $i \in I$. Multiplying μ by a suitable scalar, if need be, $(I, \mathcal{P}I, \mu)$ witnesses that (i) is true.

(d)(iii)⇒(iv) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra with a countably additive functional which is not completely additive, then $U = L^\infty(\mathfrak{A})$ is a Dedekind complete Riesz space (363Mb) and $U_c^\sim \neq U^\times$, by 363K (recalling, as in (c) above, that the functional must be bounded).

(e)(iv)⇒(iii) Let U be a Dedekind complete Riesz space such that $U^\times \neq U_c^\sim$. Take $f \in U_c^\sim \setminus U^\times$; replacing f by $|f|$ if need be, we may suppose that $f \geq 0$ is sequentially order-continuous but not order-continuous (355H, 355I). Let A be a non-empty downwards-directed set in U, with infimum 0, such that $\inf_{u\in A} f(u) > 0$ (351Ga). Take $e \in A$, and consider the solid linear subspace U_e of U generated by e; write g for the restriction of f to U_e. Because the embedding of U_e in U is order-continuous, $g \in (U_e)_c^\sim$; because $A \cap U_e$ is downwards-directed and has infimum 0, and

$$\inf_{u\in A\cap U_e} g(u) = \inf_{u\in A} f(u) > 0,$$

$g \notin U_e^\times$. But U_e is a Riesz space with order unit e, and is Dedekind complete because U is; so it can be identified with $L^\infty(\mathfrak{A})$ for some Boolean algebra $\mathfrak{A}$ (363N), and $\mathfrak{A}$ is Dedekind complete, by 363M.

Accordingly we have a Dedekind complete Boolean algebra $\mathfrak{A}$ such that $L^\infty(\mathfrak{A})^\sim_c \ne L^\infty(\mathfrak{A})^\times$. By 363K, there is a (bounded) countably additive functional on $\mathfrak{A}$ which is not completely additive, and (iii) is true.

363X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and U a Banach algebra. Let $\nu : \mathfrak{A} \to U$ be a bounded additive function and $T : L^\infty(\mathfrak{A}) \to U$ the corresponding bounded linear operator. Show that T is multiplicative iff $\nu(a \cap b) = \nu a \times \nu b$ for all a, $b \in \mathfrak{A}$.

>(b) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and $T : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{A})$ a linear operator. Show that the following are equiveridical: (i) there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $T = T_\pi$ (ii) $T(u \times v) = Tu \times Tv$ for all u, $v \in L^\infty(\mathfrak{A})$ (iii) T is a Riesz homomorphism and $Te_{\mathfrak{A}} = e_{\mathfrak{B}}$, where $e_{\mathfrak{A}}$ is the standard order unit of $L^\infty(\mathfrak{A})$.

(c) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and $T : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$ a Riesz homomorphism. Show that there are a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ and a $v \ge 0$ in $L^\infty(\mathfrak{B})$ such that $Tu = v \times T_\pi u$ for every $u \in L^\infty(\mathfrak{A})$, where T_π is the operator associated with π (363F).

(d) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Show that $L^\infty(\mathfrak{C})$, regarded as a subspace of $L^\infty(\mathfrak{A})$ (363Ga), is order-dense in $L^\infty(\mathfrak{A})$ iff $\mathfrak{C}$ is order-dense in $\mathfrak{A}$.

>(e) Let (X, Σ, μ) be a measure space with measure algebra $\mathfrak{A}$, and $\mathcal{L}^\infty$ the space of bounded Σ-measurable real-valued functions on X. A **linear lifting** of μ is a positive linear operator $T : L^\infty(\mathfrak{A}) \to \mathcal{L}^\infty$ such that $T(\chi 1_{\mathfrak{A}}) = \chi X$ and $(Tu)^\bullet = u$ for every $u \in L^\infty(\mathfrak{A})$, writing $f \mapsto f^\bullet$ for the canonical map from $\mathcal{L}^\infty$ to $L^\infty(\mathfrak{A})$ (363H-363I). (i) Show that if $\theta : \mathfrak{A} \to \Sigma$ is a lifting in the sense of 341A then T_θ, as defined in 363F, is a linear lifting. (ii) Show that if $T : L^\infty(\mathfrak{A}) \to \mathcal{L}^\infty$ is a linear lifting, then there is a corresponding lower density $\underline{\theta} : \mathfrak{A} \to \Sigma$ defined by setting $\underline{\theta}a = \{x : T(\chi a)(x) = 1\}$ for each $a \in \mathfrak{A}$. (iii) Show that $\underline{\theta}$, as defined in (ii), is a lifting iff T is a Riesz homomorphism iff T is multiplicative.

(f) Let U be any commutative ring with multiplicative identity 1. Show that the set A of **idempotents** in U (that is, elements $a \in U$ such that $a^2 = a$) is a Boolean algebra with identity 1, writing $a \cap b = ab$, $1 \setminus a = 1 - a$ for a, $b \in A$.

(g) Let $\mathfrak{A}$ be a Boolean algebra. Show that $\mathfrak{A}$ is isomorphic to the Boolean algebras of multiplicative idempotents of $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. (i) Show that for any $u \in L^\infty(\mathfrak{A})$, $\alpha \in \mathbb{R}$ there are elements $[\![u \ge \alpha]\!]$, $[\![u > \alpha]\!] \in \mathfrak{A}$, where $[\![u \ge \alpha]\!]$ is the largest $a \in \mathfrak{A}$ such that $u \times \chi a \ge \alpha\chi a$, and $[\![u > \alpha]\!] = \sup_{\beta>\alpha} [\![u \ge \beta]\!]$. (ii) Show that in the context of 363Hb, if u corresponds to $f^\bullet$ for $f \in \mathcal{L}^\infty$, then $[\![u \ge \alpha]\!] = \{x : f(x) \ge \alpha\}^\bullet$, $[\![u > \alpha]\!] = \{x : f(x) > \alpha\}^\bullet$. (iii) Show that if $A \subseteq L^\infty$ is non-empty and $v \in L^\infty$, then $v = \sup A$ iff $[\![v > \alpha]\!] = \sup_{u\in A} [\![u > \alpha]\!]$ for every $\alpha \in \mathbb{R}$; in particular, $v = u$ iff $[\![v > \alpha]\!] = [\![u > \alpha]\!]$ for every α. (iv) Show that a function $\phi : \mathbb{R} \to \mathfrak{A}$ is of the form $\phi(\alpha) = [\![u > \alpha]\!]$ iff (α) $\phi(\alpha) = \sup_{\beta>\alpha} \phi(\beta)$ for every $\alpha \in \mathbb{R}$ (β) there is an M such that $\phi(M) = 0$, $\phi(-M) = 1$. (v) Put (iii) and (iv) together to give a proof that L^∞ is Dedekind σ-complete if $\mathfrak{A}$ is.

(i) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $U \subseteq L^\infty(\mathfrak{A})$ a (sequentially) order-closed Riesz subspace containing $\chi 1$. Show that U can be identified with $L^\infty(\mathfrak{B})$ for some (sequentially) order-closed subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$. (*Hint*: set $\mathfrak{B} = \{b : \chi b \in U\}$ and use 363N.)

363Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. Given the linear structure, ordering, multiplication and norm of $S(\mathfrak{A})$ as described in §361, show that a norm completion of $S(\mathfrak{A})$ will serve for $L^\infty(\mathfrak{A})$ in the sense that all the results of 363B-363Q can be proved with no use of the axiom of choice except an occasional appeal to countably many choices in sequential forms of the theorems.

(b) Let $\mathfrak{A}$ be a Boolean algebra. Show that $\mathfrak{A}$ is ccc iff $L^\infty(\mathfrak{A})$ has the countable sup property (241Ye, 353Ye).

(c) Let X be an extremally disconnected topological space, and $\mathrm{RO}(X)$ its regular open algebra. Show that there is a natural isomorphism between $L^\infty(\mathrm{RO}(X))$ and $C_b(X)$.

(d) Let $\mathfrak{A}$ be a Boolean algebra. (i) If $u \in L^\infty = L^\infty(\mathfrak{A})$, show that $|u| = e$, the standard order unit of L^∞, iff $\max(\|u+v\|_\infty, \|u-v\|_\infty) > 1$ whenever $v \in L^\infty \setminus \{0\}$. (ii) Show that if u, $v \in L^\infty$ then $|u| \wedge |v| = 0$ iff $\|\alpha u + v + w\|_\infty \le \max(\|\alpha u + w\|_\infty, \|v + w\|_\infty)$ whenever $\alpha = \pm 1$ and $w \in L^\infty$. (iii) Show that if $T : L^\infty \to L^\infty$ is a normed space automorphism then there are a Boolean automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ and a $w \in L^\infty$ such that $|w| = e$ and $Tu = w \times T_\pi u$ for every $u \in L^\infty$.

(e) Let X be a set, Σ an algebra of subsets of X, and $\mathcal{I}$ an ideal in Σ, and $\mathcal{L}^\infty$ the set of bounded functions $f : X \to \mathbb{R}$ such that whenever $\alpha < \beta$ in $\mathbb{R}$ there is an $E \in \Sigma$ such that $\{x : f(x) \le \alpha\} \subseteq E \subseteq \{x : f(x) \le \beta\}$, as in 363H. (i) Show that $\mathcal{L}^\infty = \{g\phi : g \in C(Z)\}$, where Z is the Stone space of Σ and $\phi : X \to Z$ is a function (to be described). (ii) Show that $L^\infty(\Sigma/\mathcal{I})$ can be identified, as Banach lattice and Banach algebra, with $\mathcal{L}^\infty/\mathcal{V}$, where $\mathcal{V}$ is the set of those functions $f \in \mathcal{L}^\infty$ such that for every $\epsilon > 0$ there is a member of $\mathcal{I}$ including $\{x : |f(x)| \ge \epsilon\}$.

(f) Let (X, Σ, μ) be a complete probability space with measure algebra $\mathfrak{A}$. Let $\langle \mathfrak{B}_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$ such that $\mathfrak{A}$ is the closed subalgebra of itself generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$, and set $\Sigma_n = \{F : F^\bullet \in \mathfrak{B}_n\}$ for each n. Let $P_n : L^1(\mu) \to L^1(\mu\restriction\Sigma_n)$ be the conditional expectation operator for each n, so that $P_n\restriction L^\infty(\mu)$ is a positive linear operator from $L^\infty(\mu) \cong L^\infty(\mathfrak{A})$ to $L^\infty(\mu\restriction\Sigma_n) \cong L^\infty(\mathfrak{B}_n)$. Suppose that we are given for each n a lifting $\theta_n : \mathfrak{B}_n \to \Sigma_n$ and that $\theta_{n+1} b = \theta_n b$ whenever $n \in \mathbb{N}$ and $b \in \mathfrak{B}_n$. Let $T_n : L^\infty(\mathfrak{B}_n) \to \mathcal{L}^\infty$ be the corresponding linear liftings (363Xe), and $\mathcal{F}$ any non-principal ultrafilter on $\mathbb{N}$. (i) Show that for any $u \in L^\infty(\mathfrak{A})$, $\langle T_n P_n u \rangle_{n\in\mathbb{N}}$ converges almost everywhere. (ii) For $u \in L^\infty(\mathfrak{A})$ set $(Tu)(x) = \lim_{n\to\mathcal{F}} (T_n P_n u)(x)$ for $x \in X$, $u \in L^\infty(\mathfrak{A})$. Show that T is a linear lifting for μ. (iii) Use 363Xe(ii) and 341J to show that there is a lifting θ of μ extending every θ_n. (iv) Use this as the countable-cofinality inductive step in a proof of the Lifting Theorem (using partial liftings rather than partial lower densities, as suggested in 341Li).

(g) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a bounded countably additive functional. Suppose that $\langle u_n \rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in $L^\infty(\mathfrak{A})$ such that $\inf_{n\in\mathbb{N}} \sup_{m\ge n} u_m$ and $\sup_{n\in\mathbb{N}} \inf_{m\ge n} u_m$ are defined in $L^\infty(\mathfrak{A})$ and equal to u say. Show that $\int u\, d\nu = \lim_{n\to\infty} \int u_n d\nu$.

(h) Let Σ be the family of those sets $E \subseteq [0,1]$ such that $\mu(\operatorname{int} E) = \mu\overline{E}$, where μ is Lebesgue measure. (i) Show that Σ is an algebra of subsets of $[0,1]$ and that every member of Σ is Lebesgue measurable. (ii) Show that if we identify $L^\infty(\Sigma)$ with a set of real-valued functions on $[0,1]$, as in 363H, then we get just the space of Riemann integrable functions. (iii) Show that if we write ν for $\mu\restriction\Sigma$, then $⨍ d\nu$, as defined in 363L, is just the Riemann integral.

(i) Let X be a compact Hausdorff space. Let us say that a linear subspace U of $C(X)$ is **ℓ^∞-complemented** in $C(X)$ if there is a linear subspace V such that $C(X) = U \oplus V$ and $\|u + v\|_\infty = \max(\|u\|_\infty, \|v\|_\infty)$ for all $u \in U$, $v \in V$. Show that there is a one-to-one correspondence between such subspaces U and open-and-closed subsets E of X, given by setting $U = \{u : u \in C(X),\ u(x) = 0\ \forall\ x \in X \setminus E\}$. Hence show that if $\mathfrak{A}$ is any Boolean algebra, there is a canonical isomorphism between $\mathfrak{A}$ and the partially ordered set of ℓ^∞-complemented subspaces of $L^\infty(\mathfrak{A})$.

363 Notes and comments As with $S(\mathfrak{A})$, I have chosen a definition of $L^\infty(\mathfrak{A})$ in terms of the Stone space of $\mathfrak{A}$; but as with $S(\mathfrak{A})$, this is optional (363Ya). By and large the basic properties of L^∞ are derived very naturally from those of S. The spaces $L^\infty(\mathfrak{A})$, for general Boolean algebras $\mathfrak{A}$, are not in fact particularly important; they have too few properties not shared by all the spaces $C(X)$ for compact Hausdorff X. The point at which it becomes helpful to interpret $C(X)$ as $L^\infty(\mathfrak{A})$ is when $C(X)$ is Dedekind σ-complete. The spaces X for which this is true are difficult to picture, and alternative representations of L^∞ along the lines of 363H-363I can be easier on the imagination.

For Dedekind σ-complete $\mathfrak{A}$, there is an alternative description of members of $L^\infty(\mathfrak{A})$ in terms of objects '$[\![u > \alpha]\!]$' (363Xh); I will return to this idea in the next section. For the moment I remark only that it gives an alternative approach to 363M not necessarily depending on the representation of L^∞ as a quotient $\mathcal{L}^\infty/\mathcal{V}$ nor on an analysis of a Stone space. I used a version of such an argument in the proof of 363M which I gave in Fremlin 74a, 43D.

I spend so much time on 363M not only because Dedekind completeness is one of the basic properties of any lattice, but because it offers an abstract expression of one of the central results of Chapter 24. In 243H I showed that $L^\infty(\mu)$ is always Dedekind σ-complete, and that it is Dedekind complete if μ is localizable. We can now relate this to the results of 321H and 322Be: the measure algebra of any measure is Dedekind σ-complete, and the measure algebra of a localizable measure is Dedekind complete.

The ideas of the proof of 363M can of course be rearranged in various ways. One uses 353Yb: for completely regular spaces X, $C(X)$ is Dedekind complete iff X is extremally disconnected; while for compact Hausdorff spaces, X is extremally disconnected iff it is the Stone space of a Dedekind complete algebra. With the right modification of the concept 'extremally disconnected' (314Yf), the same approach works for Dedekind σ-completeness.

363R is the 'Nachbin-Kelley theorem'; it is commonly phrased 'a normed space U has the Hahn-Banach extension property iff it is isomorphic, as normed space, to $C(X)$ for some compact extremally disconnected Hausdorff space X', but the expression in terms of L^∞ spaces seems natural in the present context. The implication in one direction (Part A of the proof) calls for nothing but a check through one of the standard proofs of the Hahn-Banach theorem to make sure that the argument applies in the generalized form. Part B of the proof has ideas in it; I have tried to set it out in a way suggesting that if you can remember the construction of the set X the rest is just a matter of a little ingenuity.

One way of trying to understand the multiple structures of L^∞ spaces is by looking at the corresponding automorphisms. We observe, for instance, that an operator T from $L^\infty(\mathfrak{A})$ to itself is a Banach algebra automorphism iff it is a Banach lattice automorphism preserving the standard order unit iff it corresponds to an automorphism of the algebra $\mathfrak{A}$ (363Xb). Of course there are Banach space automorphisms of L^∞ which do not respect the order or multiplicative structure; but they have to be closely related to algebra isomorphisms (363Yd).

I devote a couple of exercises (363Xe, 363Yf) to indications of how the ideas here are relevant to the Lifting Theorem. If you found the formulae of the proof of 341G obscure it may help to work through the parallel argument.

A lecture by W.A.J.Luxemburg on the equivalence between (i) and (iv) in 363S was one of the turning points in my mathematical apprenticeship. I introduce it here, even though the real importance of the Banach-Ulam problem lies in the metamathematical ideas it has nourished, because these formulations provide a focus for questions which arise naturally in this volume and which otherwise might prove distracting. The next group of significant ideas in this context will appear in §438.

364 L^0

My next objective is to develop an abstract construction corresponding to the $L^0(\mu)$ spaces of §241. These generalized L^0 spaces will form the basis of the work of the rest of this chapter and also the next; partly because their own properties are remarkable, but even more because they form a framework for the study of Archimedean Riesz spaces in general (see §368). There seem to be significant new difficulties, and I take the space to describe an approach which can be made essentially independent of the route through Stone spaces used in the last three sections (364Ya). I embark directly on a definition in the new language (364A), and relate it to the constructions of §241 (364B-364D, 364I) and §§361-363 (364J). The ideas of Chapter 27 can also be expressed in this language; I make a start on developing the machinery for this in 364F, with the formula '$[\![u \in E]\!]$', 'the region in which u belongs to E', and some exercises (364Xd-364Xf). Following through the questions addressed in §363, I discuss Dedekind completeness in L^0 (364L-364M), properties of its multiplication (364N), the expression of the original algebra in terms of L^0 (364O), the action of Boolean homomorphisms on L^0 (364P) and product spaces (364R). In 364S-364V I describe representations of the L^0 space of a regular open algebra.

364A The set $L^0(\mathfrak{A})$ **(a) Definition** Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. I will write $L^0(\mathfrak{A})$ for the set of all functions $\alpha \mapsto [\![u > \alpha]\!] : \mathbb{R} \to \mathfrak{A}$ such that

(α) $[\![u > \alpha]\!] = \sup_{\beta > \alpha} [\![u > \beta]\!]$ in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$,

(β) $\inf_{\alpha \in \mathbb{R}} [\![u > \alpha]\!] = 0$,

(γ) $\sup_{\alpha \in \mathbb{R}} [\![u > \alpha]\!] = 1$.

(b) My reasons for using the notation '$[\![u > \alpha]\!]$' rather than '$u(\alpha)$' will I hope become clear in the next few paragraphs. For the moment, if you think of $\mathfrak{A}$ as a σ-algebra of sets and of $L^0(\mathfrak{A})$ as the family of $\mathfrak{A}$-measurable real-valued functions, then $[\![u > \alpha]\!]$ corresponds to the set $\{x : u(x) > \alpha\}$ (364Ia).

(c) Some readers will recognise the formula '$[\![\ldots]\!]$' as belonging to the language of forcing, so that $[\![u > \alpha]\!]$ could be read as 'the Boolean value of the proposition "$u > \alpha$"'. But a vocalisation closer to my intention might be 'the region where $u > \alpha$'.

(d) Note that condition (α) of (a) automatically ensures that $[\![u > \alpha]\!] \subseteq [\![u > \alpha']\!]$ whenever $\alpha' \le \alpha$ in $\mathbb{R}$.

(e) In fact it will sometimes be convenient to note that the conditions of (a) can be replaced by

(α') $[\![u > \alpha]\!] = \sup_{q \in \mathbb{Q}, q > \alpha} [\![u > q]\!]$ for every $\alpha \in \mathbb{R}$,

(β') $\inf_{n \in \mathbb{N}} [\![u > n]\!] = 0$,

(γ') $\sup_{n \in \mathbb{N}} [\![u > -n]\!] = 1$;

the point being that we need look only at suprema and infima of countable subsets of $\mathfrak{A}$.

***(f)** Indeed, because the function $\alpha \mapsto [\![u > \alpha]\!]$ is determined by its values on $\mathbb{Q}$, we have the option of declaring $L^0(\mathfrak{A})$ to be the set of functions $\alpha \mapsto [\![u > \alpha]\!] : \mathbb{Q} \to \mathfrak{A}$ such that

(α'') $[\![u > q]\!] = \sup_{q' \in \mathbb{Q}, q' > q} [\![u > q']\!]$ for every $q \in \mathbb{Q}$,

(β') $\inf_{n \in \mathbb{N}} [\![u > n]\!] = 0$,

(γ') $\sup_{n \in \mathbb{N}} [\![u > -n]\!] = 1$.

However I shall hold this in reserve until I come to forcing constructions in Chapter 55 of Volume 5.

(g) In order to integrate this construction into the framework of the rest of this book, I match it with an alternative route to the same object, based on σ-algebras and σ-ideals of sets, as follows.

364B Proposition Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ.

(a) Write $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ for the space of all Σ-measurable functions from X to $\mathbb{R}$. Then $\mathcal{L}^0$, with its linear structure, ordering and multiplication inherited from $\mathbb{R}^X$, is a Dedekind σ-complete f-algebra with multiplicative identity.

(b) Set

$$\mathcal{W} = \mathcal{W}_{\mathcal{I}} = \{f : f \in \mathcal{L}^0, \{x : f(x) \neq 0\} \in \mathcal{I}\}.$$

Then

(i) $\mathcal{W}$ is a sequentially order-closed solid linear subspace and ideal of $\mathcal{L}^0$;

(ii) the quotient space $\mathcal{L}^0/\mathcal{W}$, with its inherited linear, order and multiplicative structures, is a Dedekind σ-complete Riesz space and an f-algebra with a multiplicative identity;

(iii) for f, $g \in \mathcal{L}^0$, $f^\bullet \le g^\bullet$ in $\mathcal{L}^0/\mathcal{W}$ iff $\{x : f(x) > g(x)\} \in \mathcal{I}$, and $f^\bullet = g^\bullet$ in $\mathcal{L}^0/\mathcal{W}$ iff $\{x : f(x) \neq g(x)\} \in \mathcal{I}$.

proof (Compare 241A-241H.)

(a) The point is just that $\mathcal{L}^0$ is a sequentially order-closed Riesz subspace and subalgebra of $\mathbb{R}^X$. The facts we need to know – that constant functions belong to $\mathcal{L}^0$, that $f+g$, αf, $f \times g$, $\sup_{n\in\mathbb{N}} f_n$ belong to $\mathcal{L}^0$ whenever f, g, f_n do and $\{f_n : n \in \mathbb{N}\}$ is bounded above – are all covered by 121E-121F. Its multiplicative identity is of course the constant function χX.

(b)(i) The necessary verifications are all elementary.

(ii) Because $\mathcal{W}$ is a solid linear subspace of the Riesz space $\mathcal{L}^0$, the quotient inherits a Riesz space structure (351J, 352Jb); because $\mathcal{W}$ is an ideal of the ring $(\mathcal{L}^0, +, \times)$, $\mathcal{L}^0/\mathcal{W}$ inherits a multiplication; it is a commutative algebra because $\mathcal{L}^0$ is; and has a multiplicative identity $e = \chi X^\bullet$ because χX is the identity of $\mathcal{L}^0$.

To check that $\mathcal{L}^0/\mathcal{W}$ is an f-algebra it is enough to observe that, for any non-negative f, g , $h \in \mathcal{L}^0$,

$$f^\bullet \times g^\bullet = (f \times g)^\bullet \ge 0,$$

and if $f^\bullet \wedge g^\bullet = 0$ then $\{x : f(x) > 0\} \cap \{x : g(x) > 0\} \in \mathcal{I}$, so that $\{x : f(x)h(x) > 0\} \cap \{x : g(x) > 0\} \in \mathcal{I}$ and

$$(f^\bullet \times h^\bullet) \wedge g^\bullet = (h^\bullet \times f^\bullet) \wedge g^\bullet = 0.$$

Finally, $\mathcal{L}^0/\mathcal{W}$ is Dedekind σ-complete, by 353J(a-iii).

(iii) For f, $g \in \mathcal{L}^0$,

$$f^\bullet \le g^\bullet \iff (f-g)^+ \in \mathcal{W} \iff \{x : f(x) > g(x)\} = \{x : (f-g)^+(x) \neq 0\} \in \mathcal{I}$$

(using the fact that the canonical map from $\mathcal{L}^0$ to $\mathcal{L}^0/\mathcal{W}$ is a Riesz homomorphism, so that $((f-g)^+)^\bullet = (f^\bullet - g^\bullet)^+$). Similarly

$$f^\bullet = g^\bullet \iff f - g \in \mathcal{W} \iff \{x : f(x) \neq g(x)\} = \{x : (f-g)(x) \neq 0\} \in \mathcal{I}.$$

364C Theorem Let X be a set and Σ a σ-algebra of subsets of X. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \Sigma \to \mathfrak{A}$ a surjective Boolean homomorphism, with kernel a σ-ideal $\mathcal{I}$; define $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ and $\mathcal{W} = \mathcal{W}_{\mathcal{I}}$ as in 364B, so that $U = \mathcal{L}^0/\mathcal{W}$ is a Dedekind σ-complete f-algebra with multiplicative identity.

(a) We have a canonical bijection $T : U \to L^0 = L^0(\mathfrak{A})$ defined by the formula

$$[\![Tf^\bullet > \alpha]\!] = \pi\{x : f(x) > \alpha\}$$

for every $f \in \mathcal{L}^0$ and $\alpha \in \mathbb{R}$.

(b)(i) For any u, $v \in U$,

$$[\![T(u+v) > \alpha]\!] = \sup_{q\in\mathbb{Q}} [\![Tu > q]\!] \cap [\![Tv > \alpha - q]\!]$$

for every $\alpha \in \mathbb{R}$.

(ii) For any $u \in U$ and $\gamma > 0$,

$$[\![T(\gamma u) > \alpha]\!] = [\![Tu > \tfrac{\alpha}{\gamma}]\!]$$

for every $\alpha \in \mathbb{R}$.

(iii) For any u, $v \in U$,

$$u \le v \iff [\![Tu > \alpha]\!] \subseteq [\![Tv > \alpha]\!] \text{ for every } \alpha \in \mathbb{R}.$$

(iv) For any u, $v \in U^+$,

$$[\![T(u \times v) > \alpha]\!] = \sup_{q\in\mathbb{Q},q>0} [\![Tu > q]\!] \cap [\![Tv > \tfrac{\alpha}{q}]\!]$$

for every $\alpha \ge 0$.

(v) Writing $e = (\chi X)^\bullet$ for the multiplicative identity of U, we have

$$[\![Te > \alpha]\!] = 1 \text{ if } \alpha < 1,\ 0 \text{ if } \alpha \ge 1.$$

proof (a)(i) Given $f \in \mathcal{L}^0$, set $\zeta_f(\alpha) = \pi\{x : f(x) > \alpha\}$ for $\alpha \in \mathbb{R}$. Then it is easy to see that ζ_f satisfies the conditions $(\alpha)'$-$(\gamma)'$ of 364Ae, because π is sequentially order-continuous (313Qb). Moreover, if $f^\bullet = g^\bullet$ in U, then

$$\zeta_f(\alpha) \mathbin{\triangle} \zeta_g(\alpha) = \pi(\{x : f(x) > \alpha\} \triangle \{x : g(x) > \alpha\}) = 0$$

for every $\alpha \in \mathbb{R}$, because

$$\{x : f(x) > \alpha\} \triangle \{x : g(x) > \alpha\} \subseteq \{x : f(x) \ne g(x)\} \in \mathcal{I},$$

and $\zeta_f = \zeta_g$. So we have a well-defined member Tu of L^0 defined by the given formula, for any $u \in U$.

(ii) Next, given $w \in L^0$, there is a $u \in \mathcal{L}^0/\mathcal{W}$ such that $Tu = w$. **P** For each $q \in \mathbb{Q}$, choose $F_q \in \Sigma$ such that $\pi F_q = [\![w > q]\!]$ in $\mathfrak{A}$. Note that if $q' \ge q$ then

$$\pi(F_{q'} \setminus F_q) = [\![u > q']\!] \setminus [\![u > q]\!] = 0,$$

so $F_{q'} \setminus F_q \in \mathcal{I}$. Set

$$H = \bigcup_{q\in\mathbb{Q}} F_q \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{q\in\mathbb{Q},q\ge n} F_q \in \Sigma,$$

and for $x \in X$ set

$$\begin{aligned} f(x) &= \sup\{q : q \in \mathbb{Q},\ x \in F_q\} \text{ if } x \in H,\\ &= 0 \text{ otherwise.} \end{aligned}$$

(H is chosen just to make the formula here give a finite value for every x.) We have

$$\begin{aligned} \pi H &= \sup_{q\in\mathbb{Q}} [\![w > q]\!] \setminus \inf_{n\in\mathbb{N}} \sup_{q\in\mathbb{Q},q\ge n} [\![w > q]\!] \\ &= 1_{\mathfrak{A}} \setminus \inf_{n\in\mathbb{N}} [\![w > n]\!] = 1_{\mathfrak{A}} \setminus 0_{\mathfrak{A}} = 1_{\mathfrak{A}}, \end{aligned}$$

so $X \setminus H \in \mathcal{I}$. Now, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned} \{x : f(x) > \alpha\} &= \bigcup_{q\in\mathbb{Q},q>\alpha} F_q \cup (X \setminus H) \text{ if } \alpha < 0,\\ &= \bigcup_{q\in\mathbb{Q},q>\alpha} F_q \setminus (X \setminus H) \text{ if } \alpha \ge 0, \end{aligned}$$

and in either case belongs to Σ; so that $f \in \mathcal{L}^0$ and $f^\bullet$ is defined in L^0. Next, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned} [\![Tf^\bullet > \alpha]\!] &= \pi\{x : f(x) > \alpha\} = \pi\Big(\bigcup_{q\in\mathbb{Q},q>\alpha} F_q\Big) \\ &= \sup_{q\in\mathbb{Q},q>\alpha} [\![w > q]\!] = [\![w > \alpha]\!], \end{aligned}$$

and $Tf^\bullet = w$. **Q**

(iii) Thus T is surjective. To see that it is injective, observe that if f, $g \in \mathcal{L}^0$, then

$$\begin{aligned} Tf^\bullet = Tg^\bullet &\Longrightarrow [\![Tf^\bullet > \alpha]\!] = [\![Tg^\bullet > \alpha]\!] \text{ for every } \alpha \in \mathbb{R} \\ &\Longrightarrow \pi\{x : f(x) > \alpha\} = \pi\{x : g(x) > \alpha\} \text{ for every } \alpha \in \mathbb{R} \\ &\Longrightarrow \{x : f(x) > \alpha\} \triangle \{x : g(x) > \alpha\} \in \mathcal{I} \text{ for every } \alpha \in \mathbb{R} \\ &\Longrightarrow \{x : f(x) \neq g(x)\} = \bigcup_{q\in\mathbb{Q}} (\{x : f(x) > q\} \triangle \{x : g(x) > q\}) \in \mathcal{I} \\ &\Longrightarrow f^\bullet = g^\bullet. \end{aligned}$$

So we have the claimed bijection.

(b)(i) Let f, $g \in \mathcal{L}^0$ be such that $u = f^\bullet$ and $v = g^\bullet$, so that $u + v = (f + g)^\bullet$. For any $\alpha \in \mathbb{R}$,

$$\begin{aligned} [\![T(u+v) > \alpha]\!] &= \pi\{x : f(x) + g(x) > \alpha\} \\ &= \pi(\bigcup_{q\in\mathbb{Q}} \{x : f(x) > q\} \cap \{x : g(x) > \alpha - q\}) \\ &= \sup_{q\in\mathbb{Q}} \pi\{x : f(x) > q\} \cap \pi\{x : g(x) > \alpha - q\} \end{aligned}$$

(because π is a sequentially order-continuous Boolean homomorphism)

$$= \sup_{q\in\mathbb{Q}} [\![Tu > q]\!] \cap [\![Tv > \alpha - q]\!].$$

(ii) Let $f \in \mathcal{L}^0$ be such that $f^\bullet = u$, so that $(\gamma f)^\bullet = \gamma u$. For any $\alpha \in \mathbb{R}$,

$$[\![T(\gamma u) > \alpha]\!] = \pi\{x : \gamma f(x) > \alpha\} = \pi\{x : f(x) > \frac{\alpha}{\gamma}\} = [\![Tu > \tfrac{\alpha}{\gamma}]\!].$$

(iii) Let f, $g \in \mathcal{L}^0$ be such that $f^\bullet = u$ and $g^\bullet = v$. Then

$$u \le v \iff \{x : f(x) > g(x)\} \in \mathcal{I}$$

(see 364B(b-iii))

$$\begin{aligned} &\iff \bigcup_{q\in\mathbb{Q}} \{x : f(x) > q \ge g(x)\} \in \mathcal{I} \\ &\iff \{x : f(x) > \alpha\} \setminus \{x : g(x) > \alpha\} \in \mathcal{I} \text{ for every } \alpha \in \mathbb{R} \\ &\iff \pi\{x : f(x) > \alpha\} \setminus \pi\{x : g(x) > \alpha\} = 0 \text{ for every } \alpha \\ &\iff [\![Tu > \alpha]\!] \subseteq [\![Tv > \alpha]\!] \text{ for every } \alpha. \end{aligned}$$

(iv) Now suppose that u, $v \ge 0$, so that they can be expressed as $f^\bullet$, $g^\bullet$ where f, $g \ge 0$ in $\mathcal{L}^0$ (351J), and $u \times v = (f \times g)^\bullet$. If $\alpha \ge 0$, then

$$\begin{aligned} [\![T(u \times v) > \alpha]\!] &= \pi(\bigcup_{q\in\mathbb{Q}, q>0} \{x : f(x) > q\} \cap \{x : g(x) > \frac{\alpha}{q}\}) \\ &= \sup_{q\in\mathbb{Q}, q>0} \pi\{x : f(x) > q\} \cap \pi\{x : g(x) > \frac{\alpha}{q}\} \\ &= \sup_{q\in\mathbb{Q}, q>0} [\![Tu > q]\!] \cap [\![Tv > \tfrac{\alpha}{q}]\!]. \end{aligned}$$

(v) This is trivial, because

$$\begin{aligned} [\![T(\chi X)^\bullet > \alpha]\!] &= \pi\{x : (\chi X)(x) > \alpha\} \\ &= \pi X = 1 \text{ if } \alpha < 1, \\ &= \pi\emptyset = 0 \text{ if } \alpha \ge 1. \end{aligned}$$

364D Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then $L^0 = L^0(\mathfrak{A})$ has the structure of a Dedekind σ-complete f-algebra with multiplicative identity e, defined by saying

$$[\![u + v > \alpha]\!] = \sup_{q\in\mathbb{Q}} [\![u > q]\!] \cap [\![v > \alpha - q]\!],$$

whenever u, $v \in L^0$ and $\alpha \in \mathbb{R}$,

$$[\![\gamma u > \alpha]\!] = [\![u > \tfrac{\alpha}{\gamma}]\!]$$

whenever $u \in L^0$, $\gamma \in]0, \infty[$ and $\alpha \in \mathbb{R}$,

$$u \le v \iff [\![u > \alpha]\!] \subseteq [\![v > \alpha]\!] \text{ for every } \alpha \in \mathbb{R},$$

$$[\![u \times v > \alpha]\!] = \sup_{q\in\mathbb{Q},q>0} [\![u > q]\!] \cap [\![v > \tfrac{\alpha}{q}]\!]$$

whenever u, $v \ge 0$ in L^0 and $\alpha \ge 0$,

$$[\![e > \alpha]\!] = 1 \text{ if } \alpha < 1,\ 0 \text{ if } \alpha \ge 1.$$

proof (a) By the Loomis-Sikorski theorem (314M), we can find a set Z (the Stone space of $\mathfrak{A}$), a σ-algebra Σ of subsets of Z (the algebra generated by the open-and-closed sets and the ideal $\mathcal{M}$ of meager sets) and a surjective sequentially order-continuous Boolean homomorphism $\pi : \Sigma \to \mathfrak{A}$ (corresponding to the identification between $\mathfrak{A}$ and the quotient $\Sigma/\mathcal{M}$). Consequently, defining $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ and $\mathcal{W} = \mathcal{W}_\mathcal{M}$ as in 364B, we have a bijection between the Dedekind σ-complete f-algebra $\mathcal{L}^0/\mathcal{W}$ and L^0 (364Ca). Of course this endows L^0 itself with the structure of a Dedekind σ-complete f-algebra; and 364Cb tells us that the description of the algebraic operations above is consistent with this structure.

(b) In fact the f-algebra structure is completely defined by the description offered. For while scalar multiplication is not described for $\gamma \le 0$, the assertion that L^0 is a Riesz space implies that $0u = 0$ and that $\gamma u = (-\gamma)(-u)$ for $\gamma < 0$; so if we have formulae to describe $u + v$ and γu for $\gamma > 0$, this suffices to define the linear structure of L^0. Note that we have an element $\underline{0}$ in L^0 defined by setting

$$[\![\underline{0} > \alpha]\!] = 0 \text{ if } \alpha \ge 0,\ 1 \text{ if } \alpha < 0,$$

and the formula for $u + v$ shows us that

$$[\![\underline{0} + u > \alpha]\!] = \sup_{q\in\mathbb{Q}} [\![\underline{0} > q]\!] \cap [\![u > \alpha - q]\!] = \sup_{q\in\mathbb{Q},q<0} [\![u > \alpha - q]\!] = [\![u > \alpha]\!]$$

for every α, so that $\underline{0}$ is the zero of L^0. As for multiplication, if L^0 is to be an f-algebra we must have

$$[\![u \times v > \alpha]\!] \supseteq [\![\underline{0} > \alpha]\!] = 1$$

whenever u, $v \in (L^0)^+$ and $\alpha < 0$, because $u \times v \ge \underline{0}$. So the formula offered is sufficient to determine $u \times v$ for non-negative u and v; and for others we know that

$$u \times v = (u^+ \times v^+) - (u^+ \times v^-) - (u^- \times v^+) + (u^- \times v^-),$$

so the whole of the multiplication of L^0 is defined.

364E The rest of this section will be devoted to understanding the structure just established. I start with a pair of elementary facts.

Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.

(a) If u, $v \in L^0 = L^0(\mathfrak{A})$ and α, $\beta \in \mathbb{R}$,

$$[\![u + v > \alpha + \beta]\!] \subseteq [\![u > \alpha]\!] \cup [\![v > \beta]\!].$$

(b) If u, $v \ge 0$ in L^0 and α, $\beta \ge 0$ in $\mathbb{R}$,

$$[\![u \times v > \alpha\beta]\!] \subseteq [\![u > \alpha]\!] \cup [\![v > \beta]\!].$$

proof (a) For any $q \in \mathbb{Q}$, either $q \ge \alpha$ and $[\![u > q]\!] \subseteq [\![u > \alpha]\!]$, or $q \le \alpha$ and $[\![v > \alpha + \beta - q]\!] \subseteq [\![v > \beta]\!]$; thus in all cases

$$[\![u > q]\!] \cap [\![v > \alpha + \beta - q]\!] \subseteq [\![u > \alpha]\!] \cup [\![v > \beta]\!];$$

taking the supremum over q, we have the result.

(b) The same idea works, replacing $\alpha + \beta - q$ by $\alpha\beta/q$ for $q > 0$.

364F Yet another description of L^0 is sometimes appropriate, and leads naturally to an important construction (364H).

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then there is a bijection between $L^0 = L^0(\mathfrak{A})$ and the set Φ of sequentially order-continuous Boolean homomorphisms from the algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ to $\mathfrak{A}$, defined by saying that $u \in L^0$ corresponds to $\phi \in \Phi$ iff $[\![u > \alpha]\!] = \phi(]\alpha, \infty[)$ for every $\alpha \in \mathbb{R}$.

proof (a) If $\phi \in \Phi$, then the map $\alpha \mapsto \phi(]\alpha, \infty[)$ satisfies the conditions of 364Ae, so corresponds to an element u_ϕ of L^0.

(b) If ϕ, $\psi \in \Phi$ and $u_\phi = u_\psi$, then $\phi = \psi$. **P** Set $\mathcal{A} = \{E : E \in \mathcal{B}, \phi(E) = \psi(E)\}$. Then $\mathcal{A}$ is a σ-subalgebra of $\mathcal{B}$, because ϕ and ψ are both sequentially order-continuous Boolean homomorphisms, and contains $]\alpha, \infty[$ for every $\alpha \in \mathbb{R}$. Now $\mathcal{A}$ contains $]-\infty, \alpha]$ for every α, and therefore includes $\mathcal{B}$ (121J). But this means that $\phi = \psi$. **Q**

(c) Thus $\phi \mapsto u_\phi$ is injective. But it is also surjective. **P** As in 364D, take a set Z, a σ-algebra Σ of subsets of Z and a surjective sequentially order-continuous Boolean homomorphism $\pi : \Sigma \to \mathfrak{A}$; let $T : \mathcal{L}^0_\Sigma/\mathcal{W}_{\pi^{-1}[\{0\}]} \to L^0$ be the bijection described in 364C. If $u \in L^0$, there is an $f \in \mathcal{L}^0_\Sigma$ such that $Tf^\bullet = u$. Now consider $\phi E = \pi f^{-1}[E]$ for $E \in \mathcal{B}$. $f^{-1}[E]$ always belongs to Σ (121Ef), so ϕE is always well-defined; $E \mapsto f^{-1}[E]$ and π are sequentially order-continuous, so ϕ also is; and

$$\phi(]\alpha, \infty[) = \pi\{z : f(z) > \alpha\} = [\![u > \alpha]\!]$$

for every α, so $u = u_\phi$. **Q**

Thus we have the declared bijection.

364G Definition In the context of 364F, I will write $[\![u \in E]\!]$, 'the region where u takes values in E', for $\phi(E)$, where $\phi : \mathcal{B} \to \mathfrak{A}$ is the homomorphism corresponding to $u \in L^0$. Thus $[\![u > \alpha]\!] = [\![u \in]\alpha, \infty[\,]\!]$. In the same spirit I write $[\![u \ge \alpha]\!]$ for $[\![u \in [\alpha, \infty[\,]\!] = \inf_{\beta<\alpha} [\![u > \beta]\!]$, $[\![u \ne 0]\!] = [\![|u| > 0]\!] = [\![u > 0]\!] \cup [\![u < 0]\!]$ and so on, so that (for instance) $[\![u = \alpha]\!] = [\![u \in \{\alpha\}]\!] = [\![u \ge \alpha]\!] \setminus [\![u > \alpha]\!]$ for $u \in L^0$ and $\alpha \in \mathbb{R}$.

364H Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $E \subseteq \mathbb{R}$ a Borel set, and $h : E \to \mathbb{R}$ a Borel measurable function. Then whenever $u \in L^0 = L^0(\mathfrak{A})$ is such that $[\![u \in E]\!] = 1$, there is an element $\bar{h}(u)$ of L^0 defined by saying that $[\![\bar{h}(u) \in F]\!] = [\![u \in h^{-1}[F]]\!]$ for every Borel set $F \subseteq \mathbb{R}$.

proof All we have to observe is that $F \mapsto [\![u \in h^{-1}[F]]\!]$ is a sequentially order-continuous Boolean homomorphism. (The condition '$[\![u \in E]\!] = 1$' ensures that $[\![u \in h^{-1}[\mathbb{R}]]\!] = 1$.)

364I Examples Perhaps I should spell out the most important contexts in which we apply these ideas, even though they have in effect already been mentioned.

(a) Let X be a set and Σ a σ-algebra of subsets of X. Then we may identify $L^0(\Sigma)$ with the space $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ of Σ-measurable real-valued functions on X. (This is the case $\mathfrak{A} = \Sigma$ of 364C.) For $f \in \mathcal{L}^0$, $[\![f \in E]\!]$ (364G) is just $f^{-1}[E]$, for any Borel set $E \subseteq \mathbb{R}$; and if h is a Borel measurable function, $\bar{h}(f)$ (364H) is just the composition hf, for any $f \in \mathcal{L}^0$.

(b) Now suppose that $\mathcal{I}$ is a σ-ideal of Σ and that $\mathfrak{A} = \Sigma/\mathcal{I}$. Then, as in 364C, we identify $L^0(\mathfrak{A})$ with a quotient $\mathcal{L}^0/\mathcal{W}_\mathcal{I}$. For $f \in \mathcal{L}^0$, $[\![f^\bullet \in E]\!] = f^{-1}[E]^\bullet$, and $\bar{h}(f^\bullet) = (hf)^\bullet$, for any Borel set E and any Borel measurable function $h : \mathbb{R} \to \mathbb{R}$.

(c) In particular, if (X, Σ, μ) is a measure space with measure algebra $\mathfrak{A}$, then $L^0(\mathfrak{A})$ becomes identified with $L^0(\mu)$ as defined in §241.

The same remarks as in 363I apply here; the space $\mathcal{L}^0(\mu)$ of 241A is larger than the space $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ considered here. But for every $f \in \mathcal{L}^0(\mu)$ there is a $g \in \mathcal{L}^0_\Sigma$ such that $g =_{\text{a.e.}} f$ (241Bk), so that $L^0(\mu) = \mathcal{L}^0(\mu)/=_{\text{a.e.}}$ can be identified with $\mathcal{L}^0_\Sigma/\mathcal{N}$, where $\mathcal{N}$ is the set of functions in $\mathcal{L}^0$ which are zero almost everywhere (241Yc).

364J Embedding S and L^∞ in L^0: Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.

(a) We have a canonical embedding of $L^\infty = L^\infty(\mathfrak{A})$ as an order-dense solid linear subspace of $L^0 = L^0(\mathfrak{A})$; it is the solid linear subspace generated by the multiplicative identity e of L^0. Consequently $S = S(\mathfrak{A})$ also is embedded as an order-dense Riesz subspace and subalgebra of L^0.

(b) This embedding respects the linear, lattice and multiplicative structures of L^∞ and S, and the definition of $[\![u > \delta]\!]$, for $u \in S^+$ and $\delta \geq 0$, given in 361Eg.

(c) For $a \in \mathfrak{A}$, χa, when regarded as a member of L^0, can be described by the formula

$$\begin{aligned}[\![\chi a > \alpha]\!] &= 1 \text{ if } \alpha < 0,\\ &= a \text{ if } 0 \leq \alpha < 1,\\ &= 0 \text{ if } 1 \leq \alpha.\end{aligned}$$

The function $\chi : \mathfrak{A} \to L^0$ is additive, injective, order-continuous and a lattice homomorphism.

(d) For every $u \in (L^0)^+$ there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in S such that $u_0 \geq 0$ and $\sup_{n\in\mathbb{N}} u_n = u$.

proof Let Z, Σ, $\mathcal{M}$, $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$, $\mathcal{W} = \mathcal{W}_\mathcal{M}$ and π be as in the proof of 364D. I defined L^∞ to be the space $C(Z)$ of continuous real-valued functions on Z (363A); but because $\mathfrak{A}$ is Dedekind σ-complete, there is an alternative representation as $\mathcal{L}^\infty/\mathcal{W} \cap \mathcal{L}^\infty$, where $\mathcal{L}^\infty$ is the space of bounded Σ-measurable functions from Z to $\mathbb{R}$ (363Hb). Put like this, we clearly have an embedding of $L^\infty \cong \mathcal{L}^\infty/\mathcal{W} \cap \mathcal{L}^\infty$ in $L^0 \cong \mathcal{L}^0/\mathcal{W}$; and this embedding represents L^∞ as a Riesz subspace and subalgebra of L^0, because $\mathcal{L}^\infty$ is a Riesz subspace and subalgebra of $\mathcal{L}^0$. L^∞ becomes the solid linear subspace of L^0 generated by $(\chi Z)^\bullet = e$, because $\mathcal{L}^\infty$ is the solid linear subspace of $\mathcal{L}^0$ generated by χZ. To see that L^∞ is order-dense in L^0, we have only to note that $f = \sup_{n\in\mathbb{N}} f \wedge n\chi Z$ in $\mathcal{L}^0$ for every $f \in \mathcal{L}^0$, and therefore (because the map $f \mapsto f^\bullet$ is sequentially order-continuous) $u = \sup_{n\in\mathbb{N}} u \wedge ne$ in L^0 for every $u \in L^0$.

To identify χa, we have the formula $\chi(\pi E) = (\chi E)^\bullet$, as in 363H(b-iii); but this means that, if $a = \pi E$,

$$\begin{aligned}[\![\chi a > \alpha]\!] = \pi\{z : \chi E(z) > \alpha\} &= \pi Z = 1 \text{ if } \alpha < 0,\\ &= \pi E = a \text{ if } 0 \leq \alpha < 1,\\ &= \pi\emptyset = 0 \text{ if } \alpha \geq 1,\end{aligned}$$

using the formula in 364Ca. Evidently χ is injective.

Because S is an order-dense Riesz subspace and subalgebra of L^∞ (363C), the same embedding represents it as an order-dense Riesz subspace and subalgebra of L^0. (For 'order-dense', use 352N(c-iii).) Concerning the formula $[\![u > \delta]\!]$, suppose that $u \in S^+$ and $\delta \geq 0$; express u as $\sum_{j=0}^m \beta_j \chi b_j$, where $b_0, \dots , b_m \in \mathfrak{A}$ are disjoint and $\beta_j \geq 0$ for every j. Then we have disjoint sets $F_0, \dots , F_m \in \Sigma$ such that $\pi F_j = b_j$ for every j, and u is identified with $(\sum_{j=0}^m \beta_j \chi F_j)^\bullet$. Using 364Ca, we have

$$[\![u > \delta]\!] = \pi\{z : \textstyle\sum_{j=0}^m \beta_j \chi F_j(z) > \delta\} = \pi(\bigcup\{F_j : \beta_j > \delta\}) = \sup\{b_j : \beta_j > \delta\},$$

matching the expression in the proof of 361Eg. So the new interpretation of $[\![\dots]\!]$ matches the former definition in the special case envisaged in 361E.

Because $\chi : \mathfrak{A} \to L^\infty$ is additive, order-continuous and a lattice homomorphism (363D), and the embedding map $L^\infty \hookrightarrow L^0$ also is, $\chi : \mathfrak{A} \to L^0$ has the same properties.

Finally, if $u \geq 0$ in L^0, we can represent it as $f^\bullet$ where $f \geq 0$ in $\mathcal{L}^0$. For $n \in \mathbb{N}$ set

$$\begin{aligned}f_n(z) &= 2^{-n}k \text{ if } 2^{-n}k \leq f(z) < 2^{-n}(k + 1) \text{ where } 0 \leq k < 4^n,\\ &= 0 \text{ if } f(z) \geq 2^n;\end{aligned}$$

then $\langle f_n^\bullet\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in S^+ with supremum u.

364K Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Then $S(\mathfrak{A}^f)$ can be embedded as a Riesz subspace of $L^0(\mathfrak{A})$, which is order-dense iff $(\mathfrak{A}, \bar\mu)$ is semi-finite.

proof (Recall that $\mathfrak{A}^f$ is the ring $\{a : \bar\mu a < \infty\}$.) The embedding $\mathfrak{A}^f \hookrightarrow \mathfrak{A}$ is an injective ring homomorphism, so induces an embedding of $S(\mathfrak{A}^f)$ as a Riesz subspace of $S(\mathfrak{A})$, by 361J. Now $S(\mathfrak{A}^f)$ is order-dense in $S(\mathfrak{A})$ iff $(\mathfrak{A}, \bar\mu)$ is semi-finite. **P** (i) If $(\mathfrak{A}, \bar\mu)$ is semi-finite and $v > 0$ in $S(\mathfrak{A})$, then v is expressible as $\sum_{j=0}^n \beta_j \chi b_j$ where $\beta_j \geq 0$ for each j and some $\beta_j \chi b_j$ is non-zero; now there is a non-zero $a \in \mathfrak{A}^f$ such that $a \subseteq b_j$, so that $0 < \beta_j \chi a \in S(\mathfrak{A}^f)$ and $\beta_j \chi a \leq v$. As v is arbitrary, $S(\mathfrak{A}^f)$ is quasi-order-dense, therefor order-dense (353A). (ii) If $S(\mathfrak{A}^f)$ is order-dense in $S(\mathfrak{A})$ and $b \in \mathfrak{A} \setminus \{0\}$, there is a $u > 0$ in $S(\mathfrak{A}^f)$ such that $u \leq \chi b$; now there are $\alpha > 0$, $a \in \mathfrak{A}^f \setminus \{0\}$ such that $\alpha\chi a \leq u$, in which case $a \subseteq b$. **Q**

Now because $S(\mathfrak{A}^f) \subseteq S(\mathfrak{A})$ and $S(\mathfrak{A})$ is order-dense in $L^0(\mathfrak{A})$, we must have

$$S(\mathfrak{A}^f) \text{ is order-dense in } L^0(\mathfrak{A}) \iff S(\mathfrak{A}^f) \text{ is order-dense in } S(\mathfrak{A})$$
$$\iff (\mathfrak{A}, \bar{\mu}) \text{ is semi-finite.}$$

364L Suprema and infima in L^0 We know that any $L^0(\mathfrak{A})$ is a Dedekind σ-complete partially ordered set. There is a useful description of suprema for this ordering in (a) of the next result. We do not have such a simple formula for general infima (though see 364Xm), but facts in (b) are useful.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $L^0 = L^0(\mathfrak{A})$.

(a) Let A be a subset of L^0.

(i) A is bounded above in L^0 iff there is a sequence $\langle c_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, with infimum 0, such that $[\![u > n]\!] \subseteq c_n$ for every $u \in A$.

(ii) If A is non-empty, then A has a supremum in L^0 iff $c_\alpha = \sup_{u\in A} [\![u > \alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$ and $\inf_{n\in\mathbb{N}} c_n = 0$; and in this case $c_\alpha = [\![\sup A > \alpha]\!]$ for every α.

(iii) If A is non-empty and bounded above, then A has a supremum in L^0 iff $\sup_{u\in A} [\![u > \alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$.

(b)(i) If u, $v \in L^0$, then $[\![u \wedge v > \alpha]\!] = [\![u > \alpha]\!] \cap [\![v > \alpha]\!]$ for every $\alpha \in \mathbb{R}$.

(ii) If A is a non-empty subset of $(L^0)^+$, then $\inf A = 0$ in L^0 iff $\inf_{u\in A} [\![u > \alpha]\!] = 0$ in $\mathfrak{A}$ for every $\alpha > 0$.

proof (a)(i)($\boldsymbol{\alpha}$) If A has an upper bound u_0, set $c_n = [\![u_0 > n]\!]$ for each n; then $\langle c_n \rangle_{n\in\mathbb{N}}$ satisfies the conditions.

($\boldsymbol{\beta}$) If $\langle c_n \rangle_{n\in\mathbb{N}}$ satisfies the conditions, set

$$\begin{aligned} \phi(\alpha) &= 1 \text{ if } \alpha < 0, \\ &= \inf_{i\le n} c_i \text{ if } n \in \mathbb{N},\, \alpha \in [n, n+1[\,. \end{aligned}$$

Then it is easy to check that ϕ satisfies the conditions of 364Aa, since $\inf_{n\in\mathbb{N}} c_n = 0$. So there is a $u_0 \in L^0$ such that $\phi(\alpha) = [\![u_0 > \alpha]\!]$ for each α. Now, given $u \in A$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned} [\![u > \alpha]\!] &\subseteq 1 = [\![u_0 > \alpha]\!] \text{ if } \alpha < 0, \\ &\subseteq \inf_{i\le n} [\![u > i]\!] \subseteq \inf_{i\le n} c_i = [\![u_0 > \alpha]\!] \text{ if } n \in \mathbb{N},\, \alpha \in [n, n+1[\,. \end{aligned}$$

Thus u_0 is an upper bound for A in L^0.

(ii)($\boldsymbol{\alpha}$) Suppose that $c_\alpha = \sup_{u\in A} [\![u > \alpha]\!]$ is defined in $\mathfrak{A}$ for every α, and that $\inf_{n\in\mathbb{N}} c_n = 0$. Then, for any α,

$$\sup_{q\in\mathbb{Q},q>\alpha} c_q = \sup_{u\in A,q\in\mathbb{Q},q>\alpha} [\![u > q]\!] = \sup_{u\in A} [\![u > \alpha]\!] = c_\alpha.$$

Also, we are supposing that A contains some u_0, so that

$$\sup_{n\in\mathbb{N}} c_{-n} \supseteq \sup_{n\in\mathbb{N}} [\![u_0 > -n]\!] = 1.$$

Accordingly there is a $u^* \in L^0$ such that $[\![u^* > \alpha]\!] = c_\alpha$ for every $\alpha \in \mathbb{R}$. But now, for $v \in L^0$,

$$\begin{aligned} v \text{ is an upper bound for } A &\iff [\![u > \alpha]\!] \subseteq [\![v > \alpha]\!] \text{ for every } u \in A,\, \alpha \in \mathbb{R} \\ &\iff [\![u^* > \alpha]\!] \subseteq [\![v > \alpha]\!] \text{ for every } \alpha \\ &\iff u^* \le v, \end{aligned}$$

so that $u^* = \sup A$ in L^0.

($\boldsymbol{\beta}$) Now suppose that $u^* = \sup A$ is defined in L^0. Of course $[\![u^* > \alpha]\!]$ must be an upper bound for $\{[\![u > \alpha]\!] : u \in A\}$ for every α. **?** Suppose we have an α for which it is not the least upper bound, that is, there is a $c \subset [\![u^* > \alpha]\!]$ which is an upper bound for $\{[\![u > \alpha]\!] : u \in A\}$. Define $\phi : \mathbb{R} \to \mathfrak{A}$ by setting

$$\begin{aligned} \phi(\beta) &= c \cap [\![u^* > \beta]\!] \text{ if } \beta \ge \alpha, \\ &= [\![u^* > \beta]\!] \text{ if } \beta < \alpha. \end{aligned}$$

It is easy to see that ϕ satisfies the conditions of 364Aa (we need the distributive law 313Ba to check that $\phi(\beta) = \sup_{\gamma>\beta} \phi(\gamma)$ if $\beta \ge \alpha$), so corresponds to a member v of L^0. But we now find that v is an upper bound for A (because if $u \in A$ and $\beta \ge \alpha$ then

$$[\![u>\beta]\!] \subseteq [\![u>\alpha]\!] \cap [\![u^*>\beta]\!] \subseteq c \cap [\![u^*>\beta]\!] = [\![v>\beta]\!],)$$

that $v \le u^*$ and that $v \ne u^*$ (because $[\![v>\alpha]\!] = c \ne [\![u^*>\alpha]\!]$); but this is impossible, because u^* is supposed to be the supremum of A. **X** Thus if $u^* = \sup A$ is defined in L^0, then $\sup_{u\in A}[\![u>\alpha]\!] = [\![u^*>\alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$. Also, of course,

$$\inf_{n\in\mathbb{N}} \sup_{u\in A}[\![u>n]\!] = \inf_{n\in\mathbb{N}}[\![u^*>n]\!] = 0.$$

(iii) This is now easy. If A has a supremum, then surely it satisfies the condition, by (b). If A satisfies the condition, then we have a family $\langle c_\alpha\rangle_{\alpha\in\mathbb{R}}$ as required in (b); but also, by (a) or otherwise, there is a sequence $\langle c'_n\rangle_{n\in\mathbb{N}}$ such that $c_n \subseteq c'_n$ for every n and $\inf_{n\in\mathbb{N}} c'_n = 0$, so $\inf_{n\in\mathbb{N}} c_n$ also is 0, and both conditions in (b) are satisfied, so A has a supremum.

(b)(i) Take Z, $\mathcal{L}^0$ and π as in the proof of 364D. Express u as $f^\bullet$, v as $g^\bullet$ where f, $g \in \mathcal{L}^0$, so that $u \wedge v = (f \wedge g)^\bullet$, because the canonical map from $\mathcal{L}^0$ to L^0 is a Riesz homomorphism (351J). Then

$$\begin{aligned}[\![u\wedge v>\alpha]\!] &= \pi\{z : \min(f(z), g(z)) > \alpha\} = \pi(\{z : f(z) > \alpha\} \cap \{z : g(z) > \alpha\}) \\ &= \pi\{z : f(z) > \alpha\} \cap \pi\{z : g(z) > \alpha\} = [\![u>\alpha]\!] \cap [\![v>\alpha]\!]\end{aligned}$$

for every α.

(ii)(α) If $\inf_{u\in A}[\![u>\alpha]\!] = 0$ for every $\alpha > 0$, and v is any lower bound for A, then $[\![v>\alpha]\!]$ must be 0 for every $\alpha > 0$, so that $[\![v>0]\!] = 0$; now since $[\![0>\alpha]\!] = 0$ for $\alpha \ge 0$, 1 for $\alpha < 0$, $v \le 0$. As v is arbitrary, $\inf A = 0$.

(β) If $\alpha > 0$ is such that $\inf_{u\in A}[\![u>\alpha]\!]$ is undefined, or not equal to 0, let $c \in \mathfrak{A}$ be such that $0 \ne c \subseteq [\![u>\alpha]\!]$ for every $u \in A$, and consider $v = \alpha\chi c$. Then $[\![v>\beta]\!] = [\![\chi c > \frac{\beta}{\alpha}]\!]$ is 1 if $\beta < 0$, c if $0 \le \beta < \alpha$ and 0 if $\beta \ge \alpha$. If $u \in A$ then $[\![u>\beta]\!]$ is 1 if $\beta < 0$ (since $u \ge 0$), at least $[\![u>\alpha]\!] \supseteq c$ if $0 \le \beta < \alpha$, and always includes 0; so that $v \le u$. As u is arbitrary, $\inf A$ is either undefined in L^0 or not 0.

364M Now we have a reward for our labour, in that the following basic theorem is easy.

Theorem For a Dedekind σ-complete Boolean algebra $\mathfrak{A}$, $L^0 = L^0(\mathfrak{A})$ is Dedekind complete iff $\mathfrak{A}$ is.

proof The description of suprema in 364L(a-iii) makes it obvious that if $\mathfrak{A}$ is Dedekind complete, so that $\sup_{u\in A}[\![u>\alpha]\!]$ is always defined, then L^0 must be Dedekind complete. On the other hand, if L^0 is Dedekind complete, then so is $L^\infty(\mathfrak{A})$ (by 364J and 353J(b-i)), so that $\mathfrak{A}$ also is Dedekind complete, by 363Mb.

364N The multiplication of L^0 I have already observed that L^0 is always an f-algebra with identity; in particular (because L^0 is surely Archimedean) the map $u \mapsto u \times v$ is order-continuous for every $v \ge 0$ (353Oa), and multiplication is commutative (353Ob, or otherwise). The multiplicative identity is $\chi 1$ (364D, 364Jc). By 353Pb, or otherwise, $u \times v = 0$ iff $|u| \wedge |v| = 0$. There is one special feature of multiplication in L^0 which I can mention here.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then an element u of $L^0 = L^0(\mathfrak{A})$ has a multiplicative inverse in L^0 iff $|u|$ is a weak order unit in L^0 iff $[\![|u|>0]\!] = 1$.

proof If u is invertible, then $|u|$ is a weak order unit, by 353Pc or otherwise. In this case, setting $c = 1 \setminus [\![|u|>0]\!]$, we see that

$$[\![|u| \wedge \chi c > 0]\!] = [\![|u|>0]\!] \cap c = 0$$

(364L(b-i)), so that $|u| \wedge \chi c \le 0$ and $\chi c = 0$, that is, $c = 0$; so $[\![|u|>0]\!]$ must be 1. To complete the circuit, suppose that $[\![|u|>0]\!] = 1$. Let Z, Σ, $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$, π, $\mathcal{M}$ be as in the proof of 364D, and $S : \mathcal{L}^0 \to L^0$ the canonical map, so that $[\![Sh>\alpha]\!] = \pi\{z : h(z) > \alpha\}$ for every $h \in \mathcal{L}^0$, $\alpha \in \mathbb{R}$. Express u as Sf where $f \in \mathcal{L}^0$. Then $\pi\{z : |f(z)| > 0\} = [\![S|f|>0]\!] = 1$, so $\{z : f(z) = 0\} \in \mathcal{M}$. Set

$$g(z) = \frac{1}{f(z)} \text{ if } f(z) \ne 0, \quad g(z) = 0 \text{ if } f(z) = 0.$$

Then $\{z : f(z)g(z) \ne 1\} \in \mathcal{M}$ so

$$u \times Sg = S(f \times g) = S(\chi Z) = \chi 1$$

and u is invertible.

Remark The repeated phrase 'by 353x or otherwise' reflects the fact that the abstract methods there can all be replaced in this case by simple direct arguments based on the construction in 364B-364D.

364O Recovering the algebra: Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. For $a \in \mathfrak{A}$ write V_a for the band in $L^0 = L^0(\mathfrak{A})$ generated by χa. Then $a \mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands in L^0.

proof I copy from the argument for 363J, itself based on 361K. If $a \in \mathfrak{A}$ and $w \in L^0$ then $w \times \chi a \in V_a$. **P** If $v \in V_a^\perp$ then $|\chi a| \wedge |v| = 0$, so $\chi a \times v = 0$, so $(w \times \chi a) \times v = 0$, so $|w \times \chi a| \wedge |v| = 0$; thus $w \times \chi a \in V_a^{\perp\perp}$, which is equal to V_a because L^0 is Archimedean (353Ba). **Q** Now, if $a \in \mathfrak{A}$, $u \in V_a$ and $v \in V_{1\setminus a}$, then $|u| \wedge |v| = 0$ because $\chi a \wedge \chi(1 \setminus a) = 0$; and if $w \in L^0(\mathfrak{A})$ then

$$w = (w \times \chi a) + (w \times \chi(1 \setminus a)) \in V_a + V_{1\setminus a}.$$

So V_a and $V_{1\setminus a}$ are complementary projection bands in L^0. Next, if $U \subseteq L^0$ is a projection band, then $\chi 1$ is expressible as $u + v = u \vee v$ where $u \in U$, $v \in U^\perp$. Setting $a = [\![u > \frac{1}{2}]\!]$, $a' = [\![v > \frac{1}{2}]\!]$ we must have $a \cup a' = 1$ and $a \cap a' = 0$ (using 364L), so that $a' = 1 \setminus a$; also $\frac{1}{2}\chi a \le u$, so that $\chi a \in U$, and similarly $\chi(1 \setminus a) \in U^\perp$. In this case $V_a \subseteq U$ and $V_{1\setminus a} \subseteq U^\perp$, so U must be V_a precisely. Thus $a \mapsto V_a$ is surjective. Finally, just as in 361K, $a \subseteq b \iff V_a \subseteq V_b$, so we have a Boolean isomorphism.

364P I come at last to the result corresponding to 361J and 363F.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism.

(a) We have a multiplicative sequentially order-continuous Riesz homomorphism $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ defined by the formula

$$[\![T_\pi u > \alpha]\!] = \pi[\![u > \alpha]\!]$$

whenever $\alpha \in \mathbb{R}$ and $u \in L^0(\mathfrak{A})$.

(b) Defining $\chi a \in L^0(\mathfrak{A})$ as in 364J, $T_\pi(\chi a) = \chi(\pi a)$ in $L^0(\mathfrak{B})$ for every $a \in \mathfrak{A}$. If we regard $L^\infty(\mathfrak{A})$ and $L^\infty(\mathfrak{B})$ as embedded in $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ respectively, then T_π, as defined here, agrees on $L^\infty(\mathfrak{A})$ with T_π as defined in 363F.

(c) T_π is order-continuous iff π is order-continuous, injective iff π is injective, surjective iff π is surjective.

(d) $[\![T_\pi u \in E]\!] = \pi[\![u \in E]\!]$ for every $u \in L^0(\mathfrak{A})$ and every Borel set $E \subseteq \mathbb{R}$; consequently $\bar{h}T_\pi = T_\pi\bar{h}$ for every Borel measurable $h : \mathbb{R} \to \mathbb{R}$, writing $\bar{h}$ indifferently for the associated maps from $L^0(\mathfrak{A})$ to itself and from $L^0(\mathfrak{B})$ to itself (364H).

(e) If $\mathfrak{C}$ is another Dedekind σ-complete Boolean algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ another sequentially order-continuous Boolean homomorphism then $T_{\theta\pi} = T_\theta T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$.

proof I write T for T_π.

(a)(i) To see that Tu is well-defined in $L^0(\mathfrak{B})$ for every $u \in L^0(\mathfrak{A})$, all we need to do is to check that the map $\alpha \mapsto \pi[\![u > \alpha]\!] : \mathbb{R} \to \mathfrak{B}$ satisfies the conditions of 364Ae, and this is easy, because π preserves all countable suprema and infima.

(ii) To see that T is linear and order-preserving and multiplicative, we can use the formulae of 364D. For instance, if u, $v \in L^0(\mathfrak{A})$, then

$$\begin{aligned}[\![Tu + Tv > \alpha]\!] &= \sup_{q\in\mathbb{Q}} [\![Tu > q]\!] \cap [\![Tv > \alpha - q]\!] = \sup_{q\in\mathbb{Q}} \pi[\![u > q]\!] \cap \pi[\![v > \alpha - q]\!] \\ &= \pi(\sup_{q\in\mathbb{Q}} [\![u > q]\!] \cap [\![v > \alpha - q]\!]) = \pi[\![u + v > \alpha]\!] = [\![T(u + v) > \alpha]\!]\end{aligned}$$

for every $\alpha \in \mathbb{R}$, so that $Tu + Tv = T(u + v)$. In the same way,

$$T(\gamma u) = \gamma Tu \text{ whenever } \gamma > 0,$$

$$Tu \le Tv \text{ whenever } u \le v,$$

$$Tu \times Tv = T(u \times v) \text{ whenever } u,\, v \ge 0,$$

so that, using the distributive laws, T is linear and multiplicative.

To see that T is a sequentially order-continuous Riesz homomorphism, suppose that $A \subseteq L^0(\mathfrak{A})$ is a countable non-empty set with a supremum $u^* \in L^0(\mathfrak{A})$; then $T[A]$ is a non-empty subset of $L^0(\mathfrak{B})$ with an upper bound Tu^*, and

$$\sup_{u\in A}[\![Tu>\alpha]\!]=\sup_{u\in A}\pi[\![u>\alpha]\!]=\pi(\sup_{u\in A}[\![u>\alpha]\!])=\pi[\![u^*>\alpha]\!]$$

(using 364La)

$$=[\![Tu^*>\alpha]\!]$$

for every $\alpha\in\mathbb{R}$. So using 364La again, $Tu^*=\sup_{u\in A}Tu$. Now this is true, in particular, for doubleton sets A, so that T is a Riesz homomorphism; and also for non-decreasing sequences, so that T is sequentially order-continuous.

(b) The identification of $T(\chi a)$ with $\chi(\pi a)$ is another almost trivial verification. It follows that T agrees with the map of 363F on $S(\mathfrak{A})$, if we think of $S(\mathfrak{A})$ as a subspace of $L^0(\mathfrak{A})$. Next, if $u\in L^\infty(\mathfrak{A})\subseteq L^0(\mathfrak{A})$, and $\gamma=\|u\|_\infty$, then $|u|\le\gamma\chi 1_{\mathfrak{A}}$, so that $|Tu|\le\gamma\chi 1_{\mathfrak{B}}$, and $Tu\in L^\infty(\mathfrak{B})$, with $\|Tu\|_\infty\le\gamma$. Thus $T\restriction L^\infty(\mathfrak{A})$ has norm at most 1. As it agrees with the map of 363F on $S(\mathfrak{A})$, which is $\|\,\|_\infty$-dense in $L^\infty(\mathfrak{A})$ (363C), and both are continuous, they must agree on the whole of $L^\infty(\mathfrak{A})$.

(c)(i)($\boldsymbol{\alpha}$) Suppose that π is order-continuous, and that $A\subseteq L^0(\mathfrak{A})$ is a non-empty set with a supremum $u^*\in L^0(\mathfrak{A})$. Then for any $\alpha\in\mathbb{R}$,

$$[\![Tu^*>\alpha]\!]=\pi[\![u^*>\alpha]\!]=\pi(\sup_{u\in A}[\![u>\alpha]\!])$$

(by 364La)

$$=\sup_{u\in A}\pi[\![u>\alpha]\!]$$

(because π is order-continuous)

$$=\sup_{u\in A}[\![Tu>\alpha]\!].$$

As α is arbitrary, $Tu^*=\sup T[A]$, by 364La again. As A is arbitrary, T is order-continuous (351Ga).

($\boldsymbol{\beta}$) Now suppose that T is order-continuous and that $A\subseteq\mathfrak{A}$ is a non-empty set with supremum c in $\mathfrak{A}$. Then $\chi c=\sup_{a\in A}\chi a$ (364Jc) so

$$\chi(\pi c)=T(\chi c)=\sup\nolimits_{a\in A}T(\chi a)=\sup\nolimits_{a\in A}\chi(\pi a).$$

But now

$$\pi c=[\![\chi(\pi c)>0]\!]=\sup\nolimits_{a\in A}[\![\chi(\pi a)>0]\!]=\sup\nolimits_{a\in A}\pi a.$$

As A is arbitrary, π is order-continuous.

(ii)($\boldsymbol{\alpha}$) If π is injective and u, v are distinct elements of $L^0(\mathfrak{A})$, then there must be some α such that $[\![u>\alpha]\!]\ne[\![v>\alpha]\!]$, in which case $[\![Tu>\alpha]\!]\ne[\![Tv>\alpha]\!]$ and $Tu\ne Tv$.

($\boldsymbol{\beta}$) Now suppose that T is injective. It is easy to see that $\chi:\mathfrak{A}\to L^0(\mathfrak{A})$ is injective, so that $T\chi:\mathfrak{A}\to L^0(\mathfrak{B})$ is injective; but this is the same as $\chi\pi$ (by (b)), so π must also be injective.

(iii)($\boldsymbol{\alpha}$) Suppose that π is surjective. Let Σ be a σ-algebra of sets such that there is a sequentially order-continuous Boolean surjection $\phi:\Sigma\to\mathfrak{A}$. Then $\pi\phi:\Sigma\to\mathfrak{B}$ is surjective. So given $w\in L^0(\mathfrak{B})$, there is an $f\in\mathcal{L}^0_\Sigma$ such that $[\![w>\alpha]\!]=\pi\phi\{x:f(x)>\alpha\}$ for every $\alpha\in\mathbb{R}$ (364C). But, also by 364C, there is a $u\in L^0(\mathfrak{A})$ such that $[\![u>\alpha]\!]=\phi\{x:f(x)>\alpha\}$ for every α. And now of course $Tu=w$. As w is arbitrary, T is surjective.

($\boldsymbol{\beta}$) If T is surjective, and $b\in\mathfrak{B}$, there must be some $u\in L^0(\mathfrak{A})$ such that $Tu=\chi b$. Now set $a=[\![u>0]\!]$ and see that $\pi a=[\![\chi b>0]\!]=b$. As b is arbitrary, π is surjective.

(d) The map $E\mapsto\pi[\![u\in E]\!]$ is a sequentially order-continuous Boolean homomorphism, equal to $[\![Tu\in E]\!]$ when E is of the form $]\alpha,\infty[$; so by 364F the two are equal for all Borel sets E.

If $h:\mathbb{R}\to\mathbb{R}$ is a Borel measurable function, $u\in L^0(\mathfrak{A})$ and $E\subseteq\mathbb{R}$ is a Borel set, then

$$\begin{aligned}[\![\bar h(Tu)\in E]\!]&=[\![Tu\in h^{-1}[E]]\!]=\pi[\![u\in h^{-1}[E]]\!]\\&=\pi[\![\bar h(u)\in E]\!]=[\![T(\bar h(u))\in E]\!].\end{aligned}$$

As E and u are arbitrary, $T\bar h=\bar hT$.

(e) This is immediate from (a).

364Q Proposition Let X and Y be sets, Σ, T σ-algebras of subsets of X, Y respectively, and $\mathcal{I}$, $\mathcal{J}$ σ-ideals of Σ, T. Set $\mathfrak{A}=\Sigma/\mathcal{I}$ and $\mathfrak{B}=\mathrm{T}/\mathcal{J}$. Suppose that $\phi:X\to Y$ is a function such that $\phi^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$ and $\phi^{-1}[F]\in\mathcal{I}$ for every $F\in\mathcal{J}$.

(a) There is a sequentially order-continuous Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by saying that $\pi F^\bullet=\phi^{-1}[F]^\bullet$ for every $F\in\mathrm{T}$.

(b) Let $T_\pi:L^0(\mathfrak{B})\to L^0(\mathfrak{A})$ be the Riesz homomorphism corresponding to π, as defined in 364P. If we identify $L^0(\mathfrak{B})$ with $\mathcal{L}^0_{\mathrm{T}}/\mathcal{W}_{\mathcal{J}}$ and $L^0(\mathfrak{A})$ with $\mathcal{L}^0_\Sigma/\mathcal{W}_{\mathcal{I}}$ in the manner of 364B-364C, then $T_\pi(g^\bullet)=(g\phi)^\bullet$ for every $g\in\mathcal{L}^0_{\mathrm{T}}$.

(c) Let Z be a third set, Υ a σ-algebra of subsets of Z, $\mathcal{K}$ a σ-ideal of Υ, and $\psi:Y\to Z$ a function such that $\psi^{-1}[G]\in\mathrm{T}$ for every $G\in\Upsilon$ and $\psi^{-1}[G]\in\mathcal{J}$ for every $G\in\mathcal{K}$. Let $\theta:\mathfrak{C}\to\mathfrak{B}$ and $T_\theta:L^0(\mathfrak{C})\to L^0(\mathfrak{B})$ be the homomorphisms corresponding to ψ as in (a)-(b). Then $\pi\theta:\mathfrak{C}\to\mathfrak{A}$ and $T_\pi T_\theta:L^0(\mathfrak{C})\to L^0(\mathfrak{A})$ correspond to $\psi\phi:X\to Y$ in the same way.

(d) Now suppose that μ and ν are measures with domains Σ, T and null ideals $\mathcal{N}(\mu)$, $\mathcal{N}(\nu)$ respectively, and that $\mathcal{I}=\Sigma\cap\mathcal{N}(\mu)$ and $\mathcal{J}=\mathrm{T}\cap\mathcal{N}(\nu)$. In this case, identifying $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ with $L^0(\mu)$ and $L^0(\nu)$ as in 364Ic, we have $g\phi\in\mathcal{L}^0(\mu)$ and $T_\pi(g^\bullet)=(g\phi)^\bullet$ for every $g\in\mathcal{L}^0(\nu)$.

proof (a) The argument is essentially that of 324A-324B, somewhat simplified. Explicitly: if F_1, $F_2\in\mathrm{T}$ and $F_1^\bullet=F_2^\bullet$, then $F_1\triangle F_2\in\mathcal{J}$ so $\phi^{-1}[F_1]\triangle\phi^{-1}[F_2]=\phi^{-1}[F_1\triangle F_2]$ belongs to $\mathcal{I}$ and $\phi^{-1}[F_1]^\bullet=\phi^{-1}[F_2]^\bullet$. So the formula offered defines a map $\pi:\mathfrak{B}\to\mathfrak{A}$. It is a Boolean homomorphism, because if F_1, $F_2\in\mathrm{T}$ then

$$\begin{aligned}\pi F_1^\bullet\mathbin{\triangle}\pi F_2^\bullet&=\phi^{-1}[F_1]^\bullet\mathbin{\triangle}\phi^{-1}[F_2]^\bullet=(\phi^{-1}[F_1]\triangle\phi^{-1}[F_2])^\bullet\\&=\phi^{-1}[F_1\triangle F_2]^\bullet=\pi(F_1\triangle F_2)^\bullet=\pi(F_1^\bullet\mathbin{\triangle}F_2^\bullet),\end{aligned}$$

so $\pi(b_1\mathbin{\triangle}b_2)=\pi b_1\mathbin{\triangle}b_2$ for all b_1, $b_2\in\mathfrak{B}$. Similarly $\pi(b_1\cap b_2)=\pi b_1\cap b_2$ for all b_1, $b_2\in\mathfrak{B}$, and of course

$$\pi 1_{\mathfrak{B}}=\pi Y^\bullet=\phi^{-1}[Y]^\bullet=X^\bullet=1_{\mathfrak{A}}.$$

To see that π is sequentially order-continuous, let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{B}$. For each n we may choose an $F_n\in\mathrm{T}$ such that $F_n^\bullet=b_n$, and set $F=\bigcup_{n\in\mathbb{N}}F_n$. As the map $H\mapsto H^\bullet:\mathrm{T}\to\mathfrak{B}$ is sequentially order-continuous (313Qb), $F^\bullet=\sup_{n\in\mathbb{N}}b_n$ in $\mathfrak{B}$. Now

$$\begin{aligned}\pi(\sup_{n\in\mathbb{N}}b_n)&=\pi F^\bullet=\phi^{-1}[F]^\bullet=(\bigcup_{n\in\mathbb{N}}\phi^{-1}[F_n])^\bullet\\&=\sup_{n\in\mathbb{N}}\phi^{-1}[F_n]^\bullet=\sup_{n\in\mathbb{N}}\pi F_n^\bullet=\sup_{n\in\mathbb{N}}\pi b_n.\end{aligned}$$

So π is sequentially order-continuous, by 313Lc.

(b) Now suppose that $g:Y\to\mathbb{R}$ is T-measurable; write v for $g^\bullet$ in $\mathcal{L}^0_{\mathrm{T}}/\mathcal{W}_{\mathcal{J}}\cong L^0(\mathfrak{B})$. Set $f=g\phi$; then

$$\{x:f(x)>\alpha\}=\phi^{-1}[\{y:g(y)>\alpha\}]$$

belongs to Σ for every $\alpha\in\mathbb{R}$, so f is Σ-measurable and we can speak of $u=f^\bullet$ in $\mathcal{L}^0_\Sigma/\mathcal{W}_{\mathcal{I}}\cong L^0(\mathfrak{A})$. Now, by 364Ca,

$$\begin{aligned}[\![u>\alpha]\!]&=\{x:f(x)>\alpha\}^\bullet=\phi^{-1}[\{y:g(y)>\alpha\}]^\bullet\\&=\pi\{y:g(y)>\alpha\}^\bullet=\pi[\![v>\alpha]\!]=[\![T_\pi v>\alpha]\!]\end{aligned}$$

for every $\alpha\in\mathbb{R}$, and

$$(g\phi)^\bullet=f^\bullet=u=T_\pi v=T_\pi g^\bullet,$$

as claimed.

(c) Starting from the facts that $(\psi\phi)^{-1}[G]=\phi^{-1}[\psi^{-1}[G]]$ for every $G\in\Upsilon$ and $h(\psi\phi)=(h\psi)\phi$ for every $h\in\mathcal{L}^0_\Upsilon$, we just have to run through the formulae.

(d) If $g\in\mathcal{L}^0(\nu)$, there are a $g_0\in\mathcal{L}^0_{\mathrm{T}}$ and an $F\in\mathcal{J}$ such that $g(y)$ is defined and equal to $g_0(y)$ for every $y\in Y\setminus F$. In this case, $\phi^{-1}[F]\in\mathcal{I}$ and $g\phi(x)$ is defined and equal to $g_0\phi(x)$ for every $x\in X\setminus\phi^{-1}[F]$, so $g\phi\in\mathcal{L}^0(\mu)$ and

$$(g\phi)^\bullet=(g_0\phi)^\bullet=T_\pi(g_0^\bullet)=T_\pi(g^\bullet)$$

by (b).

364R Products: Proposition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Dedekind σ-complete Boolean algebras, with simple product $\mathfrak{A}$. If $\pi_i:\mathfrak{A}\to\mathfrak{A}_i$ is the coordinate map for each i, and $T_i:L^0(\mathfrak{A})\to L^0(\mathfrak{A}_i)$ the corresponding homomorphism, then $u\mapsto Tu=\langle T_iu\rangle_{i\in I}:L^0(\mathfrak{A})\to\prod_{i\in I}L^0(\mathfrak{A}_i)$ is a multiplicative Riesz space isomorphism, so $L^0(\mathfrak{A})$ may be identified with the f-algebra product $\prod_{i\in I}L^0(\mathfrak{A}_i)$ (352Wc).

proof Because each π_i is a surjective order-continuous Boolean homomorphism, 364P assures us that there are corresponding surjective multiplicative Riesz homomorphisms T_i. So all we need to check is that the multiplicative Riesz homomorphism $T:L^0(\mathfrak{A})\to\prod_{i\in I}L^0(\mathfrak{A}_i)$ is a bijection.

If u, $v\in L^0(\mathfrak{A})$ are distinct, there must be some $\alpha\in\mathbb{R}$ such that $[\![u>\alpha]\!]\ne[\![v>\alpha]\!]$. In this case there must be an $i\in I$ such that $\pi_i[\![u>\alpha]\!]\ne\pi_i[\![v>\alpha]\!]$, that is, $[\![T_iu>\alpha]\!]\ne[\![T_iv>\alpha]\!]$. So $T_iu\ne T_iv$ and $Tu\ne Tv$. As u, v are arbitrary, T is injective.

If $w=\langle w_i\rangle_{i\in I}$ is any member of $\prod_{i\in I}L^0(\mathfrak{A}_i)$, then for $\alpha\in\mathbb{R}$ set

$$\phi(\alpha)=\langle[\![w_i>\alpha]\!]\rangle_{i\in I}\in\mathfrak{A}.$$

It is easy to check that ϕ satisfies the conditions of 364Aa, because, for instance,

$$\sup_{\beta>\alpha}\pi_i\phi(\beta)=\sup_{\beta>\alpha}[\![w_i>\beta]\!]=[\![w_i>\alpha]\!]=\pi_i\phi(\alpha)$$

for every i, so that $\sup_{\beta>\alpha}\phi(\beta)=\phi(\alpha)$, for every $\alpha\in\mathbb{R}$; and the other two conditions are also satisfied because they are satisfied coordinate-by-coordinate. So there is a $u\in L^0(\mathfrak{A})$ such that $\phi(\alpha)=[\![u>\alpha]\!]$ for every α, that is, $\pi_i[\![u>\alpha]\!]=[\![w_i>\alpha]\!]$ for all α, i, that is, $T_iu=w_i$ for every i, that is, $Tu=w$. As w is arbitrary, T is surjective and we are done.

***364S Regular open algebras** I noted in 314P that for every topological space X there is a corresponding Dedekind complete Boolean algebra $\mathrm{RO}(X)$ of regular open sets. We have an identification of $L^0(\mathrm{RO}(X))$ as a space of equivalence classes of functions, different in kind from the representations above, as follows. This is hard work (especially if we do it in full generality), but instructive. I start with a temporary definition.

Definition Let $(X,\mathfrak{T})$ be a topological space and $f:X\to\mathbb{R}$ a function. For $x\in X$ write

$$\omega(f,x)=\inf_{G\in\mathfrak{T},x\in G}\sup_{y,z\in G}|f(y)-f(z)|$$

(allowing ∞).

***364T Theorem** Let X be any topological space, and $\mathrm{RO}(X)$ its regular open algebra. Let U be the set of functions $f:X\to\mathbb{R}$ such that $\{x:\omega(f,x)<\epsilon\}$ is dense in X for every $\epsilon>0$. Then U is a Riesz subspace of $\mathbb{R}^X$, closed under multiplication, and we have a surjective multiplicative Riesz homomorphism $T:U\to L^0(\mathrm{RO}(X))$ defined by writing

$$[\![Tf>\alpha]\!]=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f(x)>\beta\}},$$

the supremum being taken in $\mathrm{RO}(X)$, for every $\alpha\in\mathbb{R}$ and $f\in U$. The kernel of T is the set W of functions $f:X\to\mathbb{R}$ such that $\operatorname{int}\{x:|f(x)|\le\epsilon\}$ is dense for every $\epsilon>0$, so $L^0(\mathrm{RO}(X))$ can be identified, as f-algebra, with the quotient space U/W.

proof (a)(i)(α) The first thing to observe is that for any $f\in\mathbb{R}^X$ and $\epsilon>0$ the set

$$\{x:\omega(f,x)<\epsilon\}=\bigcup\{G:G\subseteq X\text{ is open and non-empty}\\ \text{and }\sup_{y,z\in G}|f(y)-f(z)|<\epsilon\}$$

is open.

(β) Next, it is easy to see that

$$\omega(f+g,x)\le\omega(f,x)+\omega(g,x),$$

$$\omega(\gamma f,x)=|\gamma|\omega(f,x),$$

$$\omega(|f|,x)\le\omega(f,x),$$

for all f, $g\in\mathbb{R}^X$ and $\gamma\in\mathbb{R}$.

(γ) Thirdly, it is useful to know that if $f\in U$ and $G\subseteq X$ is a non-empty open set, then there is a non-empty open set $G'\subseteq G$ on which f is bounded. **P** Take any $x_0\in G$ such that $\omega(f,x_0)<1$; then there is a non-empty open set G' containing x_0 such that $|f(y)-f(z)|<1$ for all y, $z\in G'$, and we may suppose that $G'\subseteq G$. But now $|f(x)|\le 1+|f(x_0)|$ for every $x\in G'$. **Q**

(ii) So if f, $g\in U$ and $\gamma\in\mathbb{R}$ then

$$\{x:\omega(f+g,x)<\epsilon\}\supseteq\{x:\omega(f,x)<\frac{1}{2}\epsilon\}\cap\{x:\omega(g,x)<\frac{1}{2}\epsilon\}$$

is the intersection of two dense open sets and is therefore dense, while

$$\{x:\omega(\gamma f,x)<\epsilon\}\supseteq\{x:\omega(f,x)<\tfrac{\epsilon}{1+|\gamma|}\},$$

$$\{x:\omega(|f|,x)<\epsilon\}\supseteq\{x:\omega(f,x)<\epsilon\}$$

are also dense. As ϵ is arbitrary, $f+g$, γf and $|f|$ all belong to U; as f, g and γ are arbitrary, U is a Riesz subspace of $\mathbb{R}^X$.

(iii) If f, $g\in U$ then $f\times g\in U$. **P** Take $\epsilon>0$ and let G_0 be a non-empty open subset of X. By the last remark in (i) above, there is a non-empty open set $G_1\subseteq G_0$ such that $|f|\vee|g|$ is bounded on G_1; say $\max(|f(x)|,|g(x)|)\le\gamma$ for every $x\in G_1$.

Set $\delta=\frac{\epsilon}{2\gamma+1}>0$. Then there is an $x\in G_1$ such that $\omega(f,x)<\delta$ and $\omega(g,x)<\delta$. Let H, H' be open sets containing x such that $|f(y)-f(z)|\le\delta$ for all y, $z\in H$ and $|g(y)-g(z)|\le\delta$ for all y, $z\in H'$. Consider $G=G_1\cap H\cap H'$. This is an open set containing x, and if y, $z\in G$ then

$$\begin{aligned}|f(y)g(y)-f(z)g(z)|&\le|f(y)-f(z)||g(z)|+|f(z)||g(y)-g(z)|\\&\le\delta\gamma+\gamma\delta.\end{aligned}$$

Accordingly

$$\omega(f\times g,x)\le 2\delta\gamma<\epsilon,$$

while $x\in G_0$. As G_0 is arbitrary, $\{x:\omega(f\times g,x)<\epsilon\}$ is dense; as ϵ is arbitrary, $f\times g\in U$. **Q**

Thus U is a subalgebra of $\mathbb{R}^X$.

(b) Now, for $f\in U$, consider the map $\phi_f:\mathbb{R}\to\mathrm{RO}(X)$ defined by setting

$$\phi_f(\alpha)=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f(x)>\beta\}}$$

for every $\alpha\in\mathbb{R}$. Then ϕ_f satisfies the conditions of 364Aa. **P** (See 314P for the calculation of suprema and infima in $\mathrm{RO}(X)$.) (i) If $\alpha\in\mathbb{R}$ then

$$\begin{aligned}\phi_f(\alpha)=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f(x)>\beta\}}&=\sup_{\gamma>\beta>\alpha}\operatorname{int}\overline{\{x:f(x)>\gamma\}}\\&=\sup_{\beta>\alpha}\sup_{\gamma>\beta}\operatorname{int}\overline{\{x:f(x)>\gamma\}}=\sup_{\beta>\alpha}\phi_f(\beta).\end{aligned}$$

(ii) If $G_0\subseteq X$ is a non-empty open set, then there is a non-empty open set $G_1\subseteq G_0$ such that f is bounded on G_1; say $|f(x)|<\gamma$ for every $x\in G_1$. If $\beta>\gamma$ then G_1 does not meet $\{x:f(x)>\beta\}$, so $G_1\cap\operatorname{int}\overline{\{x:f(x)>\gamma\}}=\emptyset$; as β is arbitrary, $G_1\cap\phi_f(\gamma)=\emptyset$ and $G_0\not\subseteq\inf_{\alpha\in\mathbb{R}}\phi_f(\alpha)$. On the other hand, $G_1\subseteq\{x:f(x)>-\gamma\}$, so

$$G_1\subseteq\operatorname{int}\overline{\{x:f(x)>-\gamma\}}\subseteq\phi_f(-\gamma)$$

and $G_0\cap\sup_{\alpha\in\mathbb{R}}\phi_f(\alpha)\ne\emptyset$. As G_0 is arbitrary, $\inf_{\alpha\in\mathbb{R}}\phi_f(\alpha)=\emptyset$ and $\sup_{\alpha\in\mathbb{R}}\phi_f(\alpha)=X$. **Q**

(c) Thus we have a map $T:U\to L^0=L^0(\mathrm{RO}(X))$ defined by setting $[\![Tf>\alpha]\!]=\phi_f(\alpha)$ whenever $\alpha\in\mathbb{R}$ and $f\in U$.

It is worth noting that

$$\{x:f(x)>\alpha+\omega(f,x)\}\subseteq[\![Tf>\alpha]\!]\subseteq\{x:f(x)+\omega(f,x)\ge\alpha\}$$

for every $f\in U$ and $\alpha\in\mathbb{R}$. **P** (i) If $f(x)>\alpha+\omega(f,x)$, set $\delta=\frac{1}{2}(f(x)-\alpha-\omega(f,x))>0$. Then there is an open set G containing x such that $|f(y)-f(z)|<\omega(f,x)+\delta$ for every y, $z\in G$, so that $f(y)>\alpha+\delta$ for every $y\in G$, and

$$x \in \operatorname{int}\{y : f(y) > \alpha + \delta\} \subseteq [\![Tf > \alpha]\!].$$

(ii) If $f(x) + \omega(f,x) < \alpha$, set $\delta = \frac{1}{2}(\alpha - f(x) - \omega(f,x)) > 0$; then there is an open neighbourhood G of x such that $|f(y) - f(z)| < \omega(f,x) + \delta$ for every y, $z \in G$, so that $f(y) < \alpha$ for every $y \in G$. Accordingly G does not meet $\{y : f(y) > \beta\}$ nor $\overline{\{y : f(y) > \beta\}}$ for any $\beta > \alpha$, $G \cap [\![Tf > \alpha]\!] = \emptyset$ and $x \notin [\![Tf > \alpha]\!]$. **Q**

(d) T is additive. **P** Let f, $g \in U$ and $\alpha < \beta \in \mathbb{R}$. Set $\delta = \frac{1}{5}(\beta - \alpha) > 0$, $H = \{x : \omega(f,x) < \delta, \omega(g,x) < \delta\}$; then H is the intersection of two dense open sets, so is itself dense and open.

(i) If $x \in H \cap [\![T(f+g) > \beta]\!]$, then $(f+g)(x) + \omega(f+g,x) \geq \beta$; but $\omega(f+g,x) \leq 2\delta$ (see (a-i-β) above), so $f(x) + g(x) \geq \beta - 2\delta > \alpha + 2\delta$ and there is a $q \in \mathbb{Q}$ such that

$$f(x) > q + \delta \geq q + \omega(f,x), \quad g(x) > \alpha - q + \delta \geq \alpha - q + \omega(g,x).$$

Accordingly

$$x \in [\![Tf > q]\!] \cap [\![Tg > \alpha - q]\!] \subseteq [\![Tf + Tg > \alpha]\!].$$

Thus $H \cap [\![T(f+g) > \beta]\!] \subseteq [\![Tf + Tg > \alpha]\!]$. Because H is dense, $[\![T(f+g) > \beta]\!] \subseteq [\![Tf + Tg > \alpha]\!]$.

(ii) If $x \in H$, then

$$\begin{aligned}
x \in \bigcup_{q \in \mathbb{Q}} &([\![Tf > q]\!] \cap [\![Tg > \beta - q]\!]) \\
&\Longrightarrow \exists\, q \in \mathbb{Q},\ f(x) + \omega(f,x) \geq q,\ g(x) + \omega(g,x) \geq \beta - q \\
&\Longrightarrow f(x) + g(x) + 2\delta \geq \beta \\
&\Longrightarrow (f+g)(x) \geq \alpha + 3\delta > \alpha + \omega(f+g,x) \\
&\Longrightarrow x \in [\![T(f+g) > \alpha]\!].
\end{aligned}$$

Thus

$$H \cap \bigcup\nolimits_{q \in \mathbb{Q}} ([\![Tf > q]\!] \cap [\![Tg > \beta - q]\!]) \subseteq [\![T(f+g) > \alpha]\!].$$

Because H is dense and $\bigcup_{q \in \mathbb{Q}} ([\![Tf > q]\!] \cap [\![Tg > \beta - q]\!])$ is open,

$$\begin{aligned}
[\![Tf + Tg > \beta]\!] &= \operatorname{int} \overline{\bigcup_{q \in \mathbb{Q}} [\![Tf > q]\!] \cap [\![Tg > \beta - q]\!]} \\
&\subseteq \operatorname{int} \overline{[\![T(f+g) > \alpha]\!]} = [\![T(f+g) > \alpha]\!].
\end{aligned}$$

(iii) Now let $\beta \downarrow \alpha$; we have

$$\begin{aligned}
[\![T(f+g) > \alpha]\!] &= \sup_{\beta > \alpha} [\![T(f+g) > \beta]\!] \subseteq [\![Tf + Tg > \alpha]\!] \\
&= \sup_{\beta > \alpha} [\![Tf + Tg > \beta]\!] \subseteq [\![T(f+g) > \alpha]\!],
\end{aligned}$$

so $[\![T(f+g) > \alpha]\!] = [\![Tf + Tg > \alpha]\!]$. As α is arbitrary, $T(f+g) = Tf + Tg$; as f and g are arbitrary, T is additive. **Q**

(e) It is now easy to see that T is linear. **P** If $\gamma > 0$, $f \in U$ and $\alpha \in \mathbb{R}$ then

$$\begin{aligned}
[\![T(\gamma f) > \alpha]\!] &= \sup_{\beta > \alpha} \operatorname{int} \overline{\{x : \gamma f(x) > \beta\}} = \sup_{\beta > \alpha} \operatorname{int} \overline{\{x : f(x) > \frac{\beta}{\gamma}\}} \\
&= \sup_{\beta > \alpha/\gamma} \operatorname{int} \overline{\{x : f(x) > \beta\}} = [\![Tf > \frac{\alpha}{\gamma}]\!] = [\![\gamma Tf > \alpha]\!].
\end{aligned}$$

As α is arbitrary, $T(\gamma f) = \gamma Tf$; because we already know that T is additive, this is enough to show that T is linear. **Q**

(f) In fact T is a Riesz homomorphism. **P** If $f \in U$ and $\alpha \geq 0$ then

$$
\begin{aligned}
[\![T(f^+) > \alpha]\!] &= \sup_{\beta>\alpha} \operatorname{int} \overline{\{x : f^+(x) > \beta\}} = \sup_{\beta>\alpha} \operatorname{int} \overline{\{x : f(x) > \beta\}} \\
&= [\![Tf > \alpha]\!] = [\![(Tf)^+ > \alpha]\!].
\end{aligned}
$$

If $\alpha < 0$ then

$$[\![T(f^+) > \alpha]\!] = \sup_{\beta>\alpha} \operatorname{int} \overline{\{x : f^+(x) > \beta\}} = X = [\![(Tf)^+ > \alpha]\!]. \textbf{ Q}$$

(g) Of course the constant function χX belongs to U, and is its multiplicative identity; and $T(\chi X)$ is the multiplicative identity of L^0, because

$$
\begin{aligned}
[\![T(\chi X) > \alpha]\!] &= \sup_{\beta>\alpha} \operatorname{int} \overline{\{x : (\chi X)(x) > \beta\}} \\
&= X \text{ if } \alpha < 1,\ \emptyset \text{ if } \alpha \ge 1.
\end{aligned}
$$

By 353Pd, or otherwise, T is multiplicative.

(h) The kernel of T is W. **P** (i) For $f \in U$,

$$
\begin{aligned}
Tf = 0 &\Longrightarrow [\![T|f| > 0]\!] = [\![|Tf| > 0]\!] = \emptyset \\
&\Longrightarrow \{x : |f(x)| > \omega(|f|, x)\} = \emptyset \\
&\Longrightarrow \operatorname{int}\{x : |f(x)| \le \epsilon\} \supseteq \{x : \omega(|f|, x) < \epsilon\} \text{ is dense for every } \epsilon > 0 \\
&\Longrightarrow f \in W.
\end{aligned}
$$

(ii) If $f \in W$, then, first,

$$\{x : \omega(f, x) < \epsilon\} \supseteq \operatorname{int}\{x : |f(x)| \le \frac{1}{3}\epsilon\}$$

is dense for every $\epsilon > 0$, so $f \in U$; and next, for any $\beta > 0$, $\overline{\{x : |f(x)| > \beta\}}$ does not meet the dense open set $\operatorname{int}\{x : |f(x)| \le \beta\}$, so

$$[\![|Tf| > 0]\!] = [\![T|f| > 0]\!] = \sup_{\beta>0} \operatorname{int} \overline{\{x : |f(x)| > \beta\}} = \emptyset$$

and $Tf = 0$. **Q**

(i) Finally, T is surjective. **P** Take any $u \in L^0$. Define $\tilde{f} : X \to [-\infty, \infty]$ by setting $\tilde{f}(x) = \sup\{\alpha : x \in [\![u > \alpha]\!]\}$ for each x, counting $\inf \emptyset$ as $-\infty$. Then

$$\{x : \tilde{f}(x) > \alpha\} = \bigcup_{\beta>\alpha} [\![u > \beta]\!]$$

is open, for every $\alpha \in \mathbb{R}$. The set

$$\{x : \tilde{f}(x) = \infty\} = \bigcap_{\alpha\in\mathbb{R}} [\![u > \alpha]\!]$$

is nowhere dense, because $\inf_{\alpha\in\mathbb{R}} [\![u > \alpha]\!] = \emptyset$ in $\mathrm{RO}(X)$; while

$$\{x : \tilde{f}(x) = -\infty\} = X \setminus \bigcup_{\alpha\in\mathbb{R}} [\![u > \alpha]\!]$$

also is nowhere dense, because $\sup_{\alpha\in\mathbb{R}} [\![u > \alpha]\!] = X$ in $\mathrm{RO}(X)$. Accordingly $E = \operatorname{int}\{x : \tilde{f}(x) \in \mathbb{R}\}$ is dense. Set $f(x) = \tilde{f}(x)$ for $x \in E$, 0 for $x \in X \setminus E$.

Let $\epsilon > 0$. If $G \subseteq X$ is a non-empty open set, there is an $\alpha \in \mathbb{R}$ such that $G \not\subseteq [\![u > \alpha]\!]$, so $G_1 = G \setminus \overline{[\![u > \alpha]\!]} \ne \emptyset$, and $\tilde{f}(x) \le \alpha$ for every $x \in G_1$. Set

$$\alpha' = \sup_{x\in G_1} \tilde{f}(x) \le \alpha < \infty.$$

Because E meets G_1, $\alpha' > -\infty$. Then $G_2 = G_1 \cap [\![u > \alpha' - \frac{1}{2}\epsilon]\!]$ is a non-empty open subset of G and $\alpha' - \frac{1}{2}\epsilon \le \tilde{f}(x) \le \alpha'$ for every $x \in G_2$. Accordingly $|f(y) - f(z)| \le \frac{1}{2}\epsilon$ for all y, $z \in G_2$, and $\omega(f, x) < \epsilon$ for all $x \in G_2$. As G is arbitrary, $\{x : \omega(f, x) < \epsilon\}$ is dense; as ϵ is arbitrary, $f \in U$.

Take $\alpha < \beta$ in $\mathbb{R}$, and set $\delta = \frac{1}{2}(\beta - \alpha)$. Then $H = E \cap \{x : \omega(f, x) < \delta\}$ is a dense open set, and

$$
\begin{aligned}
H \cap [\![Tf > \beta]\!] &\subseteq H \cap \{x : f(x) + \omega(f, x) \ge \beta\} \subseteq E \cap \{x : f(x) > \alpha\} \\
&\subseteq \{x : \tilde{f}(x) > \alpha\} \subseteq [\![u > \alpha]\!].
\end{aligned}
$$

As H is dense, $[\![Tf > \beta]\!] \subseteq [\![u > \alpha]\!]$. In the other direction

$$\begin{aligned} H \cap [\![u > \beta]\!] &\subseteq H \cap \{x : \tilde{f}(x) \geq \beta\} = H \cap \{x : f(x) \geq \beta\} \\ &\subseteq \{x : f(x) > \alpha + \omega(f, x)\} \subseteq [\![Tf > \alpha]\!], \end{aligned}$$

so $[\![u > \beta]\!] \subseteq [\![Tf > \alpha]\!]$. Just as in (d) above, this is enough to show that $Tf = u$. As u is arbitrary, T is surjective. **Q**

This completes the proof.

***364U Compact spaces** Suppose now that X is a compact Hausdorff topological space. In this case the space U of 364T is just the space of functions $f : X \to \mathbb{R}$ such that $\{x : f \text{ is continuous at } x\}$ is dense in X. **P** It is easy to see that

$$\{x : f \text{ is continuous at } x\} = \{x : \omega(f, x) = 0\} = \bigcap_{n \in \mathbb{N}} H_n$$

where $H_n = \{x : \omega(f, x) < 2^{-n}\}$ for each n. Each H_n is an open set (see part (a-i-α) of the proof of 364T), so by Baire's theorem (3A3G) $\bigcap_{n \in \mathbb{N}} H_n$ is dense iff every H_n is dense, that is, iff $f \in U$. **Q**

Now W, as defined in 364T, becomes $\{f : f \in U, \{x : f(x) = 0\} \text{ is dense}\}$. **P** (i) If $f \in W$, then $T|f| = 0$, so (by the formula in (c) of the proof of 364T) $|f(x)| \leq \omega(|f|, x)$ for every x. But $\{x : \omega(f, x) = 0\}$ is dense, because $f \in U$, so $\{x : f(x) = 0\}$ also is dense. (ii) If $f \in U$ and $\{x : f(x) = 0\}$ is dense, then

$$\omega(f, x) \geq \inf_{x \in G \text{ is open}} \sup_{y \in G} |f(y) - f(x)| \geq |f(x)|$$

for every $x \in X$. So for any $\epsilon > 0$, $\operatorname{int}\{x : |f(x)| \leq \epsilon\} \supseteq \{x : \omega(f, x) < \epsilon\}$ is dense, and $f \in W$. **Q**

In the case of extremally disconnected spaces, we can go farther.

***364V Theorem** Let X be a compact Hausdorff extremally disconnected space, and $\mathrm{RO}(X)$ its regular open algebra. Write $C^\infty = C^\infty(X)$ for the space of continuous functions $g : X \to [-\infty, \infty]$ such that $\{x : g(x) = \pm\infty\}$ is nowhere dense. Then we have a bijection $S : C^\infty \to L^0 = L^0(\mathrm{RO}(X))$ defined by saying that

$$[\![Sg > \alpha]\!] = \overline{\{x : g(x) > \alpha\}}$$

for every $\alpha \in \mathbb{R}$. Addition and multiplication in L^0 correspond to the operations $\dot{+}$, $\dot{\times}$ on C^∞ defined by saying that $g \dot{+} h$, $g \dot{\times} h$ are the unique elements of C^∞ agreeing with $g + h$, $g \times h$ on $\{x : g(x), h(x) \text{ are both finite}\}$. Scalar multiplication in L^0 corresponds to the operation

$$(\gamma g)(x) = \gamma g(x) \text{ for } x \in X,\ g \in C^\infty,\ \gamma \in \mathbb{R}$$

on C^∞ (counting $0 \cdot \infty$ as 0), while the ordering of L^0 corresponds to the relation

$$g \leq h \iff g(x) \leq h(x) \text{ for every } x \in X.$$

proof (a) For $g \in C^\infty$, set $H_g = \{x : g(x) \in \mathbb{R}\}$, so that H_g is a dense open set, and define $Rg : X \to \mathbb{R}$ by setting $(Rg)(x) = g(x)$ if $x \in H_g$, 0 if $x \in X \setminus H_g$. Then Rg is continuous at every point of H_g, so belongs to the space U of 364T-364U. Set $Sg = T(Rg)$, where $T : U \to L^0$ is the map of 364T. Then

$$[\![Sg > \alpha]\!] = \overline{\{x : g(x) > \alpha\}}$$

for every $\alpha \in \mathbb{R}$. **P** (i) $\omega(g, x) = 0$ for every $x \in H_g$, so, if $\beta > \alpha$,

$$H_g \cap [\![Sg > \beta]\!] \subseteq \{x : x \in H_g,\ (Rg)(x) \geq \beta\} \subseteq \{x : g(x) \geq \beta\}$$

by the formula in part (c) of the proof of 364T. As $[\![Sg > \beta]\!]$ is open and H_g is dense,

$$[\![Sg > \beta]\!] \subseteq \overline{H_g \cap [\![Sg > \beta]\!]} \subseteq \{x : g(x) \geq \beta\} \subseteq \overline{\{x : g(x) > \alpha\}}.$$

Now

$$[\![Sg > \alpha]\!] = \sup_{\beta > \alpha} [\![Sg > \beta]\!] = \operatorname{int} \overline{\bigcup_{\beta > \alpha} [\![Sg > \beta]\!]} \subseteq \overline{\{x : g(x) > \alpha\}}.$$

(ii) In the other direction, $H_g \cap \{x : g(x) > \alpha\} \subseteq [\![Sg > \alpha]\!]$, by the other half of the formula in the proof of 364T. Again because $\{x : g(x) > \alpha\}$ is open and H_g is dense,

$$\overline{\{x : g(x) > \alpha\}} \subseteq \overline{[\![Sg > \alpha]\!]} = [\![Sg > \alpha]\!]$$

because X is extremally disconnected (see 314S). **Q**

(b) Thus $S = TR$ defined by the formula offered. Now if g, $h \in C^\infty$ and $g \le h$, we surely have $\{x : g(x) > \alpha\} \subseteq \{x : h(x) > \alpha\}$ for every α, so $[\![Sg > \alpha]\!] \subseteq [\![Sh > \alpha]\!]$ for every α and $Sg \le Sh$. On the other hand, if $g \not\le h$ then $Sg \not\le Sh$. **P** Take x_0 such that $g(x_0) > h(x_0)$, and $\alpha \in \mathbb{R}$ such that $g(x_0) > \alpha > h(x_0)$; set $H = \{x : g(x) > \alpha > h(x)\}$; this is a non-empty open set and $H \subseteq [\![Sg > \alpha]\!]$. On the other hand, $H \cap \{x : h(x) > \alpha\} = \emptyset$ so $H \cap [\![Sh > \alpha]\!] = \emptyset$. Thus $[\![Sg > \alpha]\!] \not\subseteq [\![Sh > \alpha]\!]$ and $Sg \not\le Sh$. **Q** In particular, S is injective.

(c) S is surjective. **P** If $u \in L^0$, set

$$g(x) = \sup\{\alpha : x \in [\![u > \alpha]\!]\} \in [-\infty, \infty]$$

for every $x \in X$, taking $\sup \emptyset = -\infty$. Then, for any $\alpha \in \mathbb{R}$, $\{x : g(x) > \alpha\} = \bigcup_{\beta > \alpha} [\![u > \alpha]\!]$ is open. On the other hand,

$$\{x : g(x) < \alpha\} = \bigcup_{\beta < \alpha} \{x : x \notin [\![u > \beta]\!]\}$$

also is open, because all the sets $[\![u > \beta]\!]$ are open-and-closed. So $g : X \to [-\infty, \infty]$ is continuous. Also

$$\{x : g(x) > -\infty\} = \bigcup_{\alpha \in \mathbb{R}} [\![u > \alpha]\!],$$

$$\{x : g(x) < \infty\} = \bigcup_{\alpha \in \mathbb{R}} X \setminus [\![u > \alpha]\!]$$

are dense, so $g \in C^\infty$. Now, for any $\alpha \in \mathbb{R}$,

$$[\![Sg > \alpha]\!] = \overline{\{x : g(x) > \alpha\}} = \overline{\bigcup_{\beta > \alpha} [\![u > \beta]\!]}$$

$$= \operatorname{int} \overline{\bigcup_{\beta > \alpha} [\![u > \beta]\!]} = \sup_{\beta > \alpha} [\![u > \beta]\!] = [\![u > \alpha]\!].$$

So $Sg = u$. As u is arbitrary, S is surjective. **Q**

(d) Accordingly S is a bijection. I have already checked (in part (b)) that it is an isomorphism of the order structures. For the algebraic operations, observe that if g, $h \in C^\infty$ then there are f_1, $f_2 \in C^\infty$ such that $Sg + Sh = Sf_1$ and $Sg \times Sh = Sf_2$, that is,

$$T(Rg + Rh) = TRg + TRh = TRf_1, \quad T(Rg \times Rh) = TRg \times TRh = TRf_2.$$

But this means that

$$T(Rg + Rh - Rf_1) = T((Rg \times Rh) - Rf_2) = 0,$$

so that $Rg + Rh - Rf_1$, $(Rg \times Rh) - Rf_2$ belong to W, as defined in 364T-364U, and are zero on dense sets (364U). Since we know also that the set $G = \{x : g(x), h(x) \text{ are both finite}\}$ is a dense open set, while g, h, f_1 and f_2 are all continuous, we must have $f_1(x) = g(x) + h(x)$, $f_2(x) = g(x)h(x)$ for every $x \in G$. And of course this uniquely specifies f_1 and f_2 as members of C^∞.

Thus we do have operations $\dot{+}$, $\dot{\times}$ as described, rendering S additive and multiplicative. As for scalar multiplication, it is easy to check that $R(\gamma g) = \gamma Rg$ (at least, unless $\gamma = 0$, which is trivial), so that $S(\gamma g) = \gamma Sg$ for every $g \in C^\infty$.

364X Basic exercises >(a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. For u, $v \in L^0 = L^0(\mathfrak{A})$ set $[\![u < v]\!] = [\![v > u]\!] = [\![v - u > 0]\!]$, $[\![u \le v]\!] = [\![v \ge u]\!] = 1 \setminus [\![v < u]\!]$, $[\![u = v]\!] = [\![u \le v]\!] \cap [\![v \le u]\!]$. (i) Show that $([\![u < v]\!], [\![u = v]\!], [\![u > v]\!])$ is always a partition of unity in $\mathfrak{A}$. (ii) Show that for any u, u', v, $v' \in L^0$, $[\![u \le u']\!] \cap [\![v \le v']\!] \subseteq [\![u + v \le u' + v']\!]$ and $[\![u = u']\!] \cap [\![v = v']\!] \subseteq [\![u \times v = u' \times v']\!]$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. (i) Show that if u, $v \in L^0 = L^0(\mathfrak{A})$ and α, $\beta \in \mathbb{R}$ then $[\![u + v \ge \alpha + \beta]\!] \subseteq [\![u \ge \alpha]\!] \cup [\![v \ge \beta]\!]$. (ii) Show that if u, $v \in (L^0)^+$ and α, $\beta \ge 0$ then $[\![u \times v \ge \alpha\beta]\!] \subseteq [\![u \ge \alpha]\!] \cup [\![v \ge \beta]\!]$.

(c) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $u \in L^0(\mathfrak{A})$. Show that $\{[\![u \in E]\!] : E \subseteq \mathbb{R} \text{ is Borel}\}$ is the σ-subalgebra of $\mathfrak{A}$ generated by $\{[\![u > \alpha]\!] : \alpha \in \mathbb{R}\}$.

>(d) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. Show that for any $u \in L^0(\mathfrak{A})$ there is a unique Radon probability measure ν on $\mathbb{R}$ (the **distribution** of u) such that $\nu E = \bar{\mu}[\![u \in E]\!]$ for every Borel set $E \subseteq \mathbb{R}$. (*Hint*: 271B.)

(e) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\langle u_i\rangle_{i\in I}$ any family in $L^0(\mathfrak{A})$; for each $i\in I$ let $\mathfrak{B}_i$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{[\![u_i>\alpha]\!]:\alpha\in\mathbb{R}\}$. Show that the following are equiveridical: (i) $\bar{\mu}(\inf_{i\in J}[\![u_i>\alpha_i]\!])=\prod_{i\in J}\bar{\mu}[\![u_i>\alpha_i]\!]$ whenever $J\subseteq I$ is finite and $\alpha_i\in\mathbb{R}$ for each $i\in J$ (ii) $\langle\mathfrak{B}_i\rangle_{i\in I}$ is stochastically independent in the sense of 325L. (In this case we may call $\langle u_i\rangle_{i\in I}$ **$\bar{\mu}$-(stochastically)independent**.)

(f) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and u, v two $\bar{\mu}$-independent members of $L^0(\mathfrak{A})$. Show that the distribution of their sum is the convolution of their distributions. (*Hint*: 272T).

>(g) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and g, $h:\mathbb{R}\to\mathbb{R}$ Borel measurable functions. (i) Show that $\bar{g}\bar{h}=\overline{gh}$, where $\bar{g}$, $\bar{h}:L^0\to L^0$ are defined as in 364H. (ii) Show that $\overline{g+h}(u)=\bar{g}(u)+\bar{h}(u)$, $\overline{g\times h}(u)=\bar{g}(u)\times\bar{h}(u)$ for every $u\in L^0=L^0(\mathfrak{A})$. (iii) Show that if $\langle h_n\rangle_{n\in\mathbb{N}}$ is a sequence of Borel measurable functions on $\mathbb{R}$ and $\sup_{n\in\mathbb{N}}h_n=h$, then $\sup_{n\in\mathbb{N}}\bar{h}_n(u)=\bar{h}(u)$ for every $u\in L^0$. (iv) Show that if h is non-decreasing and continuous on the left, then $\bar{h}(\sup A)=\sup\bar{h}[A]$ whenever $A\subseteq L^0$ is a non-empty set with a supremum in L^0.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. (i) Show that $S(\mathfrak{A})$ can be identified (α) with the set of those $u\in L^0=L^0(\mathfrak{A})$ such that $\{[\![u>\alpha]\!]:\alpha\in\mathbb{R}\}$ is finite (β) with the set of those $u\in L^0$ such that $[\![u\in I]\!]=1$ for some finite $I\subseteq\mathbb{R}$. (ii) Show that $L^\infty(\mathfrak{A})$ can be identified with the set of those $u\in L^0$ such that $[\![u\in[-\alpha,\alpha]]\!]=1$ for some $\alpha\ge0$, and that $\|u\|_\infty$ is the smallest such α.

(i) Show that if $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, and $u\in L^0(\mathfrak{A})$, then for any $\alpha\in\mathbb{R}$

$$[\![u>\alpha]\!]=\inf_{\beta>\alpha}\sup\{a:a\in\mathfrak{A},\,u\times\chi a\ge\beta\chi a\}$$

(compare 363Xh).

>(j) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu:\mathfrak{A}\to\mathbb{R}$ a non-negative finitely additive functional. Let $\textstyle\int:L^\infty(\mathfrak{A})\to\mathbb{R}$ be the corresponding linear functional, as in 363L. Write U for the set of those $u\in L^0(\mathfrak{A})$ such that $\sup\{\textstyle\int v:v\in L^\infty(\mathfrak{A}),\,v\le|u|\}$ is finite. Show that $\textstyle\int$ has an extension to a non-negative linear functional on U.

(k) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $u\ge0$ in $L^0=L^0(\mathfrak{A})$. Show that $u=\sup_{q\in\mathbb{Q}}q\chi[\![u>q]\!]$ in L^0.

(l)(i) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $A\subseteq L^0(\mathfrak{A})$ a non-empty countable set with supremum w. Show that $[\![w\in G]\!]\subseteq\sup_{u\in A}[\![u\in G]\!]$ for every open set $G\subseteq\mathbb{R}$. (ii) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra and $A\subseteq L^0(\mathfrak{A})$ a non-empty set with supremum w. Show that $[\![w\in G]\!]\subseteq\sup_{u\in A}[\![u\in G]\!]$ for every open set $G\subseteq\mathbb{R}$.

(m) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $A\subseteq L^0=L^0(\mathfrak{A})$ a non-empty set which is bounded below in L^0. Suppose that $\phi_0(\alpha)=\inf_{u\in A}[\![u>\alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha\in\mathbb{R}$. Show that $v=\inf A$ is defined in L^0, and that $[\![v>\alpha]\!]=\sup_{\beta>\alpha}\phi_0(\beta)$ for every $\alpha\in\mathbb{R}$.

(n) Let (X,Σ,μ) be a measure space and $f:X\to[0,\infty[$ a function; set $A=\{g^\bullet:g\in\mathcal{L}^0(\mu),\,g\le_{\text{a.e.}}f\}$. (i) Show that if (X,Σ,μ) either is localizable or has the measurable envelope property (213Xl), then $\sup A$ is defined in $L^0(\mu)$. (ii) Show that if (X,Σ,μ) is complete and locally determined and $w=\sup A$ is defined in $L^0(\mu)$, then $w\in A$.

(o) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that if u, $v\in L^0=L^0(\mathfrak{A})$ then the following are equiveridical: (α) $[\![|v|>0]\!]\subseteq[\![|u|>0]\!]$ (β) v belongs to the band in L^0 generated by u (γ) there is a $w\in L^0$ such that $u\times w=v$.

>(p) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $a\in\mathfrak{A}$; let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Show that $L^0(\mathfrak{A}_a)$ can be identified, as f-algebra, with the band in $L^0(\mathfrak{A})$ generated by χa.

(q) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let $T:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ be the corresponding Riesz homomorphism (364P). Show that (i) the kernel of T is the sequentially order-closed solid linear subspace of $L^0(\mathfrak{A})$ generated by $\{\chi a:a\in\mathfrak{A},\,\pi a=0\}$ (ii) the set of values of T is the sequentially order-closed linear subspace of $L^0(\mathfrak{B})$ generated by $\{\chi(\pi a):a\in\mathfrak{A}\}$.

(r) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a sequentially order-continuous Boolean homomorphism, with $T:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ the associated operator. Suppose that h is a Borel measurable real-valued function defined on a Borel subset of $\mathbb{R}$. Show that $\bar{h}(Tu)=T\bar{h}(u)$ whenever $u\in L^0(\mathfrak{A})$ and $\bar{h}(u)$ is defined in the sense of 364H.

(s) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be probability algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism; let $T:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ be the corresponding Riesz homomorphism. Show that if $\langle u_i\rangle_{i\in I}$ is a family in $L^0(\mathfrak{A})$, it is $\bar{\mu}$-independent iff $\langle Tu_i\rangle_{i\in I}$ is $\bar{\nu}$-independent.

>(t) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Show that $L^0(\mathfrak{B})$ can be identified with the sequentially order-closed Riesz subspace of $L^0(\mathfrak{A})$ generated by $\{\chi b:b\in\mathfrak{B}\}$.

(u) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a sequentially order-continuous Boolean homomorphism; let $T_\pi:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ be the corresponding Riesz homomorphism. Let $\mathfrak{C}$ be the fixed-point subalgebra of π. Show that $\{u:u\in L^0(\mathfrak{A}),\,T_\pi u=u\}$ can be identified with $L^0(\mathfrak{C})$.

(v) Use the ideas of part (d) of the proof of 364T to show that the operator T there is multiplicative, without appealing to 353P.

(w) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$. Show that $L^0(\mathfrak{B})$, regarded as a subset of $L^0(\mathfrak{A})$, is order-closed in $L^0(\mathfrak{A})$.

364Y Further exercises >(a)(i) Show directly, without using the Loomis-Sikorski theorem or the Stone representation, that if $\mathfrak{A}$ is any Dedekind σ-complete Boolean algebra then the formulae of 364D define a group operation $+$ on $L^0(\mathfrak{A})$, and generally an f-algebra structure. (ii) Defining $\chi:\mathfrak{A}\to L^0(\mathfrak{A})$ by the formula in 364Jc, show that $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ can be identified with the linear span of $\{\chi a:a\in\mathfrak{A}\}$ and the solid linear subspace of $L^0(\mathfrak{A})$ generated by $e=\chi 1$. (iii) Still without using the Loomis-Sikorski theorem, explain how to define $\bar{h}:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ for continuous functions $h:\mathbb{R}\to\mathbb{R}$. (iv) Check that these ideas are sufficient to yield 364L-364R, except that in 364Pd we may have difficulty with arbitrary Borel functions h.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\boldsymbol{u}=(u_1,\dots,u_n)$ a member of $L^0(\mathfrak{A})^n$. Write $\mathcal{B}_n$ for the algebra of Borel sets in $\mathbb{R}^n$. (i) Show that there is a unique sequentially order-continuous Boolean homomorphism $E\mapsto[\![\boldsymbol{u}\in E]\!]:\mathcal{B}_n\to\mathfrak{A}$ such that $[\![\boldsymbol{u}\in E]\!]=\inf_{i\le n}[\![u_i>\alpha_i]\!]$ when $E=\prod_{i\le n}]\alpha_i,\infty[$. (ii) Show that for every sequentially order-continuous Boolean homomorphism $\phi:\mathcal{B}_n\to\mathfrak{A}$ there is a unique $\boldsymbol{u}\in L^0(\mathfrak{A})^n$ such that $\phi E=[\![\boldsymbol{u}\in E]\!]$ for every $E\in\mathcal{B}_n$.

(c) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $n\ge 1$ and $h:\mathbb{R}^n\to\mathbb{R}$ a Borel measurable function. Show that we have a corresponding function $\bar{h}:L^0(\mathfrak{A})^n\to L^0(\mathfrak{A})$ defined by saying that $[\![\bar{h}(\boldsymbol{u})\in E]\!]=[\![\boldsymbol{u}\in h^{-1}[E]]\!]$ for every Borel set $E\subseteq\mathbb{R}$ and $\boldsymbol{u}\in L^0(\mathfrak{A})^n$.

(d) Suppose that $h_1(x,y)=x+y$, $h_2(x,y)=xy$, $h_3(x,y)=\max(x,y)$ for all x, $y\in\mathbb{R}$. Show that, in the language of 364Yc, $\bar{h}_1(u,v)=u+v$, $\bar{h}_2(u,v)=u\times v$, $\bar{h}_3(u,v)=u\vee v$ for all u, $v\in L^0$.

(e) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that $\mathfrak{A}$ is ccc iff $L^0(\mathfrak{A})$ has the countable sup property.

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $T:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ a Riesz homomorphism such that $Te=e'$, where e, e' are the multiplicative identities of $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ respectively. Show that there is a unique sequentially order-continuous Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ such that $T=T_\pi$ in the sense of 364P. (*Hint*: use 353Pd. Compare 375A below.)

(g) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras and $\pi:\mathfrak{A}\to\mathfrak{B}$ a sequentially order-continuous ring homomorphism. (i) Show that we have a multiplicative sequentially order-continuous Riesz homomorphism $T_\pi:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ defined by the formula

$$[\![T_\pi u>\alpha]\!]=\pi[\![u>\alpha]\!]$$

whenever $u\in L^0(\mathfrak{A})$ and $\alpha>0$. (ii) Show that T_π is order-continuous iff π is order-continuous, injective iff π is injective, and surjective iff π is surjective. (iii) Show that if $\mathfrak{C}$ is another Dedekind σ-complete Boolean algebra and $\theta:\mathfrak{B}\to\mathfrak{C}$ another sequentially order-continuous ring homomorphism then $T_{\theta\pi}=T_\theta T_\pi:L^0(\mathfrak{A})\to L^0(\mathfrak{C})$.

(h) Suppose, in 364T, that $X=\mathbb{Q}$. (i) Show that there is an $f\in W$ such that $f(q)>0$ for every $q\in\mathbb{Q}$. (ii) Show that there is a $u\in L^0$ such that no $f\in U$ representing u can be continuous at any point of $\mathbb{Q}$.

(i) Let X and Y be topological spaces and $\phi : X \to Y$ a continuous function such that $\phi^{-1}[M]$ is nowhere dense in X for every nowhere dense subset M of Y. (Cf. 313R.) (i) Show that we have an order-continuous Boolean homomorphism π from the regular open algebra $\mathrm{RO}(Y)$ of Y to the regular open algebra $\mathrm{RO}(X)$ of X defined by the formula $\pi G = \operatorname{int} \overline{\phi^{-1}[G]}$ for every $G \in \mathrm{RO}(Y)$. (ii) Show that if U_X, U_Y are the function spaces of 364T then $g\phi \in U_X$ for every $g \in U_Y$. (iii) Show that if $T_X : U_X \to L^0(\mathrm{RO}(X))$, $T_Y : U_Y \to L^0(\mathrm{RO}(Y))$ are the canonical surjections, and $T : L^0(\mathrm{RO}(Y)) \to L^0(\mathrm{RO}(X))$ is the homomorphism corresponding to π, then $T(T_Y g) = T_X(g\phi)$ for every $g \in U_Y$. (iv) Rewrite these ideas for the special case in which X is a dense subset of Y and ϕ is the identity map, showing that in this case π and T are isomorphisms.

(j) Let X be a Baire space, $\mathrm{RO}(X)$ its algebra of regular open sets, $\mathcal{M}$ its ideal of meager sets, and $\widehat{\mathcal{B}}$ the Baire-property σ-algebra $\{G \triangle A : G \subseteq X \text{ is open}, A \in \mathcal{M}\}$, so that $\mathrm{RO}(X)$ can be identified with $\widehat{\mathcal{B}}/\mathcal{M}$ (314Yd). (i) Repeat the arguments of 364U in this context. (ii) Show that the space U of 364T-364U is a subspace of $\mathcal{L}^0 = \mathcal{L}^0_{\widehat{\mathcal{B}}}$, and that $W = U \cap \mathcal{W}$ where $\mathcal{W} = \{f : f \in \mathbb{R}^X, \{x : f(x) \neq 0\} \in \mathcal{M}\}$, so that the representations of $L^0(\mathrm{RO}(X))$ as U/W, $\mathcal{L}^0/\mathcal{W}$ are consistent.

(k) Work through the arguments of 364T and 364Yj for the case of compact Hausdorff X, seeking simplifications based on 364U.

(l) Let X be an extremally disconnected compact Hausdorff space with regular open algebra $\mathrm{RO}(X)$. Let U_0 be the space of real-valued functions $f : X \to \mathbb{R}$ such that $\operatorname{int}\{x : f \text{ is continuous at } x\}$ is dense. Show that U_0 is a Riesz subspace of the space U of 364T, and that every member of $L^0(\mathrm{RO}(X))$ is represented by a member of U_0.

(m) Let X be a Baire space. Let Q be the set of all continuous real-valued functions defined on subsets of X, and Q^* the set of all members of Q which are maximal in the sense that there is no member of Q properly extending them. (i) Show that the domain of any member of Q^* is a dense G_δ set. (ii) Show that we can define addition and multiplication and scalar multiplication on Q^* by saying that $f \dot{+} g$, $f \dot{\times} g$, $\gamma . f$ are to be the unique members of Q^* extending the partially-defined functions $f + g$, $f \times g$, γf, and that these definitions render Q^* an f-algebra if we say that $f \leq g$ iff $f(x) \leq g(x)$ for every $x \in \operatorname{dom} f \cap \operatorname{dom} g$. (iii) Show that every member of Q^* has an extension to a member of U, as defined in 364T, and that these extensions define an isomorphism between Q^* and $L^0(\mathrm{RO}(X))$, where $\mathrm{RO}(X)$ is the regular open algebra of X. (iv) Show that if X is compact, Hausdorff and extremally disconnected, then every member of Q^* has a unique extension to a member of $C^\infty(X)$, as defined in 364V.

(n) Let X be an extremally disconnected Hausdorff space, and Z any compact Hausdorff space. Show that if $D \subseteq X$ is dense and $f : D \to Z$ is continuous, there is a continuous $g : X \to Z$ extending f.

(o) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. (i) Show that for any $\boldsymbol{u} = (u_1, \dots , u_n) \in L^0(\mathfrak{A})^n$ there is a unique Radon probability measure ν on $\mathbb{R}^n$ such that $\nu(\prod_{1 \leq i \leq n}]\alpha, \infty[) = \bar{\mu}(\inf_{1 \leq i \leq n} [\![u_i > \alpha_i]\!])$ for all $\alpha_1, \dots , \alpha_n \in \mathbb{R}$, and that now $\nu E = \bar{\mu}[\![\boldsymbol{u} \in E]\!]$ for every Borel set $E \subseteq \mathbb{R}^n$. I will call ν the **distribution** of $\boldsymbol{u}$. (ii) Show that $(u_1, \dots , u_n)$ is stochastically independent iff ν is expressible as $\prod_{1 \leq i \leq n} \nu_i$ where ν_i is a Radon probability measure on $\mathbb{R}$ for each i. (iii) Write $\mathfrak{A}_{\boldsymbol{u}}$ for the closed subalgebra $\{[\![\boldsymbol{u} \in E]\!] : E \subseteq \mathbb{R}^n \text{ is a Borel set}\}$; check that $u_i \in L^0(\mathfrak{A}_{\boldsymbol{u}})$ for every i. Suppose that $(\mathfrak{B}, \bar{\nu})$ is another probability algebra and that $\boldsymbol{v} = (v_1, \dots , v_n) \in (L^0(\mathfrak{B}))^n$. Show that the following are equiveridical: (α) there is a measure-preserving isomorphism $\pi : \mathfrak{A}_{\boldsymbol{u}} \to \mathfrak{B}_{\boldsymbol{v}}$ such that $T_\pi u_i = v_i$ for every i (β) $\boldsymbol{u}$ and $\boldsymbol{v}$ have the same distribution.

364 Notes and comments This has been a long section, and so far all we have is a supposedly thorough grasp of the construction of L^0 spaces; discussion of their properties still lies ahead. The difficulties seem to stem from a variety of causes. First, L^0 spaces have a rich structure, being linear ordered spaces with multiplications; consequently all the main theorems have to check rather a lot of different aspects. Second, unlike L^∞ spaces, they are not accessible by means of the theory of normed spaces, so I must expect to do more of the work here rather than in an appendix. But this is in fact a crucial difference, because it affects the proof of the central theorem 364D. The point is that a given algebra $\mathfrak{A}$ will be expressible in the form $\Sigma/\mathcal{I}$ for a variety of algebras Σ of sets. Consequently any definition of $L^0(\mathfrak{A})$ as a quotient $\mathcal{L}^0_\Sigma/\mathcal{W}_\mathcal{I}$ must include a check that the structure produced is independent of the particular pair Σ, $\mathcal{I}$ chosen.

The same question arises with $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$. But in the case of S, I was able to use a general theory of additive functions on $\mathfrak{A}$ (see the proof of 361L), while in the case of L^∞ I could quote the result for S and a little theory of normed spaces (see the proof of 363H). The theorems of §368 will show, among other things, that a similar approach (describing L^0 as a special kind of extension of S or L^∞) can be made to work in the present situation. I have chosen, however, an alternative route using a novel technique. The price is the time required to develop skill in the technique, and to relate it to the earlier approach (364C, 364D, 364J). The reward is a construction which is based directly on the algebra $\mathfrak{A}$, independent of any representation (364A), and methods of dealing with it which are complementary to those of the previous three sections. In particular, they can be used in the absence of the full axiom of choice (364Ya).

I have deliberately chosen the notation $[\![u>\alpha]\!]$ from the theory of forcing. I do not propose to try to explain myself here, but I remark that much of the labour of this section is a necessary basis for understanding real analysis in Boolean-valued models of set theory. The idea is that just as a function $f:X\to\mathbb{R}$ can be described in terms of the sets $\{x:f(x)>\alpha\}$, so can an element u of $L^0(\mathfrak{A})$ be described in terms of the regions $[\![u>\alpha]\!]$ of $\mathfrak{A}$ where in some sense u is greater than α. This description is well adapted to discussion of the order struction of $L^0(\mathfrak{A})$ (see 364L-364M), but rather ill-adapted to discussion of its linear and multiplicative structures, which leads to a large part of the length of the work above. Once we have succeeded in describing the algebraic operations on L^0 in terms of the values of $[\![u>\alpha]\!]$, however, as in 364D, the fundamental result on the action of Boolean homomorphisms (364P) is elegant and reasonably straightforward.

The concept '$[\![u>\alpha]\!]$' can be dramatically generalized to the concept '$[\![(u_1,\dots,u_n)\in E]\!]$', where E is a Borel subset of $\mathbb{R}^n$ and $u_1,\dots,u_n\in L^0(\mathfrak{A})$ (364G, 364Yb). This is supposed to recall the notation $\Pr(X\in E)$, already used in Chapter 27. If, as sometimes seems reasonable, we wish to regard a random variable as a member of $L^0(\mu)$ rather than of $\mathcal{L}^0(\mu)$, then '$[\![u\in E]\!]$' is the appropriate translation of '$X^{-1}[E]$'. The reasons why we can reach all Borel sets E here, but then have to stop, seem to lie fairly deep; I will return to this question in 566O in Volume 5. We see that we have here another potential definition of $L^0(\mathfrak{A})$, as the set of sequentially order-continuous Boolean homomorphisms from the Borel σ-algebra of $\mathbb{R}$ to $\mathfrak{A}$. This is suitably independent of realizations of $\mathfrak{A}$, but makes the f-algebra structure of L^0 difficult to elucidate, unless we move to a further level of abstraction in the definitions, as in 364Yd.

I take the space to describe the L^0 spaces of general regular open algebras in detail (364T) partly to offer a demonstration of an appropriate technique, and partly to show that we are not limited to σ-algebras of sets and their quotients. This really is a new representation; for instance, it does not meld in any straightforward way with the constructions of 364F-364H. Of course the most important examples are compact Hausdorff spaces, for which alternative methods are available (364U-364V, 364Yj, 364Yl, 364Ym); from the point of view of applications, indeed, it is worth working through the details of the theory for compact Hausdorff spaces (364Yk). The version in 364V is derived from Vulikh 67. But I have starred everything from 364S on, because I shall not rely on this work later for anything essential.

365 L^1

Continuing my programme of developing the ideas of Chapter 24 at a deeper level of abstraction, I arrive at last at L^1. As usual, the first step is to establish a definition which can be matched both with the constructions of the previous sections and with the definition of $L^1(\mu)$ in §242 (365A-365C, 365F). Next, I give what I regard as the most characteristic internal properties of L^1 spaces, including versions of the Radon-Nikodým theorem (365E), before turning to abstract versions of theorems in §235 (365H, 365T) and the duality between L^1 and L^∞ (365L-365N). As in §§361 and 363, the construction is associated with universal mapping theorems (365I-365K) which define the Banach lattice structure of L^1. As in §§361, 363 and 364, homomorphisms between measure algebras correspond to operators between their L^1 spaces; but now the duality theory gives us two types of operators (365O-365Q), of which one class can be thought of as abstract conditional expectations (365R). For localizable measure algebras, the underlying algebra can be recovered from its L^1 space (365S), but the measure cannot.

365A Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. For $u\in L^0(\mathfrak{A})$, write

$$\|u\|_1=\int_0^\infty\bar\mu[\![|u|>\alpha]\!]\,d\alpha,$$

the integral being with respect to Lebesgue measure on $\mathbb{R}$, and allowing ∞ as a value of the integral. (Because the integrand is monotonic, it is certainly measurable.) Set $L^1_{\bar\mu}=L^1(\mathfrak{A},\bar\mu)=\{u:u\in L^0(\mathfrak{A}),\ \|u\|_1<\infty\}$.

It is convenient to note at once that if $u\in L^1_{\bar\mu}$, then $\mu[\![|u|>\alpha]\!]$ must be finite for almost every $\alpha>0$, and therefore for every $\alpha>0$, since it is a non-increasing function of α; so that $[\![u>\alpha]\!]$ also belongs to the Boolean ring $\mathfrak{A}^f=\{a:\bar\mu a<\infty\}$ for every $\alpha>0$.

365B Theorem Let (X,Σ,μ) be a measure space with measure algebra $(\mathfrak{A},\bar\mu)$. Then the canonical isomorphism between $L^0(\mu)$ and $L^0(\mathfrak{A})$ (364Ic) matches $L^1(\mu)\subseteq L^0(\mu)$, defined in §242, with $L^1(\mathfrak{A},\bar\mu)\subseteq L^0(\mathfrak{A})$, and the standard norm of $L^1(\mu)$ with $\| \|_1 : L^1(\mathfrak{A},\bar\mu)\to[0,\infty[$, as defined in 365A.

proof Take any Σ-measurable function $f:X\to\mathbb{R}$ (364B); write $f^\bullet$ for its equivalence class in $L^0(\mu)$, and u for the corresponding element of $L^0(\mathfrak{A})$. Then $[\![|u|>\alpha]\!]=\{x:|f(x)|>\alpha\}^\bullet$ in $\mathfrak{A}$ for every $\alpha\in\mathbb{R}$, and

$$\|u\|_1=\int_0^\infty \mu\{x:|f(x)|>\alpha\}\,d\alpha=\int|f|d\mu$$

by 252O. In particular, $u\in L^1(\mathfrak{A},\bar\mu)$ iff $f\in\mathcal{L}^1(\mu)$ iff $f^\bullet\in L^1(\mu)$, and in this case $\|u\|_1=\|f^\bullet\|_1$.

365C Accordingly we can apply everything we know about $L^1(\mu)$ spaces to $L^1_{\bar\mu}$ spaces. For instance:

Theorem For any measure algebra $(\mathfrak{A},\bar\mu)$, $L^1(\mathfrak{A},\bar\mu)$ is a solid linear subspace of $L^0(\mathfrak{A})$, and $\| \|_1$ is a norm on $L^1(\mathfrak{A},\bar\mu)$ under which $L^1(\mathfrak{A},\bar\mu)$ is an L-space. Consequently $L^1(\mathfrak{A},\bar\mu)$ is a perfect Riesz space with an order-continuous norm which has the Levi property, and if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing norm-bounded sequence in $L^1(\mathfrak{A},\bar\mu)$ then it converges for $\| \|_1$ to $\sup_{n\in\mathbb{N}}u_n$.

proof $(\mathfrak{A},\bar\mu)$ is isomorphic to the measure algebra of some measure space (X,Σ,μ) (321J). $L^1(\mu)$ is a solid linear subspace of $L^0(\mu)$ (242Cb), so $L^1_{\bar\mu}$ is a solid linear subspace of $L^0(\mathfrak{A})$. $L^1(\mu)$ is an L-space (354M), so $L^1_{\bar\mu}$ also is. The rest of the properties claimed are general features of L-spaces (354N, 354E, 356P).

365D Integration Let $(\mathfrak{A},\bar\mu)$ be any measure algebra.

(a) If $u\in L^1=L^1(\mathfrak{A},\bar\mu)$, then u^+ and u^- belong to L^1, and we may set

$$\int u=\|u^+\|_1-\|u^-\|_1=\int_0^\infty\bar\mu[\![u>\alpha]\!]\,d\alpha-\int_0^\infty\bar\mu[\![-u>\alpha]\!]\,d\alpha.$$

Now $\int:L^1\to\mathbb{R}$ is an order-continuous positive linear functional (356Pc), and under the translation of 365B matches the integral on $L^1(\mu)$ as defined in 242Ab. Note that if $a\in\mathfrak{A}^f$ then

$$\int\chi a=\int_0^\infty\bar\mu[\![\chi a>\alpha]\!]d\alpha=\int_0^1\bar\mu a\,d\alpha=\bar\mu a,$$

so that if $\bar\mu$ is totally finite then the integral here agrees with that of 363L on $L^\infty(\mathfrak{A})$. I will sometimes write $\int u\,d\bar\mu$ if it seems helpful to indicate the measure.

(b) Of course $\|u\|_1=\int|u|\ge|\int u|$ for every $u\in L^1$.

(c) If $u\in L^1$, $a\in\mathfrak{A}$ we may set $\int_a u=\int u\times\chi a$. (Compare 242Ac.) If $\gamma>0$ and $0\ne a\subseteq[\![u>\gamma]\!]$ then there is a $\delta>\gamma$ such that $a'=a\cap[\![u>\delta]\!]\ne0$, so that

$$\int_a u=\int_0^\infty\bar\mu(a\cap[\![u>\alpha]\!])d\alpha\ge\int_0^\gamma\bar\mu a\,d\alpha+\int_\gamma^\delta\bar\mu a'>\gamma\bar\mu a.$$

In particular, setting $a=[\![u>\gamma]\!]$, $\bar\mu[\![u>\gamma]\!]$ must be finite.

(d)(i) If $u\in L^1$ then $u\ge0$ iff $\int_a u\ge0$ for every $a\in\mathfrak{A}^f$, writing $\mathfrak{A}^f=\{a:\bar\mu a<\infty\}$, as usual. **P** If $u\ge0$ then $u\times\chi a\ge0$, $\int_a u\ge0$ for every $a\in\mathfrak{A}$. If $u\not\ge0$, then $[\![u^->0]\!]\ne0$ and there is an $\alpha>0$ such that $a=[\![u^->\alpha]\!]\ne0$. But now $\bar\mu a$ is finite ((c) above) and

$$\int u\times\chi a=-\int u^-\times\chi a=-\int\bar\mu(a\cap[\![u^-\ge\beta]\!])d\beta\le-\alpha\bar\mu a<0,$$

so $\int_a u<0$. **Q**

(ii) If u, $v\in L^1$ and $\int_a u=\int_a v$ for every $a\in\mathfrak{A}^f$ then $u=v$ (cf. 242Ce).

(iii) If $u\ge0$ in L^1 then $\int u=\sup\{\int_a u:a\in\mathfrak{A}^f\}$. **P** Of course $u\times\chi a\le u$ so $\int_a u\le u$ for every $a\in\mathfrak{A}$. On the other hand, setting $a_n=[\![u>2^{-n}]\!]$, $\langle u\times\chi a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum u, so $\int u=\lim_{n\to\infty}\int_{a_n}u$, while $\bar\mu a_n$ is finite for every n. **Q**

(e) If $u\in L^1$, $u\ge0$ and $\int u=0$ then $u=0$ (put 365B and 122Rc together). If $u\in L^1$, $u\ge0$ and $\int_a u=0$ then $u\times\chi a=0$, that is, $a\cap[\![u>0]\!]=0$.

(f) If $C \subseteq L^1$ is non-empty and upwards-directed and $\sup_{v\in C}\int v$ is finite, then $\sup C$ is defined in L^1 and $\int \sup C = \sup_{v\in C}\int v$ (356Pc).

(g) It will occasionally be convenient to adapt the conventions of §133 to the new context; so that I may write $\int u = \infty$ if $u \in L^0(\mathfrak{A})$, $u^- \in L^1$ and $u^+ \notin L^1$, while $\int u = -\infty$ if $u^+ \in L^1$ and $u^- \notin L^1$.

(h) On this convention, we can restate (f) as follows: if $C \subseteq (L^0)^+$ is non-empty and upwards-directed and has a supremum u in L^0, then $\int u = \sup_{v\in C}\int v$ in $[0,\infty]$. **P** For if $\sup_{v\in C}\int v$ is infinite, then surely $\int u = \infty$; while otherwise we can apply (f). **Q**

365E The Radon-Nikodým theorem again (a) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ an additive functional. Then the following are equiveridical:

(i) there is a $u \in L^1 = L^1(\mathfrak{A},\bar\mu)$ such that $\nu a = \int_a u$ for every $a \in \mathfrak{A}$;

(ii) ν is additive and continuous for the measure-algebra topology on $\mathfrak{A}$;

(iii) ν is completely additive.

(b) Let $(\mathfrak{A},\bar\mu)$ be any measure algebra, and $\nu : \mathfrak{A}^f \to \mathbb{R}$ a function. Then the following are equiveridical:

(i) ν is additive and bounded and $\inf_{a\in A}|\nu a| = 0$ whenever $A \subseteq \mathfrak{A}^f$ is downwards-directed and has infimum 0;

(ii) there is a $u \in L^1$ such that $\nu a = \int_a u$ for every $a \in \mathfrak{A}^f$.

proof (a) The equivalence of (ii) and (iii) is 327Bd. The equivalence of (i) and (iii) is just a translation of 327D into the new context.

(b)(i)$\Rightarrow$(ii)(α) Set $M = \sup_{a\in\mathfrak{A}^f}|\nu a| < \infty$.

Let $D \subseteq \mathfrak{A}^f$ be a maximal disjoint set. For each $d \in D$, write $\mathfrak{A}_d$ for the principal ideal of $\mathfrak{A}$ generated by d, and $\bar\mu_d$ for the restriction of $\bar\mu$ to $\mathfrak{A}_d$, so that $(\mathfrak{A}_d,\bar\mu_d)$ is a totally finite measure algebra. Set $\nu_d = \nu\restriction\mathfrak{A}_d$; then $\nu_d : \mathfrak{A}_d \to \mathbb{R}$ is completely additive. By (a), there is a $u_d \in L^1(\mathfrak{A}_d,\bar\mu_d)$ such that $\int_a u_d = \nu_d a = \nu a$ for every $a \subseteq d$.

Now $u_d^+ \in L^0(\mathfrak{A}_d)$ corresponds to a member $\tilde u_d^+$ of $L^0(\mathfrak{A})^+$ defined by saying

$$\begin{aligned}[\![\tilde u_d^+ > \alpha]\!] &= [\![u_d^+ > \alpha]\!] = [\![u_d > \alpha]\!] \text{ if } \alpha \ge 0,\\ &= 1 \text{ if } \alpha < 0.\end{aligned}$$

If $a \in \mathfrak{A}$, then

$$\int_a \tilde u_d^+ d\bar\mu = \int_0^\infty \bar\mu(a \cap [\![\tilde u_d^+ > \alpha]\!])d\alpha = \int_0^\infty \bar\mu_d(a \cap [\![u_d^+ > \alpha]\!])d\alpha = \int_{a\cap d} u_d^+ d\bar\mu_d;$$

taking $a = 1$, we see that $\|\tilde u_d^+\|_1 = \|u_d^+\|_1 = \nu[\![u_d > 0]\!]$ is finite, so that $\tilde u_d^+ \in L^1$.

(β) For any finite $I \subseteq D$, set $v_I = \sum_{d\in I}\tilde u_d^+$. Then

$$\int v_I = \nu(\sup_{d\in I}[\![u_d > 0]\!]) \le M;$$

consequently the upwards-directed set $A = \{v_I : I \subseteq D \text{ is finite}\}$ is bounded above in L^1, and we can set $v = \sup A$ in L^1. If $a \in \mathfrak{A}$, then $\int_a v_I = \sum_{d\in I}\int_{a\cap d}u_d^+$ for each finite $I \subseteq D$, so $\int_a v = \sum_{d\in D}\int_{a\cap d}u_d^+$.

Applying the same arguments to $-\nu$, there is a $w \in L^1$ such that

$$\int_a w = \sum_{d\in D}\int_{a\cap d}u_d^-$$

for every $a \in \mathfrak{A}$. Try $u = v - w$; then

$$\int_a u = \sum_{d\in D}\int_{a\cap d}u_d^+ - \int_{a\cap d}u_d^- = \sum_{d\in D}\int_{a\cap d}u_d = \sum_{d\in D}\nu(a\cap d)$$

for every $a \in \mathfrak{A}$.

(γ) Now take any $a \in \mathfrak{A}^f$. For $J \subseteq D$ set $a_J = \sup_{d\in J} a \cap d$. Let $\epsilon > 0$. Then there is a finite $I \subseteq D$ such that

$$|\int_a u - \nu a_J| = |\sum_{d\in D}\nu(a\cap d) - \sum_{d\in J}\nu(a\cap d)| \le \epsilon$$

whenever $I \subseteq J \subseteq D$ and J is finite. But now consider

$$A = \{a \setminus a_J : I \subseteq J \subseteq D,\ J \text{ is finite}\}.$$

Then $\inf A = 0$, so there is a finite J such that $I \subseteq J \subseteq D$ and

$$|\nu a - \nu a_J| = |\nu(a \setminus a_J)| \le \epsilon.$$

Consequently

$$|\nu a - \int_a u| \le |\nu a - \nu a_J| + |\int_a u - \nu a_J| \le 2\epsilon.$$

As ϵ is arbitrary, $\nu a = \int_a u$. As a is arbitrary, (ii) is proved.

(ii)⇒(i) From where we now are, this is nearly trivial. Thinking of νa as $\int u \times \chi a$, ν is surely additive and bounded. Also $|\nu a| \le \int |u| \times \chi a$. If $A \subseteq \mathfrak{A}^f$ is non-empty, downwards-directed and has infimum 0, the same is true of $\{|u| \times \chi a : a \in A\}$, because $a \mapsto |u| \times \chi a$ is order-continuous, so

$$\inf_{a\in A} |\nu a| \le \inf_{a\in A} \int |u| \times \chi a = \inf_{a\in A} \||u| \times \chi a\|_1 = 0.$$

365F It will be useful later to have spelt out the following elementary facts.

Lemma Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Write S^f for the intersection $S(\mathfrak{A}) \cap L^1(\mathfrak{A}, \bar\mu)$. Then S^f is a norm-dense and order-dense Riesz subspace of $L^1(\mathfrak{A}, \bar\mu)$, and can be identified with $S(\mathfrak{A}^f)$. The function $\chi : \mathfrak{A}^f \to S^f \subseteq L^1(\mathfrak{A}, \bar\mu)$ is an injective order-continuous additive lattice homomorphism. If $u \ge 0$ in $L^1(\mathfrak{A}, \bar\mu)$, there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $(S^f)^+$ such that $u = \sup_{n\in\mathbb{N}} u_n = \lim_{n\to\infty} u_n$.

proof As in 364K, we can think of $S(\mathfrak{A}^f)$ as a Riesz subspace of $S = S(\mathfrak{A})$, embedded in $L^0(\mathfrak{A})$. If $u \in S$, it is expressible as $\sum_{i=0}^n \alpha_i\chi a_i$ where $a_0, \dots , a_n \in \mathfrak{A}$ are disjoint and no α_i is zero. Now $|u| = \sum_{i=0}^n |\alpha_i|\chi a_i$, so $u \in L^1$ iff $\bar\mu a_i < \infty$ for every i, that is, iff $u \in S(\mathfrak{A}^f)$; thus $S^f = S(\mathfrak{A}^f)$.

Now suppose that $u \ge 0$ in L^1. By 364Jd, there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A})^+$ such that $u_0 \ge 0$ and $u = \sup_{n\in\mathbb{N}} u_n$ in L^0. Because L^1 is a solid linear subspace of L^0, every u_n belongs to L^1 and therefore to S^f. By 365C, $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u. This shows also that S^f is order-dense in L^1.

The map $\chi : \mathfrak{A}^f \to S^f$ is an injective order-continuous additive lattice homomorphism; because S^f is regularly embedded in L^1 (352Ne), χ has the same properties when regarded as a map into L^1.

For general $u \in L^1$, there are sequences in S^f converging to u^+ and to u^-, so that their difference is a sequence in S^f converging to u, and u belongs to the closure of S^f; thus S^f is norm-dense in L^1.

Remark Of course S^f here corresponds to the space of (equivalence classes of) simple functions, as in 242Mb.

365G Semi-finite algebras: Lemma Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra.

(a) $(\mathfrak{A}, \bar\mu)$ is semi-finite iff $L^1 = L^1(\mathfrak{A}, \bar\mu)$ is order-dense in $L^0 = L^0(\mathfrak{A})$.

(b) In this case, writing $S^f = S(\mathfrak{A}) \cap L^1$ (as in 365F), $\int u = \sup\{\int v : v \in S^f, 0 \le v \le u\}$ in $[0, \infty]$ for every $u \in (L^0)^+$.

proof (a) If $(\mathfrak{A}, \bar\mu)$ is semi-finite then S^f is order-dense in L^0 (364K), so L^1 must also be. If L^1 is order-dense in L^0, then so is S^f, by 365F and 352Nc, so $(\mathfrak{A}, \bar\mu)$ is semi-finite, by 364K in the other direction.

(b) Set $C = \{v : v \in S^f, 0 \le v \le u\}$. Then C is an upwards-directed set with supremum u, because S^f is order-dense in L^0. So $\int u = \sup_{v\in C} \int v$ by 365Dh.

365H Measurable transformations We have a generalization of the ideas of §235 in this abstract context.

Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be the sequentially order-continuous Riesz homomorphism associated with π (364P).

(a) Suppose that $w \ge 0$ in $L^0(\mathfrak{B})$ is such that $\int_{\pi a} w\,d\bar\nu = \bar\mu a$ whenever $a \in \mathfrak{A}$ and $\bar\mu a < \infty$. Then for any $u \in L^1(\mathfrak{A}, \bar\mu)$ and $a \in \mathfrak{A}$, $\int_{\pi a} Tu \times w\,d\bar\nu$ is defined and equal to $\int_a u\,d\bar\mu$.

(b) Suppose that $w' \ge 0$ in $L^0(\mathfrak{A})$ is such that $\int_a w' d\bar\mu = \bar\nu(\pi a)$ for every $a \in \mathfrak{A}$. Then $\int Tu\,d\bar\nu = \int u \times w'\,d\bar\mu$ whenever $u \in L^0(\mathfrak{A})$ and either integral is defined in $[-\infty, \infty]$.

Remark I am using the convention of 365Dg concerning '∞' as the value of an integral.

proof (a) If $u \in S^f = L^1_{\bar\mu} \cap S(\mathfrak{A})$ then u is expressible as $\sum_{i=0}^n \alpha_i\chi a_i$ where $a_0, \dots , a_n$ have finite measure, so that $Tu = \sum_{i=0}^n \alpha_i\chi(\pi a_i)$ and

$$\int Tu \times w\,d\bar\nu = \sum_{i=0}^n \alpha_i \int_{\pi a_i} w = \sum_{i=0}^n \alpha_i\bar\mu a_i = \int u\,d\bar\mu.$$

If $u \ge 0$ in $L^1_{\bar\mu}$ there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in S^f with supremum u, so that $Tu = \sup_{n\in\mathbb{N}} Tu_n$ and $w \times Tu = \sup_{n\in\mathbb{N}} w \times Tu_n$ in $L^0(\mathfrak{B})$, and

$$\int Tu \times w = \sup_{n\in\mathbb{N}} \int Tu_n \times w = \sup_{n\in\mathbb{N}} \int u_n = \int u.$$

(365Df tells us that in this context $Tu \times w \in L^1_{\bar\nu}$.) Finally, for general $u \in L^1_{\bar\mu}$,

$$\int Tu \times w = \int Tu^+ \times w - \int Tu^- \times w = \int u^+ - \int u^- = \int u.$$

(b) The argument follows the same lines: start with $u = \chi a$ for $a \in \mathfrak{A}$, then with $u \in S(\mathfrak{A})$, then with $u \in L^0(\mathfrak{A})^+$ and conclude with general $u \in L^0(\mathfrak{A})$. The point is that T is a Riesz homomorphism, so that at the last step

$$\begin{aligned}\int Tu &= \int (Tu)^+ - \int (Tu)^- = \int T(u^+) - \int T(u^-)\\ &= \int u^+ \times w' - \int u^- \times w' = \int (u \times w')^+ - \int (u \times w')^- = \int u \times w'\end{aligned}$$

whenever either side is defined in $[-\infty, \infty]$.

365I Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and U a Banach space. Let $\nu : \mathfrak{A}^f \to U$ be a function. Then the following are equiveridical:

(i) there is a continuous linear operator T from $L^1 = L^1(\mathfrak{A}, \bar\mu)$ to U such that $\nu a = T(\chi a)$ for every $a \in \mathfrak{A}^f$;

(ii)(α) ν is additive

(β) there is an $M \ge 0$ such that $\|\nu a\| \le M\bar\mu a$ for every $a \in \mathfrak{A}^f$.

Moreover, in this case, T is unique and $\|T\|$ is the smallest number M satisfying the condition in (ii-β).

proof (a)(i)$\Rightarrow$(ii) If $T : L^1 \to U$ is a continuous linear operator, then $\chi a \in L^1$ for every $a \in \mathfrak{A}^f$, so $\nu = T\chi$ is a function from $\mathfrak{A}^f$ to U. If a, $b \in \mathfrak{A}^f$ and $a \cap b = 0$, then $\chi(a \cup b) = \chi a + \chi b$ in $L^0 = L^0(\mathfrak{A})$ and therefore in L^1, so

$$\nu(a \cup b) = T\chi(a \cup b) = T(\chi a + \chi b) = T(\chi a) + T(\chi b) = \nu a + \nu b.$$

If $a \in \mathfrak{A}^f$ then $\|\chi a\|_1 = \bar\mu a$ (using the formula in 365A, or otherwise), so

$$\|\nu a\| = \|T(\chi a)\| \le \|T\|\|\chi a\|_1 = \|T\|\bar\mu a.$$

(b)(ii)$\Rightarrow$(i) Now suppose that $\nu : \mathfrak{A}^f \to U$ is additive and that $\|\nu a\| \le M\bar\mu a$ for every $a \in \mathfrak{A}^f$. Let $S^f = L^1 \cap S(\mathfrak{A})$, as in 365F. Then there is a linear operator $T_0 : S^f \to U$ such that $T_0(\chi a) = \nu a$ for every $a \in \mathfrak{A}^f$ (361F). Next, $\|T_0 u\| \le M\|u\|_1$ for every $u \in S^f$. **P** If $u \in S^f \cong S(\mathfrak{A}^f)$, then u is expressible as $\sum_{j=0}^m \beta_j \chi b_j$ where $b_0, \dots , b_m \in \mathfrak{A}^f$ are disjoint (361Eb). So

$$\|T_0 u\| = \|\sum_{j=0}^m \beta_j \nu b_j\| \le M \sum_{j=0}^m |\beta_j|\bar\mu b_j = M\|u\|_1. \ \mathbf{Q}$$

There is therefore a continuous linear operator $T : L^1 \to U$, extending T_0, and with $\|T\| \le \|T_0\| \le M$ (2A4I). Of course we still have $\nu = T\chi$.

(c) The argument in (b) shows that $T_0 = T{\restriction}S^f$ and T are uniquely defined from ν. We have also seen that if T, ν correspond to each other then

$$\|\nu a\| \le \|T\|\bar\mu a \text{ for every } a \in \mathfrak{A}^f,$$

$$\|T\| \le M \text{ whenever } \|\nu a\| \le M\bar\mu a \text{ for every } a \in \mathfrak{A}^f,$$

so that

$$\|T\| = \min\{M : M \ge 0,\ \|\nu a\| \le M\bar\mu a \text{ for every } a \in \mathfrak{A}^f\}.$$

365J Corollary Let (X, Σ, μ) be a measure space and U any Banach space. Set $\Sigma^f = \{E : E \in \Sigma,\ \mu E < \infty\}$. Let $\nu : \Sigma^f \to U$ be a function. Then the following are equiveridical:

(i) there is a continuous linear operator $T : L^1(\mu) \to U$ such that $\nu E = T(\chi E)^\bullet$ for every $E \in \Sigma^f$;

(ii)(α) $\nu(E \cup F) = \nu E + \nu F$ whenever E, $F \in \Sigma^f$ and $E \cap F = 0$ (β) there is an $M \ge 0$ auch that $\|\nu E\| \le M\mu E$ for every $E \in \Sigma^f$.

Moreover, in this case, T is unique and $\|T\|$ is the smallest number M satisfying the condition in (ii-β).

proof This is a direct translation of 365I. The only point to note is that if ν satisfies the conditions of (ii), and E, $F \in \Sigma^f$ are such that $E^\bullet = F^\bullet$ in the measure algebra $(\mathfrak{A}, \bar{\mu})$ of (X, Σ, μ), then $\mu(E \setminus F) = \mu(F \setminus E) = 0$, so that $\nu(E \setminus F) = \nu(F \setminus E) = 0$ (using condition (ii-β)) and

$$\nu E = \nu(E \cap F) + \nu(E \setminus F) = \nu(E \cap F) + \nu(F \setminus E) = \nu F.$$

This means that we have a function $\bar{\nu} : \mathfrak{A}^f \to U$, where

$$\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\} = \{E^\bullet : E \in \Sigma^f\},$$

defined by setting $\bar{\nu}E^\bullet = \nu E$ for every $E \in \Sigma^f$. Of course we now have $\bar{\nu}(a \cup b) = \bar{\nu}a + \bar{\nu}b$ whenever a, $b \in \mathfrak{A}^f$ and $a \cap b = 0$ (since we can express them as $a = E^\bullet$, $b = F^\bullet$ with $E \cap F = \emptyset$), and $\|\bar{\nu}a\| \le M\bar{\mu}a$ for every $a \in \mathfrak{A}^f$. Thus we have a one-to-one correspondence between functions $\nu : \Sigma^f \to U$ satisfying the conditions (ii) here, and functions $\bar{\nu} : \mathfrak{A}^f \to U$ satisfying the conditions (ii) of 365I. The rest of the argument is covered by the identification between $L^1(\mu)$ and L^1 in 365B.

365K Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, U a Banach lattice, and T a bounded linear operator from $L^1 = L^1(\mathfrak{A}, \bar{\mu})$ to U. Let $\nu : \mathfrak{A}^f \to U$ be the corresponding additive function, as in 365I.

(a) T is a positive linear operator iff $\nu a \ge 0$ in U for every $a \in \mathfrak{A}^f$; in this case, T is order-continuous.

(b) If U is Dedekind complete and $T \in \mathsf{L}^\sim(L^1; U)$, then $|T| : L^1 \to U$ corresponds to $|\nu| : \mathfrak{A}^f \to U$, where

$$|\nu|(a) = \sup\{\textstyle\sum_{i=0}^n |\nu a_i| : a_0, \dots, a_n \subseteq a \text{ are disjoint}\}$$

for every $a \in \mathfrak{A}^f$.

(c) T is a Riesz homomorphism iff ν is a lattice homomorphism.

proof As in 365F, let S^f be $L^1 \cap S(\mathfrak{A})$, identified with $S(\mathfrak{A}^f)$.

(a)(i) If T is a positive linear operator and $a \in \mathfrak{A}^f$, then $\chi a \ge 0$ in L^1, so $\nu a = T(\chi a) \ge 0$ in U.

(ii) Now suppose that $\nu a \ge 0$ in U for every $a \in \mathfrak{A}^f$, and let $u \ge 0$ in L^1, $\epsilon > 0$ in $\mathbb{R}$. Then there is a $v \in S^f$ such that $0 \le v \le u$ and $\|u - v\|_1 \le \epsilon$ (365F). Express v as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_i \in \mathfrak{A}^f$, $\alpha_i \ge 0$ for each i. Now

$$\|Tu - Tv\| \le \|T\|\|u - v\|_1 \le \epsilon\|T\|.$$

On the other hand,

$$Tv = \textstyle\sum_{i=0}^n \alpha_i \nu a_i \in U^+.$$

As U^+ is norm-closed in U (354Bc), and ϵ is arbitrary, $Tu \in U^+$. As u is arbitrary, T is a positive linear operator.

(iii) By 355Ka, T is order-continuous.

(b) The point is that $|T \restriction S^f| = |T| \restriction S^f$. **P** (i) Because the embedding $S^f \subsetneq L^1$ is positive, the map $R \mapsto R \restriction S^f$ is a positive linear operator from $\mathsf{L}^\sim(L^1; U)$ to $\mathsf{L}^\sim(S^f; U)$ (see 355Bd). So $|T \restriction S^f| \le |T| \restriction S^f$. (ii) There is a positive linear operator $T_1 : L^1 \to U$ extending $|T \restriction S^f|$, by 365J and (a) above, and now $T_1 \restriction S^f$ dominates both $T \restriction S^f$ and $-T \restriction S^f$; since $(S^f)^+$ is dense in $(L^1)^+$, $T_1 \ge T$ and $T_1 \ge -T$, so that $T_1 \ge |T|$ and

$$|T \restriction S^f| = T_1 \restriction S^f \ge |T| \restriction S^f. \ \mathbf{Q}$$

Now 361H tells us that

$$|T|(\chi a) = |T \restriction S^f|(\chi a) = |\nu| a$$

for every $a \in \mathfrak{A}^f$.

(c)(i) If T is a lattice homomorphism, then so is $\nu = T\chi$, because $\chi : \mathfrak{A}^f \to S^f$ is a lattice homomorphism.

(ii) Now suppose that χ is a lattice homomorphism. In this case $T \restriction S^f$ is a Riesz homomorphism (361Gc), that is, $|Tv| = T|v|$ for every $v \in S^f$. Because S^f is norm-dense in L^1 and the map $u \mapsto |u|$ is continuous both in L^1 and in U (354Bb), $|Tu| = T|u|$ for every $u \in L^1$, and T is a Riesz homomorphism.

365L The duality between L^1 and L^∞ Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and set $L^1 = L^1(\mathfrak{A}, \bar{\mu})$, $L^\infty = L^\infty(\mathfrak{A})$. If we identify L^∞ with the solid linear subspace of $L^0 = L^0(\mathfrak{A})$ generated by $e = \chi 1_{\mathfrak{A}}$ (364J), then we have a bilinear operator $(u, v) \mapsto u \times v : L^1 \times L^\infty \to L^1$, because $|u \times v| \le \|v\|_\infty |u|$ and L^1 is a solid linear subspace of L^0. Note that

$\|u \times v\|_1 \le \|u\|_1\|v\|_\infty$, so that the bilinear operator $(u,v) \mapsto u \times v$ has norm at most 1 (253Ab, 253E). Consequently we have a bilinear functional $(u,v) \mapsto \int u \times v : L^1 \times L^\infty \to \mathbb{R}$, which also has norm at most 1, corresponding to linear operators $S : L^1 \to (L^\infty)^*$ and $T : L^\infty \to (L^1)^*$, both of norm at most 1, defined by the formula

$$(Su)(v) = (Tv)(u) = \int u \times v \text{ for } u \in L^1,\ v \in L^\infty.$$

Because L^1 and L^∞ are both Banach lattices, we have $(L^1)^* = (L^1)^\sim$ and $(L^\infty)^* = (L^\infty)^\sim$ (356Dc). Because the norm of L^1 is order-continuous, $(L^1)^* = (L^1)^\times$ (356Dd).

365M Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and set $L^1 = L^1(\mathfrak{A}, \bar\mu)$, $L^\infty = L^\infty(\mathfrak{A})$. Let $S : L^1 \to (L^\infty)^* = (L^\infty)^\sim$, $T : L^\infty \to (L^1)^* = (L^1)^\sim = (L^1)^\times$ be the canonical maps defined by the duality between L^1 and L^∞, as in 365L. Then

(a) S and T are order-continuous Riesz homomorphisms, $S[L^1] \subseteq (L^\infty)^\times$, S is norm-preserving and $T[L^\infty]$ is order-dense in $(L^1)^\sim$;

(b) $(\mathfrak{A}, \bar\mu)$ is semi-finite iff T is injective, and in this case T is norm-preserving, while S is a normed Riesz space isomorphism between L^1 and $(L^\infty)^\times$;

(c) $(\mathfrak{A}, \bar\mu)$ is localizable iff T is bijective, and in this case T is a normed Riesz space isomorphism between L^∞ and $(L^1)^* = (L^1)^\sim = (L^1)^\times$.

proof (a)(i) If $u \ge 0$ in L^1 and $v \ge 0$ in L^∞ then $u \times v \ge 0$ and

$$(Tv)(u) = \int u \times v \ge 0.$$

As u is arbitrary, $Tv \ge 0$ in $(L^1)^\times$; as v is arbitrary, T is a positive linear operator.

If $v \in L^\infty$, set $a = [\![v > 0]\!] \in \mathfrak{A}$. (Remember that we are identifying $L^0(\mu)$, as defined in §241, with $L^0(\mathfrak{A})$, as defined in §364.) Then $v^+ = v \times \chi a$, so for any $u \ge 0$ in L^1

$$(Tv^+)(u) = \int u \times v \times \chi a = (Tv)(u \times \chi a) \le (Tv)^+(u).$$

As u is arbitrary, $Tv^+ \le (Tv)^+$. On the other hand, because T is a positive linear operator, $Tv^+ \ge Tv$ and $Tv^+ \ge 0$, so $Tv^+ \ge (Tv)^+$. Thus $Tv^+ = (Tv)^+$. As v is arbitrary, T is a Riesz homomorphism (352G).

(ii) Exactly the same arguments show that S is a Riesz homomorphism.

(iii) Given $u \in L^1$, set $a = [\![u > 0]\!]$; then

$$\|Su\| \ge (Su)(\chi a - \chi(1 \setminus a)) = \int_a u - \int_{1 \setminus a} u = \int |u| = \|u\|_1 \ge \|Su\|.$$

So S is norm-preserving.

(iv) By 355Ka, S is order-continuous.

(v) If $A \subseteq L^\infty$ is a non-empty downwards-directed set with infimum 0, and $u \in (L^1)^+$, then $\inf_{v \in A} u \times v = 0$ for every $u \in (L^1)^+$, because $v \mapsto u \times v : L^0 \to L^0$ is order-continuous. So

$$\inf_{v \in A}(Tv)(u) = \inf_{v \in A} \int u \times v = \inf_{v \in A} \|u \times v\|_1 = 0$$

and the only possible non-negative lower bound for $T[A]$ in $(L^1)^\times$ is 0. As A is arbitrary, T is order-continuous.

(vi) The ideas of (v) show also that $S[L^1] \subseteq (L^\infty)^\times$. **P** If $u \in (L^1)^+$ and $A \subseteq L^\infty$ is non-empty, downwards-directed and has infimum 0, then

$$\inf_{v \in A}(Su)(v) = \inf_{v \in A} \int u \times v = 0.$$

As A is arbitrary, Su is order-continuous. For general $u \in L^1$, $Su = Su^+ - Su^-$ belongs to $(L^\infty)^\times$. **Q**

(vii) Now suppose that $h > 0$ in $(L^1)^* = (L^1)^\times$. By 365Ka, applied to $-h$, there must be an $a \in \mathfrak{A}^f$ such that $h(\chi a) > 0$. Set $\nu b = h(\chi(a \cap b))$ for $b \in \mathfrak{A}^f$. Then ν is additive and non-negative and bounded by $\|h\|\bar\mu a$. If $A \subseteq \mathfrak{A}^f$ is a non-empty downwards-directed set with infimum 0, then $C = \{\chi b : b \in A\}$ is downwards-directed and has infimum 0 in $L^0(\mathfrak{A})$ (364Jc), so $\inf_{b \in A} \nu b = \inf_{u \in C} h(u) = 0$. By 365Eb, there is a $v \in L^1$ such that $\nu b = \int_b u$ for every $b \in \mathfrak{A}^f$. As $\int_b v \ge 0$ for every $b \in \mathfrak{A}^f$, $v \ge 0$ (365E(d-i)). Setting $b = [\![v > \|h\|]\!]$, we have

$$\int_b v \le h(\chi b) \le \|h\|\|\chi b\|_1 = \|h\|\bar\mu b;$$

so $b = 0$ (365Ec). Accordingly $0 \le v \le \|h\|\chi 1$ and $v \in L^\infty$. Consider $Tv \in (L^1)^\times$. We have $Tv \ge 0$ because T is positive; also

$$(Tv)(\chi a) = \int_a v = \nu a = h(\chi a) > 0,$$

so $Tv > 0$. Next, for every $b \in \mathfrak{A}^f$,

$$(Tv)(\chi b) = \int_b v = h(\chi(a \cap b)) \le h(\chi b).$$

By 365Ka again, $h - Tv \ge 0$, that is, $Tv \le h$. As h is arbitrary, $T[L^\infty]$ is quasi-order-dense in $(L^1)^*$, therefore order-dense (353A).

(b)(i) If $(\mathfrak{A}, \bar\mu)$ is not semi-finite, let $a \in \mathfrak{A}$ be such that $\bar\mu a = \infty$ and $\bar\mu b = \infty$ whenever $0 \ne b \subseteq a$. If $u \in L^1$, then $[\![|u| > \frac{1}{n}]\!]$ has finite measure for every $n \ge 1$, so must be disjoint from a; accordingly

$$a \cap [\![|u| > 0]\!] = \sup_{n\ge1} a \cap [\![|u| > \tfrac{1}{n}]\!] = 0.$$

This means that $\int u \times \chi a = 0$ for every $u \in L^1$. Accordingly $T(\chi a) = 0$ and T is not injective.

(ii) If $(\mathfrak{A}, \bar\mu)$ is semi-finite, take any $v \in L^\infty$. Then if $0 \le \delta < \|v\|_\infty$, $a = [\![|v| > \delta]\!] \ne 0$. Let $b \subseteq a$ be such that $0 < \bar\mu b < \infty$. Then $\chi b \in L^1$, and

$$\|Tv\| = \||Tv|\| = \|T|v|\| \ge (T|v|)(\chi b)/\|\chi b\|_1 \ge \delta$$

because $|v| \times \chi b \ge \delta\chi b$, so

$$(T|v|)(\chi b) \ge \delta\bar\mu b = \delta\|\chi b\|_1.$$

As δ is arbitrary, $\|Tv\| \ge \|v\|_\infty$. But we already know that $\|Tv\| \le \|v\|_\infty$, so the two are equal. As v is arbitrary, T is norm-preserving (and, in particular, is injective).

(iii) Still supposing that $(\mathfrak{A}, \bar\mu)$ is semi-finite, $S[L^1] = (L^\infty)^\times$. **P** Take any $h \in (L^\infty)^\times$. For $a \in \mathfrak{A}$, set $\nu a = h(\chi a^\bullet)$. By 363K, $\nu : \mathfrak{A} \to \mathbb{R}$ is completely additive. By 365Ea, there is a $u \in L^1$ such that

$$(Su)(\chi a) = \int u \times \chi a = \int_a u = \nu a = h(\chi a)$$

for every $a \in \mathfrak{A}$. Because Su and h are both linear functionals on L^∞, they must agree on $S(\mathfrak{A})$; because they are continuous and $S(\mathfrak{A})$ is dense in L^∞ (363C), $Su = h$. As h is arbitrary, S is surjective. **Q**

(c) Using (b), we know that if either T is bijective or $(\mathfrak{A}, \bar\mu)$ is localizable, then $(\mathfrak{A}, \bar\mu)$ is semi-finite. Given this, if T is bijective, then it is a Riesz space isomorphism between L^∞ and $(L^1)^\sim$, which is Dedekind complete (356B); so 363Mb tells us that $\mathfrak{A}$ is Dedekind complete and $(\mathfrak{A}, \bar\mu)$ is localizable. In the other direction, if $(\mathfrak{A}, \bar\mu)$ is localizable, then L^∞ is Dedekind complete. As T is injective, $T[L^\infty]$ is, in itself, Dedekind complete; being an order-dense Riesz subspace of $(L^1)^\sim$ (by (a) here) it must be solid (353K); as it contains $T(\chi 1)$, which is the standard order unit of the M-space $(L^1)^\sim$, it is the whole of $(L^1)^\sim$, and T is bijective.

365N Corollary If $(\mathfrak{A}, \bar\mu)$ is a localizable measure algebra, $L^\infty(\mathfrak{A})$ is a perfect Riesz space.

proof By 365M(b)-(c), we can identify L^∞ with $(L^1_{\bar\mu})^\times \cong (L^\infty)^{\times\times}$.

365O Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras. Let $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ be a measure-preserving ring homomorphism.

(a) There is a unique order-continuous norm-preserving Riesz homomorphism $T_\pi : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{B}, \bar\nu)$ such that $T_\pi(\chi a) = \chi(\pi a)$ whenever $a \in \mathfrak{A}^f$. We have $T_\pi(u \times \chi a) = T_\pi u \times \chi(\pi a)$ whenever $a \in \mathfrak{A}^f$ and $u \in L^1(\mathfrak{A}, \bar\mu)$.

(b) $\int T_\pi u = \int u$ and $\int_{\pi a} T_\pi u = \int_a u$ for every $u \in L^1(\mathfrak{A}, \bar\mu)$ and $a \in \mathfrak{A}^f$.

(c) $[\![T_\pi u > \alpha]\!] = \pi[\![u > \alpha]\!]$ for every $u \in L^1(\mathfrak{A}, \bar\mu)$ and $\alpha > 0$.

(d) T_π is surjective iff π is.

(e) If $(\mathfrak{C}, \bar\lambda)$ is another measure algebra and $\theta : \mathfrak{B}^f \to \mathfrak{C}^f$ another measure-preserving ring homomorphism, then $T_{\theta\pi} = T_\theta T_\pi : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{C}, \bar\lambda)$.

proof Throughout the proof I will write T for T_π and S^f for $S(\mathfrak{A}) \cap L^1_{\bar\mu} \cong S(\mathfrak{A}^f)$ (see 365F).

(a)(i) We have a map $\psi : \mathfrak{A}^f \to L^1_{\bar\nu}$ defined by writing $\psi a = \chi(\pi a)$ for $a \in \mathfrak{A}^f$. Because

$$\chi\pi(a \cup b) = \chi(\pi a \cup \pi b) = \chi\pi a + \chi\pi b, \quad \|\chi(\pi a)\|_1 = \bar\nu(\pi a) = \bar\mu a$$

whenever $a, b \in \mathfrak{A}^f$ and $a \cap b = 0$, we get a (unique) corresponding bounded linear operator $T : L^1_{\bar\mu} \to L^1_{\bar\nu}$ such that $T\chi = \chi\pi$ on $\mathfrak{A}^f$ (365I). Because $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ and $\chi : \mathfrak{B}^f \to L^1_{\bar\nu}$ are lattice homomorphisms, so is ψ, and T is a Riesz homomorphism (365Kc).

(ii) If $u \in S^f$, express it as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n$ are disjoint in $\mathfrak{A}^f$. Then $Tu = \sum_{i=0}^n \alpha_i \chi(\pi a_i)$ and $\pi a_0, \dots, \pi a_n$ are disjoint in $\mathfrak{B}^f$, so

$$\|Tu\|_1 = \sum_{i=0}^n |\alpha_i| \bar{\nu}(\pi a_i) = \sum_{i=0}^n |\alpha_i| \bar{\mu} a_i = \|u\|_1.$$

Because S^f is dense in $L^1_{\bar{\mu}}$ and $u \mapsto \|u\|_1$ is continuous (in both $L^1_{\bar{\mu}}$ and $L^1_{\bar{\nu}}$), $\|Tu\|_1 = \|u\|_1$ for every $u \in L^1_{\bar{\mu}}$, that is, T is norm-preserving. As noted in 365Ka, T is order-continuous.

(iii) If a, $b \in \mathfrak{A}^f$ then

$$T(\chi a \times \chi b) = T(\chi(a \cap b)) = \chi\pi(a \cap b) = \chi(\pi a \cap \pi b) = \chi\pi a \times \chi\pi b = \chi\pi a \times T(\chi b).$$

Because T is linear and $\times$ is bilinear, $T(\chi a \times u) = \chi\pi a \times Tu$ for every $u \in S^f$. Because the maps $u \mapsto u \times \chi a : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}}$, $T : L^1_{\bar{\mu}} \to L^1_{\bar{\nu}}$ and $v \mapsto v \times \chi\pi a : L^1_{\bar{\nu}} \to L^1_{\bar{\nu}}$ are all continuous, $Tu \times \chi\pi a = T(u \times \chi a)$ for every $u \in L^1_{\bar{\mu}}$.

(iv) T is unique because the formula $T(\chi a) = \chi\pi a$ defines T on the norm-dense and order-dense subspace S^f.

(b) Because T is positive,

$$\int Tu = \|Tu^+\|_1 - \|Tu^-\|_1 = \|u^+\|_1 - \|u^-\|_1 = \int u.$$

For $a \in \mathfrak{A}^f$,

$$\int_{\pi a} Tu = \int Tu \times \chi\pi a = \int T(u \times \chi a) = \int u \times \chi a = \int_a u.$$

(c) If $u \in S^f$, express it as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n$ are disjoint; then

$$\pi[\![u > \alpha]\!] = \pi(\sup_{i \in I} a_i) = \sup_{i \in I} \pi a_i = [\![Tu > \alpha]\!]$$

where $I = \{i : i \le n,\ \alpha_i > \alpha\}$. For $u \in (L^1_{\bar{\mu}})^+$, take a sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in S^f with supremum u; then $\sup_{n \in \mathbb{N}} Tu_n = Tu$, so

$$\pi[\![u > \alpha]\!] = \pi(\sup_{n \in \mathbb{N}} [\![u_n > \alpha]\!])$$

(364L(a-ii); $[\![u > \alpha]\!] \in \mathfrak{A}^f$ by 365A)

$$= \sup_{n \in \mathbb{N}} \pi[\![u_n > \alpha]\!]$$

(because π is order-continuous, see 361Ad)

$$= \sup_{n \in \mathbb{N}} [\![Tu_n > \alpha]\!] = [\![Tu > \alpha]\!]$$

because T is order-continuous. For general $u \in L^1_{\bar{\mu}}$,

$$\pi[\![u > \alpha]\!] = \pi[\![u^+ > \alpha]\!] = [\![T(u^+) > \alpha]\!] = [\![(Tu)^+ > \alpha]\!] = [\![Tu > \alpha]\!]$$

because T is a Riesz homomorphism.

(d)(i) Suppose that T is surjective and that $b \in \mathfrak{B}^f$. Then there is a $u \in L^1_{\bar{\mu}}$ such that $Tu = \chi b$. Now

$$b = [\![Tu > \tfrac{1}{2}]\!] = \pi[\![u > \tfrac{1}{2}]\!] \in \pi[\mathfrak{A}^f];$$

as b is arbitrary, π is surjective.

(ii) Suppose now that π is surjective. Then $T[L^1_{\bar{\mu}}]$ is a linear subspace of $L^1_{\bar{\nu}}$ containing χb for every $b \in \mathfrak{B}^f$, so includes $S(\mathfrak{B}^f)$. If $v \in (L^1_{\bar{\nu}})^+$ there is a sequence $\langle v_n \rangle_{n \in \mathbb{N}}$ in $S(\mathfrak{B}^f)^+$ with supremum v. For each n, choose u_n such that $Tu_n = v_n$. Setting $u'_n = \sup_{i \le n} u_i$, we get a non-decreasing sequence $\langle u'_n \rangle_{n \in \mathbb{N}}$ such that $v_n \le Tu'_n \le v$ for every $n \in \mathbb{N}$. So

$$\sup_{n \in \mathbb{N}} \|u'_n\|_1 = \sup_{n \in \mathbb{N}} \|Tu'_n\|_1 \le \|v\|_1 < \infty$$

and $u = \sup_{n \in \mathbb{N}} u'_n$ is defined in $L^1_{\bar{\mu}}$, with

$$Tu = \sup_{n \in \mathbb{N}} Tu'_n = v.$$

Thus $(L^1_{\bar{\nu}})^+ \subseteq T[L^1_{\bar{\mu}}]$; consequently $L^1_{\bar{\nu}} \subseteq T[L^1_{\bar{\mu}}]$ and T is surjective.

(e) This is an immediate consequence of the 'uniqueness' assertion in (i), because for any $a \in \mathfrak{A}^f$

$$T_\theta T_\pi(\chi a) = T_\theta\chi(\pi a) = \chi(\theta\pi a),$$

so that $T_\theta T_\pi : L^1_{\bar\mu} \to L^1_{\bar\lambda}$ is a bounded linear operator taking the right values at elements χa, and must therefore be equal to $T_{\theta\pi}$.

365P Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras, and $\pi : \mathfrak{A}^f \to \mathfrak{B}$ an order-continuous ring homomorphism.

(a) There is a unique positive linear operator $P_\pi : L^1(\mathfrak{B}, \bar\nu) \to L^1(\mathfrak{A}, \bar\mu)$ such that $\int_a P_\pi v = \int_{\pi a} v$ for every $v \in L^1(\mathfrak{B}, \bar\nu)$ and $a \in \mathfrak{A}^f$.

(b) P_π is order-continuous and norm-continuous, and $\|P_\pi\| \le 1$.

(c) If $a \in \mathfrak{A}^f$ and $v \in L^1(\mathfrak{B}, \bar\nu)$ then $P_\pi(v \times \chi\pi a) = P_\pi v \times \chi a$.

(d) If $\pi[\mathfrak{A}^f]$ is order-dense in $\mathfrak{B}$ then P_π is a norm-preserving Riesz homomorphism; in particular, P_π is injective.

(e) If $(\mathfrak{B}, \bar\nu)$ is semi-finite and π is injective, then P_π is surjective, and there is for every $u \in L^1(\mathfrak{A}, \bar\mu)$ a $v \in L^1(\mathfrak{B}, \bar\nu)$ such that $P_\pi v = u$ and $\|v\|_1 = \|u\|_1$.

(f) Suppose again that $(\mathfrak{B}, \bar\nu)$ is semi-finite. If $(\mathfrak{C}, \bar\lambda)$ is another measure algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ an order-continuous Boolean homomorphism, then $P_{\theta\pi} = P_\pi P_{\theta'} : L^1(\mathfrak{C}, \bar\lambda) \to L^1(\mathfrak{A}, \bar\mu)$, where I write θ' for the restriction of θ to $\mathfrak{B}^f$.

proof I write P for P_π.

(a)-(b) For $v \in L^1_{\bar\nu}$, $a \in \mathfrak{A}^f$ set $\nu_v(a) = \int_{\pi a} v$. Then $\nu_v : \mathfrak{A}^f \to \mathbb{R}$ is additive, bounded (by $\|v\|_1$) and if $A \subseteq \mathfrak{A}^f$ is non-empty, downwards-directed and has infimum 0, then

$$\inf_{a\in A} |\nu_v(a)| \le \inf_{a\in A} \int |v| \times \chi\pi a = 0$$

because $a \mapsto \int |v| \times \chi\pi a$ is a composition of order-continuous functions, therefore order-continuous. So 365Eb tells us that there is a $Pv \in L^1_{\bar\mu}$ such that $\int_a Pv = \nu_v(a) = \int_{\pi a} v$ for every $a \in \mathfrak{A}^f$. By 365D(d-ii), this formula defines Pv uniquely. Consequently P must be linear (since $Pv_1 + Pv_2$, αPv will always have the properties defining $P(v_1 + v_2)$, $P(\alpha v)$).

If $v \ge 0$ in $L^1_{\bar\nu}$, then $\int_a Pv = \int_{\pi a} v \ge 0$ for every $a \in \mathfrak{A}^f$, so $Pv \ge 0$ (365D(d-i)); thus P is positive. It must therefore be norm-continuous and order-continuous (355C, 355Ka).

Again supposing that $v \ge 0$, we have

$$\|Pv\|_1 = \int Pv = \sup_{a\in\mathfrak{A}^f} \int_a Pv = \sup_{a\in\mathfrak{A}^f} \int_{\pi a} v \le \|v\|_1$$

(using 365D(d-iii)). For general $v \in L^1_{\bar\nu}$,

$$\|Pv\|_1 = \||Pv|\|_1 \le \|P|v|\|_1 \le \|v\|_1.$$

(c) For any $c \in \mathfrak{A}^f$,

$$\int_c Pv \times \chi a = \int_{c\cap a} Pv = \int_{\pi(c\cap a)} v = \int_{\pi c} v \times \chi\pi a = \int_c P(v \times \chi\pi a).$$

(d) Now suppose that $\pi[\mathfrak{A}^f]$ is order-dense. Take any v, $v' \in L^1_{\bar\nu}$ such that $v \wedge v' = 0$. **?** Suppose, if possible, that $u = Pv \wedge Pv' > 0$. Take $\alpha > 0$ such that $a = [\![u > \alpha]\!]$ is non-zero. Since

$$\int_{\pi a} v = \int_a Pv \ge \int_a u > 0,$$

$b = \pi a \cap [\![v > 0]\!] \ne 0$. Let $c \in \mathfrak{A}^f$ be such that $0 \ne \pi c \subseteq b$; then $\pi(a \cap c) = \pi c \ne 0$, so $a \cap c \ne 0$, and

$$0 < \int_{a\cap c} u \le \int_{a\cap c} Pv' \le \int_{\pi c} v'.$$

But $\pi c \subseteq [\![v > 0]\!]$ and $v \wedge v' = 0$ so $\int_{\pi c} v' = 0$. **X**

So $Pv \wedge Pv' = 0$. As v, v' are arbitrary, P is a Riesz homomorphism (352G).

Next, if $v \ge 0$ in $L^1_{\bar\nu}$,

$$\int Pv = \sup_{a\in\mathfrak{A}^f} \int_a Pv = \sup_{a\in\mathfrak{A}^f} \int_{\pi a} v = \int v$$

because $\pi[\mathfrak{A}^f]$ is upwards-directed and has supremum 1 in $\mathfrak{B}$. So, for general $v \in L^1_{\bar\nu}$,

$$\|Pv\|_1 = \int |Pv| = \int P|v| = \int |v| = \|v\|_1,$$

and P is norm-preserving.

(e) Next suppose that $(\mathfrak{B}, \bar\nu)$ is semi-finite and that π is injective.

(i) If $u > 0$ in $L^1_{\bar{\mu}}$, there is a $v > 0$ in $L^1_{\bar{\nu}}$ such that $Pv \le u$ and $\int Pv \ge \int v$. **P** Let $\delta > 0$ be such that $a = [\![u > \delta]\!] \ne 0$. Then $\pi a \ne 0$. Because $(\mathfrak{B}, \bar{\nu})$ is semi-finite, there is a non-zero $b \in \mathfrak{B}^f$ such that $b \subseteq \pi a$. Set $u_1 = P(\chi b)$. Then $u_1 \ge 0$, $\int_a u_1 = \bar{\nu} b > 0$ and

$$\int_{1\setminus a} u_1 = \sup_{c\in\mathfrak{A}^f} \int_{c\setminus a} u_1 = \sup_{c\in\mathfrak{A}^f} \int_{\pi c\setminus \pi a} \chi b = 0.$$

So $u_1 \times \chi(1 \setminus a) = 0$ and $0 \ne [\![u_1 > 0]\!] \subseteq a$. Let $\gamma > 0$ be such that $[\![u_1 > \gamma]\!] \ne [\![u_1 > 0]\!]$, and set $a_1 = a \setminus [\![u_1 > \gamma]\!]$, $v = \frac{\delta}{\gamma}\chi(b \cap \pi a_1)$. Then

$$Pv = \frac{\delta}{\gamma} P(\chi b \times \chi(\pi a_1)) = \frac{\delta}{\gamma} P(\chi b) \times \chi a_1 = \frac{\delta}{\gamma} u_1 \times \chi a_1 \le \delta \chi a \le u,$$

because

$$[\![u_1 \times \chi a_1 > \gamma]\!] \subseteq [\![u_1 > \gamma]\!] \cap a_1 = 0$$

so

$$u_1 \times \chi a_1 \le \gamma \chi [\![u_1 > 0]\!] \le \gamma \chi a.$$

Also $a_1 \cap [\![u_1 > 0]\!] \ne 0$, so Pv and v are non-zero; and

$$\int Pv \ge \int_{a_1} Pv = \int_{\pi a_1} v = \int v. \textbf{ Q}$$

(ii) Now take any $u \ge 0$ in $L^1_{\bar{\mu}}$, and set $B = \{v : v \in L^1_{\bar{\nu}},\ v \ge 0,\ Pv \le u,\ \int v \le \int Pv\}$. B is not empty because it contains 0. If $C \subseteq B$ is non-empty and upwards-directed, then $\sup_{v\in C} \int v \le \int u$ is finite, so C has a supremum in $L^1_{\bar{\nu}}$ (365Df). Because P is order-continuous, $P(\sup C) = \sup P[C] \le u$; also

$$\int \sup C = \sup_{v\in C} \int v \le \sup_{v\in C} \int Pv \le \int P(\sup C).$$

Thus $\sup C \in B$. As C is arbitrary, B satisfies the conditions of Zorn's Lemma, and has a maximal element v_0 say.

? Suppose, if possible, that $Pv_0 \ne u$. By (α), there is a $v_1 > 0$ such that $Pv_1 \le u - Pv_0$, $\int v_1 \le \int Pv_1$. In this case, $v_0 < v_0 + v_1 \in B$, which is impossible. **X** Thus $Pv_0 = u$; also

$$\|v_0\|_1 = \int v_0 \le \int Pv_0 = \|Pv_0\|_1.$$

(iii) Finally, take any $u \in L^1_{\bar{\mu}}$. By (ii), there are non-negative v_1, $v_2 \in L^1_{\bar{\nu}}$ such that $Pv_1 = u^+$, $Pv_2 = u^-$, $\|v_1\|_1 \le \|u^+\|_1$ and $\|v_2\|_1 \le \|u^-\|_1$. Setting $v = v_1 - v_2$, we have $Pv = u$. Also we must have

$$\|v\|_1 \le \|v_1\|_1 + \|v_2\|_1 \le \|u^+\|_1 + \|u^-\|_1 = \|u\|_1 \le \|P\|\|v\|_1 = \|v\|_1,$$

so $\|v\|_1 = \|u\|_1$, as required.

(f) As usual, this is a consequence of the uniqueness of P. However (because I do not assume that $\pi[\mathfrak{A}^f] \subseteq \mathfrak{B}^f$) there is an extra refinement: we need to know that $\int_b P_{\theta'} w = \int_{\theta b} w$ for every $b \in \mathfrak{B}$ and $w \in L^1_{\bar{\lambda}}$. **P** Because θ is order-continuous and $(\mathfrak{B}, \bar{\nu})$ is semi-finite, $\theta b = \sup\{\theta b' : b' \in \mathfrak{B}^f,\ b' \subseteq b\}$, so if $w \ge 0$ then

$$\int_{\theta b} w = \sup_{b'\in\mathfrak{B}^f, b'\subseteq b} \int_{\theta b'} w = \sup_{b'\in\mathfrak{B}^f, b'\subseteq b} \int_{b'} P_{\theta'} w = \int_b P_{\theta'} w.$$

Expressing w as $w^+ - w^-$, we see that the same is true for every $w \in L^1_{\bar{\nu}}$. **Q**

Now we can say that $PP_{\theta'}$ is a positive linear operator from $L^1_{\bar{\lambda}}$ to $L^1_{\bar{\mu}}$ such that

$$\int_a PP_{\theta'} w = \int_{\pi a} P_{\theta'} w = \int_{\theta\pi a} w = \int_a P_{\theta\pi} w$$

whenever $a \in \mathfrak{A}^f$ and $w \in L^1_{\bar{\lambda}}$, and must be equal to $P_{\theta\pi}$.

365Q Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\mu})$ be measure algebras and $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ a measure-preserving ring homomorphism.

(a) In the language of 365O-365P above, $P_\pi T_\pi$ is the identity operator on $L^1(\mathfrak{A}, \bar{\mu})$.

(b) If π is surjective (so that it is an isomorphism between $\mathfrak{A}^f$ and $\mathfrak{B}^f$) then $P_\pi = T_\pi^{-1} = T_{\pi^{-1}}$ and $T_\pi = P_\pi^{-1} = P_{\pi^{-1}}$.

proof (a) If $u \in L^1_{\bar{\mu}}$, $a \in \mathfrak{A}^f$ then

$$\int_a P_\pi T_\pi u = \int_{\pi a} T_\pi u = \int_a u.$$

So $u=P_\pi T_\pi u$, by 365D(d-ii).

(b) From 365Od, we know that T_π is surjective, while $P_\pi T_\pi$ is the identity, so that $P_\pi=T_\pi^{-1}$, $T_\pi=P_\pi^{-1}$. As for $T_{\pi^{-1}}$, 365Oe tells us that $T_{\pi^{-1}}=T_\pi^{-1}$; so

$$P_{\pi^{-1}}=T_{\pi^{-1}}^{-1}=T_\pi.$$

365R Conditional expectations It is a nearly universal rule that any investigation of L^1 spaces must include a look at conditional expectations. In the present context, they take the following form.

(a) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra; write $\bar\nu$ for the restriction $\bar\mu{\restriction}\mathfrak{B}$. The identity map from $\mathfrak{B}$ to $\mathfrak{A}$ induces operators $T:L^1(\mathfrak{B},\bar\nu)\to L^1(\mathfrak{A},\bar\mu)$ and $P:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{B},\bar\nu)$. If we take $L^0(\mathfrak{A})$ to be defined as the set of functions from $\mathbb{R}$ to $\mathfrak{A}$ described in 364Aa, then $L^0(\mathfrak{B})$ becomes a subset of $L^0(\mathfrak{A})$ in the literal sense, and T is actually the identity operator associated with the subset $L^1(\mathfrak{B},\bar\nu)\subseteq L^1(\mathfrak{A},\bar\mu)$; $L^1(\mathfrak{B},\bar\nu)$ is a norm-closed and order-closed Riesz subspace of $L^1(\mathfrak{A},\bar\mu)$. P is a positive linear operator, while PT is the identity, so P is a projection from $L^1(\mathfrak{A},\bar\mu)$ onto $L^1(\mathfrak{B},\bar\nu)$. P is defined by the familiar formula

$$\int_b Pu=\int_b u \text{ for every } u\in L^1(\mathfrak{A},\bar\mu),\ b\in\mathfrak{B},$$

so is the conditional expectation operator in the sense of 242J. Observe that the formula in 365A now tells us that $L^1(\mathfrak{B},\bar\nu)$ is just $L^1(\mathfrak{A},\bar\mu)\cap L^0(\mathfrak{B})$. Translating 233K into this language, we see that $P(u\times v)=Pu\times v$ whenever $u\in L^1(\mathfrak{A},\bar\mu)$, $v\in L^0(\mathfrak{B})$ and $u\times v\in L^1(\mathfrak{A},\bar\mu)$.

(b) Just as in 233I-233J and 242K, we have a version of Jensen's inequality. Let $h:\mathbb{R}\to\mathbb{R}$ be a convex function and $\bar h:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ the corresponding map (364H). If $u\in L^1(\mathfrak{A},\bar\mu)$, then $h(\int u)\le\int\bar h(u)$; and if $\bar h(u)\in L^1(\mathfrak{A},\bar\mu)$, then $\bar h(Pu)\le P(\bar h(u))$. **P** I repeat the proof of 233I-233J. For each $q\in\mathbb{Q}$, take $\beta_q\in\mathbb{R}$ such that $h(t)\ge h_q(t)=h(q)+\beta_q(t-q)$ for every $t\in\mathbb{R}$, so that $h(t)=\sup_{q\in\mathbb{Q}}h_q(t)$ for every $t\in\mathbb{R}$, and $\bar h(u)=\sup_{q\in\mathbb{Q}}\bar h_q(u)$ for every $u\in L^0(\mathfrak{A})$. (This is because

$$\begin{aligned}[\![\bar h(u)>\alpha]\!]&=[\![u\in h^{-1}[\,]\alpha,\infty[\,]]\!]=[\![u\in\textstyle\bigcup_{q\in\mathbb{Q}}h_q^{-1}[\,]\alpha,\infty[\,]]\!]\\&=\sup_{q\in\mathbb{Q}}[\![u\in h_q^{-1}[\,]\alpha,\infty[\,]]\!]=\sup_{q\in\mathbb{Q}}[\![\bar h_q(u)>\alpha]\!]\end{aligned}$$

for every $\alpha\in\mathbb{R}$.) But setting $e=\chi1$, we see that $\bar h_q(u)=h(q)e+\beta_q(u-qe)$ for every $u\in L^0(\mathfrak{A})$, so that

$$\int\bar h_q(u)=h(q)+\beta_q(\int u-q)=h_q(\int u),$$

$$P(\bar h_q(u))=h(q)e+\beta_q(Pu-qe)=\bar h_q(Pu)$$

because $\int e=1$ and $Pe=e$. Taking the supremum over q, we get

$$h(\int u)=\sup_{q\in\mathbb{Q}}h_q(\int u)=\sup_{q\in\mathbb{Q}}\int\bar h_q(u)\le\int\bar h(u),$$

and if $\bar h(u)\in L^1_{\bar\mu}$ then

$$\bar h(Pu)=\sup_{q\in\mathbb{Q}}\bar h_q(Pu)=\sup_{q\in\mathbb{Q}}P(\bar h_q(u))\le P(\bar h(u)).\ \mathbf{Q}$$

Of course the result in this form can also be deduced from 233I-233J if we represent $(\mathfrak{A},\bar\mu)$ as the measure algebra of a probability space (X,Σ,μ) and set $\mathrm{T}=\{E:E\in\Sigma,\ E^\bullet\in\mathfrak{B}\}$.

(c) I note here a fact which is occasionally useful. If $u\in L^1(\mathfrak{A},\bar\mu)$ is non-negative, then $[\![Pu>0]\!]=\operatorname{upr}([\![u>0]\!],\mathfrak{B})$, the upper envelope of $[\![u>0]\!]$ in $\mathfrak{B}$ as defined in 313S. **P** We have only to observe that, for $b\in\mathfrak{B}$,

$$\begin{aligned}b\cap[\![Pu>0]\!]=0&\iff\chi b\times Pu=0\iff\int_b Pu=0\\&\iff\int_b u=0\iff b\cap[\![u>0]\!]=0.\end{aligned}$$

Taking complements, $b\supseteq[\![Pu>0]\!]$ iff $b\supseteq[\![u>0]\!]$. **Q**

(d) Suppose now that $(\mathfrak{C},\bar\lambda)$ is another probability algebra and $\pi:\mathfrak{A}\to\mathfrak{C}$ is a measure-preserving Boolean homomorphism. Then $\mathfrak{D}=\pi[\mathfrak{B}]$ is a closed subalgebra of $\mathfrak{C}$ (314F(a-i)). Let $Q:L^1(\mathfrak{C},\bar\lambda)\to L^1(\mathfrak{D},\bar\lambda{\restriction}\mathfrak{D})\subseteq$

$L^1(\mathfrak{C},\bar\lambda)$ be the conditional expectation associated with $\mathfrak{D}$, and $T_\pi : L^1(\mathfrak{A},\bar\mu) \to L^1(\mathfrak{C},\bar\lambda)$ the norm-preserving Riesz homomorphism defined by π. Then $T_\pi P = QT_\pi$. **P** Take $u \in L^1(\mathfrak{A},\bar\mu)$. Then

$$[\![T_\pi Pu > \alpha]\!] = \pi[\![Pu > \alpha]\!] \in \pi[\mathfrak{B}] = \mathfrak{D}$$

for every $\alpha \in \mathbb{R}$, so $T_\pi Pu \in L^0(\mathfrak{D})$. If $d \in \mathfrak{D}$, set $b = \pi^{-1}d \in \mathfrak{B}$; then

$$\begin{aligned}\int_d T_\pi Pu &= \int T_\pi Pu \times \chi d = \int T_\pi Pu \times T_\pi \chi b = \int T_\pi(Pu \times \chi b)\\ &= \int Pu \times \chi b = \int_b Pu = \int_b u = \int u \times \chi b\\ &= \int T_\pi(u \times \chi b) = \int T_\pi u \times T_\pi \chi b = \int T_\pi u \times \chi d = \int_d T_\pi u.\end{aligned}$$

As d is arbitrary, $T_\pi Pu$ satisfies the defining formula for $QT_\pi u$ and $T_\pi Pu = QT_\pi u$; as u is arbitrary, $T_\pi P = QT_\pi$. **Q**

365S Recovering the algebra: Proposition (a) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Then $\mathfrak{A}$ is isomorphic to the band algebra of $L^1(\mathfrak{A},\bar\mu)$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\bar\mu$, $\bar\nu$ two measures on $\mathfrak{A}$ such that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{A},\bar\nu)$ are both semi-finite measure algebras. Then $L^1(\mathfrak{A},\bar\mu)$ is isomorphic, as Banach lattice, to $L^1(\mathfrak{A},\bar\nu)$.

proof (a) Because $(\mathfrak{A},\bar\mu)$ is semi-finite, $L^1_{\bar\mu}$ is order-dense in $L^0 = L^0(\mathfrak{A})$ (365G). Consequently, $L^1_{\bar\mu}$ and L^0 have isomorphic band algebras (353D). But the band algebra of L^0 is just its algebra of projection bands (because $\mathfrak{A}$ and therefore L^0 are Dedekind complete, see 364M and 353I), which is isomorphic to $\mathfrak{A}$ (364O).

(b) Let $\pi : \mathfrak{A} \to \mathfrak{A}$ be the identity map. Regarding π as an order-continuous Boolean homomorphism from $\mathfrak{A}^f_{\bar\mu} = \{a : \bar\mu a < \infty\}$ to $(\mathfrak{A},\bar\nu)$, we have an associated positive linear operator $P = P_\pi : L^1_{\bar\nu} \to L^1_{\bar\mu}$; similarly, we have $Q = P_{\pi^{-1}} : L^1_{\bar\mu} \to L^1_{\bar\nu}$, and both P and Q have norm at most 1 (365Pb). Now 365Pf assures us that QP is the identity operator on $L^1_{\bar\nu}$ and PQ is the identity operator on $L^1_{\bar\mu}$. So P and Q are the two halves of a Banach lattice isomorphism between $L^1_{\bar\mu}$ and $L^1_{\bar\nu}$.

365T Having opened the question of varying measures on a single Boolean algebra, this seems an appropriate moment for a general description of how they interact.

Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and $\bar\mu : \mathfrak{A} \to [0,\infty]$, $\bar\nu : \mathfrak{A} \to [0,\infty]$ two functions such that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{A},\bar\nu)$ are both semi-finite (therefore localizable) measure algebras.

(a) There is a unique $u \in L^0 = L^0(\mathfrak{A})$ such that (if we allow ∞ as a value of the integral) $\int_a u\,d\bar\mu = \bar\nu a$ for every $a \in \mathfrak{A}$.

(b) For $v \in L^0(\mathfrak{A})$, $\int v\,d\bar\nu = \int u \times v\,d\bar\mu$ if either is defined in $[-\infty,\infty]$.

(c) u is strictly positive (i.e., $[\![u > 0]\!] = 1$) and, writing $\frac{1}{u}$ for the multiplicative inverse of u, $\int_a \frac{1}{u} d\bar\nu = \bar\mu a$ for every $a \in \mathfrak{A}$.

proof (a) Because $(\mathfrak{A},\bar\nu)$ is semi-finite, there is a partition of unity $D \subseteq \mathfrak{A}$ such that $\bar\nu d < \infty$ for every $d \in D$. For each $d \in D$, the functional $a \mapsto \bar\nu(a \cap d) : \mathfrak{A} \to \mathbb{R}$ is completely additive, so there is a $u_d \in L^1_{\bar\mu}$ such that $\int_a u_d d\bar\mu = \bar\nu(a \cap d)$ for every $a \in \mathfrak{A}$. Because $\int_a u_d d\bar\mu \ge 0$ for every a, $u_d \ge 0$. Because $\int_{1\setminus d} u_d = 0$, $[\![u_d > 0]\!] \subseteq d$. Now $u = \sup_{d\in D} u_d$ is defined in L^0. **P** (This is a special case of 368K below.) For $n \in \mathbb{N}$, set $c_n = \sup_{d\in D}[\![u_d > n]\!]$. If d, $d' \in D$ are distinct, then $d \cap [\![u_{d'} > n]\!] = 0$, so $d \cap c_n = [\![u_d > n]\!]$. Set $c = \inf_{n\in\mathbb{N}} c_n$. If $d \in D$, then

$$d \cap c = \inf_{n\in\mathbb{N}} d \cap c_n = \inf_{n\in\mathbb{N}} [\![u_d > n]\!] = 0.$$

But $c \subseteq c_0 \subseteq \sup D$, so $c = 0$. By 364L(a-i), $\{u_d : d \in D\}$ is bounded above in L^0, so has a supremum, because L^0 is Dedekind complete, by 364M. **Q**

For finite $I \subseteq D$ set $\tilde u_I = \sum_{d\in I} u_d = \sup_{d\in I} u_d$ (because $u_d \wedge u_c = 0$ for distinct c, $d \in D$). Then $u = \sup\{\tilde u_I : I \subseteq D, I \text{ is finite}\}$. So, for any $a \in \mathfrak{A}$,

$$\int_a u\,d\bar\mu = \sup_{I\subseteq D \text{ is finite}} \int_a \tilde u_I d\bar\mu$$

(365Dh)

$$= \sup_{I \subseteq D \text{ is finite}} \sum_{d \in I} \int_a u_d d\bar{\mu} = \sup_{I \subseteq D \text{ is finite}} \sum_{d \in I} \bar{\nu}(a \cap d) = \bar{\nu}a.$$

Note that if $a \in \mathfrak{A}$ is non-zero, then $\bar{\nu}a > 0$, so $a \cap [\![u > 0]\!] \neq 0$; consequently $[\![u > 0]\!] = 1$.

To see that u is unique, observe that if u' has the same property then for any $d \in D$

$$\int_a u \times \chi d \, d\bar{\mu} = \bar{\nu}(a \cap d) = \int_a u' \times \chi d \, d\bar{\mu}$$

for every $a \in \mathfrak{A}$, so that $u \times \chi d = u' \times \chi d$; because $\sup D = 1$ in $\mathfrak{A}$, u must be equal to u'.

(b) Use 365Hb, with π and T the identity maps.

(c) In the same way, there is a $w \in L^0$ such that $\int_a w \, d\bar{\nu} = \bar{\mu}a$ for every $a \in \mathfrak{A}$. To relate u and w, observe that applying 365Hb we get

$$\int w \times \chi a \times u \, d\bar{\mu} = \int w \times \chi a \, d\bar{\nu}$$

for every $a \in \mathfrak{A}$, that is, $\int_a w \times u \, d\bar{\mu} = \bar{\mu}a$ for every a. But from this we see that $w \times u \times \chi b = \chi b$ at least when $\bar{\mu}b < \infty$, so that $w \times u = \chi 1$ is the multiplicative identity of L^0, and $w = \frac{1}{u}$.

365U Uniform integrability Continuing the programme in 365C, we can transcribe the ideas of §§246, 247, 354 and 356 into the new context.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Set $L^1 = L^1(\mathfrak{A}, \bar{\mu})$.

(a) For a non-empty subset A of L^1, the following are equiveridical:

(i) A is uniformly integrable in the sense of 354P;

(ii) for every $\epsilon > 0$ there are an $a \in \mathfrak{A}^f$ and an $M \geq 0$ such that $\int (u - M\chi a)^+ \leq \epsilon$ for every $u \in \mathfrak{A}$;

(iii)(α) $\sup_{u \in A} |\int_a u|$ is finite for every atom $a \in \mathfrak{A}$,

(β) for every $\epsilon > 0$ there are $c \in \mathfrak{A}^f$ and $\delta > 0$ such that $|\int_a u| \leq \epsilon$ whenever $u \in A$, $a \in \mathfrak{A}$ and $\bar{\mu}(a \cap c) \leq \delta$;

(iv)(α) $\sup_{u \in A} |\int_a u|$ is finite for every atom $a \in \mathfrak{A}$,

(β) $\lim_{n \to \infty} \sup_{u \in A} |\int_{a_n} u| = 0$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;

(v) A is relatively weakly compact in L^1.

(b) If $(\mathfrak{A}, \bar{\mu})$ is a probability algebra and $A \subseteq L^1$ is uniformly integrable, then there is a solid convex norm-closed uniformly integrable set $C \supseteq A$ such that $P[C] \subseteq C$ whenever $P : L^1 \to L^1$ is the conditional expectation operator associated with a closed subalgebra of $\mathfrak{A}$.

proof 354Q, 354R, 356Q and 246D, with a little help from 246C and 246G.

365X Basic exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $u \in L^1_{\bar{\mu}}$. Show that

$$\int u = \int_0^\infty \bar{\mu}[\![u > \alpha]\!] \, d\alpha - \int_{-\infty}^0 \bar{\mu}(1 \setminus [\![u > \alpha]\!]) \, d\alpha.$$

>(b) Let $(\mathfrak{A}, \bar{\mu})$ be any measure algebra, and $u \in L^1_{\bar{\mu}}$. (i) Show that $\|u\|_1 \leq 2 \sup_{a \in \mathfrak{A}^f} |\int_a u|$. (*Hint*: 246F.) (ii) Show that for any $\epsilon > 0$ there is an $a \in \mathfrak{A}^f$ such that $|\int u - \int_b u| \leq \epsilon$ whenever $a \subseteq b \in \mathfrak{A}$.

>(c) Let U be an L-space. If $\langle u_n \rangle_{n \in \mathbb{N}}$ is any norm-bounded sequence in U^+, show that

$$\liminf_{n \to \infty} u_n = \sup_{n \in \mathbb{N}} \inf_{m \geq n} u_m$$

is defined in U, and that $\int \liminf_{n \to \infty} u_n \leq \liminf_{n \to \infty} \int u_n$.

(d) Let U be an L-space. Let $\mathcal{F}$ be a filter on U such that $\{u : u \geq 0, \|u\| \leq k\}$ belongs to $\mathcal{F}$ for some $k \in \mathbb{N}$. Show that $u_0 = \sup_{F \in \mathcal{F}} \inf F$ is defined in U, and that $\int u_0 \leq \sup_{F \in \mathcal{F}} \inf_{u \in F} \int u$.

(e) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $A \subseteq L^1_{\bar{\mu}}$ a non-empty set. Show that A is bounded above in $L^1_{\bar{\mu}}$ iff

$$\sup\{\sum_{i=0}^n \int_{a_i} u_i : a_0, \dots, a_n \text{ is a partition of unity in } \mathfrak{A},\ u_0, \dots, u_n \in A\}$$

is finite, and that in this case the supremum is $\int \sup A$. (*Hint*: given $u_0, \dots, u_n \in A$, set $b_{ij} = [\![u_i \geq u_j]\!]$, $b_i = \sup_{j \neq i} b_{ij}$, $a_i = b_i \setminus \sup_{j < i} b_j$, and show that $\int \sup_{i \leq n} u_i = \sum_{i=0}^n \int_{a_i} u_i$.)

(f) Let $(\mathfrak{A},\bar{\mu})$ be any measure algebra and $\nu:\mathfrak{A}^f\to\mathbb{R}$ a bounded additive functional. Show that the following are equiveridical: (i) there is a $u\in L^1_{\bar{\mu}}$ such that $\nu a=\int_a u$ for every $a\in\mathfrak{A}^f$; (ii) for every $\epsilon>0$ there is a $\delta>0$ such that $|\nu a|\le\epsilon$ whenever $\bar{\mu}a\le\delta$; (iii) for every $\epsilon>0$, $c\in\mathfrak{A}^f$ there is a $\delta>0$ such that $\nu a\le\epsilon$ whenever $a\subseteq c$ and $\bar{\mu}a\le\delta$; (iv) for every $\epsilon>0$ there are $c\in\mathfrak{A}^f$, $\delta>0$ such that $|\nu a|\le\epsilon$ whenever $a\in\mathfrak{A}^f$ and $\bar{\mu}(a\cap c)\le\delta$; (v) $\lim_{n\to\infty}\nu a_n=0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}^f$ with infimum 0.

(g) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let $T:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ be the Riesz homomorphism associated with π (364P). Suppose that $w\ge 0$ in $L^0(\mathfrak{B})$ is such that $\int_{\pi a}w\,d\bar{\nu}=\bar{\mu}a$ whenever $a\in\mathfrak{A}$. Show that for any $u\in L^0(\mathfrak{A},\bar{\mu})$, $\int Tu\times w\,d\bar{\nu}=\int u\,d\bar{\mu}$ whenever either is defined in $[-\infty,\infty]$.

>(h) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and $a\in\mathfrak{A}$; write $\mathfrak{A}_a$ for the principal ideal it generates. Show that if π is the identity embedding of $\mathfrak{A}^f\cap\mathfrak{A}_a$ into $\mathfrak{A}^f$, then T_π, as defined in 365O, identifies $L^1(\mathfrak{A}_a,\bar{\mu}\restriction\mathfrak{A}_a)$ with a band in $L^1_{\bar{\mu}}$.

>(i) Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with measure algebras $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$. Let $\phi:X\to Y$ be an inverse-measure-preserving function and $\pi:\mathfrak{B}\to\mathfrak{A}$ the corresponding measure-preserving homomorphism (324M). Show that $T_\pi:L^1_{\bar{\nu}}\to L^1_{\bar{\mu}}$ (365O) corresponds to the map $g^\bullet\mapsto(g\phi)^\bullet:L^1(\nu)\to L^1(\mu)$ of 242Xd.

(j) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras. Let $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ be a ring homomorphism such that, for some $\gamma>0$, $\bar{\nu}(\pi a)\le\gamma\bar{\mu}a$ for every $a\in\mathfrak{A}^f$. (i) Show that there is a unique order-continuous Riesz homomorphism $T:L^1_{\bar{\mu}}\to L^1_{\bar{\nu}}$ such that $T(\chi a)=\chi(\pi a)$ whenever $a\in\mathfrak{A}^f$, and that $\|T\|\le\gamma$. (ii) Show that $[\![Tu>\alpha]\!]=\pi[\![u>\alpha]\!]$ for every $u\in L^1_{\bar{\mu}}$, $\alpha>0$. (iii) Show that T is surjective iff π is, injective iff π is. (iv) Show that T is norm-preserving iff $\bar{\nu}(\pi a)=\bar{\mu}a$ for every $a\in\mathfrak{A}^f$.

(k) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism. Let $T:L^1_{\bar{\mu}}\to L^1_{\bar{\nu}}$ and $P:L^1_{\bar{\nu}}\to L^1_{\bar{\mu}}$ be the operators corresponding to $\pi\restriction\mathfrak{A}^f$, as described in 365O-365P, and $\tilde{T}:L^\infty(\mathfrak{A})\to L^\infty(\mathfrak{B})$ the operator corresponding to π, as described in 363F. (i) Show that $T(u\times v)=Tu\times\tilde{T}v$ for every $u\in L^1_{\bar{\mu}}$, $v\in L^\infty(\mathfrak{A})$. (ii) Show that if π is order-continuous, then $\int Pv\times u=\int v\times\tilde{T}u$ for every $u\in L^\infty(\mathfrak{A})$, $v\in L^1_{\bar{\nu}}$.

>(l) Let (X,Σ,μ) be a probability space, with measure algebra $(\mathfrak{A},\bar{\mu})$, and let T be a σ-subalgebra of Σ. Set $\nu=\mu\restriction\mathrm{T}$, $\mathfrak{B}=\{F^\bullet:F\in\mathrm{T}\}\subseteq\mathfrak{A}$, $\bar{\nu}=\bar{\mu}\restriction\mathfrak{B}$, so that $(\mathfrak{B},\bar{\nu})$ is a measure algebra. Let $\pi:\mathfrak{B}\to\mathfrak{A}$ be the identity homomorphism. Show that $T_\pi:L^1_{\bar{\nu}}\to L^1_{\bar{\mu}}$ (365O) corresponds to the canonical embedding of $L^1(\nu)$ in $L^1(\mu)$ described in 242Jb, while $P_\pi:L^1_{\bar{\mu}}\to L^1_{\bar{\nu}}$ (365P) corresponds to the conditional expectation operator described in 242Jd.

(m) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and $(\widehat{\mathfrak{A}},\hat{\mu})$ its localization (322Q). Show that the natural embedding of $\mathfrak{A}$ in $\widehat{\mathfrak{A}}$ induces a Banach lattice isomorphism between $L^1_{\bar{\mu}}$ and $L^1_{\hat{\mu}}$, so that the band algebra of $L^1_{\bar{\mu}}$ can be identified with the Dedekind completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$.

(n) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\bar{\mu}$, $\bar{\nu}$ two functions such that $(\mathfrak{A},\bar{\mu})$, $(\mathfrak{A},\bar{\nu})$ are measure algebras. Show that $L^1_{\bar{\mu}}\subseteq L^1_{\bar{\nu}}$ (as subsets of $L^0(\mathfrak{A})$) iff there is a $\gamma>0$ such that $\bar{\nu}a\le\gamma\bar{\mu}a$ for every $a\in\mathfrak{A}$. (*Hint*: show that the identity operator from $L^1_{\bar{\mu}}$ to $L^1_{\bar{\nu}}$ is bounded.)

(o) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, I_∞ the ideal of 'purely infinite' elements of $\mathfrak{A}$ together with 0, $\bar{\mu}_{sf}$ the measure on $\mathfrak{B}=\mathfrak{A}/I_\infty$ (322Xa). Let $\pi:\mathfrak{A}\to\mathfrak{B}$ be the canonical map. Show that T_π, as defined in 365O, is a Banach lattice isomorphism between $L^1_{\bar{\mu}}$ and $L^1(\mathfrak{B},\bar{\mu}_{sf})$.

(p) Let (X,Σ,μ) be a a semi-finite measure space. Show that $L^1(\mu)$ is separable iff μ is σ-finite and has countable Maharam type.

(q) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be probability algebras, $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism, and $T:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ the corresponding Riesz homomorphism. Let $\mathfrak{C}$ be a closed subalgebra of $\mathfrak{A}$ and $P:L^1(\mathfrak{A},\bar{\mu})\to L^1(\mathfrak{C},\bar{\mu}\restriction\mathfrak{C})\subseteq L^1(\mathfrak{A},\bar{\mu})$, $Q:L^1(\mathfrak{B},\bar{\nu})\to L^1(\mathfrak{B},\bar{\nu})$ the conditional expectation operators defined from $\mathfrak{C}\subsetneqq\mathfrak{A}$ and $\pi[\mathfrak{C}]\subsetneqq\mathfrak{B}$. Show that $TP=QT$.

365Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, not $\{0\}$. Show that the topological density of $L^1_{\bar{\mu}}$ (331Yf) is $\max(\omega,\tau(\mathfrak{A}),c(\mathfrak{A}))$, where $\tau(\mathfrak{A})$, $c(\mathfrak{A})$ are the Maharam type and cellularity of $\mathfrak{A}$.

365 Notes and comments You should not suppose that L^1 spaces appear in the second half of this chapter because they are of secondary importance. Indeed I regard them as the most important of all function spaces. I have delayed the discussion of them for so long because it is here that for the first time we need measure algebras in an essential way.

The actual definition of $L^1_{\bar{\mu}}$ which I give is designed for speed rather than illumination; I seek only a formula, visibly independent of any particular representation of $(\mathfrak{A},\bar{\mu})$ as the measure algebra of a measure space, from which I can prove 365B. 365C-365D and 365Ea are now elementary. In 365Eb I take a page to describe a form of the Radon-Nikodým theorem which is applicable to arbitrary measure algebras, at the cost of dealing with functionals on the ring $\mathfrak{A}^f$ rather than on the whole algebra $\mathfrak{A}$. This is less for the sake of applications than to emphasize one of the central properties of L^1: it depends only on $\mathfrak{A}^f$ and $\bar{\mu}\restriction\mathfrak{A}^f$. For alternative versions of the condition 365Eb(i) see 365Xf.

The convergence theorems (B.Levi's theorem, Fatou's lemma and Lebesgue's dominated convergence theorem) are so central to the theory of integrable functions that it is natural to look for versions in the language here. Corresponding to B.Levi's theorem is the Levi property of a norm in an L-space; note how the abstract formulation makes it natural to speak of general upwards-directed families rather than of non-decreasing sequences, though the sequential form is so often used that I have spelt it out (365C). In the same way, the integral becomes order-continuous rather than just sequentially order-continuous (365Da). Corresponding to Fatou's lemma we have 365Xc-365Xd. For abstract versions of Lebesgue's theorem I will wait until §367.

In 365H I have deliberately followed the hypotheses of 235A and 235R. Of course 365H can be deduced from these if we use the Stone representations of $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$, so that π can be represented by a function between the Stone spaces (312Q). But 365H is essentially simpler, because the technical problems concerning measurability which took up so much of §235 have been swept under the carpet. In the same way, 365Xg corresponds to 235E. Here we have a fair example of the way in which the abstract expression in terms of measure algebras can be tidier than the expression in terms of measure spaces. But in my view this is because here, at least, some of the mathematics has been left out.

365I-365K correspond closely to 361F-361H and 363E. 365M is a re-run of 243G, but with the additional refinement that I examine the action of L^1 on L^∞ (the operator S) as well as the action of L^∞ on L^1 (the operator T). Of course 365Mc is just the abstract version of 243Hb, and can easily be proved from it. Note that while the proof of 365M does not itself involve any representation of $(\mathfrak{A},\bar{\mu})$ as the measure algebra of a measure space, (a-vii) and (b-iii) of the proof of 365M depend on the Radon-Nikodým theorem through 327D and 365E. For a development of the theory of $L^1(\mathfrak{A},\bar{\mu})$ which does not (in a formal sense) depend on measure spaces, see Fremlin 74a, 63J.

Theorems 365O-365Q lie at the centre of my picture of L^1 spaces, and are supposed to show their dual nature. Starting from a semi-finite measure algebra $(\mathfrak{A},\bar{\mu})$ we have two essentially different routes to the L^1-space: we can either build it up from indicator functions of elements of finite measure, so that it is naturally embedded in $L^0(\mathfrak{A})$, or we can think of it as the order-continuous dual of $L^\infty(\mathfrak{A})$. The first is a 'covariant' construction (signalled by the formula $T_{\theta\pi}=T_\theta T_\pi$ in 365Oe) and the second is 'contravariant' (so that $P_{\theta\pi}=P_\pi P_{\theta'}$ in 365Pf). The first construction is the natural one if we are seeking to copy the ideas of §242, but the second arises inevitably if we follow the ordinary paths of functional analysis and study dual spaces whenever they appear. The link between them is the Radon-Nikodým theorem.

I have deliberately written out 365O and 365P with different hypotheses on the homomorphism π in the hope of showing that the two routes to L^1 really are different, and can be expected to tell us different things about it. I use the letter P in 365P in order to echo the language of 242J: in the most important context, in which $\mathfrak{A}$ is actually a subalgebra of $\mathfrak{B}$ and π is the identity map, P is a kind of conditional expectation operator (365R). I note that in the proof of 365Pe I have returned to first principles, using some of the ideas of the Radon-Nikodým theorem (232E), but a different approach to the exhaustion step (converting 'for every $u>0$ there is a $v>0$ such that $Pv\le u$' into 'P is surjective'). I chose the somewhat cruder method in 232E (part (c) of the proof) in order to use the weakest possible form of the axiom of choice. In the present context such scruples seem absurd.

I used the words 'covariant' and 'contravariant' above; of course this distinction depends on the side of the mirror on which we are standing; if our measure-preserving homomorphism is derived (contravariantly) from an inverse-measure-preserving transformation, then the T's become contravariant (365Xi). An important component of this work, for me, is the fact that not all measure-preserving homomorphisms between measure algebras can be represented by inverse-measure-preserving functions (343Jb, 343M).

I have already remarked (in the notes to §244) that the properties of $L^1(\mu)$ are not much affected by peculiarities in a measure space (X, Σ, μ). In this section I offer an explanation: unlike L^0 or L^∞, L^1 really depends only on $\mathfrak{A}^f$, the ring of elements of finite measure in the measure algebra. (See 365O-365Q, 365Xm and 365Xo.) Note that while the algebra $\mathfrak{A}$ is uniquely determined (given that $(\mathfrak{A}, \bar{\mu})$ is localizable, 365Sa), the measure $\bar{\mu}$ is not; if $\mathfrak{A}$ is any algebra carrying two non-isomorphic semi-finite measures, the corresponding L^1 spaces are still isomorphic (365Sb). For instance, the L^1-spaces of Lebesgue measure μ on $\mathbb{R}$, and the subspace measure $\mu_{[0,1]}$ on $[0, 1]$, are isomorphic, though their measure algebras are not.

I make no attempt here to add to the results in §§246, 247, 354 and 356 concerning uniform integrability and weak compactness. Once we have left measure spaces behind, these ideas belong to the theory of Banach lattices, and there is little to relate them to the questions dealt with in this section. But see 373Xj and 373Xn below.

366 L^p

In this section I apply the methods of this chapter to L^p spaces, where $1 < p < \infty$. The constructions proceed without surprises up to 366E, translating the ideas of §244 by the methods used in §365. Turning to the action of Boolean homomorphisms on L^p spaces, I introduce a space M^0, which can be regarded as the part of L^0 that can be determined from the ring $\mathfrak{A}^f$ of elements of $\mathfrak{A}$ of finite measure (366F), and which includes L^p whenever $1 \le p < \infty$. Now a measure-preserving ring homomorphism from $\mathfrak{A}^f$ to $\mathfrak{B}^f$ acts on the M^0 spaces in a way which includes injective Riesz homomorphisms from $L^p(\mathfrak{A}, \bar{\mu})$ to $L^p(\mathfrak{B}, \bar{\nu})$ and surjective positive linear operators from $L^p(\mathfrak{B}, \bar{\nu})$ to $L^p(\mathfrak{A}, \bar{\mu})$ (366H). The latter may be regarded as conditional expectation operators (366J). The case $p = 2$ (366K-366L) is of course by far the most important. As with the familiar spaces $L^p(\mu)$ of Chapter 24, we have complex versions $L^p_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$ with the expected properties (366M).

366A Definition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and suppose that $1 < p < \infty$. For $u \in L^0(\mathfrak{A})$, define $|u|^p \in L^0(\mathfrak{A})$ by setting

$$\begin{aligned} [\![|u|^p > \alpha]\!] &= [\![|u| > \alpha^{1/p}]\!] \text{ if } \alpha \ge 0, \\ &= 1 \text{ if } \alpha < 0. \end{aligned}$$

(In the language of 364H, $|u|^p = \bar{h}(u)$, where $h(t) = |t|^p$ for $t \in \mathbb{R}$.) Set

$$L^p_{\bar{\mu}} = L^p(\mathfrak{A}, \bar{\mu}) = \{u : u \in L^0(\mathfrak{A}),\ |u|^p \in L^1(\mathfrak{A}, \bar{\mu})\},$$

and for $u \in L^0(\mathfrak{A})$ set

$$\|u\|_p = (\int |u|^p)^{1/p} = \||u|^p\|_1^{1/p},$$

counting $\infty^{1/p}$ as ∞, so that $L^p_{\bar{\mu}} = \{u : u \in L^0(\mathfrak{A}),\ \|u\|_p < \infty\}$.

366B Theorem Let (X, Σ, μ) be a measure space, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Then the canonical isomorphism between $L^0(\mu)$ and $L^0(\mathfrak{A})$ (364Ic) makes $L^p(\mu)$, as defined in §244, correspond to $L^p(\mathfrak{A}, \bar{\mu})$.

proof What we really have to check is that if $w \in L^0(\mu)$ corresponds to $u \in L^0(\mathfrak{A})$, then $|w|^p$, as defined in 244A, corresponds to $|u|^p$ as defined in 366A. But this was noted in 364Ib.

Now, because the isomorphism between $L^0(\mu)$ and $L^0(\mathfrak{A})$ matches $L^1(\mu)$ with $L^1_{\bar{\mu}}$ (365B), we can be sure that $|w|^p \in L^1(\mu)$ iff $|u|^p \in L^1_{\bar{\mu}}$, and that in this case

$$\|w\|_p = \left(\int |w|^p\right)^{1/p} = \left(\int |u|^p\right)^{1/p} = \|u\|_p,$$

as required.

366C Corollary For any measure algebra $(\mathfrak{A}, \bar{\mu})$ and $p \in]1, \infty[$, $L^p = L^p(\mathfrak{A}, \bar{\mu})$ is a solid linear subspace of $L^0(\mathfrak{A})$. It is a Dedekind complete Banach lattice under its uniformly convex norm $\| \|_p$. Setting $q = p/(p-1)$, $(L^p)^*$ is identified with $L^q(\mathfrak{A}, \bar{\mu})$ by the duality $(u, v) \mapsto \int u \times v$. Writing $\mathfrak{A}^f$ for the ring $\{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\}$, $S(\mathfrak{A}^f)$ is norm-dense in L^p.

proof Because we can find a measure space (X, Σ, μ) such that $(\mathfrak{A}, \bar{\mu})$ is isomorphic to the measure algebra of μ (321J), this is just a digest of the results in 244B, 244E-244H, 244K, 244L and 244O[1]. (Of course $S(\mathfrak{A}^f)$ corresponds to the space S of equivalence classes of simple functions in 244Ha, just as in 365F.)

[1] Later editions only.

366D I can add a little more, corresponding to 365C and 365M.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $p\in\,]1,\infty[$.

(a) The norm $\| \|_p$ on $L^p=L^p(\mathfrak{A},\bar\mu)$ is order-continuous.

(b) L^p has the Levi property.

(c) Setting $q=p/(p-1)$, the canonical identification of $L^q=L^q(\mathfrak{A},\bar\mu)$ with $(L^p)^*$ is a Riesz space isomorphism between L^q and $(L^p)^\sim=(L^p)^\times$.

(d) L^p is a perfect Riesz space.

proof (a) Suppose that $A\subseteq L^p$ is non-empty, downwards-directed and has infimum 0. For u, $v\ge 0$ in L^p, $u\le v\Rightarrow u^p\le v^p$ (by the definition in 366A, or otherwise), so $B=\{u^p:u\in A\}$ is downwards-directed. If $v_0=\inf B$ in $L^1=L^1(\mathfrak{A},\bar\mu)$, then $v_0^{1/p}$ (defined by the formula in 366A, or otherwise) is less than or equal to every member of A, so must be 0, and $v_0=0$. Accordingly $\inf B=0$ in L^1. Because $\| \|_1$ is order-continuous (365C),

$$\inf_{u\in A}\|u\|_p=\inf_{u\in A}\|u^p\|_1^{1/p}=(\inf_{v\in B}\|v\|_1)^{1/p}=0.$$

As A is arbitrary, $\| \|_p$ is order-continuous.

(b) Now suppose that $A\subseteq(L^p)^+$ is non-empty, upwards-directed and norm-bounded. Then $B=\{u^p:u\in A\}$ is non-empty, upwards-directed and norm-bounded in L^1. So $v_0=\sup B$ is defined in L^1, and $v_0^{1/p}$ is an upper bound for A in L^p.

(c) By 356Dd, $(L^p)^*=(L^p)^\sim=(L^p)^\times$. The extra information we need is that the identification of L^q with $(L^p)^*$ is an order-isomorphism. **P** (α) If $w\in(L^q)^+$ and $u\in(L^p)^+$ then $u\times w\ge 0$ in L^1, so $(Tw)(u)=\int u\times w\ge 0$, writing $T:L^q\to(L^p)^*$ for the canonical bijection. As u is arbitrary, $Tw\ge 0$. As w is arbitrary, T is a positive linear operator. (β) If $w\in L^q$ and $Tw\ge 0$, consider $u=(w^-)^{q/p}$. Then $u\ge 0$ in L^p and $w^+\times u=0$ (because $[\![w^+>0]\!]\cap[\![u>0]\!]=[\![w^+>0]\!]\cap[\![w^->0]\!]=0$), so

$$0\le(Tw)(u)=\int w\times u=-\int w^-\times u=-\int(w^-)^q\le 0,$$

and $\int(w^-)^q=0$. But as $(w^-)^q\ge 0$ in L^1, this means that $(w^-)^q$ and w^- must be 0, that is, $w\ge 0$. As w is arbitrary, T^{-1} is positive and T is an order-isomorphism. **Q**

(d) This is an immediate consequence of (c), since $p=q/(q-1)$, so that L^p can be identified with $(L^q)^*=(L^q)^\times$. From 356M we see that it is also a consequence of (a) and (b).

366E Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and $p\in[1,\infty]$. Set $q=p/(p-1)$ if $1<p<\infty$, $q=\infty$ if $p=1$ and $q=1$ if $p=\infty$. Then

$$L^q(\mathfrak{A},\bar\mu)=\{u:u\in L^0(\mathfrak{A}),\,u\times v\in L^1(\mathfrak{A},\bar\mu)\text{ for every }v\in L^p(\mathfrak{A},\bar\mu)\}.$$

proof (a) We already know that if $u\in L^p=L^p(\mathfrak{A},\bar\mu)$ and $v\in L^q=L^q(\mathfrak{A},\bar\mu)$ then $u\times v\in L^1=L^1(\mathfrak{A},\bar\mu)$; this is elementary if $p\in\{1,\infty\}$ and otherwise is covered by 366C.

(b) So suppose that $u\in L^0\setminus L^p$. If $p=1$ then of course $\chi 1\in L^\infty=L^q$ and $u\times\chi 1\notin L^1$. If $p>1$ set

$$A=\{w:w\in S(\mathfrak{A}^f),\,0\le w\le|u|\}.$$

Because $\bar\mu$ is semi-finite, $S(\mathfrak{A}^f)$ is order-dense in L^0 (364K), and $|u|=\sup A$. Because the norm on L^p has the Levi property (365C, 366Db, 363Ba) and A is not bounded above in L^p, $\sup_{w\in A}\|w\|_p=\infty$.

For each $n\in\mathbb{N}$ choose $w_n\in A$ with $\|w_n\|_p>4^n$. Then there is a $v_n\in L^q$ such that $\|v_n\|_q=1$ and $\int w_n\times v_n\ge 4^n$. **P** (α) If $p<\infty$ this is covered by 366C, since $\|w_n\|_p=\sup\{\int w_n\times v:\|v\|_q\le 1\}$. (β) If $p=\infty$ then $[\![w_n>4^n]\!]\ne 0$; because $\bar\mu$ is semi-finite, there is a $b\subseteq[\![w_n>4^n]\!]$ such that $0<\bar\mu b<\infty$, and $\|\frac{1}{\bar\mu b}\chi b\|_1=1$, while $\int w_n\times\frac{1}{\bar\mu b}\chi b\ge 4^n$. **Q**

Because L^q is complete (363Ba, 366C), $v=\sum_{n=0}^\infty 2^{-n}|v_n|$ is defined in L^q. But now

$$\int|u|\times v\ge 2^{-n}\int w_n\times v_n\ge 2^n$$

for every n, so $u\times v\notin L^1$.

Remark This result is characteristic of perfect subspaces of L^0; see 369C and 369J.

366F The next step is to look at the action of Boolean homomorphisms, as in 365O. It will be convenient to be able to deal with all L^p spaces at once by introducing names for a pair of spaces which include all of them.

Definition Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Write

$$M^0_{\bar{\mu}} = M^0(\mathfrak{A},\bar{\mu}) = \{u : u \in L^0(\mathfrak{A}),\ \bar{\mu}[\![|u| > \alpha]\!] < \infty \text{ for every } \alpha > 0\},$$

$$M^{1,0}_{\bar{\mu}} = M^{1,0}(\mathfrak{A},\bar{\mu}) = \{u : u \in M^0_{\bar{\mu}},\ u \times \chi a \in L^1(\mathfrak{A},\bar{\mu}) \text{ whenever } \bar{\mu}a < \infty\}.$$

366G Lemma Let $(\mathfrak{A},\bar{\mu})$ be any measure algebra. Write $M^0 = M^0(\mathfrak{A},\bar{\mu})$, etc.

(a) M^0 and $M^{1,0}$ are Dedekind complete solid linear subspaces of L^0 which include L^p for every $p \in [1,\infty[$; moreover, M^0 is closed under multiplication.

(b) If $u \in M^0$ and $u \ge 0$, there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A}^f)$ such that $u = \sup_{n\in\mathbb{N}} u_n$.

(c) $M^{1,0} = \{u : u \in L^0,\ (|u| - \epsilon\chi 1)^+ \in L^1 \text{ for every } \epsilon > 0\} = L^1 + (L^\infty \cap M^0)$.

(d) If u, $v \in M^{1,0}$ and $\int_a u \le \int_a v$ whenever $\bar{\mu}a < \infty$, then $u \le v$; so if $\int_a u = \int_a v$ whenever $\bar{\mu}a < \infty$, $u = v$.

proof (a) If u, $v \in M^0$ and $\gamma \in \mathbb{R}$, then for any $\alpha > 0$

$$[\![|u+v| > \alpha]\!] \subseteq [\![|u| > \tfrac{1}{2}\alpha]\!] \cup [\![|v| > \tfrac{1}{2}\alpha]\!],$$

$$[\![|\gamma u| > \alpha]\!] \subseteq [\![|u| > \tfrac{\alpha}{1+|\gamma|}]\!],$$

$$[\![|u \times v| > \alpha]\!] \subseteq [\![|u| > \sqrt{\alpha}\,]\!] \cup [\![|v| > \sqrt{\alpha}\,]\!]$$

(364E) are of finite measure. So $u+v$, γu and $u \times v$ belong to M^0. Thus M^0 is a linear subspace of L^0 closed under multiplication. If $u \in M^0$, $|v| \le |u|$ and $\alpha > 0$, then $[\![|v| > \alpha]\!] \subseteq [\![|u| > \alpha]\!]$ has finite measure; thus $v \in M^0$ and M^0 is a solid linear subspace of L^0. It follows that $M^{1,0}$ also is. If $u \in L^p = L^p(\mathfrak{A},\bar{\mu})$, where $p < \infty$, and $\alpha > 0$, then $[\![|u| > \alpha]\!] = [\![|u|^p > \alpha^p]\!]$ has finite measure, so $u \in M^0$; moreover, if $\bar{\mu}a < \infty$, then $\chi a \in L^q$, where $q = p/(p-1)$, so $u \times \chi a \in L^1$; thus $u \in M^{1,0}$.

To see that M^0 is Dedekind complete, observe that if $A \subseteq (M^0)^+$ is non-empty and bounded above by $u_0 \in M^0$, and $\alpha > 0$, then $\{[\![u > \alpha]\!] : u \in A\}$ is bounded above by $[\![u_0 > \alpha]\!] \in \mathfrak{A}^f$, so has a supremum in $\mathfrak{A}$ (321C). Accordingly $\sup A$ is defined in L^0 (364L(a-iii)) and belongs to M^0. Finally, $M^{1,0}$, being a solid linear subspace of M^0, must also be Dedekind complete.

(b) If $u \ge 0$ in M^0, then there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S = S(\mathfrak{A})$ such that $u = \sup_{n\in\mathbb{N}} u_n$ and $u_0 \ge 0$ (364Jd). But now every u_n belongs to $S \cap M^0 = S(\mathfrak{A}^f)$, just as in 365F.

(c)(i) If $u \in M^{1,0}$ and $\epsilon > 0$, then $a = [\![|u| > \epsilon]\!] \in \mathfrak{A}^f$, so $u \times \chi a \in L^1 = L^1(\mathfrak{A},\bar{\mu})$; but $(|u| - \epsilon\chi 1)^+ \le |u| \times \chi a$, so $(|u| - \epsilon\chi 1)^+ \in L^1$.

(ii) Suppose that $u \in L^0$ and $(|u| - \epsilon\chi 1)^+ \in L^1$ for every $\epsilon > 0$. Then, given $\epsilon > 0$, $v = (|u| - \frac{1}{2}\epsilon\chi 1)^+ \in L^1$, and $\bar{\mu}[\![v > \frac{1}{2}\epsilon]\!] < \infty$; but $[\![|u| > \epsilon]\!] \subseteq [\![v > \frac{1}{2}\epsilon]\!]$, so also has finite measure. Thus $u \in M^0$. Next, if $a \in \mathfrak{A}^f$, then $|u \times \chi a| \le \chi a + (|u| - \chi 1)^+ \in L^1$, so $u \in M^{1,0}$.

(iii) Of course L^1 and $L^\infty \cap M^0$ are included in $M^{1,0}$, so their linear sum also is. On the other hand, if $u \in M^{1,0}$, then

$$u = (u^+ - \chi 1)^+ - (u^- - \chi 1)^+ + (u^+ \wedge \chi 1) - (u^- \wedge \chi 1) \in L^1 + (L^\infty \cap M^0).$$

(d) Take $\alpha > 0$ and set $a = [\![u - v > \alpha]\!]$. Because both u and v belong to $M^{1,0}_{\bar{\mu}}$, $\bar{\mu}a < \infty$ and $\int_a u \le \int_a v$, that is, $\int_a u - v \le 0$; so a must be 0 (365Dc). As α is arbitrary, $u - v \le 0$ and $u \le v$. If $\int_a u = \int_a v$ for every $a \in \mathfrak{A}^f$, then $v \le u$ so $u = v$.

366H Theorem Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras. Let $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ be a measure-preserving ring homomorphism.

(a)(i) We have a unique order-continuous Riesz homomorphism $T = T_\pi : M^0(\mathfrak{A},\bar{\mu}) \to M^0(\mathfrak{B},\bar{\nu})$ such that $T(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}^f$.

(ii) $[\![Tu > \alpha]\!] = \pi[\![u > \alpha]\!]$ for every $u \in M^0(\mathfrak{A},\bar{\mu})$ and $\alpha > 0$.

(iii) T is injective and multiplicative.

(iv) For $p \in [1,\infty]$ and $u \in M^0(\mathfrak{A},\bar{\mu})$, $\|Tu\|_p = \|u\|_p$; in particular, $Tu \in L^p(\mathfrak{B},\bar{\nu})$ iff $u \in L^p(\mathfrak{A},\bar{\mu})$. Consequently $\int Tu = \int u$ whenever $u \in L^1(\mathfrak{A},\bar{\mu})$.

(v) For $u \in M^0(\mathfrak{A},\bar{\mu})$, $Tu \in M^{1,0}(\mathfrak{B},\bar{\nu})$ iff $u \in M^{1,0}(\mathfrak{A},\bar{\mu})$.

(b)(i) We have a unique order-continuous positive linear operator $P = P_\pi : M^{1,0}(\mathfrak{B}, \bar{\nu}) \to M^{1,0}(\mathfrak{A}, \bar{\mu})$ such that $\int_a Pv = \int_{\pi a} v$ whenever $v \in M^{1,0}(\mathfrak{B}, \bar{\nu})$ and $a \in \mathfrak{A}^f$.

(ii) If $u \in M^0(\mathfrak{A}, \bar{\mu})$, $v \in M^{1,0}(\mathfrak{B}, \bar{\nu})$ and $v \times Tu \in M^{1,0}(\mathfrak{B}, \bar{\nu})$, then $P(v \times Tu) = u \times Pv$.

(iii) If $q \in [1, \infty[$ and $v \in L^q(\mathfrak{B}, \bar{\nu})$, then $Pv \in L^q(\mathfrak{A}, \bar{\mu})$ and $\|Pv\|_q \le \|v\|_q$; if $v \in L^\infty(\mathfrak{B}) \cap M^0(\mathfrak{B}, \bar{\nu})$, then $Pv \in L^\infty(\mathfrak{A})$ and $\|Pv\|_\infty \le \|v\|_\infty$.

(iv) $PTu = u$ for every $u \in M^{1,0}(\mathfrak{A}, \bar{\mu})$; in particular, $P[L^p(\mathfrak{B}, \bar{\nu})] = L^p(\mathfrak{A}, \bar{\mu})$ for every $p \in [1, \infty[$.

(c) If $(\mathfrak{C}, \lambda)$ is another measure algebra and $\theta : \mathfrak{B}^f \to \mathfrak{C}^f$ another measure-preserving ring homomorphism, then $T_{\theta\pi} = T_\theta T_\pi : M^0(\mathfrak{A}, \bar{\mu}) \to M^0(\mathfrak{C}, \bar{\lambda})$ and $P_{\theta\pi} = P_\pi P_\theta : M^{1,0}(\mathfrak{C}, \bar{\lambda}) \to M^{1,0}(\mathfrak{A}, \bar{\mu})$.

(d) Now suppose that $\pi[\mathfrak{A}^f] = \mathfrak{B}^f$, so that π is a measure-preserving isomorphism between the rings $\mathfrak{A}^f$ and $\mathfrak{B}^f$.

(i) T is a Riesz space isomorphism between $M^0(\mathfrak{A}, \bar{\mu})$ and $M^0(\mathfrak{B}, \bar{\nu})$, and its inverse is $T_{\pi^{-1}}$.

(ii) P is a Riesz space isomorphism between $M^{1,0}(\mathfrak{B}, \bar{\nu})$ and $M^{1,0}(\mathfrak{A}, \bar{\mu})$, and its inverse is $P_{\pi^{-1}}$.

(iii) The restriction of T to $M^{1,0}(\mathfrak{A}, \bar{\mu})$ is $P^{-1} = P_{\pi^{-1}}$; the restriction of $T^{-1} = T_{\pi^{-1}}$ to $M^{1,0}(\mathfrak{B}, \bar{\nu})$ is P.

(iv) For any $p \in [1, \infty[$, $T \restriction L^p(\mathfrak{A}, \bar{\mu}) = P_{\pi^{-1}} \restriction L^p(\mathfrak{A}, \bar{\mu})$ and $P \restriction L^p(\mathfrak{B}, \bar{\nu}) = T_{\pi^{-1}} \restriction L^p(\mathfrak{B}, \bar{\nu})$ are the two halves of a Banach lattice isomorphism between $L^p(\mathfrak{A}, \bar{\mu})$ and $L^p(\mathfrak{B}, \bar{\nu})$.

proof (a)(i) By 361J, π induces a multiplicative Riesz homomorphism $T_0 : S(\mathfrak{A}^f) \to S(\mathfrak{B}^f)$ which is order-continuous because π is (361Ad, 361Je). If $u \in S(\mathfrak{A}^f)$ and $\alpha > 0$, then $[\![T_0 u > \alpha]\!] = \pi[\![u > \alpha]\!]$. **P** Express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n$ are disjoint in $\mathfrak{A}^f$; then $T_0 u = \sum_{i=0}^n \alpha_i \chi(\pi a_i)$, so

$$[\![T_0 u > \alpha]\!] = \sup\{\pi a_i : i \le n,\, \alpha_i > \alpha\} = \pi(\sup\{a_i : i \le n,\, \alpha_i > \alpha\}) = \pi[\![u > \alpha]\!]. \text{ } \mathbf{Q}$$

Now if $u_0 \ge 0$ in $M^0_{\bar{\mu}}$, $\sup\{T_0 u : u \in S(\mathfrak{A}^f),\, 0 \le u \le u_0\}$ is defined in $M^0_{\bar{\nu}}$. **P** Set $A = \{u : u \in S(\mathfrak{A}^f),\, 0 \le u \le u_0\}$. Because $u_0 = \sup A$ (366Gb),

$$\sup_{u \in A} [\![Tu > \alpha]\!] = \sup_{u \in A} \pi[\![u > \alpha]\!] = \pi(\sup_{u \in A} [\![u > \alpha]\!]) = \pi[\![u_0 > \alpha]\!]$$

is defined and belongs to $\mathfrak{B}^f$ for any $\alpha > 0$. Also

$$\inf_{n \ge 1} \sup_{u \in A} [\![Tu > n]\!] = \pi(\inf_{n \ge 1} [\![u_0 > n]\!]) = 0.$$

By 364L(a-ii), $v_0 = \sup T_0[A]$ is defined in $L^0(\mathfrak{B})$, and $[\![v_0 > \alpha]\!] = \pi[\![u_0 > \alpha]\!] \in \mathfrak{B}^f$ for every $\alpha > 0$, so $v_0 \in M^0_{\bar{\nu}}$, as required. **Q**

Consequently T_0 has a unique extension to an order-continuous Riesz homomorphism $T : M^0_{\bar{\mu}} \to M^0_{\bar{\nu}}$ (355F).

(ii) If $u_0 \in M^0_{\bar{\mu}}$ and $\alpha > 0$, then

$$[\![Tu_0 > \alpha]\!] = [\![Tu_0^+ > \alpha]\!]$$

(because T is a Riesz homomorphism)

$$= \sup_{u \in S(\mathfrak{A}^f), 0 \le u \le u_0^+} [\![Tu > \alpha]\!]$$

(because T is order-continuous and $S(\mathfrak{A}^f)$ is order-dense in $M^0_{\bar{\mu}}$)

$$= \pi[\![u_0 > \alpha]\!]$$

by the argument used in (i).

(iii) I have already remarked, at the beginning of the proof of (i), that $T(u \times u') = Tu \times Tu'$ for u, $u' \in S(\mathfrak{A}^f)$. Because both T and $\times$ are order-continuous and $S(\mathfrak{A}^f)$ is order-dense in $M^0_{\bar{\mu}}$,

$$\begin{aligned} T(u_0 \times u_1) &= \sup\{T(u \times u') : u,\, u' \in S(\mathfrak{A}^f),\, 0 \le u \le u_0,\, 0 \le u' \le u_1\} \\ &= \sup_{u, u'} Tu \times Tu' = Tu_0 \times Tu_1 \end{aligned}$$

whenever u_0, $u_1 \ge 0$ in $M^0_{\bar{\mu}}$. Because T is linear and $\times$ is bilinear, it follows that T is multiplicative on $M^0_{\bar{\mu}}$.

To see that it is injective, observe that if $u \ne 0$ in $M^0_{\bar{\mu}}$ then there is some $\alpha > 0$ such that $a = [\![|u| > \alpha]\!] \ne 0$, so that $0 < \alpha \chi \pi a \le T|u| = |Tu|$ and $Tu \ne 0$.

(iv)($\boldsymbol{\alpha}$) For any $\alpha > 0$,

$$[\![|Tu|^p > \alpha]\!] = [\![T|u| > \alpha^{1/p}]\!] = \pi[\![|u| > \alpha^{1/p}]\!] = \pi[\![|u|^p > \alpha]\!].$$

So

$$\||Tu|^p\|_1 = \int_0^\infty \bar{\nu}[\![|Tu|^p > \alpha]\!]\,d\alpha = \int_0^\infty \bar{\mu}[\![|u|^p > \alpha]\!]\,d\alpha = \||u|^p\|_1.$$

If $p < \infty$ then, taking pth roots, $\|Tu\|_p = \|u\|_p$.

(β) As for the case $p = \infty$, if $u \in L^\infty(\mathfrak{A})$ and $\gamma = \|u\|_\infty > 0$ then $[\![|u| > \gamma]\!] = 0$, so $[\![|Tu| > \gamma]\!] = \pi[\![|u| > \gamma]\!] = 0$. This shows that $\|Tu\|_\infty \le \gamma$. On the other hand, if $0 < \alpha < \gamma$ then $a = [\![|u| > \alpha]\!] \ne 0$, and $\alpha\chi a \le |u|$ so $\alpha\chi(\pi a) \le |Tu|$; as $\pi a \ne 0$ (because $\bar{\nu}(\pi a) = \bar{\mu}a > 0$), $\|Tu\|_\infty > \alpha$. This shows that $\|Tu\|_\infty = \|u\|_\infty$, at least when $u \ne 0$; but the case $u = 0$ is trivial.

(γ) If $u \in L^1_{\bar{\mu}}$, then

$$\int Tu = \|(Tu)^+\|_1 - \|(Tu)^-\|_1 = \|Tu^+\|_1 - \|Tu^-\|_1 = \|u^+\|_1 - \|u^-\|_1 = \int u.$$

(v) If $u \in M^{1,0}_{\bar{\mu}}$ and $\epsilon > 0$, then $T(|u| \wedge \epsilon\chi 1_{\mathfrak{A}}) = |Tu| \wedge \epsilon\chi 1_{\mathfrak{B}}$. **P** Set $a = [\![|u| > \epsilon]\!] \in \mathfrak{A}^f$. Then $|u| \wedge \epsilon\chi 1_{\mathfrak{A}} = \epsilon\chi a + |u| - |u| \times \chi a$ and $[\![|Tu| > \epsilon]\!] = \pi a$. So

$$\begin{aligned} T(|u| \wedge \epsilon\chi 1_{\mathfrak{A}}) &= T(\epsilon\chi a) + T|u| - T(|u| \times \chi a) \\ &= \epsilon\chi(\pi a) + |Tu| - |Tu| \times \chi(\pi a) = |Tu| \wedge \epsilon\chi 1_{\mathfrak{B}}. \textbf{ Q} \end{aligned}$$

Consequently

$$T(|u| - \epsilon\chi 1_{\mathfrak{A}})^+ = T(|u| - |u| \wedge \epsilon\chi 1_{\mathfrak{A}}) = (|Tu| - \epsilon\chi 1_{\mathfrak{B}})^+.$$

But this means that $(|u| - \epsilon\chi 1_{\mathfrak{A}})^+ \in L^1_{\bar{\mu}}$ iff $(|Tu| - \epsilon\chi 1_{\mathfrak{B}})^+ \in L^1_{\bar{\nu}}$. Since this is true for every $\epsilon > 0$, 366Gc tells us that $u \in M^{1,0}_{\bar{\mu}}$ iff $Tu \in M^{1,0}_{\bar{\nu}}$.

(b)(i)(α) By 365Pa, we have an order-continuous positive linear operator $P_0 : L^1_{\bar{\nu}} \to L^1_{\bar{\mu}}$ such that $\int_a P_0 v = \int_{\pi a} v$ for every $v \in L^1_{\bar{\nu}}$ and $a \in \mathfrak{A}^f$.

(β) We now find that if $v_0 \ge 0$ in $M^{1,0}_{\bar{\nu}}$ and $B = \{v : v \in L^1_{\bar{\nu}}, 0 \le v \le v_0\}$, then $P_0[B]$ has a supremum in $L^0(\mathfrak{A})$ which belongs to $M^{1,0}_{\bar{\mu}}$. **P** Because B is upwards-directed and P_0 is order-preserving, $P_0[B]$ is upwards-directed. If $\alpha > 0$ and $v \in B$ and $a = [\![P_0 v > \alpha]\!]$, then

$$v \le (v_0 - \tfrac{\alpha}{2}\chi 1_{\mathfrak{B}})^+ + \tfrac{\alpha}{2}\chi 1_{\mathfrak{B}},$$

so

$$\begin{aligned} \alpha\bar{\mu}a \le \int_a P_0 v = \int_{\pi a} v &\le \int (v_0 - \frac{\alpha}{2}\chi 1_{\mathfrak{B}})^+ + \frac{\alpha}{2}\bar{\nu}(\pi a) \\ &= \int (v_0 - \frac{\alpha}{2}\chi 1_{\mathfrak{B}})^+ + \frac{\alpha}{2}\bar{\mu}a \end{aligned}$$

and

$$\bar{\mu}[\![P_0 v > \alpha]\!] \le \frac{2}{\alpha}\int (v_0 - \tfrac{\alpha}{2}\chi 1_{\mathfrak{B}})^+.$$

Thus $\{[\![P_0 v > \alpha]\!] : v \in B\}$ is an upwards-directed set in $\mathfrak{A}^f$ with measures bounded above in $\mathbb{R}$, and

$$c_\alpha = \sup_{v \in B} [\![P_0 v > \alpha]\!]$$

is defined in $\mathfrak{A}^f$. Also

$$\inf_{n \ge 1} \bar{\mu}c_n \le \inf_{n \ge 1} \frac{2}{n}\int (v_0 - \tfrac{n}{2}\chi 1_{\mathfrak{B}})^+ = 0.$$

So $\inf_{n \in \mathbb{N}} c_n = 0$ and $P_0[B]$ has a supremum $u_0 \in L^0(\mathfrak{A})$ (364L(a-ii)). As $[\![u_0 > \alpha]\!] = c_\alpha \in \mathfrak{A}^f$ for every $\alpha > 0$, $u_0 \in M^0_{\bar{\mu}}$. If $c \in \mathfrak{A}^f$, then

$$\int_c u_0 = \sup_{v \in B} \int_c P_0 v = \sup_{v \in B} \int_{\pi c} v \le \int_{\pi c} v_0 < \infty,$$

so $u_0 \in M^{1,0}_{\bar{\mu}}$. **Q**

(γ) Now 355F tells us that P_0 has a unique extension to an order-continuous positive linear operator $P : M^{1,0}_{\bar{\nu}} \to M^{1,0}_{\bar{\mu}}$. If $v_0 \ge 0$ in $M^{1,0}_{\bar{\nu}}$ and $a \in \mathfrak{A}^f$, then, as remarked above,

$$\int_a Pv_0 = \sup\{\int_a P_0 v : v \in L^1_{\bar\nu},\, 0 \le v \le v_0\}$$
$$= \sup\{\int_{\pi a} v : v \in L^1_{\bar\nu},\, 0 \le v \le v_0\} = \int_{\pi a} v_0;$$

because P is linear, $\int_a Pv = \int_{\pi a} v$ for every $v \in M^{1,0}_{\bar\nu}$, $a \in \mathfrak{A}^f$.

(δ) By 366Gd, P is uniquely defined by the formula

$$\int_a Pv = \int_{\pi a} v \text{ whenever } v \in M^{1,0}_{\bar\nu} \text{ and } a \in \mathfrak{A}^f.$$

(ii) Because $M^0_{\bar\mu}$ is closed under multiplication, $u \times Pv$ certainly belongs to $M^0_{\bar\mu}$.

(α) Suppose that u, $v \ge 0$. Fix $c \in \mathfrak{A}^f$ for the moment. Suppose that $u' \in S(\mathfrak{A}^f)$. Then we can express u' as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_i \in \mathfrak{A}^f$ for every $i \le n$. Accordingly

$$\int_c u' \times Pv = \sum_{i=0}^n \alpha_i \int_{c\cap a_i} Pv = \sum_{i=0}^n \alpha_i \int v \times \chi(\pi a_i) \times \chi(\pi c) = \int_{\pi c} v \times Tu'.$$

Next, we can find a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A}^f)^+$ with supremum u, and

$$\sup_{n\in\mathbb{N}} \int_c u_n \times Pv = \sup_{n\in\mathbb{N}} \int_{\pi c} v \times Tu_n = \int_{\pi c} \sup_{n\in\mathbb{N}} v \times Tu_n$$
$$= \int_{\pi c} v \times \sup_{n\in\mathbb{N}} Tu_n = \int_{\pi c} v \times Tu,$$

using the order-continuity of T, $\int$ and $\times$. But this means that $u \times Pv = \sup_{n\in\mathbb{N}} u_n \times Pv$ is integrable over c and that $\int_c u \times Pv = \int_{\pi c} v \times Tu$. As c is arbitrary, $u \times Pv = P(v \times Tu) \in M^{1,0}_{\bar\mu}$.

(β) For general u, v,

$$v^+ \times Tu^+ + v^+ \times Tu^- + v^- \times Tu^+ + v^- \times Tu^- = |v| \times T|u| = |v \times Tu| \in M^{1,0}_{\bar\nu}$$

(because T is a Riesz homomorphism), so we may apply (α) to each of the four products; combining them, we get $P(v \times Tu) = u \times Pv$, as required.

(iii) Because P is a positive operator, we surely have $|Pv| \le P|v|$, so it will be enough to show that $\|Pv\|_q \le \|v\|_q$ for $v \ge 0$ in $L^q_{\bar\nu}$.

(α) I take the case $q = 1$ first. In this case, for any $a \in \mathfrak{A}^f$, we have $\int_a Pv = \int_{\pi a} v \le \|v\|_1$. In particular, setting $a_n = [\![Pv > 2^{-n}]\!]$, $\int_{a_n} Pv \le \|v\|_1$. But $Pv = \sup_{n\in\mathbb{N}} Pv \times \chi a_n$, so

$$\|Pv\|_1 = \sup_{n\in\mathbb{N}} \int_{a_n} Pv \le \|v\|_1.$$

(β) Next, suppose that $q = \infty$, so that $v \in L^\infty(\mathfrak{B})^+$; say $\|v\|_\infty = \gamma$. **?** If $\gamma > 0$ and $a = [\![Pv > \gamma]\!] \ne 0$, then

$$\gamma\bar\mu a < \int_a Pv = \int_{\pi a} v \le \gamma\bar\nu(\pi a) = \gamma\bar\mu a. \text{ ✗}$$

So $[\![Pv > \gamma]\!] = 0$ and $Pv \in L^\infty(\mathfrak{A})$, with $\|Pv\|_\infty \le \|v\|_\infty$, at least when $\|v\|_\infty > 0$; but the case $\|v\|_\infty = 0$ is trivial.

(γ) I come at last to the 'general' case $q \in \,]1,\infty[$, $v \in L^q_{\bar\nu}$. In this case set $p = q/(q-1)$. If $u \in L^p_{\bar\mu}$ then $Tu \in L^p_{\bar\nu}$ so $Tu \times v \in L^1_{\bar\nu}$ and

$$|\int u \times Pv| \le \|u \times Pv\|_1 = \|P(Tu \times v)\|_1$$

(by (ii))

$$\le \|Tu \times v\|_1$$

(by (α) just above)

$$= \int |Tu| \times |v| \le \|Tu\|_p \|v\|_q = \|u\|_p \|v\|_q$$

by (a-iii) of this theorem. But this means that $u \mapsto \int u \times Pv$ is a bounded linear functional on $L^p_{\bar\mu}$, and is therefore represented by some $w \in L^q_{\bar\mu}$ with $\|w\|_q \le \|v\|_q$. If $a \in \mathfrak{A}^f$ then $\chi a \in L^p_{\bar\mu}$, so $\int_a w = \int_a Pv$; accordingly Pv is actually equal to w (by 366Gd) and $\|Pv\|_q = \|w\|_q \le \|v\|_q$, as claimed.

(iv) If $u \in M^{1,0}_{\bar\mu}$ and $a \in \mathfrak{A}^f$, we must have

$$\int_a PTu = \int_{\pi a} Tu = \int T(\chi a) \times Tu = \int T(\chi a \times u) = \int \chi a \times u = \int_a u,$$

using (a-iv) to see that $\int \chi a \times u$ is defined and equal to $\int T(\chi a \times u)$. As a is arbitrary, $u \in M^{1,0}_{\bar\mu}$ and $PTu = u$.

(c) As usual, in view of the uniqueness of $T_{\theta\pi}$ and $P_{\theta\pi}$, all we have to check is that

$$T_\theta T(\chi a) = T_\theta \chi(\pi a) = \chi(\theta\pi a) = T_{\theta\pi}(\chi a),$$

$$\int_a PP_\theta w = \int_{\pi a} P_\theta w = \int_{\theta\pi a} w = \int_a P_{\theta\pi} w$$

whenever $a \in \mathfrak{A}^f$ and $w \in M^{1,0}_{\bar\lambda}$.

(d)(i) By (c), $T_{\pi^{-1}}T = T_{\pi^{-1}\pi}$ must be the identity operator on $M^0_{\bar\mu}$; similarly, $TT_{\pi^{-1}}$ is the identity operator on $M^0_{\bar\nu}$. Because T and $T_{\pi^{-1}}$ are Riesz homomorphisms, they must be the two halves of a Riesz space isomorphism.

(ii) In the same way, P and $P_{\pi^{-1}}$ must be the two halves of an ordered linear space isomorphism between $M^{1,0}_{\bar\mu}$ and $M^{1,0}_{\bar\nu}$, and are therefore both Riesz homomorphisms.

(iii) By (b-iv), $PTu = u$ for every $u \in M^{1,0}_{\bar\mu}$, so $T{\restriction}M^{1,0}_{\bar\mu}$ must be P^{-1}. Similarly $P = P^{-1}_{\pi^{-1}}$ is the restriction of $T^{-1} = T_{\pi^{-1}}$ to $M^{1,0}_{\bar\nu}$.

(iv) Because $T^{-1}[L^p_{\bar\nu}] = L^p_{\bar\mu}$ (by (a-iv)), and T is a bijection between $M^0_{\bar\mu}$ and $M^0_{\bar\nu}$, $T{\restriction}L^p_{\bar\mu}$ must be a Riesz space isomorphism between $L^p_{\bar\mu}$ and $L^p_{\bar\nu}$; (a-iv) also tells us that it is norm-preserving. Now its inverse is $P{\restriction}L^p_{\bar\nu}$, by (iii) here.

366I Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Then, for any $p \in [1, \infty[$, $L^p(\mathfrak{B}, \bar\mu{\restriction}\mathfrak{B})$ can be identified, as Banach lattice, with the closed linear subspace of $L^p(\mathfrak{A}, \bar\mu)$ generated by $\{\chi b : b \in \mathfrak{B},\ \bar\mu b < \infty\}$.

proof The identity map $b \mapsto b : \mathfrak{B} \to \mathfrak{A}$ induces an injective Riesz homomorphism $T : L^0(\mathfrak{B}) \to L^0(\mathfrak{A})$ (364P) such that $Tu \in L^p_{\mathfrak{A}} = L^p(\mathfrak{A}, \bar\mu)$ and $\|Tu\|_p = \|u\|_p$ whenever $p \in [1, \infty[$ and $u \in L^p_{\mathfrak{B}} = L^p(\mathfrak{B}, \bar\mu{\restriction}\mathfrak{B})$ (366H(a-iv)). Because $S(\mathfrak{B}^f)$, the linear span of $\{\chi b : b \in \mathfrak{B},\ \bar\mu b < \infty\}$, is dense in $L^p_{\mathfrak{B}}$ (366C), the image of $L^p_{\mathfrak{B}}$ in $L^p_{\mathfrak{A}}$ must be the closure of the image of $S(\mathfrak{B}^f)$ in $L^p_{\mathfrak{A}}$, that is, the closed linear span of $\{\chi b : b \in \mathfrak{B}^f\}$ interpreted as a subset of $L^p_{\mathfrak{A}}$.

366J Corollary If $(\mathfrak{A}, \bar\mu)$ is a probability algebra, $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, and $P : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{B}, \bar\mu{\restriction}\mathfrak{B})$ is the conditional expectation operator (365R), then $\|Pu\|_p \le \|u\|_p$ whenever $p \in [1, \infty]$ and $u \in L^p(\mathfrak{A}, \bar\mu)$.

proof Because $(\mathfrak{A}, \bar\mu)$ is totally finite, $M^{1,0}(\mathfrak{A}, \bar\mu) = L^1_{\bar\mu}$, so that the operator P of 366Hb can be identified with the conditional expectation operator of 365R. Now 366H(b-iii) gives the result.

Remark Of course this is also covered by 244M.

366K Corollary Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras, and $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ a measure-preserving ring homomorphism. Let $T : L^2(\mathfrak{A}, \bar\mu) \to L^2(\mathfrak{B}, \bar\nu)$ and $P : L^2(\mathfrak{B}, \bar\nu) \to L^2(\mathfrak{A}, \bar\mu)$ be the corresponding operators, as in 366H. Then $TP : L^2(\mathfrak{B}, \bar\nu) \to L^2(\mathfrak{B}, \bar\nu)$ is an orthogonal projection, its range $TP[L^2(\mathfrak{B}, \bar\nu)]$ being isomorphic, as Banach lattice, to $L^2(\mathfrak{A}, \bar\mu)$. The kernel of TP is just

$$\{v : v \in L^2(\mathfrak{B}, \bar\nu),\ \int_{\pi a} v = 0 \text{ for every } a \in \mathfrak{A}^f\}.$$

proof Most of this is simply because T is a norm-preserving Riesz homomorphism (so that $T[L^2_{\bar\mu}]$ is isomorphic to $L^2_{\bar\mu}$), PT is the identity on $L^2_{\bar\mu}$ (so that $(TP)^2 = TP$) and $\|P\| \le 1$ (so that $\|TP\| \le 1$). These are enough to ensure that TP is a projection of norm at most 1, that is, an orthogonal projection. Also

$$TPv = 0 \iff Pv = 0 \iff \int_a Pv = 0 \text{ for every } a \in \mathfrak{A}^f$$
$$\iff \int_{\pi a} v = 0 \text{ for every } a \in \mathfrak{A}^f.$$

366L Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\pi:\mathfrak{A}^f\to\mathfrak{A}^f$ a measure-preserving ring automorphism. Then there is a corresponding Banach lattice isomorphism T of $L^2=L^2(\mathfrak{A},\bar\mu)$ defined by writing $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}^f$. Its inverse is defined by the formula

$$\int_a T^{-1}u=\int_{\pi a}u \text{ for every } u\in L^2,\ a\in\mathfrak{A}^f.$$

proof In the language of 366H, $T=T_\pi$ and $T^{-1}=P_\pi$.

***366M Complex L^p spaces (a)** Just as in §§241-244, we have 'complex' versions of all the spaces considered in this chapter. Using the representation theorems for Boolean algebras, we can get effective descriptions of these matching the ones in Chapter 24. Thus for any Boolean algebra $\mathfrak{A}$ with Stone space Z, we can identify $L^\infty_{\mathbb{C}}(\mathfrak{A})$ with the space $C(Z;\mathbb{C})$ of continuous functions from Z to $\mathbb{C}$; inside this, we have a $\|\;\|_\infty$-dense subspace $S_{\mathfrak{C}}(\mathfrak{A})$ consisting of complex linear combinations of indicator functions of open-and-closed sets. If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, identified with a quotient $\Sigma/\mathcal{M}$ where Σ is a σ-algebra of subsets of a set Z and $\mathcal{M}$ is a σ-ideal of Σ, then we can write $\mathcal{L}^0_{\mathbb{C}}$ for the set of functions from Z to $\mathbb{C}$ such that their real and imaginary parts are both Σ-measurable, $\mathcal{W}_{\mathbb{C}}$ for the set of those $f\in\mathcal{L}^0_{\mathbb{C}}$ such that $\{z:f(z)\ne0\}$ belongs to $\mathcal{M}$, and $L^0_{\mathbb{C}}=L^0_{\mathbb{C}}(\mathfrak{A})$ for the linear space quotient $\mathcal{L}^0_{\mathbb{C}}/\mathcal{W}_{\mathbb{C}}$. As in 241J, we find that we have a natural embedding of $L^0=L^0(\mathfrak{A})$ in $L^0_{\mathbb{C}}$ and functions

$$\mathcal{R}\mathrm{e}:L^0_{\mathbb{C}}\to L^0,\quad \mathcal{I}\mathrm{m}:L^0_{\mathbb{C}}\to L^0,\quad |\;|:L^0_{\mathbb{C}}\to L^0,\quad {}^-:L^0_{\mathbb{C}}\to L^0_{\mathbb{C}}$$

such that

$$u=\mathcal{R}\mathrm{e}(u)+i\,\mathcal{I}\mathrm{m}(u),\quad \mathcal{R}\mathrm{e}(u+v)=\mathcal{R}\mathrm{e}(u)+\mathcal{R}\mathrm{e}(v),\quad \mathcal{I}\mathrm{m}(u+v)=\mathcal{I}\mathrm{m}(u)+\mathcal{I}\mathrm{m}(v),$$

$$\mathcal{R}\mathrm{e}(\alpha u)=\mathcal{R}\mathrm{e}(\alpha)\,\mathcal{R}\mathrm{e}(u)-\mathcal{I}\mathrm{m}(\alpha)\,\mathcal{I}\mathrm{m}(u),\quad \mathcal{I}\mathrm{m}(\alpha u)=\mathcal{R}\mathrm{e}(\alpha)\,\mathcal{I}\mathrm{m}(u)+\mathcal{I}\mathrm{m}(\alpha)\,\mathcal{R}\mathrm{e}(u),$$

$$|\alpha u|=|\alpha||u|,\quad |u+v|\le|u|+|v|,\quad |u|=\sup_{|\gamma|=1}\mathcal{R}\mathrm{e}(\gamma u),$$

$$\bar u=\mathcal{R}\mathrm{e}(u)-i\,\mathcal{I}\mathrm{m}(u),\quad \overline{u+v}=\bar u+\bar v,\quad \overline{\alpha u}=\bar\alpha\bar u$$

for all u, $v\in L^0_{\mathbb{C}}$ and $\alpha\in\mathbb{C}$.

I seem to have omitted to mention it in 241J, but of course we also have a multiplication

$$u\times v=(\mathcal{R}\mathrm{e}(u)\times\mathcal{R}\mathrm{e}(v)-\mathcal{I}\mathrm{m}(u)\times\mathcal{I}\mathrm{m}(v))+i(\mathcal{R}\mathrm{e}(u)\times\mathcal{I}\mathrm{m}(v)+\mathcal{I}\mathrm{m}(u)\times\mathcal{R}\mathrm{e}(v)),$$

for which we have the expected formulae

$$u\times v=v\times u,\quad u\times(v\times w)=(u\times v)\times w,\quad u\times(v+w)=(u\times v)+(u\times w),$$

$$(\alpha u)\times v=u\times(\alpha v)=\alpha(u\times v),$$

$$\overline{u\times v}=\bar u\times\bar v,\quad |u\times v|=|u|\times|v|,\quad u\times\bar u=|u|^2=(\mathcal{R}\mathrm{e}(u))^2+(\mathcal{I}\mathrm{m}(u))^2$$

for u, $v\in L^0_{\mathbb{C}}$ and $\alpha\in\mathbb{C}$.

(b) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $1\le p<\infty$, we can think of $L^p_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ as the set of those $u\in L^0_{\mathbb{C}}$ such that $|u|\in L^p(\mathfrak{A},\bar\mu)$, with its norm defined by the formula $\|u\|_p=\||u|\|_p$; this will make $L^p_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ a Banach space (cf. 242Pb, 244Pb[2]), with dual $L^q(\mathfrak{A},\bar\mu)$ where $\frac1p+\frac1q=1$ if $p>1$ (244Pb again). (Similarly, if $(\mathfrak{A},\bar\mu)$ is localizable, the dual of $L^1_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ can be identified with $L^\infty_{\mathbb{C}}$, as in 365Mc.)

Writing $S_{\mathbb{C}}(\mathfrak{A}^f)$ for the space of linear combinations of indicator functions of elements of $\mathfrak{A}$ of finite measure, $S_{\mathbb{C}}(\mathfrak{A}^f)$ is dense in $L^p_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ whenever $1\le p<\infty$, as in 366C.

(c) Of course L^1- and L^2-spaces have special additional features, their integrals and inner products. Here we can set

$$\int u=\int\mathcal{R}\mathrm{e}(u)+i\int\mathcal{I}\mathrm{m}(u)$$

for $u\in L^1_{\mathbb{C}}(\mathfrak{A},\bar\mu)$, and $\int:L^1_{\mathbb{C}}(\mathfrak{A},\bar\mu)\to\mathbb{C}$ becomes a $\mathbb{C}$-linear functional. As for L^2, we see at once from the formulae above that

$$|u\times v|=|u|\times|v|\in L^1(\mathfrak{A},\bar\mu),\quad u\times v\in L^1_{\mathbb{C}}(\mathfrak{A},\bar\mu),\quad \int u\times\bar u=\|u\|_2^2$$

[2] Formerly 244O.

for u, $v \in L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$. So if we set

$$(u|v) = \int u \times \bar v$$

for u, $v \in L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$, $L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ becomes a complex Hilbert space.

(d) In the language of the present chapter we have something else to look at. If $\mathfrak{A}$, $\mathfrak{B}$ are Dedekind σ-complete Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a sequentially order-continuous Boolean homomorphism, then we have a linear operator $T_\pi : L^0_{\mathbb{C}}(\mathfrak{A}) \to L^0_{\mathbb{C}}(\mathfrak{B})$ defined by setting $T_\pi u = T^{\mathrm{real}}_\pi(\mathcal{R}\mathrm{e}(u)) + iT^{\mathrm{real}}_\pi(\mathcal{I}\mathrm{m}(u))$, where $T^{\mathrm{real}}_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ is the Riesz homomorphism described in 364P. Of course T_π, like T^{real}_π, will be multiplicative; hence, or otherwise, $T_\pi|u| = |T_\pi u|$ for every $u \in L^0_{\mathbb{C}}(\mathfrak{A})$. Observe that $T_\pi\bar u = \overline{T_\pi u}$ for every $u \in L^0_{\mathbb{C}}(\mathfrak{A})$. Also, as in 364Pe, if $\mathfrak{C}$ is another Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\phi : \mathfrak{B} \to \mathfrak{C}$ are sequentially order-continuous Boolean homomorphisms, $T_{\phi\pi} = T_\phi T_\pi$. So if $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean automorphism, T_π will be a bijection with inverse $T_{\pi^{-1}}$.

(e) Similarly, if $(\mathfrak{A},\bar\mu)$ is a measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism, $\int T_\pi u = \int u$ for every $u \in L^1_{\mathbb{C}}(\mathfrak{A},\bar\mu)$. If u, $v \in L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$, then

$$(T_\pi u|T_\pi v) = \int T_\pi u \times \overline{T_\pi v} = \int T_\pi u \times T_\pi \bar v = \int T_\pi(u \times \bar v) = \int u \times \bar v = (u|v).$$

If π is actually a measure-preserving Boolean automorphism, we shall have

$$(T_\pi u|v) = (T_{\pi^{-1}}T_\pi u|T_{\pi^{-1}}v) = (u|T^{-1}_\pi v)$$

for all u, $v \in L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$.

366X Basic exercises (a) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $p \in \,]1,\infty[$. Show that $\|u\|_p^p = p\int_0^\infty \alpha^{p-1}\bar\mu[\![|u| > \alpha]\!]\,d\alpha$ for every $u \in L^0(\mathfrak{A})$. (Cf. 263Xa.)

>(b) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra and $p \in [1,\infty]$. Show that the band algebra of $L^p_{\bar\mu}$ is isomorphic to $\mathfrak{A}$. (Cf. 365S.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $p \in \,]1,\infty[$. Show that $L^p_{\bar\mu}$ is separable iff $L^1_{\bar\mu}$ is.

(d) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. (i) Show that $L^\infty(\mathfrak{A}) \cap M^0_{\bar\mu}$ and $L^\infty(\mathfrak{A}) \cap M^{1,0}_{\bar\mu}$, as defined in 366F, are equal. (ii) Call this intersection $M^{\infty,0}_{\bar\mu}$. Show that it is a norm-closed solid linear subspace of $L^\infty(\mathfrak{A})$, therefore a Banach lattice in its own right.

(e) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra and $(\widehat{\mathfrak{A}},\hat\mu)$ its localization (322Q). Show that the natural embedding of $\mathfrak{A}$ in $\widehat{\mathfrak{A}}$ induces a Banach lattice isomorphism between $L^p_{\bar\mu}$ and $L^p_{\hat\mu}$ for every $p \in [1,\infty[$, so that the band algebra of $L^p_{\bar\mu}$ can be identified with $\widehat{\mathfrak{A}}$.

(f) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra which is not localizable (cf. 211Ye, 216D), and $(\widehat{\mathfrak{A}},\hat\mu)$ its localization. Let $\pi : \mathfrak{A} \to \widehat{\mathfrak{A}}$ be the identity embedding, so that π is an order-continuous measure-preserving Boolean homomorphism. Show that if we set $v = \chi b$ where $b \in \widehat{\mathfrak{A}} \setminus \mathfrak{A}$, then there is no $u \in L^\infty(\mathfrak{A})$ such that $\int_a u = \int_{\pi a} v$ whenever $\bar\mu a < \infty$.

(g) In 366H, show that $[\![Tu \in E]\!] = \pi[\![u \in E]\!]$ (notation: 364G) whenever $u \in M^0_{\bar\mu}$ and $E \subseteq \mathbb{R}$ is a Borel set such that $0 \notin \overline{E}$.

>(h) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and let G be the group of all measure-preserving ring automorphisms of $\mathfrak{A}^f$. Let H be the group of all Banach lattice automorphisms of $L^2_{\bar\mu}$. Show that the map $\pi \mapsto T$ of 366L is an injective group homomorphism from G to H, so that G is represented as a subgroup of H.

(i) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be any family of measure algebras, with simple product $(\mathfrak{A},\bar\mu)$ (322L). Show that for any $p \in [1,\infty[$, $L^p_{\bar\mu}$ can be identified, as normed Riesz space, with the solid linear subspace

$$\{u : \|u\| = \bigl(\sum_{i\in I}\|u(i)\|_p^p\bigr)^{1/p} < \infty\}$$

of $\prod_{i\in I} L^p_{\bar\mu_i}$.

(j) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\bar{\mu}$, $\bar{\nu}$ two functionals rendering $\mathfrak{A}$ a semi-finite measure algebra. Show that for any $p \in [1,\infty[$, $L^p_{\bar{\mu}}$ and $L^p_{\bar{\nu}}$ are isomorphic as normed Riesz spaces. (*Hint*: use 366Xe to reduce to the case in which $\mathfrak{A}$ is Dedekind complete. Take $w \in L^0(\mathfrak{A})$ such that $\int_a w\,d\bar{\mu} = \bar{\nu}a$ for every $a \in \mathfrak{A}$ (365T). Set $Tu = w^{1/p} \times u$ for $u \in L^p_{\bar{\mu}}$.)

(k) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras, and $p \in [1,\infty[$. Show that the following are equiveridical: (i) $L^p_{\bar{\mu}}$ and $L^p_{\bar{\nu}}$ are isomorphic as Banach lattices; (ii) $L^p_{\bar{\mu}}$ and $L^p_{\bar{\nu}}$ are isomorphic as Riesz spaces; (iii) $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic Dedekind completions.

(l) For a Boolean algebra $\mathfrak{A}$, state and prove results corresponding to 363C, 363Ea and 363F-363I for $L^\infty_{\mathbb{C}}(\mathfrak{A})$ as defined in 366Ma.

366Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and suppose that $0 < p < 1$. Write $L^p = L^p_{\bar{\mu}} = L^p(\mathfrak{A},\bar{\mu})$ for $\{u : u \in L^0(\mathfrak{A}),\ |u|^p \in L^1_{\bar{\mu}}\}$, and for $u \in L^p$ set $\tau(u) = \int |u|^p$. (i) Show that τ defines a Hausdorff linear space topology on L^p (see 2A5B). (ii) Show that if $A \subseteq L^p$ is non-empty, downwards-directed and has infimum 0 then $\inf_{u\in A} \tau(u) = 0$. (iii) Show that if $A \subseteq L^p$ is non-empty, upwards-directed and bounded in the linear topological space sense then A is bounded above. (iv) Show that $(L^p)^\sim = (L^p)^\times$ is just the set of continuous linear functionals from L^p to $\mathbb{R}$, and is $\{0\}$ iff $\mathfrak{A}$ has no atom of finite measure.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Show that $M^0(\mathfrak{A},\bar{\mu})$ has the countable sup property.

(c) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and define $M^{\infty,0}_{\bar{\mu}}$ as in 366Xd. Show that $(M^{\infty,0}_{\bar{\mu}})^\times$ can be identified with $L^1_{\bar{\mu}}$.

(d) In 366H, show that if $\tilde{T} : M^0(\mathfrak{A},\bar{\mu}) \to M^0(\mathfrak{B},\bar{\nu})$ is any positive linear operator such that $\tilde{T}(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}^f$, then $\tilde{T}$ is order-continuous, so is equal to T_π.

(e) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. (i) Show that there is a natural one-to-one correspondence between $M^{1,0}(\mathfrak{A},\bar{\mu})$ and the set of additive functionals $\nu : \mathfrak{A}^f \to \mathbb{R}$ such that $\nu \ll \mu$ in the double sense that for every $\epsilon > 0$ there are δ, $M > 0$ such that $|\nu a| \le \epsilon$ whenever $\mu a \le \delta$ and $|\nu a| \le \epsilon\mu a$ whenever $\mu a \ge M$. (ii) Use this description of $M^{1,0}$ to prove 366H(b-i).

(f) In 366H, show that the following are equiveridical: (α) $\pi[\mathfrak{A}^f] = \mathfrak{B}^f$; ($\beta$) $T = T_\pi$ is surjective; (γ) $P = P_\pi$ is injective; (δ) P is a Riesz homomorphism; (ϵ) there is some $q \in [1,\infty]$ such that $\|Pv\|_q = \|v\|_q$ for every $v \in L^q_{\bar{\nu}}$; (ζ) $TPv = v$ for every $v \in M^{1,0}_{\bar{\nu}}$.

(g) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and suppose that $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ is a measure-preserving ring homomorphism, as in 366H; let $T : M^0_{\bar{\mu}} \to M^0_{\bar{\nu}}$ be the associated linear operator. Show that if $0 < p < 1$ (as in 366Ya) then $L^p_{\bar{\mu}} \subseteq M^0_{\bar{\mu}}$ and $T^{-1}[L^p_{\bar{\nu}}] = L^p_{\bar{\mu}}$.

(h) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra. (i) For each Boolean automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$, let $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ be the associated Riesz space isomorphism, and let $w_\pi \in (L^1_{\bar{\mu}})^+$ be such that $\int_a w_\pi = \mu(\pi^{-1}a)$ for every $a \in \mathfrak{A}$ (365Ea). Set $Q_\pi u = T_\pi u \times \sqrt{w_\pi}$ for $u \in L^0(\mathfrak{A})$. Show that $\|Q_\pi u\|_2 = \|u\|_2$ for every $u \in L^2_{\bar{\mu}}$. (ii) Show that if π, $\phi : \mathfrak{A} \to \mathfrak{A}$ are Boolean automorphisms then $Q_{\pi\phi} = Q_\pi Q_\phi$.

(i) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\pi : \mathfrak{A}^f \to \mathfrak{A}^f$ a measure-preserving Boolean homomorphism, with associated linear operator $T_\pi : M^0_{\bar{\mu}} \to M^0_{\bar{\mu}}$. Show that the following are equiveridical: (i) there is some $p \in [1,\infty[$ such that $\{T^n_\pi{\restriction}L^p_{\bar{\mu}} : n \in \mathbb{N}\}$ is relatively compact in $\mathsf{B}(L^p_{\bar{\mu}};L^p_{\bar{\mu}})$ for the strong operator topology; (ii) for every $p \in [1,\infty[$, $\{T^n_\pi{\restriction}L^p_{\bar{\mu}} : n \in \mathbb{N}\}$ is relatively compact in $\mathsf{B}(L^p_{\bar{\mu}};L^p_{\bar{\mu}})$ for the strong operator topology; (iii) $\{\pi^n a : n \in \mathbb{N}\}$ is relatively compact in $\mathfrak{A}^f$, for the strong measure-algebra topology, for every $a \in \mathfrak{A}^f$.

(j) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that $L^0_{\mathbb{C}}(\mathfrak{A})$ can be identified with the complexification of $L^0(\mathfrak{A})$ as defined in 354Yl.

(k) Write $\mathcal{B}(\mathbb{C})$ for the Borel σ-algebra of $\mathbb{C} \cong \mathbb{R}^2$ as defined in 111Gd. Show that if $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, we have a unique function $(u,E) \mapsto [\![u \in E]\!] : L^0_{\mathbb{C}}(\mathfrak{A}) \times \mathcal{B}(\mathbb{C}) \to \mathfrak{A}$ such that (i) for any $u \in L^0_{\mathbb{C}}(\mathfrak{A})$, the function $E \mapsto [\![u \in E]\!]$ is a sequentially order-continuous Boolean homomorphism from $\mathcal{B}(\mathbb{C})$ to $\mathfrak{A}$ (ii) if E_0, $E_1 \subseteq \mathbb{R}$ are Borel sets, then $[\![u \in E_0 \times E_1]\!] = [\![\mathcal{R}\mathrm{e}(u) \in E_0]\!] \cap [\![\mathcal{I}\mathrm{m}(u) \in E_1]\!]$ for every $u \in L^0_{\mathbb{C}}(\mathfrak{A})$ (iii) if $\phi : \mathcal{B}(\mathbb{C}) \to \mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism, then there is a unique $u \in L^0_{\mathbb{C}}(\mathfrak{A})$ such that $\phi(E) = [\![u \in E]\!]$ for every $E \in \mathcal{B}(\mathbb{C})$.

(l) A function $h : \mathbb{C} \to \mathbb{C}$ is called **Borel measurable** if its real and imaginary parts are $\mathcal{B}(\mathbb{C})$-measurable, where $\mathcal{B}(\mathbb{C})$ is the Borel σ-algebra of $\mathbb{C}$. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. (i) Show that for every Borel measurable $h : \mathbb{C} \to \mathbb{C}$ and $u \in L^0_{\mathbb{C}}(\mathfrak{A})$ we have an element $\bar{h}(u) \in L^0_{\mathbb{C}}(\mathfrak{A})$ such that $[\![\bar{h}(u) \in E]\!] = [\![u \in h^{-1}[E]]\!]$ for every $E \in \mathcal{B}(\mathbb{C})$. (ii) Show that if $\pi : \mathfrak{A} \to \mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism and $T : L^0_{\mathbb{C}}(\mathfrak{A}) \to L^0_{\mathbb{C}}(\mathfrak{A})$ the corresponding linear operator (366Mc), then $T\bar{h} = \bar{h}T$ for every Borel measurable $h : \mathbb{C} \to \mathbb{C}$.

(m) Show that a normed space over $\mathbb{C}$ has the Hahn-Banach property of 363R for complex spaces iff it is isomorphic to $L^\infty_{\mathbb{C}}(\mathfrak{A})$ for some Dedekind complete Boolean algebra $\mathfrak{A}$.

366 Notes and comments The L^p spaces, for $1 \le p \le \infty$, constitute the most important family of leading examples for the theory of Banach lattices, and it is not to be wondered at that their properties reflect a wide variety of general results. Thus 366Dd and 366E can both be regarded as special cases of theorems about perfect Riesz spaces (356M and 369D). In a different direction, the concept of 'Orlicz space' (369Xd below) generalizes the L^p spaces if they are regarded as normed subspaces of L^0 invariant under measure-preserving automorphisms of the underlying algebra. Yet another generalization looks at the (non-locally-convex) spaces L^p for $0 < p < 1$ (366Ya).

In 366H and its associated results I try to emphasize the way in which measure-preserving homomorphisms of the underlying algebras induce both 'direct' and 'dual' operators on L^p spaces. We have already seen the phenomenon in 365P. I express this in a slightly different form in 366H, noting that we really do need the homomorphisms to be measure-preserving, for the dual operators as well as the direct operators, so we no longer have the shift in the hypotheses which appears between 365O and 365P. Of course all these refinements in the hypotheses are irrelevant to the principal applications of the results, and they make substantial demands on the reader; but I believe that the demands are actually demands to expand one's imagination, to encompass the different ways in which the spaces depend on the underlying measure algebras.

In the context of 366H, L^∞ is set apart from the other L^p spaces, because $L^\infty(\mathfrak{A})$ is not in general determined by the ideal $\mathfrak{A}^f$, and the hypotheses of 366H do not look outside $\mathfrak{A}^f$. 366H(a-iv) and 366H(b-iii) reach only the space $M^{\infty,0}$ as defined in 366Xd. To deal with L^∞ we need slightly stronger hypotheses. If we are given a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, rather than from $\mathfrak{A}^f$ to $\mathfrak{B}^f$, then of course the direct operator T has a natural version acting on $L^\infty(\mathfrak{A})$ and indeed on $M^{1,\infty}_{\bar{\mu}}$, as in 363F and 369Xm. If we know that $(\mathfrak{A}, \bar{\mu})$ is localizable, then $\mathfrak{A}$ can be recovered from $\mathfrak{A}^f$, and the dual operator P acts on $L^\infty(\mathfrak{B})$, as in 369Xm. But in general we can't expect this to work (366Xf).

Of course 366H can be applied to many other spaces; for reasons which will appear in §§371 and 374, the archetypes are not really L^p spaces at all, but the spaces $M^{1,0}$ (366F) and $M^{1,\infty}$.

I include 366L and 366Yh as pointers to one of the important applications of these ideas: the investigation of properties of a measure-preserving homomorphism in terms of its action on L^p spaces. The case $p = 2$ is the most useful because the group of unitary operators (that is, the normed space automorphisms) of L^2 has been studied intensively.

367 Convergence in measure

Continuing through the ideas of Chapter 24, I come to 'convergence in measure'. The basic results of §245 all translate easily into the new language (367L-367M, 367P). The associated concept of (sequential) order-convergence can also be expressed in abstract terms (367A), and I take the trouble to do this in the context of general lattices (367A-367B), since the concept can be applied in many ways (367C-367E, 367K, 367Xa-367Xn). In the particular case of L^0 spaces, which are the first aim of this section, the idea is most naturally expressed by 367F. It enables us to express some of the fundamental theorems from Volumes 1 and 2 in the language of this chapter (367I-367J).

In 367N and 367O I give two of the most characteristic properties of the topology of convergence in measure on L^0; it is one of the fundamental types of topological Riesz space. Another striking fact is the way it is determined by the Riesz space structure (367T). In 367U I set out a theorem which is the basis of many remarkable applications of the concept; for the sake of a result in §369 I give one such application (367V).

367A Order*-convergence As I have remarked before, the function spaces of measure theory have three interdependent structures: they are linear spaces, they have a variety of interesting topologies, and they are ordered spaces. Ordinary elementary functional analysis studies interactions between topologies and linear structures, in the theory of normed spaces and, more generally, of linear topological spaces. Chapter 35 in this volume looked at interactions between linear and order structures. It is natural to seek to complete the triangle with a theory of topological ordered spaces. The relative obscurity of any such theory is in part due to the difficulty of finding convincing definitions; that is, isolating concepts which lead to elegant and useful general theorems. Among the many rival ideas, however, I believe it is possible to identify one which is particularly important in the context of measure theory.

In its natural home in the theory of L^0 spaces, this notion of 'order*-convergence' has a very straightforward expression (367F). But, suitably interpreted, the same idea can be applied in other contexts, some of which will be very useful to us, and I therefore begin with a definition which is applicable in any lattice.

Definition Let P be a lattice, p an element of P and $\langle p_n\rangle_{n\in\mathbb{N}}$ a sequence in P. I will say that $\langle p_n\rangle_{n\in\mathbb{N}}$ **order*-converges** to p, or that p is the **order*-limit** of $\langle p_n\rangle_{n\in\mathbb{N}}$, if

$$\begin{aligned}p&=\inf\{q:\exists\,n\in\mathbb{N},\,q\ge(p'\vee p_i)\wedge p''\;\forall\,i\ge n\}\\&=\sup\{q:\exists\,n\in\mathbb{N},\,q\le p'\vee(p_i\wedge p'')\;\forall\,i\ge n\}\end{aligned}$$

whenever $p'\le p\le p''$ in P.

367B Lemma Let P be a lattice.

(a) A sequence in P can order*-converge to at most one point.

(b) A constant sequence order*-converges to its constant value.

(c) Any subsequence of an order*-convergent sequence is order*-convergent, with the same limit.

(d) If $\langle p_n\rangle_{n\in\mathbb{N}}$ and $\langle p'_n\rangle_{n\in\mathbb{N}}$ both order*-converge to p, and $p_n\le q_n\le p'_n$ for every n, then $\langle q_n\rangle_{n\in\mathbb{N}}$ order*-converges to p.

(e) If $\langle p_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in P, then it order*-converges to $p\in P$ iff

$$\begin{aligned}p&=\inf\{q:\exists\,n\in\mathbb{N},\,q\ge p_i\;\forall\,i\ge n\}\\&=\sup\{q:\exists\,n\in\mathbb{N},\,q\le p_i\;\forall\,i\ge n\}.\end{aligned}$$

(f) If P is a Dedekind σ-complete lattice (314Ab) and $\langle p_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in P, then it order*-converges to $p\in P$ iff

$$p=\sup_{n\in\mathbb{N}}\inf_{i\ge n}p_i=\inf_{n\in\mathbb{N}}\sup_{i\ge n}p_i.$$

proof (a) Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to both p and $\tilde p$. Set $p'=p\wedge\tilde p$, $p''=p\vee\tilde p$; then

$$p=\inf\{q:\exists\,n\in\mathbb{N},\,q\ge(p'\vee p_i)\wedge p''\;\forall\,i\ge n\}=\tilde p.$$

(b) is trivial.

(c) Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to p, and that $\langle p'_n\rangle_{n\in\mathbb{N}}$ is a subsequence of $\langle p_n\rangle_{n\in\mathbb{N}}$. Take p', p'' such that $p'\le p\le p''$, and set

$$B=\{q:\exists\,n\in\mathbb{N},\,q\le p'\vee(p_i\wedge p'')\;\forall\,i\ge n\},$$

$$B'=\{q:\exists\,n\in\mathbb{N},\,q\le p'\vee(p'_i\wedge p'')\;\forall\,i\ge n\},$$

$$C=\{q:\exists\,n\in\mathbb{N},\,q\ge(p'\vee p_i)\wedge p''\;\forall\,i\ge n\},$$

$$C'=\{q:\exists\,n\in\mathbb{N},\,q\ge(p'\vee p'_i)\wedge p''\;\forall\,i\ge n\}.$$

If $q\in B'$ and $q'\in C$, then for all sufficiently large i

$$q\le p'\vee(p'_i\wedge p'')\le(p'\vee p'_i)\wedge p''\le q'.$$

As $p=\inf C$, we must have $q\le p$; thus p is an upper bound for B'. On the other hand, $\{p'_i:i\ge n\}\subseteq\{p_i:i\ge n\}$ for every n, so $B\subseteq B'$ and p must be the least upper bound of B', since $p=\sup B$.

Similarly, $p=\inf C'$. As p' and p'' are arbitrary, $\langle p'_n\rangle_{n\in\mathbb{N}}$ order*-converges to p.

(d) Take p', p'' such that $p'\le p\le p''$, and set

$$B=\{q:\exists\,n\in\mathbb{N},\,q\le p'\vee(p_i\wedge p'')\;\forall\,i\ge n\},$$

$$B'=\{q:\exists\,n\in\mathbb{N},\,q\le p'\vee(q_i\wedge p'')\;\forall\,i\ge n\},$$

$$C = \{q : \exists\, n \in \mathbb{N},\, q \ge (p' \vee p'_i) \wedge p'' \;\forall\, i \ge n\},$$

$$C' = \{q : \exists\, n \in \mathbb{N},\, q \ge (p' \vee q_i) \wedge p'' \;\forall\, i \ge n\}.$$

If $q \in B'$ and $q' \in C$, then for all sufficiently large i

$$q \le p' \vee (q_i \wedge p'') \le (p' \vee p'_i) \wedge p'' \le q'.$$

As $p = \inf C$, we must have $q \le p$; thus p is an upper bound for B'. On the other hand, $p' \vee (p_i \wedge p'') \le p' \vee (q_i \wedge p'')$ for every i, so $B \subseteq B'$ and $p = \sup B'$. Similarly, $p = \inf C'$. As p' and p'' are arbitrary, $\langle q_n \rangle_{n \in \mathbb{N}}$ order*-converges to p.

(e) Set

$$B = \{q : \exists\, n \in \mathbb{N},\, q \le p_i \;\forall\, i \ge n\},$$

$$C = \{q : \exists\, n \in \mathbb{N},\, q \ge p_i \;\forall\, i \ge n\}.$$

(i) Suppose that $\langle p_n \rangle_{n \in \mathbb{N}}$ order*-converges to p. Let p', p'' be such that $p' \le p_n \le p''$ for every $n \in \mathbb{N}$ and $p' \le p \le p''$. Then

$$B = \{q : \exists\, n \in \mathbb{N},\, q \le p' \vee (p_i \wedge p'') \;\forall\, i \ge n\},$$

so $\sup B = p$. Similarly, $\inf C = p$, so the condition is satisfied.

(ii) Suppose that $\sup B = \inf C = p$. Take any p', p'' such that $p' \le p \le p''$ and set

$$B' = \{q : \exists\, n \in \mathbb{N},\, q \le p' \vee (p_i \wedge p'') \;\forall\, i \ge n\},$$

$$C' = \{q : \exists\, n \in \mathbb{N},\, q \ge (p' \vee p_i) \wedge p'' \;\forall\, i \ge n\}.$$

If $q \in B'$ and $q' \in C$, then for all large enough i

$$q \le p' \vee (p_i \wedge p'') \le p' \vee q' = q'$$

because $p \le q'$. As $\inf C = p$, p is an upper bound for B'. On the other hand, if $q \in B$, then $q \le p$, so $q \le p' \vee (p_i \wedge p'')$ whenever $q \le p_i$, which is so for all sufficiently large i, and $q \in B'$. Thus $B' \supseteq B$ and p must be the supremum of B'. Similarly, $p = \inf C'$; as p' and p'' are arbitrary, $\langle p_n \rangle_{n \in \mathbb{N}}$ order*-converges to p.

(f) This follows at once from (e). Setting

$$B = \{q : \exists\, n \in \mathbb{N},\, q \le p_i \;\forall\, i \ge n\}, \quad B' = \{\inf_{i \ge n} p_i : i \in \mathbb{N}\},$$

then $B' \subseteq B$ and for every $q \in B$ there is a $q' \in B'$ such that $q \le q'$; so $\sup B = \sup B'$ if either is defined. Similarly,

$$\inf\{q : \exists\, n \in \mathbb{N},\, q \ge p_i \;\forall\, i \ge n\} = \inf_{n \in \mathbb{N}} \sup_{i \ge n} p_i$$

if either is defined.

367C Proposition Let U be a Riesz space and $\langle u_n \rangle_{n \in \mathbb{N}}$, $\langle v_n \rangle_{n \in \mathbb{N}}$ two sequences in U order*-converging to u, v respectively.

(a) If $w \in U$, $\langle u_n + w \rangle_{n \in \mathbb{N}}$ order*-converges to $u + w$, and αu_n order*-converges to αu for every $\alpha \in \mathbb{R}$.

(b) $\langle u_n \vee v_n \rangle_{n \in \mathbb{N}}$ order*-converges to $u \vee v$ and $\langle u_n \wedge v_n \rangle_{n \in \mathbb{N}}$ order*-converges to $u \wedge v$.

(c) If $\langle w_n \rangle_{n \in \mathbb{N}}$ is any sequence in U, then it order*-converges to $w \in U$ iff $\langle |w_n - w| \rangle_{n \in \mathbb{N}}$ order*-converges to 0.

(d) $\langle u_n + v_n \rangle_{n \in \mathbb{N}}$ order*-converges to $u + v$.

(e) If U is Archimedean, and $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathbb{R}$ converging to $\alpha \in \mathbb{R}$, then $\langle \alpha_n u_n \rangle_{n \in \mathbb{N}}$ order*-converges to αu.

(f) Again suppose that U is Archimedean. Then a sequence $\langle w_n \rangle_{n \in \mathbb{N}}$ in U^+ is *not* order*-convergent to 0 iff there is a $\tilde{w} > 0$ such that $\tilde{w} = \sup_{i \ge n} \tilde{w} \wedge w_i$ for every $n \in \mathbb{N}$.

proof (a)(i) $\langle u_n + w \rangle_{n \in \mathbb{N}}$ order*-converges to $u + w$ because the ordering of U is translation-invariant; the map $w' \mapsto w' + w$ is an order-isomorphism.

(ii)($\boldsymbol{\alpha}$) If $\alpha > 0$, then the map $w' \mapsto \alpha w'$ is an order-isomorphism, so $\langle \alpha u_n \rangle_{n \in \mathbb{N}}$ order*-converges to αu.

($\boldsymbol{\beta}$) If $\alpha = 0$ then $\langle \alpha u_n \rangle_{n \in \mathbb{N}}$ order*-converges to $\alpha u = 0$ by 367Bb.

($\boldsymbol{\gamma}$) If $w' \le -u \le w''$ then $-w'' \le u \le -w'$ so

$$\begin{aligned} u &= \inf\{w : \exists\, n \in \mathbb{N},\, w \ge ((-w'') \vee u_i) \wedge (-w')\ \forall\, i \ge n\} \\ &= \sup\{w : \exists\, n \in \mathbb{N},\, w \le (-w'') \vee (u_i \wedge (-w'))\ \forall\, i \ge n\}. \end{aligned}$$

Turning these formulae upside down,

$$\begin{aligned} -u &= \sup\{w : \exists\, n \in \mathbb{N},\, w \le (w'' \wedge (-u_i)) \vee w'\ \forall\, i \ge n\} \\ &= \inf\{w : \exists\, n \in \mathbb{N},\, w \ge w'' \wedge ((-u_i) \vee w')\ \forall\, i \ge n\}. \end{aligned}$$

As w' and w'' are arbitrary, $\langle -u_n\rangle_{n\in\mathbb{N}}$ order*-converges to $-u$.

(δ) Putting (α) and (γ) together, $\langle \alpha u_n\rangle_{n\in\mathbb{N}}$ order*-converges to αu for every $\alpha < 0$.

(b) Suppose that $w' \le u \vee v \le w''$. Set

$$B = \{w : \exists\, n \in \mathbb{N},\, w \le w' \vee ((u_i \vee v_i) \wedge w'')\ \forall\, i \ge n\},$$

$$C = \{w : \exists\, n \in \mathbb{N},\, w \ge (w' \vee (u_i \vee v_i)) \wedge w''\ \forall\, i \ge n\},$$

$$B_1 = \{w : \exists\, n \in \mathbb{N},\, w \le (w' \wedge u) \vee (u_i \wedge w'')\ \forall\, i \ge n\},$$

$$B_2 = \{w : \exists\, n \in \mathbb{N},\, w \le (w' \wedge v) \vee (v_i \wedge w'')\ \forall\, i \ge n\},$$

$$C_1 = \{w : \exists\, n \in \mathbb{N},\, w \ge ((w' \wedge u) \vee u_i) \wedge w''\ \forall\, i \ge n\},$$

$$C_2 = \{w : \exists\, n \in \mathbb{N},\, w \ge ((w' \wedge v) \vee v_i) \wedge w''\ \forall\, i \ge n\},$$

If $w_1 \in B_1$ and $w_2 \in B_2$ then $w_1 \vee w_2 \in B$. **P** There is an $n \in \mathbb{N}$ such that $w_1 \le (w' \wedge u) \vee (u_i \wedge w'')$ for every $i \ge n$, while $w_2 \le (w' \wedge v) \vee (v_i \wedge w'')$ for every $i \ge n$. So

$$\begin{aligned} w_1 \vee w_2 &\le (w' \wedge u) \vee (w' \wedge v) \vee (u_i \wedge w'') \vee (v_i \wedge w'') \\ &= (w' \wedge (u \vee v)) \vee ((u_i \vee v_i) \wedge w'') \end{aligned}$$

(352Ec)

$$= w' \vee ((u_i \vee v_i) \wedge w'')$$

for every $i \ge n$, and $w_1 \vee w_2 \in B$. **Q**

Similarly, if $w_1 \in C_1$ and $w_2 \in C_2$ then $w_1 \vee w_2 \in C$. **P** There is an $n \in \mathbb{N}$ such that $w_1 \ge ((w' \wedge u) \vee u_i) \wedge w''$ and $w_2 \ge ((w' \wedge v) \vee v_i) \wedge w''$ for every $i \ge n$. So

$$\begin{aligned} w_1 \vee w_2 &\ge (((w' \wedge u) \vee u_i) \wedge w'') \vee (((w' \wedge v) \vee v_i) \wedge w'') \\ &= ((w' \wedge u) \vee u_i \vee (w' \wedge v) \vee v_i) \wedge w'' \\ &= ((w' \wedge (u \vee v)) \vee (u_i \vee v_i)) \wedge w'' \\ &= (w' \vee (u_i \vee v_i)) \wedge w'' \end{aligned}$$

for every $i \ge n$, so $w_1 \vee w_2 \in C$. **Q**

At the same time, of course, $w \le \tilde{w}$ whenever $w \in B$ and $\tilde{w} \in C$, since there is some $i \in \mathbb{N}$ such that

$$w \le w' \vee ((u_i \vee v_i) \wedge w'') \le (w' \vee (u_i \vee v_i)) \wedge w'' \le \tilde{w}.$$

Since

$$\sup\{w_1 \vee w_2 : w_1 \in B_1,\, w_2 \in B_2\} = (\sup B_1) \vee (\sup B_2) = u \vee v,$$

$$\inf\{w_1 \vee w_2 : w_1 \in C_1,\, w_2 \in C_2\} = (\inf C_1) \vee (\inf C_2) = u \vee v$$

(using the generalized distributive laws in 352E), we must have $\sup B = \inf C = u \vee v$. As w' and w'' are arbitrary, $\langle u_n \vee v_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $u \vee v$.

Putting this together with (a), we see that $\langle u_n \wedge v_n\rangle_{n\in\mathbb{N}} = \langle -((-u_n) \vee (-v_n))\rangle_{n\in\mathbb{N}}$ order*-converges to $-((-u) \vee (-v)) = u \wedge v$.

(c) The hard parts are over. (i) If $\langle w_n\rangle_{n\in\mathbb{N}}$ order*-converges to w, then $\langle w_n-w\rangle_{n\in\mathbb{N}}$, $\langle w-w_n\rangle_{n\in\mathbb{N}}$ and $\langle|w_n-w|\rangle_{n\in\mathbb{N}}=\langle(w_n-w)\vee(w-w_n)\rangle_{n\in\mathbb{N}}$ all order*-converge to 0, putting (a) and (b) together. (ii) If $\langle|w_n-w|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, then so do $\langle-|w_n-w|\rangle_{n\in\mathbb{N}}$ and $\langle w_n-w\rangle_{n\in\mathbb{N}}$, by (a) and 367Bd; so $\langle w_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by (a) again.

(d) $\langle|u_n-u|\rangle_{n\in\mathbb{N}}$ and $\langle|v_n-v|\rangle_{n\in\mathbb{N}}$ order*-converge to 0, by (c), so $\langle 2(|u_n-u|\vee|v_n-v|)\rangle_{n\in\mathbb{N}}$ also order*-converges to 0, by (b) and (a). But

$$0\le|(u_n+v_n)-(u+v)|\le|u_n-u|+|v_n-v|\le 2(|u_n-u|\vee|v_n-v|)$$

for every n, so $\langle|(u_n+v_n)-(u+v)|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367Bb and 367Bd, and $\langle u_n+v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u+v$.

(e) Set $\beta_n=\sup_{i\ge n}|\alpha_i-\alpha|$ for each n. Then $\langle\beta_n\rangle_{n\in\mathbb{N}}\to 0$, so $\inf_{n\in\mathbb{N}}\beta_n|u|=0$, because U is Archimedean. Consequently $\langle\beta_n|u|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367Be. But we also have $\beta_0|u_n-u|$ order*-converging to 0, by (c) and (a), so $\langle\beta_0|u_n-u|+\beta_n|u|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by (d). As $|\alpha_n u_n-\alpha u|\le\beta_0|u_n-u|+\beta_n|u|$ for every n, $\langle\alpha_n u_n\rangle_{n\in\mathbb{N}}$ order*-converges to αu, as required.

(f)(i) Suppose that $\langle w_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0. Then there are w', w'' such that $w'\le 0\le w''$ and either

$$B=\{w:\exists\,n\in\mathbb{N},\,w\le w'\vee(w_i\wedge w'')\;\forall\,i\ge n\}$$

does not have supremum 0, or

$$C=\{w:\exists\,n\in\mathbb{N},\,w\ge(w'\vee w_i)\wedge w''\;\forall\,i\ge n\}$$

does not have infimum 0. Now $0\in B$, because every $w_i\ge 0$, and every member of B is a lower bound for C; so 0 cannot be the greatest lower bound of C. Let $\tilde w>0$ be a lower bound for C.

Let $n\in\mathbb{N}$, and set

$$C_n=\{w:w\ge(w'\vee w_i)\wedge w''\;\forall\,i\ge n\}=\{w:w\ge w_i\wedge w''\;\forall\,i\ge n\}.$$

Because U is Archimedean, we know that $\inf(C_n-A_n)=0$, where $A_n=\{w_i\wedge w'':i\ge n\}$ (353F). Now $\tilde w$ is a lower bound for C_n, so

$$\begin{aligned}\inf_{i\ge n}(\tilde w-w_i)^+&\le\inf\{(w-w_i)^+:w\in C,\,i\ge n\}\\&\le\inf\{(w-(w_i\wedge w''))^+:w\in C,\,i\ge n\}\\&=\inf\{w-(w_i\wedge w''):w\in C,\,i\ge n\}=\inf(C_n-A_n)=0.\end{aligned}$$

As this is true for every $n\in\mathbb{N}$, $\tilde w$ has the property declared.

(ii) If $\tilde w>0$ is such that $\tilde w=\sup_{i\ge n}\tilde w\wedge w_i$ for every $n\in\mathbb{N}$, then

$$\{w:\exists\,n\in\mathbb{N},\,w\ge(0\vee w_i)\wedge\tilde w\;\forall\,i\ge n\}$$

cannot have infimum 0, and $\langle w_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0.

367D As examples of the use of this concept in a relatively abstract setting, I offer the following.

Proposition (a) Let U be a Banach lattice and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in U which is norm-convergent to $u\in U$. Then $\langle u_n\rangle_{n\in\mathbb{N}}$ has a subsequence which is order-bounded and order*-convergent to u. So if $\langle u_n\rangle_{n\in\mathbb{N}}$ itself is order*-convergent, its order*-limit is u.

(b) Let U be a Riesz space with an order-continuous norm. Then any order-bounded order*-convergent sequence is norm-convergent.

proof (a) Let $\langle u'_n\rangle_{n\in\mathbb{N}}$ be a subsequence of $\langle u_n\rangle_{n\in\mathbb{N}}$ such that $\|u'_n-u\|\le 2^{-n}$ for each $n\in\mathbb{N}$. Then $v_n=\sup_{i\ge n}|u'_i-u|$ is defined in U, and $\|v_n\|\le 2^{-n+1}$, for each n (354C). Because $\inf_{n\in\mathbb{N}}\|v_n\|=0$, $\inf_{n\in\mathbb{N}}v_n$ must be 0, while $u-v_n\le u'_i\le u+v_n$ whenever $i\ge n$; so $\langle u'_n\rangle_{n\in\mathbb{N}}$ order*-converges to u, by 367Be.

Now if $\langle u_n\rangle_{n\in\mathbb{N}}$ has an order*-limit, this must be u, by 367Ba and 367Bc.

(b) Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u. Then $\langle|u_n-u|\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0 (367Cc), so

$$C=\{v:\exists\,n\in\mathbb{N},\,v\ge|u_i-u|\;\forall\,i\ge n\}$$

has infimum 0 (367Be). Because U is a lattice, C is downwards-directed, so $\inf_{v\in C}\|v\|=0$. But

$$\inf_{v\in C}\|v\| \ge \inf_{n\in\mathbb{N}}\sup_{i\ge n}\|u_i - u\|,$$

so $\lim_{n\to\infty}\|u_n - u\| = 0$, that is, $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u.

367E One of the fundamental obstacles to the development of any satisfying general theory of ordered topological spaces is the erratic nature of the relations between subspace topologies of order topologies and order topologies on subspaces. The particular virtue of order*-convergence in the context of function spaces is that it is relatively robust when transferred to the subspaces we are interested in.

Proposition Let U be an Archimedean Riesz space and V a regularly embedded Riesz subspace. (For instance, V might be either solid or order-dense.) If $\langle v_n\rangle_{n\in\mathbb{N}}$ is a sequence in V and $v\in V$, then $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to v when regarded as a sequence in V, iff it order*-converges to v when regarded as a sequence in U.

proof (a) Since, in either V or U, $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to v iff $\langle |v_n - v|\rangle_{n\in\mathbb{N}}$ order*-converges to 0 (367Cc), it is enough to consider the case $v_n\ge 0$, $v=0$.

(b) If $\langle v_n\rangle_{n\in\mathbb{N}}$ is *not* order*-convergent to 0 in U, then, by 367Cf, there is a $u>0$ in U such that $u=\sup_{i\ge n} u\wedge v_i$ for every $n\in\mathbb{N}$ (the supremum being taken in U, of course). In particular, there is a $k\in\mathbb{N}$ such that $u\wedge v_k>0$. Now consider the set

$$C=\{w : w\in V,\ \exists\, n\in\mathbb{N},\ w\ge v_i\wedge v_k\ \forall\, i\ge n\}.$$

Then for any $w\in C$,

$$u\wedge v_k = \sup_{i\ge n} u\wedge v_i\wedge v_k \le w,$$

using the generalized distributive law in U, so 0 is not the greatest lower bound of C in U. But as the embedding of V in U is order-continuous, 0 is not the greatest lower bound of C in V, and $\langle v_n\rangle_{n\in\mathbb{N}}$ cannot be order*-convergent to 0 in V.

(c) Now suppose that $\langle v_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0 in V. Because V also is Archimedean (351Rc), there is a $w>0$ in V such that $w=\sup_{i\ge n} w\wedge v_i$ for every $n\in\mathbb{N}$, the suprema being taken in V. Again because V is regularly embedded in U, we have the same suprema in U, so, by 367Cf in the other direction, $\langle v_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0 in U.

367F I now spell out the connexion between the definition above and the concepts introduced in 245C.

Proposition Let X be a set, Σ a σ-algebra of subsets of X, $\mathfrak{A}$ a Boolean algebra and $\pi:\Sigma\to\mathfrak{A}$ a sequentially order-continuous surjective Boolean homomorphism; let $\mathcal{I}$ be its kernel. Write $\mathcal{L}^0$ for the space of Σ-measurable real-valued functions on X, and let $T:\mathcal{L}^0\to L^0=L^0(\mathfrak{A})$ be the canonical Riesz homomorphism (364C, 364P). Then for any $\langle f_n\rangle_{n\in\mathbb{N}}$ and f in $\mathcal{L}^0$, $\langle Tf_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tf in L^0 iff $X\setminus\{x : f(x)=\lim_{n\to\infty} f_n(x)\}\in\mathcal{I}$.

proof Set $H=\{x:\lim_{n\to\infty} f_n(x) \text{ exists } = f(x)\}$; of course $H\in\Sigma$. Set $g_n(x)=|f_n(x)-f(x)|$ for $n\in\mathbb{N}$ and $x\in X$.

(a) If $X\setminus H\in\mathcal{I}$, set $h_n(x)=\sup_{i\ge n} g_i(x)$ for $x\in H$ and $h_n(x)=0$ for $x\in X\setminus H$. Then $\langle h_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0 in $\mathcal{L}^0$, so $\inf_{n\in\mathbb{N}} Th_n=0$ in L^0, because T is sequentially order-continuous (364Pa). But as $X\setminus H\in\mathcal{I}$, $Th_n\ge Tg_i=|Tf_i-Tf|$ whenever $i\ge n$, so $\langle |Tf_n-Tf|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367Be or 367Bf, and $\langle Tf_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tf, by 367Cc.

(b) Now suppose that $\langle Tf_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tf. Set $g'_n(x)=\min(1,g_n(x))$ for $n\in\mathbb{N}$, $x\in X$; then $\langle Tg'_n\rangle_{n\in\mathbb{N}}=\langle e\wedge|Tf_n-Tf|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, where $e=T(\chi X)$. By 367Bf, $\inf_{n\in\mathbb{N}}\sup_{i\ge n} Tg'_i=0$ in L^0. But T is a sequentially order-continuous Riesz homomorphism, so $T(\inf_{n\in\mathbb{N}}\sup_{i\ge n} g'_i)=0$, that is,

$$X\setminus H=\{x:\inf_{n\in\mathbb{N}}\sup_{i\ge n} g'_i>0\}$$

belongs to $\mathcal{I}$.

367G Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.
(a) Any order*-convergent sequence in $L^0=L^0(\mathfrak{A})$ is order-bounded.
(b) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0, then it is order*-convergent to $u\in L^0$ iff

$$u=\inf_{n\in\mathbb{N}}\sup_{i\ge n} u_i=\sup_{n\in\mathbb{N}}\inf_{i\ge n} u_i.$$

proof (a) We can express $\mathfrak{A}$ as a quotient $\Sigma/\mathcal{I}$ of a σ-algebra of sets, in which case L^0 can be identified with the canonical image of $\mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ (364C). If $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order*-convergent sequence in L^0, then it is expressible as $\langle Tf_n\rangle_{n\in\mathbb{N}}$, where $T : \mathcal{L}^0 \to L^0$ is the canonical map, and 367F tells us that $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ converges for every $x \in H$, where $X \setminus H \in \mathcal{I}$. If we set $h(x) = \sup_{n\in\mathbb{N}} |f_n(x)|$ for $x \in H$, 0 for $x \in X \setminus H$, then we see that $|u_n| \le Th$ for every $n \in \mathbb{N}$, so that $\langle u_n\rangle_{n\in\mathbb{N}}$ is order-bounded in L^0.

(b) This now follows from 367Bf, because L^0 is Dedekind σ-complete.

367H Proposition Suppose that $E \subseteq \mathbb{R}$ is a Borel set and $h : E \to \mathbb{R}$ is a continuous function. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and set $Q_E = \{u : u \in L^0,\ [\![u \in E]\!] = 1\}$, where $L^0 = L^0(\mathfrak{A})$. Let $\bar{h} : Q_E \to L^0$ be the function defined by h (364H). Then $\langle \bar{h}(u_n)\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{h}(u)$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in Q_E order*-converging to $u \in Q_E$.

proof This is an easy consequence of 367F. We can represent $\mathfrak{A}$ as $\Sigma/\mathcal{I}$ where Σ is a σ-algebra of subsets of some set X and $\mathcal{I}$ is a σ-ideal of Σ (314M); let $T : \mathcal{L}^0 \to L^0(\mathfrak{A})$ be the corresponding homomorphism (364C, 367F). Now we can find Σ-measurable functions $\langle f_n\rangle_{n\in\mathbb{N}}$, f such that $Tf_n = u_n$, $Tf = u$, as in 367F; and the hypothesis $[\![u_n \in E]\!] = 1$, $[\![u \in E]\!] = 1$ means just that, adjusting f_n and f on a member of $\mathcal{I}$ if necessary, we can suppose that $f_n(x)$, $f(x) \in E$ for every $x \in X$. (I am passing over the trivial case $E = \emptyset$, $X \in \mathcal{I}$, $\mathfrak{A} = \{0\}$.) Accordingly $\bar{h}(u_n) = T(hf_n)$ and $\bar{h}(u) = T(hf)$, and (because h is continuous)

$$\{x : h(f(x)) \ne \lim_{n\to\infty} h(f_n(x))\} \subseteq \{x : f(x) \ne \lim_{n\to\infty} f_n(x)\} \in \mathcal{I},$$

so $\langle \bar{h}(u_n)\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{h}(u)$.

367I Dominated convergence We now have a suitable language in which to express an abstract version of Lebesgue's Dominated Convergence Theorem.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in $L^1 = L^1(\mathfrak{A}, \bar{\mu})$ which is order-bounded and order*-convergent in L^1, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u in L^1; in particular, $\int u = \lim_{n\to\infty} \int u_n$.

proof The norm of L^1 is order-continuous (365C), so $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u, by 367Db. As $\int$ is norm-continuous, $\int u = \lim_{n\to\infty} \int u_n$.

367J The Martingale Theorem In the same way, we can re-write theorems from §275 in this language.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\langle \mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$. For each $n \in \mathbb{N}$ let $P_n : L^1 = L^1(\mathfrak{A}, \bar{\mu}) \to L^1 \cap L^0(\mathfrak{B}_n)$ be the conditional expectation operator (365R); let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$, and P the conditional expectation operator onto $L^1 \cap L^0(\mathfrak{B})$.

(a) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a norm-bounded sequence in L^1 such that $P_n(u_{n+1}) = u_n$ for every $n \in \mathbb{N}$, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent in L^1.

(b) If $u \in L^1$ then $\langle P_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent to Pu.

proof If we represent $(\mathfrak{A}, \bar{\mu})$ as the measure algebra of a probability space, these become mere translations of 275G and 275I. (Note that this argument relies on the description of order*-convergence in L^0 in terms of a.e. convergence of functions, as in 367F; so that we need to know that order*-convergence in L^1 matches order*-convergence in L^0, which is what 367E is for.)

367K Some of the most important applications of these ideas concern spaces of continuous functions. I do not think that this is the time to go very far along this road, but one particular fact will be useful in §376.

Proposition Let X be a locally compact Hausdorff space, and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in $C(X)$, the space of continuous real-valued functions on X. Then $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C(X)$ iff $\{x : x \in X, \limsup_{n\to\infty} |u_n(x)| > 0\}$ is meager. In particular, $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 if $\lim_{n\to\infty} u_n(x) = 0$ for every x.

proof (a) The following elementary fact is worth noting: if $A \subseteq C(X)^+$ is non-empty and $\inf A = 0$ in $C(X)$, then $G = \bigcup_{u\in A}\{x : u(x) < \epsilon\}$ is dense for every $\epsilon > 0$. **P?** If not, take $x_0 \in X \setminus \overline{G}$. Because X is completely regular (3A3Bb), there is a continuous function $w : X \to [0,1]$ such that $w(x_0) = 1$ and $w(x) = 0$ for every $x \in \overline{G}$. But in this case $0 < \epsilon w \le u$ for every $u \in A$, which is impossible. **XQ**

(b) Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0. Set $v_n = |u_n| \wedge \chi X$, so that $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 (using 367C, as usual). Set

$$B = \{v : v \in C(X),\ \exists\, n \in \mathbb{N},\ v_i \le v\ \forall\, i \ge n\},$$

so that $\inf B = 0$ in $C(X)$ (367Be). For each $k \in \mathbb{N}$, set $G_k = \bigcup_{v\in B}\{x : v(x) < 2^{-k}\}$; then G_k is dense, by (a), and of course is open. So $H = \bigcup_{k\in\mathbb{N}} X \setminus G_k$ is a countable union of nowhere dense sets and is meager. But this means that

$$\begin{aligned}\{x : \limsup_{n\to\infty} |u_n(x)| > 0\} &= \{x : \limsup_{n\to\infty} v_n(x) > 0\}\\ &\subseteq \{x : \inf_{v\in B} v(x) > 0\} \subseteq H\end{aligned}$$

is meager.

(c) Now suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ does not order*-converge to 0. By 367Cf, there is a $w > 0$ in $C(X)$ such that $w = \sup_{i\ge n} w \wedge |u_i|$ for every $n \in \mathbb{N}$; that is, $\inf_{i\ge n}(w - |u_i|)^+ = 0$ for every n. Set

$$G_n = \{x : \inf_{i\ge n}(w - |u_i|)^+(x) < 2^{-n}\} = \{x : \sup_{i\ge n} |u_i(x)| > w(x) - 2^{-n}\}$$

for each n. Then

$$H = \bigcap_{n\in\mathbb{N}} G_n = \{x : \limsup_{n\to\infty} u_n(x) \ge w(x)\}$$

is the intersection of a sequence of dense open sets, and its complement is meager.

Let G be the non-empty open set $\{x : w(x) > 0\}$. Then G is not meager, by Baire's theorem (3A3Ha); so $G \cap H$ cannot be meager. But $\{x : \limsup_{n\to\infty} |u_n(x)| > 0\}$ includes $G \cap H$, so is also not meager.

Remark Unless the topology of X is discrete, $C(X)$ is not regularly embedded in $\mathbb{R}^X$, and we expect to find sequences in $C(X)$ which order*-converge to 0 in $C(X)$ but not in $\mathbb{R}^X$. But the proposition tells us that if we have a sequence in $C(X)$ which order*-converges in $\mathbb{R}^X$ to a member of $C(X)$, then it order*-converges in $C(X)$.

367L Everything above concerns a particular notion of sequential convergence. There is inevitably a suggestion that there ought to be a topological interpretation of this convergence (see 367Yb, 367Yk, 3A3P), but I have taken care to avoid spelling one out at this stage; I will return to the point in §393. (For a general discussion in the context of Boolean algebras, see Vladimirov 02, chap. 4.) I come now to something which really is a topology, and is as closely involved with order-convergence as any.

Convergence in measure Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. For $a \in \mathfrak{A}^f = \{a : \bar\mu a < \infty\}$ and $u \in L^0 = L^0(\mathfrak{A})$ set $\tau_a(u) = \int |u| \wedge \chi a$, $\tau_{a\epsilon}(u) = \bar\mu(a \cap [\![|u| > \epsilon]\!])$. Then the **topology of convergence in measure** on L^0 is defined *either* as the topology generated by the pseudometrics $(u, v) \mapsto \tau_a(u - v)$ *or* by saying that $G \subseteq L^0$ is open iff for every $u \in G$ there are $a \in \mathfrak{A}^f$ and $\epsilon > 0$ such that $v \in G$ whenever $\tau_{a\epsilon}(u - v) \le \epsilon$.

Remark The sentences above include a number of assertions which need proving. But at this point, rather than write out any of the relevant arguments, I refer you to §245. Since we know that $L^0(\mathfrak{A})$ can be identified with $L^0(\mu)$ for a suitable measure space (X, Σ, μ) (321J, 364Ic), everything we know about general spaces $L^0(\mu)$ can be applied directly to $L^0(\mathfrak{A})$ for measure algebras $(\mathfrak{A}, \bar\mu)$; and that is what I will do for the next few paragraphs. So far, all I have done is to write τ_a in place of the $\bar\tau_F$ of 245Ac, and call on the remarks in 245Bb and 245F.

367M Theorem (a) For any measure algebra $(\mathfrak{A}, \bar\mu)$, the topology $\mathfrak{T}$ of convergence in measure on $L^0 = L^0(\mathfrak{A})$ is a linear space topology, and any order*-convergent sequence in L^0 is $\mathfrak{T}$-convergent to the same limit.

(b) $(\mathfrak{A}, \bar\mu)$ is semi-finite iff $\mathfrak{T}$ is Hausdorff.

(c) $(\mathfrak{A}, \bar\mu)$ is localizable iff $\mathfrak{T}$ is Hausdorff and L^0 is complete under the uniformity corresponding to $\mathfrak{T}$.

(d) $(\mathfrak{A}, \bar\mu)$ is σ-finite iff $\mathfrak{T}$ is metrizable.

proof 245D, 245Cb, 245E. Of course we need 322B to assure us that the phrases 'semi-finite', 'localizable', 'σ-finite' here correspond to the same phrases used in §245, and 367F to identify order*-convergence in L^0 with the order-convergence studied in §245.

367N Proposition Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure.

(a) If $A \subseteq L^0$ is a non-empty, downwards-directed set with infimum 0, then for every neighbourhood G of 0 in L^0 there is a $u \in A$ such that $v \in G$ whenever $|v| \le u$.

(b) If $U \subseteq L^0$ is an order-dense Riesz subspace, it is topologically dense.

(c) In particular, $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ are topologically dense.

proof (a) Let $a \in \mathfrak{A}^f$, $\epsilon > 0$ be such that $u \in G$ whenever $\int |u| \wedge \chi a \le \epsilon$ (see 245Bb). Since $\{u \wedge \chi a : u \in A\}$ is a downwards-directed set in $L^1 = L^1_{\bar\mu}$ with infimum 0 in L^1, there must be a $u \in A$ such that $\int u \wedge \chi a \le \epsilon$ (365Da). But now $[-u, u] \subseteq G$, as required.

(b) Write $\overline{U}$ for the closure of U. Then $(L^0)^+ \subseteq \overline{U}$. **P** If $v \in (L^0)^+$, then $\{u : u \in U,\ u \le v\}$ is an upwards-directed set with supremum u, so $A = \{v - u : u \in U,\ u \le v\}$ is a downwards-directed set with infimum 0 (351Db). By (a), every neighbourhood of 0 meets A, and (because subtraction is continuous) every neighbourhood of v meets U, that is, $v \in \overline{U}$. **Q**

Since $\overline{U}$ is a linear subspace of L^0 (2A5Ec), it includes $(L^0)^+ - (L^0)^+ = L^0$ (352D).

(c) By 364Ja, $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ are order-dense Riesz subspaces of L^0.

367O Theorem Let U be a Banach lattice and $(\mathfrak{A}, \bar\mu)$ a measure algebra. Give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure. If $T : U \to L^0$ is a positive linear operator, then it is continuous.

proof Take any open set $G \subseteq L^0$. **?** Suppose, if possible, that $T^{-1}[G]$ is not open. Then we can find u, $\langle u_n\rangle_{n\in\mathbb{N}} \in U$ such that $Tu \in G$ and $\|u_n - u\| \le 2^{-n}$, $Tu_n \notin G$ for every n. Set $H = G - Tu$; then H is an open set containing 0 but not $T(u_n - u)$, for any $n \in \mathbb{N}$. Since $\sum_{n=0}^\infty n\|u_n - u\| < \infty$, $v = \sum_{n=0}^\infty n|u_n - u|$ is defined in U, and $|T(u_n - u)| \le \frac{1}{n}Tv$ for every $n \ge 1$. But by 367Na (or otherwise) we know that there is some n such that $w \in H$ whenever $|w| \le \frac{1}{n}Tv$, so that $T(u_n - u) \in H$ for some n, which is impossible. **X**

367P Proposition Let $(\mathfrak{A}, \bar\mu)$ be a σ-finite measure algebra.

(a) A sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $L^0 = L^0(\mathfrak{A})$ converges in measure to $u \in L^0$ iff every subsequence of $\langle u_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence which order*-converges to u.

(b) A set $F \subseteq L^0$ is closed for the topology of convergence in measure iff $u \in F$ whenever there is a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in F order*-converging to $u \in L^0$.

proof 245K, 245L.

367Q As an example of the power of the language we now have available, I give abstract versions of some martingale convergence theorems.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra; for each closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, let $P_{\mathfrak{B}}$ be the corresponding conditional expectation operator.

(a) If $\mathbb{B}$ is a non-empty downwards-directed family of closed subalgebras of $\mathfrak{A}$ with intersection $\mathfrak{C}$, then for every $u \in L^1$, $P_{\mathfrak{C}}u$ is the $\|\ \|_1$-limit of $P_{\mathfrak{B}}u$ as $\mathfrak{B}$ decreases through $\mathbb{B}$, in the sense that

for every $\epsilon > 0$ there is a $\mathfrak{B}_0 \in \mathbb{B}$ such that $\|P_{\mathfrak{B}}u - P_{\mathfrak{C}}u\|_1 \le \epsilon$ whenever $\mathfrak{B} \in \mathbb{B}$ and $\mathfrak{B} \subseteq \mathfrak{B}_0$.

(b) If $\mathbb{B}$ is a non-empty upwards-directed family of closed subalgebras of $\mathfrak{A}$ and $\mathfrak{C}$ is the closed subalgebra generated by $\bigcup\mathbb{B}$, then for every $u \in L^1$, $P_{\mathfrak{C}}u$ is the $\|\ \|_1$-limit of $P_{\mathfrak{B}}u$ as $\mathfrak{B}$ increases through $\mathbb{B}$, in the sense that

for every $\epsilon > 0$ there is a $\mathfrak{B}_0 \in \mathbb{B}$ such that $\|P_{\mathfrak{B}}u - P_{\mathfrak{C}}u\|_1 \le \epsilon$ whenever $\mathfrak{B} \in \mathbb{B}$ and $\mathfrak{B} \supseteq \mathfrak{B}_0$.

(c) Suppose that $\mathbb{B}$ is a non-empty upwards-directed family of closed subalgebras of $\mathfrak{A}$, and $\langle u_{\mathfrak{B}}\rangle_{\mathfrak{B}\in\mathbb{B}}$ is a $\|\ \|_1$-bounded family in L^1 such that $u_{\mathfrak{B}} = P_{\mathfrak{B}}u_{\mathfrak{C}}$ whenever $\mathfrak{B}$, $\mathfrak{C} \in \mathbb{B}$ and $\mathfrak{B} \subseteq \mathfrak{C}$. Then there is a $u \in L^1$ which is the limit $\lim_{\mathfrak{B}\to\mathcal{F}(\mathbb{B}\uparrow)} u_{\mathfrak{B}}$ for the topology of convergence in measure, where $\mathcal{F}(\mathbb{B}\uparrow)$ is the filter on $\mathbb{B}$ generated by $\{\{\mathfrak{C} : \mathfrak{B} \subseteq \mathfrak{C} \in \mathbb{B}\} : \mathfrak{B} \in \mathbb{B}\}$.

proof (a) Take $u \in L^1$.

(i) Note first that $\{P_{\mathfrak{B}}u : \mathfrak{B} \in \mathbb{B}\}$ is uniformly integrable (246D, or directly), therefore relatively weakly compact in L^1 (247C). Consequently there must be a $v \in L^1$ which is a weak cluster point of $P_{\mathfrak{B}}u$ as $\mathfrak{B}$ decreases through $\mathbb{B}$, in the sense that v belongs to the weak closure $\overline{\{P_{\mathfrak{B}}u : \mathfrak{B} \in \mathbb{B},\ \mathfrak{B} \subseteq \mathfrak{B}_0\}}$ for every $\mathfrak{B}_0 \in \mathbb{B}$.

It follows that $v = P_{\mathfrak{C}}u$. **P** For every $\mathfrak{B}_0 \in \mathbb{B}$, $L^1 \cap L^0(\mathfrak{B}_0)$, identified with $L^1(\mathfrak{B}_0, \bar\mu\restriction\mathfrak{B}_0)$, is a norm-closed linear subspace of L^1 containing $P_{\mathfrak{B}}u$ whenever $\mathfrak{B} \subseteq \mathfrak{B}_0$. It is therefore weakly closed (3A5Ee) and contains v. Consequently $[\![v > \alpha]\!] \in \mathfrak{B}_0$ for every $\alpha \in \mathbb{R}$. As $\mathfrak{B}_0$ is arbitrary, $[\![v > \alpha]\!] \in \mathfrak{C}$ for every $\alpha \in \mathbb{R}$, and $v \in L^1(\mathfrak{C}, \bar\mu\restriction\mathfrak{C})$. Next, if $c \in \mathfrak{C}$, then

$$\int_c v \in \overline{\{\int_c P_{\mathfrak{B}}u : \mathfrak{B} \in \mathbb{B}\}} = \{\int_c u\};$$

so $v = P_{\mathfrak{C}}u$. **Q**

(ii) Now take $\epsilon > 0$. Then there is a $\mathfrak{B}_0 \in \mathbb{B}$ such that $\|P_{\mathfrak{B}}u - P_{\mathfrak{B}_0}u\|_1 \le \frac{1}{2}\epsilon$ whenever $\mathfrak{B} \in \mathbb{B}$ and $\mathfrak{B} \subseteq \mathfrak{B}_0$. **P?** Otherwise, we can find a non-increasing sequence $\langle \mathfrak{B}_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{B}$ such that $\|P_{\mathfrak{B}_{n+1}}u - P_{\mathfrak{B}_n}u\|_1 > \frac{1}{2}\epsilon$ for every $n \in \mathbb{N}$. By the reverse martingale theorem (275K), $\langle P_{\mathfrak{B}_n}u \rangle_{n\in\mathbb{N}}$ is order*-convergent to w say. But as $\{P_{\mathfrak{B}_n}u : n \in \mathbb{N}\}$ is uniformly integrable, $\langle P_{\mathfrak{B}_n}u \rangle_{n\in\mathbb{N}}$ is $\| \|_1$-convergent to w (246Ja), and $\lim_{n\to\infty} \|P_{\mathfrak{B}_{n+1}}u - P_{\mathfrak{B}_n}u\|_1 = 0$. **XQ**

At this point, however, observe that $C = \{w : \|w - P_{\mathfrak{B}_0}u\|_1 \le \frac{1}{2}\epsilon\}$ is convex and $\| \|_1$-closed, therefore weakly closed, in L^1. Since it contains $P_{\mathfrak{B}}u$ whenever $\mathfrak{B} \in \mathbb{B}$ and $\mathfrak{B} \subseteq \mathfrak{B}_0$, it contains $v = P_{\mathfrak{C}}u$. Consequently

$$\|P_{\mathfrak{B}}u - P_{\mathfrak{C}}u\|_1 \le \|P_{\mathfrak{B}}u - P_{\mathfrak{B}_0}u\|_1 + \|P_{\mathfrak{B}_0}u - v\|_1 \le \epsilon$$

whenever $\mathfrak{B} \in \mathbb{B}$ and $\mathfrak{B} \subseteq \mathfrak{B}_0$. As ϵ and u are arbitrary, (a) is true.

(b) We can use the same method. Again take any $u \in L^1$.

(i) This time, observe that $P_{\mathfrak{B}}u$ must have a weak cluster point v as $\mathfrak{B}$ increases through $\mathbb{B}$. Since $P_{\mathfrak{B}}u$ belongs to $L^1 \cap L^0(\mathfrak{C})$ for every $\mathfrak{B} \in \mathbb{B}$, so does v. Next, if $b \in \mathfrak{B}_0 \in \mathbb{B}$, then $\int_b P_{\mathfrak{B}}u = \int_b u$ whenever $\mathfrak{B} \supseteq \mathfrak{B}_0$, so $\int_b v = \int_b u$. Thus $\mathfrak{D} = \{b : b \in \mathfrak{A}, \int_b v = \int_b u\}$ includes $\bigcup \mathbb{B}$. But $\mathfrak{D}$ is closed for the measure algebra topology of $\mathfrak{A}$, so $\mathfrak{D} \supseteq \mathfrak{C}$ and $\int_c v = \int_c u$ for every $c \in \mathfrak{C}$. Thus once again we have $v = P_{\mathfrak{C}}u$.

(ii) Now repeat the argument of (a-ii) almost word for word, but taking '$\mathfrak{B} \supseteq \mathfrak{B}'$' in place of every '$\mathfrak{B} \subseteq \mathfrak{B}'$', and quoting the ordinary martingale theorem instead of the reverse martingale theorem.

(c)(i) If $\langle \mathfrak{B}_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathbb{B}$, then $\langle u_{\mathfrak{B}_n} \rangle_{n\in\mathbb{N}}$ is order*-convergent, by Doob's martingale theorem (367Ja).

(ii) It follows that the image $\mathcal{G}$ of $\mathcal{F}(\mathbb{B}\uparrow)$ under the map $\mathfrak{B} \mapsto u_{\mathfrak{B}} : \mathbb{B} \to L^0$ is Cauchy for the linear space topology $\mathfrak{T}$ of convergence in measure. **P?** Otherwise, set $\tau(v) = \int |v| \wedge \chi 1$ for $v \in L^0$, there is an $\epsilon > 0$ such that $\sup_{v,v'\in C} \tau(v - v') > 2\epsilon$ for every $C \in \mathcal{G}$; in which case, for any $\mathfrak{B} \in \mathbb{B}$, there must be a $\mathfrak{C} \in \mathbb{B}$ such that $\tau(u_{\mathfrak{C}} - u_{\mathfrak{B}}) \ge \epsilon$. But now there will be a non-decreasing sequence $\langle \mathfrak{B}_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{B}$ such that $\tau(u_{\mathfrak{B}_{n+1}} - u_{\mathfrak{B}_n}) \ge \epsilon$ for every $n \in \mathbb{N}$ and $\langle u_{\mathfrak{B}_n} \rangle_{n\in\mathbb{N}}$ cannot be order*-convergent. **XQ**

(iii) By 367Mc, $u = \lim \mathcal{G} = \lim_{\mathfrak{B}\to\mathcal{F}(\mathbb{B}\uparrow)} u_{\mathfrak{B}}$ is defined in L^0 for $\mathfrak{T}$. But as u belongs to the $\mathfrak{T}$-closure of the $\| \|_1$-bounded set $\{u_{\mathfrak{B}} : \mathfrak{B} \in \mathbb{B}\}$, $u \in L^1$, by 245J(b-i).

367R It will be useful later to be able to quote the following straightforward facts.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Give $\mathfrak{A}$ its measure-algebra topology (323A) and $L^0 = L^0(\mathfrak{A})$ the topology of convergence in measure.

(a) The map $\chi : \mathfrak{A} \to L^0$ is a homeomorphism between $\mathfrak{A}$ and its image in L^0.

(b) If $\mathfrak{A}$ has countable Maharam type, then L^0 is separable.

(c) Suppose that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ which is closed for the measure-algebra topology. Then $L^0(\mathfrak{B})$ is closed in $L^0(\mathfrak{A})$.

proof (a) Of course χ is injective (364Jc). The measure-algebra topology of $\mathfrak{A}$ is defined by the pseudometrics $\rho_a(b, c) = \bar{\mu}(a \cap (b \triangle c))$, while the topology of L^0 is defined by the pseudometrics $\sigma_a(u, v) = \int |u - v| \wedge \chi a$, in both cases taking a to run over elements of $\mathfrak{A}$ of finite measure; as $\sigma_a(\chi b, \chi c)$ is always equal to $\rho_a(b, c)$, we have the result.

(b) By 331O, $\mathfrak{A}$ is separable in its measure-algebra topology; let $B \subseteq \mathfrak{A}$ be a countable dense set. Set

$$B^* = \{\sum_{i=0}^n \alpha_i \chi b_i : n \in \mathbb{N}, \alpha_0, \dots, \alpha_n \in \mathbb{Q}, b_0, \dots, b_n \in B\}.$$

B^* is a countable subset of L^0; let V be its closure. Then V includes $S(\mathfrak{A})$. **P** For any $n \in \mathbb{N}$, the function $(\alpha_0, \dots, \alpha_n, a_0, \dots, a_n) \mapsto \sum_{i=0}^n \alpha_i \chi a_i : \mathbb{R}^{n+1} \times \mathfrak{A}^{n+1} \to L^0$ is continuous, just because $\chi : \mathfrak{A} \to L^0$ and addition and scalar multiplication in L^0 are continuous ((a) above, 367M). So

$$D_n = \{(\alpha_0, \dots, \alpha_n, a_0, \dots, a_n) : \sum_{i=0}^n \alpha_i \chi a_i \in V\}$$

is a closed subset of $\mathbb{R}^{n+1} \times \mathfrak{A}^{n+1}$ including $\mathbb{Q}^{n+1} \times B^{n+1}$. But $\mathbb{Q}^{n+1} \times B^{n+1}$ is dense in $\mathbb{R}^{n+1} \times \mathfrak{A}^{n+1}$ (3A3Ie), so $D_n = \mathbb{R}^{n+1} \times \mathfrak{A}^{n+1}$, that is, $\sum_{i=0}^n \alpha_i \chi a_i \in V$ whenever $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ and $a_0, \dots, a_n \in V$. As n is arbitrary, $S(\mathfrak{A}) \subseteq V$. **Q**

Since $S(\mathfrak{A})$ is dense in L^0 (367Nc), $V = L^0$, B^* is dense in L^0 and L^0 is separable.

(c)(i) Note first that $\mathfrak{B}$ is order-closed in $\mathfrak{A}$ (323D(c-i)), so that $L^0(\mathfrak{B})$, defined as in 364A, is a subset of $L^0(\mathfrak{A})$ (cf. 364Xt). Applying 364P to the identity map $\mathfrak{B} \subsetneqq \mathfrak{A}$, we see that the map $L^0(\mathfrak{B}) \subsetneqq L^0(\mathfrak{A})$ identifies the operations of addition, scalar multiplication and supremum in $L^0(\mathfrak{B})$ with the restrictions of the corresponding operations on $L^0(\mathfrak{A})$.

Suppose that $u \in L^0(\mathfrak{A})$ is in the closure of $L^0(\mathfrak{B})$, and $\alpha \in \mathbb{R}$; let $n \in \mathbb{N}$ be such that $|\alpha| < n$, and fix $a \in \mathfrak{A}^f$ for the moment. For each $k \in \mathbb{N}$, choose $v_k \in L^0(\mathfrak{B})$ such that $\int |u - v_k| \times \chi a \le 2^{-k}$ (367L). Consider $v'_k = \operatorname{med}(-n\chi 1, v_k, n\chi 1)$ for $k \in \mathbb{N}$, and $v = \inf_{k\in\mathbb{N}} \sup_{j\ge k} v'_k$. We do not need to ask whether the operations here are being performed in $L^0(\mathfrak{A})$ or in $L^0(\mathfrak{B})$, and v will belong to $L^0(\mathfrak{B})$. Accordingly, now necessarily working in $L^0(\mathfrak{A})$, we shall have

$$v \times \chi a = \inf_{k\in\mathbb{N}} \sup_{j\ge k} v'_k \times \chi a.$$

Now observe that, for each k, $w_k = \sup_{j\ge k} |u - v_j| \times \chi a$ is defined in $L^1(\mathfrak{A}, \bar\mu)$ and $\int w_k \le 2^{-k+1}$. Set $u' = \operatorname{med}(-n\chi 1, u, n\chi 1)$. For $j \ge k$,

$$\begin{aligned}|u' \times \chi a - v'_j \times \chi a| &= |\operatorname{med}(-n\chi 1, u \times \chi a, n\chi 1) - \operatorname{med}(-n\chi 1, v_j \times \chi a, n\chi 1)| \\ &\le |u - v_j| \times \chi a \le w_k.\end{aligned}$$

So, for any $m \in \mathbb{N}$,

$$\begin{aligned}u' \times \chi a - v \times \chi a &= \sup_{k\in\mathbb{N}} \inf_{j\ge k} u' \times \chi a - v'_k \times \chi a \\ &= \sup_{k\ge m} \inf_{j\ge k} u' \times \chi a - v'_k \times \chi a \le \sup_{k\ge m} w_k, \\ v \times \chi a - u' \times \chi a &= \inf_{k\in\mathbb{N}} \sup_{j\ge k} v'_k \times \chi a - u' \times \chi a \\ &\le \sup_{j\ge m} v'_k \times \chi a - u' \times \chi a \le \sup_{j\ge m} w_k.\end{aligned}$$

Putting these together,

$$|u' \times \chi a - v \times \chi a| \le \sup_{j\ge m} w_k$$

for every $m \in \mathbb{N}$, and $u' \times \chi a = v \times \chi a$. But this means that $a \cap [\![u' > \alpha]\!] = a \cap [\![v > \alpha]\!]$; at the same time, because $-n < \alpha < n$, $[\![u' > \alpha]\!] = [\![u > \alpha]\!]$.

Thus we see that for every $a \in \mathfrak{A}^f$ there is a $b \in \mathfrak{B}$ such that $a \cap (b \triangle [\![u > \alpha]\!]) = 0$. It follows at once that $[\![u > \alpha]\!]$ belongs to the closure of $\mathfrak{B}$, which is $\mathfrak{B}$ itself. As α is arbitrary, $u \in L^0(\mathfrak{B})$; as u is arbitrary, $L^0(\mathfrak{B})$ is closed.

367S Proposition Let $E \subseteq \mathbb{R}$ be a Borel set, and $h : E \to \mathbb{R}$ a continuous function. Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $\bar h : Q_E \to L^0 = L^0(\mathfrak{A})$ the associated function, where $Q_E = \{u : u \in L^0,\ [\![u \in E]\!] = 1\}$ (364H). Then $\bar h$ is continuous for the topology of convergence in measure.

proof (Compare 245Dd.) Express $(\mathfrak{A}, \bar\mu)$ as the measure algebra of a measure space (X, Σ, μ). Take any $u \in Q_E$, any $a \in \mathfrak{A}$ such that $\bar\mu a < \infty$, and any $\epsilon > 0$. Express u as $f^\bullet$ where $f : X \to \mathbb{R}$ is a measurable function, and a as $F^\bullet$ where $F \in \Sigma$. Then $f(x) \in E$ a.e.(x). Set $\eta = \epsilon(2 + \mu F)$. For each $n \in \mathbb{N}$, write E_n for

$$\{t : t \in E,\ |h(s) - h(t)| \le \eta \text{ whenever } s \in E \text{ and } |s - t| \le 2^{-n}\}.$$

Then $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of Borel sets with union E, so there is an n such that $\mu\{x : x \in F,\ f(x) \notin E_n\} \le \eta$.

Now suppose that $v \in Q_E$ is such that $\int |v - u| \wedge \chi a \le 2^{-n}\eta$. Express v as $g^\bullet$ where $g : X \to \mathbb{R}$ is a measurable function. Then $g(x) \in E$ for almost every x, and

$$\int_F \min(1, |g(x) - f(x)|)\mu(dx) \le 2^{-n}\eta,$$

so $\mu\{x : x \in F,\ |f(x) - g(x)| > 2^{-n}\} \le \eta$, and

$$\begin{aligned}&\{x : x \in F,\ |h(g(x)) - h(f(x))| > \eta\} \\ &\qquad \subseteq \{x : x \in F,\ f(x) \notin E_n\} \cup \{x : g(x) \notin E\} \\ &\qquad\qquad \cup \{x : x \in F,\ |f(x) - g(x)| > 2^{-n}\}\end{aligned}$$

has measure at most 2η. But this means that

$$\int |\bar h(v) - \bar h(u)| \wedge \chi a = \int_F \min(1, |hg(x) - hf(x)|)\mu(dx) \le 2\eta + \eta\mu F = \epsilon.$$

As u, a and ϵ are arbitrary, $\bar h$ is continuous.

367T Intrinsic description of convergence in measure It is a remarkable fact that the topology of convergence in measure, not only on L^0 but on its order-dense Riesz subspaces, can be described in terms of the Riesz space structure alone, without referring at all to the underlying measure algebra or to integration. (Compare 324H.) There is more than one way of doing this. As far as I know, none is outstandingly convincing; I present a formulation which seems to me to exhibit some, at least, of the essence of the phenomenon.

Proposition Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and U an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$. Suppose that $A \subseteq U$ and $u^* \in U$. Then u^* belongs to the closure of A for the topology of convergence in measure iff

there is an order-dense Riesz subspace V of U such that

for every $v \in V^+$ there is a non-empty downwards-directed $B \subseteq U$, with infimum 0, such that

for every $w \in B$ there is a $u \in A$ such that

$$|u - u^*| \wedge v \le w.$$

proof (a) Suppose first that $u^* \in \overline{A}$. Take V to be $U \cap L^1_{\bar{\mu}}$; then V is an order-dense Riesz subspace of L^0, by 352Nc, and is therefore order-dense in U. (This is where I use the hypothesis that $(\mathfrak{A},\bar{\mu})$ is semi-finite, so that $L^1_{\bar{\mu}}$ is order-dense in L^0, by 365Ga.)

Take any $v \in V^+$. For each $n \in \mathbb{N}$, set $a_n = [\![v > 2^{-n}]\!] \in \mathfrak{A}^f$. Because $u^* \in \overline{A}$, there is a $u_n \in A$ such that $\bar{\mu}b_n \le 2^{-n}$, where

$$b_n = a_n \cap [\![|u_n - u^*| > 2^{-n}]\!] = [\![|u_n - u^*| \wedge v > 2^{-n}]\!].$$

Set $c_n = \sup_{i\ge n} b_i$; then $\bar{\mu}c_n \le 2^{-n+1}$ for each n, so $\inf_{n\in\mathbb{N}} c_n = 0$ and $\inf_{n\in\mathbb{N}} w_n = 0$ in L^0, where $w_n = v \times \chi c_n + 2^{-n}\chi 1$. Also $|u_n - u^*| \wedge v \le w_n$ for each n.

The w_n need not belong to U, so we cannot set $B = \{w_n : n \in \mathbb{N}\}$. But if instead we write

$$B = \{w : w \in U,\ w \ge v \wedge w_n \text{ for some } n \in \mathbb{N}\},$$

then B is non-empty and downwards-directed (because $\langle w_n\rangle_{n\in\mathbb{N}}$ is non-increasing); and

$$\begin{aligned}\inf B &= v - \sup\{v - w : w \in B\}\\ &= v - \sup\{w : w \in U,\ w \le (v - w_n)^+ \text{ for some } n \in \mathbb{N}\}\\ &= v - \sup_{n\in\mathbb{N}}(v - w_n)^+\end{aligned}$$

(because U is order-dense in L^0)

$$= 0.$$

Since for every $w \in B$ there is an n such that $v \wedge |u_n - u^*| \le v \wedge w_n \le w$, B witnesses that the condition is satisfied.

(b) Now suppose that the condition is satisfied. Fix $a \in \mathfrak{A}^f$ and $\epsilon > 0$. Because V is order-dense in U and therefore in L^0, there is a $v \in V$ such that $0 \le v \le \chi a$ and $\int v \ge \bar{\mu}a - \epsilon$. Let B be a downwards-directed set, with infimum 0, such that for every $w \in B$ there is a $u \in A$ with $v \wedge |u - u^*| \le w$. Then there is a $w \in B$ such that $\int w \wedge v \le \epsilon$. Now there is a $u \in A$ such that $|u - u^*| \wedge v \le w$, so that

$$\int |u - u^*| \wedge \chi a \le \epsilon + \int |u - u^*| \wedge v \le \epsilon + \int w \wedge v \le 2\epsilon.$$

As a and ϵ are arbitrary, $u^* \in \overline{A}$.

***367U Theorem** Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra; write L^1 for $L^1(\mathfrak{A},\bar{\mu})$. Let $P : (L^1)^{**} \to L^1$ be the linear operator corresponding to the band projection from $(L^1)^{**} = (L^1)^{\times\sim}$ onto $(L^1)^{\times\times}$ and the canonical isomorphism between L^1 and $(L^1)^{\times\times}$. For $A \subseteq L^1$ write A^* for the weak* closure of the image of A in $(L^1)^{**}$. Then for every $A \subseteq L^1$

$$P[A^*] \subseteq \overline{\Gamma(A)},$$

where $\Gamma(A)$ is the convex hull of A and $\overline{\Gamma(A)}$ is the closure of $\Gamma(A)$ in $L^0 = L^0(\mathfrak{A})$ for the topology of convergence in measure.

proof (a) The statement of the theorem includes a number of assertions: that $(L^1)^* = (L^1)^\times$; that $(L^1)^{**} = ((L^1)^*)^\sim$; that the natural embedding of L^1 into $(L^1)^{**} = (L^1)^{\times\sim}$ identifies L^1 with $(L^1)^{\times\times}$; and that $(L^1)^{\times\times}$ is a band in $(L^1)^{\times\sim}$. For proofs of these see 365C, 356D and 356B.

Now for the new argument. First, observe that the statement of the theorem involves the measure algebra $(\mathfrak{A},\bar\mu)$ and the space L^0 only in the definition of 'convergence in measure'; everything else depends only on the Banach lattice structure of L^1. And since we are concerned only with the question of whether members of $P[A^*]$, which is surely a subset of L^1, belong to $\overline{\Gamma(A)}$, 367T shows that this also can be answered in terms of the Riesz space structure of L^1. What this means is that we can suppose that $(\mathfrak{A},\bar\mu)$ is localizable. **P** Let $(\widehat{\mathfrak{A}},\tilde\mu)$ be the localization of $(\mathfrak{A},\bar\mu)$ (322Q). The natural expression of $\mathfrak{A}$ as an order-dense subalgebra of $\widehat{\mathfrak{A}}$ identifies $\mathfrak{A}^f=\{a:a\in\mathfrak{A},\,\bar\mu a<\infty\}$ with $\widehat{\mathfrak{A}}^f$ (322P), so that $L^1_{\bar\mu}$ becomes identified with $L^1_{\tilde\mu}$, by 365Od. Thus we can think of L^1 as $L^1_{\tilde\mu}$, and $(\widehat{\mathfrak{A}},\tilde\mu)$ is localizable. **Q**

(b) Take $\phi\in A^*$ and set $u_0=P\phi$; I have to show that $u_0\in\overline{\Gamma(A)}$. Write R for the canonical map from L^1 to $(L^1)^{**}$, so that ϕ belongs to the weak* closure of $R[A]$.

Consider first the case $u_0=0$. Take any $c\in\mathfrak{A}^f$ and $\epsilon>0$. We know that $(L^1)^*=(L^1)^\sim=(L^1)^\times$ can be identified with $L^\infty=L^\infty(\mathfrak{A})$ (365Mc), so that $\phi\in(L^\infty)^*=(L^\infty)^\sim$ must be in the band orthogonal to $(L^\infty)^\times$. Now we can identify $(L^\infty)^\sim$ with the Riesz space M of bounded additive functionals on $\mathfrak{A}$, and if we do so then $(L^\infty)^\times$ corresponds to the space M_τ of completely additive functionals (363K). Writing $P_\tau:M\to M_\tau$ for the band projection, we must have $P_\tau(\nu)=0$, where $\nu\in M$ is defined by setting $\nu a=\phi(\chi a)$ for each $a\in\mathfrak{A}$; consequently $P_\tau(|\nu|)=0$ and there is an upwards-directed family $C\subseteq\mathfrak{A}$, with supremum 1, such that $|\nu|(a)=0$ for every $a\in C$ (362D). Since $\bar\mu c=\sup_{a\in C}\bar\mu(a\cap c)$, there is an $a\in C$ such that $\bar\mu(c\setminus a)\le\epsilon$.

Consider the map $Q:L^1\to L^1$ defined by setting $Qw=w\times\chi a$ for every $w\in L^1$. Then its adjoint $Q':L^\infty\to L^\infty$ (3A5Ed) can be defined by the same formula: $Q'v=v\times\chi a$ for every $v\in L^\infty$. Since $|\phi|\in(L^\infty)^\sim$ corresponds to $|\nu|\in M$, we have

$$|\phi(Q'v)|\le\|v\|_\infty|\phi|(\chi a)=\|v\|_\infty|\nu|(a)=0$$

for every $v\in L^\infty$, and $Q''\phi=0$, where $Q'':(L^\infty)^*\to(L^\infty)^*$ is the adjoint of Q'. Since Q'' is continuous for the weak* topology on $(L^\infty)^*$, $0\in\overline{Q''[R[A]]}$, where $\overline{Q''R[A]}$ is the closure for the weak* topology of $(L^\infty)^*$. But of course $Q''R=RQ$, while the weak* topology of $(L^\infty)^*$ corresponds, on the image $R[L^1]$ of L^1, to the weak topology of L^1; so that 0 belongs to the closure of $Q[A]$ for the weak topology of L^1.

Because Q is linear, $Q[\Gamma(A)]$ is convex. Since 0 belongs to the closure of $Q[\Gamma(A)]$ for the weak topology of L^1, it belongs to the closure of $Q[\Gamma(A)]$ for the norm topology (3A5Ee). So there is a $w\in\Gamma(A)$ such that $\|w\times\chi a\|_1\le\epsilon^2$. But this means that $\bar\mu(a\cap[\![|w|\ge\epsilon]\!])\le\epsilon$ and $\bar\mu(c\cap[\![|w|\ge\epsilon]\!])\le 2\epsilon$. Since c and ϵ are arbitrary, $0\in\overline{\Gamma(A)}$.

(c) This deals with the case $u_0=0$. Now the general case follows at once if we set $B=A-u_0$ and observe that $\phi-Ru_0\in B^*$, so

$$0=P(\phi-Ru_0)\in\overline{\Gamma(B)}=\overline{\Gamma(A)-u_0}=\overline{\Gamma(A)}-u_0.$$

Remark This is a version of a theorem from Bukhvalov 95.

***367V Corollary** Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Let $\mathcal{C}$ be a family of convex subsets of $L^0=L^0(\mathfrak{A})$, all closed for the topology of convergence in measure, with the finite intersection property, and suppose that for every non-zero $a\in\mathfrak{A}$ there are a non-zero $b\subseteq a$ and a $C\in\mathcal{C}$ such that $\sup_{u\in C}\int_b|u|<\infty$. Then $\bigcap\mathcal{C}\ne\emptyset$.

proof Because $\mathcal{C}$ has the finite intersection property, there is an ultrafilter $\mathcal{F}$ on L^0 including $\mathcal{C}$. Set

$$I=\{a:a\in\mathfrak{A},\,\inf_{F\in\mathcal{F}}\sup_{u\in F}\int_a|u|<\infty\};$$

because $\mathcal{F}$ is a filter, I is an ideal in $\mathfrak{A}$, and the condition on $\mathcal{C}$ tells us that I is order-dense. For each $a\in I$, define $Q_a:L^0\to L^0$ by setting $Q_au=u\times\chi a$. Then there is an $F\in\mathcal{F}$ such that $Q_a[F]$ is a norm-bounded set in L^1, so $\phi_a=\lim_{u\to\mathcal{F}}RQ_au$ is defined in $(L^\infty)^*$ for the weak* topology on $(L^\infty)^*$, writing R for the canonical map from L^1 to $(L^\infty)^*\cong(L^1)^{**}$. If $P:(L^\infty)^*\to L^1$ is the map corresponding to the band projection $\tilde P$ from $(L^\infty)^\sim$ onto $(L^\infty)^\times$, as in 367U, and $C\in\mathcal{C}$, then 367U tells us that $P(\phi_a)$ must belong to the closure of the convex set $Q_a[C]$ for the topology of convergence in measure. Moreover, if $a\subseteq b\in I$, so that $Q_a=Q_aQ_b$, then $P(\phi_a)=Q_aP(\phi_b)$. **P** $Q_a\restriction L^1$ is a band projection on L^1, so its adjoint Q'_a is a band projection on $L^\infty\cong(L^1)^\sim$ (356C) and Q''_a is a band projection on $(L^\infty)^*\cong(L^\infty)^\sim$. This means that Q''_a will commute with $\tilde P$ (352Sb). But also Q''_a is continuous for the weak* topology of $(L^\infty)^*$, so

$$Q''_a(\phi_b)=\lim_{u\to\mathcal{F}}Q''_aRQ_bu=\lim_{u\to\mathcal{F}}RQ_aQ_bu=\phi_a,$$

and

$$P(\phi_a)=R^{-1}\tilde P(\phi_a)=R^{-1}\tilde PQ''_a(\phi_b)=R^{-1}Q''_a\tilde P(\phi_b)=Q_aR^{-1}\tilde P(\phi_b)=Q_aP(\phi_b).\ \mathbf{Q}$$

Generally, if a, $b\in I$, then

$$Q_aP(\phi_b)=Q_aQ_bP(\phi_b)=Q_{a\cap b}P(\phi_b)=P(\phi_{a\cap b})=Q_bP(\phi_a).$$

What this means is that if we take a partition D of unity included in I (313K), so that $L^0\cong\prod_{d\in D}L^0(\mathfrak{A}_d)$ (315F(iii), 364R), and define $w\in L^0$ by saying that $w\times\chi d=P(\phi_d)$ for every $d\in D$, then we shall have $w\times\chi a\times\chi d=P(\phi_a)\times\chi d$ whenever $a\in I$ and $d\in D$, so $w\times\chi a=P(\phi_a)$ for every $a\in I$. But now, given $a\in\mathfrak{A}^f$ and $\epsilon>0$ and $C\in\mathcal{C}$, there is a $b\in I$ such that $\bar\mu(a\setminus b)\le\epsilon$; $w\times\chi b\in\overline{Q_b[C]}$, so there is a $u\in C$ such that $\bar\mu(b\cap[\![|w-u|\ge\epsilon]\!])\le\epsilon$; and $\bar\mu(a\cap[\![|w-u|\ge\epsilon]\!])\le 2\epsilon$. As a and ϵ are arbitrary and C is closed, $w\in C$; as C is arbitrary, $w\in\bigcap\mathcal{C}$ and $\bigcap\mathcal{C}\ne\emptyset$.

***367W Independence** I have given myself very little room in this chapter to discuss stochastic independence. There are direct translations of results from §272 in 364Xe-364Xf. However the language here is adapted to a significant result not presented in §272. I had better begin by repeating a definition from 364Xe. Let $(\mathfrak{A},\bar\mu)$ be a probability algebra. Then a family $\langle u_i\rangle_{i\in I}$ in $L^0(\mathfrak{A})$ is **stochastically independent** if $\bar\mu(\inf_{i\in J}[\![u_i>\alpha_i]\!])=\prod_{i\in J}\bar\mu[\![u_i>\alpha_i]\!]$ whenever $J\subseteq I$ is a non-empty finite set and $\alpha_i\in\mathbb{R}$ for every $i\in I$. (The direct translation of the definition in 272Ac would rather be '$\bar\mu(\inf_{i\in J}[\![u_i\le\alpha_i]\!])=\prod_{i\in J}\bar\mu[\![u_i\le\alpha_i]\!]$ whenever $J\subseteq I$ is a non-empty finite set and $\alpha_i\in\mathbb{R}$ for every $i\in I$', intepreting $[\![u_i\le\alpha_i]\!]$ as in 364Xa. Of course 272F tells us that this comes to the same thing.) Now the new fact is the following.

Proposition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and I any set. Give $L^0=L^0(\mathfrak{A})$ its topology of convergence in measure. Then the collection of independent families $\langle u_i\rangle_{i\in I}$ is a closed set in $(L^0)^I$.

proof Suppose that $\langle u_i\rangle_{i\in I}\in(L^0)^I$ is not independent. Then there are a finite set $J\subseteq I$ and a family $\langle\alpha_i\rangle_{i\in J}$ of real numbers such that $\bar\mu(\inf_{i\in J}[\![u_i>\alpha_i]\!])\ne\prod_{i\in J}\bar\mu[\![u_i>\alpha_i]\!]$. Set $a_i=[\![u_i>\alpha_i]\!]$ for each i. Let $\delta>0$ be such that $\gamma\ne\prod_{i\in J}\gamma_i$ whenever $|\gamma-\bar\mu(\inf_{i\in J}a_i)|\le 2\delta\#(J)$ and $|\gamma_i-\bar\mu a_i|\le 2\delta$ for every $i\in J$. Let $\eta\in\,]0,1]$ be such that $\bar\mu[\![u_i>\alpha_i+2\eta]\!]\ge\bar\mu a_i-\delta$ for every $i\in J$.

Now if $\langle v_i\rangle_{i\in I}\in(L^0)^I$ and $\bar\mu[\![|v_i-u_i|>\eta]\!]\le\delta$ for each $i\in J$, $\langle v_i\rangle_{i\in I}$ is not independent. $\mathbf{P}$ For each $i\in J$, consider $b_i=[\![v_i>\alpha_i+\eta]\!]$, $a'_i=[\![u_i>\alpha_i+2\eta]\!]$. We have

$$a'_i=[\![u_i>\alpha_i+2\eta]\!]\subseteq[\![v_i>\alpha_i+\eta]\!]\cup[\![u_i-v_i>\eta]\!]\subseteq b_i\cup[\![|u_i-v_i|>\eta]\!]$$

(364Ea), and

$$b_i=[\![v_i>\alpha_i+\eta]\!]\subseteq[\![u_i>\alpha_i]\!]\cup[\![v_i-u_i>\eta]\!]\subseteq a_i\cup[\![|v_i-u_i|>\eta]\!],$$

so

$$b_i\mathbin{\triangle}a_i=(b_i\setminus a_i)\cup(a_i\setminus b_i)\subseteq[\![|v_i-u_i|>\eta]\!]\cup(a_i\setminus a'_i)$$

has measure at most 2δ. It follows that $(\inf_{i\in J}b_i)\mathbin{\triangle}(\inf_{i\in J}a_i)$ has measure at most $2\delta\#(J)$, and $|\bar\mu(\inf_{i\in J}b_i)-\bar\mu(\inf_{i\in J}a_i)|\le 2\delta\#(J)$. At the same time, for each $i\in J$, $|\bar\mu b_i-\bar\mu a_i|\le 2\delta$. By the choice of δ, $\bar\mu(\inf_{i\in J}b_i)\ne\prod_{i\in J}\bar\mu b_i$, and $\langle v_i\rangle_{i\in I}$ is not independent. $\mathbf{Q}$

This shows that the set of non-independent families is open in $(L^0)^I$, so that the set of independent families is closed, as claimed.

367X Basic exercises >(a) Let P be a lattice. (i) Show that if $p\in P$ and $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in P, then $\langle p_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to p iff $p=\sup_{n\in\mathbb{N}}p_n$. (ii) Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is a sequence in P order*-converging to $p\in P$. Show that $p=\sup_{n\in\mathbb{N}}p\wedge p_n=\inf_{n\in\mathbb{N}}p\vee p_n$. (iii) Let $\langle p_n\rangle_{n\in\mathbb{N}}$, $\langle q_n\rangle_{n\in\mathbb{N}}$ be two sequences in P which are order*-convergent to p, q respectively. Show that if $p_n\le q_n$ for every n then $p\le q$. (iv) Let $\langle p_n\rangle_{n\in\mathbb{N}}$ be a sequence in P. Show that $\langle p_n\rangle_{n\in\mathbb{N}}$ order*-converges to $p\in P$ iff $\langle p_n\vee p\rangle_{n\in\mathbb{N}}$ and $\langle p_n\wedge p\rangle_{n\in\mathbb{N}}$ order*-converge to p.

(b) Let P and Q be lattices, and $f:P\to Q$ an order-preserving function. Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence which order*-converges to p in P. Show that $\langle f(p_n)\rangle_{n\in\mathbb{N}}$ order*-converges to $f(p)$ in Q if *either* f is order-continuous *or* P is Dedekind σ-complete and f is sequentially order-continuous.

(c) Let P be *either* a Boolean algebra *or* a Riesz space. Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is a sequence in P such that $\langle p_{2n}\rangle_{n\in\mathbb{N}}$ and $\langle p_{2n+1}\rangle_{n\in\mathbb{N}}$ are both order*-convergent to $p\in P$. Show that $\langle p_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to p. (*Hint*: 313B, 352E.)

>(d) Let $\mathfrak{A}$ be a Boolean algebra and $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ two sequences in $\mathfrak{A}$ order*-converging to a, b respectively. Show that $\langle a_n\cup b_n\rangle_{n\in\mathbb{N}}$, $\langle a_n\cap b_n\rangle_{n\in\mathbb{N}}$, $\langle a_n\setminus b_n\rangle_{n\in\mathbb{N}}$, $\langle a_n\triangle b_n\rangle_{n\in\mathbb{N}}$ order*-converge to $a\cup b$, $a\cap b$, $a\setminus b$ and $a\triangle b$ respectively.

(e) Let $\mathfrak{A}$ be a Boolean algebra and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$. Show that $\langle a_n\rangle_{n\in\mathbb{N}}$ does not order*-converge to 0 iff there is a non-zero $a\in\mathfrak{A}$ such that $a=\sup_{i\ge n}a\wedge a_i$ for every $n\in\mathbb{N}$.

>(f)(i) Let U be a Riesz space and $\langle u_n\rangle_{n\in\mathbb{N}}$ an order*-convergent sequence in U^+ with limit u. Show that $h(u)\le\liminf_{n\to\infty}h(u_n)$ for every $h\in(U^\times)^+$. (ii) Let U be a Riesz space and $\langle u_n\rangle_{n\in\mathbb{N}}$ an order-bounded order*-convergent sequence in U with limit u. Show that $h(u)=\lim_{n\to\infty}h(u_n)$ for every $h\in U^\times$. (Compare 356Xd.)

>(g) Let U be a Riesz space with a Fatou norm $\|\;\|$. (i) Show that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order*-convergent sequence in U with limit u, then $\|u\|\le\liminf_{n\to\infty}\|u_n\|$. (*Hint*: $\langle|u_n|\wedge|u|\rangle_{n\in\mathbb{N}}$ is order*-convergent to $|u|$.) (ii) Show that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a norm-convergent sequence in U it has an order*-convergent subsequence. (*Hint*: if $\sum_{n=0}^\infty\|u_n\|<\infty$ then $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0.)

(h) Let U and V be Archimedean Riesz spaces and $T:U\to V$ an order-continuous Riesz homomorphism. Show that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U which order*-converges to $u\in U$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tu in V.

(i) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a regularly embedded subalgebra. Show that if $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}$ and $b\in\mathfrak{B}$, then $\langle b_n\rangle_{n\in\mathbb{N}}$ order*-converges to b in $\mathfrak{B}$ iff it order*-converges to b in $\mathfrak{A}$.

(j) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ two sequences in $L^0(\mathfrak{A})$ which are order*-convergent to u, v respectively. Show that $\langle u_n\times v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u\times v$. Show that if u, u_n all have multiplicative inverses u^{-1}, u_n^{-1} then $\langle u_n^{-1}\rangle_{n\in\mathbb{N}}$ order*-converges to u^{-1}.

(k) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\mathcal{I}$ a σ-ideal of $\mathfrak{A}$. Show that for any $\langle a_n\rangle_{n\in\mathbb{N}}$, $a\in\mathfrak{A}$, $\langle a_n^\bullet\rangle_{n\in\mathbb{N}}$ order*-converges to $a^\bullet$ in $\mathfrak{A}/\mathcal{I}$ iff $\inf_{n\in\mathbb{N}}\sup_{m\ge n}a_m\triangle a\in\mathcal{I}$.

>(l) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle h_n\rangle_{n\in\mathbb{N}}$ a sequence of Borel measurable functions from $\mathbb{R}$ to itself such that $h(t)=\lim_{n\to\infty}h_n(t)$ is defined for every $t\in\mathbb{R}$. Show that $\langle\bar h_n(u)\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar h(u)$ for every $u\in L^0=L^0(\mathfrak{A})$, where $\bar h_n$, $\bar h:L^0\to L^0$ are defined as in 364H.

(m) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in $L^1=L^1_{\bar\mu}$ which is order*-convergent to $u\in L^1$. Show that $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u iff $\{u_n:n\in\mathbb{N}\}$ is uniformly integrable iff $\|u\|_1=\lim_{n\to\infty}\|u_n\|_1$. (*Hint*: 245H, 246J.)

(n) Let U be an L-space and $\langle u_n\rangle_{n\in\mathbb{N}}$ a norm-bounded sequence in U. Show that there are a $v\in U$ and a subsequence $\langle v_n\rangle_{n\in\mathbb{N}}$ of $\langle u_n\rangle_{n\in\mathbb{N}}$ such that $\langle\frac{1}{n+1}\sum_{i=0}^n w_i\rangle_{n\in\mathbb{N}}$ order*-converges to v for every subsequence $\langle w_n\rangle_{n\in\mathbb{N}}$ of $\langle v_n\rangle_{n\in\mathbb{N}}$. (*Hint*: 276H.)

(o) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $p\in[1,\infty[$. For $v\in(L^p)^+=(L^p_{\bar\mu})^+$ define $\rho_v:L^0\times L^0\to[0,\infty[$ by setting $\rho_v(u_1,u_2)=\||u_1-u_2|\wedge v\|_p$ for all u_1, $u_2\in U$. Show that each ρ_v is a pseudometric and that the topology on $L^0(\mathfrak{A})$ defined by $\{\rho_v:v\in(L^p)^+\}$ is the topology of convergence in measure.

>(p) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and give $L^0(\mathfrak{A})$ its topology of convergence in measure. Show that $u\mapsto|u|$, $(u,v)\mapsto u\vee v$ and $(u,v)\mapsto u\times v$ are continuous.

(q) Let $(\mathfrak{A},\bar\mu)$ be a σ-finite measure algebra. Suppose we have a double sequence $\langle u_{ij}\rangle_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in $L^0=L^0(\mathfrak{A})$ such that $\langle u_{ij}\rangle_{j\in\mathbb{N}}$ order*-converges to u_i in L^0 for each i, while $\langle u_i\rangle_{i\in\mathbb{N}}$ order*-converges to u. Show that there is a strictly increasing sequence $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that $\langle u_{i,n(i)}\rangle_{i\in\mathbb{N}}$ order*-converges to u.

(r) Let (X,Σ,μ) be a semi-finite measure space. Show that $L^0(\mu)$ is separable for the topology of convergence in measure iff μ is σ-finite and has countable Maharam type. (Cf. 365Xp.)

(s) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. (i) Show that if $\langle a_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $a\in\mathfrak{A}$, then $\langle a_n\rangle_{n\in\mathbb{N}}\to a$ for the measure-algebra topology. (ii) Show that if $(\mathfrak{A},\bar\mu)$ is σ-finite, then (α) a sequence converges to a for the topology of $\mathfrak{A}$ iff every subsequence has a sub-subsequence which is order*-convergent to to a (β) a set $F\subseteq\mathfrak{A}$ is closed for the topology of $\mathfrak{A}$ iff $a\in F$ whenever there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in F which is order*-convergent to $a\in\mathfrak{A}$.

(t) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra which is not σ-finite. Show that there is a set $A \subseteq L^0(\mathfrak{A})$ such that the limit of any order*-convergent sequence in A belongs to A, but A is not closed for the topology of convergence in measure.

(u) Let U be a Banach lattice with an order-continuous norm. (i) Show that a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ is norm-convergent to $u \in U$ iff every subsequence has a sub-subsequence which is order-bounded and order*-convergent to u. (ii) Show that a set $F \subseteq U$ is closed for the norm topology iff $u \in F$ whenever there is an order-bounded sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in F order*-converging to $u \in U$.

(v) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. For $u \in L^0 = L^0(\mathfrak{A})$ let ν_u be the distribution of u (364Xd). Show that $u \mapsto \nu_u$ is continuous when L^0 is given the topology of convergence in measure and the space of probability distributions on $\mathbb{R}$ is given the vague topology (274Ld).

(w) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\langle u_n \rangle_{n\in\mathbb{N}}$ a stochastically independent sequence in $L^0(\mathfrak{A})$, all with the Cauchy distribution $\nu_{C,1}$ with centre 0 and scale parameter 1 (285Xm). For each n let C_n be the convex hull of $\{u_i : i \ge n\}$, and $\overline{C_n}$ its closure for the topology of convergence in measure. Show that every $u \in \overline{C_0}$ has distribution $\nu_{C,1}$. (*Hint*: consider first $u \in C_0$.) Show that $\overline{C_0}$ is bounded for the topology of convergence in measure. Show that $\bigcap_{n\in\mathbb{N}} \overline{C_n} = \emptyset$.

(x) If U is a linear space and $C \subseteq U$ is a convex set, a function $f : C \to \mathbb{R}$ is **convex** if $f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha)f(y)$ whenever x, $y \in C$ and $\alpha \in [0,1]$. Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra and $C \subseteq L^1_{\bar{\mu}}$ a non-empty convex norm-bounded set which is closed in $L^0(\mathfrak{A})$ for the topology of convergence in measure. Show that any convex function $f : C \to \mathbb{R}$ which is lower semi-continuous for the topology of convergence in measure is bounded below and attains its infimum.

(y) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1]$. Show that there are a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $L^1 = L^1_{\bar{\mu}}$ and u, $v \in L^1$ such that u_n and v are independent for every n, $\langle u_n \rangle_{n\in\mathbb{N}}$ converges weakly to u, but u and v are not independent.

(z) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure. (i) Show that a set $A \subseteq L^0$ is bounded in the sense of 3A5N iff for every $a \in \mathfrak{A}^f$ and $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $\bar{\mu}(a \cap [\![|u| > n]\!]) \le \epsilon$ for every $u \in A$. (ii) Show that if $(\mathfrak{A}, \bar{\mu})$ is semi-finite, then a set $A \subseteq L^0$ is bounded in this sense iff $\{\alpha_n x_n : n \in \mathbb{N}\}$ is order-bounded for every sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in A and every sequence $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$ converging to 0.

367Y Further exercises (a) Give an example of an Archimedean Riesz space U and an order-bounded sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in U which is order*-convergent to 0, but such that there is no non-increasing sequence $\langle v_n \rangle_{n\in\mathbb{N}}$, with infimum 0, such that $u_n \le v_n$ for every $n \in \mathbb{N}$.

(b) Let P be any lattice. (i) Show that there is a topology on P for which a set $A \subseteq P$ is closed iff $p \in A$ whenever there is a sequence in A which is order*-convergent to p. Show that any closed set for this topology is sequentially order-closed. (ii) Now let Q be another lattice, with the topology defined in the same way, and $f : P \to Q$ an order-preserving function. Show that if f is topologically continuous it is sequentially order-continuous.

(c) Give an example of a distributive lattice P with p, $q \in P$ and a sequence $\langle p_n \rangle_{n\in\mathbb{N}}$, order*-convergent to p, such that $\langle p_n \wedge q \rangle_{n\in\mathbb{N}}$ is not order*-convergent to $p \wedge q$.

(d) Let us say that a lattice P is $(2,\infty)$**-distributive** if (α) whenever A, $B \subseteq P$ are non-empty sets with infima p, q respectively, then $\inf\{a \vee b : a \in A, b \in B\} = p \vee q$ (β) whenever A, $B \subseteq P$ are non-empty sets with suprema p, q respectively, then $\sup\{a \wedge b : a \in A, b \in B\} = p \wedge q$. Show that, in this case, if $\langle p_n \rangle_{n\in\mathbb{N}}$ order*-converges to p and $\langle q_n \rangle_{n\in\mathbb{N}}$ order*-converges to q, $\langle p_n \vee q_n \rangle_{n\in\mathbb{N}}$ order*-converges to $p \vee q$.

(e)(i) Give an example of a Riesz space U with an order-dense Riesz subspace V of U and a sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ in V such that $\langle v_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V but does not order*-converge in U. (ii) Give an example of a Riesz space U with an order-dense Riesz subspace V of U and a sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ in V, order-bounded in V, such that $\langle v_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in U but does not order*-converge in V.

(f) Let U be an Archimedean f-algebra. Show that if $\langle u_n \rangle_{n\in\mathbb{N}}$, $\langle v_n \rangle_{n\in\mathbb{N}}$ are sequences in U order*-converging to u, v respectively, then $\langle u_n \times v_n \rangle_{n\in\mathbb{N}}$ order*-converges to $u \times v$.

(g) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $r \ge 1$. Let $E \subseteq \mathbb{R}^r$ be a Borel set and write $Q_E = \{(u_1, \dots , u_r) : [\![(u_1, \dots , u_r) \in E]\!] = 1\} \subseteq L^0(\mathfrak{A})^r$ (364Yb). Let $h : E \to \mathbb{R}$ be a continuous function and $\bar{h} : Q_E \to L^0 = L^0(\mathfrak{A})$ the corresponding map (364Yc). Show that if $\langle \mathbf{u}_n \rangle_{n\in\mathbb{N}}$ is a sequence in Q_E which is order*-convergent to $\mathbf{u} \in Q_E$ (in the lattice $(L^0)^r$), then $\langle \bar{h}(\mathbf{u}_n)\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\bar{h}(\mathbf{u})$.

(h) Let X be a completely regular Baire space (definition: 314Yd), and $\langle u_n \rangle_{n\in\mathbb{N}}$ a sequence in $C(X)$. Show that $\langle u_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C(X)$ iff $\{x : \limsup_{n\to\infty} |u_n(x)| > 0\}$ is meager in X.

(i)(i) Give an example of a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $C([0, 1])$ such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in [0, 1]$, but $\{u_n : n \in \mathbb{N}\}$ is not order-bounded in $C([0, 1])$. (ii) Give an example of an order-bounded sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $C(\mathbb{Q})$ such that $\lim_{n\to\infty} u_n(q) = 0$ for every $q \in \mathbb{Q}$, but $\sup_{i\ge n} u_i = \chi\mathbb{Q}$ in $C(\mathbb{Q})$ for every $n \in \mathbb{N}$. (iii) Give an example of a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $C([0, 1])$ such that $\langle u_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C([0, 1])$, but $\lim_{n\to\infty} u_n(q) > 0$ for every $q \in \mathbb{Q} \cap [0, 1]$.

(j) Write out an alternative proof of 367J/367Yh based on the fact that, for a Baire space X, $C(X)$ can be identified with an order-dense Riesz subspace of a quotient of the space of $\widehat{\mathcal{B}}$-measurable functions, where $\widehat{\mathcal{B}}$ is the Baire-property algebra of X.

(k) Let $\mathfrak{A}$ be a ccc weakly (σ, ∞)-distributive Boolean algebra. Show that there is a topology on $\mathfrak{A}$ such that the closure of any $A \subseteq \mathfrak{A}$ is precisely the set of limits of order*-convergent sequences in A.

(l) Give an example of a set X and a double sequence $\langle u_{mn} \rangle_{m,n\in\mathbb{N}}$ in $\mathbb{R}^X$ such that $\lim_{n\to\infty} u_{mn}(x) = u_m(x)$ exists for every $m \in \mathbb{N}$ and $x \in X$, $\lim_{m\to\infty} u_m(x) = 0$ for every $x \in X$, but there is no sequence $\langle v_k \rangle_{k\in\mathbb{N}}$ in $\{u_{mn} : m, n \in \mathbb{N}\}$ such that $\lim_{k\to\infty} v_k(x) = 0$ for every x.

(m) Let U be a Banach lattice with an order-continuous norm. For $v \in U^+$ define $\rho_v : U \times U \to [0, \infty[$ by setting $\rho_v(u_1, u_2) = \| |u_1 - u_2| \wedge v\|$ for all u_1, $u_2 \in U$. Show that every ρ_v is a pseudometric on U, and that $\{\rho_v : v \in U^+\}$ defines a Hausdorff linear space topology on U.

(n) Let U be any Riesz space. For $h \in (U^\sim_c)^+$ (356Ab), $v \in U^+$ define $\rho_{vh} : U \times U \to [0, \infty[$ by setting $\rho_{vh}(u_1, u_2) = h(|u_1 - u_2| \wedge v)$ for all u_1, $u_2 \in U$. Show that each ρ_{vh} is a pseudometric on U, and that $\{\rho_{vh} : h \in (U^\sim_c)^+,\ v \in U^+\}$ defines a linear space topology on U.

(o) Let $(\mathfrak{A}, \bar{\mu})$ be a σ-finite measure algebra. Show that the function $(\alpha, u) \mapsto [\![u > \alpha]\!] : \mathbb{R} \times L^0 \to \mathfrak{A}$ is Borel measurable when $L^0 = L^0(\mathfrak{A})$ is given the topology of convergence in measure and $\mathfrak{A}$ is given its measure-algebra topology. (*Hint*: if $a \in \mathfrak{A}$, $\gamma \ge 0$ then $\{(\alpha, u) : \bar{\mu}(a \cap [\![u > \alpha]\!]) > \gamma\}$ is open.)

(p) Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$. Show that there is no Hausdorff topology $\mathfrak{T}$ on $L^0(\mathfrak{G})$ such that $\langle u_n \rangle_{n\in\mathbb{N}}$ is $\mathfrak{T}$-convergent to u whenever $\langle u_n \rangle_{n\in\mathbb{N}}$ is order*-convergent to u. (*Hint*: Let H be any $\mathfrak{T}$-open set containing 0. Enumerate $\mathbb{Q}$ as $\langle q_n \rangle_{n\in\mathbb{N}}$. Find inductively a non-decreasing sequence $\langle G_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{G}$ such that $\chi G_n \in H$, $q_n \in G_n$ for every n. Conclude that $\chi\mathbb{R} \in \overline{H}$.)

(q) Give an example of a Banach lattice with a norm which is not order-continuous, but in which every order-bounded order*-convergent sequence is norm-convergent.

(r) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $r \ge 1$. Let $E \subseteq \mathbb{R}^r$ be a Borel set and write $Q_E = \{(u_1, \dots , u_r) : [\![(u_1, \dots , u_r) \in E]\!] = 1\} \subseteq L^0(\mathfrak{A})^r$ (364Yb). Let $h : E \to \mathbb{R}$ be a continuous function and $\bar{h} : Q_E \to L^0 = L^0(\mathfrak{A})$ the corresponding map (364Yc). Show that if $\bar{h}$ is continuous if L^0 is given its topology of convergence in measure and $(L^0)^r$ the product topology.

(s) Show that 367U is true for all measure algebras, whether semi-finite or not.

(t) In 367Qc, show that $u = \lim_{\mathfrak{B}\to\mathcal{F}(\mathbb{B}\uparrow)} u_{\mathfrak{B}}$ for the norm topology of L^1 iff $\{u_{\mathfrak{B}} : \mathfrak{B} \in \mathbb{B}\}$ is uniformly integrable, and that in this case $u_{\mathfrak{B}} = P_{\mathfrak{B}}u$ for every $\mathfrak{B} \in \mathbb{B}$.

367 Notes and comments I have given a very general definition of 'order*-convergence'. The general theory of convergence structures on ordered spaces is complex and full of traps for the unwary. I have tried to lay out a safe path to the results which are important in the context of this book. But the propositions here are necessarily full of little conditions (e.g., the requirement that U should be Archimedean in 367E) whose significance may not be immediately obvious. In particular, the definition is very much better adapted to distributive lattices than to others (367Yc, 367Yd). It is useful in the study of Riesz spaces and Boolean algebras largely because these satisfy strong distributive laws (313B, 352E). The special feature which distinguishes the definition here from other definitions of order-convergence is the fact that it can be applied to sequences which are not order-bounded. For order-bounded sequences there are useful simplifications (367Be-f), but the Martingale Theorem (367J), for instance, if we want to express it in terms of its natural home in the Riesz space L^1, refers to sequences which are hardly ever order-bounded.

The * in the phrase 'order*-convergent' is supposed to be a warning that it may not represent exactly the concept you expect. I think nearly any author using the phrase 'order-convergent' would accept sequences fulfilling the conditions of 367Bf; but beyond this no standard definitions have taken root.

The fact that order*-convergent sequences in an L^0 space are order-bounded (367G) is actually one of the characteristic properties of L^0. Related ideas will be important in the next section (368A, 368M).

It is one of the outstanding characteristics of measure algebras in this context that they provide non-trivial linear space topologies on their L^0 spaces, related in striking ways to the order structure. Not all L^0 spaces have such topologies (367Yp). A topology corresponding to 'convergence in measure' can be defined on $L^0(\mathfrak{A})$ for any Maharam algebra $\mathfrak{A}$; see 393K below.

367T shows that the topology of convergence in measure on $L^0(\mathfrak{A})$ is (at least for semi-finite measure algebras) determined by the Riesz space structure of L^0; and that indeed the same is true of its order-dense Riesz subspaces. This fact is important for a full understanding of the representation theorems in §369 below. If a Riesz space U can be embedded as an order-dense subspace of any such L^0, then there is already a 'topology of convergence in measure' on U, independent of the embedding. It is therefore not surprising that there should be alternative descriptions of the topology of convergence in measure on the important subspaces of L^0 (367Xo, 367Ym).

For σ-finite measure algebras, the topology of convergence in measure is easily described in terms of order-convergence (367P). For other measure algebras, the formula fails (367Xt). 367Yp shows that trying to apply the same ideas to Riesz spaces in general gives rise to some very curious phenomena.

367V enables us to prove results which would ordinarily be associated with some form of compactness. Of course compactness is indeed involved, as the proof through 367U makes clear; but it is weak* compactness in $(L^1)^{**}$, rather than in the space immediately to hand.

I hardly mention 'uniform integrability' in this section, not because it is uninteresting, but because I have nothing to add at this point to 246J and the exercises in §246. But I do include translations of Lebesgue's Dominated Convergence Theorem (367I) and the Martingale Theorem (367J) to show how these can be expressed in the language of this chapter.

368 Embedding Riesz spaces in L^0

In this section I turn to the representation of Archimedean Riesz spaces as function spaces. Any Archimedean Riesz space U can be represented as an order-dense subspace of $L^0(\mathfrak{A})$, where $\mathfrak{A}$ is its band algebra (368E). Consequently we get representations of Archimedean Riesz spaces as quotients of subspaces of $\mathbb{R}^X$ (368F) and as subspaces of $C^\infty(X)$ (368G), and a notion of 'Dedekind completion' (368I-368J). Closely associated with these is the fact that we have a very general extension theorem for order-continuous Riesz homomorphisms into L^0 spaces (368B). I give a characterization of L^0 spaces in terms of lateral completeness (368M, 368Yd), and I discuss weakly (σ,∞)-distributive Riesz spaces (368N-368S).

368A Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $A \subseteq (L^0)^+$ a set with no upper bound in L^0, where $L^0 = L^0(\mathfrak{A})$. If *either* A is countable *or* $\mathfrak{A}$ is Dedekind complete, there is a $v > 0$ in L^0 such that $nv = \sup_{u\in A} u \wedge nv$ for every $n \in \mathbb{N}$.

proof The hypothesis 'A is countable or $\mathfrak{A}$ is Dedekind complete' ensures that $c_\alpha = \sup_{u\in A} [\![u > \alpha]\!]$ is defined for each α. By 364L(a-i), $c = \inf_{n\in\mathbb{N}} c_n = \inf_{\alpha\in\mathbb{R}} c_\alpha$ is non-zero. Now for any $n \ge 1$, $\alpha \in \mathbb{R}$

$$[\![\sup_{u\in A}(u \wedge n\chi c) > \alpha]\!] = \sup_{u\in A} [\![u > \alpha]\!] \cap [\![\chi c > \tfrac{\alpha}{n}]\!] = [\![\chi c > \tfrac{\alpha}{n}]\!],$$

because if $\alpha \ge 0$ then

$$\sup_{u\in A} [\![u > \alpha]\!] = c_\alpha \supseteq c \supseteq [\![\chi c > \tfrac{\alpha}{n}]\!],$$

while if $\alpha < 0$ then (because A is a non-empty subset of $(L^0)^+$)

$$\sup_{u\in A} [\![u > \alpha]\!] = 1 = [\![\chi c > \tfrac{\alpha}{n}]\!].$$

So $\sup_{u\in A} u \wedge n\chi c = n\chi c$ for every $n \ge 1$, and we can take $v = \chi c$. (The case $n = 0$ is of course trivial.)

368B Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, U an Archimedean Riesz space, V an order-dense Riesz subspace of U and $T : V \to L^0 = L^0(\mathfrak{A})$ an order-continuous Riesz homomorphism. Then T has a unique extension to an order-continuous Riesz homomorphism $\tilde{T} : U \to L^0$.

proof (a) The key to the proof is the following: if $u \ge 0$ in U, then $\{Tv : v \in V, 0 \le v \le u\}$ is bounded above in L^0. **P?** Suppose, if possible, otherwise. Then by 368A there is a $w > 0$ in L^0 such that $nw = \sup_{v\in A} nw \wedge Tv$ for every $n \in \mathbb{N}$, where $A = \{v : v \in V, 0 \le v \le u\}$. In particular, there is a $v_0 \in A$ such that $w_0 = w \wedge Tv_0 > 0$. Because U is Archimedean, $\inf_{k\ge1} \frac{1}{k}u = 0$, so $v_0 = \sup_{k\ge1}(v_0 - \frac{1}{k}u)^+$. Because V is order-dense in U, $v_0 = \sup B$ where

$$B = \{v : v \in V, 0 \le v \le (v_0 - \frac{1}{k}u)^+ \text{ for some } k \ge 1\}.$$

Because T is order-continuous, $Tv_0 = \sup T[B]$ in L^0, and there is a $v_1 \in B$ such that $w_1 = w_0 \wedge Tv_1 > 0$. Let $k \ge 1$ be such that $v_1 \le (v_0 - \frac{1}{k}u)^+$. Then for any $m \in \mathbb{N}$,

$$mv_1 \wedge u \le (mv_1 \wedge kv_0) + (mv_1 \wedge (u - kv_0)^+)$$

(352Fa)

$$\le kv_0 + (m + k)(v_1 \wedge (\frac{1}{k}u - v_0)^+) = kv_0.$$

So for any $v \in A$, $m \in \mathbb{N}$,

$$mw_1 \wedge Tv = mw_1 \wedge mTv_1 \wedge Tv \le T(mv_1 \wedge v) \le T(mv_1 \wedge u) \le T(kv_0) = kTv_0.$$

But this means that, for $m \in \mathbb{N}$,

$$mw_1 = mw_1 \wedge mw = \sup_{v\in A} mw_1 \wedge (mw \wedge Tv) = \sup_{v\in A} mw_1 \wedge Tv \le kTv_0,$$

which is impossible because L^0 is Archimedean and $w_1 > 0$. **XQ**

(b) Because L^0 is Dedekind complete, $\sup\{Tv : v \in V, 0 \le v \le u\}$ is defined in L^0 for every $u \in U$. By 355F, T has a unique extension to an order-continuous Riesz homomorphism from U to L^0.

368C Corollary Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind complete Boolean algebras and U, V order-dense Riesz subspaces of $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ respectively. Then any Riesz space isomorphism between U and V extends uniquely to a Riesz space isomorphism between $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$; and in this case $\mathfrak{A}$ and $\mathfrak{B}$ must be isomorphic as Boolean algebras.

proof If $T : U \to V$ is a Riesz space isomorphism, then 368B tells us that we have (unique) order-continuous Riesz homomorphisms $\tilde{T} : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ and $\tilde{T}' : L^0(\mathfrak{B}) \to L^0(\mathfrak{A})$ extending T, T^{-1} respectively. Now $\tilde{T}'\tilde{T} : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is an order-continuous Riesz homomorphism agreeing with the identity on U, so must be the identity on $L^0(\mathfrak{A})$; similarly $\tilde{T}\tilde{T}'$ is the identity on $L^0(\mathfrak{B})$, and $\tilde{T}$ is a Riesz space isomorphism. To see that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, recall that by 364O they can be identified with the algebras of projection bands of $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$, which must be isomorphic.

368D Corollary Suppose that $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, and that U is an order-dense Riesz subspace of $L^0(\mathfrak{A})$ which is isomorphic, as Riesz space, to $L^0(\mathfrak{B})$ for some Dedekind complete Boolean algebra $\mathfrak{B}$. Then $U = L^0(\mathfrak{A})$ and $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ (so, in particular, is Dedekind complete).

proof The identity mapping $U \to U$ is surely an order-continuous Riesz homomorphism, so by 368B extends to an order-continuous Riesz homomorphism $\tilde{T} : L^0(\mathfrak{A}) \to U$. Now $\tilde{T}$ must be injective, because if $u \ne 0$ in $L^0(\mathfrak{A})$ there is a $u' \in U$ such that $0 < u' \le |u|$, so that $0 < u' \le |\tilde{T}u|$. So we must have $U = L^0(\mathfrak{A})$ and $\tilde{T}$ the identity map. By 364O again, $\mathfrak{A} \cong \mathfrak{B}$.

368E Theorem Let U be any Archimedean Riesz space, and $\mathfrak{A}$ its band algebra (353B). Then U can be embedded as an order-dense Riesz subspace of $L^0(\mathfrak{A})$.

proof (a) If $U=\{0\}$ then $\mathfrak{A}=\{0\}$, $L^0=L^0(\mathfrak{A})=\{0\}$ and the result is trivial; I shall therefore suppose henceforth that U is non-trivial. Note that by 352Q $\mathfrak{A}$ is Dedekind complete.

Let $C\subseteq U^+\setminus\{0\}$ be a maximal disjoint set (in the sense of 352C); to obtain such a set apply Zorn's lemma to the family of all disjoint subsets of $U^+\setminus\{0\}$. Now I can write down the formula for the embedding $T:U\to L^0$ immediately, though there will be a good deal of work to do in justification: for $u\in U$ and $\alpha\in\mathbb{R}$, $[\![Tu>\alpha]\!]$ will be the band in U generated by

$$\{e\wedge(u-\alpha e)^+:e\in C\}.$$

(For once, I allow myself to use the formula $[\![\ldots]\!]$ without checking immediately that it represents a member of L^0; all I claim for the moment is that $[\![Tu>\alpha]\!]$ is a member of $\mathfrak{A}$ determined by u and α.)

(b) Before getting down to the main argument, I make some remarks which will be useful later.

(i) If $u>0$ in U, then there is some $e\in C$ such that $u\wedge e>0$, since otherwise we ought to have added u to C. Thus $C^\perp=\{0\}$.

(ii) If $u\in U$ and $e\in C$ and $\alpha\in\mathbb{R}$, then $v=e\wedge(\alpha e-u)^+$ belongs to $[\![Tu>\alpha]\!]^\perp$. **P** If $e'\in C$, then either $e'\neq e$ so

$$v\wedge e'\wedge(u-\alpha e')^+\le e\wedge e'=0,$$

or $e'=e$ and

$$v\wedge e'\wedge(u-\alpha e')^+\le(\alpha e-u)^+\wedge(u-\alpha e)^+=0.$$

Accordingly $[\![Tu>\alpha]\!]$ is included in the band $\{v\}^\perp$ and $v\in[\![Tu>\alpha]\!]^\perp$. **Q**

(c) Now I must confirm that the formula given for $[\![Tu>\alpha]\!]$ is consistent with the conditions laid down in 364Aa. **P** Take $u\in U$.

(i) If $\alpha\le\beta$ then

$$0\le e\wedge(u-\beta e)^+\le e\wedge(u-\alpha e)^+\in[\![Tu>\alpha]\!]$$

so $e\wedge(u-\beta e)^+\in[\![Tu>\alpha]\!]$, for every $e\in C$, and $[\![Tu>\beta]\!]\subseteq[\![Tu>\alpha]\!]$.

(ii) Given $\alpha\in\mathbb{R}$, set $W=\sup_{\beta>\alpha}[\![Tu>\beta]\!]$ in $\mathfrak{A}$, that is, the band in U generated by $\{e\wedge(u-\beta e)^+:e\in C,\,\beta>\alpha\}$. Then for each $e\in C$,

$$\sup_{\beta>\alpha}e\wedge(u-\beta e)^+=e\wedge(u-\inf_{\beta>\alpha}\beta e)^+=e\wedge(u-\alpha e)^+$$

using the general distributive laws in U (352E), the translation-invariance of the order (351D) and the fact that U is Archimedean (to see that $\alpha e=\inf_{\beta>\alpha}\beta e$). So $e\wedge(u-\alpha e)^+\in W$; as e is arbitrary, $[\![Tu>\alpha]\!]\subseteq W$ and $[\![Tu>\alpha]\!]=W$.

(iii) Now set $W=\inf_{n\in\mathbb{N}}[\![Tu>n]\!]$. For any $e\in C$, $n\in\mathbb{N}$ we have

$$e\wedge(ne-u)^+\in[\![Tu>n]\!]^\perp\subseteq W^\perp,$$

so that

$$e\wedge(e-\tfrac{1}{n}u^+)^+\le e\wedge(e-\tfrac{1}{n}u)^+\in W^\perp$$

for every $n\ge 1$ and

$$e=\sup_{n\ge 1}e\wedge(e-\tfrac{1}{n}u^+)^+\in W^\perp.$$

Thus $C\subseteq W^\perp$ and $W\subseteq C^\perp=\{0\}$. So we have $\inf_{n\in\mathbb{N}}[\![Tu>n]\!]=0$.

(iv) Finally, set $W=\sup_{n\in\mathbb{N}}[\![Tu>-n]\!]$. Then

$$e\wedge(e-\tfrac{1}{n}u^-)^+\le e\wedge(e+\tfrac{1}{n}u)^+\le e\wedge(u+ne)^+\in W$$

for every $n\ge 1$ and $e\in C$, so

$$e=\sup_{n\ge 1}e\wedge(e-\tfrac{1}{n}u^-)^+\in W$$

for every $e\in C$ and $W^\perp=\{0\}$, $W=U$. Thus all three conditions of 364Aa are satisfied. **Q**

(d) Thus we have a well-defined map $T : U \to L^0$. I show next that $T(u+v) = Tu + Tv$ for all u, $v \in U$. **P** I rely on the formulae in 364D and 364Ea, and on partitions of unity in $\mathfrak{A}$, constructed as follows. Fix $n \ge 1$ for the moment. Then we know that

$$\sup_{i\in\mathbb{Z}} [\![Tu > \tfrac{i}{n}]\!] = 1, \quad \inf_{i\in\mathbb{Z}} [\![Tu > \tfrac{i}{n}]\!] = 0.$$

So setting

$$V_i = [\![Tu > \tfrac{i}{n}]\!] \setminus [\![Tu > \tfrac{i+1}{n}]\!] = [\![Tu > \tfrac{i}{n}]\!] \cap [\![Tu > \tfrac{i+1}{n}]\!]^\perp,$$

$\langle V_i\rangle_{i\in\mathbb{Z}}$ is a partition of unity in $\mathfrak{A}$. Similarly, $\langle W_i\rangle_{i\in\mathbb{Z}}$ is a partition of unity, where

$$W_i = [\![Tv > \tfrac{i}{n}]\!] \cap [\![Tv > \tfrac{i+1}{n}]\!]^\perp.$$

Now, for any i, j, $k \in \mathbb{Z}$ such that $i + j \ge k$,

$$V_i \cap W_j \subseteq [\![Tu > \tfrac{i}{n}]\!] \cap [\![Tv > \tfrac{j}{n}]\!] \subseteq [\![Tu + Tv > \tfrac{i+j}{n}]\!] \subseteq [\![Tu + Tv > \tfrac{k}{n}]\!];$$

thus

$$[\![Tu + Tv > \tfrac{k}{n}]\!] \supseteq \sup_{i+j\ge k} V_i \cap W_j.$$

On the other hand, if $q \in \mathbb{Q}$ and $k \in \mathbb{Z}$, there is an $i \in \mathbb{Z}$ such that $\frac{i}{n} \le q < \frac{i+1}{n}$, so that

$$[\![Tu > q]\!] \cap [\![Tv > \tfrac{k+1}{n} - q]\!] \subseteq [\![Tu > \tfrac{i}{n}]\!] \cap [\![Tv > \tfrac{k-i}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j;$$

thus for any $k \in \mathbb{Z}$

$$[\![Tu + Tv > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j \subseteq [\![Tu + Tv > \tfrac{k}{n}]\!].$$

Also, if $0 < w \in V_i \cap W_j$ and $e \in C$ then

$$w \wedge e \wedge (u - \frac{i+1}{n}e)^+ = w \wedge e \wedge (v - \frac{j+1}{n}e)^+ = 0,$$

so that

$$w \wedge e \wedge (u + v - \frac{i+j+2}{n}e)^+ = 0$$

because

$$(u + v - \frac{i+j+2}{n}e)^+ \le (u - \frac{i+1}{n}e)^+ + (v - \frac{j+1}{n}e)^+$$

by 352Fc. But this means that $V_i \cap W_j \cap [\![T(u+v) > \frac{i+j+2}{n}]\!] = \{0\}$. Turning this round,

$$[\![T(u+v) > \tfrac{k+1}{n}]\!] \cap \sup_{i+j\le k-1} V_i \cap W_j = 0,$$

and because $\sup_{i,j\in\mathbb{Z}} V_i \cap W_j = U$ in $\mathfrak{A}$,

$$[\![T(u+v) > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j.$$

Finally, if $i + j \ge k$ and $0 < w \in V_i \cap V_j$, then there is an $e \in C$ such that $w_1 = w \wedge e \wedge (u - \frac{i}{n}e)^+ > 0$; there is an $e' \in C$ such that $w_2 = w_1 \wedge e' \wedge (v - \frac{j}{n}e')^+ > 0$; of course $e = e'$, and

$$\begin{aligned} 0 < w_2 \le e \wedge (u - \frac{i}{n}e)^+ \wedge (v - \frac{j}{n})^+ &\le e \wedge (u + v - \frac{i+j}{n}e)^+ \\ &\in [\![T(u+v) > \frac{i+j}{n}]\!] \subseteq [\![T(u+v) > \frac{k}{n}]\!] \end{aligned}$$

using 352Fc. This shows that $w \notin [\![T(u+v) > \frac{k}{n}]\!]^\perp$; as w is arbitrary, $V_i \cap W_j \subseteq [\![T(u+v) > \frac{k}{n}]\!]$; so we get

$$\sup_{i+j\ge k} V_i \cap W_j \subseteq [\![T(u+v) > \tfrac{k}{n}]\!].$$

Putting these four facts together, we see that

$$[\![T(u+v) > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j \subseteq [\![Tu + Tv > \tfrac{k}{n}]\!],$$

$$[\![Tu + Tv > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j \subseteq [\![T(u+v) > \tfrac{k}{n}]\!]$$

for all $n \ge 1$ and $k \in \mathbb{Z}$. But this means that we must have

$$[\![T(u+v)>\beta]\!] \subseteq [\![Tu+Tv>\alpha]\!], \quad [\![Tu+Tv>\beta]\!] \subseteq [\![T(u+v)>\alpha]\!]$$

whenever $\alpha<\beta$. Consequently

$$\begin{aligned}[\![Tu+Tv>\alpha]\!] &= \sup_{\beta>\alpha} [\![Tu+Tv>\beta]\!] \subseteq [\![T(u+v)>\alpha]\!] \\ &= \sup_{\beta>\alpha} [\![T(u+v)>\beta]\!] \subseteq [\![Tu+Tv>\alpha]\!]\end{aligned}$$

and $[\![Tu+Tv>\alpha]\!] = [\![T(u+v)>\alpha]\!]$ for every α, that is, $T(u+v)=Tu+Tv$. **Q**

(e) The hardest part is over. If $u\in U$, $\gamma>0$ and $\alpha\in\mathbb{R}$, then for any $e\in C$

$$\min(1,\frac{1}{\gamma})(e\wedge(\gamma u-\alpha e)^+) \le e\wedge(u-\frac{\alpha}{\gamma}e)^+ \le \max(1,\frac{1}{\gamma})(e\wedge(\gamma u-\alpha e)^+),$$

so

$$[\![T(\gamma u)>\alpha]\!] = [\![Tu>\tfrac{\alpha}{\gamma}]\!] = [\![\gamma Tu>\alpha]\!];$$

as α is arbitrary, $\gamma Tu = T(\gamma u)$; as γ and u are arbitrary, T is linear. (We need only check linearity for $\gamma>0$ because we know from the additivity of T that $T(-u)=-Tu$ for every u.)

(f) To see that T is a Riesz homomorphism, take any $u\in U$ and $\alpha\in\mathbb{R}$ and consider the band $[\![Tu>\alpha]\!]\cup[\![-Tu>\alpha]\!] = [\![|Tu|>\alpha]\!]$ (by 364L(a-ii)). This is the band generated by $\{e\wedge(u-\alpha e)^+ : e\in C\}\cup\{e\wedge(-u-\alpha e)^+ : e\in C\}$. But this must also be the band generated by

$$\{(e\wedge(u-\alpha e)^+)\vee(e\wedge(-u-\alpha e)^+) : e\in C\} = \{e\wedge(|u|-\alpha e)^+ : e\in C\},$$

which is $[\![T|u|>\alpha]\!]$. Thus $[\![|Tu|>\alpha]\!] = [\![T|u|>\alpha]\!]$ for every α and $|Tu|=T|u|$. As u is arbitrary, T is a Riesz homomorphism.

(g) To see that T is injective, take any non-zero $u\in U$. Then there must be some $e\in C$ such that $|u|\wedge e\ne 0$, and some $\alpha>0$ such that $|u|\wedge e\not\le\alpha e$, so that $e\wedge(|u|-\alpha e)^+\ne 0$ and $[\![T|u|>\alpha]\!]\ne\{0\}$ and $T|u|\ne 0$ and $Tu\ne 0$. Thus T embeds U as a Riesz subspace of L^0.

(h) Finally, I must check that $T[U]$ is order-dense in L^0. **P** Let $p>0$ in L^0. Then there is some $\alpha>0$ such that $V=[\![p>\alpha]\!]\ne 0$. Take $u>0$ in V. Let $e\in C$ be such that $u\wedge e>0$. Then $v=u\wedge\alpha e>0$. Now $e\wedge(v-\alpha e)^+=0$; but also $e'\wedge v=0$ for every $e'\in C$ distinct from e, so that $[\![Tv>\alpha]\!]=\{0\}$. Next, $v\in V$, so $e'\wedge(v-\beta e')^+\in V$ whenever $e'\in C$ and $\beta\ge 0$, and $[\![Tv>\beta]\!]\subseteq V$ for every $\beta\ge 0$. Accordingly we have

$$\begin{aligned}[\![Tv>\beta]\!] &= \{0\}\subseteq[\![p>\beta]\!] \text{ if } \beta\ge\alpha, \\ &\subseteq V\subseteq[\![p>\beta]\!] \text{ if } 0\le\beta<\alpha, \\ &= U = [\![p>\beta]\!] \text{ if } \beta<0,\end{aligned}$$

and $Tv\le p$. Also $Tv>0$, by (g). As p is arbitrary, $T[U]$ is order-dense in L^0. **Q**

368F Corollary A Riesz space U is Archimedean iff it is isomorphic to a Riesz subspace of some reduced power $\mathbb{R}^X|\mathcal{F}$, where X is a set and $\mathcal{F}$ is a filter on X such that $\bigcap_{n\in\mathbb{N}}F_n\in\mathcal{F}$ whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$.

proof (a) If U is an Archimedean Riesz space, then by 368E there is a space of the form $L^0=L^0(\mathfrak{A})$ such that U can be embedded into L^0. As in the proof of 364D, L^0 is isomorphic to some space of the form $\mathcal{L}^0(\Sigma)/\mathcal{W}$, where Σ is a σ-algebra of subsets of some set X and $\mathcal{W}=\{f : f\in\mathcal{L}^0,\ \{x : f(x)\ne 0\}\in\mathcal{I}\}$, $\mathcal{I}$ being a σ-ideal of Σ. But now $\mathcal{F}=\{A : A\cup E=X \text{ for some } E\in\mathcal{I}\}$ is a filter on X such that $\bigcap_{n\in\mathbb{N}}F_n\in\mathcal{F}$ for every sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$. (I am passing over the trivial case $X\in\mathcal{I}$, since then U must be $\{0\}$.) And $\mathcal{L}^0/\mathcal{W}$ is (isomorphic to) the image of $\mathcal{L}^0$ in $\mathbb{R}^X|\mathcal{F}$, since $\mathcal{W}=\{f : f\in\mathcal{L}^0,\ \{x : f(x)=0\}\in\mathcal{F}\}$. Thus U is isomorphic to a Riesz subspace of $\mathbb{R}^X|\mathcal{F}$.

(b) On the other hand, if $\mathcal{F}$ is a filter on X closed under countable intersections, then $\mathcal{W}=\{f : f\in\mathbb{R}^X,\ \{x : f(x)=0\}\in\mathcal{F}\}$ is a sequentially order-closed solid linear subspace of the Dedekind σ-complete Riesz space $\mathbb{R}^X$, so that $\mathbb{R}^X|\mathcal{F}=\mathbb{R}^X/\mathcal{W}$ is Dedekind σ-complete (353J(a-iii)) and all its Riesz subspaces must be Archimedean (353Ha, 351Rc).

368G Corollary Every Archimedean Riesz space U is isomorphic to an order-dense Riesz subspace of some space $C^\infty(X)$ (definition: 364V), where X is an extremally disconnected compact Hausdorff space.

proof Let Z be the Stone space of the band algebra $\mathfrak{A}$ of U. Because $\mathfrak{A}$ is Dedekind complete (352Q), Z is extremally disconnected and $\mathfrak{A}$ can be identified with the regular open algebra $\mathrm{RO}(Z)$ of Z (314S). By 364V, $L^0(\mathrm{RO}(Z))$ can be identified with $C^\infty(Z)$. So an embedding of U as an order-dense Riesz subspace of $L^0(\mathfrak{A})$ (368E) can be regarded as an embedding of U as an order-dense Riesz subspace of $C^\infty(Z)$.

368H Corollary Any Dedekind complete Riesz space U is isomorphic to an order-dense solid linear subspace of $L^0(\mathfrak{A})$ for some Dedekind complete Boolean algebra $\mathfrak{A}$.

proof Embed U in $L^0 = L^0(\mathfrak{A})$ as in 368E; because U is order-dense in L^0 and (in itself) Dedekind complete, it is solid (353K).

368I Corollary Let U be an Archimedean Riesz space. Then U can be embedded as an order-dense Riesz subspace of a Dedekind complete Riesz space V in such a way that the solid linear subspace of V generated by U is V itself, and this can be done in essentially only one way. If W is any other Dedekind complete Riesz space and $T : U \to W$ is an order-continuous positive linear operator, there is a unique positive linear operator $\tilde{T} : V \to W$ extending T.

proof By 368E, we may suppose that U is actually an order-dense Riesz subspace of $L^0(\mathfrak{A})$, where $\mathfrak{A}$ is a Dedekind complete Boolean algebra. In this case, we can take V to be the solid linear subspace generated by U, that is, $\{v : |v| \le u \text{ for some } u \in U\}$; being a solid linear subspace of the Dedekind complete Riesz space $L^0(\mathfrak{A})$, V is Dedekind complete, and of course U is order-dense in V.

If W is any other Dedekind complete Riesz space and $T : U \to W$ is an order-continuous positive linear operator, then for any $v \in V^+$ there is a $u_0 \in U$ such that $v \le u_0$, so that Tu_0 is an upper bound for $\{Tu : u \in U,\, 0 \le u \le v\}$; as W is Dedekind complete, $\sup_{u \in U, 0 \le u \le v} Tu$ is defined in W. By 355Fa, T has a unique extension to an order-continuous positive linear operator from V to W.

In particular, if V_1 is another Dedekind complete Riesz space in which U can be embedded as an order-dense Riesz subspace, this embedding of U extends to an embedding of V; since V is Dedekind complete, its copy in V_1 must be a solid linear subspace, so if V_1 is the solid linear subspace of itself generated by U, we get an identification between V and V_1, uniquely determined by the embeddings of U in V and V_1.

368J Definition If U is an Archimedean Riesz space, a **Dedekind completion** of U is a Dedekind complete Riesz space V together with an embedding of U in V as an order-dense Riesz subspace of V such that the solid linear subspace of V generated by U is V itself. 368I tells us that every Archimedean Riesz space U has an essentially unique Dedekind completion, so that we may speak of 'the' Dedekind completion of U.

368K This is a convenient point at which to give a characterization of the Riesz spaces $L^0(\mathfrak{A})$.

Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Suppose that $A \subseteq L^0(\mathfrak{A})^+$ is disjoint. If *either* A is countable *or* $\mathfrak{A}$ is Dedekind complete, A is bounded above in $L^0(\mathfrak{A})$.

proof If $A = \emptyset$, this is trivial; suppose that A is not empty. For $n \in \mathbb{N}$, set $a_n = \sup_{u \in A} [\![u > n]\!]$; this is always defined; set $a = \inf_{n \in \mathbb{N}} a_n$. Now $a = 0$. **P?** Otherwise, there must be a $u \in A$ such that $a' = a \cap [\![u > 0]\!] \ne 0$, since $a \subseteq a_0$. But now, for any n, and any $v \in A \setminus \{u\}$,

$$a' \cap [\![v > n]\!] \subseteq [\![u > 0]\!] \cap [\![v > 0]\!] = 0,$$

so that $a' \subseteq [\![u > n]\!]$. As n is arbitrary, $\inf_{n \in \mathbb{N}} [\![u > n]\!] \ne 0$, which is impossible. **XQ**

By 364L(a-i), A is bounded above.

368L Definition A Riesz space U is called **laterally complete** or **universally complete** if A is bounded above whenever $A \subseteq U^+$ is disjoint.

368M Theorem Let U be an Archimedean Riesz space. Then the following are equiveridical:

(i) there is a Dedekind complete Boolean algebra $\mathfrak{A}$ such that U is isomorphic to $L^0(\mathfrak{A})$;

(ii) U is Dedekind σ-complete and laterally complete;

(iii) whenever V is an Archimedean Riesz space, V_0 is an order-dense Riesz subspace of V and $T : V_0 \to U$ is an order-continuous Riesz homomorphism, there is a positive linear operator $\tilde{T} : V \to U$ extending T.

proof (a)(i)⇒(ii) and **(i)⇒(iii)** are covered by 368K and 368B.

(b)(ii)⇒(i) Assume (ii). By 368E, we may suppose that U is actually an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$ for a Dedekind complete Boolean algebra $\mathfrak{A}$.

(α) If $u \in U^+$ and $a \in \mathfrak{A}$ then $u \times \chi a \in U$. **P** Set $A = \{v : v \in U,\, 0 \le v \le \chi a\}$, and let $C \subseteq A$ be a maximal disjoint set; then $w = \sup C$ is defined in U, and is also the supremum in L^0. Set $b = [\![w > 0]\!]$. As $w \le \chi a$, $b \subseteq a$. **?** If $b \ne a$, then $\chi(a \setminus b) > 0$, and there is a $v' \in U$ such that $0 < v' \le \chi(a \setminus b)$; but now $v' \in A$ and $v' \wedge w = 0$, so $v' \wedge v = 0$ for every $v \in C$, and we ought to have added v' to C. **X** Thus $[\![w > 0]\!] = a$.

Now consider $u' = \sup_{n\in\mathbb{N}} u \wedge nw$; as U is Dedekind σ-complete, $u' \in U$. Since $[\![u' > 0]\!] \subseteq a$, $u' \le u \times \chi a$. On the other hand,

$$u \times \chi[\![w > \tfrac{1}{n}]\!] \times \chi[\![u \le n]\!] \le u \wedge n^2 w \le u'$$

for every $n \ge 1$, so, taking the supremum over n, $u \times \chi a \le u'$. Accordingly

$$u \times \chi a = u' \in U,$$

as required. **Q**

(β) If $w \ge 0$ in L^0, there is a $u \in U$ such that $\frac{1}{2}w \le u \le w$. **P** Set

$$A = \{u : u \in U,\, 0 \le u \le w\},$$

$$C = \{a : a \in \mathfrak{A},\, a \subseteq [\![u - \tfrac{1}{2}w \ge 0]\!] \text{ for some } u \in A\}.$$

Then $\sup A = w$, so C is order-dense in $\mathfrak{A}$. (If $a \in \mathfrak{A} \setminus \{0\}$, either $a \cap [\![w > 0]\!] = 0$ and $a \subseteq [\![0 - \frac{1}{2}w \ge 0]\!]$, so $a \in C$, or there is a $u \in U$ such that $0 < u \le w \times \chi a$. In the latter case there is some n such that $2^n u \le w$ and $2^{n+1}u \not\le w$, and now $c = a \cap [\![2^n u - \frac{1}{2}w \ge 0]\!]$ is a non-zero member of C included in a.) Let $D \subseteq C$ be a partition of unity and for each $d \in D$ choose $u_d \in A$ such that $d \subseteq [\![u_d - \frac{1}{2}w \ge 0]\!]$. By (α), $u_d \times \chi d \in U$ for every $d \in D$, so $u = \sup_{d\in D} u_d \times \chi d \in U$. Now $u \le w$, but also $[\![u - \frac{1}{2}w \ge 0]\!] \supseteq d$ for every $d \in D$, so is equal to 1, and $u \ge \frac{1}{2}w$, as required. **Q**

(γ) Given $w \ge 0$ in L^0, we can therefore choose $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ inductively such that $v_0 = 0$ and

$$u_n \in U, \quad \frac{1}{2}(w - v_n) \le u_n \le w - v_n, \quad v_{n+1} = v_n + u_n$$

for every $n \in \mathbb{N}$. Now $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in U and $w - v_n \le 2^{-n}w$ for every n, so $w = \sup_{n\in\mathbb{N}} v_n \in U$.

As w is arbitrary, $(L^0)^+ \subseteq U$ and $U = L^0$ is of the right form.

(c)(iii)⇒(i) As in (b), we may suppose that U is an order-dense Riesz subspace of L^0. But now apply condition (iii) with $V = L^0$, $V_0 = U$ and T the identity operator. There is an extension $\tilde{T} : L^0 \to U$. If $v \ge 0$ in L^0, $\tilde{T}v \ge Tu = u$ whenever $u \in U$ and $u \le v$, so $\tilde{T}v \ge v$, since $v = \sup\{u : u \in U,\, 0 \le u \le v\}$ in L^0. Similarly, $\tilde{T}(\tilde{T}v - v) \ge \tilde{T}v - v$. But as $\tilde{T}v \in U$, $\tilde{T}(\tilde{T}v) = T(\tilde{T}v) = \tilde{T}v$ and $\tilde{T}(\tilde{T}v - v) = 0$, so $v = \tilde{T}v \in U$. As v is arbitrary, $U = L^0$.

368N Weakly (σ,∞)-distributive Riesz spaces We are now ready to look at the class of Riesz spaces corresponding to the weakly (σ,∞)-distributive Boolean algebras of §316.

Definition Let U be a Riesz space. Then U is **weakly (σ,∞)-distributive** if whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty downwards-directed subsets of U^+, each with infimum 0, and $\bigcup_{n\in\mathbb{N}} A_n$ has an upper bound in U, then

$$\{u : u \in U, \text{ for every } n \in \mathbb{N} \text{ there is a } v \in A_n \text{ such that } v \le u\}$$

has infimum 0 in U.

Remark Because the definition looks only at sequences $\langle A_n\rangle_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} A_n$ is order-bounded, we can invert it, as follows: a Riesz space U is weakly (σ,∞)-distributive iff whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty upwards-directed subsets of U^+, all with supremum u_0, then

$$\{u : u \in U^+, \text{ for every } n \in \mathbb{N} \text{ there is a } v \in A_n \text{ such that } u \leq v\}$$

also has supremum u_0.

368O Lemma Let U be an Archimedean Riesz space. Then the following are equiveridical:

(i) U is not weakly (σ, ∞)-distributive;

(ii) there are a $u > 0$ in U and a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of non-empty downwards-directed sets, all with infimum 0, such that $\sup_{n \in \mathbb{N}} u_n = u$ whenever $u_n \in A_n$ for every $n \in \mathbb{N}$.

proof (ii)⇒(i) is immediate from the definition of 'weakly (σ, ∞)-distributive'. For (i)⇒(ii), suppose that U is not weakly (σ, ∞)-distributive. Then there is a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of non-empty downwards-directed sets, all with infimum 0, such that $\bigcup_{n \in \mathbb{N}} A_n$ is bounded above, but

$$A = \{w : w \in U, \text{ for every } n \in \mathbb{N} \text{ there is a } v \in A_n \text{ such that } v \leq w\}$$

does not have infimum 0. Let $u > 0$ be a lower bound for A, and set $A'_n = \{u \wedge v : v \in A_n\}$ for each $n \in \mathbb{N}$. Then each A'_n is a non-empty downwards-directed set with infimum 0. Let $\langle u_n \rangle_{n \in \mathbb{N}}$ be a sequence such that $u_n \in A'_n$ for every n. Express each u_n as $u \wedge v_n$ where $v_n \in A_n$. Let B be the set of upper bounds of $\{v_n : n \in \mathbb{N}\}$. Then $\inf_{w \in B, n \in \mathbb{N}} w - v_n = 0$, because U is Archimedean (353F), while $B \subseteq A$, so $u \leq w$ for every $w \in B$. If u' is any upper bound for $\{u_n : n \in \mathbb{N}\}$, then

$$u - u' \leq u - u \wedge v_n = (u - v_n)^+ \leq (w - v_n)^+ = w - v_n$$

whenever $n \in \mathbb{N}$ and $w \in B$. So $u' \geq u$. Thus $u = \sup_{n \in \mathbb{N}} u_n$. As $\langle u_n \rangle_{n \in \mathbb{N}}$ is arbitrary, u and $\langle A'_n \rangle_{n \in \mathbb{N}}$ witness that (ii) is true.

368P Proposition (a) A regularly embedded Riesz subspace of an Archimedean weakly (σ, ∞)-distributive Riesz space is weakly (σ, ∞)-distributive.

(b) An Archimedean Riesz space with a weakly (σ, ∞)-distributive order-dense Riesz subspace is weakly (σ, ∞)-distributive.

(c) If U is a Riesz space such that $U^\times$ separates the points of U, then U is weakly (σ, ∞)-distributive; in particular, $U^\sim$ and $U^\times$ are weakly (σ, ∞)-distributive for every Riesz space U.

proof (a) Suppose that U is an Archimedean Riesz space and that $V \subseteq U$ is a regularly embedded Riesz subspace which is not weakly (σ, ∞)-distributive. Then 368O tells us that there are a $v > 0$ in V and a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of non-empty downwards-directed subsets of V, all with infimum 0 in V, such that $\sup_{n \in \mathbb{N}} v_n = v$ in V whenever $v_n \in A_n$ for every $n \in \mathbb{N}$. Because V is regularly embedded in U, $\inf A_n = 0$ in U for every n and $\sup_{n \in \mathbb{N}} v_n = v$ in U for every sequence $\langle v_n \rangle_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} A_n$, so U is not weakly (σ, ∞)-distributive. Turning this round, we have (a).

(b) Let U be an Archimedean Riesz space which is not weakly (σ, ∞)-distributive, and V an order-dense Riesz subspace of U. By 368O again, there are a $u^* > 0$ in U and a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of non-empty downwards-directed sets in U, all with infimum 0, such that $\sup_{n \in \mathbb{N}} u_n = u^*$ whenever $u_n \in A_n$ for every n. Let $v \in V$ be such that $0 < v \leq u^*$. Set

$$B_n = \{w : w \in V, \text{ there is some } u \in A_n \text{ such that } v \wedge u \leq w \leq v\}$$

for each $n \in \mathbb{N}$. Because A_n is downwards-directed, $w \wedge w' \in B_n$ for all w, $w' \in B_n$; $v \in B_n$, so $B_n \neq \emptyset$; and $\inf B_n = 0$ in V. **P** Setting

$$C = \{w : w \in V^+, \text{ there is some } u \in A_n \text{ such that } w \leq (v - u)^+\},$$

then (because V is order-dense) any upper bound for C in U is also an upper bound of $\{(v - u)^+ : u \in A_n\}$. But

$$\sup_{u \in A_n} (v - u)^+ = (v - \inf A_n)^+ = v,$$

so $v = \sup C$ in U and $\inf B_n = \inf\{v - w : w \in C\} = 0$ in U and in V. **Q**

Now if $v_n \in B_n$ for every $n \in \mathbb{N}$, we can choose $u_n \in A_n$ such that $v \wedge u_n \leq v_n \leq v$ for every n, so that

$$v = v \wedge u^* = v \wedge \sup_{n \in \mathbb{N}} u_n = \sup_{n \in \mathbb{N}} v \wedge u_n \leq \sup_{n \in \mathbb{N}} v_n \leq v,$$

and $v = \sup_{n \in \mathbb{N}} v_n$. Thus $\langle B_n \rangle_{n \in \mathbb{N}}$ witnesses that V is not weakly (σ, ∞)-distributive.

(c) Now suppose that $U^\times$ separates the points of U. In this case U is surely Archimedean (356G). **?** If U is not weakly (σ, ∞)-distributive, there are a $u > 0$ in U and a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of non-empty downwards-directed sets, all with infimum 0, such that $\sup_{n \in \mathbb{N}} u_n = u$ whenever $u_n \in A_n$ for each n. Take $f \in U^\times$ such that $f(u) \neq 0$;

replacing f by $|f|$ if necessary, we may suppose that $f > 0$. Set $\delta = f(u) > 0$. For each $n \in \mathbb{N}$, there is a $u_n \in A_n$ such that $f(u_n) \le 2^{-n-2}\delta$. But in this case $\langle \sup_{i<n} u_i\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum u, so

$$f(u) = \lim_{n\to\infty} f(\sup_{i\le n} u_i) \le \sum_{i=0}^{\infty} f(u_i) \le \tfrac{1}{2}\delta < f(u),$$

which is absurd. **X** Thus U is weakly (σ,∞)-distributive.

For any Riesz space U, U acts on $U^\sim$ as a subspace of $U^{\sim\times}$ (356F); as U surely separates the points of $U^\sim$, so does $U^{\sim\times}$. So $U^\sim$ is weakly (σ,∞)-distributive. Now $U^\times$ is a band in $U^\sim$ (356B), so is regularly embedded, and must also be weakly (σ,∞)-distributive, by (a) above.

368Q Theorem (a) For any Boolean algebra $\mathfrak{A}$, $\mathfrak{A}$ is weakly (σ,∞)-distributive iff $S(\mathfrak{A})$ is weakly (σ,∞)-distributive iff $L^\infty(\mathfrak{A})$ is weakly (σ,∞)-distributive.

(b) For a Dedekind σ-complete Boolean algebra $\mathfrak{A}$, $L^0(\mathfrak{A})$ is weakly (σ,∞)-distributive iff $\mathfrak{A}$ is weakly (σ,∞)-distributive.

proof (a)(i) ? Suppose, if possible, that $\mathfrak{A}$ is weakly (σ,∞)-distributive but $S = S(\mathfrak{A})$ is not. By 368O, as usual, we have a $u > 0$ in S and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets in S, all with infimum 0, such that $u = \sup_{n\in\mathbb{N}} u_n$ whenever $u_n \in A_n$ for every n. Let $\alpha > 0$ be such that $c = [\![u > \alpha]\!] \ne 0$ (361Eg), and consider

$$B_n = \{[\![v > \alpha]\!] : v \in A_n\} \subseteq \mathfrak{A}$$

for each $n \in \mathbb{N}$. Then each B_n is downwards-directed (because A_n is), and $\inf B_n = 0$ in $\mathfrak{A}$ (because if b is a lower bound of B_n, $\alpha\chi b \le v$ for every $v \in A_n$). Because $\mathfrak{A}$ is weakly (σ,∞)-distributive, there must be some $a \in \mathfrak{A}$ such that $a \not\supseteq c$ but there is, for every $n \in \mathbb{N}$, a $b_n \in B_n$ such that $a \supseteq b_n$. Take $v_n \in A_n$ such that $b_n = [\![v_n > \alpha]\!]$, so that

$$v_n \le \alpha\chi 1 \vee \|v_n\|_\infty \chi b_n \le \alpha\chi 1 \vee \|u\|_\infty \chi a.$$

Since $u = \sup_{n\in\mathbb{N}} v_n$, $u \le \alpha\chi 1 \vee \|u\|_\infty\chi a$. But in this case

$$c = [\![u > \alpha]\!] \subseteq a,$$

contradicting the choice of a. **X**

Thus S must be weakly (σ,∞)-distributive if $\mathfrak{A}$ is.

(ii) Now suppose that S is weakly (σ,∞)-distributive, and let $\langle B_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty downwards-directed subsets of $\mathfrak{A}$, all with infimum 0. Set $A_n = \{\chi b : b \in B_n\}$ for each n; then $A_n \subseteq S$ is non-empty, downwards-directed and has infimum 0 in S, because $\chi : \mathfrak{A} \to S$ is order-continuous (361Ef). Set

$$A = \{v : v \in S, \text{ for every } n \in \mathbb{N} \text{ there is a } u \in A_n \text{ such that } u \le v\},$$

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in B_n \text{ such that } a \subseteq b\}.$$

? If 0 is not the greatest lower bound of B, take a non-zero lower bound c. Because S is weakly (σ,∞)-distributive, $\inf A = 0$, and there is a $v \in A$ such that $\chi c \not\le v$. Express v as $\sum_{i=0}^{n} \alpha_i\chi a_i$, where $\langle a_i\rangle_{i\le n}$ is disjoint, and set $a = \sup\{a_i : i \le n,\ \alpha_i \ge 1\}$; then $\chi a \le v$, so $c \not\subseteq a$. For each n there is a $b_n \in B_n$ such that $\chi b_n \le v$. But in this case $b_n \subseteq a$ for each $n \in \mathbb{N}$, so that $a \in B$; which means that c is not a lower bound for B. **X**

Thus $\inf B = 0$ in $\mathfrak{A}$. As $\langle B_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive.

(iii) Thus S is weakly (σ,∞)-distributive iff $\mathfrak{A}$ is. But S is order-dense in $L^\infty = L^\infty(\mathfrak{A})$ (363C), therefore regularly embedded (352Ne), so 368Pa-b tell us that S is weakly (σ,∞)-distributive iff L^∞ is.

(b) In the same way, because S can be regarded as an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$ (364Ja), L^0 is weakly (σ,∞)-distributive iff S is, that is, iff $\mathfrak{A}$ is.

368R Corollary An Archimedean Riesz space is weakly (σ,∞)-distributive iff its band algebra is weakly (σ,∞)-distributive.

proof Let U be an Archimedean Riesz space and $\mathfrak{A}$ its band algebra. By 368E, U is isomorphic to an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$. By 368P, U is weakly (σ,∞)-distributive iff L^0 is; and by 368Qb L^0 is weakly (σ,∞)-distributive iff $\mathfrak{A}$ is.

368S Corollary If $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra, any regularly embedded Riesz subspace (in particular, any solid linear subspace and any order-dense Riesz subspace) of $L^0(\mathfrak{A})$ is weakly (σ,∞)-distributive.

proof By 322F, $\mathfrak{A}$ is weakly (σ,∞)-distributive; by 368Qb, $L^0(\mathfrak{A})$ is weakly (σ,∞)-distributive; by 368Pa, any regularly embedded Riesz subspace is weakly (σ,∞)-distributive.

368X Basic exercises (a) Let X be an uncountable set and Σ the countable-cocountable σ-algebra of subsets of X. Show that there is a family $A \subseteq L^0 = L^0(\Sigma)$ such that $u \wedge v = 0$ for all distinct $u, v \in A$ but A has no upper bound in L^0. Show moreover that if $w > 0$ in L^0 then there is an $n \in \mathbb{N}$ such that $nw \neq \sup_{u \in A} u \wedge nw$.

(b) Let U be a linear space, $\mathfrak{A}$ a Dedekind complete Boolean algebra, and $p : U \to L^0 = L^0(\mathfrak{A})$ a function such that $p(u+v) \leq p(u) + p(v)$ and $p(\alpha u) = \alpha p(u)$ whenever $u, v \in U$ and $\alpha \geq 0$. Suppose that $V \subseteq U$ is a linear subspace and $T : V \to L^0$ is a linear operator such that $Tv \leq p(v)$ for every $v \in V$. Show that there is a linear operator $\tilde{T} : U \to L^0$, extending T, such that $\tilde{T}u \leq p(u)$ for every $u \in U$. (*Hint*: part A of the proof of 363R.)

(c) Let $\mathfrak{A}$ be any Boolean algebra, and $\widehat{\mathfrak{A}}$ its Dedekind completion (314U). Show that $L^\infty(\widehat{\mathfrak{A}})$ can be identified with the Dedekind completions of $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$.

(d) Explain how to prove 368K from 368A.

(e) Show that any product of weakly (σ, ∞)-distributive Riesz spaces is weakly (σ, ∞)-distributive.

(f) Let $\mathfrak{A}$ be a Dedekind complete weakly (σ, ∞)-distributive Boolean algebra. Show that a set $A \subseteq L^0 = L^0(\mathfrak{A})$ is order-bounded iff $\langle 2^{-n} u_n \rangle_{n \in \mathbb{N}}$ order*-converges to 0 in L^0 whenever $\langle u_n \rangle_{n \in \mathbb{N}}$ is a sequence in A. (*Hint*: use 368A. If $v > 0$ and $v = \sup_{u \in A} v \wedge 2^{-n} u$ for every n, we can find a $w > 0$ and a sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in A such that $w \leq 2^{-n} u_n$ for every n.)

(g) Give a direct proof of 368S, using the ideas of 322F, but not relying on it or on 368Q.

368Y Further exercises (a) (i) Use 364T-364U to show that if X is any compact Hausdorff space then $C(X)$ can be regarded as an order-dense Riesz subspace of $L^0(\mathrm{RO}(X))$, where $\mathrm{RO}(X)$ is the regular open algebra of X. (ii) Use 353M to show that any Archimedean Riesz space with order unit can be embedded as an order-dense Riesz subspace of some $L^0(\mathrm{RO}(X))$. (iii) Let U be an Archimedean Riesz space and $C \subseteq U^+$ a maximal disjoint set, as in part (a) of the proof of 368E. For $e \in C$ let U_e be the solid linear subspace of U generated by e, and let V be the solid linear subspace of U generated by C. Show that V can be embedded as an order-dense Riesz subspace of $\prod_{e \in C} U_e$ and therefore in $\prod_{e \in C} L^0(\mathrm{RO}(X_e)) \cong L^0(\prod_{e \in C} \mathrm{RO}(X_e))$ for a suitable family of regular open algebras $\mathrm{RO}(X_e)$. (iv) Now use 368B to complete a proof of 368E.

(b) Let U be any Archimedean Riesz space. Let $\mathcal{V}$ be the family of pairs (A, B) of non-empty subsets of U such that B is the set of upper bounds of A and A is the set of lower bounds of B. Show that $\mathcal{V}$ can be given the structure of a Dedekind complete Riesz space defined by the formulae

$$(A_1, B_1) + (A_2, B_2) = (A, B) \text{ iff } A_1 + A_2 \subseteq A,\ B_1 + B_2 \subseteq B,$$

$$\alpha(A, B) = (\alpha A, \alpha B) \text{ if } \alpha > 0,$$

$$(A_1, B_1) \leq (A_2, B_2) \text{ iff } A_1 \subseteq A_2.$$

Show that $u \mapsto (\left]-\infty, u\right], [u, \infty[)$ defines an embedding of U as an order-dense Riesz subspace of $\mathcal{V}$, so that $\mathcal{V}$ may be identified with the Dedekind completion of U.

(c) Work through the proof of 364T when X is compact, Hausdorff and extremally disconnected, and show that it is easier than the general case. Hence show that 368Yb can be used to shorten the proof of 368E sketched in 368Ya.

(d) Let U be a Riesz space. Show that the following are equiveridical: (i) U is isomorphic, as Riesz space, to $L^0(\mathfrak{A})$ for some Dedekind σ-complete Boolean algebra $\mathfrak{A}$ (ii) U is Dedekind σ-complete and has a weak order unit and whenever $A \subseteq U^+$ is countable and disjoint then A is bounded above in U.

(e) Let U be a weakly (σ, ∞)-distributive Riesz space and V a Riesz subspace of U which is *either* solid *or* order-dense. Show that V is weakly (σ, ∞)-distributive.

(f) Show that $C([0,1])$ is not weakly (σ, ∞)-distributive. (Compare 316J.)

(g) Let $\mathfrak{A}$ be a ccc weakly (σ, ∞)-distributive Boolean algebra. Suppose we have a double sequence $\langle a_{ij} \rangle_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ in $\mathfrak{A}$ such that $\langle a_{ij} \rangle_{j \in \mathbb{N}}$ order*-converges to a_i in $\mathfrak{A}$ for each i, while $\langle a_i \rangle_{i \in \mathbb{N}}$ order*-converges to a. Show that there is a strictly increasing sequence $\langle n(i) \rangle_{i \in \mathbb{N}}$ such that $\langle a_{i,n(i)} \rangle_{i \in \mathbb{N}}$ order*-converges to a.

(h) Let U be a weakly (σ,∞)-distributive Riesz space with the countable sup property. Suppose we have an order-bounded double sequence $\langle u_{ij}\rangle_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in U such that $\langle u_{ij}\rangle_{j\in\mathbb{N}}$ order*-converges to u_i in U for each i, while $\langle u_i\rangle_{i\in\mathbb{N}}$ order*-converges to u. Show that there is a strictly increasing sequence $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that $\langle u_{i,n(i)}\rangle_{i\in\mathbb{N}}$ order*-converges to u.

(i) Let $\mathfrak{A}$ be a ccc weakly (σ,∞)-distributive Dedekind complete Boolean algebra. Show that there is a topology on $L^0 = L^0(\mathfrak{A})$ such that the closure of any $A \subseteq L^0$ is precisely the set of order*-limits of sequences in A. (Cf. 367Yk.)

(j) Let U be a weakly (σ,∞)-distributive Riesz space and $f : U \to \mathbb{R}$ a positive linear functional; write f_τ for the component of f in $U^\times$. (i) Show that for any $u \in U^+$ there is an upwards-directed $A \subseteq [0,u]$, with supremum u, such that $f_\tau(u) = \sup_{v\in A} f(v)$. (See 356Xe, 362D.) (ii) Show that if f is strictly positive, so is f_τ. (Compare 391D.)

368 Notes and comments 368A-368B are manifestations of a principle which will reappear in §375: Dedekind complete L^0 spaces are in some sense 'maximal'. If we have an order-dense subspace U of such an L^0, then any Archimedean Riesz space including U as an order-dense subspace can itself be embedded in L^0 (368B). In fact this property characterizes Dedekind complete L^0 spaces (368M). Moreover, any Archimedean Riesz space U can be embedded in this way (368E); by 368C, the L^0 space (though not the embedding) is unique up to isomorphism. If U and V are Archimedean Riesz spaces, each embedded as an order-dense Riesz subspace of a Dedekind complete L^0 space, then any order-continuous Riesz homomorphism from U to V extends uniquely to the L^0 spaces (368B). If one Dedekind complete L^0 space is embedded as an order-dense Riesz subspace of another, they must in fact be the same (368D). Thus we can say that every Archimedean Riesz space U can be extended to a Dedekind complete L^0 space, in a way which respects order-continuous Riesz homomorphisms, and that this extension is maximal, in that U cannot be order-dense in any larger space.

The proof of 368E which I give is long because I am using a bare-hands approach. Alternative methods shift the burdens. For instance, if we take the trouble to develop a direct construction of the 'Dedekind completion' of a Riesz space (368Yb), then we need prove the theorem only for Dedekind complete Riesz spaces. A more substantial aid is the representation theorem for Archimedean Riesz spaces with order unit (353M); I sketch an argument in 368Ya. The drawback to this approach is the proof of Theorem 364T, which seems to be quite as long as the direct proof of 368E which I give here. Of course we need 364T only for compact Hausdorff spaces, which are usefully easier than the general case (364U, 368Yc).

368G is a version of Ogasawara's representation theorem for Archimedean Riesz spaces. Both this and 368F can be regarded as expressions of the principle that an Archimedean Riesz space is 'nearly' a space of functions.

I have remarked before on the parallels between the theories of Boolean algebras and Archimedean Riesz spaces. The notion of 'weak (σ,∞)-distributivity' is one of the more striking correspondences. (Compare, for instance, 316Xi(i) with 368Pa.) What is really important to us, of course, is the fact that the function spaces of measure theory are mostly weakly (σ,∞)-distributive, by 368S. Of course this is easy to prove directly (368Xg), but I think that the argument through 368Q gives a better idea of what is really happening here. Some of the features of 'order*-convergence', as defined in §367, are related to weak (σ,∞)-distributivity (compare 367Yi, 367Yp); in 368Yi I describe a topology which can be thought of as an abstract version of the topology of convergence in measure on the L^0 space of a σ-finite measure algebra (367M).

369 Banach function spaces

In this section I continue the work of §368 with results which involve measure algebras. The first step is a modification of the basic representation theorem for Archimedean Riesz spaces. If U is any Archimedean Riesz space, it can be represented as a subspace of $L^0 = L^0(\mathfrak{A})$, where $\mathfrak{A}$ is its band algebra (368E); now if $U^\times$ separates the points of U, there is a measure rendering $\mathfrak{A}$ a localizable measure algebra (369A, 369Xa). Moreover, we get a simultaneous representation of $U^\times$ as a subspace of L^0 (369C-369D), the duality between U and $U^\times$ corresponding exactly to the familiar duality between L^p and L^q. In particular, every L-space can be represented as an L^1-space (369E).

Still drawing inspiration from the classical L^p spaces, we have a general theory of 'associated Fatou norms' (369F-369M, 369R). I include notes on the spaces $M^{1,\infty}$, $M^{\infty,1}$ and $M^{1,0}$ (369N-369Q), which will be particularly useful in the next chapter.

369A Theorem Let U be a Riesz space such that $U^\times$ separates the points of U. Then U can be embedded as an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$ for some localizable measure algebra $(\mathfrak{A},\bar\mu)$.

proof (a) Consider the canonical map $S : U \to U^{\times\times}$. We know that this is a Riesz homomorphism onto an order-dense Riesz subspace of $U^{\times\times}$ (356I). Because $U^\times$ separates the points of U, S is injective. Let $\mathfrak{A}$ be the band algebra of $U^{\times\times}$ and $T : U^{\times\times} \to L^0$ an injective Riesz homomorphism onto an order-dense Riesz subspace V of L^0, as in 368E. The composition $TS : U \to L^0$ is now an injective Riesz homomorphism, so embeds U as a Riesz subspace of L^0, which is order-dense because V is order-dense in L^0 and $TS[U]$ is order-dense in V (352Nc). Thus all that we need to find is a measure $\bar\mu$ on $\mathfrak{A}$ rendering it a localizable measure algebra.

(b) Note that V is isomorphic, as Riesz space, to $U^{\times\times}$, which is Dedekind complete (356B), so V must be solid in L^0 (353K). Also $V^\times$ must separate the points of V (356L).

Let D be the set of those $d \in \mathfrak{A}$ such that the principal ideal $\mathfrak{A}_d$ is measurable in the sense that there is some $\bar\nu$ for which $(\mathfrak{A}_d,\bar\nu)$ is a totally finite measure algebra. Then D is order-dense in $\mathfrak{A}$. **P** Take any non-zero $a \in \mathfrak{A}$. Because V is order-dense, there is a non-zero $v \in V$ such that $v \le \chi a$. Take $h \ge 0$ in $V^\times$ such that $h(v) > 0$. Then there is a v' such that $0 < v' \le v$ and $h(w) > 0$ whenever $0 < w \le v'$ in V (356H). Let $\alpha > 0$ be such that $d = [\![v' > \alpha]\!] \ne 0$. Then $\chi b \le \frac{1}{\alpha}v' \in V$ whenever $b \in \mathfrak{A}_d$. Set $\bar\nu b = h(\chi b) \in [0,\infty[$ for $b \in \mathfrak{A}_d$. Because the map $b \mapsto \chi b : \mathfrak{A} \to L^0$ is additive and order-continuous, the map $b \mapsto \chi b : \mathfrak{A}_d \to V$ also is, and $\bar\nu = h\chi$ must be additive and order-continuous; in particular, $\bar\nu(\sup_{n\in\mathbb{N}} b_n) = \sum_{n=0}^\infty \bar\nu b_n$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_d$. Moreover, if $b \in \mathfrak{A}_d$ is non-zero, then $0 < \alpha\chi b \le v'$, so $\bar\nu b = h(\chi b) > 0$. Thus $(\mathfrak{A}_d,\bar\nu)$ is a totally finite measure algebra, and $d \in D$, while $0 \ne d \subseteq a$. As a is arbitrary, D is order-dense. **Q**

(c) By 313K, there is a partition of unity $C \subseteq D$. For each $c \in C$, let $\bar\nu_c : \mathfrak{A}_c \to [0,\infty[$ be a functional such that $(\mathfrak{A}_c,\bar\nu_c)$ is a totally finite measure algebra. Define $\bar\mu : \mathfrak{A} \to [0,\infty]$ by setting $\bar\mu a = \sum_{c\in C} \bar\nu_c(a \cap c)$ for every $a \in \mathfrak{A}$. Then $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra. **P** (i) $\bar\mu 0 = \sum_{c\in C} \bar\nu 0 = 0$. (ii) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, then

$$\bar\mu a = \sum_{c\in C} \bar\nu_c(a \cap c) = \sum_{c\in C,n\in\mathbb{N}} \bar\nu_c(a_n \cap c) = \sum_{n=0}^\infty \bar\mu a_n.$$

(iii) If $a \in \mathfrak{A} \setminus \{0\}$, then there is a $c \in C$ such that $a \cap c \ne 0$, so that $\bar\mu a \ge \bar\nu_c(a \cap c) > 0$. Thus $(\mathfrak{A},\bar\mu)$ is a measure algebra. (iv) Moreover, in (iii), $\bar\mu(a \cap c) = \bar\nu_c(a \cap c)$ is finite. So $(\mathfrak{A},\bar\mu)$ is semi-finite. (v) $\mathfrak{A}$ is Dedekind complete, being a band algebra (352Q), so $(\mathfrak{A},\bar\mu)$ is localizable. **Q**

369B Corollary Let U be a Banach lattice with order-continuous norm. Then U can be embedded as an order-dense solid linear subspace of $L^0(\mathfrak{A})$ for some localizable measure algebra $(\mathfrak{A},\bar\mu)$.

proof By 356Dd, $U^\times = U^*$, which separates the points of U, by the Hahn-Banach theorem (3A5Ae). So 369A tells us that U can be embedded as an order-dense Riesz subspace of an appropriate $L^0(\mathfrak{A})$. But also U is Dedekind complete (354Ee), so its copy in $L^0(\mathfrak{A})$ must be solid, as in 368H.

369C The representation in 369A is complemented by the following result, which is a kind of generalization of 365M and 366Dc.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and $U \subseteq L^0 = L^0(\mathfrak{A})$ an order-dense Riesz subspace. Set

$$V = \{v : v \in L^0,\ v \times u \in L^1 \text{ for every } u \in U\},$$

writing L^1 for $L^1(\mathfrak{A},\bar\mu) \subseteq L^0$. Then V is a solid linear subspace of L^0, and we have an order-continuous injective Riesz homomorphism $T : V \to U^\times$ defined by setting

$$(Tv)(u) = \int u \times v \text{ for all } u \in U,\ v \in V.$$

The image of V is order-dense in $U^\times$. If $(\mathfrak{A},\bar\mu)$ is localizable, then T is surjective, so is a Riesz space isomorphism between V and $U^\times$.

proof (a)(i) Because $\times : L^0 \times L^0 \to L^0$ is bilinear and L^1 is a linear subspace of L^0, V is a linear subspace of L^0. If $u \in U$, $v \in V$, $w \in L^0$ and $|w| \le |v|$, then

$$|w \times u| = |w| \times |u| \le |v| \times |u| = |v \times u| \in L^1;$$

as L^1 is solid, $w \times u \in L^1$; as u is arbitrary, $w \in V$; this shows that V is solid.

(ii) By the definition of V, $(Tv)(u)$ is defined in $\mathbb{R}$ for all $u \in U$, $v \in V$. Because $\times$ is bilinear and $\int$ is linear, $Tv : U \to \mathbb{R}$ is linear for every $v \in V$, and T is a linear functional from V to the space of linear operators from U to $\mathbb{R}$.

(iii) If $u \geq 0$ in U and $v \geq 0$ in V, then $u \times v \geq 0$ in L^1 and $(Tv)(u) = \int u \times v \geq 0$. This shows that T is a positive linear operator from V to $U^\sim$.

(iv) If $v \geq 0$ in V and $A \subseteq U$ is a non-empty downwards-directed set with infimum 0 in U, then $\inf A = 0$ in L^0, because U is order-dense (352Nb). Consequently $\inf_{u \in A} u \times v = 0$ in L^0 and in L^1 (364Ba, 353Oa), and

$$\inf_{u \in A}(Tv)(u) = \inf_{u \in A} \int u \times v = 0$$

(because $\int$ is order-continuous). As A is arbitrary, Tv is order-continuous. As v is arbitrary, $T[V] \subseteq U^\times$.

(v) If $v \in V$ and $u_0 \geq 0$ in U, set $a = [\![v > 0]\!]$. Then $v^+ = v \times \chi a$. Set $A = \{u : u \in U, 0 \leq u \leq u_0 \times \chi a\}$. Because U is order-dense in L^0, $u_0 \times \chi a = \sup A$ in L^0. Because $\times$ and $\int$ are order-continuous,

$$\begin{aligned}(Tv)^+(u_0) &\geq \sup_{u \in A}(Tv)(u) = \sup_{u \in A} \int v \times u \\ &= \int v \times u_0 \times \chi a = \int v^+ \times u_0 = (Tv^+)(u_0).\end{aligned}$$

As u_0 is arbitrary, $(Tv)^+ \geq Tv^+$. But because T is a positive linear operator, we must have $Tv^+ \geq (Tv)^+$, so that $Tv^+ = (Tv)^+$. As v is arbitrary, T is a Riesz homomorphism.

(vi) Now T is injective. **P** If $v \neq 0$ in V, there is a $u > 0$ in U such that $u \leq |v|$, because U is order-dense. In this case $u \times |v| > 0$ so $\int u \times |v| > 0$. Accordingly $|Tv| = T|v| \neq 0$ and $Tv \neq 0$. **Q**

(b) Putting (a-i) to (a-vi) together, we see that T is an injective Riesz homomorphism from V to $U^\times$. All this is easy. The point of the theorem is the fact that $T[V]$ is order-dense in $U^\times$.

P Take $h > 0$ in $U^\times$. Let U_1 be the solid linear subspace of L^0 generated by U. Then U is an order-dense Riesz subspace of U_1, $h : U \to \mathbb{R}$ is an order-continuous positive linear functional, and $\sup\{h(u) : u \in U, 0 \leq u \leq v\}$ is defined in $\mathbb{R}$ for every $v \geq 0$ in U_1; so we have an extension $\tilde{h}$ of h to U_1 such that $\tilde{h} \in U_1^\times$ (355F).

Set $S_1 = S(\mathfrak{A}) \cap U_1$; then S_1 is an order-dense Riesz subspace of U_1, because $S(\mathfrak{A})$ is order-dense in L^0 and U_1 is solid in L^0. Note that S_1 is the linear span of $\{\chi c : c \in I\}$, where $I = \{c : c \in \mathfrak{A}, \chi c \in U_1\}$, and that I is an ideal in $\mathfrak{A}$.

Because $h \neq 0$, $\tilde{h} \neq 0$; there must therefore be a $u_0 \in S_1$ such that $\tilde{h}(u_0) > 0$, and a $d \in I$ such that $\tilde{h}(\chi d) > 0$. For $a \in \mathfrak{A}$, set $\nu a = \tilde{h}\chi(d \cap a)$. Because $\cap$, χ and $\tilde{h}$ are all order-continuous, so is ν, and $\nu : \mathfrak{A} \to \mathbb{R}$ is a non-negative completely additive functional.

By 365Ea, there is a $v \in L^1$ such that

$$\int_a v = \nu a = \tilde{h}\chi(d \cap a)$$

for every $a \in \mathfrak{A}$; of course $v \geq 0$. We have $\int u \times v \leq \tilde{h}(u)$ whenever $u = \chi a$ for $a \in I$, and therefore for every $u \in S_1^+$. If $u \in U^+$, then $A = \{u' : u' \in S_1, 0 \leq u' \leq u\}$ is upwards-directed, $\sup A = u$ and

$$\sup_{u' \in A} \int v \times u' \leq \sup_{u' \in A} \tilde{h}(u') = \tilde{h}(u) = h(u)$$

is finite, so $v \times u = \sup_{u \in A'} v \times u'$ belongs to L^1 (365Df) and $\int v \times u \leq h(u)$. As u is arbitrary, $v \in V$ and $Tv \leq h$. At the same time, $\int_d v = \tilde{h}(\chi d) > 0$, so $Tv > 0$. As h is arbitrary, $T[V]$ is order-dense. **Q**

It follows that T is order-continuous (352Nb), as can also be easily proved by the argument of (a-iv) above.

(c) Now suppose that $(\mathfrak{A}, \bar{\mu})$ is localizable, that is, that $\mathfrak{A}$ is Dedekind complete. $T^{-1} : T[V] \to V$ is a Riesz space isomorphism, so certainly an order-continuous Riesz homomorphism; because V is a solid linear subspace of L^0, T^{-1} is still an injective order-continuous Riesz homomorphism when regarded as a map from $T[V]$ to L^0. Since $T[V]$ is order-dense in $U^\times$, T^{-1} has an extension to an order-continuous Riesz homomorphism $Q : U^\times \to L^0$ (368B). But $Q[U^\times] \subseteq V$. **P** Take $h \geq 0$ in $U^\times$ and $u \geq 0$ in U. Then $B = \{g : g \in T[V], 0 \leq g \leq h\}$ is upwards-directed and has supremum h. For $g \in B$, we know that $u \times T^{-1}g \in L^1$ and $\int u \times T^{-1}g = g(u)$, by the definition of T. But this means that

$$\sup_{g \in B} \int u \times T^{-1}g = \sup_{g \in B} g(u) = h(u) < \infty.$$

Since $\{u \times T^{-1}g : g \in B\}$ is upwards-directed, it follows that

$$u \times Qh = \sup_{g\in B} u \times Qg = \sup_{g\in B} u \times T^{-1}g \in L^1$$

by 365Df again. As u is arbitrary, $Qh \in V$. As h is arbitrary (and Q is linear), $Q[U^\times] \subseteq V$. **Q**

Also Q is injective. **P** If $h \in U^\times$ is non-zero, there is a $v \in V$ such that $0 < Tv \le |h|$, so that

$$|Qh| = Q|h| \ge QTv = v > 0$$

and $Qh \neq 0$. **Q** Since QT is the identity on V, Q and T must be the two halves of a Riesz space isomorphism between V and $U^\times$.

369D Corollary Let U be any Riesz space such that $U^\times$ separates the points of U. Then there is a localizable measure algebra $(\mathfrak{A}, \bar\mu)$ such that the pair $(U, U^\times)$ can be represented by a pair (V, W) of order-dense Riesz subspaces of $L^0 = L^0(\mathfrak{A})$ such that $W = \{w : w \in L^0,\ v \times w \in L^1$ for every $v \in V\}$, writing L^1 for $L^1(\mathfrak{A}, \bar\mu)$. In this case, $U^{\times\times}$ becomes represented by $\tilde V = \{v : v \in L^0,\ v \times w \in L^1$ for every $w \in W\} \supseteq V$.

proof Put 369A and 369C together. The construction of 369A finds $(\mathfrak{A}, \bar\mu)$ and an order-dense V which is isomorphic to U, and 369C identifies W with $V^\times$ and $W^\times$ with $\tilde V$. To check that W is order-dense, take any $u > 0$ in L^0. There is a $v \in V$ such that $0 < v \le u$. There is an $h \in (V^\times)^+$ such that $h(v) > 0$, so there is a $w \in W^+$ such that $w \times v \neq 0$, that is, $w \wedge v \neq 0$. But now $w \wedge v \in W$, because W is solid, and $0 < w \wedge v \le u$.

Remark Thus the canonical embedding of U in $U^{\times\times}$ (356I) is represented by the embedding $V \subsetneq \tilde V$; U, or V, is 'perfect' iff $V = \tilde V$.

369E Kakutani's theorem (KAKUTANI 41) If U is any L-space, there is a localizable measure algebra $(\mathfrak{A}, \bar\mu)$ such that U is isomorphic, as Banach lattice, to $L^1 = L^1(\mathfrak{A}, \bar\mu)$.

proof U is a perfect Riesz space, and $U^\times = U^*$ has an order unit $\int$ defined by saying that $\int u = \|u\|$ for $u \ge 0$ (356P). By 369D, we can find a localizable measure algebra $(\mathfrak{A}, \bar\mu)$ and an identification of the pair $(U, U^\times)$, as dual Riesz spaces, with a pair (V, W) of subspaces of $L^0 = L^0(\mathfrak{A})$; and V will be $\{v : v \times w \in L^1$ for every $w \in W\}$. But W, like $U^\times$, must have an order unit; call it e. Because W is order-dense, $[\![e > 0]\!]$ must be 1 and e must have a multiplicative inverse $\frac{1}{e}$ in L^0 (364N). This means that V must be $\{v : v \times e \in L^1\}$, so that $v \mapsto v \times e$ is a Riesz space isomorphism between V and L^1, which gives a Riesz space isomorphism between U and L^1. Moreover, if we write $\| \|'$ for the norm on V corresponding to the norm of U, we have

$$\|u\| = \int |u| \text{ for } u \in U, \quad \|v\|' = \int |v| \times e = \int |v \times e| \text{ for } v \in V.$$

Thus the Riesz space isomorphism between U and L^1 is norm-preserving, and U and L^1 are isomorphic as Banach lattices.

369F The L^p spaces are leading examples for a general theory of normed subspaces of L^0, which I proceed to sketch in the rest of the section.

Definition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. An **extended Fatou norm** on $L^0 = L^0(\mathfrak{A})$ is a function $\tau : L^0 \to [0, \infty]$ such that

(i) $\tau(u + v) \le \tau(u) + \tau(v)$ for all $u, v \in L^0$;

(ii) $\tau(\alpha u) = |\alpha|\tau(u)$ for all $u \in L^0$, $\alpha \in \mathbb{R}$ (counting $0 \cdot \infty$ as 0, as usual);

(iii) $\tau(u) \le \tau(v)$ whenever $|u| \le |v|$ in L^0;

(iv) $\sup_{u\in A} \tau(u) = \tau(v)$ whenever $A \subseteq (L^0)^+$ is a non-empty upwards-directed set with supremum v in L^0;

(v) $\tau(u) > 0$ for every non-zero $u \in L^0$;

(vi) whenever $u > 0$ in L^0 there is a $v \in L^0$ such that $0 < v \le u$ and $\tau(v) < \infty$.

369G Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Then $L^\tau = \{u : u \in L^0,\ \tau(u) < \infty\}$ is an order-dense solid linear subspace of L^0, and τ, restricted to L^τ, is a Fatou norm under which L^τ is a Banach lattice. If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing norm-bounded sequence in $(L^\tau)^+$, then it has a supremum in L^τ; if $\mathfrak{A}$ is Dedekind complete, then L^τ has the Levi property.

proof (a) By (i), (ii) and (iii) of 369F, L^τ is a solid linear subspace of L^0; by (vi), it is order-dense. Hypotheses (i), (ii), (iii) and (v) show that τ is a Riesz norm on L^τ, while (iv) shows that it is a Fatou norm.

(b)(i) Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing norm-bounded sequence in $(L^\tau)^+$. Then $u=\sup_{n\in\mathbb{N}}u_n$ is defined in L^0. **P?** Otherwise, there is a $v>0$ in L^0 such that $kv=\sup_{n\in\mathbb{N}}kv\wedge u_n$ for every $k\in\mathbb{N}$ (368A). By (v)-(vi) of 369F, there is a v' such that $0<v'\le v$ and $0<\tau(v')<\infty$. Now $kv'=\sup_{n\in\mathbb{N}}kv'\wedge u_n$ for every k, so

$$k\tau(v')=\tau(kv')=\sup_{n\in\mathbb{N}}\tau(kv'\wedge u_n)\le\sup_{n\in\mathbb{N}}\tau(u_n)$$

for every k, using 369F(iv), and $\sup_{n\in\mathbb{N}}\tau(u_n)=\infty$, contrary to hypothesis. **XQ** By 369F(iv) again, $\tau(u)=\sup_{n\in\mathbb{N}}\tau(u_n)<\infty$, so that $u\in L^\tau$ and $u=\sup_{n\in\mathbb{N}}u_n$ in L^τ.

(ii) It follows that L^τ is complete under τ. **P** Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a sequence in L^τ such that $\tau(u_{n+1}-u_n)\le 2^{-n}$ for every $n\in\mathbb{N}$. Set $v_{mn}=\sum_{i=m}^{n}|u_{i+1}-u_i|$ for $m\le n$; then $\tau(v_{mn})\le 2^{-m+1}$ for every n, so by (i) just above $v_m=\sup_{n\in\mathbb{N}}v_{mn}$ is defined in L^τ, and $\tau(v_m)\le 2^{-m+1}$. Now $v_m=|u_{m+1}-u_m|+v_{m+1}$ for each m, so $\langle u_m-v_m\rangle_{m\in\mathbb{N}}$ is non-decreasing and $\langle u_m+v_m\rangle_{m\in\mathbb{N}}$ is non-increasing, while $u_m-v_m\le u_m\le u_m+v_m$ for every m. Accordingly $u=\sup_{m\in\mathbb{N}}u_m-v_m$ is defined in L^τ and $|u-u_m|\le v_m$ for every m. But this means that $\lim_{m\to\infty}\tau(u-u_m)\le\lim_{m\to\infty}\tau(v_m)=0$ and $u=\lim_{m\to\infty}u_m$ in L^τ. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, L^τ is complete. **Q**

(c) Now suppose that $\mathfrak{A}$ is Dedekind complete and $A\subseteq(L^\tau)^+$ is a non-empty upwards-directed norm-bounded set in L^τ. By the argument of (b-i) above, using the other half of 368A, $\sup A$ is defined in L^0 and belongs to L^τ. As A is arbitrary, L^τ has the Levi property.

369H Associate norms: Definition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0=L^0(\mathfrak{A})$. Define $\tau':L^0\to[0,\infty]$ by setting

$$\tau'(u)=\sup\{\|u\times v\|_1:v\in L^0,\,\tau(v)\le 1\}$$

for every $u\in L^0$; then τ' is the **associate** of τ. (The word suggests a symmetric relationship; it is justified by the next theorem.)

369I Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0=L^0(\mathfrak{A})$. Then
(i) its associate τ' is also an extended Fatou norm on L^0;
(ii) τ is the associate of τ';
(iii) $\|u\times v\|_1\le\tau(u)\tau'(v)$ for all u, $v\in L^0$.

proof (a) Before embarking on the proof that τ' is an extended Fatou seminorm on L^0, I give the greater part of the argument needed to show that $\tau=\tau''$, where

$$\tau''(u)=\sup\{\|u\times w\|_1:w\in L^0,\,\tau'(w)\le 1\}$$

for every $u\in L^0$.

(α) Set

$$B=\{u:u\in L^1,\,\tau(u)\le 1\},$$

writing L^1 for $L^1(\mathfrak{A},\bar\mu)$. Then B is a convex set in L^1 and is closed for the norm topology of L^1. **P** Suppose that u belongs to the closure of B in L^1. Then for each $n\in\mathbb{N}$ we can choose $u_n\in B$ such that $\|u-u_n\|_1\le 2^{-n}$. Set $v_{mn}=\inf_{m\le i\le n}|u_i|$ for $m\le n$, and

$$v_m=\inf_{n\ge m}v_{mn}=\inf_{n\ge m}|u_n|\le|u|$$

for $m\in\mathbb{N}$. The sequence $\langle v_m\rangle_{m\in\mathbb{N}}$ is non-decreasing, $\tau(v_m)\le\tau(u_m)\le 1$ for every m, and

$$\||u|-v_m\|_1\le\sup_{n\ge m}\||u|-v_{mn}\|_1\le\sum_{i=m}^{\infty}\||u|-|u_i|\|_1\le\sum_{i=m}^{\infty}\|u-u_i\|_1\to 0$$

as $m\to\infty$. So $|u|=\sup_{m\in\mathbb{N}}v_m$ in L^0,

$$\tau(u)=\tau(|u|)=\sup_{m\in\mathbb{N}}\tau(v_m)\le 1$$

and $u\in B$. **Q**

(β) Now take any $u_0\in L^0$ such that $\tau(u_0)>1$. Then, writing $\mathfrak{A}^f$ for $\{a:\bar\mu a<\infty\}$,

$$A=\{u:u\in S(\mathfrak{A}^f),\,0\le u\le u_0\}$$

is an upwards-directed set with supremum u_0 (this is where I use the hypothesis that $(\mathfrak{A},\bar\mu)$ is semi-finite, so that $S(\mathfrak{A}^f)$ is order-dense in L^0), and $\sup_{u\in A}\tau(u)=\tau(u_0)>1$. Take $u_1\in A$ such that $\tau(u_1)>1$, that is, $u_1\notin B$. By the Hahn-Banach theorem (3A5Cc), there is a continuous linear functional $f : L^1 \to \mathbb{R}$ such that $f(u_1) > 1$ but $f(u) \le 1$ for every $u \in B$. Because $(L^1)^* = (L^1)^\sim$ (356Dc), $|f|$ is defined in $(L^1)^*$, and of course

$$|f|(u_1) \ge f(u_1) > 1, \quad |f|(u) = \sup\{f(v) : |v| \le u\} \le 1$$

whenever $u \in B$ and $u \ge 0$. Set $c = [\![u_1 > 0]\!]$, so that $\bar{\mu}c < \infty$, and define

$$\nu a = |f|(\chi(a \cap c))$$

for every $a \in \mathfrak{A}$. Then ν is a completely additive real-valued functional on $\mathfrak{A}$, so there is a $w \in L^1$ such that $\nu a = \int_a w$ for every $a \in \mathfrak{A}$ (365Ea). Because $\nu a \ge 0$ for every a, $w \ge 0$. Now

$$\int_a w = |f|(\chi a \times \chi c)$$

for every $a \in \mathfrak{A}$, so

$$\int w \times u = |f|(u \times \chi c) \le |f|(u) \le 1$$

for every $u \in S(\mathfrak{A})^+ \cap B$. But if $\tau(v) \le 1$, then

$$A_v = \{u : u \in S(\mathfrak{A})^+ \cap B,\, u \le |v|\}$$

is an upwards-directed set with supremum $|v|$, so that

$$\|w \times v\|_1 = \sup_{u \in A_v} \int w \times u \le 1.$$

Thus $\tau'(w) \le 1$. On the other hand,

$$\|w \times u_0\|_1 \ge \int w \times u_0 \ge \int w \times u_1 = |f|(u_1) > 1,$$

so $\tau''(u_0) > 1$.

(γ) This shows that, for $u \in L^0$,

$$\tau''(u) \le 1 \Longrightarrow \tau(u) \le 1.$$

(c) Now I return to the proof that τ' is an extended Fatou norm. It is easy to check that it satisfies conditions (i)-(iv) of 369F; in effect, these depend only on the fact that $\| \|_1$ is an extended Fatou norm. For (v)-(vi), take $v > 0$ in L^0. Then there is a u such that $0 \le u \le v$ and $0 < \tau(u) < \infty$; set $\alpha = 1/\tau(u)$. Then $\tau(2\alpha u) > 1$, so that $\tau''(2\alpha u) > 1$ and there is a $w \in L^0$ such that $\tau'(w) \le 1$, $\|2\alpha u \times w\|_1 > 1$. But now set $v_1 = v \wedge |w|$; then

$$v \ge v_1 \ge u \wedge |w| > 0,$$

while $\tau'(v_1) < \infty$. Also $v \wedge \alpha u \ne 0$ so

$$\tau'(v) \ge \|v \times \alpha u\|_1 > 0.$$

As v is arbitrary, τ' satisfies 369F(v)-(vi).

(d) Accordingly τ'' also is an extended Fatou norm. Now in (a) I showed that

$$\tau''(u) \le 1 \Longrightarrow \tau(u) \le 1.$$

It follows easily that $\tau(u) \le \tau''(u)$ for every u (since otherwise there would be some α such that

$$\tau''(\alpha u) = \alpha\tau''(u) < 1 < \alpha\tau(u) = \tau(\alpha u)).$$

On the other hand, we surely have

$$\tau(u) \le 1 \Longrightarrow \|u \times v\|_1 \le 1 \text{ whenever } \tau'(v) \le 1 \Longrightarrow \tau''(u) \le 1,$$

so we must also have $\tau''(u) \le \tau(u)$ for every u. Thus $\tau'' = \tau$, as claimed.

(e) Of course we have $\|u \times v\|_1 \le 1$ whenever $\tau(u) \le 1$ and $\tau'(v) \le 1$. It follows easily that $\|u \times v\|_1 \le \tau(u)\tau'(v)$ whenever u, $v \in L^0$ and both $\tau(u)$, $\tau'(v)$ are non-zero. But if one of them is zero, then $u \times v = 0$, because both τ and τ' satisfy (v) of 369F, so the result is trivial.

369J Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$, with associate θ. Then

$$L^\theta = \{v : v \in L^0,\, u \times v \in L^1(\mathfrak{A}, \bar{\mu}) \text{ for every } u \in L^\tau\}.$$

proof (a) If $v \in L^\theta$ and $u \in L^\tau$, then $\|u \times v\|_1$ is finite, by 369I(iii), so $u \times v \in L^1 = L^1(\mathfrak{A}, \bar{\mu})$.

(b) If $v \notin L^\theta$ then for every $n \in \mathbb{N}$ there is a u_n such that $\tau(u_n) \le 1$ and $\|u_n \times v\|_1 \ge 2^n$. Set $w_n = \sum_{i=0}^n 2^{-i}|u_i|$ for each n. Then $\langle w_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence and $\tau(w_n) \le 2$ for each n, so $w = \sup_{n\in\mathbb{N}} w_n$ is defined in L^τ, by 369G; now $\int w \times |v| \ge n+1$ for every n, so $w \times v \notin L^1$.

369K Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$, with associate θ. Then L^θ may be identified, as normed Riesz space, with $(L^\tau)^\times \subseteq (L^\tau)^*$, and L^τ is a perfect Riesz space.

proof Putting 369J and 369C together, we have an identification between L^θ and $(L^\tau)^\times$. Now 369I tells us that τ is the associate of θ, so that we can identify L^τ with $(L^\theta)^\times$, and L^τ is perfect, as in 369D.

By the definition of θ, we have, for any $v \in L^\theta$,

$$\theta(v) = \sup_{\tau(u)\le 1} \|u \times v\|_1$$
$$= \sup_{\tau(u)\le 1, \|w\|_\infty \le 1} \int u \times v \times w = \sup_{\tau(u)\le 1} \int u \times v,$$

which is the norm of the linear functional on L^τ corresponding to v.

369L L^p I remarked above that the L^p spaces are leading examples for this theory; perhaps I should spell out the details. Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra and $p \in [1, \infty]$. Then $\| \,\|_p$ is an extended Fatou norm. **P** Conditions (i)-(iii) and (v) of 369F are true just because $L^p = L^p_{\bar{\mu}}$ is a solid linear subspace of $L^0(\mathfrak{A})$ on which $\| \,\|_p$ is a Riesz norm, (iv) is true because $\| \,\|_p$ is a Fatou norm with the Levi property (363Ba, 365C, 366D), and (vi) is true because $S(\mathfrak{A}^f)$ is included in L^p and order-dense in $L^0 = L^0(\mathfrak{A})$ (364K). **Q**

As usual, set $q = p/(p-1)$ if $1 < p < \infty$, ∞ if $p = 1$, and 1 if $p = \infty$. Then $\| \,\|_q$ is the associate extended Fatou norm of $\| \,\|_p$. **P** By 365Mb and 366C, $\|v\|_q = \sup\{\|u \times v\|_1 : \|u\|_p \le 1\}$ for every $v \in L^q = L^q_{\bar{\mu}}$. But as L^q is order-dense in L^0,

$$\|v\|_q = \sup_{w\in L^q, |w|\le v} \|w\|_q = \sup\{\int |u| \times |w| : w \in L^q,\, w \le |v|,\, \|u\|_p \le 1\}$$
$$= \sup\{\int |u| \times |v| : \|u\|_p \le 1\}$$

for every $v \in L^0$. **Q**

369M Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Then

(a) the embedding $L^\tau \subseteq L^0$ is continuous for the norm topology of L^τ and the topology of convergence in measure on L^0;

(b) $\tau : L^0 \to [0, \infty]$ is lower semi-continuous, that is, all the balls $\{u : \tau(u) \le \gamma\}$ are closed for the topology of convergence in measure;

(c) if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0 which is order*-convergent to $u \in L^0$ (definition: 367A), then $\tau(u)$ is at most $\liminf_{n\to\infty} \tau(u_n)$.

proof (a) This is a special case of 367O.

(b) Set $B_\gamma = \{u : \tau(u) \le \gamma\}$. If $u \in L^0 \setminus B_\gamma$, then

$$A = \{|u| \times \chi a : a \in \mathfrak{A}^f\}$$

is an upwards-directed set with supremum $|u|$, so there is an $a \in \mathfrak{A}^f$ such that $\tau(u \times \chi a) > \gamma$. **?** If u is in the closure of B_γ for the topology of convergence in measure, then for every $k \in \mathbb{N}$ there is a $v_k \in B_\gamma$ such that $\bar{\mu}(a \cap [\![|u - v_k| > 2^{-k}]\!]) \le 2^{-k}$ (see the formulae in 367L). Set

$$v'_k = |u| \wedge \inf_{i\ge k} |v_i|$$

for each k, and $v^* = \sup_{k\in\mathbb{N}} v'_k$. Then $\tau(v'_k) \le \tau(v_k) \le \gamma$ for each k, and $\langle v_k\rangle_{k\in\mathbb{N}}$ is non-decreasing, so $\tau(v^*) \le \gamma$. But

$$a \cap [\![|u| - v^* > 2^{-k}]\!] \subseteq a \cap \sup_{i\ge k} [\![|u - v_i| > 2^{-k}]\!]$$

has measure at most $\sum_{i=k}^{\infty} 2^{-i}$ for each k, so $a \cap [\![|u| - v^* > 0]\!]$ must be 0, that is, $|u| \times \chi a \le v^*$ and $\tau(|u| \times \chi a) \le \gamma$; contrary to the choice of a. **X** Thus u cannot belong to the closure of B_γ. As u is arbitrary, B_γ is closed.

(c) If $\langle u_n \rangle_{n\in\mathbb{N}}$ order*-converges to u, it converges in measure (367Ma). If $\gamma > \liminf_{n\to\infty} \tau(u_n)$, there is a subsequence of $\langle u_n \rangle_{n\in\mathbb{N}}$ in B_γ, and $\tau(u) \le \gamma$, by (b). As γ is arbitary, $\tau(u) \le \liminf_{n\to\infty} \tau(u_n)$.

369N I now turn to another special case which we have already had occasion to consider in other contexts.

Definition Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Set

$$M_{\bar\mu}^{\infty,1} = M^{\infty,1}(\mathfrak{A}, \bar\mu) = L^1(\mathfrak{A}, \bar\mu) \cap L^\infty(\mathfrak{A}),$$

$$M_{\bar\mu}^{1,\infty} = M^{1,\infty}(\mathfrak{A}, \bar\mu) = L^1(\mathfrak{A}, \bar\mu) + L^\infty(\mathfrak{A}),$$

and

$$\|u\|_{\infty,1} = \max(\|u\|_1, \|u\|_\infty)$$

for $u \in L^0(\mathfrak{A})$.

Remark I hope that the notation I have chosen here will not completely overload your short-term memory. The idea is that in $M^{p,q}$ the symbol p is supposed to indicate the 'local' nature of the space, that is, the nature of $u \times \chi a$ where $u \in M^{p,q}$ and $\bar\mu a < \infty$, while q indicates the nature of $|u| \wedge \chi 1$ for $u \in M^{p,q}$. Thus $M^{1,\infty}$ is the space of u such that $u \times \chi a \in L^1$ for every $a \in \mathfrak{A}^f$ amd $|u| \wedge \chi 1 \in L^\infty$; in $M^{1,0}$ we demand further that $|u| \wedge \chi 1 \in M^0$ (366F); while in $M^{\infty,1}$ we ask that $|u| \wedge \chi 1 \in L^1$ and that $u \times \chi a \in L^\infty$ for every $a \in \mathfrak{A}^f$.

369O Proposition Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra.

(a) $\| \,\|_{\infty,1}$ is an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$.

(b) Its associate $\| \,\|_{1,\infty}$ may be defined by the formulae

$$\begin{aligned}\|u\|_{1,\infty} &= \min\{\|v\|_1 + \|w\|_\infty : v \in L^1,\, w \in L^\infty,\, v + w = u\} \\ &= \min\{\alpha + \int (|u| - \alpha\chi 1)^+ : \alpha \ge 0\} \\ &= \int_0^\infty \min(1, \bar\mu[\![|u| > \alpha]\!]) d\alpha\end{aligned}$$

for every $u \in L^0$, writing $L^1 = L^1(\mathfrak{A}, \bar\mu)$, $L^\infty = L^\infty(\mathfrak{A})$.

(c)

$$\{u : u \in L^0,\, \|u\|_{1,\infty} < \infty\} = M^{1,\infty} = M^{1,\infty}(\mathfrak{A}, \bar\mu),$$

$$\{u : u \in L^0,\, \|u\|_{\infty,1} < \infty\} = M^{\infty,1} = M^{\infty,1}(\mathfrak{A}, \bar\mu).$$

(d) Writing $\mathfrak{A}^f = \{a : \bar\mu a < \infty\}$, $S(\mathfrak{A}^f)$ is norm-dense in $M^{\infty,1}$ and $S(\mathfrak{A})$ is norm-dense in $M^{1,\infty}$.

(e) For any $p \in [1, \infty]$,

$$\|u\|_{1,\infty} \le \|u\|_p \le \|u\|_{\infty,1}$$

for every $u \in L^0$.

Remark By writing 'min' rather than 'inf' in the formulae of part (b) I mean to assert that the infima are attained.

proof (a) This is easy; all we need to know is that $\| \,\|_1$ and $\| \,\|_\infty$ are extended Fatou norms.

(b) We have four functionals on L^0 to look at; let me give them names:

$$\tau_1(u) = \sup\{\|u \times v\|_1 : \|v\|_{\infty,1} \le 1\},$$

$$\tau_2(u) = \inf\{\|u'\|_1 + \|u''\|_\infty : u = u' + u''\},$$

$$\tau_3(u) = \inf_{\alpha\ge 0}(\alpha + \int (|u| - \alpha\chi 1)^+),$$

$$\tau_4(u) = \int_0^\infty \min(1, \bar\mu[\![|u| > \alpha]\!]) d\alpha.$$

(I write 'inf' here to avoid the question of attainment for the moment.) Now we have the following.

(i) $\tau_1(u) \le \tau_2(u)$. **P** If $\|v\|_{\infty,1} \le 1$ and $u = u' + u''$, then

$$\|u \times v\|_1 \le \|u' \times v\|_1 + \|u'' \times v\|_1 \le \|u'\|_1 \|v\|_\infty + \|u''\|_\infty \|v\|_1 \le \|u'\|_1 + \|u''\|_\infty.$$

Taking the supremum over v and the infinum over u' and u'', $\tau_1(u) \le \tau_2(u)$. **Q**

(ii) $\tau_2(u) \le \tau_4(u)$. **P** If $\tau_4(u) = \infty$ this is trivial. Otherwise, take w such that $\|w\|_\infty \le 1$ and $u = |u| \times w$. Set $\alpha_0 = \inf\{\alpha : \bar\mu[\![|u| > \alpha]\!] \le 1\}$, and try

$$u' = w \times (|u| - \alpha_0 \chi 1)^+, \quad u'' = w \times (|u| \wedge \alpha_0 \chi 1).$$

Then $u = u' + u''$,

$$\begin{aligned} \|u'\|_1 &= \int_0^\infty \bar\mu[\![|u'| > \alpha]\!] d\alpha = \int_0^\infty \bar\mu[\![|u| > \alpha + \alpha_0]\!] d\alpha \\ &= \int_{\alpha_0}^\infty \bar\mu[\![|u| > \alpha]\!] d\alpha = \int_{\alpha_0}^\infty \min(1, \bar\mu[\![|u| > \alpha]\!]) d\alpha, \end{aligned}$$

$$\|u''\|_\infty \le \alpha_0 = \int_0^{\alpha_0} \min(1, [\![|u| > \alpha]\!]) d\alpha,$$

so

$$\tau_2(u) \le \|u'\|_1 + \|u''\|_\infty \le \tau_4(u). \ \mathbf{Q}$$

(iii) $\tau_4(u) \le \tau_3(u)$. **P** For any $\alpha \ge 0$,

$$\begin{aligned} \tau_4(u) &= \int_0^\alpha \min(1, \bar\mu[\![|u| > \beta]\!]) d\beta + \int_\alpha^\infty \min(1, \bar\mu[\![|u| > \beta]\!]) d\beta \\ &\le \alpha + \int_0^\infty \bar\mu[\![|u| > \alpha + \beta]\!] d\beta \\ &= \alpha + \int_0^\infty \bar\mu[\![(|u| - \alpha\chi 1)^+ > \beta]\!] d\beta = \alpha + \int (|u| - \alpha\chi 1)^+. \end{aligned}$$

Taking the infimum over α, $\tau_4(u) \le \tau_3(u)$. **Q**

(iv) $\tau_3(u) \le \tau_1(u)$.

P$(\boldsymbol{\alpha})$ It is enough to consider the case $0 < \tau_1(u) < \infty$, because if $\tau_1(u) = 0$ then $u = 0$ and evidently $\tau_3(0) = 0$, while if $\tau_1(u) = \infty$ the required inequality is trivial. Furthermore, since $\tau_3(u) = \tau_3(|u|)$ and $\tau_1(u) = \tau_1(|u|)$, it is enough to consider the case $u \ge 0$.

$(\boldsymbol{\beta})$ Note next that if $\bar\mu a < \infty$, then $\|\frac{1}{\max(1,\bar\mu a)} \chi a\|_{\infty,1} \le 1$, so that $\int_a u \le \max(1, \bar\mu a)\tau_1(u)$.

$(\boldsymbol{\gamma})$ Set $c = [\![u > 2\tau_1(u)]\!]$. If $a \subseteq c$ and $\bar\mu a < \infty$, then

$$2\tau_1(u)\bar\mu a \le \int_a u \le \max(1, \bar\mu a)\tau_1(u),$$

so $\bar\mu a \le \frac{1}{2}$. As $(\mathfrak{A}, \bar\mu)$ is semi-finite, it follows that $\bar\mu c \le \frac{1}{2}$ (322Eb).

$(\boldsymbol{\delta})$ I may therefore write

$$\alpha_0 = \inf\{\alpha : \alpha \ge 0, \ \bar\mu[\![u > \alpha]\!] \le 1\}.$$

Now $[\![u > \alpha_0]\!] = \sup_{\alpha > \alpha_0} [\![u > \alpha]\!]$, so

$$\bar\mu[\![u > \alpha_0]\!] = \sup_{\alpha > \alpha_0} \bar\mu[\![u > \alpha]\!] \le 1.$$

$(\boldsymbol{\epsilon})$ If $\alpha \ge \alpha_0$ then

$$(u - \alpha_0\chi 1)^+ \le (\alpha - \alpha_0)\chi[\![u > \alpha_0]\!] + (u - \alpha\chi 1)^+,$$

so

$$\begin{aligned} \alpha_0 + \int (u - \alpha_0\chi 1)^+ &\le \alpha_0 + (\alpha - \alpha_0)\bar\mu[\![u > \alpha_0]\!] + \int (u - \alpha\chi 1)^+ \\ &\le \alpha + \int (u - \alpha\chi 1)^+. \end{aligned}$$

If $0 \le \alpha < \alpha_0$ then, for every $\beta \in [0, \alpha_0 - \alpha[$,

$$(u - \alpha_0\chi 1)^+ + \beta[\![u > \alpha + \beta]\!] \le (u - \alpha\chi 1)^+,$$

while $\bar{\mu}[\![u > \alpha + \beta]\!] > 1$, so

$$\int (u - \alpha_0\chi 1)^+ + \beta + \alpha \le \alpha + \int (u - \alpha\chi 1)^+;$$

taking the supremum over β,

$$\alpha_0 + \int (u - \alpha_0\chi 1)^+ \le \alpha + \int (u - \alpha\chi 1)^+.$$

Thus $\alpha_0 + \int (u - \alpha_0\chi 1)^+ = \tau_3(u)$.

(ζ) If $\alpha_0 = 0$, take $v = \chi[\![u > 0]\!]$; then $\|v\|_{\infty,1} = \bar{\mu}[\![u > 0]\!] \le 1$ and

$$\tau_3(u) = \int u = \|u \times v\|_1 \le \tau_1(u).$$

(η) If $\alpha_0 > 0$, set $\gamma = \bar{\mu}[\![u > \alpha_0]\!]$. Take any $\beta \in [0, \alpha_0[$. Then $\bar{\mu}([\![u > \beta]\!] \setminus [\![u > \alpha_0]\!]) > 1 - \gamma$, so there is a $b \subseteq [\![u > \beta]\!] \setminus [\![u > \alpha_0]\!]$ such that $1 - \gamma < \bar{\mu}b < \infty$. Set $v = \chi[\![u > \alpha_0]\!] + \frac{1-\gamma}{\bar{\mu}b}\chi b$. Then $\|v\|_{\infty,1} = 1$ so

$$\tau_1(u) \ge \int u \times v \ge \int (u - \alpha_0\chi 1)^+ + \alpha_0\gamma + \beta\frac{1-\gamma}{\bar{\mu}b}\bar{\mu}b = \tau_3(u) - (1-\gamma)(\alpha_0 - \beta).$$

As β is arbitrary, $\tau_1(u) \ge \tau_3(u)$ in this case also. **Q**

(v) Thus $\tau_1(u) = \tau_2(u) = \tau_3(u) = \tau_4(u)$ for every $u \in L^0$, and I may write $\|u\|_{1,\infty}$ for their common value; being the associate of $\| \|_{\infty,1}$, $\| \|_{1,\infty}$ is an extended Fatou norm. As for the attainment of the infima, the argument of (iv-ϵ) above shows that, at least when $0 < \|u\|_{1,\infty} < \infty$, there is an α_0 such that $\alpha_0 + \int (|u| - \alpha_0)^+ = \|u\|_{1,\infty}$. This omits the cases $\|u\|_{1,\infty} \in \{0, \infty\}$; but in either of these cases we can set $\alpha_0 = 0$ to see that the infimum is attained for trivial reasons. For the other infimum, observe that the argument of (ii) produces u', u'' such that $u = u' + u''$ and $\|u'\|_1 + \|u''\|_\infty \le \tau_4(u)$.

(c) This is now obvious from the definition of $\| \|_{\infty,1}$ and the characterization of $\| \|_{1,\infty}$ in terms of $\| \|_1$ and $\| \|_\infty$.

(d) To see that $S = S(\mathfrak{A})$ is norm-dense in $M^{1,\infty}$, we need only note that S is dense in L^∞ and $S \cap L^1$ is dense in L^1; so that given $v \in L^1$, $w \in L^\infty$ and $\epsilon > 0$ there are v', $w' \in S$ such that

$$\|(v + w) - (v' + w')\|_{1,\infty} \le \|v - v'\|_1 + \|w - w'\|_\infty \le \epsilon.$$

As for $M^{\infty,1}$, if $u \ge 0$ in $M^{\infty,1}$ and $r \in \mathbb{N}$, set $v_r = \sup_{k \in \mathbb{N}} 2^{-r}k\chi[\![u > 2^{-r}k]\!]$; then each $v_r \in S^f = S(\mathfrak{A}^f)$, $\|u - v_r\|_\infty \le 2^{-r}$, and $\langle v_r \rangle_{r \in \mathbb{N}}$ is a non-decreasing sequence with supremum u, so that $\lim_{r\to\infty} \int v_r = \int u$ and $\lim_{r\to\infty} \|u - v_r\|_{\infty,1} = 0$. Thus $(S^f)^+$ is dense in $(M^{\infty,1})^+$. As usual, it follows that $S^f = (S^f)^+ - (S^f)^+$ is dense in $M^{\infty,1} = (M^{\infty,1})^+ - (M^{\infty,1})^+$.

(e)(i) If $p = 1$ or $p = \infty$ this is immediate from the definition of $\| \|_{\infty,1}$ and the characterization of $\| \|_{1,\infty}$ in (b). So suppose henceforth that $1 < p < \infty$.

(ii) If $\|u\|_{\infty,1} \le 1$ then $\|u\|_p \le 1$. **P** Because $\|u\|_\infty \le 1$, $|u|^p \le |u|$, so that $\int |u|^p \le \|u\|_1 \le 1$ and $\|u\|_p \le 1$. **Q** On considering scalar multiples of u, we see at once that $\|u\|_p \le \|u\|_{\infty,1}$ for every $u \in L^0$.

(ii) Now set $q = p/(p-1)$. Then

$$\|u\|_p = \sup\{\|u \times v\|_1 : \|v\|_q \le 1\}$$

(369L)

$$\ge \sup\{\|u \times v\|_1 : \|v\|_{\infty,1} \le 1\} = \|u\|_{1,\infty}$$

because $\| \|_{1,\infty}$ is the associate of $\| \|_{\infty,1}$. This completes the proof.

369P In preparation for some ideas in §372, I go a little farther with $M^{1,0}$, as defined in 366F.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) $M^{1,0} = M^{1,0}(\mathfrak{A}, \bar{\mu})$ is a norm-closed solid linear subspace of $M^{1,\infty} = M^{1,\infty}(\mathfrak{A}, \bar{\mu})$.

(b) The norm $\| \|_{1,\infty}$ is order-continuous on $M^{1,0}$.

(c) $S(\mathfrak{A}^f)$ and $L^1(\mathfrak{A}, \bar{\mu})$ are norm-dense and order-dense in $M^{1,0}$.

proof (a) Of course $M^{1,0}$, being a solid linear subspace of $L^0 = L^0(\mathfrak{A})$ included in $M^{1,\infty}$, is a solid linear subspace of $M^{1,\infty}$. To see that it is norm-closed, take any point u of its closure. Then for any $\epsilon > 0$ there is a $v \in M^{1,0}$ such that $\|u - v\|_{1,\infty} \le \epsilon$; now $(|u - v| - \epsilon\chi 1)^+ \in L^1 = L^1_{\bar\mu}$, so $[\![|u - v| > 2\epsilon]\!]$ has finite measure; also $[\![|v| > \epsilon]\!]$ has finite measure, so

$$[\![|u| > 3\epsilon]\!] \subseteq [\![|u - v| > 2\epsilon]\!] \cup [\![|v| > \epsilon]\!]$$

(364Ea) has finite measure. As ϵ is arbitrary, $u \in M^{1,0}$; as u is arbitrary, $M^{1,0}$ is closed.

(b) Suppose that $A \subseteq M^{1,0}$ is non-empty and downwards-directed and has infimum 0. Let $\epsilon > 0$. Set $B = \{(u - \epsilon\chi 1)^+ : u \in A\}$. Then $B \subseteq L^1$ (by 366Gc); B is non-empty and downwards-directed and has infimum 0. Because $\| \,\|_1$ is order-continuous (365C), $\inf_{v \in B} \|v\|_1 = 0$ and there is a $u \in A$ such that $\|(u - \epsilon\chi 1)^+\|_1 \le \epsilon$, so that $\|u\|_{1,\infty} \le 2\epsilon$. As ϵ is arbitrary, $\inf_{u \in A} \|u\|_{1,\infty} = 0$; as A is arbitrary, $\| \,\|_{1,\infty}$ is order-continuous on $M^{1,0}$.

(c) By 366Gb, $S(\mathfrak{A}^f)$ is order-dense in $M^{1,0}$. Because the norm of $M^{1,0}$ is order-continuous, $S(\mathfrak{A}^f)$ is also norm-dense (354Ef). Now $S(\mathfrak{A}^f) \subseteq L^1 \subseteq M^{1,0}$, so L^1 must also be norm-dense and order-dense.

369Q Corollary Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra. Set $M^{1,\infty} = M^{1,\infty}(\mathfrak{A}, \bar\mu)$, etc.
(a) $(M^{1,\infty})^\times$ and $(M^{1,0})^\times$ can both be identified with $M^{\infty,1}$.
(b) $(M^{\infty,1})^\times$ can be identified with $M^{1,\infty}$; $M^{1,\infty}$ and $M^{\infty,1}$ are perfect Riesz spaces.

proof Everything is covered by 369O and 369K except the identification of $(M^{1,0})^\times$ with $M^{\infty,1}$. For this I return to 369C. Of course $M^{1,0}$ is order-dense in L^0, because it includes L^1, or otherwise. Setting

$$V = \{v : v \in L^0,\ u \times v \in L^1 \text{ for every } u \in M^{1,0}\},$$

369C identifies V with $(M^{1,0})^\times$. Of course $M^{\infty,1} \subseteq V$ just because $M^{1,0} \subseteq M^{1,\infty}$.

Also $V \subseteq M^{\infty,1}$. **P** Because $L^1 \subseteq M^{1,0}$ and $\| \,\|_\infty$ is the associate of $\| \,\|_1$, $V \subseteq L^\infty$. **?** If there is a $v \in V \setminus L^1$, then (because $(\mathfrak{A}, \bar\mu)$ is semi-finite, so that $|v| = \sup_{a \in \mathfrak{A}^f} |v| \times \chi a$) $\sup_{a \in \mathfrak{A}^f} \int_a |v| = \infty$. For each $n \in \mathbb{N}$ choose $a_n \in \mathfrak{A}^f$ such that $\int_{a_n} |v| \ge 4^n$, and set $u = \sup_{n \in \mathbb{N}} 2^{-n}\chi a_n \in M^{1,0}$; then $\int u \times |v| \ge 2^n$ for each n, so again $v \notin V$. **X** Thus $V \subseteq L^1$ and $V \subseteq M^{\infty,1}$. **Q**

So $M^{\infty,1} = V$ can be identified with $(M^{1,0})^\times$.

369R The detailed formulae of 369O are of course special to the norms $\| \,\|_1$, $\| \,\|_\infty$, but the general phenomenon is not.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra, and τ_1, τ_2 two extended Fatou norms on $L^0 = L^0(\mathfrak{A})$ with associates τ_1', τ_2'. Then we have an extended Fatou norm τ defined by the formula

$$\tau(u) = \min\{\tau_1(v) + \tau_2(w) : v,\ w \in L^0,\ v + w = u\}$$

for every $u \in L^0$, and its associate τ' is given by the formula

$$\tau'(u) = \max(\tau_1'(u), \tau_2'(u))$$

for every $u \in L^0$. Moreover, the corresponding function spaces are

$$L^\tau = L^{\tau_1} + L^{\tau_2}, \quad L^{\tau'} = L^{\tau_1'} \cap L^{\tau_2'}.$$

proof (a) For the moment, define τ by setting

$$\tau(u) = \inf\{\tau_1(v) + \tau_2(w) : v + w = u\}$$

for $u \in L^0$. It is easy to check that, for u, $u' \in L^0$ and $\alpha \in \mathbb{R}$,

$$\tau(u + u') \le \tau(u) + \tau(u'), \quad \tau(\alpha u) = |\alpha|\tau(u), \quad \tau(u) \le \tau(u') \text{ if } |u| \le |u'|.$$

(For the last, remember that in this case $u = u' \times z$ where $\|z\|_\infty \le 1$.)

(b) Take any non-empty, upwards-directed set $A \subseteq (L^0)^+$, with supremum u_0. Suppose that $\gamma = \sup_{u \in A} \tau(u) < \infty$. For $u \in A$ and $n \in \mathbb{N}$ set

$$C_{un} = \{v : v \in L^0,\ 0 \le v \le u_0,\ \tau_1(v) + \tau_2(u - v)^+ \le \gamma + 2^{-n}\}.$$

Then

(i) every C_{un} is non-empty (because $\tau(u) \le \gamma$);

(ii) every C_{un} is convex (because if v_1, $v_2 \in C_{un}$ and $\alpha \in [0,1]$ and $v = \alpha v_1 + (1-\alpha)v_2$, then

$$(u-v)^+ = (\alpha(u-v_1) + (1-\alpha)(u-v_2))^+ \le \alpha(u-v_1)^+ + (1-\alpha)(u-v_2)^+,$$

so

$$\begin{aligned}\tau_1(v) + \tau_2(u-v)^+ &\le \alpha\tau_1(v_1) + (1-\alpha)\tau_1(v_2) + \alpha\tau_2(u-v_1)^+ + (1-\alpha)\tau_2(u-v_2)^+ \\ &\le \gamma + 2^{-n});\end{aligned}$$

(iii) if u, $u' \in A$, m, $n \in \mathbb{N}$ and $u \le u'$, $m \le n$ then $C_{u'n} \subseteq C_{um}$;

(iv) every C_{un} is closed for the topology of convergence in measure. **P?** Suppose otherwise. Then we can find a v in the closure of C_{un} for the topology of convergence in measure, but such that $\tau_1(v) + \tau_2(u-v)^+ > \gamma + 2^{-n}$. In this case

$$\tau_1(v) = \sup\{\tau_1(v \times \chi a) : a \in \mathfrak{A}^f\}, \quad \tau_2(u-v)^+ = \sup\{\tau_2((u-v)^+ \times \chi a) : a \in \mathfrak{A}^f\},$$

so there is an $a \in \mathfrak{A}^f$ such that

$$\tau_1(v \times \chi a) + \tau_2((u-v)^+ \times \chi a) > \gamma + 2^{-n}.$$

Now there is a sequence $\langle v_k \rangle_{k\in\mathbb{N}}$ in C_{un} such that $\bar\mu(a \cap [\![|v - v_k| \ge 2^{-k}]\!]) \le 2^{-k}$ for every k. Setting

$$v'_k = \inf_{i\ge k} v_i, \quad w_k = \inf_{i\ge k}(u - v_i)^+$$

we have

$$\tau_1(v'_k) + \tau_2(w_k) \le \tau_1(v_k) + \tau_2(u-v_k)^+ \le \gamma + 2^{-n}$$

for each k, and $\langle v'_k \rangle_{k\in\mathbb{N}}$, $\langle w_k \rangle_{k\in\mathbb{N}}$ are non-decreasing. So setting $v^* = \sup_{k\in\mathbb{N}} v \wedge v'_k$, $w^* = \sup_{k\in\mathbb{N}}(u-v)^+ \wedge w_k$, we get

$$\tau_1(v^*) + \tau_2(w^*) \le \gamma + 2^{-n}.$$

But $v^* \ge v \times \chi a$ and $w^* \ge (u-v)^+ \times \chi a$, so

$$\tau_1(v \times \chi a) + \tau_2((u-v)^+ \times \chi a) \le \gamma + 2^{-n},$$

contrary to the choice of a. **XQ**

Applying 367V, we find that $\bigcap_{u\in A, n\in\mathbb{N}} C_{un}$ is non-empty. If v belongs to the intersection, then

$$\tau_1(v) + \tau_2(u-v)^+ \le \gamma$$

for every $u \in A$; since $\{(u-v)^+ : u \in A\}$ is an upwards-directed set with supremum $(u_0 - v)^+$, and τ_2 is an extended Fatou norm,

$$\tau_1(v) + \tau_2(u_0 - v)^+ \le \gamma.$$

(c) This shows both that the infimum in the definition of $\tau(u)$ is always attained (since this is trivial if $\tau(u) = \infty$, and otherwise we consider $A = \{|u|\}$), and also that $\tau(\sup A) = \sup_{u\in A}\tau(u)$ whenever $A \subseteq (L^0)^+$ is a non-empty upwards-directed set with a supremum. Thus τ satisfies conditions (i)-(iv) of 369F. Condition (vi) there is trivial, since (for instance) $\tau(v) \le \tau_1(v)$ for every v. As for 369F(v), suppose that $u > 0$ in L^0. Take u_1 such that $0 < u_1 \le u$ and $\tau'_1(u_1) \le 1$, u_2 such that $0 < u_2 \le u_1$ and $\tau'_2(u_2) \le 1$. In this case, if $u_2 = v + w$, we must have

$$\tau_1(v) + \tau_2(w) \ge \|v \times u_1\|_1 + \|w \times u_2\|_1 \ge \|u_2 \times u_2\|_1;$$

so that

$$\tau(u) \ge \|u_2 \times u_2\|_1 > 0.$$

Thus all the conditions of 369F are satisfied, and τ is an extended Fatou norm on L^0.

(d) The calculation of τ' is now very easy. Since surely we have $\tau \le \tau_i$ for both i, we must have $\tau' \ge \tau'_i$ for both i. On the other hand, if u, $z \in L^0$, then there are v, w such that $u = v + w$ and $\tau(u) = \tau_1(v) + \tau_2(w)$, so that

$$\|u \times z\|_1 \le \|v \times z\|_1 + \|w \times z\|_1 \le \tau_1(v)\tau'_1(z) + \tau_2(w)\tau'_2(z) \le \tau(u)\max(\tau'_1(z), \tau'_2(z));$$

as u is arbitrary, $\tau'(z) \le \max(\tau'_1(z), \tau'_2(z))$. So $\tau' = \max(\tau'_1, \tau'_2)$, as claimed.

(e) Finally, it is obvious that

$$L^{\tau'} = \{z : \tau'(z) < \infty\} = \{z : \tau'_1(z) < \infty, \tau'_2(z) < \infty\} = L^{\tau'_1} \cap L^{\tau'_2},$$

while the fact that the infimum in the definition of τ is always attained means that $L^\tau \subseteq L^{\tau_1}+L^{\tau_2}$, so that we have equality here also.

369X Basic exercises >(a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that the following are equiveridical: (i) there is a function $\bar{\mu}$ such that $(\mathfrak{A},\bar{\mu})$ is a semi-finite measure algebra; (ii) $(L^\infty)^\times$ separates the points of $L^\infty = L^\infty(\mathfrak{A})$; (iii) for every non-zero $a \in \mathfrak{A}$ there is a completely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu a \neq 0$; (iv) there is some order-dense Riesz subspace U of $L^0 = L^0(\mathfrak{A})$ such that $U^\times$ separates the points of U; (v) for every order-dense Riesz subspace U of L^0 there is an order-dense Riesz subspace V of U such that $V^\times$ separates the points of V.

(b) Let us say that a function $\phi : \mathbb{R} \to \,]-\infty,\infty]$ is **convex** if $\phi(\alpha x + (1-\alpha)y) \ge \alpha\phi(x) + (1-\alpha)\phi(y)$ for all x, $y \in \mathbb{R}$ and $\alpha \in [0,1]$, interpreting $0 \cdot \infty$ as 0, as usual. For any convex function $\phi : \mathbb{R} \to \,]-\infty,\infty]$ which is not always infinite, set $\phi^*(y) = \sup_{x\in\mathbb{R}} xy - \phi(x)$ for every $y \in \mathbb{R}$. (i) Show that $\phi^* : \mathbb{R} \to \,]-\infty,\infty]$ is convex and lower semi-continuous and not always infinite. (*Hint*: 233Xh.) (ii) Show that if ϕ is lower semi-continuous then $\phi = \phi^{**}$. (*Hint*: It is easy to check that $\phi^{**} \le \phi$. For the reverse inequality, set $I = \{x : \phi(x) < \infty\}$, and consider $x \in \operatorname{int} I$, $x \in I \setminus \operatorname{int} I$ and $x \notin I$ separately; 233Ha is useful for the first.)

>(c) For the purposes of this exercise and the next, say that a **Young's function** is a non-negative non-constant lower semi-continuous convex function $\phi : [0,\infty[\, \to [0,\infty]$ such that $\phi(0) = 0$ and $\phi(x)$ is finite for some $x > 0$. (**Warning!** the phrase 'Young's function' has other meanings.) (i) Show that in this case ϕ is non-decreasing and continuous on the left and ϕ^*, defined by saying that $\phi^*(y) = \sup_{x\ge0} xy - \phi(x)$ for every $y \ge 0$, is again a Young's function. (ii) Show that $\phi^{**} = \phi$. Say that ϕ and ϕ^* are **complementary**. (iii) Compute ϕ^* in the cases (α) $\phi(x) = x$ (β) $\phi(x) = \max(0, x-1)$ (γ) $\phi(x) = x^2$ (δ) $\phi(x) = x^p$ where $1 < p < \infty$.

>(d) Let ϕ, $\psi = \phi^*$ be complementary Young's functions in the sense of 369Xc, and $(\mathfrak{A},\bar{\mu})$ a semi-finite measure algebra. Set

$$B = \{u : u \in L^0,\ \textstyle\int \bar{\phi}(|u|) \le 1\}, \quad C = \{v : v \in L^0,\ \textstyle\int \bar{\psi}(|v|) \le 1\}.$$

(For finite-valued ϕ, $\bar{\phi} : (L^0)^+ \to L^0$ is given by 364H. Devise an appropriate convention for the case in which ϕ takes the value ∞.) (i) Show that B and C are order-closed solid convex sets, and that $\int |u \times v| \le 2$ for all $u \in B$, $v \in C$. (*Hint*: for 'order-closed', use 364Xg(iv).) (ii) Show that there is a unique extended Fatou norm τ_ϕ on L^0 for which B is the unit ball. (iii) Show that if $u \in L^0 \setminus B$ there is a $v \in C$ such that $\int |u \times v| > 1$. (*Hint*: start with the case in which $u \in S(\mathfrak{A})^+$.) (iv) Show that $\tau_\psi \le \tau'_\phi \le 2\tau_\psi$, where τ_ψ is the extended Fatou norm corresponding to ψ and τ'_ϕ is the associate of τ_ϕ, so that τ_ψ and τ'_ϕ can be interpreted as equivalent norms on the same Banach space.

(U and V are complementary **Orlicz spaces**; I will call τ_ϕ, τ_ψ **Orlicz norms**.)

(e) Let U be a Riesz space such that $U^\times$ separates the points of U, and suppose that $\|\,\|$ is a Fatou norm on U. (i) Show that there is a localizable measure algebra $(\mathfrak{A},\bar{\mu})$ with an extended Fatou norm τ on $L^0(\mathfrak{A})$ such that U can be identified, as normed Riesz space, with an order-dense Riesz subspace of L^τ. (ii) Hence, or otherwise, show that $\|u\| = \sup_{f\in U^\times,\,\|f\|\le1} |f(u)|$ for every $u \in U$. (iii) Show that if U is Dedekind complete and has the Levi property, then U becomes identified with L^τ itself, and in particular is a Banach lattice (cf. 354Xn).

(f) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0(\mathfrak{A})$. Show that the norm of L^τ is order-continuous iff the norm topology of L^τ agrees with the topology of convergence in measure on any order-bounded subset of L^τ.

(g) Let $(\mathfrak{A},\bar{\mu})$ be a σ-finite measure algebra of countable Maharam type, and τ an extended Fatou norm on $L^0(\mathfrak{A})$ such that the norm of L^τ is order-continuous. Show that L^τ is separable in its norm topology.

(h) Let $(\mathfrak{A},\bar{\mu})$ be an atomless semi-finite measure algebra. Show that $\|u\|_{1,\infty} = \max\{\int_a |u| : a \in \mathfrak{A},\ \bar{\mu}a \le 1\}$ for every $u \in L^0(\mathfrak{A})$. (*Hint*: take $a \supseteq [\![|u| > \alpha_0]\!]$ in part (b-iv) of the proof of 369O.)

(i) Let $(\mathfrak{A},\bar{\mu})$ be any semi-finite measure algebra. Show that if τ_ϕ is any Orlicz norm on $L^0 = L^0(\mathfrak{A})$, then there is a $\gamma > 0$ such that $\|u\|_{1,\infty} \le \gamma\tau_\phi(u) \le \gamma^2\|u\|_{\infty,1}$ for every $u \in L^0$, so that $M^{\infty,1}_{\bar{\mu}} \subseteq L^{\tau_\phi} \subseteq M^{1,\infty}_{\bar{\mu}}$.

(j) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra. Show that the subspaces $M^{1,\infty}_{\bar{\mu}}$, $M^{\infty,1}_{\bar{\mu}}$ of $L^0(\mathfrak{A})$ can be expressed as a complementary pair of Orlicz spaces, and that the norm $\|\,\|_{\infty,1}$ can be represented as an Orlicz norm, but $\|\,\|_{1,\infty}$ cannot.

>(k) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and U a Banach space. (i) Suppose that $\nu:\mathfrak{A}\to U$ is an additive function such that $\|\nu a\|\le\min(1,\bar\mu a)$ for every $a\in\mathfrak{A}$. Show that there is a unique bounded linear operator $T:M^{1,\infty}_{\bar\mu}\to U$ such that $T(\chi a)=\nu a$ for every $a\in\mathfrak{A}$. (ii) Suppose that $\nu:\mathfrak{A}^f\to U$ is an additive function such that $\|\nu a\|\le\max(1,\bar\mu a)$ for every $a\in\mathfrak{A}^f$. Show that there is a unique bounded linear operator $T:M^{\infty,1}_{\bar\mu}\to U$ such that $T(\chi a)=\nu a$ for every $a\in\mathfrak{A}^f$.

(l) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, and $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ a measure-preserving ring homomorphism, as in 366H, with associated maps $T:M^0_{\bar\mu}\to M^0_{\bar\nu}$ and $P:M^{1,0}_{\bar\nu}\to M^{1,0}_{\bar\mu}$. Show that $\|Tu\|_{\infty,1}=\|u\|_{\infty,1}$ for every $u\in M^{\infty,1}_{\bar\mu}$ and $\|Pv\|_{\infty,1}\le\|v\|_{\infty,1}$ for every $v\in M^{\infty,1}_{\bar\nu}$.

(m) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism. (i) Show that there is a unique Riesz homomorphism $T:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$ such that $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$ and $\|Tu\|_{1,\infty}=\|u\|_{1,\infty}$ for every $u\in M^{1,\infty}_{\bar\mu}$. (ii) Now suppose that $(\mathfrak{A},\bar\mu)$ is localizable and π is order-continuous. Show that there is a unique positive linear operator $P:M^{1,\infty}_{\bar\nu}\to M^{1,\infty}_{\bar\mu}$ such that $\int_a Pv=\int_{\pi a}v$ for every $a\in\mathfrak{A}^f$ and $v\in M^{1,\infty}_{\bar\nu}$, and that $\|Pv\|_\infty\le\|v\|_\infty$ for every $v\in L^\infty(\mathfrak{B})$, $\|Pv\|_{1,\infty}\le\|v\|_{1,\infty}$ for every $v\in M^{1,\infty}_{\bar\nu}$. (Compare 365P.)

(n) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, and $\phi:[0,\infty[\,\to[0,\infty]$ a Young's function; write τ_ϕ for the corresponding Orlicz norm on either $L^0(\mathfrak{A})$ or $L^0(\mathfrak{B})$. Let $\pi:\mathfrak{A}\to\mathfrak{B}$ be a measure-preserving Boolean homomorphism, with associated map $T:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$, as in 369Xm. (i) Show that $\tau_\phi(Tu)=\tau_\phi(u)$ for every $u\in M^{1,\infty}_{\bar\mu}$. (ii) Show that if $(\mathfrak{A},\bar\mu)$ is localizable, π is order-continuous and $P:M^{1,\infty}_{\bar\nu}\to M^{1,\infty}_{\bar\mu}$ is the map of 369Xm(ii), then $\tau_\phi(Pv)\le\tau_\phi(v)$ for every $v\in M^{1,\infty}_{\bar\nu}$. (*Hint*: 365R.)

>(o) Let $(\mathfrak{A},\bar\mu)$ be any semi-finite measure algebra and τ_1, τ_2 two extended Fatou norms on $L^0(\mathfrak{A})$. Show that $u\mapsto\max(\tau_1(u),\tau_2(u))$ is an extended Fatou norm.

(p) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and $(\widehat{\mathfrak{A}},\tilde\mu)$ its localization (322Q). Show that the Dedekind completion of $M^{1,\infty}(\mathfrak{A},\bar\mu)$ can be identified with $M^{1,\infty}(\widehat{\mathfrak{A}},\tilde\mu)$.

(q) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. (i) Show that if $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$ such that $\sup\{b:b\in\mathfrak{B},\,\bar\mu b<\infty\}=1$ in $\mathfrak{A}$, we have an order-continuous positive linear operator $P_{\mathfrak{B}}:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\mu\restriction\mathfrak{B}}$ such that $\int_b P_{\mathfrak{B}}u=\int_b u$ whenever $u\in M^{1,\infty}_{\bar\mu}$, $b\in\mathfrak{B}$ and $\bar\mu b<\infty$. (ii) Show that if $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$ such that $\sup\{b:b\in\mathfrak{B}_0,\,\bar\mu b<\infty\}=1$ in $\mathfrak{A}$, and $\mathfrak{B}$ is the closure of $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$, then $\langle P_{\mathfrak{B}_n}u\rangle_{n\in\mathbb{N}}$ is order*-convergent to $P_{\mathfrak{B}}u$ for every $u\in M^{1,\infty}_{\bar\mu}$. (Cf. 367J.)

(r) Let ϕ_1 and ϕ_2 be Young's functions and $(\mathfrak{A},\bar\mu)$ a semi-finite measure algebra. Set $\phi(x)=\max(\phi_1(x),\phi_2(x))$ for $x\in[0,\infty[$. (i) Show that ϕ is a Young's function. (ii) Writing τ_{ϕ_1} τ_{ϕ_2}, τ_ϕ for the corresponding extended Fatou norms on $L^0(\mathfrak{A})$ (369Xd), show that $\tau_\phi\ge\max(\tau_{\phi_1},\tau_{\phi_2})\ge\frac{1}{2}\tau_\phi$, so that $L^{\tau_\phi}=L^{\tau_{\phi_1}}\cap L^{\tau_{\phi_2}}$ and $L^{\tau_{\phi^*}}=L^{\tau_{\phi_1^*}}+L^{\tau_{\phi_2^*}}$, writing ϕ^* for the Young's function complementary to ϕ. (iii) Repeat with $\psi=\phi_1+\phi_2$ in place of ϕ.

369Y Further exercises (a) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra and $A\subseteq L^0=L^0(\mathfrak{A})$ a countable set. Show that the solid linear subspace U of L^0 generated by A is a perfect Riesz space. (*Hint*: reduce to the case in which U is order-dense. If $A=\{u_n:n\in\mathbb{N}\}$, $w\in(L^0)^+\setminus U$ find $v_n\in(L^0)^+$ such that $\int v_n\times w\ge 2^n\ge 4^n\int v_n\times|u_i|$ for every $i\le n$. Show that $v=\sup_{n\in\mathbb{N}}v_n$ is defined in L^0 and corresponds to a member of $U^\times$.)

(b) Let U be a Banach lattice and suppose that $p\in[1,\infty[$ is such that $\|u+v\|^p=\|u\|^p+\|v\|^p$ whenever u, $v\in U$ and $|u|\wedge|v|=0$. Show that U is isomorphic, as Banach lattice, to $L^p_{\bar\mu}$ for some localizable measure algebra $(\mathfrak{A},\bar\mu)$. (*Hint*: start by using 354Yb to show that the norm of U is order-continuous, as in 354Yk.)

(c) Let $\phi:[0,\infty[\,\to[0,\infty[$ be a strictly increasing Young's function such that $\sup_{t>0}\phi(2t)/\phi(t)$ is finite. Show that the associated Orlicz norms τ_ϕ are always order-continuous on their function spaces.

(d) Let $\phi:[0,\infty[\,\to[0,\infty]$ be a Young's function, and suppose that the corresponding Orlicz norm on $L^0(\mathfrak{A}_L)$, where $(\mathfrak{A}_L,\bar\mu_L)$ is the measure algebra of Lebesgue measure on $\mathbb{R}$, is order-continuous on its function space L^{τ_ϕ}. Show that there is an $M\ge 0$ such that $\phi(2t)\le M\phi(t)$ for every $t\ge 0$.

(e) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and τ_ϕ an Orlicz norm which is order-continuous on L^{τ_ϕ}. Show that if $\mathcal{F}$ is a filter on L^{τ_ϕ}, then $\mathcal{F}\to u\in L^{\tau_\phi}$ for the norm τ_ϕ iff (i) $\mathcal{F}\to u$ for the topology of convergence in measure (ii) $\limsup_{v\to\mathcal{F}}\tau_\phi(v)\le\tau_\phi(u)$. (Compare 245Xl.)

(f) Give an example of an extended Fatou norm τ on $L^0(\mathfrak{A}_L)$, where $(\mathfrak{A}_L,\bar{\mu}_L)$ is the measure algebra of Lebesgue measure on $[0,1]$, such that (i) τ gives rise to an order-continuous norm on its function space L^τ (ii) there is a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in L^τ, converging in measure to $u\in L^\tau$, such that $\lim_{n\to\infty}\tau(u_n)=\tau(u)$ but $\langle u_n\rangle_{n\in\mathbb{N}}$ does not converge to u for the norm on L^τ.

(g) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and τ an Orlicz norm on $L^0(\mathfrak{A})$. Show that L^τ has the Levi property, whether or not $\mathfrak{A}$ is Dedekind complete.

(h) Let $(\mathfrak{A},\bar{\mu})$ be any measure algebra. Show that $(M^{1,0}_{\bar{\mu}})^\times$ can be identified with $M^{\infty,1}_{\bar{\mu}}$. (*Hint*: show that neither $M^{1,0}$ nor $M^{\infty,1}$ is changed by moving first to the semi-finite version of $(\mathfrak{A},\bar{\mu})$, as described in 322Xa, and then to its localization.)

(i) Give an example to show that the result of 369R may fail if $(\mathfrak{A},\bar{\mu})$ is only semi-finite, not localizable.

369 Notes and comments The representation theorems 369A-369D give a concrete form to the notion of 'perfect' Riesz space: it is just one which can be expressed as a subspace of $L^0(\mathfrak{A})$, for some localizable measure algebra $(\mathfrak{A},\bar{\mu})$, in such a way that it is its own second dual, where the duality here is between subspaces of L^0, taking $V=\{v:u\times v\in L^1$ for every $u\in U\}$. (I see that in this expression I ought somewhere to mention that both U and V are assumed to be order-dense in L^0.) Indeed I believe that the original perfect spaces were the 'vollkommene Räume' of G.Köthe, which were subspaces of $\mathbb{R}^\mathbb{N}$, corresponding to the measure algebra $\mathcal{P}\mathbb{N}$ with counting measure, so that V or $U^\times$ was $\{v:u\times v\in\ell^1$ for every $u\in U\}$.

I have presented Kakutani's theorem on the representation of L-spaces as a corollary of 369A and 369C. As usual in such things, this is a reversal of the historical relationship; Kakutani's theorem was one of the results which led to the general theory. If we take the trouble to re-work the argument of 369A in this context, we find that the L-space condition '$\|u+v\|=\|u\|+\|v\|$ whenever u, $v\ge 0$' can be relaxed to '$\|u+v\|=\|u\|+\|v\|$ whenever $u\wedge v=0$' (369Yb). The complete list of localizable measure algebras provided by Maharam's theorem (332B, 332J) now gives us a complete list of L-spaces.

Just as perfect Riesz spaces come in dual pairs, so do some of the most important Banach lattices: those with Fatou norms and the Levi property for which the order-continuous dual separates the points. (Note that the dual of any space with a Riesz norm has these properties; see 356Da.) I leave the details of representing such spaces to you (369Xe). The machinery of 369F-369K gives a solid basis for studying such pairs.

Among the extended Fatou norms of 369F the Orlicz norms (369Xd, 369Yc-369Ye) form a significant subfamily. Because they are defined in a way which is to some extent independent of the measure algebra involved, these spaces have some of the same properties as L^p spaces in relation to measure-preserving homomorphisms (369Xm-369Xn). In §§373-374 I will elaborate on these ideas. Among the Orlicz spaces, we have a largest and a smallest; these are just $M^{1,\infty}=L^1+L^\infty$ and $M^{\infty,1}=L^1\cap L^\infty$ (369N-369O, 369Xi, 369Xj). Of course these two are particularly important.

There is an interesting phenomenon here. It is easy to see that $\|\,\|_{\infty,1}=\max(\|\,\|_1,\|\,\|_\infty)$ is an extended Fatou norm and that the corresponding Banach lattice is $L^1\cap L^\infty$; and that the same ideas work for any pair of extended Fatou norms (369Xo). To check that the dual of $L^1\cap L^\infty$ is precisely the linear sum $L^\infty+L^1$ a little more is needed, and the generalization of this fact to other extended Fatou norms (369R) seems to go quite deep. In view of our ordinary expectation that properties of these normed function spaces should be reflected in perfect Riesz spaces in general, I mention that I believe I have found an example, dependent on the continuum hypothesis, of two perfect Riesz subspaces U, V of $\mathbb{R}^\mathbb{N}$ such that their linear sum $U+V$ is not perfect.

Chapter 37

Linear operators between function spaces

As everywhere in functional analysis, the function spaces of measure theory cannot be properly understood without investigating linear operators between them. In this chapter I have collected a number of results which rely on, or illuminate, the measure-theoretic aspects of the theory. §371 is devoted to a fundamental property of linear operators on L-spaces, if considered abstractly, that is, of L^1-spaces, if considered in the language of Chapter 36, and to an introduction to the class $\mathcal{T}$ of operators which are norm-decreasing for both $\| \|_1$ and $\| \|_\infty$. This makes it possible to prove a version of Birkhoff's Ergodic Theorem for operators which need not be positive (372D). In §372 I give various forms of this theorem, for linear operators between function spaces, for measure-preserving Boolean homomorphisms between measure algebras, and for inverse-measure-preserving functions between measure spaces, with an excursion into the theory of continued fractions. In §373 I make a fuller analysis of the class $\mathcal{T}$, with a complete characterization of those u, v such that $v = Tu$ for some $T \in \mathcal{T}$. Using this we can describe 'rearrangement-invariant' function spaces and extended Fatou norms (§374). Returning to ideas left on one side in §§364 and 368, I investigate positive linear operators defined on L^0 spaces (§375). In the penultimate section of the chapter (§376), I look at operators which can be defined in terms of kernels on product spaces. Finally, in §377, I examine the function spaces of reduced products, projective limits and inductive limits of probability algebras.

371 The Chacon-Krengel theorem

The first topic I wish to treat is a remarkable property of L-spaces: if U and V are L-spaces, then every continuous linear operator $T : U \to V$ is order-bounded, and $\||T|\| = \|T\|$ (371D). This generalizes in various ways to other V (371B, 371C). I apply the result to a special type of operator between $M^{1,0}$ spaces which will be conspicuous in the next section (371F-371H).

371A Lemma Let U be an L-space, V a Banach lattice and $T : U \to V$ a bounded linear operator. Take $u \geq 0$ in U and set

$$B = \{\sum_{i=0}^n |Tu_i| : u_0, \dots, u_n \in U^+, \sum_{i=0}^n u_i = u\} \subseteq V^+.$$

Then B is upwards-directed and $\sup_{v \in B} \|v\| \leq \|T\|\|u\|$.

proof (a) Suppose that v, $v' \in B$. Then we have $u_0, \dots, u_m, u'_0, \dots, u'_n \in U^+$ such that $\sum_{i=0}^m u_i = \sum_{j=0}^n u'_j = u$, $v = \sum_{i=0}^m |Tu_i|$ and $v' = \sum_{j=0}^n |Tu'_j|$. Now there are $v_{ij} \geq 0$ in U, for $i \leq m$ and $j \leq n$, such that $u_i = \sum_{j=0}^n v_{ij}$ for $i \leq m$ and $u'_j = \sum_{i=0}^m v_{ij}$ for $j \leq n$ (352Fd). We have $u = \sum_{i=0}^m \sum_{j=0}^n v_{ij}$, so that $v'' = \sum_{i=0}^m \sum_{j=0}^n |Tv_{ij}| \in B$. But

$$v = \sum_{i=0}^m |Tu_i| = \sum_{i=0}^m |T(\sum_{j=0}^n v_{ij})| \leq \sum_{i=0}^m \sum_{j=0}^m |Tv_{ij}| = v'',$$

and similarly $v' \leq v''$. As v and v' are arbitrary, B is upwards-directed.

(b) The other part is easy. If $v \in B$ is expressed as $\sum_{i=0}^n |Tu_i|$ where $u_i \geq 0$ for every i and $\sum_{i=0}^n u_i = u$, then

$$\|v\| \leq \sum_{i=0}^n \|Tu_i\| \leq \|T\| \sum_{i=0}^n \|u_i\| = \|T\|\|u\|$$

because U is an L-space.

371B Theorem Let U be an L-space and V a Dedekind complete Banach lattice U with a Fatou norm. Then the Riesz space $\mathsf{L}^\sim(U;V) = \mathsf{L}^\times(U;V)$ is a closed linear subspace of the Banach space $\mathsf{B}(U;V)$ and is in itself a Banach lattice with a Fatou norm.

proof (a) I start by noting that $\mathsf{L}^\sim(U;V) = \mathsf{L}^\times(U;V) \subseteq \mathsf{B}(U;V)$ just because V has a Riesz norm and U is a Banach lattice with an order-continuous norm (355Kb, 355C).

(b) The first new step is to check that $\||T|\| \leq \|T\|$ for any $T \in \mathsf{L}^\sim(U;V)$. **P** Start with any $u \in U^+$. Set

$$B = \{\sum_{i=0}^n |Tu_i| : u_0, \dots, u_n \in U^+, \sum_{i=0}^n u_i = u\} \subseteq V^+,$$

as in 371A. If $u_0, \dots, u_n \geq 0$ are such that $\sum_{i=0}^n u_i = u$, then $|Tu_i| \leq |T|u_i$ for each i, so that $\sum_{i=0}^n |Tu_i| \leq \sum_{i=0}^n |T|u_i = |T|u$; thus B is bounded above by $|T|u$ and $\sup B \leq |T|u$. On the other hand, if $|v| \leq u$ in U, then $v^+ + v^- + (u - |v|) = u$, so $|Tv^+| + |Tv^-| + |T(u - |v|)| \in B$ and

$$|Tv| = |Tv^+ + Tv^-| \leq |Tv^+| + |Tv^-| \leq \sup B.$$

As v is arbitrary, $|T|u \le \sup B$ and $|T|u = \sup B$. Consequently

$$\||T|u\| \le \|\sup B\| = \sup_{w \in B} \|w\| \le \|T\|\|u\|$$

because V has a Fatou norm and B is upwards-directed.

For general $u \in U$,

$$\||T|u\| \le \||T||u|\| \le \|T\|\||u|\| = \|T\|\|u\|.$$

This shows that $\||T|\| \le \|T\|$. **Q**

(c) Now if $|S| \le |T|$ in $\mathsf{L}^\sim(U;V)$, and $u \in U$, we must have

$$\|Su\| \le \||S||u|\| \le \||T||u|\| \le \||T|\|\||u|\| \le \|T\|\|u\|;$$

as u is arbitrary, $\|S\| \le \|T\|$. This shows that the norm of $\mathsf{L}^\sim(U;V)$, inherited from $\mathsf{B}(U;V)$, is a Riesz norm.

(d) Suppose next that $T \in \mathsf{B}(U;V)$ belongs to the norm-closure of $\mathsf{L}^\sim(U;V)$. For each $n \in \mathbb{N}$ choose $T_n \in \mathsf{L}^\sim(U;V)$ such that $\|T - T_n\| \le 2^{-n}$. Set $S_n = |T_{n+1} - T_n| \in \mathsf{L}^\sim(U;V)$ for each n. Then

$$\|S_n\| = \|T_{n+1} - T_n\| \le 3 \cdot 2^{-n-1}$$

for each n, so $S = \sum_{n=0}^\infty S_n$ is defined in the Banach space $\mathsf{B}(U;V)$. But if $u \in U^+$, we surely have

$$Su = \sum_{n=0}^\infty S_n u \ge 0$$

in V. Moreover, if $u \in U^+$ and $|v| \le u$, then for any $n \in \mathbb{N}$

$$|T_{n+1}v - T_0 v| = |\sum_{i=0}^n (T_{i+1} - T_i)v| \le \sum_{i=0}^n S_i u \le Su,$$

and $T_0 v - Su \le T_{n+1} v \le T_0 v + Su$; letting $n \to \infty$, we see that

$$-|T_0|u - Su \le T_0 v - Su \le Tv \le T_0 v + Su \le |T_0|u + Su.$$

So $|Tv| \le |T_0|u + Su$ whenever $|v| \le u$. As u is arbitrary, $T \in \mathsf{L}^\sim(U;V)$.

This shows that $\mathsf{L}^\sim(U;V)$ is closed in $\mathsf{B}(U;V)$ and is therefore a Banach space in its own right; putting this together with (b), we see that it is a Banach lattice.

(e) Finally, the norm of $\mathsf{L}^\sim(U;V)$ is a Fatou norm. **P** Let $A \subseteq \mathsf{L}^\sim(U;V)^+$ be a non-empty, upwards-directed set with supremum $T_0 \in \mathsf{L}^\sim(U;V)$. For any $u \in U$,

$$\|T_0 u\| = \||T_0 u|\| \le \|T_0|u|\| = \|\sup_{T \in A} T|u|\|$$

by 355Ed. But $\{T|u| : T \in A\}$ is upwards-directed and the norm of V is a Fatou norm, so

$$\|T_0 u\| \le \sup_{T \in A} \|T|u|\| \le \sup_{T \in A} \|T\|\|u\|.$$

As u is arbitrary, $\|T_0\| \le \sup_{T \in A} \|T\|$. As A is arbitrary, the norm of $\mathsf{L}^\sim(U;V)$ is Fatou. **Q**

371C Theorem Let U be an L-space and V a Dedekind complete Banach lattice with a Fatou norm and the Levi property. Then $\mathsf{B}(U;V) = \mathsf{L}^\sim(U;V) = \mathsf{L}^\times(U;V)$ is a Dedekind complete Banach lattice with a Fatou norm and the Levi property. In particular, $|T|$ is defined and $\||T|\| = \|T\|$ for every $T \in \mathsf{B}(U;V)$.

proof (a) Let $T : U \to V$ be any bounded linear operator. Then $T \in \mathsf{L}^\sim(U;V)$. **P** Take any $u \ge 0$ in U. Set

$$B = \{\sum_{i=0}^n |Tu_i| : u_0, \dots, u_n \in U^+, \sum_{i=0}^n u_i = u\} \subseteq V^+$$

as in 371A. Then 371A tells us that B is upwards-directed and norm-bounded. Because V has the Levi property, B is bounded above. But just as in part (b) of the proof of 371B, any upper bound of B is also an upper bound of $\{Tv : |v| \le u\}$. As u is arbitrary, $T \in \mathsf{L}^\sim(U;V)$. **Q**

(b) Accordingly $\mathsf{L}^\sim(U;V) = \mathsf{B}(U;V)$. By 371B, this is a Banach lattice with a Fatou norm, and equal to $\mathsf{L}^\times(U;V)$. To see that it also has the Levi property, let $A \subseteq \mathsf{L}^\sim(U;V)$ be any non-empty norm-bounded upwards-directed set. For $u \in U^+$, $\{Tu : T \in A\}$ is non-empty, norm-bounded and upwards-directed in V, so is bounded above in V. By 355Ed, A is bounded above in $\mathsf{L}^\sim(U;V)$.

371D Corollary Let U and V be L-spaces. Then $\mathsf{L}^\sim(U;V) = \mathsf{L}^\times(U;V) = \mathsf{B}(U;V)$ is a Dedekind complete Banach lattice with a Fatou norm and the Levi property.

371E Remarks Note that both these theorems show that $\mathsf{L}^\sim(U;V)$ is a Banach lattice with properties similar to those of V whenever U is an L-space. They can therefore be applied repeatedly, to give facts about $\mathsf{L}^\sim(U_1;\mathsf{L}^\sim(U_2;V))$ where U_1, U_2 are L-spaces and V is a Banach lattice, for instance. I hope that this formula will recall some of those in the theory of bilinear operators and tensor products (see 253Xa-253Xb).

371F The class $\mathcal{T}^{(0)}$ For the sake of applications in the next section, I introduce now a class of operators of great intrinsic interest.

Definition Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be measure algebras. Recall that $M^{1,0}(\mathfrak{A},\bar\mu)$ is the space of those $u\in L^1(\mathfrak{A},\bar\mu)+L^\infty(\mathfrak{A})$ such that $\bar\mu[\![|u|>\alpha]\!]<\infty$ for every $\alpha>0$ (366F-366G, 369P). Write $\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ for the set of all linear operators $T:M^{1,0}(\mathfrak{A},\bar\mu)\to M^{1,0}(\mathfrak{B},\bar\nu)$ such that $Tu\in L^1(\mathfrak{B},\bar\nu)$ and $\|Tu\|_1\le\|u\|_1$ for every $u\in L^1(\mathfrak{A},\bar\mu)$, $Tu\in L^\infty(\mathfrak{B})$ and $\|Tu\|_\infty\le\|u\|_\infty$ for every $u\in L^\infty(\mathfrak{A})\cap M^{1,0}(\mathfrak{A},\bar\mu)$.

371G Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras.

(a) $\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ is a convex set in the unit ball of $\mathsf{B}(M^{1,0}(\mathfrak{A},\bar\mu);M^{1,0}(\mathfrak{B},\bar\nu))$. If $T_0:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{B},\bar\nu)$ is a linear operator of norm at most 1, and $T_0u\in L^\infty(\mathfrak{B})$ and $\|T_0u\|_\infty\le\|u\|_\infty$ for every $u\in L^1(\mathfrak{A},\bar\mu)\cap L^\infty(\mathfrak{A})$, then T_0 has a unique extension to a member of $\mathcal{T}^{(0)}$.

(b) If $T\in\mathcal{T}^{(0)}$ then T is order-bounded and $|T|$, taken in

$$\mathsf{L}^\sim(M^{1,0}(\mathfrak{A},\bar\mu);M^{1,0}(\mathfrak{B},\bar\nu))=\mathsf{L}^\times(M^{1,0}(\mathfrak{A},\bar\mu);M^{1,0}(\mathfrak{B},\bar\nu)),$$

also belongs to $\mathcal{T}^{(0)}$.

(c) If $T\in\mathcal{T}^{(0)}$ then $\|Tu\|_{1,\infty}\le\|u\|_{1,\infty}$ for every $u\in M^{1,0}(\mathfrak{A},\bar\mu)$.

(d) If $T\in\mathcal{T}^{(0)}$, $p\in[1,\infty[$ and $w\in L^p(\mathfrak{A},\bar\mu)$ then $Tw\in L^p(\mathfrak{B},\bar\nu)$ and $\|Tw\|_p\le\|w\|_p$.

(e) If $(\mathfrak{C},\bar\lambda)$ is another measure algebra then $ST\in\mathcal{T}^{(0)}_{\bar\mu,\bar\lambda}$ whenever $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ and $S\in\mathcal{T}^{(0)}_{\bar\nu,\bar\lambda}$.

proof I write $M^{1,0}_{\bar\mu}$, $L^p_{\bar\nu}$ for $M^{1,0}_{\bar\mu}$, $L^p(\mathfrak{B},\bar\nu)$, etc.

(a)(i) If $T\in\mathcal{T}^{(0)}$ and $u\in M^{1,0}_{\bar\mu}$ then there are $v\in L^1_{\bar\mu}$, $w\in L^\infty_{\bar\mu}$ such that $u=v+w$ and $\|v\|_1+\|w\|_\infty=\|u\|_{1,\infty}$ (369Ob); so that

$$\|Tu\|_{1,\infty}\le\|Tv\|_1+\|Tw\|_\infty\le\|v\|_1+\|w\|_\infty\le\|u\|_{1,\infty}.$$

As u is arbitrary, T is in the unit ball of $\mathsf{B}(M^{1,0}_{\bar\mu};M^{1,0}_{\bar\nu})$.

(ii) Because the unit balls of $\mathsf{B}(L^1_{\bar\mu};L^1_{\bar\nu})$ and $\mathsf{B}(L^\infty_{\bar\mu};L^\infty_{\bar\nu})$ are convex, so is $\mathcal{T}^{(0)}$.

(iii) Now suppose that $T_0:L^1_{\bar\mu}\to L^1_{\bar\nu}$ is a linear operator of norm at most 1 such that $\|T_0u\|_\infty\le\|u\|_\infty$ for every $u\in L^1_{\bar\mu}\cap L^\infty_{\bar\mu}$. By the argument of (i), T_0 is a bounded operator for the $\|\;\|_{1,\infty}$ norms; since $L^1_{\bar\mu}$ is dense in $M^{1,0}_{\bar\mu}$ (369Pc), T_0 has a unique extension to a bounded linear operator $T:M^{1,0}_{\bar\mu}\to M^{1,0}_{\bar\nu}$. Of course $\|Tu\|_1=\|T_0u\|_1\le\|u\|_1$ for every $u\in L^1_{\bar\mu}$.

Now suppose that $u\in L^\infty_{\bar\mu}\cap M^{1,0}_{\bar\mu}$; set $\gamma=\|u\|_\infty$. Let $\epsilon>0$, and set

$$v=(u^+-\epsilon\chi1)^+-(u^--\epsilon\chi1)^+;$$

then $|v|\le|u|$ and $\|u-v\|_\infty\le\epsilon$ and $v\in L^1_{\bar\mu}\cap L^\infty_{\bar\mu}$. Accordingly

$$\|Tu-Tv\|_{1,\infty}\le\|u-v\|_{1,\infty}\le\epsilon,\quad \|Tv\|_\infty=\|T_0v\|_\infty\le\|v\|_\infty\le\gamma.$$

So if we set $w=(|Tu-Tv|-\epsilon\chi1)^+\in L^1_{\bar\nu}$, $\|w\|_1\le\epsilon$; while

$$|Tu|\le|Tv|+w+\epsilon\chi1\le(\gamma+\epsilon)\chi1+w,$$

so

$$\|(|Tu|-(\gamma+\epsilon)\chi1)^+\|_1\le\|w\|_1\le\epsilon.$$

As ϵ is arbitrary, $|Tu|\le\gamma\chi1$, that is, $\|Tu\|_\infty\le\|u\|_\infty$. As u is arbitrary, $T\in\mathcal{T}^{(0)}$.

(b) Because $M^{1,0}_{\bar\mu}$ has an order-continuous norm (369Pb), $\mathsf{L}^\sim(M^{1,0}_{\bar\mu};M^{1,0}_{\bar\nu})=\mathsf{L}^\times(M^{1,0}_{\bar\mu};M^{1,0}_{\bar\nu})$ (355Kb). Take any $T\in\mathcal{T}^{(0)}$ and consider $T_0=T\restriction L^1_{\bar\mu}:L^1_{\bar\mu}\to L^1_{\bar\nu}$. This is an operator of norm at most 1. By 371D, T_0 is order-bounded, and $\||T_0|\|\le1$, where $|T_0|$ is taken in $\mathsf{L}^\sim(L^1_{\bar\mu};L^1_{\bar\mu})=\mathsf{B}(L^1_{\bar\mu};L^1_{\bar\nu})$. Now if $u\in L^1_{\bar\mu}\cap L^\infty_{\bar\nu}$,

$$||T_0|u|\le|T_0||u|=\sup_{|u'|\le|u|}|T_0u'|\le\|u\|_\infty\chi1,$$

so $\||T_0|u\|_\infty \le \|u\|_\infty$. By (a), there is a unique $S \in \mathcal{T}^{(0)}$ extending $|T_0|$. Now $Su^+ \ge 0$ for every $u \in L^1_{\bar\mu}$, so $Su^+ \ge 0$ for every $u \in M^{1,0}_{\bar\mu}$ (since the function $u \mapsto (Su^+)^+ - Su^+ : M^{1,0}_{\bar\mu} \to M^{1,0}_{\bar\nu}$ is continuous and zero on the dense set $L^1_{\bar\mu}$), that is, S is a positive operator; also $S|u| \ge |Tu|$ for every $u \in L^1_{\bar\mu}$, so $Sv \ge S|u| \ge |Tu|$ whenever u, $v \in M^{1,0}_{\bar\mu}$ and $|u| \le v$. This means that $T : M^{1,0}_{\bar\mu} \to M^{1,0}_{\bar\nu}$ is order-bounded. Because $M^{1,0}_{\bar\nu}$ is Dedekind complete (366Ga), $|T|$ is defined in $\mathsf{L}^\sim(M^{1,0}_{\bar\mu}; M^{1,0}_{\bar\mu})$.

If $v \ge 0$ in $L^1_{\bar\mu}$, then

$$|T|v = \sup_{|u|\le v} Tu = \sup_{|u|\le v} T_0 u = |T_0|v = Sv.$$

Thus $|T|$ agrees with S on $L^1_{\bar\mu}$. Because $M^{1,0}_{\bar\mu}$ is a Banach lattice (or otherwise), $|T|$ is a bounded operator, therefore continuous (2A4Fc), so $|T| = S \in \mathcal{T}^{(0)}$, which is what we needed to know.

(c) We can express u as $v + w$ where $\|v\|_1 + \|w\|_\infty = \|u\|_{1,\infty}$; now $w = u - v \in M^{1,0}_{\bar\mu}$, so we can speak of Tw, and

$$\|Tu\|_{1,\infty} = \|Tv + Tw\|_{1,\infty} \le \|Tv\|_1 + \|Tw\|_\infty \le \|v\|_1 + \|w\|_\infty = \|u\|_{1,\infty},$$

as required.

(d) (This is a modification of 244M.)

(i) Suppose that T, p, w are as described, and that in addition T is positive. The function $t \mapsto |t|^p$ is convex (233Xc), so we can find families $\langle\beta_q\rangle_{q\in\mathbb{Q}}$, $\langle\gamma_q\rangle_{q\in\mathbb{Q}}$ of real numbers such that $|t|^p = \sup_{q\in\mathbb{Q}} \beta_q + \gamma_q(t - q)$ for every $t \in \mathbb{R}$ (233Hb). Then $|u|^p = \sup_{q\in\mathbb{Q}} \beta_q\chi 1 + \gamma_q(u - q\chi 1)$ for every $u \in L^0$. (The easiest way to check this is perhaps to think of L^0 as a quotient of a space of functions, as in 364C; it is also a consequence of 364Xg(iii).) We know that $|w|^p \in L^1_{\bar\mu}$, so we may speak of $T(|w|^p)$; while $w \in M^{1,0}_{\bar\mu}$ (366Ga), so we may speak of Tw.

For any $q \in \mathbb{Q}$, $0^p \ge \beta_q - q\gamma_q$, that is, $q\gamma_q - \beta_q \ge 0$, while $\gamma_q w - |w|^p \le (q\gamma_q - \beta_q)\chi 1$ and $\|(\gamma_q w - |w|^p)^+\|_\infty \le q\gamma_q - \beta_q$. Now this means that

$$\begin{aligned} T(\gamma_q w - |w|^p) &\le T(\gamma_q w - |w|^p)^+ \le \|T(\gamma_q w - |w|^p)^+\|_\infty \chi 1 \\ &\le \|(\gamma_q w - |w|^p)^+\|_\infty \chi 1 \le (q\gamma_q - \beta_q)\chi 1. \end{aligned}$$

Turning this round again,

$$\beta_q \chi 1 + \gamma_q(Tw - q\chi 1) \le T(|w|^p).$$

Taking the supremum over q, $|Tw|^p \le T(|w|^p)$, so that $\int |Tw|^p \le \int |w|^p$ (because $\|Tv\|_1 \le \|v\|_1$ for every $v \in L^1$). Thus $Tw \in L^p$ and $\|Tw\|_p \le \|w\|_p$.

(ii) For a general $T \in \mathcal{T}^{(0)}$, we have $|T| \in \mathcal{T}^{(0)}$, by (b), and $|Tw| \le |T||w|$, so that $\|Tw\|_p \le \||T||w|\|_p \le \|w\|_p$, as required.

(e) This is elementary, because

$$\|STu\|_1 \le \|Tu\|_1 \le \|u\|_1, \quad \|STv\|_\infty \le \|Tu\|_\infty \le \|u\|_\infty$$

whenever $u \in L^1_{\bar\mu}$ and $v \in L^\infty_{\bar\mu} \cap M^{1,0}_{\bar\mu}$.

371H Remark In the context of 366H, $T_\pi{\restriction}M^{1,0}_{\bar\mu} \in \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$, while $P_\pi \in \mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$. Thus 366H(a-iv) and 366H(b-iii) are special cases of 371Gd.

371X Basic exercises >(a) Let U be an L-space, V a Banach lattice with an order-continuous norm and $T : U \to V$ a bounded linear operator. Let B be the unit ball of U. Show that $|T|[B] \subseteq \overline{T[B]}$.

(b) Let U and V be Banach spaces. (i) Show that the space $\mathsf{K}(U;V)$ of compact linear operators from U to V (definition: 3A5La) is a closed linear subspace of $\mathsf{B}(U;V)$. (ii) Show that if U is an L-space and V is a Banach lattice with an order-continuous norm, then $\mathsf{K}(U;V)$ is a norm-closed Riesz subspace of $\mathsf{L}^\sim(U;V)$. (See Krengel 63.)

(c) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra and set $U = L^1(\mathfrak{A}, \bar\mu)$. Show that $\mathsf{L}^\sim(U;U) = \mathsf{B}(U;U)$ is a Banach lattice with a Fatou norm and the Levi property. Show that its norm is order-continuous iff $\mathfrak{A}$ is finite. (*Hint*: consider operators $u \mapsto u \times \chi a$, where $a \in \mathfrak{A}$.)

>(d) Let U be a Banach lattice, and V a Dedekind complete M-space. Show that $\mathsf{L}^\sim(U;V) = \mathsf{B}(U;V)$ is a Banach lattice with a Fatou norm and the Levi property.

(e) Let U and V be Riesz spaces, of which V is Dedekind complete, and let $T \in \mathsf{L}^\sim(U;V)$. Define $T' \in \mathsf{L}^\sim(V^\sim;U^\sim)$ by writing $T'(h) = hT$ for $h \in V^\sim$. (i) Show that $|T|' \geq |T'|$ in $\mathsf{L}^\sim(V^\sim;U^\sim)$. (ii) Show that $|T|'h = |T'|h$ for every $h \in V^\times$. (*Hint*: show that if $u \in U^+$ and $h \in (V^\times)^+$ then $(|T'|h)(u)$ and $h(|T|u)$ are both equal to $\sup\{\sum_{i=0}^n g_i(Tu_i) : |g_i| \leq h,\, u_i \geq 0,\, \sum_{i=0}^n u_i = u\}$.)

>(f) Using 371D, but nothing about uniformly integrable sets beyond the definition (354P), show that if U and V are L-spaces, $A \subseteq U$ is uniformly integrable in U, and $T : U \to V$ is a bounded linear operator, then $T[A]$ is uniformly integrable in V.

371Y Further exercises (a) Let U and V be Banach spaces. (i) Show that the space $\mathsf{K}_w(U;V)$ of weakly compact linear operators from U to V (definition: 3A5Lb) is a closed linear subspace of $\mathsf{B}(U;V)$. (ii) Show that if U is an L-space and V is a Banach lattice with an order-continuous norm, then $\mathsf{K}_w(U;V)$ is a norm-closed Riesz subspace of $\mathsf{L}^\sim(U;V)$.

(b) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, U a Banach space, and $T : L^1(\mathfrak{A},\bar\mu) \to U$ a bounded linear operator. Show that T is a compact linear operator iff $\{\frac{1}{\bar\mu a}T(\chi a) : a \in \mathfrak{A},\, 0 < \bar\mu a < \infty\}$ is relatively compact in U.

(c) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and set $L^1 = L^1(\mathfrak{A},\bar\mu)$. Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a stochastically independent sequence of elements of $\mathfrak{A}$ of measure $\frac{1}{2}$, and define $T : L^1 \to \mathbb{R}^\mathbb{N}$ by setting $Tu(n) = \int u - 2\int_{a_n} u$ for each n. Show that $T \in \mathsf{B}(L^1;\boldsymbol{c}_0) \setminus \mathsf{L}^\sim(L^1;\boldsymbol{c}_0)$, where $\boldsymbol{c}_0$ is the Banach lattice of sequences converging to 0. (See 272Ye[1].)

(d) Regarding T of 371Yc as a map from L^1 to ℓ^∞, show that $|T'| \neq |T|'$ in $\mathsf{L}^\sim((\ell^\infty)^*, L^\infty(\mathfrak{A}))$.

(e)(i) In ℓ^2 define e_i by setting $e_i(i) = 1$, $e_i(j) = 0$ if $j \neq i$. Show that if $T \in \mathsf{L}^\sim(\ell^2;\ell^2)$ then $(|T|e_i|e_j) = |(Te_i|e_j)|$ for all i, $j \in \mathbb{N}$. (ii) Show that for each $n \in \mathbb{N}$ there is an orthogonal $(2^n \times 2^n)$-matrix $\mathbf{A}_n$ such that every coefficient of $\mathbf{A}_n$ has modulus $2^{-n/2}$. (*Hint*: $\mathbf{A}_{n+1} = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{A}_n & \mathbf{A}_n\\ -\mathbf{A}_n & \mathbf{A}_n\end{pmatrix}$.) (iii) Show that there is a linear isometry $S : \ell^2 \to \ell^2$ such that $|(Se_i|e_j)| = 2^{-n/2}$ if $2^n \leq i,\, j < 2^{n+1}$. (iv) Show that $S \notin \mathsf{L}^\sim(\ell^2;\ell^2)$.

371 Notes and comments The 'Chacon-Krengel theorem', properly speaking (Chacon & Krengel 64), is 371D in the case in which $U = L^1(\mu)$, $V = L^1(\nu)$; of course no new ideas are required in the generalizations here, which I have copied from Fremlin 74a.

Anyone with a training in functional analysis will automatically seek to investigate properties of operators $T : U \to V$ in terms of properties of their adjoints $T' : V^* \to U^*$, as in 371Xe and 371Yd. When U is an L-space, then U^* is a Dedekind complete M-space, and it is easy to see that this forces T' to be order-bounded, for any Banach lattice V (371Xd). But since no important L-space is reflexive, this approach cannot reach 371B-371D without a new idea of some kind. It can however be adapted to the special case in 371Gb (Dunford & Schwartz 57, VIII.6.4).

In fact the results of 371B-371C are characteristic of L-spaces (Fremlin 74b). To see that they fail in the simplest cases in which U is not an L-space and V is not an M-space, see 371Yc-371Ye.

372 The ergodic theorem

I come now to one of the most remarkable topics in measure theory. I cannot do it justice in the space I have allowed for it here, but I can give the basic theorem (372D, 372F) and a variety of the corollaries through which it is regularly used (372E, 372G-372J), together with brief notes on one of its most famous and characteristic applications (to continued fractions, 372L-372N) and on 'ergodic' and 'mixing' transformations (372O-372S). In the first half of the section (down to 372G) I express the arguments in the abstract language of measure algebras and their associated function spaces, as developed in Chapter 36; the second half, from 372H onwards, contains translations of the results into the language of measure spaces and measurable functions, the more traditional, and more readily applicable, forms.

[1] Formerly 272Yd.

372A Lemma Let U be a reflexive Banach space and $T : U \to U$ a bounded linear operator of norm at most 1. Then

$$V = \{u + v - Tu : u,\, v \in U,\, Tv = v\}$$

is dense in U.

proof Of course V is a linear subspace of U. **?** Suppose, if possible, that it is not dense. Then there is a non-zero $h \in U^*$ such that $h(v) = 0$ for every $v \in V$ (3A5Ad). Take $u \in U$ such that $h(u) \neq 0$. Set

$$u_n = \frac{1}{n+1}\sum_{i=0}^{n} T^i u$$

for each $n \in \mathbb{N}$, taking T^0 to be the identity operator; because

$$\|T^i u\| \le \|T^i\|\|u\| \le \|T\|^i\|u\| \le \|u\|$$

for each i, $\|u_n\| \le \|u\|$ for every n. Note also that $T^{i+1}u - T^i u \in V$ for every i, so that $h(T^{i+1}u - T^i u) = 0$; accordingly $h(T^i u) = h(u)$ for every i, and $h(u_n) = h(u)$ for every n.

Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Because U is reflexive, $v = \lim_{n\to\mathcal{F}} u_n$ is defined in U for the weak topology on U (3A5Gc). Now $Tv = v$. **P** For each $n \in \mathbb{N}$,

$$Tu_n - u_n = \frac{1}{n+1}\sum_{i=0}^{n}(T^{i+1}u - T^i u) = \frac{1}{n+1}(T^{n+1}u - u)$$

has norm at most $\frac{2}{n+1}\|u\|$. So $\langle Tu_n - u_n\rangle_{n\in\mathbb{N}} \to 0$ for the norm topology U and therefore for the weak topology, and surely $\lim_{n\to\mathcal{F}} Tu_n - u_n = 0$. On the other hand (because T is continuous for the weak topology, 2A5If)

$$Tv = \lim_{n\to\mathcal{F}} Tu_n = \lim_{n\to\mathcal{F}}(Tu_n - u_n) + \lim_{n\to\mathcal{F}} u_n = 0 + v = v,$$

where all the limits are taken for the weak topology. **Q**

But this means that $v \in V$, while

$$h(v) = \lim_{n\to\mathcal{F}} h(u_n) = h(u) \neq 0,$$

contradicting the assumption that $h \in V^\circ$. **X**

372B Lemma Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $T : L^1 \to L^1$ a positive linear operator of norm at most 1, where $L^1 = L^1(\mathfrak{A}, \bar\mu)$. Take any $u \in L^1$ and $m \in \mathbb{N}$, and set

$$a = [\![u > 0]\!] \cup [\![u + Tu > 0]\!] \cup [\![u + Tu + T^2u > 0]\!] \cup \ldots \cup [\![u + Tu + \ldots + T^m u > 0]\!].$$

Then $\int_a u \ge 0$.

proof Set $u_0 = u$, $u_1 = u + Tu, \ldots, u_m = u + Tu + \ldots + T^m u$, $v = \sup_{i\le m} u_i$, so that $a = [\![v > 0]\!]$. Consider $u + T(v^+)$. We have $T(v^+) \ge Tv \ge Tu_i$ for every $i \le m$ (because T is positive), so that $u + T(v^+) \ge u + Tu_i = u_{i+1}$ for $i < m$, and $u + T(v^+) \ge \sup_{1\le i\le m} u_i$. Also $u + T(v^+) \ge u$ because $T(v^+) \ge 0$, so $u + T(v^+) \ge v$. Accordingly

$$\int_a u \ge \int_a v - \int_a T(v^+) = \int v^+ - \int_a T(v^+) \ge \|v^+\|_1 - \|Tv^+\|_1 \ge 0$$

because $\|T\| \le 1$.

372C Maximal Ergodic Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $T : L^1 \to L^1$ a linear operator, where $L^1 = L^1(\mathfrak{A}, \bar\mu)$, such that $\|Tu\|_1 \le \|u\|_1$ for every $u \in L^1$ and $\|Tu\|_\infty \le \|u\|_\infty$ for every $u \in L^1 \cap L^\infty(\mathfrak{A})$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^{n} T^i$ for each $n \in \mathbb{N}$. Then for any $u \in L^1$, $u^* = \sup_{n\in\mathbb{N}} A_n u$ is defined in $L^0(\mathfrak{A})$, and $\alpha\bar\mu[\![u^* > \alpha]\!] \le \|u\|_1$ for every $\alpha > 0$.

proof (a) To begin with, suppose that T is positive and that $u \ge 0$ in L^1. Note that if $v \in L^1 \cap L^\infty$, then $\|T^i v\|_\infty \le \|v\|_\infty$ for every $i \in \mathbb{N}$, so $\|A_n v\|_\infty \le \|v\|_\infty$ for every n; in particular, $A_n(\chi a) \le \chi 1$ for every n and every a of finite measure.

For $m \in \mathbb{N}$ and $\alpha > 0$, set

$$a_{m\alpha} = [\![\sup_{i\le m} A_i u > \alpha]\!].$$

Then $\alpha\bar\mu a_{m\alpha} \le \|u\|_1$. **P** Set $a = a_{m\alpha}$, $w = u - \alpha\chi a$. Of course $\sup_{i\le m} A_i u$ belongs to L^1, so $\bar\mu a$ is finite and $w \in L^1$. For any $i \le m$,

$$A_i w = A_i u - \alpha A_i(\chi a) \ge A_i u - \alpha\chi 1,$$

so $[\![A_i w > 0]\!] \supseteq [\![A_i u > \alpha]\!]$. Accordingly $a \subseteq b$, where

$$b = \sup_{i \le m} [\![A_i w > 0]\!] = \sup_{i \le m} [\![w + Tw + \ldots + T^i w > 0]\!].$$

By 372B, $\int_b w \ge 0$. But this means that

$$\alpha\bar{\mu}a = \alpha\int_b \chi a = \int_b u - \int_b w \le \int_b u \le \|u\|_1,$$

as claimed. **Q**

It follows that if we set $c_\alpha = \sup_{n\in\mathbb{N}} a_{n\alpha}$, $\bar{\mu}c_\alpha \le \alpha^{-1}\|u\|_1$ for every $\alpha > 0$ and $\inf_{\alpha>0} c_\alpha = 0$. But this is exactly the criterion in 364L(a-ii) for $u^* = \sup_{n\in\mathbb{N}} A_n u$ to be defined in L^0. And $[\![u^* > \alpha]\!] = c_\alpha$, so $\alpha\bar{\mu}[\![u^* > \alpha]\!] \le \|u\|_1$ for every $\alpha > 0$, as required.

(b) Now consider the case of general T, u. In this case T is order-bounded and $\||T|\| \le 1$, where $|T|$ is the modulus of T in $\mathsf{L}^\sim(L^1; L^1) = \mathsf{B}(L^1; L^1)$ (371D). If $w \in L^1 \cap L^\infty$, then

$$||T|w| \le |T||w| = \sup_{|w'| \le |w|} |Tw'| \le \|w\|_\infty \chi 1,$$

so $\||T|w\|_\infty \le \|w\|_\infty$. Thus $|T|$ also satisfies the conditions of the theorem. Setting $B_n = \frac{1}{n+1}\sum_{i=0}^n |T|^i$, $B_n \ge A_n$ in $\mathsf{L}^\sim(L^1; L^1)$ and $B_n|u| \ge A_n u$ for every n. But by (a), $v = \sup_{n\in\mathbb{N}} B_n|u|$ is defined in L^0 and $\alpha\bar{\mu}[\![v > \alpha]\!] \le \||u|\|_1 = \|u\|_1$ for every $\alpha > 0$. Consequently $u^* = \sup_{n\in\mathbb{N}} A_n u$ is defined in L^0 and $u^* \le v$, so that $\alpha\bar{\mu}[\![u^* > \alpha]\!] \le \|u\|_1$ for every $\alpha > 0$.

372D We are now ready for a very general form of the Ergodic Theorem. I express it in terms of the space $M^{1,0}$ from 366F and the class $\mathcal{T}^{(0)}$ of operators from 371F. If these formulae are unfamiliar, you may like to glance at the statement of 372F before looking them up.

The Ergodic Theorem: first form Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and set $M^{1,0} = M^{1,0}(\mathfrak{A}, \bar{\mu})$, $\mathcal{T}^{(0)} = \mathcal{T}^{(0)}_{\bar{\mu},\bar{\mu}} \subseteq \mathsf{B}(M^{1,0}; M^{1,0})$ as in 371F-371G. Take any $T \in \mathcal{T}^{(0)}$, and set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i : M^{1,0} \to M^{1,0}$ for every n. Then for any $u \in M^{1,0}$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent (definition: 367A) and $\| \|_{1,\infty}$-convergent to a member Pu of $M^{1,0}$. The operator $P : M^{1,0} \to M^{1,0}$ is a projection onto the linear subspace $\{u : u \in M^{1,0}, Tu = u\}$, and $P \in \mathcal{T}^{(0)}$.

proof (a) It will be convenient to start with some elementary remarks. First, every A_n belongs to $\mathcal{T}^{(0)}$, by 371Ge and 371Ga. Next, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order-bounded in $L^0 = L^0(\mathfrak{A})$ for any $u \in M^{1,0}$; this is because if $u = v + w$, where $v \in L^1 = L^1(\mathfrak{A}, \bar{\mu})$ and $w \in L^\infty = L^\infty(\mathfrak{A})$, then $\langle A_n v\rangle_{n\in\mathbb{N}}$ and $\langle A_n(-v)\rangle_{n\in\mathbb{N}}$ are bounded above, by 372C, while $\langle A_n w\rangle_{n\in\mathbb{N}}$ is norm- and order-bounded in L^∞. Accordingly I can uninhibitedly speak of $P^*(u) = \inf_{n\in\mathbb{N}} \sup_{i\ge n} A_i u$ and $P_*(u) = \sup_{n\in\mathbb{N}} \inf_{i\ge n} A_i u$ for any $u \in M^{1,0}$, these both being defined in L^0.

(b) Write V_1 for the set of those $u \in M^{1,0}$ such that $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent in L^0; that is, $P^*(u) = P_*(u)$ (367Be). It is easy to see that V_1 is a linear subspace of $M^{1,0}$ (use 367Ca and 367Cd). Also it is closed for $\| \|_{1,\infty}$.

P We know that $|T|$, taken in $\mathsf{L}^\sim(M^{1,0}; M^{1,0})$, belongs to $\mathcal{T}^{(0)}$ (371Gb); set $B_n = \frac{1}{n+1}\sum_{i=0}^n |T|^i$ for each i.

Suppose that $u_0 \in \overline{V}_1$. Then for any $\epsilon > 0$ there is a $u \in V_1$ such that $\|u_0 - u\|_{1,\infty} \le \epsilon^2$. Write $Pu = P^*(u) = P_*(u)$ for the order*-limit of $\langle A_n u\rangle_{n\in\mathbb{N}}$. Express $u_0 - u$ as $v + w$ where $v \in L^1$, $w \in L^\infty$ and $\|v\|_1 + \|w\|_\infty \le 2\epsilon^2$.

Set $v^* = \sup_{n\in\mathbb{N}} B_n|v|$. Then $\bar{\mu}[\![v^* > \epsilon]\!] \le 2\epsilon$, by 372C. Next, if $w^* = \sup_{n\in\mathbb{N}} B_n|w|$, we surely have $w^* \le 2\epsilon^2\chi 1$. Now

$$|A_n u_0 - A_n u| = |A_n v + A_n w| \le B_n|v| + B_n|w| \le v^* + w^*$$

for every $n \in \mathbb{N}$, that is,

$$A_n u - v^* - w^* \le A_n u_0 \le A_n u + v^* + w^*$$

for every n. Because $\langle A_n u\rangle_{n\in\mathbb{N}}$ order*-converges to Pu,

$$Pu - v^* - w^* \le P_*(u_0) \le P^*(u_0) \le Pu + v^* + w^*,$$

and $P^*(u_0) - P_*(u_0) \le 2(v^* + w^*)$. On the other hand,

$$\bar{\mu}[\![2(v^* + w^*) > 2\epsilon + 4\epsilon^2]\!] \le \bar{\mu}[\![v^* > \epsilon]\!] + \bar{\mu}[\![w^* > 2\epsilon^2]\!] = \bar{\mu}[\![v^* > \epsilon]\!] \le 2\epsilon$$

(using 364Ea for the first inequality). So

$$\bar{\mu}[\![P^*(u_0) - P_*(u_0) > 2\epsilon(1 + 2\epsilon)]\!] \le 2\epsilon.$$

Since ϵ is arbitrary, $\langle A_n u_0\rangle_{n\in\mathbb{N}}$ order*-converges to $P^*(u_0) = P_*(u_0)$, and $u_0 \in V_1$. As u_0 is arbitrary, V_1 is closed. **Q**

(c) Similarly, the set V_2 of those $u \in M^{1,0}$ for which $\langle A_n u\rangle_{n\in\mathbb{N}}$ is norm-convergent is a linear subspace of $M^{1,0}$, and it also is closed. **P** This is a standard argument. If $u_0 \in \overline{V}_2$ and $\epsilon > 0$, there is a $u \in V_2$ such that $\|u_0 - u\|_{1,\infty} \le \epsilon$. There is an $n \in \mathbb{N}$ such that $\|A_i u - A_j u\|_{1,\infty} \le \epsilon$ for all $i, j \ge n$, and now $\|A_i u_0 - A_j u_0\|_{1,\infty} \le 3\epsilon$ for all $i, j \ge n$, because every A_i has norm at most 1 in $\mathsf{B}(M^{1,0}; M^{1,0})$ (371Gc). As ϵ is arbitrary, $\langle A_i u_0\rangle_{n\in\mathbb{N}}$ is Cauchy; because $M^{1,0}$ is complete, it is convergent, and $u_0 \in V_2$. As u_0 is arbitrary, V_2 is closed. **Q**

(d) Now let V be $\{u + v - Tu : u \in M^{1,0} \cap L^\infty, v \in M^{1,0}, Tv = v\}$. Then $V \subseteq V_1 \cap V_2$. **P** If $u \in M^{1,0} \cap L^\infty$, then for any $n \in \mathbb{N}$

$$A_n(u - Tu) = \tfrac{1}{n+1}(u - T^{n+1}u) \to 0$$

for $\| \|_\infty$, and therefore is both order*-convergent and convergent for $\| \|_{1,\infty}$; so $u - Tu \in V_1 \cap V_2$. On the other hand, if $Tv = v$, then of course $A_n v = v$ for every n, so again $v \in V_1 \cap V_2$. **Q**

(e) Consequently $L^2 = L^2(\mathfrak{A}, \bar\mu) \subseteq V_1 \cap V_2$. **P** $L^2 \cap V_1 \cap V_2$ is a linear subspace; but also it is closed for the norm topology of L^2, because the identity map from L^2 to $M^{1,0}$ is continuous (369Oe). We know also that $T\restriction L^2$ is an operator of norm at most 1 from L^2 to itself (371Gd). Consequently $W = \{u + v - Tu : u, v \in L^2, Tv = v\}$ is dense in L^2 (372A). On the other hand, given $u \in L^2$ and $\epsilon > 0$, there is a $u' \in L^2 \cap L^\infty$ such that $\|u - u'\|_2 \le \epsilon$ (take $u' = (u \wedge \gamma\chi 1) \vee (-\gamma\chi 1)$ for any γ large enough), and now $\|(u - Tu) - (u' - Tu')\|_2 \le 2\epsilon$. Thus $W' = \{u' + v - Tu' : u' \in L^2 \cap L^\infty, v \in L^2, Tv = v\}$ is dense in L^2. But $W' \subseteq V_1 \cap V_2$, by (d) above. Thus $L^2 \cap V_1 \cap V_2$ is dense in L^2, and is therefore the whole of L^2. **Q**

(f) $L^2 \supseteq S(\mathfrak{A}^f)$ is dense in $M^{1,0}$, by 369Pc, so $V_1 = V_2 = M^{1,0}$. This shows that $\langle A_n u\rangle_{n\in\mathbb{N}}$ is norm-convergent and order*-convergent for every $u \in M^{1,0}$. By 367Da, the limits are the same. Write Pu for the common value of the limits.

(g) Of course we now have

$$\|Pu\|_\infty \le \sup_{n\in\mathbb{N}} \|A_n u\|_\infty \le \|u\|_\infty$$

for every $u \in L^\infty \cap M^{1,0}$, while

$$\|Pu\|_1 \le \liminf_{n\to\infty} \|A_n u\|_1 \le \|u\|_1$$

for every $u \in L^1$, by Fatou's Lemma. So $P \in \mathcal{T}^{(0)}$. If $u \in M^{1,0}$ and $Tu = u$, then surely $Pu = u$, because $A_n u = u$ for every u. On the other hand, for any $u \in M^{1,0}$, $TPu = Pu$. **P** Because $\langle A_n u\rangle_{n\in\mathbb{N}}$ is norm-convergent to Pu,

$$\begin{aligned}\|TPu - Pu\|_{1,\infty} &= \lim_{n\to\infty} \|TA_n u - A_n u\|_{1,\infty} \\ &= \lim_{n\to\infty} \frac{1}{n+1}\|T^{n+1}u - u\|_{1,\infty} = 0. \ \mathbf{Q}\end{aligned}$$

Thus, writing $U = \{u : Tu = u\}$, $P[M^{1,0}] = U$ and $Pu = u$ for every $u \in U$.

372E Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $\pi : \mathfrak{A}^f \to \mathfrak{A}^f$ a measure-preserving ring homomorphism, where $\mathfrak{A}^f = \{a : \bar\mu a < \infty\}$. Let $T : M^{1,0} \to M^{1,0}$ be the corresponding Riesz homomorphism, where $M^{1,0} = M^{1,0}(\mathfrak{A}, \bar\mu)$ (366H, in particular part (a-v)). Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i$ for $n \in \mathbb{N}$. Then for every $u \in M^{1,0}$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_{1,\infty}$-convergent to some v such that $Tv = v$.

proof By 366H(a-iv), $T \in \mathcal{T}^{(0)}$, as defined in 371F. So the result follows at once from 372D.

372F The Ergodic Theorem: second form Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and let $T : L^1 \to L^1$, where $L^1 = L^1(\mathfrak{A}, \bar\mu)$, be a linear operator of norm at most 1 such that $Tu \in L^\infty = L^\infty(\mathfrak{A})$ and $\|Tu\|_\infty \le \|u\|_\infty$ whenever $u \in L^1 \cap L^\infty$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i : L^1 \to L^1$ for every n. Then for any $u \in L^1$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent to an element Pu of L^1. The operator $P : L^1 \to L^1$ is a projection of norm at most 1 onto the linear subspace $\{u : u \in L^1, Tu = u\}$.

proof By 371Ga, there is an extension of T to a member $\tilde T$ of $\mathcal{T}^{(0)}$. So 372D tells us that $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent to some $Pu \in L^1$ for every $u \in L^1$, and $P : L^1 \to L^1$ is a projection of norm at most 1, because P is the restriction of a projection $\tilde P \in \mathcal{T}^{(0)}$. Also we still have $TPu = Pu$ for every $u \in L^1$, and $Pu = u$ whenever $Tu = u$, so the set of values $P[L^1]$ of P must be exactly $\{u : u \in L^1, Tu = u\}$.

Remark In 372D and 372F I have used the phrase 'order*-convergent' from §367 without always being specific about the partially ordered set in which it is to be interpreted. But, as remarked in 367E, the notion is robust enough for the omission to be immaterial here. Since both $M^{1,0}$ and L^1 are solid linear subspaces of L^0, a sequence in $M^{1,0}$ is order*-convergent to a member of $M^{1,0}$ (when order*-convergence is interpreted in the partially ordered set $M^{1,0}$) iff it is order*-convergent to the same point (when convergence is interpreted in the set L^0); and the same applies to L^1 in place of $M^{1,0}$.

372G Corollary Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $T:L^1\to L^1$ be the corresponding Riesz homomorphism, where $L^1=L^1(\mathfrak{A},\bar\mu)$. Set $A_n=\frac{1}{n+1}\sum_{i=0}^n T^i$ for $n\in\mathbb{N}$. Then for every $u\in L^1$, $\langle A_nu\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\|\;\|_1$-convergent. If we set $Pu=\lim_{n\to\infty}A_nu$ for each u, P is the conditional expectation operator corresponding to the fixed-point subalgebra $\mathfrak{C}=\{a:\pi a=a\}$ of $\mathfrak{A}$.

proof (a) The first part is just a special case of 372E; the point is that because $(\mathfrak{A},\bar\mu)$ is totally finite, $L^\infty(\mathfrak{A})\subseteq L^1$, so $M^{1,0}(\mathfrak{A},\bar\mu)=L^1$. Also (because $\bar\mu 1=1$) $\|u\|_\infty\le\|u\|_1$ for every $u\in L^\infty$, so the norm $\|\;\|_{1,\infty}$ is actually equal to $\|\;\|_1$.

(b) For the last sentence, recall that $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ (cf. 333R). By 372D or 372F, P is a projection operator onto the subspace $\{u:Tu=u\}$. Now $[\![Tu>\alpha]\!]=\pi[\![u>\alpha]\!]$ (365Oc), so $Tu=u$ iff $[\![u>\alpha]\!]\in\mathfrak{C}$ for every $\alpha\in\mathbb{R}$, that is, iff u belongs to the canonical image of $L^1(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$ in L^1 (365R). To identify Pu further, observe that if $u\in L^1$ and $a\in\mathfrak{C}$ then

$$\int_a Tu=\int_{\pi a}Tu=\int_a u$$

(365Ob). Consequently $\int_a T^iu=\int_a u$ for every $i\in\mathbb{N}$, $\int_a A_nu=\int_a u$ for every $n\in\mathbb{N}$, and $\int_a Pu=\int_a u$ (because Pu is the limit of $\langle A_nu\rangle_{n\in\mathbb{N}}$ for $\|\;\|_1$). But this is enough to define Pu as the conditional expectation of u on $\mathfrak{C}$ (365R).

372H The Ergodic Theorem is most often expressed in terms of transformations of measure spaces. In the next few corollaries I will formulate such expressions. The translation is straightforward.

Corollary Let (X,Σ,μ) be a measure space and $\phi:X\to X$ an inverse-measure-preserving function. Let f be a real-valued function which is integrable over X. Then

$$g(x)=\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$$

is defined for almost every $x\in X$, and $g\phi(x)=g(x)$ for almost every x.

proof Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of (X,Σ,μ), and $\pi:\mathfrak{A}\to\mathfrak{A}$, $T:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ the homomorphisms corresponding to ϕ, as in 364Qd. Set $u=f^\bullet$ in $L^1(\mathfrak{A},\bar\mu)$. Then for any $i\in\mathbb{N}$, $T^iu=(f\phi^i)^\bullet$ (364Q(c)-(d)), so setting $A_n=\frac{1}{n+1}\sum_{i=0}^n T^i$, $A_nu=g_n^\bullet$, where $g_n(x)=\frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$ whenever this is defined. Now we know from 372F or 372E that $\langle A_nu\rangle_{n\in\mathbb{N}}$ is order*-convergent to some v such that $Tv=v$, so $\langle g_n\rangle_{n\in\mathbb{N}}$ must be convergent almost everywhere (367F), and taking $g=\lim_{n\to\infty}g_n$ where this is defined, $g^\bullet=v$. Accordingly $(g\phi)^\bullet=Tv=v=g^\bullet$ and $g\phi=_{\text{a.e.}}g$, as claimed.

372I The following facts will be useful in the next version of the theorem, and elsewhere.

Lemma Let (X,Σ,μ) be a measure space with measure algebra $(\mathfrak{A},\bar\mu)$. Let $\phi:X\to X$ be an inverse-measure-preserving function and $\pi:\mathfrak{A}\to\mathfrak{A}$ the associated homomorphism, as in 343A and 364Qd. Set $\mathfrak{C}=\{c:c\in\mathfrak{A},\ \pi c=c\}$, $\mathrm{T}=\{E:E\in\Sigma,\ \phi^{-1}[E]\triangle E\text{ is negligible}\}$ and $\mathrm{T}_0=\{E:E\in\Sigma,\ \phi^{-1}[E]=E\}$. Then T and T_0 are σ-subalgebras of Σ; $\mathrm{T}_0\subseteq\mathrm{T}$, $\mathrm{T}=\{E:E\in\Sigma,\ E^\bullet\in\mathfrak{C}\}$, and $\mathfrak{C}=\{E^\bullet:E\in\mathrm{T}_0\}$.

proof It is easy to see that T and T_0 are σ-subalgebras of Σ and that $\mathrm{T}_0\subseteq\mathrm{T}=\{E:E^\bullet\in\mathfrak{C}\}$. So we have only to check that if $c\in\mathfrak{C}$ there is an $E\in\mathrm{T}_0$ such that $E^\bullet=c$. **P** Start with any $F\in\Sigma$ such that $F^\bullet=c$. Now $F\triangle\phi^{-i}[F]$ is negligible for every $i\in\mathbb{N}$, because $(\phi^{-i}[F])^\bullet=\pi^ic=c$. So if we set

$$\begin{aligned}E&=\bigcup_{n\in\mathbb{N}}\bigcap_{i\ge n}\phi^{-i}[F]\\&=\{x:\text{ there is an }n\in\mathbb{N}\text{ such that }\phi^i(x)\in F\text{ for every }i\ge n\},\end{aligned}$$

$E^\bullet=c$. On the other hand, it is easy to check that $E\in\mathrm{T}_0$. **Q**

372J The Ergodic Theorem: third form Let (X,Σ,μ) be a probability space and $\phi : X \to X$ an inverse-measure-preserving function. Let f be a real-valued function which is integrable over X. Then

$$g(x) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x))$$

is defined for almost every $x \in X$; $g\phi =_{\text{a.e.}} g$, and g is a conditional expectation of f on the σ-algebra $\mathrm{T} = \{E : E \in \Sigma,\ \phi^{-1}[E]\triangle E$ is negligible$\}$. If *either* f is Σ-measurable and defined everywhere in X *or* $\phi[E]$ is negligible for every negligible set E, then g is a conditional expectation of f on the σ-algebra $\mathrm{T}_0 = \{E : E \in \Sigma,\ \phi^{-1}[E] = E\}$.

proof (a) We know by 372H that g is defined almost everywhere and that $g\phi =_{\text{a.e.}} g$. In the language of the proof of 372H, $g^\bullet = v$ is the conditional expectation of $u = f^\bullet$ on the closed subalgebra

$$\mathfrak{C} = \{a : a \in \mathfrak{A},\ \pi a = a\} = \{F^\bullet : F \in \mathrm{T}\} = \{F^\bullet : F \in \mathrm{T}_0\},$$

by 372G and 372I. So v must be expressible as $h^\bullet$ where $h : X \to \mathbb{R}$ is T_0-measurable and is a conditional expectation of f on T_0 (and also on T). Since every set of measure zero belongs to T, $g = h$ $\mu\restriction\mathrm{T}$-a.e., and g also is a conditional expectation of f on T.

(b) Suppose now that f is defined everywhere and Σ-measurable. Here I come to a technical obstruction. The definition of 'conditional expectation' in 233D asks for g to be $\mu\restriction\mathrm{T}_0$-integrable, and since μ-negligible sets do not need to be $\mu\restriction\mathrm{T}_0$-negligible we have some more checking to do, to confirm that $\{x : x \in \operatorname{dom} g,\ g(x) = h(x)\}$ is $\mu\restriction\mathrm{T}_0$-conegligible as well as μ-conegligible.

(i) For $n \in \mathbb{N}$, set $\Sigma_n = \{\phi^{-n}[E] : E \in \Sigma\}$; then Σ_n is a σ-subalgebra of Σ, including T_0. Set $\Sigma_\infty = \bigcap_{n\in\mathbb{N}} \Sigma_n$, still a σ-algebra including T_0. Now any negligible set $E \in \Sigma_\infty$ is $\mu\restriction\mathrm{T}_0$-negligible. **P** For each $n \in \mathbb{N}$ choose $F_n \in \Sigma$ such that $E = \phi^{-n}[F_n]$. Because ϕ is inverse-measure-preserving, every F_n is negligible, so that

$$E^* = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N},j\ge m} \phi^{-j}[F_n]$$

is negligible. Of course $E = \bigcap_{m\in\mathbb{N}} \phi^{-m}[F_m]$ is included in E^*. Now

$$\phi^{-1}[E^*] = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N},j\ge m} \phi^{-j-1}[F_n] = \bigcap_{m\ge 1} \bigcup_{n\in\mathbb{N},j\ge m} \phi^{-j}[F_n] = E^*$$

because

$$\bigcup_{n\in\mathbb{N},j\ge 1} \phi^{-j}[F_n] \subseteq \bigcup_{n\in\mathbb{N},j\ge 0} \phi^{-j}[F_n].$$

So $E^* \in \mathrm{T}_0$ and E is included in a negligible member of T_0, which is what we needed to know. **Q**

(ii) We are assuming that f is Σ-measurable and defined everywhere, so that $g_n = \frac{1}{n+1}\sum_{i=0}^{n} f\circ\phi^i$ is Σ-measurable and defined everywhere. If we set $g^* = \limsup_{n\to\infty} g_n$, then $g^* : X \to [-\infty,\infty]$ is Σ_∞-measurable. **P** For any $m \in \mathbb{N}$, $f\circ\phi^i$ is Σ_m-measurable for every $i \ge m$, since $\{x : f(\phi^i(x)) > \alpha\} = \phi^{-m}[\{x : f(\phi^{i-m}(x)) > \alpha\}]$ for every α. Accordingly

$$g^* = \limsup_{n\to\infty} \frac{1}{n+1}\sum_{i=m}^{n} f\circ\phi^i$$

is Σ_m-measurable. As m is arbitrary, g^* is Σ_∞-measurable. **Q**

Since h is surely Σ_∞-measurable, and $h = g^*$ μ-a.e., (i) tells us that $h = g^*$ $\mu\restriction\mathrm{T}_0$-a.e. But similarly $h = \liminf_{n\to\infty} g_n$ $\mu\restriction\mathrm{T}_0$-a.e., so we must have $h = g$ $\mu\restriction\mathrm{T}_0$-a.e.; and g, like h, is a conditional expectation of f on T_0.

(c) Finally, suppose that $\phi[E]$ is negligible for every negligible set E. Then every μ-negligible set is $\mu\restriction\mathrm{T}_0$-negligible. **P** If E is μ-negligible, then $\phi[E]$, $\phi^2[E] = \phi[\phi[E]], \dots$ are all negligible, so $E^* = \bigcup_{n\in\mathbb{N}} \phi^n[E]$ is negligible, and there is a measurable negligible set $F \supseteq E^*$. Now $F_* = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \phi^{-n}[F]$ is a negligible set in T_0 including E, so E is $\mu\restriction\mathrm{T}_0$-negligible. **Q** Consequently $g = h$ $\mu\restriction\mathrm{T}_0$-a.e., and in this case also g is a conditional expectation of f on T_0.

372K Remark Parts (b)-(c) of the proof above are dominated by the technical question of the exact definition of 'conditional expectation of f on T_0', and it is natural to be impatient with such details. The kind of example I am concerned about is the following. Let $C \subseteq [0,1]$ be the Cantor set (134G), and $\phi : [0,1] \to [0,1]$ a Borel measurable function such that $\phi[C] = [0,1]$ and $\phi(x) = x$ for $x \in [0,1]\setminus C$. (For instance, we could take ϕ agreeing with the Cantor function on C (134H).) Because C is negligible, ϕ is inverse-measure-preserving for Lebesgue measure μ, and if f is any Lebesgue integrable function then $g(x) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x))$ is defined and equal to $f(x)$ for every $x \in \operatorname{dom} f \setminus C$. But for $x \in C$ we can, by manipulating ϕ, arrange for $g(x)$ to be almost anything; and if f is undefined on C then g will also be undefined on C. On the other hand, C is not $\mu\restriction\mathrm{T}_0$-negligible, because the only member of T_0 including C is $[0,1]$. So we cannot be sure of being able to form $\int g\,d(\mu\restriction\mathrm{T}_0)$.

If instead of Lebesgue measure itself we took its restriction $\mu_{\mathcal{B}}$ to the algebra of Borel subsets of $[0,1]$, then ϕ would still be inverse-measure-preserving for $\mu_{\mathcal{B}}$, but we should now have to worry about the possibility that $f\restriction C$ was non-measurable, so that $g\restriction C$ came out to be non-measurable, even if everywhere defined, and g was not $\mu_{\mathcal{B}}\restriction\mathrm{T}_0$-virtually measurable.

In the statement of 372J I have offered two ways of being sure that the problem does not arise: check that $\phi[E]$ is negligible whenever E is negligible (so that all negligible sets are $\mu\restriction\mathrm{T}_0$-negligible), or check that f is defined everywhere and Σ-measurable. Even if these conditions are not immediately satisfied in a particular application, it may be possible to modify the problem so that they are. For instance, completing the measure will leave ϕ inverse-measure-preserving (234Ba[2]), will not change the integrable functions but will make them all measurable (212F, 212Bc), and may enlarge T_0 enough to make a difference. If our function f is measurable (because the measure is complete, or otherwise) we can extend it to a measurable function defined everywhere (121I) and the corresponding extension of g will be $\mu\restriction\mathrm{T}_0$-integrable. Alternatively, if the difficulty seems to lie in the behaviour of ϕ rather than in the behaviour of f (as in the example above), it may help to modify ϕ on a negligible set.

372L Continued fractions A particularly delightful application of the results above is to a question which belongs as much to number theory as to analysis. It takes a bit of space to describe, but I hope you will agree with me that it is well worth knowing in itself, and that it also illuminates some of the ideas above.

(a) Set $X=[0,1]\setminus\mathbb{Q}$. For $x\in X$, set $\phi(x)=<\frac{1}{x}>$, the fractional part of $\frac{1}{x}$, and $k_1(x)=\frac{1}{x}-\phi(x)=\lfloor\frac{1}{x}\rfloor$, the integer part of $\frac{1}{x}$; then $\phi(x)\in X$ for each $x\in X$, so we may define $k_n(x)=k_1(\phi^{n-1}(x))$ for every $n\ge 1$. The strictly positive integers $k_1(x)$, $k_2(x)$, $k_3(x),\dots$ are the **continued fraction coefficients** of x. Of course $k_{n+1}(x)=k_n(\phi(x))$ for every $n\ge 1$. Now define $\langle p_n(x)\rangle_{n\in\mathbb{N}}$, $\langle q_n(x)\rangle_{n\in\mathbb{N}}$ inductively by setting

$$p_0(x)=0,\quad p_1(x)=1,\quad p_n(x)=p_{n-2}(x)+k_n(x)p_{n-1}(x)\text{ for }n\ge 1,$$

$$q_0(x)=1,\quad q_1(x)=k_1(x),\quad q_n(x)=q_{n-2}(x)+k_n(x)q_{n-1}(x)\text{ for }n\ge 1.$$

The **continued fraction approximations** or **convergents** to x are the quotients $p_n(x)/q_n(x)$.

(I do not discuss rational x, because for my purposes here these are merely distracting. But if we set $k_1(0)=\infty$, $\phi(0)=0$ then the formulae above produce the conventional values for $k_n(x)$ for rational $x\in[0,1[$. As for the p_n and q_n, use the formulae above until you get to $x=p_n(x)/q_n(x)$, $\phi^n(x)=0$, $k_{n+1}(x)=\infty$, and then set $p_m(x)=p_n(x)$, $q_m(x)=q_n(x)$ for $m\ge n$.)

(b) The point is that the quotients $r_n(x)=p_n(x)/q_n(x)$ are, in a strong sense, good rational approximations to x. (See 372Xl(v).) We have $r_n(x)<x<r_{n+1}(x)$ for every even n (372Xl). If $x=\pi-3$, then the first few coefficients are

$$k_1=7,\quad k_2=15,\quad k_3=1,$$

$$r_1=\frac{1}{7},\quad r_2=\frac{15}{106},\quad r_3=\frac{16}{113};$$

the first and third of these corresponding to the classical approximations $\pi\simeq\frac{22}{7}$, $\pi\simeq\frac{355}{113}$. Or if we take $x=e-2$, we get

$$k_1=1,\quad k_2=2,\quad k_3=1,\quad k_4=1,\quad k_5=4,\quad k_6=1,\quad k_7=1,$$

$$r_1=1,\quad r_2=\frac{2}{3},\quad r_3=\frac{3}{4},\quad r_4=\frac{5}{7},\quad r_5=\frac{23}{32},\quad r_6=\frac{28}{39},\quad r_7=\frac{51}{71};$$

note that the obvious approximations $\frac{17}{24}$, $\frac{86}{120}$ derived from the series for e are not in fact as close as the even terms $\frac{5}{7}$, $\frac{28}{39}$ above, and involve larger numbers[3].

(c) Now we need a variety of miscellaneous facts about these coefficients, which I list here.

[2] Formerly 235Hc.

[3] There is a remarkable expression for the continued fraction expansion of e, due essentially to Euler; $k_{3m-1}=2m$, $k_{3m}=k_{3m+1}=1$ for $m\ge 2$. See Cohn 06.

(i) For any $x \in X$, $n \ge 1$ we have

$$p_{n-1}(x)q_n(x) - p_n(x)q_{n-1}(x) = (-1)^n, \quad \phi^n(x) = \frac{p_n(x)-xq_n(x)}{xq_{n-1}(x)-p_{n-1}(x)}$$

(induce on n), so

$$x = \frac{p_n(x)+p_{n-1}(x)\phi^n(x)}{q_n(x)+q_{n-1}(x)\phi^n(x)}.$$

(ii) Another easy induction on n shows that for any finite string $\mathbf{m} = (m_1, \dots , m_n)$ of strictly positive integers the set $D_{\mathbf{m}} = \{x : x \in X,\ k_i(x) = m_i \text{ for } 1 \le i \le n\}$ is an interval in X on which ϕ^n is monotonic, being strictly increasing if n is even and strictly decreasing if n is odd. (For the inductive step, note just that

$$D_{(m_1,\dots ,m_n)} = [\tfrac{1}{m_1+1}, \tfrac{1}{m_1}] \cap \phi^{-1}[D_{(m_2,\dots ,m_n)}].)$$

(iii) We also need to know that the intervals $D_{\mathbf{m}}$ of (ii) are small; specifically, that if $\mathbf{m} = (m_1, \dots , m_n)$, the length of $D_{\mathbf{m}}$ is at most 2^{-n+1}. **P** All the coefficients p_i, q_i, for $i \le n$, take constant values p_i^*, q_i^* on $D_{\mathbf{m}}$, since they are determined from the coefficients k_i which are constant on $D_{\mathbf{m}}$ by definition. Now every $x \in D_{\mathbf{m}}$ is of the form $(p_n^* + tp_{n-1}^*)/(q_n^* + tq_{n-1}^*)$ for some $t \in X$ (see (i) above) and therefore lies between p_{n-1}^*/q_{n-1}^* and p_n^*/q_n^*. But the distance between these is

$$\Bigl|\frac{p_n^*q_{n-1}^*-p_{n-1}^*q_n^*}{q_n^*q_{n-1}^*}\Bigr| = \frac{1}{q_n^*q_{n-1}^*},$$

by the first formula in (i). Next, noting that $q_i^* \ge q_{i-1}^* + q_{i-2}^*$ for each $i \ge 2$, we see that $q_i^*q_{i-1}^* \ge 2q_{i-1}^*q_{i-2}^*$ for $i \ge 2$, and therefore that $q_n^*q_{n-1}^* \ge 2^{n-1}$, so that the length of $D_{\mathbf{m}}$ is at most 2^{-n+1}. **Q**

372M Theorem Set $X = [0, 1] \setminus \mathbb{Q}$, and define $\phi : X \to X$ as in 372L. Then for every Lebesgue integrable function f on X,

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x)) = \frac{1}{\ln 2}\int_0^1 \frac{f(t)}{1+t}dt$$

for almost every $x \in X$.

proof (a) The integral just written, and the phrase 'almost every', refer of course to Lebesgue measure; but the first step is to introduce another measure, so I had better give a name μ_L to Lebesgue measure on X. Let ν be the indefinite-integral measure on X defined by saying that $\nu E = \frac{1}{\ln 2}\int_E \frac{1}{1+x}\mu_L(dx)$ whenever this is defined. The coefficient $\frac{1}{\ln 2}$ is of course chosen to make $\nu X = 1$. Because $\frac{1}{1+x} > 0$ for every $x \in X$, $\operatorname{dom}\nu = \operatorname{dom}\mu_L$ and ν has just the same negligible sets as μ_L (234Lc[4]); I can therefore safely use the terms 'measurable set', 'almost everywhere' and 'negligible' without declaring which measure I have in mind each time.

(b) Now ϕ is inverse-measure-preserving when regarded as a function from (X, ν) to itself. **P** For each $k \ge 1$, set $I_k = \Bigl[\frac{1}{k+1}, \frac{1}{k}\Bigl[$. On $X \cap I_k$, $\phi(x) = \frac{1}{x} - k$. Observe that $\phi\restriction I_k : X \cap I_k \to X$ is bijective and differentiable relative to its domain in the sense of 262Fb. Consider, for any measurable $E \subseteq X$,

$$\begin{aligned}\int_E \frac{1}{(y+k)(y+k+1)}\mu_L(dy) &= \int_{I_k\cap\phi^{-1}[E]} \frac{1}{(\phi(x)+k)(\phi(x)+k+1)}|\phi'(x)|\mu_L(dx)\\ &= \int_{I_k\cap\phi^{-1}[E]} \frac{x^2}{x+1}\frac{1}{x^2}\mu_L(dx) = \ln 2 \cdot \nu(I_k \cap \phi^{-1}[E]),\end{aligned}$$

using 263D (or more primitive results, of course). But

$$\sum_{k=1}^\infty \frac{1}{(y+k)(y+k+1)} = \sum_{k=1}^\infty \frac{1}{y+k} - \frac{1}{y+k+1} = \frac{1}{y+1}$$

for every $y \in [0, 1]$, so

$$\nu E = \frac{1}{\ln 2}\sum_{k=1}^{\infty}\int_E \frac{1}{(y+k)(y+k+1)}\mu_L(dy) = \sum_{k=1}^{\infty}\nu(I_k \cap \phi^{-1}[E]) = \nu\phi^{-1}[E].$$

[4] Formerly 234D.

As E is arbitrary, ϕ is inverse-measure-preserving. **Q**

(c) The next thing we need to know is that if $E \subseteq X$ and $\phi^{-1}[E] = E$ then E is either negligible or conegligible. **P** I use the sets $D_{\mathbf{m}}$ of 372L(c-ii).

(i) For any string $\mathbf{m} = (m_1, \dots, m_n)$ of strictly positive integers, we have

$$x = \frac{p_n^* + p_{n-1}^*\phi^n(x)}{q_n^* + q_{n-1}^*\phi^n(x)}$$

for every $x \in D_{\mathbf{m}}$, where p_n^*, etc., are defined from $\mathbf{m}$ as in 372L(c-iii). Recall also that ϕ^n is strictly monotonic on $D_{\mathbf{m}}$. So for any interval $I \subseteq [0,1]$ (open, closed or half-open) with endpoints $\alpha < \beta$, $\phi^{-n}[I] \cap D_{\mathbf{m}}$ will be of the form $X \cap J$, where J is an interval with endpoints $(p_n^* + p_{n-1}^*\alpha)/(q_n^* + q_{n-1}^*\alpha)$, $(p_n^* + p_{n-1}^*\beta)/(q_n^* + q_{n-1}^*\beta)$ in some order. This means that we can estimate $\mu_L(\phi^{-n}[I] \cap D_{\mathbf{m}})/\mu_L D_{\mathbf{m}}$, because it is

$$\frac{\left|\frac{p_n^*+p_{n-1}^*\alpha}{q_n^*+q_{n-1}^*\alpha} - \frac{p_n^*+p_{n-1}^*\beta}{q_n^*+q_{n-1}^*\beta}\right|}{\left|\frac{p_n^*}{q_n^*} - \frac{p_n^*+p_{n-1}^*}{q_n^*+q_{n-1}^*}\right|} = \frac{(\beta-\alpha)q_n^*(q_n^*+q_{n-1}^*)}{(q_n^*+q_{n-1}^*\alpha)(q_n^*+q_{n-1}^*\beta)} \ge \frac{(\beta-\alpha)q_n^*}{q_n^*+q_{n-1}^*} \ge \frac{1}{2}(\beta - \alpha).$$

Now look at

$$\mathcal{A} = \{E : E \subseteq [0,1] \text{ is Lebesgue measurable, } \mu_L(\phi^{-n}[E] \cap D_{\mathbf{m}}) \ge \frac{1}{2}\mu_L E \cdot \mu_L D_{\mathbf{m}}\}.$$

Clearly the union of two disjoint members of $\mathcal{A}$ belongs to $\mathcal{A}$. Because $\mathcal{A}$ contains every subinterval of $[0,1]$ it includes the algebra $\mathcal{E}$ of subsets of $[0,1]$ consisting of finite unions of intervals. Next, the union of any non-decreasing sequence in $\mathcal{A}$ belongs to $\mathcal{A}$, and the intersection of a non-increasing sequence likewise. But this means that $\mathcal{A}$ must include the σ-algebra generated by $\mathcal{E}$ (136G), that is, the Borel σ-algebra. But also, if $E \in \mathcal{A}$ and $H \subseteq [0,1]$ is negligible, then

$$\mu_L(\phi^{-n}[E \triangle H] \cap D_{\mathbf{m}}) = \mu_L(\phi^{-n}[E] \cap D_{\mathbf{m}}) \ge \frac{1}{2}\mu_L E \cdot \mu_L D_{\mathbf{m}} = \frac{1}{2}\mu_L(E \triangle H) \cdot \mu_L D_{\mathbf{m}}$$

and $E \triangle H \in \mathcal{A}$. And this means that every Lebesgue measurable subset of $[0,1]$ belongs to $\mathcal{A}$ (134Fb).

(ii) **?** Now suppose, if possible, that E is a measurable subset of X and that $\phi^{-1}[E] = E$ and E is neither negligible nor conegligible in X. Set $\gamma = \frac{1}{2}\mu_L E > 0$. By Lebesgue's density theorem (223B) there is some $x \in X \setminus E$ such that $\lim_{\delta \downarrow 0} \psi(\delta) = 0$, where $\psi(\delta) = \frac{1}{2\delta}\mu_L(E \cap [x-\delta, x+\delta])$ for $\delta > 0$. Take n so large that $\psi(\delta) < \frac{1}{2}\gamma$ whenever $0 < \delta \le 2^{-n+1}$, and set $m_i = k_i(x)$ for $i \le n$, so that $x \in D_{\mathbf{m}}$. Taking the least δ such that $D_{\mathbf{m}} \subseteq [x-\delta, x+\delta]$, we must have $\delta \le 2^{-n+1}$, because the length of $D_{\mathbf{m}}$ is at most 2^{-n+1} (372L(c-iii)), while $\mu_L D_{\mathbf{m}} \ge \delta$, because $D_{\mathbf{m}}$ is an interval. Accordingly

$$\mu_L(E \cap D_{\mathbf{m}}) \le \mu_L(E \cap [x-\delta, x+\delta]) = 2\delta\psi(\delta) < \gamma\delta \le \gamma\mu_L D_{\mathbf{m}}.$$

But we also have

$$\mu_L(E \cap D_{\mathbf{m}}) = \mu_L(\phi^{-n}[E] \cap D_{\mathbf{m}}) \ge \gamma\mu_L D_{\mathbf{m}},$$

by (i). **X**

This proves the result. **Q**

(d) The final fact we need in preparation is that $\phi[E]$ is negligible for every negligible $E \subseteq X$. This is because ϕ is differentiable relative to its domain (see 263D(ii)).

(e) Now let f be any μ_L-integrable function. Because $\frac{1}{1+x} \le 1$ for every x, f is also ν-integrable (235K[5]); consequently, using (b) above and 372J,

$$g(x) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$$

is defined for almost every $x \in X$, and is a conditional expectation of f (with respect to the measure ν) on the σ-algebra $\mathrm{T}_0 = \{E : E \text{ is measurable}, \phi^{-1}[E] = E\}$. But we have just seen that T_0 consists only of negligible and conegligible sets, so g must be essentially constant; since $\int g\,d\nu = \int f d\nu$, we must have

[5] Formerly 235M.

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x)) = \int f d\nu = \frac{1}{\ln 2}\int_0^1 \frac{f(t)}{1+t}\mu_L(dt)$$

for almost every x (using 235K to calculate $\int f d\nu$).

372N Corollary For almost every $x \in [0,1]\setminus\mathbb{Q}$,

$$\lim_{n\to\infty}\frac{1}{n}\#(\{i : 1 \le i \le n,\, k_i(x) = k\}) = \frac{1}{\ln 2}(2\ln(k+1) - \ln k - \ln(k+2))$$

for every $k \ge 1$, where $k_1(x), \dots$ are the continued fraction coefficients of x.

proof In 372M, set $f = \chi(X \cap [\frac{1}{k+1}, \frac{1}{k}])$. Then (for $i \ge 1$) $f(\phi^i(x)) = 1$ if $k_i(x) = k$ and zero otherwise. So

$$\begin{aligned}
\lim_{n\to\infty}&\frac{1}{n}\#(\{i : 1 \le i \le n,\, k_i(x) = k\})\\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(\phi^i(x)) = \lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x))\\
&= \frac{1}{\ln 2}\int_0^1 \frac{f(t)}{1+t}dt = \frac{1}{\ln 2}\int_{1/k+1}^{1/k}\frac{1}{1+t}dt\\
&= \frac{1}{\ln 2}(\ln(1+\frac{1}{k}) - \ln(1+\frac{1}{k+1})) = \frac{1}{\ln 2}(2\ln(k+1) - \ln k - \ln(k+2)),
\end{aligned}$$

for almost every $x \in X$.

372O Mixing and ergodic transformations This seems an appropriate moment for some brief notes on three special types of measure-preserving homomorphism or inverse-measure-preserving function.

Definitions (a)(i) Let $\mathfrak{A}$ be a Boolean algebra. Then a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is **ergodic** if whenever $a, b \in \mathfrak{A}\setminus\{0\}$ there are $m, n \in \mathbb{N}$ such that $\pi^m a \cap \pi^n b \ne 0$.

(ii) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Then π is **mixing** (sometimes called **strongly mixing**) if $\lim_{n\to\infty}\bar\mu(\pi^n a \cap b) = \bar\mu a \cdot \bar\mu b$ for all $a, b \in \mathfrak{A}$.

(iii) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Then π is **weakly mixing** if $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}|\bar\mu(\pi^n a \cap b) - \bar\mu a \cdot \bar\mu b| = 0$ for all $a, b \in \mathfrak{A}$.

(b) Let (X, Σ, μ) be a probability space and $\phi : X \to X$ an inverse-measure-preserving function.

(i) ϕ is **ergodic** (also called **metrically transitive**, **indecomposable**) if every measurable set E such that $\phi^{-1}[E] = E$ is either negligible or conegligible.

(ii) ϕ is **mixing** if $\lim_{n\to\infty}\mu(F \cap \phi^{-n}[E]) = \mu E \cdot \mu F$ for all $E, F \in \Sigma$.

(iii) ϕ is **weakly mixing** if if $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}|\mu(F \cap \phi^{-n}[E]) - \mu E \cdot \mu F| = 0$ for all $E, F \in \Sigma$.

372P For the principal applications of the idea in 372O(a-i), we have an alternative definition in terms of fixed-point subalgebras.

Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism, with fixed-point subalgebra $\mathfrak{C}$.
(a) If π is ergodic, then $\mathfrak{C} = \{0, 1\}$.
(b) If π is an automorphism, then π is ergodic iff $\sup_{n\in\mathbb{Z}}\pi^n a = 1$ for every $a \in \mathfrak{A}\setminus\{0\}$.
(c) If π is an automorphism and $\mathfrak{A}$ is Dedekind σ-complete, then π is ergodic iff $\mathfrak{C} = \{0, 1\}$.

proof (a) If $c \in \mathfrak{C}$, then $\pi^m c = c$ is disjoint from $\pi^n(1\setminus c) = 1\setminus c$ for all $m, n \in \mathbb{N}$, so one of c, $1\setminus c$ must be zero.

(b)(i) If π is ergodic and $a \ne 0$ and $b \cap \pi^n a = 0$ for every $n \in \mathbb{Z}$, then $\pi^m b \cap \pi^n a = \pi^m(b \cap \pi^{n-m}a) = 0$ for all m, $n \in \mathbb{N}$, so $b = 0$. As b is arbitrary, $\sup_{n\in\mathbb{Z}}\pi^n a = 1$; as a is arbitrary, π satisfies the condition.

(ii) If π satisfies the condition, and a, $b \in \mathfrak{A} \setminus \{0\}$, then there is an $m \in \mathbb{Z}$ such that $\pi^m a \cap b \neq 0$; setting $n = \max(-m, 0)$, $\pi^{m+n} a \cap \pi^n b \neq 0$, while $m + n$ and n both belong to $\mathbb{N}$. As a and b are arbitrary, π is ergodic.

(c) If π is ergodic then $\mathfrak{C} = \{0, 1\}$, by (a). If $\mathfrak{C} = \{0, 1\}$ and $a \in \mathfrak{A} \setminus \{0\}$, consider $c = \sup_{n\in\mathbb{Z}} \pi^n a$, which is defined because $\mathfrak{A}$ is Dedekind σ-complete. Being an automorphism, π is order-continuous (313Ld), so $\pi c = \sup_{n\in\mathbb{Z}} \pi^{n+1} a = c$ and $c \in \mathfrak{C}$. Since $c \supseteq a$ is non-zero, $c = 1$. As a is arbitrary, π is ergodic, by (b).

372Q The following facts are equally straightforward.

Proposition (a) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism, and $T : L^0 = L^0(\mathfrak{A}) \to L^0$ the Riesz homomorphism such that $T(\chi a) = \chi\pi a$ for every $a \in \mathfrak{A}$.

(i) If π is mixing, it is weakly mixing.

(ii) If π is weakly mixing, it is ergodic.

(iii) The following are equiveridical: (α) π is ergodic; (β) the only $u \in L^0$ such that $Tu = u$ are the multiples of $\chi 1$; (γ) for every $u \in L^1 = L^1(\mathfrak{A}, \bar\mu)$, $\langle \frac{1}{n+1} \sum_{i=0}^n T^i u\rangle_{n\in\mathbb{N}}$ order*-converges to $(\int u)\chi 1$.

(iv) The following are equiveridical: (α) π is mixing; (β) $\lim_{n\to\infty}(T^n u|v) = \int u \int v$ for all u, $v \in L^2(\mathfrak{A}, \bar\mu)$.

(v) The following are equiveridical: (α) π is weakly mixing; (β) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |(T^k u|v) - \int u \int v| = 0$ for all u, $v \in L^2(\mathfrak{A}, \bar\mu)$.

(b) Let (X, Σ, μ) be a probability space, with measure algebra $(\mathfrak{A}, \bar\mu)$. Let $\phi : X \to X$ be an inverse-measure-preserving function and $\pi : \mathfrak{A} \to \mathfrak{A}$ the associated homomorphism such that $\pi E^\bullet = (\phi^{-1}[E])^\bullet$ for every $E \in \Sigma$.

(i) The following are equiveridical: (α) ϕ is ergodic; (β) π is ergodic; (γ) for every μ-integrable real-valued function f, $\langle \frac{1}{n+1} \sum_{i=0}^n f(\phi^i(x))\rangle_{n\in\mathbb{N}}$ converges to $\int f$ for almost every $x \in X$.

(ii) ϕ is mixing iff π is, and in this case ϕ is weakly mixing.

(iii) ϕ is weakly mixing iff π is, and in this case ϕ is ergodic.

proof (a)(i)-(ii) Immediate from the definitions.

(iii)(α)⇒(β) $Tu = u$ iff $\pi[\![u > \alpha]\!] = [\![u > \alpha]\!]$ for every α; if π is ergodic, this means that $[\![u > \alpha]\!] \in \{0, 1\}$ for every α, by 372Pa, and u must be of the form $\gamma\chi 1$, where $\gamma = \inf\{\alpha : [\![u > \alpha]\!] = 0\}$.

(β)⇒(γ) If (β) is true and $u \in L^1$, then we know from 372G that $\langle \frac{1}{n+1} \sum_{i=0}^n T^i u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent to some v such that $Tv = v$; by (β), v is of the form $\gamma\chi 1$; and

$$\gamma = \int v = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n \int T^i u = \int u.$$

(γ)⇒(α) Assuming (γ), take any $a \in \mathfrak{A}$ such that $\pi a = a$, and consider $u = \chi a$. Then $T^i u = \chi a$ for every i, so

$$\chi a = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n T^i u = (\int u)\chi 1 = \bar\mu a \cdot \chi 1,$$

and a must be either 0 or 1. By 372Pc, π is ergodic.

(iv)(α)⇒(β) Since π is mixing,

$$\begin{aligned}\lim_{n\to\infty}(T^n \chi a|\chi b) &= \lim_{n\to\infty}(\chi\pi^n a|\chi b) = \lim_{n\to\infty} \bar\mu(\pi^n a \cap b) \\ &= \bar\mu a \cdot \bar\mu b = \int \chi a \int \chi b\end{aligned}$$

for all a, $b \in \mathfrak{A}$. Because $(u, v) \mapsto (T^n u|v)$ and $(u, v) \mapsto \int u \int v$ are both bilinear,

$$\lim_{n\to\infty}(T^n u|v) = \int u \int v$$

for all u, $v \in S(\mathfrak{A})$. For general u, $v \in L^2(\mathfrak{A}, \bar\mu)$, take any $\epsilon > 0$. Then there are u', $v' \in S(\mathfrak{A})$ such that

$$(\|u - u'\|_2 + \|v - v'\|_2) \max(\|u\|_2, \|v\|_2 + \|v - v'\|_2) \le \epsilon$$

(366C), so that

$$
\begin{aligned}
|(T^n u|v) - (T^n u'|v')| &\le |(T^n u|v-v')| + |(T^n u - T^n u'|v')| \\
&\le \|T^n u\|_2 \|v-v'\|_2 + \|T^n u - T^n u'\|_2 \|v'\|_2 \\
&\le \|u\|_2 \|v-v'\|_2 + \|u-u'\|_2 (\|v\|_2 + \|v-v'\|_2)
\end{aligned}
$$

(366H(a-iv))

$$
\begin{aligned}
&\le \epsilon, \\
|\int u \int v - \int u' \int v'| &\le |\int u| |\int v - v'| + |\int u - u'| |\int v'| \\
&\le \|u\|_2 \|v-v'\|_2 + \|u-u'\|_2 \|v'\|_2 \le \epsilon
\end{aligned}
$$

for every n, and

$$\limsup_{n\to\infty} |(T^n u|v) - \int u \int v| \le 2\epsilon + \lim_{n\to\infty} |(T^n u'|v') - \int u' \int v'| = 2\epsilon.$$

As ϵ is arbitrary, $\lim_{n\to\infty}(T^n u|v) = \int u \int v$, as required.

$(\beta)\Rightarrow(\alpha)$ This is elementary, as (α) is just the case $u = \chi a$, $v = \chi b$ of (β).

(v) The argument is essentially the same as in (iv); (α) is a special case of (β); if (α) is true, then by linearity (β) is true when u, $v \in S(\mathfrak{A})$, and the functional $(u, v) \mapsto \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |(T^k u|v) - \int u \int v|$ is continuous.

(b)(i)$(\alpha)\Rightarrow(\beta)$ If $\pi a = a$ there is an E such that $\phi^{-1}[E] = E$ and $E^\bullet = a$, by 372I; now $\bar{\mu} a = \mu E \in \{0, 1\}$, so $a \in \{0, 1\}$. Thus the fixed-point subalgebra of π is $\{0, 1\}$; by 372Pc again, π is ergodic.

$(\beta)\Rightarrow(\gamma)$ Set $u = f^\bullet \in L^1$. In the language of (a), $T^i u = (f\phi^i)^\bullet$ for each i, as in the proof of 372H, so that

$$(\tfrac{1}{n+1} \textstyle\sum_{i=0}^n f\phi^i)^\bullet = \tfrac{1}{n+1} \textstyle\sum_{i=0}^n T^i u$$

is order*-convergent to $(\int u)\chi 1 = (\int f)\chi 1$, and $\frac{1}{n+1} \sum_{i=0}^n f\phi^i \to \int f$ a.e.

$(\gamma)\Rightarrow(\alpha)$ If $\phi^{-1}[E] = E$ then, applying (γ) to $f = \chi E$, we see that $\chi E =_{\text{a.e.}} \mu E \cdot \chi X$, so that E is either negligible or conegligible.

(ii)-(iii) Simply translating the definitions, we see that π is mixing, or weakly mixing, iff ϕ is. So the results here are reformulations of (a-i) and (a-ii).

372R Remarks (a) The reason for introducing 'ergodic' homomorphisms in this section is of course 372G/372J; if π in 372G, or ϕ in 372J, is ergodic, then the limit Pu or g must be (essentially) constant, being a conditional expectation on a trivial subalgebra.

(b) In the definition 372O(b-i) I should perhaps emphasize that we look only at *measurable* sets E. We certainly expect that there will generally be many sets E for which $\phi^{-1}[E] = E$, since any union of orbits of ϕ will have this property.

(c) Part (c) of the proof of 372M was devoted to showing that the function ϕ there was ergodic; see also 372Xm. For another ergodic transformation see 372Xr. For examples of mixing transformations see 333P, 372Xp, 372Xq, 372Xt, 372Xw and 372Xx.

(d) It seems to be difficult to display explicitly a weakly mixing transformation which is not mixing. There is an example in Chacon 69, and I give another in 494F in Volume 4. In a certain sense, however, 'most' measure-preserving automorphisms of the Lebesgue probability algebra are weakly mixing but not mixing; I will return to this in 494E.

372S There is a useful sufficient condition for a homomorphism or function to be mixing.

Proposition (a) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. If $\bigcap_{n\in\mathbb{N}} \pi^n[\mathfrak{A}] = \{0, 1\}$, then π is mixing.

(b) Let (X, Σ, μ) be a probability space, and $\phi : X \to X$ an inverse-measure-preserving function. Set

$$\mathrm{T} = \{E : \text{for every } n \in \mathbb{N} \text{ there is an } F \in \Sigma \text{ such that } E = \phi^{-n}[F]\}.$$

If every member of T is either negligible or conegligible, ϕ is mixing.

proof (a) Let $T : L^0 = L^0(\mathfrak{A}) \to L^0$ be the Riesz homomorphism associated with π. Take any a, $b \in \mathfrak{A}$ and any non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Then $\langle T^n(\chi a)\rangle_{n\in\mathbb{N}}$ is a bounded sequence in the reflexive space $L^2_{\bar\mu} = L^2(\mathfrak{A}, \bar\mu)$, so $v = \lim_{n\to\mathcal{F}} T^n(\chi a)$ is defined for the weak topology of $L^2_{\bar\mu}$. Now for each $n \in \mathbb{N}$ set $\mathfrak{B}_n = \pi^n[\mathfrak{A}]$. This is a closed subalgebra of $\mathfrak{A}$ (314F(a-i)), and contains $\pi^i a$ for every $i \ge n$. So if we identify $L^2(\mathfrak{B}_n, \bar\mu\restriction\mathfrak{B}_n)$ with the corresponding subspace of $L^2_{\bar\mu}$ (366I), it contains $T^i(\chi a)$ for every $i \ge n$; but also it is norm-closed, therefore weakly closed (3A5Ee), so contains v. This means that $[\![v > \alpha]\!]$ must belong to $\mathfrak{B}_n$ for every α and every n. But in this case $[\![v > \alpha]\!] \in \bigcap_{n\in\mathbb{N}} \mathfrak{B}_n = \{0, 1\}$ for every α, and v is of the form $\gamma\chi 1$. Also

$$\gamma = \int v = \lim_{n\to\mathcal{F}} \int T^n(\chi a) = \bar\mu a.$$

So

$$\lim_{n\to\mathcal{F}} \bar\mu(\pi^n a \cap b) = \lim_{n\to\mathcal{F}} \int T^n(\chi a) \times \chi b = \int v \times \chi b = \gamma\bar\mu b = \bar\mu a \cdot \bar\mu b.$$

But this is true of every non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$, so we must have $\lim_{n\to\infty} \bar\mu(\pi^n a \cap b) = \bar\mu a \cdot \bar\mu b$ (3A3Lc). As a and b are arbitrary, π is mixing.

(b) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of (X, Σ, μ), and $\pi : \mathfrak{A} \to \mathfrak{A}$ the measure-preserving homomorphism corresponding to ϕ. The point is that if $a \in \bigcap_{n\in\mathbb{N}} \pi^n[\mathfrak{A}]$, there is an $E \in \mathrm{T}$ such that $E^\bullet = a$. **P** For each $n \in \mathbb{N}$ there is an $a_n \in \mathfrak{A}$ such that $\pi^n a_n = a$; say $a_n = F_n^\bullet$ where $F_n \in \Sigma$. Then $\phi^{-n}[F_n]^\bullet = a$. Set

$$E = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \phi^{-n}[F_n], \quad E_k = \bigcup_{m\ge k} \bigcap_{n\ge m} \phi^{-(n-k)}[F_n]$$

for each k; then $E^\bullet = a$ and

$$\phi^{-k}[E_k] = \bigcup_{m\ge k} \bigcap_{n\ge m} \phi^{-n}[F_n] = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \phi^{-n}[F_n] = E$$

for every k, so $E \in \mathrm{T}$. **Q**

So $\bigcap_{n\in\mathbb{N}} \mathfrak{A}_n = \{0, 1\}$ and π and ϕ are mixing.

372X Basic exercises (a) Let U be any reflexive Banach space, and $T : U \to U$ an operator of norm at most 1. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i$ for each $n \in \mathbb{N}$. Show that $Pu = \lim_{n\to\infty} A_n u$ is defined (as a limit for the norm topology) for every $u \in U$, and that $P : U \to U$ is a projection onto $\{u : Tu = u\}$. (*Hint*: show that $\{u : Pu \text{ is defined}\}$ is a closed linear subspace of U containing $Tu - u$ for every $u \in U$.)

(This is a version of the **mean ergodic theorem**.)

>(b) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $T \in \mathcal{T}^{(0)}_{\bar\mu,\bar\mu}$; set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i$ for $n \in \mathbb{N}$. Take any $p \in [1, \infty[$ and $u \in L^p = L^p(\mathfrak{A}, \bar\mu)$. Show that $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_p$-convergent to some $v \in L^p$. (*Hint*: put 372Xa together with 372D.)

(c) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $P : L^1 \to L^1$ be the operator defined as in 365P/366Hb, where $L^1 = L^1_{\bar\mu}$, so that $\int_a Pu = \int_{\pi a} u$ for $u \in L^1$ and $a \in \mathfrak{A}$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n P^i : L^1 \to L^1$ for each i. Show that for any $u \in L^1$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent to the conditional expectation of u on the subalgebra $\{a : \pi a = a\}$.

(d) Show that if f is any Lebesgue integrable function on $\mathbb{R}$, and $y \in \mathbb{R} \setminus \{0\}$, then

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^n f(x + ky) = 0$$

for almost every $x \in \mathbb{R}$.

(e) Let (X, Σ, μ) be a measure space and $\phi : X \to X$ an inverse-measure-preserving function. Set $\mathrm{T} = \{E : E \in \Sigma,\ \mu(\phi^{-1}[E]\triangle E) = 0\}$, $\mathrm{T}_0 = \{E : E \in \Sigma,\ \phi^{-1}[E] = E\}$. (i) Show that $\mathrm{T} = \{E\triangle F : E \in \mathrm{T}_0,\ F \in \Sigma,\ \mu F = 0\}$. (ii) Show that a set $A \subseteq X$ is $\mu\restriction\mathrm{T}_0$-negligible iff $\phi^n[A]$ is μ-negligible for every $n \in \mathbb{N}$.

>(f) Let ν be a Radon probability measure on $\mathbb{R}$ such that $\int |t|\nu(dt)$ is finite (cf. 271F). On $X = \mathbb{R}^{\mathbb{N}}$ let λ be the product measure obtained when each factor is given the measure ν. Define $\phi : X \to X$ by setting $\phi(x)(n) = x(n+1)$ for $x \in X$, $n \in \mathbb{N}$. (i) Show that ϕ is inverse-measure-preserving. (*Hint*: 254G. See also 372Xw below.) (iii) Set $\gamma = \int t\nu(dt)$, the expectation of the distribution ν. By considering $\frac{1}{n+1}\sum_{i=0}^n f \circ \phi^i$, where $f(x) = x(0)$ for $x \in X$, show that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n x(i) = \gamma$ for λ-almost every $x \in X$.

>(g) Use the Ergodic Theorem to prove Kolmogorov's Strong Law of Large Numbers (273I), as follows. Given a complete probability space (Ω, Σ, μ) and an independent identically distributed sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of measurable functions from Ω to $\mathbb{R}$, set $X = \mathbb{R}^{\mathbb{N}}$ and $f(\omega) = \langle f_n(\omega)\rangle_{n\in\mathbb{N}}$ for $\omega \in \Omega$. Show that if we give each copy of $\mathbb{R}$ the distribution of f_0 then f is inverse-measure-preserving for μ and the product measure λ on X. Now use 372Xf.

>(h) Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of real-valued random variables with finite expectation such that $(f_0, f_1, \dots, f_n)$ has the same joint distribution as $(f_1, f_2, \dots, f_{n+1})$ for every $n \in \mathbb{N}$. Show that $\langle \frac{1}{n+1}\sum_{i=0}^n f_i\rangle_{n\in\mathbb{N}}$ converges a.e. (*Hint*: Let (X, Σ, μ) be the underlying probability space. Reduce to the case in which every f_i is measurable and defined everywhere in X. Define $\theta : X \to \mathbb{R}^{\mathbb{N}}$ by setting $\theta(x)(n) = f_n(x)$ for $x \in X$, $n \in \mathbb{N}$. Let λ be the image measure $\mu\theta^{-1}$. Set $\phi(z)(n) = z(n + 1)$ for $z \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Show that ϕ is inverse-measure-preserving for λ, and apply 372J.)

(i) Show that the continued fraction coefficients of $\frac{1}{\sqrt{2}}$ are 1, 2, 2, 2, ...

>(j) For $x \in X = [0, 1] \setminus \mathbb{Q}$ let $k_1(x), k_2(x), \dots$ be its continued-fraction coefficients. Show that $x \mapsto \langle k_{n+1}(x) - 1\rangle_{n\in\mathbb{N}}$ is a bijection between X and $\mathbb{N}^{\mathbb{N}}$ which is a homeomorphism if X is given its usual topology (as a subset of $\mathbb{R}$) and $\mathbb{N}^{\mathbb{N}}$ is given its usual product topology (each copy of $\mathbb{N}$ being given its discrete topology).

(k) Set $x = \frac{1}{2}(\sqrt{5} - 1)$. Show that, in the notation of 372L, $k_n(x) = 1$ and $q_n(x) = p_{n-1}(x)$ for every $n \ge 1$ and that $\langle p_n(x)\rangle_{n\in\mathbb{N}}$ is the Fibonacci sequence.

(l) For any irrational $x \in [0, 1]$ let $k_1(x), k_2(x), \dots$ be its continued-fraction coefficients and $p_n(x)$, $q_n(x)$ the numerators and denominators of its continued-fraction approximations, as described in 372L. Write $r_n(x) = p_n(x)/q_n(x)$. (i) Show that x lies between $r_n(x)$ and $r_{n+1}(x)$ for every $n \in \mathbb{N}$. (ii) Show that $r_{n+1}(x)-r_n(x) = (-1)^n/q_n(x)q_{n+1}(x)$ for every $n \in \mathbb{N}$. (iii) Show that $|x - r_n(x)| \le 1/q_n(x)^2k_{n+1}(x)$ for every $n \ge 1$. (iv) Hence show that for almost every $\gamma \in \mathbb{R}$, the set $\{(p, q) : p \in \mathbb{Z},\ q \ge 1,\ |\gamma - \frac{p}{q}| \le \epsilon/q^2\}$ is infinite for every $\epsilon > 0$. (v) Show that if $n \ge 1$, p, $q \in \mathbb{N}$ and $0 < q \le q_n(x)$, then $|x - \frac{p}{q}| \ge |x - r_n(x)|$, with equality only when $p = p_n(x)$ and $q = q_n(x)$.

(m) In 372M, let T_1 be the family $\{E :$ for every $n \in \mathbb{N}$ there is a measurable set $F \subseteq X$ such that $\phi^{-n}[F] = E\}$. Show that every member of T_1 is either negligible or conegligible. (*Hint*: the argument of part (c) of the proof of 372M still works.) Hence show that ϕ is mixing for the measure ν.

(n) Let $(\mathfrak{A}, \bar\mu)$ be an atomless probability algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is homogeneous; (ii) there is an ergodic measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$; (iii) there is a mixing measure-preserving automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$. (*Hint*: 333P.)

(o) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. (i) Show that if $n \ge 1$ then π is mixing iff π^n is mixing. (ii) Show that if $n \ge 1$ then π is weakly mixing iff π^n is weakly mixing. (iii) Show that if $n \ge 1$ and π^n is ergodic then π is ergodic. (iv) Show that if π is an automorphism then it is ergodic, or mixing, or weakly mixing, iff π^{-1} is.

>(p) Consider the **tent map** $\phi_\alpha(x) = \alpha \min(x, 1 - x)$ for $x \in [0, 1]$, $\alpha \in [0, 2]$. Show that ϕ_2 is inverse-measure-preserving and mixing for Lebesgue measure on $[0, 1]$. (*Hint*: show that $\phi_2^{n+1}(x) = \phi_2(<2^nx>)$ for $n \ge 1$, and hence that $\mu(I \cap \phi_2^{-n}[J]) = \mu I \cdot \mu J$ whenever I is of the form $[2^{-n}k, 2^{-n}(k + 1)]$ and J is an interval.)

(q) Consider the **logistic map** $\psi_\beta(x) = \beta x(1 - x)$ for $x \in [0, 1]$, $\beta \in [0, 4]$. Show that ψ_4 is inverse-measure-preserving and mixing for the Radon measure on $[0, 1]$ with density function $t \mapsto \frac{1}{\pi\sqrt{t(1-t)}}$. (*Hint*: show that the transformation $t \mapsto \sin^2 \frac{\pi t}{2}$ matches it with the tent map.) Show that for almost every x,

$$\lim_{n\to\infty} \frac{1}{n+1}\#(\{i : i \le n,\ \psi_4^i(x) \le \alpha\}) = \frac{2}{\pi} \arcsin \sqrt{\alpha}$$

for every $\alpha \in [0, 1]$.

(r) Let μ be Lebesgue measure on $[0, 1[$, and fix an irrational number $\alpha \in [0, 1[$. (i) Set $\phi(x) = x +_1 \alpha$ for every $x \in [0, 1[$, where $x +_1 \alpha$ is whichever of $x+\alpha$, $x+\alpha-1$ belongs to $[0, 1[$. Show that ϕ is inverse-measure-preserving. (ii)

Show that if $I \subseteq [0,1[$ is an interval then $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \chi I(\phi^i(x)) = \mu I$ for almost every $x \in [0,1[$. (*Hint*: this is Weyl's Equidistribution Theorem (281N).) (iii) Show that ϕ is ergodic. (*Hint*: take the conditional expectation operator P of 372G, and look at $P(\chi I^\bullet)$ for intervals I.) (iv) Show that ϕ^n is ergodic for any $n \in \mathbb{Z} \setminus \{0\}$. (v) Show that ϕ is not weakly mixing.

(s) Let p, $q \in [1,\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. (i) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a mixing measure-preserving homomorphism, and $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ the corresponding homomorphism. Show that $\lim_{n\to\infty} \int T^n u \times v = \int u \int v$ whenever $u \in L^p(\mathfrak{A},\bar{\mu})$ and $v \in L^q(\mathfrak{A},\bar{\mu})$. (*Hint*: start with u, $v \in S(\mathfrak{A})$.) (ii) Let (X,Σ,μ) be a probability space and $\phi : X \to X$ a mixing inverse-measure-preserving function. Show that $\lim_{n\to\infty} \int f(\phi^n(x))g(x)dx = \int f \int g$ whenever $f \in \mathcal{L}^p(\mu)$ and $g \in \mathcal{L}^q(\mu)$.

(t) Give $[0,1[$ Lebesgue measure μ, and let $k \ge 2$ be an integer. Define $\phi : [0,1[\to [0,1[$ by setting $\phi(x) = {<}kx{>}$, the fractional part of kx. Show that ϕ is inverse-measure-preserving. Show that ϕ is mixing. (*Hint*: if $I = [k^{-n}i, k^{-n}(i+1)[$, $J = [k^{-n}j, k^{-n}(j+1)[$ then $\mu(I \cap \phi^{-m}[J]) = \mu I \cdot \mu J$ for all $m \ge n$.)

(u) Let (X,Σ,μ) be a probability space and $\phi : X \to X$ an ergodic inverse-measure-preserving function. Let f be a μ-virtually measurable function defined almost everywhere in X such that $\int f d\mu = \infty$. Show that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f\phi^i$ is infinite a.e. (*Hint*: look at the corresponding limits for $f_m = f \wedge m\chi X$.)

(v) For irrational $x \in [0,1]$, write $k_1(x), k_2(x), \dots$ for the continued-fraction coefficients of x. Show that the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} k_i(x)$ is infinite for almost every x. (*Hint*: take ϕ, ν as in 372M, and show that $\int k_1 d\nu = \infty$.)

(w) Let (X,Σ,μ) be any probability space, and let λ be the product measure on $X^{\mathbb{N}}$. Define $\phi : X^{\mathbb{N}} \to X^{\mathbb{N}}$ by setting $\phi(x)(n) = x(n+1)$. Show that ϕ is inverse-measure-preserving. Show that ϕ satisfies the conditions of 372S, so is mixing.

(x) Let (X,Σ,μ) be any probability space, and λ the product measure on $X^{\mathbb{Z}}$. Define $\phi : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ by setting $\phi(x)(n) = x(n+1)$. Show that ϕ is inverse-measure-preserving. Show that ϕ is mixing. (*Hint*: show that if C, C' are basic cylinder sets then $\mu(C \cap \phi^{-n}[C']) = \mu C \cdot \mu C'$ for all n large enough.) Show that ϕ does not ordinarily satisfy the conditions of 372S. (Compare 333P.)

(y)(i) Let $\mathfrak{A}$ be a Boolean algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism, and $\phi : \mathfrak{A} \to \mathfrak{A}$ an automorphism. Show that if π is ergodic then $\phi\pi\phi^{-1}$ is ergodic. (ii) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism, and $\phi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean automorphism. Show that if π is mixing, or weakly mixing, then so is $\phi\pi\phi^{-1}$.

372Y Further exercises (a) In 372D, show that the null space of the limit operator P is precisely the closure in $M^{1,0}$ of the subspace $\{Tu - u : u \in M^{1,0}\}$.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, $T \in \mathcal{T}^{(0)}_{\bar{\mu},\bar{\mu}}$, $p \in]1,\infty[$ and $u \in L^p(\mathfrak{A},\bar{\mu})$. Set $u^* = \sup_{n\in\mathbb{N}} \frac{1}{n+1}\sum_{i=0}^{n} |T^i u|$. (i) Show that for any $\gamma > 0$,

$$\bar{\mu}[\![u^* > \gamma]\!] \le \frac{2}{\gamma}\int_{[\![|u|>\gamma/2]\!]} |u|.$$

(*Hint*: apply 372C to $(|u| - \frac{1}{2}\gamma\chi 1)^+$.) (ii) Show that $\|u^*\|_p \le 2(\frac{p}{p-1})^{1/p}\|u\|_p$. (*Hint*: show that $\int_{[\![|u|>\alpha]\!]} |u| = \alpha\bar{\mu}[\![|u| > \alpha]\!] + \int_\alpha^\infty \bar{\mu}[\![|u| > \beta]\!]d\beta$; see 365A. Use 366Xa to show that

$$\|u^*\|_p^p \le 2p\int_0^\infty \gamma^{p-2}\int_{\gamma/2}^\infty \bar{\mu}[\![|u| > \beta]\!]d\beta d\gamma + 2^p\|u\|_p^p,$$

and reverse the order of integration. Compare 275Yd.) (This is **Wiener's Dominated Ergodic Theorem**.)

(c) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and T an operator in $\mathcal{T}^{(0)}_{\bar{\mu},\bar{\mu}}$. Take $u \in L^1 = L^1(\mathfrak{A},\bar{\mu})$ such that $h(|u|) \in L^1$, where $h(t) = t\ln t$ for $t \ge 1$, 0 for $t \le 1$, and $\bar{h}$ is the corresponding function from $L^0(\mathfrak{A})$ to itself. Set $u^* = \sup_{n\in\mathbb{N}} \frac{1}{n+1}\sum_{i=0}^{n} |T^i u|$. Show that $u^* \in L^1$. (*Hint*: use the method of 372Yb to show that $\int_2^\infty \bar{\mu}[\![u^* > \gamma]\!]d\gamma \le 2\int \bar{h}(u)$.)

(d) Let U be a Banach space, $(\mathfrak{A},\bar{\mu})$ a semi-finite measure algebra and $\langle T_n\rangle_{n\in\mathbb{N}}$ a sequence of continuous linear operators from U to $L^0 = L^0(\mathfrak{A})$ with its topology of convergence in measure. Suppose that $\sup_{n\in\mathbb{N}} T_n u$ is defined in L^0 for every $u \in U$. Show that $\{u : u \in U,\ \langle T_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent in $L^0\}$ is a norm-closed linear subspace of U.

(e) In 372G, suppose that $\mathfrak{A}$ is atomless. Show that there is always an $a \in \mathfrak{A}$ such that $\bar{\mu}a \le \frac{1}{2}$ and $\inf_{i\le n}\pi^i a \ne 0$ for every n, so that (except in trivial cases) $\langle A_n(\chi a)\rangle_{n\in\mathbb{N}}$ will not be $\|\ \|_\infty$-convergent.

(f) Let (X,Σ,μ) be a measure space with measure algebra $(\mathfrak{A},\bar{\mu})$. Let Φ be a family of inverse-measure-preserving functions from X to itself, and for $\phi \in \Phi$ let $\pi_\phi : \mathfrak{A} \to \mathfrak{A}$ be the associated homomorphism. Set $\mathfrak{C} = \{c : c \in \mathfrak{A},\ \pi_\phi c = c$ for every $\phi \in \Phi\}$, $\mathrm{T} = \{E : E \in \Sigma,\ \phi^{-1}[E]\triangle E$ is negligible for every $\phi \in \Phi\}$ and $\mathrm{T}_0 = \{E : E \in \Sigma,\ \phi^{-1}[E] = E$ for every $\phi \in \Phi\}$. Show that (i) T and T_0 are σ-subalgebras of Σ (ii) $\mathrm{T}_0 \subseteq \mathrm{T}$ (iii) $\mathrm{T} = \{E : E \in \Sigma,\ E^\bullet \in \mathfrak{C}\}$ (iv) if Φ is countable and $\phi\psi = \psi\phi$ for all ϕ, $\psi \in \Phi$, then $\mathfrak{C} = \{E^\bullet : E \in \mathrm{T}_0\}$.

(g) Show that an irrational $x \in\ ]0,1[$ has an eventually periodic sequence of continued fraction coefficients iff it is a solution of a quadratic equation with integral coefficients.

(h) In the language of 372L-372N and 372Xl, show the following. (i) For any $x \in X$, $n \ge 2$, $q_n(x)q_{n-1}(x) \ge 2^{n-1}$, $p_n(x)p_{n+1}(x) \ge 2^{n-1}$, so that $q_{n+1}(x)p_n(x) \ge 2^{n-1}$ and $|1 - x/r_n(x)| \le 2^{-n+1}$, $|\ln x - \ln r_n(x)| \le 2^{-n+2}$. Also $|x - r_n(x)| \ge 1/q_n(x)q_{n+2}(x)$. (ii) For any $x \in X$, $n \ge 1$, $p_{n+1}(x) = q_n(\phi(x))$ and $q_n(x)\prod_{i=0}^{n-1} r_{n-i}(\phi^i(x)) = 1$. (iii) For any $x \in X$, $n \ge 1$, $|\ln q_n(x) + \sum_{i=0}^{n-1}\ln\phi^i(x)| \le 4$. (iv) For almost every $x \in X$,

$$\lim_{n\to\infty}\frac{1}{n}\ln q_n(x) = -\frac{1}{\ln 2}\int_0^1 \frac{\ln t}{1+t}dt = \frac{\pi^2}{12\ln 2}.$$

(*Hint*: 225Xi, 282Xo.) (v) For almost every $x \in X$, $\lim_{n\to\infty}\frac{1}{n}\ln|x - r_n(x)| = -\frac{\pi^2}{6\ln 2}$. (vi) For almost every $x \in X$, $11^{-n} \le |x - r_n(x)| \le 10^{-n}$ and $3^n \le q_n(x) \le 4^n$ for all but finitely many n.

(i)(i) Let (X,Σ,μ) and (Y,T,ν) be probability spaces, with c.l.d. product $(X\times Y,\Lambda,\lambda)$. Suppose that $\phi : X \to X$ is a weakly mixing inverse-measure-preserving function and $\psi : Y \to Y$ is an ergodic inverse-measure-preserving function. Define $\theta : X\times Y \to X\times Y$ by setting $\theta(x,y) = (\phi(x),\psi(y))$ for all x, y. Show that θ is an ergodic inverse-measure-preserving function. (ii) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be probability algebras, with probability algebra free product $(\mathfrak{C},\bar{\lambda})$. Suppose that $\phi : \mathfrak{A} \to \mathfrak{A}$ is a weakly mixing measure-preserving Boolean homomorphism and $\psi : \mathfrak{B} \to \mathfrak{B}$ is an ergodic measure-preserving Boolean homomorphism. Let $\theta : \mathfrak{C} \to \mathfrak{C}$ be the measure-preserving Boolean homomorphism such that $\theta(a\otimes b) = \phi a \otimes \psi b$ for all $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ (325Xe). Show that θ is ergodic.

(j) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any family of probability spaces, with product (X,Λ,λ). Suppose that for each $i \in I$ we are given an inverse-measure-preserving function $\phi_i : X_i \to X_i$. (i) Show that there is a corresponding inverse-measure-preserving function $\phi : X \to X$ given by setting $\phi(x)(i) = \phi_i(x(i))$ for $x \in X$, $i \in I$. (ii) Show that if every ϕ_i is mixing so is ϕ. (iii) Show that if every ϕ_i is weakly mixing so is ϕ.

(k) Give an example of an ergodic measure-preserving automorphism $\phi : [0,1[\ \to [0,1[$ such that ϕ^2 is not ergodic. (*Hint*: set $\phi(x) = \frac{1}{2}(1 + \phi_0(2x))$ for $x < \frac{1}{2}$, $x - \frac{1}{2}$ for $x \ge \frac{1}{2}$. See also 388Xe.)

(l) Show that there is an ergodic $\phi : [0,1] \to [0,1]$ such that $(\xi_1,\xi_2) \mapsto (\phi(\xi_1),\phi(\xi_2)) : [0,1]^2 \to [0,1]^2$ is not ergodic. (*Hint*: 372Xr.)

(m) Let M be an $r\times r$ matrix with integer coefficients and non-zero determinant, where $r \ge 1$. Let $\phi : [0,1[^r \to [0,1[^r$ be the function such that $\phi(x) - Mx \in \mathbb{Z}^r$ for every $x \in [0,1[^r$. Show that ϕ is inverse-measure-preserving for Lebesgue measure on $[0,1[^r$.

(n)(i) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a weakly mixing measure-preserving Boolean homomorphism, and $T = T_\pi : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}}$ the corresponding linear operator (365O). Show that if $u \in L^1_{\bar{\mu}}$ is such that $\{T^n u : n \in \mathbb{N}\}$ is relatively compact for the norm topology, then $u = \alpha\chi 1$ for some α. (ii) Let μ be Lebesgue measure on $[0,1[$, $(\mathfrak{A},\bar{\mu})$ its measure algebra, $\alpha \in [0,1[$ an irrational number, $\phi(x) = x +_1 \alpha$ for $x \in [0,1[$ (as in 372Xr), and $T : L^1(\mu) \to L^1(\mu)$ the linear operator defined by setting $Tg^\bullet = (g\phi)^\bullet$ for $g \in \mathcal{L}^1(\mu)$. Show that $\{T^n : n \in \mathbb{Z}\}$ is relatively compact for the strong operator topology on $\mathsf{B}(L^1(\mu);L^1(\mu))$.

(o) In 372M, show that for any measurable set $E \subseteq X$, $\lim_{n\to\infty} \mu_L \phi^{-n}[E] = \nu E$. (*Hint*: recall that ϕ is mixing for ν (372Xm). Hence show that $\lim_{n\to\infty} \int_{\phi^{-n}[E]} g\,d\nu = \nu E \cdot \int g\,d\nu$ for any integrable g. Apply this to a Radon-Nikodým derivative of μ_L with respect to ν.) (I understand that this result is due to Gauss.)

(p) (i) Show that there are a Boolean algebra $\mathfrak{A}$ and an automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ which is not ergodic, but has fixed-point algebra $\{0, 1\}$. (ii) Show that there are a σ-finite measure algebra $(\mathfrak{A}, \bar{\mu})$ and a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ which is not ergodic, but has fixed-point algebra $\{0, 1\}$.

(q) For a Boolean algebra $\mathfrak{A}$ and a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$, write T_π for the corresponding operator from $L^\infty(\mathfrak{A})$ to itself, as defined in 363F. (i) Suppose that $\mathfrak{A}$ is a Boolean algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism, $u \in L^\infty(\mathfrak{A})$ and $T_\pi u = u$. Show that if either π is ergodic or $\mathfrak{A}$ is Dedekind σ-complete and the fixed-point subalgebra of π is $\{0, 1\}$, then u must be a multiple of $\chi 1$. (ii) Find a Boolean algebra $\mathfrak{A}$, an automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ with fixed-point algebra $\{0, 1\}$, and a $u \in L^\infty(\mathfrak{A})$, not a multiple of $\chi 1$, such that $T_\pi u = u$.

(r) Set $\mathcal{F}_d = \{I : I \subseteq \mathbb{N},\ \lim_{n\to\infty} \frac{1}{n}\#(I \cap n) = 1\}$. (i) Show that $\mathcal{F}_d$ is a filter on $\mathbb{N}$. (ii) Show that for a bounded sequence $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$, the following are equiveridical: (α) $\lim_{n\to\mathcal{F}_d} \alpha_n = 0$; ($\beta$) $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |\alpha_k| = 0$; ($\gamma$) $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \alpha_k^2 = 0$. ($\mathcal{F}_d$ is called the **(asymptotic) density filter**.)

(s) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\phi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. (i) Show that there are a probability algebra $(\mathfrak{C}, \bar{\lambda})$, a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$, and a measure-preserving automorphism $\tilde{\phi} : \mathfrak{C} \to \mathfrak{C}$ such that $\tilde{\phi}\pi = \pi\phi$ and $\mathfrak{C}$ is the closure of $\bigcup_{n\in\mathbb{N}} \tilde{\phi}^{-n}[\pi[\mathfrak{A}]]$ for the measure-algebra topology. (*Hint*: 328J.) (ii) Show that $\tilde{\phi}$ is ergodic, or mixing, or weakly mixing iff ϕ is.

372 Notes and comments I have chosen an entirely conventional route to the Ergodic Theorem here, through the Mean Ergodic Theorem (372Xa) or, rather, the fundamental lemma underlying it (372A), and the Maximal Ergodic Theorem (372B-372C). What is not to be found in every presentation is the generality here. I speak of arbitrary $T \in \mathcal{T}^{(0)}$, the operators which are contractions both for $\| \|_1$ and for $\| \|_\infty$, not requiring T to be positive, let alone correspond to a measure-preserving homomorphism. (I do not mention $\mathcal{T}^{(0)}$ in the statement of 372C, but of course it is present in spirit.) The work we have done up to this point puts this extra generality within easy reach, but as the rest of the section shows, it is not needed for the principal examples. Only in 372Xc do I offer an application not associated in the usual way with a measure-preserving homomorphism or an inverse-measure-preserving function.

The Ergodic Theorem is an 'almost-everywhere pointwise convergence theorem', like the strong law(s) of large numbers and the martingale theorem(s) (§273, §275). Indeed Kolmogorov's form of the strong law can be derived from the Ergodic Theorem (372Xg). There are some very strong family resemblances. For instance, the Maximal Ergodic Theorem corresponds to the most basic of all the martingale inequalities (275D). Consequently we have similar results, obtained by similar methods, concerning the domination of sequences starting from members of L^p (372Yb, 275Yd), though the inequalities are not identical. (Compare also 372Yc with 275Ye.) There are some tantalising reflections of these traits in results surrounding Carleson's theorem on the pointwise convergence of square-integrable Fourier series (see §286 *notes*), but Carleson's theorem seems to be much harder than the others. Other forms of the strong law (273D, 273H) do not appear to fit into quite the same pattern, but I note that here, as with the Ergodic Theorem, we begin with a study of square-integrable functions (see part (e) of the proof of 372D).

After 372D, there is a contraction and concentration in the scope of the results, starting with a simple replacement of $M^{1,0}$ with L^1 (372F). Of course it is almost as easy to prove 372D from 372F as the other way about; I give precedence to 372D only because $M^{1,0}$ is the space naturally associated with the class $\mathcal{T}^{(0)}$ of operators to which these methods apply. Following this I turn to the special family of operators to which the rest of the section is devoted, those associated with measure-preserving homomorphisms (372E), generally on probability spaces (372G). This is the point at which we can begin to identify the limit as a conditional expectation as well as an invariant element.

Next comes the translation into the language of measure spaces and inverse-measure-preserving functions, all perfectly straightforward in view of 372I. These turn 372E into 372H and 372G into the main part of 372J.

In 372J-372K I find myself writing at some length about a technical problem. The root of the difficulty is in the definition of 'conditional expectation'. Now it is generally accepted that any pure mathematician has 'Humpty Dumpty's privilege': 'When *I* use a word, it means just what I choose it to mean – neither more nor less'. With any privilege come duties and responsibilities; here, the duty to be self-consistent, and the responsibility to try to use terms in ways which will not mystify or mislead the unprepared reader. Having written down a definition of 'conditional expectation' in Volume 2, I must either stick to it, or go back and change it, or very carefully explain exactly what modification I wish to make here. I don't wish to suggest that absolute consistency – in terminology or anything else – is supreme among mathematical virtues. Surely it is better to give local meanings to words, or tolerate ambiguities, than to suppress ideas which cannot be formulated effectively otherwise, and among 'ideas' I wish to include the analogies and resonances which a suitable language can suggest. But I do say that it is always best to be conscious of what one is doing – I go farther: one of the things which mathematics is for, is to raise our consciousness of what our thoughts really are. So I believe it is right to pause occasionally over such questions.

In 372L-372N (see also 372Xl, 372Xv, 372Xm, 372Xk, 372Yh, 372Yo) I make an excursion into number theory. This is a remarkable example of the power of advanced measure theory to give striking results in other branches of mathematics. Everything here is derived from Billingsley 65, who goes farther than I have space for, and gives references to more. Here let me point to 372Xj; almost accidentally, the construction offers a useful formula for a homeomorphism between two of the most important spaces of descriptive set theory, which will be important to us in Volume 4.

I end the section by introducing three terms, 'ergodic', 'mixing' and 'weakly mixing' transformations, not because I wish to use them for any new ideas (apart from the elementary 372P-372S, these must wait for §§385-387 below and §494 in Volume 4), but because it may help if I immediately classify some of the inverse-measure-preserving functions we have seen (372Xp-372Xr, 372Xt, 372Xw, 372Xx). Of course in any application of any ergodic theorem it is of great importance to be able to identify the limits promised by the theorem, and the point about an ergodic transformation is just that our averages converge to constant limits (372Q). Actually proving that a given inverse-measure-preserving function is ergodic is rarely quite trivial (see 372M, 372Xq, 372Xr), though a handful of standard techniques cover a large number of cases, and it is usually obvious when a map is *not* ergodic, so that if an invariant region does not leap to the eye one has a good hope of ergodicity. The extra concept of 'weakly mixing' transformation is hardly relevant to anything in this volume (though see 372Yi-372Yj), but is associated with a remarkable topological fact about automorphism groups of probability algebras, to come in 494E.

I ought to remark on the odd shift between the definitions of 'ergodic Boolean homomorphism' and 'ergodic inverse-measure-preserving function' in 372O. The point is that the version in 372O(b-i) is the standard formulation in this context, but that its natural translation into the version 'a Boolean homomorphism from a probability algebra to itself is ergodic if its fixed-point subalgebra is trivial', although perfectly satisfactory in that context, allows unwelcome phenomena if applied to general Boolean algebras (372Yp, 372Yq). The definition in 372O(a-i) is rather closer to the essential idea of ergodicity of a dynamical system, which asks that the system should always evolve along a path which approximates all possible states. In practice, however, we shall nearly always be dealing with automorphisms of Dedekind σ-complete algebras, for which we can use the fixed-point criterion of 372Pc.

I take the opportunity to mention two famous functions from $[0,1]$ to itself, the 'tent' and 'logistic' maps (372Xp, 372Xq). In the formulae ϕ_α, ψ_β I include redundant parameters; this is because the real importance of these functions lies in the way their behaviour depends, in bewildering complexity, on these parameters. It is only at the extreme values $\alpha=2$, $\beta=4$ that the methods of this volume can tell us anything interesting.

373 Decreasing rearrangements

I take a section to discuss operators in the class $\mathcal{T}^{(0)}$ of 371F-371H and §372 and two associated classes $\mathcal{T}$, $\mathcal{T}^\times$ (373A). These turn out to be intimately related to the idea of 'decreasing rearrangement' (373C). In 373D-373F I give elementary properties of decreasing rearrangements; then in 373G-373O I show how they may be used to characterize the set $\{Tu : T \in \mathcal{T}\}$ for a given u. The argument uses a natural topology on the set $\mathcal{T}$ (373K). I conclude with remarks on the possible values of $\int Tu \times v$ for $T \in \mathcal{T}$ (373P-373Q) and identifications between $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$, $\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ and $\mathcal{T}^\times_{\bar\mu,\bar\nu}$ (373R-373T).

373A Definition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. Recall that $M^{1,\infty}(\mathfrak{A},\bar\mu)=L^1(\mathfrak{A},\bar\mu)+L^\infty(\mathfrak{A})$ is the set of those $u\in L^0(\mathfrak{A})$ such that $(|u|-\alpha\chi1)^+$ is integrable for some α, its norm $\| \|_{1,\infty}$ being defined by the formulae

$$\begin{aligned}\|u\|_{1,\infty}&=\min\{\|v\|_1+\|w\|_\infty : v\in L^1,\,w\in L^\infty,\,v+w=u\}\\&=\min\{\alpha+\int(|u|-\alpha\chi1)^+ : \alpha\ge0\}\end{aligned}$$

(369Ob).

(a) $\mathcal{T}_{\bar\mu,\bar\nu}$ will be the space of linear operators $T : M^{1,\infty}(\mathfrak{A},\bar\mu) \to M^{1,\infty}(\mathfrak{B},\bar\nu)$ such that $\|Tu\|_1 \le \|u\|_1$ for every $u \in L^1(\mathfrak{A},\bar\mu)$ and $\|Tu\|_\infty \le \|u\|_\infty$ for every $u \in L^\infty(\mathfrak{A})$. (Compare the definition of $\mathcal{T}^{(0)}$ in 371F.)

(b) If $\mathfrak{B}$ is Dedekind complete, so that $M^{1,\infty}(\mathfrak{A},\bar\mu)$, being a solid linear subspace of the Dedekind complete space $L^0(\mathfrak{B})$, is Dedekind complete, $\mathcal{T}^\times_{\bar\mu,\bar\nu}$ will be $\mathcal{T}_{\bar\mu,\bar\nu} \cap \mathsf{L}^\times(M^{1,\infty}(\mathfrak{A},\bar\mu); M^{1,\infty}(\mathfrak{A},\bar\mu))$.

373B Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras.

(a) $\mathcal{T} = \mathcal{T}_{\bar\mu,\bar\nu}$ is a convex set in the unit ball of $\mathsf{B}(M^{1,\infty}(\mathfrak{A},\bar\mu); M^{1,\infty}(\mathfrak{B},\bar\nu))$.

(b) If $T \in \mathcal{T}$ then $T{\restriction}M^{1,0}(\mathfrak{A},\bar\mu)$ belongs to $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$, as defined in 371F. So if $T \in \mathcal{T}$, $p \in [1,\infty[$ and $u \in L^p(\mathfrak{A},\bar\mu)$ then $Tu \in L^p(\mathfrak{B},\bar\nu)$ and $\|Tu\|_p \le \|u\|_p$.

(c) If $\mathfrak{B}$ is Dedekind complete and $T \in \mathcal{T}$, then $T \in \mathsf{L}^\sim(M^{1,\infty}(\mathfrak{A},\bar\mu); M^{1,\infty}(\mathfrak{B},\bar\nu))$ and $T_1 \in \mathcal{T}$ whenever $T_1 \in \mathsf{L}^\sim(M^{1,\infty}(\mathfrak{A},\bar\mu); M^{1,\infty}(\mathfrak{B},\bar\nu))$ and $|T_1| \le |T|$; in particular, $|T| \in \mathcal{T}$.

(d) If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a measure-preserving Boolean homomorphism, then we have a corresponding operator $T \in \mathcal{T}$ defined by saying that $T(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$. If π is order-continuous, then so is T.

(e) If $(\mathfrak{C},\bar\lambda)$ is another measure algebra and $T \in \mathcal{T}$, $S \in \mathcal{T}_{\bar\nu,\bar\lambda}$ then $ST \in \mathcal{T}_{\bar\mu,\bar\lambda}$.

proof (a) As 371G, parts (a-i) and (a-ii) of the proof.

(b) If $u \in M^{1,0}_{\bar\mu}$ and $\epsilon > 0$, then u is expressible as $u' + u''$ where $\|u''\|_\infty \le \epsilon$ and $u' \in L^1_{\bar\mu}$. (Set

$$u'' = (u^+ \wedge \epsilon\chi 1) - (u^- \wedge \epsilon\chi 1).)$$

So

$$(|Tu| - \epsilon\chi 1)^+ \le |Tu - Tu''| \in L^1_{\bar\nu}.$$

As ϵ is arbitrary, $Tu \in M^{1,0}_{\bar\nu}$; as u is arbitrary, $T{\restriction}M^{1,0}_{\bar\mu} \in \mathcal{T}^{(0)}$. Now the rest is a consequence of 371Gd.

(c) Because $M^{1,\infty}_{\bar\nu}$ is a solid linear subspace of $L^0(\mathfrak{B})$, which is Dedekind complete because $\mathfrak{B}$ is, $\mathsf{L}^\sim(M^{1,\infty}_{\bar\mu}; M^{1,\infty}_{\bar\nu})$ is a Riesz space (355Ea).

Take any $u \ge 0$ in $M^{1,\infty}_{\bar\mu}$. Let $\alpha \ge 0$ be such that $(u - \alpha\chi 1)^+ \in L^1_{\bar\mu}$. Because $T{\restriction}L^1_{\bar\mu}$ belongs to $\mathsf{B}(L^1_{\bar\mu}; L^1_{\bar\nu}) = \mathsf{L}^\sim(L^1_{\bar\mu}; L^1_{\bar\nu})$ (371D), $w_0 = \sup\{Tv : v \in L^1_{\bar\mu},\ 0 \le v \le (u - \alpha\chi 1)^+\}$ is defined in $L^1_{\bar\nu}$. Now if $v \in M^{1,\infty}_{\bar\mu}$ and $0 \le v \le u$, we must have

$$Tv = T(v - \alpha\chi 1)^+ + T(v \wedge \alpha\chi 1) \le w_0 + \alpha\chi 1 \in M^{1,\infty}_{\bar\nu}.$$

Thus $\{Tv : 0 \le v \le u\}$ is bounded above in $M^{1,\infty}_{\bar\nu}$. As u is arbitrary, $T \in \mathsf{L}^\sim(M^{1,\infty}_{\bar\mu}; M^{1,\infty}_{\bar\nu})$ (355Ba).

Now take T_1 such that $|T_1| \le |T|$ in $\mathsf{L}^\sim(M^{1,\infty}_{\bar\mu}; M^{1,\infty}_{\bar\nu})$. 371D also tells us that $\||T{\restriction}L^1_{\bar\mu}|\| = \|T{\restriction}L^1_{\bar\mu}\|$, so that

$$\begin{aligned}\|T_1 u\|_1 &= \||T_1 u|\|_1 \le \||T_1||u|\|_1 \le \||T||u|\|_1\\ &= \|\sup_{|v|\le|u|} Tv\|_1 \le \|\sup_{|v|\le|u|} v\|_1 = \|u\|_1\end{aligned}$$

for every $u \in L^1_{\bar\mu}$ (using a formula in 355Eb for the first equality). At the same time, if $u \in L^\infty(\mathfrak{A})$, then

$$\begin{aligned}|T_1 u| &\le |T_1||u| \le |T||u| = \sup_{|v|\le|u|} Tv\\ &\le \sup_{|v|\le|u|} \|Tv\|_\infty \chi 1 \le \sup_{|v|\le|u|} \|v\|_\infty \chi 1 = \|u\|_\infty \chi 1,\end{aligned}$$

so $\|T_1 u\|_\infty \le \|u\|_\infty$. Thus $T_1 \in \mathcal{T}$.

(d) By 365O and 363F, we have norm-preserving positive linear operators $T_1 : L^1_{\bar\mu} \to L^1_{\bar\nu}$ and $T_\infty : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$ defined by saying that $T_1(\chi a) = \chi(\pi a)$ whenever $\bar\mu a < \infty$ and $T_\infty(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$. If $u \in S(\mathfrak{A}^f) = L^1_{\bar\mu} \cap S(\mathfrak{A})$ (365F), then $T_1 u = T_\infty u$, because both T_1 and T_∞ are linear and they agree on $\{\chi a : \bar\mu a < \infty\}$. If $u \ge 0$ in $M^{\infty,1}_{\bar\mu} = L^1_{\bar\mu} \cap L^\infty(\mathfrak{A})$, there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A}^f)$ such that $u = \sup_{n\in\mathbb{N}} u_n$ and

$$\lim_{n\to\infty} \|u - u_n\|_1 = \lim_{n\to\infty} \|u - u_n\|_\infty = 0$$

(see the proof of 369Od), so that

$$T_1 u = \sup_{n\in\mathbb{N}} T_1 u_n = \sup_{n\in\mathbb{N}} T_\infty u_n = T_\infty u.$$

Accordingly T_1 and T_∞ agree on $L^1_{\bar\mu}\cap L^\infty(\mathfrak A)$. But this means that if $u\in M^{1,\infty}_{\bar\mu}$ is expressed as $v+w=v'+w'$, where v, $v'\in L^1_{\bar\mu}$ and w, $w'\in L^\infty(\mathfrak A)$, we shall have

$$T_1v'+T_\infty w'=T_1v+T_\infty w+T_1(v'-v)-T_\infty(w-w')=T_1v+T_\infty w,$$

because $v'-v=w-w'\in M^{\infty,1}_{\bar\mu}$. Accordingly we have an operator $T:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$ defined by setting

$$T(v+w)=T_1v+T_\infty w \text{ whenever } v\in L^1_{\bar\mu},\ w\in L^\infty(\mathfrak A).$$

This formula makes it easy to check that T is linear and positive, and it clearly belongs to $\mathcal T$.

To see that T is uniquely defined, observe that $T\restriction L^1_{\bar\mu}$ and $T\restriction L^\infty(\mathfrak A)$ are uniquely defined by the values T takes on $S(\mathfrak A^f)$, $S(\mathfrak A)$ respectively, because these spaces are dense for the appropriate norms.

Now suppose that π is order-continuous. Then T_1 and T_∞ are also order-continuous (365Oa, 363Ff). If $A\subseteq M^{1,\infty}_{\bar\mu}$ is non-empty and downwards-directed and has infimum 0, take $u_0\in A$ and $\gamma>0$ such that $(u_0-\gamma\chi1)^+\in L^1_{\bar\mu}$. Set

$$A_1=\{(u-\gamma\chi1)^+:u\in A,\ u\le u_0\},\quad A_\infty=\{u\wedge\gamma\chi1:u\in A\}.$$

Then $A_1\subseteq L^1_{\bar\mu}$ and $A_\infty\subseteq L^\infty(\mathfrak A)$ are both downwards-directed and have infimum 0, so $\inf T_1[A_1]=\inf T_\infty[A_\infty]=0$ in $L^0(\mathfrak B)$. But this means that $\inf(T_1[A_1]+T_\infty[A_\infty])=0$ (351Dc). Now any $w\in T_1[A_1]+T_\infty[A_\infty]$ is expressible as $T(u-\gamma\chi1)^++T(u'\wedge\gamma\chi1)$ where u, $u'\in A$; because A is downwards-directed, there is a $v\in A$ such that $v\le u\wedge u'$, in which case $Tv\le w$. Accordingly $T[A]$ must also have infimum 0. As A is arbitrary, T is order-continuous.

(e) is obvious, as usual.

373C Decreasing rearrangements The following concept is fundamental to any understanding of the class $\mathcal T$. Let $(\mathfrak A,\bar\mu)$ be a measure algebra. Write $M^{0,\infty}_{\bar\mu}=M^{0,\infty}(\mathfrak A,\bar\mu)$ for the set of those $u\in L^0(\mathfrak A)$ such that $\bar\mu[\![|u|>\alpha]\!]$ is finite for some $\alpha\in\mathbb R$. (See 369N for the ideology of this notation.) It is easy to see that $M^{0,\infty}(\mathfrak A,\bar\mu)$ is a solid linear subspace of $L^0(\mathfrak A)$. Let $(\mathfrak A_L,\bar\mu_L)$ be the measure algebra of Lebesgue measure on $[0,\infty[$. For $u\in M^{0,\infty}(\mathfrak A,\bar\mu)$ its **decreasing rearrangement** is $u^*\in M^{0,\infty}(\mathfrak A_L,\bar\mu_L)$, defined by setting $u^*=g^\bullet$, where

$$g(t)=\inf\{\alpha:\alpha\ge0,\ \bar\mu[\![|u|>\alpha]\!]\le t\}$$

for every $t>0$. (This is always finite because $\inf_{\alpha\in\mathbb R}\bar\mu[\![|u|>\alpha]\!]=0$, by 364Aa($\beta$) and 321F.)

I will maintain this usage of the symbols $\mathfrak A_L$, $\bar\mu_L$, u^* for the rest of this section.

373D Lemma Let $(\mathfrak A,\bar\mu)$ be a measure algebra.

(a) For any $u\in M^{0,\infty}(\mathfrak A,\bar\mu)$, its decreasing rearrangement u^* may be defined by the formula

$$[\![u^*>\alpha]\!]=[0,\bar\mu[\![|u|>\alpha]\!][^\bullet \text{ for every } \alpha\ge0,$$

that is,

$$\bar\mu_L[\![u^*>\alpha]\!]=\bar\mu[\![|u|>\alpha]\!] \text{ for every } \alpha\ge0.$$

(b) If $|u|\le|v|$ in $M^{0,\infty}(\mathfrak A,\bar\mu)$, then $u^*\le v^*$; in particular, $|u|^*=u^*$.

(c)(i) If $u=\sum_{i=0}^n\alpha_i\chi a_i$, where $a_0\supseteq a_1\supseteq\ldots\supseteq a_n$ and $\alpha_i\ge0$ for each i, then $u^*=\sum_{i=0}^n\alpha_i\chi\,[0,\bar\mu a_i[^\bullet$.

(ii) If $u=\sum_{i=0}^n\alpha_i\chi a_i$ where $a_0,\ldots,a_n$ are disjoint and $|\alpha_0|\ge|\alpha_1|\ge\ldots\ge|\alpha_n|$, then $u^*=\sum_{i=0}^n|\alpha_i|\chi\,[\beta_i,\beta_{i+1}[^\bullet$, where $\beta_i=\sum_{j<i}\bar\mu a_i$ for $i\le n+1$.

(d) If $E\subseteq\,]0,\infty[$ is any Borel set, and $u\in M^0(\mathfrak A,\bar\mu)$, then $\bar\mu_L[\![u^*\in E]\!]=\bar\mu[\![|u|\in E]\!]$.

(e) Let $h:[0,\infty[\,\to[0,\infty[$ be a non-decreasing function such that $h(0)=0$, and write $\bar h$ for the corresponding functions on $L^0(\mathfrak A)^+$ and $L^0(\mathfrak A_L)^+$ (364H). Then $(\bar h(u))^*=\bar h(u^*)$ whenever $u\ge0$ in $M^0(\mathfrak A,\bar\mu)$. If h is continuous on the left, $(\bar h(u))^*=\bar h(u^*)$ whenever $u\ge0$ in $M^{0,\infty}(\mathfrak A,\bar\mu)$.

(f) If $u\in M^{0,\infty}(\mathfrak A,\bar\mu)$ and $\alpha\ge0$, then

$$(u^*-\alpha\chi1)^+=((|u|-\alpha\chi1)^+)^*.$$

(g) If $u\in M^{0,\infty}(\mathfrak A,\bar\mu)$, then for any $t>0$

$$\int_0^t u^*=\inf_{\alpha\ge0}\alpha t+\int(|u|-\alpha\chi1)^+.$$

(h) If $A\subseteq(M^{0,\infty}(\mathfrak A,\bar\mu))^+$ is non-empty and upwards-directed and has supremum $u_0\in M^{0,\infty}(\mathfrak A,\bar\mu)$, then $u_0^*=\sup_{u\in A}u^*$.

proof (a) Set

$$g(t)=\inf\{\alpha:\bar\mu[\![|u|>\alpha]\!]\le t\}$$

as in 373C. If $\alpha\ge 0$,

$$g(t)>\alpha\iff\bar\mu[\![|u|>\beta]\!]>t\text{ for some }\beta>\alpha\iff\bar\mu[\![|u|>\alpha]\!]>t$$

(because $[\![|u|>\alpha]\!]=\sup_{\beta>\alpha}[\![|u|>\beta]\!]$), so

$$[\![u^*>\alpha]\!]=\{t:g(t)>\alpha\}^\bullet=[0,\bar\mu[\![|u|>\alpha]\!][^\bullet.$$

Of course this formula defines u^*.

(b) This is obvious, either from the definition in 373C or from (a) just above.

(c)(i) Setting $v=\sum_{i=0}^n\alpha_i\chi\,[0,\bar\mu a_i[^\bullet$, we have

$$\begin{aligned}[\![v>\alpha]\!]&=0\text{ if }\sum_{i=0}^n\alpha_i\le\alpha,\\&=[0,\bar\mu a_j[^\bullet\text{ if }\sum_{i=0}^{j-1}\alpha_i\le\alpha<\sum_{i=0}^{j}\alpha_i,\\&=[0,\bar\mu a_0[^\bullet\text{ if }0\le\alpha<\alpha_0,\end{aligned}$$

and in all cases is equal to $[0,\bar\mu[\![|u|>\alpha]\!][^\bullet$.

(ii) A similar argument applies. (If any a_j has infinite measure, then a_i is irrelevant for $i>j$.)

(d) Fix $\gamma>0$ for the moment, and consider

$$\mathcal{A}=\{E:E\subseteq\,]\gamma,\infty[\text{ is a Borel set, }\bar\mu_L[\![u^*\in E]\!]=\bar\mu[\![|u|\in E]\!]\},$$

$$\mathcal{I}=\{]\alpha,\infty[\,:\alpha\ge\gamma\}.$$

Then $\mathcal{I}\subseteq\mathcal{A}$ (by (a)), $I\cap J\in\mathcal{I}$ for all I, $J\in\mathcal{I}$, $E\setminus F\in\mathcal{A}$ whenever E, $F\in\mathcal{A}$ and $F\subseteq E$ (because $u\in M^0_{\bar\mu}$, so $\bar\mu[\![|u|\in E]\!]<\infty$), and $\bigcup_{n\in\mathbb{N}}E_n\in\mathcal{A}$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$. So, by the Monotone Class Theorem (136B), $\mathcal{A}$ includes the σ-algebra of subsets of $]\gamma,\infty[$ generated by $\mathcal{I}$; but this must contain $E\cap]\gamma,\infty[$ for every Borel set $E\subseteq\mathbb{R}$.

Accordingly, for any Borel set $E\subseteq\,]0,\infty[$,

$$\bar\mu_L[\![u^*\in E]\!]=\sup_{n\in\mathbb{N}}\bar\mu_L[\![u^*\in E\cap\,]2^{-n},\infty[\,]\!]=\bar\mu[\![|u|\in E]\!].$$

(e) For any $\alpha>0$, $E_\alpha=\{t:h(t)>\alpha\}$ is a Borel subset of $]0,\infty[$. If $u\in M^0_{\bar\mu}$ then, using (d) above,

$$\bar\mu_L[\![\bar h(u^*)>\alpha]\!]=\bar\mu_L[\![u^*\in E_\alpha]\!]=\bar\mu[\![u\in E_\alpha]\!]=\bar\mu[\![\bar h(u)>\alpha]\!]=\bar\mu_L[\![(\bar h(u))^*>\alpha]\!].$$

As both $(\bar h(u))^*$ and $\bar h(u^*)$ are equivalence classes of non-increasing functions, they must be equal.

If h is continuous on the left, then $E_\alpha=\,]\gamma,\infty[$ for some γ, so we no longer need to use (d), and the argument works for any $u\in(M^{0,\infty}_{\bar\mu})^+$.

(f) Apply (e) with $h(\beta)=\max(0,\beta-\alpha)$.

(g) Express u^* as $g^\bullet$, where

$$g(s)=\inf\{\alpha:\bar\mu[\![|u|>\alpha]\!]\le s\}$$

for every $s>0$. Because g is non-increasing, it is easy to check that, for $t>0$,

$$\int_0^t g=tg(t)+\int_0^\infty\max(0,g(s)-g(t))ds\le\alpha t+\int_0^\infty\max(0,g(s)-\alpha)ds$$

for every $\alpha\ge 0$; so that

$$\int_0^t u^*=\min_{\alpha\ge0}\alpha t+\int(u^*-\alpha\chi1)^+.$$

Now

$$\begin{aligned}\int(u^*-\alpha\chi1)^+&=\int_0^\infty\bar\mu_L[\![(u^*-\alpha\chi1)^+>\beta]\!]d\beta\\&=\int_0^\infty\bar\mu[\![(|u|-\alpha\chi1)^+>\beta]\!]d\beta=\int(|u|-\alpha\chi1)^+\end{aligned}$$

for every $\alpha \ge 0$, using (f) and 365A, and

$$\int_0^t u^* = \min_{\alpha\ge 0} \alpha t + \int (|u| - \alpha\chi 1)^+.$$

(h)

$$\bar{\mu}[\![u_0 > \alpha]\!] = \bar{\mu}(\sup_{u\in A} [\![u > \alpha]\!]) = \sup_{u\in A} \bar{\mu}[\![u > \alpha]\!]$$

for any $\alpha > 0$, using 364L(a-ii) and 321D. So

$$[\![u_0^* > \alpha]\!] = [0, \bar{\mu}[\![u_0 > \alpha]\!][^\bullet = \sup_{u\in A} [0, \bar{\mu}[\![u > \alpha]\!][^\bullet = [\![\sup_{u\in A} u^* > \alpha]\!]$$

for every α, and $u_0^* = \sup_{u\in A} u^*$.

373E Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Then $\int |u \times v| \le \int u^* \times v^*$ for all u, $v \in M^{0,\infty}(\mathfrak{A}, \bar{\mu})$.

proof (a) Consider first the case u, $v \ge 0$ in $S(\mathfrak{A})$. Then we may express u, v as $\sum_{i=0}^m \alpha_i \chi a_i$, $\sum_{j=0}^n \beta_j \chi b_j$ where $a_0 \supseteq a_1 \supseteq \ldots \supseteq a_m$, $b_0 \supseteq \ldots \supseteq b_n$ in $\mathfrak{A}$ and α_i, $\beta_j \ge 0$ for all i, j (361Ec). Now u^*, v^* are given by

$$u^* = \sum_{i=0}^m \alpha_i \chi\, [0, \bar{\mu}a_i[^\bullet, \quad v^* = \sum_{j=0}^n \beta_j \chi\, [0, \bar{\mu}b_j[^\bullet$$

(373Dc). So

$$\begin{aligned}\int u \times v &= \sum_{i=0}^m \sum_{j=0}^n \alpha_i \beta_j \bar{\mu}(a_i \cap b_j) \le \sum_{i=0}^m \sum_{j=0}^n \alpha_i \beta_j \min(\bar{\mu}a_i, \bar{\mu}b_j) \\ &= \sum_{i=0}^m \sum_{j=0}^n \alpha_i \beta_j \mu_L([0, \bar{\mu}a_i[\cap [0, \bar{\mu}b_j[) = \int u^* \times v^*.\end{aligned}$$

(b) For the general case, we have non-decreasing sequences $\langle u_n \rangle_{n\in\mathbb{N}}$, $\langle v_n \rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A})^+$ with suprema $|u|$, $|v|$ respectively (364Jd), so that

$$|u \times v| = |u| \times |v| = \sup_{n\in\mathbb{N}} |u| \times v_n = \sup_{m,n\in\mathbb{N}} u_m \times v_n = \sup_{n\in\mathbb{N}} u_n \times v_n$$

and

$$\int |u \times v| = \int \sup_{n\in\mathbb{N}} u_n \times v_n = \sup_{n\in\mathbb{N}} \int u_n \times v_n \le \sup_{n\in\mathbb{N}} \int u_n^* \times v_n^* \le \int u^* \times v^*,$$

using 373Db.

373F Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and u any member of $M^{0,\infty}(\mathfrak{A}, \bar{\mu})$.

(a) For any $p \in [1, \infty]$, $u \in L^p(\mathfrak{A}, \bar{\mu})$ iff $u^* \in L^p(\mathfrak{A}_L, \bar{\mu}_L)$, and in this case $\|u\|_p = \|u^*\|_p$.

(b)(i) $u \in M^0(\mathfrak{A}, \bar{\mu})$ iff $u^* \in M^0(\mathfrak{A}_L, \bar{\mu}_L)$;

(ii) $u \in M^{1,\infty}(\mathfrak{A}, \bar{\mu})$ iff $u^* \in M^{1,\infty}(\mathfrak{A}_L, \bar{\mu}_L)$, and in this case $\|u\|_{1,\infty} = \|u^*\|_{1,\infty}$;

(iii) $u \in M^{1,0}(\mathfrak{A}, \bar{\mu})$ iff $u^* \in M^{1,0}(\mathfrak{A}_L, \bar{\mu}_L)$;

(iv) $u \in M^{\infty,1}(\mathfrak{A}, \bar{\mu})$ iff $u^* \in M^{\infty,1}(\mathfrak{A}_L, \bar{\mu}_L)$, and in this case $\|u\|_{\infty,1} = \|u^*\|_{\infty,1}$.

proof (a)(i) Consider first the case $p = 1$. In this case

$$\int |u| = \int_0^\infty \bar{\mu}[\![|u| > \alpha]\!] d\alpha = \int_0^\infty \bar{\mu}_L[\![u^* > \alpha]\!] d\alpha = \int u^*.$$

(ii) If $1 < p < \infty$, then by 373De we have $(|u|^p)^* = (u^*)^p$, so that

$$\|u\|_p^p = \int |u|^p = \int (|u|^p)^* = \int (u^*)^p = \|u^*\|_p^p$$

if either $\|u\|_p$ or $\|u^*\|_p$ is finite. (iii) As for $p = \infty$,

$$\|u\|_\infty \le \gamma \iff [\![|u| > \gamma]\!] = 0 \iff [\![u^* > \gamma]\!] = 0 \iff \|u^*\|_\infty \le \gamma.$$

(b)(i)

$$\begin{aligned} u \in M^0_{\bar{\mu}} &\iff \bar{\mu}[\![|u| > \alpha]\!] < \infty \text{ for every } \alpha > 0 \\ &\iff \bar{\mu}_L[\![u^* > \alpha]\!] < \infty \text{ for every } \alpha > 0 \iff u^* \in M^0_{\bar{\mu}_L}.\end{aligned}$$

(ii) For any $\alpha \ge 0$,

$$\int(|u|-\alpha\chi 1)^+ = \int(u^*-\alpha\chi 1)^+$$

as in the proof of 373Dg. So $\|u\|_{1,\infty}=\|u^*\|_{1,\infty}$ if either is finite, by the formula in 369Ob.

(iii) This follows from (i) and (ii), because $M^{1,0}=M^0\cap M^{1,\infty}$.

(iv) Allowing ∞ as a value of an integral, we have

$$\begin{aligned}\|u\|_{1,\infty} &= \min\{\alpha+\int(|u|-\alpha\chi 1)^+ : \alpha\ge 0\}\\ &= \min\{\alpha+\int(u^*-\alpha\chi 1)^+ : \alpha\ge 0\}=\|u^*\|_{1,\infty}\end{aligned}$$

by 369Ob; in particular, $u\in M^{1,\infty}_{\bar\mu}$ iff $u^*\in M^{1,\infty}_{\bar\mu_L}$.

373G Lemma Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. If

either $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $T\in\mathcal{T}_{\bar\mu,\bar\nu}$

or $u\in M^{1,0}(\mathfrak{A},\bar\mu)$ and $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$,

then $\int_0^t(Tu)^*\le\int_0^t u^*$ for every $t\ge 0$.

proof Set $T_1=T\restriction L^1_{\bar\mu}$, so that $\|T_1\|\le 1$ in $\mathsf{B}(L^1_{\bar\mu};L^1_{\bar\nu})$, and $|T_1|$ is defined in $\mathsf{B}(L^1_{\bar\mu};L^1_{\bar\nu})$, also with norm at most 1. If $\alpha\ge 0$, then we can express u as u_1+u_2 where $|u_1|\le(|u|-\alpha\chi 1)^+$ and $|u_2|\le\alpha\chi 1$. (Let $w\in L^\infty(\mathfrak{A})$ be such that $\|w\|_\infty\le 1$, $u=|u|\times w$; set $u_2=w\times(|u|\wedge\alpha\chi 1)$.) So if $\int(|u|-\alpha\chi 1)^+<\infty$,

$$|Tu|\le|Tu_1|+|Tu_2|\le|T_1||u_1|+\alpha\chi 1$$

and

$$\int(|Tu|-\alpha\chi 1)^+\le\int|T_1||u_1|\le\int|u_1|\le\int(|u|-\alpha\chi 1)^+.$$

The formula of 373Dg now tells us that $\int_0^t(Tu)^*\le\int_0^t u^*$ for every t.

373H Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\theta:\mathfrak{A}^f\to\mathbb{R}$ an additive functional, where $\mathfrak{A}^f=\{a:\bar\mu a<\infty\}$.

(a) The following are equiveridical:

(α) $\lim_{t\downarrow 0}\sup_{\bar\mu a\le t}|\theta a|=\lim_{t\to\infty}\frac{1}{t}\sup_{\bar\mu a\le t}|\theta a|=0$,

(β) there is some $u\in M^{1,0}(\mathfrak{A},\bar\mu)$ such that $\theta a=\int_a u$ for every $a\in\mathfrak{A}^f$,

and in this case u is uniquely defined.

(b) Now suppose that $(\mathfrak{A},\bar\mu)$ is localizable. Then the following are equiveridical:

(α) $\lim_{t\downarrow 0}\sup_{\bar\mu a\le t}|\theta a|=0$, $\quad\limsup_{t\to\infty}\frac{1}{t}\sup_{\bar\mu a\le t}|\theta a|<\infty$,

(β) there is some $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$ such that $\theta a=\int_a u$ for every $a\in\mathfrak{A}^f$,

and again this u is uniquely defined.

proof (a)(i) Assume (α). For a, $c\in\mathfrak{A}^f$, set $\theta_c(a)=\theta(a\cap c)$. Then for each $c\in\mathfrak{A}^f$, there is a unique $u_c\in L^1_{\bar\mu}$ such that $\theta_c a=\int_a u_c$ for every $a\in\mathfrak{A}^f$ (365Eb). Because u_c is unique we must have $u_c=u_d\times\chi c$ whenever $c\subseteq d\in\mathfrak{A}^f$. Next, given $\alpha>0$, there is a $t_0\ge 0$ such that $|\theta a|\le\alpha\bar\mu a$ whenever $a\in\mathfrak{A}^f$ and $\bar\mu a\ge t_0$; so that $\bar\mu[\![u_c>\alpha]\!]\le t_0$ for every $c\in\mathfrak{A}^f$, and $e(\alpha)=\sup_{c\in\mathfrak{A}^f}[\![u_c^+>\alpha]\!]$ is defined in $\mathfrak{A}^f$. Of course $e(\alpha)=[\![u^+_{e(1)}>\alpha]\!]$ for every $\alpha\ge 1$, so $\inf_{\alpha\in\mathbb{R}}e(\alpha)=0$, and $v_1=\sup_{c\in\mathfrak{A}^f}u_c^+$ is defined in $L^0=L^0(\mathfrak{A})$ (364L(a-ii) again). Because $[\![v_1>\alpha]\!]=e(\alpha)\in\mathfrak{A}^f$ for each $\alpha>0$, $v_1\in M^0_{\bar\mu}$. For any $a\in\mathfrak{A}^f$,

$$v_1\times\chi a=\sup_{c\in\mathfrak{A}^f}u_c^+\times\chi a=u_a^+,$$

so $v_1\in M^{1,0}_{\bar\mu}$ and $\int_a v_1=\int_a u_a^+$ for every $a\in\mathfrak{A}^f$.

Similarly, $v_2=\sup_{c\in\mathfrak{A}^f}u_c^-$ is defined in $M^{1,0}_{\bar\mu}$ and $\int_a v_2=\int_a u_a^-$ for every $a\in\mathfrak{A}^f$. So we can set $u=v_1-v_2\in M^{1,0}_{\bar\mu}$ and get

$$\int_a u=\int_a u_a=\theta a$$

for every $a\in\mathfrak{A}^f$. Thus (β) is true.

(ii) Assume (β). If $\epsilon > 0$, there is a $\delta > 0$ such that $\int_a (|u| - \epsilon\chi 1)^+ \le \epsilon$ whenever $\bar\mu a \le \delta$ (365Ea), so that $|\int_a u| \le \epsilon(1 + \bar\mu a)$ whenever $\bar\mu a \le \delta$. As ϵ is arbitrary, $\lim_{t\downarrow 0} \sup_{\bar\mu a \le t} |\int_a u| = 0$. Moreover, whenever $t > 0$ and $\bar\mu a \le t$, $\frac{1}{t}|\int_a u| \le \epsilon + \frac{1}{t}\int(|u| - \epsilon\chi 1)^+$. Thus

$$\limsup_{t\to\infty} \tfrac{1}{t} \sup_{\bar\mu a \le t} |\int_a u| \le \epsilon.$$

As ϵ is arbitrary, θ satisfies the conditions in (α).

(iii) The uniqueness of u is a consequence of 366Gd.

(b) The argument for (b) uses the same ideas.

(i) Assume (α). Again, for each $c \in \mathfrak{A}^f$, we have $u_c \in L^1_{\bar\mu}$ such that $\theta_c a = \int_a u_c$ for every $a \in \mathfrak{A}^f$; again, set $e(\alpha) = \sup_{c\in\mathfrak{A}^f} [\![u_c^+ > \alpha]\!]$, which is defined because $\mathfrak{A}$ is supposed to be Dedekind complete. This time, there are t_0, $\gamma \ge 0$ such that $|\theta a| \le \gamma\bar\mu a$ whenever $a \in \mathfrak{A}^f$ and $\bar\mu a \ge t_0$; so that $\bar\mu[\![u_c > \gamma]\!] \le t_0$ for every $c \in \mathfrak{A}^f$, and $\bar\mu e(\gamma) < \infty$. Accordingly

$$\inf_{\alpha\ge\gamma} e(\alpha) = \inf_{\alpha\ge\gamma} [\![u^+_{e(\gamma)} > \alpha]\!] = 0,$$

and once more $v_1 = \sup_{c\in\mathfrak{A}^f} u_c^+$ is defined in $L^0 = L^0(\mathfrak{A})$. As before, $v_1 \times \chi a = u_a^+ \in L^1_{\bar\mu}$ for any $a \in \mathfrak{A}^f$, Because $[\![v_1 > \gamma]\!] = e(\gamma) \in \mathfrak{A}^f$, $v_1 \in M^{1,\infty}_{\bar\mu}$. Similarly, $v_2 = \sup_{c\in\mathfrak{A}^f} u_c^-$ is defined in $M^{1,\infty}_{\bar\mu}$, with $v_2 \times \chi a = u_a^-$ for every $a \in \mathfrak{A}^f$. So $u = v_1 - v_2 \in M^{1,\infty}_{\bar\mu}$, and

$$\int_a u = \int_a u_a = \theta a$$

for every $a \in \mathfrak{A}^f$.

(ii) Assume (β). Take $\gamma \ge 0$ such that $\beta = \int(|u| - \gamma\chi 1)^+$ is finite. If $\epsilon > 0$, there is a $\delta > 0$ such that $\int_a(|u| - \gamma\chi 1)^+ \le \epsilon$ whenever $\bar\mu a \le \delta$, so that $|\int_a u| \le \epsilon + \gamma\bar\mu a$ whenever $\bar\mu a \le \delta$. As ϵ is arbitrary, $\lim_{t\downarrow 0}\sup_{\bar\mu a \le t} |\int_a u| = 0$. Moreover, whenever $t > 0$ and $\bar\mu a \le t$, then $\frac{1}{t}|\int_a u| \le \gamma + \frac{1}{t}\int(|u| - \epsilon\chi 1)^+$. Thus

$$\limsup_{t\to\infty} \tfrac{1}{t}\sup_{\bar\mu a \le t} |\int_a u| \le \gamma < \infty,$$

and the function $a \mapsto \int_a u$ satisfies the conditions in (β).

(iii) u is uniquely defined because $u \times \chi a$ must be u_a, as defined in (i), for every $a \in \mathfrak{A}^f$, and $(\mathfrak{A}, \bar\mu)$ is semi-finite.

373I Lemma Suppose that u, v, $w \in M^{0,\infty}(\mathfrak{A}_L, \bar\mu_L)$ are all equivalence classes of non-negative non-increasing functions. If $\int_0^t u \le \int_0^t v$ for every $t \ge 0$, then $\int u \times w \le \int v \times w$.

proof For $n \in \mathbb{N}$, $i \le 4^n$ set $a_{ni} = [\![w > 2^{-n}i]\!]$; set $w_n = \sum_{i=1}^{4^n} 2^{-n}\chi a_{ni}$. Then each a_{ni} is of the form $[0,t]^\bullet$, so

$$\int u \times w_n = \sum_{i=1}^{4^n} 2^{-n}\int_{a_{ni}} u \le \sum_{i=1}^{4^n} 2^{-n}\int_{a_{ni}} v = \int v \times w_n.$$

Also $\langle w_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum w, so

$$\int u \times w = \sup_{n\in\mathbb{N}} \int u \times w_n \le \sup_{n\in\mathbb{N}} \int v \times w_n = \int v \times w.$$

373J Corollary Suppose that $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ are measure algebras and $v \in M^{0,\infty}(\mathfrak{B}, \bar\nu)$. If

either $u \in M^{1,0}(\mathfrak{A}, \bar\mu)$ and $T \in \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$

or $u \in M^{1,\infty}(\mathfrak{A}, \bar\mu)$ and $T \in \mathcal{T}_{\bar\mu,\bar\nu}$

then $\int |Tu \times v| \le \int u^* \times v^*$.

proof Put 373E, 373G and 373I together.

373K The very weak operator topology of $\mathcal{T}$ Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be two measure algebras. For $u \in M^{1,\infty}(\mathfrak{A}, \bar\mu)$, $w \in M^{\infty,1}(\mathfrak{B}, \bar\nu)$ set

$$\rho_{uw}(S, T) = |\int Su \times w - \int Tu \times w| \text{ for all } S,\, T \in \mathcal{T} = \mathcal{T}_{\bar\mu,\bar\nu}.$$

Then ρ_{uw} is a pseudometric on $\mathcal{T}$. I will call the topology generated by $\{\rho_{uw} : u \in M^{1,\infty}(\mathfrak{A}, \bar\mu),\, w \in M^{\infty,1}(\mathfrak{B}, \bar\nu)\}$ (2A3F) the **very weak operator topology** on $\mathcal{T}$.

373L Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a localizable measure algebra. Then $\mathcal{T}=\mathcal{T}_{\bar\mu,\bar\nu}$ is compact in its very weak operator topology.

proof Let $\mathcal{F}$ be an ultrafilter on $\mathcal{T}$. If $u\in M^{1,\infty}_{\bar\mu}$, $w\in M^{\infty,1}_{\bar\nu}$ then

$$|\int Tu\times w|\le\int u^*\times w^*<\infty$$

for every $T\in\mathcal{T}$ (373J); $\int u^*\times w^*$ is finite because $u^*\in M^{1,\infty}$ and $w^*\in M^{\infty,1}$ (373F).

In particular, $\{\int Tu\times w:T\in\mathcal{T}\}$ is bounded. Consequently $h_u(w)=\lim_{T\to\mathcal{F}}\int Tu\times w$ is defined in $\mathbb{R}$ (2A3Se). Because $w\mapsto\int Tu\times w$ is additive for every $T\in\mathcal{T}$, so is h_u. Also

$$|h_u(w)|\le\int u^*\times w^*\le\|u^*\|_{1,\infty}\|w^*\|_{\infty,1}=\|u\|_{1,\infty}\|w\|_{\infty,1}$$

for every $w\in M^{\infty,1}_{\bar\nu}$.

$|h_u(\chi b)|\le\int_0^t u^*$ whenever $b\in\mathfrak{B}^f$ and $\bar\nu b\le t$. So

$$\lim_{t\downarrow 0}\sup_{\bar\nu b\le t}|h_u(\chi b)|\le\lim_{t\downarrow 0}\int_0^t u^*=0,$$

$$\limsup_{t\to\infty}\tfrac{1}{t}\sup_{\bar\nu b\le t}|h_u(\chi b)|\le\limsup_{t\to\infty}\tfrac{1}{t}\int_0^t u^*<\infty.$$

Of course $b\mapsto h_u(\chi b)$ is additive, so by 373H there is a unique $Su\in M^{1,\infty}_{\bar\nu}$ such that $h_u(\chi b)=\int_b Su$ for every $b\in\mathfrak{B}^f$. Since both h_u and $w\mapsto\int Su\times w$ are linear and continuous on $M^{\infty,1}_{\bar\nu}$, and $S(\mathfrak{B}^f)$ is dense in $M^{\infty,1}_{\bar\nu}$ (369Od),

$$\int Su\times w=h_u(w)=\lim_{T\to\mathcal{F}}\int Tu\times w$$

for every $w\in M^{\infty,1}_{\bar\nu}$. And this is true for every $u\in M^{1,\infty}_{\bar\mu}$.

For any particular $w\in M^{\infty,1}_{\bar\nu}$, all the maps $u\mapsto\int Tu\times w$ are linear, so $u\mapsto\int Su\times w$ also is; that is, $S:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$ is linear.

Now $S\in\mathcal{T}$. **P** (α) If $u\in L^1_{\bar\mu}$ and b, $c\in\mathfrak{B}^f$, then

$$\begin{aligned}\int_b Su-\int_c Su&=\lim_{T\to\mathcal{F}}\int Tu\times(\chi b-\chi c)\le\sup_{T\in\mathcal{T}}\int Tu\times(\chi b-\chi c)\\&\le\sup_{T\in\mathcal{T}}\|Tu\|_1\|\chi b-\chi c\|_\infty\le\|u\|_1.\end{aligned}$$

But, setting $e=[\![Su>0]\!]$, we have

$$\begin{aligned}\int|Su|&=\int_e Su-\int_{1\setminus e}Su\\&=\sup_{b\in\mathfrak{B}^f,b\subseteq e}\int_b Su+\sup_{c\in\mathfrak{B}^f,c\subseteq 1\setminus e}(-Su)\le\|u\|_1.\end{aligned}$$

(β) If $u\in L^\infty(\mathfrak{A})$, then

$$|\int_b Su|\le\sup_{T\in\mathcal{T}}|\int Tu\times\chi b|\le\sup_{T\in\mathcal{T}}\|Tu\|_\infty\bar\nu b\le\|u\|_\infty\bar\nu b$$

for every $b\in\mathfrak{B}^f$. So $[\![Su>\|u\|_\infty]\!]=[\![-Su>\|u\|_\infty]\!]=0$ and $\|Su\|_\infty\le\|u\|_\infty$. (Note that both parts of this argument depend on knowing that $(\mathfrak{B},\bar\nu)$ is semi-finite, so that we cannot be troubled by purely infinite elements of $\mathfrak{B}$.) **Q**

Of course we now have $\lim_{T\to\mathcal{F}}\rho_{uw}(T,S)=0$ for all $u\in M^{1,0}_{\bar\mu}$, $w\in M^{\infty,1}_{\bar\nu}$, so that $S=\lim\mathcal{F}$ in $\mathcal{T}$. As $\mathcal{F}$ is arbitrary, $\mathcal{T}$ is compact (2A3R).

373M Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a localizable measure algebra, and u any member of $M^{1,\infty}(\mathfrak{A},\bar\mu)$. Then $B=\{Tu:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}$ is compact in $M^{1,\infty}(\mathfrak{B},\bar\nu)$ for the topology $\mathfrak{T}_s(M^{1,\infty}(\mathfrak{B},\bar\nu),M^{\infty,1}(\mathfrak{B},\bar\nu))$.

proof The point is just that the map $T\mapsto Tu:\mathcal{T}_{\bar\mu,\bar\nu}\to M^{1,0}_{\bar\nu}$ is continuous for the very weak operator topology on $\mathcal{T}_{\bar\mu,\bar\nu}$ and $\mathfrak{T}_s(M^{1,\infty}_{\bar\nu},M^{\infty,1}_{\bar\nu})$. So B is a continuous image of a compact set, therefore compact (2A3Nb).

373N Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, $(\mathfrak{B},\bar\nu)$ a localizable measure algebra and u any member of $M^{1,\infty}(\mathfrak{A},\bar\mu)$; set $B=\{Tu:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}$. If $\langle v_n\rangle_{n\in\mathbb{N}}$ is any non-decreasing sequence in B, then $\sup_{n\in\mathbb{N}}v_n$ is defined in $M^{1,\infty}(\mathfrak{B},\bar\nu)$ and belongs to B.

proof By 373M, $\langle v_n\rangle_{n\in\mathbb{N}}$ must have a cluster point $v\in B$ for $\mathfrak{T}_s(M^{1,\infty}_{\bar\nu},M^{\infty,1}_{\bar\nu})$. Now for any $b\in\mathfrak{B}^f$, $\int_b v$ must be a cluster point of $\langle\int_n v_n\rangle_{n\in\mathbb{N}}$, because $w\mapsto\int_b w$ is continuous for $\mathfrak{T}_s(M^{1,\infty}_{\bar\nu},M^{\infty,1}_{\bar\nu})$. But $\langle\int_b v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence, so its only possible cluster point is its supremum; thus $\int_b v=\lim_{n\to\infty}\int_b v_n$. Consequently $v\times\chi b$ must be the supremum of $\{v_n\times\chi b:n\in\mathbb{N}\}$ in L^1. And this is true for every $b\in\mathfrak{B}^f$; as $(\mathfrak{B},\bar\nu)$ is semi-finite, v is the supremum of $\langle v_n\rangle_{n\in\mathbb{N}}$ in $L^0(\mathfrak{B})$ and in $M^{1,\infty}_{\bar\nu}$.

373O Theorem Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be measure algebras and $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$, $v\in M^{1,\infty}(\mathfrak{B},\bar\nu)$. Then the following are equiveridical:

(i) there is a $T\in\mathcal{T}_{\bar\mu,\bar\nu}$ such that $Tu=v$,

(ii) $\int_0^t v^*\le\int_0^t u^*$ for every $t\ge 0$.

In particular, given $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$, there are $S\in\mathcal{T}_{\bar\mu,\bar\mu_L}$, $T\in\mathcal{T}_{\bar\mu_L,\bar\mu}$ such that $Su=u^*$, $Tu^*=u$.

proof (i)$\Rightarrow$(ii) is Lemma 373G. Accordingly I shall devote the rest of the proof to showing that (ii)$\Rightarrow$(i).

(a) If $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ are measure algebras and $u\in M^{1,\infty}_{\bar\mu}$, $v\in M^{1,\infty}_{\bar\nu}$, I will say that $v\preccurlyeq u$ if there is a $T\in\mathcal{T}_{\bar\mu,\bar\nu}$ such that $Tu=v$, and that $v\sim u$ if $v\preccurlyeq u$ and $u\preccurlyeq v$. (Properly speaking, I ought to write $(u,\bar\mu)\preccurlyeq(v,\bar\nu)$, because we could in principle have two different measures on the same algebra. But I do not think any confusion is likely to arise in the argument which follows.) By 373Be, $\preccurlyeq$ is transitive and $\sim$ is an equivalence relation. Now we have the following facts.

(b) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and u_1, $u_2\in M^{1,\infty}_{\bar\mu}$ are such that $|u_1|\le|u_2|$, then $u_1\preccurlyeq u_2$. **P** There is a $w\in L^\infty(\mathfrak{A})$ such that $u_1=w\times u_2$ and $\|w\|_\infty\le 1$. Set $Tv=w\times v$ for for $v\in M^{1,\infty}_{\bar\mu}$; then $T\in\mathcal{T}_{\bar\mu,\bar\mu}$ and $Tu_2=u_1$. **Q** So $u\sim|u|$ for every $u\in M^{1,\infty}_{\bar\mu}$.

(c) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $u\ge 0$ in $S(\mathfrak{A})$, then $u\preccurlyeq u^*$. **P** If $u=0$ this is trivial. Otherwise, express u as $\sum_{i=0}^n\alpha_i\chi a_i$ where $a_0,\dots,a_n$ are disjoint and non-zero and $\alpha_0>\alpha_1\ldots>\alpha_n>0\in\mathbb{R}$. If $\bar\mu a_i=\infty$ for any i, take m to be minimal subject to $\bar\mu a_m=\infty$; otherwise, set $m=n$. Then $u^*=\sum_{i=0}^m\alpha_i\chi\,[\beta_i,\beta_{i+1}[^\bullet$, where $\beta_0=0$, $\beta_j=\sum_{i=0}^{j-1}\bar\mu a_i$ for $1\le j\le m+1$.

For $i<m$, and for $i=m$ if $\bar\mu a_m<\infty$, define $h_i:M^{1,\infty}_{\bar\mu}\to\mathbb{R}$ by setting

$$h_i(v)=\frac{1}{\bar\mu a_i}\int_{a_i}v$$

for every $v\in M^{1,\infty}_{\bar\mu}$. If $\bar\mu a_m=\infty$, then we need a different idea to define h_m, as follows. Let I be $\{a:a\in\mathfrak{A},\ \bar\mu(a\cap a_m)<\infty\}$. Then I is an ideal of $\mathfrak{A}$ not containing a_m, so there is a Boolean homomorphism $\pi:\mathfrak{A}\to\{0,1\}$ such that $\pi a=0$ for $a\in I$ and $\pi a_m=1$ (311D). This induces a corresponding $\|\,\|_\infty$-continuous linear operator $h:L^\infty(\mathfrak{A})\to L^\infty(\{0,1\})\cong\mathbb{R}$, as in 363F. Now $h(\chi a)=0$ whenever $\bar\mu a<\infty$, and accordingly $h(v)=0$ whenever $v\in M^{\infty,1}_{\bar\mu}$, since $S(\mathfrak{A}^f)$ is dense in $M^{\infty,1}_{\bar\mu}$ for $\|\,\|_{\infty,1}$ and therefore also for $\|\,\|_\infty$. But this means that h has a unique extension to a linear functional $h_m:M^{1,\infty}_{\bar\mu}\to\mathbb{R}$ such that $h_m(v)=0$ for every $v\in L^1_{\bar\mu}$, while $h_m(\chi a_m)=1$ and $|h(v)|\le\|v\|_\infty$ for every $v\in L^\infty(\mathfrak{A})$.

Having defined h_i for every $i\le m$, define $T:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\mu_L}$ by setting

$$Tv=\textstyle\sum_{i=0}^m h_i(v)\chi\,[\beta_i,\beta_{i+1}[^\bullet$$

for every $v\in M^{1,\infty}_{\bar\mu}$.

For any $i\le m$, $v\in L^1_{\bar\mu}$,

$$\int_{\beta_i}^{\beta_{i+1}}|Tv|=|h_i(v)|\bar\mu a_i\le\int_{a_i}|v|;$$

summing over i, $\|Tv\|_1\le\|v\|_1$. Similarly, for any $i\le m$, $v\in L^\infty(\mathfrak{B})$, $|h_i(v)|\le\|v\|_\infty$, so $\|Tv\|_\infty\le\|v\|_\infty$.

Thus $T\in\mathcal{T}_{\bar\mu,\bar\mu_L}$. Since $u^*=Tu$, we conclude that $u^*\preccurlyeq u$, as claimed. **Q**

(d) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $u\ge 0$ in $M^{1,\infty}_{\bar\mu}$, then $u^*\preccurlyeq u$. **P** Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A})$ with $u_0\ge 0$ and $\sup_{n\in\mathbb{N}}u_n=u$. Then $\langle u_n^*\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $M^{1,\infty}_{\bar\mu_L}$ with supremum u^*, by 373Db and 373Dh. Also $u_n^*\preccurlyeq u_n\preccurlyeq u$ for every n, by (b) and (c) of this proof. By 373N, $u^*\preccurlyeq u$. **Q**

(e) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $u\ge 0$ in $S(\mathfrak{A})$, then $u\preccurlyeq u^*$. **P** The argument is very similar to that of (c). Again, the result is trivial if $u=0$; suppose that $u>0$ and define α_i, a_i, m, β_i as before. This time, set $a_i'=a_i$ for $i<m$, $a_m'=\sup_{m\le j\le n}a_j$, $\tilde u=\sum_{i=0}^m\alpha_i\chi a_i'$; then $u\le\tilde u$ amd $\tilde u^*=u^*$. Set

$$h_i(v) = \frac{1}{\beta_{i+1}-\beta_i}\int_{\beta_i}^{\beta_{i+1}} v$$

if $i \le m$, $\beta_{i+1} < \infty$ (that is, $\bar\mu a_i < \infty$) and $v \in M^{1,\infty}_{\bar\mu_L}$; and if $\bar\mu a_m = \infty$, set

$$h_m(v) = \lim_{k\to\mathcal{F}} \frac{1}{k}\int_0^k v$$

for some non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. As before, we have

$$|h_i(v)|\bar\mu a_i' \le \int_{\beta_i}^{\beta_{i+1}} |v|,$$

whenever $v \in L^1_{\bar\mu_L}$, $i \le m$, while $|h_i(v)| \le \|v\|_\infty$ whenever $v \in L^\infty(\mathfrak{A}_L)$ and $i \le m$. So we can define $T \in \mathcal{T}_{\bar\mu_L,\bar\mu}$ by setting $Tv = \sum_{i=0}^m h_i(v)\chi a_i'$ for every $v \in M^{1,\infty}_{\bar\mu_L}$, and get

$$u \preccurlyeq \tilde u = Tu^* \preccurlyeq u^*. \ \mathbf{Q}$$

(f) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $u \ge 0$ in $M^{1,\infty}_{\bar\mu}$, then $u \preccurlyeq u^*$. $\mathbf{P}$ This time I seek to copy the ideas of (d); there is a new obstacle to circumvent, since $(\mathfrak{A},\bar\mu)$ might not be localizable. Set

$$\alpha_0 = \inf\{\alpha : \alpha \ge 0,\ \bar\mu[\![u > \alpha]\!] < \infty\}, \quad e = [\![u > \alpha_0]\!].$$

Then $e = \sup_{n\in\mathbb{N}} [\![u > \alpha_0 + 2^{-n}]\!]$ is a countable supremum of elements of finite measure, so the principal ideal $\mathfrak{A}_e$, with its induced measure $\bar\mu_e$, is σ-finite. Now let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A})$ with $u_0 \ge 0$ and $\sup_{n\in\mathbb{N}} u_n = u$; set $\tilde u = u \times \chi e$ and $\tilde u_n = u_n \times \chi e$, regarded as members of $S(\mathfrak{A}_e)$, for each n. In this case

$$\tilde u_n \preccurlyeq \tilde u_n^* \preccurlyeq u^*$$

for every n. Because $(\mathfrak{A}_e,\bar\mu_e)$ is σ-finite, therefore localizable, 373N tells us that $\tilde u \preccurlyeq u^*$.

Let $S \in \mathcal{T}_{\bar\mu_L,\bar\mu_e}$ be such that $Su^* = \tilde u$. As in part (e), choose a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ and set

$$h(v) = \lim_{k\to\mathcal{F}} \frac{1}{k}\int_0^k v$$

for $v \in M^{1,\infty}_{\bar\mu_L}$. Now define $T : M^{1,\infty}_{\bar\mu_L} \to M^{1,\infty}_{\bar\mu}$ by setting

$$Tv = Sv + h(v)\chi(1 \setminus e),$$

here regarding Sv as a member of $M^{1,\infty}_{\bar\mu}$. (I am taking it to be obvious that $M^{1,\infty}_{\bar\mu_e}$ can be identified with $\{w \times \chi e : w \in M^{1,\infty}_{\bar\mu}\}$.) Then it is easy to see that $T \in \mathcal{T}_{\bar\mu_L,\bar\mu}$. Also $u \le Tu^*$, because

$$h(u^*) = \inf\{\alpha : \bar\mu_L[\![u^* > \alpha]\!] < \infty\} = \alpha_0,$$

while $u \times \chi(1 \setminus e) \le \alpha_0\chi(1 \setminus e)$. So we get $u \preccurlyeq Tu^* \preccurlyeq u^*$. $\mathbf{Q}$

(g) Now suppose that u, $v \ge 0$ in $M^{1,\infty}_{\bar\mu_L}$, that $\int_0^t u^* \ge \int_0^t v^*$ for every $t \ge 0$, and that v is of the form $\sum_{i=1}^n \alpha_i\chi a_i$ where $\alpha_1 > \ldots > \alpha_n > 0$, $a_1,\ldots,a_n \in \mathfrak{A}_L$ are disjoint and $\bar\mu_L a_i < \infty$ for each i. Then $v \preccurlyeq u$. $\mathbf{P}$ Induce on n. If $n = 0$ then $v = 0$ and the result is trivial. For the inductive step to $n \ge 1$, if $v^* \le u^*$ we have

$$v \sim v^* \preccurlyeq u^* \sim u,$$

using (b), (d) and (f) above. Otherwise, look at $\phi(t) = \frac{1}{t}\int_0^t u^*$ for $t > 0$. We have

$$\phi(t) \ge \frac{1}{t}\int_0^t v^* = \alpha_1$$

for $t \le \beta = \bar\mu a_1$, while $\lim_{t\to\infty}\phi(t) < \alpha_1$, because $(\lim_{t\to\infty}\phi(t))\chi 1 \le u^*$ and $v^* \le \alpha_1\chi 1$ and $v^* \not\le u^*$. Becaasue ϕ is continuoue, there is a $\gamma \ge \beta$ such that $\phi(\gamma) = \alpha_1$. Define $T_0 \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ by setting

$$T_0 w = (\frac{1}{\gamma}\int_0^\gamma w)\chi\,[0,\gamma[^\bullet + (w \times \chi\,[\gamma,\infty[^\bullet)$$

for every $w \in M^{1,\infty}_{\bar\mu_L}$. Then $T_0u^* \preccurlyeq u^* \sim u$, and

$$T_0u^* \times \chi\,[0,\gamma[^\bullet = (\frac{1}{\gamma}\int_0^\gamma u^*)\chi\,[0,\gamma[^\bullet = \alpha_1\chi\,[0,\gamma[^\bullet.$$

We need to know that $\int_0^t T_0u^* \ge \int_0^t v^*$ for every t; this is because

$$\int_0^t T_0u^* = \alpha_1 t \ge \int_0^t v^* \text{ whenever } t \le \gamma,$$
$$= \int_0^\gamma T_0u^* + \int_\gamma^t T_0u^* = \int_0^t u^* \ge \int_0^t v^* \text{ whenever } t \ge \gamma.$$

Set

$$u_1 = T_0u^* \times \chi\,[\beta,\infty[^\bullet, \quad v_1 = v^* \times \chi\,[\beta,\infty[^\bullet.$$

Then u_1^*, v_1^* are just translations of T_0u^*, v^* to the left, so that

$$\int_0^t u_1^* = \int_\beta^{\beta+t} T_0u^* = \int_0^{\beta+t} T_0u^* - \alpha_1\beta \ge \int_0^{\beta+t} v^* - \alpha_1\beta = \int_\beta^{\beta+t} v^* = \int_0^t v_1^*$$

for every $t \ge 0$. Also $v_1 = \sum_{i=2}^n \alpha_i\chi\,[\beta_{i-1},\beta_i[^\bullet$ where $\beta_i = \sum_{j=1}^i \bar\mu a_j$ for each j. So by the inductive hypothesis, $v_1 \preccurlyeq u_1$.

Let $S \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ be such that $Su_1 = v_1$, and define $T \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ by setting

$$Tw = w \times \chi\,[0,\beta[^\bullet + S(w \times \chi\,[\beta,\infty[^\bullet) \times \chi\,[\beta,\infty[^\bullet$$

for every $w \in M^{1,\infty}_{\bar\mu_L,\bar\mu_L}$. Then $TT_0u^* = v^*$, so $v \sim v^* \preccurlyeq u^* \sim u$, as required. **Q**

(h) We are nearly home. If u, $v \ge 0$ in $M^{1,\infty}_{\bar\mu_L}$ and $\int_0^t v^* \le \int_0^t u^*$ for every $t \ge 0$, then $v \preccurlyeq u$. **P** Let $\langle v_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A}_L^f)^+$ with supremum v. Then $v_n^* \le v^*$ for each n, so (g) tells us that $v_n \preccurlyeq u$ for every n. By 373N, for the last time, $v \preccurlyeq u$. **Q**

(i) Finally, suppose that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are arbitrary measure algebras and that $u \in M^{1,\infty}_{\bar\mu}$, $v \in M^{1,\infty}_{\bar\nu}$ are such that $\int_0^t v^* \le \int_0^t u^*$ for every $t \ge 0$. By (b), $v \preccurlyeq |v|$; by (f), $|v| \preccurlyeq |v|^*$; by 373Db, $|v|^* = v^*$; by (h) of this proof, $v^* \preccurlyeq u^*$; by (d), $u^* = |u|^* \preccurlyeq |u|$; and by (b) again, $|u| \preccurlyeq u$.

373P Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a semi-finite measure algebra. Then for any $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $v \in M^0(\mathfrak{B},\bar\nu)$, there is a $T \in \mathcal{T} = \mathcal{T}_{\bar\mu,\bar\nu}$ such that $\int Tu \times v = \int u^* \times v^*$.

proof (a) It is convenient to dispose immediately of some elementary questions.

(i) We need only find a $T \in \mathcal{T}$ such that $\int |Tu \times v| \ge \int u^* \times v^*$. **P** Take $v_0 \in L^\infty(\mathfrak{B})$ such that $|Tu \times v| = v_0 \times Tu \times v$ and $\|v_0\|_\infty \le 1$, and set $T_1w = v_0 \times Tw$ for $w \in M^{1,\infty}_{\bar\mu}$; then $T_1 \in \mathcal{T}$ and

$$\int T_1u \times v = \int |Tu \times v| \ge \int u^* \times v^* \ge \int T_1u \times v$$

by 373J. **Q**

(ii) Consequently it will be enough to consider $v \ge 0$, since of course $\int |Tu \times v| = \int |Tu \times |v||$, while $|v|^* = v^*$.

(iii) It will be enough to consider $u = u^*$. **P** If we can find $T \in \mathcal{T}_{\bar\mu_L,\bar\nu}$ such that $\int Tu^* \times v = \int (u^*)^* \times v^*$, then we know from 373O that there is an $S \in \mathcal{T}_{\bar\mu,\bar\mu_L}$ such that $Su = u^*$, so that $TS \in \mathcal{T}$ and

$$\int TSu \times v = \int (u^*)^* \times v^* = \int u^* \times v^*. \text{ **Q**}$$

(iv) It will be enough to consider localizable $(\mathfrak{B},\bar\nu)$. **P** Assuming that $v \ge 0$, following (ii) above, set $e = [\![v > 0]\!] = \sup_{n\in\mathbb{N}} [\![v > 2^{-n}]\!]$, and let $\bar\nu_e$ be the restriction of $\bar\nu$ to the principal ideal $\mathfrak{B}_e$ generated by e. Then if we write $\tilde v$ for the member of $L^0(\mathfrak{B}_e)$ corresponding to v (so that $[\![\tilde v > \alpha]\!] = [\![v > \alpha]\!]$ for every $\alpha > 0$), $\tilde v^* = v^*$. Also $(\mathfrak{B}_e,\bar\nu_e)$ is σ-finite, therefore localizable. Now if we can find $T \in \mathcal{T}_{\bar\mu,\bar\nu_e}$ such that $\int Tu \times \tilde v = \int u^* \times \tilde v^*$, then ST will belong to $\mathcal{T}_{\bar\mu,\bar\nu}$, where $S : L^0(\mathfrak{B}_e) \to L^0(\mathfrak{B})$ is the canonical embedding defined by the formula

$$[\![Sw > \alpha]\!] = [\![w > \alpha]\!] \text{ if } \alpha \ge 0,$$
$$= [\![w > \alpha]\!] \cup (1 \setminus e) \text{ if } \alpha < 0,$$

and

$$\int STu \times v = \int Tu \times \tilde v = \int u^* \times \tilde v^* = \int u^* \times v^*. \text{ **Q**}$$

(b) So let us suppose henceforth that $\bar\mu = \bar\mu_L$, $u = u^*$ is the equivalence class of a non-increasing non-negative function, $v \ge 0$ and $(\mathfrak{B},\bar\nu)$ is localizable.

For n, $i \in \mathbb{N}$ set

$$b_{ni} = [\![v > 2^{-n}i]\!], \quad \beta_{ni} = \bar{\nu}b_{ni}, \quad c_{ni} = b_{ni} \setminus b_{n,i+1}, \quad \gamma_{ni} = \bar{\nu}c_{ni} = \beta_{ni} - \beta_{n,i+1}$$

(because $\beta_{ni} < \infty$ if $i > 0$; this is really where I use the hypothesis that $v \in M^0$). For $n \in \mathbb{N}$ set

$$K_n = \{i : i \geq 1, \gamma_{ni} > 0\},$$

$$T_n w = \sum_{i \in K_n} \Bigl(\frac{1}{\gamma_{ni}} \int_{\beta_{n,i+1}}^{\beta_{ni}} w\Bigr)\chi c_{ni}$$

for $w \in M^{1,\infty}_{\bar{\mu}_L}$; this is defined in $L^0(\mathfrak{B})$ because K_n is countable and $\langle c_{ni}\rangle_{i\in\mathbb{N}}$ is disjoint. Of course $T_n : M^{1,\infty}_{\bar{\mu}_L} \to L^0(\mathfrak{B})$ is linear. If $w \in L^\infty(\mathfrak{A}_L)$ then

$$\|T_n w\|_\infty = \sup_{i\in K_n}\Bigl|\frac{1}{\gamma_{ni}}\int_{\beta_{n,i+1}}^{\beta_{ni}} w\Bigr| \leq \|w\|_\infty,$$

and if $w \in L^1_{\bar{\mu}_L}$ then

$$\|T_n w\|_1 = \sum_{i\in K_n}\Bigl|\frac{1}{\gamma_{ni}}\int_{\beta_{n,i+1}}^{\beta_{ni}} w\Bigr|\bar{\nu}c_{ni} = \sum_{i\in K_n}\Bigl|\int_{\beta_{n,i+1}}^{\beta_{ni}} w\Bigr| \leq \|w\|_1;$$

so $T_n w \in M^{1,\infty}_{\bar{\nu}}$ for every $w \in M^{1,\infty}_{\bar{\mu}_L}$, and $T_n \in \mathcal{T}$. It will be helpful to observe that

$$\int_{c_{ni}} T_n w = \int_{\beta_{n,i+1}}^{\beta_{ni}} w$$

whenever $i \geq 1$, since if $i \notin K_n$ then both sides are 0.

Note next that for every n, $i \in \mathbb{N}$,

$$b_{ni} = b_{n+1,2i}, \quad \beta_{ni} = \beta_{n+1,2i}, \quad c_{ni} = c_{n+1,2i} \cup c_{n+1,2i+1}, \quad \gamma_{ni} = \gamma_{n+1,2i} + \gamma_{n+1,2i+1},$$

so that, for $i \geq 1$,

$$\int_{c_{ni}} T_n u = \int_{\beta_{n,i+1}}^{\beta_{ni}} u = \int_{c_{ni}} T_{n+1} u.$$

This means that if T is any cluster point of $\langle T_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{T}$ for the very weak operator topology (and such a cluster point exists, by 373L), $\int_{c_{mi}} Tu$ must be a cluster point of $\langle\int_{c_{mi}} T_n u\rangle_{n\in\mathbb{N}}$, and therefore equal to $\int_{c_{mi}} T_m u$, for every $m \in \mathbb{N}$, $i \geq 1$.

Consequently, if $m \in \mathbb{N}$,

$$\int |Tu \times v| \geq \sum_{i=0}^\infty \int_{c_{mi}} |Tu| \times v \geq \sum_{i=0}^\infty 2^{-m}i \int_{c_{mi}} |Tu|$$

(because $c_{mi} \subseteq [\![v > 2^{-m}i]\!]$)

$$\geq \sum_{i=1}^\infty 2^{-m}i\Bigl|\int_{c_{mi}} Tu\Bigr| = \sum_{i=1}^\infty 2^{-m}i \int_{c_{mi}} T_m u$$

$$= \sum_{i=0}^\infty 2^{-m}i \int_{\beta_{m,i+1}}^{\beta_{mi}} u \geq \int u \times (v^* - 2^{-m}\chi 1)^+$$

because

$$[\beta_{m,i+1}, \beta_{mi}]^\bullet \subseteq [\![v^* \leq 2^{-m}(i+1)]\!] = [\![(v^* - 2^{-m}\chi 1)^+ \leq 2^{-m}i]\!]$$

for each $i \in \mathbb{N}$. But letting $m \to \infty$, we have

$$\int |Tu \times v| \geq \lim_{m\to\infty} \int u \times (v^* - 2^{-m}\chi 1)^+ = \int u \times v^*$$

because $\langle u \times (v^* - 2^{-m}\chi 1)^+\rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence with supremum $u \times v^*$. In view of the reductions in (a) above, this is enough to complete the proof.

373Q Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, $(\mathfrak{B}, \bar{\nu})$ a semi-finite measure algebra, $u \in M^{1,\infty}(\mathfrak{A}, \bar{\mu})$ and $v \in M^{0,\infty}(\mathfrak{B}, \bar{\nu})$. Then

$$\int u^*\times v^* = \sup\{\int |Tu\times v| : T\in\mathcal{T}_{\bar\mu,\bar\nu}\} = \sup\{\int Tu\times v : T\in\mathcal{T}_{\bar\mu,\bar\nu}\}.$$

proof There is a non-decreasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}$ such that $\bar\nu c_n<\infty$ for every n and $v^*=\sup_{n\in\mathbb{N}}(v\times\chi c_n)^*$. **P** For each rational $q>0$, we can find a countable non-empty set $B_q\subseteq\mathfrak{B}$ such that

$$b\subseteq [\![|v|>q]\!],\ \bar\nu b<\infty \text{ for every } b\in B_q,$$

$$\sup_{b\in B_q}\bar\nu b = \bar\nu[\![|v|>q]\!]$$

(because $(\mathfrak{B},\bar\nu)$ is semi-finite). Let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\bigcup_{q\in\mathbb{Q},q>0}B_q$ and set $c_n=\sup_{i\le n}b_i$, $v_n=v\times\chi c_n$ for each n. Then $\langle|v_n|\rangle_{n\in\mathbb{N}}$ and $\langle v_n^*\rangle_{n\in\mathbb{N}}$ are non-decreasing and $\sup_{n\in\mathbb{N}}v_n^*\le v^*$ in $L^0(\mathfrak{A}_L)$. But in fact $\sup_{n\in\mathbb{N}}v_n^*=v^*$, because

$$\bar\mu_L[\![v^*>q]\!]=\bar\mu[\![|v|>q]\!]=\sup_{n\in\mathbb{N}}\bar\mu[\![v_n>q]\!]=\sup_{n\in\mathbb{N}}\bar\mu_L[\![v_n^*>q]\!]=\bar\mu_L[\![\sup_{n\in\mathbb{N}}v_n^*>q]\!]$$

for every rational $q>0$, by 373Da. **Q**

For each $n\in\mathbb{N}$ we have a $T_n\in\mathcal{T}_{\bar\mu,\bar\nu}$ such that $\int T_nu\times v_n=\int u^*\times v_n^*$ (373P). Set $S_nw=T_nw\times\chi c_n$ for $n\in\mathbb{N}$, $w\in M^{1,\infty}_{\bar\mu}$; then every S_n belongs to $\mathcal{T}_{\bar\mu,\bar\nu}$, so

$$\begin{aligned}\sup\{\int Tu\times v : T\in\mathcal{T}_{\bar\mu,\bar\mu}\} &\ge \sup_{n\in\mathbb{N}}\int S_nu\times v = \sup_{n\in\mathbb{N}}\int T_nu\times v_n\\ &= \sup_{n\in\mathbb{N}}\int u^*\times v_n^* = \int u^*\times v^*\\ &\ge \sup\{\int |Tu\times v| : T\in\mathcal{T}_{\bar\mu,\bar\mu}\} \ge \sup\{\int Tu\times v : T\in\mathcal{T}_{\bar\mu,\bar\mu}\}\end{aligned}$$

by 373J, as usual.

373R Order-continuous operators: Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, $(\mathfrak{B},\bar\nu)$ a localizable measure algebra, and $T_0\in\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$. Then there is a $T\in\mathcal{T}^\times=\mathcal{T}^\times_{\bar\mu,\bar\nu}$ extending T_0. If $(\mathfrak{A},\bar\mu)$ is semi-finite, T is uniquely defined.

proof (a) Suppose first that $T_0\in\mathcal{T}^{(0)}$ is non-negative, regarded as a member of $\mathsf{L}^\sim(M^{1,0}_{\bar\mu};M^{1,0}_{\bar\nu})$. In this case T_0 has an extension to an order-continuous positive linear operator $T:M^{1,\infty}_{\bar\mu}\to L^0(\mathfrak{B})$ defined by saying that $Tw=\sup\{T_0u : u\in M^{1,0}_{\bar\mu},\ 0\le u\le w\}$ for every $w\ge 0$ in $M^{1,\infty}_{\bar\mu}$. **P** I use 355F. $M^{1,0}_{\bar\mu}$ is a solid linear subspace of $M^{1,\infty}_{\bar\mu}$. T_0 is order-continuous when its codomain is taken to be $M^{1,0}_{\bar\nu}$, as noted in 371Gb, and therefore if its codomain is taken to be $L^0(\mathfrak{B})$, because $M^{1,0}$ is a solid linear subspace in L^0, so the embedding is order-continuous. If $w\ge 0$ in $M^{1,\infty}_{\bar\mu}$, let $\gamma\ge 0$ be such that $u_1=(w-\gamma\chi 1)^+$ is integrable. If $u\in M^{1,0}_{\bar\mu}$ and $0\le u\le w$, then $(u-\gamma\chi 1)^+\le u_1$, so

$$T_0u=T_0(u-\gamma\chi 1)^+ + T_0(u\wedge\gamma\chi 1)\le T_0u_1+\gamma\chi 1\in L^0(\mathfrak{B}).$$

Thus $\{T_0u : u\in M^{1,0}_{\bar\nu},\ 0\le u\le w\}$ is bounded above in $L^0(\mathfrak{B})$, for any $w\ge 0$ in $M^{1,\infty}_{\bar\mu}$. $L^0(\mathfrak{B})$ is Dedekind complete, because $(\mathfrak{B},\bar\nu)$ is localizable, so $\sup\{T_0u : 0\le u\le w\}$ is defined in $L^0(\mathfrak{B})$; and this is true for every $w\in(M^{1,\infty}_{\bar\mu})^+$. Thus the conditions of 355F are satisfied and we have the result. **Q**

(b) Now suppose that T_0 is any member of $\mathcal{T}^{(0)}$. Then T_0 has an extension to a member of $\mathcal{T}^\times$. **P** $|T_0|$, $T_0^+=\frac12(|T_0|+T_0)$ and $T_0^-=\frac12(|T_0|-T_0)$, taken in $\mathsf{L}^\sim(M^{1,0}_{\bar\mu};M^{1,0}_{\bar\nu})$, all belong to $\mathcal{T}^{(0)}$ (371G), so have extensions S, S_1 and S_2 to order-continuous positive linear operators from $M^{1,\infty}_{\bar\mu}$ to $L^0(\mathfrak{B})$ as defined in (a). Now for any $w\in L^1_{\bar\mu}$,

$$\|Sw\|_1=\||T_0|w\|_1\le\|w\|_1,$$

and for any $w\in L^\infty(\mathfrak{A})$,

$$|Sw|\le S|w|=\sup\{|T_0|u : u\in M^{1,0}_{\bar\mu},\ 0\le u\le w\}\le\|w\|_\infty\chi 1,$$

so $\|Sw\|_\infty\le\|w\|_\infty$. Thus $S\in\mathcal{T}$; similarly, S_1 and S_2 can be regarded as operators from $M^{1,\infty}_{\bar\mu}$ to $M^{1,\infty}_{\bar\nu}$, and as such belong to $\mathcal{T}$. Next, for $w\ge 0$ in $M^{1,\infty}_{\bar\mu}$,

$$\begin{aligned}S_1w+S_2w &= \sup\{T_0^+u : u\in M^{1,0}_{\bar\mu},\ 0\le u\le w\}+\sup\{T_0^-u : u\in M^{1,0}_{\bar\mu},\ 0\le u\le w\}\\ &= \sup\{T_0^+u+T_0^-u : u\in M^{1,0}_{\bar\mu},\ 0\le u\le w\}=Sw.\end{aligned}$$

But this means that

$$S = S_1 + S_2 \geq |S_1 - S_2|$$

and $T = S_1 - S_2 \in \mathcal{T}$, by 373Bc; while of course T extends $T_0^+ - T_0^- = T_0$. Finally, because S_1 and S_2 are order-continuous, $T \in \mathsf{L}^\times(M_{\bar\mu}^{1,\infty}; M_{\bar\nu}^{1,\infty})$, so $T \in \mathcal{T}^\times$. **Q**

(c) If $(\mathfrak{A}, \bar\mu)$ is semi-finite, then $M_{\bar\mu}^{1,0}$ is order-dense in $M_{\bar\mu}^{1,\infty}$ (because it includes $L_{\bar\mu}^1$, which is order-dense in $L^0(\mathfrak{A})$); so that the extension T is unique, by 355Fe.

373S Adjoints in $\mathcal{T}^{(0)}$: Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras, and T any member of $\mathcal{T}_{\bar\mu,\bar\nu}^{(0)}$. Then there is a unique operator $T' \in \mathcal{T}_{\bar\nu,\bar\mu}^{(0)}$ such that $\int_a T'(\chi b) = \int_b T(\chi a)$ for every $a \in \mathfrak{A}^f$, $b \in \mathfrak{B}^f$, and now $\int u \times T'v = \int Tu \times v$ whenever $u \in M^{1,0}(\mathfrak{A}, \bar\mu)$, $v \in M^{1,0}(\mathfrak{B}, \bar\nu)$ are such that $\int u^* \times v^* < \infty$.

proof (a) For each $v \in M_{\bar\nu}^{1,0}$ we can define $T'v \in M_{\bar\mu}^{1,0}$ by the formula

$$\int_a T'v = \int T(\chi a) \times v$$

for every $a \in \mathfrak{A}^f$. **P** Set $\theta a = \int T(\chi a) \times v$ for each $a \in \mathfrak{A}^f$; because $\int (\chi a)^* \times v^* < \infty$, the integral is defined and finite (373J). Of course $\theta : \mathfrak{A}^f \to \mathbb{R}$ is additive because χ is additive and T, $\times$ and $\int$ are linear. Also

$$\lim_{t\downarrow 0} \sup_{\bar\mu a \leq t} |\theta a| \leq \lim_{t\downarrow 0} \int_0^t v^* = 0,$$

$$\lim_{t\to\infty} \frac{1}{t} \sup_{\bar\mu a \leq t} |\theta a| \leq \lim_{t\to\infty} \tfrac{1}{t} \int_0^t v^* = 0$$

because $v \in M_{\bar\nu}^{1,0}$, so $v^* \in M_{\bar\mu_L}^{1,0}$. By 373Ha, there is a unique $T'v \in M_{\bar\mu}^{1,0}$ such that $\int_a T'v = \theta a$ for every $a \in \mathfrak{A}^f$. **Q**

(b) Because the formula uniquely determines $T'v$, we see that $T' : M_{\bar\nu}^{1,0} \to M_{\bar\mu}^{1,0}$ is linear. Now $T' \in \mathcal{T}_{\bar\nu,\bar\mu}^{(0)}$. **P** (i) If $v \in L_{\bar\nu}^1$, then (because $T'v \in M_{\bar\mu}^{1,0}$) $|T'v| = \sup_{a\in\mathfrak{A}^f} |T'v| \times \chi a$, and

$$\begin{aligned} \|T'v\|_1 &= \int |T'v| = \sup_{a\in\mathfrak{A}^f} \int_a |T'v| = \sup_{b,c\in\mathfrak{A}^f} (\int_b T'v - \int_c T'v) \\ &= \sup_{b,c\in\mathfrak{A}^f} \int T(\chi b - \chi c) \times v \leq \sup_{b,c\in\mathfrak{A}^f} \int (\chi b - \chi c)^* \times v^* \\ &= \int v^* = \|v\|_1. \end{aligned}$$

(ii) Now suppose that $v \in L^\infty(\mathfrak{B}) \cap M_{\bar\nu}^{1,0}$, and set $\gamma = \|v\|_\infty$. **?** If $a = [\![|T'v| > \gamma]\!] \neq 0$, then $T'v \neq 0$ so $v \neq 0$ and $\gamma > 0$ and $\bar\mu a < \infty$, because $T'v \in M_{\bar\mu}^{1,0}$. Set $b = [\![(T'v)^+ > \gamma]\!]$, $c = [\![(T'v)^- > \gamma]\!]$; then

$$\begin{aligned} \gamma\bar\mu a &< \int_a |T'v| = \int_b T'v - \int_c T'v = \int T(\chi b - \chi c) \times v \\ &\leq \gamma \|T(\chi b - \chi c)\|_1 \leq \gamma \|\chi b - \chi c\|_1 = \gamma\bar\mu a, \end{aligned}$$

which is impossible. **X** Thus $[\![|T'v| > \gamma]\!] = 0$ and $\|T'v\|_\infty \leq \gamma = \|v\|_\infty$.

Putting this together with (i), we see that $T' \in \mathcal{T}_{\bar\nu,\bar\mu}^{(0)}$. **Q**

(c) Let $|T|$ be the modulus of T in $\mathsf{L}^\sim(M_{\bar\mu}^{1,0}; M_{\bar\nu}^{1,0})$, so that $|T| \in \mathcal{T}_{\bar\mu,\bar\mu}^{(0)}$, by 371Gb. If $u \geq 0$ in $M_{\bar\mu}^{1,0}$, $v \geq 0$ in $M_{\bar\nu}^{1,0}$ are such that $\int u^* \times v^* < \infty$, let $\langle u_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A}^f)^+$ with supremum u. In this case $|T|u = \sup_{n\in\mathbb{N}} |T|u_n$, so $\int |T|u \times v = \sup_{n\in\mathbb{N}} \int |T|u_n \times v$ and

$$|\int Tu \times v - \int Tu_n \times v| \leq \int |T|(u - u_n) \times v \to 0$$

as $n \to \infty$, because

$$\int |T|u \times v \leq \int u^* \times v^* < \infty.$$

At the same time,

$$|\int u \times T'v - \int u_n \times T'v| \leq \int (u - u_n) \times |T'v| \to 0$$

because $\int u \times |T'v| \le \int u^* \times v^* < \infty$. So

$$\int Tu \times v = \lim_{n\to\infty} \int Tu_n \times v = \lim_{n\to\infty} \int u_n \times T'v = \int u \times T'v,$$

the middle equality being valid because each u_n is a linear combination of indicator functions.

Because T and T' are linear, it follows at once that $\int u \times T'v = \int Tu \times v$ whenever $u \in M^{1,0}_{\bar\mu}$, $v \in M^{1,0}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$.

(d) Finally, to see that T' is uniquely defined by the formula in the statement of the theorem, observe that this surely defines $T'(\chi b)$ for every $b \in \mathfrak{B}^f$, by the remarks in (a). Consequently it defines T' on $S(\mathfrak{B}^f)$. Since $S(\mathfrak{B}^f)$ is order-dense in $M^{1,0}_{\bar\mu}$, and any member of $\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ must belong to $\mathsf{L}^\times(M^{1,0}_{\bar\nu};M^{1,0}_{\bar\mu})$ (371Gb), the restriction of T' to $S(\mathfrak{B}^f)$ determines T' (355J).

373T Corollary Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras. Then for any $T \in \mathcal{T}^\times_{\bar\mu,\bar\nu}$ there is a unique $T' \in \mathcal{T}^\times_{\bar\nu,\bar\mu}$ such that $\int u \times T'v = \int Tu \times v$ whenever $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$, $v \in M^{1,\infty}(\mathfrak{B},\bar\nu)$ are such that $\int u^* \times v^* < \infty$.

proof The restriction $T{\restriction}M^{1,0}_{\bar\mu}$ belongs to $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ (373Bb), so there is a unique $S \in \mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ such that $\int u \times Sv = \int Tu \times v$ whenever $u \in M^{1,0}_{\bar\mu}$, $v \in M^{1,0}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$ (373S). Now there is a unique $T' \in \mathcal{T}^\times_{\bar\nu,\bar\mu}$ extending S (373R). If $u \ge 0$ in $M^{1,\infty}_{\bar\mu}$, $v \ge 0$ in $M^{1,\infty}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$, then $\int u \times T'v = \int Tu \times v$. **P** If $T \ge 0$, then both are

$$\sup\{\int u_0 \times T'v_0 : u_0 \in M^{1,0}_{\bar\mu},\ v \in M^{1,0}_{\bar\nu},\ 0 \le u_0 \le u,\ 0 \le v_0 \le v\}$$

because both T and T' are (order-)continuous. In general, we can apply the same argument to T^+ and T^-, taken in $\mathsf{L}^\sim(M^{1,\infty}_{\bar\mu};M^{1,\infty}_{\bar\nu})$, since these belong to $\mathcal{T}^\times_{\bar\mu,\bar\nu}$, by 373B and 355H, and we shall surely have $T' = (T^+)' - (T^-)'$. **Q**

As in 373S, it follows that $\int u \times T'v = \int Tu \times v$ whenever $u \in M^{1,\infty}_{\bar\mu}$, $v \in M^{1,\infty}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$.

373U Corollary Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous measure-preserving Boolean homomorphism. Then the associated map $T \in \mathcal{T}^\times_{\bar\mu,\bar\nu}$ (373Bd) has an adjoint $P \in \mathcal{T}^\times_{\bar\nu,\bar\mu}$ defined by the formula $\int_a P(\chi b) = \bar\nu(b \cap \pi a)$ for $a \in \mathfrak{A}^f$, $b \in \mathfrak{B}^f$.

proof The adjoint $P = T'$ must have the property that

$$\int_a P(\chi b) = \int \chi a \times P(\chi b) = \int T(\chi a) \times \chi b = \int \chi(\pi a) \times \chi b = \bar\nu(\pi a \cap b)$$

for every $a \in \mathfrak{A}^f$, $b \in \mathfrak{B}^f$. To see that this defines P uniquely, let $S \in \mathcal{T}^\times_{\bar\nu,\bar\mu}$ be any other operator with the same property. By 373Hb, $S(\chi b) = P(\chi b)$ for every $b \in \mathfrak{B}^f$, so S and P agree on $S(\mathfrak{B}^f)$. Because both P and S are supposed to belong to $\mathsf{L}^\times(M^{1,\infty}_{\bar\nu};M^{1,\infty}_{\bar\mu})$, and $S(\mathfrak{B}^f)$ is order-dense in $M^{1,\infty}_{\bar\nu}$, $S = P$, by 355J.

373X Basic exercises (a) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a *ring* homomorphism such that $\bar\nu\pi a \le \bar\mu a$ for every $a \in \mathfrak{A}$. (i) Show that there is a unique $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ such that $T(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$, and that T is a Riesz homomorphism. (ii) Show that T is (sequentially) order-continuous iff π is.

>(b) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\mu)$ be measure algebras, and $\phi : \mathbb{R} \to \mathbb{R}$ a convex function such that $\phi(0) \le 0$. Show that if $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ and $T \ge 0$, then $\bar\phi(Tu) \le T(\bar\phi(u))$ whenever $u \in M^{1,\infty}_{\bar\mu}$ is such that $\bar\phi(u) \in M^{1,\infty}_{\bar\mu}$. (*Hint*: 233J, 365Rb.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Show that if $w \in L^\infty(\mathfrak{A})$ and $\|w\|_\infty \le 1$ then $u \mapsto u \times w : M^{1,\infty}_{\bar\mu} \to M^{1,\infty}_{\bar\mu}$ belongs to $\mathcal{T}^\times_{\bar\mu,\bar\mu}$.

(d) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. Show that if $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$, $\mathfrak{B}$ respectively, and $\langle T_i\rangle_{i\in I}$ is any family in $\mathcal{T}_{\bar\mu,\bar\nu}$, and *either* I is countable *or* $\mathfrak{B}$ is Dedekind complete, then we have an operator $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ such that $Tu \times \chi b_i = T_i(u \times \chi a_i) \times \chi b_i$ for every $u \in M^{1,\infty}_{\bar\mu,\bar\nu}$, $i \in I$.

>(e) Let I, J be sets and write $\mu = \bar\mu$, $\nu = \bar\nu$ for counting measure on I, J respectively. Show that there is a natural one-to-one correspondence between $\mathcal{T}^\times_{\bar\mu,\bar\nu}$ and the set of matrices $\langle a_{ij}\rangle_{i\in I,j\in J}$ such that $\sum_{i\in I}|a_{ij}| \le 1$ for every $j \in J$, $\sum_{j\in J}|a_{ij}| \le 1$ for every $i \in I$.

>(f) Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces, with measure algebras $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$, and product measure λ on $X \times Y$. Let $h : X \times Y \to \mathbb{R}$ be a measurable function such that $\int |h(x,y)|dx \le 1$ for ν-almost every $y \in Y$ and $\int |h(x,y)|dy \le 1$ for μ-almost every $x \in X$. Show that there is a corresponding $T \in \mathcal{T}^\times_{\bar\mu,\bar\nu}$ defined by writing $T(f^\bullet) = g^\bullet$ whenever $f \in \mathcal{L}^1(\mu) + \mathcal{L}^\infty(\mu)$ and $g(y) = \int h(x,y)f(x)dx$ for almost every y.

>(g) Let μ be Lebesgue measure on $\mathbb{R}$, and $(\mathfrak{A},\bar{\mu})$ its measure algebra. Show that for any μ-integrable function h with $\int|h|d\mu \le 1$ we have a corresponding $T \in \mathcal{T}^{\times}_{\bar{\mu},\bar{\mu}}$ defined by setting $T(f^\bullet) = (h*f)^\bullet$ whenever $g \in \mathcal{L}^1(\mu)+\mathcal{L}^\infty(\mu)$, writing $h*f$ for the convolution of h and f (255E). Explain how this may be regarded as a special case of 373Xf.

>(h) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $u \in L^0(\mathfrak{A})^+$; let ν_u be its distribution (364Xd). Show that each of u^*, ν_u is uniquely determined by the other.

(i) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism; let $T: M^{1,\infty}_{\bar{\mu}} \to M^{1,\infty}_{\bar{\nu}}$ be the corresponding operator (373Bd). Show that $(Tu)^* = u^*$ for every $u \in M^{1,\infty}_{\bar{\mu}}$.

(j) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra, and A a subset of $L^1_{\bar{\mu}}$. Show that the following are equiveridical: (i) A is uniformly integrable; (ii) $\{u^* : u \in A\}$ is uniformly integrable in $L^1_{\bar{\mu}_L}$; (iii) $\lim_{t\downarrow 0}\sup_{u\in A}\int_0^t u^* = 0$.

(k) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $A \subseteq (M^0_{\bar{\mu}})^+$ a non-empty downwards-directed set. Show that $(\inf A)^* = \inf_{u\in A} u^*$ in $L^0(\mathfrak{A}_L)$.

(l) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Show that $\|u\|_{1,\infty} = \int_0^1 u^*$ for every $u \in M^{1,\infty}(\mathfrak{A},\bar{\mu})$.

(m) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and ϕ a Young's function (369Xc). Write $U_{\phi,\bar{\mu}} \subseteq L^0(\mathfrak{A})$, $U_{\phi,\bar{\nu}} \subseteq L^0(\mathfrak{B})$ for the corresponding Orlicz spaces. (i) Show that if $T \in \mathcal{T}_{\bar{\mu},\bar{\nu}}$ and $u \in U_{\phi,\bar{\mu}}$, then $Tu \in U_{\phi,\bar{\nu}}$ and $\|Tu\|_\phi \le \|u\|_\phi$. (ii) Show that $u \in U_{\phi,\bar{\mu}}$ iff $u^* \in U_{\phi,\bar{\mu}_L}$, and in this case $\|u\|_\phi = \|u^*\|_\phi$.

>(n) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and $(\mathfrak{B},\bar{\nu})$ a totally finite measure algebra. Show that if $A \subseteq L^1_{\bar{\mu}}$ is uniformly integrable, then $\{Tu : u \in A,\, T \in \mathcal{T}_{\bar{\mu},\bar{\nu}}\}$ is uniformly integrable in $L^1_{\bar{\nu}}$.

(o)(i) Give examples of u, $v \in L^1(\mathfrak{A}_L)$ such that $(u+v)^* \not\le u^*+v^*$. (ii) Show that if $(\mathfrak{A},\bar{\mu})$ is any measure algebra and u, $v \in M^{0,\infty}_{\bar{\mu}}$, then $\int_0^t (u+v)^* \le \int_0^t u^*+v^*$ for every $t \ge 0$.

(p) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be two measure algebras. For $u \in M^{1,0}_{\bar{\mu}}$, $w \in M^{\infty,1}_{\bar{\nu}}$ set

$$\rho_{uw}(S,T) = |\int (Su-Tu)\times w| \text{ for all } S,\, T \in \mathcal{T}^{(0)} = \mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}.$$

The topology generated by the pseudometrics ρ_{uw} is the **very weak operator topology** on $\mathcal{T}^{(0)}$. Show that $\mathcal{T}^{(0)}$ is compact in this topology.

(q) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras and let $u \in M^{1,0}_{\bar{\mu}}$. (i) Show that $B = \{Tu : T \in \mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}\}$ is compact for the topology $\mathfrak{T}_s(M^{1,0}_{\bar{\nu}}, M^{\infty,1}_{\bar{\nu}})$. (ii) Show that any non-decreasing sequence in B has a supremum in $L^0(\mathfrak{B})$ which belongs to B.

(r) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $u \in M^{1,0}_{\bar{\mu}}$, $v \in M^{1,0}_{\bar{\nu}}$. Show that the following are equiveridical: (i) there is a $T \in \mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}$ such that $Tu = v$; (ii) $\int_0^t u^* \le \int_0^t v^*$ for every $t \ge 0$.

(s) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras. Suppose that u_1, $u_2 \in M^{1,\infty}_{\bar{\mu}}$ and $v \in M^{1,\infty}_{\bar{\nu}}$ are such that $\int_0^t v^* \le \int_0^t (u_1+u_2)^*$ for every $t \ge 0$. Show that there are v_1, $v_2 \in M^{1,\infty}_{\bar{\nu}}$ such that $v_1+v_2 = v$ and $\int_0^t v_i^* \le \int_0^t u_i^*$ for both i, every $t \ge 0$.

>(t) Set $g(t) = t/(t+1)$ for $t \ge 0$, and set $v = g^\bullet$, $u = \chi[0,1]^\bullet \in L^\infty(\mathfrak{A}_L)$. Show that $\int u^*\times v^* = 1 > \int Tu\times v$ for every $T \in \mathcal{T}_{\bar{\mu}_L,\bar{\mu}_L}$.

(u) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and for $T \in \mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}$ define $T' \in \mathcal{T}^{(0)}_{\bar{\nu},\bar{\mu}}$ as in 373S. Show that $T'' = T$.

(v) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and give $\mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}$, $\mathcal{T}^{(0)}_{\bar{\nu},\bar{\mu}}$ their very weak operator topologies (373Xp). Show that the map $T \mapsto T' : \mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}} \to \mathcal{T}^{(0)}_{\bar{\nu},\bar{\mu}}$ is an isomorphism for the convex, order and topological structures of the two spaces. (By the 'convex structure' of a convex set C in a linear space I mean the operation $(x,y,t) \mapsto tx+(1-t)y : C\times C\times[0,1] \to C$.)

373Y Further exercises (a) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$. Set $u=f^\bullet$ and $v=g^\bullet$ in $L^0(\mathfrak{A})$, where $f(t)=t$, $g(t)=1-2|t-\frac{1}{2}|$ for $t\in[0,1]$. Show that $u^*=v^*$, but that there is no measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $T_\pi v=u$, writing $T_\pi:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ for the operator induced by π, as in 364P. (*Hint*: show that $\{[\![v>\alpha]\!]:\alpha\in\mathbb{R}\}$ does not τ-generate $\mathfrak{A}$.)

(b) Let $(\mathfrak{A},\bar\mu)$ be a totally finite homogeneous measure algebra of uncountable Maharam type. Let $u,v\in(M^{1,\infty}_{\bar\mu})^+$ be such that $u^*=v^*$. Show that there is a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $T_\pi u=v$.

(c) Let u, $v\in M^{1,\infty}_{\bar\mu_L}$ be such that $u=u^*$, $v=v^*$ and $\int_0^t v\le\int_0^t u$ for every $t\ge 0$. Show that there is a non-negative $T\in\mathcal{T}_{\bar\mu_L,\bar\mu_L}$ such that $Tu=v$ and $\int_0^t Tw\le\int_0^t w$ for every $w\ge 0$ in $M^{1,\infty}$. Show that any such T must belong to $\mathcal{T}^\times_{\bar\mu_L,\bar\mu_L}$.

(d) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $u\in M^{1,\infty}_{\bar\mu}$. (i) Suppose that $w\in S(\mathfrak{B}^f)$. Show directly (without quoting the result of 373O, but possibly using some of the ideas of the proof) that for every $\gamma<\int u^*\times w^*$ there is a $T\in\mathcal{T}_{\bar\mu,\bar\nu}$ such that $\int Tu\times w\ge\gamma$. (ii) Suppose that $(\mathfrak{B},\bar\nu)$ is localizable and that $v\in M^{1,\infty}_{\bar\nu}\setminus\{Tu:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}$. Show that there is a $w\in S(\mathfrak{B}^f)$ such that $\int v\times w>\sup\{\int Tu\times w:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}$. (*Hint*: use 373M and the Hahn-Banach theorem in the following form: if U is a linear space with the topology $\mathfrak{T}_s(U,V)$ defined by a linear subspace V of $\mathrm{L}(U;\mathbb{R})$, $C\subseteq U$ is a non-empty closed convex set, and $v\in U\setminus C$, then there is an $f\in V$ such that $f(v)>\sup_{u\in C}f(u)$.) (iii) Hence prove 373O for localizable $(\mathfrak{B},\bar\nu)$. (iv) Now prove 373O for general $(\mathfrak{B},\bar\nu)$.

(e)(i) Define $v\in L^\infty(\mathfrak{A}_L)$ as in 373Xt. Show that there is no $T\in\mathcal{T}^\times_{\bar\mu_L,\bar\mu_L}$ such that $Tv=v^*$. (ii) Set $h(t)=1+\max(0,\frac{\sin t}{t})$ for $t>0$, $w=h^\bullet\in L^\infty(\mathfrak{A}_L)$. Show that there is no $T\in\mathcal{T}^\times_{\bar\mu_L,\bar\mu_L}$ such that $Tw^*=w$.

(f) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$. Show that $\mathcal{T}_{\bar\mu,\bar\mu_L}=\mathcal{T}^\times_{\bar\mu,\bar\mu_L}$ can be identified, as convex ordered space, with $\mathcal{T}^\times_{\bar\mu_L,\bar\mu}$, and that this is a proper subset of $\mathcal{T}_{\bar\mu_L,\bar\mu}$.

(g) Show that the adjoint operation of 373T is not as a rule continuous for the very weak operator topologies of $\mathcal{T}^\times_{\bar\mu,\bar\nu}$, $\mathcal{T}^\times_{\bar\nu,\bar\mu}$.

373 Notes and comments 373A-373B are just alternative expressions of concepts already treated in 371F-371H. My use of the simpler formula $\mathcal{T}_{\bar\mu,\bar\nu}$ symbolizes my view that $\mathcal{T}$, rather than $\mathcal{T}^{(0)}$ or $\mathcal{T}^\times$, is the most natural vehicle for these ideas; I used $\mathcal{T}^{(0)}$ in §§371-372 only because that made it possible to give theorems which applied to all measure algebras, without demanding localizability (compare 371Gb with 373Bc).

The obvious examples of operators in $\mathcal{T}$ are those derived from measure-preserving Boolean homomorphisms, as in 373Bd, and their adjoints (373U). Note that the latter include conditional expectation operators. In return, we find that operators in $\mathcal{T}$ share some of the characteristic properties of the operators derived from Boolean homomorphisms (373Bb, 373Xb, 373Xm). Other examples are multiplication operators (373Xc), operators obtained by piecing others together (373Xd) and kernel operators of the type described in 373Xe-373Xf, including convolution operators (373Xg). (For a general theory of kernel operators, see §376 below.)

Most of the section is devoted to the relationships between the classes $\mathcal{T}$ of operators and the 'decreasing rearrangements' of 373C. If you like, the decreasing rearrangement u^* of u describes the 'distribution' of $|u|$ (373Xh); but for $u\notin M^0$ it loses some information (373Xt, 373Ye). It is important to be conscious that even when $u\in L^0(\mathfrak{A}_L)$, u^* is not necessarily obtained by 'rearranging' the elements of the algebra $\mathfrak{A}_L$ by a measure-preserving automorphism (which would, of course, correspond to an automorphism of the measure space $([0,\infty[\,,\mu_L)$, by 344C). I will treat 'rearrangements' of this narrower type in the next section; for the moment, see 373Ya. Apart from this, the basic properties of decreasing rearrangements are straightforward enough (373D-373F). The only obscure area concerns the relationship between $(u+v)^*$ and u^*, v^* (see 373Xo).

In 373G I embark on results involving both decreasing rearrangements and operators in $\mathcal{T}$, leading to the characterization of the sets $\{Tu:T\in\mathcal{T}\}$ in 373O. In one direction this is easy, and is the content of 373G. In the other direction it depends on a deeper analysis, and the easiest method seems to be through studying the 'very weak operator topology' on $\mathcal{T}$ (373K-373L), even though this is an effective tool only when one of the algebras involved is localizable (373L). A functional analyst is likely to feel that the method is both natural and illuminating; but from the point of view of a measure theorist it is not perfectly satisfactory, because it is essentially non-constructive. While it tells us that there are operators $T\in\mathcal{T}$ acting in the required ways, it gives only the vaguest of hints concerning what they actually look like.

Of course the very weak operator topology is interesting in its own right; and see also 373Xp-373Xq.

The proof of 373O can be thought of as consisting of three steps. Given that $\int_0^t v^* \le \int_0^t u^*$ for every t, then I set out to show that v is expressible as $T_1 v^*$ (parts (c)-(d) of the proof), that v^* is expressible as $T_2 u^*$ (part (g)) and that u^* is expressible as $T_3 u$ (parts (e)-(f)), each T_i belonging to an appropriate $\mathcal{T}$. In all three steps the general case follows easily from the case of u, $v \in S(\mathfrak{A})$, $S(\mathfrak{B})$. If we are willing to use a more sophisticated version of the Hahn-Banach theorem than those given in 3A5A and 363R, there is an alternative route (373Yd). I note that the central step above, from u^* to v^*, can be performed with an order-continuous T_2 (373Yc), but that in general neither of the other steps can (373Ye), so that we cannot use $\mathcal{T}^\times$ in place of $\mathcal{T}$ here.

A companion result to 373O, in that it also shows that $\{Tu : T \in \mathcal{T}\}$ is large enough to reach natural bounds, is 373P; given u and v, we can find T such that $\int Tu \times v$ is as large as possible. In this form the result is valid only for $v \in M^{(0)}$ (373Xt). But if we do not demand that the supremum should be attained, we can deal with other v (373Q).

We already know that every operator in $\mathcal{T}^{(0)}$ is a difference of order-continuous operators, just because $M^{1,0}$ has an order-continuous norm (371Gb). It is therefore not surprising that members of $\mathcal{T}^{(0)}$ can be extended to members of $\mathcal{T}^\times$, at least when the codomain $M^{1,\infty}_{\bar\nu}$ is Dedekind complete (373R). It is also very natural to look for a correspondence between $\mathcal{T}_{\bar\mu,\bar\nu}$ and $\mathcal{T}_{\bar\nu,\bar\mu}$, because if $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ we shall surely have an adjoint operator $(T{\restriction}L^1_{\bar\mu})'$ from $(L^1_{\bar\nu})^*$ to $(L^1_{\bar\mu})^*$, and we can hope that this will correspond to some member of $\mathcal{T}_{\bar\nu,\bar\mu}$. But when we come to the details, the normed-space properties of a general member of $\mathcal{T}$ are not enough (373Yf), and we need some kind of order-continuity. For members of $\mathcal{T}^{(0)}$ this is automatically present (373S), and now the canonical isomorphism between $\mathcal{T}^{(0)}$ and $\mathcal{T}^\times$ gives us an isomorphism between $\mathcal{T}^\times_{\bar\mu,\bar\nu}$ and $\mathcal{T}^\times_{\bar\nu,\bar\mu}$ when $\bar\mu$ and $\bar\nu$ are localizable (373T).

374 Rearrangement-invariant spaces

As is to be expected, many of the most important function spaces of analysis are symmetric in various ways; in particular, they share the symmetries of the underlying measure algebras. The natural expression of this is to say that they are 'rearrangement-invariant' (374E). In fact it turns out that in many cases they have the stronger property of '$\mathcal{T}$-invariance' (374A). In this section I give a brief account of the most important properties of these two kinds of invariance. In particular, $\mathcal{T}$-invariance is related to a kind of transfer mechanism, enabling us to associate function spaces on different measure algebras (374C-374D). As for rearrangement-invariance, the salient fact is that on the most important measure algebras many rearrangement-invariant spaces are $\mathcal{T}$-invariant (374K, 374M).

374A $\mathcal{T}$-invariance: Definitions Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Recall that I write

$$M^{1,\infty}_{\bar\mu} = L^1_{\bar\mu} + L^\infty(\mathfrak{A}) \subseteq L^0(\mathfrak{A}),$$

$$M^{\infty,1}_{\bar\mu} = L^1_{\bar\mu} \cap L^\infty(\mathfrak{A}),$$

$$M^{0,\infty}_{\bar\mu} = \{u : u \in L^0(\mathfrak{A}),\ \inf_{\alpha>0} \bar\mu[\![|u| > \alpha]\!] < \infty\},$$

(369N, 373C).

(a) I will say that a subset A of $M^{1,\infty}_{\bar\mu}$ is **$\mathcal{T}$-invariant** if $Tu \in A$ whenever $u \in A$ and $T \in \mathcal{T} = \mathcal{T}_{\bar\mu,\bar\mu}$ (definition: 373Aa).

(b) An extended Fatou norm τ on L^0 is **$\mathcal{T}$-invariant** or **fully symmetric** if $\tau(Tu) \le \tau(u)$ whenever $u \in M^{1,\infty}_{\bar\mu}$ and $T \in \mathcal{T}$.

(c) As in §373, I will write $(\mathfrak{A}_L, \bar\mu_L)$ for the measure algebra of Lebesgue measure on $[0, \infty[$, and $u^* \in M^{0,\infty}_{\bar\mu_L}$ for the decreasing rearrangement of any u belonging to any $M^{0,\infty}_{\bar\mu}$ (373C).

374B The first step is to show that the associate of a $\mathcal{T}$-invariant norm is $\mathcal{T}$-invariant.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$. Let L^τ be the Banach lattice defined from τ (369G), and τ' the associate extended Fatou norm (369H-369I). Then

(i) $M^{\infty,1}_{\bar\mu} \subseteq L^\tau \subseteq M^{1,\infty}_{\bar\mu}$;

(ii) τ' is also $\mathcal{T}$-invariant, and $\int u^* \times v^* \le \tau(u)\tau'(v)$ for all u, $v \in M^{0,\infty}_{\bar\mu}$.

proof (a) I check first that $L^\tau \subseteq M^{0,\infty}_{\bar\mu}$. **P** Take any $u \in L^0(\mathfrak{A}) \setminus M^{0,\infty}_{\bar\mu}$. There is surely some $w > 0$ in L^τ, and we can suppose that $w = \chi a$ for some a of finite measure. Now, for any $n \in \mathbb{N}$,

$$(|u| \wedge n\chi 1)^* = n\chi 1 \ge nw^*$$

in $L^0(\mathfrak{A}_L)$, because $\bar\mu[\![|u| > n]\!] = \infty$. So there is a $T \in \mathcal{T}_{\bar\mu,\bar\mu}$ such that $T(|u| \wedge n\chi 1) = nw$, by 373O, and

$$\tau(u) \ge \tau(|u| \wedge n\chi 1) \ge \tau(T(|u| \wedge n\chi 1)) = \tau(nw) = n\tau(w).$$

As n is arbitrary, $\tau(u) = \infty$. As u is arbitrary, $L^\tau \subseteq M^{0,\infty}_{\bar\mu}$. **Q**

(b) Next, $\int u^* \times v^* \le \tau(u)\tau'(v)$ for all u, $v \in M^{0,\infty}_{\bar\mu}$. **P** If $u \in M^{1,\infty}_{\bar\mu}$, then

$$\int u^* \times v^* = \sup\{\int |Tu \times v| : T \in \mathcal{T}_{\bar\mu,\bar\mu}\}$$

(373Q)

$$\le \sup\{\tau(Tu)\tau'(v) : T \in \mathcal{T}_{\bar\mu,\bar\mu}\} = \tau(u)\tau'(v).$$

Generally, setting $u_n = |u| \wedge n\chi 1$, $\langle u_n^* \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum u^* (373Db, 373Dh), so

$$\int u^* \times v^* = \sup_{n\in\mathbb{N}} \int u_n^* \times v^* \le \sup_{n\in\mathbb{N}} \tau(u_n)\tau'(v) = \tau(u)\tau'(v). \textbf{ Q}$$

(c) Consequently, $L^\tau \subseteq M^{1,\infty}_{\bar\mu}$. **P** If $\mathfrak{A} = \{0\}$, this is trivial. Otherwise, take $u \in L^\tau$. There is surely some non-zero a such that $\tau'(\chi a) < \infty$; now, setting $v = \chi a$,

$$\int_0^{\bar\mu a} u^* = \int u^* \times v^* \le \tau(u)\tau'(v) < \infty$$

by (b) above. But this means that $u^* \in M^{1,\infty}_{\bar\mu}$, so that $u \in M^{1,\infty}_{\bar\mu}$ (373F(b-ii)). **Q**

(d) Next, τ' is $\mathcal{T}$-invariant. **P** Suppose that $v \in M^{1,\infty}_{\bar\mu}$, $T \in \mathcal{T}_{\bar\mu,\bar\mu}$, $u \in L^0(\mathfrak{A})$ and $\tau(u) \le 1$. Then $u \in M^{1,\infty}_{\bar\mu}$, by (c), so

$$\int |u \times Tv| \le \int u^* \times v^* \le \tau(u)\tau'(v) \le \tau'(v),$$

using 373J for the first inequality. Taking the supremum over u, we see that $\tau'(Tv) \le \tau'(v)$; as T and v are arbitrary, τ' is $\mathcal{T}$-invariant. **Q**

(e) Finally, putting (d) and (c) together, $L^{\tau'} \subseteq M^{1,\infty}_{\bar\mu}$, so that $L^\tau \supseteq M^{\infty,1}_{\bar\mu}$, using 369J and 369O.

374C For any $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$ there are corresponding norms on $L^0(\mathfrak{A})$ for any semi-finite measure algebra, as follows.

Theorem Let θ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$, and $(\mathfrak{A}, \bar\mu)$ a semi-finite measure algebra.

(a) There is a $\mathcal{T}$-invariant extended Fatou norm τ on $L^0(\mathfrak{A})$ defined by setting

$$\tau(u) = \theta(u^*) \text{ if } u \in M^{0,\infty}_{\bar\mu},$$
$$= \infty \text{ if } u \in L^0(\mathfrak{A}) \setminus M^{0,\infty}_{\bar\mu}.$$

(b) Writing θ', τ' for the associates of θ and τ, we now have

$$\tau'(v) = \theta'(v^*) \text{ if } v \in M^{0,\infty}_{\bar\mu},$$
$$= \infty \text{ if } v \in L^0(\mathfrak{A}) \setminus M^{0,\infty}_{\bar\mu}.$$

(c) If θ is an order-continuous norm on the Banach lattice L^θ, then τ is an order-continuous norm on L^τ.

proof (a)(i) The argument seems to run better if I use a different formula to define τ: set

$$\tau(u) = \sup\{\int |u \times Tw| : T \in \mathcal{T}_{\bar\mu_L,\bar\mu},\, w \in L^0(\mathfrak{A}_L),\, \theta'(w) \le 1\}$$

for $u \in L^0(\mathfrak{A})$. (By 374B(i), $w \in M^{1,\infty}_{\bar\mu_L}$ whenever $\theta'(w) \le 1$, so there is no difficulty in defining Tw.) Now $\tau(u) = \theta(u^*)$ for every $u \in M^{0,\infty}_{\bar\mu}$. **P** ($\alpha$) If $w \in L^0(\mathfrak{A}_L)$ and $\theta'(w) \le 1$, then $w \in M^{1,\infty}_{\bar\mu_L}$, so there is an $S \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ such that $Sw = w^*$ (373O). Accordingly $\theta'(w^*) \le \theta'(w)$ (because θ' is $\mathcal{T}$-invariant, by 374B); now

$$\int |u \times Tw| \le \int u^* \times w^* \le \theta(u^*)\theta'(w^*) \le \theta(u^*)\theta'(w) \le \theta(u^*);$$

as w is arbitrary, $\tau(u) \le \theta(u^*)$. (β) If $w \in L^0(\mathfrak{A}_L)$ and $\theta'(w) \le 1$, then

$$\int |u^* \times w| \le \int (u^*)^* \times w^*$$

(373E)

$$= \int u^* \times w^* = \sup\{\int |u \times Tw| : T \in \mathcal{T}_{\bar{\mu}_L,\bar{\mu}}\}$$

(373Q)

$$\le \tau(u).$$

But because θ is the associate of θ' (369I(ii)), this means that $\theta(u^*) \le \tau(u)$. **Q**

(ii) Now τ is an extended Fatou norm on $L^0(\mathfrak{A})$. **P** Of the conditions in 369F, (i)-(iv) are true just because $\tau(u) = \sup_{v \in B} \int |u \times v|$ for some set $B \subseteq L^0$. As for (v) and (vi), observe that if $u \in M^{\infty,1}_{\bar{\mu}}$ then $u^* \in M^{\infty,1}_{\bar{\mu}_L}$ (373F(b-iv)), so that $\tau(u) = \theta(u^*) < \infty$, by 374B(i), while also

$$u \ne 0 \Longrightarrow u^* \ne 0 \Longrightarrow \tau(u) = \theta(u^*) > 0.$$

As $M^{\infty,1}_{\bar{\mu}}$ is order-dense in $L^0(\mathfrak{A})$ (this is where I use the hypothesis that $(\mathfrak{A}, \bar{\mu})$ is semi-finite), 369F(v)-(vi) are satisfied, and τ is an extended Fatou norm. **Q**

(iii) τ is $\mathcal{T}$-invariant. **P** Take $u \in M^{1,\infty}_{\bar{\mu}}$ and $T \in \mathcal{T}_{\bar{\mu},\bar{\mu}}$. There are $S_0 \in \mathcal{T}_{\bar{\mu}_L,\bar{\mu}}$ and $S_1 \in \mathcal{T}_{\bar{\mu},\bar{\mu}_L}$ such that $S_0 u^* = u$, $S_1 Tu = (Tu)^*$ (373O); now $S_1 T S_0 \in \mathcal{T}_{\bar{\mu}_L,\bar{\mu}_L}$ (373Be), so

$$\tau(Tu) = \theta((Tu)^*) = \theta(S_1 T S_0 u^*) \le \theta(u^*) = \tau(u)$$

because θ is $\mathcal{T}$-invariant. **Q**

(iv) We can now return to the definition of τ. I have already remarked that $\tau(u) = \theta(u^*)$ if $u \in M^{0,\infty}_{\bar{\mu}}$. For other u, we must have $\tau(u) = \infty$ just because τ is a $\mathcal{T}$-invariant extended Fatou norm (374B(i)). So the definitions in the statement of the theorem and (i) above coincide.

(b) We surely have $\tau'(v) = \infty$ if $v \in L^0(\mathfrak{A}) \setminus M^{0,\infty}_{\bar{\mu}}$, by 374B, because τ', like τ, is a $\mathcal{T}$-invariant extended Fatou norm. So take $v \in M^{0,\infty}_{\bar{\mu}}$.

(i) If $u \in L^0(\mathfrak{A})$ and $\tau(u) \le 1$, then

$$\int |v \times u| \le \int v^* \times u^* \le \theta'(v^*)\theta(u^*) = \theta'(v^*)\tau(u) \le \theta'(v^*);$$

as u is arbitrary, $\tau'(v) \le \theta'(v^*)$.

(ii) If $w \in L^0(\mathfrak{A}_L)$ and $\theta(w) \le 1$, then

$$\int |v^* \times w| \le \int v^* \times w^* = \sup\{\int |v \times Tw| : T \in \mathcal{T}_{\bar{\mu}_L,\bar{\mu}}\}$$

(373Q)

$$\le \sup\{\tau'(v)\tau(Tw) : T \in \mathcal{T}_{\bar{\mu}_L,\bar{\mu}}\} = \sup\{\tau'(v)\theta((Tw)^*) : T \in \mathcal{T}_{\bar{\mu}_L,\bar{\mu}}\}$$
$$\le \sup\{\tau'(v)\theta(STw) : T \in \mathcal{T}_{\bar{\mu}_L,\bar{\mu}},\ S \in \mathcal{T}_{\bar{\mu},\bar{\mu}_L}\}$$

(because, given T, we can find an S such that $STw = (Tw)^*$, by 373O)

$$\le \sup\{\tau'(v)\theta(Tw) : T \in \mathcal{T}_{\bar{\mu}_L,\bar{\mu}_L}\} \le \tau'(v).$$

As w is arbitrary, $\theta'(v^*) \le \tau'(v)$ and the two are equal. This completes the proof of (b).

(c)(i) The first step is to note that $L^\tau \subseteq M^0_{\bar{\mu}}$. **P?** Suppose that $u \in L^\tau \setminus M^0_{\bar{\mu}}$, that is, that $\bar{\mu}[\![|u| > \alpha]\!] = \infty$ for some $\alpha > 0$. Then $u^* \ge \alpha\chi 1$ in $L^0(\mathfrak{A}_L)$, so $L^\infty(\mathfrak{A}_L) \subseteq L^\theta$. For each $n \in \mathbb{N}$, set $v_n = \chi\,[n, \infty[^\bullet$. Then $v_n^* = v_0$, so we can find a $T_n \in \mathcal{T}_{\bar{\mu}_L,\bar{\mu}_L}$ such that $T_n v_n = v_0$ (373O), and $\theta(v_n) \ge \theta(v_0)$ for every n. But as $\langle v_n \rangle_{n \in \mathbb{N}}$ is a decreasing sequence with infimum 0, this means that θ is not an order-continuous norm. **XQ**

(ii) Now suppose that $A \subseteq L^\tau$ is non-empty and downwards-directed and has infimum 0. Then $\inf_{u \in A} \bar{\mu}[\![u > \alpha]\!] = 0$ for every $\alpha > 0$ (put 364L(b-ii) and 321F together). But this means that $B = \{u^* : u \in A\}$ must have infimum 0; since B is surely downwards-directed, $\inf_{v \in B} \theta(v) = 0$, that is, $\inf_{u \in A} \tau(u) = 0$. As A is arbitrary, τ is an order-continuous norm.

374D What is more, every $\mathcal{T}$-invariant extended Fatou norm can be represented in this way.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$. Then there is a $\mathcal{T}$-invariant extended Fatou norm θ on $L^0(\mathfrak{A}_L)$ such that $\tau(u)=\theta(u^*)$ for every $u\in M^{0,\infty}_{\bar\mu}$.

proof I use the method of 374C. If $\mathfrak{A}=\{0\}$ the result is trivial; assume that $\mathfrak{A}\ne\{0\}$.

(a) Set

$$\theta(w)=\sup\{\int|w\times Tv|:T\in\mathcal{T}_{\bar\mu,\bar\mu_L},\,v\in L^0(\mathfrak{A}),\,\tau'(v)\le 1\}$$

for $w\in L^0(\mathfrak{A}_L)$. Note that

$$\theta(w)=\sup\{\int w^*\times v^*:v\in L^0(\mathfrak{A}),\,\tau'(v)\le 1\}$$

for every $w\in M^{0,\infty}_{\bar\mu_L}$, by 373Q again.

θ is an extended Fatou norm on $L^0(\mathfrak{A}_L)$. **P** As in 374C, the conditions 369F(i)-(iv) are elementary. If $w>0$ in $L^0(\mathfrak{A}_L)$, take any $v\in L^0(\mathfrak{A})$ such that $0<\tau'(v)\le 1$; then $w^*\times v^*\ne 0$ so $\theta(w)\ge\int w^*\times v^*>0$. So 369F(v) is satisfied. As for 369F(vi), if $w>0$ in $L^0(\mathfrak{A}_L)$, take a non-zero $a\in\mathfrak{A}$ of finite measure such that $\alpha=\tau(\chi a)<\infty$. Let $\beta>0$, $b\in\mathfrak{A}_L$ be such that $0<\bar\mu_L b\le\bar\mu a$ and $\beta\chi b\le w$; then

$$\theta(\chi b)=\sup_{\tau'(v)\le 1}\int(\chi b)^*\times v^*\le\sup_{\tau'(v)\le 1}\int(\chi a)^*\times v^*\le\tau(\chi a)<\infty$$

by 374B(ii). So $\theta(\beta\chi b)<\infty$ and 369F(vi) is satisfied. Thus θ is an extended Fatou norm. **Q**

(b) θ is $\mathcal{T}$-invariant. **P** If $T\in\mathcal{T}_{\bar\mu_L,\bar\mu_L}$ and $w\in M^{1,\infty}_{\bar\mu_L}$, then

$$\theta(Tw)=\sup_{\tau'(v)\le 1}\int(Tw)^*\times v^*\le\sup_{\tau'(v)\le 1}\int w^*\times v^*=\theta(w)$$

by 373G and 373I. **Q**

(c) $\theta(u^*)=\tau(u)$ for every $u\in M^{0,\infty}_{\bar\mu}$. **P** We have

$$\tau(u)=\sup_{\tau'(v)\le 1}\int|u\times v|\le\sup_{\tau'(v)\le 1}\int u^*\times v^*\le\tau(u),$$

using 369I, 373E and 374B. So

$$\theta(u^*)=\sup_{\tau'(v)\le 1}\int u^*\times v^*=\tau(u)$$

by the remark in (a) above. **Q**

374E I turn now to rearrangement-invariance. Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) I will say that a subset A of $L^0=L^0(\mathfrak{A})$ is **rearrangement-invariant** if $T_\pi u\in A$ whenever $u\in A$ and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a measure-preserving Boolean automorphism, writing $T_\pi:L^0\to L^0$ for the isomorphism corresponding to π (364P).

(b) I will say that an extended Fatou norm τ on L^0 is **rearrangement-invariant** if $\tau(T_\pi u)=\tau(u)$ whenever $u\in L^0$ and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a measure-preserving automorphism.

374F Remarks (a) If $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a sequentially order-continuous measure-preserving Boolean homomorphism, then $T_\pi\restriction M^{1,\infty}_{\bar\mu}$ belongs to $\mathcal{T}_{\bar\mu,\bar\mu}$; this is obvious from the definition of $M^{1,\infty}=L^1+L^\infty$ and the basic properties of T_π (364P). Accordingly, any $\mathcal{T}$-invariant extended Fatou norm τ on $L^0(\mathfrak{A})$ must be rearrangement-invariant, since (by 374B) we shall have $\tau(u)=\tau(T_\pi(u))=\infty$ when $u\notin M^{1,\infty}_{\bar\mu}$. Similarly, any $\mathcal{T}$-invariant subset of $M^{1,\infty}_{\bar\mu}$ will be rearrangement-invariant.

(b) I seek to describe cases in which rearrangement-invariance implies $\mathcal{T}$-invariance. This happens only for certain measure algebras; in order to shorten the statements of the main theorems I introduce a special phrase.

374G Definition I say that a measure algebra $(\mathfrak{A},\bar\mu)$ is **quasi-homogeneous** if for any non-zero a, $b\in\mathfrak{A}$ there is a measure-preserving Boolean automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $\pi a\cap b\ne 0$.

374H Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Then the following are equiveridical:

(i) $(\mathfrak{A},\bar\mu)$ is quasi-homogeneous;

(ii) *either* $\mathfrak{A}$ is purely atomic and every atom of $\mathfrak{A}$ has the same measure *or* there is a $\kappa \ge \omega$ such that the principal ideal $\mathfrak{A}_a$ is homogeneous, with Maharam type κ, for every $a \in \mathfrak{A}$ of non-zero finite measure.

proof (i)⇒(ii) Suppose that $(\mathfrak{A}, \bar{\mu})$ is quasi-homogeneous.

(α) Suppose that $\mathfrak{A}$ has an atom a. In this case, for any $b \in \mathfrak{A} \setminus \{0\}$ there is an automorphism π of $(\mathfrak{A}, \bar{\mu})$ such that $\pi a \cap b \ne 0$; now πa must be an atom, so $\pi a = \pi a \cap b$ and πa is an atom included in b. As b is arbitrary, $\mathfrak{A}$ is purely atomic; moreover, if b is an atom, then it must be equal to πa and therefore of the same measure as a, so all atoms of $\mathfrak{A}$ have the same measure.

(β) Now suppose that $\mathfrak{A}$ is atomless. In this case, if $a \in \mathfrak{A}$ has finite non-zero measure, $\mathfrak{A}_a$ is homogeneous. **P?** Otherwise, there are non-zero b, $c \subseteq a$ such that the principal ideals $\mathfrak{A}_b$, $\mathfrak{A}_c$ are homogeneous and of different Maharam types, by Maharam's theorem (332B, 332H). But now there is supposed to be an automorphism π such that $\pi b \cap c \ne 0$, in which case $\mathfrak{A}_b$, $\mathfrak{A}_{\pi b}$, $\mathfrak{A}_{\pi b \cap c}$ and $\mathfrak{A}_c$ must all have the same Maharam type. **XQ**

Consequently, if a, $b \in \mathfrak{A}$ are both of non-zero finite measure, the Maharam types of $\mathfrak{A}_a$, $\mathfrak{A}_{a \cup b}$ and $\mathfrak{A}_b$ must all be the same infinite cardinal κ.

(ii)⇒(i) Assume (ii), and take a, $b \in \mathfrak{A} \setminus \{0\}$. If $a \cap b \ne 0$ we can take π to be the identity automorphism and stop. So let us suppose that $a \cap b = 0$.

(α) If $\mathfrak{A}$ is purely atomic and every atom has the same measure, then there are atoms $a_0 \subseteq a$, $b_0 \subseteq b$. Set

$$\begin{aligned} \pi c &= c \text{ if } c \supseteq a_0 \cup b_0 \text{ or } c \cap (a_0 \cup b_0) = 0, \\ &= c \triangle (a_0 \cup b_0) \text{ otherwise.} \end{aligned}$$

Then it is easy to check that π is a measure-preserving automorphism of $\mathfrak{A}$ such that $\pi a_0 = b_0$, so that $\pi a \cap b \ne 0$.

(β) If $\mathfrak{A}_c$ is Maharam-type-homogeneous with the same infinite Maharam type κ for every non-zero c of finite measure, set $\gamma = \min(1, \bar{\mu}a, \bar{\mu}b) > 0$. Because $\mathfrak{A}$ is atomless, there are $a_0 \subseteq a$, $b_0 \subseteq b$ with $\bar{\mu}a_0 = \bar{\mu}b_0 = \gamma$ (331C). Now $\mathfrak{A}_{a_0}$ and $\mathfrak{A}_{b_0}$ are homogeneous with the same Maharam type and the same magnitude, so by Maharam's theorem (331I) there is a measure-preserving isomorphism $\pi_0 : \mathfrak{A}_{a_0} \to \mathfrak{A}_{b_0}$. Define $\pi : \mathfrak{A} \to \mathfrak{A}$ by setting

$$\pi c = (c \setminus (a_0 \cup b_0)) \cup \pi_0(c \cap a_0) \cup \pi_0^{-1}(c \cap b_0)$$

for $c \in \mathfrak{A}$; then it is easy to see that π is a measure-preserving automorphism of $\mathfrak{A}$ and that $\pi a \cap b \ne 0$.

Remark We shall return to these ideas in Chapter 38. In particular, the construction of π from π_0 in the last part of the proof will be of great importance; in the language of 381R, $\pi = (\overleftarrow{a_0 \,{}_{\pi_0}\, b_0})$.

374I Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra. Then

(a) whenever $a, b \in \mathfrak{A}$ have the same finite measure, the principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_b$ are isomorphic as measure algebras;

(b) there is a subgroup Γ of the additive group $\mathbb{R}$ such that (α) $\bar{\mu}a \in \Gamma$ whenever $a \in \mathfrak{A}$ and $\bar{\mu}a < \infty$ (β) whenever $a \in \mathfrak{A}$, $\gamma \in \Gamma$ and $0 \le \gamma \le \bar{\mu}a$ then there is a $c \subseteq a$ such that $\bar{\mu}c = \gamma$.

proof If $\mathfrak{A}$ is purely atomic, with all its atoms of measure γ_0, set $\Gamma = \gamma_0\mathbb{Z}$, and the results are elementary. If $\mathfrak{A}$ is atomless, set $\Gamma = \mathbb{R}$; then (a) is a consequence of Maharam's theorem, and (b) is a consequence of 331C, already used in the proof of 374H.

374J Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra and u, $v \in M^{0,\infty}_{\bar{\mu}}$. Let $\mathrm{Aut}_{\bar{\mu}}$ be the group of measure-preserving automorphisms of $\mathfrak{A}$. Then

$$\int u^* \times v^* = \sup_{\pi \in \mathrm{Aut}_{\bar{\mu}}} \int |u \times T_\pi v|,$$

where $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is the isomorphism corresponding to π.

proof (a) Suppose first that u, v are non-negative and belong to $S(\mathfrak{A}^f)$, where $\mathfrak{A}^f$ is the ring $\{a : \bar{\mu}a < \infty\}$, as usual. Then they can be expressed as $u = \sum_{i=0}^m \alpha_i \chi a_i$, $v = \sum_{j=0}^n \beta_j \chi b_j$ where $\alpha_0 \ge \dots \alpha_m \ge 0$, $\beta_0 \ge \dots \ge \beta_n \ge 0$, $a_0, \dots, a_m$ are disjoint and of finite measure, and $b_0, \dots, b_n$ are disjoint and of finite measure. Extending each list by a final term having a coefficient of 0, if need be, we may suppose that $\sup_{i \le m} a_i = \sup_{j \le n} b_j$.

Let $(t_0, \dots, t_s)$ enumerate in ascending order the set

$$\{0\} \cup \{\textstyle\sum_{i=0}^k \bar{\mu}a_i : k \le m\} \cup \{\textstyle\sum_{j=0}^k \bar{\mu}b_j : k \le n\}.$$

Then every t_r belongs to the subgroup Γ of 374Ib, and $t_s = \sum_{i=0}^{m} \bar{\mu}a_i = \sum_{j=0}^{n} \bar{\mu}b_j$. For $1 \le r \le s$ let $k(r)$, $l(r)$ be minimal subject to the requirements $t_r \le \sum_{i=0}^{k(r)} \bar{\mu}a_i$, $t_r \le \sum_{j=0}^{l(r)} \bar{\mu}b_j$. Then $\bar{\mu}a_i = \sum_{k(r)=i} t_r - t_{r-1}$, so (using 374Ib) we can find a disjoint family $\langle c_r \rangle_{1\le r\le s}$ such that $c_r \subseteq a_{k(r)}$ and $\bar{\mu}c_r = t_r - t_{r-1}$ for each r. Similarly, there is a disjoint family $\langle d_r \rangle_{1\le r\le s}$ such that $d_r \subseteq b_{l(r)}$ and $\bar{\mu}d_r = t_r - t_{r-1}$ for each r. Now the principal ideals $\mathfrak{A}_{c_r}$, $\mathfrak{A}_{d_r}$ are isomorphic for every r, by 374Ia; let $\pi_r : \mathfrak{A}_{d_r} \to \mathfrak{A}_{c_r}$ be measure-preserving isomorphisms. Define $\pi : \mathfrak{A} \to \mathfrak{A}$ by setting

$$\pi a = (a \setminus \sup_{1\le r\le s} d_r) \cup \sup_{1\le r\le s} \pi_r(a \cap d_r);$$

because

$$\sup_{r\le s} c_r = \sup_{i\le m} a_i = \sup_{j\le n} b_j = \sup_{r\le s} d_r,$$

$\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism.

Now

$$u = \sum_{r=1}^{s} \alpha_{k(r)} \chi c_r, \quad v = \sum_{r=1}^{s} \beta_{l(r)} \chi d_r,$$

$$u^* = \sum_{r=1}^{s} \alpha_{k(r)} \chi\, [t_{r-1}, t_r[^\bullet, \quad v^* = \sum_{r=1}^{s} \beta_{l(r)} \chi\, [t_{r-1}, t_r[^\bullet,$$

so

$$\int u \times T_\pi v = \sum_{r=1}^{s} \alpha_{k(r)}\beta_{l(r)}\bar{\mu}c_r = \sum_{r=1}^{s} \alpha_{k(r)}\beta_{l(r)}(t_r - t_{r-1}) = \int u^* \times v^*.$$

(b) Now take any u_0, $v_0 \in M_{\bar{\mu}}^{0,\infty}$. Set

$$A = \{u : u \in S(\mathfrak{A}^f),\ 0 \le u \le |u_0|\}, \quad B = \{v : v \in S(\mathfrak{A}^f),\ 0 \le v \le |v_0|\}.$$

Then A is an upwards-directed set with supremum $|u_0|$, because $(\mathfrak{A}, \bar{\mu})$ is semi-finite, so $\{u^* : u \in A\}$ is an upwards-directed set with supremum $|u_0|^* = u_0^*$ (373Db, 373Dh). Similarly $\{v^* : v \in B\}$ is upwards-directed and has supremum v_0^*, so $\{u^* \times v^* : u \in A,\ v \in B\}$ is upwards-directed and has supremum $u_0^* \times v_0^*$.

Consequently, if $\gamma < \int u_0^* \times v_0^*$, there are $u \in A$, $v \in B$ such that $\gamma \le \int u^* \times v^*$. Now, by (a), there is a $\pi \in \mathrm{Aut}_{\bar{\mu}}$ such that

$$\gamma \le \int u \times T_\pi v \le \int |u_0| \times T_\pi |v_0| = \int |u_0 \times T_\pi v_0|$$

because T_π is a Riesz homomorphism. As γ is arbitrary,

$$\int u_0^* \times v_0^* \le \sup_{\pi\in\mathrm{Aut}_{\bar{\mu}}} \int |u_0 \times T_\pi v_0|.$$

But the reverse inequality is immediate from 373J.

374K Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra, and τ a rearrangement-invariant extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Then τ is $\mathcal{T}$-invariant.

proof Write τ' for the associate of τ. Then 374J tells us that for any u, $v \in M_{\bar{\mu}}^{0,\infty}$,

$$\int u^* \times v^* = \sup_{\pi\in\mathrm{Aut}_{\bar{\mu}}} \int |T_\pi u \times v| \le \sup_{\pi\in\mathrm{Aut}_{\bar{\mu}}} \tau(T_\pi u)\tau'(v) = \tau(u)\tau'(v),$$

writing u^*, v^* for the decreasing rearrangements of u and v, and $\mathrm{Aut}_{\bar{\mu}}$ for the group of measure-preserving automorphisms of $(\mathfrak{A}, \bar{\mu})$. But now, if $u \in M_{\bar{\mu}}^{1,\infty}$ and $T \in \mathcal{T}_{\bar{\mu},\bar{\mu}}$,

$$\tau(Tu) = \sup\{\int |Tu \times v| : \tau'(v) \le 1\}$$

(369I)

$$\le \sup\{\int u^* \times v^* : \tau'(v) \le 1\}$$

(373J)

$$\le \tau(u).$$

As T, u are arbitrary, τ is $\mathcal{T}$-invariant.

374L Lemma Let $(\mathfrak{A},\bar\mu)$ be a quasi-homogeneous semi-finite measure algebra. Suppose that u, $v\in(M^{0,\infty}_{\bar\mu})^+$ are such that $\int u^*\times v^*=\infty$. Then there is a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $\int u\times T_\pi v=\infty$.

proof I take three cases separately.

(a) Suppose that $\mathfrak{A}$ is purely atomic; then u, $v\in L^\infty(\mathfrak{A})$ and u^*, $v^*\in L^\infty(\mathfrak{A}_L)$, so neither u^* nor v^* can belong to $L^1_{\bar\mu_L}$ and neither u nor v can belong to $L^1_{\bar\mu}$. Let γ be the common measure of the atoms of $\mathfrak{A}$. For each $n\in\mathbb{N}$, set

$$\alpha_n=\inf\{\alpha:\alpha\ge 0,\,\bar\mu[\![u>\alpha]\!]\le 3^n\gamma\},\quad \tilde a_n=[\![u>\tfrac12\alpha_n]\!].$$

Then $\bar\mu[\![u>\alpha_n]\!]\le 3^n\gamma$; also $\alpha_n>0$, since otherwise u would belong to $L^1_{\bar\mu}$, so $\bar\mu\tilde a_n\ge 3^n\gamma$. We can therefore choose $\langle a'_n\rangle_{n\in\mathbb{N}}$ inductively such that $a'_n\subseteq\tilde a_n$ and $\bar\mu a'_n=3^n\gamma$ for each n (using 374Ib). For each $n\ge 1$, set $a''_n=a'_n\setminus\sup_{i<n}a'_i$; then $\bar\mu a''_n\ge\dfrac12\cdot 3^{-n}\gamma$, so we can choose an $a_n\subseteq a''_n$ such that $\bar\mu a_n=3^{n-1}\gamma$.

Also, of course, $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is non-increasing. We now see that

$$\langle a_n\rangle_{n\ge1}\text{ is disjoint},\quad u\ge\frac12\alpha_n\chi a_n\text{ for every }n\ge1,$$

$$u^*\le\|u\|_\infty\chi\,[0,\gamma[^\bullet\vee\sup\nolimits_{n\in\mathbb{N}}\alpha_n\chi\,[3^n\gamma,3^{n+1}\gamma[^\bullet.$$

Similarly, there are a non-increasing sequence $\langle\beta_n\rangle_{n\in\mathbb{N}}$ in $[0,\infty[$ and a disjoint sequence $\langle b_n\rangle_{n\ge1}$ in $\mathfrak{A}$ such that

$$\bar\mu b_n=3^{n-1}\gamma,\quad v\ge\frac12\beta_n\chi b_n\text{ for every }n\ge1,$$

$$v^*\le\|v\|_\infty\chi\,[0,\gamma[^\bullet\vee\sup\nolimits_{n\in\mathbb{N}}\beta_n\chi\,[3^n\gamma,3^{n+1}\gamma[^\bullet.$$

We are supposing that

$$\begin{aligned}\infty=\int u^*\times v^*&=\gamma\|u\|_\infty\|v\|_\infty+\sum_{n=0}^\infty 2\cdot3^n\gamma\alpha_n\beta_n\\&=\gamma\|u\|_\infty\|v\|_\infty+2\gamma\alpha_0\beta_0+2\gamma\sum_{n=0}^\infty 3^{2n+1}(\alpha_{2n+1}\beta_{2n+1}+3\alpha_{2n+2}\beta_{2n+2})\\&\le\gamma\|u\|_\infty\|v\|_\infty+2\gamma\alpha_0\beta_0+24\sum_{n=0}^\infty 3^{2n}\gamma\alpha_{2n+1}\beta_{2n+1},\end{aligned}$$

so $\sum_{n=0}^\infty 3^{2n}\alpha_{2n+1}\beta_{2n+1}=\infty$.

At this point, recall that we are dealing with a purely atomic algebra in which every atom has measure γ. Let A_n, B_n be the sets of atoms included in a_n, b_n for each $n\ge1$, and $A=\bigcup_{n\ge1}A_n\cup B_n$. Then $\#(A_n)=\#(B_n)=3^{n-1}$ for each $n\ge1$. We therefore have a permutation $\phi:A\to A$ such that $\phi[B_{2n+1}]=A_{2n+1}$ for every n. (The point is that $A\setminus\bigcup_{n\in\mathbb{N}}A_{2n+1}$ and $A\setminus\bigcup_{n\in\mathbb{N}}B_{2n+1}$ are both countably infinite.) Define $\pi:\mathfrak{A}\to\mathfrak{A}$ by setting

$$\pi c=(c\setminus\sup A)\cup\sup\nolimits_{a\in A,a\subseteq c}\phi a$$

for $c\in\mathfrak{A}$. Then π is well-defined (because A is countable), and it is easy to check that it is a measure-preserving Boolean automorphism (because it is just a permutation of the atoms); and $\pi b_{2n+1}=a_{2n+1}$ for every n. Consequently

$$\int u\times T_\pi v\ge\sum\nolimits_{n=0}^\infty\frac14\alpha_{2n+1}\beta_{2n+1}\bar\mu a_{2n+1}=\frac14\gamma\sum\nolimits_{n=0}^\infty 3^{2n}\alpha_{2n+1}\beta_{2n+1}=\infty.$$

So we have found a suitable automorphism.

(b) Next, consider the case in which $(\mathfrak{A},\bar\mu)$ is atomless and of finite magnitude γ. Of course $\gamma>0$. For each $n\in\mathbb{N}$ set

$$\alpha_n=\inf\{\alpha:\alpha\ge0,\,\bar\mu[\![u>\alpha]\!]\le3^{-n}\gamma\},\quad\tilde a_n=[\![u>\frac12\alpha_n]\!].$$

Then $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and

$$u^*\le\sup\nolimits_{n\in\mathbb{N}}\alpha_{n+1}\chi\,[3^{-n-1}\gamma,3^{-n}\gamma[^\bullet.$$

This time, $\bar\mu\tilde a_n\ge3^{-n}\gamma$, and we are in an atomless measure algebra, so we can choose $a'_n\subseteq\tilde a_n$ such that $\bar\mu a'_n=3^{-n}\gamma$; taking $a''_n=a'_n\setminus\sup_{i>n}a'_i$, $\bar\mu a''_n\ge\dfrac12\cdot3^{-n}\gamma$, and we can choose $a_n\subseteq a''_n$ such that $\bar\mu a_n=3^{-n-1}\gamma$ for every n. As before, $u \ge \frac{1}{2}\alpha_n\chi a_n$ for every n, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint.

In the same way, we can find $\langle\beta_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ such that $\langle b_n\rangle_{n\in\mathbb{N}}$ is disjoint,

$$v^* \le \sup_{n\in\mathbb{N}}\beta_{n+1}\chi\,[3^{-n-1}\gamma, 3^{-n}\gamma[^\bullet, \quad v \ge \sup_{n\in\mathbb{N}}\frac{1}{2}\beta_n\chi b_n$$

and $\bar\mu b_n = 3^{-n-1}\gamma$ for each n. In this case, we have

$$\infty = \int u^* \times v^* \le \sum_{n=0}^{\infty} 2\cdot 3^{-n-1}\gamma\alpha_{n+1}\beta_{n+1},$$

and $\sum_{n=0}^{\infty} 3^{-n}\alpha_n\beta_n$ is infinite.

Now all the principal ideals $\mathfrak{A}_{a_n}$, $\mathfrak{A}_{b_n}$ are homogeneous and of the same Maharam type, so there are measure-preserving isomorphisms $\pi_n : \mathfrak{A}_{b_n} \to \mathfrak{A}_{a_n}$; similarly, setting $\tilde a = 1\setminus\sup_{n\in\mathbb{N}} a_n$ and $\tilde b = 1\setminus\sup_{n\in\mathbb{N}} b_n$, there is a measure-preserving isomorphism $\tilde\pi : \mathfrak{A}_{\tilde b} \to \mathfrak{A}_{\tilde a}$. Define $\pi : \mathfrak{A}\to\mathfrak{A}$ by setting

$$\pi c = \tilde\pi(c\cap\tilde b)\cup\sup_{n\in\mathbb{N}}\pi_n(c\cap a_n)$$

for every $c\in\mathfrak{A}$; then π is a measure-preserving automorphism of $\mathfrak{A}$, and $\pi b_n = a_n$ for each n. In this case,

$$\int u\times T_\pi v \ge \frac{1}{4}\sum_{n=0}^{\infty} 3^{-n-1}\gamma\alpha_n\beta_n = \infty,$$

and again we have a suitable automorphism.

(c) Thirdly, consider the case in which $\mathfrak{A}$ is atomless and not totally finite; take κ to be the common Maharam type of all the principal ideals $\mathfrak{A}_a$ where $0 < \bar\mu a < \infty$. In this case, set

$$\alpha_n = \inf\{\alpha : \bar\mu[\![u>\alpha]\!] \le 3^n\}, \quad \beta_n = \inf\{\alpha : \bar\mu[\![v>\alpha]\!] \le 3^n\}$$

for each $n\in\mathbb{Z}$. This time

$$u^* \le \sup_{n\in\mathbb{Z}}\alpha_n\chi\,[3^n, 3^{n+1}[^\bullet, \quad v^* \le \sup_{n\in\mathbb{Z}}\beta_n\chi\,[3^n, 3^{n+1}[^\bullet,$$

so

$$\infty = \int u^*\times v^* = 2\sum_{n=-\infty}^{\infty} 3^n\alpha_n\beta_n \le 8\sum_{n=-\infty}^{\infty} 3^{2n}\alpha_{2n}\beta_{2n}.$$

For each $n\in\mathbb{Z}$, $3^n \le \bar\mu[\![u>\frac{1}{2}\alpha_n]\!]$, so there is an a_n'' such that

$$a_n'' \subseteq [\![u>\tfrac{1}{2}\alpha_n]\!], \quad \bar\mu a_n'' = 3^n.$$

Set $a_n' = a_n''\setminus\sup_{-\infty<i<n} a_i''$; then $\bar\mu a_n' \ge \frac{1}{2}\cdot 3^n$ for each n; choose $a_n\subseteq a_n'$ such that $\bar\mu a_n = 3^{n-1}$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint and $u\ge\frac{1}{2}\alpha_n\chi a_n$ for each n.

Similarly, there is a disjoint sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ such that

$$\bar\mu b_n = 3^{n-1}, \quad v \ge \frac{1}{2}\beta_n\chi b_n$$

for each $n\in\mathbb{N}$.

Set $d^* = \sup_{n\in\mathbb{Z}} a_n\cup\sup_{n\in\mathbb{Z}} b_n$. Then

$$\tilde a = d^*\setminus\sup_{n\in\mathbb{Z}} a_{2n}, \quad \tilde b = d^*\setminus\sup_{n\in\mathbb{Z}} b_{2n}$$

both have magnitude ω and Maharam type κ. So there is a measure-preserving isomorphism $\tilde\pi : \mathfrak{A}_{\tilde b}\to\mathfrak{A}_{\tilde a}$ (332J). At the same time, for each $n\in\mathbb{Z}$ there is a measure-preserving isomorphism $\pi_n : \mathfrak{A}_{b_{2n}}\to\mathfrak{A}_{a_{2n}}$. So once again we can assemble these to form a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$, defined by the formula

$$\pi c = (c\setminus d^*)\cup\tilde\pi(c\cap\tilde b)\cup\sup_{n\in\mathbb{Z}}\pi_n(c\cap b_{2n}).$$

Just as in (a) and (b) above,

$$\int u\times T_\pi v \ge \sum_{n=-\infty}^{\infty}\frac{1}{4}\cdot 3^{2n-1}\alpha_{2n}\beta_{2n} = \infty.$$

Thus we have a suitable π in any of the cases allowed by 374H.

374M Proposition Let $(\mathfrak{A},\bar\mu)$ be a quasi-homogeneous localizable measure algebra, and $U\subseteq L^0 = L^0(\mathfrak{A})$ a solid linear subspace which, regarded as a Riesz space, is perfect. If U is rearrangement-invariant and $M^{\infty,1}_{\bar\mu}\subseteq U\subseteq M^{1,\infty}_{\bar\mu}$, then U is $\mathcal{T}$-invariant.

proof Set $V = \{v : u \times v \in L^1$ for every $u \in U\}$, so that V is a solid linear subspace of L^0 which can be identified with $U^\times$ (369C), and U becomes $\{u : u \times v \in L^1$ for every $v \in V\}$; note that $M^{\infty,1}_{\bar\mu} \subseteq V \subseteq M^{1,\infty}_{\bar\mu}$ (using 369Q).

If $u \in U^+$, $v \in V^+$ and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism, then $T_\pi u \in U$, so $\int v \times T_\pi u < \infty$; by 374L, $\int u^* \times v^*$ is finite. But this means that if $u \in U$, $v \in V$ and $T \in \mathcal{T}_{\bar\mu,\bar\mu}$,

$$\int |Tu \times v| \le \int u^* \times v^* < \infty.$$

As v is arbitrary, $Tu \in U$; as T and u are arbitrary, U is $\mathcal{T}$-invariant.

374X Basic exercises >(a) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and $A \subseteq M^{1,\infty}_{\bar\mu}$ a $\mathcal{T}$-invariant set. (i) Show that A is solid. (ii) Show that if A is a linear subspace and not $\{0\}$, then it includes $M^{\infty,1}_{\bar\mu}$. (iii) Show that if $u \in A$, $v \in M^{0,\infty}_{\bar\mu}$ and $\int_0^t v^* \le \int_0^t u^*$ for every $t > 0$, then $v \in A$. (iv) Show that if $(\mathfrak{B}, \bar\nu)$ is any other measure algebra, then $B = \{Tu : u \in A, T \in \mathcal{T}_{\bar\mu,\bar\nu}\}$ and $C = \{v : v \in M^{1,\infty}_{\bar\nu}, Tv \in A$ for every $T \in \mathcal{T}_{\bar\nu,\bar\mu}\}$ are $\mathcal{T}$-invariant subsets of $M^{1,\infty}_{\bar\nu}$, and that $B \subseteq C$. Give two examples in which $B \subset C$. Show that if $(\mathfrak{A}, \bar\mu) = (\mathfrak{A}_L, \bar\mu_L)$ then $B = C$.

>(b) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Show that the extended Fatou norm $\| \|_p$ on $L^0(\mathfrak{A})$ is $\mathcal{T}$-invariant for every $p \in [1, \infty]$. (*Hint*: 371Gd.)

(c) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be semi-finite measure algebras, and ϕ a Young's function (369Xc). Let τ_ϕ, $\tilde\tau_\phi$ be the corresponding Orlicz norms on $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$. Show that $\tilde\tau_\phi(Tu) \le \tau_\phi(u)$ for every $u \in L^0(\mathfrak{A})$, $T \in \mathcal{T}_{\bar\mu,\bar\nu}$. (*Hint*: 369Xn, 373Xm.) In particular, τ_ϕ is $\mathcal{T}$-invariant.

(d) Show that if $(\mathfrak{A}, \bar\mu)$ is a semi-finite measure algebra and τ is a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$, then the Banach lattice L^τ defined from τ is $\mathcal{T}$-invariant.

(e) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$ which is an order-continuous norm on L^τ. Show that $L^\tau \subseteq M^{1,0}_{\bar\mu}$.

(f) Let θ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$ and $(\mathfrak{A}, \bar\mu)$, $(\mathfrak{B}, \bar\nu)$ two semi-finite measure algebras. Let τ_1, τ_2 be the extended Fatou norms on $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ defined from θ by the method of 374C. Show that $\tau_2(Tu) \le \tau_1(u)$ whenever $u \in M^{1,\infty}_{\bar\mu}$ and $T \in \mathcal{T}_{\bar\mu,\bar\nu}$.

>(g) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, not $\{0\}$, and set $\tau(u) = \sup_{0<\bar\mu a<\infty} \frac{1}{\sqrt{\bar\mu a}} \int_a |u|$ for $u \in L^0(\mathfrak{A})$. Show that τ is a $\mathcal{T}$-invariant extended Fatou norm. Find examples of $(\mathfrak{A}, \bar\mu)$ for which τ is, and is not, order-continuous on L^τ.

(h) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be semi-finite measure algebras and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$. (i) Show that there is a $\mathcal{T}$-invariant extended Fatou norm θ on $L^0(\mathfrak{B})$ defined by setting $\theta(v) = \sup\{\tau(Tv) : T \in \mathcal{T}_{\bar\nu,\bar\mu}\}$ for $v \in M^{1,\infty}_{\bar\nu}$. (ii) Show that when $(\mathfrak{A}, \bar\mu) = (\mathfrak{A}_L, \bar\mu_L)$ then $\theta(v) = \tau(v^*)$ for every $v \in M^{0,\infty}_{\bar\nu}$. (iii) Show that when $(\mathfrak{B}, \bar\nu) = (\mathfrak{A}_L, \bar\mu_L)$ then $\tau(u) = \theta(u^*)$ for every $u \in M^{0,\infty}_{\bar\mu}$.

(i) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Suppose that L^τ is a $\mathcal{T}$-invariant subset of L^0. Show that there is a $\mathcal{T}$-invariant extended Fatou norm $\tilde\tau$ which is equivalent to τ in the sense that, for some $M > 0$, $\tilde\tau(u) \le M\tau(u) \le M^2\tilde\tau(u)$ for every $u \in L^0$. (*Hint*: show first that $\int u^* \times v^* < \infty$ for every $u \in L^\tau$ and $v \in L^{\tau'}$, then that $\sup_{\tau(u)\le 1, \tau'(v)\le 1} \int u^* \times v^* < \infty$.)

(j) Suppose that τ is a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$, and that $0 < w = w^* \in M^{1,\infty}_{\bar\mu_L}$. Let $(\mathfrak{A}, \bar\mu)$ be any semi-finite measure algebra. Show that the function $u \mapsto \tau(w \times u^*)$ extends to a $\mathcal{T}$-invariant extended Fatou norm θ on $L^0(\mathfrak{A})$. (*Hint*: $\tau(w \times u^*) = \sup\{\tau(w \times Tu) : T \in \mathcal{T}_{\bar\mu,\bar\mu_L}\}$ for $u \in M^{1,\infty}_{\bar\mu_L}$.) (When $\tau = \| \|_p$ these norms are called **Lorentz norms**; see LINDENSTRAUSS & TZAFRIRI 79, p. 121.)

(k) Let $(\mathfrak{A}, \bar\mu)$ be $\mathcal{P}\mathbb{N}$ with counting measure. Identify $L^0(\mathfrak{A})$ with $\mathbb{R}^{\mathbb{N}}$. Let U be $\{u : u \in \mathbb{R}^{\mathbb{N}}, \{n : u(n) \ne 0\}$ is finite$\}$. Show that U is a perfect Riesz space, and is rearrangement-invariant but not $\mathcal{T}$-invariant.

(l) Let $(\mathfrak{A},\bar\mu)$ be an atomless quasi-homogeneous localizable measure algebra, and $U\subseteq L^0(\mathfrak{A})$ a rearrangement -invariant solid linear subspace which is a perfect Riesz space. Show that $U\subseteq M^{1,\infty}_{\bar\mu}$ and that U is $\mathcal{T}$-invariant. (*Hint*: assume $U\ne\{0\}$. Show that (i) $\chi a\in U$ whenever $\bar\mu a<\infty$ (ii) $V=\{v:v\times u\in L^1\ \forall\ u\in U\}$ is rearrangement -invariant (iii) U, $V\subseteq M^{1,\infty}$.)

374Y Further exercises (a) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra and $U\subseteq M^{1,\infty}_{\bar\mu}$ a non-zero $\mathcal{T}$-invariant Riesz subspace which, regarded as a Riesz space, is perfect. (i) Show that U includes $M^{\infty,1}_{\bar\mu}$. (ii) Show that its dual $\{v:v\in L^0(\mathfrak{A}),\ v\times u\in L^1_{\bar\mu}\ \forall\ u\in U\}$ (which in this exercise I will denote by $U^\times$) is also $\mathcal{T}$-invariant, and is $\{v:v\in M^{0,\infty}_{\bar\mu},\ \int u^*\times v^*<\infty\ \forall\ u\in U\}$. (iii) Show that for any localizable measure algebra $(\mathfrak{B},\bar\nu)$ the set $V=\{v:v\in M^{1,\infty}_{\bar\nu},\ Tv\in U\ \forall\ T\in\mathcal{T}_{\bar\nu,\bar\mu}\}$ is a perfect Riesz subspace of $L^0(\mathfrak{B})$, and that $V^\times=\{v:v\in M^{1,\infty}_{\bar\nu},\ Tv\in U^\times\ \forall\ T\in\mathcal{T}_{\bar\nu,\bar\mu}\}$. (iv) Show that if, in (i)-(iii), $(\mathfrak{A},\bar\mu)=(\mathfrak{A}_L,\bar\mu_L)$, then $V=\{v:v\in M^{0,\infty},\ v^*\in U\}$. (v) Show that if, in (iii), $(\mathfrak{B},\bar\nu)=(\mathfrak{A}_L,\bar\mu_L)$, then $U=\{u:u\in M^{0,\infty}_{\bar\mu},\ u^*\in V\}$.

(b) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and suppose that $1\le q\le p<\infty$. Let $w_{pq}\in L^0(\mathfrak{A}_L)$ be the equivalence class of the function $t\mapsto t^{(q-p)/p}$. (i) Show that for any $u\in L^0(\mathfrak{A})$,

$$\int w_{pq}\times(u^*)^q=p\int_0^\infty t^{q-1}(\bar\mu[\![|u|>t]\!])^{q/p}dt.$$

(ii) Show that we have an extended Fatou norm $\|\ \|_{p,q}$ on $L^0(\mathfrak{A})$ defined by setting

$$\|u\|_{p,q}=\bigl(p\int_0^\infty t^{q-1}(\bar\mu[\![|u|>t]\!])^{q/p}dt\bigr)^{1/q}$$

for every $u\in L^0(\mathfrak{A})$. (*Hint*: use 374Xj with $w=w_{pq}^{1/q}$, $\|\ \|=\|\ \|_q$.) (iii) Show that if $(\mathfrak{B},\bar\nu)$ is another semi-finite measure algebra and $T\in\mathcal{T}_{\bar\mu,\bar\nu}$, then $\|Tu\|_{p,q}\le\|u\|_{p,q}$ for every $u\in M^{1,\infty}_{\bar\mu}$. (iv) Show that $\|\ \|_{p,q}$ is an order-continuous norm on $L^{\|\ \|_{p,q}}$.

(c) Let $(\mathfrak{A},\bar\mu)$ be a homogeneous measure algebra of uncountable Maharam type, and u, $v\ge0$ in $M^0_{\bar\mu}$ such that $u^*=v^*$. Show that there is a measure-preserving automorphism π of $\mathfrak{A}$ such that $T_\pi u=v$, where $T_\pi:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ is the isomorphism corresponding to π.

(d) In $L^0(\mathfrak{A}_L)$ let u be the equivalence class of the function $f(t)=te^{-t}$. Show that there is no Boolean automorphism π of $\mathfrak{A}_L$ such that $T_\pi u=u^*$. (*Hint*: show that $\mathfrak{A}_L$ is τ-generated by $\{[\![u^*>\alpha]\!]:\alpha>0\}$.)

(e) Let $(\mathfrak{A},\bar\mu)$ be a quasi-homogeneous semi-finite measure algebra and $C\subseteq L^0(\mathfrak{A})$ a solid convex order-closed rearrangement-invariant set. Show that $C\cap M^{1,\infty}_{\bar\mu}$ is $\mathcal{T}$-invariant.

374 Notes and comments I gave this section the title 'rearrangement-invariant spaces' because it looks good on the Contents page, and it follows what has been common practice since Luxemburg 67b; but actually I think that it's $\mathcal{T}$-invariance which matters, and that rearrangement-invariant spaces are significant largely because the important ones are $\mathcal{T}$-invariant. The particular quality of $\mathcal{T}$-invariance which I have tried to bring out here is its transferability from one measure algebra (or measure space, of course) to another. This is what I take at a relatively leisurely pace in 374B-374D and 374Xf, and then encapsulate in 374Xh and 374Ya. The special place of the Lebesgue algebra $(\mathfrak{A}_L,\bar\mu_L)$ arises from its being more or less the simplest algebra over which every $\mathcal{T}$-invariant set can be described; see 374Xa.

I don't think this work is particularly easy, and (as in §373) there are rather a lot of unattractive names in it; but once one has achieved a reasonable familiarity with the concepts, the techniques used can be seen to amount to half a dozen ideas – non-trivial ideas, to be sure – from §§369 and 373. From §369 I take concepts of duality: the symmetric relationship between a perfect Riesz space $U\subseteq L^0$ and the representation of its dual (369C-369D), and the notion of associate extended Fatou norms (369H-369K). From §373 I take the idea of 'decreasing rearrangement' and theorems guaranteeing the existence of useful members of $\mathcal{T}_{\bar\mu,\bar\nu}$ (373O-373Q). The results of the present section all depend on repeated use of these facts, assembled in a variety of patterns.

There is one new method here, but an easy one: the construction of measure-preserving automorphisms by joining isomorphisms together, as in the proofs of 374H and 374J. I shall return to this idea, in greater generality and more systematically investigated, in §381. I hope that the special cases here will give no difficulty.

While $\mathcal{T}$-invariance is a similar phenomenon for both extended Fatou norms and perfect Riesz spaces (see 374Xh, 374Ya), the former seem easier to deal with. The essential difference is I think in 374B(i); with a $\mathcal{T}$-invariant extended

Fatou norm, we are necessarily confined to $M^{1,\infty}$, the natural domain of the methods used here. For perfect Riesz spaces we have examples like $\mathbb{R}^{\mathbb{N}} \cong L^0(\mathcal{P}\mathbb{N})$ and its dual, the space of eventually-zero sequences (374Xk); these are rearrangement-invariant but not $\mathcal{T}$-invariant, as I have defined it. This problem does not arise over atomless algebras (374Xl).

I think it is obvious that for algebras which are not quasi-homogeneous (374G) rearrangement-invariance is going to be of limited interest; there will be regions between which there is no communication by means of measure-preserving automorphisms, and the best we can hope for is a discussion of quasi-homogeneous components, if they exist, corresponding to the partition of unity used in the proof of 332J. There is a special difficulty concerning rearrangement-invariance in $L^0(\mathfrak{A}_L)$: two elements can have the same decreasing rearrangement without being rearrangements of each other in the strict sense (373Ya, 374Yd). The phenomenon of 373Ya is specific to algebras of countable Maharam type (374Yc). You will see that some of the labour of 374L is because we have to make room for the pieces to move in. 374J is easier just because in that context we can settle for a supremum, rather than an actual infinity, so the rearrangement needed (part (a) of the proof) can be based on a region of finite measure.

375 Kwapien's theorem

In §368 and the first part of §369 I examined maps from various types of Riesz space into L^0 spaces. There are equally striking results about maps out of L^0 spaces. I start with some relatively elementary facts about positive linear operators from L^0 spaces to Archimedean Riesz spaces in general (375A-375D), and then turn to a remarkable analysis, due essentially to S.Kwapien, of the positive linear operators from a general L^0 space to the L^0 space of a semi-finite measure algebra (375J), with a couple of simple corollaries.

375A Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and W an Archimedean Riesz space. If $T : L^0(\mathfrak{A}) \to W$ is a positive linear operator, it is sequentially order-continuous.

proof (a) The first step is to observe that if $\langle u_n \rangle_{n\in\mathbb{N}}$ is any non-increasing sequence in $L^0 = L^0(\mathfrak{A})$ with infimum 0, and $\epsilon > 0$, then $\{n(u_n - \epsilon u_0) : n \in \mathbb{N}\}$ is bounded above in L^0. **P** For $k \in \mathbb{N}$ set $a_k = \sup_{n\in\mathbb{N}} [\![n(u_n - \epsilon u_0) > k]\!]$; set $a = \inf_{k\in\mathbb{N}} a_k$. **?** Suppose, if possible, that $a \neq 0$. Because $u_n \le u_0$, $n(u_n - \epsilon u_0) \le nu_0$ for every n and

$$a \subseteq a_0 \subseteq [\![u_0 > 0]\!] = [\![\epsilon u_0 > 0]\!] = \sup_{n\in\mathbb{N}} [\![\epsilon u_0 - u_n > 0]\!].$$

So there is some $m \in \mathbb{N}$ such that $a' = a \cap [\![\epsilon u_0 - u_m > 0]\!] \neq 0$. Now, for any $n \ge m$, any $k \in \mathbb{N}$,

$$a' \cap [\![n(u_n - \epsilon u_0) > k]\!] \subseteq [\![\epsilon u_0 - u_m > 0]\!] \cap [\![u_m - \epsilon u_0 > 0]\!] = 0.$$

But $a' \subseteq \sup_{n\in\mathbb{N}} [\![n(u_n - \epsilon u_0) > k]\!]$, so in fact

$$a' \subseteq \sup_{n\le m} [\![n(u_n - \epsilon u_0) > k]\!] = [\![v > k]\!],$$

where $v = \sup_{n\le m} n(u_n - \epsilon u_0)$. And this means that $\inf_{k\in\mathbb{N}} [\![v > k]\!] \supseteq a' \neq 0$, which is impossible. **X** Accordingly $a = 0$; by 364L(a-i), $\{n(u_n - \epsilon u_0) : n \in \mathbb{N}\}$ is bounded above. **Q**

(b) Now suppose that $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in L^0 with infimum 0, and that $w \in W$ is a lower bound for $\{Tu_n : n \in \mathbb{N}\}$. Take any $\epsilon > 0$. By (a), $\{n(u_n - \epsilon u_0) : n \in \mathbb{N}\}$ has an upper bound v in L^0. Because T is positive,

$$w \le Tu_n = T(u_n - \epsilon u_0) + T(\epsilon u_0) \le T(\frac{1}{n}v) + T(\epsilon u_0) = \frac{1}{n}Tv + \epsilon Tu_0$$

for every $n \ge 1$. Because W is Archimedean, $w \le \epsilon Tu_0$. But this is true for every $\epsilon > 0$, so (again because W is Archimedean) $w \le 0$. As w is arbitrary, $\inf_{n\in\mathbb{N}} Tu_n = 0$. As $\langle u_n \rangle_{n\in\mathbb{N}}$ is arbitrary, T is sequentially order-continuous (351Gb).

375B Proposition Let $\mathfrak{A}$ be an atomless Dedekind σ-complete Boolean algebra. Then $L^0(\mathfrak{A})^\times = \{0\}$.

proof ? Suppose, if possible, that $h : L^0(\mathfrak{A}) \to \mathbb{R}$ is a non-zero order-continuous positive linear functional. Then there is a $u > 0$ in L^0 such that $h(v) > 0$ whenever $0 < v \le u$ (356H). Because $\mathfrak{A}$ is atomless, there is a disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ such that $a_n \subseteq [\![u > 0]\!]$ for each n, so that $u_n = u \times \chi a_n > 0$, while $u_m \wedge u_n = 0$ if $m \neq n$. Now however

$$v = \sup_{n\in\mathbb{N}} \frac{n}{h(u_n)} u_n$$

is defined in L^0, by 368K, and $h(v) \ge n$ for every n, which is impossible. **X**

375C Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, W an Archimedean Riesz space, and $T : L^0(\mathfrak{A}) \to W$ an order-continuous Riesz homomorphism. Then $V = T[L^0(\mathfrak{A})]$ is an order-closed Riesz subspace of W.

proof The kernel U of T is a band in $L^0 = L^0(\mathfrak{A})$ (352Oe), and must be a projection band (353I), because L^0 is Dedekind complete (364M). Since $U+U^\perp = L^0$, $T[U]+T[U^\perp] = V$, that is, $T[U^\perp] = V$; since $U \cap U^\perp = \{0\}$, T is an isomorphism between $U^\perp$ and V. Now suppose that $A \subseteq V$ is upwards-directed and has a least upper bound $w \in W$. Then $B = \{u : u \in U^\perp, Tu \in A\}$ is upwards-directed and $T[B] = A$. The point is that B is bounded above in L^0. **P** **?** If not, then $\{u^+ : u \in B\}$ cannot be bounded above, so there is a $u_0 > 0$ in L^0 such that $nu_0 = \sup_{u\in B} nu_0 \wedge u^+$ for every $n \in \mathbb{N}$ (368A). Since $B \subseteq U^\perp$, $u_0 \in U^\perp$ and $Tu_0 > 0$. But now, because T is an order-continuous Riesz homomorphism,

$$nTu_0 = \sup_{u\in B} T(nu_0 \wedge u^+) = \sup_{v\in A} nTu_0 \wedge v^+ \le w^+$$

for every $n \in \mathbb{N}$, which is impossible. **XQ**

Set $u^* = \sup B$; then $Tu^* = \sup A = w$ and $w \in V$. As A is arbitrary, V is order-closed.

375D Corollary Let W be a Riesz space and V an order-dense Riesz subspace which is isomorphic to $L^0(\mathfrak{A})$ for some Dedekind complete Boolean algebra $\mathfrak{A}$. Then $V = W$.

proof By 353Q, W is Archimedean. So we can apply 375C to an isomorphism $T : L^0(\mathfrak{A}) \to V$ to see that V is order-closed in W.

375E Theorem Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, $(\mathfrak{B}, \bar\nu)$ any measure algebra, and $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ an order-continuous positive linear operator. Then T is continuous for the topologies of convergence in measure.

proof **?** Otherwise, we can find $w \in L^0(\mathfrak{A})$, $b \in \mathfrak{B}^f$ and $\epsilon > 0$ such that whenever $a \in \mathfrak{A}^f$ and $\delta > 0$ there is a $u \in L^0(\mathfrak{A})$ such that $\bar\mu(a \cap [\![|u-w| > \delta]\!]) \le \delta$ and $\bar\nu(b \cap [\![|Tu - Tw| > \epsilon]\!]) \ge \epsilon$ (367M, 2A3H). Of course it follows that whenever $a \in \mathfrak{A}^f$ and $\delta > 0$ there is a $u \in L^0(\mathfrak{A})$ such that $\bar\mu(a \cap [\![|u| > \delta]\!]) \le \delta$ and $\bar\nu(b \cap [\![|Tu| > \epsilon]\!]) \ge \epsilon$ (367M). Choose $\langle a_n\rangle_{n\in\mathbb{N}}$ and $\langle u_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $a_0 = 0$. Given that $a_n \in \mathfrak{A}^f$, let $u_n \in L^0(\mathfrak{A})$ be such that $\bar\mu(a_n \cap [\![|u_n| > 2^{-n}]\!]) \le 2^{-n}$ and $\bar\nu(b \cap [\![|Tu_n| > \epsilon]\!]) \ge \epsilon$. Of course it follows that $\bar\nu(b \cap [\![T|u_n| > \epsilon]\!]) \ge \epsilon$. Because $(\mathfrak{A}, \bar\mu)$ is semi-finite, $|u_n| = \sup_{a\in\mathfrak{A}^f} |u_n| \times \chi a$); because T is order-continuous, $T|u_n| = \sup_{a\in\mathfrak{A}^f} T(|u_n| \times \chi a)$, and we can find $a_{n+1} \in \mathfrak{A}^f$ such that $\bar\nu(b \cap [\![T(|u_n| \times \chi a_{n+1}) > \epsilon]\!]) \ge \frac{1}{2}\epsilon$. Enlarging a_{n+1} if necessary, arrange that $a_{n+1} \supseteq a_n$. Continue.

At the end of the induction, set $v_n = 2^n|u_n| \times \chi a_{n+1}$; then $\bar\mu(a_n \cap [\![v_n > 1]\!]) \le 2^{-n}$, for each $n \in \mathbb{N}$. It follows that $\{v_n : n \in \mathbb{N}\}$ is bounded above. **P** For $k \in \mathbb{N}$, set $c_k = \sup_{n\in\mathbb{N}} [\![v_n > k]\!]$. Then $c_k \subseteq \sup_{n\in\mathbb{N}} a_n$. If $n \in \mathbb{N}$ and $\delta > 0$, let $m \ge n$ be such that $2^{-m+1} \le \delta$, and $k \ge 1$ such that $\bar\mu(a_n \cap [\![\sup_{m<n} v_m > k]\!]) \le \delta$. Then

$$\bar\mu(a_n \cap c_k) \le \bar\mu(a_n \cap [\![\sup_{m<n} v_m > k]\!]) + \sum_{i=m}^\infty \bar\mu(a_i \cap [\![v_i > 1]\!]) \le 2\delta.$$

As δ is arbitrary, $a_n \cap \inf_{k\in\mathbb{N}} c_k = 0$; as n is arbitrary, $\inf_{k\in\mathbb{N}} c_k = 0$; by 364L(a-i) again, $\{v_n : n \in \mathbb{N}\}$ is bounded above. **Q**

Set $v = \sup_{n\in\mathbb{N}} v_n$. Then $2^{-n}v \ge |u_n| \times \chi a_{n+1}$, so $2^{-n}Tv \ge T(|u_n| \times \chi a_{n+1})$ and $\bar\nu(b \cap [\![2^{-n}Tv > \epsilon]\!]) \ge \frac{1}{2}\epsilon$, for each $n \in \mathbb{N}$. But $\inf_{n\in\mathbb{N}} 2^{-n}Tv = 0$, so $\inf_{n\in\mathbb{N}} [\![2^{-n}Tv > \epsilon]\!] = 0$ (364L(b-ii)) and $\inf_{n\in\mathbb{N}} \bar\nu(b \cap [\![2^{-n}Tv > \epsilon]\!]) = 0$. **X**

So we have the result.

375F I come now to the deepest result of this section, concerning positive linear operators from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$ where $\mathfrak{B}$ is a measure algebra. I approach through a couple of lemmas which are striking enough in their own right.

The following temporary definition will be useful.

Definition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. I will say that a function $\phi : \mathfrak{A} \to \mathfrak{B}$ is a **σ-subhomomorphism** if

$\phi(a \cup a') = \phi(a) \cup \phi(a')$ for all a, $a' \in \mathfrak{A}$,

$\inf_{n\in\mathbb{N}} \phi(a_n) = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

Now we have the following easy facts.

375G Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism.

(a) $\phi(0) = 0$, $\phi(a) \subseteq \phi(a')$ whenever $a \subseteq a'$, and $\phi(a) \setminus \phi(a') \subseteq \phi(a \setminus a')$ for every a, $a' \in \mathfrak{A}$.

(b) If $\bar\mu$, $\bar\nu$ are measures such that $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ are totally finite measure algebras, then for every $\epsilon > 0$ there is a $\delta > 0$ such that $\bar\nu\phi(a) \le \epsilon$ whenever $\bar\mu a \le \delta$.

proof (a) This is elementary. Set every $a_n = 0$ in the second clause of the definition 375F to see that $\phi(0) = 0$. The other two parts are immediate consequences of the first clause.

(b) (Compare 232Ba, 327Bb.) **?** Suppose, if possible, otherwise. Then for every $n \in \mathbb{N}$ there is an $a_n \in \mathfrak{A}$ such that $\bar\mu a_n \le 2^{-n}$ and $\bar\nu\phi(a_n) \ge \epsilon$. Set $c_n = \sup_{i\ge n} a_i$ for each n; then $\langle c_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0 (since $\bar\mu c_n \le 2^{-n+1}$ for each n), but $\bar\nu\phi(c_n) \ge \epsilon$ for every n, so $\inf_{n\in\mathbb{N}} \phi c_n$ cannot be 0. **X**

375H Lemma Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be totally finite measure algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism. Then for every non-zero $b_0 \in \mathfrak{B}$ there are a non-zero $b \subseteq b_0$ and an $m \in \mathbb{N}$ such that $b \cap \inf_{j\le m} \phi(a_j) = 0$ whenever $a_0, \dots, a_m \in \mathfrak{A}$ are disjoint.

proof (a) Suppose first that $\mathfrak{A}$ is atomless and that $\bar\mu 1 = 1$.

Set $\epsilon = \frac{1}{5}\bar\nu b_0$ and let $m \ge 1$ be such that $\bar\nu\phi(a) \le \epsilon$ whenever $\bar\mu a \le \frac{1}{m}$ (375Gb). We need to know that $(1-\frac{1}{m})^m \le \frac{1}{2}$; this is because (if $m \ge 2$) $\ln m - \ln(m-1) \ge \frac{1}{m}$, so $m\ln(1-\frac{1}{m}) \le -1 \le -\ln 2$.

Set

$$C = \{\inf_{j\le m} \phi(a_j) : a_0, \dots, a_m \in \mathfrak{A} \text{ are disjoint}\}.$$

? Suppose, if possible, that $b_0 \subseteq \sup C$. Then there are $c_0, \dots, c_k \in C$ such that $\bar\nu(b_0 \cap \sup_{i\le k} c_i) \ge 4\epsilon$. For each $i \le k$ choose disjoint $a_{i0}, \dots, a_{im} \in \mathfrak{A}$ such that $c_i = \inf_{j\le m} \phi(a_{ij})$. Let D be the set of atoms of the finite subalgebra of $\mathfrak{A}$ generated by $\{a_{ij} : i \le k,\ j \le m\}$, so that D is a finite partition of unity in $\mathfrak{A}$, and every a_{ij} is the join of the members of D it includes. Set $p = \#(D)$, and for each $d \in D$ take a maximal disjoint set $E_d \subseteq \{e : e \subseteq d,\ \bar\mu e = \frac{1}{pm}\}$, so that $\bar\mu(d \setminus \sup E_d) < \frac{1}{pm}$; set

$$d^* = 1 \setminus \sup(\bigcup_{d\in D} E_d) = \sup_{d\in D}(d \setminus \sup E_d),$$

so that $\bar\mu d^*$ is a multiple of $\frac{1}{pm}$ and is less than $\frac{1}{m}$. Let E^* be a disjoint set of elements of measure $\frac{1}{pm}$ with union d^*, and take $E = E^* \cup \bigcup_{d\in D} E_d$, so that E is a partition of unity in $\mathfrak{A}$, $\bar\mu e = \frac{1}{pm}$ for every $e \in E$, and $a_{ij} \setminus d^*$ is the join of the members of E it includes for every $i \le k$ and $j \le m$.

Set

$$\mathcal{K} = \{K : K \subseteq E,\ \#(K) = p\}, \quad M = \#(\mathcal{K}) = \frac{(mp)!}{p!(mp-p)!}.$$

For every $K \in \mathcal{K}$, $\bar\mu(\sup K) = \frac{1}{m}$ so $\bar\nu\phi(\sup K) \le \epsilon$. So if we set

$$v = \sum_{K\in\mathcal{K}} \chi\phi(\sup K),$$

$\int v \le \epsilon M$. On the other hand,

$$\bar\nu(b_0 \cap \sup_{i\le k} c_i) \ge 4\epsilon, \quad \bar\nu\phi(d^*) \le \epsilon,$$

so $\bar\nu b_1 \ge 3\epsilon$, where

$$b_1 = b_0 \cap \sup_{i\le k} c_i \setminus \phi(d^*).$$

Accordingly $\int v \le \frac{1}{3}M\bar\nu b_1$ and

$$b_2 = b_1 \cap [\![v < \tfrac{1}{2}M]\!]$$

is non-zero.

Because $b_2 \subseteq b_1$, there is an $i \le k$ such that $b_2 \cap c_i \ne 0$. Now

$$b_2 \cap c_i \subseteq c_i \setminus \phi(d^*) = \inf_{j\le m} \phi(a_{ij}) \setminus \phi(d^*) \subseteq \inf_{j\le m} \phi(a_{ij} \setminus d^*).$$

But every $a_{ij} \setminus d^*$ is the join of the members of E it includes, so

$$\begin{aligned} b_2 \cap c_i &\subseteq \inf_{j\le m} \phi(a_{ij} \setminus d^*) \subseteq \inf_{j\le m} \phi(\sup\{e : e \in E,\ e \subseteq a_{ij}\}) \\ &= \inf_{j\le m} \sup\{\phi(e) : e \in E,\ e \subseteq a_{ij}\} \\ &= \sup\{\inf_{j\le m} \phi(e_j) : e_0, \dots, e_m \in E \text{ and } e_j \subseteq a_{ij} \text{ for every } j\}. \end{aligned}$$

So there are $e_0, \dots, e_m \in E$ such that $e_j \subseteq a_{ij}$ for each j and $b_3 = b_2 \cap \inf_{j\le m} \phi(e_j) \ne 0$. Because $a_{i0}, \dots, a_{im}$ are disjoint, $e_0, \dots, e_m$ are distinct; set $J = \{e_0, \dots, e_m\}$. Then whenever $K \in \mathcal{K}$ and $K \cap J \ne \emptyset$, $b_3 \subseteq \phi(\sup K)$.

So let us calculate the size of $\mathcal{K}_1 = \{K : K \in \mathcal{K},\ K \cap J \ne \emptyset\}$. This is

$$M - \frac{(mp-m-1)!}{p!(mp-p-m-1)!} = M\big(1 - \frac{(mp-p)(mp-p-1)\dots(mp-p-m)}{mp(mp-1)\dots(mp-m)}\big)$$
$$\ge M\big(1 - (\frac{mp-p}{mp})^{m+1}\big) \ge \frac{1}{2}M.$$

But this means that $b_3 \subseteq [\![v \ge \frac{1}{2}M]\!]$, while also $b_3 \subseteq [\![v < \frac{1}{2}M]\!]$; which is surely impossible. **X**

Accordingly $b_0 \not\subseteq \sup C$, and we can take $b = b_0 \setminus \sup C$.

(b) Now for the general case. Let A be the set of atoms of $\mathfrak{A}$, and set $d = 1 \setminus \sup A$. Then the principal ideal $\mathfrak{A}_d$ is atomless, so there are a non-zero $b_1 \subseteq b_0$ and an $n \in \mathbb{N}$ such that $b_1 \cap \inf_{j \le n} \phi(a_j) = 0$ whenever $a_0, \dots , a_n \in \mathfrak{A}_d$ are disjoint. **P** If $\bar\mu d > 0$ this follows from (a), if we apply it to $\phi\restriction\mathfrak{A}_d$ and $(\bar\mu d)^{-1}\bar\mu\restriction\mathfrak{A}_d$. If $\bar\mu d = 0$ then we can just take $b_1 = b_0$ and $n = 0$. **Q**

Let $\delta > 0$ be such that $\bar\nu\phi(a) < \bar\nu b_1$ whenever $\bar\mu a \le \delta$. Let $A_1 \subseteq A$ be a finite set such that $\bar\mu(\sup A_1) \ge \bar\mu(\sup A)-\delta$, and set $r = \#(A)$, $d^* = \sup(A \setminus A_1)$. Then $\bar\mu d^* \le \delta$ so $b = b_1 \setminus \phi(d^*) \ne 0$. Try $m = n + r$. If $a_0, \dots , a_m$ are disjoint, then at most r of them can meet $\sup A_1$, so (re-ordering if necessary) we can suppose that $a_0, \dots , a_n$ are disjoint from $\sup A_1$, in which case $a_j \setminus d^* \subseteq d$ for each $j \le m$. But in this case (because $b \cap \phi(d^*) = 0$)

$$b \cap \inf_{j\le m} \phi(a_j) \subseteq b \cap \inf_{j\le n} \phi(a_j) = b \cap \inf_{j\le n} \phi(a_j \cap d) = 0$$

by the choice of n and b_1.

Thus in the general case also we can find appropriate b and m.

375I Lemma Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be totally finite measure algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism. Then for every non-zero $b_0 \in \mathfrak{B}$ there are a non-zero $b \subseteq b_0$ and a finite partition of unity $C \subseteq \mathfrak{A}$ such that $a \mapsto b \cap \phi(a \cap c)$ is a ring homomorphism for every $c \in C$.

proof By 375H, we can find b_1, m such that $0 \ne b_1 \subseteq b_0$ and $b_1 \cap \inf_{j\le m} \phi(a_j) = 0$ whenever $a_0, \dots , a_m \in \mathfrak{A}$ are disjoint. Do this with the smallest possible m. If $m = 0$ then $b_1 \cap \phi(1) = 0$, so we can take $b = b_1$, $C = \{1\}$. Otherwise, because m is minimal, there must be disjoint $c_1, \dots , c_m \in \mathfrak{A}$ such that $b = b_1 \cap \inf_{1\le j\le m} \phi(c_j) \ne 0$. Set $c_0 = 1 \setminus \sup_{1\le j\le m} c_j$, $C = \{c_0, c_1, \dots , c_m\}$; then C is a partition of unity in $\mathfrak{A}$. Set $\pi_j(a) = b \cap \phi(a \cap c_j)$ for each $a \in \mathfrak{A}$ and $j \le m$. Then we always have $\pi_j(a \cup a') = \pi_j(a) \cup \pi_j(a')$ for all a, $a' \in \mathfrak{A}$, because ϕ is a subhomomorphism.

To see that every π_j is a ring homomorphism, we need only check that $\pi_j(a \cap a') = 0$ whenever $a \cap a' = 0$. (Compare 312H(iv).) In the case $j = 0$, we actually have $\pi_0(a) = 0$ for every a, because $b \cap \phi(c_0) = b_1 \cap \inf_{0\le j\le m} \phi(c_j) = 0$ by the choice of b_1 and m. When $1 \le j \le m$, if $a \cap a' = 0$, then

$$\pi_j(a) \cap \pi_j(a') = b_1 \cap \inf_{1\le i\le m, i\ne j} \phi(c_j) \cap \phi(a) \cap \phi(a')$$

is again 0, because a, a', $c_1, \dots , c_{j-1}, c_{j+1}, \dots , c_m$ are disjoint. So we have a suitable pair b, C.

375J Theorem Let $\mathfrak{A}$ be any Dedekind σ-complete Boolean algebra and $(\mathfrak{B}, \bar\nu)$ a semi-finite measure algebra. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be a positive linear operator. Then we can find B, $\langle A_b\rangle_{b\in B}$ such that B is a partition of unity in $\mathfrak{B}$, each A_b is a finite partition of unity in $\mathfrak{A}$, and $u \mapsto T(u \times \chi a) \times \chi b$ is a Riesz homomorphism whenever $b \in B$ and $a \in A_b$.

proof (a) Write B^* for the set of potential members of B; that is, the set of those $b \in \mathfrak{B}$ such that there is a finite partition of unity $A \subseteq \mathfrak{A}$ such that T_{ab} is a Riesz homomorphism for every $a \in A$, writing $T_{ab}(u) = T(u \times \chi a) \times \chi b$. If I can show that B^* is order-dense in $\mathfrak{B}$, this will suffice, since there will then be a partition of unity $B \subseteq B^*$.

(b) So let b_0 be any non-zero member of $\mathfrak{B}$; I seek a non-zero member of B^* included in b_0. Of course there is a non-zero $b_1 \subseteq b_0$ with $\bar\nu b_1 < \infty$. Let $\gamma > 0$ be such that $b_2 = b_1 \cap [\![T(\chi 1) \le \gamma]\!]$ is non-zero. Define $\mu : \mathfrak{A} \to [0, \infty[$ by setting $\mu a = \int_{b_2} T(\chi a)$ for every $a \in \mathfrak{A}$. Then μ is countably additive, because χ, T and $\int$ are all additive and sequentially order-continuous (using 375A). Set $\mathcal{N} = \{a : \mu a = 0\}$; then $\mathcal{N}$ is a σ-ideal of $\mathfrak{A}$, and $(\mathfrak{C}, \bar\mu)$ is a totally finite measure algebra, where $\mathfrak{C} = \mathfrak{A}/\mathcal{N}$ and $\bar\mu a^\bullet = \mu a$ for every $a \in \mathfrak{A}$ (just as in 321H).

(c) We have a function ϕ from $\mathfrak{C}$ to the principal ideal $\mathfrak{B}_{b_2}$ defined by saying that $\phi a^\bullet = b_2 \cap [\![T(\chi a) > 0]\!]$ for every $a \in \mathfrak{A}$. **P** If a_1, $a_2 \in \mathfrak{A}$ are such that $a_1^\bullet = a_2^\bullet$ in $\mathfrak{C}$, this means that $a_1 \triangle a_2 \in \mathcal{N}$; now

$$[\![T(\chi a_1) > 0]\!] \triangle [\![T(\chi a_2) > 0]\!] \subseteq [\![|T(\chi a_1) - T(\chi a_2)| > 0]\!]$$
$$\subseteq [\![T(|\chi a_1 - \chi a_2|) > 0]\!] = [\![T\chi(a_1 \triangle a_2) > 0]\!]$$

is disjoint from b_2 because $\int_{b_2} T\chi(a_1 \triangle a_2) = 0$. Accordingly $b_2 \cap [\![T(\chi a_1) > 0]\!] = b_2 \cap [\![T(\chi a_2) > 0]\!]$ and we can take this common value for $\phi(a_1^\bullet) = \phi(a_2^\bullet)$. **Q**

(d) Now ϕ is a σ-subhomomorphism. **P** (i) For any a_1, $a_2 \in \mathfrak{A}$ we have

$$[\![T\chi(a_1 \cup a_2) > 0]\!] = [\![T(\chi a_1) > 0]\!] \cup [\![T(\chi a_2) > 0]\!]$$

because

$$T(\chi a_1) \vee T(\chi a_2) \le T\chi(a_1 \cup a_2) \le T(\chi a_1) + T(\chi a_2).$$

So $\phi(c_1 \cup c_2) = \phi(c_1) \cup \phi(c_2)$ for all c_1, $c_2 \in \mathfrak{C}$. (ii) If $\langle c_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{C}$ with infimum 0, choose $a_n \in \mathfrak{A}$ such that $a_n^\bullet = c_n$ for each n, and set $\tilde a_n = \inf_{i\le n} a_i \setminus \inf_{i\in\mathbb{N}} a_i$ for each n; then $\tilde a_n^\bullet = c_n$ so $\phi(c_n) = [\![T(\chi\tilde a_n) > 0]\!]$ for each n, while $\langle\tilde a_n\rangle_{n\in\mathbb{N}}$ is non-increasing and $\inf_{n\in\mathbb{N}} \tilde a_n = 0$. **?** Suppose, if possible, that $b' = \inf_{n\in\mathbb{N}} \phi(c_n) \ne 0$; set $\epsilon = \frac{1}{2}\bar\nu b'$. Then $\bar\nu(b_2 \cap [\![T(\chi\tilde a_n) > 0]\!]) \ge 2\epsilon$ for every $n \in \mathbb{N}$. For each n, take $\alpha_n > 0$ such that $\bar\nu(b_2 \cap [\![T(\chi\tilde a_n) > \alpha_n]\!]) \ge \epsilon$. Then $u = \sup_{n\in\mathbb{N}} n\alpha_n^{-1}\chi\tilde a_n$ is defined in $L^0(\mathfrak{A})$ (because $\sup_{n\in\mathbb{N}} [\![n\alpha_n^{-1}\chi\tilde a_n > k]\!] \subseteq \tilde a_m$ if $k \ge \max_{i\le m} i\alpha_i^{-1}$, so $\inf_{k\in\mathbb{N}}\sup_{n\in\mathbb{N}} [\![n\alpha_n^{-1}\chi\tilde a_n > k]\!] = 0$). But now

$$\bar\nu(b_2 \cap [\![Tu > n]\!]) \ge \bar\nu(b_2 \cap [\![T(\chi\tilde a_n) > \alpha_n]\!]) \ge \epsilon$$

for every n, so $\inf_{n\in\mathbb{N}} [\![Tu > n]\!] \ne 0$, which is impossible. **X** Thus $\inf_{n\in\mathbb{N}} \phi(c_n) = 0$; as $\langle c_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ϕ is a σ-subhomomorphism. **Q**

(e) By 375I, there are a non-zero $b \in \mathfrak{B}_{b_2}$ and a finite partition of unity $C \subseteq \mathfrak{C}$ such that $d \mapsto b \cap \phi(d \cap c)$ is a ring homomorphism for every $c \in C$. There is a partition of unity $A \subseteq \mathfrak{A}$, of the same size as C, such that $C = \{a^\bullet : a \in A\}$. Now T_{ab} is a Riesz homomorphism for every $a \in A$. **P** It is surely a positive linear operator. If u_1, $u_2 \in L^0(\mathfrak{A})$ and $u_1 \wedge u_2 = 0$, set $e_i = [\![u_i > 0]\!]$ for each i, so that $e_1 \cap e_2 = 0$. Observe that $u_i = \sup_{n\in\mathbb{N}} u_i \wedge n\chi e_i$, so that

$$[\![T_{ab}u_i > 0]\!] = \sup_{n\in\mathbb{N}} [\![T_{ab}(u_i \wedge n\chi e_i) > 0]\!] \subseteq [\![T_{ab}(\chi e_i) > 0]\!] = b \cap [\![T\chi(e_i \cap a) > 0]\!]$$

for both i (of course T_{ab}, like T, is sequentially order-continuous). But this means that

$$\begin{aligned}[\![T_{ab}u_1 > 0]\!] \cap [\![T_{ab}u_2 > 0]\!] &\subseteq b \cap [\![T\chi(e_1 \cap a) > 0]\!] \cap [\![T\chi(e_2 \cap a) > 0]\!]\\ &= b \cap \phi(e_1^\bullet \cap a^\bullet) \cap \phi(e_2^\bullet \cap a^\bullet) = 0\end{aligned}$$

because $a^\bullet \in C$, so $d \mapsto b \cap \phi(d \cap a^\bullet)$ is a ring homomorphism, while $e_1^\bullet \cap e_2^\bullet = 0$. So $T_{ab}u_1 \wedge T_{ab}u_2 = 0$. As u_1 and u_2 are arbitrary, T_{ab} is a Riesz homomorphism (352G(iv)). **Q**

(f) Thus $b \in B^*$. As b_0 is arbitrary, B^* is order-dense, and we're home.

375K Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and U a Dedekind complete Riesz space such that $U^\times$ separates the points of U. If $T : L^0(\mathfrak{A}) \to U$ is a positive linear operator, there is a sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ of Riesz homomorphisms from $L^0(\mathfrak{A})$ to U such that $T = \sum_{n=0}^\infty T_n$, in the sense that $Tu = \sup_{n\in\mathbb{N}} \sum_{i=0}^n T_i u$ for every $u \ge 0$ in $L^0(\mathfrak{A})$.

proof By 369A, U can be embedded as an order-dense Riesz subspace of $L^0(\mathfrak{B})$ for some localizable measure algebra $(\mathfrak{B}, \bar\nu)$; being Dedekind complete, it is solid in $L^0(\mathfrak{B})$ (353K). Regard T as an operator from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$, and take B, $\langle A_b\rangle_{b\in B}$ as in 375J. Note that $L^0(\mathfrak{B})$ can be identified with $\prod_{b\in B} L^0(\mathfrak{B}_b)$ (364R, 322L). For each $b \in B$ let $f_b : A_b \to \mathbb{N}$ be an injection. If $b \in B$ and $n \in f_b[A_b]$, set $T_{nb}(u) = \chi b \times T(u \times \chi f_b^{-1}(n))$; otherwise set $T_{nb} = 0$. Then $T_{nb} : L^0(\mathfrak{A}) \to L^0(\mathfrak{B}_b)$ is a Riesz homomorphism; because A_b is a finite partition of unity, $\sum_{n=0}^\infty T_{nb}u = \chi b \times Tu$ for every $u \in L^0(\mathfrak{A})$. But this means that if we set $T_n u = \langle T_{nb}u\rangle_{b\in B}$,

$$T_n : L^0(\mathfrak{A}) \to \prod_{b\in B} L^0(\mathfrak{B}_b) \cong L^0(\mathfrak{B})$$

is a Riesz homomorphism for each n; and $T = \sum_{n=0}^\infty T_n$. Of course every T_n is an operator from $L^0(\mathfrak{A})$ to U because $|T_n u| \le T|u| \in U$ for every $u \in L^0(\mathfrak{A})$.

375L Corollary (a) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, $(\mathfrak{B}, \bar\nu)$ is a semi-finite measure algebra, and there is any non-zero positive linear operator from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$, then there is a non-trivial sequentially order-continuous ring homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(b) If $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ are homogeneous probability algebras and $\tau(\mathfrak{A}) > \tau(\mathfrak{B})$, then $\mathrm{L}^\sim(L^0(\mathfrak{A}); L^0(\mathfrak{B})) = \{0\}$.

proof (a) It is probably quickest to look at the proof of 375J: starting from a non-zero positive linear operator $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$, we move to a non-zero σ-subhomomorphism $\phi : \mathfrak{A}/\mathcal{N} \to \mathfrak{B}$ and thence to a non-zero ring homomorphism from $\mathfrak{A}/\mathcal{N}$ to $\mathfrak{B}$, corresponding to a non-zero ring homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, which is sequentially order-continuous because it is dominated by ϕ. Alternatively, quoting 375J, we have a non-zero Riesz homomorphism $T_1 : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$, and it is easy to check that $a \mapsto [\![T(\chi a) > 0]\!]$ is a non-zero sequentially order-continuous ring homomorphism.

(b) Use (a) and 331J.

375X Basic exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and W an Archimedean Riesz space. Let $T : L^0(\mathfrak{A}) \to W$ be a positive linear operator. Show that T is order-continuous iff $T\chi : \mathfrak{A} \to W$ is order-continuous.

(b) Let $\mathfrak{A}$ be an atomless Dedekind σ-complete Boolean algebra and W a Banach lattice. Show that the only order-continuous positive linear operator from $L^0(\mathfrak{A})$ to W is the zero operator.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and W a Riesz space. Let $T : L^0(\mathfrak{A}) \to W$ be an order-continuous Riesz homomorphism such that $T[L^0(\mathfrak{A})]$ is order-dense in W. Show that T is surjective.

>(d) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism as defined in 375F. Show that ϕ is sequentially order-continuous.

>(e) Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure on $[0, 1]$ and $\mathfrak{G}$ the regular open algebra of $\mathbb{R}$. (i) Show that there is no non-zero positive linear operator from $L^0(\mathfrak{G})$ to $L^0(\mathfrak{A})$. (*Hint*: suppose $T : L^0(\mathfrak{G}) \to L^0(\mathfrak{A})$ were such an operator. Reduce to the case $T(\chi 1) \le \chi 1$. Let $\langle b_n\rangle_{n\in\mathbb{N}}$ enumerate an order-dense subset of $\mathfrak{G}$ (316Yo). For each $n \in \mathbb{N}$ take non-zero $b'_n \subseteq b_n$ such that $\int T(\chi b'_n) \le 2^{-n-2}\int T(\chi 1)$ and consider $T\chi(\sup_{n\in\mathbb{N}} b'_n)$. See also 375Yf-375Ye.) (ii) Show that there is no non-zero positive linear operator from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{G})$. (*Hint*: suppose $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{G})$ were such an operator. For each $n \in \mathbb{N}$ choose $a_n \in \mathfrak{A}$, $\alpha_n > 0$ such that $\bar{\mu}a_n \le 2^{-n}$ and if $b_n \subseteq [\![T(\chi 1) > 0]\!]$ then $b_n \cap [\![T(\chi a_n) > \alpha_n]\!] \ne 0$. Consider Tu where $u = \sum_{n=0}^{\infty} n\alpha_n^{-1}\chi a_n$.)

(f) In 375K, show that for any $u \in L^0(\mathfrak{A})$

$$\inf_{n\in\mathbb{N}} \sup_{m\ge n} [\![|Tu - \textstyle\sum_{i=0}^{m} T_i u| > 0]\!] = 0.$$

>(g) Prove directly, without quoting 375F-375L, that if $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra then every positive linear functional from $L^0(\mathfrak{A})$ to $\mathbb{R}$ is a finite sum of Riesz homomorphisms.

(h) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ a Riesz homomorphism. Show that there are a sequentially order-continuous ring homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ and a $w \in L^0(\mathfrak{B})^+$ such that $Tu = w \times T_\pi u$ for every $u \in L^0(\mathfrak{A})$, where $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ is defined as in 364Yg.

375Y Further exercises (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ a linear operator. (i) Show that if T is order-bounded, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $L^0(\mathfrak{B})$ (definition: 367A) whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $L^0(\mathfrak{A})$. (ii) Show that if $\mathfrak{B}$ is ccc and weakly (σ, ∞)-distributive and $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $L^0(\mathfrak{B})$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $L^0(\mathfrak{A})$, then T is order-bounded.

(b) Show that the following are equiveridical: (i) there is a purely atomic probability space (X, Σ, μ) such that $\Sigma = \mathcal{P}X$ and $\mu\{x\} = 0$ for every $x \in X$; (ii) there are a set X and a Riesz homomorphism $f : \mathbb{R}^X \to \mathbb{R}$ which is not order-continuous; (iii) there are a Dedekind complete Boolean algebra $\mathfrak{A}$ and a positive linear operator $f : L^0(\mathfrak{A}) \to \mathbb{R}$ which is not order-continuous; (iv) there are a Dedekind complete Boolean algebra $\mathfrak{A}$ and a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \{0, 1\}$ which is not order-continuous; (v) there are a Dedekind complete Riesz space U and a sequentially order-continuous Riesz homomorphism $f : U \to \mathbb{R}$ which is not order-continuous; *(vi) there are an atomless Dedekind complete Boolean algebra $\mathfrak{A}$ and a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \{0, 1\}$ which is not order-continuous. (Compare 363S.)

(c) Give an example of an atomless Dedekind σ-complete Boolean algebra $\mathfrak{A}$ such that $L^0(\mathfrak{A})^\sim \ne \{0\}$.

(d) Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure on $[0,1]$, and set $L^0 = L^0(\mathfrak{A})$. Show that there is a positive linear operator $T : L^0 \to L^0$ such that $T[L^0]$ is not order-closed in L^0.

(e) Show that the following are equiveridical: (i) there is a probability space (X, Σ, μ) such that $\Sigma = \mathcal{P}X$ and $\mu\{x\} = 0$ for every $x \in X$; (ii) there are localizable measure algebras $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ and a positive linear operator $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ which is not order-continuous.

(f) Let $\mathfrak{A}$, $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras of which $\mathfrak{B}$ is weakly σ-distributive. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be a positive linear operator. Show that $a \mapsto [\![T(\chi a) > 0]\!] : \mathfrak{A} \to \mathfrak{B}$ is a σ-subhomomorphism.

(g) Let $\mathfrak{A}$, $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras of which $\mathfrak{B}$ is weakly σ-distributive. Let $\phi : \mathfrak{A} \to \mathfrak{B}$ be a σ-subhomomorphism such that $\pi a \neq 0$ whenever $a \in \mathfrak{A} \setminus \{0\}$. Show that $\mathfrak{A}$ is weakly σ-distributive.

(h) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind complete Boolean algebras, and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism such that $\phi 1_{\mathfrak{A}} = 1_{\mathfrak{B}}$. Show that there is a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\pi a \subseteq \phi a$ for every $a \in \mathfrak{A}$.

(i) Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$, and $L^0 = L^0(\mathfrak{G})$. Give an example of a non-zero positive linear operator $T : L^0 \to L^0$ such that there is no non-zero Riesz homomorphism $S : L^0 \to L^0$ with $S \leq T$.

375Z Problem Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$, and $L^0 = L^0(\mathfrak{G})$. If $T : L^0 \to L^0$ is a positive linear operator, must $T[L^0]$ be order-closed?

375 Notes and comments Both this section, and the earlier work on linear operators into L^0 spaces, can be regarded as describing different aspects of a single fact: L^0 spaces are very large. The most explicit statements of this principle are 368E and 375D: every Archimedean Riesz space can be embedded into a Dedekind complete L^0 space, but no such L^0 space can be properly embedded as an order-dense Riesz subspace of any other Archimedean Riesz space. Consequently there are many maps into L^0 spaces (368B). But by the same token there are few maps out of them (375B, 375Lb), and those which do exist have a variety of special properties (375A, 375J).

The original version of Kwapien's theorem (Kwapien 73) was the special case of 375J in which $\mathfrak{A}$ is the Lebesgue measure algebra. The ideas of the proof here are mostly taken from Kalton Peck & Roberts 84. I have based my account on the concept of 'subhomomorphism' (375F); this seems to be an effective tool when $\mathfrak{B}$ is weakly (σ,∞)-distributive (375Yf), but less useful in other cases. The case $\mathfrak{B} = \{0,1\}$, $L^0(\mathfrak{B}) \cong \mathbb{R}$ is not entirely trivial and is worth working through on its own (375Xg).

376 Kernel operators

The theory of linear integral equations is in large part the theory of operators T defined from formulae of the type

$$(Tf)(y) = \int k(x,y)f(x)dx$$

for some function k of two variables. I make no attempt to study the general theory here. However, the concepts developed in this book make it easy to discuss certain aspects of such operators defined between the 'function spaces' of measure theory, meaning spaces of equivalence classes of functions, and indeed allow us to do some of the work in the abstract theory of Riesz spaces, omitting all formal mention of measures (376D, 376H, 376P). I give a very brief account of two theorems characterizing kernel operators in the abstract (376E, 376H), with corollaries to show the form these theorems can take in the ordinary language of integral kernels (376J, 376N). To give an idea of the kind of results we can hope for in this area, I go a bit farther with operators with domain L^1 (376Mb, 376P, 376S).

I take the opportunity to spell out versions of results from §253 in the language of this volume (376B-376C).

376A Kernel operators To give an idea of where this section is going, I will try to describe the central idea in a relatively concrete special case. Let (X, Σ, μ) and (Y, T, ν) be σ-finite measure spaces; you can take them both to be $[0,1]$ with Lebesgue measure if you like. Let λ be the product measure on $X \times Y$. If $k \in \mathcal{L}^1(\lambda)$, then $\int k(x,y)dx$ is defined for almost every y, by Fubini's theorem; so if $f \in \mathcal{L}^\infty(\mu)$ then $g(y) = \int k(x,y)f(x)dx$ is defined for almost every y. Also

$$\int g(y)dy = \int k(x,y)f(x)dxdy$$

is defined, because $(x,y)\mapsto k(x,y)f(x)$ is λ-virtually measurable, defined λ-a.e. and is dominated by a multiple of the integrable function k. Thus k defines a function from $\mathcal{L}^\infty(\mu)$ to $\mathcal{L}^1(\nu)$. Changing f on a set of measure 0 will not change g, so we can think of this as an operator from $L^\infty(\mu)$ to $\mathcal{L}^1(\nu)$; and of course we can move immediately to the equivalence class of g in $L^1(\nu)$, so getting an operator T_k from $L^\infty(\mu)$ to $L^1(\nu)$. This operator is plainly linear; also it is easy to check that $\pm T_k\le T_{|k|}$, so that $T_k\in\mathsf{L}^\sim(L^\infty(\mu);L^1(\nu))$, and that $\|T_k\|\le\int|k|$. Moreover, changing k on a λ-negligible set does not change T_k, so that in fact we can speak of T_w for any $w\in L^1(\lambda)$.

I think it is obvious, even before investigating them, that operators representable in this way will be important. We can immediately ask what their properties will be and whether there is any straightforward way of recognising them. We can look at the properties of the map $w\mapsto T_w:L^1(\lambda)\to\mathsf{L}^\sim(L^\infty(\mu);L^1(\nu))$. And we can ask what happens when $L^\infty(\mu)$ and $L^1(\nu)$ are replaced by other function spaces, defined by extended Fatou norms or otherwise. Theorems 376E and 376H are answers to questions of this kind.

It turns out that the formula $g(y)=\int k(x,y)f(x)dx$ gives rise to a variety of technical problems, and it is much easier to characterize Tu in terms of its action on the dual. In the language of the special case above, if $h\in\mathcal{L}^\infty(\nu)$, then we shall have

$$\int k(x,y)f(x)h(y)d(x,y)=\int g(y)h(y)dy;$$

since $g^\bullet\in L^1(\nu)$ is entirely determined by the integrals $\int g(y)h(y)dy$ as h runs over $\mathcal{L}^\infty(\nu)$, we can define the operator T in terms of the functional $(f,h)\mapsto\int k(x,y)f(x)h(y)d(x,y)$. This enables us to extend the results from the case of σ-finite spaces to general strictly localizable spaces; perhaps more to the point in the present context, it gives them natural expressions in terms of function spaces defined from measure algebras rather than measure spaces, as in 376E.

Before going farther along this road, however, I give a couple of results relating the theorems of §253 to the methods of this volume.

376B The canonical map $L^0\times L^0\to L^0$: Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, and $(\mathfrak{C},\bar\lambda)$ their localizable measure algebra free product (325E). Then we have a bilinear operator $(u,v)\mapsto u\otimes v:L^0(\mathfrak{A})\times L^0(\mathfrak{B})\to L^0(\mathfrak{C})$ with the following properties.

(a) For any $u\in L^0(\mathfrak{A})$, $v\in L^0(\mathfrak{B})$ and $\alpha\in\mathbb{R}$,

$$[\![u\otimes\chi 1_{\mathfrak{B}}>\alpha]\!]=[\![u>\alpha]\!]\otimes 1_{\mathfrak{B}},\quad [\![\chi 1_{\mathfrak{A}}\otimes v>\alpha]\!]=1_{\mathfrak{A}}\otimes[\![v>\alpha]\!]$$

where for $a\in\mathfrak{A}$, $b\in\mathfrak{B}$ I write $a\otimes b$ for the corresponding member of $\mathfrak{A}\otimes\mathfrak{B}$ (315N), identified with a subalgebra of $\mathfrak{C}$ (325Dc).

(b)(i) For any $u\in L^0(\mathfrak{A})^+$, the map $v\mapsto u\otimes v:L^0(\mathfrak{B})\to L^0(\mathfrak{C})$ is an order-continuous multiplicative Riesz homomorphism.

(ii) For any $v\in L^0(\mathfrak{B})^+$, the map $u\mapsto u\otimes v:L^0(\mathfrak{A})\to L^0(\mathfrak{C})$ is an order-continuous multiplicative Riesz homomorphism.

(c) In particular, $|u\otimes v|=|u|\otimes|v|$ for all $u\in L^0(\mathfrak{A})$ and $v\in L^0(\mathfrak{B})$.

(d) For any $u\in L^0(\mathfrak{A})^+$ and $v\in L^0(\mathfrak{B})^+$, $[\![u\otimes v>0]\!]=[\![u>0]\!]\otimes[\![v>0]\!]$.

proof The canonical maps $a\mapsto a\otimes 1_{\mathfrak{B}}$, $b\mapsto 1_{\mathfrak{A}}\otimes b$ from $\mathfrak{A}$, $\mathfrak{B}$ to $\mathfrak{C}$ are order-continuous Boolean homomorphisms (325Da), so induce order-continuous multiplicative Riesz homomorphisms from $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ to $L^0(\mathfrak{C})$ (364P); write $\tilde u$, $\tilde v$ for the images of $u\in L^0(\mathfrak{A})$, $v\in L^0(\mathfrak{B})$. Observe that $|\tilde u|=|u|^\sim$, $|\tilde v|=|v|^\sim$ and $(\chi 1_{\mathfrak{A}})^\sim=(\chi 1_{\mathfrak{B}})^\sim=\chi 1_{\mathfrak{C}}$. Now set $u\otimes v=\tilde u\times\tilde v$. The properties listed in (a)-(c) are just a matter of putting the definition in 364Pa together with the fact that $L^0(\mathfrak{C})$ is an f-algebra (364D). As for $[\![u\otimes v>0]\!]=[\![\tilde u\times\tilde v>0]\!]$, this is (for non-negative u, v) just

$$[\![\tilde u>0]\!]\cap[\![\tilde v>0]\!]=([\![u>0]\!]\otimes 1_{\mathfrak{B}})\cap(1_{\mathfrak{A}}\otimes[\![v>0]\!])=[\![u>0]\!]\otimes[\![v>0]\!].$$

376C For L^1 spaces we have a similar result, with additions corresponding to the Banach lattice structures of the three spaces.

Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras with localizable measure algebra free product $(\mathfrak{C},\bar\lambda)$.

(a) If $u\in L^1_{\bar\mu}=L^1(\mathfrak{A},\bar\mu)$ and $v\in L^1_{\bar\nu}=L^1(\mathfrak{B},\bar\nu)$ then $u\otimes v\in L^1_{\bar\lambda}=L^1(\mathfrak{C},\bar\lambda)$ and

$$\int u\otimes v=\int u\int v,\quad \|u\otimes v\|_1=\|u\|_1\|v\|_1.$$

(b) Let W be a Banach space and $\phi:L^1_{\bar\mu}\times L^1_{\bar\nu}\to W$ a bounded bilinear operator. Then there is a unique bounded linear operator $T:L^1_{\bar\lambda}\to W$ such that $T(u\otimes v)=\phi(u,v)$ for all $u\in L^1_{\bar\mu}$ and $v\in L^1_{\bar\nu}$, and $\|T\|=\|\phi\|$.

(c) Suppose, in (b), that W is a Banach lattice. Then

(i) T is positive iff $\phi(u,v)\ge 0$ for all u, $v\ge 0$;

(ii) T is a Riesz homomorphism iff $u\mapsto\phi(u,v_0):L^1_{\bar\mu}\to W$ and $v\mapsto\phi(u_0,v):L^1_{\bar\nu}\to W$ are Riesz homomorphisms for all $v_0\ge 0$ in $L^1_{\bar\nu}$ and $u_0\ge 0$ in $L^1_{\bar\mu}$.

proof (a) I refer to the proof of 325D. Let (X,Σ,μ) and (Y,T,ν) be the Stone spaces of $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ (321K), so that $(\mathfrak{C},\bar\lambda)$ can be identified with the measure algebra of the c.l.d. product measure λ on $X\times Y$ (part (a) of the proof of 325D), and $L^1_{\bar\mu}$, $L^1_{\bar\nu}$, $L^1_{\bar\lambda}$ can be identified with $L^1(\mu)$, $L^1(\nu)$ and $L^1(\lambda)$ (365B). Now if $f\in\mathcal{L}^0(\mu)$ and $g\in\mathcal{L}^0(\nu)$ then $f\otimes g\in\mathcal{L}^0(\lambda)$ (253Cb), and it is easy to check that $(f\otimes g)^\bullet\in L^0(\bar\lambda)$ corresponds to $f^\bullet\otimes g^\bullet$ as defined in 376B. (Look first at the cases in which one of f, g is a constant function with value 1.) By 253E, we have a canonical map $(f^\bullet,g^\bullet)\mapsto(f\otimes g)^\bullet$ from $L^1(\mu)\times L^1(\nu)$ to $L^1(\lambda)$, with $\int f\otimes g=\int f\int g$ (253D); so that if $u\in L^1_{\bar\mu}$ and $v\in L^1_{\bar\nu}$ we must have $u\otimes v\in L^1_{\bar\lambda}$, with $\int u\otimes v=\int u\int v$. As in 253E, it follows that $\|u\otimes v\|_1=\|u\|_1\|v\|_1$.

(b) In view of the situation described in (a) above, this is now just a translation of the same result about $L^1(\mu)$, $L^1(\nu)$ and $L^1(\lambda)$, which is Theorem 253F.

(c) Identifying the algebraic free product $\mathfrak{A}\otimes\mathfrak{B}$ with its canonical image in $\mathfrak{C}$ (325Dc), I write $(\mathfrak{A}\otimes\mathfrak{B})^f$ for $\{c:c\in\mathfrak{A}\otimes\mathfrak{B},\ \bar\lambda c<\infty\}$, so that $(\mathfrak{A}\otimes\mathfrak{B})^f$ is a subring of $\mathfrak{C}$. Recall that any member of $\mathfrak{A}\otimes\mathfrak{B}$ is expressible as $\sup_{i\le n}a_i\otimes b_i$ where $a_0,\dots,a_n$ are disjoint (315Oa); evidently this will belong to $(\mathfrak{A}\otimes\mathfrak{B})^f$ iff $\bar\mu a_i\cdot\bar\nu b_i$ is finite for every i.

The next fact to lift from previous theorems is in part (e) of the proof of 253F: the linear span M of $\{\chi(a\otimes b):a\in\mathfrak{A}^f,\ b\in\mathfrak{B}^f\}$ is norm-dense in $L^1_{\bar\lambda}$. Of course M can also be regarded as the linear span of $\{\chi c:c\in(\mathfrak{A}\otimes\mathfrak{B})^f\}$, or $S(\mathfrak{A}\otimes\mathfrak{B})^f$. (Strictly speaking, this last remark relies on 361J; the identity map from $(\mathfrak{A}\otimes\mathfrak{B})^f$ to $\mathfrak{C}$ induces an injective Riesz homomorphism from $S(\mathfrak{A}\otimes\mathfrak{B})^f$ into $S(\mathfrak{C})\subseteq L^0(\mathfrak{C})$. To see that $\chi c\in M$ for every $c\in(\mathfrak{A}\otimes\mathfrak{B})^f$, we need to know that c can be expressed as a disjoint union of members of $\mathfrak{A}\otimes\mathfrak{B}$, as noted above.)

(i) If T is positive then of course $\phi(u,v)=T(u\otimes v)\ge 0$ whenever u, $v\ge 0$, since $u\otimes v\ge 0$. On the other hand, if ϕ is non-negative on $U^+\times V^+$, then, in particular, $T\chi(a\otimes b)=\phi(\chi a,\chi b)\ge 0$ whenever $\bar\mu a\cdot\bar\nu b<\infty$. Consequently $T(\chi c)\ge 0$ for every $c\in(\mathfrak{A}\otimes\mathfrak{B})^f$ and $Tw\ge 0$ whenever $w\ge 0$ in $M\cong S(\mathfrak{A}\otimes\mathfrak{B})^f$, as in 361Ga.

Now this means that $T|w|\ge 0$ whenever $w\in M$. But as M is norm-dense in $L^1_{\bar\lambda}$, $w\mapsto T|w|$ is continuous and W^+ is closed, it follows that $T|w|\ge 0$ for every $w\in L^1_{\bar\lambda}$, that is, that T is positive.

(ii) If T is a Riesz homomorphism then of course $u\mapsto\phi(u,v_0)=T(u\otimes v_0)$ and $v\mapsto\phi(u_0,v)=T(u_0\otimes v)$ are Riesz homomorphisms for v_0, $u_0\ge 0$. On the other hand, if all these maps are Riesz homomorphisms, then, in particular,

$$\begin{aligned}T\chi(a\otimes b)\wedge T\chi(a'\otimes b')&=\phi(\chi a,\chi b)\wedge\phi(\chi a',\chi b')\\&\le\phi(\chi a,\chi b+\chi b')\wedge\phi(\chi a',\chi b+\chi b')\\&=\phi(\chi a\wedge\chi a',\chi b+\chi b')=0\end{aligned}$$

whenever a, $a'\in\mathfrak{A}^f$, b, $b'\in\mathfrak{B}^f$ and $a\cap a'=0$. Similarly, $T\chi(a\otimes b)\wedge T\chi(a'\otimes b')=0$ if $b\cap b'=0$. But this means that $T\chi c\wedge T\chi c'=0$ whenever c, $c'\in(\mathfrak{A}\otimes\mathfrak{B})^f$ and $c\cap c'=0$. **P** Express c, c' as $\sup_{i\le m}a_i\otimes b_i$, $\sup_{j\le n}a'_j\otimes b'_j$ where a_i, a'_j, b_i, b'_j all have finite measure. Now if $i\le m$, $j\le n$, $(a_i\cap a'_j)\otimes(b_i\cap b'_j)=(a_i\otimes b_i)\cap(a'_j\otimes b'_j)=0$, so one of $a_i\cap a'_j$, $b_i\cap b'_j$ must be zero, and in either case $T\chi(a_i\otimes b_i)\wedge T\chi(a'_j\otimes b'_j)=0$. Accordingly

$$\begin{aligned}T\chi c\wedge T\chi c'&\le(\sum_{i=0}^m T\chi(a_i\otimes b_i))\wedge(\sum_{j=0}^n T\chi(a'_j\times b'_j))\\&\le\sum_{i=0}^m\sum_{j=0}^n T\chi(a_i\otimes b_i)\wedge T\chi(a'_j\otimes b'_j)=0,\end{aligned}$$

using 352Fa for the second inequality. **Q**

This implies that $T\restriction M$ must be a Riesz homomorphism (361Gc), that is, $T|w|=|Tw|$ for all $w\in M$. Again because M is dense in $L^1_{\bar\lambda}$, $T|w|=|Tw|$ for every $w\in L^1_{\bar\lambda}$, and T is a Riesz homomorphism.

376D Abstract integral operators: Definition The following concept will be used repeatedly in the theorems below; it is perhaps worth giving it a name. Let U be a Riesz space and V a Dedekind complete Riesz space, so that $\mathrm{L}^\times(U;V)$ is a Dedekind complete Riesz space (355H). If $f\in U^\times$ and $v\in V$ write $P_{fv}u=f(u)v$ for each $u\in U$;

then $P_{fv} \in \mathsf{L}^\times(U;V)$. **P** If $f \geq 0$ in $U^\times$ and $v \geq 0$ in $V^\times$ then P_{fv} is a positive linear operator from U to V which is order-continuous because if $A \subseteq U$ is non-empty, downwards-directed and has infimum 0, then (as V is Archimedean)

$$\inf_{u\in A} P_{fv}(u) = \inf_{u\in A} f(u)v = 0.$$

Of course $(f,g) \mapsto P_{fg}$ is bilinear, so $P_{fv} \in \mathsf{L}^\times(U;V)$ for every $f \in U^\times$, $v \in V$. **Q** Now I call a linear operator from U to V an **abstract integral operator** if it is in the band in $\mathsf{L}^\times(U;V)$ generated by $\{P_{fv} : f \in U^\times, v \in V\}$.

The first result describes these operators when U, V are expressed as subspaces of $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ for measure algebras $\mathfrak{A}$, $\mathfrak{B}$ and V is perfect.

376E Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, with localizable measure algebra free product $(\mathfrak{C},\bar\lambda)$, and $U \subseteq L^0(\mathfrak{A})$, $V \subseteq L^0(\mathfrak{B})$ order-dense Riesz subspaces. Write W for the set of those $w \in L^0(\mathfrak{C})$ such that $w \times (u \otimes v)$ is integrable for every $u \in U$ and $v \in V$. Then we have an operator $w \mapsto T_w : W \to \mathsf{L}^\times(U;V^\times)$ defined by setting

$$T_w(u)(v) = \int w \times (u \otimes v)$$

for every $w \in W$, $u \in U$ and $v \in V$. The map $w \mapsto T_w$ is a Riesz space isomorphism between W and the band of abstract integral operators in $\mathsf{L}^\times(U;V^\times)$.

proof (a) The first thing to check is that the formula offered does define a member $T_w(u)$ of $V^\times$ for any $w \in W$ and $u \in U$. **P** Of course $T_w(u)$ is a linear operator because $\int$ is linear and $\otimes$ and $\times$ are bilinear. It belongs to $V^\sim$ because, writing $g(v) = \int |w| \times (|u| \otimes v)$, g is a positive linear operator and $|T_w(u)(v)| \leq g(|v|)$ for every v. (I am here using 376Bc to see that $|w \times (u \otimes Fv)| = |w| \times (|u| \otimes |v|)$.) Also $g \in V^\times$ because $v \mapsto |u| \otimes v$, $w' \mapsto |w| \times w'$ and $\int$ are all order-continuous; so $T_w(u)$ also belongs to $V^\times$. **Q**

(b) Next, for any given $w \in W$, the map $T_w : U \to V^\times$ is linear (again because $\otimes$ and $\times$ are bilinear). It is helpful to note that W is a solid linear subspace of $L^0(\mathfrak{C})$. Now if $w \geq 0$ in W, then $T_w \in \mathsf{L}^\times(U;V^\times)$. **P** If u, $v \geq 0$ then $u \otimes v \geq 0$, $w \times (u \otimes v) \geq 0$ and $T_w(u)(v) \geq 0$; as v is arbitrary, $T_w(u) \geq 0$ whenever $u \geq 0$; as u is arbitrary, T_w is positive. If $A \subseteq U$ is non-empty, downwards-directed and has infimum 0, then $T_w[A]$ is downwards-directed, and for any $v \in V^+$

$$(\inf T_w[A])(v) = \inf_{u\in A} T_w(u)(v) = \inf_{u\in A} \int w \times (u \otimes v) = 0$$

because $u \mapsto u \otimes v$ is order-continuous. So $\inf T_w[A] = 0$; as A is arbitrary, T_w is order-continuous. **Q**

For general $w \in W$, we now have $T_w = T_{w^+} - T_{w^-} \in L^\times(U;V^\times)$.

(c) Ths shows that $w \mapsto T_w$ is a map from W to $L^\times(U;V^\times)$. Running through the formulae once again, it is linear, positive and order-continuous; this last because, given a non-empty downwards-directed $C \subseteq W$ with infimum 0, then for any $u \in U^+$, $v \in V^+$

$$(\inf_{w\in C} T_w)(u)(v) \leq \inf_{w\in C} \int w \times (u \otimes v) = 0$$

(because $\int$ and $\times$ are order-continuous); as v is arbitrary, $(\inf_{w\in C} T_w)(u) = 0$; as u is arbitrary, $\inf_{w\in C} T_w = 0$.

(d) All this is easy, being nothing but a string of applications of the elementary properties of $\otimes$, $\times$ and $\int$. But I think a new idea is needed for the next fact: the map $w \mapsto T_w : W \to \mathsf{L}^\times(U;V^\times)$ is a Riesz homomorphism. **P** Write $\mathfrak{D}$ for the set of those $d \in \mathfrak{C}$ such that $T_w \wedge T_{w'} = 0$ whenever w, $w' \in W^+$, $[\![w > 0]\!] \subseteq d$ and $[\![w' > 0]\!] \subseteq 1_{\mathfrak{C}} \setminus d$. (i) If d_1, $d_2 \in \mathfrak{D}$, w, $w' \in W^+$, $[\![w > 0]\!] \subseteq d_1 \cup d_2$ and $[\![w' > 0]\!] \cap (d_1 \cup d_2) = 0$, then set $w_1 = w \times \chi d_1$, $w_2 = w - w_1$. In this case

$$[\![w_1 > 0]\!] \subseteq d_1, \quad [\![w_2 > 0]\!] \subseteq d_2,$$

so

$$T_{w_1} \wedge T_{w'} = T_{w_2} \wedge T_{w'} = 0, \quad T_w \wedge T_{w'} \leq (T_{w_1} \wedge T_{w'}) + (T_{w_2} \wedge T_{w'}) = 0.$$

As w, w' are arbitrary, $d_1 \cup d_2 \in \mathfrak{D}$. Thus $\mathfrak{D}$ is closed under $\cup$. (ii) The symmetry of the definition of $\mathfrak{D}$ means that $1_{\mathfrak{C}} \setminus d \in \mathfrak{D}$ whenever $d \in \mathfrak{D}$. (iii) Of course $0 \in \mathfrak{D}$, just because $T_w = 0$ if $w \in W^+$ and $[\![w > 0]\!] = 0$; so $\mathfrak{D}$ is a subalgebra of $\mathfrak{C}$. (iv) If $D \subseteq \mathfrak{D}$ is non-empty and upwards-directed, with supremum c in $\mathfrak{C}$, and if w, $w' \in W^+$ are such that $[\![w > 0]\!] \subseteq c$, $[\![w' > 0]\!] \cap c = 0$, then consider $\{w \times \chi d : d \in D\}$. This is upwards-directed, with supremum w; so $T_w = \sup_{d\in D} T_{w\times\chi d}$, because the map $q \mapsto T_q$ is order-continuous. Also $T_{w\times\chi d} \wedge T_{w'} = 0$ for every $d \in D$, so $T_w \wedge T_{w'} = 0$. As w, w' are arbitrary, $c \in \mathfrak{D}$; as D is arbitrary, $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{C}$. (v) If $a \in \mathfrak{A}$ and

w, $w' \in W^+$ are such that $[\![w > 0]\!] \subseteq a \otimes 1_{\mathfrak{B}}$ and $[\![w' > 0]\!] \cap (a \otimes 1_{\mathfrak{B}}) = 0$, then any $u \in U^+$ is expressible as $u_1 + u_2$ where $u_1 = u \times \chi a$, $u_2 = u \times \chi(1_{\mathfrak{A}} \setminus a)$. Now

$$T_w(u_2)(v) = \int w \times (u_2 \otimes v) = \int w \times \chi(a \otimes 1_{\mathfrak{B}}) \times (u \otimes v) \times \chi((1_{\mathfrak{A}} \setminus a) \otimes 1_{\mathfrak{B}}) = 0$$

for every $v \in V$, so $T_w(u_2) = 0$. Similarly, $T_{w'}(u_1) = 0$. But this means that

$$(T_w \wedge T_{w'})(u) \le T_w(u_2) + T_{w'}(u_1) = 0.$$

As u is arbitrary, $T_w \wedge T_{w'} = 0$; as w and w' are arbitrary, $a \otimes 1_{\mathfrak{B}} \in \mathfrak{D}$. (vi) Now suppose that $b \in \mathfrak{B}$ and that w, $w' \in W^+$ are such that $[\![w > 0]\!] \subseteq 1_{\mathfrak{A}} \otimes b$ and $[\![w' > 0]\!] \cap (1_{\mathfrak{A}} \otimes b) = 0$. If $u \in U^+$ and $v \in V^+$ then

$$(T_w \wedge T_{w'})(u)(v) \le \int w \times (u \otimes (v \times \chi(1_{\mathfrak{B}} \setminus b))) + \int w' \times (u \otimes (v \times \chi b)) = 0.$$

As u, v are arbitrary, $T_w \wedge T_{w'} = 0$; as w and w' are arbitrary, $1_{\mathfrak{A}} \otimes b \in \mathfrak{D}$. (vii) This means that $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{C}$ including $\mathfrak{A} \otimes \mathfrak{B}$, and is therefore the whole of $\mathfrak{C}$ (325D(c-ii)). (viii) Now take any w, $w' \in W$ such that $w \wedge w' = 0$, and consider $c = [\![w > 0]\!]$. Then $[\![w' > 0]\!] \subseteq 1_{\mathfrak{C}} \setminus c$ and $c \in \mathfrak{D}$, so $T_w \wedge T_{w'} = 0$. This is what we need to be sure that $w \mapsto T_w$ is a Riesz homomorphism (352G). **Q**

(e) The map $w \mapsto T_w$ is injective. **P** (i) If $w > 0$ in W, then consider

$$A = \{a : a \in \mathfrak{A},\ \exists\, u \in U,\ \chi a \le u\}, \qquad B = \{b : b \in \mathfrak{B},\ \exists\, v \in V,\ \chi b \le v\}.$$

Because U and V are order-dense in $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ respectively, A and B are order-dense in $\mathfrak{A}$ and $\mathfrak{B}$. Also both are upwards-directed. So $\sup_{a \in A, b \in B} a \otimes b = 1_{\mathfrak{C}}$ and $0 < \int w = \sup_{a \in A, b \in B} \int_{a \otimes b} w$. Take $a \in A$, $b \in B$ such that $\int_{a \otimes b} w > 0$; then there are $u \in U$, $v \in V$ such that $\chi a \le u$ and $\chi b \le v$, so that

$$T_w(u)(v) \ge \int_{a \otimes b} w > 0$$

and $T_w > 0$. (ii) For general non-zero $w \in W$, we now have $|T_w| = T_{|w|} > 0$ so $T_w \ne 0$. **Q**

Thus $w \mapsto T_w$ is an order-continuous injective Riesz homomorphism.

(f) Write $\tilde{W}$ for $\{T_w : w \in W\}$, so that $\tilde{W}$ is a Riesz subspace of $\mathsf{L}^\times(U; V^\times)$ isomorphic to W, and $\widehat{W}$ for the band it generates in $\mathsf{L}^\times(U; V^\times)$. Then $\tilde{W}$ is order-dense in $\widehat{W}$. **P** Suppose that $S > 0$ in $\widehat{W} = \tilde{W}^{\perp\perp}$ (353Ba). Then $S \notin \tilde{W}^\perp$, so there is a $w \in W$ such that $S \wedge T_w > 0$. Set $w_1 = w \wedge \chi 1_{\mathfrak{C}}$. Then $w = \sup_{n \in \mathbb{N}} w \wedge n w_1$, so $T_w = \sup_{n \in \mathbb{N}} T_w \wedge n T_{w_1}$ and $R = S \wedge T_{w_1} > 0$.

Set $U_1 = U \cap L^1(\mathfrak{A}, \bar{\mu})$. Because U is an order-dense Riesz subspace of $L^0(\mathfrak{A})$, U_1 is an order-dense Riesz subspace of $L^1_{\bar{\mu}} = L^1(\mathfrak{A}, \bar{\mu})$, therefore also norm-dense. Similarly $V_1 = V \cap L^1(\mathfrak{B}, \bar{\nu})$ is a norm-dense Riesz subspace of $L^1_{\bar{\nu}} = L^1(\mathfrak{B}, \bar{\nu})$. Define $\phi_0 : U_1 \times V_1 \to \mathbb{R}$ by setting $\phi_0(u, v) = R(u)(v)$ for $u \in U_1$ and $v \in V_1$. Then ϕ_0 is bilinear, and

$$\begin{aligned} |\phi_0(u, v)| = |R(u)(v)| &\le |R(u)|(|v|) \le R(|u|)(|v|) \le T_{w_1}(|u|)(|v|) \\ &= \int w_1 \times (|u| \otimes |v|) \le \int |u| \otimes |v| = \|u\|_1 \|v\|_1 \end{aligned}$$

for all $u \in U_1$, $v \in V_1$, because $0 \le R \le T_{w_1}$ in $\mathsf{L}^\times(U; V^\times)$. Because U_1, V_1 are norm-dense in $L^1_{\bar{\mu}}$, $L^1_{\bar{\nu}}$ respectively, ϕ_0 has a unique extension to a continuous bilinear operator $\phi : L^1_{\bar{\mu}} \times L^1_{\bar{\nu}} \to \mathbb{R}$. (To reduce this to standard results on linear operators, think of R as a function from U_1 to V_1^*; since every member of V_1^* has a unique extension to a member of $(L^1_{\bar{\nu}})^*$, we get a corresponding function $R_1 : U_1 \to (L^1_{\bar{\nu}})^*$ which is continuous and linear, so has a unique extension to a continuous linear operator $R_2 : L^1_{\bar{\mu}} \to (L^1_{\bar{\nu}})^*$, and we set $\phi(u, v) = R_2(u)(v)$.)

By 376C, there is a unique $h \in (L^1_{\bar{\lambda}})^* = L^1(\mathfrak{C}, \bar{\lambda})^*$ such that $h(u \otimes v) = \phi(u, v)$ for every $u \in L^1_{\bar{\mu}}$ and $v \in L^1_{\bar{\nu}}$. Because $(\mathfrak{C}, \bar{\lambda})$ is localizable, this h corresponds to a $w' \in L^\infty(\mathfrak{C})$ (365Mc), and

$$\int w' \times (u \otimes v) = h(u \otimes v) = \phi_0(u, v) = R(u)(v)$$

for every $u \in U_1$, $v \in V_1$.

Because U_1 is norm-dense in $L^1_{\bar{\mu}}$, U_1^+ is dense in $(L^1_{\bar{\mu}})^+$, and similarly V_1^+ is dense in $(L^1_{\bar{\nu}})^+$, so $U_1^+ \times V_1^+$ is dense in $(L^1_{\bar{\mu}})^+ \times (L^1_{\bar{\nu}})^+$; now ϕ_0 is non-negative on $U_1^+ \times V_1^+$, so ϕ (being continuous) is non-negative on $(L^1_{\bar{\mu}})^+ \times (L^1_{\bar{\nu}})^+$. By 376Cc, $h \ge 0$ in $(L^1_{\bar{\lambda}})^*$ and $w' \ge 0$ in $L^\infty(\mathfrak{C})$. In the same way, because $\phi_0(u, v) \le T_w(u)(v)$ for $u \in U_1^+$ and $v \in V_1^+$, $w' \le w_1 \le w$ in $L^0(\mathfrak{C})$, so $w' \in W$. We have

$$T_{w'}(u)(v) = \int w' \times (u \otimes v) = R(u)(v)$$

for all $u \in U_1$, $v \in V_1$. If $u \in U_1^+$, then $T_{w'}(u)$ and $R(u)$ are both order-continuous, so must be identical, since V_1 is order-dense in V. This means that $T_{w'}$ and R agree on U_1. But as both are themselves order-continuous linear operators, and U_1 is order-dense in U, they must be equal.

Thus $0 < T_{w'} \le S$ in $\mathsf{L}^\times(U;V^\times)$. As S is arbitrary, $\tilde W$ is quasi-order-dense in $\widehat W$, therefore order-dense (353A). **Q**

(g) Because $w \mapsto T_w : W \mapsto \tilde W$ is an injective Riesz homomorphism, we have an inverse map $Q : \tilde W \to L^0(\mathfrak{C})$, setting $Q(T_w) = w$; this is a Riesz homomorphism, and it is order-continuous because W is solid in $L^0(\mathfrak{C})$, so that the embedding $W \subsetneq L^0(\mathfrak{C})$ is order-continuous. By 368B, Q has an extension to an order-continuous Riesz homomorphism $\tilde Q : \widehat W \to L^0(\mathfrak{C})$. Because $Q(S) > 0$ whenever $S > 0$ in $\tilde W$, $\tilde Q(S) > 0$ whenever $S > 0$ in $\widehat W$, so $\tilde Q$ is injective. Now $\tilde Q(S) \in W$ for every $S \in \widehat W$. **P** It is enough to look at non-negative S. In this case, $\tilde Q(S)$ must be $\sup\{\tilde Q(T_w) : w \in W, T_w \le S\} = \sup C$, where $C = \{w : T_w \le S\} \subseteq W$. Take $u \in U^+$ and $v \in V^+$. Then $\{w \times (u \otimes v) : w \in C\}$ is upwards-directed, because C is, and

$$\sup_{w\in C} \int w \times (u \otimes v) = \sup_{w\in C} T_w(u)(v) \le S(u)(v) < \infty.$$

So $\tilde Q(S) \times (u \otimes v) = \sup_{w\in C} w \times (u \otimes v)$ belongs to $L^1_{\tilde\lambda}$ (365Df). As u and v are arbitrary, $\tilde Q(S) \in W$. **Q**

(h) Of course this means that $\tilde W = \widehat W$ and $\tilde Q = Q$, that is, that $w \mapsto T_w : W \mapsto \widehat W$ is a Riesz space isomorphism.

(i) I have still to check on the identification of $\widehat W$ as the band Z of abstract integral operators in $\mathsf{L}^\times(U;V^\times)$. Write $P_{fg}(u) = f(u)g$ for $f \in U^\times$, $g \in V^\times$ and $u \in U$.

Set

$$U^\# = \{u : u \in L^0(\mathfrak{A}),\ u \times u' \in L^1_{\bar\mu} \text{ for every } u' \in U\},$$

$$V^\# = \{v : v \in L^0(\mathfrak{B}),\ v \times v' \in L^1_{\bar\nu} \text{ for every } v' \in V\}.$$

From 369C we know that if we set $f_u(u') = \int u \times u'$ for $u \in U^\#$ and $u' \in U$, then $f_u \in U^\times$ for every $u \in U^\#$, and $u \mapsto f_u$ is an isomorphism between $U^\#$ and an order-dense Riesz subspace of $U^\times$. Similarly, setting $g_v(v') = \int v \times v'$ for $v \in V^\#$ and $v' \in V$, $v \mapsto g_v$ is an isomorphism between $V^\#$ and an order-dense Riesz subspace of $V^\times$.

If $u \in U^\#$ and $v \in V^\#$ then

$$\int (u \otimes v) \times (u' \otimes v') = \int (u \times u') \otimes (v \times v') = (\int u \times u')(\int v \times v') = f_u(u')g_v(v')$$

for every $u' \in U$, $v' \in V$, so $u \otimes v \in W$ and $T_{u\otimes v} = P_{f_u g_v}$.

Now take $f \in (U^\times)^+$ and $g \in (V^\times)^+$. Set $A = \{u : u \in U^\#, u \ge 0, f_u \le f\}$ and $B = \{v : v \in V^\#, v \ge 0, g_v \le g\}$. These are upwards-directed, so $C = \{u \otimes v : u \in A, v \in B\}$ is upwards-directed in $L^0(\mathfrak{C})$. Because $\{f_u : u \in U^\#\}$ is order-dense in $U^\times$, $f = \sup_{u\in A} f_u$; by 355Ed, $f(u') = \sup_{u\in A} f_u(u')$ for every $u' \in U^+$. Similarly, $g(v') = \sup_{v\in B} f_v(v')$ for every $v' \in V^+$.

? Suppose, if possible, that C is not bounded above in $L^0(\mathfrak{C})$. Because $\mathfrak{C}$ and $L^0(\mathfrak{C})$ are Dedekind complete,

$$c = \inf_{n\in\mathbb{N}} \sup_{u\in A, v\in B} [\![u \otimes v \ge n]\!]$$

must be non-zero (364L(a-i)). Because U and V are order-dense in $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ respectively,

$$1_{\mathfrak{A}} = \sup\{[\![u' > 0]\!] : u' \in U\}, \quad 1_{\mathfrak{B}} = \sup\{[\![v' > 0]\!] : v' \in V\},$$

and there are $u' \in U^+$, $v' \in V^+$ such that $c \cap [\![u' > 0]\!] \otimes [\![v' > 0]\!] \ne 0$, so that $\int_c u' \otimes v' > 0$. But now, for any $n \in \mathbb{N}$,

$$\begin{aligned} f(u')g(v') &\ge \sup_{u\in A, v\in B} f_u(u')g_v(v') \\ &= \sup_{u\in A, v\in B} \int (u \otimes v) \times (u' \otimes v') \\ &\ge \sup_{u\in A, v\in B} \int ((u \otimes v) \wedge n\chi c) \times (u' \otimes v') \\ &= \int \sup_{u\in A, v\in B} ((u \otimes v) \wedge n\chi c) \times (u' \otimes v') \end{aligned}$$

(because $w \mapsto \int w \times (u' \otimes v')$ is order-continuous)

$$= \int (n\chi c) \times (u' \otimes v') = n \int_c u' \otimes v',$$

which is impossible. **X**

Thus C is bounded above in $L^0(\mathfrak{C})$, and has a supremum $w \in L^0(\mathfrak{C})$. If $u' \in U^+$, $v' \in V^+$ then

$$\begin{aligned}\int w \times (u' \otimes v') &= \sup_{u \in A, v \in B} \int (u \otimes v) \times (u' \otimes v') \\ &= \sup_{u \in A, v \in B} f_u(u')g_v(v') = f(u')g(v') = P_{fg}(u')(v').\end{aligned}$$

Thus $w \in W$ and

$$P_{fg} = T_w \in \tilde{W} \subseteq \widehat{W}.$$

And this is true for any non-negative $f \in U^\times$ and $g \in V^\times$. Of course it follows that $P_{fg} \in \widehat{W}$ for every $f \in U^\times$, $g \in V^\times$; as $\widehat{W}$ is a band, it must include Z.

(j) Finally, $\widehat{W} \subseteq Z$. **P** Since $Z = Z^{\perp\perp}$, it is enough to show that $\widehat{W} \cap Z^\perp = \{0\}$. Take any $T > 0$ in $\widehat{W}$. There are $u_0' \in U^+$, $v_0' \in V^+$ such that $T(u_0')(v_0') > 0$. So there is a $v \in V^\#$ such that $0 \le g_v \le T(u_0')$ and $g_v(v_0') > 0$, that is, $\int v \times v_0' > 0$. Because V is order-dense in $L^0(\mathfrak{B})$, there is a $v_1' \in V$ such that $0 < v_1' \le v_0' \times \chi[\![v > 0]\!]$, so that

$$0 < \int v \times v_1' = g_v(v_1') \le T(u_0')(v_1')$$

and $[\![v_1' > 0]\!] \subseteq [\![v > 0]\!]$.

Now consider the functional $u' \mapsto h(u') = T(u')(v_1') : U \to \mathbb{R}$. This belongs to $(U^\times)^+$ and $h(u_0') > 0$, so there is a $u \in U^\#$ such that $0 \le f_u \le h$ and $f_u(u_0') > 0$. This time, $\int u \times u_0' > 0$ so (because U is order-dense in $L^0(\mathfrak{A})$) there is a $u_1' \in U$ such that $h(u_1') > 0$ and $[\![u_1' > 0]\!] \subseteq [\![u > 0]\!]$.

We can express T as T_w where $w \in W^+$. In this case, we have

$$\int w \times (u_1' \otimes v_1') = T(u_1')(v_1') = h(u_1') > 0,$$

so

$$\begin{aligned}0 \ne [\![w > 0]\!] \cap [\![u_1' \otimes v_1' > 0]\!] &= [\![w > 0]\!] \cap ([\![u_1' > 0]\!] \otimes [\![v_1' > 0]\!]) \\ &\subseteq [\![w > 0]\!] \cap ([\![u > 0]\!] \otimes [\![v > 0]\!]) = [\![w > 0]\!] \cap [\![u \otimes v > 0]\!],\end{aligned}$$

and $w \wedge (u \otimes v) > 0$, so

$$T_w \wedge P_{f_u g_v} = T_w \wedge T_{u \otimes v} = T_{w \wedge (u \otimes v)} > 0.$$

Thus $T \notin Z^\perp$. Accordingly $\widehat{W} \cap Z^\perp = \{0\}$ and $\widehat{W} \subseteq Z^{\perp\perp} = Z$. **Q**

Since we already know that $Z \subseteq \widehat{W}$, this completes the proof.

376F Corollary Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be localizable measure algebras, with localizable measure algebra free product $(\mathfrak{C}, \bar{\lambda})$. Let $U \subseteq L^0(\mathfrak{A})$, $V \subseteq L^0(\mathfrak{B})$ be perfect order-dense solid linear subspaces, and $T : U \to V$ a linear operator. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) there is a $w \in L^0(\mathfrak{C})$ such that $\int w \times (u \otimes v')$ is defined and equal to $\int Tu \times v'$ whenever $u \in U$ and $v' \in L^0(\mathfrak{B})$ is such that $v' \times v$ is integrable for every $v \in V$.

proof Setting $V^\# = \{v' : v' \in L^0(\mathfrak{B}),\ v \times v' \in L^1 \text{ for every } v \in V\}$, we know that we can identify $V^\#$ with $V^\times$ and V with $(V^\#)^\times$ (369C). So the equivalence of (i) and (ii) is just 376E applied to $V^\#$ in place of V.

376G Lemma Let U be a Riesz space, V an Archimedean Riesz space, $T : U \to V$ a linear operator, $f \in (U^\sim)^+$ and $e \in V^+$. Suppose that $0 \le Tu \le f(u)e$ for every $u \in U^+$. Then if $\langle u_n \rangle_{n \in \mathbb{N}}$ is a sequence in U such that $\lim_{n\to\infty} g(u_n) = 0$ whenever $g \in U^\sim$ and $|g| \le f$, $\langle Tu_n \rangle_{n \in \mathbb{N}}$ order*-converges to 0 in V (definition: 367A).

proof Let V_e be the solid linear subspace of V generated by e; then $Tu \in V_e$ for every $u \in U$. We can identify V_e with an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, with e corresponding to χX (353M). For $x \in X$, set $g_x(u) = (Tu)(x)$ for every $u \in U$; then $0 \le g_x(u) \le f(u)$ for $u \ge 0$, so $|g_x| \le f$ and $\lim_{n\to\infty}(Tu_n)(x) = 0$. As x is arbitrary, $\langle Tu_n \rangle_{n \in \mathbb{N}}$ order*-converges to 0 in $C(X)$, by 367K, and therefore in V_e, because V_e is order-dense in $C(X)$ (367E). But V_e, regarded as a subspace of V, is solid, so 367E tells us also that $\langle Tu_n \rangle_{n \in \mathbb{N}}$ order*-converges to 0 in V.

376H Theorem Let U be a Riesz space and V a weakly (σ,∞)-distributive Dedekind complete Riesz space (definition: 368N). Suppose that $T \in \mathsf{L}^\times(U;V)$. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U^+ and $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V;

(iii) whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U and $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

proof For $f \in U^\times$, $v \in V$ and $u \in U$ set $P_{fv}(u) = f(u)v$. Write $Z \subseteq \mathsf{L}^\times(U;V)$ for the band of abstract integral operators.

(a)(i)⇒(iii) Suppose that $T \in Z^+$, and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U such that $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$. Note that $\{P_{fv} : f \in U^{\times+},\ v \in V^+\}$ is upwards-directed, so that $T = \sup\{T\wedge P_{fv} : f \in U^{\times+},\ v \in V^+\}$ (352Va).

Take $u^* \in U^+$ such that $|u_n| \le u^*$ for every n, and set $w = \inf_{n\in\mathbb{N}}\sup_{m\ge n} Tu_m$, which is defined because $|Tu_n| \le Tu^*$ for every n. Now $w \le (T-P_{fv})^+(u^*)$ for every $f \in U^{\times+}$ and $v \in V^+$. **P** Setting $T_1 = T\wedge P_{fv}$, $w_0 = (T-P_{fv})^+(u^*)$ we have

$$Tu_n - T_1u_n \le |T-T_1|(u^*) = (T-P_{fv})^+(u^*) = w_0$$

for every $n \in \mathbb{N}$, so $Tu_n \le w_0 + T_1u_n$. On the other hand, $0 \le T_1u \le f(u)v$ for every $u \in U^+$, so by 376G we must have $\inf_{n\in\mathbb{N}}\sup_{m\ge n} T_1u_m = 0$. Accordingly

$$w \le w_0 + \inf\nolimits_{n\in\mathbb{N}}\sup\nolimits_{m\ge n} T_1u_m = w_0. \text{ **Q**}$$

But as $\inf\{(T-P_{fv})^+ : f \in U^{\times+},\ v \in V^+\} = 0$, $w \le 0$. Similarly (or applying the same argument to $\langle -u_n\rangle_{n\in\mathbb{N}}$), $\sup_{n\in\mathbb{N}}\inf_{n\in\mathbb{N}} Tu_n \ge 0$ and $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to zero.

For general $T \in Z$, this shows that $\langle T^+u_n\rangle_{n\in\mathbb{N}}$ and $\langle T^-u_n\rangle_{n\in\mathbb{N}}$ both order*-converge to 0, so $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367Cd. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, (iii) is satisfied.

(b)(iii)⇒(ii) is trivial.

(c)(ii)⇒(i) ? Now suppose, if possible, that (ii) is satisfied, but that $T \notin Z$. Because $\mathsf{L}^\times(U;V)$ is Dedekind complete (355H), Z is a projection band (353I), so T is expressible as $T_1 + T_2$ where $T_1 \in Z$, $T_2 \in Z^\perp$ and $T_2 \ne 0$. At least one of T_2^+, T_2^- is non-zero; replacing T by $-T$ if need be, we may suppose that $T_2^+ > 0$.

Because T_2^+, like T, belongs to $\mathsf{L}^\times(U;V)$, its kernel U_0 is a band in U, which cannot be the whole of U, and there is a $u_0 > 0$ in $U_0^\perp$. In this case $T_2^+u_0 > 0$; because $T_2^+ \wedge (T_2^- + |T_1|) = 0$, there is a $u_1 \in [0,u_0]$ such that $T_2^+(u_0-u_1) + (T_2^- + |T_1|)(u_1) \not\ge T_2^+u_0$, so that

$$Tu_1 \ge T_2u_1 - |T_1|(u_1) \not\le 0$$

and $Tu_1 \ne 0$. Now this means that the sequence $(Tu_1, Tu_1, \dots)$ is not order*-convergent to zero, so there must be some $f \in U^\times$ such that $(f(u_1), f(u_1), \dots)$ does not converge to 0, that is, $f(u_1) \ne 0$; replacing f by $|f|$ if necessary, we may suppose that $f \ge 0$ and that $f(u_1) > 0$.

By 356H, there is a u_2 such that $0 < u_2 \le u_1$ and $g(u_2) = 0$ whenever $g \in U^\times$ and $g\wedge f = 0$. Because $0 < u_2 \le u_0$, $u_2 \in U_0^\perp$ and $v_0 = T_2^+u_2 > 0$. Consider $P_{fv_0} \in Z$. Because $T_2 \in Z^\perp$, $T_2^+ \wedge P_{fv_0} = 0$; set $S = P_{fv_0} + T_2^-$, so that $T_2^+ \wedge S = 0$. Then

$$\inf\nolimits_{u\in[0,u_2]} T_2^+(u_2-u) + Su = 0, \quad \sup\nolimits_{u\in[0,u_2]} T_2^+u - Su = v_0$$

(use 355Ec for the first equality, and then subtract both sides from v_0). Now $Su \ge f(u)v_0$ for every $u \ge 0$, so that for any $\epsilon > 0$

$$\sup\nolimits_{u\in[0,u_2],f(u)\ge\epsilon} T_2^+u - Su \le (1-\epsilon)v_0$$

and accordingly

$$\sup\nolimits_{u\in[0,u_2],f(u)\le\epsilon} T_2^+u = v_0,$$

since the join of these two suprema is surely at least v_0, while the second is at most v_0. Note also that

$$v_0 = \sup\nolimits_{u\in[0,u_2],f(u)\le\epsilon} T_2^+u = \sup\nolimits_{0\le u'\le u\le u_2,f(u)\le\epsilon} T_2u' = \sup\nolimits_{0\le u'\le u_2,f(u')\le\epsilon} T_2u'.$$

For $k \in \mathbb{N}$ set $A_k = \{u : 0 \le u \le u_2,\ f(u) \le 2^{-k}\}$. We know that

$$B_k = \{\sup\nolimits_{u\in I} T_2u : I \subseteq A_k \text{ is finite}\}$$

is an upwards-directed set with supremum v_0 for each k. Because V is weakly (σ,∞)-distributive, we can find a sequence $\langle v'_k\rangle_{k\in\mathbb{N}}$ such that $v'_k \in B_k$ for every k and $v_1 = \inf_{k\in\mathbb{N}} v'_k > 0$. For each k let $I_k \subseteq A_k$ be a finite set such that $v'_k = \sup_{u\in I_k} T_2u$.

Because each I_k is finite, we can build a sequence $\langle u'_n\rangle_{n\in\mathbb{N}}$ in $[0,u_2]$ enumerating each in turn, so that $\lim_{n\to\infty} f(u'_n) = 0$ (since $f(u) \le 2^{-k}$ if $u \in I_k$) while $\sup_{m\ge n} T_2u'_m \ge v_1$ for every n (since $\{u'_m : m \ge n\}$ always includes some I_k). Now $\langle T_2u'_n\rangle_{n\in\mathbb{N}}$ does not order*-converge to 0.

However, $\lim_{n\to\infty} g(u'_n) = 0$ for every $g \in U^\times$. **P** Express $|g|$ as $g_1 + g_2$ where g_1 belongs to the band of $U^\times$ generated by f and $g_2 \wedge f = 0$ (353Hc). Then $g_2(u'_n) = g_2(u_2) = 0$ for every n, by the choice of u_2. Also $g_1 = \sup_{n\in\mathbb{N}} g_1 \wedge nf$ (352Vb); so, given $\epsilon > 0$, there is an $m \in \mathbb{N}$ such that $(g_1 - mf)^+(u_2) \le \epsilon$ and $(g_1 - mf)^+(u'_n) \le \epsilon$ for every $n \in \mathbb{N}$. But this means that

$$|g(u'_n)| \le |g|(u'_n) \le \epsilon + mf(u'_n)$$

for every n, and $\limsup_{n\to\infty} |g(u'_n)| \le \epsilon$; as ϵ is arbitrary, $\lim_{n\to\infty} g(u'_n) = 0$. **Q**

Now, however, part (a) of this proof tells us that $\langle T_1u'_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0, because $T_1 \in Z$, while $\langle Tu'_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0, by hypothesis; so $\langle T_2u'_n\rangle_{n\in\mathbb{N}} = \langle Tu'_n - T_1u'_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0. **X**

This contradiction shows that every operator satisfying the condition (ii) must be in Z.

376I The following elementary remark will be useful for the next corollary and also for Theorem 376S.

Lemma Let (X,Σ,μ) be a σ-finite measure space and U an order-dense solid linear subspace of $L^0(\mu)$. Then there is a non-decreasing sequence $\langle X_n\rangle_{n\in\mathbb{N}}$ of measurable subsets of X, with union X, such that $\chi X_n^\bullet \in U$ for every $n \in \mathbb{N}$.

proof Write $\mathfrak{A}$ for the measure algebra of μ, so that $L^0(\mu)$ can be identified with $L^0(\mathfrak{A})$ (364Ic). $A = \{a : a \in \mathfrak{A}\setminus\{0\},\ \chi a \in U\}$ is order-dense in $\mathfrak{A}$, so includes a partition of unity $\langle a_i\rangle_{i\in I}$. Because μ is σ-finite, $\mathfrak{A}$ is ccc (322G) and I is countable, so we can take I to be a subset of $\mathbb{N}$. Choose $E_i \in \Sigma$ such that $E_i^\bullet = a_i$ for $i \in I$; set $E = X \setminus \bigcup_{i\in I} E_i$, $X_n = E \cup \bigcup_{i\in I, i\le n} E_i$ for $n \in \mathbb{N}$.

376J Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces, with product measure λ on $X\times Y$. Let $U \subseteq L^0(\mu)$, $V \subseteq L^0(\nu)$ be perfect order-dense solid linear subspaces, and $T : U \to V$ a linear operator. Write $\mathcal{U} = \{f : f \in \mathcal{L}^0(\mu),\ f^\bullet \in U\}$, $\mathcal{V}^\# = \{h : h \in \mathcal{L}^0(\nu),\ h^\bullet \times v \in L^1$ for every $v \in V\}$. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) there is a $k \in \mathcal{L}^0(\lambda)$ such that

(α) $\int |k(x,y)f(x)h(y)|d(x,y) < \infty$ for every $f \in \mathcal{U}$, $h \in \mathcal{V}^\#$,

(β) if $f \in \mathcal{U}$ and we set $g(y) = \int k(x,y)f(x)dx$ wherever this is defined, then $g \in \mathcal{L}^0(\nu)$ and $Tf^\bullet = g^\bullet$;

(iii) $T \in \mathrm{L}^\sim(U;V)$ and whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U^+ and $\lim_{n\to\infty} h(u_n) = 0$ for every $h \in U^\times$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

Remark I write '$d(x,y)$' above to indicate integration with respect to the product measure λ. Recall that in the terminology of §251, λ can be taken to be either the 'primitive' or 'c.l.d.' product measure (251K).

proof The idea is of course to identify $L^0(\mu)$ and $L^0(\nu)$ with $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$, where $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are the measure algebras of μ and ν, so that their localizable measure algebra free product can be identified with the measure algebra of λ (325E), while $V^\# = \{h^\bullet : h \in \mathcal{V}^\#\}$ can be identified with $V^\times$, because (T,T,ν) is localizable (see the last sentence in 369C).

(a)(i)$\Rightarrow$(ii) By 376F, there is a $w \in L^0(\lambda)$ such that $\int w\times(u\otimes v')$ is defined and equal to $\int Tu\times v'$ whenever $u \in U$ and $v' \in V^\#$. Express w as $k^\bullet$ where $k \in \mathcal{L}^0(\lambda)$. If $f \in \mathcal{U}$ and $h \in \mathcal{V}^\#$ then $\int |k(x,y)f(x)h(y)|d(x,y) = \int |w\times(f^\bullet\otimes h^\bullet|$ is finite, so (ii-α) is satisfied.

Now take any $f \in \mathcal{U}$, and set $g(y) = \int k(x,y)f(x)dx$ whenever this is defined in $\mathbb{R}$. Write $\mathcal{F}$ for the set of those $F \in \mathrm{T}$ such that $\chi F \in \mathcal{V}^\#$. Then for any $F \in \mathcal{F}$, g is defined almost everywhere in F and $g\restriction F$ is ν-virtually measurable. **P** $\int k(x,y)f(x)\chi F(y)d(x,y)$ is defined in $\mathbb{R}$, so by Fubini's theorem (252B, 252C) $g_F(y) = \int k(x,y)f(x)\chi F(y)dx$ is defined for almost every y, and is ν-virtually measurable; now $g\restriction F = g_F\restriction F$. **Q** Next, there is a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ with union Y, by 376I, because V is perfect and order-dense, so $V^\#$ must also be order-dense in $L^0(\nu)$.

For each $n \in \mathbb{N}$, there is a measurable set $F'_n \subseteq F_n \cap \operatorname{dom} g$ such that $g\restriction F_n$ is measurable and $F_n \setminus F'_n$ is negligible. Setting $G = \bigcup_{n\in\mathbb{N}} F'_n$, G is conegligible and $g\restriction G$ is measurable, so $g \in \mathcal{L}^0(\nu)$.

If $\tilde g \in L^0(\nu)$ represents $Tu \in L^0(\nu)$, then for any $F \in \mathcal{F}$

$$\int_F \tilde{g} = \int Tu \times (\chi F)^\bullet = \int_F g.$$

In particular, this is true whenever $F \in \mathrm{T}$ and $F \subseteq F_n$. So g and $\tilde{g}$ agree almost everywhere in F_n, for each n, and $g =_{\text{a.e.}} \tilde{g}$. Thus g also represents Tu, as required in (ii-β).

(b)(ii)⇒(i) Set $w = k^\bullet$ in $L^0(\lambda)$. If $f \in \mathfrak{U}$ and $h \in \mathcal{V}^\#$ the hypothesis (α) tells us that $(x, y) \mapsto k(x, y)f(x)h(y)$ is integrable (because it surely belongs to $\mathcal{L}^0(\lambda)$). By Fubini's theorem,

$$\int k(x, y)f(x)h(y)d(x, y) = \int g(y)h(y)dy$$

where $g(y) = \int k(x, y)f(x)dx$ for almost every y, so that $Tf^\bullet = g^\bullet$, by (β). But this means that, setting $u = f^\bullet$ and $v' = h^\bullet$,

$$\int w \times (u \otimes v') = \int Tu \times v';$$

and this is true for every $u \in U$, $v' \in V^\#$.

Thus T satisfies the condition 376F(ii), and is an abstract integral operator.

(b)(i)⇒(iii) Because V is weakly (σ, ∞)-distributive (368S), this is covered by 376H(i)⇒(iii).

(c)(iii)⇒(i) Suppose that T satisfies (iii). The point is that T^+ is order-continuous. **P?** Otherwise, let $A \subseteq U$ be a non-empty downwards-directed set, with infimum 0, such that $v_0 = \inf_{u \in A} T^+(u) > 0$. Let $\langle X_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence of sets of finite measure covering X, and set $a_n = X_n^\bullet$ for each n. For each n, $\inf_{u \in A} [\![u > 2^{-n}]\!] = 0$, so we can find $\tilde{u}_n \in A$ such that $\bar{\mu}(a_n \cap [\![\tilde{u}_n > 2^{-n}]\!]) \le 2^{-n}$. Set $u_n = \inf_{i \le n} \tilde{u}_i$ for each n; then $\langle u_n \rangle_{n \in \mathbb{N}}$ is non-increasing and has infimum 0; also, $[0, u_n]$ meets A for each n, so that $v_0 \le \sup\{Tu : 0 \le u \le u_n\}$ for each n. Because V is weakly (σ, ∞)-distributive, we can find a sequence $\langle I_n \rangle_{n \in \mathbb{N}}$ of finite sets such that $I_n \subseteq [0, u_n]$ for each n and $v_1 = \inf_{n \in \mathbb{N}} \sup_{u \in I_n} (Tu)^+ > 0$. Enumerating $\bigcup_{n \in \mathbb{N}} I_n$ as $\langle u'_n \rangle_{n \in \mathbb{N}}$, as in part (c) of the proof of 376H, we see that $\langle u'_n \rangle_{n \in \mathbb{N}}$ is order-bounded and $\lim_{n \to \infty} f(u'_n) = 0$ for every $f \in U^\times$ (indeed, $\langle u'_n \rangle_{n \in \mathbb{N}}$ order*-converges to 0 in U), while $\langle Tu'_n \rangle_{n \in \mathbb{N}} \not\to^* 0$ in V. **XQ**

Similarly, T^- is order-continuous, so $T \in \mathsf{L}^\times(U; V)$. Accordingly T is an abstract integral operator by condition (ii) of 376H.

376K As an application of the ideas above, I give a result due to N.Dunford (376N) which was one of the inspirations underlying the theory. Following the method of Zaanen 83, I begin with a couple of elementary lemmas.

Lemma Let U and V be Riesz spaces. Then there is a Riesz space isomorphism $T \mapsto T' : \mathsf{L}^\times(U; V^\times) \to \mathsf{L}^\times(V; U^\times)$ defined by the formula

$$(T'v)(u) = (Tu)(v) \text{ for every } u \in U,\ v \in V.$$

If we write $P_{fg}(u) = f(u)g$ for $f \in U^\times$, $g \in V^\times$ and $u \in U$, then $P_{fg} \in \mathsf{L}^\times(U; V^\times)$ and $P'_{fg} = P_{gf}$ in $\mathsf{L}^\times(V; U^\times)$. Consequently T is an abstract integral operator iff T' is.

proof All the ideas involved have already appeared. For positive $T \in \mathsf{L}^\times(U; V^\times)$ the functional $(u, v) \mapsto (Tu)(v)$ is bilinear and order-continuous in each variable separately; so (just as in the first part of the proof of 376E) corresponds to a $T' \in \mathsf{L}^\times(V; U^\times)$. The map $T \mapsto T' : \mathsf{L}^\times(U; V^\times)^+ \to \mathsf{L}^\times(V; U^\times)^+$ is evidently an additive, order-preserving bijection, so extends to an isomorphism between $\mathsf{L}^\times(U; V^\times)$ and $\mathsf{L}^\times(V; U^\times)$ given by the same formula. I remarked in part (i) of the proof of 376E that every P_{fg} belongs to $\mathsf{L}^\times(U; V^\times)$, and the identification $P'_{fg} = P_{gf}$ is just a matter of checking the formulae. Of course it follows at once that the bands of abstract integral operators must also be matched by the map $T \mapsto T'$.

376L Lemma Let U be a Banach lattice with an order-continuous norm. If $w \in U^+$ there is a $g \in (U^\times)^+$ such that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|u\| \le \epsilon$ whenever $0 \le u \le w$ and $g(u) \le \delta$.

proof (a) As remarked in 356D, $U^* = U^\sim = U^\times$. Set

$$A = \{v : v \in U \text{ and there is an } f \in (U^\times)^+ \text{ such that } f(u) > 0 \text{ whenever } 0 < u \le |v|\}.$$

Then $v' \in A$ whenever $|v'| \le |v| \in A$ and $v + v' \in A$ for all v, $v' \in A$ (if $f(u) > 0$ whenever $0 < u \le |v|$ and $f'(u) > 0$ whenever $0 < u \le |v'|$, then $(f + f')(u) > 0$ whenever $0 < u \le |v + v'|$); moreover, if $v_0 > 0$ in U, there is a $v \in A$ such that $0 < v \le v_0$. **P** Because $U^\times = U^*$ separates the points of U, there is a $g > 0$ in $U^\times$ such that $g(v_0) > 0$; now by 356H there is a $v \in]0, v_0]$ such that g is strictly positive on $]0, v]$, so that $v \in A$. **Q** But this means that A is an order-dense solid linear subspace of U.

(b) In fact $w \in A$. **P** $w = \sup B$, where $B = A \cap [0, w]$. Because B is upwards-directed, $w \in \overline{B}$ (354Ea), and there is a sequence $\langle u'_n \rangle_{n\in\mathbb{N}}$ in B converging to w for the norm. For each n, choose $f_n \in (U^\times)^+$ such that $f_n(u) > 0$ whenever $0 < u \le u'_n$. Set

$$f = \sum_{n=0}^{\infty} \frac{1}{2^n(1+\|f_n\|)} f_n$$

in $U^* = U^\times$. Then whenever $0 < u \le w$ there is some $n \in \mathbb{N}$ such that $u \wedge u'_n > 0$, so that $f_n(u) > 0$ and $f(u) > 0$. So f witnesses that $w \in A$. **Q**

(c) Take $g \in (U^\times)^+$ such that $g(u) > 0$ whenever $0 < u \le w$. This g serves. **P?** Otherwise, there is some $\epsilon > 0$ such that for every $n \in \mathbb{N}$ we can find a $u_n \in [0, w]$ with $g(u_n) \le 2^{-n}$ and $\|u_n\| \ge \epsilon$. Set $v_n = \sup_{i\ge n} u_i$; then $0 \le v_n \le w$, $g(v_n) \le 2^{-n+1}$ and $\|v_n\| \ge \epsilon$ for every $n \in \mathbb{N}$. But $\langle v_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, so $v = \inf_{n\in\mathbb{N}} v_n$ must be non-zero, while $0 \le v \le w$ and $g(v) = 0$; which is impossible. **XQ**

Thus we have found an appropriate g.

376M Theorem (a) Let U be a Banach lattice with an order-continuous norm and V a Dedekind complete M-space. Then every bounded linear operator from U to V is an abstract integral operator.

(b) Let U be an L-space and V a Banach lattice with order-continuous norm. Then every bounded linear operator from U to $V^\times$ is an abstract integral operator.

proof (a) By 355Kb and 355C, $\mathsf{L}^\times(U;V) = \mathsf{L}^\sim(U;V) \subseteq \mathsf{B}(U;V)$; but since norm-bounded sets in V are also order-bounded, $\{Tu : |u| \le u_0\}$ is bounded above in V for every $T \in \mathsf{B}(U;V)$ and $u_0 \in U^+$, and $\mathsf{B}(U;V) = \mathsf{L}^\times(U;V)$.

I repeat ideas from the proof of 376H. (I cannot quote 376H directly as I am not assuming that V is weakly (σ,∞)-distributive.) **?** Suppose, if possible, that $\mathsf{B}(U;V)$ is not the band Z of abstract integral operators. In this case there is a $T > 0$ in $Z^\perp$. Take $u_1 \ge 0$ such that $v_0 = Tu_1$ is non-zero. Let $f \ge 0$ in $U^\times$ be such that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|u\| \le \epsilon$ whenever $0 \le u \le u_1$ and $f(u) \le \delta$ (376L). Then, just as in part (c) of the proof of 376H,

$$\sup_{u\in[0,u_1], f(u)\le\delta} Tu = v_0$$

for every $\delta > 0$. But there is a $\delta > 0$ such that $\|T\|\|u\| \le \frac{1}{2}\|v_0\|$ whenever $0 \le u \le u_1$ and $f(u) \le \delta$; in which case $\|\sup_{u\in[0,u_1], f(u)\le\delta} Tu\| \le \frac{1}{2}\|v_0\|$, which is impossible. **X**

Thus $Z = \mathsf{B}(U;V)$, as required.

(b) Because V has an order-continuous norm, $V^* = V^\times = V^\sim$; and the norm of V^* is a Fatou norm with the Levi property (356Da). So $\mathsf{B}(U;V^*) = \mathsf{L}^\times(U;V^\times)$, by 371C. By 376K, this is canonically isomorphic to $\mathsf{L}^\times(V;U^\times)$. Now $U^\times = U^*$ is an M-space (356Pb). By (a), every member of $\mathsf{L}^\times(V;U^\times)$ is an abstract integral operator; but the isomorphism between $\mathsf{L}^\times(V;U^\times)$ and $\mathsf{L}^\times(U;V^\times)$ matches the abstract integral operators in each space (376K), so every member of $\mathsf{B}(U;V^*)$ is also an abstract integral operator, as claimed.

376N Corollary: Dunford's theorem Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces and $T : L^1(\mu) \to L^p(\nu)$ a bounded linear operator, where $1 < p \le \infty$. Then there is a measurable function $k : X \times Y \to \mathbb{R}$ such that $Tf^\bullet = g_f^\bullet$, where $g_f(y) = \int k(x,y)f(x)dx$ almost everywhere, for every $f \in \mathcal{L}^1(\mu)$.

proof Set $q = \frac{p}{p-1}$ if p is finite, 1 if $p = \infty$. We can identify $L^p(\nu)$ with $V^\times$, where $V = L^q(\nu) \cong L^p(\nu)^\times$ (366Dc, 365Mc) has an order-continuous norm because $1 \le q < \infty$. By 376Mb, T is an abstract integral operator. By 376F/376J, T is represented by a kernel, as claimed.

376O Under the right conditions, weakly compact operators are abstract integral operators.

Lemma Let U be a Riesz space, and W a solid linear subspace of $U^\sim$. If $C \subseteq U$ is relatively compact for the weak topology $\mathfrak{T}_s(U,W)$ (3A5E), then for every $g \in W^+$ and $\epsilon > 0$ there is a $u^* \in U^+$ such that $g(|u| - u^*)^+ \le \epsilon$ for every $u \in C$.

proof Let W_g be the solid linear subspace of W generated by g. Then W_g is an Archimedean Riesz space with order unit, so $W_g^\times$ is a band in the L-space $W_g^* = W_g^\sim$ (356Na), and is therefore an L-space in its own right (354O). For $u \in U$, $h \in W_g^\times$ set $(Tu)(h) = h(u)$; then T is an order-continuous Riesz homomomorphism from U to $W_g^\times$ (356F).

Now W_g is perfect. **P** I use 356K. W_g is Dedekind complete because it is a solid linear subspace of the Dedekind complete space $U^\sim$. $W_g^\times$ separates the points of W because $T[U]$ does. If $A \subseteq W_g$ is upwards-directed and $\sup_{h\in A}\phi(h)$ is finite for every $\phi\in W_g^\times$, then A acts on $W_g^\times$ as a set of bounded linear functionals which, by the Uniform Boundedness Theorem (3A5Ha), is uniformly bounded; that is, there is some $M\ge 0$ such that $\sup_{h\in A}|\phi(h)|\le M\|\phi\|$ for every $\phi\in W_g^\times$. Because g is the standard order unit of W_g, we have $\|\phi\|=|\phi|(g)$ and $|\phi(h)|\le M|\phi|(g)$ for every $\phi\in W_g^\times$ and $h\in A$. In particular,

$$h(u)\le|h(u)|=|(Tu)(h)|\le M|Tu|(g)=M(Tu)(g)=Mg(u)$$

for every $h\in A$ and $u\in U^+$. But this means that $h\le Mg$ for every $h\in A$ and A is bounded above in W_g. Thus all the conditions of 356K are satisfied and W_g is perfect. **Q**

Accordingly T is continuous for the topologies $\mathfrak{T}_s(U,W)$ and $\mathfrak{T}_s(W_g^\times,W_g^{\times\times})$, because every element ϕ of $W_g^{\times\times}$ corresponds to a member of $W_g\subseteq W$, so 3A5Ec applies.

Now we are supposing that C is relatively compact for $\mathfrak{T}_s(U,W)$, that is, is included in some compact set C'; accordingly $T[C']$ is compact and $T[C]$ is relatively compact for $\mathfrak{T}_s(W_g^\times,W_g^{\times\times})$. Since $W_g^\times$ is an L-space, $T[C]$ is uniformly integrable (356Q); consequently (ignoring the trivial case $C=\emptyset$) there are $\phi_0,\dots,\phi_n\in T[C]$ such that $\|(|\phi|-\sup_{i\le n}|\phi_i|)^+\|\le\epsilon$ for every $\phi\in T[C]$ (354Rb), so that $(|\phi|-\sup_{i\le n}|\phi_i|)^+(g)\le\epsilon$ for every $\phi\in T[C]$.

Translating this back into terms of C itself, and recalling that T is a Riesz homomorphism, we see that there are $u_0,\dots,u_n\in C$ such that $g(|u|-\sup_{i\le n}|u_i|)^+\le\epsilon$ for every $u\in C$. Setting $u^*=\sup_{i\le n}|u_i|$ we have the result.

376P Theorem Let U be an L-space and V a perfect Riesz space. If $T:U\to V$ is a linear operator such that $\{Tu:u\in U,\ \|u\|\le 1\}$ is relatively compact for the weak topology $\mathfrak{T}_s(V,V^\times)$, then T is an abstract integral operator.

proof (a) For any $g\ge 0$ in $V^\times$, $M_g=\sup_{\|u\|\le 1}g(|Tu|)$ is finite. **P** By 376O, there is a $v^*\in V^+$ such that $g(|Tu|-v^*)^+\le 1$ whenever $\|u\|\le 1$; now $M_g\le g(v^*)+1$. **Q** Considering $\|u\|^{-1}u$, we see that $g(|Tu|)\le M_g\|u\|$ for every $u\in U$.

Next, we find that $T\in\mathsf{L}^\sim(U;V)$. **P** Take $u\in U^+$. Set

$$B=\{\sum_{i=0}^n|Tu_i|:u_0,\dots,u_n\in U^+,\ \sum_{i=0}^n u_i=u\}\subseteq V^+.$$

Then B is upwards-directed. (Cf. 371A.) If $g\ge 0$ in $V^\times$,

$$\begin{aligned}\sup_{v\in B}g(v)&=\sup\{\sum_{i=0}^n g(|Tu_i|):\sum_{i=0}^n u_i=u\}\\&\le\sup\{\sum_{i=0}^n M_g\|u_i\|:\sum_{i=0}^n u_i=u\}=M_g\|u\|\end{aligned}$$

is finite. By 356K, B is bounded above in V; and of course any upper bound for B is also an upper bound for $\{Tu':0\le u'\le u\}$. As u is arbitrary, T is order-bounded. **Q**

Because U is a Banach lattice with an order-continuous norm, $T\in\mathsf{L}^\times(U;V)$ (355Kb).

(b) Since we can identify $\mathsf{L}^\times(U;V)$ with $\mathsf{L}^\times(U;V^{\times\times})$, we have an adjoint operator $T'\in\mathsf{L}^\times(V^\times;U^\times)$, as in 376K. Now if $g\ge 0$ in $V^\times$ and $\langle g_n\rangle_{n\in\mathbb{N}}$ is a sequence in $[0,g]$ such that $\lim_{n\to\infty}g_n(v)=0$ for every $v\in V$, $\langle T'g_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $U^\times$. **P** For any $\epsilon>0$, there is a $v^*\in V^+$ such that $g(|Tu|-v^*)^+\le\epsilon$ whenever $\|u\|\le 1$; consequently

$$\begin{aligned}\|T'g_n\|&=\sup_{\|u\|\le 1}(T'g_n)(u)=\sup_{\|u\|\le 1}g_n(Tu)\\&\le g_n(v^*)+\sup_{\|u\|\le 1}g_n(|Tu|-v^*)^+\\&\le g_n(v^*)+\sup_{\|u\|\le 1}g(|Tu|-v^*)^+\le g_n(v^*)+\epsilon\end{aligned}$$

for every $n\in\mathbb{N}$. As $\lim_{n\to\infty}g_n(v^*)=0$, $\limsup_{n\to\infty}\|T'g_n\|\le\epsilon$; as ϵ is arbitrary, $\langle\|T'g_n\|\rangle_{n\in\mathbb{N}}\to 0$. But as $U^\times$ is an M-space (356Pb), it follows that $\langle T'g_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0. **Q**

By 368Pc, $U^\times$ is weakly (σ,∞)-distributive. By 376H, T' is an abstract integral operator, so T also is, by 376K.

376Q Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces and $T:L^1(\mu)\to L^1(\nu)$ a weakly compact linear operator. Then there is a function $k:X\times Y\to\mathbb{R}$ such that $Tf^\bullet=g_f^\bullet$, where $g_f(y)=\int k(x,y)f(x)dx$ almost everywhere, for every $f\in\mathcal{L}^1(\mu)$.

proof This follows from 376P and 376J, just as in 376N.

376R So far I have mentioned actual kernel functions $k(x,y)$ only as a way of giving slightly more concrete form to the abstract kernels of 376E. But of course they can provide new structures and insights. I give one result as an example. The following lemma is useful.

Lemma Let (X,Σ,μ) be a measure space, (Y,T,ν) a σ-finite measure space, and λ the c.l.d. product measure on $X\times Y$. Suppose that k is a λ-integrable real-valued function. Then for any $\epsilon>0$ there is a finite partition $E_0,\dots,E_n$ of X into measurable sets such that $\|k-k_1\|_1\le\epsilon$, where

$$k_1(x,y)=\frac{1}{\mu E_i}\int_{E_i}k(t,y)dt \text{ whenever } x\in E_i,\ 0<\mu E_i<\infty$$

and the integral is defined in $\mathbb{R}$,

$$=0 \text{ in all other cases.}$$

proof Once again I refer to the proof of 253F: there are sets $H_0,\dots,H_r$ of finite measure in X, sets $F_0,\dots,F_r$ of finite measure in Y, and $\alpha_0,\dots,\alpha_r$ such that $\|k-k_2\|_1\le\frac{1}{2}\epsilon$, where $k_2=\sum_{j=0}^{r}\alpha_i\chi(H_j\times F_j)$. Let $E_0,\dots,E_n$ be the partition of X generated by $\{H_i:i\le r\}$. Then for any $i\le n$, $\int_{E_i\times Y}|k-k_1|$ is defined and is at most $2\int_{E_i\times Y}|k-k_2|$. **P** If $\mu E_i=0$, this is trivial, as both are zero. If $\mu E_i=\infty$, then again the result is elementary, since both k_1 and k_2 are zero on $E_i\times Y$. So let us suppose that $0<\mu E_i<\infty$. In this case $\int_{E_i}k(t,y)dt$ must be defined for almost every y, by Fubini's theorem. So k_1 is defined almost everywhere in $E_i\times Y$, and

$$\int_{E_i\times Y}|k-k_1|=\int_Y\int_{E_i}|k(x,y)-k_1(x,y)|dxdy.$$

Now take some fixed $y\in Y$ such that

$$\beta=\frac{1}{\mu E_i}\int_{E_i}k(t,y)dt$$

is defined. Then $\beta=k_1(x,y)$ for every $x\in E_i$. For every $x\in E_i$, we must have $k_2(x,y)=\alpha$ where $\alpha=\sum\{\alpha_j:E_i\subseteq H_j,\ y\in F_j\}$. But in this case, because $\int_{E_i}k(x,y)-\beta\,dx=0$, we have

$$\int_{E_i}\max(0,k(x,y)-\beta)dx=\int_{E_i}\max(0,\beta-k(x,y))dx=\frac{1}{2}\int_{E_i}|k(x,y)-k_1(x,y)|dx.$$

If $\beta\ge\alpha$,

$$\int_{E_i}\max(0,k(x,y)-\beta)dx\le\int_{E_i}\max(0,k(x,y)-\alpha)dx\le\int_{E_i}|k(x,y)-k_2(x,y)|dx;$$

if $\beta\le\alpha$,

$$\int_{E_i}\max(0,\beta-k(x,y))dx\le\int_{E_i}\max(0,\alpha-k(x,y))dx\le\int_{E_i}|k(x,y)-k_2(x,y)|dx;$$

in either case,

$$\frac{1}{2}\int_{E_i}|k(x,y)-k_1(x,y)|dx\le\int_{E_i}|k(x,y)-k_2(x,y)|dx.$$

This is true for almost every y, so integrating with respect to y we get the result. **Q**

Now, summing over i, we get

$$\int|k-k_1|\le 2\int|k-k_2|\le\epsilon,$$

as required.

376S Theorem Let (X,Σ,μ) be a complete locally determined measure space, (Y,T,ν) a σ-finite measure space, and λ the c.l.d. product measure on $X\times Y$. Let τ be an extended Fatou norm on $L^0(\nu)$ and write $\mathcal{L}^{\tau'}$ for $\{g:g\in\mathcal{L}^0(\nu),\ \tau'(g^\bullet)<\infty\}$, where τ' is the associate extended Fatou norm of τ (369H-369I). Suppose that $k\in\mathcal{L}^0(\lambda)$ is such that $k\times(f\otimes g)$ is integrable whenever $f\in\mathcal{L}^1(\mu)$ and $g\in\mathcal{L}^{\tau'}$. Then we have a corresponding linear operator $T:L^1(\mu)\to L^\tau$ defined by saying that $\int(Tf^\bullet)\times g^\bullet=\int k\times(f\otimes g)$ whenever $f\in\mathcal{L}^1(\mu)$ and $g\in\mathcal{L}^{\tau'}$.

For $x\in X$ set $k_x(y)=k(x,y)$ whenever this is defined. Then $k_x\in\mathcal{L}^0(\nu)$ for almost every x; set $v_x=k_x^\bullet\in L^0(\nu)$ for such x. In this case $x\mapsto\tau(v_x)$ is measurable and defined and finite almost everywhere, and $\|T\|=\operatorname{ess\,sup}_x\tau(v_x)$.

Remarks The discussion of extended Fatou norms in §369 regarded them as functionals on spaces of the form $L^0(\mathfrak{A})$. I trust that no-one will be offended if I now speak of an extended Fatou norm on $L^0(\nu)$, with the associated function spaces L^τ, $L^{\tau'}\subseteq L^0$, taking for granted the identification in 364Ic.

Recall that $(f\otimes g)(x,y)=f(x)g(y)$ for $x\in\operatorname{dom}f$ and $y\in\operatorname{dom}g$ (253B).

By 'ess $\sup_x \tau(v_x)$' I mean

$$\inf\{M : M \ge 0,\ \{x : v_x \text{ is defined and } \tau(v_x) \le M\} \text{ is conegligible}\}$$

(see 243D).

proof (a) To see that the formula $(f,g) \mapsto \int k \times (f \otimes g)$ gives rise to an operator in $\mathsf{L}^\times(U;(L^{\tau'})^\times)$, it is perhaps quickest to repeat the argument of parts (a) and (b) of the proof of 376E. (We are not quite in a position to quote 376E, as stated, because the localizable measure algebra free product there might be strictly larger than the measure algebra of λ; see 325B.) The first step, of course, is to note that changing f or g on a negligible set does not affect the integral $\int k \times (f \otimes g)$, so that we have a bilinear functional on $L^1 \times L^{\tau'}$; and the other essential element is the fact that the maps $f^\bullet \mapsto (f \otimes \chi Y)^\bullet$, $g^\bullet \mapsto (\chi X \otimes g)^\bullet$ are order-continuous (put 325A and 364Pc together).

By 369K, we can identify $(L^{\tau'})^\times$ with L^τ, so that T becomes an operator in $\mathsf{L}^\times(U;L^\tau)$. Note that it must be norm-bounded (355C).

(b) By 376I, there is a non-decreasing sequence $\langle Y_n\rangle_{n\in\mathbb{N}}$ of measurable sets in Y, covering Y, such that $\chi Y_n \in \mathcal{L}^{\tau'}$ for every n. Set $X_0 = \{x : x \in X,\ k_x \in \mathcal{L}^0(\nu)\}$. Then X_0 is conegligible in X. **P** Let $E \in \Sigma$ be any set of finite measure. Then for any $n \in \mathbb{N}$, $k \times (\chi E \otimes \chi Y_n)$ is integrable, that is, $\int_{E\times Y_n} k$ is defined and finite; so by Fubini's theorem $\int_{Y_n} k_x$ is defined and finite for almost every $x \in E$. Consequently, for almost every $x \in E$, $k_x \times \chi Y_n \in \mathcal{L}^0(\nu)$ for every $n \in \mathbb{N}$, that is, $k_x \in \mathcal{L}^0(\nu)$, that is, $x \in X_0$.

Thus $E \setminus X_0$ is negligible for every set E of finite measure. Because μ is complete and locally determined, X_0 is conegligible. **Q**

This means that v_x and $\tau(v_x)$ are defined for almost every x.

(c) $\tau(v_x) \le \|T\|$ for almost every x. **P** Take any $E \in \Sigma$ of finite measure, and $n \in \mathbb{N}$. Then $k \times \chi(E \times Y_n)$ is integrable. For each $r \in \mathbb{N}$, there is a finite partition $E_{r0}, \dots, E_{r,m(r)}$ of E into measurable sets such that $\int_{E\times Y_n} |k - k^{(r)}| \le 2^{-r}$, where

$$\begin{aligned} k^{(r)}(x,y) &= \frac{1}{\mu E_{ri}} \int_{E_{ri}} k(t,y)dt \text{ whenever } y \in Y_n,\ x \in E_{ri},\ \mu E_{ri} > 0 \\ &\qquad\qquad\qquad\qquad \text{and the integral is defined in } \mathbb{R} \\ &= 0 \text{ otherwise} \end{aligned}$$

(376R). Now $k^{(r)}$ also is integrable over $E \times Y_n$, so $k_x^{(r)} \in \mathcal{L}^0(\nu)$ for almost every $x \in E$, writing $k_x^{(r)}(y) = k^{(r)}(x,y)$, and we can speak of $v_x^{(r)} = (k_x^{(r)})^\bullet$ for almost every x. Note that $k_x^{(r)} = k_{x'}^{(r)}$ whenever x, x' belong to the same E_{ri}.

If $\mu E_{ri} > 0$, then $v_x^{(r)}$ must be defined for every $x \in E_{ri}$. If $v' \in L^{\tau'}$ is represented by $g \in \mathcal{L}^{\tau'}$ then

$$\begin{aligned} \int k \times (\chi E_{ri} \otimes (g \times \chi Y_n)) &= \int_{E_{ri}\times Y_n} k(t,y)g(y)d(t,y) \\ &= \mu E_{ri} \int k^{(r)}(x,y)g(y)dy = \mu E_{ri} \int v_x^{(r)} \times v' \end{aligned}$$

for any $x \in E_{ri}$. But this means that

$$\mu E_{ri} \int v_x^{(r)} \times v' = \int T(\chi E_{ri}^\bullet) \times v' \times \chi Y_n^\bullet$$

for every $v' \in L^{\tau'}$, so

$$v_x^{(r)} = \frac{1}{\mu E_{ri}} T(\chi E_{ri}^\bullet) \times \chi Y_n^\bullet, \quad \tau(v_x^{(r)}) \le \frac{1}{\mu E_{ri}} \|T\| \|\chi E_{ri}^\bullet\|_1 = \|T\|$$

for every $x \in E_{ri}$. This is true whenever $\mu E_{ri} > 0$, so in fact $\tau(v_x^{(r)}) \le \|T\|$ for almost every $x \in E$.

Because $\sum_{r\in\mathbb{N}} \int_{E\times Y_n} |k - k^{(r)}| < \infty$, we must have $k(x,y) = \lim_{r\to\infty} k^{(r)}(x,y)$ for almost every $(x,y) \in E \times Y_n$. Consequently, for almost every $x \in E$, $k(x,y) = \lim_{r\to\infty} k^{(r)}(x,y)$ for almost every $y \in Y_n$, that is, $\langle v_x^{(r)}\rangle_{r\in\mathbb{N}}$ order*-converges to $v_x \times \chi Y_n^\bullet$ (in $L^0(\nu)$) for almost every $x \in E$. But this means that, for almost every $x \in E$,

$$\tau(v_x \times \chi Y_n^\bullet) \le \liminf_{r\to\infty} \tau(v_x^{(r)}) \le \|T\|$$

(369Mc). Now

$$\tau(v_x) = \lim_{n\to\infty} \tau(v_x \times \chi Y_n^\bullet) \le \|T\|$$

for almost every $x \in E$.

As in (b), this implies (since E is arbitrary) that $\tau(v_x) \le \|T\|$ for almost every $x \in X$. **Q**

(d) I now show that $x \mapsto \tau(v_x)$ is measurable. **P** Take $\gamma \in [0,\infty[$ and set $A = \{x : x \in X_0,\ \tau(v_x) \le \gamma\}$. Suppose that $\mu E < \infty$. Let G be a measurable envelope of $A \cap E$ (132Ee). Set $\tilde{k}(x,y) = k(x,y)$ when $x \in G$ and $(x,y) \in \operatorname{dom} k$, 0 otherwise. If $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^{\tau'}$, then

$$\int \tilde{k}(x,y)f(x)g(y)d(x,y) = \int_{G\times Y} k(x,y)f(x)g(y)d(x,y) = \int_G f(x)\int_Y k(x,y)g(y)dydx$$

is defined.

Take any $g \in \mathcal{L}^{\tau'}$. For $x \in X_0$, set $h(x) = \int |\tilde{k}(x,y)g(y)|dy$. Then h is finite almost everywhere and measurable. For $x \in A \cap E$,

$$\int |\tilde{k}(x,y)g(y)|dy = \int |v_x \times g^\bullet| \le \gamma\tau'(g^\bullet).$$

So the measurable set $G' = \{x : h(x) \le \gamma\tau'(g^\bullet)\}$ includes $A \cap E$, and $\mu(G \setminus G') = 0$. Consequently

$$|\int \tilde{k}(x,y)f(x)g(y)d(x,y)| \le \int_G |f(x)|h(x)dx \le \gamma\|f\|_1\tau'(g^\bullet),$$

and this is true whenever $f \in \mathcal{L}^1(\mu)$.

Now we have an operator $\tilde{T} : L^1(\mu) \to L^\tau$ defined by the formula

$$\int(\tilde{T}f^\bullet) \times g^\bullet = \int \tilde{k} \times (f \otimes g) \text{ when } f \in \mathcal{L}^1(\nu) \text{ and } g \in \mathcal{L}^{\tau'},$$

and the formula just above tells us that $|\int \tilde{T}u \times v'| \le \gamma\|u\|_1\tau'(v')$ for every $u \in L^1(\nu)$ and $v' \in L^{\tau'}$; that is, $\tau(\tilde{T}u) \le \gamma\|u\|_1$ for every $u \in L^1(\mu)$; that is, $\|\tilde{T}\| \le \gamma$. But now (c) tells us that $\tau(\tilde{v}_x) \le \gamma$ for almost every $x \in X$, where $\tilde{v}_x$ is the equivalence class of $y \mapsto \tilde{k}(x,y)$, that is, $\tilde{v}_x = v_x$ for $x \in G \cap X_0$, 0 for $x \in X \setminus G$. So $\tau(v_x) \le \gamma$ for almost every $x \in G$, and $G \setminus A$ is negligible. But this means that $A \cap E$ is measurable. As E is arbitrary, A is measurable; as γ is arbitrary, $x \mapsto \tau(v_x)$ is measurable. **Q**

(e) Finally, the ideas in (d) show that $\|T\| \le \operatorname{ess\,sup}_x \tau(v_x)$. **P** Set $M = \operatorname{ess\,sup}_x \tau(v_x)$. If $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^{\tau'}$, then

$$\int |k(x,y)f(x)g(y)|d(x,y) \le \int |f(x)|\tau(v_x)\tau'(g^\bullet)dx \le M\|f\|_1\tau'(g^\bullet);$$

as g is arbitrary, $\tau(Tf^\bullet) \le M\|f\|_1$; as f is arbitrary, $\|T\| \le M$. **Q**

376X Basic exercises >(a) Let μ be Lebesgue measure on $\mathbb{R}$. Let h be a μ-integrable real-valued function with $\|h\|_1 \le 1$, and set $k(x,y) = h(y-x)$ whenever this is defined. Show that if f is in either $\mathcal{L}^1(\mu)$ or $\mathcal{L}^\infty(\mu)$ then $g(y) = \int k(x,y)f(x)dx$ is defined for almost every $y \in \mathbb{R}$, and that this formula gives rise to an operator $T \in \mathcal{T}^\times_{\bar{\mu},\bar{\mu}}$ as defined in 373Ab. (*Hint*: 255H.)

(b) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras with localizable measure algebra free product $(\mathfrak{C},\bar{\lambda})$, and take $p \in [1,\infty]$. Show that if $u \in L^p(\mathfrak{A},\bar{\mu})$ and $v \in L^p(\mathfrak{B},\bar{\nu})$ then $u \otimes v \in L^p(\mathfrak{C},\bar{\lambda})$ and $\|u \otimes v\|_p = \|u\|_p\|v\|_p$.

>(c) Let U, V, W be Riesz spaces, of which V and W are Dedekind complete, and suppose that $T \in \mathsf{L}^\times(U;V)$ and $S \in \mathsf{L}^\times(V;W)$. Show that if either S or T is an abstract integral operator, so is ST.

(d) Let h be a Lebesgue integrable function on $\mathbb{R}$, and f a square-integrable function. Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable functions such that (α) $|f_n| \le f$ for every n (β) $\lim_{n\to\infty}\int_E f_n = 0$ for every measurable set E of finite measure. Show that $\lim_{n\to\infty}(h * f_n)(y) = 0$ for almost every $y \in \mathbb{R}$, where $h * f_n$ is the convolution of h and f_n. (*Hint*: 376Xa, 376H.)

(e) Let U and V be Riesz spaces, of which V is Dedekind complete. Suppose that $W \subseteq U^\sim$ is a solid linear subspace, and that T belongs to the band in $\mathsf{L}^\sim(U;V)$ generated by operators of the form $u \mapsto f(u)v$, where $f \in W$ and $v \in V$. Show that whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U such that $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in W$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

(f) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and $U \subseteq L^0 = L^0(\mathfrak{A})$ an order-dense Riesz subspace such that $U^\times$ separates the points of U. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be an order-bounded sequence in U. Show that the following are equiveridical: (i) $\lim_{n\to\infty} f(|u_n|) = 0$ for every $f \in U^\times$; (ii) $\langle u_n\rangle_{n\in\mathbb{N}} \to 0$ for the topology of convergence in measure on L^0. (*Hint*: by 367T, condition (ii) is intrinsic to U, so we can replace $(\mathfrak{A},\bar{\mu})$ by a localizable algebra and use the representation in 369D.)

(g) Let U be a Banach lattice with an order-continuous norm, and V a weakly (σ,∞)-distributive Riesz space. Show that for $T\in\mathsf{L}^\sim(U;V)$ the following are equiveridical: (i) T belongs to the band in $\mathsf{L}^\sim(U;V)$ generated by operators of the form $u\mapsto f(u)v$ where $f\in U^\sim$, $v\in V$; (ii) $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U^+ which is norm-convergent to 0; (iii) $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U which is weakly convergent to 0.

(h) Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces, with product measure λ on $X\times Y$, and measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$. Suppose that $k\in\mathcal{L}^0(\lambda)$. Show that the following are equiveridical: (i)(α) if $f\in\mathcal{L}^1(\mu)$ then $g_f(y)=\int k(x,y)f(x)dx$ is defined for almost every y and $g_f\in\mathcal{L}^1(\nu)$ (β) there is an operator $T\in\mathcal{T}^\times_{\bar\mu,\bar\nu}$ defined by setting $Tf^\bullet=g_f^\bullet$ for every $f\in\mathcal{L}^1(\mu)$; (ii) $\int|k(x,y)|dy\le 1$ for almost every $x\in X$, $\int|k(x,y)|dx\le 1$ for almost every $y\in Y$.

>(i)(i) Show that there is a compact linear operator from ℓ^2 to itself which is not in $\mathsf{L}^\sim(\ell^2;\ell^2)$. (*Hint*: start from the operator S of 371Ye.) (ii) Show that the identity operator on ℓ^2 is an abstract integral operator.

>(j) Let μ be Lebesgue measure on $[0,1]$. (i) Give an example of a measurable function $k:[0,1]^2\to\mathbb{R}$ such that, for any $f\in\mathcal{L}^2(\mu)$, $g_f(y)=\int k(x,y)f(x)dx$ is defined for every y and $\|g_f\|_2=\|f\|_2$, but k is not integrable, so the linear isometry on $L^2=L^2(\mu)$ defined by k does not belong to $\mathsf{L}^\sim(L^2;L^2)$. (ii) Show that the identity operator on L^2 is not an abstract integral operator.

(k) Let (X,Σ,μ) be a σ-finite measure space and (Y,T,ν) a complete locally determined measure space. Let $U\subseteq L^0(\mu)$, $V\subseteq L^0(\nu)$ be solid linear subspaces, of which V is order-dense; write $V^\#=\{v:v\in L^0(\nu),\ v\times v'$ is integrable for every $v'\in V\}$, $\mathcal{U}=\{f:f\in\mathcal{L}^0(\nu),\ f^\bullet\in U\}$, $\mathcal{V}=\{g:g\in\mathcal{L}^0(\nu),\ g^\bullet\in V\}$, $\mathcal{V}^\#=\{h:h\in\mathcal{L}^0(\nu),\ h^\bullet\in V^\#\}$. Let λ be the c.l.d. product measure on $X\times Y$, and $k\in\mathcal{L}^0(\lambda)$ a function such that $k\times(f\otimes g)$ is integrable for whenever $f\in\mathcal{U}$ and $g\in\mathcal{V}$. (i) Show that for any $f\in\mathcal{U}$, $h_f(y)=\int k(x,y)f(x)dx$ is defined for almost every $y\in Y$, and that $h_f\in\mathcal{V}^\#$. (ii) Show that we have a map $T\in\mathsf{L}^\times(U;V^\#)$ defined *either* by writing $Tf^\bullet=h_f^\bullet$ for every $f\in\mathcal{U}$ *or* by writing $\int(Tf^\bullet)\times g^\bullet=\int k\times(f\otimes g)$ for every $f\in\mathcal{U}$ and $g\in\mathcal{V}$.

(l) Let (X,Σ,μ), (Y,T,ν) and (Z,Λ,λ) be σ-finite measure spaces, and U, V, W perfect order-dense solid linear subspaces of $L^0(\mu)$, $L^0(\nu)$ and $L^0(\lambda)$ respectively. Suppose that $T:U\to V$ and $S:V\to W$ are abstract integral operators corresponding to kernels $k_1\in\mathcal{L}^0(\mu\times\nu)$, $k_2\in\mathcal{L}^0(\nu\times\lambda)$, writing $\mu\times\nu$ for the (c.l.d. or primitive) product measure on $X\times Y$. Show that $ST:U\to W$ is represented by the kernel $k\in\mathcal{L}^0(\mu\times\lambda)$ defined by setting $k(x,z)=\int k_1(x,y)k_2(y,z)dy$ whenever this integral is defined.

(m) Let U be a perfect Riesz space. Show that a set $C\subseteq U$ is relatively compact for $\mathfrak{T}_s(U,U^\times)$ iff for every $g\in(U^\times)^+$, $\epsilon>0$ there is a $u^*\in U$ such that $g(|u|-u^*)^+\le\epsilon$ for every $u\in C$. (*Hint*: 376O and the proof of 356Q.)

>(n) Let μ be Lebesgue measure on $[0,1]$, and ν counting measure on $[0,1]$. Set $k(x,y)=1$ if $x=y$, 0 otherwise. Show that 376S fails in this context (with, e.g., $\tau=\|\,\|_\infty$).

(o) Suppose, in 376Xk, that $U=L^\tau$ for some extended Fatou norm on $L^0(\mu)$ and that $V=L^1(\nu)$, so that $V^\#=L^\infty(\nu)$. Set $k_y(x)=k(x,y)$ whenever this is defined, $w_y=k_y^\bullet$ whenever $k_y\in\mathcal{L}^0(\mu)$. Show that $w_y\in L^{\tau'}$ for almost every $y\in Y$, and that the norm of T in $\mathsf{B}(L^\tau;L^\infty)$ is $\operatorname{ess\,sup}_y\tau'(w_y)$. (*Hint*: do the case of totally finite Y first.)

376Y Further exercises (a) Let U, V and W be linear spaces (over any field F) and $\phi:U\times V\to W$ a bilinear operator. Let W_0 be the linear subspace of W generated by $\phi[U\times V]$. Show that the following are equiveridical: (i) for every linear space Z over F and every bilinear $\psi:U\times V\to Z$, there is a (unique) linear operator $T:W_0\to Z$ such that $T\phi=\psi$ (ii) whenever $u_0,\dots,u_n\in U$ are linearly independent and $v_0,\dots,v_n\in V$ are non-zero, $\sum_{i=0}^n\phi(u_i,v_i)\ne0$ (iii) whenever $u_0,\dots,u_n\in U$ are non-zero and $v_0,\dots,v_n\in V$ are linearly independent, $\sum_{i=0}^n\phi(u_i,v_i)\ne0$ (iv) for any Hamel bases $\langle u_i\rangle_{i\in I}$, $\langle v_j\rangle_{j\in J}$ of U and V, $\langle\phi(u_i,v_j)\rangle_{i\in I,j\in J}$ is a Hamel basis of W_0 (v) for some pair $\langle u_i\rangle_{i\in I}$, $\langle v_j\rangle_{j\in J}$ of Hamel bases of U and V, $\langle\phi(u_i,v_j)\rangle_{i\in I,j\in J}$ is a Hamel basis of W_0.

(b) Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, and $(\mathfrak{C},\bar\lambda)$ their localizable measure algebra free product. Show that $\otimes:L^0(\mathfrak{A})\times L^0(\mathfrak{B})\to L^0(\mathfrak{C})$ satisfies the equivalent conditions of 376Ya.

(c) Let (X,Σ,μ) and (Y,T,ν) be semi-finite measure spaces and λ the c.l.d. product measure on $X\times Y$. Show that the map $(f,g)\mapsto f\otimes g:\mathcal{L}^0(\mu)\times\mathcal{L}^0(\nu)\to\mathcal{L}^0(\lambda)$ induces a map $(u,v)\mapsto u\otimes v:L^0(\mu)\times L^0(\nu)\to L^0(\lambda)$ possessing all the properties described in 376B and 376Ya, subject to a suitable interpretation of the formula $\otimes:\mathfrak{A}\times\mathfrak{B}\to\mathfrak{C}$.

(d) Let $(\mathfrak{B}_{\omega_1},\bar\nu_{\omega_1})$ be the measure algebra of $\{0,1\}^{\omega_1}$ with its usual measure, and $\langle a_\xi\rangle_{\xi<\omega_1}$ a stochastically independent (definition: 325Xf) family of elements of measure $\frac{1}{2}$ in $\mathfrak{B}_{\omega_1}$. Set $U=L^2(\mathfrak{B}_{\omega_1},\bar\nu_{\omega_1})$ and $V=\{v:v\in\mathbb{R}^{\omega_1},\ \{\xi:v(\xi)\ne0\}$ is countable$\}$. Define $T:U\to\mathbb{R}^{\omega_1}$ by setting $Tu(\xi)=2\int_{a_\xi}u-\int u$ for $\xi<\omega_1$, $u\in U$. Show that (i) $Tu\in V$ for every $u\in U$ (ii) $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U such that $\lim_{n\to\infty}f(u_n)=0$ for every $f\in U^\times$ (iii) $T\notin\mathsf{L}^\sim(U;V)$.

(e) Let U be a Riesz space with the countable sup property (definition: 241Ye) such that $U^\times$ separates the points of U, and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in U. Show that the following are equiveridical: (i) $\lim_{n\to\infty}f(v\wedge|u_n|)=0$ for every $f\in U^\times$, $v\in U^+$; (ii) every subsequence of $\langle u_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence which is order*-convergent to 0.

(f) Let U be an Archimedean Riesz space and $\mathfrak{A}$ a weakly (σ,∞)-distributive Dedekind complete Boolean algebra. Suppose that $T:U\to L^0=L^0(\mathfrak{A})$ is a linear operator such that $\langle|Tu_n|\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in L^0 whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is order-bounded and order*-convergent to 0 in U. Show that $T\in\mathsf{L}^\sim_c(U;L^0)$ (definition: 355G), so that if U has the countable sup property then $T\in\mathsf{L}^\times(U;L^0)$.

(g) Suppose that (Y,T,ν) is a probability space in which $\mathrm{T}=\mathcal{P}Y$ and $\nu\{y\}=0$ for every $y\in Y$. (See 363S.) Take $X=Y$ and let μ be counting measure on X; let λ be the c.l.d. product measure on $X\times Y$, and set $k(x,y)=1$ if $x=y$, 0 otherwise. Show that we have an operator $T:L^\infty(\mu)\to L^\infty(\nu)$ defined by setting $Tf=g^\bullet$ whenever $f\in L^\infty(\mu)\cong\ell^\infty(X)$ and $g(y)=\int k(x,y)f(x)dx=f(y)$ for every $y\in Y$. Show that T satisfies the conditions (ii) and (iii) of 376J but does not belong to $\mathsf{L}^\times(L^\infty(\mu);L^\infty(\nu))$.

(h) Give an example of an abstract integral operator $T:\ell^2\to L^1(\mu)$, where μ is Lebesgue measure on $[0,1]$, such that $\langle Te_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent in $L^1(\mu)$, where $\langle e_n\rangle_{n\in\mathbb{N}}$ is the standard orthonormal sequence in ℓ^2.

(i) Set $k(m,n)=1/\pi(n-m+\frac{1}{2})$ for m, $n\in\mathbb{Z}$. (i) Show that $\sum_{n=-\infty}^\infty k(m,n)^2=1$ and $\sum_{n=-\infty}^\infty k(m,n)k(m',n)=0$ for all distinct m, $m'\in\mathbb{Z}$. (*Hint*: find the Fourier series of $x\mapsto e^{i(m+\frac{1}{2})x}$ and use 282K.) (ii) Show that there is a norm-preserving linear operator T from $\ell^2=\ell^2(\mathbb{Z})$ to itself given by the formula $(Tu)(n)=\sum_{m=-\infty}^\infty k(m,n)u(m)$. (iii) Show that T^2 is the identity operator on ℓ^2. (iv) Show that $T\notin\mathsf{L}^\sim(\ell^2;\ell^2)$. (*Hint*: consider $\sum_{m,n=-\infty}^\infty|k(m,n)|x(m)x(n)$ where $x(n)=1/\sqrt{|n|}\ln|n|$ for $|n|\ge2$.) (T is a form of the **Hilbert transform.**)

(j) Let U be an L-space and V a Banach lattice with an order-continuous norm. Let $T\in\mathsf{L}^\sim(U;V)$. Show that the following are equiveridical: (i) T is an abstract integral operator; (ii) $T[C]$ is norm-compact in V whenever C is weakly compact in U. (*Hint*: start with the case in which C is order-bounded, and remember that it is weakly sequentially compact.)

(k) Let (X,Σ,μ) be a complete locally determined measure space and (Y,T,ν), (Z,Λ,λ) two σ-finite measure spaces. Suppose that τ, θ are extended Fatou norms on $L^0(\nu)$, $L^0(\lambda)$ respectively, and that $T:L^1(\mu)\to L^\tau$ is an abstract integral operator, with corresponding kernel $k\in\mathcal{L}^0(\mu\times\nu)$, while $S\in\mathsf{L}^\times(L^\tau;L^\theta)$, so that $ST:L^1(\mu)\to L^\theta$ is an abstract integral operator (376Xc); let $\tilde k\in\mathcal{L}^0(\mu\times\lambda)$ be the corresponding kernel. For $x\in X$ set $v_x=k_x^\bullet$ when this is defined in L^τ, as in 376S, and similarly take $w_x=\tilde k_x^\bullet\in L^\theta$. Show that $Sv_x=w_x$ for almost every $x\in X$.

376 Notes and comments I leave 376Yb to the exercises because I do not rely on it for any of the work here, but of course it is an essential aspect of the map $\otimes:L^0(\mathfrak{A})\times L^0(\mathfrak{B})\to L^0(\mathfrak{C})$ I discuss in this section. The conditions in 376Ya are characterizations of the 'tensor product' of two linear spaces, a construction of great importance in abstract linear algebra (and, indeed, in modern applied linear algebra; it is by no means trivial even in the finite-dimensional case). In particular, note that conditions (ii), (iii) of 376Ya apply to arbitrary subspaces of U and V if they apply to U and V themselves.

The principal ideas used in 376B-376C have already been set out in §§253 and 325. Here I do little more than list the references. I remark however that it is quite striking that $L^1(\mathfrak{C},\bar\lambda)$ should have no fewer than three universal mapping theorems attached to it (376Cb, 376C(c-i) and 376C(c-ii)).

The real work of this section begins in 376E. As usual, much of the proof is taken up with relatively straightforward verifications, as in parts (a) and (b), while part (i) is just a manoeuvre to show that it doesn't matter if $\mathfrak{A}$ and $\mathfrak{B}$

aren't Dedekind complete, because $\mathfrak{C}$ is. But I think that parts (d), (f) and (j) have ideas in them. In particular, part (f) is a kind of application of the Radon-Nikodým theorem (through the identification of $L^1(\mathfrak{C},\bar{\lambda})^*$ with $L^\infty(\mathfrak{C})$).

I have split 376E from 376H because the former demands the language of measure algebras, while the latter can be put into the language of pure Riesz space theory. Asking for a weakly (σ,∞)-distributive space V in 376H is a way of applying the ideas to $V = L^0$ as well as to Banach function spaces. (When $V = L^0$, indeed, variations on the hypotheses are possible, using 376Yf.) But it is a reminder of one of the directions in which it is often possible to find generalizations of ideas beginning in measure theory.

The condition '$\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$' (376H(ii)) seems natural in this context, and gives marginally greater generality than some alternatives (because it does the right thing when $U^\times$ does not separate the points of U), but it is not the only way of expressing the idea; see 376Xf and 376Ye. Note that the conditions (ii) and (iii) of 376H are significantly different. In 376H(iii) we could easily have $|u_n| = u^*$ for every n; for instance, if $u_n = 2\chi a_n - \chi 1$ for some stochastically independent sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of elements of measure $\frac{1}{2}$ in a probability algebra (272Ye).

If you have studied compact linear operators between Banach spaces (definition: 3A5La), you will have encountered the condition '$Tu_n \to 0$ strongly whenever $u_n \to 0$ weakly'. The conditions in 376H and 376J are of this type. If a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in a Riesz space U is order-bounded and order*-convergent to 0, then $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$ (367Xf). Visibly this latter condition is associated with weak convergence, and order*-convergence is (in Banach lattices) closely related to norm convergence (367D). In the context of 376H, an abstract integral operator is one which transforms convergent sequences of a weak type into convergent sequences of a stronger type. The relationship between the classes of (weakly) compact operators and abstract integral operators is interesting, but outside the scope of this book; I leave you with 376P-376Q and 376Y, and a pair of elementary examples to guard against extravagant conjecture (376Xi).

376O belongs to an extensive general theory of weak compactness in perfect Riesz spaces, based on adaptations of the concept of 'uniform integrability'. I give the next step in 376Xm. For more information see Fremlin 74a, chap. 8.

Note that 376Mb and 376P overlap when $V^\times$ in 376Mb is reflexive – for instance, when V is an L^p space for some $p \in \,]1,\infty[$ – since then every bounded linear operator from L^1 to $V^\times$ must be weakly compact. For more information on the representation of operators see Dunford & Schwartz 57, particularly Table VI in the notes to Chapter VI.

As soon as we leave formulations in terms of the spaces $L^0(\mathfrak{A})$ and their subspaces, and return to the original conception of a kernel operator in terms of integrating functions against sections of a kernel, we are necessarily involved in the pathology of Fubini's theorem for general measure spaces. In general, the repeated integrals $\iint k(x,y)dxdy$, $\iint k(x,y)dydx$ need not be equal, and something has to give (376Xn). Of course this particular worry disappears if the spaces are σ-finite, as in 376J. In 376S I take the trouble to offer a more general condition, mostly as a reminder that the techniques developed in Volume 2 do enable us sometimes to go beyond the σ-finite case. Note that this is one of the many contexts in which anything we can prove about probability spaces will be true of all σ-finite spaces; but that we cannot make the next step, to all strictly localizable spaces.

376S verges on the theory of integration of vector-valued functions, which I don't wish to enter here; but it also seems to have a natural place in the context of this chapter. It is of course a special property of L^1 spaces. The formula $\|T_k\| = \operatorname{ess\,sup}_x \tau(k_x^\bullet)$ shows that $\|T_{|k|}\| = \|T_k\|$; now we know fron 376E that $T_{|k|} = |T_k|$, so we get a special case of the Chacon-Krengel theorem (371D). Reversing the roles of X and Y, we find ourselves with an operator from L^τ to L^∞ (376Xo), which is the other standard context in which $\|T\| = \||T|\|$ (371Xd). I include two exercises on L^2 spaces (376Xj, 376Yi) designed to emphasize the fact that $\mathsf{B}(U;V)$ is included in $\mathsf{L}^\sim(U;V)$ only in very special cases.

The history of the theory here is even more confusing than that of mathematics in general, because so many of the ideas were developed in national schools in very imperfect contact with each other. My own account gives no hint of how this material arose; I ought in particular to note that 376N is one of the oldest results, coming (essentially) from Dunford 36. For further references, see Zaanen 83, chap. 13.

*377 Function spaces of reduced products

In §328 I introduced 'reduced products' of probability algebras. In this section I seek to describe the function spaces of reduced products as images of subspaces of products of function spaces of the original algebras. I add a group of universal mapping theorems associated with the constructions of projective and inductive limits of directed families of probability algebras (377G-377H).

377A Proposition If $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a non-empty family of Boolean algebras with simple product $\mathfrak{A}$, then $L^\infty(\mathfrak{A})$ can be identified, as normed space and f-algebra, with the subspace W_∞ of $\prod_{i\in I}L^\infty(\mathfrak{A}_i)$ consisting of families $u=\langle u_i\rangle_{i\in I}$ such that $\|u\|_\infty=\sup_{i\in I}\|u_i\|_\infty$ is finite.

proof (a) I begin by noting that W_∞ is, in itself, an Archimedean f-algebra and $\|\,\|_\infty$ is a Riesz norm on W_∞. **P** W_∞ is a solid linear subspace of $\prod_{i\in I}L^\infty(\mathfrak{A}_i)$, so inherits a Riesz space structure (352K, 352Ja). Now it is easy to check that $e=\langle\chi 1_{\mathfrak{A}_i}\rangle_{i\in I}$ is an order unit in W_∞ and that $\|\,\|_\infty$ is the corresponding order-unit norm (354F-354G). Finally, because W_∞ is the solid linear subspace of $\prod_{i\in I}L^\infty(\mathfrak{A}_i)$ generated by e, and e is the multiplicative identity of $\prod_{i\in I}L^\infty(\mathfrak{A}_i)$, W_∞ is closed under multiplication, and is an f-algebra. **Q**

(b) We have a natural function $\theta:\mathfrak{A}\to W_\infty$ defined by saying that $\theta a=\langle\chi a_i\rangle_{i\in I}$ whenever $a=\langle a_i\rangle_{i\in I}\in\mathfrak{A}$. Clearly θ is additive and $\|\theta a\|_\infty\le 1$ for every $a\in\mathfrak{A}$; moreover, $\theta a\wedge\theta b=0$ when a, $b\in\mathfrak{A}$ are disjoint. By 363E, we have a corresponding Riesz homomorphism $T:L^\infty(\mathfrak{A})\to W_\infty$ of norm at most 1.

(c) In fact $\|Tw\|_\infty=\|w\|_\infty$ for every $w\in L^\infty(\mathfrak{A})$. **P** If $w=0$, this is trivial. If $w\in S(\mathfrak{A})\setminus\{0\}$, express it as $\sum_{k=0}^n\alpha_k\chi a^{(k)}$ where $\langle a^{(k)}\rangle_{k\le n}$ is a disjoint family of non-zero elements. Expressing each $a^{(k)}$ as $\langle a_{ki}\rangle_{i\in I}$,

$$Tw=\langle\textstyle\sum_{k=0}^n\alpha_k\chi a_{ki}\rangle_{i\in I}.$$

There must be a j such that $|\alpha_j|=\|w\|_\infty$; now there is an i such that $a_{ji}\ne0$; as $\langle a_{ki}\rangle_{k\le n}$ is disjoint,

$$\|Tw\|_\infty\ge\|\textstyle\sum_{k=0}^n\alpha_k\chi a_{ki}\|_\infty\ge|\alpha_j|=\|w\|_\infty.$$

If now w is any member of $L^\infty(\mathfrak{A})$,

$$\begin{aligned}\|w\|_\infty&=\sup\{\|w'\|_\infty:w'\in S(\mathfrak{A}),\,|w'|\le|w|\}\\&=\sup\{\|Tw'\|_\infty:w'\in S(\mathfrak{A}),\,|w'|\le|w|\}\le\|Tw\|_\infty\end{aligned}$$

because T is a Riesz homomorphism. **Q**

Thus T is norm-preserving, therefore injective.

(d) Next, T is surjective. **P** Suppose that $\langle u_i\rangle_{i\in I}\in W_\infty^+$ is non-negative, and that $\epsilon>0$. Let $n\in\mathbb{N}$ be such that $n\epsilon\ge\sup_{i\in I}\|u_i\|_\infty$, and for $k\le n$, $i\in I$ set $a_{ki}=[\![u_i>k\epsilon]\!]$. Set $w=\epsilon\sum_{k=1}^n\chi(\langle a_{ki}\rangle_{i\in I})$. Then $w\in L^\infty(\mathfrak{A})$ and $Tw=\langle v_i\rangle_{i\in I}$, where $v_i=\epsilon\sum_{k=1}^n\chi a_{ki}$, so that $v_i\le u_i$ and $\|u_i-v_i\|_\infty\le\epsilon$, for every $i\in I$. Thus $\|Tw-\langle u_i\rangle_{i\in I}\|_\infty\le\epsilon$.

As $\langle u_i\rangle_{i\in I}$ and ϵ are arbitrary, $T[L^\infty(\mathfrak{A})]\cap W_\infty^+$ is norm-dense in W_∞^+. But T is an isometry and $L^\infty(\mathfrak{A})$ is norm-complete, so $T[L^\infty(\mathfrak{A})]$ is closed in W_∞ and includes W_∞^+ and therefore W_∞; that is, T is surjective. **Q**

So T is a norm-preserving bijective Riesz homomorphism, that is, a normed Riesz space isomorphism. Finally, by 353Pd or otherwise, T is multiplicative, so is an f-algebra isomorphism.

377B Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, and $(\mathfrak{B},\bar\nu)$ a probability algebra. Let $\mathfrak{A}$ be the simple product of $\langle\mathfrak{A}_i\rangle_{i\in I}$, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I})\le\sup_{i\in I}\bar\mu_ia_i$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}$. Let W_0 be the subspace of $\prod_{i\in I}L^0(\mathfrak{A}_i)$ consisting of families $\langle u_i\rangle_{i\in I}$ such that $\inf_{k\in\mathbb{N}}\sup_{i\in I}\bar\mu_i[\![|u_i|>k]\!]=0$.

(a) W_0 is a solid linear subspace and a subalgebra of $\prod_{i\in I}L^0(\mathfrak{A}_i)$, and there is a unique Riesz homomorphism $T:W_0\to L^0(\mathfrak{B})$ such that $T(\langle\chi a_i\rangle_{i\in I})=\chi\pi(\langle a_i\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}$. Moreover, T is multiplicative, and $[\![Tu>0]\!]\subseteq\pi(\langle[\![u_i>0]\!]\rangle_{i\in I})$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_0.

(b) If $h:\mathbb{R}\to\mathbb{R}$ is a continuous function, and we write $\bar h$ for the corresponding maps from L^0 to itself for any of the spaces $L^0=L^0(\mathfrak{A}_i)$, $L^0=L^0(\mathfrak{B})$ (364H), then $\langle\bar h(u_i)\rangle_{i\in I}\in W_0$ and $T(\langle\bar h(u_i)\rangle_{i\in I})=\bar h(Tu)$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_0.

proof (a) For $u=\langle u_i\rangle_{i\in I}\in\prod_{i\in I}L^0(\mathfrak{A}_i)$ and $k\in\mathbb{N}$, set $\gamma_k(u)=\sup_{i\in I}\bar\mu_i[\![|u_i|>k]\!]$.

(i) W_0 is a solid linear subspace and subalgebra of the f-algebra $\prod_{i\in I}L^0(\mathfrak{A}_i)$. **P** For $k\in\mathbb{N}$ and u, $v\in\prod_{i\in I}L^0(\mathfrak{A}_i)$,

$$\gamma_k(u) \le \gamma_k(v) \text{ whenever } |u| \le |v|,$$

$$\gamma_{2k}(u+v) \le \gamma_k(u) + \gamma_k(v),$$

$$\gamma_{k^2}(u \times v) \le \gamma_k(u) + \gamma_k(v)$$

for all u, $v \in \prod_{i\in I} L^0(\mathfrak{A}_i)$ and $k \in \mathbb{N}$. So W_0 is solid, is closed under addition, and is closed under multiplication. **Q**

(ii) Let $W_\infty \subseteq W_0$ be the set of families $\langle u_i\rangle_{i\in I} \in \prod_{i\in I} L^\infty(\mathfrak{A}_i)$ such that $\sup_{i\in I} \|u_i\|_\infty$ is finite; by 377A, we can identify W_∞ with $L^\infty(\mathfrak{A})$. We therefore have a corresponding multiplicative Riesz homomorphism $S : W_\infty \to L^\infty(\mathfrak{B})$ such that $S(\langle\chi a_i\rangle_{i\in I}) = \chi\pi(\langle a_i\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I} \in \mathfrak{A}$ (363F); note that $S(\langle\chi 1_{\mathfrak{A}_i}\rangle_{i\in I}) = \chi 1_{\mathfrak{B}}$.

(iii) If $u = \langle u_i\rangle_{i\in I} \in W_\infty$ and $k \in \mathbb{N}$, then $[\![Su > k]\!] \subseteq \pi(\langle[\![u_i > k]\!]\rangle_{i\in I})$. **P** Setting $a_i = [\![u_i > k]\!]$, we have $u_i \times \chi(1_{\mathfrak{A}_i} \setminus a_i) \le k\chi 1_{\mathfrak{A}_i}$ for every i. Set $a = \langle a_i\rangle_{i\in I}$. Since S is a multiplicative Riesz homomorphism,

$$\begin{aligned} Su \times \chi(1_{\mathfrak{B}} \setminus \pi a) &= Su \times \chi\pi(\langle 1_{\mathfrak{A}_i} \setminus a_i\rangle_{i\in I}) = S(\langle u_i\rangle_{i\in I}) \times S(\langle\chi(1_{\mathfrak{A}_i} \setminus a_i)\rangle_{i\in I}) \\ &= S(\langle u_i\rangle_{i\in I} \times \langle\chi(1_{\mathfrak{A}_i} \setminus a_i)\rangle_{i\in I}) = S(\langle u_i \times \chi(1_{\mathfrak{A}_i} \setminus a_i)\rangle_{i\in I}) \\ &\le S(\langle k\chi 1_{\mathfrak{A}_i}\rangle_{i\in I}) = k\chi 1_{\mathfrak{B}} \end{aligned}$$

and $[\![Su > k]\!] \subseteq \pi a$, as claimed. **Q**

(iv) If $u = \langle u_i\rangle_{i\in I} \in W_0^+$, then $\sup\{Sv : v \in W_\infty, 0 \le v \le u\}$ is defined in $L^0(\mathfrak{B})$. **P** Set $A_u = S[W_\infty \cap [0, u]\,]$. Because $W_\infty \cap [0, u]$ is upwards-directed, so is A. If $v = \langle v_i\rangle_{i\in I} \in W_\infty \cap [0, u]$, then $[\![Sv > k]\!] \subseteq \pi(\langle[\![v_i > k]\!]\rangle_{i\in I})$, by (iii), so

$$\bar\nu[\![Sv > k]\!] \le \sup_{i\in I} \bar\mu_i[\![v_i > k]\!] \le \gamma_k(u).$$

Thus $\bar\nu[\![w > k]\!] \le \gamma_k(u)$ for every $w \in A$. Since $u \in W_0$, $\lim_{k\to\infty} \gamma_k(u) = 0$; so 364L(a-ii) tells us that $\sup A_u$ is defined in $L^0(\mathfrak{B})$. **Q**

By 355F, there is a Riesz homomorphism $T : W_0 \to L^0(\mathfrak{B})$ extending S and such that $Tu = A_u$ for every $u \in W_0^+$. By 353Pd, T is multiplicative.

(v) Because T is multiplicative, we can repeat the calculations of (iii), with T in place of S, to see that

$$[\![Tu > k]\!] \subseteq \pi(\langle[\![u_i > k]\!]\rangle_{i\in I})$$

whenever $u = \langle u_i\rangle_{i\in I} \in W_0$; in particular, $[\![Tu > 0]\!] \subseteq \pi(\langle[\![u_i > 0]\!]\rangle_{i\in I})$.

(vi) To see that T is uniquely defined, let $T' : W_0 \to L^0(\mathfrak{B})$ be another Riesz homomorphism agreeing with T on families of the form $\langle\chi a_i\rangle_{i\in I}$. Then T and T' agree on $W_\infty \cong L^\infty(\mathfrak{A})$, by the uniqueness guaranteed in 363Fa, and T' also is multiplicative, by 353Pd again. As in (v), we therefore have

$$[\![Tu > k]\!] \cup [\![T'u > k]\!] \subseteq \pi(\langle[\![u_i > k]\!]\rangle_{i\in I}), \quad \bar\nu([\![Tu > k]\!] \cup [\![T'u > k]\!]) \le \gamma_k(u)$$

whenever $u \in W_0$ and $k \in \mathbb{N}$.

Suppose that $u \in W_0^+$ and $\epsilon > 0$. Then there is a $k \in \mathbb{N}$ such that $\gamma_k(u) \le \epsilon$. Set $v_i = u_i \wedge k\chi 1_{\mathfrak{A}_i}$ for each i, and $v = \langle v_i\rangle_{i\in I}$. Then $Tv = T'v$, so

$$\bar\nu[\![|Tu - T'u| > 0]\!] \le \bar\nu([\![Tu - Tv > 0]\!] \cup [\![T'u - T'v > 0]\!]) \le \gamma_0(u - v) = \gamma_k(u) \le \epsilon.$$

As ϵ is arbitrary, $Tu = T'u$; as u is arbitrary, $T = T'$.

(b)(i) If $\epsilon > 0$, there is a $k \in \mathbb{N}$ such that $\bar\mu_i[\![|u_i| > k]\!] \le \epsilon$ for every $i \in I$. Now there is an $l \in \mathbb{N}$ such that $|h(t)| \le l$ whenever $|t| \le k$. So $[\![|\bar h(u_i)| > l]\!] \subseteq [\![|u_i| > k]\!]$ and $\bar\mu_i[\![|\bar h(u_i)| > l]\!] \le \epsilon$ for every $i \in I$. As ϵ is arbitrary, $\langle\bar h(u_i)\rangle_{i\in I} \in W_0$.

(ii) Again take any $\epsilon > 0$. Let $k \in \mathbb{N}$ be such that $\bar\mu_i a_i \le \epsilon$ for every $i \in I$, where $a_i = [\![|u_i| > k]\!]$. By the Stone-Weierstrass theorem in the form 281E, there is a polynomial $g : \mathbb{R} \to \mathbb{R}$ such that $|g(t) - h(t)| \le \epsilon$ whenever $|t| \le k$. Setting $v_i = \bar h(u_i)$, $v'_i = \bar g(u_i)$, $v = \langle u_i\rangle_{i\in I}$ and $v' = \langle v'_i\rangle_{i\in I}$, we have $[\![|v_i - v'_i| > \epsilon]\!] \subseteq a_i$ for every i (use 364Ib for a quick check of the calculation). Because T is multiplicative (and $T(\langle\chi 1_{\mathfrak{A}_i}\rangle_{i\in I}) = \chi 1_{\mathfrak{B}}$), $Tv' = \bar g(Tu)$. So

$$\begin{aligned} [\![|Tv - \bar h(Tu)| > 2\epsilon]\!] &\subseteq [\![T|v - v'| > \epsilon]\!] \cup [\![|\bar g(Tu) - \bar h(Tu)| > \epsilon]\!] \\ &\subseteq \pi(\langle[\![|v_i - v'_i| > \epsilon]\!]\rangle_{i\in I}) \cup [\![|Tu| > k]\!] \end{aligned}$$

(using (b))

$$\subseteq \pi(\langle a_i\rangle_{i\in I})$$

(see (a-v) above), which has measure at most ϵ. As ϵ is arbitrary, $Tv = \bar{h}(Tu)$, as claimed.

377C Theorem Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, $(\mathfrak{B},\bar{\nu})$ a probability algebra, and $\pi : \prod_{i\in I}\mathfrak{A}_i \to \mathfrak{B}$ a Boolean homomorphism such that $\bar{\nu}\pi(\langle a_i\rangle_{i\in I}) \le \sup_{i\in I}\bar{\mu}_i a_i$ whenever $\langle a_i\rangle_{i\in I} \in \prod_{i\in I}\mathfrak{A}_i$. Let $W_0 \subseteq \prod_{i\in I} L^0(\mathfrak{A}_i)$ and $T : W_0 \to L^0(\mathfrak{B})$ be as in 377B. Suppose *either* that every $\mathfrak{A}_i$ is atomless *or* that there is an ultrafilter $\mathcal{F}$ on I such that $\bar{\nu}\pi(\langle a_i\rangle_{i\in I}) = \lim_{i\to\mathcal{F}}\bar{\mu}_i a_i$ whenever $\langle a_i\rangle_{i\in I}$ in $\prod_{i\in I}\mathfrak{A}_i$. For $1 \le p \le \infty$ let W_p be the subspace of $\prod_{i\in I} L^0(\mathfrak{A}_i)$ consisting of families $\langle u_i\rangle_{i\in I}$ such that $\sup_{i\in I}\|u_i\|_p$ is finite. Then $T[W_p] \subseteq L^p(\mathfrak{B},\bar{\nu})$, and $\|Tu\|_p \le \sup_{i\in I}\|u_i\|_p$ whenever $u = \langle u_i\rangle_{i\in I}$ belongs to W_p.

proof (a) I should begin by explaining why $W_1 \subseteq W_0$. All we need to observe is that if $u = \langle u_i\rangle_{i\in I}$ belongs to W_1, so that $\gamma = \sup_{i\in I}\|u_i\|_1$ is finite, then

$$\inf_{k\ge1}\sup_{i\in I}\bar{\mu}_i[\![u_i > k]\!] \le \inf_{k\ge1}\frac{\gamma}{k} = 0,$$

so $u \in W_0$. Of course we now have $W_p \subseteq W_1$ for $p \ge 1$, because every $(\mathfrak{A}_i,\bar{\mu}_i)$ is a probability algebra.

(b) I start real work on the proof with a note on the case in which every $\mathfrak{A}_i$ is atomless. Suppose that this is so, and that we are given a family $\langle a_i\rangle_{i\in I} \in \prod_{i\in I}\mathfrak{A}_i$ and $\gamma \in \mathbb{Q}\cap[0,1]$. Then there is a family $\langle a'_i\rangle_{i\in I}$ such that $a'_i \subseteq a_i$ and $\bar{\mu}_i a'_i = \gamma\mu_i a_i$ for every $i \in I$, and

$$\gamma\bar{\nu}\pi(\langle a_i\rangle_{i\in I}) \le \bar{\nu}\pi(\langle a'_i\rangle_{i\in I}).$$

P For each $i \in I$, we can find a non-decreasing family $\langle a_{it}\rangle_{t\in[0,1]}$ in $\mathfrak{A}_i$ such that $a_{i1} = a_i$ and $\bar{\mu}_i a_{it} = t\bar{\mu} a_i$ for every $t \in [0,1]$. Set $b(t) = \pi(\langle a_{it}\rangle_{i\in I})$ and $\beta(t) = \bar{\nu}b(t)$ for $t \in [0,1]$; then $\beta(s) \le \beta(t) \le \beta(s)+t-s$ for $0 \le s \le t \le 1$, because

$$\beta(t)-\beta(s) = \bar{\nu}\pi(\langle a_{it}\setminus a_{is}\rangle_{i\in I}) \le \sup_{i\in I}\bar{\mu}_i(a_{it}\setminus a_{is}) = (t-s)\sup_{i\in I}\bar{\mu}_i a_i \le t-s.$$

Let $n \ge 1$ be such that $\dfrac{1}{n} \le \epsilon$ and $m = n\gamma$ is an integer, and set $\alpha_i = \beta(\frac{i+1}{n}) - \beta(\frac{i}{n})$ for $i < n$; then

$$\sum_{i=0}^{n-1}\alpha_i = \beta(1) = \bar{\nu}b(1).$$

Consider the possible values of $\gamma_K = \sum_{k\in K}\alpha_k$ for sets $K \in [n]^m$. (I am thinking of n as the set $\{0,1,\dots,n-1\}$.) The average value of γ_K over all m-element subsets of n is just $\frac{m}{n}\beta(1) = \gamma\beta(1)$, so there is some K such that $\gamma_K \ge \gamma\beta(1)$.

Set

$$a'_i = \sup_{k\in K} a_{i,(k+1)/n}\setminus a_{i,k/n}$$

for $i \in I$. Then $\bar{\mu}_i a'_i = \gamma\bar{\mu}_i a_i$ for every i, while

$$\bar{\nu}\pi(\langle a'_i\rangle_{i\in I}) = \sup_{k\in K}\bar{\nu}(b(\tfrac{k+1}{n})\setminus b(\tfrac{k}{n})) = \sum_{k\in K}\alpha_k$$

is at least $\gamma\beta(1)$, as required. **Q**

(c) We find now that under either of the hypotheses proposed,

$$\sum_{k=0}^{n}\gamma_k\bar{\nu}\pi(\langle a_{ki}\rangle_{i\in I}) \le \sup_{i\in I}\sum_{k=0}^{n}\gamma_k\mu_i a_{ki}$$

whenever $\gamma_0,\dots,\gamma_n \ge 0$ are rational and $\langle a_{ki}\rangle_{k\le n}$ is a disjoint family in $\mathfrak{A}_i$ for each $i \in I$.

P(i) Consider first the case in which every $\mathfrak{A}_i$ is atomless and every γ_k is between 0 and 1. In this case, given $\epsilon > 0$, (b) above tells us that we can find $a'_{ki} \subseteq a_{ki}$, for $i \in I$ and $k \le n$, such that $\bar{\mu}_i a'_{ki} = \gamma_k\bar{\mu}_i a_{ki}$ and

$$\gamma_k\bar{\nu}\pi(\langle a_{ki}\rangle_{i\in I}) \le \bar{\nu}\pi(\langle a'_{ki}\rangle_{i\in I}).$$

Set $c_i = \sup_{k\le n} a'_{ki}$ for $i \in I$; then

$$\sum_{k=0}^{n}\gamma_k\bar{\nu}\pi(\langle a_{ki}\rangle_{i\in I}) \le \sum_{k=0}^{n}\bar{\nu}\pi(\langle a'_{ki}\rangle_{i\in I}) = \bar{\nu}\pi(\sup_{k\le n}\langle a'_{ki}\rangle_{i\in I}) = \bar{\nu}\pi(\langle c_i\rangle_{i\in I})$$

$$\le \sup_{i\in I}\bar{\mu}_i c_i = \sup_{i\in I}\sum_{k=0}^{n}\bar{\mu}_i a'_{ki} = \sup_{i\in I}\sum_{k=0}^{n}\gamma_k\bar{\mu}_i a_{ki},$$

as required.

(ii) Because T is linear, it follows at once that the result is true for any rational $\gamma_0, \dots, \gamma_n \ge 0$, if every $\mathfrak{A}_i$ is atomless.

(iii) Now consider the case in which there is an ultrafilter $\mathcal{F}$ on I such that $\bar\nu\pi(\langle a_i\rangle_{i\in I}) = \lim_{i\to\mathcal{F}} \bar\mu_i a_i$ for every $\langle a_i\rangle_{i\in I}$. In this case, given $\epsilon > 0$, the set

$$J = \{j : j \in I,\, \bar\nu(\langle a_{ki}\rangle_{i\in I}) \le \bar\mu_j a_{kj} + \epsilon \text{ for every } k \le n\}$$

belongs to $\mathcal{F}$ and is not empty. Take any $j \in J$; then

$$\sum_{k=0}^n \gamma_k \bar\nu\pi(\langle a_{ki}\rangle_{i\in I}) \le \sum_{k=0}^n \gamma_k(\bar\mu_j a_{kj} + \epsilon) \le \epsilon\sum_{k=0}^n \gamma_k + \sup_{i\in I} \sum_{k=0}^n \gamma_k \bar\mu_i a_{ki}.$$

As ϵ is arbitrary, we again have the result. **Q**

(d) Next, $\int Tu \le \sup_{i\in I} \int u_i$ whenever $u = \langle u_i\rangle_{i\in I}$ belongs to W_∞^+. **P** Let $\epsilon > 0$ and let $n \in \mathbb{N}$ be such that $\|u_i\|_\infty \le n\epsilon$ for every $i \in I$. For $i \in I$ and $k \le n$, set $a_{ki} = [\![u_i > k\epsilon]\!] \setminus [\![u_i > (k+1)\epsilon]\!]$; for $i \in I$, set $u'_i = \sum_{k=0}^n k\epsilon\chi a_{ki}$; then $u'_i \le u_i \le u'_i + \epsilon\chi 1_{\mathfrak{A}_i}$. Setting $u' = \langle u'_i\rangle_{i\in I}$, $Tu \le Tu' + \epsilon\chi 1_{\mathfrak{B}}$, so

$$\int Tu - \epsilon \le \int Tu' = \int \sum_{k=0}^n k\epsilon\chi\pi(\langle a_{ki}\rangle_{i\in I})$$

$$= \sum_{k=0}^n k\epsilon\bar\nu\pi(\langle a_{ki}\rangle_{i\in I}) \le \sup_{i\in I} \sum_{k=0}^n k\epsilon\bar\mu_i a_{ki}$$

(by (c))

$$= \sup_{i\in I} \int u'_i \le \sup_{i\in I} \int u_i.$$

As ϵ is arbitrary, we have the result. **Q**

(d) It follows that $Tu \in L^1(\mathfrak{B},\bar\nu)$ and $\int Tu \le \sup_{i\in I} \int u_i$ whenever $u = \langle u_i\rangle_{i\in I}$ belongs to W_1^+ and $u \ge 0$. **P** Set $\gamma = \sup_{i\in I} \int u_i$. Let $\epsilon > 0$. Set $\gamma' = \gamma/\epsilon$. For $i \in I$ set $v_i = u_i \wedge \gamma'\chi 1_{\mathfrak{A}_i}$; set $v = \langle v_i\rangle_{i\in I}$. Then $v \in W_\infty$ and

$$\int Tv \le \sup_{i\in I} \int v_i \le \sup_{i\in I} \int u_i = \gamma$$

by (c) above. Also $[\![Tu - Tv > 0]\!] \subseteq \pi(\langle [\![u_i > \gamma']\!]\rangle_{i\in I})$, by 377Ba, so

$$\bar\nu[\![Tu - Tv > 0]\!] \le \sup_{i\in I} \bar\mu_i[\![u_i > \gamma']\!] \le \epsilon.$$

Thus for each $n \in \mathbb{N}$ we can find a $w_n \in L^\infty(\mathfrak{B})$ such that $0 \le w_n \le Tu$, $\int w_n \le \gamma$ and $\bar\nu[\![Tu - w_n > 0]\!] \le 2^{-n}$. Set $w'_n = \inf_{i\ge n} w_i$ for each n; then $\langle w'_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum Tu in $L^0(\mathfrak{B})$, while $\int w'_n \le \gamma$ for every n. Consequently $Tu \in L^1(\mathfrak{B},\bar\nu)$ and $\int Tu \le \gamma$, as claimed. **Q**

(e) Because T is a Riesz homomorphism, $Tu \in L^1(\mathfrak{B},\bar\nu)$ and $\|Tu\|_1 = \int T|u|$ is at most $\sup_{i\in I} \int |u_i| = \sup_{i\in I} \|u_i\|_1$ for every $u \in W_1$.

(f) Now suppose that $p \in \left]1,\infty\right[$ and that $u = \langle u_i\rangle_{i\in I}$ belongs to W_p. In this case, $\langle |u_i|^p\rangle_{i\in I}$ belongs to W_1, so $T(\langle |u_i|^p\rangle_{i\in I}) \in L^1(\mathfrak{B},\bar\nu)$ and $\int T(\langle |u_i|^p\rangle_{i\in I}) \le \sup_{i\in I} \int |u_i|^p$. By 377Bb, with $h(t) = |t|^p$, $T(\langle |u_i|^p\rangle_{i\in I}) = |Tu|^p$. So $Tu \in L^p(\mathfrak{B},\bar\nu)$ and

$$\|Tu\|_p = (\int |Tu|^p)^{1/p} \le \sup_{i\in I}(\int |u_i|^p)^{1/p} = \sup_{i\in I} \|u_i\|_p$$

as claimed.

377D The original motivation for the work of this section was to understand the function spaces associated with the reduced products of §328. For these we have various simplifications in addition to that observed in 377C.

Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, $\mathcal{F}$ an ultrafilter on I, and $(\mathfrak{B},\bar\nu)$ a probability algebra. Let $\mathfrak{A}$ be the simple product $\prod_{i\in I} \mathfrak{A}_i$ and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I}) = \lim_{i\to\mathcal{F}} \bar\mu_i a_i$ whenever $\langle a_i\rangle_{i\in I} \in \mathfrak{A}$. Let $W_0 \subseteq \prod_{i\in I} L^0(\mathfrak{A}_i)$ and $T : W_0 \to L^0(\mathfrak{B})$ be as in 377B-377C.

(a) If $u = \langle u_i\rangle_{i\in I}$ belongs to W_0 and $\{i : i \in I,\, u_i = 0\} \in \mathcal{F}$, then $Tu = 0$.

(b) For $1 \le p \le \infty$, write W_p for the set of those families $\langle u_i\rangle_{i\in I} \in \prod_{i\in I} L^p(\mathfrak{A}_i,\bar\mu_i)$ such that $\sup_{i\in I} \|u_i\|_p$ is finite. Then $Tu \in L^p(\mathfrak{B},\bar\nu)$ and $\|Tu\|_p \le \lim_{i\to\mathcal{F}} \|u_i\|_p$ whenever $u = \langle u_i\rangle_{i\in I}$ belongs to W_p.

(c) Let W_{ui} be the subspace of $\prod_{i\in I} L^1(\mathfrak{A}_i,\bar\mu_i)$ consisting of families $\langle u_i\rangle_{i\in I}$ such that $\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|u_i|-k\chi 1_{\mathfrak{A}_i})^+=0$. Then $\int Tu=\lim_{i\to\mathcal{F}}\int u_i$ and $\|Tu\|_1=\lim_{i\to\mathcal{F}}\|u_i\|_1$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_{ui}.

(d) Suppose now that $\pi[\mathfrak{A}]=\mathfrak{B}$.

(i) $T[W_0]=L^0(\mathfrak{B})$.

(ii) $T[W_{ui}]=L^1(\mathfrak{B},\bar\nu)$.

(iii) If $p\in[1,\infty]$, then $T[W_p]=L^p(\mathfrak{B},\bar\nu)$ and for every $w\in L^p(\mathfrak{B},\bar\nu)$ there is a $u=\langle u_i\rangle_{i\in I}$ in W_p such that $Tu=w$ and $\sup_{i\in I}\|u_i\|_p=\|w\|_p$.

proof (a) Setting

$$a_i=1_{\mathfrak{A}_i}\text{ if }u_i\ne 0,$$
$$=0\text{ if }u_i=0,$$

$\langle a_i\rangle_{i\in I}\in\mathfrak{A}$ and $\bar\nu\pi(\langle a_i\rangle_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu_i a_i=0$, so $\pi(\langle a_i\rangle_{i\in I})=0$. Accordingly

$$Tu=T(\langle u_i\times\chi a_i\rangle_{i\in I})=Tu\times T(\langle\chi a_i\rangle_{i\in I})=Tu\times\chi\pi(\langle a_i\rangle_{i\in I})=0.$$

(b) Suppose that $u=\langle u_i\rangle_{i\in I}\in W_p$ and that $J\in\mathcal{F}$. Set

$$v_i=u_i\text{ if }i\in J,$$
$$=0\text{ if }i\in I\setminus J;$$

then, putting (a) and 377C together,

$$\|Tu\|_p=\|Tv\|_p\le\sup_{i\in I}\|v_i\|_p=\sup_{i\in J}\|u_i\|_p.$$

As J is arbitrary, $\|Tu\|_p\le\lim_{i\to\mathcal{F}}\|u_i\|_p$.

(c)(i) Clearly W_{ui} is a solid linear subspace of W_1. Suppose that $u=\langle u_i\rangle_{i\in I}\in W_{ui}^+$ and $\epsilon>0$. Let $n\ge 1$ be such that $\int(u_i-n\epsilon\chi 1_{\mathfrak{A}_i})^+\le\epsilon$ for every $i\in I$. For $i\in I$ and $k\le n$, set $a_{ki}=[\![u_i>k\epsilon]\!]$; set $v_i=\sum_{k=1}^n k\epsilon\chi a_{ki}$, so that

$$v_i\le u_i\le v_i+\epsilon\chi 1_{\mathfrak{A}_i}+(u_i-n\epsilon\chi 1_{\mathfrak{A}_i})^+,\quad \int u_i\le\int v_i+2\epsilon.$$

If $v=\langle v_i\rangle_{i\in I}$, then

$$\begin{aligned}\int Tu&=\|Tu\|_1\le\lim_{i\to\mathcal{F}}\|u_i\|_1=\lim_{i\to\mathcal{F}}\int u_i\\&\le 2\epsilon+\lim_{i\to\mathcal{F}}\int v_i=2\epsilon+\sum_{k=1}^n k\epsilon\lim_{i\to\mathcal{F}}\bar\mu_i a_{ki}\\&=2\epsilon+\sum_{k=1}^n k\epsilon\bar\nu\pi(\langle a_{ki}\rangle_{i\in I})=2\epsilon+\int\sum_{k=1}^n k\epsilon\chi\pi(\langle a_{ki}\rangle_{i\in I})\\&=2\epsilon+\int Tv\le 2\epsilon+\int Tu.\end{aligned}$$

As ϵ is arbitrary, $\int Tu=\lim_{i\to\mathcal{F}}\int u_i$.

(ii) It follows at once that $\int Tu=\lim_{i\to\mathcal{F}}\int u_i$ and that

$$\|Tu\|_1=\int|Tu|=\int T|u|=\lim_{i\to\mathcal{F}}\int|u_i|=\lim_{i\to\mathcal{F}}\|u_i\|_1$$

whenever $u=\langle u_i\rangle_{i\in I}\in W_{ui}$.

(d)(i)($\boldsymbol{\alpha}$) Let $T_\pi:L^\infty(\mathfrak{A})\to L^\infty(\mathfrak{B})$ be the Riesz homomorphism associated with the Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$. Since π is surjective, 363Fd tells us that T_π is surjective. Identifying W_∞ with $L^\infty(\mathfrak{A})$, and $T\restriction W_\infty$ with T_π, as in part (a) of the the proof of 377B, we see that $T[W_\infty]=L^\infty(\mathfrak{B})$. Moreover, 363Fd tells us also that if $w\in L^\infty$ there is a $v\in L^\infty(\mathfrak{A})$ such that $T_\pi v=w$ and $\|v\|_\infty=\|w\|_\infty$; translating this into terms of W_∞, we have a $u=\langle u_i\rangle_{i\in I}\in W_\infty$ such that $Tu=w$ and $\sup_{i\in I}\|u_i\|_\infty=\|w\|_\infty$.

It will be useful to know that if $b\in\mathfrak{B}$ and $\epsilon>0$ there is a family $\langle a_i\rangle_{i\in I}\in\mathfrak{A}$ such that $\pi(\langle a_i\rangle_{i\in I})=b$ and $\sup_{i\in I}\bar\mu_i a_i\le\bar\nu b+\epsilon$. **P** By hypothesis, there is a family $\langle a_i'\rangle_{i\in I}\in\mathfrak{A}$ such that $\pi(\langle a_i'\rangle_{i\in I})=b$, and $\bar\nu b=\lim_{i\to\mathcal{F}}\bar\mu_i a_i$. Set

$$a_i = a_i' \text{ if } \bar\mu_i a_i \le \bar\nu b + \epsilon,$$
$$= 0 \text{ for other } i \in I.$$

Then $\lim_{i\to\mathcal{F}} \bar\mu_i(a_i' \triangle a_i) = 0$ so $\pi(\langle a_i' \triangle a_i\rangle_{i\in I}) = 0$ and $\pi(\langle a_i\rangle_{i\in I}) = b$, while $\bar\mu_i a_i \le \bar\nu b + \epsilon$ for every $i \in I$. **Q**

(β) Now suppose that $w \in L^0(\mathfrak{B})^+$. For each $n \in \mathbb{N}$, set $w_n = w \wedge n\chi 1_{\mathfrak{B}}$ and let $u^{(n)} = \langle u_{ni}\rangle_{i\in I} \in W_\infty$ be such that $Tu^{(n)} = w_{n+1} - w_n$ and $\|u_{ni}\|_\infty \le 1$ for every $i \in I$. Next, for each n, set $b_n = [\![w_{n+1} - w_n > 0]\!]$, and let $\langle a_{ni}\rangle_{i\in I} \in \mathfrak{A}$ be such that $\pi(\langle a_{ni}\rangle_{i\in I}) = b_n$ and $\sup_{i\in I} \bar\mu_i a_{ni} \le \bar\nu b_n + 2^{-n}$. If we set $a'_{ni} = \inf_{m\le n} a_{mi}$ and $u'_{ni} = u_{ni} \times \chi a'_{ni}$, we shall have

$$T(\langle u'_{ni}\rangle_{i\in I}) = T(\langle u_{ni}\rangle_{i\in I}) \times \chi\pi(\langle a'_{ni}\rangle_{i\in I})$$
$$= (w_{n+1} - w_n) \times \inf_{m\le n} \chi b_m = w_{n+1} - w_n$$

for every n. Also, for each $i \in I$, $\langle a'_{ni}\rangle_{n\in\mathbb{N}}$ is non-increasing and

$$\lim_{n\to\infty} \bar\mu_i a'_{ni} \le \lim_{n\to\infty} \bar\nu b_n + 2^{-n} = 0.$$

So $v_i = \sup_{n\in\mathbb{N}} \sum_{m=0}^n u'_{ni}$ is defined in $L^0(\mathfrak{A}_i)$, and

$$\inf_{k\in\mathbb{N}} \sup_{i\in I} \bar\mu_i [\![v_i > k]\!] \le \inf_{k\in\mathbb{N}} \sup_{i\in I} \bar\mu_i a'_{ki} = 0.$$

Thus $v = \langle v_i\rangle_{i\in I}$ belongs to W_0 and we can speak of Tv. Of course

$$Tv \ge \sum_{m=0}^n T(\langle u'_{ni}\rangle_{i\in I}) = w_{n+1}$$

for every n, so $Tv \ge w$. On the other hand, for any $n \in \mathbb{N}$,

$$[\![v_i - \sum_{m=0}^n u'_{ni} > 0]\!] \subseteq a'_{ni}$$

for every i, so $[\![Tv - w_{n+1} > 0]\!] \subseteq b_n$, by 377B; as $\inf_{n\in\mathbb{N}} b_n = 0$, $Tv = \sup_{n\in\mathbb{N}} w_n = w$.

(γ) Thus $T[W_0] \supseteq L^0(\mathfrak{B})^+$; as T is linear, $T[W_0] = L^0(\mathfrak{B})$.

(ii) Now suppose that $w \in L^1(\mathfrak{B}, \bar\nu)^+$. In this case, repeat the process of (i-β) above. This time, observe that as $\chi b_{n+1} \le w_{n+1} - w_n$ for every n, $\sum_{n=0}^\infty \bar\nu b_n \le 1 + \int w$ is finite. Consequently, in the first place,

$$\sum_{n=0}^\infty \int u'_{ni} \le \sum_{n=0}^\infty \bar\mu_i a_{ni} \le \sum_{n=0}^\infty \bar\nu b_n + 2^{-n}$$

is finite, and $v_i \in L^1(\mathfrak{A}_i, \bar\mu_i)$, for every $i \in I$. But also, for any $k \in \mathbb{N}$ and $i \in I$,

$$\int (v_i - k\chi 1_{\mathfrak{A}_i})^+ \le \sum_{n=k}^\infty \int u'_{ni} \le \sum_{n=k}^\infty \bar\nu b_n + 2^{-n} \to 0$$

$k \to \infty$. So $v \in W_{ui}$ and $w \in T[W_{ui}]$. Because W_{ui} is a linear subspace of W_0, $T[W_{ui}] = L^1(\mathfrak{B}, \bar\nu)$.

(iii)(α) If $p = \infty$ the result has already been dealt with in (i-α) above.

(β) For the case $p = 1$, take $w \in L^1(\mathfrak{B}, \bar\nu)$. Let $v = \langle v_i\rangle_{i\in I} \in W_{ui}$ be such that $Tv = w$. For $i \in I$ set

$$u_i = \frac{\|w\|_1}{\|v_i\|_1} v_i \text{ if } \|v_i\|_1 > \|w\|_1,$$
$$= v_i \text{ otherwise.}$$

Then

$$\bar\mu [\![(|u_i| - k > 0]\!] \le \bar\mu [\![|v_i| - k > 0]\!]$$

for all $k \in \mathbb{N}$ and $i \in I$, so $u = \langle u_i\rangle_{i\in I} \in W_{ui}$. Since $\lim_{i\to\mathcal{F}} \|v_i\|_1 = \|w\|_1$, by (c) above, $\lim_{i\to\mathcal{F}} \|u_i - v_i\|_1 = 0$ and $Tu = Tv = w$, by (b). And of course $\|u_i\|_1 \le \|w\|_1$ for every i.

(γ) Now suppose that $1 < p < \infty$ and that $w \in L^p(\mathfrak{B}, \bar\nu)$. By ($\beta$), there is a $v = \langle v_i\rangle_{i\in I} \in W_1$ such that $Tv = |w|^p$ and $\sup_{i\in I} \|v_i\|_1 = \|w\|_p^p$. Set $v_i' = |v_i|^{1/p}$ for each i; then $v' = \langle v_i'\rangle_{i\in I} \in W_p$ and $Tv' = |w|$, by 377Bb. Next, w is expressible as $|w| \times \tilde w$, where $\tilde w \in L^\infty(\mathfrak{B})$ and $\|\tilde w\|_\infty \le 1$. There is a $\tilde v = \langle \tilde v_i\rangle_{i\in I} \in W_\infty$ such that $T\tilde v = \tilde w$ and $\sup_{i\in I} \|\tilde v_i\|_\infty = 1$. Set $u_i = v_i' \times \tilde v_i$ for each i; then $u = \langle u_i\rangle_{i\in I}$ belongs to W_p, $\|u_i\|_p \le \|w\|_p$ for every i, and $Tu = w$.

377E Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be probability algebras, I a set and $\mathcal{F}$ an ultrafilter on I. Let $\pi:\mathfrak{A}^I\to\mathfrak{B}$ be a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu a_i$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}^I$. Let W_0 be the set of families in $L^0(\mathfrak{A})^I$ which are bounded for the topology of convergence in measure on $L^0(\mathfrak{A})$.

(a)(i) W_0 is a solid linear subspace and a subalgebra of $L^0(\mathfrak{A})^I$, and there is a unique multiplicative Riesz homomorphism $T:W_0\to L^0(\mathfrak{B})$ such that $T(\langle\chi a_i\rangle_{i\in I})=\chi\pi(\langle a_i\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}^I$.

(ii) $[\![Tu>0]\!]\subseteq\pi(\langle[\![u_i>0]\!]\rangle_{i\in I})$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_0.

(iii) If $h:\mathbb{R}\to\mathbb{R}$ is a continuous function, and we write $\bar h$ for the corresponding maps from L^0 to itself for either of the spaces $L^0=L^0(\mathfrak{A})$, $L^0=L^0(\mathfrak{B})$, then $\langle\bar h(u_i)\rangle_{i\in I}\in W_0$ and $T(\langle\bar h(u_i)\rangle_{i\in I})=\bar h(Tu)$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_0.

(b)(i) For $1\le p\le\infty$ let W_p be the subspace of $L^p(\mathfrak{A},\bar\mu)^I$ consisting of $\|\,\|_p$-bounded families. Then $T[W_p]\subseteq L^p(\mathfrak{B},\bar\nu)$, and $\|Tu\|_p\le\lim_{i\to\mathcal{F}}\|u_i\|_p$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_p.

(ii) Let W_{ui} be the subspace of $L^1(\mathfrak{A}_i,\bar\mu_i)^I$ consisting of uniformly integrable families. Then $\int Tu=\lim_{i\to\mathcal{F}}\int u_i$ and $\|Tu\|_1=\lim_{i\to\mathcal{F}}\|u_i\|_1$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_{ui}.

(c)(i) We have a measure-preserving Boolean homomorphism $\tilde\pi:\mathfrak{A}\to\mathfrak{B}$ defined by setting $\tilde\pi a=\pi(\langle a\rangle_{i\in I})$ for each $a\in\mathfrak{A}$.

(ii) Let $P_{\tilde\pi}:L^1(\mathfrak{B},\bar\nu)\to L^1(\mathfrak{A},\bar\mu)$ be the conditional-expectation operator corresponding to $\tilde\pi:\mathfrak{A}\to\mathfrak{B}$ (365P). If $\langle u_i\rangle_{i\in I}$ is a uniformly integrable family in $L^1(\mathfrak{A})$, then $P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})$ is the limit $\lim_{i\to\mathcal{F}}u_i$ for the weak topology of $L^1(\mathfrak{A},\bar\mu)$.

(iii) Suppose that $1<p<\infty$ and that $\langle u_i\rangle_{i\in I}$ is a bounded family in $L^p(\mathfrak{A},\bar\mu)$. Then $P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})$ is the limit $\lim_{i\to\mathcal{F}}u_i$ for the weak topology of $L^p(\mathfrak{A},\bar\mu)$.

proof (a) The point is that a family $\langle u_i\rangle_{i\in I}$ in $L^0(\mathfrak{A})$ is bounded for the topology of convergence in measure iff $\inf_{k\in\mathbb{N}}\sup_{i\in I}\bar\mu[\![|u_i|>k]\!]=0$. **P** (i) If $\langle u_i\rangle_{i\in I}$ is bounded in this sense, take any $\epsilon>0$. Then $G=\{u:u\in L^0(\mathfrak{A}),\ \bar\mu[\![|u|>1]\!]\le\epsilon\}$ is a neighbourhood of 0 in $L^0(\mathfrak{A})$, so there is a $k\in\mathbb{N}$ such that $u_i\in kG$, that is, $\bar\mu[\![|u_i|>k]\!]\le\epsilon$, for every $i\in I$. So $\{u_i:i\in I\}$ satisfies the condition. (ii) If $\{u_i:i\in I\}$ satisfies the condition, and G is a neighbourhood of 0 in $L^0(\mathfrak{A})$, then there is an $\epsilon>0$ such that G includes $\{u:\bar\mu[\![|u|>\epsilon]\!]\le\epsilon\}$ (367L). Now there is a $k\in\mathbb{N}$ such that $\bar\mu[\![|u_i|>k]\!]\le\epsilon$ for every $i\in I$, in which case, setting $n=\lceil k/\epsilon\rceil$, we have $u_i\in nG$ for every $i\in I$. As G is arbitrary, A is bounded. **Q**

So we just have a special case of 377B.

(b) Similarly, the condition '$\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|u_i|-k\chi 1_{\mathfrak{A}_i})^+=0$' translates into '$\{u_i:i\in I\}$ is uniformly integrable' (cf. 246Bd), so we are looking at a special case of 377Db-377Dc.

(c)(i) $\tilde\pi$ is a Boolean homomorphism just because the function taking $a\in\mathfrak{A}$ into the constant family with value a is a Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{A}^I$. The formula '$\bar\nu\pi(\langle a_i\rangle_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu a_i$' now ensures that $\tilde\pi$ is measure-preserving.

(ii) By the defining formula for $P_{\tilde\pi}$ (365Pa),

$$\begin{aligned}\int_a P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})&=\int T(\langle u_i\rangle_{i\in I})\times\chi\tilde\pi(a)=\int T(\langle u_i\rangle_{i\in I})\times\chi\pi(\langle a\rangle_{i\in I})\\&=\int T(\langle u_i\rangle_{i\in I})\times T(\langle\chi a\rangle_{i\in I})\\&=\int T(\langle u_i\times\chi a\rangle_{i\in I})=\lim_{i\to\mathcal{F}}\int u_i\times\chi a\end{aligned}$$

(because $\{u_i\times\chi a:i\in I\}$ is uniformly integrable)

$$=\lim_{i\to\mathcal{F}}\int_a u_i$$

for every $a\in\mathfrak{A}$. It follows that $P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})=\lim_{i\to\mathcal{F}}u_i$. **P** We have

$$\int P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})\times v=\lim_{i\to\mathcal{F}}\int u_i\times v$$

whenever $v=\chi a$, for any $a\in\mathfrak{A}$; by linearity, whenever $v\in S(\mathfrak{A})$, the space of $\mathfrak{A}$-simple functions; and by continuity, whenever $v\in L^\infty(\mathfrak{A})$ (because $\{u_i:i\in I\}$ is $\|\,\|_1$-bounded, and $S(\mathfrak{A})$ is $\|\,\|_\infty$-dense in $L^\infty(\mathfrak{A})$). Since $L^\infty(\mathfrak{A})$ can be identified with the dual of $L^1(\mathfrak{A},\bar\mu)$ (365Mc), we have the required weak convergence. **Q**

(iii) If $\{u_i:i\in I\}$ is $\|\,\|_p$-bounded, where $1<p<\infty$, then it is uniformly integrable. **P** Set $q=\frac{p}{p-1}$. If $k\ge1$,

$$\inf_{k\ge 1}\sup_{i\in I}\int(|u_i|-k\chi 1_{\mathfrak{A}})^+\le\inf_{k\ge 1}\frac{1}{k^{p-1}}\sup_{i\in I}\|u_i\|_p^p=0.\ \mathbf{Q}$$

So

$$\int P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})\times v=\lim_{i\to\mathcal{F}}\int u_i\times v$$

for every $v\in S(\mathfrak{A})$, and therefore for every $v\in L^q(\mathfrak{A},\bar\mu)$, since v can be $\|\,\|_q$-approximated by members of $S(\mathfrak{A})$ (366C). Since $L^q(\mathfrak{A},\bar\mu)$ can be identified with $L^p(\mathfrak{A},\bar\mu)^*$, we again have weak convergence.

377F Finally, I come to a result which depends on the special properties of reduced products of probability algebras.

Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{A}',\bar\mu')$ be probability algebras, I a set and $\mathcal{F}$ an ultrafilter on I; let $(\mathfrak{B},\bar\nu)$ and $(\mathfrak{B}',\bar\nu')$ be the reduced powers $(\mathfrak{A},\bar\mu)^I|\mathcal{F}$, $(\mathfrak{A}',\bar\mu')^I|\mathcal{F}$ as described in 328A-328C, with corresponding homomorphisms $\pi:\mathfrak{A}^I\to\mathfrak{B}$ and $\pi':\mathfrak{A}'^I\to\mathfrak{B}'$.

(a) Writing W_0, W_0' for the spaces of topologically bounded families in $L^0(\mathfrak{A})^I$, $L^0(\mathfrak{A}')^I$ respectively, we have unique Riesz homomorphisms $T:W_0\to L^0(\mathfrak{B})$ and $T':W_0'\to L^0(\mathfrak{B}')$ such that $T(\langle\chi a_i\rangle_{i\in I})=\chi\pi(\langle a_i\rangle_{i\in I})$, $T'(\langle\chi a_i'\rangle_{i\in I})=\chi\pi'(\langle a_i'\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}^I$ and $\langle a_i'\rangle_{i\in I}\in(\mathfrak{A}')^I$.

(b) Suppose that $S:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{A}',\bar\mu')$ is a bounded linear operator. Then we have a unique bounded linear operator $\hat S:L^1(\mathfrak{B},\bar\nu)\to L^1(\mathfrak{B}',\bar\nu')$ such that $\hat ST(\langle u_i\rangle_{i\in I})=T'(\langle Su_i\rangle_{i\in I})$ whenever $\langle u_i\rangle_{i\in I}$ is a uniformly integrable family in $L^1(\mathfrak{A},\bar\mu)$.

(c) The map $S\mapsto\hat S$ is a norm-preserving Riesz homomorphism from $\mathsf{B}(L^1(\mathfrak{A},\bar\mu);L^1(\mathfrak{A}',\bar\mu'))$ to $\mathsf{B}(L^1(\mathfrak{B},\bar\nu);L^1(\mathfrak{B}',\bar\nu'))$.

proof (a) Once again, this is nothing but a specialization of the corresponding fragments of 377Ba and 377Ea.

(b) Write W_{ui} for the space of uniformly integrable families in $L^1_{\bar\mu}=L^1(\mathfrak{A},\bar\mu)$. If $\langle u_i\rangle_{i\in I}\in W_{ui}$, then $\langle Su_i\rangle_{i\in I}$ is uniformly integrable in $L^1_{\bar\mu'}=L^1(\mathfrak{A}',\bar\mu')$ (because $\{u_i:i\in I\}$ and $\{Su_i:i\in I\}$ are relatively weakly compact, as in 247D), so belongs to W_0', and we can speak of $T'(\langle Su_i\rangle_{i\in I})$. If moreover $T(\langle u_i\rangle_{i\in I})=0$, then $\lim_{i\to\mathcal{F}}\|u_i\|_1=0$ (377E(b-ii)), so $\lim_{i\to\mathcal{F}}\|Su_i\|_1=0$ and $T'(\langle Su_i\rangle_{i\in I})=0$. Finally, $T[W_{ui}]=L^1_{\bar\nu}=L^1(\mathfrak{B},\bar\nu)$ by 377D(d-ii). So the given formula defines a linear operator $\hat S:L^1_{\bar\nu}\to L^1_{\bar\nu'}=L^1(\mathfrak{B}',\bar\nu')$. Next, if $w\in L^1_{\bar\nu}$, we can take any family $\langle u_i\rangle_{i\in I}\in W_{ui}$ such that $T(\langle u_i\rangle_{i\in I})=w$, and

$$\|\hat Sw\|_1=\|T'(\langle Su_i\rangle_{i\in I})\|_1=\lim_{i\to\mathcal{F}}\|Su_i\|_1$$

(377E(b-ii))

$$\le\|S\|\lim_{i\to\mathcal{F}}\|u_i\|_1=\|S\|\|w\|_1.$$

As w is arbitrary, $\hat S$ is a bounded linear operator, and $\|\hat S\|\le\|S\|$. On the other hand, if $u\in L^1_{\bar\mu}$ and $\|u\|_1\le 1$, $\|T(\langle u\rangle_{i\in I})\|_1\le 1$ so

$$\|\hat S\|\ge\|\hat ST(\langle u\rangle_{i\in I})\|=\|T'(\langle Su\rangle_{i\in I})\|_1=\|Su\|_1;$$

as u is arbitrary, $\|\hat S\|\ge\|S\|$.

(c)(i) Recall from 371D that the Banach space $\mathsf{B}(L^1_{\bar\mu};L^1_{\bar\mu'})$ of continuous linear operators is also the Dedekind complete Riesz space $L^\sim(L^1_{\bar\mu};L^1_{\bar\mu'})$ of order-bounded linear operators, and its norm is a Riesz norm; similarly, $\mathsf{B}(L^1_{\bar\nu};L^1_{\bar\nu'})=L^\sim(L^1_{\bar\nu};L^1_{\bar\nu'})$. We have already seen that $S\mapsto\hat S$ is norm-preserving, and it is clearly linear. If $w\in(L^1_{\bar\nu})^+$, then, by 377D(d-ii), $w=T(\langle u_i\rangle_{i\in I})$ for a family $\langle u_i\rangle_{i\in I}\in W_{ui}$; since T is a Riesz homomorphism, $w=w^+=T(\langle u_i^+\rangle_{i\in I}$; so that if $S\ge 0$ we shall have $\hat Sw=T'(\langle Su_i^+\rangle_{i\in I})\ge 0$. This shows that $\hat S\ge 0$ whenever $S\ge 0$, so that $S\mapsto\hat S$ is a positive linear operator.

(ii) To show that $S\mapsto\hat S$ is a Riesz homomorphism, I argue as follows. Take any bounded linear operator $S:L^1_{\bar\mu}\to L^1_{\bar\mu'}$ and $\epsilon>0$. Then

$$B=\{\sum_{k=0}^n|Sv_k|:v_0,\dots,v_k\in(L^1_{\bar\mu})^+,\ \sum_{k=0}^n v_k=\chi 1_{\mathfrak{A}}\}$$

is an upwards-directed set in $L^1_{\bar\mu'}$ with supremum $|S|(\chi 1_{\mathfrak{A}})$ (371A, part (b) of the proof of 371B). So we can find $v_0,\dots,v_n\in(L^1_{\bar\mu})^+$ such that $\sum_{k=0}^n v_k=\chi 1_{\mathfrak{A}}$ and $\|v'\|_1\le\epsilon$, where $v'=|S|(\chi 1_{\mathfrak{A}})-\sum_{k=0}^n|Sv_k|\ge 0$.

Next, if $0 \le u \le \chi 1_{\mathfrak{A}}$ in $L^1_{\bar\mu}$, set $u' = \chi 1_{\mathfrak{A}} - u$; we have

$$\begin{aligned}|S|(\chi 1_{\mathfrak{A}}) - v' &= \sum_{k=0}^{n} |Sv_k| \le \sum_{k=0}^{n} |S(u \times v_k)| + \sum_{k=0}^{n} |S(u' \times v_k)| \\ &\le |S|(u) + |S|(u') = |S|(\chi 1_{\mathfrak{A}}).\end{aligned}$$

So $|S|(u) - \sum_{k=0}^{n} |S(u \times v_k)| \le v'$ and $\| |S|(u) - \sum_{k=0}^{n} |S(u \times v_k)| \|_1 \le \epsilon$.

Now take any $w \in L^1_{\bar\nu}$ such that $0 \le w \le \chi 1_{\mathfrak{B}}$. Again because T is a Riesz homomorphism and $T(\langle \chi 1_{\mathfrak{A}} \rangle_{i\in I}) = \chi 1_{\mathfrak{B}}$, we can express w as $T(\langle u_i \rangle_{i\in I})$ where $0 \le u_i \le \chi 1_{\mathfrak{A}}$ for every i. Consequently, setting $v_i' = |S|u_i - \sum_{k=0}^{n} |S(u_i \times v_k)|$ for each i, and $w' = T'(\langle v_i' \rangle_{i\in I})$,

$$\begin{aligned}|S|^\frown(w) &= T'(\langle |S|u_i \rangle_{i\in I}) = T'(\langle \sum_{k=0}^{n} |S(u_i \times v_k)| + v_i' \rangle_{i\in I}) \\ &= \sum_{k=0}^{n} |T'(\langle S(u_i \times v_k) \rangle_{i\in I})| + T'(\langle v_i' \rangle_{i\in I}) \\ &= \sum_{k=0}^{n} |\hat S T(\langle u_i \times v_k \rangle_{i\in I})| + w' = \sum_{k=0}^{n} |\hat S\bigl(T(\langle u_i \rangle_{i\in I}) \times T(\langle v_k \rangle_{i\in I})\bigr)| + w' \\ &\le \sum_{k=0}^{n} |\hat S|\bigl(T(\langle u_i \rangle_{i\in I}) \times T(\langle v_k \rangle_{i\in I})\bigr) + w' = |\hat S|(w) + w'\end{aligned}$$

because

$$\textstyle\sum_{k=0}^{n} T(\langle v_k \rangle_{i\in I}) = T(\langle \chi 1_{\mathfrak{A}} \rangle_{i\in I}) = \chi 1_{\mathfrak{B}}.$$

But we also have $\|w'\|_1 = \lim_{i\to\mathcal{F}} \|v_i'\|_1 \le \epsilon$, while $|\hat S| \le |S|^\frown$. So we conclude that $\| |S|^\frown(w) - |\hat S|(w) \|_1 \le \epsilon$; as ϵ is arbitrary, $|S|^\frown(w) = |\hat S|(w)$.

This is true whenever $0 \le w \le \chi 1_{\mathfrak{B}}$. But as both $|S|^\frown$ and $|\hat S|$ are continuous linear operators, and $L^\infty(\mathfrak{B})$ is dense in $L^1_{\bar\nu}$, $|S|^\frown = |\hat S|$. As S is arbitrary, we have a Riesz homomorphism (352G).

377G Projective limits: Proposition Let $(I, \le)$, $\langle (\mathfrak{A}_i, \bar\mu_i) \rangle_{i\in I}$ and $\langle \pi_{ij} \rangle_{i\le j}$ be such that $(I, \le)$ is a non-empty upwards-directed partially ordered set, every $(\mathfrak{A}_i, \bar\mu_i)$ is a probability algebra, $\pi_{ij} : \mathfrak{A}_j \to \mathfrak{A}_i$ is a measure-preserving Boolean homomorphism whenever $i \le j$ in I, and $\pi_{ik} = \pi_{ij}\pi_{jk}$ whenever $i \le j \le k$. Let $(\mathfrak{C}, \bar\lambda, \langle \pi_i \rangle_{i\in I})$ be the corresponding projective limit (328I). Write $L^1_{\bar\mu_i}$ for $L^1(\mathfrak{A}_i, \bar\mu_i)$ and $L^1_{\bar\lambda}$ for $L^1(\mathfrak{C}, \bar\lambda)$. For $i \le j$ in I, let $T_{ij} : L^1_{\bar\mu_j} \to L^1_{\bar\mu_i}$ and $P_{ij} : L^1_{\bar\mu_i} \to L^1_{\bar\mu_j}$ be the norm-preserving Riesz homomorphism and the positive linear operator corresponding to $\pi_{ij} : \mathfrak{A}_j \to \mathfrak{A}_i$ (365O, 365P), and $T_i : L^1_{\bar\lambda} \to L^1_{\bar\mu_i}$, $P_i : L^1_{\bar\mu_i} \to L^1_{\bar\lambda}$ the operators corresponding to $\pi_i : \mathfrak{C} \to \mathfrak{A}_i$. Let X be any set.

(a) Suppose that for each $i \in I$ we are given a function $S_i : L^1_{\bar\mu_i} \to X$ such that $S_i T_{ij} = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : L^1_{\bar\lambda} \to X$ such that $S = S_i T_i$ for every $i \in I$.

(b) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar\mu_i}$ such that $T_{ij} S_j = S_i$ whenever $i \le j$ in I. Then there is a unique function $S : X \to L^1_{\bar\lambda}$ such that $T_i S = S_i$ for every $i \in I$.

(c) Suppose that X is a topological space, and for each $i \in I$ we are given a norm-continuous function $S_i : L^1_{\bar\mu_i} \to X$ such that $S_j P_{ij} = S_i$ whenever $i \le j$ in I. Then there is a unique function $S : L^1_{\bar\lambda} \to X$ such that $SP_i = S_i$ for every $i \in I$.

(d) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar\mu_i}$ such that $P_{ij} S_i = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : X \to L^1_{\bar\lambda}$ such that $S = P_i S_i$ for every $i \in I$.

proof: preliminary remarks (i) It will be helpful to recall some basic facts from §§328 and 365. If $i \le j$ in I, then by the definition of 'projective limit' we have $\pi_{ij}\pi_j = \pi_i$ so $T_{ij} T_j = T_i$ and $P_j P_{ij} = P_i$. Also $P_{ij} T_{ij}$ is the identity operator on $L^1_{\bar\mu_j}$, and $P_i T_i$ is the identity operator on $L^1_{\bar\lambda}$.

(ii) At a deeper level, we have useful concretizations of $(\mathfrak{C}, \bar\lambda)$, as follows. Fix $i \in I$ for the moment. For $j \ge i$, set $\mathfrak{B}_j = \pi_{ij}[\mathfrak{A}_j]$, $\bar\nu_j = \bar\mu_i \restriction \mathfrak{B}_j$; then $\mathfrak{B}_j$ is a closed subalgebra of $\mathfrak{A}_i$, isomorphic (as probability algebra) to $\mathfrak{A}_j$. If $u \in L^1_{\bar\mu_i}$ and $b \in \mathfrak{B}_j$, set $b' = \pi_{ij}^{-1} b \in \mathfrak{A}_j$; then

$$\int_b u = \int_{\pi_{ij}b'} u = \int_{b'} P_{ij}u = \int_b T_{ij}P_{ij}u;$$

thus $T_{ij}P_{ij}$ is the conditional expectation $P_{\mathfrak{B}_j} : L^1_{\bar\mu_i} \to L^1(\mathfrak{B}_j, \bar\nu_j)$, identifying $L^1(\mathfrak{B}_j, \bar\nu_j)$ with $L^1_{\bar\mu_i} \cap L^0(\mathfrak{B}_j)$ as in 365Ra.

If $k \ge j$, then $\pi_{ik} = \pi_{ij}\pi_{jk}$ so $\mathfrak{B}_k \subseteq \mathfrak{B}_j$; thus $\mathbb{B} = \{\mathfrak{B}_j : j \ge i\}$ is downwards-directed. Set $\mathfrak{D} = \bigcap \mathbb{B}$, $\bar\nu = \bar\mu \restriction \mathfrak{D}$.

For $k \ge i$, set $\phi_k = \pi_{ik}^{-1} \restriction \mathfrak{D} : \mathfrak{D} \to \mathfrak{A}_k$; then ϕ_k is a measure-preserving Boolean homomorphism, and $\phi_j = \pi_{jk}\phi_k$ whenever $i \le j \le k$. We can therefore define $\phi_j : \mathfrak{D} \to \mathfrak{A}_j$, for any $j \in I$, by saying that $\phi_j = \pi_{jk}\phi_k$ whenever $k \in I$ is greater than or equal to both i and j, and we shall have $\phi_j = \pi_{jk}\phi_k$ whenever $j \le k$ in I. **P** If $j \in I$ and k_0, k_1 are two upper bounds of $\{i, j\}$ in I, take an upper bound k of $\{k_0, k_1\}$; then

$$\pi_{jk_0}\phi_{k_0} = \pi_{jk_0}\pi_{k_0k}\phi_k = \pi_{jk}\phi_k = \pi_{jk_1}\pi_{k_1k}\phi_k = \pi_{jk_1}\phi_{k_1},$$

so ϕ_j is well-defined. If j, $k \in I$ and $j \le k$, let k' be an upper bound of $\{i, k\}$; then

$$\pi_{jk}\phi_k = \pi_{jk}\pi_{kk'}\phi_{k'} = \pi_{jk'}\phi_{k'} = \phi_j. \text{ **Q**}$$

Of course every ϕ_j is a measure-preserving Boolean homomorphism.

By the definition of $(\mathfrak{C}, \bar\lambda)$, there is a measure-preserving Boolean homomorphism $\phi : \mathfrak{D} \to \mathfrak{C}$ such that $\pi_j\phi = \phi_j$ for every $j \in I$. In this case, $\pi_i\phi = \phi_i$ is the identity embedding of $\mathfrak{D}$ in $\mathfrak{A}_i$, and $\pi_i[\mathfrak{C}] = \mathfrak{D}$. Accordingly $P_{\mathfrak{D}} = T_iP_i$. By the generalized reverse martingale theorem 367Qa, T_iP_i is the limit of $P_{\mathfrak{B}}$ as $\mathfrak{B}$ decreases in $\mathbb{B}$, in the sense that for every $u \in L^1(\mathfrak{A}_i)$ and $\epsilon > 0$ there is a $j \ge i$ in I such that

$$\|T_iP_iu - T_{ik}P_{ik}u\|_1 = \|P_{\mathfrak{D}}u - P_{\mathfrak{B}_k}u\|_1 \le \epsilon$$

whenever $k \ge j$ in I. If we write $\mathcal{F}(I\uparrow)$ for the filter on I generated by $\{\{k : k \ge j\} : j \in I\}$, we have

$$T_iP_iu = \lim_{j \to \mathcal{F}(I\uparrow)} T_{ij}P_{ij}u,$$

for the norm in $L^1_{\bar\mu_i}$, for every $u \in L^1_{\bar\mu_i}$.

Now let us turn to (a)-(d) as listed above.

(a) All we have to know is that

$$S_iT_i = S_iT_{ij}T_j = S_jT_j$$

whenever $i \le j$ in I; because I is upwards-directed, $S_iT_i = S_jT_j$ for all i, $j \in I$, and we have a sound definition for S.

(b) The point is that $T_iP_iS_i = S_i$ for every $i \in I$. **P** For $j \ge i$,

$$T_{ij}P_{ij}S_i = T_{ij}P_{ij}T_{ij}S_j = T_{ij}S_j = S_i.$$

If $x \in X$,

$$T_iP_iS_ix = \lim_{j \to \mathcal{F}(I\uparrow)} T_{ij}P_{ij}S_ix = S_ix. \text{ **Q**}$$

If now $i \le j$ in I,

$$P_iS_i = P_jP_{ij}T_{ij}S_j = P_jS_j.$$

As I is upwards-directed, $P_iS_i = P_jS_j$ for all i, $j \in I$; write S for this common value. Then

$$T_iS = T_iP_iS_i = S_i$$

for every $i \in I$. As T_i is injective for every $i \in I$, the formula uniquely defines the function S.

(c) This time, we have $S_iT_iP_i = S_i$ for every $i \in I$. **P** For any $u \in L^1_{\bar\lambda}$,

$$S_iT_iP_iu = \lim_{j \to \mathcal{F}(I\uparrow)} S_iT_{ij}P_{ij}u$$

(because S_i is continuous)

$$= \lim_{j \to \mathcal{F}(I\uparrow)} S_jP_{ij}T_{ij}P_{ij}u = \lim_{j \to \mathcal{F}(I\uparrow)} S_jP_{ij}u = S_iu. \text{ **Q**}$$

If $i \le j$ in I,

$$S_iT_i = S_jP_{ij}T_{ij}T_j = S_jT_j;$$

consequently $S_iT_i = S_jT_j$ for all i, $j \in I$, and we can call this common function S. In this case, $SP_i = S_iT_iP_i = S_i$ for every $i \in I$. Since $P_i[L^1_{\bar\mu_i}] = L^1_{\bar\lambda}$, this defines S uniquely.

(d) As in (a), all we have to check is that if $i \le j$ in I then

$$P_jS_j = P_jP_{ij}S_i = P_iS_i.$$

377H Inductive limits: Proposition Let $(I, \le)$, $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$ and $\langle\pi_{ji}\rangle_{i\le j}$ be such that $(I, \le)$ is a non-empty upwards-directed partially ordered set, every $(\mathfrak{A}_i, \bar\mu_i)$ is a probability algebra, $\pi_{ji} : \mathfrak{A}_i \to \mathfrak{A}_j$ is a measure-preserving Boolean homomorphism whenever $i \le j$ in I, and $\pi_{ki} = \pi_{kj}\pi_{ji}$ whenever $i \le j \le k$. Let $(\mathfrak{C}, \bar\lambda, \langle\pi_i\rangle_{i\in I})$ be the corresponding inductive limit (328H). Write $L^1_{\bar\mu_i}$ for $L^1(\mathfrak{A}_i, \bar\mu_i)$ and $L^1_{\bar\lambda}$ for $L^1(\mathfrak{C}, \bar\lambda)$. For $i \le j$ in I, let $T_{ji} : L^1_{\bar\mu_i} \to L^1_{\bar\mu_j}$ and $P_{ji} : L^1_{\bar\mu_j} \to L^1_{\bar\mu_i}$ be the Riesz homomorphism and the positive linear operator corresponding to $\pi_{ji} : \mathfrak{A}_i \to \mathfrak{A}_j$, and $T_i : L^1_{\bar\mu_i} \to L^1_{\bar\lambda}$, $P_i : L^1_{\bar\lambda} \to L^1_{\bar\mu_i}$ the operators corresponding to $\pi_i : \mathfrak{A}_i \to \mathfrak{C}$. Let X be a set.

(a) Suppose that for each $i \in I$ we are given a function $S_i : L^1_{\bar\mu_i} \to X$ such that $S_jT_{ji} = S_i$ whenever $i \le j$ in I. Then there is a function $S : L^1_{\bar\lambda} \to X$ such that $S_i = ST_i$ for every $i \in I$.

(b) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar\mu_i}$ such that $T_{ji}S_i = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : X \to L^1_{\bar\lambda}$ such that $T_iS_i = S$ for every $i \in I$.

(c) Suppose that for each $i \in I$ we are given a function $S_i : L^1_{\bar\mu_i} \to X$ such that $S_iP_{ji} = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : L^1_{\bar\lambda} \to X$ such that $S = S_iP_i$ for every $i \in I$.

(d) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar\mu_i}$ such that $P_{ji}S_j = S_i$ whenever $i \le j$ in I, and that

$$\inf_{k\in\mathbb{N}} \sup_{i\in I} \int (|S_ix| - k\chi 1_{\mathfrak{A}_i})^+ = 0$$

for every $x \in X$. Then there is a unique function $S : X \to L^1_{\bar\lambda}$ such that $S_i = P_iS$ for every $i \in I$.

proof We can follow the same programme as in the proof of 377G, but with a couple of new twists.

preliminary remarks (i) If $i \le j$ in I, then by the definition of 'inductive limit' we have $\pi_j\pi_{ji} = \pi_i$ so $T_jT_{ji} = T_i$ and $P_{ji}P_j = P_i$. $P_{ji}T_{ji}$ and P_iT_i are the identity operator on $L^1_{\bar\mu_i}$.

(ii) Let $\mathcal{F}(I\uparrow)$ be the filter on I generated by $\{\{k : k \ge j\} : j \in I\}$. Then $\lim_{i\to\mathcal{F}(I\uparrow)} T_iP_iu = u$ for every $u \in L^1_{\bar\lambda}$. **P** Setting $\mathfrak{B}_i = T_i[\mathfrak{A}_i]$ for each $i \in I$, $\mathbb{B} = \{\mathfrak{B}_i : i \in I\}$ is an upwards-directed family of closed subalgebras of $\mathfrak{C}$; set $\mathfrak{D} = \overline{\bigcup\mathbb{B}}$ and $\bar\nu = \bar\lambda\restriction\mathfrak{D}$, so that $(\mathfrak{D}, \bar\nu)$ is a probability algebra. Since $\pi_i : \mathfrak{A}_i \to \mathfrak{D}$ is a measure-preserving Boolean homomorphism and $\pi_i = \pi_j\pi_{ji}$ whenever $i \le j$ in I, there is a measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{D}$ such that $\phi\pi_i = \pi_i$ for every i. But this means that $\mathfrak{C} = \mathfrak{D}$.

As in 377G, we can identify each $T_iP_i : L^1_{\bar\lambda} \to L^1_{\bar\lambda}$ with the conditional expectation $P_{\mathfrak{B}_i}$. This time, 367Qb tells us that $P_{\mathfrak{B}}u \to P_{\mathfrak{D}}u = u$ as $\mathfrak{B}$ increases through $\mathbb{B}$, that is, $u = \lim_{i\to\mathcal{F}(I\uparrow)} T_iP_iu$, for every $u \in L^1_{\bar\lambda}$. **Q**

(a) The point is that if i, $j \in I$, $u \in L^1_{\bar\mu_i}$, $v \in L^1_{\bar\mu_j}$ and $T_iu = T_jv$, then $S_iu = S_jv$. **P** Let $k \in I$ be such that $i \le k$ and $j \le k$. Then

$$T_kT_{ki}u = T_iu = T_jv = T_kT_{kj}v;$$

since T_k is injective, $T_{ki}u = T_{kj}v$. Accordingly

$$S_iu = S_kT_{ki}u = S_kT_{kj}v = S_jv. \textbf{ Q}$$

There is therefore a function $S' : \bigcup_{i\in I} S_i[L^1_{\bar\mu_i}] \to X$ defined by saying that $S(T_iu) = S_iu$ whenever $i \in I$ and $u \in L^1_{\bar\mu_i}$; extending S' arbitrarily to a function $S : L^1_{\bar\lambda} \to X$, we get the result.

(b) All we have to do is to check that if $i \le j$ in I, then

$$T_iS_i = T_jT_{ji}S_i = T_jS_j.$$

(c) In this case, we have

$$S_jP_j = S_iP_{ji}P_j = S_iP_i$$

whenever $i \le j$ in I.

(d)(i) For each $x \in X$, $\{T_iS_ix : i \in I\} \subseteq L^1_{\bar\lambda}$ is uniformly integrable. **P** If $k \in \mathbb{N}$ and $i \in I$,

$$|T_iS_ix| \le T_i(|S_ix| - k\chi 1_{\mathfrak{A}_i})^+ + T_i(k\chi 1_{\mathfrak{A}_i}) \le T_i(|S_ix| - k\chi 1_{\mathfrak{A}_i})^+ + k\chi 1_{\mathfrak{C}},$$

so

$$\int(|T_iS_ix|-k\chi 1_{\mathfrak{C}})^+ \le \int T_i(|S_ix|-k\chi 1_{\mathfrak{A}_i})^+ = \int(|S_ix|-k\chi 1_{\mathfrak{A}_i})^+.$$

Accordingly

$$\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|T_iS_ix|-k\chi 1_{\mathfrak{C}})^+ \le \inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|S_ix|-k\chi 1_{\mathfrak{A}_i})^+ = 0. \quad \mathbf{Q}$$

(ii) Fix an ultrafilter $\mathcal{G}$ on I including $\mathcal{F}(I\uparrow)$. For each $x\in X$, $\{T_iS_ix : i\in I\}$ is relatively weakly compact in $L^1_{\bar\lambda}$, so $Sx=\lim_{i\to\mathcal{G}}T_iS_ix$ is defined for the weak topology on $L^1_{\bar\lambda}$. Now for any $i\in I$,

$$P_iSx = \lim_{j\to\mathcal{G}}P_iT_jS_jx$$

(for the weak topology on $L^1_{\bar\mu_i}$)

$$= \lim_{j\to\mathcal{G}}P_{ji}P_jT_jS_jx$$

(because $\{j : j\ge i\}\in\mathcal{G}$)

$$= \lim_{j\to\mathcal{G}}P_{ji}S_jx = S_ix.$$

(iii) To see that S is uniquely defined, it is enough to recall that

$$Sx=\lim_{i\to\mathcal{F}(I\uparrow)}T_iP_iSx=\lim_{i\to\mathcal{F}(I\uparrow)}T_iS_ix$$

is uniquely defined by the family $\langle S_ix\rangle_{i\in I}$, for every $x\in X$.

377X Basic exercises (a) In 377B, show that $\langle u_i\rangle_{i\in I}\in\prod_{i\in I}L^0(\mathfrak{A}_i)$ belongs to W_0 iff $\{u_i^* : i\in I\}$ is bounded above in $L^0(\mathfrak{A}_L)$, where $\mathfrak{A}_L$ is the measure algebra of Lebesgue measure on $[0,\infty[$, and u_i^* is the decreasing rearrangement of u_i for each i (373C).

(b) In 377D, suppose that $u=\langle u_i\rangle_{i\in I}$ and $v=\langle v_i\rangle_{i\in I}$ belong to W_2, and that at least one of $|u|^2$, $|v|^2$ belongs to W_{ui}. Show that $(Tu|Tv)=\lim_{i\to\mathcal{F}}(u_i|v_i)$.

(c) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, and suppose that we have $u_i\in L^1(\mathfrak{A}_i,\bar\mu_i)$ for each i. Show that the following are equiveridical: (i) $\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|u_i|-k\chi 1_{\mathfrak{A}_i})^+=0$; (ii) $\{u_i^* : i\in I\}$ is uniformly integrable in $L^1(\mu_L)$, where μ_L is Lebesgue measure on $[0,\infty[$, and u_i^* is the decreasing rearrangement of u_i for each $i\in I$.

(d) Take any $p\in\,]1,\infty[$. Show that 377G remains true if we replace every 'L^1' by 'L^p'.

(e) Take any $p\in\,]1,\infty[$. Show that 377H remains true if we replace every 'L^1' by 'L^p' and in part (d) we replace '$\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|S_ix|-k\chi 1_{\mathfrak{A}_i})^+=0$' by '$\sup_{i\in I}\|S_ix\|_p<\infty$'.

(f) In 377Ha, suppose that X has a metric ρ under which it is complete, and that $\langle S_i\rangle_{i\in I}$ is uniformly equicontinuous in the sense that for every $\epsilon>0$ there is a $\delta>0$ such that $\rho(S_iu,S_iv)\le\epsilon$ whenever $i\in I$, u, $v\in L^1_{\bar\mu_i}$ and $\|u-v\|_1\le\delta$. Show that there is a unique continuous function $S:L^1_{\bar\lambda}\to X$ such that $S_i=ST_i$ for every $i\in I$.

377Y Further exercises (a) Find a non-empty family $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ of probability algebras, a probability algebra $(\mathfrak{B},\bar\nu)$, a Boolean homomorphism $\pi:\prod_{i\in I}\mathfrak{A}_i\to\mathfrak{B}$ such that $\bar\nu\pi(\langle a_i\rangle_{i\in I})\le\sup_{i\in I}\bar\mu_ia_i$ whenever $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$, and an element $u=\langle u_i\rangle_{i\in I}$ of W_0^+, as described in 377B, such that $\|Tu\|_1>\sup_{i\in I}\|u_i\|_1$, where $T:W_0\to L^0(\mathfrak{B})$ is the Riesz homomorphism of 377B-377C. (*Hint*: $\#(I)=2$.)

(b) Show that if, in 377Gc, we omit the hypothesis that the S_i are to be continuous, then the result can fail.

(c) Let $\langle U_i\rangle_{i\in I}$ be a non-empty family of L-spaces and $\mathcal{F}$ an ultrafilter on I. (i) Show that $\prod_{i\in I}U_i$ is a Dedekind complete Riesz space (see 352K) in which $W_\infty=\{\langle u_i\rangle_{i\in I} : \sup_{i\in I}\|u_i\|<\infty\}$ is a solid linear subspace. (ii) Let $W_0\subseteq W_\infty$ be $\{\{\langle u_i\rangle_{i\in I} : \sup_{i\in I}\|u_i\|<\infty, \lim_{i\to\mathcal{F}}\|u_i\|=0\}$; show that W_0 is a solid linear subspace of W_∞. (iii) Let U be the quotient Riesz space W_∞/W_0 (352U). Show that U is an L-space under the norm $\|\langle u_i\rangle^\bullet_{i\in I}\|=\lim_{i\to\mathcal{F}}\|u_i\|$ for $\langle u_i\rangle_{i\in I}\in W_\infty$.

(d) Let V be a normed space, and suppose that for every finite-dimensional subspace V_0 of V there are an L-space U and a norm-preserving linear map $T : V_0 \to U$. Show that there are an L-space U and a norm-preserving linear map $T : V \to U$.

377 Notes and comments Although my main target in this section has been to understand the function spaces of reduced products of probability algebras, I have as usual felt that the ideas are clearer if each is developed in a context closer to the most general case in which it is applicable. Only in part (b) of the proof of 377C, I think, does this involve us in extra work.

The new techniques of this section are forced on us by the fact that we are looking at Boolean homomorphisms $\pi : \prod_{i\in I}\mathfrak{A}_i \to \mathfrak{B}$ which are not normally sequentially order-continuous. While we have a natural Riesz homomorphism from $L^\infty(\prod_{i\in I}\mathfrak{A}_i)$ to $L^\infty(\mathfrak{B})$, as in 363F, we cannot expect a similar operator from the whole of $L^0(\prod_{i\in I}\mathfrak{A}_i) \cong \prod_{i\in I} L^0(\mathfrak{A}_i)$ to $L^0(\mathfrak{B})$. However the condition '$\bar\nu\pi(\langle a_i\rangle_{i\in I}) \le \sup_{i\in I}\bar\mu_i a_i$' ensures that there is a space $W_0 \subseteq \prod_{i\in I} L^0(\mathfrak{A}_i)$ on which an operator to $L^0(\mathfrak{B})$ can be defined, and which is large enough to give us a method of investigating the spaces $L^p(\mathfrak{B},\bar\nu)$ as images of subspaces W_p of products $\prod_{i\in I} L^p(\mathfrak{A}_i,\bar\mu_i)$.

In 377E, the case $p = 1$ is special because we can identify W_{ui} as the space of relatively weakly compact families in $L^1(\mathfrak{A},\bar\mu)$, and for such a family $u = \langle u_i\rangle_{i\in I}$ we have $\|Tu\|_1 = \lim_{i\to\mathcal{F}}\|u_i\|_1$. So the Banach space $L^1(\mathfrak{B},\bar\nu)$ is a kind of reduced power, describable in terms of the normed space $L^1(\mathfrak{A},\bar\nu)$. For other L^p spaces we need to know something more, e.g., the lattice structure, if we are to identify those $u \in W_p$ such that $Tu = 0$. The difference becomes significant when we come to look at morphisms of $L^p(\mathfrak{B},\bar\nu)$ corresponding to morphisms of $L^p(\mathfrak{A},\bar\mu)$, as in 377F.

In 377G-377H I give a string of results which are visibly mass-produced. What is striking is that in eight cases out of eight we have a straightforward formula corresponding to the idea that $(\mathfrak{C},\bar\lambda)$ is a limit of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$. What is curious is that in two of the eight cases (377Gc, 377Hd) we have to impose different special conditions on the functions S_i which the target S is supposed to approximate, and in just one case (377Ha) the target S is not uniquely defined in the absence of further constraints (377Xf). I think the ideas take up enough room when given only in their application to L^1 spaces, but of course there are versions, only slightly modified, which apply to other L^p spaces (377Xd-377Xe).

The repeated conditions of the form

$$\inf_{k\in\mathbb{N}}\sup_{i\in I}\bar\mu_i[\![|u_i| > k]\!] = 0,$$

$$\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|u_i| - k\chi 1_{\mathfrak{A}_i})^+ = 0,$$

(377B, 377Dc, 377Hd) both have expressions in terms of decreasing rearrangements (377Xa, 377Xc). The latter is clearly associated with uniform integrability and weak compactness, and unsurprisingly we use it to show that a weak limit will be defined. The former is there to ensure that a set appearing in an L^0 space will be bounded above, so that we can apply 355F to extend a Riesz homomorphism.

Chapter 38

Automorphism groups

As with any mathematical structure, every measure algebra has an associated symmetry group, the group of all measure-preserving automorphisms. In this chapter I set out to describe some of the remarkable features of these groups. I begin with elementary results on automorphisms of general Boolean algebras (§381), introducing definitions for the rest of the chapter. In §382 I give a general theorem on the expression of an automorphism as the product of involutions (382M), with a description of the normal subgroups of certain groups of automorphisms (382R). Applications of these ideas to measure algebras are in §383. I continue with a discussion of circumstances under which these automorphism groups determine the underlying algebras and/or have few outer automorphisms (§384).

One of the outstanding open problems of the subject is the 'isomorphism problem', the classification of automorphisms of measure algebras up to conjugacy in the automorphism group. I offer two sections on 'entropy', the most important numerical invariant enabling us to distinguish some non-conjugate automorphisms (§§385-386). For Bernoulli shifts on the Lebesgue measure algebra (385Q-385S), the isomorphism problem is solved by Ornstein's theorem (387I, 387K); I present a complete proof of this theorem in §§386-387. Finally, in §388, I give Dye's theorem, describing the full subgroups generated by single automorphisms of measure algebras of countable Maharam type.

381 Automorphisms of Boolean algebras

I begin the chapter with a preparatory section of definitions (381B) and mostly elementary facts. A fundamental method of constructing automorphisms is in 381C-381D. The idea of 'support' of an endomorphism is explored in 381E-381G, a first look at 'periodic' and 'aperiodic' parts is in 381H, and basic facts about 'full subgroups' are in 381I-381J. We start to go deeper with the notion of 'recurrence', treated in 381L-381P. I describe how these phenomena appear when we represent an endomorphism as a map on the Stone space of an algebra (381Q). I end by introducing a 'cycle notation' for certain automorphisms.

381A The group Aut $\mathfrak{A}$ For any Boolean algebra $\mathfrak{A}$, I write $\operatorname{Aut}\mathfrak{A}$ for the set of automorphisms of $\mathfrak{A}$, that is, the set of bijective Boolean homomorphisms $\pi:\mathfrak{A}\to\mathfrak{A}$. This is a group, being a subgroup of the group of all permutations of $\mathfrak{A}$ (use 312G). Note that every member of $\operatorname{Aut}\mathfrak{A}$ is order-continuous; this is because it must be an isomorphism of the order structure of $\mathfrak{A}$ (313Ld).

381B The primary aim of this chapter is to study automorphisms of probability algebras. In the context of the present section, this means that for a first reading you can take it that all algebras are Dedekind complete. The methods can however be used in many other contexts, at the price of complicating some of the statements of the lemmas. It is also interesting, and occasionally important, to apply some of the ideas to general Boolean homomorphisms. In the following definitions I try to set out a language to make this possible.

Definitions (a) If $\mathfrak{A}$ is a Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a Boolean homomorphism, I say that $a\in\mathfrak{A}$ **supports** π if $\pi d=d$ for every $d\subseteq 1\setminus a$.

(b) Let $\mathfrak{A}$ be a Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a Boolean homomorphism. If $\min\{a:a\in\mathfrak{A}\text{ supports }\pi\}$ is defined in $\mathfrak{A}$, I will call it the **support** $\operatorname{supp}\pi$ of π.

(c) If $\mathfrak{A}$ is a Boolean algebra, an automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ is **periodic**, with **period** $n\ge 1$, if $\mathfrak{A}\ne\{0\}$, π^n is the identity operator and 1 is the support of π^i whenever $1\le i<n$.

(d) If $\mathfrak{A}$ is a Boolean algebra, a Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ is **aperiodic** if the support of π^n is 1 for every $n\ge 1$. I remark immediately that if π is aperiodic, so is π^n for every $n\ge 1$ (see 381Xc). Note that if $\mathfrak{A}=\{0\}$ then the trivial automorphism of $\mathfrak{A}$ is counted as aperiodic.

(e) If $\mathfrak{A}$ is a Boolean algebra, a subgroup G of $\operatorname{Aut}\mathfrak{A}$ is **full** if whenever $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, $\langle\pi_i\rangle_{i\in I}$ is a family in G, and $\pi\in\operatorname{Aut}\mathfrak{A}$ is such that $\pi d=\pi_i d$ whenever $i\in I$ and $d\subseteq a_i$, then $\pi\in G$.

(f) If $\mathfrak{A}$ is a Boolean algebra, a subgroup G of $\operatorname{Aut}\mathfrak{A}$ is **countably full** if whenever $\langle a_i\rangle_{i\in I}$ is a countable partition of unity in $\mathfrak{A}$, $\langle\pi_i\rangle_{i\in I}$ is a family in G, and $\pi\in\operatorname{Aut}\mathfrak{A}$ is such that $\pi d=\pi_i d$ whenever $i\in I$ and $d\subseteq a_i$, then $\pi\in G$.

(g) If $\mathfrak{A}$ is a Boolean algebra, $a \in \mathfrak{A}$ and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism, I say that π is **recurrent on** a if for every non-zero $b \subseteq a$ there is a $k \geq 1$ such that $a \cap \pi^k b \neq 0$. If $\pi \in \operatorname{Aut}\mathfrak{A}$ and π and π^{-1} are both recurrent on a, I say that π is **doubly recurrent** on a.

381C Before setting out to explore the concepts just listed, I give a fundamental result on piecing automorphisms together from fragments.

Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ two partitions of unity in $\mathfrak{A}$. Assume

either that I is finite

or that I is countable and $\mathfrak{A}$ is Dedekind σ-complete

or that $\mathfrak{A}$ is Dedekind complete.

Suppose that for each $i \in I$ we have an isomorphism $\pi_i : \mathfrak{A}_{a_i} \to \mathfrak{A}_{b_i}$ between the corresponding principal ideals. Then there is a unique $\pi \in \operatorname{Aut}\mathfrak{A}$ such that $\pi d = \pi_i d$ whenever $i \in I$ and $d \subseteq a_i$.

proof By 315F, we may identify $\mathfrak{A}$ with each of the products $\prod_{i\in I}\mathfrak{A}_{a_i}$, $\prod_{i\in I}\mathfrak{A}_{b_i}$; now π corresponds to the isomorphism between the two products induced by the π_i.

381D Corollary Let $\mathfrak{A}$ be a homogeneous Boolean algebra, and A, B two partitions of unity in $\mathfrak{A}$, neither containing 0. Let $\theta : A \to B$ be a bijection. Suppose

either that A, B are finite

or that A, B are countable and $\mathfrak{A}$ is Dedekind σ-complete

or that $\mathfrak{A}$ is Dedekind complete.

Then there is an automorphism of $\mathfrak{A}$ extending θ.

proof For every $a \in A$, the principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_{\theta a}$ are isomorphic to the whole algebra $\mathfrak{A}$, and therefore to each other; let $\pi_a : \mathfrak{A}_a \to \mathfrak{A}_{\theta a}$ be an isomorphism. Now apply 381C.

381E Lemma Let $\mathfrak{A}$ be a Boolean algebra, and π, ϕ, $\psi : \mathfrak{A} \to \mathfrak{A}$ Boolean homomorphisms of which π is injective.

(a) If $a \in \mathfrak{A}$ supports ϕ then $\phi a = a$ and $\phi d \subseteq a$ for every $d \subseteq a$.

(b) If $a \in \mathfrak{A}$ supports both ϕ and ψ then it supports $\phi\psi$.

(c) Let A be the set of elements of $\mathfrak{A}$ supporting ϕ. Then A is non-empty and closed under $\cap$; also $b \in A$ whenever $b \supseteq a \in A$. If ϕ is order-continuous, then $\inf B \in A$ whenever $B \subseteq A$ has an infimum in $\mathfrak{A}$.

(d) If $a \in \mathfrak{A}$ supports $\pi\phi$, then ϕa supports $\pi\phi$.

(e) If π commutes with ϕ, and $a \in \mathfrak{A}$ is such that πa supports ϕ, then a supports ϕ.

(f) If ϕ is supported by a and ψ is supported by b, where $a \cap b = 0$, then $\phi\psi = \psi\phi$.

(g) For any $n \geq 1$ and $a \in \mathfrak{A}$, a supports π^n iff πa supports π^n. Consequently $\pi(\operatorname{supp}\pi^n) = \operatorname{supp}\pi^n$ if π^n has a support.

(h) If $\pi \in \operatorname{Aut}\mathfrak{A}$ and $a \in \mathfrak{A}$ supports π, then a supports π^{-1}.

(i) If $\pi \in \operatorname{Aut}\mathfrak{A}$ and $a \in \mathfrak{A}$, then a supports π iff $d \subseteq a$ whenever $d \cap \pi d = 0$ iff $d \cap \pi d \neq 0$ whenever $0 \neq d \subseteq 1 \setminus a$.

(j) If $\pi \in \operatorname{Aut}\mathfrak{A}$ and $a \in \mathfrak{A}$ supports ϕ, then πa supports $\pi\phi\pi^{-1}$.

(k) If $a \in \mathfrak{A}$ supports ϕ, and π_1, $\pi_2 \in \operatorname{Aut}\mathfrak{A}$ agree on $\mathfrak{A}_a$, then $\pi_1\phi\pi_1^{-1} = \pi_2\phi\pi_2^{-1}$.

proof (a) $\phi(1 \setminus a) = 1 \setminus a$, so $\phi a = a$, and if $d \subseteq a$ then $\phi d \subseteq \phi a = a$.

(b) If $d \cap a = 0$ then $\phi d = d = \psi d$ so $\phi\psi d = d$.

(c) Of course $1 \in A$, because $\phi 0 = 0$; and it is also obvious that if $b \supseteq a \in A$ then $b \in A$. If a, $b \in A$ and $d \cap a \cap b = 0$, then $\phi d = \phi(d \setminus a) \cup \phi(d \setminus b) = d$. If ϕ is order-continuous, $B \subseteq A$ is non-empty and $c = \inf B$ is defined in $\mathfrak{A}$, then for any $d \subseteq 1 \setminus c$ we have

$$d = d \setminus c = \sup_{b\in B} d \setminus b,$$

and

$$\phi d = \sup_{b\in B} \phi(d \setminus b) = \sup_{b\in B} d \setminus b = d.$$

So in this case c supports ϕ.

(d) If $d \cap \phi a = 0$ then $\pi d \cap a = \pi d \cap \pi\phi a = 0$, so $\pi\phi\pi d = \pi d$ and (because π is injective) $\phi\pi d = d$.

(e) If $d \cap a = 0$ then $\pi d \cap \pi a = 0$, so $\pi\phi d = \phi\pi d = \pi d$ and $\phi d = d$.

(f) If $d \subseteq a$ then $\phi d \subseteq a$ and $\psi d = d$ so $\psi\phi d = \phi d = \phi\psi d$; if $d \subseteq b$ then $\psi\phi d = \phi\psi d = \psi d$; and if $d \subseteq 1 \setminus (a \cup b)$ then $\psi\phi d = \phi\psi d = d$.

(g) Because π is injective, so is π^{n-1}. So if a supports $\pi^n = \pi^{n-1}\pi$, so does πa, by (d). On the other hand, π commutes with π^n, so if πa supports π^n so does a, by (e).

If $c = \operatorname{supp}\pi^n$ then πc supports π^n so $c \subseteq \pi c$. Consequently $\pi^i c \subseteq \pi^{i+1}c$ for every $i \in \mathbb{N}$ and $c \subseteq \pi c \subseteq \pi^n c$. But as $\pi^n c = c$, by (a), $\pi c = c$.

(h) If $d \cap a = 0$ then $\pi d = d$ so $d = \pi^{-1}d$.

(i)(α) If a supports π and $d \cap \pi d = 0$, then

$$d \cap (d \setminus a) = d \cap \pi(d \setminus a) \subseteq d \cap \pi d = 0,$$

so $d \subseteq a$.

(β) If $d \subseteq a$ whenever $d \cap \pi d = 0$, and $0 \ne d' \subseteq 1 \setminus a$, then of course $d' \cap \pi d' \ne 0$.

(γ) If a does not support π, there is a $c \subseteq 1 \setminus a$ such that $\pi c \ne c$. So one of $c \setminus \pi c$, $\pi c \setminus c$ is non-zero. If $c \setminus \pi c \ne 0$, take this for d; then $d \subseteq 1 \setminus a$ and $\pi d \cap d \subseteq \pi c \setminus \pi c = 0$. Otherwise, take $d = \pi^{-1}(\pi c \setminus c)$; then $0 \ne d \subseteq c \subseteq 1 \setminus a$, while

$$d \cap \pi d = (c \setminus \pi^{-1}c) \cap (\pi c \setminus c) = 0.$$

(j) If $d \cap \pi a = 0$ then $\pi^{-1}d \cap a = 0$ so $\phi\pi^{-1}d = \pi^{-1}d$ and $\pi\phi\pi^{-1}d = d$.

(k) For $d \subseteq a$, $\pi_2^{-1}\pi_1 d = \pi_2^{-1}\pi_2 d = d$, so $\pi_2^{-1}\pi_1$ is supported by $1 \setminus a$. By (f), $\phi\pi_2^{-1}\pi_1 = \pi_2^{-1}\pi_1\phi$, so

$$\pi_1\phi\pi_1^{-1} = \pi_2\pi_2^{-1}\pi_1\phi\pi_1^{-1} = \pi_2\phi\pi_2^{-1}\pi_1\pi_1^{-1} = \pi_2\phi\pi_2^{-1}.$$

381F Corollary If $\mathfrak{A}$ is a Dedekind complete Boolean algebra, then every order-continuous Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ has a support.

proof By 381Ec, $\inf\{a : a \in \mathfrak{A}$ supports $\phi\}$ is the support of ϕ.

381G Corollary Let $\mathfrak{A}$ be a Boolean algebra, and suppose that $\pi \in \operatorname{Aut}\mathfrak{A}$ has a support e.

(a) $\pi e = e$.

(b) $e = \sup\{d : d \in \mathfrak{A},\ d \cap \pi d = 0\}$.

(c) e is the support of π^{-1}.

(d) For any $\phi \in \operatorname{Aut}\mathfrak{A}$, ϕe is the support of $\phi\pi\phi^{-1}$.

proof (a) 381Ea.

(b) 381Ei.

(c) 381Eh.

(d) By 381Ej, ϕe supports $\phi\pi\phi^{-1}$. At the same time, if $a \in \mathfrak{A}$ supports $\phi\pi\phi^{-1}$, then $\phi^{-1}a$ supports π, so $e \subseteq \phi^{-1}a$ and $a \supseteq \phi e$. Thus ϕe is the smallest element of $\mathfrak{A}$ supporting $\phi\pi\phi^{-1}$ and is the support of $\phi\pi\phi^{-1}$.

381H Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an injective Boolean homomorphism such that π^n has a support for every $n \in \mathbb{N}$. Then there is a partition of unity $\langle c_i \rangle_{1 \le i \le \omega}$ in $\mathfrak{A}$ such that $\pi c_i \subseteq c_i$ for every i, $\pi \restriction \mathfrak{A}_{c_n}$ is periodic with period n whenever $n \in \mathbb{N} \setminus \{0\}$ and $c_n \ne 0$, and $\pi \restriction \mathfrak{A}_{c_\omega}$ is aperiodic.

proof Set

$$\begin{aligned} c_1 &= 1 \setminus \operatorname{supp}\pi, \\ c_n &= \inf_{i<n} \operatorname{supp}\pi^i \setminus \operatorname{supp}\pi^n \text{ for } n \ge 2, \\ c_\omega &= \inf_{n\in\mathbb{N}} \operatorname{supp}\pi^n. \end{aligned}$$

Then $\langle c_i \rangle_{1 \le i \le \omega}$ is a partition of unity. By 381Eg, $\pi c_n = c_n$ for every n, so $\pi c_\omega \subseteq c_\omega$. If $d \subseteq c_n$, where $1 \le n \in \mathbb{N}$, then $d \cap \operatorname{supp}\pi^n = 0$ so $\pi^n d = d$. If $1 \le i < j \le \omega$ and $0 \ne a \subseteq c_j$, then $a \subseteq \operatorname{supp}\pi^i$ so there is a $d \subseteq a$ such that $(\pi \restriction \mathfrak{A}_{c_n})^i d = \pi^i d \ne d$; thus if $n \in \mathbb{N} \setminus \{0\}$ (and $c_n \ne 0$) $\pi \restriction \mathfrak{A}_{c_n}$ is periodic with period n, while $\pi \restriction \mathfrak{A}_{c_\omega}$ is aperiodic.

Remark The hypothesis 'every π^n has a support' will be satisfied if $\mathfrak{A}$ is Dedekind complete and π is order-continuous (381F). For other sufficient conditions see 382E.

381I Full and countably full subgroups If $\mathfrak{A}$ is a Boolean algebra, it is obvious that the intersection of any family of (countably) full subgroups of $\operatorname{Aut}\mathfrak{A}$ is again (countably) full. We may therefore speak of the (countably) full subgroup of $\mathfrak{A}$ generated by an element of $\operatorname{Aut}\mathfrak{A}$.

Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) Let G be a subgroup of $\operatorname{Aut}\mathfrak{A}$. Let H be the set of those $\pi \in \operatorname{Aut}\mathfrak{A}$ such that for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and a $\phi \in G$ such that $\pi c = \phi c$ for every $c \subseteq b$. Then H is a full subgroup of $\operatorname{Aut}\mathfrak{A}$, the smallest full subgroup of $\mathfrak{A}$ including G.

(b) Suppose that $\mathfrak{A}$ is Dedekind σ-complete and π, $\phi \in \operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) ϕ belongs to the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(ii) there is a partition of unity $\langle a_n\rangle_{n\in\mathbb{Z}}$ in $\mathfrak{A}$ such that $\phi c = \pi^n c$ whenever $n \in \mathbb{Z}$ and $c \subseteq a_n$.

(c) Suppose that $\mathfrak{A}$ is Dedekind complete, and π, $\phi \in \operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(ii) for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and an $n \in \mathbb{Z}$ such that $\phi c = \pi^n c$ for every $c \subseteq b$;

(iii) ϕ belongs to the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(iv) $\inf_{n\in\mathbb{Z}} \operatorname{supp}(\pi^n\phi) = 0$.

proof (a)(i) $\pi_2\pi_1 \in H$ for all π_1, $\pi_2 \in H$. **P** Let $a \in \mathfrak{A}$ be non-zero; then there are a non-zero $b \subseteq a$ and a $\phi_1 \in G$ such that π_1 and ϕ_1 agree on the principal ideal $\mathfrak{A}_b$. Next, there are a non-zero $c \subseteq \pi_1 b$ and a $\phi_2 \in G$ such that π_2 and ϕ_2 agree on $\mathfrak{A}_c$. Set $d = \pi_1^{-1}c$; then $d \in \mathfrak{A}_a \setminus \{0\}$, and $\phi_2\phi_1$ is a member of G agreeing with $\pi_2\pi_1$ on $\mathfrak{A}_d$. As a is arbitrary, $\pi_2\pi_1 \in H$. **Q**

(ii) $\pi^{-1} \in H$ for every $\pi \in H$. **P** If $a \in \mathfrak{A} \setminus \{0\}$, there are a non-zero $b \subseteq \pi^{-1}a$ and a $\phi \in G$ such that π and ϕ agree on $\mathfrak{A}_b$; now $0 \neq \pi b \subseteq a$ and π^{-1} and ϕ^{-1} agree on $\mathfrak{A}_{\pi b}$. As a is arbitrary, $\pi^{-1} \in H$. **Q** Of course $H \supseteq G$, so H is a subgroup of $\operatorname{Aut}\mathfrak{A}$.

(iii) Suppose now that $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, $\langle \pi_i\rangle_{i\in I}$ is a family in H, and $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that π agrees with π_i on $\mathfrak{A}_{a_i}$ for every $i \in I$. Then $\pi \in H$. **P** If $a \in \mathfrak{A} \setminus \{0\}$, there is an $i \in I$ such that $b = a \cap a_i$ is non-zero; now π agrees with π_i on b. **Q** So H is a full subgroup of $\operatorname{Aut}\mathfrak{A}$.

(iv) If H' is any full subgroup of $\operatorname{Aut}\mathfrak{A}$ including G, then $H' \supseteq H$. **P** If $\pi \in H$, then $B = \{b :$ there is a $\phi \in G$ agreeing with π on $\mathfrak{A}_b\}$ is an order-dense subset of $\mathfrak{A}$, so there is a partition $\langle a_i\rangle_{i\in I}$ of unity in $\mathfrak{A}$ such that $a_i \in B$ for every i. For each $i \in I$, let $\pi_i \in G$ be such that π and π_i agree on $\mathfrak{A}_{a_i}$; then $\langle (a_i, \pi_i)\rangle_{i\in I}$ witnesses that $\pi \in H'$. As π is arbitrary, $H \subseteq H'$. **Q**

(b) (ii)$\Rightarrow$(i) is trivial. In the other direction, let G be the family of all those automorphisms ψ of $\mathfrak{A}$ such that there is a partition of unity $\langle a_n\rangle_{n\in\mathbb{Z}}$ in $\mathfrak{A}$ such that $\psi c = \pi^n c$ whenever $n \in \mathbb{Z}$ and $c \subseteq a_n$. Then G is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ containing π.

P Of course $\pi \in G$ (set $a_1 = 1$, $a_n = 0$ for $n \neq 1$).

Take ψ_1, $\psi_2 \in G$. Let $\langle a_n\rangle_{n\in\mathbb{Z}}$, $\langle a'_n\rangle_{n\in\mathbb{Z}}$ be partitions of unity in $\mathfrak{A}$ such that $\psi_1 c = \pi^n c$ whenever $n \in \mathbb{Z}$ and $c \subseteq a_n$, while $\psi_2 c = \pi^n c$ whenever $n \in \mathbb{Z}$ and $c \subseteq a'_n$. Then $\langle a'_n \cap \psi_2^{-1}a_m\rangle_{m,n\in\mathbb{Z}}$ is a partition of unity. If $c \subseteq a'_n \cap \psi_2^{-1}a_m$, then $\psi_2 c = \pi^n c \subseteq a_m$, so $\psi_1\psi_2 c = \pi^{m+n}c$. So if we set $b_n = \sup_{i\in\mathbb{Z}} a'_i \cap \psi_2^{-1}a_{n-i}$ for each $n \in \mathbb{Z}$, $\langle b_n\rangle_{n\in\mathbb{Z}}$ is a partition of unity in $\mathfrak{A}$ witnessing that $\psi_1\psi_2 \in G$. At the same time, $\langle \psi_1 a_{-n}\rangle_{n\in\mathbb{Z}}$ is a partition of unity witnessing that $\psi_1^{-1} \in G$. As ψ_1 and ψ_2 are arbitrary, G is a subgroup of $\operatorname{Aut}\mathfrak{A}$.

Now suppose that $\langle a_i\rangle_{i\in I}$ is a countable partition of unity in $\mathfrak{A}$ and that $\psi \in \operatorname{Aut}\mathfrak{A}$ is such that for every $i \in I$ there is a $\psi_i \in G$ such that $\psi c = \psi_i c$ for every $c \subseteq a_i$. For each $i \in I$ let $\langle a_{in}\rangle_{n\in\mathbb{Z}}$ be a partition of unity such that $\psi_i c = \pi^n c$ whenever $c \subseteq a_{in}$. Then $\langle a_i \cap a_{in}\rangle_{i\in I,n\in\mathbb{Z}}$ is a partition of unity such that $\psi c = \pi^n c$ whenever $c \subseteq c_i \cap a_{in}$. So setting $b_n = \sup_{i\in I} a_i \cap a_{in}$ for each $n \in \mathbb{Z}$, $\langle b_n\rangle_{n\in\mathbb{Z}}$ is a partition of unity witnessing that $\psi \in G$. As ψ is arbitrary, G is countably full. **Q**

Accordingly G must include (indeed, must coincide with) the countably full subgroup generated by π, and (i)$\Rightarrow$(ii).

(c)(i)$\Rightarrow$(ii) is a special case of (a).

(ii)$\Rightarrow$(iii) For $n \in \mathbb{Z}$, let B_n be the set of those $b \in \mathfrak{A}$ such that $\phi c = \pi^n c$ for every $c \subseteq b$. Set $b_n = \sup B_n$ for each n; then if $c \subseteq b_n$,

$$\phi c = \phi(\sup_{b\in B_n} b \cap c) = \sup_{b\in B_n} \phi(b \cap c) = \sup_{b\in B_n} \pi^n(b \cap c) = \pi^n c.$$

Set

$$\begin{aligned} a_n &= b_n \setminus \sup_{0\le i<n} b_i \text{ if } n \in \mathbb{N}, \\ &= b_n \setminus \sup_{i>n} b_i \text{ if } n \in \mathbb{Z}\setminus\mathbb{N}; \end{aligned}$$

then $\langle a_n\rangle_{n\in\mathbb{Z}}$ is disjoint,

$$\sup_{n\in\mathbb{Z}} a_n = \sup_{n\in\mathbb{Z}} b_n = \sup(\bigcup_{n\in\mathbb{Z}} B_n) = 1,$$

and $\phi c = \pi^n c$ for every $c \subseteq a_n$, $n \in \mathbb{Z}$. Thus ϕ satisfies condition (ii) of (a) and belongs to the countably full subgroup generated by π.

(iii)⇒(i) is trivial.

(ii)⇔(iv) The point is that, for $n \in \mathbb{Z}$ and $b \in \mathfrak{A}$,

$$\begin{aligned} \phi c = \pi^n c \text{ for every } c \subseteq b &\iff \pi^{-n}\phi c = c \text{ for every } c \subseteq b \\ &\iff b \cap \operatorname{supp}(\pi^{-n}\phi) = 0. \end{aligned}$$

So we have

$$\begin{aligned} \text{(ii)} &\iff \forall\, a \in \mathfrak{A}\setminus\{0\}\ \exists\, n \in \mathbb{Z},\, b \text{ such that } 0 \ne b \subseteq a \text{ and } b \cap \operatorname{supp}(\pi^{-n}\phi) = 0 \\ &\iff \forall\, a \in \mathfrak{A}\setminus\{0\}\ \exists\, n \in \mathbb{Z},\, a \setminus \operatorname{supp}(\pi^{-n}\phi) \ne 0 \\ &\iff \inf_{n\in\mathbb{Z}} \operatorname{supp}(\pi^{-n}\phi) = 0, \end{aligned}$$

as required.

381J Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\pi \in \operatorname{Aut}\mathfrak{A}$. Suppose that ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π.

(a) If $c \in \mathfrak{A}$ is such that $\pi c = c$, then $\phi c = c$.

(b) If $a \in \mathfrak{A}$ supports π then it supports ϕ.

proof (a) Let G be the set of all $\psi \in \operatorname{Aut}\mathfrak{A}$ such that $\psi c = c$. Then G is a subgroup of $\operatorname{Aut}\mathfrak{A}$ containing π. Also G is full. **P** If $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, $\langle\psi_i\rangle_{i\in I}$ is a family in G, and $\psi \in \operatorname{Aut}\mathfrak{A}$ is such that $\psi d = \psi_i d$ whenever $d \subseteq a_i$, then

$$\psi c = \sup_{i\in I} \psi(c \cap a_i) = \sup_{i\in I} \psi_i(c \cap a_i) = \sup_{i\in I} \psi_i c \cap \psi_i a_i = \sup_{i\in I} c \cap \psi_i a_i = c.$$

So $\psi \in G$; as ψ is arbitrary, G is full. **Q** So $\phi \in G$ and $\phi c = c$, as claimed.

(b) If $c \cap a = 0$ then $\pi c = c$ so $\phi c = c$.

381K Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a sequentially order-continuous Boolean homomorphism.

(a) If $a \in \mathfrak{A}$ and $a^* = \inf_{k\in\mathbb{N}} \sup_{i\ge k} \pi^i a$, then $\pi a^* = a^*$.

(b) If $a \in \mathfrak{A}$ is such that $a \subseteq \sup_{i\ge 1} \pi^i a$, then $\sup_{i\ge k} \pi^i a = \sup_{i\in\mathbb{N}} \pi^i a$ for every $k \in \mathbb{N}$.

proof (a) Because π is sequentially order-continuous,

$$\pi a^* = \inf_{k\in\mathbb{N}} \sup_{i\ge k} \pi^{i+1} a$$

(313Lc)

$$= \inf_{k\in\mathbb{N}} \sup_{i\ge k+1} \pi^i a = \inf_{k\ge 1} \sup_{i\ge k} \pi^i a = \inf_{k\in\mathbb{N}} \sup_{i\ge k} \pi^i a = a^*.$$

(b) Induce on k. For $k = 0$ the result is just the hypothesis. For the inductive step to $k + 1$, because π is sequentially order-continuous, so is π^k (313Ic), so

$$\begin{aligned} \sup_{i\ge k+1} \pi^i a &= \sup_{i\ge 1} \pi^k \pi^i a = \pi^k(\sup_{i\ge 1} \pi^i a) \\ &= \pi^k(\sup_{i\in\mathbb{N}} \pi^i a) = \sup_{i\ge k} \pi^i a = \sup_{i\in\mathbb{N}} \pi^i a, \end{aligned}$$

and the induction continues.

381L Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. Then for any $a \in \mathfrak{A}$, the following are equiveridical:

(i) π is recurrent on a;

(ii) $a \subseteq \sup_{n\ge1} \pi^{-n}a$;

(iii) there is some $k \ge 1$ such that $a \subseteq \sup_{n\ge k} \pi^{-n}a$;

(iv) $a \subseteq \sup_{n\ge k} \pi^{-n}a$ for every $k \in \mathbb{N}$.

proof (i)⇒(ii) If (i) is true, set $b = a \setminus \sup_{n\ge1} \pi^{-n}a$. Then $a \cap \pi^n b = 0$ for every $n \ge 1$, so $b = 0$, that is, $a \subseteq \sup_{n\ge1} \pi^{-n}a$.

(ii)⇒(i) If (ii) is true and $0 \ne b \subseteq a$, then there is some $n \ge 1$ such that $b \cap \pi^{-n}a \ne 0$, that is, $\pi^n b \cap a \ne 0$; as b is arbitrary, π is recurrent on a.

(iv)⇒(ii)⇔(iii) are trivial.

(ii)⇒(iv) Apply 381Kb to π^{-1}.

381M It is with the idea of 'recurrence' that we start to get genuine surprises. The first fundamental construction is that of 'induced automorphism' in the following sense.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $a \in \mathfrak{A}$. Suppose that $\pi \in \operatorname{Aut}\mathfrak{A}$ is doubly recurrent on a. Then we have a Boolean automorphism $\pi_a : \mathfrak{A}_a \to \mathfrak{A}_a$ defined by saying that $\pi_a d = \pi^n d$ whenever $n \ge 1$ and $d \subseteq a \cap \pi^{-n}a \setminus \sup_{1\le i<n} \pi^{-i}a$; I will call π_a the **induced automorphism** on $\mathfrak{A}_a$.

proof For $n \ge 1$ set

$$d_n = a \cap \pi^{-n}a \setminus \sup_{1\le i<n} \pi^{-i}a.$$

If $1 \le m < n$ then

$$d_n \subseteq \pi^{-n}a \setminus \pi^{-m}a, \quad d_m \subseteq \pi^{-m}a$$

so $d_m \cap d_n = 0$. Also

$$d_m \subseteq a, \quad \pi^{n-m}d_n \cap a = \pi^{n-m}(d_n \cap \pi^{-(n-m)}a) = 0$$

so

$$\pi^n d_n \cap \pi^m d_m = \pi^m(\pi^{n-m}d_n \cap d_m) = 0.$$

Finally, $\sup_{n\ge1} d_n = a \cap \sup_{n\ge1} \pi^{-n}a = a$, because π is recurrent on a (using (a)).

It follows that $\langle d_n\rangle_{n\ge1}$ is a partition of unity in $\mathfrak{A}_a$. Since $\langle \pi^n d_n\rangle_{n\ge1}$ also is a disjoint family in $\mathfrak{A}_a$, and

$$\begin{aligned}\sup_{n\ge1} \pi^n d_n &= \sup_{n\ge1}(\pi^n a \cap a \setminus \sup_{1\le i<n} \pi^{n-i}a)\\ &= a \cap \sup_{n\ge1}(\pi^n a \setminus \sup_{1\le i<n} \pi^i a) = a \cap \sup_{n\ge1} \pi^n a = a,\end{aligned}$$

(because π^{-1} is recurrent on a), $\langle \pi^n d_n\rangle_{n\ge1}$ is another partition of unity. So we have an automorphism $\pi_a : \mathfrak{A}_a \to \mathfrak{A}_a$ defined by setting $\pi_a d = \pi^n d$ if $d \subseteq d_n$ (381C).

381N Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $a \in \mathfrak{A}$. Suppose that $\pi \in \operatorname{Aut}\mathfrak{A}$ is doubly recurrent on a. Let $\pi_a \in \operatorname{Aut}\mathfrak{A}_a$ be the induced automorphism.

(a) π^{-1} is doubly recurrent on a, and the induced automorphism $(\pi^{-1})_a$ is $(\pi_a)^{-1}$.

(b) For every $n \in \mathbb{N}$ there is a partition of unity $\langle b_i\rangle_{i\ge n}$ in $\mathfrak{A}_a$ such that $\pi_a^n b = \pi^i b$ whenever $i \ge n$ and $b \subseteq b_i$.

(c) If $n \ge 1$ and $0 \ne b \subseteq a \cap \pi^{-n}a$, there are a non-zero $b' \subseteq b$ and a j such that $1 \le j \le n$ and $\pi^n d = \pi_a^j d$ for every $d \subseteq b'$.

(d) Suppose that $m \ge 1$ is such that $a \cap \pi^i a = 0$ for $1 \le i < m$. Then for any $n \ge 1$ we have a disjoint family $\langle b_{ni}\rangle_{1\le i\le\lfloor n/m\rfloor}$, with supremum $a \cap \pi^{-n}a$, such that $\pi^n d = \pi_a^i d$ whenever $1 \le i \le \lfloor \frac{n}{m} \rfloor$ and $d \subseteq b_{ni}$.

(e) Suppose that $b \subseteq a$. Then π is doubly recurrent on b iff π_a is doubly recurrent on b, and in this case $\pi_b = (\pi_a)_b$, where $(\pi_a)_b$ is the automorphism of $\mathfrak{A}_b$ induced by π_a.

(f) Suppose that $c \in \mathfrak{A}$ is such that $\pi c = c$. Then π is doubly recurrent on $a \cap c$, and $\pi_{a\cap c} = \pi_a \restriction \mathfrak{A}_{a\cap c}$; in particular, $\pi_a(a \cap c) = a \cap c$.

(g) If π is aperiodic, so is π_a.

(h) Suppose that $a \cap \pi a = 0$, and that $b \subseteq a$ is such that $b \cap \pi_a b = 0$. Then b, πb and $\pi^2 b$ are all disjoint.

(i) There is an automorphism $\tilde{\pi}_a \in \operatorname{Aut}\mathfrak{A}$ defined by setting $\tilde{\pi}_a d = \pi_a d$ for $d \subseteq a$, $\tilde{\pi}_a d = d$ for $d \subseteq 1 \setminus a$, and $\tilde{\pi}_a$ belongs to the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π.

proof Set $d_n = a \cap \pi^{-n} a \setminus \sup_{1 \le i < n} \pi^{-i} a$ for $n \ge 1$, so that $\langle d_n \rangle_{n \ge 1}$ and $\langle \pi^n d_n \rangle_{n \ge 1}$ are partitions of unity in $\mathfrak{A}_a$, and $\pi_a b = \pi^n b$ for $b \subseteq d_n$.

(a) By the symmetry in the definition of 'doubly recurrent', π^{-1} is doubly recurrent on a iff π is. In this case,

$$\pi^n d_n = \pi^n a \cap a \setminus \sup_{1 \le i < n} \pi^{n-i} a = a \cap \pi^n a \cap a \setminus \sup_{1 \le i < n} \pi^i a$$

so $(\pi^{-1})_a b = \pi^{-n} b = (\pi_a)^{-1} b$ for every $b \subseteq \pi^n d_n$; as $\langle \pi_n d_n \rangle_{n \in \mathbb{N}}$ is a partition of unity in $\mathfrak{A}_a$, $(\pi^{-1})_a = (\pi_a)^{-1}$.

(b) Induce on n. For $n = 0$ we can take $b_0 = a$ and $b_i = 0$ for $i > 0$. For the inductive step to $n+1$, let $\langle b_i \rangle_{i \ge n}$ be a partition of unity in $\mathfrak{A}_a$ such that $\pi_a^n b = \pi^i b$ for $b \subseteq b_i$. Then $\langle \pi_a^{-1} b_i \rangle_{i \ge n}$ and $\langle d_k \cap \pi_a^{-1} b_i \rangle_{k \ge 1, i \ge n}$ are partitions of unity in $\mathfrak{A}_a$. If $b \subseteq d_k \cap \pi_a^{-1} b_i$, then $\pi_a b = \pi^k b \subseteq b_i$, so $\pi_a^{n+1} b = \pi^{k+i} b$. This means that if we set $b'_j = \sup_{k \ge 1, i \ge n, k+i=j} d_k \cap \pi_a^{-1} b_i$ for $j \ge n+1$, $\langle b'_j \rangle_{j \ge n+1}$ will be a partition of unity in $\mathfrak{A}_a$, and $\pi_a^{n+1} b = \pi^j b$ whenever $b \subseteq b_j$. So the induction continues.

(c) Induce on n. If $b \cap \pi^{-i} a = 0$ for $1 \le i < n$ then we can take $b' = b$ and $j = 1$. Otherwise, take the first $i \ge 1$ such that $b_1 = b \cap \pi^{-i} a \ne 0$. Then $\pi_a d = \pi^i d$ for every $d \subseteq b_1$. Also $\pi^{n-i} \pi^i b_1 \subseteq a$, so, by the inductive hypothesis, there are a non-zero $c \subseteq \pi^i b_1$ and a j such that $1 \le j \le n-i$ and $\pi^{n-i} d = \pi_a^j d$ for every $d \subseteq c$. Setting $b' = \pi^{-i} c \subseteq b_1$, we have $0 \ne b' \subseteq b$ and

$$\pi^n d = \pi^{n-i} \pi^i d = \pi_a^j \pi_a d = \pi_a^{j+1} d$$

whenever $d \subseteq b'$. So the induction continues.

(d) Again induce on n. If $1 \le n < m$ then $a \cap \pi^{-n} a = 0$ and the result is trivial. If $n = m$, then $a \cap \pi^{-n} a = d_n$ and $\pi_a d = \pi^n d$ for every $d \subseteq d_n$, so we can set $b_{n1} = d_n$. For the inductive step to $n > m$, we have

$$\begin{aligned} a \cap \pi^{-n} a &= d_n \cup \sup_{m \le k < n} (d_k \cap \pi^{-n} a) = d_n \cup \sup_{m \le k < n} (d_k \cap \pi^{-k}(a \cap \pi^{-(n-k)} a)) \\ &= d_n \cup \sup_{\substack{m \le k \le n-m \\ 1 \le j \le \lfloor (n-k)/m \rfloor}} (d_k \cap \pi^{-k} b_{n-k,j}) \end{aligned}$$

by the inductive hypothesis, while $\langle d_k \cap \pi^{-k} b_{n-k,j} \rangle_{m \le k \le n-m, 1 \le j \le \lfloor (n-k)/m \rfloor}$ is disjoint. Now if $m \le k \le n-m$ and $1 \le j \le \lfloor \frac{n-k}{m} \rfloor$ and $d \subseteq d_k \cap \pi^{-k} b_{n-k,j}$, we have $\pi_a d = \pi^k d \subseteq b_{n-k,j}$, so $\pi^n d = \pi^{n-k} \pi_a d = \pi_a^{j+1} d$; while if $d \subseteq d_n$ then $\pi^n d = \pi_a d$. So we can set

$$b_{n1} = d_n, \quad b_{ni} = \sup_{m \le k \le n-m} d_k \cap b_{n-k,i-1}$$

for $2 \le i \le \lfloor \frac{n}{m} \rfloor$, and the induction will continue.

(e) Applying (b) and (d) to π and π^{-1}, and using 381L and (a), we see that π is doubly recurrent on b iff π_a is doubly recurrent on b.

In this case, set $D = \{d : d \in \mathfrak{A}_b,\ \pi_b d = (\pi_a)_b d\}$. Then D is order-dense in $\mathfrak{A}_b$. **P** Take any non-zero $c \in \mathfrak{A}_b$. Since $b \subseteq \sup_{n \ge 1} \pi^{-n} b$, there is an $n \ge 1$ such that $c' = c \cap \pi^{-n} b \setminus \sup_{1 \le i < n} \pi^{-i} b$ is non-zero. Next, there is a non-zero $d \subseteq c'$ such that for every $m \le n$ either $d \subseteq \pi^{-m} a$ or $d \cap \pi^{-m} a = 0$. Enumerate $\{m : m \le n,\ d \subseteq \pi^{-m} a\}$ in ascending order as $(m_0, \dots, m_k)$ (note that as $c' \subseteq a \cap \pi^{-n} a$, we must have $m_0 = 0$ and $m_k = n$). Set $d_i = \pi^{m_i} d$ for $i \le k$, so that

$$d_0 = d, \quad \pi^{m_{i+1} - m_i} d_i = d_{i+1} \subseteq a,$$

while

$$\pi^j d_i = \pi^{m_i + j} d \subseteq 1 \setminus a$$

for $1 \le j < m_{i+1} - m_i$; that is, $d_{i+1} = \pi_a d_i$ for $i < k$. Thus

$$\pi_a^k d = \pi^{m_k} d = \pi^n d \subseteq b,$$

while

$$\pi_a^i d = d_i = \pi^{m_i} d \subseteq \pi^{m_i} c' \subseteq 1 \setminus b$$

for every $i < k$, and

$$(\pi_a)_b d = \pi_a^k d = \pi^n d = \pi_b d,$$

so that $d \in D$. As c is arbitrary, D is order-dense. **Q**

Because π_b and $(\pi_a)_b$ are both order-continuous Boolean homomorphisms on $\mathfrak{A}_b$, and every member of $\mathfrak{A}_b$ is a supremum of some subset of D (313K), $\pi_b = (\pi_a)_b$, as required.

(f) We have

$$a \cap c \subseteq \sup_{n\ge1} \pi^{-n} a \cap c = \sup_{n\ge1} \pi^{-n} a \cap \pi^{-n} c = \sup_{n\ge1} \pi^{-n}(a \cap c),$$

so π is recurrent on $a \cap c$; similarly, π^{-1} is recurrent on $a \cap c$. If $n \ge 1$ and

$$d \subseteq a \cap c \cap \pi^{-n}(a \cap c) \setminus \sup_{1\le i<n} \pi^{-i}(a \cap c) = c \cap a \cap \pi^{-n} a \setminus \sup_{1\le i<n} \pi^{-i} a,$$

then $\pi_{a\cap c} d = \pi^n d = \pi_a d$. So π_a extends $\pi_{a\cap c}$, as claimed.

(g) If $0 \ne b \subseteq a$, and $n \ge 1$, then (b) tells us that there are a non-zero $c \subseteq b$ and an $i \ge n$ such that $\pi_a^n d = \pi^i d$ for every $d \subseteq c$. Now we are supposing that $\operatorname{supp} \pi^i = 1$, so there is a $d \subseteq c$ such that $\pi^i d \ne d$, that is, $\pi_a^n d \ne d$. As b is arbitrary, $\operatorname{supp} \pi_a^n = a$; as n is arbitrary, π_a is aperiodic.

(h) Of course $\pi b \subseteq \pi a$ is disjoint from $b \subseteq a$; it follows that $\pi b \cap \pi^2 b = \pi(b \cap \pi b) = 0$. If $c = b \cap \pi^{-2} b$, then $c \subseteq a \cap \pi^{-2} a \setminus \pi^{-1} a$, so

$$\pi^2 b \cap b = \pi^2 c = \pi_a c \subseteq \pi_a b$$

is disjoint from b and must be 0. So b, πb and $\pi^2 b$ are all disjoint.

(i) By 381C, the formula defines an automorphism $\tilde\pi_a$. Setting $d_0 = 1 \setminus a$, $\langle d_n \rangle_{n\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}$ and $\tilde\pi_a d = \pi^n d$ for $d \subseteq d_n$, so $\tilde\pi_a$ belongs to the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π.

381O Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism. Then the following are equiveridical:

(i) π is recurrent on every $a \in \mathfrak{A}$;

(ii) for every non-zero $a \in \mathfrak{A}$ there is a $k \ge 1$ such that $a \cap \pi^k a \ne 0$;

(iii) $a = \sup_{k\ge1} a \cap \pi^k a$ for every $a \in \mathfrak{A}$.

proof (i)⇒(ii) If (i) is true, and $a \in \mathfrak{A} \setminus \{0\}$, then taking $b = a$ in the definition 381Bg we see that there is a $k \ge 1$ such that $a \cap \pi^k a \ne 0$.

(ii)⇒(iii) Suppose (ii) is true. **?** If $a \in \mathfrak{A}$ is not the supremum of $\{a \cap \pi^k a : k \ge 1\}$, let $b \subseteq a$ be non-zero and disjoint from $\pi^k a$ for every $k \ge 1$. Then $b \cap \pi^k b = 0$ for every $k \ge 1$, which is impossible. **X**

(iii)⇒(i) Suppose (iii) is true. If $0 \ne b \subseteq a$ then $b = \sup_{k\ge1} b \cap \pi^k b$, so there is certainly some $k \ge 1$ such that $b \cap \pi^k b \ne 0$, in which case $a \cap \pi^k b \ne 0$. As b is arbitrary, π is recurrent on a; as a is arbitrary, (i) is true.

Remark The condition 'recurrent on every $a \in \mathfrak{A}$' looks, and is, very restrictive; but it is satisfied by the homomorphisms we care about most (386A).

381P Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism which is recurrent on every $a \in \mathfrak{A}$. Then π is aperiodic iff $\mathfrak{A}$ is relatively atomless (definition: 331A) over the fixed-point algebra $\mathfrak{C} = \{c : c \in \mathfrak{A}, \pi c = c\}$. In particular, if π is ergodic, it is aperiodic iff $\mathfrak{A}$ is atomless.

proof It is elementary to check that $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$.

(a) Suppose that π is not aperiodic. Then there is a least $n \ge 1$ such that 1 is not the support of π^n; that is, there is a non-zero $a \in \mathfrak{A}$ such that $\pi^n d = d$ for every $d \subseteq a$. Now if $0 \ne b \subseteq a$ and $1 \le i < n$ there is a non-zero $b' \subseteq b$ such that $b' \cap \pi^i b' = 0$. **P** We are supposing that the support of π^i is 1, so there is a $d \subseteq b$ such that $d \ne \pi^i d$. If $d \setminus \pi^i d \ne 0$, take $b' = d \setminus \pi^i d$. Otherwise, try $b' = d \setminus \pi^{n-i} d$; then

$$\pi^i b' = \pi^i d \setminus \pi^n d = \pi^i d \setminus d \ne 0,$$

so $b' \ne 0$, while $b' \cap \pi^i b' \subseteq d \setminus \pi^n d = 0$. **Q**

We can therefore find a non-zero $b \subseteq a$ such that $b \cap \pi^i b = 0$ whenever $1 \le i < n$. Now b is a relative atom of $\mathfrak{A}$ over $\mathfrak{C}$. **P** If $d \subseteq b$, set $c = \sup_{0\le i<n} \pi^i d$. Then $\pi c = \sup_{1\le i\le n} \pi^i d = c$, so $c \in \mathfrak{C}$, while $b \cap \pi^i d = 0$ for $1 \le i < n$, so $d = b \cap c$. **Q** Thus b witnesses that $\mathfrak{A}$ is not relatively atomless over $\mathfrak{C}$.

(b)(i) Note that if $a \in \mathfrak{A}$ and $a \subseteq \pi a$ then $a = \pi a$. **P?** Otherwise, set $b = \pi a \setminus a$. Then $\pi^n b = \pi^{n+1} a \setminus \pi^n a$ for every n; also $a \subseteq \pi a \subseteq \pi^2 a \subseteq \ldots$, so $\langle \pi^n b \rangle_{n\in\mathbb{N}}$ is disjoint. But in this case π cannot be recurrent on b. **XQ**

(ii) Suppose that $\mathfrak{A}$ is not relatively atomless over $\mathfrak{C}$. Then there is a relative atom $a \in \mathfrak{A}$; as π is recurrent on a, there is a first $n \geq 1$ such that $a \cap \pi^n a \neq 0$. Then $\pi^n b = b$ for every $b \subseteq a \cap \pi^n a$. **P** Because a is a relative atom over $\mathfrak{C}$, there is a $c \in \mathfrak{C}$ such that $b = a \cap c$. Now $\pi^n b = \pi^n a \cap c \supseteq b$. Set $b_1 = \sup_{0\leq i<n} \pi^i b$; then $\pi b_1 = \sup_{1\leq i\leq n} \pi^i b \supseteq b_1$. So $b_1 = \pi b_1$, by (i), and $\pi^n b \subseteq \sup_{i<n} \pi^i b$. Next,

$$\pi^n b \cap \pi^i b = \pi^i(\pi^{n-i} b \cap b) \subseteq \pi^i(\pi^{n-i} a \cap a) = 0$$

for $0 < i < n$, so $\pi^n b \subseteq b$ and $\pi^n b = b$. **Q** Thus $a \cap \pi^n a$ witnesses that π is not aperiodic.

(c) Finally, if π is ergodic, then $\mathfrak{C} = \{0, 1\}$ (372Pa), so that 'relatively atomless over $\mathfrak{C}$' becomes 'atomless'.

381Q As far as possible I will express the ideas of this chapter in 'pure' Boolean algebra terms, without shifting to measure spaces or Stone spaces. However there is a crucial argument in §382 for which the Stone representation is an invaluable aid, and anyone studying the subject has to be able to use it.

Proposition Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. For $a \in \mathfrak{A}$ let $\widehat{a}$ be the corresponding open-and-closed subset of Z; recall that $\widehat{a}$ can be identified with the Stone space of $\mathfrak{A}_a$ (312T). For a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ let $f_\pi : Z \to Z$ be the continuous function such that $\widehat{\pi a} = f_\pi^{-1}[\widehat{a}]$ for every $a \in \mathfrak{A}$ (312Q).

(a) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ is a Boolean homomorphism represented by a continuous function $g : \widehat{b} \to \widehat{a}$, then $\pi \in \operatorname{Aut}\mathfrak{A}$ agrees with ϕ on $\mathfrak{A}_a$ iff f_π agrees with g on $\widehat{b}$.

(b) If $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism, then $a \in \mathfrak{A}$ supports π iff $\widehat{a} \supseteq \{z : f_\pi(z) \neq z\}$. So a is the support of π iff $\widehat{a} = \overline{\{z : f_\pi(z) \neq z\}}$.

(c) Suppose that $\mathfrak{A}$ is Dedekind complete and π, $\phi \in \operatorname{Aut}\mathfrak{A}$. Let G be the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π. Then

$$\begin{aligned}\phi \in G &\iff \bigcup_{n\in\mathbb{Z}} \operatorname{int}\{x : f_\phi(z) = f_\pi^n(z)\} \text{ is dense in } Z\\ &\iff \{z : f_\phi(z) \in \{f_\pi^n(z) : n \in \mathbb{Z}\}\} \text{ is comeager in } Z.\end{aligned}$$

(d) A Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is recurrent on $a \in \mathfrak{A}$ iff $\widehat{a} \subseteq \overline{\bigcup_{n\geq 1} f_\pi^n[\widehat{a}]}$.

(e) Suppose that $\mathfrak{A}$ is Dedekind σ-complete, $\pi \in \operatorname{Aut}\mathfrak{A}$ is recurrent on $a \in \mathfrak{A}$, and that $\pi_a \in \operatorname{Aut}\mathfrak{A}_a$ is the induced automorphism (381M). Let f_{π_a} be the corresponding autohomeomorphism of $\widehat{a}$. For $k \geq 1$, set $G_k = \{z : z \in \widehat{a}, f^k(z) \in \widehat{a}, f^i(z) \notin \widehat{a} \text{ for } 1 \leq i < k\}$. Then $\bigcup_{k\geq 1} G_k = \widehat{a} \cap \bigcup_{k\geq 1} f^{-k}[\widehat{a}]$ is a dense open subset of $\widehat{a}$ and $f_{\pi_a}(z) = f_\pi^k(z)$ whenever $k \geq 1$ and $z \in G_k$.

proof Recall that $f_{\pi\phi} = f_\phi f_\pi$ for all Boolean homomorphisms π, $\phi : \mathfrak{A} \to \mathfrak{A}$ (312R).

(a) The point is that $\{\widehat{d} : d \subseteq a\}$ is a base for the Hausdorff topology of of $\widehat{a}$. So if $g \neq f_\pi \restriction \widehat{b}$, there are a $z \in \widehat{b}$ such that $f_\pi(z) \neq g(z)$ and a $d \subseteq a$ such that $g(z) \in \widehat{d}$ and $f_\pi(z) \notin \widehat{d}$. In this case,

$$z \in g^{-1}[\widehat{d}] \setminus f_\pi^{-1}[\widehat{d}] = \widehat{\phi d} \setminus \widehat{\pi d},$$

and $\phi \neq \pi \restriction \mathfrak{A}_a$. On the other hand, if $g = f_\pi \restriction \widehat{b}$, then

$$\widehat{\pi d} = f_\pi^{-1}[\widehat{d}] = g^{-1}[\widehat{d}] = \widehat{\phi d}$$

for every $d \subseteq a$, and $\phi = \pi \restriction \mathfrak{A}_a$.

(b)

$$\begin{aligned}a \in \mathfrak{A} \text{ supports } \pi &\iff \pi \text{ agrees with the identity on } 1 \setminus a\\ &\iff f_\pi(z) = z \text{ for every } z \in \widehat{\pi(1 \setminus a)} = Z \setminus \widehat{a}\\ &\iff \widehat{a} \supseteq \{z : f_\pi(z) \neq z\}\\ &\iff \widehat{a} \supseteq \overline{\{z : f_\pi(z) \neq z\}}.\end{aligned}$$

So the smallest such a, if there is one, must have $\widehat{a} = \overline{\{z : f_\pi(z) \neq z\}}$.

(c) If $\phi \in G$, let $\langle a_n \rangle_{n\in\mathbb{Z}}$ be a partition of unity in $\mathfrak{A}$ such that $\phi b = \pi^n b$ whenever $n \in \mathbb{Z}$ and $b \subseteq a_n$ (381I). Then $g(z) = f_\pi^n(z)$ whenever $z \in \widehat{\phi a_n}$ ((a) above). As $\sup_{n\in\mathbb{Z}} \phi a_n = 1$ in $\mathfrak{A}$,

$$\bigcup_{n\in\mathbb{Z}} \operatorname{int}\{x : f_\phi(z) = f_\pi^n(z)\} \supseteq \bigcup_{n\in\mathbb{Z}} \widehat{\phi a_n}$$

is dense (313Ca).

If $\bigcup_{n\in\mathbb{Z}} \operatorname{int}\{x : f_\phi(z) = f_\pi^n(z)\}$ is dense, it is a dense open subset of $\{z : f_\phi(z) \in \{f_\pi^n(z) : n \in \mathbb{Z}\}$, so the latter is comeager.

If $\{z : f_\phi(z) \in \{f_\pi^n(z) : n \in \mathbb{Z}\}\}$ is comeager, set $F_n = \{z : f_\phi(z) = f_\pi^n(z)\}$ for each n. Then $F_n \setminus \operatorname{int} F_n$ is nowhere dense for each n, and $Z \setminus \bigcup_{n\in\mathbb{Z}} F_n$ is meager, so $\bigcup_{n\in\mathbb{Z}} \operatorname{int} F_n$ is comeager, therefore dense (by Baire's theorem, 3A3G). If $a \in \mathfrak{A}$ is non-zero, there are an $n \in \mathbb{Z}$ such that $\widehat{\phi a} \cap \operatorname{int} F_n \neq \emptyset$ and a $b \in \mathfrak{A}$ such that $\emptyset \neq \widehat{b} \subseteq \widehat{\phi a} \cap F_n$, in which case $0 \neq \phi^{-1} b \subseteq a$ and $\phi c = \pi^n c$ for every $c \subseteq b$. By 381I(c-ii), $\phi \in G$. So the cycle is complete.

(d)

$$\begin{aligned}
\pi \text{ is recurrent on } a \iff & \text{ whenever } 0 \neq b \subseteq a \text{ there is a } k \geq 1 \\
& \qquad\qquad \text{such that } a \cap \pi^k b \neq 0 \\
\iff & \text{ whenever } 0 \neq b \subseteq a \text{ there is a } k \geq 1 \\
& \qquad\qquad \text{such that } \widehat{a} \cap (f_\pi^k)^{-1}[\widehat{b}] \neq \emptyset \\
\iff & \text{ whenever } 0 \neq b \subseteq a \text{ there is a } k \geq 1 \\
& \qquad\qquad \text{such that } f_\pi^k[\widehat{a}] \cap \widehat{b} \neq \emptyset \\
\iff & \ \widehat{a} \cap \bigcup_{k\geq 1} f_\pi^k[\widehat{a}] \text{ is dense in } \widehat{a} \\
\iff & \ \widehat{a} \subseteq \overline{\bigcup_{k\geq 1} f_\pi^k[\widehat{a}]}.
\end{aligned}$$

(e) Set $d_k = a \cap \pi^{-k} a \setminus \sup_{1\leq i<k} \pi^{-i} a$, so that $\pi^k d_k = a \cap \pi^k a \setminus \sup_{1\leq i<k} \pi^i a$. Since π^k and π_a agree on $\mathfrak{A}_{d_k}$, (a) tells us that f_π^k and f_{π_a} agree on

$$\widehat{\pi^k d_k} = \widehat{\pi_a d_k} = f_{\pi_a}^{-1}[\widehat{d_k}] = G_k.$$

Because $\sup_{k\geq 1} \pi^k d_k = a$, $\bigcup_{k\geq 1} G_k$ is dense in $\widehat{a}$.

381R Cyclic automorphisms I end the section by describing a notation which is often useful.

Definition Let $\mathfrak{A}$ be a Boolean algebra.

(a) Suppose that a, b are disjoint members of $\mathfrak{A}$ and that $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that $\pi a = b$. I will write $(\overleftarrow{a \,{}_\pi\, b})$ for the member ψ of $\operatorname{Aut}\mathfrak{A}$ defined by setting

$$\begin{aligned}
\psi d &= \pi d \text{ if } d \subseteq a, \\
&= \pi^{-1} d \text{ if } d \subseteq b, \\
&= d \text{ if } d \subseteq 1 \setminus (a \cup b).
\end{aligned}$$

Observe that in this case (if $a \neq 0$) ψ is an involution, that is, has order 2 in the group $\operatorname{Aut}\mathfrak{A}$; I will call such a ψ an **exchanging involution**, and say that it **exchanges** a with b.

(b) More generally, if $a_1, \ldots, a_n$ are disjoint elements of $\mathfrak{A}$ and $\pi_i \in \operatorname{Aut}\mathfrak{A}$ are such that $\pi_i a_i = a_{i+1}$ for each $i < n$, then I will write

$$(\overleftarrow{a_1 \,{}_{\pi_1}\, a_2 \,{}_{\pi_2}\, \ldots \,{}_{\pi_{n-1}}\, a_n})$$

for that $\psi \in \operatorname{Aut}\mathfrak{A}$ such that

$$\begin{aligned}
\psi d &= \pi_i d \text{ if } 1 \leq i < n,\ d \subseteq a_i, \\
&= \pi_1^{-1}\pi_2^{-1} \ldots \pi_{n-1}^{-1} d \text{ if } d \subseteq a_n, \\
&= d \text{ if } d \subseteq 1 \setminus \sup_{i\leq n} a_i.
\end{aligned}$$

(c) It will occasionally be convenient to use the same notation when each π_i is a Boolean isomorphism between the principal ideals $\mathfrak{A}_{a_i}$ and $\mathfrak{A}_{a_{i+1}}$, rather than an automorphism of the whole algebra $\mathfrak{A}$.

Remark The point of this notation is that we can expect to use the standard techniques for manipulating cycles that are (I suppose) familiar to you from elementary group theory; the principal change is that we have to keep track of the subscripted automorphisms ${}_{\pi}$. The following results are typical.

381S Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\psi = (\overleftarrow{a\,{}_{\pi}\,b})$ is an exchanging involution in $\operatorname{Aut}\mathfrak{A}$, then

$$\psi = (\overleftarrow{a\,{}_{\psi}\,b}) = (\overleftarrow{b\,{}_{\psi}\,a}) = (\overleftarrow{b\,{}_{\pi^{-1}}\,a})$$

has support $a \cup b$.

(b) If $\pi = (\overleftarrow{a\,{}_{\pi}\,b})$ is an exchanging involution in $\operatorname{Aut}\mathfrak{A}$, then for any $\phi \in \operatorname{Aut}\mathfrak{A}$,

$$\phi\pi\phi^{-1} = (\overleftarrow{\phi a\,{}_{\phi\pi\phi^{-1}}\,\phi b})$$

is another exchanging involution.

(c) If $\pi = (\overleftarrow{a\,{}_{\pi}\,b})$ and $\phi = (\overleftarrow{c\,{}_{\phi}\,d})$ are exchanging involutions, and a, b, c, d are all disjoint, then π and ϕ commute, and $\psi = \pi\phi = \phi\pi$ is another exchanging involution, being $(\overleftarrow{a \cup c\,{}_{\psi}\,b \cup d})$.

(d) If G is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$, $a_1, \dots, a_n \in \mathfrak{A}$ are disjoint, and $\pi_1, \dots, \pi_{n-1} \in G$, then

$$(\overleftarrow{a_1\,{}_{\pi_1}\,a_2\,{}_{\pi_2}\,\dots\,{}_{\pi_{n-1}}\,a_n}) \in G.$$

proof (a) Check the action of ψ on the principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_b$, $\mathfrak{A}_{1 \setminus (a \cup b)}$.

(b) $\phi a \cap \phi b = \phi(a \cap b) = 0$ and

$$\phi\pi\phi^{-1}\phi a = \phi\pi a = \phi b,$$

so $\psi = (\overleftarrow{\phi a\,{}_{\phi\pi\phi^{-1}}\,\phi b})$ is well-defined. Now check the action of ψ on the principal ideals $\mathfrak{A}_{\phi a}$, $\mathfrak{A}_{\phi b}$, $\mathfrak{A}_{1 \setminus \phi(a \cup b)}$.

(c) Check the action of ψ on each of the principal ideals $\mathfrak{A}_a, \dots, \mathfrak{A}_e$, where $e = 1 \setminus (a \cup b \cup c \cup d)$.

(d) Immediate from the definitions in 381Rb and 381Be.

381T Remark I must emphasize that while, after a little practice, calculations of this kind become easy and safe, they are absolutely dependent on all the cycles present involving only members of one list of disjoint elements of $\mathfrak{A}$. If, for instance, a, b, c are disjoint, then

$$(\overleftarrow{a\,{}_{\pi}\,b})(\overleftarrow{b\,{}_{\phi}\,c}) = (\overleftarrow{a\,{}_{\pi}\,b\,{}_{\phi}\,c}).$$

But if $a \cap c \neq 0$ then there is no expression for the product in this language. Secondly, of course, we must be scrupulous in checking, at every use of the notation $(\overleftarrow{a_1\,{}_{\pi_1}\,\dots\,a_n})$, that $a_1, \dots, a_n$ are disjoint and that $\pi_i a_i = a_{i+1}$ for $i < n$. Thirdly, a significant problem can arise if the automorphisms involved don't match. Consider for instance the product

$$\psi = (\overleftarrow{a\,{}_{\pi}\,b})(\overleftarrow{a\,{}_{\phi}\,b}).$$

Then we have $\psi d = \pi^{-1}\phi d$ if $d \subseteq a$, $\pi\phi^{-1}d$ if $d \subseteq b$; ψ is not necessarily expressible as a product of 'disjoint' cycles. Clearly there are indefinitely complex variations possible on this theme. A possible formal expression of a sufficient condition to avoid these difficulties is the following. Restrict yourself to calculations involving a fixed list $a_1, \dots, a_n$ of disjoint elements of $\mathfrak{A}$ for which you can describe a family of isomorphisms $\phi_{ij} : \mathfrak{A}_{a_i} \to \mathfrak{A}_{a_j}$ such that ϕ_{ii} is always the identity on $\mathfrak{A}_{a_i}$, $\phi_{jk}\phi_{ij} = \phi_{ik}$ for all i, j, k, and whenever $a_i\,{}_{\pi}\,a_j$ appears in a cycle of the calculation, then π agrees with ϕ_{ij} on $\mathfrak{A}_{a_i}$. Of course this would be intolerably unwieldy if it were really necessary to exhibit all the ϕ_{ij} every time. I believe however that it is usually easy enough to form a mental picture of the actions of the isomorphisms involved sufficiently clear to offer confidence that such ϕ_{ij} are indeed present; and in cases of doubt, then *after* performing the formal operations it is always straightforward to check that the calculations are valid, by looking at the actions of the automorphisms on each relevant principal ideal.

381X Basic exercises (a) Let X be a set and Σ an algebra of subsets of X containing all singleton sets. Show that $\operatorname{Aut}\Sigma$ can be identified with the group of permutations $f : X \to X$ such that $f[E]$ and $f^{-1}[E]$ belong to Σ for every $E \in \Sigma$.

(b) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ partitions of unity in $\mathfrak{A}$, $\mathfrak{B}$ respectively. Assume *either* that I is finite *or* that I is countable and $\mathfrak{B}$ is Dedekind σ-complete *or* that $\mathfrak{B}$ is Dedekind complete. Suppose that for each $i \in I$ we have a Boolean homomorphism $\pi_i : \mathfrak{A}_{a_i} \to \mathfrak{B}_{b_i}$. (i) Show that there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ extending every π_i. (ii) Show that π is injective iff every π_i is. (iii) Show that if *either* I is finite *or* I is countable and $\mathfrak{A}$ is Dedekind σ-complete *or* $\mathfrak{A}$ is Dedekind complete, then π is surjective iff every π_i is. (iv) Show that π is order-continuous, or sequentially order-continuous, iff every π_i is.

(c) Let $\mathfrak{A}$ be a Boolean algebra. Show that if $\pi \in \operatorname{Aut}\mathfrak{A}$ and $k \in \mathbb{Z}\setminus\{0\}$, then π is aperiodic iff π^k is.

(d) In 381H, show that the family $\langle c_i\rangle_{1\le i\le\omega}$ is uniquely determined.

>(e) Let (X,Σ,μ) be a countably separated measure space (definition: 343D), $\mathfrak{A}$ its measure algebra, $f : X \to X$ an inverse-measure-preserving function and $\pi : \mathfrak{A} \to \mathfrak{A}$ the induced homomorphism (343A). (i) Show that the support of π is $\{x : x \in X,\ f(x) \ne x\}^\bullet$. (ii) Show that π is periodic, with period $n \ge 1$, iff $\mu X > 0$, $f^n(x) = x$ for almost every x and $\{x : f^i(x) = x\}$ is negligible for $1 \le i < n$.

(f) Let (X,Σ,μ) be a localizable measure space, with measure algebra $(\mathfrak{A},\bar\mu)$. Suppose that π and ϕ are automorphisms of $\mathfrak{A}$, and that π is represented by a measure space automorphism $f : X \to X$. Show that the following are equiveridical: (i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π; (ii) there is a function $g : X \to X$, representing ϕ, such that $g(x) \in \{f^n(x) : n \in \mathbb{Z}\}$ for every $x \in X$. (*Hint*: for (ii)$\Rightarrow$(i), consider measurable envelopes of sets $F \cap g[A_n]$, where $A_n = \{x : g(x) = f^n(x)\}$ and $\mu F < \infty$.)

(g) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\pi : \mathfrak{A} \to \mathfrak{A}$ an automorphism with fixed-point subalgebra $\mathfrak{C}$. Show that π is periodic, with period $n \ge 1$, iff $\pi\restriction\mathfrak{A}_c$ has order n in the group $\operatorname{Aut}\mathfrak{A}_c$ whenever $c \in \mathfrak{C}\setminus\{0\}$. Show that π is aperiodic iff $\pi\restriction\mathfrak{A}_c$ has infinite order in the group $\operatorname{Aut}\mathfrak{A}_c$ whenever $c \in \mathfrak{C}\setminus\{0\}$.

(h) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, G a subgroup of $\operatorname{Aut}\mathfrak{A}$ and $\phi \in \operatorname{Aut}\mathfrak{A}$. Show that ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by G iff $\inf_{\pi\in G}\operatorname{supp}(\pi\phi) = 0$.

(i) Let $\mathfrak{A}$ be a Boolean algebra. Let us say that a subgroup G of $\operatorname{Aut}\mathfrak{A}$ is **finitely full** if whenever $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$, $\langle\pi_i\rangle_{i\in I}$ is a family in G, and $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that $\pi a = \pi_i a_i$ whenever $i \in I$ and $a \subseteq a_i$, then $\pi \in G$. Show that if $\pi, \phi \in \operatorname{Aut}\mathfrak{A}$ then ϕ belongs to the finitely full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π iff there are an $n \in \mathbb{N}$ and a partition of unity $\langle a_i\rangle_{-n\le i\le n}$ in $\mathfrak{A}$ such that $\phi d = \pi^i d$ whenever $|i| \le n$ and $d \subseteq a_i$.

(j) Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism which is recurrent on $a \in \mathfrak{A}$. Show that for any non-zero $b \subseteq a$ and any $n \in \mathbb{N}$ there is a $k \ge n$ such that $a \cap \pi^k b \ne 0$.

(k) Let $\mathfrak{A}$ be a Boolean algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism, and $a \in \mathfrak{A}$. Show that the following are equiveridical: (i) π is recurrent on every $b \subseteq a$; (ii) for every non-zero $b \subseteq a$ there is an $n \ge 1$ such that $b \cap \pi^n b \ne 0$; (iii) $b = \sup_{n\ge1} b \cap \pi^n b$ for every $b \subseteq a$.

>(l) Let (X,Σ,μ) be a measure space, $\mathfrak{A}$ its measure algebra, $f : X \to X$ a measure space automorphism, and π the corresponding automorphism of $\mathfrak{A}$. (i) Show that if $E \in \Sigma$ then π is doubly recurrent on $a = E^\bullet$ iff $E \setminus \bigcup_{n\ge1} f^{-n}[E]$ and $E \setminus \bigcup_{n\ge1} f^n[E]$ are negligible. (ii) Show that in this case there is a measurable $F \subseteq E$ such that $E \setminus F$ is negligible and $\{n : n \in \mathbb{Z},\ f^n(x) \in F\}$ is unbounded above and below in $\mathbb{Z}$ for every $x \in F$. (iii) For $x \in F$ let $k(x) = \min\{n : n \ge 1,\ f^n(x) \in F\}$. Show that $x \mapsto f^{k(x)}(x) : F \to F$ represents the induced automorphism π_a on the principal ideal $\mathfrak{A}_a$.

(m) For a Boolean algebra $\mathfrak{A}$, a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is **nowhere aperiodic** if $\inf\{a : a \in \mathfrak{A},\ a$ supports π^n for some $n \ge 1\} = 0$. Show that if $\mathfrak{A}$ is Dedekind σ-complete and $\pi \in \operatorname{Aut}\mathfrak{A}$ is nowhere aperiodic and doubly recurrent on $a \in \mathfrak{A}$, then the induced automorphism π_a is nowhere aperiodic.

(n) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $\pi \in \operatorname{Aut}\mathfrak{A}$ an automorphism and $\mathfrak{C}$ the fixed-point subalgebra of π. Suppose that π is doubly recurrent on $a \in \mathfrak{A}$ and that π_a is the induced automorphism on $\mathfrak{A}_a$. Show that the fixed-point subalgebra of π_a is $\{c \cap a : c \in \mathfrak{C}\}$, so that if π is ergodic, so is π_a.

(o) Let $\mathfrak{A}$ be a Boolean algebra with Stone space Z, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism corresponding to $f : Z \to Z$. (i) Show that π is periodic, with period $n \ge 1$, iff $Z \ne \emptyset$, $f^n(z) = z$ for every $z \in Z$ and $\{z : f^i(z) = z\}$ is nowhere dense whenever $1 \le i < n$. (ii) Show that π is aperiodic iff $\{z : f^n(z) = z,\ f^n(w) \ne z$ for every $w \ne z\}$ is nowhere dense for every $n \ge 1$.

(p) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, G a subgroup of $\operatorname{Aut}\mathfrak{A}$ and G^* the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by G. Suppose that every member of G has a support. Show that every member of G^* has a support.

381Y Further exercises (a) (i) Give an example to show that the word 'injective' in the statement of 381H is essential. (ii) Give an example to show that, in 381H, we can have $\pi c_\omega \neq c_\omega$.

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Let us say that G is **full** if whenever $\phi : \mathfrak{A} \to \mathfrak{A}$ is an order-continuous Boolean homomorphism, and there is a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that for every $i \in I$ there is a $\pi_i \in G$ such that $\phi a = \pi_i a$ for every $a \subseteq a_i$, then $\phi \in G$. Show that if ϕ and π are order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself, then the following are equiveridical: (i) ϕ belongs to the full semigroup generated by π; (ii) for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and an $n \in \mathbb{N}$ such that $\phi d = \pi^n d$ for every $d \subseteq b$; (iii) there is a partition of unity $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\phi a = \pi^n a$ whenever $n \in \mathbb{N}$ and $a \subseteq a_n$.

(c) Give an example of a Dedekind σ-complete Boolean algebra $\operatorname{Aut}\mathfrak{A}$ and an automorphism π of $\mathfrak{A}$ such that the countably full subgroup generated by π is not full.

(d) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and let G be the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by a subset A of $\operatorname{Aut}\mathfrak{A}$. Show that if *either* A is countable *or* $\mathfrak{A}$ is ccc, then G is full.

381 Notes and comments There are no long individual proofs in this section, and in so far as there is any delicacy in the arguments it is as often as not because (as in 381E) I am taking facts which are easy to prove for automorphisms of Dedekind complete algebras and separating out the parts which happen to be true in greater generality. However the parts are numerous enough for the sum to be not entirely predictable. The most important ideas are surely in 381M-381N.

In 381Q I give indications, including the minimum necessary for an application in the next section, of how to express the concepts here in terms of continuous functions on Stone spaces. When we come, in §383 and onwards, to look specifically at measure algebras, many of our homomorphisms will be derived from inverse-measure-preserving functions, and the results will be more effective if we can display them in terms of functions on measure spaces. Some appropriate translations are in 381Xe-381Xl. But these I will avoid in the proofs of the main theorems because not all automorphisms of measure algebras can be represented by automorphisms of the measure spaces we start from (343Jc). Of course Lebesgue measure is different, in ways explored in §344, and classical ergodic theory has not needed to make a clear distinction here. One of my purposes in this volume is to set out a framework in which transformations of measure *spaces* take their proper place as an inspiration for the theory rather than a foundation.

382 Factorization of automorphisms

My aim in this chapter is to investigate the automorphism groups of measure algebras, but as usual I prefer to begin with results which can be expressed in the language of general Boolean algebras. The principal theorems in this section are 382M, giving a sufficient condition for every member of a full group of automorphism to be a product of involutions, and 382R, describing the normal subgroups of full groups. The former depends on Dedekind σ-completeness and the presence of 'separators' (382Aa); the latter needs a Dedekind complete algebra and a group with 'many involutions' (382O). Both concepts are chosen with a view to the next section, where the results will be applied to groups of measure-preserving automorphisms.

382A Definitions Let $\mathfrak{A}$ be a Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$.

(a) I say that $a \in \mathfrak{A}$ is a **separator** for π if $a \cap \pi a = 0$ and $\pi b = b$ whenever $b \in \mathfrak{A}$ and $b \cap \pi^n a = 0$ for every $n \in \mathbb{Z}$.

(b) I say that $a \in \mathfrak{A}$ is a **transversal** for π if $\sup_{n\in\mathbb{Z}} \pi^n a = 1$ and $\pi^n b = b$ whenever $n \in \mathbb{Z}$ and $b \subseteq a \cap \pi^n a$.

382B Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. If every power of π has a separator and π^n is the identity, where $n \geq 1$, then π has a transversal.

proof (a) For $0 \le j < n$ let $a_j \in \mathfrak{A}$ be a separator for π^j. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $A = \{\pi^i a_j : 0 \le i, j < n\}$. Because $\pi[A] = A$, $\pi[\mathfrak{B}] = \mathfrak{B}$. (The set $\{a : a \in \mathfrak{B}, \pi a \in \mathfrak{B}, \pi^{-1}a \in \mathfrak{B}\}$ is a subalgebra of $\mathfrak{A}$ including A, so must be $\mathfrak{B}$.) Because A is finite, so is $\mathfrak{B}$; let B be the set of atoms of $\mathfrak{B}$. Then $\pi{\restriction}B$ is a permutation of the finite set B.

(b) Let $\mathcal{C}$ be the set of orbits of $\pi{\restriction}B$, that is, the family of sets of the form $\{\pi^k b : k \in \mathbb{Z}\}$ for $b \in B$. If $b \in C \in \mathcal{C}$, set $m = \#(C)$; then $d = \pi^m d$ for every $d \subseteq b$. **P** If $m = n$ this is trivial. Otherwise, b is either disjoint from, or included in, $\pi^i a_m$ whenever $0 \le i < n$, and therefore for every $i \in \mathbb{Z}$. But we have $a_m \cap \pi^m a_m = 0$, so $\pi^i a_m \cap \pi^{i+m} a_m = 0$ for every i, and $b = \pi^m b$ must be disjoint from $\pi^i a_m$, for every i. By the other clause in the definition of 'separator', $\pi^m d = d$ for every $d \subseteq b$. **Q**

(c) For each $C \in \mathcal{C}$, choose $b_C \in C$. Set $c = \sup_{C\in\mathcal{C}} b_C$. Then c is a transversal for π. **P** If $C \in \mathcal{C}$, we have $\pi^n b_C = b_C$, so $k_C = \#(C)$ is a factor of n. Now

$$\sup_{0\le k<n} \pi^k c = \sup_{C\in\mathcal{C},0\le k<n} \pi^k b_C = \sup_{C\in\mathcal{C}} \sup C = \sup(\bigcup \mathcal{C}) = \sup B = 1.$$

So certainly $\sup_{k\in\mathbb{Z}} \pi^k c = 1$. Now suppose that $k \in \mathbb{Z} \setminus \{0\}$ and $d \subseteq c \cap \pi^k c$. Set $B_0 = \{b : b \in B,\ d \cap b \ne 0\}$. If $b \in B_0$, then $b \cap c \ne 0$, so $b = b_C$ where $C \in \mathcal{C}$ is the orbit of $\pi{\restriction}B$ containing b. Next, $d \cap b \cap \pi^k c \ne 0$, so $\pi^{-k}(d \cap b) \cap c \ne 0$ and there is a $b' \in B$ such that $\pi^{-k}(d \cap b) \cap b' \ne 0$; in this case we must have $b' = \pi^{-k} b \in C$. But as $b' \cap c \supseteq \pi^{-k}(d \cap b) \cap c$ is non-zero, $b' = b_C = b$. Thus $b = \pi^k b$ and k is a multiple of $\#(C)$. Since $\pi^{\#(C)}(d \cap b) = d \cap b$, by (b), $\pi^k(d \cap b) = d \cap b$.

This is true for every $b \in B$ meeting d; so

$$\pi^k d = \pi^k(\sup_{b\in B_0} d \cap b) = \sup_{b\in B_0} \pi^k(d \cap b) = \sup_{b\in B_0} d \cap b = d.$$

As k and d are arbitrary, c is a transversal for π. **Q**

382C Corollary If $\mathfrak{A}$ is a Boolean algebra and $\pi \in \mathfrak{A}$ is an involution, then π is an exchanging involution iff it has a separator iff it has a transversal.

proof If π exchanges a and πa then of course a is a separator for π. If π has a separator, then every power of π has a separator, so 382B tells us that π has a transversal. If a is a transversal for π then $a \cup \pi a = \sup_{n\in\mathbb{Z}} \pi^n a = 1$ and $\pi b = b$ whenever $b \subseteq a \cap \pi a$, so π exchanges $a \setminus \pi a$ and $\pi a \setminus a$.

382D Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) π has a separator;

(ii) there is an $a \in \mathfrak{A}$ such that $a \cap \pi a = 0$ and $a \cup \pi a \cup \pi^2 a$ supports π;

(iii) there is a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\sup_{n\in\mathbb{N}} \pi a_n \setminus a_n$ supports π;

(iv) there is a partition of unity (a', a'', b', b'', c, e) in $\mathfrak{A}$ such that

$$\pi a' = b', \quad \pi a'' = b'', \quad \pi b'' = c, \quad \pi(b' \cup c) = a' \cup a'', \quad \pi d = d \text{ for every } d \subseteq e.$$

proof (i)$\Rightarrow$(ii) Suppose that a is a separator for π. Set $a^+ = \sup_{n\ge1} \pi^n a$, $a^- = \sup_{n\ge1} \pi^{-n} a$; we are supposing that $a \cap \pi a = 0$ and that $a \cup a^+ \cup a^-$ supports π. For $n \in \mathbb{N}$ set $a_n = \pi^n a \setminus \sup_{0\le i<n} \pi^i a$, so that $\langle a_n \rangle_{n\in\mathbb{N}}$ is disjoint and has supremum $a \cup a^+$. Set $b_1 = \sup_{n\in\mathbb{N}} a_{2n} \setminus \pi^{-1} a$. Since $a \cap \pi^{-1} a = \pi^{-1}(a \cap \pi a) = 0$, $a \subseteq b_1 \subseteq a \cup a^+$. For any $n \in \mathbb{N}$,

$$\pi(a_{2n} \setminus \pi^{-1} a) = \pi^{2n+1} a \setminus (a \cup \sup_{1\le i\le 2n} \pi^i a) = a_{2n+1},$$

so $b_1 \cap \pi b_1 = 0$. Note that $\pi b_1 \subseteq a^+$, while $a^+ \setminus \pi^{-1} a \subseteq b_1 \cup \pi b_1$.

Set $c = a \setminus a^+$. Then

$$\pi^i c \cap \pi^j c = \pi^j(c \cap \pi^{i-j} c) \subseteq \pi^j(a \setminus \pi^{i-j} a^+) \subseteq \pi^j(a \setminus \pi^{i-j}\pi^{j-i} a) = 0$$

whenever $i < j$ in $\mathbb{Z}$, so $\langle \pi^k c \rangle_{k\in\mathbb{Z}}$ is disjoint. We have

$$\begin{aligned}\sup_{n\ge1} \pi^{-n} c &= \sup_{n\ge1}(\pi^{-n} a \setminus \sup_{i>-n} \pi^i a) = \sup_{n\ge1}(\pi^{-n} a \setminus \sup_{0\le i<n} \pi^{-i} a) \setminus (a \cup a^+)\\ &= \sup_{n\ge1} \pi^{-n} a \setminus (a \cup a^+) = a^- \setminus (a \cup a^+).\end{aligned}$$

If $k \ge 1$ and $i \ge 0$ then

$$\pi^{-k}c\cap\pi^i a=\pi^{-k}(c\cap\pi^{i+k}a)\subseteq\pi^{-k}(c\cap a^+)=0;$$

as i is arbitrary, $\pi^{-k}c\cap b_1=0$. So if we set $b=b_1\cup\sup_{k\ge1}\pi^{-2k}c$,

$$\begin{aligned}b\cap\pi b&=(b_1\cap\pi b_1)\cup(\sup_{k\ge1}b_1\cap\pi^{1-2k}c)\cup(\sup_{k\ge1}\pi^{-2k}c\cap\pi b_1)\cup(\sup_{j,k\ge1}\pi^{-2j}c\cap\pi^{1-2k}c)\\&\subseteq 0\cup0\cup\pi(\sup_{k\ge1}\pi^{-2k-1}c\cap b_1)\cup0=0.\end{aligned}$$

Since

$$\begin{aligned}b\cup\pi b\cup\pi^{-1}b&\supseteq b_1\cup\pi b_1\cup\pi^{-1}a\cup\sup_{n\ge1}\pi^{-n}c\\&\supseteq a\cup a^+\cup(a^-\setminus(a\cup a^+))=a\cup a^+\cup a^-\end{aligned}$$

supports π, $\pi^{-1}b$ witnesses that (ii) is true.

(ii)$\Rightarrow$(iii) If $a\in\mathfrak{A}$ is such that $a\cap\pi a=0$ and $a\cup\pi a\cup\pi^2a$ supports π, then $\pi^{n+1}a=\pi^{n+1}a\setminus\pi^n a$ for every n, so we can set $a_n=\pi^{n-1}a$ for each n to obtain a sequence witnessing (iii).

(iii)$\Rightarrow$(i) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is such that $\sup_{n\in\mathbb{N}}\pi a_n\setminus a_n$ supports π, set $b_n=\sup_{k\in\mathbb{Z}}\pi^k(\pi a_n\setminus a_n)$, $c_n=b_n\setminus\sup_{0\le i<n}b_i$ for each $n\in\mathbb{N}$. Then $\pi b_n=b_n$ and $\pi c_n=c_n$ for every $n\in\mathbb{N}$, while $\langle c_n\rangle_{n\in\mathbb{N}}$ is disjoint. Set $a=\sup_{n\in\mathbb{N}}c_n\cap a_n\setminus\pi^{-1}a_n$. Then

$$\begin{aligned}a\cap\pi a&=\sup_{m,n\in\mathbb{N}}(c_m\cap a_m\setminus\pi^{-1}a_m)\cap(\pi c_n\cap\pi a_n\setminus a_n)\\&=\sup_{m,n\in\mathbb{N}}(c_m\cap a_m\setminus\pi^{-1}a_m)\cap(c_n\cap\pi a_n\setminus a_n)\\&=\sup_{n\in\mathbb{N}}c_n\cap(a_n\setminus\pi^{-1}a_n)\cap(\pi a_n\setminus a_n)=0.\end{aligned}$$

Next,

$$\begin{aligned}\sup_{k\in\mathbb{Z}}\pi^k a&=\sup_{n\in\mathbb{N},k\in\mathbb{Z}}c_n\cap\pi^k a_n\setminus\pi^{k-1}a_n\\&=\sup_{n\in\mathbb{N},k\in\mathbb{Z}}c_n\cap\pi^{k+1}a_n\setminus\pi^k a_n=\sup_{n\in\mathbb{N}}c_n\cap b_n\\&=\sup_{n\in\mathbb{N}}c_n=\sup_{n\in\mathbb{N}}b_n\supseteq\sup_{n\in\mathbb{N}}\pi a_n\setminus a_n\end{aligned}$$

supports π. So a is a separator for π.

(ii)$\Rightarrow$(iv) Let a be such that $a\cap\pi a=0$ and $a\cup\pi a\cup\pi^2a$ supports π. Set $c=\pi^2a\setminus(a\cup\pi a)$, $b''=\pi^{-1}c\subseteq\pi a$, $b'=\pi a\setminus b''$, $a''=\pi^{-1}b''\subseteq a$, $a'=a\setminus a''$ and $e=1\setminus(a\cup\pi a\cup\pi^2a)$. Then $(a,\pi a,c,e)$ and (a',a'',b',b'',c,e) are partitions of unity in $\mathfrak{A}$; $\pi a''=b''$; $\pi a'=\pi a\setminus b''=b'$; $\pi b''=c$; $\pi d=d$ for every $d\subseteq e$; so

$$\pi(b'\cup c)=\pi(1\setminus(a\cup b''\cup e))=1\setminus(\pi a\cup\pi b''\cup\pi e)=1\setminus(\pi a\cup c\cup e)=a=a'\cup a''.$$

(iv)$\Rightarrow$(ii) If a',a'',b',b'',c,e witness (iv), then $a=a'\cup a''$ witnesses (ii).

382E Corollary (a) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\pi\in\operatorname{Aut}\mathfrak{A}$ has a separator, then π has a support.

(b) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra then every $\pi\in\operatorname{Aut}\mathfrak{A}$ has a separator.

proof (a) Taking $a\in\mathfrak{A}$ such that $a\cap\pi a=0$ and $e=a\cup\pi a\cup\pi^2a$ supports π, we see that e must actually be the support of π (381Ei, 381Ea).

(b) If $\mathfrak{A}$ is Dedekind complete and $\pi\in\operatorname{Aut}\mathfrak{A}$, let P be the set $\{d:d\in\mathfrak{A},\ d\cap\pi d=0\}$. Then P has a maximal element. **P** Of course $P\ne\emptyset$, as $0\in P$. If $Q\subseteq P$ is non-empty and upwards-directed, set $a=\sup Q$, which is defined because $\mathfrak{A}$ is Dedekind complete; then $\pi a=\sup\pi[Q]$ (since π, being an automorphism, is surely order-continuous). If d_1, $d_2\in Q$, there is a $d\in Q$ such that $d_1\cup d_2\subseteq d$, so $d_1\cap\pi d_2\subseteq d\cap\pi d=0$. By 313Bc, $a\cap\pi a=0$. This means that $a\in P$ and is an upper bound for Q in P. As Q is arbitrary, Zorn's Lemma tells us that P has a maximal element. **Q**

Let $b \in P$ be maximal. Then $b \cap \pi b = 0$. Set $e = b \cup \pi a \cup \pi^{-1} b$. **?** If e does not support π, let $d \subseteq 1 \setminus e$ be such that $d \cap \pi d = 0$ (381Ei). Then $d \cap \pi b \subseteq d \cap e = 0$, while also $b \cap \pi d \subseteq \pi(\pi^{-1} b \cap d) \subseteq \pi(e \cap d) = 0$; so $(b \cup d) \cap \pi(b \cup d) = 0$, and $b \subset b \cup d \in P$, which is impossible. **X** So if we set $a = \pi^{-1} b$ we have a witness of 382D(ii), and π has a separator.

Remark 382Eb and 382D(i)⇔(ii) together amount to 'Frolík's theorem' (FROLÍK 68).

382F Corollary Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra.

(a) Every involution in $\operatorname{Aut}\mathfrak{A}$ is an exchanging involution.

(b) If $\pi \in \operatorname{Aut}\mathfrak{A}$ is periodic with period $n \geq 2$, there is an $a \in \mathfrak{A}$ such that $(a, \pi a, \pi^2 a, \dots, \pi^{n-1} a)$ is a partition of unity in $\mathfrak{A}$; that is (in the language of 381R) π is of the form $(\overleftarrow{a_1 \,{}_\pi\, a_2 \,{}_\pi\, \dots \,{}_\pi\, a_n})$ where $(a_1, \dots, a_n)$ is a partition of unity in $\mathfrak{A}$.

proof (a) By 382Eb, every involution has a separator; now use 382C.

(b) Again because every automorphism has a separator, 382B tells us that π has a transversal a. In this case, $a \cap \pi^k a$ must be disjoint from the support of π^k for every $k \in \mathbb{Z}$; since $\operatorname{supp} \pi^k = 1$ for $0 < k < n$, $a \cap \pi^k a = 0$ for $0 < k < n$; of course it follows that $\pi^i a \cap \pi^j a = \pi^i(a \cap \pi^{j-i} a) = 0$ if $0 \leq i < j < n$. So $a, \pi a, \dots, \pi^{n-1} a$ are disjoint; since $\sup_{0 \leq i < n} \pi^i a = \sup_{i \in \mathbb{Z}} \pi^i a = 1$, they constitute a partition of unity.

382G Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$.

(a) Suppose that $\langle a_n \rangle_{n \in \mathbb{N}}$ is a family in $\mathfrak{A}$ such that $\pi a_n = a_n$ and $\pi \restriction \mathfrak{A}_{a_n}$ has a transversal for every n. Set $a = \sup_{n \in \mathbb{N}} a_n$; then $\pi a = a$ and $\pi \restriction \mathfrak{A}_a$ has a transversal.

(b) If a is a transversal for π it is a transversal for π^{-1}.

(c) Suppose that $a \in \mathfrak{A}$. Set

$$a^* = \sup_{n \in \mathbb{Z}} (\pi^n a \setminus \sup_{i > n} \pi^i a), \quad a_* = \sup_{n \in \mathbb{Z}} (\pi^n a \setminus \sup_{i < n} \pi^i a).$$

Then $\pi a^* = a^*$, $\pi a_* = a_*$ and $\pi \restriction \mathfrak{A}_{a^*}$, $\pi \restriction \mathfrak{A}_{a_*}$ both have transversals.

proof (a) Of course $\pi a = \sup_{n \in \mathbb{N}} \pi a_n = a$, so we can speak of $\pi \restriction \mathfrak{A}_a$. For each $n \in \mathbb{N}$, let b_n be a transversal for $\pi \restriction \mathfrak{A}_{a_n}$. Set $b = \sup_{n \in \mathbb{N}} (b_n \setminus \sup_{i < n} a_i)$. Then b is a transversal for $\pi \restriction \mathfrak{A}_a$. **P** Of course $b \in \mathfrak{A}_a$. Now

$$\begin{aligned} \sup_{k \in \mathbb{Z}} \pi^k b &= \sup_{k \in \mathbb{Z}} \sup_{n \in \mathbb{N}} (\pi^k b_n \setminus \sup_{i < n} \pi^k a_i) = \sup_{n \in \mathbb{N}} \sup_{k \in \mathbb{Z}} (\pi^k b_n \setminus \sup_{i < n} a_i) \\ &= \sup_{n \in \mathbb{N}} \big((\sup_{k \in \mathbb{Z}} \pi^k b_n) \setminus \sup_{i < n} a_i \big) = \sup_{n \in \mathbb{N}} (a_n \setminus \sup_{i < n} a_i) = \sup_{n \in \mathbb{N}} a_n = a. \end{aligned}$$

Next, suppose that $k \in \mathbb{Z}$ and

$$\begin{aligned} d \subseteq b \cap \pi^k b &= \sup_{m, n \in \mathbb{N}} (b_m \setminus \sup_{i < m} a_i) \cap (\pi^k b_n \setminus \sup_{j < n} \pi^k a_j) \\ &= \sup_{m, n \in \mathbb{N}} (b_m \cap a_m \setminus \sup_{i < m} a_i) \cap (\pi^k b_n \cap a_n \setminus \sup_{j < n} a_j) = \sup_{n \in \mathbb{N}} (b_n \cap \pi^k b_n \setminus \sup_{i < n} a_i). \end{aligned}$$

Setting $d_n = d \cap b_n \cap \pi^k b_n$ for each n, we have

$$d = \sup_{n \in \mathbb{N}} d_n = \sup_{n \in \mathbb{N}} \pi^k d_n = \pi^k d.$$

As k and d are arbitrary, b is a transversal for $\pi \restriction \mathfrak{A}_a$. **Q**

(b) We have only to note that the definition in 382Ab is symmetric between π and π^{-1}.

(c)

$$\begin{aligned} \pi a^* &= \sup_{n \in \mathbb{Z}} (\pi^{n+1} a \setminus \sup_{i > n} \pi^{i+1} a) \\ &= \sup_{n \in \mathbb{Z}} (\pi^{n+1} a \setminus \sup_{i > n+1} \pi^i a) = \sup_{n \in \mathbb{Z}} (\pi^n a \setminus \sup_{i > n} \pi^i a) = a^*. \end{aligned}$$

Set $b_n = \pi^n a \setminus \sup_{i > n} \pi^i a$ for each n, $b = \sup_{n \in \mathbb{Z}} \pi^{-n} b_n \subseteq a$. Writing b^* for $\sup_{n \in \mathbb{Z}} \pi^n b$, we have $b^* \supseteq \sup_{n \in \mathbb{Z}} b_n = a^*$. Note that $\pi^{-n} b_n \cap \pi^i a = 0$ for every $i \geq 1$. So if $m < n$ in $\mathbb{Z}$,

$$\pi^m b \cap \pi^n b \subseteq \pi^m (\sup_{i \in \mathbb{Z}} \pi^{-i} b_i \cap \pi^{n-m} a) = 0.$$

Thus $\langle \pi^i b \rangle_{i \in \mathbb{Z}}$ is disjoint, and b is a transversal for $\pi \restriction \mathfrak{A}_{a^*}$.

Now

$$a_* = \sup_{n\in\mathbb{Z}}(\pi^n a \setminus \sup_{i<n}\pi^i a) = \sup_{n\in\mathbb{Z}}(\pi^{-n} a \setminus \sup_{i>n}\pi^{-i} a).$$

So $\pi^{-1}a_* = a_*$ and $\pi^{-1}\restriction\mathfrak{A}_{a_*}$ has a transversal. It follows at once that $\pi a_* = a_*$ and (using (b)) that $\pi\restriction\mathfrak{A}_{a_*}$ has a transversal.

382H Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. If π has a transversal, it is expressible as the product of at most two exchanging involutions both belonging to the countably full subgroup of $\mathfrak{A}$ generated by π.

proof Let a be a transversal for π. For $n \ge 1$, set $a_n = a \cap \pi^n a \setminus \sup_{1\le i<n}\pi^i a$; set $a_0 = a \setminus \sup_{i\ge 1}\pi^i a$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint and $\sup_{n\in\mathbb{N}} a_n = a$. We have $\pi^n b = b$ whenever $b \subseteq a_n$, while $\langle \pi^i a_0\rangle_{i\ge 1}$ is disjoint, so $\langle \pi^i a_0\rangle_{i\in\mathbb{Z}}$ is disjoint. For any $n \ge 1$, a_n is disjoint from $\pi^i a_n$ for $0 < i < n$, so $\langle \pi^i a_n\rangle_{i<n}$ is disjoint. If $0 \le i < m$ and $0 \le j < n$ and $i \le j$ and $\pi^i a_m \cap \pi^j a_n$ is non-zero, then $1 \le n - j + i \le n$ and

$$\begin{aligned} a_n \cap \pi^{n-j+i}a &= \pi^{n-j+i}a \cap \pi^n a_n = \pi^{n-j}(\pi^i a \cap \pi^j a_n) \\ &\supseteq \pi^{n-j}(\pi^i a_m \cap \pi^j a_n) \ne 0, \end{aligned}$$

so $i = j$; in this case $a_m \cap a_n \ne 0$ so $m = n$. If $0 \le i < n$ and $j \in \mathbb{Z}$ and $b = \pi^i a_n \cap \pi^j a_0$, then $\pi^n b = b$ and $\pi^n b$ is disjoint from b, so $b = 0$. This shows that all the $\pi^i a_n$ for $0 \le i < n$, and the $\pi^j a_0$ for $j \in \mathbb{Z}$, are disjoint. Also, because $\pi^n a_n = a_n$ for $n \ge 1$,

$$\sup_{0\le i<n}\pi^i a_n \cup \sup_{j\in\mathbb{Z}}\pi^j a_0 = \sup_{n\in\mathbb{N},j\in\mathbb{Z}}\pi^j a_n \cup \sup_{j\in\mathbb{Z}}\pi^j a_0 = \sup_{j\in\mathbb{Z}}\pi^j a = 1.$$

For any $n \ge 1$,

$$\langle \pi^{-2j}\pi^j a_n\rangle_{0\le j<n} = \langle \pi^{-j} a_n\rangle_{0\le j<n} = \langle \pi^{n-j} a_n\rangle_{0\le j<n},$$

$$\langle \pi^{1-2j}\pi^j a_n\rangle_{0\le j<n} = \langle \pi^{1-j} a_n\rangle_{0\le j<n} = \langle \pi^{n+1-j} a_n\rangle_{0\le j<n}$$

are disjoint and cover $\sup_{0\le j<n}\pi^j a_n$; while of course

$$\langle \pi^{-2j}\pi^j a_0\rangle_{j\in\mathbb{Z}} = \langle \pi^{-j} a_0\rangle_{j\in\mathbb{Z}},$$

$$\langle \pi^{1-2j}\pi^j a_0\rangle_{j\in\mathbb{Z}} = \langle \pi^{1-j} a_0\rangle_{j\in\mathbb{Z}}$$

are disjoint and cover $\sup_{j\in\mathbb{Z}}\pi^j a_0$. So we can define ϕ_1, $\phi_2 \in \operatorname{Aut}\mathfrak{A}$ by setting

$$\begin{aligned} \phi_1 d = \pi^{-2j}d &\text{ if } j \in \mathbb{Z} \text{ and } d \subseteq \pi^j a_0 \\ &\text{ or if } 0 \le j < n \text{ and } d \subseteq \pi^j a_n \\ \phi_2 d = \pi^{1-2j}d &\text{ if } j \in \mathbb{Z} \text{ and } d \subseteq \pi^j a_0 \\ &\text{ or if } 0 \le j < n \text{ and } d \subseteq \pi^j a_n. \end{aligned}$$

Note that if $n \ge 1$ and $k \in \mathbb{Z}$ is arbitrary, then we have $\pi^k a_n = \pi^j a_n$ where $0 \le j < n$ and $j \equiv k \bmod n$, so if $d \subseteq \pi^k a_n$ then

$$\phi_1 d = \pi^{-2j}d = \pi^{-2k}d, \quad \phi_2 d = \pi^{1-2j}d = \pi^{1-2k}d$$

because $\pi^n d = d$. So if $d \subseteq \pi^j a_n$ for any $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, we have $\phi_1 d = \pi^{-2j}d \subseteq \pi^{-j}a_n$ and

$$\phi_2\phi_1 d = \pi^{1-2(-j)}\pi^{-2j}d = \pi d.$$

Because $\sup_{n\in\mathbb{N},j\in\mathbb{Z}}\pi^j a_n = 1$, $\phi_2\phi_1 = \pi$. Of course both ϕ_1 and ϕ_2 belong to the countably full subgroup generated by π. Next, ϕ_1 exchanges

$$\sup_{j\ge 1}\pi^j a_0 \cup \sup_{\substack{n\ge 2\\ 0<j\le\lfloor(n-1)/2\rfloor}}\pi^j a_n,$$

$$\sup_{j\le -1}\pi^j a_0 \cup \sup_{\substack{n\ge 2\\ -\lfloor(n-1)/2\rfloor\le j<0}}\pi^j a_n,$$

so is either the identity or an exchanging involution. In the same way, ϕ_2 exchanges

$$\sup_{j\ge 1}\pi^j a_0 \cup \sup_{\substack{n\ge 2\\ 1\le j\le\lfloor n/2\rfloor}}\pi^j a_n,$$

$$\sup_{j\le 0}\pi^j a_0 \cup \sup_{\substack{n\ge 2\\ -\lfloor n/2\rfloor<j\le 0}}\pi^j a_n,$$

so it too is either the identity or an exchanging involution. Thus we have a factorization of the desired type.

382I Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator.

(a) Every member of G has a support.

(b) Suppose $\pi\in G$ and $n\ge 1$ are such that π^n is the identity. Then π has a transversal.

(c) Let $\pi\in G$, and set $e^*=\inf_{n\ge 1}\operatorname{supp}(\pi^n)$. Then $\pi\restriction\mathfrak{A}_{1\setminus e^*}$ has a transversal.

(d) If $e\in\mathfrak{A}$ is such that $\pi e=e$ for every $\pi\in G$, then $\{\pi\restriction\mathfrak{A}_e:\pi\in G\}$ is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}_e$, and $\pi\restriction\mathfrak{A}_e$ has a separator for every $\pi\in G$.

proof (a) 382Ea.

(b) Induce on n. If $n=1$ then 1 is a transversal for π. For the inductive step to $n>1$, let $a\in\mathfrak{A}$ be such that $a\cap\pi a=0$ and $\pi b=b$ whenever $b\cap\pi^i a=0$ for every $i\in\mathbb{Z}$. Let $\mathfrak{B}$ be the (finite) subalgebra of $\mathfrak{A}$ generated by $\{\pi^i a:0\le i<n\}$. Then $\pi^n a=a\in\mathfrak{B}$, so $\{b:\pi b\in\mathfrak{B}\}$ is a subalgebra of $\mathfrak{A}$ containing $\pi^i a$ whenever $i<n$, and includes $\mathfrak{B}$; thus $\pi b\in\mathfrak{B}$ for every $b\in\mathfrak{B}$. As π is injective, $\pi\restriction\mathfrak{B}\in\operatorname{Aut}\mathfrak{B}$. Let E be the set of atoms of $\mathfrak{B}$; then $\pi\restriction E$ is a permutation of E.

Let $C\subseteq E$ be an orbit of π. Then $\pi(\sup C)=\sup C$, and $\pi\restriction\mathfrak{A}_{\sup C}$ has a transversal. **P** Take $e\in C$, $k=\#(C)$. Then $\pi^i e\in C\setminus\{e\}$, so $e\cap\pi^i e=0$, whenever $1\le i<k$. As π^n is the identity, k is a factor of n. If $k=1$, then e itself is a transversal for $\pi\restriction\mathfrak{A}_{\sup C}=\pi\restriction\mathfrak{A}_e$. If $k>1$, define $\phi\in\operatorname{Aut}\mathfrak{A}$ by setting $\phi d=\pi^k(e\cap d)\cup(d\setminus e)$ for every $d\in\mathfrak{A}$. Then $\phi\in G$, because G is countably full, and $\phi^{n/k}$ is the identity. By the inductive hypothesis, ϕ has a transversal $c\in\mathfrak{A}$. There is some $m\in\mathbb{Z}$ such that $e'=e\cap\phi^m c\ne 0$. Now

$$\sup_{i\in\mathbb{Z}}\pi^{ki}e'=\sup_{i\in\mathbb{Z}}\phi^i e'=\sup_{i\in\mathbb{Z}}(e\cap\phi^{m+i}c)=e\cap\sup_{i\in\mathbb{Z}}\phi^i c=e,$$

so

$$\sup_{j\in\mathbb{Z}}\pi^j e'=\sup_{0\le j<k}\pi^j(\sup_{i\in\mathbb{Z}}\pi^{ki}e')=\sup_{0\le j<k}\pi^j e=\sup C.$$

Also, if $0\le j<k$ and $i\in\mathbb{Z}$ and

$$0\ne d\subseteq e'\cap\pi^{ki+j}e'\subseteq e\cap\pi^{ki+j}e=e\cap\pi^j e,$$

we must have $j=0$ and $d\subseteq e'\cap\phi^i e'$, in which case $\pi^{ki+j}d=\phi^i d=d$. So e' is a transversal for $\pi\restriction\mathfrak{A}_{\sup C}$. **Q**

Let $\mathcal{C}$ be the set of orbits of $\pi\restriction E$, and for $C\in\mathcal{C}$ let c_C be a transversal for $\pi\restriction\mathfrak{A}_{\sup C}$. Then $\sup_{C\in\mathcal{C}}c_C$ is a transversal for π (382Ga). Thus the induction proceeds.

(c) Set $e_0=1\setminus\operatorname{supp}\pi$, $e_n=\inf_{1\le i\le n}\operatorname{supp}(\pi^i)\setminus\operatorname{supp}(\pi^{n+1})$ for $n\ge 1$. Then $\langle e_n\rangle_{n\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}_{1\setminus e^*}$, and $\pi^{n+1}a=a$ whenever $a\subseteq e_n$. Also $\pi e_n=e_n$ for each n, by 381Eg. By (b), $\pi\restriction\mathfrak{A}_{e_n}$ has a transversal for every n; so $\pi\restriction\mathfrak{A}_{1\setminus e^*}$ has a transversal (382Ga again).

(d)(i) Write G_e for $\{\pi\restriction\mathfrak{A}_e:\pi\in G\}$. If $\langle a_i\rangle_{i\in I}$ is a countable partition of unity in $\mathfrak{A}_e$, $\langle\pi_i\rangle_{i\in I}$ a family in G, and $\phi\in\operatorname{Aut}\mathfrak{A}_e$ is such that $\phi d=\pi_i d$ whenever $i\in I$ and $d\subseteq a_i$, set $J=I\cup\{\infty\}$ for some object $\infty\notin I$, $a_\infty=1\setminus e$ and π_∞ the identity in $\operatorname{Aut}\mathfrak{A}$; then we have a $\tilde\phi\in\operatorname{Aut}\mathfrak{A}$ defined by setting $\tilde\phi d=\phi(d\cap e)\cup(d\setminus e)$ for every $d\in\mathfrak{A}$, and $\langle a_i\rangle_{i\in J}$, $\langle\pi_i\rangle_{i\in J}$ witness that $\tilde\phi\in G$, so $\phi=\tilde\phi\restriction\mathfrak{A}_e$ belongs to G_e. As $\langle a_i\rangle_{i\in I}$ and $\langle\pi_i\rangle_{i\in I}$ are arbitrary, G_e is countably full.

(ii) If $\pi\in G$, let a be a separator for π, and consider $a'=a\cap e$. Then $a'\cap\pi a'=0$ and $\sup_{k\in\mathbb{Z}}\pi^k a'=\sup_{k\in\mathbb{Z}}\pi^k a\cap e=e$, so a' is a separator for $\pi\restriction\mathfrak{A}_e$.

382J Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator, and $\pi\in G$ an aperiodic automorphism. Then there is a non-increasing sequence $\langle e_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $e_0=1$ and

(i) π is doubly recurrent on e_n, and in fact $\sup_{i\ge 1}\pi^i e_n=\sup_{i\ge 1}\pi^{-i}e_n=1$,

(ii) e_{n+1}, $\pi_{e_n}e_{n+1}$ and $\pi^2_{e_n}e_{n+1}$ are disjoint for every $n \in \mathbb{N}$, where $\pi_{e_n} \in \operatorname{Aut}\mathfrak{A}_{e_n}$ is the automorphism induced by π (381M).

proof Construct $\langle a_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Start with $a_0 = 1$. Given that $\sup_{i\ge1}\pi^i a_n = \sup_{i\ge1}\pi^{-i}a_n = 1$, then of course π is doubly recurrent on a_n (381L). Now there is an $a_{n+1} \subseteq a_n$ such that $a_{n+1}\cap\pi_{a_n}a_{n+1} = 0$ and $a_{n+1}\cup\pi_{a_n}a_{n+1}\cup\pi_{a_n}^2 a_{n+1} = a_n$. **P** We have a $\tilde\pi_{a_n} \in \operatorname{Aut}\mathfrak{A}$ defined by setting $\tilde\pi_{a_n}d = \pi_{a_n}d$ for $d \subseteq a_n$, $\tilde\pi_{a_n}d = d$ for $d \subseteq 1\setminus a_n$. Because π is aperiodic, so is π_{a_n} (381Ng); in particular, the support of π_{a_n} is a_n and this must also be the support of $\tilde\pi_{a_n}$. Because G is countably full, $\tilde\pi_{a_n} \in G$ (381Ni), so $\tilde\pi_{a_n}$ has a separator. By 382D, there is an $a_{n+1} \in \mathfrak{A}$ such that $a_{n+1}\cap\tilde\pi_{a_n}a_{n+1} = 0$ and $a_{n+1}\cup\tilde\pi_{a_n}a_{n+1}\cup\tilde\pi_{a_n}^2 a_{n+1}$ supports $\tilde\pi_{a_n}$, that is,

$$a_n = a_{n+1}\cup\tilde\pi_{a_n}a_{n+1}\cup\tilde\pi_{a_n}^2 a_{n+1} = a_{n+1}\cup\pi_{a_n}a_{n+1}\cup\pi_{a_n}^2 a_{n+1}. \quad \mathbf{Q}$$

Now

$$\sup_{i\ge1}\pi^i a_{n+1} = \sup_{i\ge1}\pi^i(\sup_{j\ge0}\pi^j a_{n+1}) \supseteq \sup_{i\ge1}\pi^i(\sup_{j\ge0}\pi_{a_n}^j a_{n+1})$$

(381Nb)

$$= \sup_{i\ge1}\pi^i a_n = 1.$$

Similarly, because we can identify $\pi_{a_n}^{-1}$ with $(\pi^{-1})_{a_n}$ (381Na), and

$$a_{n+1}\cup\pi_{a_n}^{-1}a_{n+1}\cup\pi_{a_n}^{-2}a_{n+1} = \pi_{a_n}^{-2}(a_{n+1}\cup\tilde\pi_{a_n}a_{n+1}\cup\tilde\pi_{a_n}^2 a_{n+1}) = a_n,$$

we have

$$\sup_{i\ge1}\pi^{-i}ia_{n+1} = \sup_{i\ge1}\pi^{-i}(\sup_{j\ge0}\pi^{-j}a_{n+1})$$
$$\supseteq \sup_{i\ge1}\pi^{-i}(\sup_{j\ge0}\pi_{a_n}^{-j}a_{n+1}) = \sup_{i\ge1}\pi^{-i}a_n = 1,$$

and the induction continues.

At the end of the induction, set $e_n = a_{2n}$ for every n. Then, for each n, we have

$$0 = a_{2n+1}\cap\pi_{e_n}a_{2n+1} = e_{n+1}\cap\pi_{a_{2n+1}}e_{n+1}.$$

Since we can identify $\pi_{a_{2n+1}}$ with $(\pi_{e_n})_{a_{2n+1}}$ (381Ne), we can apply 381Nh to π_{e_n} to see that e_{n+1}, $\pi_{e_n}e_{n+1}$ and $\pi_{e_n}^2 e_{n+1}$ are all disjoint.

382K Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Suppose that we have an aperiodic $\pi \in \operatorname{Aut}\mathfrak{A}$ and a non-increasing sequence $\langle e_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $e_0 = 1$ and

$$\sup_{i\ge1}\pi^i e_n = \sup_{i\ge1}\pi^{-i}e_n = 1, \quad e_{n+1},\ \pi_{e_n}(e_{n+1}) \text{ and } \pi_{e_n}^2(e_{n+1}) \text{ are disjoint}$$

for every $n \in \mathbb{N}$, writing $\pi_{e_n} \in \operatorname{Aut}\mathfrak{A}_{e_n}$ for the induced automorphism. Let G be the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π. Then there is a $\phi \in G$ such that ϕ is either the identity or an exchanging involution and $\inf_{n\ge1}\operatorname{supp}(\pi\phi)^n = 0$.

proof (a) We need to check that every member of G has a support. **P** If $\phi \in G$, there is a partition $\langle a_n\rangle_{n\in\mathbb{Z}}$ of unity such that $\phi a = \pi^n a$ whenever $n \in \mathbb{Z}$ and $a \subseteq a_n$ (381Ib). If $a \subseteq a_0$, then $\phi a = a$, so $1\setminus a_0$ supports ϕ. On the other hand, if $a\setminus a_0 \ne 0$, there is an $n \ne 0$ such that $a\cap a_n \ne 0$. As $\operatorname{supp}\pi^n = 1$, there is a non-zero $d \subseteq a\cap a_n$ such that $0 = d\cap\pi^n d = d\cap\phi d$. Thus $1\setminus a_0 = \sup\{d : d\cap\phi d = 0\}$ is the support of ϕ (381Ei). **Q**

(b) For each $n \in \mathbb{N}$, write π_n for π_{e_n} and $\tilde\pi_n \in G$ for the corresponding automorphism of $\mathfrak{A}$, as in 381Ni. Set

$$u_n' = \pi_n^{-1}e_{n+1}, \quad u_n'' = \pi_n e_{n+1}.$$

Then all the u_n', u_n'' are disjoint. **P**

$$u_n'\cap u_n'' = \pi_n^{-1}(e_{n+1}\cap\pi_n^2 e_{n+1}) = 0$$

for each n. And if $m < n$, then $u_n'\cup u_n'' \subseteq e_n \subseteq e_{m+1}$ is disjoint from

$$u_m'\cup u_m'' \subseteq \pi_m^{-1}(e_{m+1})\cup\pi_m(e_{m+1}). \quad \mathbf{Q}$$

(c) By 381C, there is an automorphism $\phi_1 \in \operatorname{Aut}\mathfrak{A}$ defined by setting

$$\begin{aligned}\phi_1 d &= \pi_n\pi_{n+1}^{-1}\pi_n d = \tilde{\pi}_n\tilde{\pi}_{n+1}^{-1}\tilde{\pi}_n d \text{ if } n \in \mathbb{N},\ d \subseteq u'_n,\\ &= \pi_n^{-1}\pi_{n+1}\pi_n^{-1}d = \tilde{\pi}_n^{-1}\tilde{\pi}_{n+1}\tilde{\pi}_n^{-1}d \text{ if } n \in \mathbb{N},\ d \subseteq u''_n,\\ &= d \text{ if } d \cap \sup_{n\in\mathbb{N}}(u'_n \cup u''_n) = 0;\end{aligned}$$

$\phi_1 \in G$ and ϕ_1^2 is the identity and ϕ_1 exchanges $\sup_{n\in\mathbb{N}} u'_n$ with $\sup_{n\in\mathbb{N}} u''_n$, so is either the identity or an exchanging involution. Set $c_0 = \inf_{k\ge1}\operatorname{supp}(\pi\phi_1)^k$ and $c_1 = \sup_{i\in\mathbb{Z}}\pi^i c_0$, so that $\pi c_1 = c_1$ and $\phi_1 c_1 = c_1$ (381J).

(d) For $l \ge 1$, set

$$v'_l = \pi^{-l}c_0 \setminus \sup_{-l<i\le l}\pi^i c_0, \quad v''_l = \pi^l c_0 \setminus \sup_{-l\le i<l}\pi^i c_0.$$

Then v'_k, v''_k, v'_l and v''_l are disjoint whenever $1 \le k < l$. For j, $l \ge 1$, set

$$d'_{lj} = v'_l \cap \pi^{-j}v''_l \setminus \sup_{1\le i<j}\pi^{-i}v''_l,$$

$$d''_{lj} = v''_l \cap \pi^j v'_l \setminus \sup_{1\le i<j}\pi^i v'_l,$$

$$d_{lj} = d'_{lj} \cap \pi^{-j}d''_{lj};$$

now define $\phi_2 \in \operatorname{Aut}\mathfrak{A}$ by setting

$$\begin{aligned}\phi_2 d &= \pi^j d \text{ if } d \subseteq d_{lj} \text{ for some } j,\ l \ge 1,\\ &= \pi^{-j}d \text{ if } d \subseteq \pi^j d_{lj} \text{ for some } j,\ l \ge 1,\\ &= d \text{ if } d \cap \sup_{j,l\ge1}(d_{lj} \cup \pi^j d_{lj}) = 0,\end{aligned}$$

so that $\phi_2 \in G$, ϕ_2^2 is the identity and

$$\operatorname{supp}\phi_2 = \sup_{l,j\ge1} d_{lj} \cup \pi^j d_{lj} \subseteq \sup_{l\ge1} v'_l \cup v''_l \subseteq c_1.$$

As ϕ_2 exchanges $\sup_{j,l\ge1} d_{lj} \subseteq \sup_{j,l\ge1} d'_{lj}$ with $\sup_{j,l\ge1}\pi^j d_{lj} \subseteq \sup_{j,l\ge1} d''_{lj}$, it too is either trivial or an exchanging involution.

(e) Define $\phi \in \operatorname{Aut}\mathfrak{A}$ by setting

$$\begin{aligned}\phi d &= \phi_1 d \text{ if } d \subseteq 1 \setminus c_1,\\ &= \phi_2 d \text{ if } d \subseteq c_1.\end{aligned}$$

It is easy to check that ϕ is either the identity or an exchanging involution. Set $c_2 = \inf_{n\ge1}\operatorname{supp}(\pi\phi)^n$.

(f) I wish to show that $c_2 = 0$. The rest of the argument does not strictly speaking require the Stone representation (382Yb), but I think that most readers will find it easier to follow when expressed in terms of the Stone space Z of $\mathfrak{A}$. Let f, g_1, g_2 and g be the autohomeomorphisms of Z corresponding to π, ϕ_1, ϕ_2 and ϕ; write $\widehat{a} \subseteq Z$ for the open-and-closed set corresponding to $a \in \mathfrak{A}$. For each $n \in \mathbb{N}$, let $f_n : \widehat{e_n} \to \widehat{e_n}$ be the autohomeomorphism corresponding to π_{e_n}. Since

$\operatorname{supp}\pi^k = 1$ for every $k \ge 1$,
$\sup_{i\ge k}\pi^i e_n = \sup_{i\ge k}\pi^{-i}e_n = 1$ for every $n \in \mathbb{N}$, $k \in \mathbb{Z}$ (381L),
$c_0 = \inf_{k\ge1}\operatorname{supp}(\pi\phi_1)^k$,
$c_1 = \sup_{i\in\mathbb{Z}}\pi^i c_0$,
$\operatorname{supp}\phi_2 \setminus \sup_{l\ge1}(v'_l \cup v''_l) = 0$,
$c_2 = \inf_{k\ge1}\operatorname{supp}(\pi\phi)^k$,

the sets

$\{z : f^k(z) = z\}$, for $k \ge 1$,
$Z \setminus \bigcup_{i\ge k} f^{-i}[\widehat{e_n}]$, for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$,
$Z \setminus \bigcup_{i\le k} f^{-i}[\widehat{e_n}]$, for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$,
$\widehat{c_0} \triangle \bigcap_{k\ge1}\{z : (g_1 f)^k(z) \ne z\}$,
$\widehat{c_1} \triangle \bigcup_{i\in\mathbb{Z}} f^{-i}[\widehat{c_0}]$,
$\widehat{\operatorname{supp}\phi_2} \setminus \bigcup_{l\ge1}(\widehat{v'_l} \cup \widehat{v''_l})$,

$\widehat{c_2}\triangle\{z : (gf)^k(x) \neq x \text{ for every } k \geq 1\}$,

as well as the sets

$\{z : g_1(z) \notin \{f^i(z) : i \in \mathbb{Z}\}\}$,

$\{z : g_2(z) \notin \{f^i(z) : i \in \mathbb{Z}\}\}$

are all meager (using 381Qb), and their union Y is meager. Set $Y' = \bigcup_{i\in\mathbb{Z}} f^{-i}[Y]$; then Y' also is meager, and $X = Z \setminus Y'$ is comeager, therefore dense, by Baire's theorem (3A3G). Of course $f^i(x) \in X$ whenever $x \in X$ and $i \in \mathbb{Z}$.

(g) Fix $x \in X \cap \widehat{c_1}$ for the time being. Because $f^k(x) \neq x$ for any $k \geq 1$, the map $i \mapsto f^i(x) : \mathbb{Z} \to X$ is injective. Because $g_k(f^i(z)) \in \{f^{i+j}(z) : j \in \mathbb{Z}\}$ for every $i \in \mathbb{Z}$ and both $k \in \{1, 2\}$, we can define g_1^x, $g_2^x : \mathbb{Z} \to \mathbb{Z}$ by saying that $g_k^x(i) = j$ if $g_k(f^i(x)) = f^j(x)$. Similarly, f is represented on $\{f^i(x) : i \in \mathbb{Z}\}$ by s, where $s(i) = i + 1$ for every $i \in \mathbb{Z}$.

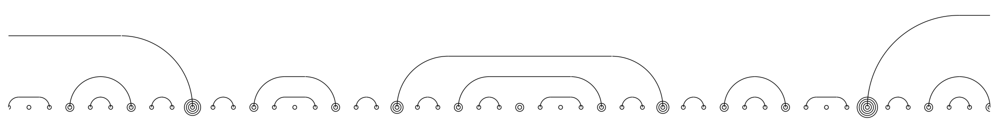
(i) For $n \in \mathbb{N}$, set

$$E_n = \{i : i \in \mathbb{Z},\ f^i(x) \in \widehat{e_n}\},$$

$$U'_n = \{i : f^i(x) \in \widehat{u'_n}\}, \quad U''_n = \{i : f^i(x) \in \widehat{u''_n}\}.$$

Because $x \in \bigcup_{i\geq k} f^{-i}[\widehat{e_n}] \cap \bigcup_{i\leq k} f^{-i}[\widehat{e_n}]$ for every k, E_n is unbounded above and below. If $i \in E_n$, then $f_n(f^i(x)) = f^{k+i}(x)$ where $k \geq 1$ is the first such that $f^{k+i}(x) \in \widehat{e_n}$ (381Qe), that is, such that $k + i \in E_n$. Turning this round, $f_n^{-1}(f^i(x)) = f^j(x)$ where j is the greatest member of E_n less than i. In particular, $i \in U'_n$ iff i is the next point of E_n above a point of E_{n+1}, and $i \in U''_n$ iff i is the next point of E_n below a point of E_{n+1}. If $i \in U'_n$, then $f_n^{-1}f^i(x) = f^j(x)$ where $j \in E_{n+1}$ is the next point of E_n below i, and $f_{n+1}f_n^{-1}f^i(x) = f^k(x)$ where k is the next point of E_{n+1} above j. Since g_1 must agree with $f_n^{-1}f_{n+1}f_n^{-1}$ on $\widehat{u'_n}$ (381Qa), $g_1f^i(x) = f_n^{-1}f_{n+1}f_n^{-1}f_i(x) = f^l(x)$ where l is the next point of E_n below $f^k(x)$. This means that g_1^x exchanges pairs $i < l$ exactly when i, $l \in E_n$ are the first and last points in $E_n \cap]j, k[$ where j, k are successive points of E_{n+1}. In this case, there is no point of E_{n+1} in the interval $[i, l]$. Accordingly, if $i' < l'$ and g_1^x exchanges i' and l' and either i' or l' is in $]i, l[$, we must have i', $l' \in E_m$ for some $m < n$; and as the interval $[i', l']$ cannot meet $E_{m+1} \supseteq E_n$, it is included in $]i, l[$. Thus g_1^x fixes $]i, l[$ in the sense that if $i < i' < l$ then $g_1^x(i') = l'$ for some $l' \in]i, l[$. It follows that $g_1^x s$ fixes $[i, l[$. In this case, of course, every point of $[i, l[$ must be fixed by some power of $g_1^x s$.

The following diagram attempts to show how g_1^x links pairs of integers. The points of E_n, as n increases, are shown as progressively multiplied circles.



Pairs of points exchanged by g_1^x

Note that because e_{n+1}, $\phi_{e_n}e_{n+1}$ and $\pi_{e_n}^2 e_{n+1}$ are always disjoint, there are always at least two points of E_n between any two successive points of E_{n+1}.

(ii) Set $C_0 = \{i : f^i(x) \in \widehat{c_0}\}$. Then

$$C_0 = \mathbb{Z} \setminus \bigcup\{[i, l[\ : i < l = g_1^x(i)\}.$$

P Because X does not meet $\widehat{c_0} \triangle \bigcap_{k\geq 1}\{z : (g_1f)^k(z) \neq z\}$,

$$C_0 = \{i : (g_1f)^k f^i(x) \neq f^i(x) \text{ for every } k \geq 1\} = \{i : (g_1^x s)^k(i) \neq i \text{ for every } k \geq 1\}.$$

If $i < l = g_1^x(i)$ then (i) tells us that every point of $[i, l[$ is fixed by some power of $g_1^x s$ and cannot belong to C_0. Conversely, if $j \in \mathbb{Z}$ does not belong to any such interval $[i, l[$, then $g_1^x(i) > j$ for every $i > j$, so $g_1^x s(i) > j$ for every $i \geq j$ and $j \notin C_0$. **Q**

Because X does not meet $\widehat{c_1} \setminus \bigcup_{i\in\mathbb{Z}} f^{-i}[\widehat{c_0}]$, C_0 is not empty. Now C_0 has no greatest member. **P** Let $j_0 \in C_0$. Then $j_0 \notin [i, l[$ for any pair i, l exchanged by g_1^x. If $j_0 + 1 \in C_0$ we can stop. Otherwise, there are i_0, l_0 exchanged by g_1^x such that $i_0 \leq j_0 + 1 < l_0$. **?** If $l_0 \notin C_0$ there are i_1, l_1 exchanged by g_1^x such that $i_1 \leq l_0 < l_1$. But in this case $i_1 \leq j_0 < l_1$. **X** Thus $j_0 < l_0 \in C_0$ and j_0 cannot be the greatest member of C_0. **Q**

Similarly, C_0 has no least member. **P** If $j_0 \in C_0$ but $j_0 - 1 \notin C_0$, take i_0, l_0 exchanged by g_1^x such that $i_0 \leq j_0 - 1 < l_0$. **?** If $i_0 - 1 \notin C_0$, take i_1, l_1 exchanged by g_1^x such that $i_1 \leq i_0 - 1 < l_1$; then $i_1 \leq j_0 = l_0 < l_1$. **X** So $i_0 - 1$ is a member of C_0 less than j_0. **Q**

Thus C_0 is unbounded above and below.

(iii) For $l \ge 1$,

$$\widehat{v'_l} = f^l[\widehat{c_0}] \setminus \bigcup_{-l\le j<l} f^j[\widehat{c_0}], \quad \widehat{v''_l} = f^{-l}[\widehat{c_0}] \setminus \bigcup_{-l<j\le l} f^j[\widehat{c_0}];$$

so setting

$$V'_l = \{i : f^i(x) \in \widehat{v'_l}\}, \quad V''_l = \{i : f^i(x) \in \widehat{v''_l}\},$$

we see that

$$V'_l = \{i : i-l \in C_0,\ i+j \notin C_0 \text{ if } -l<j\le l\} = \{i+l : i \in C_0,\ C_0 \cap\,]i, i+2l] = \emptyset\},$$

$$V''_l = \{i : i+l \in C_0,\ i+j \notin C_0 \text{ if } -l\le j< l\} = \{i-l : i \in C_0,\ C_0 \cap [i-2l, i[\, = \emptyset\};$$

that is to say, if j, k are successive members of C_0, and $j+l<k-l$, then $j+l \in V'_l$ and $k-l \in V''_l$. Looking at this from the other direction, if j and k are successive members of C_0, and $l_0 = \lfloor\frac{k-j-1}{2}\rfloor$, then if $1 \le l \le l_0$ we have exactly one $i' \in V'_l \cap [j,k]$ and exactly one $i'' \in V''_l \cap [j,k]$ and $i'<i''$, while if $l>l_0$ then neither V'_l nor V''_l meets $[j,k]$.

(iv) Now the point is that every V'_l is unbounded above. **P** Because there are at least two points of E_n between any two points of E_{n+1}, successive points of E_n always differ by at least 3^n, for every n. Take n such that $3^n \ge 2l+1$. For any $i_0 \in \mathbb{Z}$, there are an $i_1 \in C_0$ such that $i_1 \ge i_0$, and a $j \in E_{n+1}$ such that $j \ge i_1$; let k be the next point of E_{n+1} above j. Then we have points j', k' of $E_n \cap\,]j,k[$ such that C_0 is disjoint from $[j',k'[$. So if we take $i = \max(C_0 \cap\,]-\infty, j'[)$ and $i' = \min(C_0 \cap [j', \infty[)$, $i'-i \ge k'-j' \ge 2l+1$ and $i+l \in V'_l$, while $i+l \ge i \ge i_1 \ge i_0$. As i_0 is arbitrary, V'_l is unbounded above. **Q** Similarly, turning the argument upside down, V''_l is unbounded below.

(v) Next consider

$$\begin{aligned} D'_{lj} = \{i : f^i(x) \in \widehat{d'_{lj}}\} &= V'_l \cap (V''_l + j) \setminus \bigcup_{1\le i<j} V''_l + i \\ &= \{i : i \in V'_l,\ i-j = \max(V''_l \cap\,]-\infty, i[\}, \\ D''_{lj} = \{i : f^i(x) \in \widehat{d'_{lj}}\} &= V''_l \cap (V'_l + j) \setminus \bigcup_{1\le i<j} V'_l + i \\ &= \{i : i \in V''_l,\ i+j = \min(V'_l \cap\,]i, \infty[\}, \\ D_{lj} = \{i : f^i(x) \in \widehat{d_{lj}}\} &= D'_{lj} \cap (D''_{lj} + j). \end{aligned}$$

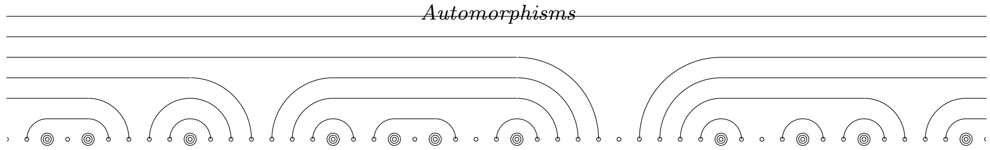
Since ϕ_2 agrees with π^j on $\mathfrak{A}_{d_{lj}}$, g_2 agrees with f^j on $\widehat{\pi^j d_{lj}}$, and $g_2^x(i) = i+j$ whenever $f^i(x) \in f^{-j}[\widehat{d_{lj}}]$, that is, whenever $i+j \in D_{lj}$. This means that g_2^x exchanges pairs $i''<i'$ exactly when, for some l, i'' is the greatest member of V''_l less than i' and i' is the least member of V'_l greater than i''. Since X does not meet $\widehat{\operatorname{supp}\phi_2} \setminus \bigcup_{l\ge1}(\widehat{v'_l} \cup \widehat{v''_l})$, g_2^x does not move any other i.

But, starting from any $l \ge 1$ and $i' \in V'_l$, let i'' be the greatest element of V''_l less than i'. Then $i'-l$ and $i''+l$ belong to C_0, and if k, k' are any successive members of C_0 such that $i''<k<k'<i'$ then there is no member of V''_l in $[k,k']$ and therefore no member of V'_l in $[k,k']$. So i' is the least member of V'_l greater than i'', and $g_2^x(i') = i''$. Similarly, every member of every V''_l is moved by g_2^x.

At the same time we see that if $i'' \in V''_l$ and $i' \in V'_l$ are exchanged by g_2^x, and $m>l$, then there can be no interval of C_0 of length $2m+1$ or greater between i'' and i', so there is no point of $V''_m \cup V'_m$ in $[i'', i']$. For the same reason, if $m<l$ then no pair of points in $V''_m \cup V'_m$ exchanged by g_2^x can bracket either i'' or i'. So g_2^x leaves the interval $[i'', i']$ invariant. Accordingly $g_2 s$ leaves $[i'', i'[$ invariant.

The next diagram attempts to illustrate g_2^x. Members of C_0 are shown as multiple circles[1].

[1] I have made no attempt to arrange these in a configuration compatible with the process by which C_0 was constructed; the diagram aims only to show how the links would be formed from a particular set.



Pairs of points exchanged by g_2^x

At this point observe that 0 belongs to some $g_2^x s$-invariant interval. **P** Let k, k' be successive members of C_0 such that $k \le 0 < k'$. Take l such that $k' - k \le 2l$. Let i' be the least member of V_l' greater than 0, and i'' the greatest member of V_l'' less than 0; since neither V_l' nor V_l'' meets $[k, k']$, i'' and i' are exchanged by g_2^x, while $0 \in [i'', i'[$. **Q** This means that there is a $k \ge 1$ such that $(g_2^x s)^k(0) = 0$, that is, $(g_2 f)^k(x) = x$.

(vi) We know that g agrees with g_2 on $\widehat{\phi_2 c_1} = \widehat{c_1}$. Since $x \in \widehat{c_1}$ and $f^{-1}[\widehat{c_1}] = \widehat{c_1}$, $(gf)^k(x) = x$. Because X does not meet $\widehat{c_2} \triangle \{z : (gf)^k(x) \ne x$ for every $k \ge 1\}$, $x \notin \widehat{c_2}$.

This is true for every $x \in X \cap \widehat{c_1}$. Since X is dense in Z, $\widehat{c_1} \cap \widehat{c_2}$ is empty, that is, $c_1 \cap c_2 = 0$.

(h) Since $\pi\phi$ agrees with $\pi\phi_1$ on $\mathfrak{A}_{1 \setminus c_1}$, and $c_1 = \pi\phi c_1$, $\operatorname{supp}(\pi\phi)^k \setminus c_1 = \operatorname{supp}(\pi\phi_1)^k \setminus c_1$ for every k, and

$$c_2 \setminus c_1 = \inf_{k \ge 1} \operatorname{supp}(\pi\phi_1)^k \setminus c_1 \subseteq \inf_{k \ge 1} \operatorname{supp}(\pi\phi_1)^k \setminus c_0 = 0.$$

Putting this together with (g), we see that $c_2 = 0$, as required.

382L Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator. If $\pi \in G$, there is a $\phi \in G$ such that ϕ is either the identity or an exchanging involution and $\pi\phi$ has a transversal.

proof (a) We may suppose that G is the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π. π^n has a support for every $n \ge 1$ (382Ia); set $e = \inf_{n \ge 1} \operatorname{supp}\pi^n$, so that $\pi e = e$ and $\pi{\restriction}\mathfrak{A}_{1 \setminus e}$ has a transversal (382Ic), while $\pi{\restriction}\mathfrak{A}_e$ is aperiodic (381H). By 381J, $\psi e = e$ for every $\psi \in G$; by 382Id, $G_e = \{\psi{\restriction}\mathfrak{A}_e : \psi \in G\}$ is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}_e$ and every member of G_e has a separator.

(b) Applying 382J to $\pi{\restriction}\mathfrak{A}_e$, we can find $\langle e_n \rangle_{n \ge 1}$ such that $e_0 = e$, $\langle e_n \rangle_{n \in \mathbb{N}}$ is non-increasing, $\sup_{i \ge 1} \pi^i e_n = \sup_{i \ge 1} \pi^{-i} e_n = e$ for every n, and e_{n+1}, $\pi_{e_n} e_{n+1}$ and $\pi_{e_n}^2 e_{n+1}$ are disjoint for every n. (By 381Ne or otherwise, we can compute π_{e_n} either in $\operatorname{Aut}\mathfrak{A}$ or in $\operatorname{Aut}\mathfrak{A}_e$. Note that $\pi_e = \pi{\restriction}\mathfrak{A}_e$, by 381Nf or otherwise.) Now 382K tells us that there is a $\phi \in G_e$ such that ϕ is either the identity or an exchanging involution, and $\inf_{n \ge 1} \operatorname{supp}(\pi_e\phi)^n = 0$.

(c) Take $\tilde{\phi} \in \operatorname{Aut}\mathfrak{A}$ to agree with ϕ on $\mathfrak{A}_e$ and with the identity on $\mathfrak{A}_{1 \setminus e}$, so that $\tilde{\phi}$ is either the identity or an exchanging involution. Now $\pi\tilde{\phi}{\restriction}\mathfrak{A}_{1 \setminus e} = \pi{\restriction}\mathfrak{A}_{1 \setminus e}$ and $\pi\tilde{\phi}{\restriction}\mathfrak{A}_e = \pi\phi{\restriction}\mathfrak{A}_e$ both have transversals (using 382I again). So $\pi\tilde{\phi}$ has a transversal (382Ga).

382M Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator. If $\pi \in G$, it can be expressed as the product of at most three exchanging involutions belonging to G.

proof By 382L, there is a $\phi \in G$, either the identity or an exchanging involution, such that $\pi\phi$ has a transversal. By 382H, $\pi\phi$ is the product of at most two exchanging involutions in G, so $\pi = \pi\phi\phi^{-1}$ is the product of at most three exchanging involutions.

382N Corollary If $\mathfrak{A}$ is a Dedekind complete Boolean algebra and G is a full subgroup of $\operatorname{Aut}\mathfrak{A}$, every $\pi \in G$ is expressible as the product of at most three involutions all belonging to G and all supported by $\operatorname{supp}\pi$.

proof We may suppose that G is the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π. By 382Eb, every member of G has a separator. By 382M, π is the product of at most three involutions all belonging to G; by 381Jb, they are all supported by $\operatorname{supp}\pi$.

382O Definition Let $\mathfrak{A}$ be a Boolean algebra, and G a subgroup of the automorphism group $\operatorname{Aut}\mathfrak{A}$. I will say that G **has many involutions** if for every non-zero $a \in \mathfrak{A}$ there is an involution $\pi \in G$ which is supported by a.

382P Lemma Let $\mathfrak{A}$ be an atomless homogeneous Boolean algebra. Then $\operatorname{Aut}\mathfrak{A}$ has many involutions, and in fact every non-zero element of $\mathfrak{A}$ is the support of an exchanging involution.

proof If $a \in \mathfrak{A} \setminus \{0\}$, then there is a b such that $0 \ne b \subset a$. Let $\psi : \mathfrak{A}_b \to \mathfrak{A}_{a \setminus b}$ be an isomorphism; define $\pi \in \operatorname{Aut}\mathfrak{A}$ to agree with ψ on $\mathfrak{A}_b$, with ψ^{-1} on $\mathfrak{A}_{a \setminus b}$, and with the identity on $\mathfrak{A}_{1 \setminus a}$. Then π is an exchanging involution with support a.

382Q Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Then every non-zero element of $\mathfrak{A}$ is the support of an exchanging involution belonging to G.

proof By the definition 382O,

$$C = \{\operatorname{supp}\pi : \pi \in G \text{ is an involution}\}$$

is order-dense in $\mathfrak{A}$. So if $a \in \mathfrak{A} \setminus \{0\}$ there is a disjoint $B \subseteq C$ such that $\sup B = a$ (313K). For each $b \in B$ let $\pi_b \in G$ be an involution with support b. Define $\pi \in G$ by setting $\pi d = \pi_b d$ for $d \subseteq b \in B$, $\pi d = d$ if $d \cap a = 0$; then $\pi \in G$ is an involution with support a. By 382Fa it is an exchanging involution.

382R Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Then a subset H of G is a normal subgroup of G iff it is of the form

$$\{\pi : \pi \in G, \operatorname{supp}\pi \in I\}$$

for some ideal $I \lhd \mathfrak{A}$ which is G**-invariant**, that is, such that $\pi a \in I$ for every $a \in I$ and $\pi \in G$.

proof (a) I deal with the easy implication first. Let $I \lhd \mathfrak{A}$ be a G-invariant ideal and set $H = \{\pi : \pi \in G, \operatorname{supp}\pi \in I\}$. Because the support of the identity automorphism ι is $0 \in I$, $\iota \in H$. If ϕ, $\psi \in H$ and $\pi \in G$, then

$$\operatorname{supp}(\phi\psi) \subseteq \operatorname{supp}\phi \cup \operatorname{supp}\psi \in I,$$

$$\operatorname{supp}(\psi^{-1}) = \operatorname{supp}\psi \in I,$$

$$\operatorname{supp}(\pi\psi\pi^{-1}) = \pi(\operatorname{supp}\psi) \in I$$

(381E), and $\phi\psi$, ψ^{-1}, $\pi\psi\pi^{-1}$ all belong to H; so $H \lhd G$.

(b) For the rest of the proof, therefore, I suppose that H is a normal subgroup of G and seek to express it in the given form. We can in fact describe the ideal I immediately, as follows. Set

$$J = \{a : a \in \mathfrak{A},\ \pi \in H \text{ whenever } \pi \in G \text{ is an involution and } \operatorname{supp}\pi \subseteq a\};$$

then $0 \in J$ and $a \in J$ whenever $a \subseteq b \in J$. Also $\pi a \in J$ whenever $a \in J$ and $\pi \in G$. **P** If $\phi \in G$ is an involution and $\operatorname{supp}\phi \subseteq \pi a$ then $\phi_1 = \pi^{-1}\phi\pi$ is an involution in G and

$$\operatorname{supp}\phi_1 = \pi^{-1}(\operatorname{supp}\phi) \subseteq a,$$

so $\phi_1 \in H$ and $\phi = \pi\phi_1\pi^{-1} \in H$. As ϕ is arbitrary, $\pi a \in J$. **Q**

I do not know how to prove directly that J is an ideal, so let us set

$$I = \{a_0 \cup a_1 \cup \ldots \cup a_n : a_0, \ldots, a_n \in J\};$$

then $I \lhd \mathfrak{A}$, and $\pi a \in I$ for every $a \in I$ and $\pi \in G$.

(c) If $a \in \mathfrak{A}$, $\psi \in H$ and $a \cap \psi a = 0$ then $a \in J$. **P** If $a = 0$, this is trivial. Otherwise, let $\pi \in G$ be an involution with $\operatorname{supp}\pi \subseteq a$; say $\pi = (\overleftarrow{b\,{}_{\pi}\,c})$ where $b \cup c \subseteq a$. By 382Q there is an involution $\pi_1 \in G$ such that $\operatorname{supp}\pi_1 = b$; say $\pi_1 = (\overleftarrow{b'\,{}_{\pi_1}\,b''})$ where $b' \cup b'' = b$. Set

$$c' = \pi b', \quad c'' = \pi b'' = c \setminus c',$$

$$\pi_2 = \pi_1\pi\pi_1\pi^{-1} = (\overleftarrow{b'\,{}_{\pi_1}\,b''})(\overleftarrow{c'\,{}_{\pi\pi_1\pi^{-1}}\,c''}), \quad \pi_3 = (\overleftarrow{b'\,{}_{\pi}\,c'}),$$

$$\phi = \pi_2^{-1}\psi\pi_2\psi^{-1} \in H,$$

$$\bar\pi = \pi_3^{-1}\phi\pi_3\phi^{-1} = \pi_3^{-1}\pi_2^{-1}\psi\pi_2\psi^{-1}\pi_3\psi\pi_2^{-1}\psi^{-1}\pi_2 \in H.$$

Now

$$\operatorname{supp}(\psi\pi_2\psi^{-1}) = \psi(\operatorname{supp}\pi_2) = \psi(b \cup c) \subseteq \psi a$$

is disjoint from

$$\operatorname{supp}\pi_3 = b' \cup c' \subseteq a,$$

so π_3 commutes with $\psi\pi_2\psi^{-1}$, and

$$
\begin{aligned}
\bar{\pi} &= \pi_3^{-1}\pi_2^{-1}\pi_3\psi\pi_2\psi^{-1}\psi\pi_2^{-1}\psi^{-1}\pi_2 \\
&= \pi_3^{-1}\pi_2^{-1}\pi_3\pi_2 \\
&= (\overleftarrow{b'\,{}_{\pi}\,c'})(\overleftarrow{b'\,{}_{\pi_1}\,b''})(\overleftarrow{c'\,{}_{\pi\pi_1\pi^{-1}}\,c''})(\overleftarrow{b'\,{}_{\pi}\,c'})(\overleftarrow{b'\,{}_{\pi_1}\,b''})(\overleftarrow{c'\,{}_{\pi\pi_1\pi^{-1}}\,c''}) \\
&= (\overleftarrow{b'\,{}_{\pi}\,c'})(\overleftarrow{b''\,{}_{\pi}\,c''}) \\
&= \pi.
\end{aligned}
$$

So $\pi \in H$. As π is arbitrary, $a \in J$. **Q**

(d) If $\pi = (\overleftarrow{a\,{}_{\pi}\,b})$ is an involution in G and $a \in J$, then $\pi \in H$. **P** By 382Q again, there is an involution $\psi \in G$ such that $\operatorname{supp}\psi = a$; because $a \in J$, $\psi \in H$. Express ψ as $(\overleftarrow{a'\,{}_{\psi}\,a''})$ where $a' \cup a'' = a$. Set $b' = \pi a'$ and $b'' = \pi a''$, so that $\pi = (\overleftarrow{a'\,{}_{\pi}\,b'})(\overleftarrow{a''\,{}_{\pi}\,b''})$, and

$$\psi_1 = \psi\pi\psi\pi^{-1} = (\overleftarrow{a'\,{}_{\psi}\,a''})(\overleftarrow{b'\,{}_{\pi\psi\pi^{-1}}\,b''}) \in H.$$

As $\psi_1(a' \cup b') = a'' \cup b''$ is disjoint from $a' \cup b'$, $a' \cup b' \in J$, by (c), and $\pi_1 = (\overleftarrow{a'\,{}_{\pi}\,b'}) \in H$; similarly, $a'' \cup b'' \in J$, so $\pi_2 = (\overleftarrow{a''\,{}_{\pi}\,b''}) \in H$ and $\pi = \pi_1\pi_2$ belongs to H. **Q**

(e) If $\pi \in G$ is an involution and $\operatorname{supp}\pi \in I$, then $\pi \in H$. **P** Express π as $(\overleftarrow{a\,{}_{\pi}\,b})$. Let $a_0, \dots, a_n \in J$ be such that $a \cup b \subseteq a_0 \cup \dots \cup a_n$. Set

$$c_j = a \cap a_j \setminus \sup_{i<j} a_i, \quad b_j = \pi c_j, \quad \pi_j = (\overleftarrow{c_j\,{}_{\pi}\,b_j})$$

for $j \le n$; then every c_j belongs to J, so every π_j belongs to H (by (d)) and $\pi = \pi_0 \dots \pi_n \in H$. **Q**

(f) If $\pi \in G$ and $\operatorname{supp}\pi \in I$ then $\pi \in H$. **P** By 382N, π is a product of involutions in G all with supports included in $\operatorname{supp}\pi$; by (e), they all belong to H, so π also does. **Q**

(g) We are nearly home. So far we know that I is a G-invariant ideal and that $\pi \in H$ whenever $\pi \in G$ and $\operatorname{supp}\pi \in I$. On the other hand, $\operatorname{supp}\pi \in I$ for every $\pi \in H$. **P** By 382Eb, π has a separator; take a', a'', b', b'', c from 382D(iv). Then

$$a' \cap \pi a' = b' \cap \pi b' = \dots = c \cap \pi c = 0,$$

so $a', \dots, c$ all belong to J, by (c), and $\operatorname{supp}\pi = a' \cup \dots \cup c$ belongs to I. **Q**

So H is precisely the set of members of G with supports in I, as required.

382S Corollary Let $\mathfrak{A}$ be a homogeneous Dedekind complete Boolean algebra. Then $\operatorname{Aut}\mathfrak{A}$ is simple.

proof If $\mathfrak{A}$ is $\{0\}$ or $\{0,1\}$ this is trivial. Otherwise, let H be a normal subgroup of $\operatorname{Aut}\mathfrak{A}$. Then by 382R and 382P there is an invariant ideal I of $\mathfrak{A}$ such that $H = \{\pi : \operatorname{supp}\pi \in I\}$. But if H is non-trivial so is I; say $a \in I \setminus \{0\}$. If $a = 1$ then certainly $1 \in I$ and $H = \operatorname{Aut}\mathfrak{A}$. Otherwise, there is a $\pi \in \operatorname{Aut}\mathfrak{A}$ such that $\pi a = 1 \setminus a$ (as in 381D), so $1 \setminus a \in I$, and again $1 \in I$ and $H = \operatorname{Aut}\mathfrak{A}$.

Remark I ought to remark that in fact $\operatorname{Aut}\mathfrak{A}$ is simple for any homogeneous Dedekind σ-complete Boolean algebra; see Štěpánek & Rubin 89, Theorem 5.9b.

382X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. Suppose that $\pi \in \operatorname{Aut}\mathfrak{A}$ is represented by $f_\pi : Z \to Z$. For $z \in Z$, write $\operatorname{Orb}_\pi(z) = \{f_\pi^n(z) : n \in \mathbb{Z}\}$. (i) Show that $a \in \mathfrak{A}$ is a separator for π iff $f_\pi^{-1}[\widehat{a}] \cap \widehat{a}$ is empty and $\{z : \operatorname{Orb}_\pi(z) \cap \widehat{a}\} \ne \emptyset\}$ is dense in $\{z : f_\pi(z) \ne z\}$. (ii) Show that $a \in \mathfrak{A}$ is a transversal for π iff $\{z : \operatorname{Orb}_\pi(z) \cap \widehat{a} \ne \emptyset\}$ is dense in Z and $\#(\operatorname{Orb}_\pi(z) \cap \widehat{a}) \le 1$ for every z.

>(b) Let X be any set. Show that any automorphism of the Boolean algebra $\mathcal{P}X$ is expressible as a product of at most two involutions.

>(c) (Miller 04) Let X be a set and Σ a σ-algebra of subsets of X. Suppose that (X, Σ) is countably separated in the sense that there is a countable subset of Σ separating the points of X (cf. 343D). Let G be the group of permutations $f : X \to X$ such that $\Sigma = \{f^{-1}[E] : E \in \Sigma\}$. Show that every automorphism of the Boolean algebra Σ has a separator, so that every member of G is expressible as the product of at most three involutions belonging to G.

(d) Recall that in any group G, a **commutator** in G is an element of the form $ghg^{-1}h^{-1}$ where g, $h \in G$. Show that if $\mathfrak{A}$ is a Dedekind complete Boolean algebra and G is a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions then every involution in G is a commutator in G, so that every element of G is expressible as a product of three commutators.

(e) Give an example of a Dedekind complete Boolean algebra $\mathfrak{A}$ such that not every member of $\operatorname{Aut}\mathfrak{A}$ is a product of commutators in $\operatorname{Aut}\mathfrak{A}$.

(f) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and suppose that $\operatorname{Aut}\mathfrak{A}$ has many involutions. Show that if $H \triangleleft \operatorname{Aut}\mathfrak{A}$ then every member of H is expressible as the product of at most three involutions belonging to H.

(g) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Show that the partially ordered set $\mathcal{H}$ of normal subgroups of G is a distributive lattice, that is, $H \cap K_1K_2 = (H \cap K_1)(H \cap K_2)$, $H(K_1 \cap K_2) = HK_1 \cap HK_2$ for all H, K_1, $K_2 \in \mathcal{H}$.

(h) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Show that if H is the normal subgroup of G generated by a finite subset of G, then it is the normal subgroup generated by a single involution.

(i) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Show (i) that there is an involution $\pi \in G$ such that every member of G is expressible as a product of conjugates of π in G (ii) any proper normal subgroup of G is included in a maximal proper normal subgroup of G.

(j) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra. Show that if $\pi : \mathfrak{A} \to \mathfrak{A}$ is an ergodic measure-preserving automorphism it has no transversal.

(k) Show that if $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra with countable Maharam type (definition: 331F), then every automorphism of $\mathfrak{A}$ has a separator. (*Hint*: show that if $b \in \mathfrak{A}$ then $\{a : a \mathbin{\triangle} \pi a \subseteq b\}$ is an order-closed subalgebra.)

(l) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. Show that π has a separator iff there is a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that π is supported by $\sup_{n\in\mathbb{N}} a_n \mathbin{\triangle} \pi a_n$.

(m) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Show that for every $n \geq 2$ and every $a \in \mathfrak{A} \setminus \{0\}$ there is a $\pi \in G$ with period n and support a.

382Y Further exercises (a) Find a Dedekind σ-complete Boolean algebra with an involution which is not an exchanging involution.

(b) Devise an expression of the ideas of parts (f)-(h) of the proof of 382K which does not involve the Stone representation. (*Hint*: show that there is a non-increasing sequence in $\mathfrak{A}^+$ which makes enough decisions to play the role of the Boolean homomorphism $x : \mathfrak{A} \to \mathbb{Z}_2$.)

(c) Let $\mathcal{B}$ be the algebra of Borel subsets of $\mathbb{R}$. Show that $\operatorname{Aut}\mathcal{B}$ has exactly three proper normal subgroups. (*Hint*: re-work the proof of 382R, paying particular attention to calls on Lemma 382Q. You will need to know that if $E \in \mathcal{B}$ is uncountable then the subspace σ-algebra on E is isomorphic to $\mathcal{B}$; see §424 in Volume 4.)

(d) Find a Dedekind σ-complete Boolean algebra $\mathfrak{A}$ with an automorphism which cannot be expressed either as a product of finitely many involutions in $\operatorname{Aut}\mathfrak{A}$, or as a product of finitely many commutators in $\operatorname{Aut}\mathfrak{A}$. (This seems to require a certain amount of ingenuity.)

382 Notes and comments The ideas of 382A and 382G-382N are adapted from Miller 04, and (most conspicuously in part (g) of the proof of 382K) betray their origin in a study of Borel automorphisms of $\mathbb{R}$ (see 382Xc). The magic number of three involutions appears in Ryzhikov 93 and Truss 89. The idea of the method presented here is to shift from a 'separator' to a 'transversal'. Since there are many automorphisms without transversals (382Xj), something quite surprising has to happen. The diagrams in the proof of 382K are supposed to show the two steps involved in the argument. We are trying to draw non-overlapping links to build a function g^x such that every point of $\mathbb{Z}$ will belong to a finite orbit of $g^x s$. This must be done by some uniform, translation-invariant, process based on configurations already present; in particular, we are *not* permitted to single out any point of $\mathbb{Z}$ as a centre for the construction. The first attempt is based on the sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ of sets corresponding to the decreasing sequence $\langle e_n\rangle_{n\in\mathbb{N}}$. The construction of such a sequence (382J) requires that there be many separators, which is why these results cannot be applied to all Boolean algebras, or even to all homogeneous ones. If this first attempt fails, however, the points not recurrent under $g_1^x s$ provide a set C_0 with arbitrarily large gaps both to left and to right, from which the second method can build an adequate family of links.

Of course the search for these factorizations was inspired by the well-known corresponding fact for algebras $\mathcal{P}X$ (382Xb). In those algebras we can use the axiom of choice unscrupulously to pick out a point of each orbit, thereby forming a transversal in one step without considering separators, and then apply 382H in its original simple form. Perhaps the principal psychological barrier we need to overcome in 382K is raised in the phrase 'fix $x \in X \cap \widehat{c_1}$'. What I could have said is 'fix an orbit of f meeting $\widehat{c_1}$, and order it by the transitive closure of the relation f'; because the whole point of the subsequent argument is that we do not have a marker to work from.

This volume is concerned with measure algebras, and all the most important measure algebras are Dedekind complete. I take the trouble to express the ideas down to Theorem 382M in terms of σ-complete algebras partly because this is the natural boundary of the arguments given and partly because in Volume 4 I will look at Borel automorphisms, as in 382Xc, and 382M as stated may then be illuminating. But note that in 382N σ-completeness is insufficient (382Yd). In 382S I allow myself for once to present a result with a stronger hypothesis than is required for the conclusion; the point being that homogeneous semi-finite measure algebras are necessarily Dedekind complete (383E), and the arguments for the more general case do not seem to tell us anything which we can use elsewhere in this treatise.

It is natural to ask whether the number 'three' in 382M is best possible (cf. 382Xb). It seems to be quite difficult to exhibit an automorphism requiring three involutions; examples may be found in Anzai 51 and Ornstein & Sheilds 73[2].

Just as well-known facts about symmetry groups lead us to the factorization theorem 382M, they suggest that automorphism groups of Boolean algebras may often have few normal subgroups; and once again we find that the form of the theorem changes significantly. However the root of the phenomenon remains the fact that our groups are multiply transitive. 382O-382S are derived from Štěpánek & Rubin 89 and Fathi 78. An obvious question arising from 382S is: does *every* homogeneous Boolean algebra have a simple automorphism group? This leads into deep water. As remarked after 382S, every homogeneous Dedekind σ-complete algebra has a simple automorphism group. Using the continuum hypothesis, it is possible to construct a homogeneous Boolean algebra which does not have a simple automorphism group; but as far as I am aware no such construction is known which does not rely on some special axiom outside ordinary set theory. See Štěpánek & Rubin 89, §5.

383 Automorphism groups of measure algebras

I turn now to the group of measure-preserving automorphisms of a measure algebra, seeking to apply the results of the last section. The principal theorems are 383D, which is a straightforward special case of 382N, and 383I, corresponding to 382S. I give another example of the use of 382R to describe the normal subgroups of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ (383J). I conclude with an important fact about conjugacy in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and $\mathrm{Aut}\,\mathfrak{A}$ (383L).

383A Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. I will write $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ for the set of all measure-preserving automorphisms of $\mathfrak{A}$. This is a group, being a subgroup of the group $\mathrm{Aut}\,\mathfrak{A}$ of all Boolean automorphisms of $\mathfrak{A}$.

383B Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ two partitions of unity in $\mathfrak{A}$. Assume

either that I is countable

or that $(\mathfrak{A},\bar\mu)$ is localizable.

Suppose that for each $i\in I$ we have a measure-preserving isomorphism $\pi_i:\mathfrak{A}_{a_i}\to\mathfrak{A}_{b_i}$ between the corresponding principal ideals. Then there is a unique $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\pi c=\pi_i c$ whenever $i\in I$ and $c\subseteq a_i$.

proof (Compare 381C.) By 322L, we may identify $\mathfrak{A}$ with each of the simple products $\prod_{i\in I}\mathfrak{A}_{a_i}$, $\prod_{i\in I}\mathfrak{A}_{b_i}$; now π corresponds to the isomorphism between the two products induced by the π_i.

383C Corollary If $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra, then, in the language of 381Be, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is a full subgroup of $\mathrm{Aut}\,\mathfrak{A}$.

[2] I am indebted to P.Biryukov and G.Hjorth for the references.

383D Theorem Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Then every measure-preserving automorphism of $\mathfrak{A}$ is expressible as the product of at most three measure-preserving involutions.

proof This is immediate from 383C and 382N.

383E Lemma If $(\mathfrak{A},\bar\mu)$ is a homogeneous semi-finite measure algebra, it is σ-finite, therefore localizable.

proof If $\mathfrak{A}=\{0\}$, this is trivial. Otherwise there is an $a\in\mathfrak{A}$ such that $0<\bar\mu a<\infty$. The principal ideal $\mathfrak{A}_a$ is ccc (322G), so $\mathfrak{A}$ also is, and $(\mathfrak{A},\bar\mu)$ must be σ-finite, by 322G in the opposite direction.

383F Lemma Let $(\mathfrak{A},\bar\mu)$ be a homogeneous semi-finite measure algebra.

(a) If $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are partitions of unity in $\mathfrak{A}$ with $\bar\mu a_i=\bar\mu b_i$ for every i, there is a $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\pi a_i=b_i$ for each i.

(b) If $(\mathfrak{A},\bar\mu)$ is totally finite, then whenever $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$ with $\bar\mu a_i=\bar\mu b_i$ for every i, there is a $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\pi a_i=b_i$ for each i.

proof (a) By 383E, $(\mathfrak{A},\bar\mu)$ is σ-finite, therefore localizable. For each $i\in I$, the principal ideals $\mathfrak{A}_{a_i}$, $\mathfrak{A}_{b_i}$ are homogeneous, of the same measure and the same Maharam type (being $\tau(\mathfrak{A})$ if $a_i\ne 0$, 0 if $a_i=0$). Because they are ccc, they are of the same magnitude, as defined in 332Ga, and there is a measure-preserving isomorphism $\pi_i:\mathfrak{A}_{a_i}\to\mathfrak{A}_{b_i}$ (332J). By 383B there is a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $\pi d=\pi_i d$ for every $i\in I$, $d\subseteq a_i$; and this π serves.

(b) Set $a^*=1\setminus\sup_{i\in I}a_i$, $b^*=1\setminus\sup_{i\in I}b_i$. We must have

$$\bar\mu a^*=\bar\mu 1-\textstyle\sum_{i\in I}\bar\mu a_i=\bar\mu 1-\sum_{i\in I}\bar\mu b_i=\bar\mu b^*,$$

so adding a^*, b^* to the families we obtain partitions of unity to which we can apply the result of (a).

383G Lemma (a) If $(\mathfrak{A},\bar\mu)$ is an atomless semi-finite measure algebra, then $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ have many involutions.

(b) If $(\mathfrak{A},\bar\mu)$ is an atomless localizable measure algebra, then every element of $\mathfrak{A}$ is the support of some involution in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

proof (a) If $a\in\mathfrak{A}\setminus\{0\}$, then by 332A there is a non-zero $b\subseteq a$, of finite measure, such that the principal ideal $\mathfrak{A}_b$ is (Maharam-type-)homogeneous. Now because $\mathfrak{A}$ is atomless, there is a $c\subseteq b$ such that $\bar\mu c=\frac12\bar\mu b$ (331C), so that $\mathfrak{A}_c$ and $\mathfrak{A}_{b\setminus c}$ are isomorphic measure algebras. If $\theta:\mathfrak{A}_c\to\mathfrak{A}_{b\setminus c}$ is any measure-preserving isomorphism, then $\pi=(\overleftarrow{c\,{}_{\theta}\,b\setminus c})$ is an involution in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ (and therefore in $\mathrm{Aut}\,\mathfrak{A}$) supported by a.

(b) Use 383C, (a) and 382Q.

383H Corollary Let $(\mathfrak{A},\bar\mu)$ be an atomless localizable measure algebra. Then

(a) the lattice of normal subgroups of $\mathrm{Aut}\,\mathfrak{A}$ is isomorphic to the lattice of $\mathrm{Aut}\,\mathfrak{A}$-invariant ideals of $\mathfrak{A}$;

(b) the lattice of normal subgroups of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is isomorphic to the lattice of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$.

proof Use 382R. Taking G to be either $\mathrm{Aut}\,\mathfrak{A}$ or $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, and $\mathcal{I}$ to be the family of G-invariant ideals in $\mathfrak{A}$, we have a map $I\mapsto H_I=\{\pi:\pi\in G,\ \mathrm{supp}\,\pi\in I\}$ from $\mathcal{I}$ to the family $\mathcal{H}$ of normal subgroups of G. Of course this map is order-preserving; 382R tells us that it is surjective; and 383Gb tells us that it is injective and its inverse is order-preserving, since if $a\in I\setminus J$ there is a $\pi\in G$ with $\mathrm{supp}\,\pi=a$, so that $\pi\in H_I\setminus H_J$. Thus we have an order-isomorphism between $\mathcal{H}$ and $\mathcal{I}$.

383I Normal subgroups of $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ 382R provides the machinery for a full description of the normal subgroups of $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ when $(\mathfrak{A},\bar\mu)$ is an atomless localizable measure algebra, as we know that they correspond exactly to the invariant ideals of $\mathfrak{A}$. The general case is complicated. But the following are easy enough.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a homogeneous semi-finite measure algebra.

(a) $\mathrm{Aut}\,\mathfrak{A}$ is simple.

(b) If $(\mathfrak{A},\bar\mu)$ is totally finite, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is simple.

(c) If $(\mathfrak{A},\bar\mu)$ is not totally finite, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ has exactly one non-trivial proper normal subgroup.

proof (a) $\mathfrak{A}$ is Dedekind complete (383E), so this is a special case of 382S.

(b)-(c) The point is that the only possible $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$ are $\{0\}$, $\mathfrak{A}^f$ and $\mathfrak{A}$. **P** If $\mathfrak{A}$ is $\{0\}$ or $\{0,1\}$ this is trivial. Otherwise, $\mathfrak{A}$ is atomless. Let $I \vartriangleleft \mathfrak{A}$ be an invariant ideal.

(i) If $I \not\subseteq \mathfrak{A}^f$, take $a \in I$ with $\bar\mu a = \infty$. By 383E, $\mathfrak{A}$ is σ-finite, so a has the same magnitude ω as 1. By 332I, there is a partition of unity $\langle e_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with $\bar\mu e_n = 1$ for every n; setting $b = \sup_{n\in\mathbb{N}} e_{2n}$, $b' = 1\setminus b$, we see that both b and b' are of infinite measure. Similarly we can divide a into c and c', both of infinite measure. Now by 332J the principal ideals $\mathfrak{A}_b$, $\mathfrak{A}_{b'}$, $\mathfrak{A}_c$, $\mathfrak{A}_{1\setminus c}$ are all isomorphic as measure algebras, so that there are automorphisms π, $\phi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that

$$\pi c = b, \quad \phi c = b'.$$

But this means that both b and b' belong to I, so that $1 = b \cup b' \in I$ and $I = \mathfrak{A}$.

(ii) If $I \subseteq \mathfrak{A}^f$ and $I \neq \{0\}$, take any non-zero $a \in I$. If b is any member of $\mathfrak{A}$, then (because $\mathfrak{A}$ is atomless) b can be partitioned into $b_0, \dots, b_n$, all of measure at most $\bar\mu a$. Then for each i there is a $b_i' \subseteq a$ such that $\bar\mu b_i' = \bar\mu b_i$; since this common measure is finite, $\bar\mu(1\setminus b_i') = \bar\mu(1\setminus b_i)$. By 332J and 383Fa, there is a $\pi_i \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ such that $\pi_i b_i' = b_i$, so that b_i belongs to I. Accordingly $b \in I$. As b is arbitrary, $I = \mathfrak{A}^f$.

Thus the only invariant ideals of $\mathfrak{A}$ are $\{0\}$, $\mathfrak{A}^f$ and $\mathfrak{A}$. **Q**

By 383Hb we therefore have either one, two or three normal subgroups of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$, according to whether $\bar\mu 1$ is zero, finite and not zero, or infinite.

Remark For the Lebesgue probability algebra, (b) is due to Fathi 78. The extension to algebras of uncountable Maharam type is from Choksi & Prasad 82.

383J The language of §352 offers a way of describing another case.

Proposition Let $(\mathfrak{A}, \bar\mu)$ be an atomless totally finite measure algebra. For each infinite cardinal κ, let e_κ be the Maharam-type-κ component of $\mathfrak{A}$, and let K be $\{\kappa : e_\kappa \neq 0\}$. Let $\mathcal{H}$ be the lattice of normal subgroups of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Then

(i) if K is finite, $\mathcal{H}$ is isomorphic, as partially ordered set, to $\mathcal{P}K$;

(ii) if K is infinite, then $\mathcal{H}$ is isomorphic, as partially ordered set, to the lattice of solid linear subspaces of ℓ^∞.

proof (a) Let $\mathcal{I}$ be the family of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$, so that $\mathcal{H} \cong \mathcal{I}$, by 383Hb. For a, $b \in \mathfrak{A}$, say that $a \preccurlyeq b$ if there is some $k \in \mathbb{N}$ such that $\bar\mu(a \cap e_\kappa) \le k\bar\mu(b \cap e_\kappa)$ for every $\kappa \in K$. Then an ideal I of $\mathfrak{A}$ is $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$-invariant iff $a \in I$ whenever $a \preccurlyeq b \in I$. **P** (α) Suppose that I is $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ invariant and that $b \in I$, $\bar\mu(a \cap e_\kappa) \le k\bar\mu(b \cap e_\kappa)$ for every $\kappa \in K$. Then for each κ we can find $a_{\kappa 1}, \dots, a_{\kappa k}$ such that $a \cap e_\kappa = \sup_{i\le k} a_{\kappa i}$ and $\bar\mu a_{\kappa i} \le \bar\mu(b \cap e_\kappa)$ for every i. Now there are measure-preserving automorphisms $\pi_{\kappa i}$ of the principal ideal $\mathfrak{A}_{e_\kappa}$ such that $\pi_{\kappa i} a_{\kappa i} \subseteq b$. Setting $\pi_i d = \sup_{\kappa\in K} \pi_{\kappa i}(d \cap e_\kappa)$ for every $d \in \mathfrak{A}$, and $a_i = \sup_{\kappa\in K} a_{\kappa i}$, we have $\pi_i \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and $\pi_i a_i \subseteq b$, so $a_i \in I$ for each i; also $a = \sup_{i\le k} a_i$, so $a \in I$. (β) On the other hand, if $a \in \mathfrak{A}$ and $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$, then

$$\bar\mu(\pi a \cap e_\kappa) = \bar\mu\pi(a \cap e_\kappa) = \bar\mu(a \cap e_\kappa)$$

for every $\kappa \in K$, because $\pi e_\kappa = e_\kappa$, so that $\pi a \preccurlyeq a$. So if I satisfies the condition, $\pi[I] \subseteq I$ for every $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and $I \in \mathcal{I}$. **Q**

(b) Consequently, for $I \in \mathcal{I}$ and $\kappa \in K$, $e_\kappa \in I$ iff there is some $a \in I$ such that $a \cap a_\kappa \neq 0$, since in this case $e_\kappa \preccurlyeq a$. (This is where I use the hypothesis that $(\mathfrak{A}, \bar\mu)$ is totally finite.) It follows that if K is finite, any $I \in \mathcal{I}$ is the principal ideal generated by $\sup\{e_\kappa : e_\kappa \in I\}$. Conversely, of course, all such ideals are $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$-invariant. Thus $\mathcal{I}$ is in a natural order-preserving correspondence with $\mathcal{P}K$, and $\mathcal{H} \cong \mathcal{P}K$.

(c) Now suppose that K is infinite; enumerate it as $\langle\kappa_n\rangle_{n\in\mathbb{N}}$. Define $\theta : \mathfrak{A} \to \ell^\infty$ by setting

$$\theta a = \langle\bar\mu(a \cap e_{\kappa_n})/\bar\mu(e_{\kappa_n})\rangle_{n\in\mathbb{N}}$$

for $a \in \mathfrak{A}$; so that

$$a \preccurlyeq b \text{ iff there is some } k \text{ such that } \theta a \le k\theta b,$$

$$\theta a \le \theta(a \cup b) \le \theta a + \theta b \le 2\theta(a \cup b)$$

for all a, $b \in \mathfrak{A}$, while $\theta(1_{\mathfrak{A}})$ is the standard order unit $\chi\mathbb{N}$ of ℓ^∞. Let $\mathcal{U}$ be the family of solid linear subspaces of ℓ^∞ and define functions $I \mapsto V_I : \mathcal{I} \to \mathcal{U}$, $U \mapsto J_U : \mathcal{U} \to \mathcal{I}$ by saying

$$V_I = \{f : f \in \ell^\infty,\ |f| \le k\theta a \text{ for some } a \in I,\ k \in \mathbb{N}\},$$

$$J_U = \{a : a \in \mathfrak{A},\ \theta a \in U\}.$$

The properties of θ just listed ensure that $V_I \in \mathcal{U}$ and $J_U \in \mathcal{I}$ for every $I \in \mathcal{I}$, $U \in \mathcal{U}$. Of course both $I \mapsto V_I$ and $U \mapsto J_U$ are order-preserving. If $I \in \mathcal{I}$, then

$$J_{V_I} = \{a : \exists\, b \in I,\ a \preccurlyeq b\} = I.$$

Finally, $V_{J_U} = U$ for every $U \in \mathcal{U}$. **P**

$$V_{J_U} = \{f : \exists\, a \in \mathfrak{A},\ k \in \mathbb{N},\ |f| \le k\theta a \in U\} \subseteq U$$

because U is a solid linear subspace. But also, given $g \in U$, there is an $a \in \mathfrak{A}$ such that $\bar\mu(a \cap e_{\kappa_n}) = \min(1, |g(n)|)\bar\mu(e_{\kappa_n})$ for every n (because $\mathfrak{A}$ is atomless); in which case

$$\theta a \le |g| \le \max(1, \|g\|_\infty)\theta a$$

so $a \in J_U$ and $g \in V_{J_U}$. Thus $U = V_{J_U}$. **Q** So the functions $I \mapsto V_I$ and $U \mapsto J_U$ are the two halves of an order-isomorphism between $\mathcal{I}$ and $\mathcal{U}$, and $\mathcal{H} \cong \mathcal{I} \cong \mathcal{U}$, as claimed.

383K Later in this chapter I will give a good deal of space to the question of when two automorphisms of a measure algebra are conjugate. Because, on any measure algebra $(\mathfrak{A}, \bar\mu)$, we have two groups $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ with claims on our attention, we have two different conjugacy relations to examine. To clear the ground, I give a result showing that in a significant number of cases the two coincide.

Proposition Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an ergodic measure-preserving Boolean homomorphism. If $\phi \in \mathrm{Aut}\,\mathfrak{A}$ is such that $\phi\pi\phi^{-1}$ is measure-preserving, then ϕ is measure-preserving.

proof Consider the functional $\nu : \mathfrak{A} \to \mathbb{R}$ defined by saying that $\nu a = \bar\mu(\phi a)$ for every $a \in \mathfrak{A}$. Because $\bar\mu$ is completely additive (321F) and strictly positive, so is ν. We therefore have a $c = [\![\nu > \bar\mu]\!]$ in $\mathfrak{A}$ such that $\nu a > \bar\mu a$ whenever $0 \ne a \subseteq c$ and $\nu a \le \bar\mu a$ whenever $a \cap c = 0$ (326T). Now $\pi c = c$. **P?** Otherwise, because π is measure-preserving,

$$\bar\mu(\pi c \setminus c) = \bar\mu(\pi c) - \bar\mu(c \cap \pi c) = \bar\mu c - \bar\mu(c \cap \pi c) = \bar\mu(c \setminus \pi c) = \frac{1}{2}\bar\mu(c \,\triangle\, \pi c) > 0.$$

Next,

$$\nu\pi c = \bar\mu(\phi\pi c) = \bar\mu(\phi\pi\phi^{-1}\phi c) = \nu c,$$

so we also have $\nu(\pi c \setminus c) = \nu(c \setminus \pi c)$. But now observe that

$$\nu(\pi c \setminus c) \le \bar\mu(\pi c \setminus c), \quad \nu(c \setminus \pi c) > \bar\mu(c \setminus \pi c$$

by the choice of c, which is impossible. **XQ**

Because π is ergodic, c must be 0 or 1 (372Pa). But as $\nu\pi 1 = \nu 1 = \bar\mu 1$, we cannot have $0 \ne 1 \subseteq c$, so $c = 0$. This means that $\nu a \le \bar\mu a$ for every $a \in \mathfrak{A}$; once again, $\nu 1 = \bar\mu 1$, so in fact $\nu a = \bar\mu a$ for every a, that is, ϕ is measure-preserving.

383L Corollary Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, and π_1, $\pi_2 \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ two ergodic measure-preserving automorphisms. If they are conjugate in $\mathrm{Aut}\,\mathfrak{A}$ then they are conjugate in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

proof There is a $\phi \in \mathrm{Aut}\,\mathfrak{A}$ such that $\phi\pi_1\phi^{-1} = \pi_2$; now 383K tells us that $\phi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

383X Basic exercises (a) Let (X, Σ, μ) be a countably separated measure space, and write $\mathrm{Aut}_\mu\Sigma$ for the group of automorphisms $\phi : \Sigma \to \Sigma$ such that $\mu\phi(E) = \mu E$ for every $E \in \Sigma$. Show that every member of $\mathrm{Aut}_\mu\Sigma$ is expressible as a product of at most three involutions belonging to $\mathrm{Aut}_\mu\Sigma$. (*Hint*: 382Xc.)

>(b) Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra. For each infinite cardinal κ, let e_κ be the Maharam-type-κ component of $\mathfrak{A}$. (i) Show that $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is a simple group iff *either* there is just one infinite cardinal κ such that $e_\kappa \ne 0$, that e_κ has finite measure and all the atoms of $\mathfrak{A}$ (if any) have different measures *or* $\mathfrak{A}$ is purely atomic and there is just one pair of atoms of the same measure *or* $\mathfrak{A}$ is purely atomic and all its atoms have different measures. (ii) Show that $\mathrm{Aut}\,\mathfrak{A}$ is a simple group iff *either* $(\mathfrak{A}, \bar\mu)$ is σ-finite and there is just one infinite cardinal κ such that $e_\kappa \ne 0$ and $\mathfrak{A}$ has at most one atom *or* $\mathfrak{A}$ is purely atomic and has at most two atoms.

(c) Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra. (i) Show that $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is simple iff it is isomorphic to one of the groups $\{\iota\}$, $\mathbb{Z}_2$ or $\mathrm{Aut}_{\bar\nu_\kappa}\mathfrak{B}_\kappa$ where κ is an infinite cardinal and $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ is the measure algebra of the usual measure on $\{0,1\}^\kappa$. (ii) Show that $\mathrm{Aut}\,\mathfrak{A}$ is simple iff it is isomorphic to one of the groups $\{\iota\}$, $\mathbb{Z}_2$ or $\mathrm{Aut}\,\mathfrak{B}_\kappa$.

(d) Show that if $(\mathfrak{A},\bar{\mu})$ is a semi-finite measure algebra of magnitude greater than $\mathfrak{c}$, its automorphism group $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is not simple.

(e) Let $(\mathfrak{A},\bar{\mu})$ be an atomless localizable measure algebra. For each infinite cardinal κ write e_κ for the Maharam-type-κ component of $\mathfrak{A}$. For π, $\psi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ show that π belongs to the normal subgroup of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ generated by ψ iff there is a $k\in\mathbb{N}$ such that

$$\operatorname{mag}(e_\kappa\cap\operatorname{supp}\pi)\le k\operatorname{mag}(e_\kappa\cap\operatorname{supp}\psi)\text{ for every infinite cardinal }\kappa,$$

writing $\operatorname{mag}a$ for the magnitude of a, and setting $k\zeta=\zeta$ if $k>0$ and ζ is an infinite cardinal.

>(f) Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of Lebesgue measure on $\mathbb{R}$. For $n\in\mathbb{N}$ set $e_n=[-n,n]^\bullet\in\mathfrak{A}$. Let $G\le\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ be the group consisting of measure-preserving automorphisms π such that $\operatorname{supp}\pi\subseteq e_n$ for some n. Show that G is simple. (*Hint*: show that G is the union of an increasing sequence of simple subgroups.)

(g) Let $(\mathfrak{A},\bar{\mu})$ be an atomless totally finite measure algebra. Let $\mathcal{H}$ be the lattice of normal subgroups of $\operatorname{Aut}\mathfrak{A}$. Show that $\mathcal{H}$ is isomorphic, as partially ordered set, to $\mathcal{P}K$ for some countable set K.

(h) Let $(\mathfrak{A},\bar{\mu})$ be an atomless localizable measure algebra which is not σ-finite, and suppose that $\tau(\mathfrak{A}_a)=\tau(\mathfrak{A}_b)$ whenever a, $b\in\mathfrak{A}$ and $0<\bar{\mu}a\le\bar{\mu}b<\infty$. Let κ be the magnitude of $\mathfrak{A}$. (i) Show that the lattice $\mathcal{H}$ of normal subgroups of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is well-ordered, with least member $\{\iota\}$ and one member H_ζ for each infinite cardinal ζ less than or equal to κ^+, setting

$$H_\zeta=\{\pi:\pi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A},\ \operatorname{mag}(\operatorname{supp}\pi)<\zeta\},$$

where $\operatorname{mag}a$ is the magnitude of a. (ii) Show that the lattice $\mathcal{H}'$ of normal subgroups of $\operatorname{Aut}\mathfrak{A}$ is well-ordered, with least member $\{\iota\}$ and one member H'_ζ for each uncountable cardinal ζ less than or equal to κ^+, setting

$$H'_\zeta=\{\pi:\pi\in\operatorname{Aut}\mathfrak{A},\ \operatorname{mag}(\operatorname{supp}\pi)<\zeta\}.$$

(i) Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1]$. Give an example of two measure-preserving automorphisms of $\mathfrak{A}$ which are conjugate in $\operatorname{Aut}\mathfrak{A}$ but not in $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$.

(j) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra. For π, $\phi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ set

$$\rho(\pi,\phi)=\sup_{a\in\mathfrak{A}}\bar{\mu}(\pi a\mathbin{\triangle}\phi a),\quad \sigma(\pi,\phi)=\bar{\mu}(\operatorname{supp}(\pi^{-1}\phi)).$$

(i) Show that ρ and σ are metrics on $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, and that $\rho\le\sigma\le\frac{3}{2}\rho$. (*Hint*: 382Eb.) (ii) Show that $\rho(\psi\pi,\psi\phi)=\rho(\pi\psi,\phi\psi)=\rho(\pi,\phi)$, $\rho(\pi^{-1},\phi^{-1})=\rho(\pi,\phi)$, $\rho(\pi\psi,\phi\theta)\le\rho(\pi,\phi)+\rho(\psi,\theta)$, $\sigma(\psi\pi,\psi\phi)=\sigma(\pi\psi,\phi\psi)=\sigma(\pi,\phi)$, $\sigma(\pi^{-1},\phi^{-1})=\sigma(\pi,\phi)$, $\sigma(\pi\psi,\phi\theta)\le\sigma(\pi,\phi)+\sigma(\psi,\theta)$ for all π, ϕ, ψ, $\theta\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$. (iii) Show that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is complete under ρ and σ.

(k) Let (X,Σ,μ) be a measure space and $(\mathfrak{A},\bar{\mu})$ its measure algebra. Let S be the set of functions which are isomorphisms between conegligible measurable subsets of X with their subspace measures. (i) Show that the composition of two members of S belongs to S. (ii) Show that there is a map $f\mapsto\pi_f:S\to\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ defined by saying that $\pi_f(E^\bullet)=f^{-1}[E]^\bullet$ for every $E\in\Sigma$, and that $\pi_{fg}=\pi_f\pi_g$, $\pi_f^{-1}=\pi_{f^{-1}}$ for all f, $g\in S$. (iii) Show that $\{\pi_f:f\in S\}$ is a countably full subgroup of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$.

>(l) Let (X,Σ,μ) be a measure space and $(\mathfrak{A},\bar{\mu})$ its measure algebra. Let Φ be the group of measure space automorphisms of (X,Σ,μ). For $f\in\Phi$, let $\pi_f\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ be the corresponding automorphism, defined by setting $\pi_f(E^\bullet)=(f^{-1}[E])^\bullet$ for every $E\in\Sigma$. (i) Show that $f\mapsto\pi_f$ is a group homomorphism from Φ to $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$. (ii) Show that if $F\subseteq\Phi$ and the subgroup of Φ generated by F is Ψ, then the subgroup of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ generated by $\{\pi_f:f\in F\}$ is $\{\pi_f:f\in\Psi\}$. (iii) Show that if (X,Σ,μ) is countably separated (definition: 343D) and $F\subseteq\Phi$ is countable, then the full subgroup of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ generated by $\{\pi_f:f\in F\}$ is $\{\pi_g:g\in F^*\}$, where

$$F^*=\{g:g\in\Phi,\ g(x)\in\{f(x):x\in F\}\text{ for every }x\in X\}.$$

383Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be an atomless totally finite measure algebra. Show that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{A}$ have the same (cardinal) number of normal subgroups.

(b) Let X be a set. Show that $\operatorname{Aut}\mathcal{P}X$ has one normal subgroup if $\#(X) \le 1$, two if $\#(X) = 2$, three if $\#(X) = 3$ or $5 \le \#(X) \le \omega$, four if $\#(X) = 4$ or $\#(X) = \omega$, five if $\#(X) = \omega_1$.

383 Notes and comments This section is short because there are no substantial new techniques to be developed. 383D is simply a matter of checking that the hypotheses of 382N are satisfied (and these hypotheses were of course chosen with 383D in mind), and 383I is similarly direct from 382R. 383I-383J, 383Xe and 383Xh are variations on a theme. In a general Boolean algebra $\mathfrak{A}$ with a group G of automorphisms, we have a transitive, reflexive relation $\preccurlyeq_G$ defined by saying that $a \preccurlyeq_G b$ if there are $\pi_1, \dots, \pi_k \in G$ such that $a \subseteq \sup_{i \le k} \pi_i b$; the point about localizable measure algebras is that the functions 'Maharam type' and 'magnitude' enable us to describe this relation when $G = \operatorname{Aut}_{\bar\mu}\mathfrak{A}$, and the essence of 382R is that in that context π belongs to the normal subgroup of G generated by ψ iff $\operatorname{supp}\pi \preccurlyeq_G \operatorname{supp}\psi$.

Some of the most interesting questions concerning automorphism groups of measure algebras can be expressed in the form 'how can we determine when a given pair of automorphisms are conjugate?' Generally, people have concentrated on conjugacy in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$. But the same question can be asked in $\operatorname{Aut}\mathfrak{A}$. In particular, it is possible for two members of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ to be conjugate in $\operatorname{Aut}\mathfrak{A}$ but not in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ (383Xi). However this phenomenon does not occur for *ergodic* automorphisms, or even for ergodic measure-preserving Boolean homomorphisms (383K-383L).

Most of the work of this chapter is focused on atomless measure algebras. There are various extra complications which appear if we allow atoms. The most striking are in the next section; here I mention only 383Xb and 383Yb.

384 Outer automorphisms

Continuing with the investigation of the abstract group-theoretic nature of the automorphism groups $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, I devote a section to some remarkable results concerning isomorphisms between them. Under any of a variety of conditions, any isomorphism between two groups $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{B}$ must correspond to an isomorphism between the underlying Boolean algebras (384E, 384F, 384J, 384M); consequently $\operatorname{Aut}\mathfrak{A}$ has few, or no, outer automorphisms (384G, 384K, 384O). I organise the section around a single general result (384D).

384A Lemma Let $\mathfrak{A}$ be a Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ which has many involutions (definition: 382O). Then for every non-zero $a \in \mathfrak{A}$ there is an automorphism $\psi \in G$, of order 4, which is supported by a.

proof Let $\pi \in G$ be an involution supported by a. Let $b \subseteq a$ be such that $\pi b \ne b$. Then at least one of $b \setminus \pi b$, $\pi b \setminus b = \pi(b \setminus \pi b)$ is non-zero, so in fact both are. Let ϕ be an involution supported by $b \setminus \pi b$. Then $\pi\phi\pi = \pi\phi\pi^{-1}$ is an involution supported by $\pi b \setminus b$, so commutes with ϕ, and the product $\phi\pi\phi\pi$ is an involution. But this means that $\psi = \phi\pi$ has order 4, and of course it is supported by a because ϕ and π both are.

384B A note on supports Since in this section we shall be looking at more than one automorphism group at a time, I shall need to call on the following elementary extension of a fact in §381. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\theta : \mathfrak{A} \to \mathfrak{B}$ a Boolean isomorphism. If $\pi \in \operatorname{Aut}\mathfrak{A}$ is supported by $a \in \mathfrak{A}$, then $\theta\pi\theta^{-1} \in \operatorname{Aut}\mathfrak{B}$ is supported by θa. (Use the same argument as in 381Ej.) Accordingly, if a is the support of π then θa will be the support of $\theta\pi\theta^{-1}$, as in 381Gd.

384C Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Boolean algebras, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. If θ_1, $\theta_2 : \mathfrak{A} \to \mathfrak{B}$ are distinct isomorphisms, then there is a $\phi \in G$ such that $\theta_1\phi\theta_1^{-1} \ne \theta_2\phi\theta_2^{-1}$.

proof Because $\theta_1 \ne \theta_2$, $\theta = \theta_2^{-1}\theta_1$ is not the identity automorphism on $\mathfrak{A}$, and there is some non-zero $a \in \mathfrak{A}$ such that $\theta a \cap a = 0$. Let $\pi \in G$ be an involution supported by a; then $\theta\pi\theta^{-1}$ is supported by θa, so cannot be equal to π, and $\theta_1\pi\theta_1^{-1} \ne \theta_2\pi\theta_2^{-1}$.

384D Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind complete Boolean algebras and G and H subgroups of $\operatorname{Aut}\mathfrak{A}$, $\operatorname{Aut}\mathfrak{B}$ respectively, both having many involutions. Let $q : G \to H$ be an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in G$.

proof (a) The first half of the proof is devoted to setting up some structures in the group G. Let $\pi \in G$ be any involution. Set

$$C_\pi = \{\phi : \phi \in G,\ \phi\pi = \pi\phi\},$$

the centralizer of π in G;

$$U_\pi = \{\phi : \phi \in C_\pi,\, \phi = \phi^{-1},\, \phi\psi\phi\psi^{-1} = \psi\phi\psi^{-1}\phi \text{ for every } \psi \in C_\pi\},$$

the set of involutions in C_π commuting with all their conjugates in C_π, together with the identity,

$$V_\pi = \{\phi : \phi \in G,\, \phi\psi = \psi\phi \text{ for every } \psi \in U_\pi\},$$

the centralizer of U_π in G,

$$S_\pi = \{\phi^2 : \phi \in V_\pi\},$$

$$W_\pi = \{\phi : \phi \in G,\, \phi\psi = \psi\phi \text{ for every } \psi \in S_\pi\},$$

the centralizer of S_π in G.

(b) The point of this list is to provide a purely group-theoretic construction corresponding to the support of π in $\mathfrak{A}$. In the next few paragraphs of the proof (down to (f)), I set out to describe the objects just introduced in terms of their action on $\mathfrak{A}$. First, note that π is an exchanging involution (382Fa); express it as $(\overleftarrow{a' \,{}_\pi\, a''})$, so that the support of π is $a_\pi = a' \cup a''$.

(c) I start with two elementary properties of C_π:

(i) $\phi(a_\pi) = a_\pi$ for every $\phi \in C_\pi$. **P** As remarked in 381Gd, the support of $\pi = \phi\pi\phi^{-1}$ is $\phi(a_\pi)$, so this must be a_π. **Q**

(ii) If $\phi \in C_\pi$ and ϕ is not supported by a_π, there is a non-zero $d \subseteq 1 \setminus a_\pi$ such that $d \cap \phi d = 0$, by 381Ei.

(d) Now for the properties of U_π:

(i) If $\phi \in U_\pi$, then ϕ is supported by a_π.

P$(\boldsymbol{\alpha})$**?** Suppose first that there is a $d \subseteq 1 \setminus a_\pi$ such that $d \cap (\phi d \cup \phi^2 d) = 0$. Let $\psi \in G$ be an involution supported by d. Then $\operatorname{supp}\psi \cap \operatorname{supp}\pi = 0$, so $\psi \in C_\pi$. There is a $c \subseteq d$ such that $\psi c \ne c$, so

$$\psi\phi\psi^{-1}\phi c = \psi\phi^2 c = \phi^2 c$$

because $d \cap (\phi c \cup \phi^2 c) = 0$, while

$$\phi\psi\phi\psi^{-1} c = \phi^2\psi^{-1} c$$

because $d \cap \phi\psi^{-1}c = 0$; but this means that $\psi\phi\psi^{-1}\phi c \ne \phi\psi\phi\psi^{-1}c$, so ϕ and $\psi\phi\psi^{-1}$ do not commute, and $\phi \notin U_\pi$. **X**

$(\boldsymbol{\beta})$**?** Suppose that ϕ^2 is not supported by a_π. Then, as remarked in (c-ii), there is a non-zero $d \subseteq 1 \setminus a_\pi$ such that $\phi^2 d \cap d = 0$. Now $d \not\subseteq \phi^2 d$, so $d \not\subseteq \phi d$; set $d' = d \setminus \phi d$. Then $d' \cap \phi d' = d' \cap \phi^2 d' = 0$ and $0 \ne d' \subseteq 1 \setminus a_\pi$; but this is impossible, by (α). **X**

$(\boldsymbol{\gamma})$ Thus $\phi^2 d = d$ for every $d \subseteq 1 \setminus a_\pi$. **?** Suppose, if possible, that ϕ is not supported by a_π. Then there is a non-zero $d \subseteq 1 \setminus a_\pi$ such that $\phi d \cap d = 0$. By 384A, there is a $\psi \in G$, of order 4, supported by d. Because $d \cap a_\pi = 0$, $\psi \in C_\pi$. Because $\psi \ne \psi^{-1}$, there is a $c \subseteq d$ such that $\psi c \ne \psi^{-1}c$; but now $\phi c \cap d = \phi\psi^{-1}c \cap d = 0$, so

$$\psi\phi\psi^{-1}\phi c = \psi\phi^2 c = \psi c \ne \psi^{-1}c = \phi^2\psi^{-1}c = \phi\psi\phi\psi^{-1}c,$$

and ϕ does not commute with its conjugate $\psi\phi\psi^{-1}$, contradicting the assumption that $\phi \in U_\pi$. **X**

So ϕ is supported by a_π, as claimed. **Q**

(ii) If $u \in \mathfrak{A}$ and $\pi u = u$, then $\pi_u \in U_\pi$, where

$$\pi_u d = \pi d \text{ if } d \subseteq u, \quad \pi_u d = d \text{ if } d \cap u = 0,$$

that is, $\pi_u = (\overleftarrow{a' \cap u \,{}_\pi\, a'' \cap u})$. **P** For any $\psi \in \operatorname{Aut}\mathfrak{A}$,

$$\psi\pi_u\psi^{-1} = (\overleftarrow{\psi(a' \cap u) \,{}_{\psi\pi\psi^{-1}}\, \psi(a'' \cap u)})$$

(381Sb). (α) Accordingly

$$\pi\pi_u\pi^{-1} = (\overleftarrow{a'' \cap u \,{}_\pi\, a' \cap u}) = \pi_u$$

and $\pi_u \in C_\pi$. (β) If $\psi \in C_\pi$, then

$$\pi = \psi\pi\psi^{-1} = (\overleftarrow{\psi a' \,{}_{\psi\pi\psi^{-1}}\, \psi a''}) = (\overleftarrow{\psi a' \,{}_\pi\, \psi a''}).$$

So

$$\psi\pi_u\psi^{-1} = (\overleftarrow{\psi(a'\cap u)\,{}_{\psi\pi\psi^{-1}}\,\psi(a''\cap u)}) = (\overleftarrow{\psi a'\cap\psi u\,{}_{\pi}\,\psi a''\cap\psi u}) = \pi_{\psi u}.$$

Now if $\pi v = v$ then $\pi_u\pi_v = \pi_{u\triangle v} = \pi_v\pi_u$; in particular, $\pi_{\psi u}\pi_u = \pi_u\pi_{\psi u}$. As ψ is arbitrary, $\pi_u \in U_\pi$. **Q**

In particular, of course, $\pi = \pi_1$ belongs to U_π.

(e) The two parts of (d) lead directly to the properties we need of V_π.

(i) $V_\pi \subseteq C_\pi$, because $\pi \in U_\pi$. Consequently $\phi a_\pi = a_\pi$ for every $\phi \in V_\pi$.

(ii) If $\phi \in V_\pi$ then $\phi d \subseteq d \cup \pi d$ for every $d \subseteq a_\pi$. **P?** Suppose, if possible, otherwise. Set $u_0 = d \cup \pi d$, so that $\pi u_0 = u_0$, and $u = \phi u_0 \setminus u_0 \neq 0$; also $u \subseteq \phi a_\pi = a_\pi$. Since $\pi\phi u_0 = \phi\pi u_0 = \phi u_0$, $\pi u = u$. Set $v = u \cap a'$, so that $u = v \cup \pi v$ and $v \neq \pi v$. Because $u \cap \phi v \subseteq \phi(u_0 \cap u) = 0$,

$$\pi_u\phi v = \phi v \neq \phi\pi v = \phi\pi_u v,$$

which is impossible. **XQ**

(iii) It follows that $\phi^2 d = d$ whenever $\phi \in V_\pi$ and $d \subseteq a_\pi$. **P** Let e be the support of ϕ. Recall that $e = \sup\{c : c \cap \phi c = 0\}$ (381Gb), so that $d \cap e = \sup\{c : c \subseteq d,\ c \cap \phi c = 0\}$. Now if $c \subseteq a_\pi$ and $c \cap \phi c = 0$, we know that $\phi c \subseteq c \cup \pi c$, so in fact $\phi c \subseteq \pi c$. This shows that $\phi(d \cap e) \subseteq \pi(d \cap e)$. Also, because $\pi\phi = \phi\pi$, by (i), we have

$$\phi^2(d \cap e) \subseteq \phi\pi(d \cap e) = \pi\phi(d \cap e) \subseteq \pi^2(d \cap e) = d \cap e.$$

Of course $\phi^2(d \setminus e) = d \setminus e$, so $\phi^2 d \subseteq d$. This is true for every $d \subseteq a_\pi$. But as also $\phi^2 a_\pi = \phi a_\pi = a_\pi$, $\phi^2 d = d$ for every $d \subseteq a_\pi$. **Q**

(iv) The final thing we need to know about V_π is that $\phi \in V_\pi$ whenever $\phi \in G$ and $\operatorname{supp}\phi \cap a_\pi = 0$; this is immediate from (d-i) above.

(f) From (e-iii), we see that if $\phi \in S_\pi$ then $\operatorname{supp}\phi \cap a_\pi = 0$. But we also see from (e-iv) that if $0 \neq c \subseteq 1 \setminus a_\pi$ there is an involution in S_π supported by c; for there is a member ψ of G, of order 4, supported by c, and now $\psi \in V_\pi$ so $\psi^2 \in S_\pi$, while ψ^2 is an involution.

(g) Consequently, W_π is just the set of members of G supported by a_π. **P** (i) If $\operatorname{supp}\phi \subseteq a_\pi$ and $\psi \in S_\pi$, then $\operatorname{supp}\psi \cap a_\pi = 0$, as noted in (e), so $\phi\psi = \psi\phi$; as ψ is arbitrary, $\phi \in W_\pi$. (ii) If $\operatorname{supp}\phi \not\subseteq a_\pi$, then take a non-zero $d \subseteq 1 \setminus a_\pi$ such that $\phi d \cap d = 0$. Let $\psi \in S_\pi$ be an involution supported by d; then if $c \subseteq d$ is such that $\psi c \neq c$,

$$\phi\psi c \neq \phi c = \psi\phi c,$$

and $\phi\psi \neq \psi\phi$ so $\phi \notin W_\pi$. **Q**

(h) We can now return to consider the isomorphism $q : G \to H$. If $\pi \in G$ is an involution, then $q(\pi) \in H$ is an involution, and it is easy to check that

$$q[C_\pi] = C_{q(\pi)},$$

$$q[U_\pi] = U_{q(\pi)},$$

$$q[V_\pi] = V_{q(\pi)},$$

$$q[S_\pi] = S_{q(\pi)},$$

$$q[W_\pi] = W_{q(\pi)},$$

defining $C_{q(\pi)}, \dots, W_{q(\pi)} \subseteq H$ as in (a) above. So we see that, for any $\phi \in G$,

$$\operatorname{supp}\phi \subseteq \operatorname{supp}\pi \iff \operatorname{supp}q(\phi) \subseteq \operatorname{supp}q(\pi).$$

(i) Define $\theta : \mathfrak{A} \to \mathfrak{B}$ by writing

$$\theta a = \sup\{\operatorname{supp}q(\pi) : \pi \in G \text{ is an involution and } \operatorname{supp}\pi \subseteq a\}$$

for every $a \in \mathfrak{A}$. Evidently θ is order-preserving. Now if $a \in \mathfrak{A}$, $\pi \in G$ is an involution and $\operatorname{supp}\pi \not\subseteq a$, $\operatorname{supp}q(\pi) \not\subseteq \theta a$. **P** There is a $\phi \in G$, of order 4, supported by $\operatorname{supp}\pi \setminus a$. Now ϕ^2 is an involution supported by $\operatorname{supp}\pi$, so $\operatorname{supp}q(\phi^2) \subseteq \operatorname{supp}q(\pi)$. On the other hand, if $\pi' \in G$ is an involution supported by a, then $\phi \in V_{\pi'}$ and $\phi^2 \in S_{\pi'}$, so $q(\phi^2) \in S_{q(\pi')}$ and $\operatorname{supp}q(\phi^2) \cap \operatorname{supp}q(\pi') = 0$. As π' is arbitrary, $\operatorname{supp}q(\phi^2) \cap \theta a = 0$; so

$$\operatorname{supp} q(\pi) \setminus \theta a \supseteq \operatorname{supp} q(\phi^2) \neq 0. \ \mathbf{Q}$$

(j) In the same way, we can define $\theta^* : \mathfrak{B} \to \mathfrak{A}$ by setting

$$\theta^* b = \sup\{\operatorname{supp} q^{-1}(\pi) : \pi \in H \text{ is an involution and } \operatorname{supp}\pi \subseteq b\}$$

for every $b \in \mathfrak{B}$. Now $\theta^*\theta a = a$ for every $a \in \mathfrak{A}$. **P** (α) If $0 \neq u \subseteq a$, there is an involution $\pi \in G$ supported by u. Now $q(\pi)$ is an involution in H supported by θa, so

$$u \cap \theta^*\theta a \supseteq u \cap \operatorname{supp} q^{-1}q(\pi) = \operatorname{supp}\pi \neq 0.$$

As u is arbitrary, $a \subseteq \theta^*\theta a$. ($\beta$) If $\pi \in H$ is an involution supported by θa, then $\phi = q^{-1}(\pi)$ is an involution in G with $\operatorname{supp} q(\phi) = \operatorname{supp}\pi \subseteq \theta a$, so $\operatorname{supp}\phi \subseteq a$, by (i) above; as π is arbitrary, $\theta^*\theta a \subseteq a$. **Q**

Similarly, $\theta\theta^* b = b$ for every $b \in \mathfrak{B}$. But this means that θ and θ^* are the two halves of an order-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. By 312M, both are Boolean homomorphisms.

(k) If $\pi \in G$ is an involution, then $\theta(\operatorname{supp}\pi) = \operatorname{supp} q(\pi)$. **P** By the definition of θ, $\operatorname{supp} q(\pi) \subseteq \theta(\operatorname{supp}\pi)$. On the other hand,

$$\operatorname{supp} q(\pi) = \theta\theta^*(\operatorname{supp} q(\pi)) \supseteq \theta(\operatorname{supp} q^{-1}q(\pi)) = \theta(\operatorname{supp}\pi). \ \mathbf{Q}$$

Similarly, if $\pi \in H$ is an involution, $\theta^{-1}(\operatorname{supp}\pi) = \theta^*(\operatorname{supp}\pi) = \operatorname{supp} q^{-1}(\pi)$.

(l) We are nearly home. Let us confirm that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in G$. **P?** Otherwise, $\psi = q(\phi)^{-1}\theta\phi\theta^{-1}$ is not the identity automorphism on $\mathfrak{B}$, and there is a non-zero $b \in \mathfrak{B}$ such that $\psi b \cap b = 0$, that is, $\theta\phi\theta^{-1}b \cap q(\phi)b = 0$. Let $\pi \in H$ be an involution supported by b. Then $q^{-1}(\pi)$ is supported by $\theta^{-1}b$, by (j), so $\phi\theta^{-1}b$ supports $\phi q^{-1}(\pi)\phi^{-1}$ and $\theta\phi\theta^{-1}b$ supports $q(\phi q^{-1}(\pi)\phi^{-1}) = q(\phi)\pi q(\phi)^{-1}$. On the other hand, $q(\phi)b$ also supports $q(\phi)\pi q(\phi)^{-1}$, which is not the identity automorphism; so these two elements of $\mathfrak{B}$ cannot be disjoint. **XQ**

(m) Finally, θ is unique by 384C.

Remark The ideas of the proof here are taken from Eigen 82.

384E The rest of this section may be regarded as a series of corollaries of this theorem. But I think it will be apparent that they are very substantial results.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be atomless homogeneous Boolean algebras, and $q : \operatorname{Aut}\mathfrak{A} \to \operatorname{Aut}\mathfrak{B}$ an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$.

proof (a) Let $\widehat{\mathfrak{A}}$ be the Dedekind completion of $\mathfrak{A}$ (314U). Then every $\phi \in \operatorname{Aut}\mathfrak{A}$ has a unique extension to a Boolean homomorphism $\hat\phi : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{A}}$ (314Tb). Because the extension is unique, we must have $(\phi\psi)\hat{} = \hat\phi\hat\psi$ for all ϕ, $\psi \in \operatorname{Aut}\mathfrak{A}$; consequently, $\hat\phi$ and $\widehat{\phi^{-1}}$ are inverses of each other, and $\hat\phi \in \operatorname{Aut}\widehat{\mathfrak{A}}$ for each $\phi \in \operatorname{Aut}\mathfrak{A}$; moreover, $\phi \mapsto \hat\phi$ is a group homomorphism. Of course it is injective, so we have a subgroup $G = \{\hat\phi : \phi \in \operatorname{Aut}\mathfrak{A}\}$ of $\operatorname{Aut}\widehat{\mathfrak{A}}$ which is isomorphic to $\operatorname{Aut}\mathfrak{A}$. Clearly

$$G = \{\phi : \phi \in \operatorname{Aut}\widehat{\mathfrak{A}},\ \phi u \in \mathfrak{A} \text{ for every } u \in \mathfrak{A}\}.$$

If $a \in \widehat{\mathfrak{A}}$ is non-zero, then there is a non-zero $u \subseteq a$ belonging to $\mathfrak{A}$. Because $\mathfrak{A}$ is atomless and homogeneous, there is an involution $\pi \in \operatorname{Aut}\mathfrak{A}$ supported by u (382P); now $\hat\pi \in G$ is an involution supported by a. As a is arbitrary, G has many involutions.

Similarly, writing $\widehat{\mathfrak{B}}$ for the Dedekind completion of $\mathfrak{B}$, we have a subgroup $H = \{\hat\psi : \psi \in \operatorname{Aut}\mathfrak{B}\}$ of $\operatorname{Aut}\widehat{\mathfrak{B}}$ isomorphic to $\operatorname{Aut}\mathfrak{B}$, and with many involutions. Let $\hat q : G \to H$ be the corresponding isomorphism, so that $\hat q(\hat\phi) = \widehat{q(\phi)}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$.

By 384D, there is a Boolean isomorphism $\hat\theta : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{B}}$ such that $\hat q(\phi) = \hat\theta\phi\hat\theta^{-1}$ for every $\phi \in G$. Note that

$$\hat\theta(\operatorname{supp}\phi) = \operatorname{supp}(\hat\theta\phi\hat\theta^{-1}) = \operatorname{supp}\hat q(\phi)$$

for every $\phi \in G$, so that $\hat\theta(\operatorname{supp}\hat q^{-1}(\pi)) = \operatorname{supp}\pi$ for every $\pi \in H$.

(b) If $u \in \mathfrak{A}$, then $\hat\theta u \in \mathfrak{B}$. **P** It is enough to consider the case $u \notin \{0,1\}$, since surely $\hat\theta 0 = 0$ and $\hat\theta 1 = 1$. Take any $w \in \mathfrak{B}$ which is neither 0 nor 1; then there is an involution in $\operatorname{Aut}\mathfrak{B}$ with support w (382P again); the corresponding member π of H is still an involution with support w. Its image $\hat q^{-1}(\pi)$ in G is an involution with support $a = \hat\theta^{-1}w \in \widehat{\mathfrak{A}}$; of course $0 \neq a \neq 1$. Take non-zero u_1, $u_3 \in \mathfrak{A}$ such that $u_1 \subset a$ and $u_3 \subseteq 1 \setminus a$; set

$u_2 = 1 \setminus (u_1 \cup u_3)$. Because $\mathfrak{A}$ is homogeneous, there are ϕ, $\psi \in G$ such that $\phi u_1 = u$, $\psi u_1 = u_1$, $\psi u_2 = u_3$; set $\phi_2 = \phi\psi$. Then we have

$$u = \phi u_1 \subseteq \phi(\operatorname{supp} \hat{q}^{-1}(\pi)) = \operatorname{supp}(\phi\hat{q}^{-1}(\pi)\phi^{-1}) \subseteq \phi(u_1 \cup u_2) = u \cup \phi u_2,$$

$$u = \phi_2 u_1 \subseteq \phi_2(\operatorname{supp} \hat{q}^{-1}(\pi)) = \operatorname{supp}(\phi_2\hat{q}^{-1}(\pi)\phi_2^{-1}) \subseteq u \cup \phi_2 u_2 = u \cup \phi u_3,$$

so

$$\phi(\operatorname{supp} \hat{q}^{-1}(\pi)) \cap \phi_2(\operatorname{supp} \hat{q}^{-1}(\pi)) = u,$$

and

$$\begin{aligned}\hat{\theta}u &= \hat{\theta}(\phi(\operatorname{supp} \hat{q}^{-1}(\pi))) \cap \hat{\theta}(\phi_2(\operatorname{supp} \hat{q}^{-1}(\pi)))\\ &= \hat{\theta}(\operatorname{supp} \phi\hat{q}^{-1}(\pi)\phi^{-1}) \cap \hat{\theta}(\operatorname{supp} \phi_2\hat{q}^{-1}(\pi)\phi_2^{-1})\\ &= \hat{\theta}(\operatorname{supp} \hat{q}^{-1}(\hat{q}(\phi)\pi\hat{q}(\phi)^{-1})) \cap \hat{\theta}(\operatorname{supp} \hat{q}^{-1}(\hat{q}(\phi_2)\pi\hat{q}(\phi_2)^{-1}))\\ &= \operatorname{supp}(\hat{q}(\phi)\pi\hat{q}(\phi)^{-1}) \cap \operatorname{supp}(\hat{q}(\phi_2)\pi\hat{q}(\phi_2)^{-1})\end{aligned}$$

(see the last sentence of (a) above)

$$= \hat{q}(\phi)(\operatorname{supp} \pi) \cap \hat{q}(\phi_2)(\operatorname{supp} \pi) = \hat{q}(\phi)w \cap \hat{q}(\phi_2)w \in \mathfrak{B}$$

because both $\hat{q}(\phi)$ and $\hat{q}(\phi_2)$ belong to H. **Q**

Similarly, $\hat{\theta}^{-1}v \in \mathfrak{A}$ for every $v \in \mathfrak{B}$, and $\theta = \hat{\theta}\restriction\mathfrak{A}$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

We now have

$$q(\phi) = \hat{q}(\hat{\phi})\restriction\mathfrak{B} = (\hat{\theta}\hat{\phi}\hat{\theta}^{-1})\restriction\mathfrak{B} = \theta\phi\theta^{-1}$$

for every $\phi \in \operatorname{Aut}\mathfrak{A}$. Finally, θ is unique by 384C, as before.

384F Corollary If $\mathfrak{A}$ and $\mathfrak{B}$ are atomless homogeneous Boolean algebras with isomorphic automorphism groups, they are isomorphic as Boolean algebras.

Remark Of course a one-element Boolean algebra $\{0\}$ and a two-element Boolean algebra $\{0,1\}$ have isomorphic automorphism groups without being isomorphic.

384G Corollary If $\mathfrak{A}$ is a homogeneous Boolean algebra, then $\operatorname{Aut}\mathfrak{A}$ has no outer automorphisms.

proof If $\mathfrak{A} = \{0,1\}$ this is trivial. Otherwise, $\mathfrak{A}$ is atomless, so if q is any automorphism of $\operatorname{Aut}\mathfrak{A}$, there is a Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{A}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$, and q is an inner automorphism.

384H Definitions Complementary to the notion of ‘many involutions’ is the following concept.

(a) A Boolean algebra $\mathfrak{A}$ is **rigid** if the only automorphism of $\mathfrak{A}$ is the identity automorphism.

(b) A Boolean algebra $\mathfrak{A}$ is **nowhere rigid** if no non-trivial principal ideal of $\mathfrak{A}$ is rigid.

384I Lemma Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is nowhere rigid;

(ii) for every $a \in \mathfrak{A} \setminus \{0\}$ there is a $\phi \in \operatorname{Aut}\mathfrak{A}$, not the identity, supported by a;

(iii) for every $a \in \mathfrak{A} \setminus \{0\}$ there are distinct b, $c \subseteq a$ such that the principal ideals $\mathfrak{A}_b$, $\mathfrak{A}_c$ they generate are isomorphic;

(iv) the automorphism group $\operatorname{Aut}\mathfrak{A}$ has many involutions.

proof (a)(ii)⇒(i) If $a \in \mathfrak{A} \setminus \{0\}$, let $\phi \in \operatorname{Aut}\mathfrak{A}$ be a non-trivial automophism supported by a; then $\phi\restriction\mathfrak{A}_a$ is a non-trivial automorphism of the principal ideal $\mathfrak{A}_a$, so $\mathfrak{A}_a$ is not rigid.

(b)(i)⇒(iii) There is a non-trivial automorphism ψ of $\mathfrak{A}_a$; now if $b \in \mathfrak{A}_a$ is such that $\psi b = c \ne b$, $\mathfrak{A}_b$ is isomorphic to $\psi[\mathfrak{A}_b] = \mathfrak{A}_c$.

(c)(iii)⇒(iv) Take any non-zero $a \in \mathfrak{A}$. By (iii), there are distinct b, $c \subseteq a$ such that $\mathfrak{A}_b$, $\mathfrak{A}_c$ are isomorphic. At least one of $b \setminus c$, $c \setminus b$ is non-zero; suppose the former. Let $\psi : \mathfrak{A}_b \to \mathfrak{A}_c$ be an isomorphism, and set $d = b \setminus c$, $d' = \psi(b \setminus c)$; then $d' \subseteq c$, so $d' \cap d = 0$, and $\phi = (\overleftarrow{d \,{}_\psi\, d'})$ is an involution supported by a.

(d)(iv)⇒(ii) is trivial.

384J Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be nowhere rigid Dedekind complete Boolean algebras and $q : \operatorname{Aut}\mathfrak{A} \to \operatorname{Aut}\mathfrak{B}$ an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$.

proof Put 384I(i)⇒(iv) and 384D together.

384K Corollary Let $\mathfrak{A}$ be a nowhere rigid Dedekind complete Boolean algebra. Then $\operatorname{Aut}\mathfrak{A}$ has no outer automorphisms.

384L Examples I note the following examples of nowhere rigid algebras.

(a) A non-trivial homogeneous Boolean algebra is nowhere rigid.
(b) Any principal ideal of a nowhere rigid Boolean algebra is nowhere rigid.
(c) A simple product of nowhere rigid Boolean algebras is nowhere rigid.
(d) Any atomless semi-finite measure algebra is nowhere rigid.
(e) A free product of nowhere rigid Boolean algebras is nowhere rigid.
(f) The Dedekind completion of a nowhere rigid Boolean algebra is nowhere rigid.

Indeed, the difficulty is to find an atomless Boolean algebra which is not nowhere rigid; for a variety of constructions of rigid algebras, see Bekkali & Bonnet 89.

384M Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be atomless localizable measure algebras, and $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, $\operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$ the corresponding groups of measure-preserving automorphisms. Let $q : \operatorname{Aut}_{\bar{\mu}}\mathfrak{A} \to \operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$ be an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$.

proof The point is just that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ has many involutions. **P** Let $a \in \mathfrak{A} \setminus \{0\}$. Then there is a non-zero $b \subseteq a$ such that the principal ideal $\mathfrak{A}_b$ is Maharam-type-homogeneous. Take $c \subseteq b$ and $d \subseteq b \setminus c$ such that $\bar{\mu}c = \bar{\mu}d = \min(1, \frac{1}{2}\bar{\mu}b)$ (331C). The principal ideals $\mathfrak{A}_c$, $\mathfrak{A}_d$ are now isomorphic as measure algebras (331I); let $\psi : \mathfrak{A}_c \to \mathfrak{A}_d$ be a measure-preserving isomorphism. Then $(\overleftarrow{c\,\underset{\psi}{}\,d}) \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is an involution supported by a. **Q**

Similarly, $\operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$ has many involutions, and the result follows at once from 384D.

384N To make proper use of the last theorem we need the following result.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be localizable measure algebras and $\theta : \mathfrak{A} \to \mathfrak{B}$ a Boolean isomorphism. For each infinite cardinal κ let e_κ be the Maharam-type-κ component of $\mathfrak{A}$ (332Gb) and for each $\gamma \in \left]0, \infty\right[$ let A_γ be the set of atoms of $\mathfrak{A}$ of measure γ. Then the following are equiveridical:

(i) for every $\phi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, $\theta\phi\theta^{-1} \in \operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$;

(ii)(α) for every infinite cardinal κ there is an $\alpha_\kappa > 0$ such that $\bar{\nu}(\theta a) = \alpha_\kappa\bar{\mu}a$ for every $a \subseteq e_\kappa$,

(β) for every $\gamma \in \left]0, \infty\right[$ there is an $\alpha_\gamma > 0$ such that $\bar{\nu}(\theta a) = \alpha_\gamma\bar{\mu}a$ for every $a \in A_\gamma$.

proof (a)(i)⇒(ii)(α) Let κ be an infinite cardinal. The point is that if a, $a' \subseteq e_\kappa$ and $\bar{\mu}a = \bar{\mu}a' < \infty$ then $\bar{\nu}(\theta a) = \bar{\nu}(\theta a')$. **P** The principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_{a'}$ are isomorphic as measure algebras; moreover, by 332J, the principal ideals $\mathfrak{A}_{e_\kappa \setminus a}$, $\mathfrak{A}_{e_\kappa \setminus a'}$ are isomorphic. We therefore have a $\phi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\phi a = a'$. Consequently $\psi\theta a = \theta a'$, where $\psi = \theta\phi\theta^{-1} \in \operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$, and $\bar{\nu}(\theta a) = \bar{\nu}(\theta a')$. **Q**

If $e_\kappa = 0$ we can take $\alpha_\kappa = 1$. Otherwise fix on some $c_0 \subseteq e_\kappa$ such that $0 < \bar{\mu}c_0 < \infty$; take $b \subseteq \theta c_0$ such that $0 < \bar{\nu}b < \infty$, and set $c = \theta^{-1}b$, $\alpha_\kappa = \bar{\nu}b/\bar{\mu}c$. Then we shall have $\bar{\nu}(\theta a) = \bar{\nu}(\theta c) = \alpha_\kappa\bar{\mu}a$ whenever $a \subseteq e_\kappa$ and $\bar{\mu}a = \bar{\mu}c$. But we can find for any $n \ge 1$ a partition $c_{n1}, \dots, c_{nn}$ of c into elements of measure $\frac{1}{n}\bar{\mu}c$; since $\bar{\nu}(\theta c_{ni}) = \bar{\nu}(\theta c_{nj})$ for all i, $j \le n$, we must have $\bar{\nu}(\theta c_{ni}) = \frac{1}{n}\bar{\nu}(\theta c) = \alpha_\kappa\bar{\mu}c_{ni}$ for all i. So if $a \subseteq e_\kappa$ and $\bar{\mu}a = \frac{1}{n}\bar{\mu}c$, $\bar{\nu}(\theta a) = \bar{\nu}(\theta c_{n1}) = \alpha_\kappa\bar{\mu}a$. Now suppose that $a \subseteq e_\kappa$ and $\bar{\mu}a = \frac{k}{n}\bar{\mu}c$ for some k, $n \ge 1$; then a can be partitioned into k elements of measure $\frac{1}{n}\bar{\mu}c$, so in this case also $\bar{\nu}(\theta a) = \alpha_\kappa\bar{\mu}a$. Finally, for any $a \subseteq e_\kappa$, set

$$D = \{d : d \subseteq a,\ \bar{\mu}d \text{ is a rational multiple of } \bar{\mu}c\},$$

and let $D' \subseteq D$ be a maximal upwards-directed set. Then $\sup D' = a$, so $\theta[D']$ is an upwards-directed set with supremum θa, and

$$\bar{\nu}(\theta a) = \sup_{d \in D'} \bar{\nu}(\theta d) = \sup_{d \in D'} \alpha_\kappa\bar{\mu}d = \alpha_\kappa\bar{\mu}a.$$

(β) Let $\gamma \in \left]0, \infty\right[$. If $A_\gamma = \emptyset$ take $\alpha_\gamma = 1$. Otherwise, fix on any $c \in A_\gamma$ and set $\alpha_\gamma = \bar{\nu}(\theta c)/\gamma$. If $a \in A_\gamma$ then there is a $\phi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ exchanging the atoms a and c, so that $\theta\phi\theta^{-1} \in \operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$ exchanges the atoms θa and θc, and

$$\bar{\nu}(\theta a) = \bar{\nu}(\theta c) = \alpha_\gamma \bar{\mu} a.$$

(b)(ii)⇒(i) Now suppose that the conditions (α) and (β) are satisfied, that $\phi \in \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ and that $a \in \mathfrak{A}$. For each infinite cardinal κ, we have $\phi e_\kappa = e_\kappa$, so

$$\bar{\nu}(\theta\phi(e_\kappa \cap a)) = \alpha_\kappa \bar{\mu}(\phi(e_\kappa \cap a)) = \alpha_\kappa \bar{\mu}(e_\kappa \cap a) = \bar{\nu}(\theta(e_\kappa \cap a)).$$

Similarly, if we write $a_\gamma = \sup A_\gamma$, then for each $\gamma \in \,]0,\infty[$ we have $\phi[A_\gamma] = A_\gamma$ and $\phi a_\gamma = a_\gamma$, and for $c \subseteq a_\gamma$ we have

$$\bar{\mu} c = \gamma \#(\{e : e \in A_\gamma,\, e \subseteq c\});$$

so

$$\begin{aligned}\bar{\nu}(\theta\phi(a_\gamma \cap a)) &= \alpha_\gamma \gamma \#(\{e : e \in A_\gamma,\, e \subseteq \phi a\}) \\ &= \alpha_\gamma \gamma \#(\{e : e \in A_\gamma,\, e \subseteq a\}) \\ &= \sum_{e \in A_\gamma, e \subseteq a} \bar{\nu}(\theta e) = \bar{\nu}(\theta(a_\gamma \cap a)).\end{aligned}$$

Putting these together,

$$\begin{aligned}\bar{\nu}(\theta\phi a) &= \sum_{\kappa \text{ is an infinite cardinal}} \bar{\nu}(\theta\phi(e_\kappa \cap a)) + \sum_{\gamma \in]0,\infty[} \bar{\nu}(\theta\phi(a_\gamma \cap a)) \\ &= \sum_{\kappa \text{ is an infinite cardinal}} \bar{\nu}(\theta(e_\kappa \cap a)) + \sum_{\gamma \in]0,\infty[} \bar{\nu}(\theta(a_\gamma \cap a)) = \bar{\nu}(\theta a).\end{aligned}$$

But this means that

$$\bar{\nu}(\theta\phi\theta^{-1} b) = \bar{\nu}(\theta\theta^{-1} b) = \bar{\nu} b$$

for every $b \in \mathfrak{B}$, and $\theta\phi\theta^{-1}$ is measure-preserving, as required by (i).

384O Corollary If $(\mathfrak{A}, \bar{\mu})$ is an atomless totally finite measure algebra, $\mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ has no outer automorphisms.

proof Let $q : \mathrm{Aut}_{\bar{\mu}} \mathfrak{A} \to \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ be any automorphism. By 384M, there is a corresponding $\theta \in \mathrm{Aut}\, \mathfrak{A}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$. By 384N, there is for each infinite cardinal κ an $\alpha_\kappa > 0$ such that $\bar{\mu}(\theta a) = \alpha_\kappa \bar{\mu} a$ whenever $a \subseteq e_\kappa$, the Maharam-type-κ component of $\mathfrak{A}$. But since $\theta e_\kappa = e_\kappa$ and $\bar{\mu} e_\kappa < \infty$ for every κ, we must have $\alpha_\kappa = 1$ whenever $e_\kappa \neq 0$; as $\mathfrak{A}$ is atomless,

$$\begin{aligned}\bar{\mu}(\theta a) &= \sum_{\kappa \text{ is an infinite cardinal}} \bar{\mu}(\theta(a \cap e_\kappa)) \\ &= \sum_{\kappa \text{ is an infinite cardinal}} \alpha_\kappa \bar{\mu}(a \cap e_\kappa) \\ &= \sum_{\kappa \text{ is an infinite cardinal}} \bar{\mu}(a \cap e_\kappa) = \bar{\mu} a\end{aligned}$$

for every $a \in \mathfrak{A}$. Thus $\theta \in \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ and q is an inner automorphism.

384P The results above are satisfying and complete in their own terms, but leave open a number of obvious questions concerning whether some of the hypotheses can be relaxed. Atoms can produce a variety of complications (see 384Ya-384Yd below). To show that we really do need to assume that our algebras are Dedekind complete or localizable, I offer the following.

Example (a) There are an atomless localizable measure algebra $(\mathfrak{A}, \bar{\mu})$ and an atomless semi-finite measure algebra $(\mathfrak{B}, \bar{\nu})$ such that $\mathrm{Aut}\, \mathfrak{A} \cong \mathrm{Aut}\, \mathfrak{B}$, $\mathrm{Aut}_{\bar{\mu}} \mathfrak{A} \cong \mathrm{Aut}_{\bar{\nu}} \mathfrak{B}$ but $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic.

proof Let $(\mathfrak{A}_0, \bar{\mu}_0)$ be an atomless homogeneous probability algebra; for instance, the measure algebra of Lebesgue measure on the unit interval. Let $(\mathfrak{A}, \bar{\mu})$ be the simple product measure algebra $(\mathfrak{A}_0, \bar{\mu}_0)^{\omega_1}$ (322L); then $(\mathfrak{A}, \bar{\mu})$ is an atomless localizable measure algebra. In $\mathfrak{A}$ let I be the set

$$\{a : a \in \mathfrak{A} \text{ and the principal ideal } \mathfrak{A}_a \text{ is ccc}\};$$

then I is an ideal of $\mathfrak{A}$, the σ-ideal generated by the elements of finite measure (cf. 322G). Set

$$\mathfrak{B} = \{a : a \in \mathfrak{A}, \text{ either } a \in I \text{ or } 1 \setminus a \in I\}.$$

Then $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$, so if we set $\bar{\nu} = \bar{\mu} \restriction \mathfrak{B}$ then $(\mathfrak{B}, \bar{\nu})$ is a measure algebra in its own right.

The definition of I makes it plain that it is invariant under all Boolean automorphisms of $\mathfrak{A}$; so $\mathfrak{B}$ also is invariant under all automorphisms, and we have a homomorphism $\phi \mapsto q(\phi) = \phi \restriction \mathfrak{B} : \operatorname{Aut} \mathfrak{A} \to \operatorname{Aut} \mathfrak{B}$. On the other hand, because $\mathfrak{B}$ is order-dense in $\mathfrak{A}$, and $\mathfrak{A}$ is Dedekind complete, every automorphism of $\mathfrak{B}$ can be extended to an automorphism of $\mathfrak{A}$ (see part (a) of the proof of 384E). So q is actually an isomorphism between $\operatorname{Aut} \mathfrak{A}$ and $\operatorname{Aut} \mathfrak{B}$. Moreover, still because $\mathfrak{B}$ is order-dense, $q(\phi)$ is measure-preserving iff ϕ is measure-preserving, so $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ is isomorphic to $\operatorname{Aut}_{\bar{\nu}} \mathfrak{B}$. But of course there is no Boolean isomorphism, let alone a measure algebra isomorphism, between $\mathfrak{A}$ and $\mathfrak{B}$, because $\mathfrak{A}$ is Dedekind complete while $\mathfrak{B}$ is not.

Remark Thus the hypothesis 'Dedekind complete' in 384D and 384J (and 'localizable' in 384M), and the hypothesis 'homogeneous' in 384E-384F, are essential.

(b) There is an atomless semi-finite measure algebra $(\mathfrak{C}, \bar{\lambda})$ such that $\operatorname{Aut} \mathfrak{C}$ has an outer automorphism.

proof In fact we can take $\mathfrak{C}$ to be the simple product of $\mathfrak{A}$ and $\mathfrak{B}$ above. I claim that the isomorphism between $\operatorname{Aut} \mathfrak{A}$ and $\operatorname{Aut} \mathfrak{B}$ gives rise to an outer automorphism of $\operatorname{Aut} \mathfrak{C}$; this seems very natural, but I think there is a fair bit to check, so I take the argument in easy stages.

(i) We may identify the Dedekind completion of $\mathfrak{C} = \mathfrak{A} \times \mathfrak{B}$ with $\mathfrak{A} \times \mathfrak{A}$. For $\phi \in \operatorname{Aut} \mathfrak{C}$, we have a corresponding $\hat{\phi} \in \operatorname{Aut}(\mathfrak{A} \times \mathfrak{A})$. Now $\mathfrak{B} \times \mathfrak{A}$ is invariant under $\hat{\phi}$. **P** Consider first $\phi(0,1) = (a_1, b_1) \in \mathfrak{C}$. The corresponding principal ideal $\mathfrak{C}_{(a_1,b_1)} \cong \mathfrak{A}_{a_1} \times \mathfrak{B}_{b_1}$ of $\mathfrak{C}$ must be isomorphic to the principal ideal $\mathfrak{C}_{(0,1)} \cong \mathfrak{B}$; so that if $(a,b) \in \mathfrak{C}$ and $(a,b) \subseteq (a_1, b_1)$, then just one of the principal ideals $\mathfrak{C}_{(a,b)} \cong \mathfrak{A}_a \times \mathfrak{B}_b$, $\mathfrak{C}_{(a_1 \setminus a, b_1 \setminus b)} \cong \mathfrak{A}_{a_1 \setminus a} \times \mathfrak{B}_{b_1 \setminus b}$ is ccc. But this can only happen if $\mathfrak{A}_{a_1}$ is ccc and $\mathfrak{B}_{b_1}$ is not; that is, if a_1 and $1 \setminus b_1$ belong to I. Consequently $\hat{\phi}(0,a) \subseteq (a_1, b_1)$ belongs to $\mathfrak{B} \times \mathfrak{A}$ for every $a \in \mathfrak{A}$. We also find that

$$\phi(1,0) = (1,1) \setminus \phi(0,1) = (1 \setminus a_1, 1 \setminus b_1) \in \mathfrak{B} \times \mathfrak{A}.$$

Now if $b \in I$, then

$$\mathfrak{C}_{\phi(b,0)} \cong \mathfrak{C}_{(b,0)} \cong \mathfrak{A}_b$$

is ccc and

$$\phi(b,0) \in I \times I \subseteq \mathfrak{B} \times \mathfrak{A};$$

while

$$\phi(1 \setminus b, 0) = (1 \setminus a_1, 1 \setminus b_1) \setminus \phi(b,0) \in \mathfrak{B} \times \mathfrak{A}.$$

This shows that $\phi(b,0) \in \mathfrak{B} \times \mathfrak{A}$ for every $b \in \mathfrak{B}$. So

$$\hat{\phi}(b,a) = \hat{\phi}(b,0) \cup \hat{\phi}(0,a) \in \mathfrak{B} \times \mathfrak{A}$$

for every $b \in \mathfrak{B}$ and $a \in \mathfrak{A}$. **Q**

(ii) Let $\theta : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A} \times \mathfrak{A}$ be the involution defined by setting $\theta(a,b) = (b,a)$ for all a, $b \in \mathfrak{A}$. Take $\phi \in \operatorname{Aut} \mathfrak{C}$ and consider $\psi = \theta \hat{\phi} \theta^{-1} \in \operatorname{Aut}(\mathfrak{A} \times \mathfrak{A})$. If $c = (a,b) \in \mathfrak{C}$, then $\theta^{-1} c = (b,a) \in \mathfrak{B} \times \mathfrak{A}$, so $\hat{\phi} \theta^{-1} c \in \mathfrak{B} \times \mathfrak{A}$, by (i), and $\psi c \in \mathfrak{A} \times \mathfrak{B} = \mathfrak{C}$. This shows that $\psi \restriction \mathfrak{C}$ is a homomorphism from $\mathfrak{C}$ to itself. Of course $\psi^{-1} = \theta \hat{\phi}^{-1} \theta^{-1}$ has the same property. So we have a map $q : \operatorname{Aut} \mathfrak{C} \to \operatorname{Aut} \mathfrak{C}$ given by setting

$$q(\phi) = \theta \hat{\phi} \theta^{-1} \restriction \mathfrak{C}$$

for $\phi \in \operatorname{Aut} \mathfrak{C}$. Evidently q is an automorphism.

(iii) **?** Suppose, if possible, that q were an inner automorphism. Let $\chi \in \operatorname{Aut} \mathfrak{C}$ be such that $q(\phi) = \chi \phi \chi^{-1}$ for every $\phi \in \operatorname{Aut} \mathfrak{C}$. Then

$$\hat{\chi} \hat{\phi} \hat{\chi}^{-1} = \widehat{q(\phi)} = \theta \hat{\phi} \theta^{-1}$$

for every $\phi \in \operatorname{Aut} \mathfrak{C}$. Since $G = \{\hat{\phi} : \phi \in \operatorname{Aut} \mathfrak{C}\}$ is a subgroup of $\operatorname{Aut}(\mathfrak{A} \times \mathfrak{A})$ with many involutions, the 'uniqueness' assertion of 384D tells us that $\hat{\chi} = \theta$. But

$$\theta[\mathfrak{C}] = \mathfrak{B} \times \mathfrak{A} \neq \mathfrak{C} = \chi[\mathfrak{C}] = \hat{\chi}[\mathfrak{C}],$$

so this cannot be. **X**

Thus q is the required outer automorphism of $\operatorname{Aut}\mathfrak{C}$.

Remark Thus the hypothesis 'homogeneous' in 384E, and the hypothesis 'Dedekind complete' in 384J, are necessary.

384Q Example Let μ be Lebesgue measure on $\mathbb{R}$, and $(\mathfrak{A},\bar\mu)$ its measure algebra. Then $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ has an outer automorphism. **P** Set $f(x)=2x$ for $x\in\mathbb{R}$. Then $E\mapsto f^{-1}[E]=\frac{1}{2}E$ is a Boolean automorphism of the domain Σ of μ, and $\mu(\frac{1}{2}E)=\frac{1}{2}\mu E$ for every $E\in\Sigma$ (263A, or otherwise). So we have a corresponding $\theta\in\operatorname{Aut}\mathfrak{A}$ defined by setting $\theta E^\bullet=(\frac{1}{2}E)^\bullet$ for every $E\in\Sigma$, and $\bar\mu(\theta a)=\frac{1}{2}\bar\mu a$ for every $a\in\mathfrak{A}$. By 384N, we have an automorphism q of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ defined by setting $q(\phi)=\theta\phi\theta^{-1}$ for every measure-preserving automorphism ϕ. But q is now an outer automorphism of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, because (by 384D) the only possible automorphism of $\mathfrak{A}$ corresponding to q is θ, and θ is not measure-preserving. **Q**

Thus the hypothesis 'totally finite' in 384O cannot be omitted.

384X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is nowhere rigid; (ii) for every $a\in\mathfrak{A}\setminus\{0\}$ and $n\in\mathbb{N}$ there are disjoint non-zero $b_0,\dots,b_n\subseteq a$ such that the principal ideals $\mathfrak{A}_{b_i}$ they generate are all isomorphic; (iii) for every $a\in\mathfrak{A}\setminus\{0\}$ and $n\ge 1$ there is a $\phi\in\operatorname{Aut}\mathfrak{A}$, of order n, supported by a.

(b) Let $\mathfrak{A}$ be an atomless homogeneous Boolean algebra and $\mathfrak{B}$ a nowhere rigid Boolean algebra, and suppose that $\operatorname{Aut}\mathfrak{A}$ is isomorphic to $\operatorname{Aut}\mathfrak{B}$. Show that there is an invariant order-dense subalgebra of $\mathfrak{B}$ which is isomorphic to $\mathfrak{A}$.

(c) Let $\mathfrak{A}$ and $\mathfrak{B}$ be nowhere rigid Boolean algebras. Show that if $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{B}$ are isomorphic, then the Dedekind completions $\widehat{\mathfrak{A}}$ and $\widehat{\mathfrak{B}}$ are isomorphic.

(d) Find two non-isomorphic atomless totally finite measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ such that $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $\operatorname{Aut}_{\bar\nu}\mathfrak{B}$ are isomorphic. (This is easy.)

(e) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras and $\theta:\mathfrak{A}\to\mathfrak{B}$ a Boolean isomorphism. Show that the following are equiveridical: (i) for every $\phi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, $\theta\phi\theta^{-1}\in\operatorname{Aut}_{\bar\nu}\mathfrak{B}$; (ii)($\alpha$) for every infinite cardinal κ there is an $\alpha_\kappa>0$ such that $\bar\nu(\theta a)=\alpha_\kappa\bar\mu a$ whenever $a\in\mathfrak{A}$ and the principal ideal $\mathfrak{A}_a$ is Maharam-type-homogeneous with Maharam type κ; (β) for every $\gamma\in\left]0,\infty\right[$ there is an $\alpha_\gamma>0$ such that $\bar\nu(\theta a)=\alpha_\gamma\bar\mu a$ whenever $a\in\mathfrak{A}$ is an atom of measure γ.

(f) Let $q:\operatorname{Aut}\mathfrak{C}\to\operatorname{Aut}\mathfrak{C}$ be the automorphism of 384Pb. Show that $q(\phi)$ is measure-preserving whenever ϕ is measure-preserving, so that $q\restriction\operatorname{Aut}_{\bar\lambda}\mathfrak{C}$ is an outer automorphism of $\operatorname{Aut}_{\bar\lambda}\mathfrak{C}$.

384Y Further exercises (a) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras such that $\operatorname{Aut}\mathfrak{A}\cong\operatorname{Aut}\mathfrak{B}$. Show that *either* $\mathfrak{A}\cong\mathfrak{B}$ *or* one of $\mathfrak{A}$, $\mathfrak{B}$ has just one atom and the other is atomless.

(b) Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be localizable measure algebras such that $\operatorname{Aut}_{\bar\mu}\mathfrak{A}\cong\operatorname{Aut}_{\bar\nu}\mathfrak{B}$. Show that *either* $(\mathfrak{A},\bar\mu)\cong(\mathfrak{B},\bar\nu)$ *or* there is some $\gamma\in\left]0,\infty\right[$ such that one of $\mathfrak{A}$, $\mathfrak{B}$ has just one atom of measure γ and the other has none *or* there are γ, $\gamma'\in\left]0,\infty\right[$ such that the number of atoms of $\mathfrak{A}$ of measure γ is equal to the number of atoms of $\mathfrak{B}$ of measure γ', but not to the number of atoms of $\mathfrak{A}$ of measure γ'.

(c) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Show that there is an outer automorphism of $\operatorname{Aut}\mathfrak{A}$ iff $\mathfrak{A}$ has exactly six atoms.

(d) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. For each infinite cardinal κ let e_κ be the Maharam-type-κ component of $\mathfrak{A}$ and for each $\gamma\in\left]0,\infty\right[$ let A_γ be the set of atoms of $\mathfrak{A}$ of measure γ. Show that there is an outer automorphism of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ iff

either there is an infinite cardinal κ such that $\bar\mu e_\kappa=\infty$

or there are distinct γ, $\delta\in\left]0,\infty\right[$ such that $\#(A_\gamma)=\#(A_\delta)\ge 2$

or there is a $\gamma\in\left]0,\infty\right[$ such that $\#(A_\gamma)=6$

or there are γ, $\delta\in\left]0,\infty\right[$ such that $\#(A_\gamma)=2<\#(A_\delta)<\omega$.

384 Notes and comments Let me recapitulate the results above. If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras, any isomorphism between $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{B}$ corresponds to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ if *either* $\mathfrak{A}$ and $\mathfrak{B}$ are atomless and homogeneous (384E) *or* they are nowhere rigid and Dedekind complete (384J). If $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are atomless localizable measure algebras, then any automorphism between $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $\operatorname{Aut}_{\bar\nu}\mathfrak{B}$ corresponds to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ (384M) which if $\bar\mu=\bar\nu$ is totally finite will be measure-preserving (384O).

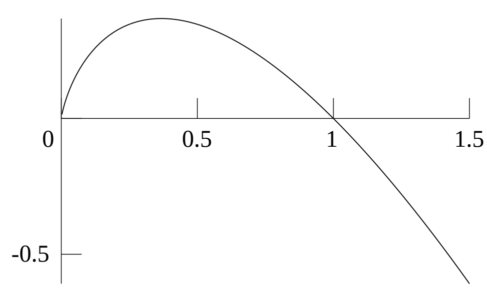
These results may appear a little less surprising if I remark that the elementary Boolean algebras $\mathcal{P}X$ give rise to some of the same phenomena. The automorphism group of $\mathcal{P}X$ can be identified with the group S_X of all permutations of X, and this has no outer automorphisms unless X has just six elements. Some of the ideas of the fundamental theorem 384D can be traced through in the purely atomic case also, though of course there are significant changes to be made, and some serious complications arise, of which the most striking surround the remarkable fact that S_6 *does* have an outer automorphism (Burnside 1911, §162; Rotman 84, Theorem 7.8). I have not attempted to incorporate these into the main results. For localizable measure algebras, where the only rigid parts are atoms, the complications are superable, and I think I have listed them all (384Ya-384Yd).

385 Entropy

Perhaps the most glaring problem associated with the theory of measure-preserving homomorphisms and automorphisms is the fact that we have no generally effective method of determining when two homomorphisms are the same, in the sense that two structures $(\mathfrak{A},\bar\mu,\pi)$ and $(\mathfrak{B},\bar\nu,\phi)$ are isomorphic, where $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are measure algebras and $\pi:\mathfrak{A}\to\mathfrak{A}$, $\phi:\mathfrak{B}\to\mathfrak{B}$ are Boolean homomorphisms. Of course the first part of the problem is to decide whether $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are isomorphic; but this is solved (at least for localizable algebras) by Maharam's theorem (see 332J). The difficulty lies in the homomorphisms. Even when we know that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are both isomorphic to the Lebesgue measure algebra, the extraordinary variety of constructions of homomorphisms – corresponding in part to the variety of measure spaces with such measure algebras, each with its own natural inverse-measure-preserving functions – means that the question of which are isomorphic to each other is continually being raised. In this section I give the most elementary ideas associated with the concept of 'entropy', up to the Kolmogorov-Sinaǐ theorem (385P). This is an invariant which can be attached to any measure-preserving homomorphism on a probability algebra, and therefore provides a useful method for distinguishing non-isomorphic homomorphisms.

The main work of the section deals with homomorphisms on measure algebras, but as many of the most important ones arise from inverse-measure-preserving functions on measure spaces. I comment on the extra problems arising in the isomorphism problem for such functions (385T-385V). I should remark that some of the lemmas will be repeated in stronger forms in the next section.

385A Notation Throughout this section, I will use the letter q to denote the function from $[0,\infty[$ to $\mathbb{R}$ defined by saying that $q(t)=-t\ln t=t\ln\frac{1}{t}$ if $t>0$, $q(0)=0$.



The function q

We shall need the following straightforward facts concerning q.

(a) q is continuous on $[0,\infty[$ and differentiable on $]0,\infty[$; $q'(t)=-1-\ln t$ and $q''(t)=-\frac{1}{t}$ for $t>0$. Because q'' is negative, q is concave, that is, $-q$ is convex. q has a unique maximum at $(\frac{1}{e},\frac{1}{e})$.

(b) If $s\ge 0$ and $t>0$ then $q'(s+t)\le q'(t)$; consequently

$$q(s+t)=q(s)+\int_0^t q'(s+\tau)d\tau\le q(s)+q(t)$$

for s, $t\ge 0$. It follows that $q(\sum_{i=0}^n s_i)\le\sum_{i=0}^n q(s_i)$ for all $s_0,\dots,s_n\ge 0$ and (because q is continuous) $q(\sum_{i=0}^\infty s_i)\le\sum_{i=0}^\infty q(s_i)$ for every non-negative summable series $\langle s_i\rangle_{i\in\mathbb{N}}$.

(c) If s, $t \ge 0$ then $q(st) = sq(t) + tq(s)$; more generally, if $n \ge 1$ and $s_i \ge 0$ for $i \le n$ then

$$q(\prod_{i=0}^n s_i) = \sum_{j=0}^n q(s_j)\prod_{i\ne j} s_i.$$

(d) The function $t \mapsto q(t) + q(1-t)$ has a unique maximum at $(\frac12, \ln 2)$. ($\frac{d}{dt}(q(t)+q(1-t)) = \ln\frac{1-t}{t}$.) It follows that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|t - \frac12| \le \epsilon$ whenever $q(t) + q(1-t) \ge \ln 2 - \delta$.

(e) If $0 \le t \le \frac12$, then $q(1-t) \le q(t)$. **P** Set $f(t) = q(t) - q(1-t)$. Then

$$f''(t) = -\frac1t + \frac{1}{1-t} = \frac{2t-1}{t(1-t)} \le 0$$

for $0 < t \le \frac12$, while $f(0) = f(\frac12) = 0$, so $f(t) \ge 0$ for $0 \le t \le \frac12$. **Q**

(f)(i) If $\mathfrak A$ is a Dedekind σ-complete Boolean algebra, I will write $\bar q$ for the function from $L^0(\mathfrak A)^+$ to $L^0(\mathfrak A)$ defined from q (364H). Note that because $0 \le q(t) \le 1$ for $t \in [0,1]$, $0 \le \bar q(u) \le \chi 1$ if $0 \le u \le \chi 1$.

(ii) By (b), $\bar q(u+v) \le \bar q(u) + \bar q(v)$ for all u, $v \ge 0$ in $L^0(\mathfrak A)$. (Represent $\mathfrak A$ as the measure algebra of a measure space, so that $\bar q(f^\bullet) = (qf)^\bullet$, as in 364Ib.)

(iii) Similarly, if u, $v \in L^0(\mathfrak A)^+$, then $\bar q(u\times v) = u \times \bar q(v) + v \times \bar q(u)$.

385B Lemma Let $(\mathfrak A, \bar\mu)$ be a probability algebra, $\mathfrak B$ a closed subalgebra of $\mathfrak A$, and $P : L^1(\mathfrak A,\bar\mu) \to L^1(\mathfrak A,\bar\mu)$ the corresponding conditional expectation operator (365R). Then $\int \bar q(u) \le q(\int u)$ and $P(\bar q(u)) \le \bar q(Pu)$ for every $u \in L^\infty(\mathfrak A)^+$.

proof Apply the remarks in 365Rb to $-q$. ($\bar q(u) \in L^\infty \subseteq L^1$ for every $u \in (L^\infty)^+$ because q is bounded on every bounded interval in $[0,\infty[$.)

385C Definition Let $(\mathfrak A, \bar\mu)$ be a probability algebra. If A is a partition of unity in $\mathfrak A$, its **entropy** is $H(A) = \sum_{a\in A} q(\bar\mu a)$, where q is the function defined in 385A.

Remarks (a) In the definition of 'partition of unity' (311Gc) I allowed 0 to belong to the family. In the present context this is a mild irritant, and when convenient I shall remove 0 from the partitions of unity considered here (as in 385F below). But because $q(0) = 0$, it makes no difference; $H(A) = H(A\setminus\{0\})$ whenever A is a partition of unity. So if you wish you can read 'partition of unity' in this section to mean 'partition of unity not containing 0', if you are willing to make an occasional amendment in a formula. In important cases, in fact, A is of the form $\{a_i : i \in I\}$ or $\{a_i : i \in I\}\setminus\{0\}$, where $\langle a_i\rangle_{i\in I}$ is an indexed partition of unity, with $a_i \cap a_j = 0$ for $i \ne j$, but no restriction in the number of i with $a_i = 0$; in this case, we still have $H(A) = \sum_{i\in I} q(\bar\mu a_i)$.

(b) Many authors prefer to use $\log_2$ in place of ln. This makes sense in terms of one of the intuitive approaches to entropy as the 'information' associated with a partition. See Petersen 83, §5.1.

385D Definition Let $(\mathfrak A, \bar\mu)$ be a probability algebra, $\mathfrak B$ a closed subalgebra of $\mathfrak A$ and A a partition of unity in $\mathfrak A$. Let $P : L^1(\mathfrak A,\bar\mu) \to L^1(\mathfrak A,\bar\mu)$ be the conditional expectation operator associated with $\mathfrak B$. Then the **conditional entropy of A on $\mathfrak B$** is

$$H(A|\mathfrak B) = \sum_{a\in A}\int \bar q(P\chi a),$$

where $\bar q$ is defined as in 385Af.

385E Elementary remarks (a) In the formula

$$\sum_{a\in A}\int \bar q(P\chi a),$$

we have $0 \le P(\chi a) \le \chi 1$ for every a, so $\bar q(P\chi a) \ge 0$ and every term in the sum is non-negative; accordingly $H(A|\mathfrak B)$ is well-defined in $[0,\infty]$.

(b) $H(A) = H(A|\{0,1\})$, since if $\mathfrak B = \{0,1\}$ then $P(\chi a) = \bar\mu a\chi 1$, so that $\int \bar q(P\chi a) = q(\bar\mu a)$. If $A \subseteq \mathfrak B$, $H(A|\mathfrak B) = 0$, since $P(\chi a) = \chi a$, $\bar q(P\chi a) = 0$ for every a.

385F Definition If $\mathfrak A$ is a Boolean algebra and A, $B \subseteq \mathfrak A$ are partitions of unity, I write $A \vee B$ for the partition of unity $\{a \cap b : a \in A, b \in B\}\setminus\{0\}$. (See 385Xq.)

385G Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra. Let $A\subseteq\mathfrak{A}$ be a partition of unity.

(a) If B is another partition of unity in $\mathfrak{A}$, then

$$H(A|\mathfrak{B})\le H(A\vee B|\mathfrak{B})\le H(A|\mathfrak{B})+H(B|\mathfrak{B}).$$

(b) If $\mathfrak{B}$ is purely atomic and D is the set of its atoms, then $H(A\vee D)=H(D)+H(A|\mathfrak{B})$.

(c) If $\mathfrak{C}\subseteq\mathfrak{B}$ is a smaller closed subalgebra of $\mathfrak{A}$, then $H(A|\mathfrak{C})\ge H(A|\mathfrak{B})$. In particular, $H(A)\ge H(A|\mathfrak{B})$.

(d) Suppose that $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$ such that $\mathfrak{B}=\overline{\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n}$. If $H(A)<\infty$ then

$$H(A|\mathfrak{B})=\lim_{n\to\infty}H(A|\mathfrak{B}_n).$$

In particular, if $A\subseteq\mathfrak{B}$ then $\lim_{n\to\infty}H(A|\mathfrak{B}_n)=0$.

proof Write P for the conditional expectation operator on $L^1(\mathfrak{A},\bar\mu)$ associated with $\mathfrak{B}$.

(a)(i) If B is infinite, enumerate it as $\langle b_j\rangle_{j\in\mathbb{N}}$; if it is finite, enumerate it as $\langle b_j\rangle_{j\le n}$ and set $b_j=0$ for $j>n$. For any $a\in A$,

$$\chi a=\sum_{j=0}^\infty\chi(a\cap b_j),\quad P(\chi a)=\sum_{j=0}^\infty P\chi(a\cap b_j),$$

$$\begin{aligned}\bar q(P\chi a)&=\lim_{n\to\infty}\bar q(\sum_{j=0}^n P\chi(a\cap b_j))\\&\le\lim_{n\to\infty}\sum_{j=0}^n\bar q(P\chi(a\cap b_j))=\sum_{j=0}^\infty\bar q(P\chi(a\cap b_j))\end{aligned}$$

where all the infinite sums are to be regarded as order*-limits of the corresponding finite sums (see §367), and the middle inequality is a consequence of 385A(f-ii). Accordingly

$$\begin{aligned}H(A\vee B|\mathfrak{B})&=\sum_{a\in A,b\in B,a\cap b\ne0}\int\bar q(P\chi(a\cap b))\\&=\sum_{a\in A}\sum_{j=0}^\infty\int\bar q(P\chi(a\cap b_j))\ge\sum_{a\in A}\int\bar q(P\chi a)=H(A|\mathfrak{B}).\end{aligned}$$

(ii) Suppose for the moment that A and B are both finite. For $a\in\mathfrak{A}$ set $u_a=P(\chi a)$. If a, $b\in\mathfrak{A}$ we have $0\le u_{a\cap b}\le u_b$ in $L^0(\mathfrak{B})$, so we may choose $v_{ab}\in L^0(\mathfrak{B})$ such that $0\le v_{ab}\le\chi 1$ and $u_{a\cap b}=v_{ab}\times u_b$.

For any $b\in B$, $\sum_{a\in A}u_{a\cap b}=u_b$ (because $\sum_{a\in A}\chi(a\cap b)=\chi b$), so $u_b\times\sum_{a\in A}v_{ab}=u_b$. Since $[\![|\bar q(u_b)|>0]\!]\subseteq[\![u_b>0]\!]$, $\bar q(u_b)\times\sum_{a\in A}v_{ab}=\bar q(u_b)$.

For any $a\in A$,

$$\bar q(u_a)=\bar q(\sum_{b\in B}u_{a\cap b})=\bar q(\sum_{b\in B}u_b\times v_{ab})=\bar q(P(\sum_{b\in B}\chi b\times v_{ab}))$$

(because $v_{ab}\in L^0(\mathfrak{B})$ for every b, so $P(\chi b\times v_{ab})=P(\chi b)\times v_{ab}$)

$$\ge P(\bar q(\sum_{b\in B}\chi b\times v_{ab}))$$

(385B)

$$=P(\sum_{b\in B}\chi b\times\bar q(v_{ab}))$$

(because B is disjoint)

$$=\sum_{b\in B}u_b\times\bar q(v_{ab})$$

(because $\bar q(v_{ab})\in L^0(\mathfrak{B})$ for every b).

Putting these together,

$$H(A \vee B|\mathfrak{B}) = \sum_{a\in A,b\in B} \int \bar{q}(u_{a\cap b}) = \sum_{a\in A,b\in B} \int \bar{q}(u_b \times v_{ab})$$

$$= \sum_{a\in A,b\in B} \int u_b \times \bar{q}(v_{ab}) + \sum_{a\in A,b\in B} \int v_{ab} \times \bar{q}(u_b)$$

(385A(f-iii))

$$\leq \sum_{a\in A} \int \bar{q}(u_a) + \sum_{b\in B} \int \bar{q}(u_b) = H(A|\mathfrak{B}) + H(B|\mathfrak{B}).$$

(iii) For general partitions of unity A and B, take any finite set $C \subseteq A \vee B$. Then $C \subseteq \{a \cap b : a \in A_0,\, b \in B_0\}$ where $A_0 \subseteq A$ and $B_0 \subseteq B$ are finite. Set

$$A' = A_0 \cup \{1 \setminus \sup A_0\}, \quad B' = B_0 \cup \{1 \setminus \sup B_0\},$$

so that A' and B' are finite partitions of unity and $C \subseteq A' \vee B'$. Now

$$\sum_{c\in C} \int \bar{q}(P\chi c) \leq \sum_{c\in A'\vee B'} \int \bar{q}(P\chi c) = H(A' \vee B'|\mathfrak{B}) \leq H(A'|\mathfrak{B}) + H(B'|\mathfrak{B})$$

(by (ii))

$$\leq H(A' \vee A|\mathfrak{B}) + H(B' \vee B|\mathfrak{B})$$

(by (i))

$$= H(A|\mathfrak{B}) + H(B|\mathfrak{B}).$$

As C is arbitrary,

$$H(A \vee B|\mathfrak{B}) = \sum_{c\in A\vee B} \int \bar{q}(P\chi c) \leq H(A|\mathfrak{B}) + H(B|\mathfrak{B}).$$

(b) It follows from 385Ab that $\sum_{d\in D} q(\bar{\mu}(a \cap d)) \geq q(\bar{\mu}a)$ for any $a \in A$.

Now, because $\mathfrak{B}$ is purely atomic and D is its set of atoms,

$$P(\chi a) = \sum_{d\in D} \frac{\bar{\mu}(a\cap d)}{\bar{\mu}d}\chi d, \quad \bar{q}(P(\chi a)) = \sum_{d\in D} q(\frac{\bar{\mu}(a\cap d)}{\bar{\mu}d})\chi d$$

for every $a \in A$,

$$H(A|\mathfrak{B}) = \sum_{a\in A,d\in D} q(\frac{\bar{\mu}(a\cap d)}{\bar{\mu}d})\bar{\mu}d.$$

Putting these together,

$$H(A \vee D) = \sum_{a\in A,d\in D} q(\bar{\mu}(a \cap d)) = \sum_{a\in A,d\in D} q(\frac{\bar{\mu}(a\cap d)}{\bar{\mu}d})\bar{\mu}d + \frac{\bar{\mu}(a\cap d)}{\bar{\mu}d}q(\bar{\mu}d)$$

(385Ac)

$$= H(A|\mathfrak{B}) + \sum_{d\in D} q(\bar{\mu}d) = H(A|\mathfrak{B}) + H(D).$$

(c) Write $P_{\mathfrak{C}}$ for the conditional expectation operator corresponding to $\mathfrak{C}$. If $a \in \mathfrak{A}$,

$$\bar{q}(P_{\mathfrak{C}}\chi a) = \bar{q}(P_{\mathfrak{C}}P\chi a) \geq P_{\mathfrak{C}}\bar{q}(P\chi a)$$

by 385B. So

$$H(A|\mathfrak{C}) = \sum_{a\in A} \int \bar{q}(P_{\mathfrak{C}}\chi a) \geq \sum_{a\in A} \int P_{\mathfrak{C}}\bar{q}(P\chi a) = \sum_{a\in A} \int \bar{q}(P\chi a) = H(A|\mathfrak{B}).$$

Taking $\mathfrak{C} = \{0, 1\}$, we get $H(A) \geq H(A|\mathfrak{B})$.

(d) Let P_n be the conditional expectation operator corresponding to $\mathfrak{B}_n$, for each n. Fix $a \in A$. Then $P(\chi a)$ is the order*-limit of $\langle P_n(\chi a)\rangle_{n\in\mathbb{N}}$, by Lévy's martingale theorem (367Jb). Consequently (because q is continuous) $\langle \bar{q}(P_n\chi a)\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\bar{q}(P\chi a)$ for every $a \in A$ (367H). Also, because $0 \le P_n\chi a \le \chi 1$ for every n, $0 \le \bar{q}(P_n\chi a) \le \frac{1}{e}\chi 1$ for every n. By the Dominated Convergence Theorem (367I), $\lim_{n\to\infty}\int \bar{q}(P_n\chi a) = \int \bar{q}(P\chi a)$.

By 385B, we also have

$$0 \le \int \bar{q}(P_n\chi a) \le q(\int P_n(\chi a)) = q(\int \chi a) = q(\bar{\mu}a)$$

for every $a \in A$ and $n \in \mathbb{N}$; since also

$$0 \le \int \bar{q}(P\chi a) = q(\bar{\mu}a),$$

we have $|\int \bar{q}(P_n\chi a) - \int \bar{q}(P\chi a)| \le q(\bar{\mu}a)$ for every $a \in A$, $n \in \mathbb{N}$.

Now we are supposing that $H(A)$ is finite. Given $\epsilon > 0$, we can find a finite set $I \subseteq A$ such that $\sum_{a\in A\setminus I} q(\bar{\mu}a) \le \epsilon$, and an $n_0 \in \mathbb{N}$ such that

$$\sum_{a\in I}|\int \bar{q}(P_n\chi a) - \int \bar{q}(P\chi a)| \le \epsilon$$

for every $n \ge n_0$; in which case

$$\sum_{a\in A\setminus I}|\int \bar{q}(P_n\chi a) - \int \bar{q}(P\chi a)| \le \sum_{a\in A\setminus I} q(\bar{\mu}a) \le \epsilon$$

and $|H(A|\mathfrak{B}_n) - H(A|\mathfrak{B})| \le 2\epsilon$ for every $n \ge n_0$. As ϵ is arbitrary, $H(A|\mathfrak{B}) = \lim_{n\to\infty} H(A|\mathfrak{B}_n)$.

385H Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and A, B two partitions of unity in $\mathfrak{A}$. Then $H(A) \le H(A \vee B) \le H(A) + H(B)$.

proof Take $\mathfrak{B} = \{0, 1\}$ in 385Ga.

385I Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. If $A \subseteq \mathfrak{A}$ is a partition of unity, then $H(\pi[A]) = H(A)$.

proof $\sum_{a\in A} q(\bar{\mu}\pi a) = \sum_{a\in A} q(\bar{\mu}a)$.

385J Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Let A be the set of its atoms. Then the following are equiveridical:

(i) *either* $\mathfrak{A}$ is not purely atomic *or* $\mathfrak{A}$ is purely atomic and $H(A) = \infty$;

(ii) there is a partition of unity $B \subseteq \mathfrak{A}$ such that $H(B) = \infty$;

(iii) for every $\gamma \in \mathbb{R}$ there is a finite partition of unity $C \subseteq \mathfrak{A}$ such that $H(C) \ge \gamma$.

proof (i)⇒(ii) We need examine only the case in which $\mathfrak{A}$ is not purely atomic. Let $a \in \mathfrak{A}$ be a non-zero element such that the principal ideal $\mathfrak{A}_a$ is atomless. By 331C we can choose inductively a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $a_n \subseteq a$ and $\bar{\mu}a_n = 2^{-n-1}\bar{\mu}a$. Now, for each $n \in \mathbb{N}$, choose a disjoint set B_n such that

$$\#(B_n) = 2^{2^n}, \quad b \subseteq a_n \text{ and } \bar{\mu}b = 2^{-2^n}\bar{\mu}a_n \text{ for each } b \in B_n.$$

Set

$$B = \bigcup_{n\in\mathbb{N}} B_n \cup \{1 \setminus a\}.$$

Then B is a partition of unity in $\mathfrak{A}$ and

$$H(B) \ge \sum_{n=0}^{\infty}\sum_{b\in B_n} q(\bar{\mu}B_n) = \sum_{n=0}^{\infty} 2^{2^n} q\Big(\frac{\bar{\mu}a}{2^{n+1+2^n}}\Big)$$
$$= \sum_{n=0}^{\infty}\frac{\bar{\mu}a}{2^{n+1}}\ln\Big(\frac{2^{n+1+2^n}}{\bar{\mu}a}\Big) \ge \sum_{n=0}^{\infty}\frac{\bar{\mu}a}{2^{n+1}}2^n\ln 2 = \infty.$$

(ii)⇒(iii) Enumerate B as $\langle b_i\rangle_{i\in\mathbb{N}}$. For each $n \in \mathbb{N}$, $C_n = \{b_i : i \le n\} \cup \{1 \setminus \sup_{i\le n} b_i\}$ is a finite partition of unity, and

$$\lim_{n\to\infty} H(C_n) \ge \lim_{n\to\infty}\sum_{i=0}^{n} q(\bar{\mu}b_i) = H(B) = \infty.$$

(iii)⇒(i) We need only consider the case in which $\mathfrak{A}$ is purely atomic. In this case, $A \vee C = A$ for every partition of unity $C \subseteq \mathfrak{A}$, so $H(C) \le H(A)$ for every C (385H), and $H(A)$ must be infinite.

385K Definition Let $\mathfrak{A}$ be a Boolean algebra. If $\pi:\mathfrak{A}\to\mathfrak{A}$ is an order-continuous Boolean homomorphism, $A\subseteq\mathfrak{A}$ is a partition of unity and $n\ge 1$, write $D_n(A,\pi)$ for the partition of unity generated by $\{\pi^i a: a\in A,\, 0\le i<n\}$, that is, $\{\inf_{i<n}\pi^i a_i: a_i\in A$ for every $i<n\}\setminus\{0\}$. It will occasionally be convenient to take $D_0(A,\pi)=\{1\}$ (or $\emptyset$ in the trivial case $\mathfrak{A}=\{0\}$). Observe that $D_1(A,\pi)=A\setminus\{0\}$ and

$$D_{n+1}(A,\pi)=D_n(A,\pi)\vee\pi^n[A]=A\vee\pi[D_n(A,\pi)]$$

for every $n\in\mathbb{N}$.

385L Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $A\subseteq\mathfrak{A}$ be a partition of unity. Then $\lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi))=\inf_{n\ge1}\frac{1}{n}H(D_n(A,\pi))$ is defined in $[0,\infty]$.

proof (a) Set $\alpha_0=0$, $\alpha_n=H(D_n(A,\pi))$ for $n\ge1$. Then $\alpha_{m+n}\le\alpha_m+\alpha_n$ for all m, $n\ge0$. **P** If m, $n\ge1$, $D_{m+n}(A,\pi)=D_m(A,\pi)\vee\pi^m[D_n(A,\pi)]$. So 385Ga tells us that

$$H(D_{m+n}(A,\pi))\le H(D_m(A,\pi))+H(\pi^m[D_n(A,\pi)])=H(D_m(A,\pi))+H(D_n(A,\pi))$$

because π is measure-preserving. **Q**

(b) If $\alpha_1=\infty$ then of course $H(D_n(A,\pi))\ge H(A)=\infty$ for every $n\ge1$, by 385H, so $\inf_{n\ge1}\frac{1}{n}H(D_n(A,\pi))$ $=\infty=\lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi))$. Otherwise, $\alpha_n\le n\alpha_1$ is finite for every n. Set $\alpha=\inf_{n\ge1}\frac{1}{n}\alpha_n$. If $\epsilon>0$ there is an $m\ge1$ such that $\frac{1}{m}\alpha_m\le\alpha+\epsilon$. Set $M=\max_{j<m}\alpha_j$. Now, for any $n\ge m$, there are $k\ge1$, $j<m$ such that $n=km+j$, so that

$$\alpha_n\le k\alpha_m+\alpha_j,\quad \frac{1}{n}\alpha_n\le\frac{k}{n}\alpha_m+\frac{M}{n}\le\frac{1}{m}\alpha_m+\frac{M}{n}.$$

Accordingly $\limsup_{n\to\infty}\frac{1}{n}\alpha_n\le\alpha+\epsilon$. As ϵ is arbitrary,

$$\alpha\le\liminf_{n\to\infty}\frac{1}{n}\alpha_n\le\limsup_{n\to\infty}\frac{1}{n}\alpha_n\le\alpha$$

and $\lim_{n\to\infty}\frac{1}{n}\alpha_n=\alpha$ is defined in $[0,\infty]$.

Remark See also 385Yb and 386Lc below.

385M Definition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. For any partition of unity $A\subseteq\mathfrak{A}$, set

$$h(\pi,A)=\inf_{n\ge1}\tfrac{1}{n}H(D_n(A,\pi))=\lim_{n\to\infty}\tfrac{1}{n}H(D_n(A,\pi))$$

(385L). Now the **entropy** of π is

$$h(\pi)=\sup\{h(\pi,A):A\subseteq\mathfrak{A}\text{ is a finite partition of unity}\}.$$

Remarks (a) We always have

$$h(\pi,A)\le H(D_1(A,\pi))=H(A).$$

(b) Observe that if π is the identity automorphism then $D_n(A,\pi)=A\setminus\{0\}$ for every A and n, so that $h(\pi)=0$.

385N Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and A, B two partitions of unity in $\mathfrak{A}$. Let $\pi:\mathfrak{A}\to\mathfrak{A}$ be a measure-preserving Boolean homomorphism. Then $h(\pi,A)\le h(\pi,B)+H(A|\mathfrak{B})$, where $\mathfrak{B}$ is the closed subalgebra of $\mathfrak{A}$ generated by B.

proof We may suppose that $0\notin B$, since removing 0 from B changes neither $D_n(B,\pi)$ nor $\mathfrak{B}$. For each $n\in\mathbb{N}$, set $A_n=\pi^n[A]$ and $B_n=\pi^n[B]$. Let $\mathfrak{B}_n=\pi^n[\mathfrak{B}]$ be the closed subalgebra of $\mathfrak{A}$ generated by B_n, and $\mathfrak{B}_n^*$ the closed subalgebra of $\mathfrak{A}$ generated by $D_n(B,\pi)$. Then $H(A_n|\mathfrak{B}_n)=H(A|\mathfrak{B})$ for each n. **P** The point is that, because $\mathfrak{B}$ is purely atomic and B is its set of atoms,

$$H(A|\mathfrak{B})=\sum_{a\in A,b\in B}q(\frac{\bar\mu(a\cap b)}{\bar\mu b})\bar\mu b$$

as in the proof of 385Gb. Similarly,

$$H(A_n|\mathfrak{B}_n)=\sum_{a\in A,b\in B}q(\frac{\bar\mu(\pi^n a\cap\pi^n b)}{\bar\mu(\pi^n b)})\bar\mu(\pi^n b)=H(A|\mathfrak{B}).\ \mathbf{Q}$$

Accordingly, for any $n \geq 1$,

$$H(D_n(A,\pi)|\mathfrak{B}_n^*) \leq \sum_{i=0}^{n-1} H(A_i|\mathfrak{B}_n^*)$$

(by 385Ga)

$$\leq \sum_{i=0}^{n-1} H(A_i|\mathfrak{B}_i)$$

(by 385Gc)

$$= nH(A|\mathfrak{B}).$$

Now

$$h(\pi, A) = \lim_{n\to\infty} \frac{1}{n} H(D_n(A,\pi)) \leq \limsup_{n\to\infty} \frac{1}{n} H(D_n(A,\pi) \vee D_n(B,\pi))$$

(385Ga)

$$\leq \limsup_{n\to\infty} \frac{1}{n} H(D_n(B,\pi)) + \frac{1}{n} H(D_n(A,\pi)|\mathfrak{B}_n^*)$$

(385Gb)

$$\leq h(\pi, B) + H(A|\mathfrak{B}).$$

385O Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism, and $A \subseteq \mathfrak{A}$ a partition of unity such that $H(A) < \infty$. Then $h(\pi, A) \leq h(\pi)$.

proof If A is finite, this is immediate from the definition of $h(\pi)$; so suppose that A is infinite. Enumerate A as $\langle a_i \rangle_{i\in\mathbb{N}}$. For each $n \in \mathbb{N}$ let $\mathfrak{B}_n$ be the subalgebra of $\mathfrak{A}$ generated by $a_0, \dots, a_n$; set $\mathfrak{B} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n}$. Then $A \subseteq \mathfrak{B}$, so

$$\lim_{n\to\infty} H(A|\mathfrak{B}_n) = H(A\mathfrak{B}) = 0$$

by 385Eb and 385Gd. Accordingly, using 385N,

$$h(\pi, A) \leq h(\pi, B_n) + H(A|\mathfrak{B}_n) \leq h(\pi) + H(A|\mathfrak{B}_n) \to h(\pi)$$

as $n \to \infty$, and $h(\pi, A) \leq h(\pi)$.

385P Theorem (Kolmogorov 58, Sinaǐ 59) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism.

(i) Suppose that $A \subseteq \mathfrak{A}$ is a partition of unity such that $H(A) < \infty$ and the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} \pi^n[A]$ is $\mathfrak{A}$ itself. Then $h(\pi) = h(\pi, A)$.

(ii) Suppose that π is an automorphism, and that $A \subseteq \mathfrak{A}$ is a partition of unity such that $H(A) < \infty$ and the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{Z}} \pi^n[A]$ is $\mathfrak{A}$ itself. Then $h(\pi) = h(\pi, A)$.

proof I take the two arguments together. In both cases, by 385O, we have $h(\pi, A) \leq h(\pi)$, so I have to show that if $B \subseteq \mathfrak{A}$ is any finite partition of unity, then $h(\pi, B) \leq h(\pi, A)$. For (i), let A_n be the partition of unity generated by $\bigcup_{0\leq j<n} \pi^j[A]$; for (ii), let A_n be the partition of unity generated by $\bigcup_{-n\leq j<n} \pi^j[A]$. Then $h(\pi, A_n) = h(\pi, A)$ for every n. **P** In case (i), we have $D_m(A_n, \pi) = D_{m+n}(A, \pi)$ for every m, so that

$$\lim_{m\to\infty} \frac{1}{m} H(D_m(A_n,\pi)) = \lim_{m\to\infty} \frac{1}{m} H(D_{m+n}(A,\pi))$$

$$= \lim_{m\to\infty} \frac{1}{m} H(D_m(A,\pi)).$$

In case (ii), we have $D_m(A_n, \pi) = \pi^{-n}[D_{m+2n}(A, \pi)]$ for every m, so that

$$\lim_{m\to\infty}\frac{1}{m}H(D_m(A_n,\pi)) = \lim_{m\to\infty}\frac{1}{m}H(D_{m+2n}(A,\pi))$$
$$= \lim_{m\to\infty}\frac{1}{m}H(D_m(A,\pi)). \textbf{ Q}$$

Let $\mathfrak{A}_n$ be the purely atomic closed subalgebra of $\mathfrak{A}$ generated by A_n; our hypothesis is that the closed subalgebra generated by $\bigcup_{n\in\mathbb{N}} A_n$ is $\mathfrak{A}$ itself, that is, that $\bigcup_{n\in\mathbb{N}}\mathfrak{A}_n$ is dense. But this means that $\lim_{n\to\infty} H(B|\mathfrak{A}_n) = 0$ (385Gd). Since

$$h(\pi, B) \le h(\pi, A_n) + H(B|\mathfrak{A}_n) = h(\pi, A) + H(B|\mathfrak{A}_n)$$

for every n (385N), we have the result.

385Q Bernoulli shifts Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism.

(a) π is a **one-sided Bernoulli shift** if there is a closed subalgebra $\mathfrak{A}_0$ in $\mathfrak{A}$ such that (i) $\langle \pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ is stochastically independent (that is, $\bar\mu(\inf_{j\le k}\pi^j a_j) = \prod_{j=0}^k \bar\mu a_j$ for all $a_0, \dots , a_k \in \mathfrak{A}_0$; see 325L) (ii) the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}}\pi^k[\mathfrak{A}_0]$ is $\mathfrak{A}$ itself. In this case $\mathfrak{A}_0$ is a **root algebra** for π.

(b) π is a **two-sided Bernoulli shift** if it is an automorphism and there is a closed subalgebra $\mathfrak{A}_0$ in $\mathfrak{A}$ such that (i) $\langle \pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{Z}}$ is independent (ii) the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{Z}}\pi^k[\mathfrak{A}_0]$ is $\mathfrak{A}$ itself. In this case $\mathfrak{A}_0$ is a **root algebra** for π.

It is important to be aware that a Bernoulli shift can have many, and (in the case of a two-sided shift) very different, root algebras; this is the subject of §387 below.

385R Theorem Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Bernoulli shift, either one- or two-sided, with root algebra $\mathfrak{A}_0$.

(i) If $\mathfrak{A}_0$ is purely atomic, then $h(\pi) = H(A)$, where A is the set of atoms of $\mathfrak{A}_0$.

(ii) If $\mathfrak{A}_0$ is not purely atomic, then $h(\pi) = \infty$.

proof (a) The point is that for any partition of unity $C \subseteq \mathfrak{A}_0 \setminus \{0\}$, $h(\pi, C) = H(C)$. **P** For any $n \ge 1$, $D_n(C, \pi)$ is the partition of unity consisting of elements of the form $\inf_{j<n}\pi^j c_j$, where $c_0, \dots , c_{n-1} \in C$. So

$$H(D_n(C,\pi)) = \sum_{c_0,\dots,c_{n-1}\in C} q(\bar\mu(\inf_{j<n}\pi^j c_j)) = \sum_{c_0,\dots,c_{n-1}\in C} q(\prod_{j=0}^{n-1}\bar\mu c_j))$$
$$= \sum_{c_0,\dots,c_{n-1}\in C}\sum_{j=0}^{n-1} q(\bar\mu c_j)\prod_{i\ne j}\bar\mu c_i$$

(385Ac)

$$= \sum_{j=0}^{n-1}\sum_{c\in C} q(\bar\mu c) = nH(C).$$

So

$$h(\pi, C) = \lim\nolimits_{n\to\infty}\frac{1}{n}H(D_n(C,\pi)) = H(C). \textbf{ Q}$$

(b) If $\mathfrak{A}_0$ is purely atomic and $H(A) < \infty$, the result can now be read off from 385P, because the closed subalgebra of $\mathfrak{A}$ generated by A is $\mathfrak{A}_0$ and the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}}\pi^k[A]$ or $\bigcup_{k\in\mathbb{Z}}\pi^k[A]$ is $\mathfrak{A}$; so $h(\pi) = h(\pi, A) = H(A)$.

(c) Otherwise, 385J tells us that there are finite partitions of unity $C \subseteq \mathfrak{A}_0$ such that $H(C)$ is arbitrarily large. Since $h(\pi) \ge h(\pi, C) = H(C)$ for any such C, by (a) and the definition of $h(\pi)$, $h(\pi)$ must be infinite, as claimed.

385S Remarks (a) The standard construction of a Bernoulli shift is from a product space, as follows. If (X,Σ,μ_0) is any probability space, write μ for the product measure on $X^{\mathbb{N}}$; let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ, and $\mathfrak{A}_0\subseteq\mathfrak{A}$ the set of equivalence classes of sets of the form $\{x:x(0)\in E\}$ where $E\in\Sigma$, so that $(\mathfrak{A}_0,\bar\mu\restriction\mathfrak{A}_0)$ can be identified with the measure algebra of μ_0. We have an inverse-measure-preserving function $f:X^{\mathbb{N}}\to X^{\mathbb{N}}$ defined by setting

$$f(x)(n)=x(n+1)\text{ for every }x\in X^{\mathbb{N}},\ n\in\mathbb{N},$$

and f induces, as usual, a measure-preserving homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$. Now π is a one-sided Bernoulli shift with root algebra $\mathfrak{A}_0$. **P** (i) If $a_0,\dots,a_k\in\mathfrak{A}_0$, express each a_j as $\{x:x(0)\in E_j\}^\bullet$, where $E_j\in\Sigma$. Now

$$\pi^j a_j=\{x:(f^j(x))(0)\in E_j\}^\bullet=\{x:x(j)\in E_j\}^\bullet$$

for each j, so

$$\bar\mu(\inf_{j\le k}\pi^j a_j)=\mu(\bigcap_{j\le k}\{x:x(j)\in E_j\})=\prod_{j=0}^k\mu_0E_j=\prod_{j=0}^k\bar\mu a_j.$$

Thus $\langle\pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ is independent. (ii) The closed subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}}\pi^k[\mathfrak{A}_0]$ must contain $\{x:x(k)\in E\}^\bullet$ for every $k\in\mathbb{N}$, $E\in\Sigma$, so must contain $W^\bullet$ for every W in the σ-algebra generated by sets of the form $\{x:x(k)\in E\}$; but every set measured by μ is equivalent to such a set W. So $\mathfrak{A}'=\mathfrak{A}$. **Q**

(b) The same method gives us two-sided Bernoulli shifts. Again let (X,Σ,μ_0) be a probability space, and this time write μ for the product measure on $X^{\mathbb{Z}}$; again let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ, and $\mathfrak{A}_0\subseteq\mathfrak{A}$ the set of equivalence classes of sets of the form $\{x:x(0)\in E\}$ where $E\in\Sigma$, so that $(\mathfrak{A}_0,\bar\mu\restriction\mathfrak{A}_0)$ can once more be identified with the measure algebra of μ_0. This time, we have a measure space automorphism $f:X^{\mathbb{Z}}\to X^{\mathbb{Z}}$ defined by setting

$$f(x)(n)=x(n+1)\text{ for every }x\in X^{\mathbb{Z}},\ n\in\mathbb{Z},$$

and f induces a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$. The arguments used above show that π is a two-sided Bernoulli shift with root algebra $\mathfrak{A}_0$.

It follows that if $(\mathfrak{A},\bar\mu)$ is an atomless homogeneous probability algebra it has a two-sided Bernouilli shift. **P** We can identify $(\mathfrak{A},\bar\mu)$ with the measure algebra of the usual measure on $\{0,1\}^{\kappa\times\mathbb{Z}}\cong(\{0,1\}^\kappa)^{\mathbb{Z}}$, where κ is the Maharam type of $\mathfrak{A}$. **Q**

(c) I remarked above that a Bernoulli shift will normally have many root algebras. But it is important to know that, up to isomorphism, any probability algebra is the root algebra of just one Bernoulli shift of each type.

P(i) Given a probability algebra $(\mathfrak{A}_0,\bar\mu_0)$ then we can identify it with the measure algebra of a probability space (X,Σ,μ_0) (321J), and now the constructions of (a) and (b) provide Bernoulli shifts with root algebras isomorphic to $(\mathfrak{A}_0,\bar\mu_0)$.

(ii) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be probability algebras with one-sided Bernoulli shifts π, ϕ with root algebras $\mathfrak{A}_0$, $\mathfrak{B}_0$, and suppose that $\theta_0:\mathfrak{A}_0\to\mathfrak{B}_0$ is a measure-preserving isomorphism. Then $(\mathfrak{A},\bar\mu)$ can be identified with the probability algebra free product of $\langle\pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ (325L), while $(\mathfrak{B},\bar\nu)$ can be identified with the probability algebra free product of $\langle\pi^k[\mathfrak{B}_0]\rangle_{k\in\mathbb{N}}$. For each $k\in\mathbb{N}$, $\phi^k\theta_0(\pi^k)^{-1}$ is a measure-preserving isomorphism between $\pi^k[\mathfrak{A}_0]$ and $\phi^k[\mathfrak{B}_0]$. Assembling these, we have a measure-preserving isomorphism $\theta:\mathfrak{A}\to\mathfrak{B}$ such that $\theta a=\phi^k\theta_0(\pi^k)^{-1}a$ whenever $k\in\mathbb{N}$ and $a\in\pi^k[\mathfrak{A}_0]$, that is, $\theta\pi^k a=\phi^k\theta_0a$ for every $a\in\mathfrak{A}_0$, $k\in\mathbb{N}$. Of course θ extends θ_0.

If we set

$$\mathfrak{C}=\{a:a\in\mathfrak{A},\ \theta\pi a=\phi\theta a\},$$

then $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$. If $a\in\mathfrak{A}_0$ and $k\in\mathbb{N}$, then

$$\theta\pi(\pi^k a)=\theta\pi^{k+1}a=\phi^{k+1}\theta_0a=\phi(\phi^k\theta_0a)=\phi\theta(\pi^k a),$$

so $\pi^k a\in\mathfrak{C}$. Thus $\phi^k[\mathfrak{A}_0]\subseteq\mathfrak{C}$ for every $k\in\mathbb{N}$, and $\mathfrak{C}=\mathfrak{A}$.

This means that $\theta:\mathfrak{A}\to\mathfrak{B}$ is such that $\phi=\theta\pi\theta^{-1}$; θ is an isomorphism between the structures $(\mathfrak{A},\bar\mu,\pi)$ and $(\mathfrak{B},\bar\nu,\phi)$ extending the isomorphism θ_0 from $\mathfrak{A}_0$ to $\mathfrak{B}_0$.

(iii) Now suppose that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are probability algebras with two-sided Bernoulli shifts π, ϕ with root algebras $\mathfrak{A}_0$, $\mathfrak{B}_0$, and suppose that $\theta_0:\mathfrak{A}_0\to\mathfrak{B}_0$ is a measure-preserving isomorphism. Repeating (ii) word for word, but changing each $\mathbb{N}$ into $\mathbb{Z}$, we find that θ_0 has an extension to a measure-preserving isomorphism $\theta:\mathfrak{A}\to\mathfrak{B}$ such that $\theta\pi=\phi\theta$, so that once more the structures $(\mathfrak{A},\bar\mu,\pi)$ and $(\mathfrak{B},\bar\nu,\phi)$ are isomorphic. **Q**

(d) The classic problem to which the theory of this section was directed was the following: suppose we have two two-sided Bernoulli shifts π and ϕ, one based on a root algebra with two atoms of measure $\frac{1}{2}$ and the other on a root algebra with three atoms of measure $\frac{1}{3}$; are they isomorphic? The Kolmogorov-Sinaĭ theorem tells us that they are not, because $h(\pi) = \ln 2$ and $h(\phi) = \ln 3$ are different. The question of which Bernoulli shifts *are* isomorphic is addressed, and (for countably-generated algebras) solved, in §387 below.

(e) We shall need to know that any Bernoulli shift (either one- or two-sided) is ergodic. In fact, it is mixing. **P** Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Bernoulli shift with root algebra $\mathfrak{A}_0$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}} \pi^k[\mathfrak{A}_0]$ (if π is one-sided) or by $\bigcup_{k\in\mathbb{Z}} \pi^k[\mathfrak{A}_0]$ (if π is two-sided). If b, $c \in \mathfrak{B}$, there is some $n \in \mathbb{N}$ such that both belong to the algebra $\mathfrak{B}_n$ generated by $\bigcup_{j\le n} \pi^j[\mathfrak{A}_0]$ (if π is one-sided) or by $\bigcup_{|j|\le n} \pi^j[\mathfrak{A}_0]$ (if π is two-sided). If now $k > 2n$, $\pi^k b$ belongs to the algebra generated by $\bigcup_{j>n} \pi^j[\mathfrak{A}_0]$. But this is independent of $\mathfrak{B}_n$ (cf. 325Xg, 272K), so

$$\bar{\mu}(c \cap \pi^k b) = \bar{\mu}c \cdot \bar{\mu}(\pi^k b) = \bar{\mu}c \cdot \bar{\mu}b.$$

And this is true for every $k \ge n$. Generally, if b, $c \in \mathfrak{A}$ and $\epsilon > 0$, there are b', $c' \in \mathfrak{B}$ such that $\bar{\mu}(b \triangle b') \le \epsilon$ and $\bar{\mu}(c \triangle c') \le \epsilon$, so that

$$\begin{aligned}\limsup_{k\to\infty} |\bar{\mu}(c \cap \pi^k b) - \bar{\mu}c \cdot \bar{\mu}b| &\le \limsup_{k\to\infty} |\bar{\mu}(c' \cap \pi^k b') - \bar{\mu}c' \cdot \bar{\mu}b'| \\ &\qquad + \bar{\mu}(c \triangle c') + \bar{\mu}(\pi^k b \triangle \pi^k b') + |\bar{\mu}c \cdot \bar{\mu}b - \bar{\mu}c' \cdot \bar{\mu}b'| \\ &\le 0 + \epsilon + \epsilon + |\bar{\mu}c - \bar{\mu}c'| + |\bar{\mu}b - \bar{\mu}b'| \le 4\epsilon.\end{aligned}$$

As ϵ, b and c are arbitrary, π is mixing. By 372Qa, it is ergodic. **Q**

(f) The following elementary remark will be useful. If $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism, and $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is a closed subalgebra such that $\langle \pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ is independent, then $\langle \pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{Z}}$ is independent. **P** If $J \subseteq \mathbb{Z}$ is finite and $\langle a_j\rangle_{j\in J}$ is a family in $\mathfrak{A}_0$, take $n \in \mathbb{N}$ such that $-n \le j$ for every $j \in J$; then

$$\bar{\mu}(\inf_{j\in J} \pi^j a_j) = \bar{\mu}(\inf_{j\in J} \pi^{n+j} a_j) = \prod_{j\in J} \bar{\mu}a_j. \quad \mathbf{Q}$$

(g) It is I hope obvious, but perhaps I should explicitly say: if $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, $\phi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism, and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a (one- or two-sided) Bernouilli shift with a root algebra $\mathfrak{A}_0$, then $\phi\pi\phi^{-1}$ is a Bernouilli shift and $\phi[\mathfrak{A}_0]$ is a root algebra for $\phi\pi\phi^{-1}$.

385T Isomorphic homomorphisms (a) In this section I have spoken of 'isomorphic homomorphisms' without offering a formal definition. I hope that my intention was indeed obvious, and that the next sentence will merely confirm what you have already assumed. If $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ are measure algebras, and $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_2$, $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ are functions, then I say that $(\mathfrak{A}_1, \bar{\mu}_1, \pi_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2, \pi_2)$ are isomorphic if there is a measure-preserving isomorphism $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $\pi_2 = \phi\pi_1\phi^{-1}$. In this context, using Maharam's theorem or otherwise, we can expect to be able to decide whether $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ are or are not isomorphic; and if they are, we have a good hope of being able to describe a measure-preserving isomorphism $\theta : \mathfrak{A}_1 \to \mathfrak{A}_2$. In this case, of course, $(\mathfrak{A}_2, \bar{\mu}_2, \pi_2)$ will be isomorphic to $(\mathfrak{A}_1, \bar{\mu}_1, \pi_2')$ where $\pi_2' = \theta^{-1}\pi_2\theta$. So now we have to decide whether $(\mathfrak{A}_1, \bar{\mu}_1, \pi_1)$ is isomorphic to $(\mathfrak{A}_1, \bar{\mu}_1, \pi_2')$; and when π_1, π_2 are measure-preserving Boolean automorphisms, this is just the question of whether π_1, π_2' are conjugate in the group $\mathrm{Aut}_{\bar{\mu}_1}(\mathfrak{A}_1)$ of measure-preserving automorphisms of $\mathfrak{A}_1$. Thus the isomorphism problem, as stated here, is very close to the classical group-theoretic problem of identifying the conjugacy classes in $\mathrm{Aut}_{\bar{\mu}}(\mathfrak{A})$ for a measure algebra $(\mathfrak{A}, \bar{\mu})$. But we also want to look at measure-preserving homomorphisms which are not automorphisms, so there would be something left even if the conjugacy problem were solved. (In effect, we are studying conjugacy in the semigroup of all measure-preserving Boolean homomorphisms, not just in its group of invertible elements.)

The point of the calculation of the entropy of a homomorphism is that it is an invariant under this kind of isomorphism; so that if π_1, π_2 have different entropies then $(\mathfrak{A}_1, \bar{\mu}_1, \pi_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2, \pi_2)$ cannot be isomorphic. Of course the properties of being 'ergodic' or 'mixing' (see 372O) are also invariant.

(b) All the main work of this section has been done in terms of measure algebras; part of my purpose in this volume has been to insist that this is often the right way to proceed, and to establish a language which makes the arguments smooth and natural. But of course a large proportion of the most important homomorphisms arise in the context of measure spaces, and I take a moment to discuss such applications. Suppose that we have two quadruples $(X_1, \Sigma_1, \mu_1, f_1)$ and $(X_2, \Sigma_2, \mu_2, f_2)$ where, for each i, (X_i, Σ_i, μ_i) is a measure space and $f_i : X_i \to X_i$ is an inverse-measure-preserving function. Then we have associated structures $(\mathfrak{A}_1, \bar{\mu}_1, \pi_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2, \pi_2)$ where $(\mathfrak{A}_i, \bar{\mu}_i)$ is the measure algebra of (X_i,Σ_i,μ_i) and $\pi_i:\mathfrak{A}_i\to\mathfrak{A}_i$ is the measure-preserving homomorphism defined by the usual formula $\pi_iE^\bullet=f_i^{-1}[E]^\bullet$. Now we can call (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) isomorphic if there is a measure space isomorphism $g:X_1\to X_2$ such that $f_2=gf_1g^{-1}$. In this case $(\mathfrak{A}_1,\bar\mu_1,\pi_1)$ and $(\mathfrak{A}_2,\bar\mu_2,\pi_2)$ are isomorphic under the obvious isomorphism $\phi(E^\bullet)=g[E]^\bullet$ for every $E\in\Sigma_1$.

It is not the case that if the $(\mathfrak{A}_i,\bar\mu_i,\pi_i)$ are isomorphic, then the (X_i,Σ_i,μ_i,f_i) are; in fact we do not even need to have an isomorphism of the measure spaces (for instance, one could be Lebesgue measure, and the other the Stone space of the Lebesgue measure algebra). Even when $(\mathfrak{A}_1,\bar\mu_1,\pi_1)$ and $(\mathfrak{A}_2,\bar\mu_2,\pi_2)$ are actually identical, f_1 and f_2 need not be isomorphic. There are two examples in §343 of a probability space (X,Σ,μ) with a measure space automorphism $f:X\to X$ such that $f(x)\ne x$ for every $x\in X$ but the corresponding automorphism on the measure algebra is the identity (343I, 343J); writing ι for the identity map from X to itself, (X,Σ,μ,ι) and (X,Σ,μ,f) are non-isomorphic but give rise to the same $(\mathfrak{A},\bar\mu,\pi)$.

(c) Even with Lebesgue measure, we can have a problem in a formal sense. Take (X,Σ,μ) to be $[0,1]$ with Lebesgue measure, and set $f(0)=1$, $f(1)=0$, $f(x)=x$ for $x\in\,]0,1[$; then f is not isomorphic to the identity function on X, but induces the identity automorphism on the measure algebra. But in this case we can sort things out just by discarding the negligible set $\{0,1\}$, and for Lebesgue measure such a procedure is effective in a wide variety of situations. To formalize it I offer the following definition.

385U Definition Let (X_1,Σ_1,μ_1) and (X_2,Σ_2,μ_2) be measure spaces, and $f_1:X_1\to X_1$, $f_2:X_2\to X_2$ two inverse-measure-preserving functions. I will say that the structures (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) are **almost isomorphic** if there are conegligible sets $X_i'\subseteq X_i$ such that $f_i[X_i']\subseteq X_i'$ for both i and the structures $(X_i',\Sigma_i',\mu_i',f_i')$ are isomorphic in the sense of 385Tb, where Σ_i' is the algebra of relatively measurable subsets of X_i', μ_i' is the subspace measure on X_i' and $f_i'=f_i{\restriction}X_i'$.

385V I leave the elementary properties of this notion to the exercises (385Xn-385Xp), but I spell out the result for which the definition is devised. I phrase it in the language of §§342-343; if the terms are not immediately familiar, start by imagining that both (X_i,Σ_i,μ_i) are measurable subspaces of $\mathbb{R}$ endowed with some Radon measure (342J, 343H), or indeed that both are $[0,1]$ with Lebesgue measure.

Proposition Let (X_1,Σ_1,μ_1) and (X_2,Σ_2,μ_2) be perfect, complete, strictly localizable and countably separated measure spaces, and $(\mathfrak{A}_1,\bar\mu_1)$, $(\mathfrak{A}_2,\bar\mu_2)$ their measure algebras. Suppose that $f_1:X_1\to X_1$, $f_2:X_2\to X_2$ are inverse-measure-preserving functions and that $\pi_1:\mathfrak{A}_1\to\mathfrak{A}_1$, $\pi_2:\mathfrak{A}_2\to\mathfrak{A}_2$ are the measure-preserving Boolean homomorphisms they induce. If $(\mathfrak{A}_1,\bar\mu_1,\pi_1)$ and $(\mathfrak{A}_2,\bar\mu_2,\pi_2)$ are isomorphic, then (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) are almost isomorphic.

proof Because $(\mathfrak{A}_1,\bar\mu_1)$ and $(\mathfrak{A}_2,\bar\mu_2)$ are isomorphic, we surely have $\mu_1X_1=\mu_2X_2$. If both are zero, we can take $X_1'=X_2'=\emptyset$ and stop; so let us suppose that $\mu_1X_1>0$. Let $\phi:\mathfrak{A}_1\to\mathfrak{A}_2$ be a measure-preserving automorphism such that $\pi_2=\phi\pi_1\phi^{-1}$. Because both μ_1 and μ_2 are complete and strictly localizable and compact (343K), there are inverse-measure-preserving functions $g_1:X_1\to X_2$ and $g_2:X_2\to X_1$ representing ϕ^{-1}, ϕ respectively (343B). Now $g_1g_2:X_2\to X_2$, $g_2g_1:X_1\to X_1$, $f_2g_1:X_1\to X_2$ and $g_1f_1:X_1\to X_2$ represent, respectively, the identity automorphism on $\mathfrak{A}_2$, the identity automorphism on $\mathfrak{A}_1$, the homomorphism $\phi^{-1}\pi_2=\pi_1\phi^{-1}:\mathfrak{A}_2\to\mathfrak{A}_1$ and the homomorphism $\pi_1\phi^{-1}$ again. Next, because both μ_1 and μ_2 are countably separated, the sets $E_1=\{x:g_2g_1(x)=x\}$, $H=\{x:f_2g_1(x)=g_1f_1(x)\}$ and $E_2=\{y:g_1g_2(y)=y\}$ are all conegligible (343F). As in part (b) of the proof of 344I, $g_1{\restriction}E_1$ and $g_2{\restriction}E_2$ are the two halves of a bijection, a measure space isomorphism if E_1 and E_2 are given their subspace measures. Set $G_0=E_1\cap H$, and for $n\in\mathbb{N}$ set $G_{n+1}=G_n\cap f_1^{-1}[G_n]$. Then every G_n is conegligible, so $X_1'=\bigcap_{n\in\mathbb{N}}G_n$ is conegligible. Because X_1' is a conegligible subset of E_1, $h=g_1{\restriction}X_1'$ is a measure space isomorphism between X_1' and $X_2'=g_1[X_1']$, which is conegligible in X_2. Because $f_1[G_{n+1}]\subseteq G_n$ for each n, $f_1[X_1']\subseteq X_1'$. Because $X_1'\subseteq H$, $g_1f_1(x)=f_2g_1(x)$ for every $x\in X_1'$. Next, if $y\in X_2'$, $g_2(y)\in X_1'$, so

$$f_2(y)=f_2g_1g_2(y)=g_1f_1g_2(y)\in g_1[f_1[X_1']]\subseteq g_1[X_1']=X_2'.$$

Accordingly we have $f_2'=hf_1'h^{-1}$, where $f_i'=f_i{\restriction}X_i'$ for both i.

Thus h is an isomorphism between (X_1',f_1') and (X_2',f_2'), and (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) are almost isomorphic.

385X Basic exercises (a) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $A\subseteq\mathfrak{A}$ a partition of unity. Show that if $\#(A)=n$ then $H(A)\le\ln n$.

>(b) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and A a partition of unity in $\mathfrak{A}$, enumerated as $\langle a_n\rangle_{n\in\mathbb{N}}$. Set $a_n^*=\sup_{i>n}a_i$, $A_n=\{a_0,\ldots,a_n,a_n^*\}$ for each n. Show that $H(A_n|\mathfrak{B})\le H(A_{n+1}|\mathfrak{B})$ for every n, and that $H(A|\mathfrak{B})=\lim_{n\to\infty}H(A_n|\mathfrak{B})$.

(c) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and A a partition of unity in $\mathfrak{A}$. Show that $H(A|\mathfrak{B})=0$ iff $A\subseteq\mathfrak{B}$.

(d) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and A a partition of unity in $\mathfrak{A}$. Show that $H(A|\mathfrak{B})=H(A)$ iff $\bar{\mu}(a\cap b)=\bar{\mu}a\cdot\bar{\mu}b$ for every $a\in A$, $b\in\mathfrak{B}$. (*Hint*: for 'only if', start with the case $\mathfrak{B}=\{0,b,1\setminus b,1\}$ and use 385Gc.)

(e) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and A, B two partitions of unity in $\mathfrak{A}$. Show that $H(A\vee B)=H(A)+H(B)$ iff $\bar{\mu}(a\cap b)=\bar{\mu}a\cdot\bar{\mu}b$ for all $a\in A$, $b\in B$. Show that $H(A\vee B)=H(A)$ iff every member of A is included in some member of B, that is, iff $A=A\vee B$.

(f) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, with probability algebra free product $(\mathfrak{C},\bar{\lambda})$ (325K). Suppose that $\pi_i:\mathfrak{A}_i\to\mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for each $i\in I$, and that $\pi:\mathfrak{C}\to\mathfrak{C}$ is the measure-preserving Boolean homomorphism they induce. Show that $h(\pi)=\sum_{i\in I}h(\pi_i)$. (*Hint*: use 385Gb and 385Gd to show that $h(\pi)$ is the supremum of $h(\pi,A)$ as A runs over the finite partitions of unity in $\bigotimes_{i\in I}\mathfrak{A}_i$. Use this to reduce to the case $I=\{0,1\}$. Now show that if $A_i\subseteq\mathfrak{A}_i$ is a finite partition of unity for each i, and $A=\{a_0\otimes a_1:a_0\in A_0,\,a_1\in A_1\}$, then $H(A)=H(A_0)+H(A_1)$, so that $h(\pi,A)=h(\pi_0,A_0)+h(\pi_1,A_1)$.)

>(g) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving automorphism. Show that $h(\pi^{-1})=h(\pi)$.

(h) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Show that $h(\pi^k)=kh(\pi)$ for any $k\in\mathbb{N}$. (*Hint*: if $A\subseteq\mathfrak{A}$ is a partition of unity, $h(\pi^k,A)\le h(\pi^k,D_k(A,\pi))=kh(\pi,A)$.)

>(i) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. (i) Suppose there is a partition of unity $A\subseteq\mathfrak{A}$ such that (α) $\bar{\mu}(a\cap\pi b)=\bar{\mu}a\cdot\bar{\mu}b$ for every $a\in A$, $b\in\mathfrak{A}$ (β) $\mathfrak{A}$ is the closed subalgebra of itself generated by $\bigcup_{n\in\mathbb{N}}\pi^n[A]$. Show that π is a one-sided Bernoulli shift, and that $h(\pi)=H(A)$. (ii) Suppose that π is a one-sided Bernoulli shift of finite entropy. Show that there is a partition of unity satisfying (α) and (β).

>(j) Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1[$. Fix an integer $k\ge2$, and define $f:[0,1[\to[0,1[$ by setting $f(x)=<kx>$, the fractional part of kx, for every $x\in[0,1[$; let $\pi:\mathfrak{A}\to\mathfrak{A}$ be the corresponding homomorphism. (Cf. 372Xt.) Show that π is a one-sided Bernoulli shift and that $h(\pi)=\ln k$. (*Hint*: in 385Xi, set $A=\{a_0,\ldots,a_{k-1}\}$ where $a_i=\left[\frac{i}{k},\frac{i+1}{k}\right[^\bullet$ for $i<k$.)

>(k) Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1]$. Set $f(x)=2\min(x,1-x)$ for $x\in[0,1]$ (see 372Xp). Show that the corresponding homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ is a one-sided Bernoulli shift and that $h(\pi)=\ln2$. (*Hint*: in 385Xi, set $A=\{a,1\setminus a\}$ where $a=[0,\frac{1}{2}]^\bullet$.)

(l) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a two-sided Bernoulli shift. Show that π^{-1} is a two-sided Bernoulli shift and π and π^{-1} are conjugate in $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$, so that π is a product of two involutions in $\mathrm{Aut}_{\bar{\mu}}(\mathfrak{A})$.

(m) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, and $(\mathfrak{C},\bar{\lambda})$ their probability algebra free product. Suppose that for each $i\in I$ we have a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_i\to\mathfrak{A}_i$, and that $\pi:\mathfrak{C}\to\mathfrak{C}$ is the measure-preserving homomorphism induced by $\langle\pi_i\rangle_{i\in I}$ (325Xe). (i) Show that if every π_i is a one-sided Bernoulli shift so is π. (ii) Show that if every π_i is a two-sided Bernoulli shift so is π.

(n) Show that the relation 'almost isomorphic to' (385U) is an equivalence relation.

(o) Show that the concept of 'almost isomorphism' described in 385U is not changed if we amend the definition to require that the subspaces X_1', X_2' should be measurable.

(p) Show that if (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) are almost isomorphic quadruples as described in 385U, then $(\mathfrak{A}_1,\bar{\mu}_1,\pi_1)$ and $(\mathfrak{A}_2,\bar{\mu}_2,\pi_2)$ are isomorphic, where for each i $(\mathfrak{A}_i,\bar{\mu}_i)$ is the measure algebra of (X_i,Σ_i,μ_i) and $\pi_i:\mathfrak{A}_i\to\mathfrak{A}_i$ is the measure-preserving Boolean homomorphism derived from $f_i:X_i\to X_i$.

(q) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and write $\mathcal{A}$ for the set of partitions of unity in $\mathfrak{A}$ not containing 0, ordered by saying that $A \leq B$ if every member of B is included in some member of A. (i) Show that $\mathcal{A}$ is a Dedekind complete lattice, and can be identified with the lattice of purely atomic closed subalgebras of $\mathfrak{A}$. Show that for A, $B \in \mathcal{A}$, $A \vee B$, as defined in 385F, is $\sup\{A, B\}$ in $\mathcal{A}$. (ii) Show that $H(A \vee B) + H(A \wedge B) \leq H(A) + H(B)$ for all A, $B \in \mathcal{A}$, where $\vee$, $\wedge$ are the lattice operations on $\mathcal{A}$. (iii) Set $\mathcal{A}_1 = \{A : A \in \mathcal{A}, H(A) < \infty\}$. For A, $B \in \mathcal{A}_1$ set $\rho(A, B) = 2H(A \vee B) - H(A) - H(B)$. Show that ρ is a metric on $\mathcal{A}_1$ (the **entropy metric**). (iv) Show that if $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism, then $|h(\pi, A) - h(\pi, B)| \leq \rho(A, B)$ for all A, $B \in \mathcal{A}_1$. (iv) Show that the lattice operations $\vee$ and $\wedge$ are ρ-continuous on $\mathcal{A}_1$. (v) Show that $H : \mathcal{A}_1 \to [0, \infty[$ is order-continuous. (vi) Show that if $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$, then $A \mapsto H(A|\mathfrak{B})$ is order-continuous on $\mathcal{A}_1$.

(r) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\mathfrak{B}$ a topologically dense subalgebra of $\mathfrak{A}$. (i) Show that if $\langle a_i \rangle_{i \leq n}$ is a partition of unity in $\mathfrak{A}$ and $\epsilon > 0$, there is a partition $\langle b_i \rangle_{i \leq n}$ of unity in $\mathfrak{B}$ such that $\bar{\mu}(a_i \triangle b_i) \leq \epsilon$ for every $i \leq n$. (ii) Show that if A is a finite partition of unity in $\mathfrak{A}$ and $\epsilon > 0$ then there is a finite partition of unity $D \subseteq \mathfrak{B}$ such that $H(A \vee D) \leq H(A) + \epsilon$. (iii) Show that if $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism, then $h(\pi) = \sup\{h(\pi, D) : D \subseteq \mathfrak{B}$ is a finite partition of unity$\}$. (*Hint*: 385N, 385Gb.)

(s) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an ergodic measure-preserving Boolean homomorphism. Show that if $h(\pi) > 0$ then $\mathfrak{A}$ is atomless.

385Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and write $\mathfrak{P}$ for the lattice of closed subalgebras of $\mathfrak{A}$. Show that if A is any partition of unity in $\mathfrak{A}$ of finite entropy, then the order-preserving function $\mathfrak{B} \mapsto -H(A|\mathfrak{B}) : \mathfrak{P} \to]-\infty, 0]$ is order-continuous.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, A a partition of unity in $\mathfrak{A}$ of finite entropy, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Show that $h(\pi, A) = \lim_{n\to\infty} H(A|\mathfrak{B}_n)$, where $\mathfrak{B}_n$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{1 \leq i \leq n} \pi^i[A]$. (*Hint*: use 385Gb to show that $H(A|\mathfrak{B}_n) = H(D_{n+1}(A, \pi)) - H(D_n(A, \pi))$ and observe that $\lim_{n\to\infty} H(A|\mathfrak{B}_n)$ is defined.)

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Suppose that there is a partition of unity A of finite entropy such that the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i \geq 1} \pi^i[A]$ is $\mathfrak{A}$. Show that $h(\pi) = 0$. (*Hint*: use 385Yb and 385Pa.)

(d) Let μ be Lebesgue measure on $[0, 1[$, and take any $\alpha \in]0, 1[$. Let $f : [0, 1[\to [0, 1[$ be the measure space automorphism defined by saying that $f(x)$ is to be one of $x + \alpha$, $x + \alpha - 1$. Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of $([0, 1[, \mu)$ and $\pi : \mathfrak{A} \to \mathfrak{A}$ the measure-preserving automorphism corresponding to f. Show that $h(\pi) = 0$. (*Hint*: if $\alpha \in \mathbb{Q}$, use 385Xh; otherwise use 385Yc with $A = \{a, 1 \setminus a\}$ where $a = \left[0, \frac{1}{2}\right[^{\bullet}$.)

(e) Set $X = [0, 1] \setminus \mathbb{Q}$, let ν be the measure on X defined by setting $\nu E = \frac{1}{\ln 2} \int_E \frac{1}{1+x} dx$ for every Lebesgue measurable set $E \subseteq X$, and for $x \in X$ let $f(x)$ be the fractional part $<\frac{1}{x}>$ of $\frac{1}{x}$. Recall that f is inverse-measure-preserving for ν (see 372M). Let $(\mathfrak{A}, \bar{\nu})$ be the measure algebra of (X, ν) and $\pi : \mathfrak{A} \to \mathfrak{A}$ the homomorphism corresponding to f. Show that $h(\pi) = \pi^2/6 \ln 2$. (*Hint*: use the Kolmogorov-Sinaĭ theorem and 372Yh(v).)

(f) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\phi : \mathfrak{A} \to \mathfrak{A}$ a one-sided Bernoulli shift. Show that there are a probability algebra $(\mathfrak{C}, \bar{\lambda})$, a two-sided Bernouilli shift $\tilde{\phi} : \mathfrak{C} \to \mathfrak{C}$, and a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\tilde{\phi}\pi = \pi\phi$. (*Hint*: 328J.)

(g) Consider the triplets $([0, 1[, \mu_1, f_1)$ and $([0, 1], \mu_2, f_2)$ where μ_1, μ_2 are Lebesgue measure on $[0, 1[$, $[0, 1]$ respectively, $f_1(x) = <2x>$ for each $x \in [0, 1[$, and $f_2(x) = 2\min(x, 1 - x)$ for each $x \in [0, 1]$. Show that these structures are almost isomorphic in the sense of 385U, and give a formula for an almost-isomorphism.

(h) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\mathcal{A}_1$ the set of partitions of unity of finite entropy not containing 0, as in 385Xq. Show that $\mathcal{A}_1$ is complete under the entropy metric. (*Hint*: show that if $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}_1$ and $\sup_{n\in\mathbb{N}} H(A_n) < \infty$, then the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} A_n$ is purely atomic.)

385 Notes and comments In preparing this section I have been heavily influenced by PETERSEN 83. I have taken almost the shortest possible route to Theorem 385P, the original application of the theory, ignoring both the many extensions of these ideas and their intuitive underpinning in the concept of the quantity of 'information' carried by a partition. For both of these I refer you to PETERSEN 83. The techniques described there are I think sufficiently powerful to make possible the calculation of the entropy of any of the measure-preserving homomorphisms which have yet appeared in this treatise.

Of course the idea of entropy of a partition, or of a homomorphism, can be translated into the language of probability spaces and inverse-measure-preserving functions; if (X, Σ, μ) is a probability space, with measure algebra $(\mathfrak{A}, \bar{\mu})$, then partitions of unity in $\mathfrak{A}$ correspond (subject to decisions on how to treat negligible sets) to countable partitions of X into measurable sets, and an inverse-measure-preserving function $f : X \to X$ gives rise to a measure-preserving homomorphism $\pi_f : \mathfrak{A} \to \mathfrak{A}$; so we can define the entropy of f to be $h(\pi_f)$. The whole point of the language I have sought to develop in this volume is that we can do this when and if we choose; in particular, we are not limited to those homomorphisms which are representable by inverse-measure-preserving functions. But of course a large proportion of the most important examples do arise in this way (see 385Xj, 385Xk). The same two examples are instructive from another point of view: the case $k = 2$ of 385Xj is (almost) isomorphic to the tent map of 385Xk. The similarity is obvious, but exhibiting an actual isomorphism is I think another matter (385Yg).

I must say 'almost' isomorphic here because the doubling map on $[0, 1[$ is everywhere two-to-one, while the tent map is not, so they cannot be isomorphic in any exact sense. This is the problem grappled with in 385T-385V. In some moods I would say that a dislike of such contortions is a sign of civilized taste. Certainly it is part of my motivation for working with measure algebras whenever possible. But I have to say also that new ideas in this topic arise more often than not from actual measure spaces, and that it is absolutely necessary to be able to operate in the more concrete context.

386 More about entropy

In preparation for the next two sections, I present a number of basic facts concerning measure-preserving homomorphisms and entropy. Compared with the work to follow, they are mostly fairly elementary, but the Halmos-Rokhlin-Kakutani lemma (386C) and the Shannon-McMillan-Breiman theorem (386E), in their full strengths, go farther than one might expect.

386A I start by returning to the notion of 'recurrence' from 381L-381P, in its original home.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Then π is recurrent on every $a \in \mathfrak{A}$.

proof If $a \in \mathfrak{A}$ is non-zero, then $\sum_{k=0}^{\infty} \bar{\mu}(\pi^k a) = \infty > \mu 1$, so there are $i < j$ such that $0 \neq \pi^i a \cap \pi^j a = \pi^i(a \cap \pi^{j-i} a)$ and $a \cap \pi^{j-i} a \neq 0$. Thus (ii) of 381O is satisfied; by 381O, π is recurrent on every $a \in \mathfrak{A}$.

386B Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $\mathfrak{C}$ be its fixed-point subalgebra $\{c : c \in \mathfrak{A}, \pi c = c\}$. Then

$$\sup_{k \geq n} \pi^k a = \operatorname{upr}(a, \mathfrak{C}) = \inf\{c : a \subseteq c \in \mathfrak{C}\} \in \mathfrak{C}$$

for any $a \in \mathfrak{A}$ and $n \in \mathbb{N}$.

proof By 386A and 381O, $a \subseteq \sup_{k \geq 1} \pi^k a$. Set $a^* = \sup_{k \in \mathbb{N}} \pi^k a$; by 381Kb, $a^* = \sup_{k \geq n} \pi^k a$ for every n; by 381Ka, $a^* \in \mathfrak{C}$. Also, of course, $a^* \subseteq c$ whenever $a \subseteq c \in \mathfrak{C}$, so $a^* = \operatorname{upr}(a, \mathfrak{C})$.

386C The Halmos-Rokhlin-Kakutani lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism, with fixed-point subalgebra $\mathfrak{C}$. Then the following are equiveridical:

(i) π is aperiodic;

(ii) $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$ (definition: 331A);

(iii) whenever $n \geq 1$ and $0 \leq \gamma < \frac{1}{n}$ there is an $a \in \mathfrak{A}$ such that a, πa, $\pi^2 a, \dots, \pi^{n-1} a$ are disjoint and $\bar{\mu}(a \cap c) = \gamma \bar{\mu} c$ for every $c \in \mathfrak{C}$;

(iv) whenever $n \geq 1$, $0 \leq \gamma < \frac{1}{n}$ and $B \subseteq \mathfrak{A}$ is finite, there is an $a \in \mathfrak{A}$ such that a, πa, $\pi^2 a, \dots, \pi^{n-1} a$ are disjoint and $\bar{\mu}(a \cap b) = \gamma \bar{\mu} b$ for every $b \in B$.

proof Note that $\mathfrak{C}$ is (order-)closed because π is (order-)continuous (324Kb).

(i)⇒(ii) Put 386A and 381P together.

(ii)⇒(iii) Set $\delta = \frac{1}{n}(\frac{1}{n} - \gamma) > 0$. By 331B, there is a $d \in \mathfrak{A}$ such that $\bar{\mu}(c \cap d) = \delta\bar{\mu}c$ for every $c \in \mathfrak{C}$. Set $d_k = \pi^k d \setminus \sup_{i<k} \pi^i d$ for $k \in \mathbb{N}$. Note that

$$d_{j+k} = \pi^{j+k} d \setminus \sup_{i<j+k} \pi^i d \subseteq \pi^{j+k} d \setminus \sup_{i<k} \pi^{j+i} d = \pi^j d_k$$

whenever j, $k \in \mathbb{N}$. Next, $\pi^i d_j \cap d_k \subseteq \sup_{m \le i} d_m$ for any i, j, $k \in \mathbb{N}$ such that $i + j \neq k$. **P** (α) If $k \le i$ this is obvious. (β) If $i < k < i + j$ then

$$\pi^i d_j \cap d_k \subseteq \pi^i d_j \cap \pi^i d_{k-i} = \pi^i (d_j \cap d_{k-i}) = 0.$$

(γ) If $i + j < k$, then

$$\pi^i d_j \cap d_k \subseteq \pi^{i+j} d \cap d_k = 0. \text{ **Q**}$$

Setting $c^* = \sup_{i\in\mathbb{N}} d_i = \sup_{i\in\mathbb{N}} \pi^i d$, we have $c^* \in \mathfrak{C}$, by 386B, so that $\bar{\mu}(d \setminus c^*) = \delta\bar{\mu}(1 \setminus c^*)$; but as $d \subseteq c^*$, $c^* = 1$.

Set $a^* = \sup_{m\in\mathbb{N}} d_{mn}$ (the mn here is a product, not a double subscript!), $d^* = \sup_{i<n} d_i = \sup_{i<n} \pi^i d$. Then

$$\bar{\mu}(c \cap d^*) \le \sum_{i=0}^{n-1} \bar{\mu}(c \cap \pi^i d) = \sum_{i=0}^{n-1} \bar{\mu}\pi^i(c \cap d) = n\bar{\mu}(c \cap d) = n\delta\bar{\mu}c$$

for every $c \in \mathfrak{C}$. Next, $\pi^i d_{mn} \supseteq d_{mn+i}$ for all m and i, so

$$\sup_{i<n} \pi^i a^* = \sup_{i\in\mathbb{N}} d_i = 1.$$

Consequently

$$\bar{\mu}c \le \sum_{i=0}^{n-1} \bar{\mu}(c \cap \pi^i a^*) = n\bar{\mu}(c \cap a^*),$$

$$\bar{\mu}(c \cap a^* \setminus d^*) \ge \bar{\mu}(c \cap a^*) - \bar{\mu}(c \cap d^*) \ge (\tfrac{1}{n} - n\delta)\bar{\mu}c = \gamma\bar{\mu}c$$

for every $c \in \mathfrak{C}$.

By 331B again (applied to the principal ideal of $\mathfrak{A}$ generated by $a^* \setminus d^*$) there is an $a \subseteq a^* \setminus d^*$ such that $\bar{\mu}(a \cap c) = \gamma\bar{\mu}c$ for every $c \in \mathfrak{C}$. For $0 < i < n$,

$$\pi^i a^* \cap a^* = \sup_{k,l\in\mathbb{N}} \pi^i d_{kn} \cap d_{ln} \subseteq \sup_{m\le i} d_m \subseteq d^*,$$

so $\pi^i a \cap a = 0$; accordingly a, $\pi a, \dots ,\pi^{n-1} a$ are all disjoint and (iii) is satisfied.

(iii)⇒(iv) Note that $\mathfrak{A}$ is certainly atomless, since for every $k \ge 1$ we can find a $c \in \mathfrak{A}$ such that $c, \pi c, \dots , \pi^{k-1} c$ are disjoint and $\bar{\mu}c = \frac{\bar{\mu}1}{k+1}$, so that we have a partition of unity consisting of sets of measure $\frac{\bar{\mu}1}{k+1}$. Let B' be the set of atoms of the (finite) subalgebra of $\mathfrak{A}$ generated by B, and $m = \#(B')$. Let $\delta > 0$ and r, $k \in \mathbb{N}$ be such that

$$3\delta \le (1 - n\gamma)\bar{\mu}b \text{ for every } b \in B', \quad m(\bar{\mu}1)^2 < r\delta^2, \quad k\delta \ge \bar{\mu}1.$$

By (iii), there is a $c \in \mathfrak{A}$ such that $c, \pi c, \dots , \pi^{nr(k+1)-1} c$ are disjoint and $\bar{\mu}(\sup_{i<nr(k+1)} \pi^i c) = 1 - \delta$. For $j < r$, set $e_j = \sup_{l\le k, i<n} \pi^{n(k+1)j+nl+i} c$, $d_j = \sup_{l<k} \pi^{n(k+1)j+nl} c$. Observe that d_j, $\pi d_j, \dots , \pi^{n-1} d_j$ are disjoint, and that $\pi^i d_j \subseteq e_j$ for $i < 2n$. Set $e = \sup_{j<r} e_j = \sup_{i<nr(k+1)} \pi^i c$, so that $\bar{\mu}e = 1 - \delta$.

Suppose we choose $d \in \mathfrak{A}$ by the following random process. Take $s(0), \dots , s(r-1)$ independently in $\{0, \dots , n-1\}$, so that $\Pr(s(j) = l) = \frac{1}{n}$ for each $l < n$, and set $d = \sup_{j<r} \pi^{s(j)} d_j$. Because we certainly have $\pi^i \pi^{s(j)} d_j \subseteq e_j$ whenever $i < n$, d, $\pi d, \dots , \pi^{n-1} d$ will be disjoint. Now for any $b \in \mathfrak{A}$,

$$\Pr\bigl(\bar{\mu}(d \cap b) \le \frac{1}{n}(\bar{\mu}b - 3\delta)\bigr) < \frac{1}{m}.$$

P We can express the random variable $\bar{\mu}(d \cap b)$ as $X = \sum_{j=0}^{r-1} X_j$, where $X_j = \bar{\mu}(\pi^{s(j)} d_j \cap b)$. Then the X_j are independent random variables. For each j, X_j takes values between 0 and $\bar{\mu}d_j = k\bar{\mu}c \le \dfrac{\bar{\mu}1}{nr}$, and has expectation $\frac{1}{n}\bar{\mu}(e'_j \cap b)$, where

$$e'_j = \sup_{i<n} \pi^i d_j = \sup_{l<k, i<n} \pi^{n(k+1)j+nl+i} c.$$

So X has expectation $\frac{1}{n}\bar{\mu}(e' \cap b)$ where $e' = \sup_{j<r} e'_j$. Now

$$e_j \setminus e'_j = \sup_{i<n} \pi^{n(k+1)j+nk+i} c$$

has measure $n\bar{\mu}c \le \dfrac{n\bar{\mu}1}{nr(k+1)}$ for each j, so $\bar{\mu}(e \setminus e') \le \frac{\bar{\mu}1}{k+1}$ and $\bar{\mu}(1 \setminus e') \le 2\delta$; thus $\mathbb{E}(X) \ge \frac{1}{n}(\bar{\mu}b - 2\delta)$, while

$$\mathrm{Var}(X)=\sum_{j=0}^{r-1}\mathrm{Var}(X_j)\le r\bigl(\frac{\bar\mu 1}{nr}\bigr)^2=\frac{(\bar\mu 1)^2}{n^2r}.$$

But this means that

$$\frac{(\bar\mu 1)^2}{n^2r}\ge\bigl(\frac{\delta}{n}\bigr)^2\Pr\bigl(X\le\frac{1}{n}(\bar\mu b-3\delta)\bigr),$$

and

$$\Pr\bigl(X\le\frac{1}{n}(\bar\mu b-3\delta)\bigr)\le\frac{(\bar\mu 1)^2}{r\delta^2}<\frac{1}{m}$$

by the choice of r. **Q**

This is true for every $b\in B'$, while $\#(B')=m$. There must therefore be some choice of $s(0),\dots,s(r-1)$ such that, taking $d^*=\sup_{j<r}\pi^{s(j)}d_j$,

$$\bar\mu(d^*\cap b)\ge\frac{1}{n}(\bar\mu b-3\delta)\ge\gamma\bar\mu b$$

for every $b\in B'$, while $d^*,\pi d^*,\dots,\pi^{n-1}d^*$ are disjoint. Because $\mathfrak{A}$ is atomless, there is a $d\subseteq d^*$ such that $\bar\mu(d\cap b)=\gamma\bar\mu b$ for every $b\in B'$. Since every member of B is a disjoint union of members of B', $\bar\mu(d\cap b)=\gamma\bar\mu b$ for every $b\in B$.

(iv)⇒(i) If $a\in\mathfrak{A}\setminus\{0,1\}$ and $n\ge 1$ then (iv) tells us that there is a $b\in\mathfrak{A}$ such that $b,\pi b,\dots,\pi^n b$ are all disjoint and $\bar\mu(1\setminus\sup_{i\le n}\pi^i b)<\bar\mu a$. Now there must be some $i<n$ such that $d=\pi^i b\cap a\ne 0$, in which case

$$d\cap\pi^n d\subseteq\pi^i b\cap\pi^{i+n}b=\pi^i(b\cap\pi^n b)=0,$$

and $\pi^n d\ne d$. As n and a are arbitrary, π is aperiodic.

386D Corollary An ergodic measure-preserving Boolean homomorphism on an atomless totally finite measure algebra is aperiodic.

proof By 372P, this is (ii)⇒(i) of 386C in the case $\mathfrak{C}=\{0,1\}$ (compare 381P).

386E I turn now to a celebrated result which is a kind of strong law of large numbers.

The Shannon-McMillan-Breiman theorem Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism and $A\subseteq\mathfrak{A}$ a partition of unity of finite entropy. For each $n\ge 1$, set

$$w_n=\frac{1}{n}\sum_{d\in D_n(A,\pi)}\ln(\frac{1}{\bar\mu d})\chi d,$$

where $D_n(A,\pi)$ is the partition of unity generated by $\{\pi^i a: a\in A,\ i<n\}$, as in 385K. Then $\langle w_n\rangle_{n\in\mathbb{N}}$ is norm-convergent in $L^1=L^1(\mathfrak{A},\bar\mu)$ to w say; moreover, $\langle w_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to w (definition: 367A). If $T:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ is the Riesz homomorphism defined by π, so that $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$ (364P), then $Tw=w$.

proof (Petersen 83) We may suppose that $0\notin A$.

(a) For each $n\in\mathbb{N}$, let $\mathfrak{B}_n$ be the subalgebra of $\mathfrak{A}$ generated by $\{\pi^i a: a\in A,\ 1\le i\le n\}$, B_n the set of its atoms, and P_n the corresponding conditional expectation operator on L^1 (365R). Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$, and P the corresponding conditional expectation operator. Observe that $B_n=\pi[D_n(A,\pi)]$ and that, in the language of 385F, $D_{n+1}(A,\pi)=A\vee B_n$. Let $\mathfrak{C}$ be the fixed-point subalgebra of π and Q the associated conditional expectation. Set $L^0=L^0(\mathfrak{A})$, and let $\bar{\ln}$ be the function from $\{v:[\![v>0]\!]=1\}$ to L^0 corresponding to $\ln:]0,\infty[\to\mathbb{R}$ (364H).

(b) It will save a moment later if I note a simple fact here: if $v\in L^1$, then $\langle\frac{1}{n}T^n v\rangle_{n\ge 1}$ is order*-convergent and $\|\ \|_1$-convergent to 0. **P** We know from the ergodic theorem (372G) that $\langle\tilde v_n\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\|\ \|_1$-convergent to Qv, where $\tilde v_n=\frac{1}{n+1}\sum_{i=0}^n T^i v$. Now $\frac{1}{n}T^n v=\frac{n+1}{n}\tilde v_n-\tilde v_{n-1}$ is order*-convergent and $\|\ \|_1$-convergent to $Qv-Qv=0$ (using 367C for 'order*-convergent'). **Q**

(c) Set

$$v_n=\sum_{a\in A}P_n(\chi a)\times\chi a=\sum_{a\in A,b\in B_n}\frac{\bar\mu(a\cap b)}{\bar\mu b}\chi(a\cap b).$$

By Lévy's martingale theorem (275I, 367Jb),

$$\langle v_n\times\chi a\rangle_{n\in\mathbb{N}}=\langle P_n(\chi a)\times\chi a\rangle_{n\in\mathbb{N}}$$

is order*-convergent to $P(\chi a)\times\chi a$ for every $a\in A$; consequently $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $v=\sum_{a\in A}P(\chi a)\times\chi a$. It follows that $\langle\bar{\ln}\,v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{\ln}\,v$. **P** The point is that, for any $a\in A$ and $n\in\mathbb{N}$, $a\subseteq[\![P_n(\chi a)>0]\!]$, so that $[\![v_n>0]\!]=1$ for every n, and $\bar{\ln}\,v_n$ is defined. Similarly, $\bar{\ln}\,v$ is defined, and $\langle\bar{\ln}\,v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{\ln}\,v$ by 367H. **Q** As $0\le v_n\le\chi 1$ for every n, $\langle v_n\rangle_{n\in\mathbb{N}}\to v$ for $\| \|_1$, by the Dominated Convergence Theorem (367I).

Next, $\langle\bar{\ln}\,v_n\rangle_{n\in\mathbb{N}}$ is order-bounded in L^1. **P** Of course $\bar{\ln}\,v_n\le 0$ for every n, because $P_n(\chi a)\le P_n(\chi 1)\le\chi 1$ for each a, so $v_n\le\chi 1$. To see that $\{\bar{\ln}\,v_n:n\in\mathbb{N}\}$ is bounded below in L^1, we use an idea from the fundamental martingale inequality 275D. Set $v_*=\inf_{n\in\mathbb{N}}v_n$. For $\alpha>0$, $a\in A$ and $n\in\mathbb{N}$ set

$$b_{an}(\alpha)=[\![P_n(\chi a)<\alpha]\!]\cap\inf_{i<n}[\![P_i(\chi a)\ge\alpha]\!],$$

so that

$$[\![v_*<\alpha]\!]=\sup_{a\in A,n\in\mathbb{N}}a\cap b_{an}(\alpha).$$

Now $b_{an}(\alpha)\in\mathfrak{B}_n$, so

$$\bar\mu(a\cap b_{an}(\alpha))=\int_{b_{an}(\alpha)}\chi a=\int_{b_{an}(\alpha)}P_n(\chi a)\le\alpha\bar\mu(b_{an}(\alpha)),$$

and

$$\begin{aligned}\bar\mu(a\cap[\![v_*<\alpha]\!])&\le\min(\bar\mu a,\sum_{n=0}^{\infty}\bar\mu(a\cap b_{an}(\alpha)))\\&\le\min(\bar\mu a,\alpha\sum_{n=0}^{\infty}\bar\mu b_{an}(\alpha))\le\min(\bar\mu a,\alpha).\end{aligned}$$

Letting $\alpha\downarrow 0$, $\bar\mu(a\cap[\![v_*=0]\!])=0$ for every $a\in A$, so $[\![v_*>0]\!]=1$, and $\bar{\ln}\,v_*$ is defined. Moreover,

$$\bar\mu(a\cap[\![-\bar{\ln}\,v_*>-\ln\alpha]\!])=\bar\mu(a\cap[\![v_*<\alpha]\!])\le\min(\bar\mu a,\alpha)$$

for every $a\in A$, $\alpha>0$; that is,

$$\bar\mu(a\cap[\![-\bar{\ln}\,v_*>\beta]\!])\le\min(\bar\mu a,e^{-\beta})$$

for every $a\in A$ and $\beta\in\mathbb{R}$. Accordingly

$$\begin{aligned}\int(-\bar{\ln}\,v_*)&=\int_0^\infty\bar\mu[\![-\bar{\ln}\,v_*>\beta]\!]d\beta=\sum_{a\in A}\int_0^\infty\bar\mu(a\cap[\![-\bar{\ln}\,v_*>\beta]\!])d\beta\\&\le\sum_{a\in A}\int_0^\infty\min(\bar\mu a,e^{-\beta})d\beta\\&=\sum_{a\in A}\Big(\int_0^{\ln(1/\bar\mu a)}\bar\mu a\,d\beta+\int_{\ln(1/\bar\mu a)}^\infty e^{-\beta}d\beta\Big)\\&=\sum_{a\in A}\Big(\ln(\frac{1}{\bar\mu a})\bar\mu a+e^{\ln\bar\mu a}\Big)\\&=\sum_{a\in A}\ln(\frac{1}{\bar\mu a})\bar\mu a+\sum_{a\in A}\bar\mu a=H(A)+1<\infty\end{aligned}$$

because A has finite entropy. But this means that $\bar{\ln}\,v_*$ belongs to L^1, and of course it is a lower bound for $\{\bar{\ln}\,v_n:n\in\mathbb{N}\}$. **Q**

By 367I again, $\bar{\ln}\,v\in L^1$ and $\langle\bar{\ln}\,v_n\rangle_{n\in\mathbb{N}}\to\bar{\ln}\,v$ for $\| \|_1$.

(d) Fix $n\in\mathbb{N}$ for the moment. For each $d\in D_{n+1}(A,\pi)$ let d' be the unique element of B_n such that $d\subseteq d'$. Then

$$(n+1)w_{n+1} = \sum_{d\in D_{n+1}(A,\pi)} \ln(\frac{1}{\bar\mu d'})\chi d - \sum_{d\in D_{n+1}(A,\pi)} \ln(\frac{\bar\mu d}{\bar\mu d'})\chi d$$

$$= \sum_{b\in B_n} \ln(\frac{1}{\bar\mu b})\chi b - \sum_{\substack{a\in A\\ b\in B_n\\ a\cap b\neq 0}} \ln(\frac{\bar\mu(a\cap b)}{\bar\mu b})\chi(a\cap b)$$

$$= \sum_{d\in D_n(A,\pi)} \ln(\frac{1}{\bar\mu(\pi d)})\chi(\pi d) - \bar{\ln} v_n = T(nw_n) - \bar{\ln} v_n.$$

Inducing on n, starting from

$$w_1 = \sum_{a\in A}\ln(\frac{1}{\bar\mu a})\chi a = -\bar{\ln} v_0,$$

we get

$$nw_n = \sum_{i=0}^{n-1} T^i(-\bar{\ln} v_{n-i-1}), \quad w_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i(-\bar{\ln} v_{n-i-1})$$

for every $n\geq 1$.

(e) Set $w'_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i(-\bar{\ln} v)$ for $n\geq 1$. By the Ergodic Theorem, $\langle w'_n\rangle_{n\geq 1}$ is order*-convergent and $\|\,\|_1$-convergent to $w = Q(-\bar{\ln} v)$, and $Tw = w$. To estimate $w_n - w'_n$, set $u^*_n = \sup_{k\geq n}|\bar{\ln} v_k - \bar{\ln} v|$ for each $n\in\mathbb{N}$. Then $\langle u^*_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence, $u^*_0\in L^1$ (by (c) above), and $\inf_{n\in\mathbb{N}} u^*_n = 0$ because $\langle\bar{\ln} v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{\ln} v$. Now, whenever $n > m\in\mathbb{N}$,

$$|w_n - w'_n| \leq \frac{1}{n}\sum_{i=0}^{n-1} T^i|\bar{\ln} v - \bar{\ln} v_{n-i-1}|$$

$$= \frac{1}{n}\Bigl(\sum_{i=0}^{n-m-1} T^i|\bar{\ln} v - \bar{\ln} v_{n-i-1}| + \sum_{i=n-m}^{n-1} T^i|\bar{\ln} v - \bar{\ln} v_{n-i-1}|\Bigr)$$

$$\leq \frac{1}{n}\Bigl(\sum_{i=0}^{n-m-1} T^i u^*_m + \sum_{j=0}^{m-1} T^{n-1-j}|\bar{\ln} v - \bar{\ln} v_j|\Bigr)$$

$$\leq \frac{1}{n-m}\Bigl(\sum_{i=0}^{n-m-1} T^i u^*_m + \sum_{j=0}^{m-1} T^{n-1-j}u^*_0\Bigr)$$

$$= \frac{1}{n-m}\sum_{i=0}^{n-m-1} T^i u^*_m + \frac{1}{n-m}T^{n-m}\sum_{j=0}^{m-1} T^{m-1-j}u^*_0$$

$$= \frac{1}{n-m}\sum_{i=0}^{n-m-1} T^i u^*_m + \frac{1}{n-m}T^{n-m}\tilde u_m,$$

setting $\tilde u_m = \sum_{j=0}^{m-1} T^{m-1-j}u^*_0$.

Holding m fixed and letting $n\to\infty$, we know that

$$\frac{1}{n-m}\sum_{i=0}^{n-m-1} T^i u^*_m$$

is order*-convergent and $\|\,\|_1$-convergent to Qu^*_m. As for the other term, $\frac{1}{n-m}T^{n-m}\tilde u_m$ is order*-convergent and $\|\,\|_1$-convergent to 0, by (b). What this means is that

$$\limsup_{n\to\infty}|w_n - w'_n| \leq Qu^*_m,$$

$$\limsup_{n\to\infty}\|w_n - w'_n\|_1 \leq \|Qu^*_m\|_1$$

for every $m\in\mathbb{N}$. Since $\langle Qu^*_m\rangle_{m\in\mathbb{N}}$ is surely a non-decreasing sequence with infimum 0,

$$\limsup_{n\to\infty}|w_n - w'_n| = 0, \quad \limsup_{n\to\infty}\|w_n - w'_n\|_1 = 0.$$

Since w'_n is order*-convergent and $\|\,\|_1$-convergent to w, so is w_n.

386F Corollary If, in 386E, π is ergodic, then $\langle w_n\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent to $h(\pi,A)\chi 1$.

proof Because the limit w in 386E has $Tw=w$, it must be of the form $\gamma\chi 1$, because π is ergodic (372Q(a-ii)). Now $\gamma=\int w$ must be

$$\lim_{n\to\infty}\int w_n=\lim_{n\to\infty}\frac{1}{n}\sum_{d\in D_n(A,\pi)}\ln(\frac{1}{\bar\mu d})\bar\mu d=\lim_{n\to\infty}\frac{1}{n}\sum_{d\in D_n(A,\pi)}q(\bar\mu d)$$

(where q is the function of 385A)

$$=\lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi))=h(\pi,A).$$

386G Definition Set $p(t)=t\ln t$ for $t>0$, $p(0)=0$; for any Dedekind σ-complete Boolean algebra $\mathfrak{A}$, let $\bar p:L^0(\mathfrak{A})^+\to L^0(\mathfrak{A})$ be the corresponding function, as in 364H. (Thus $p=-q$ where q is the function of 385A.)

386H Lemma (Csiszár 67, Kullback 67) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and u a member of $L^1(\mathfrak{A},\bar\mu)^+$ such that $\int u=1$. Then

$$(\int|u-\chi 1|)^2\le 2\int\bar p(u).$$

proof Set $a=[\![u<1]\!]$, $\alpha=\bar\mu a$, $\beta=\int_a u$, $b=1\setminus a$. Then $\bar\mu b=1-\alpha$ and $\int_b u=1-\beta$. Surely $\beta\le\alpha<1$. If $\alpha=0$ then $u=\chi 1$ and the result is trivial; so let us suppose that $0<\alpha<1$. Because the function p is convex,

$$\int_a\bar p(u)\ge\bar\mu a\cdot p(\frac{1}{\bar\mu a}\int_a u)=\alpha p(\frac{\beta}{\alpha})=p(\beta)-\beta\ln\alpha,$$

(using 233Ib/365Rb for the inequality), and similarly

$$\int_b\bar p(u)\ge p(1-\beta)-(1-\beta)\ln(1-\alpha).$$

Also

$$\int|u-\chi 1|=\int_a(\chi 1-u)+\int_b(u-\chi 1)=\alpha-\beta+(1-\beta)-(1-\alpha)=2(\alpha-\beta),$$

so

$$\begin{aligned}\int\bar p(u)-\frac{1}{2}(\int|u-\chi 1|)^2&\ge p(\beta)-\beta\ln\alpha+p(1-\beta)-(1-\beta)\ln(1-\alpha)-2(\alpha-\beta)^2\\&=\phi(\beta)\end{aligned}$$

say. Now ϕ is continuous on $[0,1]$ and arbitrarily often differentiable on $]0,1[$,

$$\phi(\alpha)=0,$$

$$\phi'(t)=\ln t-\ln\alpha-\ln(1-t)+\ln(1-\alpha)+4(\alpha-t)\text{ for }t\in\,]0,1[,$$

$$\phi'(\alpha)=0,$$

$$\phi''(t)=\frac{1}{t}+\frac{1}{1-t}-4\ge 0\text{ for }t\in\,]0,1[.$$

So $\phi(t)\ge 0$ for $t\in[0,1]$ and, in particular, $\phi(\beta)\ge 0$; but this means that

$$\int\bar p(u)-\frac{1}{2}(\int|u-\chi 1|)^2\ge 0,$$

that is, $(\int|u-\chi 1|)^2\le 2\int\bar p(u)$, as claimed.

386I Corollary Whenever $(\mathfrak{A},\bar\mu)$ is a probability algebra and A, B are partitions of unity of finite entropy,

$$\sum_{a\in A,b\in B}|\bar\mu(a\cap b)-\bar\mu a\cdot\bar\mu b|\le\sqrt{2(H(A)+H(B)-H(A\vee B))}.$$

proof Replacing A, B by $A\setminus\{0\}$ and $B\setminus\{0\}$ if necessary, we may suppose that neither A nor B contains $\{0\}$. Let $(\mathfrak{C},\bar\lambda)$ be the probability algebra free product of $(\mathfrak{A},\bar\mu)$ with itself (325E, 325K). Set

$$u=\sum_{a\in A,b\in B}\frac{\bar\mu(a\cap b)}{\bar\mu a\cdot\bar\mu b}\chi(a\otimes b)\in L^0(\mathfrak{C});$$

then u is non-negative and integrable and $\int u=\sum_{a\in A,b\in B}\bar\mu(a\cap b)=1$. Now

$$\begin{aligned}\int\bar p(u)&=\sum_{a\in A,b\in B}\bar\mu(a\cap b)\ln\frac{\bar\mu(a\cap b)}{\bar\mu a\cdot\bar\mu b}\\&=-H(A\vee B)-\sum_{a\in A,b\in B}\bar\mu(a\cap b)\ln\bar\mu a-\sum_{a\in A,b\in B}\bar\mu(a\cap b)\ln\bar\mu b\\&=-H(A\vee B)-\sum_{a\in A}\bar\mu a\ln\bar\mu a-\sum_{b\in B}\bar\mu b\ln\bar\mu b\\&=H(A)+H(B)-H(A\vee B).\end{aligned}$$

On the other hand,

$$\int|u-\chi 1|=\sum_{a\in A,b\in B}\bar\mu a\cdot\bar\mu b\Big|\frac{\bar\mu(a\cap b)}{\bar\mu a\cdot\bar\mu b}-1\Big|=\sum_{a\in A,b\in B}|\bar\mu(a\cap b)-\bar\mu a\cdot\bar\mu b|.$$

So what we are seeking to prove is that

$$\int|u-\chi 1|\le\sqrt{2\int\bar p(u)},$$

which is 386H.

386J The next six lemmas are notes on more or less elementary facts which will be used at various points in the next section. The first two are nearly trivial.

Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ two partitions of unity in $\mathfrak{A}$. Then

$$\bar\mu(\sup_{i\in I}a_i\cap b_i)=1-\tfrac{1}{2}\sum_{i\in I}\bar\mu(a_i\triangle b_i).$$

proof

$$\begin{aligned}\bar\mu(\sup_{i\in I}a_i\cap b_i)=\sum_{i\in I}\bar\mu(a_i\cap b_i)&=\sum_{i\in I}\frac{1}{2}(\bar\mu a_i+\bar\mu b_i-\bar\mu(a_i\triangle b_i))\\&=1-\frac{1}{2}\sum_{i\in I}\bar\mu(a_i\triangle b_i).\end{aligned}$$

386K Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, $\langle B_k\rangle_{k\in\mathbb{N}}$ a non-decreasing sequence of subsets of $\mathfrak{A}$ such that $0\in B_0$, and $\langle c_i\rangle_{i\in I}$ a partition of unity in $\mathfrak{A}$ such that $c_i\in\overline{\bigcup_{k\in\mathbb{N}}B_k}$ for every $i\in I$. Then

$$\lim_{k\to\infty}\sup_{i\in I}\rho(c_i,B_k)=0,$$

writing $\rho(c,B)=\inf_{b\in B}\bar\mu(c\triangle b)$ for $c\in\mathfrak{A}$ and non-empty $B\subseteq\mathfrak{A}$, as in 3A4I.

proof Let $\epsilon>0$. Then $J=\{j:j\in I,\ \bar\mu c_j\ge\epsilon\}$ is finite. For each $j\in J$, $\lim_{k\to\infty}\rho(c_i,B_k)=0$, by 3A4I, while

$$\rho(c_i,B_k)\le\bar\mu(c_i\triangle 0)=\bar\mu c_i\le\epsilon$$

for every $i\in I\setminus J$. So

$$\limsup_{k\to\infty}\sup_{i\in I}\rho(c_i,B_k)\le\max(\epsilon,\limsup_{k\to\infty}\sup_{i\in J}\rho(c_i,B_k))=\epsilon.$$

As ϵ is arbitrary, we have the result.

386L Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Let A, B and C be partitions of unity in $\mathfrak{A}$.

(a) $H(A\vee B\vee C)+H(C)\le H(B\vee C)+H(A\vee C)$.

(b) $h(\pi, A) \le h(\pi, A \vee B) \le h(\pi, A) + h(\pi, B) \le h(\pi, A) + H(B)$.

(c) If $H(A) < \infty$,

$$\begin{aligned} h(\pi, A) &= \inf_{n \in \mathbb{N}} H(D_{n+1}(A, \pi)) - H(D_n(A, \pi)) \\ &= \lim_{n \to \infty} H(D_{n+1}(A, \pi)) - H(D_n(A, \pi)). \end{aligned}$$

(d) If $H(A) < \infty$ and $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{B}] \subseteq \mathfrak{B}$, then $h(\pi, A) \le h(\pi \restriction \mathfrak{B}) + H(A|\mathfrak{B})$.

proof (a) Let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by C, so that $\mathfrak{C}$ is purely atomic and C is the set of its atoms. Then

$$\begin{aligned} H(A \vee B \vee C) + H(C) &= H(A \vee B|\mathfrak{C}) + 2H(C) \\ &\le H(A|\mathfrak{C}) + H(B|\mathfrak{C}) + 2H(C) = H(A \vee C) + H(B \vee C) \end{aligned}$$

by 385Gb and 385Ga.

(b) We need only observe that $D_n(A \vee B, \pi) = D_n(A, \pi) \vee D_n(B, \pi)$ for every $n \in \mathbb{N}$, being the partition of unity generated by $\{\pi^i a : i < n,\, a \in A\} \cup \{\pi^i b : i < n,\, b \in B\}$. Consequently

$$\begin{aligned} h(\pi, A) = \lim_{n \to \infty} \frac{1}{n} H(D_n(A, \pi)) &\le \lim_{n \to \infty} \frac{1}{n} H(D_n(A, \pi) \vee D_n(B, \pi)) \\ &= \lim_{n \to \infty} \frac{1}{n} H(D_n(A \vee B, \pi)) = h(\pi, A \vee B) \\ &\le \lim_{n \to \infty} \frac{1}{n} (H(D_n(A, \pi) + H(D_n(B, \pi))) = h(\pi, A) + h(\pi, B) \\ &\le h(\pi, A) + H(B) \end{aligned}$$

as remarked in 385M.

(c) Set $\gamma_n = H(D_{n+1}(A, \pi)) - H(D_n(A, \pi))$ for each $n \in \mathbb{N}$. By 385H, $\gamma_n \ge 0$. From (a) we see that

$$\begin{aligned} \gamma_{n+1} &= H(A \vee \pi[D_{n+1}(A, \pi)]) - H(A \vee \pi[D_n(A, \pi)]) \\ &\le H(\pi[D_{n+1}(A, \pi)]) - H(\pi[D_n(A, \pi]) = \gamma_n \end{aligned}$$

for every $n \in \mathbb{N}$. So $\lim_{n \to \infty} \gamma_n = \inf_{n \in \mathbb{N}} \gamma_n$; write γ for the common value. Now

$$h(\pi, A) = \lim\nolimits_{n \to \infty} \frac{1}{n} H(D_n(A, \pi)) = \lim\nolimits_{n \to \infty} \frac{1}{n} \sum\nolimits_{i=0}^{n-1} \gamma_i = \gamma$$

(273Ca).

(d) Let $P : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}}$ be the conditional expectation operator corresponding to $\mathfrak{B}$. Let $\langle b_k \rangle_{k \in \mathbb{N}}$ be a sequence running over $\{[\![P(\chi a) > q]\!] : a \in A,\, q \in \mathbb{Q}\}$, so that $b_k \in \mathfrak{B}$ for every k, and for each $k \in \mathbb{N}$ let $\mathfrak{B}_k \subseteq \mathfrak{B}$ be the subalgebra generated by $\{b_i : i \le k\}$; let P_k be the conditional expectation operator corresponding to $\mathfrak{B}_k$. Writing $\mathfrak{B}_\infty \subseteq \mathfrak{B}$ for $\overline{\bigcup_{k \in \mathbb{N}} \mathfrak{B}_k}$, and P_∞ for the corresponding conditional expectation operator, then $P(\chi a) \in L^0(\mathfrak{B}_\infty)$, so $P_\infty(\chi a) = P(\chi a)$, for every $a \in A$. So

$$H(A|\mathfrak{B}) = \sum\nolimits_{a \in A} \int \bar{q}(P\chi a) = H(A|\mathfrak{B}_\infty) = \lim\nolimits_{k \to \infty} H(A|\mathfrak{B}_k),$$

by 385Gd.

For each k, let B_k be the set of atoms of $\mathfrak{B}_k$. Then

$$h(\pi, A) \le h(\pi, B_k) + H(A|\mathfrak{B}_k) \le h(\pi \restriction \mathfrak{B}) + H(A|\mathfrak{B}_k)$$

by 385N and the definition of $h(\pi \restriction \mathfrak{B})$. So

$$h(\pi, A) \le h(\pi \restriction \mathfrak{B}) + \lim\nolimits_{k \to \infty} H(A|\mathfrak{B}_k) = h(\pi \restriction \mathfrak{B}) + H(A|\mathfrak{B}).$$

386M Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra.

(a) There is a function $h:\mathfrak{A}\to\mathfrak{B}$ such that $\bar{\mu}(a\triangle h(a))=\rho(a,\mathfrak{B})$ for every $a\in\mathfrak{A}$ and $h(a)\cap h(a')=0$ whenever $a\cap a'=0$.

(b) If A is a partition of unity in $\mathfrak{A}$, then $H(A|\mathfrak{B})\le\sum_{a\in A}q(\rho(a,\mathfrak{B}))$, where q is the function of 385A.

(c) If $\mathfrak{B}$ is atomless and $\langle a_i\rangle_{i\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}$, then there is a partition of unity $\langle b_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{B}$ such that $\bar{\mu}b_i=\bar{\mu}a_i$ and $\bar{\mu}(b_i\triangle a_i)\le 2\rho(a_i,\mathfrak{B})$ for every $i\in\mathbb{N}$.

proof (a) Let $P:L^1_{\bar{\mu}}\to L^1_{\bar{\mu}}$ be the conditional expectation operator associated with $\mathfrak{B}$. For any $b\in\mathfrak{B}$,

$$\int|P(\chi a)-\chi b|=\int_{1\setminus b}P(\chi a)+\bar{\mu}b-\int_b P(\chi a)=\int_{1\setminus b}\chi a+\bar{\mu}b-\int_b\chi a$$
$$=\bar{\mu}(a\setminus b)+\bar{\mu}b-\bar{\mu}(a\cap b)=\bar{\mu}(a\triangle b).$$

If $a\in\mathfrak{A}$ set $h(a)=[\![P(\chi a)>\frac{1}{2}]\!]$. Then $|P(\chi a)-\chi h(a)|\le|P(\chi a)-\chi b|$ for any $b\in\mathfrak{B}$, so

$$\rho(a,\mathfrak{B})=\inf_{b\in\mathfrak{B}}\bar{\mu}(a\triangle b)=\inf_{b\in\mathfrak{B}}\int|P(\chi a)-\chi b|$$
$$=\int|P(\chi a)-\chi h(a)|=\bar{\mu}(a\triangle h(a)).$$

If $a\cap a'=0$, then

$$P(\chi a)+P(\chi a')=P\chi(a\cup a')\le\chi 1,$$

so

$$h(a)\cap h(a')=[\![P(\chi a)>\tfrac{1}{2}]\!]\cap[\![P(\chi a')>\tfrac{1}{2}]\!]\subseteq[\![P(\chi a)+P(\chi a')>1]\!]=0,$$

by 364Ea.

(b) By 385Ae, $q(t)\le q(1-t)$ whenever $\frac{1}{2}\le t\le 1$. Consequently $q(t)\le q(\min(t,1-t))$ for every $t\in[0,1]$, and $\bar{q}(u)\le\bar{q}(u\wedge(\chi 1-u))$ whenever $u\in L^0(\mathfrak{A})$ and $0\le u\le\chi 1$. Fix $a\in A$ for the moment. We have

$$\bar{q}(P(\chi a))\le\bar{q}(P(\chi a)\wedge(\chi 1-P(\chi a))=\bar{q}(|P(\chi a)-\chi h(a)|).$$

Consequently

$$\int\bar{q}(P\chi a)\le\int\bar{q}(|P(\chi a)-\chi h(a)|)\le q\Bigl(\int|P(\chi a)-\chi h(a)|\Bigr)$$

(because q is concave)

$$=q(\rho(a,\mathfrak{B})).$$

Summing over a,

$$H(A|\mathfrak{B})=\textstyle\sum_{a\in A}\int\bar{q}(P\chi a)\le\sum_{a\in A}q(\rho(a,\mathfrak{B})).$$

(c) Set $b'_i=h(a_i)$ for each $i\in\mathbb{N}$. Then $\langle b'_i\rangle_{i\in\mathbb{N}}$ is disjoint. Next, for each $i\in\mathbb{N}$, take $b''_i\in\mathfrak{B}$ such that $b''_i\subseteq b'_i$ and $\bar{\mu}b''_i=\min(\bar{\mu}a_i,\bar{\mu}b'_i)$; then $\langle b''_i\rangle_{i\in\mathbb{N}}$ is disjoint and $\bar{\mu}b''_i\le\bar{\mu}a_i$ for every i. We can therefore find a partition of unity $\langle b_i\rangle_{i\in\mathbb{N}}$ such that $b_i\supseteq b''_i$ and $\bar{\mu}b_i=\bar{\mu}a_i$ for every i. (Use 331C to choose $\langle d_i\rangle_{i\in\mathbb{N}}$ inductively so that $d_i\subseteq 1\setminus(\sup_{j<i}d_j\cup\sup_{j\in\mathbb{N}}b''_j)$ and $\bar{\mu}d_i=\bar{\mu}a_i-\bar{\mu}b''_i$ for each i, and set $b_i=b''_i\cup d_i$.)

Take any $i\in\mathbb{N}$. If $\bar{\mu}b'_i>\bar{\mu}a_i$, then

$$\bar{\mu}(a_i\triangle b_i)=\bar{\mu}(a_i\triangle b''_i)\le\bar{\mu}(a_i\triangle b'_i)+\bar{\mu}(b'_i\triangle b''_i)$$
$$=\bar{\mu}(a_i\triangle b'_i)+\bar{\mu}b'_i-\bar{\mu}a_i\le 2\bar{\mu}(a_i\triangle b'_i)=2\rho(a_i,\mathfrak{B}).$$

If $\bar{\mu}b'_i\le\bar{\mu}a_i$, then

$$\bar{\mu}(a_i\triangle b_i)\le\bar{\mu}(a_i\triangle b'_i)+\bar{\mu}(b'_i\triangle b_i)$$
$$=\bar{\mu}(a_i\triangle b'_i)+\bar{\mu}a_i-\bar{\mu}b'_i\le 2\bar{\mu}(a_i\triangle b'_i)=2\rho(a_i,\mathfrak{B}).$$

386N Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving automorphism. Suppose that $B\subseteq\mathfrak{A}$. For $k\in\mathbb{N}$, let $\mathfrak{B}_k$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b:b\in B,\,|j|\le k\}$, and let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b:b\in B,\,j\in\mathbb{Z}\}$.

(a) $\mathfrak{B}$ is the topological closure $\overline{\bigcup_{k\in\mathbb{N}}\mathfrak{B}_k}$.

(b) $\pi[\mathfrak{B}]=\mathfrak{B}$.

(c) If $\mathfrak{C}$ is any closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{C}]=\mathfrak{C}$, and $a\in\mathfrak{B}_k$, then

$$\rho(a,\mathfrak{C})\le(2k+1)\textstyle\sum_{b\in B}\rho(b,\mathfrak{C}).$$

proof (a) Because $\langle\mathfrak{B}_k\rangle_{k\in\mathbb{N}}$ is non-decreasing, $\bigcup_{k\in\mathbb{N}}\mathfrak{B}_k$ is a subalgebra of $\mathfrak{A}$, so its closure also is (323J), and must be $\mathfrak{B}$.

(b) Of course $\pi^{-1}[\mathfrak{B}_{k+1}]$ is a closed subalgebra of $\mathfrak{A}$ containing $\pi^j b$ whenever $|j|\le k$ and $b\in B$, so includes $\mathfrak{B}_k$; thus $\pi[\mathfrak{B}_k]\subseteq\mathfrak{B}_{k+1}\subseteq\mathfrak{B}$ for every k, and

$$\pi[\mathfrak{B}]=\pi[\overline{\textstyle\bigcup_{k\in\mathbb{N}}\mathfrak{B}_k}]\subseteq\overline{\textstyle\bigcup_{k\in\mathbb{N}}\pi[\mathfrak{B}_k]}\subseteq\overline{\mathfrak{B}}\subseteq\mathfrak{B}$$

because π is continuous (324Kb). Similarly, $\pi^{-1}[\mathfrak{B}]\subseteq\mathfrak{B}$ and $\pi[\mathfrak{B}]=\mathfrak{B}$.

(c) For each $b\in B$, choose $c_b\in\mathfrak{C}$ such that $\bar{\mu}(b\,\triangle\,c_b)=\rho(c_b,\mathfrak{C})$ (386Ma). Set

$$e=\sup_{|j|\le k}\sup_{b\in B}\pi^j(b\,\triangle\,c_b);$$

then

$$\bar{\mu}e\le(2k+1)\textstyle\sum_{b\in B}\bar{\mu}(b\,\triangle\,c_b)=(2k+1)\textstyle\sum_{b\in B}\rho(b,\mathfrak{C}).$$

Now

$$\mathfrak{B}'=\{d:d\in\mathfrak{A},\exists\,c\in\mathfrak{C}\text{ such that }d\setminus e=c\setminus e\}$$

is a subalgebra of $\mathfrak{A}$. By 314F(a-i), applied to the order-continuous homomorphism $c\mapsto c\setminus e:\mathfrak{C}\to\mathfrak{A}_{1\setminus e}$, $\{c\setminus e:c\in\mathfrak{C}\}$ is an order-closed subalgebra of the principal ideal $\mathfrak{A}_{1\setminus e}$; by 313Id, applied to the order-continuous function $d\mapsto d\setminus e:\mathfrak{A}\to\mathfrak{A}_{1\setminus e}$, $\mathfrak{B}'$ is order-closed. If $b\in B$ and $|j|\le k$, then $\pi^j b\,\triangle\,\pi^j c_b\subseteq e$, so $\pi^j b\in\mathfrak{B}'$; accordingly $\mathfrak{B}'\supseteq\mathfrak{B}_k$. Now $a\in\mathfrak{B}_k$, so there is a $c\in\mathfrak{C}$ such that $a\,\triangle\,c\subseteq e$, and

$$\rho(a,\mathfrak{C})\le\bar{\mu}(a\,\triangle\,c)\le\bar{\mu}e\le(2k+1)\textstyle\sum_{b\in B}\rho(b,\mathfrak{C}),$$

as claimed.

386O Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and suppose *either* that $\mathfrak{A}$ is not purely atomic *or* that it is purely atomic and $H(D_0)=\infty$, where D_0 is the set of atoms of $\mathfrak{A}$. Then whenever $A\subseteq\mathfrak{A}$ is a partition of unity and $H(A)\le\gamma\le\infty$, there is a partition of unity B, refining A, such that $H(B)=\gamma$.

proof (a) By 385J, there is a partition of unity D_1 such that $H(D_1)=\infty$. Set $D=D_1\vee A$; then we still have $H(D)=\infty$. Enumerate D as $\langle d_i\rangle_{i\in\mathbb{N}}$. Choose $\langle B_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. $B_0=A$. Given that B_k is a partition of unity, then if $H(B_k\vee\{d_k,1\setminus d_k\})\le\gamma$, set $B_{k+1}=B_k\vee\{d_k,1\setminus d_k\}$; otherwise set $B_{k+1}=B_k$.

Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}}B_k$. Note that, for each $d\in D$,

$$\{c:c\in\mathfrak{A},\,d\subseteq c\text{ or }d\cap c=0\}$$

is a closed subalgebra of $\mathfrak{A}$ including every B_k, so includes $\mathfrak{B}$. If $b\in\mathfrak{B}\setminus\{0\}$, there is surely some $d\in D$ such that $b\cap d\ne0$, so $b\supseteq d$; thus $\mathfrak{B}$ must be purely atomic. Let B be the set of atoms of $\mathfrak{B}$. Because $A=B_0\subseteq\mathfrak{B}$, B refines A.

(b) $H(B)\le\gamma$. **P** For each $k\in\mathbb{N}$, let $\mathfrak{B}_k$ be the closed subalgebra of $\mathfrak{A}$ generated by B_k, so that $\mathfrak{B}=\overline{\bigcup_{k\in\mathbb{N}}\mathfrak{B}_k}$. Suppose that $b_0,\dots,b_n$ are distinct members of B. Then for each $k\in\mathbb{N}$ we can find disjoint $b_{0k},\dots,b_{nk}\in\mathfrak{B}_k$ such that $\bar{\mu}(b_{ik}\,\triangle\,b_i)\le\rho(b_i,\mathfrak{B}_k)$ for every $i\le n$ (386Ma). Accordingly $\bar{\mu}b_i=\lim_{k\to\infty}\bar{\mu}b_{ik}$ for each i, and

$$\textstyle\sum_{i=0}^n q(\bar{\mu}b_i)=\lim_{k\to\infty}\sum_{i=0}^n q(\bar{\mu}b_{ik})\le\sup_{k\in\mathbb{N}}H(B_k)\le\gamma.$$

As $b_0,\dots,b_n$ are arbitrary, $H(B)\le\gamma$. **Q**

(c) $H(B)\ge\gamma$. **P?** Suppose otherwise. We know that

$$\lim_{k\to\infty}H(\{d_k,1\setminus d_k\})=\lim_{k\to\infty}q(\bar{\mu}d_k)+q(1-\bar{\mu}d_k)=0.$$

Let $m\in\mathbb{N}$ be such that $H(B)+H(\{d_k,1\setminus d_k\})\le\gamma$ for every $k\ge m$. Because B refines B_k, we must have

$$H(B_k \vee \{d_k, 1 \setminus d_k\}) \le H(B_k) + H(\{d_k, 1 \setminus d_k\}) \le \gamma,$$

so that $B_{k+1} = B_k \vee \{d_k, 1 \setminus d_k\}$ for every $k \ge m$. But this means that $d_k \in B$ for every $k \ge m$, so that

$$\gamma > H(B) \ge \sum_{k=m}^{\infty} q(\bar{\mu} d_k) = \infty,$$

which is impossible. **XQ**

Thus B has the required properties.

386X Basic exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism with fixed-point subalgebra $\mathfrak{C}$. Take any $a \in \mathfrak{A}$ and set $a_n = \pi^n a \setminus \sup_{1 \le i < n} \pi^i a$ for $n \ge 1$. Show that $\sum_{n=1}^{\infty} n \bar{\mu}(a \cap a_n) = \bar{\mu}(\mathrm{upr}(a, \mathfrak{C}))$. (*Hint*: for $0 \le j < k$ set $a_{jk} = \pi^j (a \cap a_{k-j})$. Show that if $r \in \mathbb{N}$, then $\langle a_{jk} \rangle_{j \le r < k}$ is disjoint.)

(b) Let (X, Σ, μ) be a totally finite measure space and $f : X \to X$ an inverse-measure-preserving function. Take $E \in \Sigma$ and set $F = \{x : \exists n \ge 1,\ f^n(x) \in E\}$. (i) Show that $E \setminus F$ is negligible. (ii) For $x \in E \cap F$ set $k_x = \min\{n : n \ge 1,\ f^n(x) \in E\}$. Show that $\int_E k_x \mu(dx) = \mu F$. (This is a simple form of the **Recurrence Theorem**.)

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $\langle B_k \rangle_{k \in \mathbb{N}}$ a non-decreasing sequence of subsets of $\mathfrak{A}$ such that $0 \in B_0$, and $\langle c_i \rangle_{i \in I}$ a partition of unity in $\mathfrak{A}$. Show that

$$\lim_{k \to \infty} \sum_{i \in I} \rho(c_i, B_k) = \sum_{i \in I} \rho(c_i, B)$$

where $B = \overline{\bigcup_{k \in \mathbb{N}} B_k}$.

>(d) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism and A a partition of unity in $\mathfrak{A}$. Show that $h(\pi, D_n(A, \pi)) = h(\pi, A) = h(\pi, \pi[A])$ for any $n \ge 1$.

(e) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Suppose that $B \subseteq \mathfrak{A}$. For $k \in \mathbb{N}$, let $\mathfrak{B}_k$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b : b \in B,\ j \le k\}$, and let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b : b \in B,\ j \in \mathbb{N}\}$. Show that

$$\mathfrak{B} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{B}_k}, \quad \pi[\mathfrak{B}] \subseteq \mathfrak{B},$$

and that if $\mathfrak{C}$ is any subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{C}] \subseteq \mathfrak{C}$, and $a \in \mathfrak{B}_k$, then $\rho(a, \mathfrak{C}) \le (k+1) \sum_{b \in B} \rho(b, \mathfrak{C})$.

(f) Prove 332L without using Maharam's theorem.

386Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving Boolean homomorphism. Set $\mathfrak{C} = \{c : \pi c = c\}$. Show that whenever $n \ge 1$, $0 \le \gamma < \frac{1}{n}$ and $B \subseteq \mathfrak{A}$ is finite, there is an $a \in \mathfrak{A}$ such that $a, \pi a, \pi^2 a, \dots, \pi^{n-1} a$ are disjoint and $\bar{\mu}(a \cap b \cap c) = \gamma \bar{\mu}(b \cap c)$ for every $b \in B$, $c \in \mathfrak{C}$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $\mathfrak{P}$ be the set of all closed subalgebras of $\mathfrak{A}$ which are invariant under π, ordered by inclusion. Show that $\mathfrak{B} \mapsto h(\pi{\restriction}\mathfrak{B}) : \mathfrak{P} \to [0, \infty]$ is order-preserving and **order-continuous on the left**, in the sense that if $\mathfrak{Q} \subseteq \mathfrak{P}$ is non-empty and upwards-directed then $h(\pi{\restriction}\sup \mathfrak{Q}) = \sup_{\mathfrak{B} \in \mathfrak{Q}} h(\pi{\restriction}\mathfrak{B})$.

386 Notes and comments I have taken the trouble to give sharp forms of the Halmos-Rokhlin-Kakutani lemma (386C) and the Cziszár-Kullback inequality (386H); while it is possible to get through the principal results of the next two sections with rather less, the formulae become better focused if we have the exact expressions available. Of course one can always go farther still (386Ya). Ornstein's theorem in §387 (though not Sinaĭ's, as stated there) can be deduced from the special case of the Shannon-McMillan-Breiman theorem (386E) in which the homomorphism π is a Bernoulli shift.

387 Ornstein's theorem

I come now to the most important of the handful of theorems known which enable us to describe automorphisms of measure algebras up to isomorphism: two two-sided Bernoulli shifts (on algebras of countable Maharam type) of the same entropy are isomorphic (387I, 387K). This is hard work. It requires both delicate ϵ-δ analysis and substantial skill with the manipulation of measure-preserving homomorphisms. The proof is based on two difficult lemmas (387C and 387F), and includes Sinaǐ's theorem (387E, 387L), describing the Bernoulli shifts which arise as factors of a given ergodic automorphism.

387A The following definitions offer a language in which to express the ideas of this section.

Definitions Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism.

(a) A **Bernoulli partition** for π is a partition of unity $\langle a_i \rangle_{i \in I}$ such that

$$\bar{\mu}(\inf_{j \le k} \pi^j a_{i(j)}) = \prod_{j=0}^{k} \bar{\mu} a_{i(j)}$$

whenever $k \in \mathbb{N}$ and $i(0), \dots , i(k) \in I$.

(b) If π is an automorphism, a Bernoulli partition $\langle a_i \rangle_{i \in I}$ for π is **(two-sidedly) generating** if the closed subalgebra generated by $\{\pi^j a_i : i \in I, \, j \in \mathbb{Z}\}$ is $\mathfrak{A}$ itself.

(c) A **factor** of $(\mathfrak{A}, \bar{\mu}, \pi)$ is a triple $(\mathfrak{B}, \bar{\mu} \restriction \mathfrak{B}, \pi \restriction \mathfrak{B})$ where $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{B}] = \mathfrak{B}$.

387B Remarks Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism and $\langle a_i \rangle_{i \in I}$ a Bernoulli partition for π.

(a) $\langle \pi^k[\mathfrak{A}_0] \rangle_{k \in \mathbb{N}}$ is independent, where $\mathfrak{A}_0$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{a_i : i \in I\}$. **P** Suppose that $c_j \in \pi^j[\mathfrak{A}_0]$ for $j \le k$. Then each $\pi^{-j} c_j \in \mathfrak{A}_0$ is expressible as $\sup_{i \in I_j} a_i$ for some $I_j \subseteq I$. Now

$$\begin{aligned}
\bar{\mu}(\inf_{j \le k} c_j) &= \bar{\mu}(\sup_{i_0 \in I_0, \dots , i_k \in I_k} \inf_{j \le k} \pi^j a_{i_j}) \\
&= \sum_{i_0 \in I_0, \dots , i_k \in I_k} \bar{\mu}(\inf_{j \le k} \pi^j a_{i_j}) = \sum_{i_0 \in I_0, \dots , i_k \in I_k} \prod_{j=0}^{k} \bar{\mu} a_{i_j} \\
&= \prod_{j=0}^{k} \sum_{i \in I_j} \bar{\mu} a_i = \prod_{j=0}^{k} \bar{\mu}(\sup_{i \in I_j} a_i) = \prod_{j=0}^{k} \bar{\mu} c_j.
\end{aligned}$$

As $c_0, \dots , c_k$ are arbitrary, $\langle \pi^k[\mathfrak{A}_0] \rangle_{k \in \mathbb{N}}$ is independent. **Q**

(b) If π is an automorphism, then $\langle \pi^k[\mathfrak{A}_0] \rangle_{k \in \mathbb{Z}}$ is independent, by 385Sf.

(c) Setting $A = \{a_i : i \in I\} \setminus \{0\}$, we have $h(\pi, A) = H(A)$, as in part (a) of the proof of 385R, so $h(\pi) \ge H(A)$.

(d) If $H(A) > 0$, then $\mathfrak{A}$ is atomless. **P** As A contains at least two elements of non-zero measure, $\gamma = \max_{a \in A} \bar{\mu} a < 1$. Because $\langle a_i \rangle_{i \in I}$ is a Bernoulli partition, every member of $D_k(A, \pi)$ has measure at most γ^k, for any $k \in \mathbb{N}$. Thus any atom of $\mathfrak{A}$ could have measure at most $\inf_{k \in \mathbb{N}} \gamma^k = 0$. **Q**

(e) If $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{B}] \subseteq \mathfrak{B}$, then $h(\pi \restriction \mathfrak{B}) \le h(\pi)$, just because $h(\pi \restriction \mathfrak{B})$ is calculated from the action of π on a smaller set of partitions. If $\mathfrak{C}^+$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j a_i : i \in I, \, j \in \mathbb{N}\}$, then $\pi[\mathfrak{C}^+] \subseteq \mathfrak{C}^+$ (compare 386Nb), and $\pi \restriction \mathfrak{C}^+$ is a one-sided Bernoulli shift with root algebra $\mathfrak{A}_0$ and entropy $H(A)$, so that

$$H(A) = h(\pi \restriction \mathfrak{C}^+)$$

by the Kolmogorov-Sinaǐ theorem (385P, 385R).

(f) If π is an automorphism, and $\mathfrak{C}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j a_i : i \in I, \, j \in \mathbb{Z}\}$, then $\pi[\mathfrak{C}] = \mathfrak{C}$ (386Nb) and $\pi \restriction \mathfrak{C}$ is a two-sided Bernoulli shift with root algebra $\mathfrak{A}_0$.

(g) Thus every Bernoulli partition for π gives rise to a factor of $(\mathfrak{A},\bar\mu,\pi)$ which is a one-sided Bernoulli shift, and if π is an automorphism we can extend this to the corresponding two-sided Bernoulli shift. If π has a generating Bernoulli partition then it is itself a Bernoulli shift.

(h) Now suppose that $(\mathfrak{B},\bar\nu)$ is another probability algebra, $\phi:\mathfrak{B}\to\mathfrak{B}$ is a measure-preserving Boolean homomorphism, and $\langle b_i\rangle_{i\in I}$ is a Bernoulli partition for ϕ such that $\bar\nu b_i=\bar\mu a_i$ for every i. We have a unique measure-preserving Boolean homomorphism $\theta^+:\mathfrak{C}^+\to\mathfrak{B}$ such that $\theta^+(\pi^j a_i)=\phi^j b_i$ for every $i\in I$, $j\in\mathbb{N}$. (Apply 324P.) Now $\theta^+\pi=\phi\theta^+$. (The set $\{a:\theta^+\pi a=\phi\theta^+ a\}$ is a closed subalgebra of $\mathfrak{C}^+$ containing every $\pi^j a_i$.)

(i) If, in (h) above, π and ϕ are both automorphisms, then the same arguments show that we have a unique measure-preserving Boolean homomorphism $\theta:\mathfrak{C}\to\mathfrak{B}$ such that $\theta a_i=b_i$ for every $i\in I$ and $\theta\pi=\phi\theta$.

387C Lemma Let $(\mathfrak{A},\bar\mu)$ be an atomless probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ an ergodic measure-preserving automorphism. Let $\langle a_i\rangle_{i\in\mathbb{N}}$ be a partition of unity in $\mathfrak{A}$, of finite entropy, and $\langle\gamma_i\rangle_{i\in\mathbb{N}}$ a sequence of non-negative real numbers such that

$$\sum_{i=0}^\infty\gamma_i=1,\quad \sum_{i=0}^\infty q(\gamma_i)\le h(\pi),$$

where q is the function of 385A. Then for any $\epsilon>0$ we can find a partition $\langle a'_i\rangle_{i\in\mathbb{N}}$ of unity in $\mathfrak{A}$ such that

(i) $\{i:a'_i\ne 0\}$ is finite,

(ii) $\sum_{i=0}^\infty|\gamma_i-\bar\mu a'_i|\le\epsilon$,

(iii) $\sum_{i=0}^\infty\bar\mu(a'_i\triangle a_i)\le\epsilon+6\sqrt{\sum_{i=0}^\infty|\bar\mu a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}}$

where $A=\{a_i:i\in\mathbb{N}\}\setminus\{0\}$,

(iv) $H(A')\le h(\pi,A')+\epsilon$

where $A'=\{a'_i:i\in\mathbb{N}\}\setminus\{0\}$.

proof (a) Of course $h(\pi,A)\le H(A)$, as remarked in 385M, so the square root $\sqrt{2(H(A)-h(\pi,A))}$ gives no difficulty. Set $\beta=\sqrt{\sum_{i=0}^\infty|\bar\mu a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}}$, $\delta=\min(\frac14,\frac1{24}\epsilon)$.

There is a sequence $\langle\bar\gamma_i\rangle_{i\in\mathbb{N}}$ of non-negative real numbers such that $\{i:\bar\gamma_i>0\}$ is finite, $\sum_{i=0}^\infty\bar\gamma_i=1$, $\sum_{i=0}^\infty|\bar\gamma_i-\gamma_i|\le 2\delta^2$ and $\sum_{i=0}^\infty q(\bar\gamma_i)\le h(\pi)$. **P** Take $k\in\mathbb{N}$ such that $\sum_{i=k}^\infty\gamma_i\le\delta^2$, and set $\bar\gamma_i=\gamma_i$ for $i<k$, $\bar\gamma_k=\sum_{i=k}^\infty\gamma_i$ and $\bar\gamma_i=0$ for $i>k$; then $q(\bar\gamma_k)\le\sum_{i=k}^\infty q(\gamma_i)$ (385Ab), so

$$\sum_{i=0}^\infty q(\bar\gamma_i)\le\sum_{i=0}^\infty q(\gamma_i)\le h(\pi),$$

while

$$\sum_{i=0}^\infty|\bar\gamma_i-\gamma_i|\le\bar\gamma_k+\sum_{i=k}^\infty\gamma_i\le 2\delta^2.\ \mathbf{Q}$$

Because $\sum_{i=0}^\infty q(\bar\gamma_i)$ is finite, there is a partition of unity C in $\mathfrak{A}$, of finite entropy, such that $\sum_{i=0}^\infty q(\bar\gamma_i)\le h(\pi,C)+3\delta$; replacing C by $C\vee A$ if need be (note that $C\vee A$ still has finite entropy, by 385H), we may suppose that C refines A.

There is a sequence $\langle\gamma'_i\rangle_{i\in\mathbb{N}}$ of non-negative real numbers such that $\sum_{i=0}^\infty\gamma'_i=1$, $\{i:\gamma'_i>0\}$ is finite, $\sum_{i=0}^\infty|\gamma'_i-\gamma_i|\le 4\delta^2$ and

$$\sum_{i=0}^\infty q(\gamma'_i)=h(\pi,C)+3\delta.$$

P Take $k\in\mathbb{N}$ such that $\bar\gamma_i=0$ for $i>k$. Take $r\ge1$ such that $\delta^2\ln(\frac{r}{\delta^2})\ge h(\pi,C)+3\delta$ and set

$$\begin{aligned}\tilde\gamma_i&=(1-\delta^2)\bar\gamma_i\text{ for }i\le k\\&=\frac1r\delta^2\text{ for }k+1\le i\le k+r\\&=0\text{ for }i>k+r.\end{aligned}$$

Then

$$\sum_{i=0}^\infty|\tilde\gamma_i-\bar\gamma_i|=2\delta^2,\quad\sum_{i=0}^\infty|\tilde\gamma_i-\gamma_i|\le4\delta^2,$$

$$\sum_{i=0}^{k+r}q(\bar\gamma_i)\le h(\pi,C)+3\delta\le\delta^2\ln(\tfrac{r}{\delta^2})=rq(\tfrac{\delta^2}{r})\le\sum_{i=0}^{k+r}q(\tilde\gamma_i).$$

Now the function

$$\alpha \mapsto \sum_{i=0}^{k+r} q(\alpha\bar{\gamma}_i + (1-\alpha)\tilde{\gamma}_i) : [0,1] \to \mathbb{R}$$

is continuous, so there is some $\alpha \in [0,1]$ such that

$$\sum_{i=0}^{k+r} q(\alpha\bar{\gamma}_i + (1-\alpha)\tilde{\gamma}_i) = h(\pi, C) + 3\delta,$$

and we can set $\gamma_i' = \alpha\bar{\gamma}_i + (1-\alpha)\tilde{\gamma}_i$ for every i; of course

$$\sum_{i=0}^{\infty} |\gamma_i' - \gamma_i| \le \alpha \sum_{i=0}^{\infty} |\bar{\gamma}_i - \gamma_i| + (1-\alpha)\sum_{i=0}^{\infty} |\tilde{\gamma}_i - \gamma_i| \le 4\delta^2. \ \mathbf{Q}$$

Set $M = \{i : \gamma_i' \ne 0\}$, so that M is finite.

(b) Let $\eta \in]0, \delta]$ be so small that

(i) $|q(s) - q(t)| \le \frac{\delta}{1+\#(M)}$ whenever s, $t \in [0,1]$ and $|s-t| \le 3\eta$,

(ii) $\sum_{c \in C} q(\min(\bar{\mu}c, 2\eta)) \le \delta$,

(iii) $\eta \le \frac{1}{6}$.

(Actually, (iii) is a consequence of (i). For (ii) we must of course rely on the fact that $\sum_{c\in C} q(\bar{\mu}c)$ is finite.)

Let ν be the probability measure on M defined by saying that $\nu\{i\} = \gamma_i'$ for every $i \in M$, and λ the product measure on $M^{\mathbb{N}}$. Define $X_{ij} : M^{\mathbb{N}} \to \{0,1\}$, for $i \in M$ and $j \in \mathbb{N}$, and $Y_j : M^{\mathbb{N}} \to \mathbb{R}$, for $j \in \mathbb{N}$, by setting

$$\begin{aligned} X_{ij}(\omega) &= 1 \text{ if } \omega(j) = i, \\ &= 0 \text{ otherwise,} \\ Y_j(\omega) &= \ln(\gamma'_{\omega(j)}) \text{ for every } \omega \in M^{\mathbb{N}}. \end{aligned}$$

Then, for each $i \in M$, $\langle X_{ij}\rangle_{j\in\mathbb{N}}$ is an independent sequence of random variables, all with expectation γ_i', and $\langle Y_j\rangle_{j\in\mathbb{N}}$ also is an independent sequence of random variables, all with expectation

$$\sum_{i\in M} \gamma_i' \ln \gamma_i' = -\sum_{i=0}^{\infty} q(\gamma_i') = -h(\pi, C) - 3\delta.$$

Let $n \ge 1$ be so large that

(iv) $\bar{\mu}[\![w_n - h(\pi, C)\chi 1 \ge \delta]\!] < \eta$, where

$$w_n = \frac{1}{n}\sum_{d \in D_n(C,\pi)} \ln(\frac{1}{\bar{\mu}d})\chi d;$$

(v)

$$\Pr\bigl(\sum_{i\in M} |\frac{1}{n}\sum_{j=0}^{n-1} X_{ij} - \gamma_i'| \le \eta\bigr) \ge 1 - \delta,$$

$$\Pr\bigl(|\frac{1}{n}\sum_{j=0}^{n-1} Y_j + h(\pi, C) + 3\delta| \le \delta\bigr) \ge 1 - \delta;$$

(vi) $e^{n\delta} \ge 2, \quad \frac{1}{n+1} \le \eta, \quad q(\frac{1}{n+1}) + q(\frac{n}{n+1}) \le \delta;$

these will be true for all sufficiently large n, using the Shannon-McMillan-Breiman theorem (386E) for (iv) and the strong law of large numbers (in any of the forms 273D, 273H or 273I) for (v).

(c) There is a family $\langle b_{ji}\rangle_{j<n, i\in M}$ such that

(α) for each $j < n$, $\langle b_{ji}\rangle_{i\in M}$ is a partition of unity in $\mathfrak{A}$,

(β) $\bar{\mu}(\inf_{j<n} b_{j,i(j)}) = \prod_{j=0}^{n-1} \gamma'_{i(j)}$ for every $i(0), \dots, i(n-1) \in M$,

(γ) $\sum_{i\in M} \bar{\mu}(b_{ji} \cap \pi^j a_i) \ge 1 - \beta^2 - 4\delta^2$ for every $j < n$.

P Construct $\langle b_{ji}\rangle_{i\in M}$ for $j = n-1$, $n-2, \dots, 0$, as follows. Given b_{ji}, for $k < j < n$, such that

$$\bar{\mu}(\inf_{j\le k} \pi^j a_{i(j)} \cap \inf_{k<j<n} b_{j,i(j)}) = \bar{\mu}(\inf_{j\le k} \pi^j a_{i(j)}) \cdot \prod_{j=k+1}^{n-1} \gamma'_{i(j)}$$

for every $i(0), \dots, i(n-1) \in M$ (of course this hypothesis is trivial for $k = n-1$), let B_k be the set of atoms of the (finite) subalgebra of $\mathfrak{A}$ generated by $\{b_{ji} : i \in M, k < j < n\}$. Then $\bar{\mu}(b \cap d) = \bar{\mu}b \cdot \bar{\mu}d$ for every $b \in B_k$, $d \in D_{k+1}(A, \pi)$.

Now

$$\sum_{i=0}^{\infty}\sum_{c\in D_k(A,\pi)}|\bar\mu(\pi^k a_i\cap c)-\gamma'_i\bar\mu c|$$
$$\le\sum_{i=0}^{\infty}\sum_{c\in D_k(A,\pi)}|\bar\mu(\pi^k a_i\cap c)-\bar\mu a_i\cdot\bar\mu c|+\sum_{i=0}^{\infty}|\bar\mu a_i-\gamma'_i|\sum_{c\in D_k(A,\pi)}\bar\mu c$$
$$\le\sum_{i=0}^{\infty}|\gamma_i-\gamma'_i|+\sum_{i=0}^{\infty}|\bar\mu a_i-\gamma_i|+\sum_{i=0}^{\infty}\sum_{c\in D_k(A,\pi)}|\bar\mu(\pi^k a_i\cap c)-\bar\mu a_i\cdot\bar\mu c|$$
$$\le 4\delta^2+\sum_{i=0}^{\infty}|\bar\mu a_i-\gamma_i|+\sqrt{2\bigl(H(\pi^k[A])+H(D_k(A,\pi))-H(D_{k+1}(A,\pi))\bigr)}$$

(by 386I, because $D_{k+1}(A,\pi)=\pi^k[A]\vee D_k(A,\pi)$)

$$\le 4\delta^2+\sum_{i=0}^{\infty}|\bar\mu a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}$$

(because $h(\pi,A)\le H(D_{k+1}(A,\pi))-H(D_k(A,\pi))$, by 386Lc)

$$=\beta^2+4\delta^2.$$

Choose a partition of unity $\langle b_{ki}\rangle_{i\in M}$ such that, for each $c\in D_k(A,\pi)$, $b\in B_k$ and $i\in M$,

$$\bar\mu(b_{ki}\cap b\cap c)=\gamma'_i\bar\mu(b\cap c),$$

$$\text{if }\bar\mu(\pi^k a_i\cap b\cap c)\ge\gamma'_i\bar\mu(b\cap c)\text{ then }b_{ki}\cap b\cap c\subseteq\pi^k a_i,$$

$$\text{if }\bar\mu(\pi^k a_i\cap b\cap c)\le\gamma'_i\bar\mu(b\cap c)\text{ then }\pi^k a_i\cap b\cap c\subseteq b_{ki}.$$

(This is where I use the hypothesis that $\mathfrak{A}$ is atomless.) Note that in these formulae we always have

$$\bar\mu(b\cap c)=\bar\mu b\cdot\bar\mu c,\quad \bar\mu(\pi^k a_i\cap b\cap c)=\bar\mu(\pi^k a_i\cap c)\cdot\bar\mu b.$$

Consequently

$$\sum_{i\in M}\bar\mu(\pi^k a_i\cap b_{ki})=\sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i\in M}\bar\mu(b\cap c\cap(\pi^k a_i\cap b_{ki}))$$
$$=\sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}\min(\bar\mu(b\cap c\cap\pi^k a_i),\gamma'_i\bar\mu(b\cap c))$$
$$\ge\sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}\bar\mu(b\cap c\cap\pi^k a_i)-|\bar\mu(b\cap c\cap\pi^k a_i)-\gamma'_i\bar\mu(b\cap c)|$$
$$=1-\sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}|\bar\mu(b\cap c\cap\pi^k a_i)-\gamma'_i\bar\mu(b\cap c)|$$
$$=1-\sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}\bar\mu b\cdot|\bar\mu(c\cap\pi^k a_i)-\gamma'_i\bar\mu c|$$
$$=1-\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}|\bar\mu(c\cap\pi^k a_i)-\gamma'_i\bar\mu c|\ge 1-\beta^2-4\delta^2.$$

Also we have

$$\bar\mu(b_{ki}\cap b\cap c)=\gamma'_i\bar\mu b\cdot\bar\mu c=\bar\mu(b_{ki}\cap b)\cdot\bar\mu c$$

for every $b\in B_k$, $c\in D_k(A,\pi)$ and $i\in M$, so the (downwards) induction proceeds. **Q**

(d) Let B be the set of atoms of the algebra generated by $\{b_{ji}:j<n,\ i\in M\}$. For $b\in B$ and $d\in D_n(C,\pi)$ set

$$I_{bd} = \{j : j < n,\ \exists\, i \in M,\ b \subseteq b_{ji},\ d \subseteq \pi^j a_i\}.$$

Then, for any $j < n$,

$$\sup\{b \cap d : b \in B,\ d \in D_n(C,\pi),\ j \in I_{bd}\} = \sup_{i \in M} b_{ji} \cap \pi^j a_i,$$

because C refines A, so every $\pi^j a_i$ is a supremum of members of $D_n(C,\pi)$. Accordingly

$$\sum_{b \in B, d \in D_n(C,\pi)} \#(I_{bd})\bar\mu(b \cap d) = \sum_{j=0}^{n-1} \sum_{i \in M} \bar\mu(b_{ji} \cap \pi^j a_i) \ge n(1 - \beta^2 - 4\delta^2).$$

Set

$$e_0 = \sup\{b \cap d : b \in B,\ d \in D_n(C,\pi),\ \#(I_{bd}) \ge n(1 - \beta - 4\delta)\};$$

then $\bar\mu e_0 \ge 1 - \beta - \delta$.

(e) Let $B' \subseteq B$ be the set of those $b \in B$ such that

$$\bar\mu b \le e^{-n(h(\pi,C)+2\delta)}, \quad \textstyle\sum_{i \in M} |\gamma'_i - \frac{1}{n}\#(\{j : j < n,\ b \subseteq b_{ji}\})| \le \eta.$$

Then $\bar\mu(\sup B') \ge 1 - 2\delta$. **P** Set

$$\begin{aligned} B'_1 &= \{b : b \in B,\ \bar\mu b \le e^{-n(h(\pi,C)+2\delta)}\} \\ &= \{b : b \in B,\ h(\pi,C) + 2\delta + \frac{1}{n}\ln(\bar\mu b) \le 0\} \\ &= \{\inf_{j<n} b_{j,i(j)} : i(0),\dots,i(n-1) \in M,\ h(\pi,C) + 2\delta + \frac{1}{n}\sum_{j=0}^{n-1} \ln \gamma'_{i(j)} \le 0\}. \end{aligned}$$

Then

$$\begin{aligned} \bar\mu(\sup B'_1) &= \Pr(h(\pi,C) + 2\delta + \frac{1}{n}\sum_{j=0}^{n-1} Y_j \le 0) \\ &\ge \Pr(|h(\pi,C) + 3\delta + \frac{1}{n}\sum_{j=0}^{n-1} Y_j| \le \delta) \ge 1 - \delta \end{aligned}$$

by the choice of n. On the other hand, setting

$$\begin{aligned} B'_2 &= \{b : b \in B,\ \sum_{i \in M} |\gamma'_i - \frac{1}{n}\#(\{j : j < n,\ b \subseteq b_{ji}\})| \le \eta\} \\ &= \{\inf_{j<n} b_{j,i(j)} : i(0),\dots,i(n-1) \in M,\ \sum_{i \in M} |\gamma'_i - \frac{1}{n}\#(\{j : i(j) = i\})| \le \eta\}, \end{aligned}$$

we have

$$\bar\mu(\sup B'_2) = \Pr(\textstyle\sum_{i \in M} |\gamma'_i - \frac{1}{n}\sum_{j=0}^{n-1} X_{ij}| \le \eta) \ge 1 - \delta$$

by the other half of clause (b-v). Since $B' = B'_1 \cap B'_2$, $\bar\mu(\sup B) \ge 1 - 2\delta$. **Q**

Let D'_0 be the set of those $d \in D_n(C,\pi)$ such that

$$\frac{1}{n}\ln(\frac{1}{\bar\mu d}) \le h(\pi,C) + \delta, \quad \text{i.e.,} \quad \bar\mu d \ge e^{-n(h(\pi,C)+\delta)};$$

by (b-iv), $\bar\mu(\sup D'_0) > 1 - \eta$. Let $D' \subseteq D'_0$ be a finite set such that $\bar\mu(\sup D') \ge 1 - \eta$. If $d \in D'$ and $b \in B'$ then

$$\bar\mu d \ge e^{-n(h(\pi,C)+\delta)} \ge e^{n\delta}\bar\mu b \ge 2\bar\mu b.$$

Since $\bar\mu(\sup D') \le 1 \le 2\bar\mu(\sup B')$ (remember that $\delta \le \frac{1}{4}$), $\#(D') \le \#(B')$.

Set $e_1 = e_0 \cap \sup B'$, so that $\bar\mu e_1 \ge 1 - \beta - 3\delta$, and

$$D'' = \{d : d \in D',\ \bar\mu(d \cap e_1) \ge \tfrac{1}{2}\bar\mu d\};$$

then

$$\bar{\mu}(\sup(D' \setminus D'')) \le 2\bar{\mu}(1 \setminus e_1) \le 2\beta + 6\delta,$$

so

$$\bar{\mu}(\sup D'') \ge 1 - 2\beta - 6\delta - \eta \ge 1 - 2\beta - 7\delta.$$

(f) If $d_1, \dots, d_k \in D''$ are distinct,

$$\bar{\mu}(\sup_{1 \le i \le k} d_i \cap e_1) \ge \frac{k}{2} \inf_{i \le k} \bar{\mu} d_i \ge k \sup_{b \in B'} \bar{\mu} b,$$

and

$$\#(\{b : b \in B',\, b \cap e_0 \cap \sup_{1 \le i \le k} d_i\} \ne 0) \ge k.$$

By the Marriage Lemma (3A1K), there is an injective function $f_0 : D'' \to B'$ such that $d \cap f_0(d) \cap e_0 \ne 0$ for every $d \in D''$. Because $\#(D') \le \#(B')$, we can extend f_0 to an injective function $f : D' \to B'$.

(g) By the Halmos-Rokhlin-Kakutani lemma, in the strong form 386C(iv), there is an $a \in \mathfrak{A}$ such that $a, \pi^{-1}a, \dots,$ $\pi^{-n+1}a$ are disjoint and $\bar{\mu}(a \cap d) = \frac{1}{n+1}\bar{\mu} d$ for every $d \in D' \cup \{1\}$. Set $\tilde{e} = \sup\{\pi^{-j}(a \cap d) : j < n,\, d \in D'\}$. Because $\langle \pi^{-j}(a \cap d) \rangle_{j<n, d \in D'}$ is disjoint,

$$\bar{\mu}\tilde{e} = \sum_{j=0}^{n-1} \sum_{d \in D'} \bar{\mu}(a \cap d) = \frac{n}{n+1} \sum_{d \in D'} \bar{\mu} d \ge (1 - \eta)^2 \ge 1 - 2\eta.$$

(h) For $i \in M$, set

$$a'_i = \sup\{\pi^{-j}(a \cap d) : j < n,\, d \in D',\, f(d) \subseteq b_{ji}\}.$$

Then the a'_i are disjoint. **P** Suppose that $i,\, i' \in M$ are distinct. If $j,\, j' < n$ and $d,\, d' \in D'$ and $f(d) \subseteq b_{ji}$, $f(d') \subseteq b_{j'i'}$, then either $j \ne j'$ or $j = j'$. In the former case,

$$\pi^{-j}(a \cap d) \cap \pi^{-j'}(a \cap d') \subseteq \pi^{-j} a \cap \pi^{-j'} a = 0.$$

In the latter case, $b_{ji} \cap b_{j'i'} = 0$, so $f(d) \ne f(d')$ and $d \ne d'$ and

$$\pi^{-j}(a \cap d) \cap \pi^{-j'}(a \cap d') \subseteq \pi^{-j}(d \cap d') = 0. \text{ } \mathbf{Q}$$

Observe that

$$\sup_{i \in M} a'_i = \sup_{j<n, d \in D'} \pi^{-j}(a \cap d) = \tilde{e}$$

because if $j < n$ and $d \in D'$ there must be some $i \in M$ such that $f(d) \subseteq b_{ji}$. Take any $m \in \mathbb{N} \setminus M$ and set $a'_m = 1 \setminus \tilde{e}$, $a'_i = 0$ for $i \in \mathbb{N} \setminus (M \cup \{m\})$; then $\langle a'_i \rangle_{i \in \mathbb{N}}$ is a partition of unity. Now

$$\begin{aligned}
\sum_{i \in M} |\bar{\mu} a'_i - \gamma'_i| &\le \sum_{i \in M} \gamma'_i |1 - n\bar{\mu}(a \cap \sup D')| + \sum_{i \in M} |\bar{\mu} a'_i - n\gamma'_i \bar{\mu}(a \cap \sup D')| \\
&\le 1 - \frac{n}{n+1} \bar{\mu}(\sup D') \\
&\qquad + \sum_{i \in M} |\sum_{j=0}^{n-1} \sum_{\substack{d \in D' \\ f(d) \subseteq b_{ji}}} \bar{\mu}(\pi^{-j}(a \cap d)) - n\gamma'_i \sum_{d \in D'} \bar{\mu}(a \cap d)| \\
&\le 1 - (1 - \eta)^2 \\
&\qquad + \sum_{d \in D'} \sum_{i \in M} |\bar{\mu}(a \cap d) \cdot \#(\{j : j < n,\, f(d) \subseteq b_{ji}\}) - n\gamma'_i \bar{\mu}(a \cap d)| \\
&\le 1 - (1 - \eta)^2 + \sum_{d \in D'} \bar{\mu}(a \cap d) n\eta
\end{aligned}$$

(see the definition of B' in (e) above)

$$\le 2\eta + n\eta\bar{\mu} a \le 3\eta.$$

So

$$\sum_{i=0}^{\infty}|\bar\mu a_i'-\gamma_i|\le\bar\mu a_m'+\sum_{i\in M}|\bar\mu a_i'-\gamma_i'|+\sum_{i=0}^{\infty}|\gamma_i'-\gamma_i|$$
$$\le 2\eta+3\eta+4\delta^2\le 6\delta\le\epsilon.$$

We shall later want to know that $|\bar\mu a_i'-\gamma_i'|\le 3\eta$ for every i; for $i\in M$ this is covered by the formulae above, for $i=m$ it is true because $\bar\mu a_m'=1-\bar\mu\tilde e\le 2\eta$ (see (g)), and for other i it is trivial.

(i) The next step is to show that $\sum_{i=0}^{\infty}\bar\mu(a_i'\cap a_i)\ge 1-3\beta-12\delta$. **P** It is enough to consider the case in which $3\beta+12\delta<1$. We know that

$$\begin{aligned}\sup_{i\in\mathbb N}a_i'\cap a_i\supseteq\sup\{\pi^{-j}(a\cap d):j<n,\,d\in D',\\ \exists\,i\in M\text{ such that }f(d)\subseteq b_{ji}\text{ and }d\subseteq\pi^j a_i\}\\ =\sup\{\pi^{-j}(a\cap d):d\in D',\,j\in I_{f(d),d}\}\end{aligned}$$

(see (d) for the definition of I_{bd}) has measure at least $\sum_{d\in D'}\#(I_{f(d),d})\bar\mu(a\cap d)$.

For $d\in D''$, we arranged that $d\cap f(d)\cap e_0\ne 0$. This means that there must be some $b\in B$ and $d'\in D_n(C,\pi)$ such that $d\cap f(d)\cap b\cap f(d')\ne 0$ and $\#(I_{bd'})\ge n(1-\beta-4\delta)$; of course $b=f(d)$ and $d'=d$ (because f is injective), so that $\#(I_{f(d),d})$ must be at least $n(1-\beta-4\delta)$. Accordingly

$$\sum_{i=0}^{\infty}\bar\mu(a_i'\cap a_i)\ge\sum_{d\in D''}n(1-\beta-4\delta)\bar\mu(a\cap d)=n(1-\beta-4\delta)\frac{1}{n+1}\bar\mu(\sup D'')$$
$$\ge(1-\eta)(1-\beta-4\delta)(1-2\beta-7\delta)\ge 1-3\beta-12\delta.\ \mathbf{Q}$$

But this means that

$$\textstyle\sum_{i=0}^{\infty}\bar\mu(a_i'\mathbin{\triangle}a_i)=2(1-\sum_{i=0}^{\infty}\bar\mu(a_i'\cap a_i))\le 6\beta+24\delta\le\epsilon+6\beta$$

(using 386J for the equality).

(j) Finally, we need to estimate $H(A')$ and $h(\pi,A')$, where $A'=\{a_i':i\in\mathbb N\}\setminus\{0\}$. For the former, we have $H(A')\le h(\pi,C)+4\delta$. **P** $|\bar\mu a_i'-\gamma_i'|\le 3\eta$ for every i, by (h) above. So by (b-i),

$$\textstyle H(A')=\sum_{i\in M\cup\{m\}}q(\bar\mu a_i')\le\delta+\sum_{i=0}^{\infty}q(\gamma_i')=h(\pi,C)+4\delta.\ \mathbf{Q}$$

(k) Consider the partition of unity

$$A''=A'\vee\{a,1\setminus a\}.$$

Let $\mathfrak D$ be the closed subalgebra of $\mathfrak A$ generated by $\{\pi^j c:j\in\mathbb Z,\,c\in A''\}$.

(i) $a\cap d\in\mathfrak D$ for every $d\in D'$. **P** Of course $a\cap\tilde e\in\mathfrak D$, because $1\setminus\tilde e=a_m'$. If $d'\in D'$ and $d'\ne d$, then (because f is injective) $f(d)\ne f(d')$; there must therefore be some $k<n$ and distinct i, $i'\in M$ such that $f(d)\subseteq b_{ki}$ and $f(d')\subseteq b_{ki'}$. But this means that $\pi^{-k}(a\cap d)\subseteq a_i'$ and $\pi^{-k}(a\cap d')\subseteq a_{i'}'$, so that $a\cap d\subseteq\pi^k a_i'$ and $a\cap d'\cap\pi^k a_i'=0$.

What this means is that if we set

$$\tilde d=a\cap\tilde e\cap\inf\{\pi^k a_i':k<n,\,i\in M,\,a\cap d\subseteq\pi^k a_i'\},$$

we get a member of $\mathfrak D$ (because every $a_i'\in\mathfrak D$, and $\pi[\mathfrak D]=\mathfrak D$) including $a\cap d$ and disjoint from $a\cap d'$ whenever $d'\in D'$ and $d'\ne d$. But as $a\cap\pi^{-j}a=0$ if $0<j<n$, $a\cap\tilde e$ must be $\sup\{a\cap d':d'\in D'\}$, and $a\cap d=\tilde d$ belongs to $\mathfrak D$. **Q**

(ii) Consequently $c\cap\tilde e\in\mathfrak D$ for every $c\in C$. **P** We have

$$\begin{aligned}c\cap\tilde e&=\sup\{c\cap\pi^{-j}(a\cap d):j<n,\,d\in D'\}\\&=\sup\{\pi^{-j}(\pi^j c\cap a\cap d):j<n,\,d\in D'\}\\&=\sup\{\pi^{-j}(a\cap d):j<n,\,d\in D',\,d\subseteq\pi^j c\}\end{aligned}$$

(because if $d\in D'$ and $j<n$ then either $d\subseteq\pi^j c$ or $d\cap\pi^j c=0$)

$$\in\mathfrak D$$

because $a \cap d \in \mathfrak{D}$ for every $d \in D'$ and $\pi^{-1}[\mathfrak{D}] = \mathfrak{D}$. **Q**

(iii) It follows that $h(\pi, A'') \geq h(\pi, C) - \delta$. **P** For any $c \in C$,

$$\rho(c, \mathfrak{D}) \leq \bar{\mu}(c \triangle (c \cap \tilde{e})) = \bar{\mu}(c \setminus \tilde{e}) \leq \min(\bar{\mu}c, 2\eta) \leq \frac{1}{3}.$$

So

$$h(\pi, C) \leq h(\pi \restriction \mathfrak{D}) + H(C|\mathfrak{D})$$

(386Ld, because $\pi[\mathfrak{D}] = \mathfrak{D}$)

$$\leq h(\pi, A'') + \sum_{c \in C} q(\rho(c, \mathfrak{D}))$$

(by the Kolmogorov-Sinaǐ theorem (385P) and 386Mb)

$$\leq h(\pi, A'') + \sum_{c \in C} q(\min(\bar{\mu}c, 2\eta))$$

(because q is monotonic on $[0, \frac{1}{3}]$)

$$\leq h(\pi, A'') + \delta$$

by the choice of η. **Q**

(iv) Finally, $h(\pi, A') \geq h(\pi, C) - 2\delta$. **P** Using 386Lb,

$$\begin{aligned} h(\pi, C) - \delta \leq h(\pi, A'') &\leq h(\pi, A') + H(\{a, 1 \setminus a\}) \\ &= h(\pi, A') + q(\bar{\mu}a) + q(1 - \bar{\mu}a) \\ &= h(\pi, A') + q(\tfrac{1}{n+1}) + q(\tfrac{n}{n+1}) \leq h(\pi, A') + \delta \end{aligned}$$

by the choice of n. **Q**

(l) Putting these together,

$$H(A') \leq h(\pi, C) + 4\delta \leq h(\pi, A') + 6\delta \leq h(\pi, A') + \epsilon,$$

and the proof is complete.

387D Corollary Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an ergodic measure-preserving automorphism. Let $\langle a_i \rangle_{i \in \mathbb{N}}$ be a partition of unity in $\mathfrak{A}$, of finite entropy, and $\langle \gamma_i \rangle_{i \in \mathbb{N}}$ a sequence of non-negative real numbers such that

$$\textstyle\sum_{i=0}^{\infty} \gamma_i = 1, \quad \sum_{i=0}^{\infty} q(\gamma_i) \leq h(\pi).$$

Then for any $\epsilon > 0$ we can find a Bernoulli partition $\langle a_i^* \rangle_{i \in \mathbb{N}}$ for π such that $\bar{\mu}a_i^* = \gamma_i$ for every $i \in \mathbb{N}$ and

$$\textstyle\sum_{i=0}^{\infty} \bar{\mu}(a_i^* \triangle a_i) \leq \epsilon + 6\sqrt{\sum_{i=0}^{\infty} |\bar{\mu}a_i - \gamma_i| + \sqrt{2(H(A) - h(\pi, A))}},$$

writing $A = \{a_i : i \in \mathbb{N}\} \setminus \{0\}$.

proof (a) Set $\beta = \sqrt{\sum_{i=0}^{\infty} |\bar{\mu}a_i - \gamma_i| + \sqrt{2(H(A) - h(\pi, A))}}$. Let $\langle \epsilon_n \rangle_{n \in \mathbb{N}}$ be a sequence of strictly positive real numbers such that

$$\textstyle\sum_{n=0}^{\infty} \epsilon_n + 6\sqrt{\epsilon_n + \sqrt{2\epsilon_n}} \leq \epsilon.$$

Using 387C, we can choose inductively, for $n \in \mathbb{N}$, partitions of unity $\langle a_{ni} \rangle_{i \in \mathbb{N}}$ such that, for each $n \in \mathbb{N}$,

$$\textstyle\sum_{i=0}^{\infty} |\gamma_i - \bar{\mu}a_{ni}| \leq \epsilon_n,$$

$$H(A_n) \leq h(\pi, A_n) + \epsilon_n < \infty$$

(writing $A_n = \{a_{ni} : i \in \mathbb{N}\} \setminus \{0\}$),

$$\textstyle\sum_{i=0}^{\infty} \bar{\mu}(a_{n+1,i} \triangle a_{ni}) \leq \epsilon_{n+1} + 6\sqrt{\epsilon_n + \sqrt{2\epsilon_n}},$$

while

$$\sum_{i=0}^{\infty} \bar{\mu}(a_{0i} \triangle a_i) \le \epsilon_0 + 6\beta.$$

On completing the induction, we see that

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \bar{\mu}(a_{n+1,i} \triangle a_{ni}) \le \sum_{n=1}^{\infty} \epsilon_n + \sum_{n=0}^{\infty} 6\sqrt{\epsilon_n + \sqrt{2\epsilon_n}} < \infty.$$

In particular, given $i \in \mathbb{N}$, $\sum_{n=0}^{\infty} \bar{\mu}(a_{n+1,i} \triangle a_{ni})$ is finite, so $\langle a_{ni} \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space $\mathfrak{A}$ (323Gc), and has a limit a_i^*, with

$$\bar{\mu}a_i^* = \lim_{n\to\infty} \bar{\mu}a_{ni} = \gamma_i$$

(323C). If $i \ne j$,

$$a_i^* \cap a_j^* = \lim_{n\to\infty} a_{ni} \cap a_{nj} = 0$$

(using 323B), so $\langle a_i^* \rangle_{i\in\mathbb{N}}$ is disjoint; since

$$\sum_{i=0}^{\infty} \bar{\mu}a_i^* = \sum_{i=0}^{\infty} \gamma_i = 1,$$

$\langle a_i^* \rangle_{i\in\mathbb{N}}$ is a partition of unity. We also have

$$\begin{aligned} \sum_{i=0}^{\infty} \bar{\mu}(a_i^* \triangle a_i) &\le \sum_{i=0}^{\infty} \bar{\mu}(a_{0i} \triangle a_i) + \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} \bar{\mu}(a_{n+1,i} \triangle a_{ni}) \\ &\le \epsilon_0 + 6\beta + \sum_{n=1}^{\infty} \epsilon_n + \sum_{n=0}^{\infty} 6\sqrt{\epsilon_n + \sqrt{2\epsilon_n}} \le \epsilon + 6\beta. \end{aligned}$$

(b) Now take any $i(0), \dots, i(k) \in \mathbb{N}$. For each $j < k$, $n \in \mathbb{N}$,

$$H(\pi^j[A_n]) + H(D_j(A_n, \pi)) - H(D_{j+1}(A_n, \pi)) \le H(A_n) - h(\pi, A_n) \le \epsilon_n$$

(using 386Lc). But this means that

$$\sum_{d\in D_j(A_n,\pi)} \sum_{i=0}^{\infty} |\bar{\mu}(d \cap \pi^j a_{ni}) - \bar{\mu}d \cdot \bar{\mu}a_{ni}| \le \sqrt{2\epsilon_n},$$

by 386I. *A fortiori*,

$$|\bar{\mu}(d \cap \pi^j a_{ni}) - \bar{\mu}d \cdot \bar{\mu}a_{ni}| \le \sqrt{2\epsilon_n}$$

for each $d \in D_j(A_n, \pi)$, $i \in \mathbb{N}$. Inducing on r, we see that

$$|\bar{\mu}(\inf_{j\le r} \pi^j a_{n,i(j)}) - \prod_{j=0}^{r} \bar{\mu}a_{n,i(j)}| \le r\sqrt{2\epsilon_n} \to 0$$

as $n \to \infty$, for any $r \le k$. Because $\bar{\mu}$, $\cap$ and π are all continuous (323C, 323B, 324Kb),

$$\begin{aligned} \bar{\mu}(\inf_{j\le k} \pi^j a_{i(j)}^*) &= \lim_{n\to\infty} \bar{\mu}(\inf_{j\le k} \pi^j a_{n,i(j)}) \\ &= \lim_{n\to\infty} \prod_{j=0}^{k} \bar{\mu}a_{n,i(j)} = \prod_{j=0}^{k} \gamma_{i(j)}. \end{aligned}$$

As $i(0), \dots, i(k)$ are arbitrary, $\langle a_i^* \rangle_{i\in\mathbb{N}}$ is a Bernoulli partition for π.

387E Sinaǐ's theorem (atomic case) (Sinaǐ 62) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an ergodic measure-preserving automorphism. Let $\langle \gamma_i \rangle_{i\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $\sum_{i=0}^{\infty} \gamma_i = 1$ and $\sum_{i=0}^{\infty} q(\gamma_i) \le h(\pi)$. Then there is a Bernoulli partition $\langle a_i^* \rangle_{i\in\mathbb{N}}$ for π such that $\bar{\mu}a_i^* = \gamma_i$ for every $i \in \mathbb{N}$.

proof Apply 387D from any starting point, e.g., $a_0 = 1$, $a_i = 0$ for $i > 0$.

387F Lemma Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra and π a measure-preserving automorphism of $\mathfrak{A}$. Let $\langle b_i \rangle_{i\in\mathbb{N}}$ and $\langle c_i \rangle_{i\in\mathbb{N}}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak{B}$, $\mathfrak{C}$ for the closed subalgebras of $\mathfrak{A}$ generated by $\{\pi^j b_i : i \in \mathbb{N},\ j \in \mathbb{Z}\}$ and $\{\pi^j c_i : i \in \mathbb{N},\ j \in \mathbb{Z}\}$. Suppose that $\mathfrak{C} \subseteq \mathfrak{B}$. Then for any $\epsilon > 0$ we can find a Bernoulli partition $\langle d_i \rangle_{i\in\mathbb{N}}$ for π such that

(i) $d_i \in \mathfrak{C}$ for every $i \in \mathbb{N}$,

(ii) $\bar\mu d_i = \bar\mu b_i$ for every $i \in \mathbb{N}$,

(iii) $\bar\mu(\phi c_i \,\triangle\, c_i) \le \epsilon$ for every $i \in \mathbb{N}$,

where $\phi : \mathfrak{B} \to \mathfrak{C}$ is the measure-preserving Boolean homomorphism such that $\phi b_i = d_i$ for every i and $\pi\phi = \phi\pi$ (387Bi).

proof (a) Set $B = \{b_i : i \in \mathbb{N}\} \setminus \{0\}$, $C = \{c_i : i \in \mathbb{N}\} \setminus \{0\}$. If only one c_i is non-zero, then $H(C) = 0$, so $H(B) = 0$ and $\mathfrak{B} = \{0, 1\}$, in which case $\mathfrak{B} = \mathfrak{C}$ and we take $d_i = b_i$ and stop. Otherwise, $\mathfrak{C}$ is atomless (387Bd).

For $k \in \mathbb{N}$, let $\mathfrak{B}_k \subseteq \mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_i : i \le k,\ |j| \le k\}$. Because $\mathfrak{C} \subseteq \mathfrak{B}$, there is an $m \in \mathbb{N}$ such that

$$\rho(c_i, \mathfrak{B}_m) \le \frac{1}{4}\epsilon \text{ for every } i \in \mathbb{N}$$

(386K). Let η, $\xi > 0$ be such that

$$\eta + 6\sqrt[4]{2\eta} \le \frac{\epsilon}{4(2m+1)}, \quad \xi \le \min(\frac{\epsilon}{4}, \frac{1}{6}), \quad \sum_{i=0}^\infty q(\min(2\xi, \bar\mu c_i)) \le \eta.$$

(The last is achievable because $\sum_{i=0}^\infty q(\bar\mu c_i)$ is finite.) Let $r \ge m$ be such that

$$\rho(c_i, \mathfrak{B}_r) \le \xi \text{ for every } i \in \mathbb{N}.$$

Let $n \ge r$ be such that

$$\frac{2r+1}{2n+2} \le \xi, \quad \bar\mu c_i \le \xi \text{ for every } i > n.$$

(b) Let $\langle b'_i \rangle_{i\in\mathbb{N}}$ be a partition of unity in $\mathfrak{C}$ such that $\bar\mu b'_i = \bar\mu b_i$ for every $i \in \mathbb{N}$. Let U be the set of atoms of the subalgebra of $\mathfrak{B}$ generated by $\{\pi^j b_i : i \le n,\ |j| \le n\} \cup \{\pi^j c_i : i \le n,\ |j| \le n\}$, and V the set of atoms of the subalgebra of $\mathfrak{C}$ generated by $\{\pi^j b'_i : i \le n,\ |j| \le n\} \cup \{\pi^j c_i : i \le n,\ |j| \le n\}$. For each $v \in V$, choose a disjoint family $\langle d_{vu} \rangle_{u\in U}$ in $\mathfrak{C}$ such that $\sup_{u\in U} d_{vu} = v$ and $\bar\mu d_{vu} = \bar\mu(v \cap u)$ for every $u \in U$. By 386C(iv), there is an $a \in \mathfrak{C}$ such that $a, \pi a, \dots, \pi^{2n} a$ are disjoint and $\bar\mu(a \cap d_{vu}) = \frac{1}{2n+2}\bar\mu(d_{vu})$ for every $u \in U$ and $v \in V$. ($\pi\restriction\mathfrak{C}$ is a Bernoulli shift, therefore ergodic, by 385Se, therefore aperiodic, by 386D.) Set $e = \sup_{|j|\le n} \pi^j a$, $\tilde e = \sup_{|j|\le n-r} \pi^j a$; then

$$\bar\mu e = (2n+1)\bar\mu a = \frac{2n+1}{2n+2}, \quad \bar\mu\tilde e = (2(n-r)+1)\bar\mu a = 1 - \frac{2r+1}{2n+2}.$$

Let $\mathfrak{C}_{\tilde e}$ be the principal ideal of $\mathfrak{C}$ generated by $\tilde e$.

(c) The family $\langle \pi^{-j}(a \cap d_{vu}) \rangle_{|j|\le n, u\in U, v\in V}$ is disjoint. **P** All we have to note is that the families $\langle d_{vu} \rangle_{u\in U, v\in V}$ and

$$\langle \pi^{-j} a \rangle_{|j|\le n} = \langle \pi^{-n}(\pi^{n+j} a) \rangle_{|j|\le n}$$

are disjoint. **Q** Consequently, if we set

$$\hat b_i = \sup_{|j|\le n} \sup_{v\in V} \sup_{u\in U, u \subseteq \pi^j b_i} \pi^{-j}(a \cap d_{vu}) \in \mathfrak{C}$$

for $i \in \mathbb{N}$, $\langle \hat b_i \rangle_{i\in\mathbb{N}}$ is disjoint, since a given triple (j, u, v) can contribute to at most one $\hat b_i$.

Of course $\hat b_i \subseteq \sup_{|j|\le n} \pi^{-j} a = e$ for every i. If $i \le n$, we also have $\bar\mu\hat b_i = \bar\mu e \cdot \bar\mu b_i$. **P** For $|j| \le n$, $\pi^j b_i$ is a union of members of U, so

$$\bar\mu\hat b_i = \sum_{j=-n}^{n} \sum_{v\in V} \sum_{u\in U, u \subseteq \pi^j b_i} \bar\mu(\pi^{-j}(a \cap d_{vu}))$$

(because $\langle \pi^{-j}(a \cap d_{vu}) \rangle_{|j|\le n, u\in U, v\in V}$ is disjoint)

$$= \sum_{j=-n}^{n} \sum_{v\in V} \sum_{u\in U, u \subseteq \pi^j b_i} \bar\mu(a \cap d_{vu}) = \frac{1}{2n+2} \sum_{j=-n}^{n} \sum_{v\in V} \sum_{u\in U, u \subseteq \pi^j b_i} \bar\mu d_{vu}$$

(by the choice of a)

$$= \frac{1}{2n+2} \sum_{j=-n}^{n} \sum_{v \in V} \sum_{u \in U, u \subseteq \pi^j b_i} \bar\mu(v \cap u)$$

(by the choice of d_{vu})

$$= \frac{1}{2n+2} \sum_{j=-n}^{n} \sum_{u \in U, u \subseteq \pi^j b_i} \bar\mu u = \frac{1}{2n+2} \sum_{j=-n}^{n} \bar\mu(\pi^j b_i)$$

(because $\pi^j b_i$ is a disjoint union of members of U when $i \le n$, $|j| \le n$)

$$= \frac{2n+1}{2n+2} \bar\mu b_i = \bar\mu e \cdot \bar\mu b_i. \textbf{ Q}$$

Again because $\mathfrak{C}$ is atomless, we can choose a partition of unity $\langle b_i^* \rangle_{i \in \mathbb{N}}$ in $\mathfrak{C}$ such that $\bar\mu b_i^* = \bar\mu b_i$ for every i, while $b_i^* \supseteq \hat b_i$ and $b_i^* \cap e = \hat b_i$ for $i \le n$.

(d) Let $\mathfrak{E}$ be the finite subalgebra of $\mathfrak{B}$ generated by $\{\pi^j b_i : i \le n, |j| \le r\} \cup \{\pi^j c_i : i \le n, |j| \le r\}$. Define $\theta : \mathfrak{E} \to \mathfrak{C}_{\tilde e}$ by setting

$$\theta b = \sup_{|j| \le n-r} \sup_{v \in V} \sup_{u \in U, u \subseteq \pi^j b} \pi^{-j}(a \cap d_{vu})$$

for $b \in \mathfrak{E}$.

(i) θ is a Boolean homomorphism. **P** The point is that if $|j| \le n-r$ and $b \in \mathfrak{E}$, then $\pi^j b$ belongs to the algebra generated by $\{\pi^k b_i : i \le n, |k| \le n\} \cup \{\pi^k c_i : i \le n, |k| \le n\}$, so is a union of members of U. Since each map

$$b \mapsto \pi^{-j}(a \cap d_{vu}) \text{ if } u \subseteq \pi^j b, \quad 0 \text{ otherwise}$$

is a Boolean homomorphism from $\mathfrak{E}$ to the principal ideal generated by $\pi^{-j}(a \cap d_{vu})$, and $\langle \pi^{-j}(a \cap d_{vu}) \rangle_{|j| \le n-r, u \in U, v \in V}$ is a partition of unity in $\mathfrak{C}_{\tilde e}$, θ also is a Boolean homomorphism. **Q**

(ii) $\bar\mu(\theta b) \le \bar\mu b$ for every $b \in \mathfrak{E}$. **P** (Compare (c) above.)

$$\bar\mu(\theta b) = \sum_{j=-n+r}^{n-r} \sum_{v \in V} \sum_{u \in U, u \subseteq \pi^j b} \bar\mu \pi^{-j}(a \cap d_{vu})$$

$$= \frac{1}{2n+2} \sum_{j=-n+r}^{n-r} \sum_{v \in V} \sum_{u \in U, u \subseteq \pi^j b} \bar\mu(v \cap u) = \frac{2n-2r+1}{2n+2} \bar\mu b \le \bar\mu b. \textbf{ Q}$$

(iii) $\theta(\pi^k b_i) = \tilde e \cap \pi^k b_i^*$ for $i \le n$, $|k| \le r$. **P** Of course $\pi^k b_i \in \mathfrak{E}$. If $|j| \le n-r$, then $|j+k| \le n$, so

$$\begin{aligned} \pi^{-j} a \cap \theta(\pi^k b_i) &= \sup_{v \in V} \sup_{u \in U, u \subseteq \pi^{j+k} b_i} \pi^{-j}(a \cap d_{vu}) \\ &= \pi^k \big(\sup_{v \in V} \sup_{u \in U, u \subseteq \pi^{j+k} b_i} \pi^{-j-k}(a \cap d_{vu}) \big) \\ &= \pi^k(\pi^{-j-k} a \cap \hat b_i) = \pi^{-j} a \cap \pi^k(e \cap b_i^*) = \pi^{-j} a \cap \pi^k b_i^* \end{aligned}$$

because $\pi^{-j} a \subseteq \pi^k e$. Taking the supremum of these pieces we have

$$\theta(\pi^k b_i) = \sup_{|j| \le n-r} \pi^{-j} a \cap \theta(\pi^k b_i) = \sup_{|j| \le n-r} \pi^{-j} a \cap \pi^k b_i^* = \tilde e \cap \pi^k b_i^*. \textbf{ Q}$$

It follows that

$$\theta(\pi^k(1 \setminus \sup_{i \le l} b_i)) = \tilde e \cap \pi^k(1 \setminus \sup_{i \le l} b_i^*))$$

if $l \le n$ and $|k| \le r$.

(iv) Finally, $\theta c_i = c_i \cap \tilde e$ for every $i \le n$. **P** If $|j| \le n-r$ and $v \in V$ then either $v \subseteq \pi^j c_i$ or $v \cap \pi^j c_i = 0$. In the former case,

$$d_{vu} = v \cap u = 0 \text{ whenever } u \in U \text{ and } u \not\subseteq \pi^j c_i,$$

so that

$$v = \sup_{u \in U} d_{vu} = \sup_{u \in U, u \subseteq \pi^j c_i} d_{vu};$$

in the latter case, $d_{vu}=v\cap u=0$ whenever $u\subseteq\pi^jc_i$. So we have

$$v\cap\pi^jc_i=\sup_{u\in U,u\subseteq\pi^jc_i}d_{vu}$$

for every $v\in V$, and

$$\begin{aligned}\theta c_i&=\sup_{|j|\le n-r}\sup_{v\in V}\sup_{u\in U,u\subseteq\pi^jc_i}\pi^{-j}(a\cap d_{vu})\\&=\sup_{|j|\le n-r}\pi^{-j}(a\cap\sup_{v\in V}\sup_{u\in U,u\subseteq\pi^jc_i}d_{vu})\\&=\sup_{|j|\le n-r}\pi^{-j}(a\cap\sup_{v\in V}(v\cap\pi^jc_i))\\&=\sup_{|j|\le n-r}\pi^{-j}(a\cap\pi^jc_i)=c_i\cap\sup_{|j|\le n-r}\pi^{-j}a=c_i\cap\tilde e.\ \mathbf{Q}\end{aligned}$$

(e) Let $\mathfrak{B}^*$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^jb_i^*:i\in\mathbb{N},|j|\in\mathbb{Z}\}$. Then for every $b\in\mathfrak{B}_r$ there is a $b^*\in\mathfrak{B}^*$ such that $\theta b=b^*\cap\tilde e$. **P** The set of b for which this is true is a subalgebra of $\mathfrak{A}$ containing π^kb_i for $i\le r$ and $|k|\le r$, by (d-iii). **Q** It follows that

$$\rho(c_i,\mathfrak{B}^*)\le2\xi\text{ for }i\in\mathbb{N}.$$

P If $i>n$ this is trivial, because $\bar\mu c_i\le\xi$, by the choice of n. Otherwise, $c_i\in\mathfrak{C}$. Take $b\in\mathfrak{B}_r$ such that $\bar\mu(b\triangle c_i)=\rho(c_i,\mathfrak{B}_r)\le\xi$ (386Ma). Let $b^*\in\mathfrak{B}^*$ be such that $\theta b=b^*\cap\tilde e$. Then

$$\begin{aligned}\rho(c_i,\mathfrak{B}^*)&\le\bar\mu(c_i\triangle b^*)\le1-\bar\mu\tilde e+\bar\mu(\tilde e\cap(c_i\triangle b^*))\\&=\tfrac{2r+1}{2n+2}+\bar\mu((\tilde e\cap c_i)\triangle\theta b)=\tfrac{2r+1}{2n+2}+\bar\mu(\theta c_i\triangle\theta b)\end{aligned}$$

(by (d-iv))

$$=\tfrac{2r+1}{2n+2}+\bar\mu(\theta(c_i\triangle b))\le\tfrac{2r+1}{2n+2}+\bar\mu(c_i\triangle b)$$

(by (d-ii))

$$\le2\xi$$

by the choice of n. **Q**

(f) Set $B^*=\{b_i^*:i\in\mathbb{N}\}\setminus\{0\}$. Then $H(B^*)=h(\pi,C)\le h(\pi,B^*)+\eta$. **P**

$$H(B^*)=H(B)=H(C)$$

(because $\bar\mu b_i^*=\bar\mu b_i$ for every i, and we supposed from the beginning that $H(C)=H(B)$)

$$=h(\pi,C)$$

(because C is a Bernoulli partition, see 387Bc)

$$\le h(\pi\restriction\mathfrak{B}^*)+H(C|\mathfrak{B}^*)$$

(386Ld)

$$\le h(\pi\restriction\mathfrak{B}^*)+\sum_{i=0}^\infty q(\rho(c_i,\mathfrak{B}^*))$$

(386Mb)

$$\le h(\pi,B^*)+\sum_{i=0}^\infty q(\min(2\xi,\bar\mu c_i))$$

(by the Kolmogorov-Sinaĭ theorem, 385P(ii), and (e) above, recalling that $\xi\le\frac16$, so that q is monotonic on $[0,2\xi]$)

$$\le h(\pi,B^*)+\eta$$

by the choice of ξ. **Q**

Note also that $H(B^*) = h(\pi, C) \le h(\pi)$.

(g) By 387D, applied to $\pi\restriction\mathfrak{C}$ and the partition $\langle b_i^*\rangle_{i\in\mathbb{N}}$ of unity in $\mathfrak{C}$ and the sequence $\langle\gamma_i\rangle_{i\in\mathbb{N}} = \langle\bar\mu b_i^*\rangle_{i\in\mathbb{N}}$, we have a Bernoulli partition $\langle d_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{C}$ such that $\bar\mu d_i = \bar\mu b_i^* = \bar\mu b_i$ for every $i \in \mathbb{N}$ and

$$\textstyle\sum_{i=0}^\infty \bar\mu(d_i \triangle b_i^*) \le \eta + 6\sqrt[4]{2\eta} \le \frac{\epsilon}{4(2m+1)}.$$

Let $\mathfrak{D} \subseteq \mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_i : i \in \mathbb{N},\ j \in \mathbb{Z}\}$. Then $(\mathfrak{B}, \pi\restriction\mathfrak{B}, \langle b_i\rangle_{i\in\mathbb{N}})$ is isomorphic to $(\mathfrak{D}, \pi\restriction\mathfrak{D}, \langle d_i\rangle_{i\in\mathbb{N}})$, with an isomorphism $\phi : \mathfrak{B} \to \mathfrak{D}$ such that $\phi\pi = \pi\phi$ and $\phi b_i = d_i$ for every $i \in \mathbb{N}$ (387Bi).

(h) Set

$$e^* = \tilde e \setminus \sup_{|j|\le m, i\in\mathbb{N}} \pi^j(d_i \triangle b_i^*).$$

Then $\phi(\pi^j b_i) \cap e^* = \theta(\pi^j b_i) \cap e^*$ whenever $i \le m$ and $|j| \le m$. **P**

$$\begin{aligned}\phi(\pi^j b_i) \cap e^* &= \pi^j(\phi b_i) \cap e^* = \pi^j d_i \cap e^* \\ &= \pi^j b_i^* \cap e^* = \pi^j b_i^* \cap \tilde e \cap e^* = \theta(\pi^j b_i) \cap e^*\end{aligned}$$

by (d-iii), because i and $|j|$ are both at most $m \le r \le n$. **Q** Since $b \mapsto \phi b \cap e^* : \mathfrak{A} \to \mathfrak{A}_{e^*}$, $b \mapsto \theta b \cap e^* : \mathfrak{C} \to \mathfrak{A}_{e^*}$ are Boolean homomorphisms, $\phi b \cap e^* = \theta b \cap e^*$ for every $b \in \mathfrak{B}_m$.

Now $\bar\mu(c_i \triangle \phi c_i) \le \epsilon$ for every $i \in \mathbb{N}$. **P** If $i > n$ then of course

$$\bar\mu(\phi c_i \triangle c_i) \le 2\bar\mu c_i \le 2\xi \le \epsilon.$$

If $i \le n$, then (by the choice of m) there is a $b \in \mathfrak{B}_m$ such that $\bar\mu(c_i, b) \le \frac14\epsilon$. So

$$\begin{aligned}\phi c_i \triangle c_i &\subseteq (\phi c_i \triangle \phi b) \cup (\phi b \triangle \theta b) \cup (\theta b \triangle \theta c_i) \cup (\theta c_i \triangle c_i) \\ &\subseteq \phi(c_i \triangle b) \cup (1 \setminus e^*) \cup \theta(b \triangle c_i)\end{aligned}$$

(using the definition of e^* and (d-iv)) has measure at most

$$\bar\mu(c_i \triangle b) + \bar\mu(1 \setminus e^*) + \bar\mu(b \triangle c_i)$$

(by (d-ii), since b and c_i both belong to $\mathfrak{C}$)

$$\begin{aligned}&\le 2\bar\mu(c_i \triangle b) + \bar\mu(1 \setminus \tilde e) + (2m+1)\sum_{i=0}^\infty \bar\mu(d_i \triangle b_i^*) \\ &\le \frac{\epsilon}{2} + \frac{2r+1}{2n+2} + \frac{\epsilon}{4} \le \epsilon,\end{aligned}$$

as required. **Q**

387G Lemma Let $(\mathfrak{A}, \bar\mu)$ be an atomless probability algebra and π a measure-preserving automorphism of $\mathfrak{A}$. Let $\langle b_i\rangle_{i\in\mathbb{N}}$ and $\langle c_i\rangle_{i\in\mathbb{N}}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak{B}$, $\mathfrak{C}$ for the closed subalgebras generated by $\{\pi^j b_i : i \in \mathbb{N},\ j \in \mathbb{Z}\}$ and $\{\pi^j c_i : i \in \mathbb{N},\ j \in \mathbb{Z}\}$. Suppose that $\mathfrak{C} \subseteq \mathfrak{B}$. Then for any $\epsilon > 0$ we can find a Bernoulli partition $\langle d_i\rangle_{i\in\mathbb{N}}$ for π such that

(i) $\bar\mu d_i = \bar\mu c_i$ for every $i \in \mathbb{N}$,

(ii) $\bar\mu(d_i \triangle c_i) \le \epsilon$ for every $i \in \mathbb{N}$,

(iii) writing $\mathfrak{D}$ for the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_i : i \in \mathbb{N},\ j \in \mathbb{Z}\}$, $\rho(b_i, \mathfrak{D}) \le \epsilon$ for every $i \in \mathbb{N}$.

proof (a) By 387F, there is a Bernoulli partition $\langle b_i^*\rangle_{i\in\mathbb{N}}$ for π such that $b_i^* \in \mathfrak{C}$ for every $i \in \mathbb{N}$, $\bar\mu b_i^* = \bar\mu b_i$ for every $i \in \mathbb{N}$, and $\bar\mu(\phi c_i \triangle c_i) \le \frac14\epsilon$ for every $i \in \mathbb{N}$, where $\phi : \mathfrak{B} \to \mathfrak{C}$ is the measure-preserving Boolean homomorphism such that $\phi b_i = b_i^*$ for every i and $\pi\phi = \phi\pi$. Note that this implies that $\pi^{-1}\phi = \phi\pi^{-1}$, and generally that $\pi^j\phi = \phi\pi^{-j}$ for every $j \in \mathbb{Z}$; so $\phi[\mathfrak{B}] \subseteq \mathfrak{C}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\phi\pi^j b_i : i \in \mathbb{N},\ j \in \mathbb{Z}\} = \{\pi^j b_i^* : i \in \mathbb{N},\ j \in \mathbb{Z}\}$ (324L), and is invariant under the action of π and π^{-1}.

Let $m \in \mathbb{N}$ be such that

$$\rho(c_i, \mathfrak{B}_m) \le \tfrac14\epsilon \text{ for every } i \in \mathbb{N},$$

where $\mathfrak{B}_m$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_i : i \in \mathbb{N},\ |j| \le m\}$ (386K). Let $\eta \in\ ]0, \epsilon]$ be such that

$$(2m+1)\sum_{i=0}^{\infty}\min(\eta,2\bar\mu b_i)\le\tfrac14\epsilon.$$

By 387F again, applied to $\pi\restriction\mathfrak C$, there is a Bernoulli partition $\langle c_i^*\rangle_{i\in\mathbb N}$ for π such that $c_i^*\in\phi[\mathfrak B]$, $\bar\mu c_i^*=\bar\mu c_i$ and $\bar\mu(\psi b_i^*\triangle b_i^*)\le\eta$ for every $i\in\mathbb N$, where $\psi:\mathfrak C\to\mathfrak C$ is the measure-preserving Boolean homomorphism such that $\psi c_i=c_i^*$ for every $i\in\mathbb N$ and $\psi\pi=\pi\psi$. Once again, $\psi[\mathfrak C]$ will be the closed subalgebra of $\mathfrak A$ generated by $\{\psi\pi^jc_i:i\in\mathbb N,\,j\in\mathbb Z\}=\{\pi^jc_i^*:i\in\mathbb N,\,j\in\mathbb Z\}$; because every c_i^* belongs to $\phi[\mathfrak B]$, $\psi[\mathfrak C]\subseteq\phi[\mathfrak B]$.

(b) Now $\bar\mu(c_i^*\triangle\phi c_i)\le\epsilon$ for every $i\in\mathbb N$. **P** There is a $b\in\mathfrak B_m$ such that $\bar\mu(c_i\triangle b)\le\frac14\epsilon$. We know that $\phi[\mathfrak B_m]$ is the closed subalgebra of $\mathfrak A$ generated by $\{\phi\pi^jb_i:i\in\mathbb N,\,|j|\le m\}=\{\pi^jb_i^*:i\in\mathbb N,\,|j|\le m\}$, and contains ϕb. Because

$$\psi(\phi b)\triangle\phi b\subseteq\ \sup_{i\in\mathbb N,|j|\le m}\psi(\pi^jb_i^*)\triangle\pi^jb_i^*=\sup_{|j|\le m}\pi^j(\sup_{i\in\mathbb N}\psi b_i^*\triangle b_i^*),$$

we have

$$\begin{aligned}\bar\mu(\psi\phi b\triangle\phi b)&\le(2m+1)\sum_{i=0}^{\infty}\bar\mu(\psi b_i^*\triangle b_i^*)\\&\le(2m+1)\sum_{i=0}^{\infty}\min(\eta,2\bar\mu b_i)\le\frac14\epsilon.\end{aligned}$$

But this means that

$$\begin{aligned}\bar\mu(c_i^*\triangle\phi c_i)&=\bar\mu(\psi c_i\triangle\phi c_i)\le\bar\mu(\psi c_i\triangle\psi\phi b)+\bar\mu(\psi\phi b\triangle\phi b)+\bar\mu(\phi b\triangle\phi c_i)\\&\le\bar\mu(c_i\triangle\phi b)+\frac{\epsilon}{4}+\bar\mu(b\triangle c_i)\le\bar\mu(c_i\triangle\phi c_i)+\bar\mu(\phi c_i\triangle\phi b)+\frac{\epsilon}{2}\\&\le\frac{\epsilon}{4}+\bar\mu(c_i\triangle b)+\frac{\epsilon}{2}\le\epsilon.\ \mathbf{Q}\end{aligned}$$

(c) Set $d_i=\phi^{-1}c_i^*$ for each i; this is well-defined because ϕ is injective and $c_i^*\in\phi[\mathfrak B]$. Write $\mathfrak D$ for the closed subalgebra of $\mathfrak A$ generated by $\{\pi^jd_i:i\in\mathbb N,\,j\in\mathbb Z\}=\{\phi^{-1}\psi\pi^jc_i:i\in\mathbb N,\,j\in\mathbb Z\}$; note that $\mathfrak D=\phi^{-1}[\psi[\mathfrak C]]$, by 324L again, because $\phi^{-1}:\phi[\mathfrak B]\to\mathfrak B$ is a measure-preserving homomorphism. Then $\bar\mu d_i=\bar\mu c_i^*=\bar\mu c_i$ for every $i\in\mathbb N$, and $\langle d_i\rangle_{i\in\mathbb N}$ is a Bernoulli partition for π. **P** If $i(0),\dots,i(n)\in\mathbb N$, then

$$\begin{aligned}\bar\mu(\inf_{j\le n}\pi^jd_{i(j)})&=\bar\mu(\inf_{j\le n}\pi^j\phi^{-1}c_{i(j)}^*)=\bar\mu(\phi(\inf_{j\le n}\pi^j\phi^{-1}c_{i(j)}^*))\\&=\bar\mu(\inf_{j\le n}\pi^jc_{i(j)}^*))=\prod_{j=0}^{n}\bar\mu c_{i(j)}^*=\prod_{j=0}^{n}\bar\mu d_{i(j)}.\ \mathbf{Q}\end{aligned}$$

Next,

$$\bar\mu(c_i\triangle d_i)=\bar\mu(\phi c_i\triangle\phi d_i)=\bar\mu(\phi c_i\triangle c_i^*)\le\epsilon$$

for every i, by (b). Finally, if $i\in\mathbb N$, then ψb_i^* belongs to $\psi[\mathfrak C]$, while $\mathfrak D=\phi^{-1}[\psi[\mathfrak C]]$, so

$$\rho(b_i,\mathfrak D)=\rho(\phi b_i,\psi[\mathfrak C])\le\bar\mu(\phi b_i\triangle\psi b_i^*)=\bar\mu(b_i^*\triangle\psi b_i^*)\le\eta\le\epsilon.$$

This completes the proof.

387H Lemma Let $(\mathfrak A,\bar\mu)$ be an atomless probability algebra and π a measure-preserving automorphism of $\mathfrak A$. Let $\langle b_i\rangle_{i\in\mathbb N}$, $\langle c_i\rangle_{i\in\mathbb N}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak B$, $\mathfrak C$ for the closed subalgebras generated by $\{\pi^jb_i:i\in\mathbb N,\,j\in\mathbb Z\}$ and $\{\pi^jc_i:i\in\mathbb N,\,j\in\mathbb Z\}$. Suppose that $\mathfrak C\subseteq\mathfrak B$. Then for any $\epsilon>0$ we can find a Bernoulli partition $\langle d_i\rangle_{i\in\mathbb N}$ for π such that

(i) $\bar\mu d_i=\bar\mu c_i$ for every $i\in\mathbb N$,

(ii) $\bar\mu(d_i\triangle c_i)\le\epsilon$ for every $i\in\mathbb N$,

(iii) the closed subalgebra of $\mathfrak A$ generated by $\{\pi^jd_i:i\in\mathbb N,\,j\in\mathbb Z\}$ is $\mathfrak B$.

proof (a) To begin with (down to the end of (c) below) suppose that $\mathfrak A=\mathfrak B$. Choose $\langle\epsilon_n\rangle_{n\in\mathbb N}$, $\langle\delta_n\rangle_{n\in\mathbb N}$, $\langle r_n\rangle_{n\in\mathbb N}$ and $\langle\langle d_{ni}\rangle_{i\in\mathbb N}\rangle_{n\in\mathbb N}$ inductively, as follows. Start with $r_0=0$ and $d_{0i}=c_i$ for every i. Given that $\langle d_{ni}\rangle_{i\in\mathbb N}$ is a Bernoulli partition with $\bar\mu d_{ni}=\bar\mu c_i$ for every i, take ϵ_n, $\delta_n>0$ such that

$$(2r_m+1)\epsilon_n\le2^{-n}\text{ for every }m\le n,$$

$$\delta_n \le 2^{-n-1}\epsilon, \quad \sum_{i=0}^{\infty} \min(\delta_n, 2\bar\mu c_i) \le \epsilon_n,$$

and use 387G to find a Bernoulli partition $\langle d_{n+1,i}\rangle_{i\in\mathbb{N}}$ for π such that

$$\bar\mu d_{n+1,i} = \bar\mu c_i, \quad \bar\mu(d_{n+1,i} \mathbin{\triangle} d_{ni}) \le \delta_n, \quad \rho(b_i, \mathfrak{D}^{(n+1)}) \le 2^{-n-1}$$

for every $i \in \mathbb{N}$, where $\mathfrak{D}^{(n+1)}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_{n+1,i} : i \in \mathbb{N},\, j \in \mathbb{Z}\}$. Let r_{n+1} be such that

$$\rho(b_i, \mathfrak{D}^{(n+1)}_{r_{n+1}}) \le 2^{-n}$$

for every $i \in \mathbb{N}$, where $\mathfrak{D}^{(n+1)}_{r_{n+1}}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_{n+1,i} : i \in \mathbb{N},\, |j| \le r_{n+1}\}$. Continue.

(b) For any $i \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} \bar\mu(d_{n+1,i} \mathbin{\triangle} d_{ni}) \le \sum_{n=0}^{\infty} \delta_n \le \epsilon,$$

so $\langle d_{ni}\rangle_{n\in\mathbb{N}}$ has a limit d_i in $\mathfrak{A}$. Of course

$$\bar\mu(c_i \mathbin{\triangle} d_i) \le \sum_{n=0}^{\infty} \bar\mu(d_{n+1,i} \mathbin{\triangle} d_{ni}) \le \epsilon$$

for every i. We must have

$$\bar\mu d_i = \lim_{n\to\infty} \bar\mu d_{ni} = \bar\mu c_i$$

for each i, and if $i \ne j$ then

$$d_i \cap d_j = \lim_{n\to\infty} d_{ni} \cap d_{nj} = 0;$$

since $\sum_{i=0}^{\infty} \bar\mu c_i = 1$, $\langle d_i\rangle_{i\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}$. For any $i(0), \dots , i(k)$ in $\mathbb{N}$,

$$\bar\mu(\inf_{j\le k} \pi^j d_{i(j)}) = \lim_{n\to\infty} \bar\mu(\inf_{j\le k} \pi^j d_{n,i(j)}) = \lim_{n\to\infty} \prod_{j=0}^{k} \bar\mu d_{n,i(j)} = \prod_{j=0}^{k} \bar\mu d_{i(j)},$$

so $\langle d_i\rangle_{i\in\mathbb{N}}$ is a Bernoulli partition.

(c) Let $\mathfrak{D}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_i : i \in \mathbb{N},\, j \in \mathbb{Z}\}$. Then $b_j \in \mathfrak{D}$ for every $j \in \mathbb{N}$. **P** Fix $m \in \mathbb{N}$. Then $\rho(b_j, \mathfrak{D}^{(m+1)}_{r_{m+1}}) \le 2^{-m}$, so there is a $b \in \mathfrak{D}^{(m+1)}_{r_{m+1}}$ such that $\bar\mu(b_j \mathbin{\triangle} b) \le 2^{-m}$. Now

$$\sum_{i=0}^{\infty} \rho(d_{m+1,i}, \mathfrak{D}) \le \sum_{i=0}^{\infty} \bar\mu(d_{m+1,i} \mathbin{\triangle} d_i) \le \sum_{i=0}^{\infty} \sum_{k=m+1}^{\infty} \bar\mu(d_{k+1,i} \mathbin{\triangle} d_{ki})$$
$$\le \sum_{k=m+1}^{\infty} \sum_{i=0}^{\infty} \min(2\bar\mu c_i, \delta_k) \le \sum_{k=m+1}^{\infty} \epsilon_k.$$

So

$$\rho(b, \mathfrak{D}) \le (2r_{m+1} + 1) \sum_{i=0}^{\infty} \rho(d_{m+1,i}, \mathfrak{D})$$

(386Nc)

$$\le \sum_{k=m+1}^{\infty} (2r_{m+1} + 1)\epsilon_k \le \sum_{k=m+1}^{\infty} 2^{-k} = 2^{-m},$$

and

$$\rho(b_j, \mathfrak{D}) \le \bar\mu(b_j \mathbin{\triangle} b) + \rho(b, \mathfrak{D}) \le 2^{-m} + 2^{-m} = 2^{-m+1}.$$

As m is arbitrary, $\rho(b_j, \mathfrak{D}) = 0$ and $b_j \in \mathfrak{D}$. **Q**

(d) This completes the proof if $\mathfrak{A} = \mathfrak{B}$. For the general case, apply the arguments above to $(\mathfrak{B}, \bar\mu\restriction\mathfrak{B}, \pi\restriction\mathfrak{B})$.

387I Ornstein's theorem (finite entropy case) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be probability algebras, and $\pi : \mathfrak{A} \to \mathfrak{A}$, $\phi : \mathfrak{B} \to \mathfrak{B}$ two-sided Bernoulli shifts of the same finite entropy. Then $(\mathfrak{A}, \bar\mu, \pi)$ and $(\mathfrak{B}, \bar\nu, \phi)$ are isomorphic.

proof (a) Let $\langle a_i\rangle_{i\in\mathbb{N}}$, $\langle b_i\rangle_{i\in\mathbb{N}}$ be (two-sided) generating Bernoulli partitions in $\mathfrak{A}$, $\mathfrak{B}$ respectively. By the Kolmogorov-Sinaĭ theorem, $\langle a_i\rangle_{i\in\mathbb{N}}$ and $\langle b_i\rangle_{i\in\mathbb{N}}$ both have entropy equal to $h(\pi) = h(\phi)$. If this entropy is zero, then $\mathfrak{A}$ and $\mathfrak{B}$ are both $\{0, 1\}$, and the result is trivial; so let us assume that $h(\pi) > 0$, so that $\mathfrak{A}$ is atomless (387Bd).

(b) By Sinaĭ's theorem (387E), there is a Bernoulli partition $\langle c_i\rangle_{i\in\mathbb{N}}$ for π such that $\bar{\mu}c_i = \bar{\nu}b_i$ for every $i \in \mathbb{N}$. By 387H, there is a Bernoulli partition $\langle d_i\rangle_{i\in\mathbb{N}}$ for π such that $\bar{\mu}d_i = \bar{\mu}c_i$ for every i and the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_i : i \in \mathbb{N},\ j \in \mathbb{Z}\}$ is $\mathfrak{A}$. But now $(\mathfrak{A}, \bar{\mu}, \pi, \langle d_i\rangle_{i\in\mathbb{N}})$ is isomorphic to $(\mathfrak{B}, \bar{\nu}, \phi, \langle b_i\rangle_{i\in\mathbb{N}})$, so $(\mathfrak{A}, \bar{\mu}, \pi)$ and $(\mathfrak{B}, \bar{\nu}, \phi)$ are isomorphic.

387J Using the same methods, we can extend the last result to the case of Bernoulli shifts of infinite entropy. The first step uses the ideas of 387C, as follows.

Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an ergodic measure-preserving automorphism. Suppose that $\langle a_i\rangle_{i\in I}$ is a finite Bernoulli partition for π, with $\#(I) = r \ge 1$ and $\bar{\mu}a_i = 1/r$ for every $i \in I$, and that $h(\pi) \ge \ln 2r$. Then for any $\epsilon > 0$ there is a Bernoulli partition $\langle b_{ij}\rangle_{i\in I, j\in\{0,1\}}$ for π such that

$$\bar{\mu}(a_i \triangle (b_{i0} \cup b_{i1})) \le \epsilon, \quad \bar{\mu}b_{i0} = \bar{\mu}b_{i1} = \frac{1}{2r}$$

for every $i \in I$.

proof (a) Let $\delta > 0$ be such that

$$\delta + 6\sqrt{4\delta} \le \epsilon.$$

Let $\eta > 0$ be such that

$$\eta < \ln 2, \quad \sqrt{8\eta} \le \delta$$

and

$$|t - \tfrac{1}{2}| \le \delta \text{ whenever } t \in [0,1] \text{ and } q(t) + q(1-t) \ge \ln 2 - 4\eta$$

(385Ad). We have

$$H(A) = rq(\frac{1}{r}) = \ln r,$$

and $\bar{\mu}d = r^{-n}$ whenever $n \in \mathbb{N}$, $d \in D_n(A, \pi)$.

Note that $\mathfrak{A}$ is atomless. **P?** If $a \in \mathfrak{A}$ is an atom, then $\sup_{j\in\mathbb{Z}} \pi^j a = 1$ (because π is ergodic, 372Pb), and $\mathfrak{A}$ is purely atomic, with atoms all of the same size as a; but this means that $H(C) \le \ln(\frac{1}{\bar{\mu}a})$ for every partition of unity $C \subseteq \mathfrak{A}$, so that

$$h(\pi, C) = \lim_{n\to\infty} \frac{1}{n} H(D_n(C, \pi)) \le \lim_{n\to\infty} \frac{1}{n} \ln(\frac{1}{\bar{\mu}a}) = 0$$

for every partition of unity C, and

$$0 = h(\pi) \ge \ln 2r \ge \ln 2. \text{ **XQ**}$$

(b) There is a finite partition of unity $C \subseteq \mathfrak{A}$ such that

$$h(\pi, C) = \ln 2r - \eta,$$

and C refines $A = \{a_i : i \in I\} \setminus \{0\}$. **P** Because $h(\pi) \ge \ln 2r$, there is a finite partition of unity C' such that $h(\pi, C') \ge \ln 2r - \eta$; replacing C' by $C' \vee A$ if need be, we may suppose that C' refines A; take such a C' of minimal size. Because $H(C') \ge h(\pi, C') > H(A)$, there must be distinct c_0, $c_1 \in C'$ included in the same member of A. Because $\mathfrak{A}$ is atomless, the principal ideal generated by c_1 has a closed subalgebra isomorphic, as measure algebra, to the measure algebra of Lebesgue measure on $[0,1]$, up to a scalar multiple of the measure; and in particular there is a family $\langle d_t\rangle_{t\in[0,1]}$ such that $d_s \subseteq d_t$ whenever $s \le t$, $d_1 = c_1$ and $\bar{\mu}d_t = t\bar{\mu}c_1$ for every $t \in [0,1]$. Let D_t be the partition of unity

$$(C' \setminus \{c_0, c_1\}) \cup \{c_0 \cup d_t, c_1 \setminus d_t\}$$

for each $t \in [0,1]$. Then

$$h(\pi, D_1) = h(\pi, (C' \setminus \{c_0, c_1\}) \cup \{c_0 \cup c_1\}) < \ln 2r - \eta,$$

by the minimality of $\#(C')$, while

$$h(\pi, D_0) = h(\pi, C') \ge \ln 2r - \eta.$$

Using 385N, we also have, for any s, $t \in [0,1]$ such that $|s - t| \le \frac{1}{e}$,

$$h(\pi, D_s) - h(\pi, D_t) \le H(D_s|\mathfrak{D}_t)$$

(where $\mathfrak{D}_t$ is the closed subalgebra generated by D_t)

$$\le q(\rho(c_0 \cup d_s, \mathfrak{D}_t)) + q(\rho(c_1 \setminus d_s, \mathfrak{D}_t))$$

(by 386Mb, because $D_s \setminus \mathfrak{D}_t \subseteq \{c_0 \cup d_s, c_1 \setminus d_s\}$)

$$\le q(\bar\mu((c_0 \cup d_s) \triangle (c_0 \cup d_t))) + q(\bar\mu((c_1 \setminus d_s) \triangle (c_1 \setminus d_t)))$$
$$= 2q(\bar\mu(d_s \triangle d_t)) = 2q(|s-t|\bar\mu c_1)$$

because q is monotonic on $[0, |s-t|\bar\mu c_1]$. But this means that $t \mapsto h(\pi, D_t)$ is continuous and there must be some t such that $h(\pi, D_t) = \ln 2r - \eta$; take $C = D_t$. **Q**

(c) Let $\xi > 0$ be such that

$$\xi \le \eta, \quad \xi \le \frac{1}{6}, \quad q(2\xi) + q(1-2\xi) \le \eta, \quad \sum_{c \in C} q(\min(2\xi, \bar\mu c)) \le \eta.$$

Let $n \in \mathbb{N}$ be such that

$$\frac{1}{n+1} \le \xi, \quad q(\frac{1}{n+1}) + q(\frac{n}{n+1}) \le \eta, \quad \bar\mu[\![w_n - h(\pi, C)\chi 1 \ge \eta]\!] \le \xi,$$

where

$$w_n = \frac{1}{n}\sum_{d \in D_n(C,\pi)} \ln(\frac{1}{\bar\mu d})\chi d.$$

(The Shannon-McMillan-Breiman theorem, 386E, assures us that any sufficiently large n has these properties.)

(d) Let D be the set of those $d \in D_n(C, \pi)$ such that

$$\bar\mu d \ge (2r)^{-n}, \quad \text{i.e.,} \quad \frac{1}{n}\ln(\frac{1}{\bar\mu d}) \le \ln 2r.$$

Then $\bar\mu(\sup D) \ge 1 - \xi$, by the choice of n, because $h(\pi, C) = \ln 2r - \eta$. Note that every member of D is included in some member of $D_n(A, \pi)$, because C refines A. If $b \in D_n(A, \pi)$, then $\bar\mu b = r^{-n}$, so $\#(\{d : d \in D,\ d \subseteq b\}) \le 2^n$; we can therefore find a function $f : D \to \{0,1\}^n$ such that f is injective on $\{d : d \in D,\ d \subseteq b\}$ for every $b \in D_n(A, \pi)$.

(e) By 386C(iv), as usual, there is an $a \in \mathfrak{A}$ such that $a, \pi^{-1}a, \dots, \pi^{-n+1}a$ are disjoint and $\bar\mu(a \cap d) = \frac{1}{n+1}\bar\mu d$ for every $d \in D_n(C, \pi)$. Set

$$e = \sup_{d \in D, j<n} \pi^{-j}(a \cap d);$$

then

$$\bar\mu e = \sum_{j=0}^{n-1}\sum_{d \in D}\bar\mu(a \cap d) = \frac{n}{n+1}\bar\mu(\sup D) \ge (1-\xi)^2 \ge 1 - 2\xi.$$

(f) Set

$$c^* = \sup\{\pi^{-j}(a \cap d) : j < n,\ d \in D,\ f(d)(j) = 1\}.$$

(I am identifying members of $\{0,1\}^n$ with functions from $\{0, \dots, n-1\}$ to $\{0,1\}$.) Set

$$A^* = A \vee \{c^*, 1 \setminus c^*\}, \quad A' = A^* \vee \{a, 1 \setminus a\} \vee \{e, 1 \setminus e\},$$

and let $\mathfrak{A}'$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j a : a \in A',\ j \in \mathbb{Z}\}$. Then $a \cap d \in \mathfrak{A}'$ for every $d \in D$. **P** Set

$$\tilde d = \operatorname{upr}(a \cap d, \mathfrak{A}') = \inf\{c : c \in \mathfrak{A}',\ c \supseteq a \cap d\} \in \mathfrak{A}'.$$

Let b be the element of $D_n(A, \pi)$ including d. Because a, b, $e \in \mathfrak{A}'$,

$$\tilde d \subseteq a \cap b \cap e = \sup_{d' \in D} a \cap b \cap d' = \sup\{a \cap d' : d' \in D,\ d' \subseteq b\}.$$

Now if $d' \in D$, $d' \subseteq b$ and $d' \ne d$, then $f(d') \ne f(d)$. Let j be such that $f(d')(j) \ne f(d)(j)$; then $\pi^{-j}(a \cap d)$ is included in one of c^*, $1 \setminus c^*$ and $\pi^{-j}(a \cap d')$ in the other. This means that one of $\pi^j c^*$, $1 \setminus \pi^j c^*$ is a member of $\mathfrak{A}'$ including $a \cap d$ and disjoint from $a \cap d'$, so that $\tilde d \cap d' = 0$. Thus $\tilde d$ must be actually equal to $a \cap d$, and $a \cap d \in \mathfrak{A}'$. **Q**

Next, $c \cap e \in \mathfrak{A}'$ for every $c \in C$. **P** $\langle \pi^{-j}(a \cap d)\rangle_{j<n,\, d \in D}$ is a disjoint family in $\mathfrak{A}'$ with supremum e. But whenever $d \in D$ and $j < n$ we must have $d \subseteq \pi^j c'$ for some $c' \in C$, so either $d \subseteq \pi^j c$ or $d \cap \pi^j c = 0$; thus $\pi^{-j}(a \cap d)$ must be either included in c or disjoint from it. Accordingly

$$c \cap e = \sup\{\pi^{-j}(a \cap d) : j < n,\ d \in D,\ d \subseteq \pi^j c\} \in \mathfrak{A}'. \textbf{ Q}$$

Consequently $h(\pi, A') \geq \ln 2r - 2\eta$. **P** For any $c \in C$,

$$\rho(c, \mathfrak{A}') \leq \bar{\mu}(c \triangle (c \cap e)) = \bar{\mu}(c \setminus e) \leq \min(\bar{\mu}c, 2\xi) \leq \frac{1}{3},$$

so

$$\ln 2r - \eta = h(\pi, C) \leq h(\pi{\restriction}\mathfrak{A}') + H(C|\mathfrak{A}')$$

(386Ld)

$$\leq h(\pi, A') + \sum_{c \in C} q(\rho(c, \mathfrak{A}'))$$

(by the Kolmogorov-Sinaĭ theorem and 386Mb)

$$\leq h(\pi, A') + \sum_{c \in C} q(\min(\bar{\mu}c, 2\xi)) \leq h(\pi, A') + \eta$$

by the choice of ξ. **Q**

Finally, $h(\pi, A^*) \geq \ln 2r - 4\eta$. **P**

$$\ln 2r - 2\eta \leq h(\pi, A') \leq h(\pi, A^*) + H(\{a, 1 \setminus a\}) + H(\{e, 1 \setminus e\})$$

(applying 386Lb twice)

$$= h(\pi, A^*) + q(\bar{\mu}a) + q(1 - \bar{\mu}a) + q(\bar{\mu}e) + q(1 - \bar{\mu}e)$$
$$\leq h(\pi, A^*) + q(\tfrac{1}{n}) + q(\tfrac{n}{n+1}) + q(2\xi) + q(1 - 2\xi)$$
$$\leq h(\pi, A^*) + \eta + \eta = h(\pi, A^*) + 2\eta. \textbf{ Q}$$

(g) We have

$$\ln 2r - 4\eta \leq h(\pi, A^*) \leq H(A^*)$$
$$\leq H(A) + H(\{c^*, 1 \setminus c^*\}) = \ln r + H(\{c^*, 1 \setminus c^*\}) \leq \ln 2r,$$

so

$$q(\bar{\mu}c^*) + q(1 - \bar{\mu}c^*) = H(\{c^*, 1 \setminus c^*\}) \geq \ln 2 - 4\eta.$$

By the choice of η, $|\bar{\mu}c^* - \frac{1}{2}| \leq \delta$.

Next,

$$\textstyle\sum_{i \in I} |\bar{\mu}(a_i \cap c^*) - \frac{1}{2r}| + |\bar{\mu}(a_i \setminus c^*) - \frac{1}{2r}| \leq 3\delta.$$

P By 386I,

$$\sum_{i \in I} |\bar{\mu}(a_i \cap c^*) - \frac{1}{r}\bar{\mu}c^*| + |\bar{\mu}(a_i \setminus c^*) - \frac{1}{r}\bar{\mu}(1 \setminus c^*)|$$
$$\leq \sqrt{2(H(A) + H(\{c^*, 1 \setminus c^*\}) - H(A^*))}$$
$$\leq \sqrt{2(\ln r + \ln 2 - \ln 2r + 4\eta)} = \sqrt{8\eta} \leq \delta.$$

So

$$\sum_{i \in I} |\bar{\mu}(a_i \cap c^*) - \frac{1}{2r}| + |\bar{\mu}(a_i \setminus c^*) - \frac{1}{2r}|$$
$$\leq \sum_{i \in I} \big(|\bar{\mu}(a_i \cap c^*) - \frac{1}{r}\bar{\mu}c^*| + \frac{1}{r}|\bar{\mu}c^* - \frac{1}{2}|$$
$$+ |\bar{\mu}(a_i \setminus c^*) - \frac{1}{r}\bar{\mu}(1 \setminus c^*)| + \frac{1}{r}|\bar{\mu}(1 \setminus c^*) - \frac{1}{2}|\big)$$
$$\leq \delta + |\bar{\mu}c^* - \frac{1}{2}| + |\bar{\mu}(1 \setminus c^*) - \frac{1}{2}| \leq 3\delta. \textbf{ Q}$$

(h) Now apply 387D to the partition of unity A^*, indexed as $\langle a^*_{ij}\rangle_{i\in I,j\in\{0,1\}}$, where $a^*_{i1}=a_i\cap c^*$ and $a^*_{i0}=a_i\setminus c^*$, and $\langle\gamma_{ij}\rangle_{i\in I,j\in\{0,1\}}$, where $\gamma_{ij}=\frac{1}{2r}$ for all i, j. We have

$$\textstyle\sum_{i\in I,j\in\{0,1\}}|\bar\mu a^*_{ij}-\gamma_{ij}|\le 3\delta$$

by (g), while

$$H(A^*)-h(\pi,A^*)\le\ln 2r-\ln 2r+4\eta=4\eta,$$

so

$$\textstyle\sum_{i\in I,j\in\{0,1\}}|\bar\mu a^*_{ij}-\gamma_{ij}|+\sqrt{2(H(A^*)-h(\pi,A^*))}\le 3\delta+\sqrt{8\eta}\le 4\delta.$$

Also

$$\textstyle\sum_{i\in I,j\in\{0,1\}}q(\gamma_{ij})=\ln 2r\le h(\pi).$$

So 387D tells us that there is a Bernoulli partition $\langle b_{ij}\rangle_{i\in I,j\in\{0,1\}}$ for π such that $\bar\mu b^*_{ij}=\frac{1}{2r}$ for all i, j and

$$\textstyle\sum_{i\in I,j\in\{0,1\}}\bar\mu(b_{ij}\mathbin{\triangle}a^*_{ij})\le\delta+6\sqrt{4\delta}\le\epsilon.$$

Now of course

$$\begin{aligned}\sum_{i\in I}\bar\mu(a_i\mathbin{\triangle}(b_{i0}\cup b_{i1}))&\le\sum_{i\in I}\bar\mu((a_i\cap c^*)\mathbin{\triangle}b_{i1})+\bar\mu((a_i\setminus c^*)\mathbin{\triangle}b_{i0})\\&=\sum_{i\in I,j\in\{0,1\}}\bar\mu(a^*_{ij}\mathbin{\triangle}b_{ij})\le\epsilon,\end{aligned}$$

as required.

387K Ornstein's theorem (infinite entropy case) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra of countable Maharam type, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a two-sided Bernoulli shift of infinite entropy. Then $(\mathfrak{A},\bar\mu,\pi)$ is isomorphic to $(\mathfrak{B}_{\mathbb{Z}},\bar\nu_{\mathbb{Z}},\phi)$, where $(\mathfrak{B}_{\mathbb{Z}},\bar\nu_{\mathbb{Z}})$ is the measure algebra of the usual measure on $[0,1]^{\mathbb{Z}}$, and ϕ is the standard two-sided Bernoulli shift on $\mathfrak{B}_{\mathbb{Z}}$ (385Sb).

proof (a) We have to find a root algebra $\mathfrak{C}$ for π which is isomorphic to the measure algebra of Lebesgue measure on $[0,1]$. The materials we have to start with are a root algebra $\mathfrak{A}_0\subseteq\mathfrak{A}$ such that *either* $\mathfrak{A}_0$ is not purely atomic *or* $H(A_0)=\infty$, where A_0 is the set of atoms of $\mathfrak{A}_0$.

Because $\mathfrak{A}$ has countable Maharam type, there is a sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_0$ such that $\{d_n:n\in\mathbb{N}\}$ is dense in the metric of $\mathfrak{A}_0$.

(b) There is a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of partitions of unity in $\mathfrak{A}_0$ such that C_{n+1} refines C_n, $H(C_n)=n\ln 2$ and d_n is a union of members of C_{n+1} for every n. **P** We have

$$\sup\{H(C):C\subseteq\mathfrak{A}_0\text{ is a partition of unity}\}=\infty$$

(385J). Choose the C_n inductively, as follows. Start with $C_0=\{0,1\}$. Given C_n with $H(C_n)=n\ln 2$, set $C'_n=C_n\vee\{d_n,1\setminus d_n\}$; then

$$H(C'_n)\le H(C_n)+H(\{d_n,1\setminus d_n\})\le(n+1)\ln 2.$$

By 386O, there is a partition of unity C_{n+1}, refining C'_n, such that $H(C_{n+1})=(n+1)\ln 2$. Continue. **Q**

(c) For each $n\in\mathbb{N}$, let $\mathfrak{C}_n$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^ja:a\in C_n,\ j\in\mathbb{Z}\}$. Then $\langle\mathfrak{C}_n\rangle_{n\in\mathbb{N}}$ is increasing. For each n, $\pi[\mathfrak{C}_n]=\mathfrak{C}_n$; because $C_n\subseteq\mathfrak{A}_0$, $\pi\restriction\mathfrak{C}_n$ is a Bernoulli shift with generating partition C_n. Accordingly

$$h(\pi\restriction\mathfrak{C}_n)=h(\pi,C_n)=H(C_n)=n\ln 2.$$

Of course $d_n\in\mathfrak{C}_{n+1}$ for every n.

Choose inductively, for each $n\in\mathbb{N}$, $\epsilon_n>0$, $r_n\in\mathbb{N}$ and a Bernoulli partition $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^n}$ in $\mathfrak{C}_n$, as follows. Start with $b_{0\emptyset}=1$. (See 3A1H for the notation I am using here.) Given that $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^n}$ is a Bernoulli partition for π which generates $\mathfrak{C}_n$, in the sense that $\mathfrak{C}_n$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^jb_{n\sigma}:\sigma\in\{0,1\}^n,\ j\in\mathbb{Z}\}$, and $\bar\mu b_{n\sigma}=2^{-n}$ for every σ, take $\epsilon_n>0$ such that

$$(2r_m+1)\epsilon_n\le 2^{-n}\text{ for every }m<n.$$

We know that

$$h(\pi\restriction\mathfrak{C}_{n+1})=(n+1)\ln 2=\ln(2\cdot 2^n).$$

So we can apply 387J to $(\mathfrak{C}_{n+1},\pi\restriction\mathfrak{C}_{n+1})$ to see that there is a Bernoulli partition $\langle b'_{n\tau}\rangle_{\tau\in\{0,1\}^{n+1}}$ for π such that

$$b'_{n\tau}\in\mathfrak{C}_{n+1},\quad \bar\mu b'_{n\tau}=2^{-n-1}$$

for every $\tau\in\{0,1\}^{n+1}$,

$$\bar\mu(b_{n\sigma}\triangle(b'_{n,\sigma^\frown<0>}\cup b'_{n,\sigma^\frown<1>}))\le 2^{-n}\epsilon_n$$

for every $\sigma\in\{0,1\}^n$. By 387H (with $\mathfrak{B}=\mathfrak{C}=\mathfrak{C}_{n+1}$), there is a Bernoulli partition $\langle b_{n+1,\tau}\rangle_{\tau\in\{0,1\}^{n+1}}$ for $\pi\restriction\mathfrak{C}_{n+1}$ such that the closed subalgebra generated by $\{\pi^j b_{n+1,\tau}:\tau\in\{0,1\}^{n+1},\, j\in\mathbb{Z}\}$ is $\mathfrak{C}_{n+1}$, $\bar\mu b_{n+1,\tau}=2^{-n-1}$ for every $\tau\in\{0,1\}^{n+1}$, and

$$\sum_{\tau\in\{0,1\}^{n+1}}\bar\mu(b_{n+1,\tau}\triangle b'_{n\tau})\le\epsilon_n.$$

For each $k\in\mathbb{N}$, let $\mathfrak{B}^{(n+1)}_k$ be the closed subalgebra of $\mathfrak{C}_{n+1}$ generated by $\{\pi^j b_{n+1,\tau}:\tau\in\{0,1\}^{n+1},\,|j|\le k\}$. Since $d_m\in\mathfrak{C}_{m+1}\subseteq\mathfrak{C}_{n+1}$ for every $m\le n$, there is an $r_n\in\mathbb{N}$ such that

$$\rho(d_m,\mathfrak{B}^{(n+1)}_{r_n})\le 2^{-n}\text{ for every }m\le n.$$

Continue.

(d) Fix $m\le n\in\mathbb{N}$ for the moment. For $\sigma\in\{0,1\}^m$, set

$$b_{n\sigma}=\sup\{b_{n\tau}:\tau\in\{0,1\}^n,\ \tau\text{ extends }\sigma\}.$$

(If $n=m$, then of course σ is the unique member of $\{0,1\}^m$ extending itself, so this formula is safe.) Then

$$\bar\mu b_{n\sigma}=2^{-n}\#(\{\tau:\tau\in\{0,1\}^n,\ \tau\text{ extends }\sigma\})=2^{-n}2^{n-m}=2^{-m}.$$

Next, if σ, $\sigma'\in\{0,1\}^m$ are distinct, there is no member of $\{0,1\}^n$ extending both, so $b_{n\sigma}\cap b_{n\sigma'}=0$; thus $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^m}$ is a partition of unity. If $\sigma(0),\dots,\sigma(k)\in\{0,1\}^m$, then

$$\begin{aligned}\bar\mu(\inf_{j\le k}\pi^j b_{n,\sigma(j)})&=\bar\mu(\sup_{\substack{\tau(0),\dots,\tau(k)\in\{0,1\}^n\\ \tau(j)\supseteq\sigma(j)\forall j\le k}}\inf_{j\le k}\pi^j b_{n,\tau(j)})\\&=\sum_{\substack{\tau(0),\dots,\tau(k)\in\{0,1\}^n\\ \tau(j)\supseteq\sigma(j)\forall j\le k}}\bar\mu(\inf_{j\le k}\pi^j b_{n,\tau(j)})\\&=\sum_{\substack{\tau(0),\dots,\tau(k)\in\{0,1\}^n\\ \tau(j)\supseteq\sigma(j)\forall j\le k}}(2^{-n})^{k+1}\\&=(2^{n-m})^{k+1}(2^{-n})^{k+1}=(2^{-m})^{k+1}=\prod_{j=0}^k\bar\mu b_{n,\sigma(j)},\end{aligned}$$

so $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^m}$ is a Bernoulli partition.

(e) If $m\le n\in\mathbb{N}$, then

$$\sum_{\sigma\in\{0,1\}^m}\bar\mu(b_{n\sigma}\triangle b_{n+1,\sigma})\le 2\epsilon_n.$$

P We have

$$\begin{aligned}\sum_{\sigma\in\{0,1\}^m}\bar\mu(b_{n\sigma}\triangle b_{n+1,\sigma})&\le\sum_{\tau\in\{0,1\}^n}\bar\mu(b_{n\tau}\triangle b_{n+1,\tau})\\&=\sum_{\tau\in\{0,1\}^n}\bar\mu(b_{n\tau}\triangle(b_{n+1,\tau^\frown<0>}\cup b_{n+1,\tau^\frown<1>}))\\&\le\sum_{\tau\in\{0,1\}^n}\bar\mu(b_{n\tau}\triangle(b'_{n,\tau^\frown<0>}\cup b'_{n,\tau^\frown<1>}))+\sum_{\upsilon\in\{0,1\}^{n+1}}\bar\mu(b'_{n\upsilon}\triangle b_{n+1,\upsilon})\\&\le\sum_{\tau\in\{0,1\}^n}2^{-n}\epsilon_n+\epsilon_n=2\epsilon_n.\ \mathbf{Q}\end{aligned}$$

(f) In particular, for any $m \in \mathbb{N}$ and $\sigma \in \{0,1\}^m$,

$$\sum_{n=m}^{\infty} \bar{\mu}(b_{n\sigma} \triangle b_{n+1,\sigma}) \le \sum_{n=m}^{\infty} 2\epsilon_n < \infty.$$

So we can define $b_\sigma = \lim_{n\to\infty} b_{n\sigma}$ in $\mathfrak{A}$. We have

$$\bar{\mu}b_\sigma = \lim_{n\to\infty} \bar{\mu}b_{n\sigma} = 2^{-m};$$

and if σ, $\sigma' \in \{0,1\}^m$ are distinct, then

$$b_\sigma \cap b_{\sigma'} = \lim_{n\to\infty} b_{n\sigma} \cap b_{n\sigma'} = 0,$$

so $\langle b_\sigma\rangle_{\sigma\in\{0,1\}^m}$ is a partition of unity in $\mathfrak{A}$. If $\sigma(0),\dots,\sigma(k) \in \{0,1\}^m$, then

$$\begin{aligned}\bar{\mu}(\inf_{j\le k} \pi^j b_{\sigma(j)}) &= \lim_{n\to\infty} \bar{\mu}(\inf_{j\le k} \pi^j b_{n,\sigma(j)})\\ &= \lim_{n\to\infty} \prod_{j=0}^{k} \bar{\mu}b_{n,\sigma(j)} = \prod_{j=0}^{k} \bar{\mu}b_{\sigma(j)},\end{aligned}$$

so $\langle b_\sigma\rangle_{\sigma\in\{0,1\}^m}$ is a Bernoulli partition for π. If $\sigma \in \{0,1\}^m$, then

$$b_{\sigma^\frown<0>} \cup b_{\sigma^\frown<1>} = \lim_{n\to\infty} b_{n,\sigma^\frown<0>} \cup b_{n,\sigma^\frown<1>} = \lim_{n\to\infty} b_{n,\sigma} = b_\sigma.$$

(g) Let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{m\in\mathbb{N}}\{b_\sigma : \sigma \in \{0,1\}^m\}$. Then $\mathfrak{C}$ is atomless and countably generated, so $(\mathfrak{C}, \bar{\mu}\restriction\mathfrak{C})$ is isomorphic to the measure algebra of Lebesgue measure on $[0,1]$. Now $\bar{\mu}(\inf_{j\le k} \pi^j e_j) = \prod_{j=0}^{k} \bar{\mu}e_j$ for all $e_0,\dots,e_k \in \mathfrak{C}$. **P** Let $\epsilon > 0$. For $m \in \mathbb{N}$, let $\mathfrak{C}_m$ be the subalgebra of $\mathfrak{C}$ generated by $\{b_\sigma : \sigma \in \{0,1\}^m\}$. $\langle\mathfrak{C}_m\rangle_{m\in\mathbb{N}}$ is non-decreasing, so $\overline{\bigcup_{m\in\mathbb{N}} \mathfrak{C}_m}$ is a closed subalgebra of $\mathfrak{A}$, and must be $\mathfrak{C}$. Now the function

$$(a_0,\dots,a_k) \to \bar{\mu}(\inf_{j\le k} \pi^j a_j) - \prod_{j=0}^{k} \bar{\mu}a_j : \mathfrak{A}^{k+1} \to \mathbb{R}$$

is continuous and zero on $\mathfrak{C}_m^{k+1}$ for every m, by 387Bb, so is zero on $\mathfrak{C}^{k+1}$, and in particular is zero at $(e_0,\dots,e_k)$, as required. **Q**

By 385Sf, $\langle\pi^j[\mathfrak{C}]\rangle_{j\in\mathbb{Z}}$ is independent.

(h) Let $\mathfrak{B}^*$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_\sigma : \sigma \in \bigcup_{m\in\mathbb{N}}\{0,1\}^m,\ j \in \mathbb{Z}\}$; then $\mathfrak{B}^*$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{j\in\mathbb{Z}} \pi^j[\mathfrak{C}]$. It follows from (e) that, for any $m \in \mathbb{N}$,

$$\begin{aligned}\sum_{\sigma\in\{0,1\}^m} \rho(b_{m\sigma}, \mathfrak{B}^*) &\le \sum_{\sigma\in\{0,1\}^m} \bar{\mu}(b_{m\sigma} \triangle b_\sigma)\\ &\le \sum_{\sigma\in\{0,1\}^m} \sum_{n=m}^{\infty} \bar{\mu}(b_{n\sigma} \triangle b_{n+1,\sigma}) \le 2\sum_{n=m}^{\infty} \epsilon_n.\end{aligned}$$

So if $b \in \mathfrak{B}_{r_m}^{(m+1)}$,

$$\rho(b, \mathfrak{B}^*) \le (2r_m + 1) \sum_{\sigma\in\{0,1\}^{m+1}} \rho(b_{m+1,\sigma}, \mathfrak{B}^*)$$

(386Nc)

$$\le 2(2r_m + 1) \sum_{n=m+1}^{\infty} \epsilon_n \le 2 \sum_{n=m+1}^{\infty} 2^{-n} = 2^{-m+1}.$$

It follows that, whenever $m \le n$ in $\mathbb{N}$,

$$\rho(d_m, \mathfrak{B}^*) \le \rho(d_m, \mathfrak{B}_{r_n}^{(n+1)}) + 2^{-n+1} \le 2^{-n} + 2^{-n+1}$$

by the choice of r_n. Letting $n \to \infty$, we see that $\rho(d_m, \mathfrak{B}^*) = 0$, that is, $d_m \in \mathfrak{B}^*$, for every $m \in \mathbb{N}$. But this means that $\mathfrak{A}_0 \subseteq \mathfrak{B}^*$, by the choice of $\langle d_m\rangle_{m\in\mathbb{N}}$. Accordingly $\pi^j[\mathfrak{A}_0] \subseteq \mathfrak{B}^*$ for every j and $\mathfrak{B}^*$ must be the whole of $\mathfrak{A}$.

(i) Thus π is a two-sided Bernoulli shift with root algebra $\mathfrak{C}$; by 385Sc, $(\mathfrak{A}, \bar{\mu}, \pi)$ is isomorphic to $(\mathfrak{B}_{\mathbb{Z}}, \bar{\nu}_{\mathbb{Z}}, \phi)$.

387L Corollary: Sinaĭ's theorem (general case) Let $(\mathfrak{A},\bar{\mu})$ be an atomless probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving automorphism. Let $(\mathfrak{B},\bar{\nu})$ be a probability algebra of countable Maharam type, and $\phi:\mathfrak{B}\to\mathfrak{B}$ a one- or two-sided Bernoulli shift with $h(\phi)\le h(\pi)$. Then $(\mathfrak{B},\bar{\nu},\phi)$ is isomorphic to a factor of $(\mathfrak{A},\bar{\mu},\pi)$.

proof (a) To begin with (down to the end of (b)) suppose that ϕ is two-sided. Let $\mathfrak{B}_0$ be a root algebra for ϕ. If $\mathfrak{B}_0$ is purely atomic, then there is a generating Bernoulli partition $\langle b_i\rangle_{i\in\mathbb{N}}$ for ϕ of entropy $h(\phi)$. By 387E, there is a Bernoulli partition $\langle c_i\rangle_{i\in\mathbb{N}}$ for π such that $\bar{\mu}c_i=\bar{\nu}b_i$ for every i. Let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j c_i : i\in\mathbb{N},\, j\in\mathbb{Z}\}$. Now $(\mathfrak{C},\bar{\mu}\restriction\mathfrak{C},\pi\restriction\mathfrak{C})$ is a factor of $(\mathfrak{A},\bar{\mu},\pi)$ isomorphic to $(\mathfrak{B},\bar{\nu},\phi)$.

(b) If $\mathfrak{B}_0$ is not purely atomic, then there is still a partition of unity $\langle b_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{B}_0$ of infinite entropy. Again, let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j c_i : i\in\mathbb{N},\, j\in\mathbb{Z}\}$, where $\langle c_i\rangle_{i\in\mathbb{N}}$ is a Bernoulli partition for π such that $\bar{\mu}c_i=\bar{\nu}b_i$ for every i. Now $\pi\restriction\mathfrak{C}$ is a Bernoulli shift of infinite entropy and $\mathfrak{C}$ has countable Maharam type, so 387K tells us that there is a closed subalgebra $\mathfrak{C}_0\subseteq\mathfrak{C}$ such that $\langle\pi^k[\mathfrak{C}_0]\rangle_{k\in\mathbb{N}}$ is independent and $(\mathfrak{C}_0,\bar{\mu}\restriction\mathfrak{C}_0)$ is isomorphic to the measure algebra of Lebesgue measure on $[0,1]$. But $(\mathfrak{B}_0,\bar{\nu}\restriction\mathfrak{B}_0)$ is a probability algebra of countable Maharam type, so is isomorphic to a closed subalgebra $\mathfrak{C}_1$ of $\mathfrak{C}_0$ (332N). Of course $\langle\pi^k[\mathfrak{C}_1]\rangle_{k\in\mathbb{N}}$ is independent, so if we take $\mathfrak{C}_1^*$ to be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{Z}}\pi^k[\mathfrak{C}_1]$, $\pi\restriction\mathfrak{C}_1^*$ will be a two-sided Bernoulli shift isomorphic to ϕ (385Sf).

(c) If ϕ is a one-sided Bernoulli shift, then 385Sa and 385Sc show that $(\mathfrak{B},\bar{\nu},\phi)$ can be represented in terms of a product measure on a space $X^{\mathbb{N}}$ and the standard shift operator on $X^{\mathbb{N}}$. Now this extends naturally to the standard two-sided Bernoulli shift represented by the product measure on $X^{\mathbb{Z}}$, as described in 385Sb (cf. 385Yf); so that $(\mathfrak{B},\bar{\nu},\phi)$ becomes represented as a factor of $(\mathfrak{B}',\bar{\nu}',\phi')$ where ϕ' is a two-sided Bernoulli shift with the same entropy as ϕ (since the entropy is determined by the root algebra, by 385R). By (a)-(b), $(\mathfrak{B}',\bar{\nu}',\phi')$ is isomorphic to a factor of $(\mathfrak{A},\bar{\mu},\pi)$, so $(\mathfrak{B},\bar{\nu},\phi)$ also is.

Remark Thus $(\mathfrak{A},\bar{\mu},\pi)$ has factors which are Bernoulli shifts based on root algebras of all countably-generated types permitted by the entropy of π.

387X Basic exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a one- or two-sided Bernoulli shift. Show that π^n is a Bernoulli shift for any $n\ge 1$. (*Hint*: if $\mathfrak{A}_0$ is a root algebra for π, the closed subalgebra generated by $\bigcup_{j<n}\pi^j[\mathfrak{A}_0]$ is a root algebra for π^n.)

(b) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and $\pi\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ a measure-preserving automorphism such that $\pi[\mathfrak{B}]=\mathfrak{B}$. Show that if π is ergodic or mixing, so is $\pi\restriction\mathfrak{B}$.

(c) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra of countable Maharam type, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a two-sided Bernoulli shift. Show that for any $n\ge 1$ there is a Bernoulli shift $\phi:\mathfrak{A}\to\mathfrak{A}$ such that $\phi^n=\pi$. (*Hint*: construct a Bernoulli shift ψ such that $h(\psi)=\frac{1}{n}h(\pi)$, and use 385Xh and Ornstein's theorem to show that π is isomorphic to ψ^n.)

(d) Let $\langle\alpha_i\rangle_{i\in\mathbb{N}}$, $\langle\beta_i\rangle_{i\in\mathbb{N}}$ be non-negative real sequences such that $\sum_{i=0}^{\infty}\alpha_i=\sum_{i=0}^{\infty}\beta_i=1$ and $\sum_{i=0}^{\infty}q(\alpha_i)=\sum_{i=0}^{\infty}q(\beta_i)$. Let μ_0, ν_0 be the measures on $\mathbb{N}$ defined by the formulae

$$\mu_0E=\sum_{i\in E}\alpha_i,\quad \nu_0E=\sum_{i\in E}\beta_i$$

for $E\subseteq\mathbb{N}$. Set $X=\mathbb{N}^{\mathbb{Z}}$ and let μ, ν be the product measures on X derived from μ_0 and ν_0. Show that there is a permutation $f:X\to X$ such that ν is precisely the image measure μf^{-1} and f is translation-invariant, that is, $f(x\theta)=f(x)\theta$ for every $x\in X$, where $\theta(n)=n+1$ for every $n\in\mathbb{Z}$.

(e) Let $(\mathfrak{A},\bar{\mu},\pi)$ and $(\mathfrak{B},\bar{\nu},\phi)$ be probability algebras of countable Maharam type with two-sided Bernoulli shifts. Suppose that each is isomorphic to a factor of the other. Show that they are isomorphic.

387Y Further exercises (a) Suppose that $(\mathfrak{A},\bar{\mu},\pi)$ and $(\mathfrak{B},\bar{\nu},\phi)$ are probability algebras with one-sided Bernoulli shifts, and that they are isomorphic. Show that they have isomorphic root algebras. (*Hint*: apply the results of §333 to $(\mathfrak{A},\bar{\mu},\pi[\mathfrak{A}])$.)

387 Notes and comments The arguments here are expanded from Smorodinsky 71 and Ornstein 74. I have sought the most direct path to 387I and 387K; of course there is a great deal more to be said (387Xc is a hint), and, in particular, extensions of the methods here provide powerful theorems enabling us to show that automorphisms are Bernoulli shifts. (See Ornstein 74.)

388 Dye's theorem

I have repeatedly said that any satisfactory classification theorem for automorphisms of measure algebras remains elusive. There is however a classification, at least for the Lebesgue measure algebra, of the 'orbit structures' corresponding to measure-preserving automorphisms; in fact, they are defined by the fixed-point subalgebras, which I described in §333. We have to work hard for this result, but the ideas are instructive.

388A Orbit structures I said that this section was directed to a classification of 'orbit structures', without saying what these might be. In fact what I will do is to classify the full subgroups generated by measure-preserving automorphisms of the Lebesgue measure algebra. One aspect of the relation with 'orbits' is the following. (Cf. 381Qc.)

Proposition Let (X,Σ,μ) be a localizable countably separated measure space (definition: 343D), with measure algebra $(\mathfrak{A},\bar{\mu})$. Suppose that f and g are measure space automorphisms from X to itself, inducing measure-preserving automorphisms π, ϕ of $\mathfrak{A}$. Then the following are equiveridical:

(i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(ii) for almost every $x\in X$, there is an $n\in\mathbb{Z}$ such that $g(x)=f^n(x)$;

(iii) for almost every $x\in X$, $\{g^n(x):n\in\mathbb{Z}\}\subseteq\{f^n(x):n\in\mathbb{Z}\}$.

proof (i)⇒(ii) Let $\langle H_k\rangle_{k\in\mathbb{N}}$ be a sequence in Σ which separates the points of X; we may suppose that $H_0=X$. By 381Ib, there is a partition of unity $\langle a_n\rangle_{n\in\mathbb{Z}}$ in $\mathfrak{A}$ such that $\phi c=\pi^n c$ for every $c\subseteq a_n$, $n\in\mathbb{Z}$. For each $n\in\mathbb{Z}$ let $E_n\in\Sigma$ be such that $E_n^\bullet=a_n$; then $Y_0=\bigcup_{n\in\mathbb{Z}}E_n$ is conegligible. The transformation f^n induces π^n, so for any $k\in\mathbb{N}$ and $n\in\mathbb{Z}$ the set

$$\begin{aligned}F_{nk}=\{x:f^n(x)\in E_n\cap H_k,\,g(x)\notin E_n\cap H_k\}\\ \cup\{x:g(x)\in E_n\cap H_k,\,f^n(x)\notin E_n\cap H_k\}\end{aligned}$$

is negligible, and $Y=g^{-1}[Y_0]\setminus\bigcup_{n\in\mathbb{Z},k\in\mathbb{N}}F_{nk}$ is conegligible. Now, for any $x\in Y$, there is some n such that $g(x)\in E_n$, so that $f^n(x)\in E_n$ and $\{k:g(x)\in H_k\}=\{k:f^n(x)\in H_k\}$ and $g(x)=f^n(x)$. As Y is conegligible, (ii) is satisfied.

(ii)⇒(iii) For $x\in X$, set $\Omega_x=\{f^n(x):n\in\mathbb{Z}\}$; we are supposing that $A_0=\{x:g(x)\notin\Omega_x\}$ is negligible. Set $A=\bigcup_{n\in\mathbb{Z}}g^{-n}[A_0]$, so that A is negligible and $g^n(x)\in X\setminus A$ for every $x\in X\setminus A$, $n\in\mathbb{Z}$.

Suppose that $x\in X\setminus A$ and $n\in\mathbb{N}$. Then $g^n(x)\in\Omega_x$. **P** Induce on n. Of course $g^0(x)=x\in\Omega_x$. For the inductive step to $n+1$, $g^n(x)\in\Omega_x\setminus A_0$, so there is a $k\in\mathbb{Z}$ such that $g^n(x)=f^k(x)$. At the same time, there is an $i\in\mathbb{Z}$ such that $g(g^n(x))=f^i(g^n(x))$, so that $g^{n+1}(x)=f^{i+k}(x)\in\Omega_x$. Thus the induction continues. **Q**

Consequently $g^{-n}(x)\in\Omega_x$ whenever $x\in X\setminus A$ and $n\in\mathbb{N}$. **P** Since $g^{-n}(x)\in X\setminus A$, there is a $k\in\mathbb{Z}$ such that $x=g^ng^{-n}(x)=f^kg^{-n}(x)$ and $g^{-n}(x)=f^{-k}(x)\in\Omega_x$. **Q**

Thus $\{g^n(x):n\in\mathbb{Z}\}\subseteq\Omega_x$ for every x in the conegligible set $X\setminus A$.

(iii)⇒(ii) is trivial.

(ii)⇒(i) Set

$$E_n=\{x:g(x)=f^n(x)\}=X\setminus\bigcup_{k\in\mathbb{N}}(g^{-1}[H_k]\triangle f^{-n}[H_k]),$$

for $n\in\mathbb{Z}$. Then (ii) tells us that $\bigcup_{n\in\mathbb{Z}}E_n$ is conegligible, so $\bigcup_{n\in\mathbb{Z}}g[E_n]$ is conegligible. But also each E_n is measurable, so $g[E_n]$ also is, and we can set $a_n=g[E_n]^\bullet$. Now for $y\in g[E_n]$, $y=f^n(g^{-1}(y))$, that is, $g^{-1}(y)=f^{-n}(y)$; so $\phi a=\pi^n a$ for every $a\subseteq a_n$. Since $\sup_{n\in\mathbb{Z}}a_n=1$ in $\mathfrak{A}$, ϕ belongs to the full subgroup generated by π.

Remark Of course the requirement 'countably separated' is essential here; for other measure spaces we can have ϕ and π actually equal without $g(x)$ and $f(x)$ being related for any particular x (see 343I and 343J).

388B Corollary Under the hypotheses of 388A, π and ϕ generate the same full subgroup of $\operatorname{Aut}\mathfrak{A}$ iff $\{f^n(x):n\in\mathbb{Z}\}=\{g^n(x):n\in\mathbb{Z}\}$ for almost every $x\in X$.

388C Extending some ideas from 381M-381N, we have the following fact.

Lemma Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving automorphism; let $\mathfrak{C}$ be its fixed-point subalgebra $\{c:\pi c=c\}$. Let $\langle d_i\rangle_{i\in I}$, $\langle e_i\rangle_{i\in I}$ be two disjoint families in $\mathfrak{A}$ such that $\bar{\mu}(c\cap d_i)=\bar{\mu}(c\cap e_i)$

for every $i \in I$ and $c \in \mathfrak{C}$. Then there is a $\phi \in G_\pi$, the full subgroup of $\mathrm{Aut}\,\mathfrak{A}$ generated by π, such that $\phi d_i = e_i$ for every $i \in I$.

proof Adding $d^* = 1 \setminus \sup_{i\in I} d_i$, $e^* = 1 \setminus \sup_{i\in I} e_i$ to the respective families, we may suppose that $\langle d_i\rangle_{i\in I}$, $\langle e_i\rangle_{i\in I}$ are partitions of unity. Define $\langle a_n\rangle_{n\in\mathbb{N}}$ inductively by the formula

$$a_n = \sup_{i\in I}(d_i \setminus \sup_{m<n} a_m) \cap \pi^{-n}(e_i \setminus \sup_{m<n} \pi^m a_m).$$

Then $a_n \cap d_i \cap a_m = 0$ whenever $m < n$ and $i \in I$, so $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint. Also

$$\pi^n a_n \subseteq \sup_{i\in I} e_i \setminus \sup_{m<n} \pi^m a_m$$

for each n, so $\langle \pi^n a_n\rangle_{n\in\mathbb{N}}$ is disjoint. Note that as $\pi^n(a_n \cap d_j) \subseteq e_j$ for each j,

$$\begin{aligned}\pi^n a_n \cap e_i &= \sup_{j\in I} \pi^n(a_n \cap d_j) \cap e_i = \sup_{j\in I} \pi^n(a_n \cap d_j) \cap e_j \cap e_i \\ &= \pi^n(a_n \cap d_i) \cap e_i = \pi^n(a_n \cap d_i)\end{aligned}$$

for every $i \in I$ and $n \in \mathbb{N}$.

? Suppose, if possible, that $a = 1 \setminus \sup_{n\in\mathbb{N}} a_n$ is non-zero. Then there is an $i \in I$ such that $a \cap d_i \neq 0$. Set $c = \sup_{n\in\mathbb{N}} \pi^n(a \cap d_i)$; then $\pi c \subseteq c$ so $c \in \mathfrak{C}$. Now

$$\begin{aligned}\sum_{n=0}^{\infty} \bar\mu(c \cap e_i \cap \pi^n a_n) &= \sum_{n=0}^{\infty} \bar\mu(c \cap \pi^n(a_n \cap d_i)) = \sum_{n=0}^{\infty} \bar\mu(\pi^n(c \cap a_n \cap d_i)) \\ &= \sum_{n=0}^{\infty} \bar\mu(c \cap a_n \cap d_i) = \bar\mu(c \cap d_i \setminus a) < \bar\mu(c \cap d_i) = \bar\mu(c \cap e_i).\end{aligned}$$

So $b = c \cap e_i \setminus \sup_{n\in\mathbb{N}} \pi^n a_n$ is non-zero, and there is an $n \in \mathbb{N}$ such that $b \cap \pi^n(a \cap d_i)$ is non-zero. But look at $a' = \pi^{-n}(b \cap \pi^n(a \cap d_i))$. We have $0 \neq a' \subseteq a \cap d_i$, so $a' \subseteq d_i \setminus \sup_{m<n} a_m$; while

$$\pi^n a' \subseteq b \subseteq e_i \setminus \sup_{m<n} \pi^m a_m.$$

But this means that $a' \subseteq a_n$, which is absurd. **X**

This shows that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}$. Since

$$\sum_{n=0}^{\infty} \bar\mu(\pi^n a_n) = \sum_{n=0}^{\infty} \bar\mu a_n = \bar\mu 1,$$

$\langle \pi^n a_n\rangle_{n\in\mathbb{N}}$ also is a partition of unity. We can therefore define $\phi \in G_\pi$ by setting $\phi d = \pi^n d$ whenever $n \in \mathbb{N}$ and $d \subseteq a_n$. Now, for any $i \in I$,

$$\phi d_i = \sup_{n\in\mathbb{N}} \phi(d_i \cap a_n) = \sup_{n\in\mathbb{N}} \pi^n(d_i \cap a_n) = \sup_{n\in\mathbb{N}} e_i \cap \pi^n a_n = e_i.$$

So we have found a suitable ϕ.

388D von Neumann automorphisms: Definitions (a) Let $\mathfrak{A}$ be a Boolean algebra and $\pi \in \mathrm{Aut}\,\mathfrak{A}$ an automorphism. π is **weakly von Neumann** if there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $a_0 = 1$ and, for every n, $a_{n+1} \cap \pi^{2^n} a_{n+1} = 0$, $a_{n+1} \cup \pi^{2^n} a_{n+1} = a_n$. In this case, π is **von Neumann** if $\langle a_n\rangle_{n\in\mathbb{N}}$ can be chosen in such a way that $\{\pi^m a_n : m, n \in \mathbb{N}\}$ τ-generates $\mathfrak{A}$, and **relatively von Neumann** if $\langle a_n\rangle_{n\in\mathbb{N}}$ can be chosen so that $\{\pi^m a_n : m, n \in \mathbb{N}\} \cup \{c : \pi c = c\}$ τ-generates $\mathfrak{A}$.

(b) There is another way of looking at automorphisms of this type which will be useful. If $\mathfrak{A}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an automorphism, then a **dyadic cycle system** for π is a finite or infinite family $\langle d_{mi}\rangle_{m\le n, i<2^m}$ or $\langle d_{mi}\rangle_{m\in\mathbb{N}, i<2^m}$ such that (α) for each m, $\langle d_{mi}\rangle_{i<2^m}$ is a partition of unity such that $\pi d_{mi} = d_{m,i+1}$ whenever $i < 2^m - 1$ (so that $\pi d_{m,2^m-1}$ must be d_{m0}) (β) $d_{m0} = d_{m+1,0} \cup d_{m+1,2^m}$ for every $m < n$ (in the finite case) or for every $m \in \mathbb{N}$ (in the infinite case). An easy induction on m shows that if $k \le m$ then

$$d_{ki} = \sup\{d_{mj} : j < 2^m,\ j \equiv i \bmod 2^k\}$$

for every $i < 2^k$.

Conversely, if d is such that $\langle \pi^j d\rangle_{j<2^n}$ is a partition of unity in $\mathfrak{A}$, then we can form a finite dyadic cycle system $\langle d_{mi}\rangle_{m\le n, i<2^m}$ by setting $d_{mi} = \sup\{\pi^j d : j < 2^n,\ j \equiv i \bmod 2^m\}$ whenever $m \le n$ and $j < 2^m$.

(c) Now an automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is weakly von Neumann iff it has an infinite dyadic cycle system $\langle d_{mi}\rangle_{m\in\mathbb{N}, i<2^m}$. (The a_m of (a) correspond to the d_{m0} of (b); starting from the definition in (a), you must check first, by induction on m, that $\langle\pi^i a_m\rangle_{i<2^m}$ is a partition of unity in $\mathfrak{A}$.) π is von Neumann iff it has a dyadic cycle system $\langle d_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ which τ-generates $\mathfrak{A}$.

388E Example The following is the basic example of a von Neumann transformation – in a sense, the only example of a measure-preserving von Neumann transformation. Let μ be the usual measure on $X=\{0,1\}^{\mathbb{N}}$, Σ its domain, and $(\mathfrak{A},\bar\mu)$ its measure algebra. Define $f:X\to X$ by setting

$$\begin{aligned} f(x)(n) &= 1-x(n) \text{ if } x(i)=0 \text{ for every } i<n,\\ &= x(n) \text{ otherwise.}\end{aligned}$$

Then f is a homeomorphism and a measure space automorphism. **P** (i) To see that f is a homeomorphism, perhaps the easiest way is to look at g, where

$$\begin{aligned} g(x)(n) &= 1-x(n) \text{ if } x(i)=1 \text{ for every } i<n,\\ &= x(n) \text{ otherwise,}\end{aligned}$$

and check that f and g are both continuous and that fg and gf are both the identity function. (ii) To see that f is inverse-measure-preserving, it is enough to check that $\mu\{x: f(x)(i)=z(i) \text{ for every } i\le n\}=2^{-n-1}$ for every $n\in\mathbb{N}$, $z\in X$ (254G). But

$$\{x: f(x)(i)=z(i) \text{ for every } i\le n\}=\{x: x(i)=g(z)(i) \text{ for every } i\le n\}.$$

(iii) Similarly, g is inverse-measure-preserving, so f is a measure space automorphism. **Q**

If $n\in\mathbb{N}$, $x\in X$ then

$$\begin{aligned} f^{2^k}(x)(n) &= 1-x(n) \text{ if } n\ge k \text{ and } x(i)=0 \text{ whenever } k\le i<n,\\ &= x(n) \text{ otherwise.}\end{aligned}$$

(Induce on k. For the inductive step, observe that if we identify X with $\{0,1\}\times X$ then $f^2(\epsilon,y)=(\epsilon,f(y))$ for every $\epsilon\in\{0,1\}$ and $y\in X$.)

Let $\pi:\mathfrak{A}\to\mathfrak{A}$ be the corresponding automorphism, setting $\pi E^\bullet=f^{-1}[E]^\bullet$ for $E\in\Sigma$. Then π is a von Neumann automorphism. **P** Set $E_n=\{x: x\in X,\ x(i)=1 \text{ for every } i<n\}$, $a_n=E_n^\bullet$. Then $f^{-2^n}[E_{n+1}]=\{x: x(i)=1$ for $i<n$, $x(n)=0\}$, so a_{n+1} and $\pi^{2^n}a_{n+1}$ split a_n for each n, and $\langle a_n\rangle_{n\in\mathbb{N}}$ witnesses that π is weakly von Neumann. Next, inducing on n, we find that $\{f^{-i}[E_n]: i<2^n\}$ runs over the basic cylinder sets of the form $\{x: x(i)=z(i)$ for every $i<n\}$ determined by coordinates less than n. Since the equivalence classes of such sets τ-generate $\mathfrak{A}$ (see part (a) of the proof of 331K), π is a von Neumann automorphism. **Q**

f is sometimes called the **odometer transformation**. For another way of looking at the functions f and g, see 445Xp in Volume 4.

388F We are now ready to approach the main results of this section.

Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ an aperiodic measure-preserving automorphism. Let $\mathfrak{C}$ be its fixed-point subalgebra. Then for any $a\in\mathfrak{A}$ there is a $b\subseteq a$ such that $\bar\mu(b\cap c)=\frac12\bar\mu(a\cap c)$ for every $c\in\mathfrak{C}$ and π_b is a weakly von Neumann automorphism, writing π_b for the induced automorphism of the principal ideal $\mathfrak{A}_b$, as in 381M.

Remark On first reading, there is something to be said for supposing here that π is ergodic, that is, that $\mathfrak{C}=\{0,1\}$.

proof I should remark straight away that π is doubly recurrent on every $b\in\mathfrak{A}$ (386A), so we have an induced automorphism $\pi_b:\mathfrak{A}_b\to\mathfrak{A}_b$ for every $b\in\mathfrak{A}$ (381M).

(a) Set $\epsilon_n=\frac12(1+2^{-n})$ for each $n\in\mathbb{N}$, so that $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ is strictly decreasing, with $\epsilon_0=1$ and $\lim_{n\to\infty}\epsilon_n=\frac12$. Now there are $\langle b_n\rangle_{n\in\mathbb{N}}$, $\langle d_{ni}\rangle_{n\in\mathbb{N},i<2^n}$ such that, for each $n\in\mathbb{N}$,

$$b_{n+1}\subseteq b_n\subseteq a,\quad \bar\mu(b_n\cap c)=\epsilon_n\bar\mu(a\cap c) \text{ for every } c\in\mathfrak{C},$$

$$\langle d_{ni}\rangle_{i<2^n} \text{ is disjoint},\quad \sup_{i<2^n} d_{ni}=b_n,$$

$$\pi_{b_n}d_{ni}=d_{n,i+1} \text{ for every } i<2^n-1,$$

$$b_{n+1}\cap d_{ni}=d_{n+1,i}\cup d_{n+1,i+2^n} \text{ for every } i<2^n.$$

P Start with $b_0 = d_{00} = a$. To construct b_{n+1} and $\langle d_{n+1,i}\rangle_{i<2^{n+1}}$, given $\langle d_{ni}\rangle_{i<2^n}$, note first that (because π_{b_n} is measure-preserving and $\pi_{b_n}(c \cap d) = c \cap \pi_{b_n} d$ for every $d \subseteq b_n$, see 381Nf) $\bar\mu(d_{n0} \cap c) = \bar\mu(d_{ni} \cap c)$ whenever $c \in \mathfrak{C}$, $i < 2^n$, so

$$\bar\mu(d_{n0} \cap c) = 2^{-n}\bar\mu(b_n \cap c) = 2^{-n}\epsilon_n\bar\mu(a \cap c)$$

for every $c \in \mathfrak{C}$, and

$$d_{n0} = b_n \setminus \sup_{i<2^n-1} \pi_{b_n} d_{ni} = \pi_{b_n} d_{n,2^n-1} = \pi_{b_n}^{2^n} d_{n0}.$$

Now π_{b_n} is aperiodic (381Ng) so $\pi_{b_n}^{2^n}$ also is (381Bd), and there is a $d_{n+1,0} \subseteq d_{n0}$ such that

$$\pi_{b_n}^{2^n} d_{n+1,0} \cap d_{n+1,0} = 0, \quad \bar\mu(d_{n+1,0} \cap c) = 2^{-n-1}\epsilon_{n+1}\bar\mu(a \cap c) \text{ for every } c \in \mathfrak{C}$$

(applying 386C(iii) to $\pi_{b_n}^{2^n}\restriction\mathfrak{A}_{d_{n0}}$, with $\gamma = \epsilon_{n+1}/2\epsilon_n$). Set $d_{n+1,j} = \pi_{b_n}^j d_{n+1,0}$ for each $j < 2^{n+1}$. Because $\pi_{b_n}^{2^n} d_{n+1,0} \subseteq d_{n0} \setminus d_{n+1,0}$, while $\langle\pi_{b_n}^j d_{n0}\rangle_{j<2^n}$ is disjoint, $\langle\pi_{b_n}^j d_{n+1,0}\rangle_{j<2^{n+1}}$ is disjoint. Set $b_{n+1} = \sup_{i<2^{n+1}} \pi_{b_n}^i d_{n+1,0}$; then $b_{n+1} \subseteq b_n$ and $\bar\mu(b_{n+1} \cap c) = \epsilon_{n+1}\bar\mu(a \cap c)$ for every $c \in \mathfrak{C}$. For $j < 2^{n+1}$, $d_{n+1,j} \subseteq d_{ni}$ where i is either j or $j - 2^n$, so $b_{n+1} \cap d_{ni} = d_{n+1,i} \cup d_{n+1,i+2^n}$.

For $i < 2^{n+1} - 1$,

$$\pi_{b_n} d_{n+1,i} = d_{n+1,i+1} \subseteq b_{n+1},$$

so we must also have

$$\pi_{b_{n+1}} d_{n+1,i} = (\pi_{b_n})_{b_{n+1}} d_{n+1,i} = d_{n+1,i+1}$$

(using 381Ne). Thus the induction continues. **Q**

(b) Set

$$b = \inf_{n\in\mathbb{N}} b_n, \quad e_{ni} = b \cap d_{ni} \text{ for } n \in \mathbb{N},\ i < 2^n.$$

Because $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-increasing,

$$\bar\mu(b \cap c) = \lim_{n\to\infty} \bar\mu(b_n \cap c) = \frac{1}{2}\bar\mu(a \cap c)$$

for every $c \in \mathfrak{C}$. Next,

$$e_{ni} = b \cap b_{n+1} \cap d_{ni} = b \cap (d_{n+1,i} \cup d_{n+1,i+2^n}) = e_{n+1,i} \cup e_{n+1,i+2^n}$$

whenever $i < 2^n$.

If $m \le n$, $j < 2^m$ then

$$b_n \cap d_{mj} = \sup\{d_{ni} : i < 2^n,\ i \equiv j \bmod 2^m\}$$

(induce on n). So

$$\bar\mu(b_n \cap d_{mj}) = 2^{n-m}\bar\mu d_{n0} = 2^{-m}\epsilon_n;$$

taking the limit as $n \to \infty$, $\bar\mu e_{mj} = 2^{-m}\bar\mu b$. Next,

$$\begin{aligned}\pi_{b_n}(b_n \cap d_{mj}) &= \sup\{d_{n,i+1} : i < 2^n,\ i \equiv j \bmod 2^m\} \\ &= \sup\{d_{ni} : i < 2^n,\ i \equiv j + 1 \bmod 2^m\} = b_n \cap d_{m,j+1},\end{aligned}$$

here interpreting $d_{n,2^n}$ as d_{n0}, $d_{m,2^m}$ as d_{m0}. Consequently $\pi_b e_{mj} \subseteq e_{m,j+1}$. **P?** Otherwise, there are a non-zero $e \subseteq d_{mj} \cap b$ and $k \ge 1$ such that $\pi^i e \cap b = 0$ for $1 \le i < k$ and $\pi^k e \subseteq b \setminus d_{m,j+1}$. Take $n \ge m$ so large that $\bar\mu e > k\bar\mu(b_n \setminus b)$, so that

$$e' = e \setminus \sup_{1\le i<k} \pi^{-i}(b_n \setminus b) \ne 0;$$

now $\pi^i e' \cap b_n = 0$ for $1 \le i < k$, while $\pi^k e' \subseteq b_n$, and

$$\pi_{b_n} e' = \pi^k e' \subseteq 1 \setminus d_{m,j+1}.$$

But this means that $\pi_{b_n}(b_n \cap d_{mj}) \not\subseteq d_{m,j+1}$, which is impossible. **XQ**

Since $\bar\mu(\pi_b e_{mj}) = \bar\mu e_{m,j+1}$, we must have $\pi_b e_{mj} = e_{m,j+1}$. And this is true whenever $m \in \mathbb{N}$ and $j < 2^m$, if we identify $e_{m,2^m}$ with e_{m0}. Thus $\langle e_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ is a dyadic cycle system for π_b and π_b is a weakly von Neumann automorphism.

388G Lemma Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra and π, ψ two measure-preserving automorphisms of $\mathfrak{A}$. Suppose that ψ belongs to the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π and that there is a $b\in\mathfrak{A}$ such that $\sup_{n\in\mathbb{Z}}\psi^n b=1$ and the induced automorphisms ψ_b, π_b on $\mathfrak{A}_b$ are equal. Then $G_\psi=G_\pi$.

proof (a) The first fact to note is that if $0\ne b'\subseteq b$, $n\in\mathbb{Z}$ and $\pi^n b'\subseteq b$, then there are $m\in\mathbb{Z}$, $b''\subseteq b'$ such that $b''\ne 0$ and $\pi^n d=\psi^m d$ for every $d\subseteq b''$. **P** (α) If $n=0$ take $b''=b'$, $m=0$. (β) Next, suppose that $n>0$. We have $0\ne b'\subseteq b\cap\pi^{-n}b$, so by 381Nc there are i, b'_1 such that $1\le i\le n$, $0\ne b'_1\subseteq b'$ and $\pi^n d=\pi_b^i d$ for every $d\subseteq b'_1$. Now by 381Nb there are a non-zero $b''\subseteq b'_1$ and an $m\ge i$ such that $\psi_b^i d=\psi^m d$ for every $d\subseteq b''$; so that $\pi^n d=\psi^m d$ for every $d\subseteq b''$. (γ) If $n<0$, then apply (β) to π^{-1} and ψ^{-1}, recalling that $(\pi^{-1})_b=\pi_b^{-1}=\psi_b^{-1}=(\psi^{-1})_b$ (381Na). **Q**

(b) Now take any non-zero $a\in\mathfrak{A}$. Then there are m, $n\in\mathbb{Z}$ such that $a_1=a\cap\psi^m b\ne 0$, $a_2=\pi a_1\cap\psi^n b\ne 0$. Set $b_1=\psi^{-m}\pi^{-1}a_2$. Because $\psi\in G_\pi$, there are a non-zero $b_2\subseteq b_1$ and a $k\in\mathbb{Z}$ such that $\psi^{-n}\pi\psi^m d=\pi^k d$ for every $d\subseteq b_2$. Now

$$\pi^k b_2=\psi^{-n}\pi\psi^m b_2\subseteq\psi^{-n}\pi\psi^m b_1=\psi^{-n}a_2\subseteq b.$$

By (a), there are a non-zero $b_3\subseteq b_2$ and an $r\in\mathbb{Z}$ such that $\pi^k d=\psi^r d$ for every $d\subseteq b_3$. Consider $a'=\psi^m b_3$. Then

$$0\ne a'\subseteq\psi^m b_1=\pi^{-1}a_2\subseteq a_1\subseteq a;$$

and, for $d\subseteq a'$, $\psi^{-m}d\subseteq b_3\subseteq b_2$, so that

$$\pi d=\psi^n(\psi^{-n}\pi\psi^m)\psi^{-m}d=\psi^n\pi^k\psi^{-m}d=\psi^{n+r-m}d.$$

As a is arbitrary, this shows that $\pi\in G_\psi$, so that $G_\pi\subseteq G_\psi$ and the two are equal.

388H Lemma Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra, $\pi:\mathfrak{A}\to\mathfrak{A}$ an aperiodic measure-preserving automorphism, and ϕ any member of the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π. Suppose that $\langle d_{mi}\rangle_{m\le n,i<2^m}$ is a finite dyadic cycle system for ϕ. Then there is a weakly von Neumann automorphism ψ, with dyadic cycle system $\langle d'_{mi}\rangle_{m\in\mathbb{N},i<2^m}$, such that $G_\psi=G_\pi$, $\psi a=\phi a$ whenever $a\cap d_{n0}=0$, and $d'_{mi}=d_{mi}$ whenever $m\le n$ and $i<2^m$.

proof Write $\mathfrak{C}$ for the closed subalgebra $\{c:\pi c=c\}$. By 388F there is a $b\subseteq d_{n0}$ such that $\bar{\mu}(b\cap c)=\frac{1}{2}\bar{\mu}(d_{n0}\cap c)$ for every $c\in\mathfrak{C}$ and $\pi_b:\mathfrak{A}_b\to\mathfrak{A}_b$ is a weakly von Neumann automorphism. Let $\langle e_{ki}\rangle_{k\in\mathbb{N},i<2^k}$ be a dyadic cycle system for π_b.

If we define $\psi_1\in\operatorname{Aut}\mathfrak{A}$ by setting

$$\psi_1 d=\pi_b d\text{ for }d\subseteq b,\quad \psi_1 d=\pi_{1\setminus b}d\text{ for }d\subseteq 1\setminus b,$$

then $\psi_1\in G_\pi$. Next, for any $c\in\mathfrak{C}$,

$$\bar{\mu}(\phi^{-2^n+1}b\cap c)=\bar{\mu}\phi^{-2^n+1}(b\cap c)=\bar{\mu}(b\cap c)=\frac{1}{2}\bar{\mu}(d_{n0}\cap c)=\bar{\mu}((d_{n0}\setminus b)\cap c)$$

because $\phi^{-2^n+1}\in G_\pi$, so $\phi^{-2^n+1}c=c$. By 388C, there is a $\psi_2\in G_\pi$ such that $\psi_2(d_{n0}\setminus b)=\phi^{-2^n+1}b$. Set $\psi_3=\phi^{-2^n+1}\psi_2^{-1}\phi^{-2^n+1}\psi_1$, so that $\psi_3\in G_\pi$ and

$$\psi_3 b=\phi^{-2^n+1}\psi_2^{-1}\phi^{-2^n+1}b=\phi^{-2^n+1}(d_{n0}\setminus b).$$

Thus $\psi_3 b$ and $\psi_2(d_{n0}\setminus b)$ are disjoint and have union $\phi^{-2^n+1}d_{n0}=d_{n1}$ (if $n=0$, we must read d_{01} as $d_{00}=1$). Accordingly we can define $\psi\in G_\pi$ by setting

$$\begin{aligned}\psi d&=\psi_3 d\text{ if }d\subseteq b,\\&=\psi_2 d\text{ if }d\subseteq d_{n0}\setminus b,\\&=\phi d\text{ if }d\cap d_{n0}=0.\end{aligned}$$

Since $\psi d_{n0}=d_{n1}$, we have $\psi d_{ni}=\phi d_{ni}$ for every $i<2^n$, and therefore $\psi^i d_{m0}=d_{mi}$ whenever $m\le n$ and $i<2^m$. Looking at ψ^{2^n}, we have

$$\psi^{2^n}d_{n0}=\phi^{2^n}d_{n0}=d_{n0},\quad \psi^{2^n}b=\phi^{2^n-1}\psi_3 b=d_{n0}\setminus b,$$

so that $\psi^{2^n}(d_{n0}\setminus b)=b$ and $\psi^{2^{n+1}}b=b$. Accordingly

$$\psi^{2^{n+1}}d=\phi^{2^n-1}\psi_2\phi^{2^n-1}\psi_3 d=\psi_1 d=\pi_b d$$

for every $d\subseteq b$. Also $\sup_{i<2^{n+1}}\psi^i b=1$, so 388G tells us that $G_\psi=G_\pi$.

Now define $\langle a_m\rangle_{m\in\mathbb{N}}$ as follows. For $m\le n$, $a_m=d_{m0}$; for $m>n$, $a_m=e_{m-n-1,0}$. Then for $m<n$ we have

$$\psi^{2^m}a_{m+1}=\psi^{2^m}d_{m+1,0}=\phi^{2^m}d_{m+1,0}=d_{m+1,2^m}=a_m\setminus a_{m+1},$$

for $m=n$ we have

$$\psi^{2^n}a_{n+1}=\psi^{2^n}e_{00}=\psi^{2^n}b=d_{n0}\setminus b=a_n\setminus a_{n+1},$$

and for $m>n$ we have

$$\begin{aligned}\psi^{2^m}a_{m+1}&=(\psi^{2^{n+1}})^{2^{m-n-1}}e_{m-n,0}=(\pi_b)^{2^{m-n-1}}e_{m-n,0}\\&=e_{m-n,2^{m-n-1}}=e_{m-n-1,0}\setminus e_{m-n,0}=a_m\setminus a_{m+1}.\end{aligned}$$

Thus $\langle a_m\rangle_{m\in\mathbb{N}}$ witnesses that ψ is a weakly von Neumann automorphism. If $d'_{mi}=\psi^i a_m$ for $m\in\mathbb{N}$, $i<2^m$ then $\langle d'_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ will be a dyadic cycle system for ψ and $d'_{mi}=d_{mi}$ for $m\le n$, as required.

388I Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$. For $a\in\mathfrak{A}$ write $\mathfrak{C}_a=\{a\cap c:c\in\mathfrak{C}\}$.

(a) Suppose that $b\in\mathfrak{A}$, $w\in\mathfrak{C}$ and $\delta>0$ are such that $\bar\mu(b\cap c)\ge\delta\bar\mu c$ whenever $c\in\mathfrak{C}$ and $c\subseteq w$. Then there is an $e\in\mathfrak{A}$ such that $e\subseteq b\cap w$ and $\bar\mu(e\cap c)=\delta\bar\mu c$ whenever $c\in\mathfrak{C}_w$.

(b) Suppose that $k\ge1$ and that $(b_0,\dots,b_r)$ is a finite partition of unity in $\mathfrak{A}$. Then there is a partition E of unity in $\mathfrak{A}$ such that

$$\bar\mu(e\cap c)=\frac{1}{k}\bar\mu c\text{ for every }e\in E,\ c\in\mathfrak{C},$$

$$\#(\{e:e\in E,\exists\, i\le r,\ b_i\cap e\notin\mathfrak{C}_e\})\le r+1.$$

proof (a) Set $a=b\cap w$ and consider the principal ideal $\mathfrak{A}_a$ generated by $\mathfrak{A}$. We know that $(\mathfrak{A}_a,\bar\mu\restriction\mathfrak{A}_a)$ is a totally finite measure algebra (322H), and that $\mathfrak{C}_a$ is a closed subalgebra of $\mathfrak{A}_a$ (333Bc); and it is easy to see that $\mathfrak{A}_a$ is relatively atomless over $\mathfrak{C}_a$.

Let $\theta:\mathfrak{C}_w\to\mathfrak{C}_a$ be the Boolean homomorphism defined by setting $\theta c=c\cap b$ for $c\in\mathfrak{C}_w$. If $c\in\mathfrak{C}_w$ and $\theta c=0$, then $c\in\mathfrak{C}$ and $\delta\bar\mu c\le\bar\mu(c\cap b)=0$, so $c=0$; thus θ is injective; since it is certainly surjective, it is a Boolean isomorphism. We can therefore define a functional $\nu=\bar\mu\theta^{-1}:\mathfrak{C}_a\to[0,\infty[$, and we shall have $\delta\nu d\le\bar\mu d$ for every $d\in\mathfrak{C}_a$. By 331B, there is an $e\in\mathfrak{A}_a$ such that $\delta\nu d=\bar\mu(d\cap e)$ for every $d\in\mathfrak{C}_a$, that is, $\delta\bar\mu c=\bar\mu(c\cap e)$ for every $c\in\mathfrak{C}_w$, as required.

(b)(i) Write D for the set of all those $e\in\mathfrak{A}$ such that $\bar\mu(c\cap e)=\frac{1}{k}\bar\mu c$ for every $c\in\mathfrak{C}$ and $b_i\cap e\in\mathfrak{C}_e$ for every $i\le r$. Then whenever $a\in\mathfrak{A}$ and $\gamma>\frac{r+1}{k}$ is such that $\mu(a\cap c)=\gamma\mu c$ for every $c\in\mathfrak{C}$, there is an $e\in D$ such that $e\subseteq a$. **P** For $d\in\mathfrak{A}$, $c\in\mathfrak{C}$ set $\nu_d(c)=\bar\mu(d\cap c)$, so that $\nu_d:\mathfrak{C}\to[0,\infty[$ is a completely additive functional. For $i\le r$ set $v_i=[\![\bar\mu\restriction\mathfrak{C}>k\nu_{a\cap b_i}]\!]$, in the notation of 326T; so that $v_i\in\mathfrak{C}$ and $\bar\mu c\ge k\mu(a\cap b_i\cap c)$ whenever $c\in\mathfrak{C}$ and $c\subseteq v_i$, while $\bar\mu c\le k\bar\mu(a\cap b_i\cap c)$ whenever $c\in\mathfrak{C}$ and $c\cap v_i=0$. Setting $v=\inf_{i\le r}v_i$, we have

$$k\gamma\bar\mu v=k\bar\mu(a\cap v)=\sum_{i=0}^r k\mu(a\cap b_i\cap v)\le(r+1)\bar\mu v.$$

Since $k\gamma>r+1$, $v=0$. So if we now set $w_i=(\inf_{j<i}v_j)\setminus v_i$ for $i\le r$ (starting with $w_0=1\setminus v_0$), $(w_0,\dots,w_r)$ is a partition of unity in $\mathfrak{C}$, and $\bar\mu c\le k\bar\mu(a\cap b_i\cap c)$ whenever $c\in\mathfrak{C}$ and $c\subseteq w_i$.

By (a), we can find for each $i\le r$ an $e_i\in\mathfrak{A}$ such that $e_i\subseteq a\cap b_i\cap w_i$ and $\bar\mu(c\cap e_i)=\frac{1}{k}\bar\mu c$ whenever $c\in\mathfrak{C}$ and $c\subseteq w_i$. Set $e=\sup_{i\le r}e_i$, so that $e\subseteq a$,

$$e\cap b_i=e\cap w_i\cap b_i=e_i=e\cap w_i\in\mathfrak{C}_e$$

for each i, and

$$\bar\mu(c\cap e)=\sum_{i=0}^r\bar\mu(c\cap e_i)=\sum_{i=0}^r\bar\mu(c\cap w_i\cap e_i)=\sum_{i=0}^r\frac{1}{k}\bar\mu(c\cap w_i)=\frac{1}{k}\bar\mu c$$

for every $c\in\mathfrak{C}$. So e has all the properties required. **Q**

(ii) Let $E_0\subseteq D$ be a maximal disjoint family, and set $m=\#(E_0)$, $a=1\setminus\sup E_0$. Then

$$\bar\mu(a\cap c)=\bar\mu c-\sum_{e\in E_0}\bar\mu(c\cap e)=(1-\frac{m}{k})\bar\mu c$$

for every $c\in\mathfrak{C}$, while a does not include any member of D. By (i), $1-\frac{m}{k}\le\frac{r+1}{k}$, that is, $k-m\le r-1$.

Applying (a) repeatedly, with $w = 1$ and $\delta = \frac{1}{k}$, we can find disjoint $d_0, \dots, d_{k-m-1} \subseteq a$ such that $\bar\mu(c \cap d_i) = \frac{1}{k}\bar\mu c$ for every $c \in \mathfrak{C}$ and $i < k - m$. So if we set $E = E_0 \cup \{d_i : i < k - m\}$ we shall have a partition of unity with the properties required.

388J Lemma Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism, with fixed-point subalgebra $\mathfrak{C}$. Suppose that ϕ is a member of the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π with a finite dyadic cycle system $\langle d_{mi}\rangle_{m\le n, i<2^m}$, and that $a \in \mathfrak{A}$ and $\epsilon > 0$. Then there is a $\psi \in G_\pi$ such that

(i) ψ has a dyadic cycle system $\langle d'_{mi}\rangle_{m\le k, i<2^m}$, with $k \ge n$ and $d'_{mi} = d_{mi}$ for $m \le n$, $i < 2^m$;

(ii) $\psi d = \phi d$ if $d \cap d_{n0} = 0$;

(iii) there is an a' in the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{d'_{ki} : i < 2^k\}$ such that $\bar\mu(a \,\triangle\, a') \le \epsilon$.

proof (a) Take $k \ge n$ so large that $2^k\epsilon \ge 2^n 2^{2^n}\bar\mu 1$. Let $\mathfrak{D}$ be the subalgebra of the principal ideal $\mathfrak{A}_{d_{n1}}$ generated by $\{d_{n1} \cap \phi^{-j}a : j < 2^n\}$; then $\mathfrak{D}$ has atoms $b_0, \dots, b_r$ where $r < 2^{2^n}$. (If $n = 0$, take $d_{01} = d_{00} = 1$.) Applying 388I to the closed subalgebra $\mathfrak{C}_{d_{n1}}$ of $\mathfrak{A}_{d_{n1}}$, we can find a partition of unity E of $\mathfrak{A}_{d_{n1}}$ such that

$$\bar\mu(e \cap c) = 2^{-k}\bar\mu c \text{ for every } e \in E,\, c \in \mathfrak{C},$$

$$E_1 = \{e : e \in E, \text{ there is some } i \le r \text{ such that } b_i \cap e \notin \mathfrak{C}_e\}$$

has cardinal at most $r + 1 \le 2^{2^n}$. Of course $\bar\mu e = 2^{-k}$ for every $e \in E$, so $\#(E) = 2^{k-n}$ and $\bar\mu(\sup E_1) \le 2^{-k}2^{2^n} \le 2^{-n}\epsilon$. Write e^* for $\sup E_1$.

(b) For $e \in E$ set $e' = \phi^{2^n-1}e$; then $\{e' : e \in E\}$ is a disjoint family, of cardinal 2^{k-n}; enumerate it as $\langle v_i\rangle_{i<2^{k-n}}$. Note that

$$\sup_{i<2^{k-n}} v_i = \phi^{2^n-1}(\sup E) = d_{n0},$$

$$\bar\mu(v_i \cap c) = \bar\mu(\phi^{-2^n+1}v_i \cap c) = 2^{-k}\bar\mu c$$

for every $c \in \mathfrak{C}$ and $i < 2^{k-n}$. There is therefore a $\psi_1 \in G_\pi$ such that

$$\psi_1 v_i = \phi^{-2^n+1}v_{i+1} \text{ for } i < 2^{k-n} - 1, \quad \psi_1 v_{2^{k-n}-1} = \phi^{-2^n+1}v_0$$

(388C). We have

$$\begin{aligned}\psi_1 d_{n0} = \psi_1(\sup_{i<2^{k-n}} v_i) &= \sup_{i<2^{k-n}} \psi_1 v_i = \sup_{i<2^{k-n}-1} \phi^{-2^n+1}v_{i+1} \cup \phi^{-2^n+1}v_0 \\ &= \sup_{i<2^{k-n}} \phi^{-2^n+1}v_i = \phi^{-2^n+1}d_{n0} = d_{n1} = \phi d_{n0}.\end{aligned}$$

So we may define $\psi \in G_\pi$ by setting

$$\begin{aligned}\psi d &= \psi_1 d \text{ if } d \subseteq d_{n0}, \\ &= \phi d \text{ if } d \cap d_{n0} = 0.\end{aligned}$$

(c) For each $i < 2^{k-n}$,

$$\psi^{2^n}v_i = \phi^{2^n-1}\psi_1 v_i = v_{i+1}$$

(identifying $v_{2^{k-n}}$ with v_0). Moreover, $\psi^j v_i \subseteq d_{nl}$ whenever $i < 2^{k-n}$ and $j \equiv l \bmod 2^n$. So $\langle\psi^j v_0\rangle_{j<2^k}$ is a partition of unity in $\mathfrak{A}$. What this means is that if we set

$$d'_{mj} = \sup\{\psi^i v_0 : i < 2^k,\, i \equiv j \bmod 2^m\}$$

for $m \le k$, then $\langle d'_{mj}\rangle_{m\le k, j<2^m}$ is a dyadic cycle system for ψ, with $d'_{mj} = d_{mj}$ if $m \le n$, $j < 2^m$.

(d) Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{d'_{kj} : j < 2^k\}$. Recall the definition of $\{v_i : i < 2^{k-n}\}$ as $\{\phi^{2^n-1}e : e \in E\}$; this implies that

$$\{\psi v_i : i < 2^{k-n}\} = \{\psi_1 v_i : i < 2^{k-n}\} = \{\phi^{-2^n+1}v_i : i < 2^{k-n}\} = E,$$

so that

$$\{\psi^{j+1}v_i : i < 2^{k-n}\} = \{\phi^j e : e \in E\}$$

for $j < 2^n$, and

$$\mathfrak{B} \supseteq \{d'_{kj} : j < 2^k\} = \{\psi^j v_i : i < 2^{k-n},\, j < 2^n\} = \{\phi^j e : e \in E,\, j < 2^n\}.$$

Set $E_0 = E \setminus E_1$. For $e \in E_0$ and $i \le r$ there is a $c_{ei} \in \mathfrak{C}$ such that $e \cap b_i = e \cap c_{ei}$. Set

$$K = \{(i,j) : 1 \le i \le r,\, j < 2^n,\, b_i \subseteq \phi^{-j} a\},$$

$$a' = \sup\{\phi^j e \cap c_{ei} : e \in E_0,\, (i,j) \in K\}.$$

Then a' is a supremum of (finitely many) members of $\mathfrak{B}$, so belongs to $\mathfrak{B}$. If $(i,j) \in K$ and $e \in E_0$, then

$$\phi^j e \cap c_{ei} = \phi^j (e \cap c_{ei}) = \phi^j (e \cap b_i) \subseteq a,$$

so $a' \subseteq a$. Next, $d_{n1} \cap \phi^{-j}(a \setminus a') \subseteq e^*$ for each $j < 2^n$. **P** Set

$$I = \{i : i \le r,\, (i,j) \in K\} = \{i : b_i \subseteq \phi^{-j} a\};$$

then $d_{n1} \cap \phi^{-j} a = \sup_{i \in I} b_i$. Now, for each $i \in I$,

$$b_i = \sup_{e \in E}(b_i \cap e) \subseteq \sup_{e \in E_0}(e \cap c_{ei}) \cup e^*,$$

so that

$$d_{n1} \cap \phi^{-j} a = \sup_{i \in I} b_i \subseteq \sup_{e \in E_0, i \in I}(e \cap c_{ei}) \cup e^* = (d_{n1} \cap \phi^{-j} a') \cup e^*. \text{ } \mathbf{Q}$$

But this means that

$$\bar\mu(d_{n,j+1} \cap a \setminus a') = \bar\mu(d_{n1} \cap \phi^{-j}(a \setminus a')) \le \bar\mu e^* \le 2^{-n}\epsilon$$

for every $j < 2^n$ (interpreting $d_{n,2^n}$ as d_{n0}, as usual), and

$$\bar\mu(a \,\triangle\, a') = \sum_{j=1}^{2^n} \bar\mu(d_{nj} \cap a \setminus a') \le \epsilon,$$

so that the final condition of the lemma is satisfied.

388K Theorem Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, with Maharam type ω, and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism. Then there is a relatively von Neumann automorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ such that ϕ and π generate the same full subgroups of $\operatorname{Aut}\mathfrak{A}$.

proof (a) The idea is to construct ϕ as the limit of a sequence $\langle \phi_n \rangle_{n \in \mathbb{N}}$ of weakly von Neumann automorphisms such that $G_{\phi_n} = G_\pi$. Each ϕ_n will have a dyadic cycle system $\langle d_{nmi} \rangle_{m \in \mathbb{N}, i < 2^m}$; there will be a strictly increasing sequence $\langle k_n \rangle_{n \in \mathbb{N}}$ such that

$$d_{n+1,m,i} = d_{n,m,i} \text{ whenever } m \le k_n,\, i < 2^m,$$

$$\phi_{n+1} a = \phi_n a \text{ whenever } a \cap d_{n,k_n,0} = 0.$$

Interpolated between the ϕ_n will be a second sequence $\langle \psi_n \rangle_{n \in \mathbb{N}}$ in G_π, with associated (finite) dyadic cycle systems $\langle d'_{nmi} \rangle_{m \le k'_n, i < 2^m}$.

(b) Before starting on the inductive construction we must fix on a countable set $B \subseteq \mathfrak{A}$ which τ-generates $\mathfrak{A}$, and a sequence $\langle b_n \rangle_{n \in \mathbb{N}}$ in B such that every member of B recurs cofinally often in the sequence. (For instance, take the sequence of first members of an enumeration of $B \times \mathbb{N}$.) As usual, I write $\mathfrak{C}$ for the closed subalgebra $\{c : \pi c = c\}$. The induction begins with $\psi_0 = \pi$, $k'_0 = 0$, $d'_{000} = 1$. Given $\psi_n \in G_\pi$ and its dyadic cycle system $\langle d'_{nmi} \rangle_{m \le k'_n, i < 2^m}$, use 388H to find a weakly von Neumann automorphism ϕ_n, with dyadic cycle system $\langle d_{nmi} \rangle_{m \in \mathbb{N}, i < 2^m}$, such that $G_{\phi_n} = G_\pi$, $d_{nmi} = d'_{nmi}$ for $m \le k'_n$ and $i < 2^m$, and $\phi_n a = \psi_n a$ whenever $a \cap d'_{n,k'_n,0} = 0$.

(c) Given the weakly von Neumann automorphism ϕ_n, with its dyadic cycle system $\langle d_{nmi} \rangle_{m \in \mathbb{N}, i < 2^m}$, such that $G_{\phi_n} = G_\pi$, then we have a partition of unity $\langle e_{nj} \rangle_{j \in \mathbb{Z}}$ such that $\pi a = \phi_n^j a$ whenever $j \in \mathbb{Z}$ and $a \subseteq e_{nj}$ (381I). Take r_n such that $\bar\mu \tilde e_n \le 2^{-n}$, where $\tilde e_n = \sup_{|j| > r_n} e_{nj}$, and $k_n > k'_n$ such that $2^{-k_n}(2r_n + 1) \le 2^{-n}$. Set

$$e^*_n = \sup_{|j| \le r_n} \phi_n^{-j} d_{n,k_n,0},$$

so that $\bar\mu e^*_n \le 2^{-n+1}$.

Now use 388J to find a $\psi_{n+1} \in G_\pi$, with a dyadic cycle system $\langle d'_{n+1,m,i} \rangle_{m \le k'_{n+1}, i < 2^m}$, such that $k'_{n+1} \ge k_n$, $d'_{n+1,m,i} = d_{nmi}$ if $m \le k_n$, $\psi_{n+1} a = \phi_n a$ if $a \cap d_{n,k_n,0} = 0$, and there is a b'_n in the algebra generated by $\mathfrak{C} \cup \{d'_{n+1,m,i} : m \le k'_{n+1},\, i < 2^m\}$ such that $\bar\mu(b_n \,\triangle\, b'_n) \le 2^{-n}$. Continue.

(d) The effect of this construction is to ensure that if $l < n$ in $\mathbb{N}$ then

$$d_{lmi} = d_{nmi} \text{ whenever } m \le k_l,\ i < 2^m,$$

$$\phi_n a = \phi_l a \text{ whenever } a \cap d_{l,k_l,0} = 0,$$

$$b'_l \text{ belongs to the subalgebra generated by } \mathfrak{C} \cup \{d_{nmi} : m \le k_n,\ i < 2^m\},$$

and, of course, $d_{n,k_n,0} \subseteq d_{l,k_l,0}$. Since $\langle k_n \rangle_{n\in\mathbb{N}}$ is strictly increasing, $\inf_{n\in\mathbb{N}} d_{n,k_n,0} = 0$. Now, for each $n \in \mathbb{N}$,

$$d_{n,k_n,1} = \phi_n d_{n,k_n,0} = \phi_{n+1} d_{n,k_n,0} \supseteq \phi_{n+1} d_{n+1,k_{n+1},0} = d_{n+1,k_{n+1},1},$$

so setting

$$a_0 = 1 \setminus d_{0,k_0,0}, \quad a_{n+1} = d_{n,k_n,0} \setminus d_{n+1,k_{n+1},0} \text{ for each } n,$$

we have

$$\phi_0 a_0 = 1 \setminus d_{0,k_0,1}, \quad \phi_{n+1} a_{n+1} = d_{n,k_n,1} \setminus d_{n+1,k_{n+1},1} \text{ for each } n,$$

and $\langle \phi_n a_n \rangle_{n\in\mathbb{N}}$ is a partition of unity. There is therefore a $\phi \in \operatorname{Aut}\mathfrak{A}$ defined by setting $\phi a = \phi_n a$ if $a \subseteq a_n$; because G_π is full, $\phi \in G_\pi$.

(e) If $m \le n$, then $a_m \cap d_{m,k_m,0} = 0$, so $\phi_n a = \phi_m a = \phi a$ for every $a \subseteq a_m$. Thus $\phi_n a = \phi a$ for every $a \subseteq \sup_{m\le n} a_m = 1 \setminus d_{n,k_n,0}$. In particular, $\phi d_{nmi} = d_{n,m,i+1}$ whenever $m \le k_n$, $1 \le i < 2^m$ (counting $d_{n,m,2^m}$ as d_{nm0}, as usual); so that in fact $\phi d_{nmi} = d_{n,m,i+1}$ whenever $m \le k_n$, $i < 2^m$.

For each n, we have $d_{nmi} = d'_{n+1,m,i} = d_{n+1,m,i}$ whenever $m \le k_n$ and $i < 2^m$. We therefore have a family $\langle d^*_{mi} \rangle_{m\in\mathbb{N},i<2^m}$ defined by saying that $d^*_{mi} = d_{nmi}$ whenever $n \in \mathbb{N}$, $m \le k_n$ and $i < 2^m$. Now, for any $m \in \mathbb{N}$, there is a $k_n > m$, so that $\langle d^*_{mi} \rangle_{i<2^m} = \langle d_{nmi} \rangle_{i<2^m}$ is a partition of unity; and

$$d^*_{mi} = d_{nmi} = d_{n,m+1,i} \cup d_{n,m+1,i+2^m} = d^*_{m+1,i} \cup d^*_{m+1,i+2^m}$$

for each $i < 2^m$. Moreover,

$$\phi d^*_{m,i} = \phi_n d_{nmi} = d_{n,m,i+1} = d^*_{m,i+1}$$

at least for $1 \le i < 2^m$ (counting $d^*_{m,2^m}$ as $d^*_{m,0}$, as usual), so that in fact $\phi d^*_{mi} = d^*_{m,i+1}$ for every $i < 2^m$. Thus $\langle d^*_{mi} \rangle_{m\in\mathbb{N},i<2^m}$ is a dyadic cycle system for ϕ, and ϕ is a weakly von Neumann automorphism.

Writing $\mathfrak{B}$ for the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{d^*_{mi} : m \in \mathbb{N},\ i < 2^m\}$, then

$$\begin{aligned} \mathfrak{C} \cup \{d'_{nmi} : m \le k'_n,\ i < 2^m\} &= \mathfrak{C} \cup \{d_{n+1,m,i} : m \le k'_n,\ i < 2^m\} \\ &= \mathfrak{C} \cup \{d^*_{mi} : m \le k'_n,\ i < 2^m\} \subseteq \mathfrak{B} \end{aligned}$$

for any $n \in \mathbb{N}$. So $b'_n \in \mathfrak{B}$ for every n. If $b \in B$ and $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $2^{-n} \le \epsilon$ and $b_n = b$, so that $\bar{\mu}(b \triangle b'_n) \le \epsilon$; as every b'_n belongs to $\mathfrak{B}$, and $\mathfrak{B}$ is closed, $b \in \mathfrak{B}$; as b is arbitrary, and B τ-generates $\mathfrak{A}$, $\mathfrak{B} = \mathfrak{A}$. Thus ϕ is a relatively von Neumann automorphism.

(f) If $n \in \mathbb{N}$ and $d \cap e^*_n = 0$, then $\phi^j d = \phi^j_n d$ and $\phi^{-j} d = \phi^{-j}_n d$ whenever $0 \le j \le r_n$. **P** Induce on j. For $j = 0$ the result is trivial. For the inductive step to $j + 1 \le r_n$, note that if $d' \cap d_{n,k_n,1} = 0$ then $\phi^{-1}_n d' \cap d_{n,k_n,0} = 0$, so

$$\phi^{-1} d' = \phi^{-1} \phi_n (\phi^{-1}_n d') = \phi^{-1} \phi (\phi^{-1}_n d') = \phi^{-1}_n d'.$$

Now we have

$$\phi^{j+1} d = \phi(\phi^j_n d) = \phi_n(\phi^j_n d) = \phi^{j+1}_n d$$

because

$$\phi^j_n d \cap d_{n,k_n,0} = \phi^j_n (d \cap \phi^{-j}_n d_{n,k_n,0}) = 0,$$

while

$$\phi^{-j-1} d = \phi^{-1}(\phi^{-j}_n d) = \phi^{-1}_n(\phi^{-j}_n d) = \phi^{-j-1}_n d$$

because

$$\phi^{-j}_n d \cap d_{n,k_n,1} = \phi^{-j}_n (d \cap \phi^{j+1}_n d_{n,k_n,0}) = 0. \quad \mathbf{Q}$$

Thus $\phi^j d = \phi^j_n d$ whenever $|j| \le r_n$.

(g) Finally, $G_\phi = G_\pi$. **P** I remarked in (d) that $\phi \in G_\pi$, so that $G_\phi \subseteq G_\pi$. To see that $\pi \in G_\phi$, take any non-zero $a \in \mathfrak{A}$. Because $\bar{\mu}(e^*_n \cup \tilde{e}_n) \le 2^{-n+1}$ for each n, there is an n such that $a' = a \setminus (e^*_n \cup \tilde{e}_n) \ne 0$. Now there is some $j \in \mathbb{Z}$ such that $a'' = a' \cap e_{nj} \ne 0$; since $a' \cap \tilde{e}_n = 0$, $|j| \le r_n$. If $d \subseteq a''$, then $\pi d = \phi^j_n d$, by the definition of e_{nj}. But also $\phi^j_n d = \phi^j d$, by (f), because $d \cap e^*_n = 0$. So $\pi d = \phi^j d$ for every $d \subseteq a''$. As a is arbitrary, $\pi \in G_\phi$ and $G_\pi \subseteq G_\phi$. **Q**

This completes the proof.

388L Theorem Let $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be totally finite measure algebras of countable Maharam type, and $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_1$, $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ measure-preserving automorphisms. For each i, let $\mathfrak{C}_i$ be the fixed-point subalgebra of π_i and G_{π_i} the full subgroup of $\operatorname{Aut}\mathfrak{A}_i$ generated by π_i. If $(\mathfrak{A}_1, \bar{\mu}_1, \mathfrak{C}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2, \mathfrak{C}_2)$ are isomorphic, so are $(\mathfrak{A}_1, \bar{\mu}_1, G_{\pi_1})$ and $(\mathfrak{A}_2, \bar{\mu}_2, G_{\pi_2})$.

proof (a) It is enough to consider the case in which $(\mathfrak{A}_1, \bar{\mu}_1, \mathfrak{C}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2, \mathfrak{C}_2)$ are actually equal; I therefore delete the subscripts and speak of a structure $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$, with two automorphisms π_1, π_2 of $\mathfrak{A}$ both with fixed-point subalgebra $\mathfrak{C}$.

(b) Suppose first that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$, that is, that both the π_i are aperiodic (381P). In this case, 388K tells us that there are relatively von Neumann automorphisms ϕ_1 and ϕ_2 of $\mathfrak{A}$ such that $G_{\pi_1} = G_{\phi_1}$ and $G_{\pi_2} = G_{\phi_2}$. But $(\mathfrak{A}, \bar{\mu}, \phi_1)$ and $(\mathfrak{A}, \bar{\mu}, \phi_2)$ are isomorphic. **P** Let $\langle d_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ and $\langle d'_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ be dyadic cycle systems for ϕ_1, ϕ_2 respectively such that $\mathfrak{C} \cup \{d_{mi} : m \in \mathbb{N},\ i < 2^m\}$ and $\mathfrak{C} \cup \{d'_{mi} : m \in \mathbb{N},\ i < 2^m\}$ both τ-generate $\mathfrak{A}$.

Writing $\mathfrak{B}_1$, $\mathfrak{B}_2$ for the subalgebras of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{d_{mi} : m \in \mathbb{N},\ i < 2^m\}$ and $\mathfrak{C} \cup \{d'_{mi} : m \in \mathbb{N},\ i < 2^m\}$ respectively, it is easy to see that these algebras are isomorphic: we just set $\theta_0 c = c$ for $c \in \mathfrak{C}$, $\theta_0 d_{mi} = d'_{mi}$ for $i < 2^m$ to obtain a measure-preserving isomorphism $\theta_0 : \mathfrak{B}_1 \to \mathfrak{B}_2$. Because these are topologically dense subalgebras of $\mathfrak{A}$, there is a unique extension of θ_0 to a measure-preserving automorphism $\theta : \mathfrak{A} \to \mathfrak{A}$ (324O). Next, we see that

$$\theta\phi_1\theta^{-1}c = c = \phi_2 c \text{ for every } c \in \mathfrak{C},$$

$$\theta\phi_1\theta^{-1}d'_{mi} = \theta\phi_1 d_{mi} = \theta d_{m,i+1} = d'_{m,i+1} = \phi_2 d'_{mi}$$

for $m \in \mathbb{N}$, $i < 2^m$ (as usual, taking $d_{m,2^m}$ to be d_{m0} and $d'_{m,2^m}$ to be d'_{m0}). But this means that $\theta\phi_1\theta^{-1}b = \phi_2 b$ for every $b \in \mathfrak{B}_2$, so (again because $\mathfrak{B}_2$ is dense in $\mathfrak{A}$) $\theta\phi_1\theta^{-1} = \phi_2$. Thus θ is an isomorphism between $(\mathfrak{A}, \bar{\mu}, \phi_1)$ and $(\mathfrak{A}, \bar{\mu}, \phi_2)$. **Q**

Of course θ is now also an isomorphism between $(\mathfrak{A}, \bar{\mu}, G_{\phi_1}) = (\mathfrak{A}, \bar{\mu}, G_{\pi_1})$ and $(\mathfrak{A}, \bar{\mu}, G_{\phi_2}) = (\mathfrak{A}, \bar{\mu}, G_{\pi_2})$.

(c) Next, consider the case in which π_1 is periodic, with period n, for some $n \ge 1$. In this case $\pi_2 \in G_{\pi_1}$. **P** Let $(d_0, \dots, d_{n-1})$ be a partition of unity in $\mathfrak{A}$ such that $\pi_1 d_i = d_{i+1}$ for $i < n-1$ and $\pi_1 d_{n-1} = d_0$ (382Fb). If $d \subseteq d_j$, then $c = \sup_{i<n} \pi_1^i d \in \mathfrak{C}$ and $d = d_j \cap c$; so any member of $\mathfrak{A}$ is of the form $\sup_{j<n} d_j \cap c_j$ for some family $c_0, \dots, c_{n-1}$ in $\mathfrak{C}$.

If $a \in \mathfrak{A} \setminus \{0\}$, take i, $j < n$ such that $a' = a \cap d_i \cap \pi_2^{-1} d_j \ne 0$. Then any $d \subseteq a'$ is of the form

$$d_i \cap c_1 = \pi_2^{-1}(d_j \cap c_2) = c_2 \cap \pi_2^{-1} d_j$$

for some c_1, $c_2 \in \mathfrak{C}$; setting $c = c_1 \cap c_2$, we have

$$d = d_i \cap c, \quad \pi_2 d = d_j \cap c = \pi_1^{j-i} d.$$

As a is arbitrary, this shows that $\pi_2 \in G_{\pi_1}$. **Q**

Now $\sup_{n\in\mathbb{Z}} \pi_2^n d_0$ belongs to $\mathfrak{C}$ and includes d_0, so must be 1. Finally, the two induced automorphisms $(\pi_1)_{d_0}$, $(\pi_2)_{d_0}$ on $\mathfrak{A}_{d_0}$ are both the identity. **P** If $0 \ne \tilde{d} \subseteq d_0$ there are a non-zero $d' \subseteq \tilde{d}$ and an $m \ge 1$ such that $(\pi_2)_{d_0} d = \pi_2^m d$ for every $d \subseteq d'$. As $\pi_2^m \in G_{\pi_1}$, there are a non-zero $d \subseteq d'$ and a $k \in \mathbb{Z}$ such that $\pi_2^m d = \pi_1^k d$. Now $\pi_1^k d \subseteq d_0$ so k is a multiple of n and $(\pi_2)_{d_0} d = \pi_1^k d = d$. This shows that $\{d : (\pi_2)_{d_0} d = d\}$ is order-dense in $\mathfrak{A}_{d_0}$ and must be the whole of $\mathfrak{A}_{d_0}$. As for π_1, we have $(\pi_1)_{d_0} d = \pi_1^n d = d$ for every $d \subseteq d_0$. **Q**

So 388G tells us that $G_{\pi_1} = G_{\pi_2}$.

(d) For the general case, we see from 381H that there is a partition of unity $\langle c_i\rangle_{1\le i\le\omega}$ in $\mathfrak{C}$ such that $\pi_1{\restriction}\mathfrak{A}_{c_\omega}$ is aperiodic and if i is finite and $c_i \ne 0$ then $\pi_1{\restriction}\mathfrak{A}_{c_i}$ is periodic with period i. For each i, let H_i be $\{\phi{\restriction}\mathfrak{A}_{c_i} : \phi \in G_{\pi_1}\}$; then H_i is a full subgroup of $\operatorname{Aut}\mathfrak{A}_{c_i}$, and

$$G_{\pi_1} = \{\phi : \phi \in \operatorname{Aut}\mathfrak{A},\ \phi{\restriction}\mathfrak{A}_{c_i} \in H_i \text{ whenever } 1 \le i \le \omega\}.$$

Similarly, writing $H'_i = \{\phi{\restriction}\mathfrak{A}_{c_i} : \phi \in G_{\pi_2}\}$,

$$G_{\pi_2} = \{\phi : \phi \in \operatorname{Aut}\mathfrak{A},\ \phi{\restriction}\mathfrak{A}_{c_i} \in H'_i \text{ whenever } 1 \le i \le \omega\}.$$

Note also that H_i, H'_i are the full subgroups of $\operatorname{Aut}\mathfrak{A}_{c_i}$ generated by $\pi_1{\restriction}\mathfrak{A}_{c_i}$, $\pi_2{\restriction}\mathfrak{A}_{c_i}$ respectively. By (b) and (c), $H_i = H'_i$ for finite i, while there is a measure-preserving automorphism $\theta : \mathfrak{A}_{c_\omega} \to \mathfrak{A}_{c_\omega}$ such that $\theta H_\omega \theta^{-1} = H'_\omega$. Now we can define a measure-preserving automorphism $\theta_1 : \mathfrak{A} \to \mathfrak{A}$ by setting $\theta_1 a = \theta a$ if $a \subseteq c_\omega$, $\theta_1 a = a$ if $a \cap c_\omega = 0$, and we shall have $\theta_1 G_{\pi_1} \theta_1^{-1} = G_{\pi_2}$. Thus $(\mathfrak{A}, \bar{\mu}, G_{\pi_1})$ and $(\mathfrak{A}, \bar{\mu}, G_{\pi_2})$ are isomorphic, as claimed.

388X Basic exercises >(a) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving automorphism. Let us say that a **pseudo-cycle** for π is a partition of unity $\langle a_i \rangle_{i<n}$, where $n \ge 1$, such that $\pi a_i = a_{i+1}$ for $i < n-1$ (so that $\pi a_{n-1} = a_0$). (i) Show that if we have pseudo-cycles $\langle a_i \rangle_{i<n}$ and $\langle b_j \rangle_{j<m}$, where m is a multiple of n, then we have a pseudo-cycle $\langle c_j \rangle_{j<m}$ with $c_0 \subseteq a_0$, so that $a_i = \sup\{c_j : j < m,\ j \equiv i \bmod n\}$ for every $i < n$. (ii) Show that π is weakly von Neumann iff it has a pseudo-cycle of length 2^n for any $n \in \mathbb{N}$.

(b) Let $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be probability algebras, and $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_2$ and $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ measure-preserving von Neumann automorphisms. Show that there is a measure-preserving Boolean isomorphism $\theta : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $\pi_2 = \theta \pi_2 \theta^{-1}$.

(c) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra of countable Maharam type, and $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ the group of measure-preserving Boolean automorphisms of $\mathfrak{A}$. Let $\pi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ be a von Neumann automorphism. (i) Show that for any ultrafilter $\mathcal{F}$ on $\mathbb{N}$ there is a $\phi_{\mathcal{F}} \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ defined by the formula $\phi_{\mathcal{F}}(a) = \lim_{n\to\mathcal{F}} \pi^n a$ for every $a \in \mathfrak{A}$, the limit being taken in the measure-algebra topology. (ii) Show that $\{\phi_{\mathcal{F}} : \mathcal{F} \text{ is an ultrafilter on } \mathbb{N}\}$ is a subgroup of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ homeomorphic to $\mathbb{Z}_2^{\mathbb{N}}$. (*Hint*: 388E.)

(d) Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a weakly von Neumann automorphism. Show that π^n is a weakly von Neumann automorphism for every $n \in \mathbb{Z} \setminus \{0\}$. (*Hint*: consider $n = 2$, $n = -1$, odd $n \ge 3$ separately. The formula of 388E may be useful.)

(e) Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a von Neumann automorphism. (i) Show that π^2 is not ergodic. (ii) Show that π^2 is relatively von Neumann. (iii) Show that π^n is von Neumann for every odd $n \in \mathbb{Z}$. (iv) Show that if $\mathfrak{A}$ is a probability algebra (when endowed with a suitable measure), π is ergodic.

388Y Further exercises (a) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ such that the quotient algebra $\mathfrak{A} = \Sigma/\mathcal{I}$ is Dedekind complete and there is a countable subset of Σ separating the points of X. Suppose that f and g are automorphisms of the structure $(X, \Sigma, \mathcal{I})$ inducing $\pi, \phi \in \operatorname{Aut}\mathfrak{A}$. Show that the following are equiveridical: (i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π; (ii) $\{x : x \in X,\ f(x) \notin \{g^n(x) : n \in \mathbb{Z}\}\} \in \mathcal{I}$; (iii) $\{x : x \in X,\ \{f^n(x) : n \in \mathbb{Z}\} \not\subseteq \{g^n(x) : n \in \mathbb{Z}\}\} \in \mathcal{I}$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a relatively von Neumann automorphism. Show that π is aperiodic and has zero entropy.

(c) Let $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be atomless probability algebras, and $(\mathfrak{A}, \bar{\mu})$ their probability algebra free product. Let $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_1$ be a measure-preserving von Neumann automorphism and $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ a mixing measure-preserving automorphism. Let $\pi : \mathfrak{A} \to \mathfrak{A}$ be the measure-preserving automorphism such that $\pi(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$ for all $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$. Show that π is an ergodic weakly von Neumann automorphism which is not a relatively von Neumann automorphism.

(d) Let μ be Lebesgue measure on $[0,1]^2$, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra; let $\mathfrak{C}$ be the closed subalgebra of elements expressible as $(E \times [0,1])^\bullet$, where $E \subseteq [0,1]$ is measurable. Suppose that $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism such that $\mathfrak{C} = \{c : \pi c = c\}$. Show that there is a family $\langle f_x \rangle_{x\in[0,1]}$ of ergodic measure space automorphisms of $[0,1]$ such that $(x, y) \mapsto (x, f_x(y))$ is a measure space automorphism of $[0,1]^2$ representing π.

(e) Show that the odometer transformation on $\{0,1\}^{\mathbb{N}}$ is expressible as the product of two Borel measurable measure-preserving involutions.

(f) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$ a relatively von Neumann automorphism; let $T = T_\pi : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}}$ be the corresponding Riesz homomorphism (365O). (i) Show that $\bigcup_{n\ge1}\{u : T^n u = u\}$ is dense in $L^1_{\bar{\mu}}$. (ii) Show that $\{T^n : n \in \mathbb{Z}\}$ is relatively compact in $\mathsf{B}(L^1_{\bar{\mu}}; L^1_{\bar{\mu}})$ for the strong operator topology.

(g) Give an example of a ccc Dedekind complete Boolean algebra $\mathfrak{A}$ and a von Neumann automorphism $\pi \in \operatorname{Aut}\mathfrak{A}$ which is not ergodic.

388 Notes and comments Dye's theorem (Dye 59) is actually Theorem 388L in the case in which π_1, π_2 are ergodic, that is, in which $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are both trivial. I take the trouble to give the generalized form here (a simplified version of that in Krieger 76) because it seems a natural target, once we have a classification of the relevant structures $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$ (333R). The essential mathematical ideas are the same in both cases. You can find the special case worked out in Hajian Ito & Kakutani 75, from which I have taken the argument used here; and you may find it useful to go through the version above, to check what kind of simplifications arise if each $\mathfrak{C}$ is taken to be $\{0, 1\}$. Essentially the difference will be that every 'aperiodic' turns into 'ergodic' (with an occasional 'atomless' thrown in) and '331B' turns into '331C'. As far as I know, there is no simplification available in the structure of the argument; of course the details become a bit easier, but with the possible exception of 388I-388J I think there is little difference.

Of course modifying a general argument to give a simpler proof of a special case is a standard exercise in this kind of mathematics. What is much more interesting is the reverse process. What kinds of theorem about ergodic automorphisms will in fact be true of all automorphisms? A variety of very powerful approaches to such questions have been developed in the last half-century, and I hope to describe some of the ideas in Volumes 4 and 5. The methods used in this section are relatively straightforward and do not require any deep theoretical underpinning beyond Maharam's lemma 331B. But an alternative approach can be found using 388Yd: in effect (at least for the Lebesgue measure algebra) any measure-preserving automorphism can be disintegrated into ergodic measure space automorphisms (the fibre maps f_x of 388Yd). It is sometimes possible to guess which theorems about ergodic transformations are 'uniformisable' in the sense that they can be applied to such a family $\langle f_x\rangle_{x\in[0,1]}$, in a systematic way, to provide a structure which can be interpreted on the product measure. The details tend to be complex, which is one of the reasons why I do not attempt to work through them here; but such disintegrations can be a most valuable aid to intuition.

In this section I use von Neumann automorphisms as an auxiliary tool: the point is, first, that two von Neumann automorphisms are isomorphic – that is, the von Neumann automorphisms on a given totally finite measure algebra $(\mathfrak{A}, \bar{\mu})$ (necessarily isomorphic to the Lebesgue measure algebra, since we must have $\mathfrak{A}$ atomless and $\tau(\mathfrak{A}) = \omega$) form a conjugacy class in the group $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ of measure-preserving automorphisms; and next, that for any ergodic measure-preserving automorphism π (on an atomless totally finite algebra of countable Maharam type) there is a von Neumann automorphism ϕ such that $G_\pi = G_\phi$ (388K). But I think they are remarkable in themselves. A (weakly) von Neumann automorphism has a 'pseudo-cycle' (388Xa) for every power of 2. For some purposes, existence is all we need to know; but in the arguments of 388H-388K we need to keep track of named pseudo-cycles in what I call 'dyadic cycle systems' (388D).

In this volume I have systematically preferred arguments which deal directly with measure algebras, rather than with measure spaces. I believe that such arguments can have a simplicity and clarity which repays the extra effort of dealing with more abstract structures. But undoubtedly it is necessary, if you are to have any hope of going farther in the subject, to develop methods of transferring intuitions and theorems between the two contexts. I offer 381Xl as an example. The description there of 'induced automorphism' requires a certain amount of manoeuvering around negligible sets, but gives a valuably graphic description. In the same way, 381Xf, 388A and 381Qc provide alternative ways of looking at full subgroups.

There are contexts in which it is useful to know whether an element of the full subgroup generated by π actually belongs to the full semigroup generated by π (381Yb); for instance, this happens in 388C.

Chapter 39

Measurable algebras

In the final chapter of this volume, I present results connected with the following question: which algebras can appear as the underlying Boolean algebras of measure algebras? Put in this form, there is a trivial answer (391A). The proper question is rather: which algebras can appear as the underlying Boolean algebras of semi-finite measure algebras? This is easily reducible to the question: which algebras can appear as the underlying Boolean algebras of probability algebras? Now in one sense Maharam's theorem (§332) gives us the answer exactly: they are the countable simple products of the measure algebras of $\{0,1\}^\kappa$ for cardinals κ. But if we approach from another direction, things are more interesting. Probability algebras share a very large number of very special properties. Can we find a selection of these properties which will be sufficient to force an abstract Boolean algebra to be a probability algebra when endowed with a suitable functional?

No fully satisfying answer to this question is known. But in exploring the possibilities we encounter some interesting and important ideas. In §391 I discuss algebras which have strictly positive additive real-valued functionals; for such algebras, weak (σ,∞)-distributivity is necessary and sufficient for the existence of a measure; so we are led to look for conditions sufficient to ensure that there is a strictly positive additive functional. A slightly different approach lies through the concept of 'submeasure'. Submeasures arise naturally in the theories of topological Boolean algebras (393J), topological Riesz spaces (393K) and vector measures (394P), and on any given algebra there is a strictly positive 'uniformly exhaustive' submeasure iff there is a strictly positive additive functional; this is the Kalton-Roberts theorem (392F).

Submeasures in general are common, but correspondingly limited in what they can tell us about a structure in the absence of further properties. Uniformly exhaustive submeasures are not far from additive functionals. An intermediate class, the 'exhaustive' submeasures, has been intensively studied, originally in the hope that they might lead to characterizations of measurable algebras, but more recently for their own sake. Just as additive functionals lead to measurable algebras, totally finite exhaustive submeasures lead to 'Maharam algebras' (§393). For many years it was not known whether every exhaustive submeasure was exhaustive (equivalently, whether every Maharam algebra was a measurable algebra); an example was eventually found by M.Talagrand, and is presented in §394.

In §395, I look at a characterization of measurable algebras in terms of the special properties which the automorphism group of a measure algebra must have (Kawada's theorem, 395Q). §396 complements the previous section by looking briefly at the subgroups of an automorphism group $\operatorname{Aut}\mathfrak{A}$ which can appear as groups of measure-preserving automorphisms.

391 Kelley's theorem

In this section I introduce the notion of 'measurable algebra' (391B), which will be the subject of the whole chapter once the trivial construction of 391A has been dealt with. I show that for weakly (σ,∞)-distributive algebras countable additivity can be left to look after itself, and all we need to find is a strictly positive finitely additive functional (391D). I give Kelley's criterion for the existence of such a functional (391H-391J).

391A Proposition Let $\mathfrak{A}$ be any Dedekind σ-complete Boolean algebra. Then there is a function $\bar\mu:\mathfrak{A}\to[0,\infty]$ such that $(\mathfrak{A},\bar\mu)$ is a measure algebra.

proof Set $\bar\mu 0=0$, $\bar\mu a=\infty$ for $a\in\mathfrak{A}\setminus\{0\}$.

391B Definition (a) I will call a Boolean algebra $\mathfrak{A}$ **measurable** if there is a functional $\bar\mu:\mathfrak{A}\to[0,\infty[$ such that $(\mathfrak{A},\bar\mu)$ is a totally finite measure algebra.

In this case, if $\bar\mu\ne0$, then it has a scalar multiple with total mass 1. So a Boolean algebra $\mathfrak{A}$ is measurable iff either it is $\{0\}$ or there is a functional $\bar\mu$ such that $(\mathfrak{A},\bar\mu)$ is a probability algebra.

(b) I will call a Boolean algebra $\mathfrak{A}$ **chargeable** if there is an additive functional $\nu:\mathfrak{A}\to[0,\infty[$ which is **strictly positive**, that is, $\nu a>0$ for every non-zero $a\in\mathfrak{A}$.

Of course a measurable algebra is chargeable.

(c) I will call a Boolean algebra **nowhere measurable** if none of its non-zero principal ideals are measurable algebras.

391C Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) The following are equiveridical: (i) there is a functional $\bar{\mu} : \mathfrak{A} \to [0, \infty]$ such that $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra; (ii) $\mathfrak{A}$ is Dedekind σ-complete and $\{a : a \in \mathfrak{A}, \mathfrak{A}_a$ is measurable$\}$ is order-dense in $\mathfrak{A}$, writing $\mathfrak{A}_a$ for the principal ideal generated by a.

(b) The following are equiveridical: (i) there is a functional $\bar{\mu} : \mathfrak{A} \to [0, \infty]$ such that $(\mathfrak{A}, \bar{\mu})$ is a localizable measure algebra; (ii) $\mathfrak{A}$ is Dedekind complete and $\{a : a \in \mathfrak{A}, \mathfrak{A}_a$ is measurable$\}$ is order-dense in $\mathfrak{A}$.

proof (a) (i)$\Rightarrow$(ii): if $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra, then $\mathfrak{A}^f = \{a : \bar{\mu}a < \infty\}$ is order-dense in $\mathfrak{A}$ and $\mathfrak{A}_a$ is measurable for every $a \in \mathfrak{A}^f$.

(ii)$\Rightarrow$(i): setting $D = \{a : a \in \mathfrak{A}, \mathfrak{A}_a$ is measurable$\}$, D is order-dense, so there is a partition of unity $C \subseteq D$ (313K). For each $c \in C$, choose $\bar{\mu}_c$ such that $(\mathfrak{A}_c, \bar{\mu}_c)$ is a totally finite measure algebra. Set $\bar{\mu}a = \sum_{c \in C} \bar{\mu}_c(a \cap c)$ for every $a \in \mathfrak{A}$; then it is easy to check that $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra.

(b) Follows immediately.

391D Theorem (Kantorovich Vulikh & Pinsker 50) Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is measurable;

(ii) $\mathfrak{A}$ is Dedekind σ-complete, weakly (σ, ∞)-distributive and chargeable.

proof (i)$\Rightarrow$(ii) Put the definition together with 322C(b)-(c) (for Dedekind completeness) and 322F (for weak (σ, ∞)-distributivity).

(ii)$\Rightarrow$(i) Given that (ii) is satisfied, let M be the L-space of bounded additive functionals on $\mathfrak{A}$, $M_\tau \subseteq M$ the band of completely additive functionals, and $P_\tau : M \to M_\tau$ the band projection (362Bd). Let $\nu : \mathfrak{A} \to [0, \infty[$ be a strictly positive additive functional, and set $\bar{\mu} = P_\tau(\nu)$. Then $\bar{\mu}$ is strictly positive. **P** If $c \in \mathfrak{A}$ is non-zero, there is an upwards-directed set A, with supremum c, such that $\bar{\mu}c = \sup_{a \in A} \nu c$ (362D); as ν is strictly positive and A contains a non-zero element, $\bar{\mu}c > 0$. **Q** Of course $\bar{\mu}$ is countably additive, so witnesses that $\mathfrak{A}$ is measurable.

391E Thus we are led naturally to the question: which Boolean algebras carry strictly positive *finitely* additive functionals? The Hahn-Banach theorem, suitably applied, gives some sort of answer to this question. For the sake of applications later on, I give two general results on the existence of additive functionals related to given functionals.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\phi : \mathfrak{A} \to [0, 1]$ a functional. Then the following are equiveridical:

(i) there is a finitely additive functional $\nu : \mathfrak{A} \to [0, 1]$ such that $\nu 1 = 1$ and $\nu a \leq \phi a$ for every $a \in \mathfrak{A}$;

(ii) whenever $\langle a_i \rangle_{i \in I}$ is a finite indexed family in $\mathfrak{A}$, $m \in \mathbb{N}$ and $\sum_{i \in I} \chi a_i \geq m\chi 1$ in $S = S(\mathfrak{A})$ (definition: 361A), then $\sum_{i \in I} \phi a_i \geq m$.

proof (a)(i)$\Rightarrow$(ii) If $\nu : \mathfrak{A} \to [0, 1]$ is a finitely additive functional such that $\nu 1 = 1$ and $\nu a \leq \phi a$ for every $a \in \mathfrak{A}$, let $h : S \to \mathbb{R}$ be the positive linear functional corresponding to ν (361G). Now if $\langle a_i \rangle_{i \in I}$ is a finite family in $\mathfrak{A}$ and $\sum_{i \in I} \chi a_i \geq m\chi 1$, then

$$\begin{aligned}\sum_{i \in I} \phi a_i &\geq \sum_{i \in I} \nu a_i = \sum_{i \in I} h(\chi a_i) \\ &= h(\sum_{i \in I} \chi a_i) \geq h(m\chi 1) = m.\end{aligned}$$

As $\langle a_i \rangle_{i \in I}$ is arbitrary, (ii) is true.

(b)(ii)$\Rightarrow$(i) Now suppose that ϕ satisfies (ii). For $u \in S$, set

$$p(u) = \inf\{\textstyle\sum_{i=0}^n \alpha_i \phi a_i : a_0, \dots, a_n \in \mathfrak{A},\ \alpha_0, \dots, \alpha_n \geq 0,\ \sum_{i=0}^n \alpha_i \chi a_i \geq u\}.$$

Then it is easy to check that $p(u+v) \leq p(u)+p(v)$ for all u, $v \in S$, and that $p(\alpha u) = \alpha p(u)$ for all $u \in S$, $\alpha \geq 0$. Also $p(\chi 1) \geq 1$. **P?** If not, there are $a_0, \dots, a_n \in \mathfrak{A}$ and $\alpha_0, \dots, \alpha_n \geq 0$ such that $\chi 1 \leq \sum_{i=0}^n \alpha_i \chi a_i$ but $\sum_{i=0}^n \alpha_i \phi a_i < 1$. Increasing each α_i slightly if necessary, we may suppose that every α_i is rational; let $m \geq 1$ and $k_0, \dots, k_n \in \mathbb{N}$ be such that $\alpha_i = k_i/m$ for each $i \leq n$.

Set $K = \{(i, j) : 0 \leq i \leq n,\ 1 \leq j \leq k_i\}$, and for $(i, j) \in K$ set $a_{ij} = a_i$. Then

$$\textstyle\sum_{(i,j) \in K} \chi a_{ij} = \sum_{i=0}^n k_i \chi a_i = m \sum_{i=0}^n \alpha_i \chi a_i \geq m\chi 1,$$

but

$$\sum_{(i,j)\in K} \phi a_{ij} = \sum_{i=0}^{n} k_i \phi a_i = m\sum_{i=0}^{n} \alpha_i \phi a_i < m,$$

which is supposed to be impossible. **XQ**

By the Hahn-Banach theorem, in the form 3A5Aa, there is a linear functional $h : S \to \mathbb{R}$ such that $h(\chi 1) = p(\chi 1) \ge 1$ and $h(u) \le p(u)$ for every $u \in S$. In particular, $h(\chi a) \le \phi b$ whenever $a \subseteq b \in \mathfrak{A}$. Set $\nu a = h(\chi a)$ for $a \in \mathfrak{A}$; then $\nu : \mathfrak{A} \to [0, \infty[$ is an additive functional, $\nu 1 \ge 1$ and $\nu a \le \phi b$ whenever $a \subseteq b$ in $\mathfrak{A}$. We do not know whether ν is positive, but if we define ν^+ as in 362Ab, we shall have a non-negative additive functional such that

$$\nu^+ a = \sup_{b \subseteq a} \nu b \le \phi a$$

for every $a \in \mathfrak{A}$, and

$$1 \le \nu 1 \le \nu^+ 1 \le \phi 1 \le 1,$$

so ν^+ witnesses that (i) is true.

391F Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\psi : A \to [0,1]$ a functional, where $A \subseteq \mathfrak{A}$. Then the following are equiveridical:

(i) there is a finitely additive functional $\nu : \mathfrak{A} \to [0,1]$ such that $\nu 1 = 1$ and $\nu a \ge \psi a$ for every $a \in A$;

(ii) whenever $\langle a_i \rangle_{i\in I}$ is a finite indexed family in A, there is a set $J \subseteq I$ such that $\#(J) \ge \sum_{i\in I} \psi a_i$ and $\inf_{i\in J} a_i \ne 0$.

Remark In (ii) here, we may have to interpret the infimum of the empty set in $\mathfrak{A}$ as 1.

proof (a) We apply 391E to ϕ, where

$$\begin{aligned} \phi a &= 1 - \psi(1 \setminus a) \text{ if } a \in \mathfrak{A} \text{ and } 1 \setminus a \in A, \\ &= 1 \text{ for other } a \in \mathfrak{A}. \end{aligned}$$

(b) Suppose that (i) here is true of ψ. Then 391E(i) is true of ϕ. **P** Let $\nu : \mathfrak{A} \to [0,1]$ be an additive functional such that $\nu 1 = 1$ and $\nu a \ge \psi a$ for every $a \in \mathfrak{A}$. If $a \in \mathfrak{A}$ and $1 \setminus a \in A$, then

$$\nu a = 1 - \nu(1 \setminus a) \le 1 - \psi(1 \setminus a) = \phi a;$$

for other $a \in \mathfrak{A}$, $\nu a \le 1 = \phi a$. **Q**

(c) Suppose that 391E(i) is true of ϕ. Then (i) here is true of ψ. **P** There is an additive functional $\nu : \mathfrak{A} \to [0,1]$ such that $\nu 1 = 1$ and $\nu a \le \phi a$ for every $a \in \mathfrak{A}$; in this case, for $a \in A$,

$$\nu a = 1 - \nu(1 \setminus a) \ge 1 - \phi(1 \setminus a) = \psi a. \text{ } \mathbf{Q}$$

(d) Suppose that (ii) here is true of ψ, and that $\langle a_i \rangle_{i\in I}$ is a finite family in $\mathfrak{A}$ such that $\sum_{i\in I} \chi a_i \ge m\chi 1$, while $\sum_{i\in I} \phi a_i = \beta$. Set $K = \{i : i \in I,\ 1 \setminus a_i \in A\}$.

$$\sum_{i\in K} \psi(1 \setminus a_i) = \sum_{i\in K}(1 - \phi a_i) = \#(K) - \sum_{i\in I} \phi a_i + \#(I \setminus K) = \#(I) - \beta,$$

so there is a set $J \subseteq K$ such that $\#(J) \ge \#(I) - \beta$ and $\inf_{i\in J}(1 \setminus a_i) = c \ne 0$. Now $c \cap a_i = 0$ for $i \in J$, so

$$m\chi c \le \sum_{i\in I} \chi(a_i \cap c) = \sum_{i\in I\setminus J} \chi(a_i \cap c) \le \#(I \setminus J)\chi c$$

and $m \le \#(I) - \#(J) \le \beta$. As $\langle a_i \rangle_{i\in I}$ is arbitrary, 391E(ii) is true of ϕ.

(e) Suppose that 391E(ii) is true of ϕ, and that $\langle a_i \rangle_{i\in I}$ is a family in A. Set

$$\beta = \sum_{i\in I} \phi(1 \setminus a_i) = \#(I) - \sum_{i\in I} \psi a_i$$

and let k be the least integer greater than β. Since $\sum_{i\in I} \phi(1 \setminus a_i) < k$, $\sum_{i\in I} \chi(1 \setminus a_i) \not\ge k\chi 1$, that is, $\sum_{i\in I} \chi a_i \not\le (\#(I) - k)\chi 1$. But this means that there must be some $J \subseteq I$ such that $\#(J) > \#(I) - k$ and $\inf_{i\in J} a_i \ne 0$. Now

$$\sum_{i\in I} \psi a_i = \#(I) - \beta \le \#(I) - (k-1) \le \#(J).$$

As $\langle a_i \rangle_{i\in I}$ is arbitrary, (ii) here is true of ψ.

(f) Since we know that 391E(i)$\Leftrightarrow$391E(ii), we can conclude that (i) and (ii) here are equiveridical.

391G Corollary Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, and $\nu_0 : \mathfrak{B} \to \mathbb{R}$ a non-negative finitely additive functional. Then there is a non-negative finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ extending ν_0.

proof (a) Suppose first that $\nu_0 1 = 1$. Set $\psi b = \nu_0 b$ for every $b \in \mathfrak{B}$. Then ψ must satisfy the condition (ii) of 391F when regarded as a functional defined on a subset of $\mathfrak{B}$; but this means that it satisfies the same condition when regarded as a functional defined on a subset of $\mathfrak{A}$. So there is a non-negative finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu 1 = 1$ and $\nu b \geq \nu_0 b$ for every $b \in \mathfrak{B}$. In this case

$$\nu b = 1 - \nu(1 \setminus b) \leq 1 - \nu_0(1 \setminus b) = \nu_0 b \leq \nu b$$

for every $b \in \mathfrak{B}$, so ν extends ν_0.

(b) For the general case, if $\nu_0 1 = 0$ then ν_0 must be the zero functional on $\mathfrak{B}$, so we can take ν to be the zero functional on $\mathfrak{A}$; and if $\nu_0 1 = \gamma > 0$, we apply (a) to $\gamma^{-1}\nu_0$.

391H Definition Let $\mathfrak{A}$ be a Boolean algebra, and $A \subseteq \mathfrak{A} \setminus \{0\}$ any non-empty set. The **intersection number** of A is the largest $\delta \geq 0$ such that whenever $\langle a_i \rangle_{i \in I}$ is a finite family in A, with $I \neq \emptyset$, there is a $J \subseteq I$ such that $\#(J) \geq \delta \#(I)$ and $\inf_{i \in J} a_i \neq 0$.

Remarks (a) It is essential to note that in the definition here the $\langle a_i \rangle_{i \in I}$ are indexed families, with repetitions allowed; see 391Xi.

(b) I spoke perhaps rather glibly of 'the largest δ such that ...'; you may prefer to write

$$\delta = \inf\{\sup_{\emptyset \neq J \subseteq \{0,\dots,n\}, \inf_{j \in J} a_j \neq 0} \frac{\#(J)}{n+1} : a_0, \dots, a_n \in A\}.$$

391I Proposition Let $\mathfrak{A}$ be a Boolean algebra and $A \subseteq \mathfrak{A} \setminus \{0\}$ any non-empty set. Write C for the set of non-negative finitely additive functionals $\nu : \mathfrak{A} \to [0,1]$ such that $\nu 1 = 1$. Then the intersection number of A is precisely $\max_{\nu \in C} \inf_{a \in A} \nu a$.

proof Write δ for the intersection number of A, and δ' for $\sup_{\nu \in C} \inf_{a \in A} \nu a$.

(a) For any $\gamma < \delta'$, we can find a $\nu \in C$ such that $\nu a \geq \gamma$ for every $a \in A$. So if we set $\psi a = \gamma$ for every $a \in A$, ψ satisfies condition (i) of 391F. But this means that if $\langle a_i \rangle_{i \in I}$ is any finite family in A, there must be a $J \subseteq I$ such that $\inf_{i \in J} a_i \neq 0$ and $\#(J) \geq \gamma \#(I)$. Accordingly $\gamma \leq \delta$; as γ is arbitrary, $\delta' \leq \delta$.

(b) Define $\psi : A \to [0,1]$ by setting $\psi a = \delta$ for every $a \in A$. If $\langle a_i \rangle_{i \in I}$ is a finite indexed family in A, there is a $J \subseteq I$ such that $\#(J) \geq \delta \#(I)$ and $\inf_{i \in J} a_i \neq 0$; but $\delta \#(I) = \sum_{i \in I} \psi a_i$, so this means that condition (ii) of 391F is satisfied. So there is a $\nu \in C$ such that $\nu a \geq \delta$ for every $a \in A$; and ν witnesses not only that $\delta' \geq \delta$, but that the supremum is a maximum.

391J Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is chargeable;

(ii) either $\mathfrak{A} = \{0\}$ or $\mathfrak{A} \setminus \{0\}$ is expressible as a countable union of sets with non-zero intersection numbers.

proof (i)$\Rightarrow$(ii) If there is a strictly positive finitely additive functional ν on $\mathfrak{A}$, and $\mathfrak{A} \neq \{0\}$, set $A_n = \{a : \nu a \geq 2^{-n}\nu 1\}$ for every $n \in \mathbb{N}$; then (applying 391I to the functional $\frac{1}{\nu 1}\nu$) we see that every A_n has intersection number at least 2^{-n}, while $\mathfrak{A} \setminus \{0\} = \bigcup_{n \in \mathbb{N}} A_n$ because ν is strictly positive, so (ii) is satisfied.

(ii)$\Rightarrow$(i) If $\mathfrak{A} \setminus \{0\}$ is expressible as $\bigcup_{n \in \mathbb{N}} A_n$, where each A_n has intersection number $\delta_n > 0$, then for each n choose a finitely additive functional ν_n on $\mathfrak{A}$ such that $\nu_n 1 = 1$ and $\nu_n a \geq \delta_n$ for every $a \in A_n$. Setting $\nu a = \sum_{n=0}^{\infty} 2^{-n} \nu_n a$ for every $a \in \mathfrak{A}$, ν is a strictly positive additive functional on $\mathfrak{A}$, and (i) is true.

391K Corollary Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is measurable iff it is Dedekind σ-complete and weakly (σ,∞)-distributive and either $\mathfrak{A} = \{0\}$ or $\mathfrak{A} \setminus \{0\}$ is expressible as a countable union of sets with non-zero intersection numbers.

proof Put 391D and 391J together.

391X Basic exercises (a) Show that a chargeable Boolean algebra is ccc, so is Dedekind complete iff it is Dedekind σ-complete.

(b) Show (i) that any subalgebra of a chargeable Boolean algebra is chargeable (ii) that a countable simple product of chargeable Boolean algebras is chargeable (iii) that any free product of chargeable Boolean algebras is chargeable.

(c)(i) Let $\mathfrak{A}$ be a Boolean algebra with a chargeable order-dense subalgebra. Show that $\mathfrak{A}$ is chargeable. (ii) Show that the Dedekind completion of a chargeable Boolean algebra is chargeable.

(d)(i) Show that the algebra of open-and-closed subsets of $\{0,1\}^I$ is chargeable for any set I. (ii) Show that the regular open algebra of $\mathbb{R}$ is chargeable.

(e)(i) Show that any principal ideal of a chargeable Boolean algebra is chargeable. (ii) Let $\mathfrak{A}$ be a chargeable Boolean algebra and $\mathcal{I}$ an order-closed ideal of $\mathfrak{A}$. Show that $\mathfrak{A}/\mathcal{I}$ is chargeable.

>(f) Show that a Boolean algebra is chargeable iff it is isomorphic to a subalgebra of a measurable algebra. (*Hint*: 324O, 392H.)

(g) Let $\mathfrak{A}$ be a Boolean algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is chargeable and weakly (σ,∞)-distributive; (ii) there is a strictly positive countably additive functional on $\mathfrak{A}$; (iii) there is a strictly positively completely additive functional on $\mathfrak{A}$.

(h) Explain how to use the Hahn-Banach theorem to prove 391G directly, without passing through 391F. (*Hint*: $S(\mathfrak{B})$ can be regarded as a subspace of $S(\mathfrak{A})$.)

>(i) Take $X=\{0,1,2,3\}$, $\mathfrak{A}=\mathcal{P}X$, $A=\{\{0,1\},\{0,2\},\{0,3\},\{1,2,3\}\}$. Show that the intersection number of A is $\frac{3}{5}$. (*Hint*: use 391I.) Show that if $a_0,\dots,a_n$ are *distinct* members of A then there is a set $J\subseteq\{0,\dots,n\}$, with $\#(J)\ge\frac{2}{3}(n+1)$, such that $\inf_{j\in J}a_j\ne 0$.

(j) Let $\mathfrak{A}$ be a Boolean algebra. For non-empty $A\subseteq\mathfrak{A}\setminus\{0\}$ write $\delta(A)$ for the intersection number of A. Show that for any non-empty $A\subseteq\mathfrak{A}\setminus\{0\}$, $\delta(A)=\inf\{\delta(I): I$ is a non-empty finite subset of $A\}$.

(k) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$. For $a_0,\dots,a_n\in\mathfrak{A}$ set $t(a_0,\dots,a_n)=\max\{m: m\in\mathbb{N},\ m\chi 1\le\sum_{i=0}^n\chi a_i\}$. Let $A\subseteq\mathfrak{A}$ be non-empty. Show that

$$\begin{aligned}\sup\{\tfrac{1}{n+1}t(a_0,\dots,a_n): a_0,\dots,a_n\in A\}\\ =\min\{\sup_{a\in A}\nu a:\nu \text{ is a non-negative additive functional on } \mathfrak{A},\ \nu 1=1\}.\end{aligned}$$

(This is the **Kelley covering number** of A.)

(l) Let $\mathfrak{A}$ be a Boolean algebra. (i) Show that the following are equiveridical: (α) there is a functional $\bar\mu$ such that $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra; (β) $L^\infty(\mathfrak{A})$ is a perfect Riesz space (definition: 356J). (ii) Show that in this case $\mathfrak{A}$ is a measurable algebra iff it is ccc.

391Y Further exercises (a) Show that in 391D and 391K we can replace 'weakly (σ,∞)-distributive' by 'weakly σ-distributive'.

(b) Show that $\mathcal{P}\mathbb{N}$ is chargeable but that the quotient algebra $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ is not ccc, therefore not chargeable.

(c)(i) Show that if X is a separable topological space, then its regular open algebra is chargeable. (ii) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces with chargeable regular open algebras. Show that their product has a chargeable regular open algebra.

(d) Let μ be Lebesgue measure on $[0,1]$, and Σ its domain. Let $\mathcal{A}$ be a non-empty family of non-empty subsets of X, with intersection number δ, and let $\mathcal{W}$ be the family of those sets $W\in\mathcal{P}X\widehat{\otimes}\Sigma$ such that $W^{-1}[\{t\}]\in\mathcal{A}$ for every $t\in[0,1]$. Set $\alpha=\inf_{W\in\mathcal{W}}\sup_{x\in X}\mu W[\{x\}]$. (i) Show that $\alpha\le\delta$. (ii) Give an example in which $\alpha<\delta$.

(e) Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, U a linear space and $\nu_0:\mathfrak{B}\to U$ an additive functional. Show that there is an additive functional $\nu:\mathfrak{A}\to U$ extending ν_0. (*Hint*: 361F.)

391 Notes and comments By the standards of this volume, this is an easy section; I note that I have hardly called on anything after Chapter 32, except for a reference to the construction $S(\mathfrak{A})$ in §361. I do ask for a bit of functional analysis (the Hahn-Banach theorem) in 391E.

391J-391K are due to Kelley 59; condition (ii) of 391J is called **Kelley's criterion**. It provides some sort of answer to the question 'which Boolean algebras carry strictly positive finitely additive functionals?', but leaves quite open the possibility that there is some more abstract criterion which is also necessary and sufficient. It is indeed a non-trivial exercise to find any ccc Boolean algebra which does not carry a strictly positive finitely additive functional. The first example published seems to have been that of Gaifman 64, which is described in Comfort & Negrepontis 82. But for the purposes of this book Gaifman's example has been superseded by Talagrand's example, presented in §394.

Kelley's criterion is a little unsatisfying. It is undoubtedly important (see 392F below), but at the same time the structure of the criterion – a special sequence of subsets of $\mathfrak{A}$ – is rather close to the structure of the conclusion; after all, one is, or can be represented by, a function from $\mathfrak{A}\setminus\{0\}$ to $\mathbb{N}$, while the other is a function from $\mathfrak{A}$ to $\mathbb{R}$. Also the actual intersection number of a family $A\subseteq\mathfrak{A}\setminus\{0\}$ can be hard to calculate; as often as not, the best method is to look at the additive functionals on $\mathfrak{A}$ (see 391Xi).

392 Submeasures

In §391 I looked at what we can deduce if a Boolean algebra carries a strictly positive finitely additive functional. There are important contexts in which we find ourselves with subadditive, rather than additive, functionals, and these are what I wish to investigate here. It turns out that, once we have found the right hypotheses, such functionals can also provide a criterion for measurability of an algebra (392G below). The argument runs through a new idea, using a result in finite combinatorics (392D).

At the end of the section I include notes on metrics associated with submeasures (392H) and on products of submeasures (392K).

392A Definition Let $\mathfrak{A}$ be a Boolean algebra. A **submeasure** on $\mathfrak{A}$ is a functional $\nu:\mathfrak{A}\to[0,\infty]$ such that

$\nu 0=0$,

$\nu a\le\nu b$ whenever $a\subseteq b$,

$\nu(a\cup b)\le\nu a+\nu b$ for all a, $b\in\mathfrak{A}$.

392B The following list mostly repeats ideas we have already used in the context of measures; but (b) and (c) are new, and will be the basis of this section.

Definitions Let $\mathfrak{A}$ be a Boolean algebra and $\nu:\mathfrak{A}\to[0,\infty]$ a submeasure.

(a) ν is **strictly positive** if $\nu a>0$ for every $a\ne0$.

(b) ν is **exhaustive** if $\lim_{n\to\infty}\nu a_n=0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$.

(c) ν is **uniformly exhaustive** if for every $\epsilon>0$ there is an $n\in\mathbb{N}$ such that there is no disjoint family $a_0,\dots,a_n$ with $\nu a_i\ge\epsilon$ for every $i\le n$.

(d) ν is **totally finite** if $\nu 1<\infty$.

(e) ν is **unital** if $\nu 1=1$.

(f) ν is **atomless** if whenever $a\in\mathfrak{A}$ and $\nu a>0$ there is a $b\subseteq a$ such that $\nu b>0$ and $\nu(a\setminus b)>0$.

(g) If ν' is another submeasure on $\mathfrak{A}$, then ν' is **absolutely continuous** with respect to ν if for every $\epsilon>0$ there is a $\delta>0$ such that $\nu'a\le\epsilon$ whenever $\nu a\le\delta$.

392C Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If there is an exhaustive strictly positive submeasure on $\mathfrak{A}$, then $\mathfrak{A}$ is ccc.

(b) A uniformly exhaustive submeasure on $\mathfrak{A}$ is exhaustive.

(c) Any non-negative additive functional on $\mathfrak{A}$ is a uniformly exhaustive submeasure.

proof These are all elementary. If $\nu:\mathfrak{A}\to[0,\infty]$ is an exhaustive strictly positive submeasure, and $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}\setminus\{0\}$, then $\{i:\nu a_i\ge2^{-n}\}$ must be finite for each n, so I is countable. (Cf. 322G.) If $\nu:\mathfrak{A}\to[0,\infty]$ is a uniformly exhaustive submeasure and $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint in $\mathfrak{A}$, then $\{i:\nu a_i\ge2^{-n}\}$ is finite for each n, so $\lim_{i\to\infty}\nu a_i=0$. If $\nu:\mathfrak{A}\to[0,\infty[$ is a non-negative additive functional, it is a submeasure, by 326Ba and 326Bf. If $\epsilon>0$, then take $n\ge\frac{1}{\epsilon}\nu1$; if $a_0,\dots,a_n$ are disjoint, then $\sum_{i=0}^n\nu a_i\le\nu1$, so $\min_{i\le n}\nu a_i<\epsilon$.

392D Lemma Suppose that k, l, $m \in \mathbb{N}$ are such that $3 \le k \le l \le m$ and $18mk \le l^2$. Let L, M be sets of sizes l, m respectively. Then there is a set $R \subseteq M \times L$ such that (i) each vertical section of R has just three members (ii) $\#(R[E]) \ge \#(E)$ whenever $E \in [M]^{\le k}$; so that for every $E \in [M]^{\le k}$ there is an injective function $f : E \to L$ such that $(x, f(x)) \in R$ for every $x \in E$.

Recall that $[M]^{\le k} = \{I : I \subseteq M, \#(I) \le k\}$, $[M]^k = \{I : I \subseteq M, \#(I) = k\}$ (3A1J).

proof (a) We need to know that $n! \ge 3^{-n}n^n$ for every $n \in \mathbb{N}$; this is immediate from the inequality

$$\sum_{i=2}^n \ln i \ge \int_1^n \ln x\,dx = n\ln n - n + 1 \text{ for every } n \ge 2.$$

(b) Let Ω be the set of those $R \subseteq M \times L$ such that each vertical section of R has just three members, so that

$$\#(\Omega) = \#([L]^3)^m = \bigl(\tfrac{l!}{3!(l-3)!}\bigr)^m.$$

Let us regard Ω as a probability space with the uniform probability.

If $F \in [L]^n$, where $3 \le n \le k$, and $x \in M$, then

$$\Pr(R[\{x\}] \subseteq F) = \tfrac{\#([F]^3)}{\#([L]^3)}$$

(because $R[\{x\}]$ is a random member of $[L]^3$)

$$= \tfrac{n(n-1)(n-2)}{l(l-1)(l-2)} \le \tfrac{n^3}{l^3}$$

as $n < l$. So if $E \in [M]^n$ and $F \in [L]^n$, then

$$\Pr(R[E] \subseteq F) = \prod_{x \in E} \Pr(R[\{x\}] \subseteq F)$$

(because the sets $R[\{x\}]$ are chosen independently)

$$\le \tfrac{n^{3n}}{l^{3n}}.$$

Accordingly

$$\Pr(\text{there is an } E \subseteq M \text{ such that } \#(R[E]) < \#(E) \le k)$$
$$\le \Pr(\text{there is a non-empty } E \subseteq M \text{ such that } \#(R[E]) \le \#(E) \le k)$$
$$= \Pr(\text{there is an } E \subseteq M \text{ such that } 3 \le \#(R[E]) \le \#(E) \le k)$$

(because if $E \ne \emptyset$ then $\#(R[E]) \ge 3$)

$$\le \sum_{n=3}^k \sum_{E \in [M]^n} \sum_{F \in [L]^n} \Pr(R[E] \subseteq F) \le \sum_{n=3}^k \#([M]^n)\#([L]^n)\frac{n^{3n}}{l^{3n}}$$

$$= \sum_{n=3}^k \frac{m!}{n!(m-n)!}\frac{l!}{n!(l-n)!}\frac{n^{3n}}{l^{3n}} \le \sum_{n=3}^k \frac{m^n l^n n^{3n}}{n!n!l^{3n}} \le \sum_{n=3}^k \frac{m^n n^n 3^{2n}}{l^{2n}}$$

(using (a))

$$= \sum_{n=3}^k \bigl(\frac{9mn}{l^2}\bigr)^n \le \sum_{n=3}^k \frac{1}{2^n} < 1.$$

There must therefore be some $R \in \Omega$ such that $\#(R[E]) \ge \#(E)$ whenever $E \subseteq M$ and $\#(E) \le k$.

(c) If now $E \in [M]^{\le k}$, the restriction $R_E = R \cap (E \times L)$ has the property that $\#(R_E[I]) \ge \#(I)$ for every $I \subseteq E$. By Hall's Marriage Lemma (3A1K) there is an injective function $f : E \to L$ such that $(x, f(x)) \in R_E \subseteq R$ for every $x \in E$.

Remark Of course this argument can be widely generalized; see references in Kalton & Roberts 83.

392E Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0, \infty]$ a uniformly exhaustive submeasure. Then for any $\epsilon \in]0, \nu 1]$ the set $A = \{a : \nu a \geq \epsilon\}$ has intersection number greater than 0.

proof (a) To begin with (down to the end of (d) below), suppose that $\nu 1 = 1$. Because ν is uniformly exhaustive, there is an $r \geq 1$ such that whenever $\langle c_i \rangle_{i \in I}$ is a disjoint family in $\mathfrak{A}$ then $\#(\{i : \nu c_i > \frac{1}{5}\epsilon\}) \leq r$, so that $\sum_{i \in I} \nu c_i \leq r + \frac{1}{5}\epsilon\#(I)$. Set $\delta = \epsilon/5r$, $\eta = \frac{1}{74}\delta^2$, so that

$$\delta - \eta \geq \tfrac{1}{18}(\delta - \eta)^2 \geq \tfrac{1}{18}(\delta^2 - 2\eta) = 4\eta.$$

(b) Let $\langle a_i \rangle_{i \in I}$ be a non-empty finite family in A. Let m be any multiple of $\#(I)$ greater than or equal to $1/\eta$. Then there are integers k, l such that

$$3\eta \leq \frac{k}{m} \leq 4\eta \leq \frac{1}{18}(\delta - \eta)^2, \quad \delta - \eta \leq \frac{l}{m} \leq \delta,$$

in which case

$$3 \leq k \leq l \leq m, \quad 18mk \leq m^2(\delta - \eta)^2 \leq l^2.$$

(c) Take a set M of the form $I \times S$ where $\#(S) = m/\#(I)$, so that $\#(M) = m$. For $x = (i, s) \in M$ set $d_x = a_i$. Let L be a set with l members. By 392D, there is a set $R \subseteq M \times L$ such that every vertical section of R has just three members and whenever $E \in [M]^{\leq k}$ there is an injective function $f_E : E \to L$ such that $(x, f_E(x)) \in R$ for every $x \in E$.

For $E \subseteq M$ set

$$b_E = \inf_{x \in E} d_x \setminus \sup_{x \in M \setminus E} d_x,$$

so that $\langle b_E \rangle_{E \subseteq M}$ is a partition of unity in $\mathfrak{A}$. For $x \in M$ and $j \in L$ set

$$c_{xj} = \sup\{b_E : x \in E \in [M]^{\leq k}, \, f_E(x) = j\}.$$

If x, y are distinct members of M and $j \in L$ then

$$c_{xj} \cap c_{yj} = \sup\{b_E : x, \, y \in E \in [M]^{\leq k}, \, f_E(x) = f_E(y) = j\} = 0,$$

because every f_E is injective. Set

$$m_j = \#(\{x : x \in M, \, c_{xj} \neq 0\})$$

for each $j \in L$. Note that $c_{xj} = 0$ if $(x, j) \notin R$, so $\sum_{j \in L} m_j \leq \#(R) = 3m$.

We have

$$\sum_{x \in M} \nu c_{xj} \leq r + \frac{1}{5}\epsilon m_j$$

for each j, by the choice of r; so

$$\sum_{x \in M, j \in L} \nu c_{xj} \leq rl + \frac{1}{5}\epsilon \sum_{j \in L} m_j \leq rl + \frac{3}{5}m\epsilon$$

$$\leq (r\delta + \frac{3}{5}\epsilon)m = \frac{4}{5}\epsilon m < \epsilon m$$

by the choice of l and δ. There must therefore be some $x \in M$ such that

$$\nu(\sup_{j \in L} c_{xj}) \leq \sum_{j \in L} \nu c_{xj} < \epsilon \leq \nu d_x,$$

and d_x cannot be included in

$$\sup_{j \in L} c_{xj} = \sup\{b_E : x \in E \in [M]^{\leq k}\}.$$

But as $\sup\{b_E : x \in E \subseteq M\}$ is just d_x, there must be an $E \subseteq M$, of cardinal greater than k, such that $b_E \neq 0$.

Recall now that $M = I \times S$, and that

$$k \geq 3\eta m = 3\eta\#(I)\#(S).$$

The set $J = \{i : \exists \, s, \, (i, s) \in E\}$ must therefore have more than $3\eta\#(I)$ members, since $E \subseteq J \times S$. But also $d_{(i,s)} = a_i$ for each $(i, s) \in E$, so that $\inf_{i \in J} a_i \supseteq b_E \neq 0$.

(d) As $\langle a_i \rangle_{i \in I}$ is arbitrary, the intersection number of A is at least $3\eta > 0$.

(e) This completes the proof in the case in which $\nu 1 = 1$. If $\nu 1 = 0$ the result is vacuous. If $\nu 1 > 0$, set $\nu' a = \frac{\min(\nu a, 1)}{\min(\nu 1, 1)}$ for each a; then it is easy to check that ν' is a uniformly exhaustive submeasure with $\nu' 1 = 1$, and

$$\{a : \nu a \ge \epsilon\} \subseteq \{a : \nu' a \ge \frac{\min(\epsilon, 1)}{\min(\nu 1, 1)}\}$$

has non-zero intersection number for any $\epsilon \in \,]0, \nu 1]$. So the result is true in the generality stated.

392F Theorem Let $\mathfrak{A}$ be a Boolean algebra with a strictly positive uniformly exhaustive submeasure. Then $\mathfrak{A}$ is chargeable, that is, has a strictly positive finitely additive functional.

proof If $\mathfrak{A} = \{0\}$ this is trivial. Otherwise, let $\nu : \mathfrak{A} \to [0, \infty]$ be a strictly positive uniformly exhaustive submeasure. For each n, $A_n = \{a : \nu a \ge \min(2^{-n}, \nu 1)\}$ has intersection number greater than 0, and $\bigcup_{n \in \mathbb{N}} A_n = \mathfrak{A} \setminus \{0\}$ because ν is strictly positive; so $\mathfrak{A}$ has a strictly positive finitely additive functional, by Kelley's theorem (391J).

392G Corollary Let $\mathfrak{A}$ be a Boolean algebra. Then it is measurable iff it is weakly (σ, ∞)-distributive and Dedekind σ-complete and has a strictly positive uniformly exhaustive submeasure.

proof Put 391D and 392F together.

392H This completes the main work of this section. However it will be convenient later to have some more facts available which belong to the same group of ideas.

Metrics from submeasures: Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν a strictly positive totally finite submeasure on $\mathfrak{A}$.

(a) We have a metric ρ on $\mathfrak{A}$ defined by the formula

$$\rho(a, b) = \nu(a \triangle b)$$

for all a, $b \in \mathfrak{A}$.

(b) The Boolean operations $\cup$, $\cap$, $\triangle$, $\setminus$ and the function $\nu : \mathfrak{A} \to \mathbb{R}$ are all uniformly continuous for ρ.

(c) The metric space completion $(\widehat{\mathfrak{A}}, \hat{\rho})$ of $(\mathfrak{A}, \rho)$ is a Boolean algebra under the natural continuous extensions of the Boolean operations, and ν has a unique continuous extension $\hat{\nu}$ to $\widehat{\mathfrak{A}}$ which is again a strictly positive submeasure.

(d) If ν is additive, then $(\widehat{\mathfrak{A}}, \hat{\nu})$ is a totally finite measure algebra.

proof (a)-(b) This is just a generalization of 323A-323B; essentially the same formulae can be used. For the triangle inequality for ρ, we have $a \triangle c \subseteq (a \triangle b) \cup (b \triangle c)$, so

$$\rho(a, c) = \nu(a \triangle c) \le \nu(a \triangle b) + \nu(b \triangle c) = \rho(a, b) + \rho(b, c).$$

For the uniform continuity of the Boolean operations, we have

$$(b * c) \triangle (b' * c') \subseteq (b \triangle b') \cup (c \triangle c')$$

so that

$$\rho(b * c, b' * c') \le \rho(b, b') + \rho(c, c')$$

for each of the operations $* = \cup$, $\cap$, $\setminus$ and $\triangle$ and all b, c, b', $c' \in \mathfrak{A}$. For the uniform continuity of the function ν itself, we have

$$\nu b \le \nu c + \nu(b \setminus c) \le \nu c + \rho(b, c),$$

so that $|\nu b - \nu c| \le \rho(b, c)$.

(c) $\mathfrak{A} \times \mathfrak{A}$ is a dense subset of $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$, so the Boolean operations on $\mathfrak{A}$, regarded as uniformly continuous functions from $\mathfrak{A} \times \mathfrak{A}$ to $\mathfrak{A} \subseteq \widehat{\mathfrak{A}}$, have unique extensions to continuous binary operations on $\widehat{\mathfrak{A}}$ (3A4G). If we look at

$$A = \{(a, b, c) : a \triangle (b \triangle c) = (a \triangle b) \triangle c\},$$

this is a closed subset of $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$, because the maps $(a, b, c) \mapsto a \triangle (b \triangle c)$, $(a, b, c) \mapsto (a \triangle b) \triangle c$ are continuous and the topology of $\widehat{\mathfrak{A}}$ is Hausdorff; since A includes the dense set $\mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$, it is the whole of $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$, that is, $a \triangle (b \triangle c) = (a \triangle b) \triangle c$ for all a, b, $c \in \widehat{\mathfrak{A}}$. All the other identities we need to show that $\widehat{\mathfrak{A}}$ is a Boolean algebra can be confirmed by the same method. Of course $\mathfrak{A}$ is now a subalgebra of $\widehat{\mathfrak{A}}$.

Because $\nu : \mathfrak{A} \to [0, \infty[$ is uniformly continuous, it has a unique continuous extension $\hat{\nu} : \widehat{\mathfrak{A}} \to [0, \infty[$. We have

$$\hat{\nu}0 = 0, \quad \hat{\nu}a \le \hat{\nu}(a \cup b) \le \hat{\nu}a + \hat{\nu}b, \quad \hat{\nu}a = \hat{\rho}(a, 0)$$

for every a, $b \in \mathfrak{A}$ and therefore for every a, $b \in \widehat{\mathfrak{A}}$, so $\hat{\nu}$ is a submeasure on $\widehat{\mathfrak{A}}$, and

$$\hat{\nu}a = 0 \Longrightarrow \hat{\rho}(a, 0) = 0 \Longrightarrow a = 0,$$

so $\hat{\nu}$ is strictly positive.

(d) We have $\nu(a \cup b) + \nu(a \cap b) = \nu a + \nu b$ for all a, $b \in \mathfrak{A}$; because all the operations are continuous, $\hat{\nu}(a \cup b) + \hat{\nu}(a \cap b) = \hat{\nu}a + \hat{\nu}b$ for all a, $b \in \widehat{\mathfrak{A}}$. In particular, since $\hat{\nu}0 = 0$, $\hat{\nu}$ is additive. Next, if $\langle a_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\widehat{\mathfrak{A}}$, $\hat{\rho}(a_m \mathbin{\triangle} a_n) = |\hat{\nu}a_m - \hat{\nu}a_n|$ for all m, $n \in \mathbb{N}$, and $\langle a_n \rangle_{n \in \mathbb{N}}$ is $\hat{\rho}$-Cauchy, therefore convergent to some $a \in \widehat{\mathfrak{A}}$. Since

$$a \cap a_n = \lim_{m \to \infty} a_m \cap a_n = a_n$$

for each n, $a \supseteq a_n$ for every n. If $b \in \widehat{\mathfrak{A}}$ is any upper bound for $\{a_n : n \in \mathbb{N}\}$, then

$$b \cap a = \lim_{n \to \infty} b \cap a_n = \lim_{n \to \infty} a_n = a$$

and $b \supseteq a$; thus a is the least upper bound of $\{a_n : n \in \mathbb{N}\}$.

So, first, if $\langle b_n \rangle_{n \in \mathbb{N}}$ is any sequence in $\widehat{\mathfrak{A}}$, and we set $a_n = \sup_{i \le n} b_i$ for each n, $\sup_{n \in \mathbb{N}} a_n$ is defined and must be equal to $\sup_{n \in \mathbb{N}} b_n$; accordingly $\widehat{\mathfrak{A}}$ is Dedekind σ-complete. Next, if $\langle b_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\widehat{\mathfrak{A}}$, and again we set $a_n = \sup_{i \le n} b_i$ for each n, $a = \sup_{n \in \mathbb{N}} a_n = \sup_{n \in \mathbb{N}} b_n$, we shall have

$$\hat{\nu}a = \lim_{n \to \infty} \hat{\nu}a_n = \lim_{n \to \infty} \sum_{i=0}^{n} \hat{\nu}b_i = \sum_{n=0}^{\infty} \hat{\nu}b_n;$$

which means that $\hat{\nu}$ is countably additive, and $(\widehat{\mathfrak{A}}, \hat{\nu})$ is a measure algebra.

392I Corollary Let $\mathfrak{A}$ be a Boolean algebra and ν a non-negative additive functional on $\mathfrak{A}$. Then there are a totally finite measure algebra $(\mathfrak{C}, \bar{\mu})$ and a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\nu a = \bar{\mu}(\pi a)$ for every $a \in \mathfrak{A}$.

proof Set $I = \{a : \nu a = 0\}$; then $I \lhd \mathfrak{A}$, so we can form the quotient algebra $\mathfrak{B} = \mathfrak{A}/I$ (312L); let $\pi : \mathfrak{A} \to \mathfrak{B}$ be the canonical map. As in part (b) of the proof of 321H, we have an additive functional $\mu : \mathfrak{B} \to [0, \infty[$ such that $\mu(\pi a) = \nu a$ for every $a \in \mathfrak{A}$, and (as in 321H) μ is strictly positive. Take $(\mathfrak{C}, \bar{\mu})$ to be $(\widehat{\mathfrak{B}}, \hat{\mu})$ as in 392Hd, so that $(\mathfrak{C}, \bar{\mu})$ is a totally finite measure algebra. If we now think of π as a map from $\mathfrak{A}$ to $\mathfrak{C}$, it will still be a Boolean homomorphism, and

$$\nu a = \mu(\pi a) = \bar{\mu}(\pi a)$$

for every $a \in \mathfrak{A}$.

392J Proposition Let $\mathfrak{A}$ be a Boolean algebra, ν an exhaustive submeasure on $\mathfrak{A}$, and $\langle a_n \rangle_{n \in \mathbb{N}}$ a sequence in $\mathfrak{A}$ such that $\inf_{n \in \mathbb{N}} \nu a_n > 0$. Then there is an infinite $I \subseteq \mathbb{N}$ such that $\nu(\inf_{i \in I \cap n} a_i) > 0$ for every $n \in \mathbb{N}$.

Remark In the formula $I \cap n$ I am identifying n with the set of its predecessors, as in 3A1H.

proof For finite $J \subseteq \mathbb{N}$ set $b_J = \inf_{i \in J} a_i$. Let $\mathcal{J}$ be the family of those $J \in [\mathbb{N}]^{<\omega}$ such that $\limsup_{n \to \infty} \nu(a_n \cap b_J) > 0$.

? Suppose, if possible, that there is no strictly increasing sequence in $\mathcal{J}$. Then $\mathcal{J}$ must have a maximal element J say. Set $a'_n = a_n \cap b_J$ for $n \in \mathbb{N}$ and $\delta = \limsup_{n \to \infty} \nu a'_n > 0$. For any $n \in \mathbb{N} \setminus J$, $J \cup \{n\} \notin \mathcal{J}$ so

$$\lim_{m \to \infty} a'_m \cap a'_n = \lim_{m \to \infty} a_m \cap b_{J \cup \{n\}} = 0.$$

We can therefore choose inductively a sequence $\langle k_n \rangle_{n \in \mathbb{N}}$ such that

$$k_n > \sup J, \quad \nu a'_{k_n} \ge \frac{3}{4}\delta, \quad \nu(a'_{k_n} \cap a'_{k_i}) \le 2^{-i-2}\delta \text{ for every } i < n$$

for each $n \in \mathbb{N}$. Now set $b_n = a_{k_n} \setminus \sup_{i<n} a_{k_i}$ for each n. Then $\langle b_n \rangle_{n \in \mathbb{N}}$ is disjoint. Also

$$\frac{3}{4}\delta \le \nu a_{k_n} \le \nu b_n + \sum_{i=0}^{n-1} \nu(a_{k_n} \cap a_{k_i}) \le \nu b_n + \sum_{i=0}^{n-1} 2^{-i-2}\delta \le \nu b_n + \frac{1}{2}\delta$$

and $\nu b_n \ge \frac{1}{4}\delta$ for every n; which is impossible. **X**

There must therefore be a strictly increasing sequence $\langle J_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{J}$. Set $I = \bigcup_{n \in \mathbb{N}} J_n$. If $n \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that $I \cap n \subseteq J_m$ and $\nu(\inf_{i \in I \cap n} a_i) \ge \nu b_{J_m} > 0$. So we have an appropriate I.

***392K Products of submeasures** There seems to be no fully satisfying general construction for products of submeasures. However the following method has some interesting features.

(a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with submeasures μ, ν respectively. On the free product $\mathfrak{A}\otimes\mathfrak{B}$ (§315), we have a functional $\mu\ltimes\nu$ defined by saying that whenever $c\in\mathfrak{A}\otimes\mathfrak{B}$ is of the form $\sup_{i\in I}a_i\otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$, then

$$\begin{aligned}(\mu\ltimes\nu)(c)&=\min_{J\subseteq I}\max(\{\mu(\sup_{i\in J}a_i)\}\cup\{\nu b_i:i\in I\setminus J\})\\&=\min\{\epsilon:\epsilon\in[0,\infty],\,\mu(\sup\{a_i:i\in I,\,\nu b_i>\epsilon\})\le\epsilon\}.\end{aligned}$$

P Every $c\in\mathfrak{A}\otimes\mathfrak{B}$ can be expressed in this form (315Oa). Of course this can be done in many different ways. But if $c=\sup_{j\in J}a'_j\otimes b'_j$ is another expression of the same kind, then $b_i=b'_j$ whenever $a_i\cap a'_j\ne0$. So

$$\begin{aligned}\sup\{a_i:i\in I,\,\nu b_i>\epsilon\}&=\sup\{a_i\cap a'_j:i\in I,\,j\in J,\,a_i\cap a'_j\ne0,\,\nu b_i>\epsilon\}\\&=\sup\{a_i\cap a'_j:i\in I,\,j\in J,\,a_i\cap a'_j\ne0,\,\nu b'_j>\epsilon\}\\&=\sup\{a'_j:j\in J,\,\nu b'_j>\epsilon\}\end{aligned}$$

for any ϵ, and the two calculations for $\mu\ltimes\nu$ give the same result. **Q**

Note that $(\mu\ltimes\nu)(a\otimes b)=\min(\mu a,\nu b)$ for all $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$.

(b) In the context of (a), $\mu\ltimes\nu$ is a submeasure.

P By definition, $(\mu\ltimes\nu)c\ge0$ for every $c\in\mathfrak{A}\otimes\mathfrak{B}$; and if $c=0$ then $c=1\otimes0$ and $(\mu\ltimes\nu)c=0$.

If c, c' are two members of $\mathfrak{A}\otimes\mathfrak{B}$, express them in the forms $c=\sup_{i\in I}a_i\otimes b_i$ and $c'=\sup_{j\in J}a'_j\otimes b'_j$ where $\langle a_i\rangle_{i\in I}$ and $\langle a'_j\rangle_{j\in J}$ are partitions of unity in $\mathfrak{A}$. Set $K=\{(i,j):a_i\cap a'_j\ne0\}\subseteq I\times J$, $a''_{ij}=a_i\cap a'_j$ for $(i,j)\in K$; then $\langle a''_{ij}\rangle_{(i,j)\in K}$ is a partition of unity in $\mathfrak{A}$, $c=\sup_{(i,j)\in K}a''_{ij}\otimes b_i$ and $c'=\sup_{(i,j)\in K}a''_{ij}\otimes b'_j$. Set $\alpha=(\mu\ltimes\nu)c$, $\beta=(\mu\ltimes\nu)c'$, $L=\{(i,j):(i,j)\in K,\,\nu b_i>\alpha\}$, $L'=\{(i,j):(i,j)\in K,\,\nu b'_j>\beta\}$, $e=\sup\{a_{ij}:(i,j)\in L\}$ and $e'=\sup\{a_{ij}:(i,j)\in L'\}$; then $\mu e\le\alpha$ and $\mu e'\le\beta$. So $\mu(e\cup e')\le\alpha+\beta$; but $e\cup e'=\sup_{(i,j)\in L\cup L'}a''_{ij}$ and

$$\nu(b_i\cup b'_j)\le\nu b_i+\nu b'_j\le\alpha+\beta$$

for all $(i,j)\in K\setminus(L\cup L')$. So $(\mu\ltimes\nu)(c\cup c')\le\alpha+\beta$.

If $c\subseteq c'$, then $b_i\subseteq b'_j$ for every $(i,j)\in K$. So $\nu b_i\le\beta$ for every $(i,j)\in K\setminus L'$ and $(\mu\ltimes\nu)c\le\beta$.

Thus $\mu\ltimes\nu$ is subadditive and order-preserving and is a submeasure. **Q**

(c) I note that only in exceptional cases will $\mu\ltimes\nu$ be matched with $\nu\ltimes\mu$ by the canonical isomorphism between $\mathfrak{A}\otimes\mathfrak{B}$ and $\mathfrak{B}\otimes\mathfrak{A}$; this product is not 'commutative'. (See 392Yc.) It is however 'associative', in the following sense. Let $(\mathfrak{A}_1,\mu_1)$, $(\mathfrak{A}_2,\mu_2)$, $(\mathfrak{A}_3,\mu_3)$ be Boolean algebras endowed with submeasures. Set

$$\lambda_{12}=\mu_1\ltimes\mu_2,\quad\lambda_{(12)3}=\lambda_{12}\ltimes\mu_3,\quad\lambda_{23}=\mu_2\ltimes\mu_3,\quad\lambda_{1(23)}=\mu_1\ltimes\lambda_{23}.$$

Then the canonical isomorphisms between $(\mathfrak{A}_1\otimes\mathfrak{A}_2)\otimes\mathfrak{A}_3$, $\mathfrak{A}_1\otimes\mathfrak{A}_2\otimes\mathfrak{A}_3$ and $\mathfrak{A}_1\otimes(\mathfrak{A}_2\otimes\mathfrak{A}_3)$ (315L) identify $\lambda_{(12)3}$ with $\lambda_{1(23)}$.

P Take any $d\in\mathfrak{A}_1\otimes\mathfrak{A}_2\otimes\mathfrak{A}_3$. Express d as $\sup_{i\in I}a_i\otimes e_i$ where $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}_1$ and $e_i\in\mathfrak{A}_2\otimes\mathfrak{A}_3$ for each i; express each e_i as $\sup_{j\in J_i}b_{ij}\otimes c_{ij}$ where $\langle b_{ij}\rangle_{j\in J_i}$ is a partition of unity in $\mathfrak{A}_2$ and $c_{ij}\in\mathfrak{A}_3$ for $i\in I$, $j\in J_i$. In this case, $\langle a_i\otimes b_{ij}\rangle_{i\in I,j\in J_i}$ is a partition of unity in $\mathfrak{A}_1\otimes\mathfrak{A}_2$ and $d=\sup_{i\in I,j\in J_i}a_i\otimes b_{ij}\otimes c_{ij}$.

Let $\epsilon>0$. For $i\in I$, set $J'_i=\{j:j\in J_i,\,\mu_3c_{ij}>\epsilon\}$, $e'_i=\sup_{j\in J'_i}b_{ij}$. Then $\lambda_{23}(\sup_{j\in J_i}b_{ij}\otimes c_{ij})\le\epsilon$ iff $\mu_2e'_i\le\epsilon$. Set $I'=\{i:\mu_2e'_i>\epsilon\}$; then $\lambda_{1(23)}d\le\epsilon$ iff $\mu_1(\sup_{i\in I'}a_i)\le\epsilon$. From the other direction, set $f=\sup\{a_i\otimes b_{ij}:i\in I,\,j\in J'_i\}$; then $\lambda_{(12)3}d\le\epsilon$ iff $\lambda_{12}f\le\epsilon$. But $f=\sup_{i\in I}a_i\otimes e'_i$, so $\lambda_{12}f\le\epsilon$ iff $\mu_1(\sup_{i\in I'}a_i)\le\epsilon$.

As ϵ and d are arbitrary, $\lambda_{(12)3}=\lambda_{1(23)}$, as claimed. **Q**

(d) If μ, μ' are submeasures on $\mathfrak{A}$, ν and ν' are submeasures on $\mathfrak{B}$, μ is absolutely continuous with respect to μ' and ν is absolutely continuous with respect to ν', then $\mu\ltimes\nu$ is absolutely continuous with respect to $\mu'\otimes\nu'$. **P** For any $\epsilon>0$ there is a $\delta>0$ such that $\mu a\le\epsilon$ whenever $\mu'a\le\delta$ and $\nu b\le\epsilon$ whenever $\nu'b\le\delta$. If now $c\in\mathfrak{A}\otimes\mathfrak{B}$ and $(\mu'\ltimes\nu')(c)\le\delta$, we have an expression $c=\sup_{i\in I}a_i\otimes b_i$ and a set $J\subseteq I$ such that $\langle a_i\rangle_{i\in I}$ is a partition of unity, $\mu'(\sup_{i\in J}a_i)\le\delta$ and $\nu'b_i\le\delta$ for every $i\in I\setminus J$; so $\mu(\sup_{i\in J}a_i)\le\epsilon$, $\nu b_i\le\epsilon$ for every $i\in I\setminus J$ and $(\mu\ltimes\nu)(c)\le\epsilon$. **Q**

(e) If μ and ν are exhaustive, so is $\mu\ltimes\nu$. **P** Let $\langle c_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}\otimes\mathfrak{B}$ such that $(\mu\ltimes\nu)c_n>\epsilon>0$ for every n. For each n, express c_n as $\sup_{i\in I_n}a_{ni}\otimes b_{ni}$ where $\langle a_{ni}\rangle_{i\in I_n}$ is a partition of unity; set $I'_n=\{i:i\in I_n,$

$\nu b_{ni} > \epsilon\}$, $a_n = \sup_{i \in I'_n} a_{ni}$; then $\mu a_n > \epsilon$. By 392J, there is an infinite $J \subseteq \mathbb{N}$ such that $\inf_{i \in J \cap n} a_i \neq 0$ for every $n \in \mathbb{N}$. Let Z be the Stone space of $\mathfrak{A}$, and write $\widehat{a} \subseteq Z$ for the open-and-closed set corresponding to $a \in \mathfrak{A}$; then there is a $z \in \bigcap_{n \in J} \widehat{a}_n$. For every $n \in J$ there is an $i_n \in I'_n$ such that $z \in \widehat{a}_{n,i_n}$. But now observe that $\nu b_{n,i_n} > \epsilon$ for every $n \in J$, so there must be distinct m, $n \in J$ such that $b_{m,i_m} \cap b_{n,i_n} \neq 0$; as $a_{m,i_m} \cap a_{n,i_n}$ is also non-zero, $c_m \cap c_n \neq 0$. As $\langle c_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\mu \ltimes \nu$ is exhaustive. **Q**

(f) We can extend the construction to infinite products, as follows. Let I be a totally ordered set and $\langle (\mathfrak{A}_i, \mu_i) \rangle_{i \in I}$ a family of Boolean algebras endowed with unital submeasures. For a finite set $J = \{i_0, \ldots, i_n\}$ where $i_0 < \ldots < i_n$ in I, let λ_J be the product submeasure $(.(\mu_{i_0} \ltimes \mu_{i_1}) \ltimes \ldots) \ltimes \mu_{i_n}$ on $\mathfrak{C}_J = \bigotimes_{j \in J} \mathfrak{A}_j$; for definiteness, on $\mathfrak{C}_\emptyset = \{0, 1\}$ take $\lambda_\emptyset$ to be the unital submeasure, while $\mathfrak{C}_{\{i\}} = \mathfrak{A}_i$ and $\lambda_{\{i\}} = \mu_i$ for each $i \in I$. Using (c) repeatedly, we see that if J, $K \in [I]^{<\omega}$ and $j < k$ for every $j \in J$, $k \in K$, then the identification of $\mathfrak{C}_{J \cup K}$ with $\mathfrak{C}_J \otimes \mathfrak{C}_K$ (315L) matches $\lambda_{J \cup K}$ with $\lambda_J \ltimes \lambda_K$. Moreover, if $K \in [I]^{<\omega}$ and J is any subset of K (not necessarily an initial segment) and $\varepsilon_{JK} : \mathfrak{C}_J \to \mathfrak{C}_K$ is the canonical embedding corresponding to the identification of $\mathfrak{C}_K$ with $\mathfrak{C}_J \otimes \mathfrak{C}_{K \setminus J}$, then $\lambda_J = \lambda_K \varepsilon_{JK}$; this also is an easy induction on $\#(K)$. What this means is that for any subset M of I we have a submeasure λ_M on $\mathfrak{C}_M = \bigcup \{\varepsilon_{JM} \mathfrak{C}_J : J \in [M]^{<\omega}\}$, being the unique functional such that $\lambda_M \varepsilon_{JM} = \lambda_J$ for every $J \in [M]^{<\omega}$. Finally, if L, M are subsets of I with $l < m$ for every $l \in L$ and $m \in M$, then $\lambda_{L \cup M}$ can be identified with $\lambda_L \ltimes \lambda_M$.

(g) I should perhaps have remarked already that if μ and ν, in (a), are additive and unital, then we have an additive function λ' on $\mathfrak{A} \otimes \mathfrak{B}$ such that $\lambda'(a \otimes b) = \mu a \cdot \nu b$ for every $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ (326E). Now, setting $\lambda = \mu \ltimes \nu$, each of λ, λ' is absolutely continuous with respect to the other. **P** If $c \in \mathfrak{A} \otimes \mathfrak{B}$, express c as $\sup_{i \in I} a_i \otimes b_i$ where $\langle a_i \rangle_{i \in I}$ is a finite partition of unity. Then $\mu(\sup\{a_i : \nu b_i > \lambda c\}) \leq \lambda c$, so $\lambda' c = \sum_{i \in I} \mu a_i \cdot \nu b_i$ is at most $2\lambda c$. On the other hand, $\mu(\sup\{a_i : \nu b_i > \sqrt{\lambda' c}\}) \leq \sqrt{\lambda' c}$, so $\lambda c \leq \sqrt{\lambda' c}$. **Q**

392X Basic exercises (a) Show that the first two clauses of the definition 392A can be replaced by '$\nu a \leq \nu(a \cup b) \leq \nu a + \nu b$ whenever $a \cap b = 0$'.

(b) Let $\mathfrak{A}$ be any Boolean algebra and ν a finite-valued submeasure on $\mathfrak{A}$. (i) Show that ν is order-continuous iff whenever $A \subseteq \mathfrak{A}$ is non-empty, downwards-directed and has infimum 0, then $\inf_{a \in A} \nu a = 0$. (ii) Show that in this case ν is exhaustive. (*Hint*: if $\langle a_n \rangle_{n \in \mathbb{N}}$ is disjoint, then $\bigcup_{n \in \mathbb{N}} \{b : b \supseteq a_i$ for every $i \geq n\}$ has infimum 0.)

(c) Let $\mathfrak{A}$ be a Boolean algebra and μ, ν two strictly positive submeasures on $\mathfrak{A}$, each of which is absolutely continuous with respect to the other. Show that they induce uniformly equivalent metrics on $\mathfrak{A}$ (392H), so that both give the same metric completion of $\mathfrak{A}$.

(d) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras with uniformly exhaustive submeasures μ, ν respectively. Show that $\mu \ltimes \nu$ is uniformly exhaustive.

392Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and $\lambda : \mathfrak{A} \to [0, 1]$ a functional such that $\lambda 0 = 0$ and $\lambda a \leq \lambda(a \cup b) \leq 2 \max(\lambda a, \lambda b)$ for all a, $b \in \mathfrak{A}$. Show that there is a submeasure ν on $\mathfrak{A}$ such that $\frac{1}{2}\lambda \leq \nu \leq \lambda$.

(b) (T.Jech) Show that a Boolean algebra $\mathfrak{A}$ is chargeable iff there are sequences $\langle A_n \rangle_{n \in \mathbb{N}}$ and $\langle k_n \rangle_{n \in \mathbb{N}}$ such that (α) $\bigcup_{n \in \mathbb{N}} A_n = \mathfrak{A} \setminus \{0\}$ (β) whenever a, $b \in \mathfrak{A}$, $n \in \mathbb{N}$ and $a \cup b \in A_n$ then at least one of a, b belongs to A_{n+1} (γ) if $n \in \mathbb{N}$ then $k_n \in \mathbb{N}$ and if $a_0, \ldots, a_{k_n} \in \mathfrak{A}$ are disjoint then some a_j does not belong to A_n.

(c) I will say that a submeasure ν on a Boolean algebra $\mathfrak{A}$ is **properly atomless** if for every $\epsilon > 0$ there is a finite partition A of unity in $\mathfrak{A}$ such that $\nu a \leq \epsilon$ for every $a \in A$. (Compare 326F.) (i) Show that if $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras with submeasures μ, ν respectively, we have a functional $\mu \rtimes \nu : \mathfrak{A} \otimes \mathfrak{B} \to [0, \infty]$ defined by saying that

$$(\mu \rtimes \nu)(\sup_{i \in I} a_i \otimes b_i) = \min_{J \subseteq I} \max(\{\nu(\sup_{i \in J} b_i)\} \cup \{\mu a_i : i \in I \setminus J\})$$

whenever $\langle b_i \rangle_{i \in I}$ is a finite partition of unity in $\mathfrak{B}$ and $a_i \in \mathfrak{A}$ for each $i \in I$. (ii) Show that if μ is a non-zero properly atomless submeasure, ν is a submeasure, and $\mu \ltimes \nu$ is absolutely continuous with respect to $\mu \rtimes \nu$, then ν is uniformly exhaustive.

(d) (See 328H.) Let $(I, \leq)$ be a non-empty upwards-directed partially ordered set, and $\langle (\mathfrak{A}_i, \bar{\mu}_i) \rangle_{i \in I}$ a family of probability algebras; suppose that $\pi_{ji} : \mathfrak{A}_i \to \mathfrak{A}_j$ is a measure-preserving Boolean homomorphism whenever $i \leq j$, and that $\pi_{ki} = \pi_{kj} \pi_{ji}$ whenever $i \leq j \leq k$. (i) Let $\mathcal{F}$ be the filter

$$\{A : A \subseteq I, \text{ there is some } i \in I \text{ such that } j \in A \text{ whenever } i \leq j\},$$

and set $\nu\langle a_i\rangle_{i\in I} = \limsup_{i\to\mathcal{F}} \bar{\mu}_i a_i$ for $\langle a_i\rangle_{i\in I} \in \prod_{i\in I} \mathfrak{A}_i$. Show that ν is a submeasure on $\mathfrak{A} = \prod_{i\in I} \mathfrak{A}_i$. (ii) Let $\mathcal{J}$ be the ideal $\{d : \nu d = 0\}$ of $\mathfrak{A}$, and $\mathfrak{D}$ the quotient algebra $\mathfrak{A}/\mathcal{J}$. Show that we have a strictly positive unital submeasure $\bar{\nu}$ on $\mathfrak{D}$ such that $\bar{\nu}d^\bullet = \nu d$ for every $d \in \mathfrak{A}$, and that $\mathfrak{D}$ is complete under the metric defined by $\bar{\nu}$. (iii) Show that for each $i \in I$ we have a Boolean homomorphism $\pi_i : \mathfrak{A}_i \to \mathfrak{D}$ defined by setting $\pi_i a = \langle a_j\rangle^\bullet_{j\in I}$, where $a_j = \pi_{ji}a$ if $j \geq i$, $0_{\mathfrak{A}_j}$ otherwise, and that $\bar{\nu}\pi_i = \bar{\mu}_i$. Show that $\pi_i = \pi_j\phi_{ji}$ whenever $i \leq j$. (iv) Show that $\mathfrak{D}_0 = \bigcup_{i\in I} \pi_i[\mathfrak{A}_i]$ is a subalgebra of $\mathfrak{D}$, and that $\bar{\nu}\restriction\mathfrak{D}_0$ is additive. (v) Let $\mathfrak{C}$ be the closure of $\mathfrak{D}_0$ in $\mathfrak{D}$, and set $\bar{\lambda} = \bar{\nu}\restriction\mathfrak{C}$. Show that $(\mathfrak{C}, \bar{\lambda})$ is a probability algebra. (vi) Now suppose that $(\mathfrak{B}, \bar{\nu})$ is a probability algebra, and that for each $i \in I$ we are given a measure-preserving Boolean homomorphism $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ such that $\phi_i = \phi_j\pi_{ji}$ whenever $i \leq j$. Show that there is a unique measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi\pi_i = \phi_i$ for every $i \in I$.

(e) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0,\infty]$ a submeasure. Show that the following are equiveridical: (i) ν is uniformly exhaustive; (ii) whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$, there is a set $I \subseteq \mathbb{N}$, not of zero asymptotic density, such that $a_i \cap a_j \neq 0$ for all i, $j \in I$; (iii) whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$, there is a set $I \subseteq \mathbb{N}$, not of zero asymptotic density, such that $\nu(\inf_{i\in I, i\leq n} a_i) > 0$ for every $n \in \mathbb{N}$.

392 Notes and comments Much of the first part of this section is a matter of generalizing earlier arguments. Thus 392C ought by now to be very easy, while 392Xb recalls the elementary theory of τ-additive functionals.

The new ideas are in the combinatorics of 392D-392E. I have cast 392D in the form of an argument in probability theory. Of course there is nothing here but simple counting, since the probability measure simply puts the same mass on each point of Ω, and every statement of the form 'Pr($R \ldots) \leq \ldots$' is just a matter of counting the elements R of Ω with the given property. But I think many of us find that the probabilistic language makes the calculations more natural; in particular, we can use intuitions associated with the notion of independence of events. Indeed I strongly recommend the method. It has been used to very great effect in the last sixty years in a wide variety of combinatorial problems. 392F and 392G together constitute the **Kalton-Roberts theorem** (Kalton & Roberts 83).

393 Maharam submeasures

Continuing our exploration of variations on the idea of 'measurable algebra', we come to sequentially order-continuous submeasures. These are associated with 'Maharam algebras' (393E), which share a great many properties with measurable algebras; for instance, the existence of a standard topology defined by the algebraic structure (393G). This topology is intimately connected with the order*-convergence of sequences introduced in §367 (393L). We can indeed characterize Maharam algebras in terms of properties of the order-sequential topology defined by this convergence (393Q).

393A Definition Let $\mathfrak{A}$ be a Boolean algebra. A **Maharam submeasure** or **continuous outer measure** on $\mathfrak{A}$ is a totally finite submeasure $\nu : \mathfrak{A} \to [0,\infty[$ such that $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

393B Lemma Let $\mathfrak{A}$ be a Boolean algebra and ν a Maharam submeasure on $\mathfrak{A}$.

(a) ν is sequentially order-continuous.

(b) ν is 'countably subadditive', that is, whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ and $a \in \mathfrak{A}$ is such that $a = \sup_{n\in\mathbb{N}} a \cap a_n$, then $\nu a \leq \sum_{n=0}^{\infty} \nu a_n$.

(c) If $\mathfrak{A}$ is Dedekind σ-complete, then ν is exhaustive.

proof (a) (Of course ν is an order-preserving function, by the definition of 'submeasure'; so we can apply the ordinary definition of 'sequentially order-continuous' in 313Hb.) (i) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum a, then $\langle a_n \setminus a\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0, so $\lim_{n\to\infty} \nu(a_n \setminus a) = 0$; but as

$$\nu a_n \leq \nu a \leq \nu a_n + \nu(a \setminus a_n)$$

for every n, it follows that $\nu a = \lim_{n\to\infty} \nu a_n$. (ii) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum a, then

$$\nu a \leq \nu a_n \leq \nu a + \nu(a_n \setminus a) \to \nu a$$

as $n \to \infty$.

(b) Set $b_n = \sup_{i \le n} a \cap a_i$; then $\nu b_n \le \sum_{i=0}^{n} \nu a_i$ for each n (inducing on n), so that

$$\nu a = \lim_{n\to\infty} \nu b_n \le \sum_{i=0}^{\infty} \nu a_i.$$

(c) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, set $b_n = \sup_{i\ge n} a_i$ for each n; then $\inf_{n\in\mathbb{N}} b_n = 0$, so

$$\limsup_{n\to\infty} \nu a_n \le \lim_{n\to\infty} \nu b_n = 0.$$

393C Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν a strictly positive Maharam submeasure on $\mathfrak{A}$. Then $\mathfrak{A}$ is ccc, Dedekind complete and weakly (σ, ∞)-distributive, and ν is order-continuous.

proof By 393Bc, ν is exhaustive; by 392Ca, $\mathfrak{A}$ is ccc; by 316Fa, $\mathfrak{A}$ is Dedekind complete; by 316Fc and 393Ba, ν is order-continuous

Now suppose that we have a sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed subsets of $\mathfrak{A}$, all with infimum 0. Let B be the set

$$\{b : b \in \mathfrak{A}, \forall\ n \in \mathbb{N}\ \exists\ a \in A_n \text{ such that } a \subseteq b\}.$$

As ν is order-continuous, $\inf_{a\in A_n} \nu a = 0$ for each n. Given $\epsilon > 0$, we can choose $\langle a_n \rangle_{n\in\mathbb{N}}$ such that $a_n \in A_n$ and $\nu a_n \le 2^{-n}\epsilon$ for each n; now $b = \sup_{n\in\mathbb{N}} a_n$ belongs to B and $\nu b \le \sum_{n=0}^{\infty} \nu a_n \le 2\epsilon$. Thus $\inf_{b\in B} \nu b = 0$. Since ν is strictly positive, $\inf B = 0$. As $\langle A_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive.

393D Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then it is measurable iff it is Dedekind σ-complete and carries a uniformly exhaustive strictly positive Maharam submeasure.

proof If $\mathfrak{A}$ is measurable, it surely satisfies the conditions, since any totally finite measure on $\mathfrak{A}$ is also a uniformly exhaustive strictly positive Maharam submeasure. If $\mathfrak{A}$ satisfies the conditions, then it is weakly (σ,∞)-distributive, by 393C, so 392G gives the result.

393E Maharam algebras (a) Definition A **Maharam algebra** is a Dedekind σ-complete Boolean algebra $\mathfrak{A}$ such that there is a strictly positive Maharam submeasure on $\mathfrak{A}$.

(b) Every measurable algebra is a Maharam algebra, while every Maharam algebra is ccc and weakly (σ,∞)-distributive (393C), therefore Dedekind complete. A Maharam algebra $\mathfrak{A}$ is measurable iff there is a strictly positive uniformly exhaustive submeasure on $\mathfrak{A}$. (Put 393C and 392G together again.)

(c)(i) A principal ideal in a Maharam algebra is a Maharam algebra; an order-closed subalgebra of a Maharam algebra is a Maharam algebra. **P** Let $\mathfrak{A}$ be a Maharam algebra and $\mathfrak{B}$ *either* a principal ideal of $\mathfrak{A}$ *or* an order-closed subalgebra of $\mathfrak{A}$. Because $\mathfrak{A}$ is Dedekind complete, so is $\mathfrak{B}$ (314Ea). Let $\nu : \mathfrak{A} \to [0,\infty[$ be a strictly positive Maharam submeasure. Then $\nu\restriction\mathfrak{B}$ is a strictly positive Maharam submeasure on $\mathfrak{A}$, so $\mathfrak{B}$ is a Maharam algebra. **Q**

(ii) The simple product of a countable family of Maharam algebras is a Maharam algebra. **P** Let $\langle \mathfrak{A}_i \rangle_{i\in I}$ be a countable family of Maharam algebras and $\mathfrak{A}$ its simple product. Then $\mathfrak{A}$ is Dedekind complete (315De). For each $i \in I$, let ν_i be a strictly positive Maharam submeasure on $\mathfrak{A}_i$. Let $\langle \epsilon_i \rangle_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I} \epsilon_i$ is finite. Set $\nu a = \sum_{i\in I} \min(\epsilon_i, \nu_i a(i))$ for $a \in \mathfrak{A}$; then ν is a strictly positive Maharam submeasure on $\mathfrak{A}$, so $\mathfrak{A}$ is a Maharam algebra. **Q**

393F Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν, ν' two Maharam submeasures on $\mathfrak{A}$ such that $\nu a = 0$ whenever $\nu' a = 0$. Then ν is absolutely continuous with respect to ν'.

proof (Compare 232Ba.) **?** Otherwise, we can find a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\nu' a_n \le 2^{-n}$ for every n, but $\epsilon = \inf_{n\in\mathbb{N}} \nu a_n > 0$. Set $b_n = \sup_{i\ge n} a_i$ for each n, $b = \inf_{n\in\mathbb{N}} b_n$. Then $\nu' b_n \le \sum_{i=n}^{\infty} 2^{-i} \le 2^{-n+1}$ for each n (393Bb), so $\nu' b = 0$; but $\nu b_n \ge \epsilon$ for each n, so $\nu b \ge \epsilon$ (393Ba), contrary to the hypothesis. **X**

393G Proposition Let $\mathfrak{A}$ be a Maharam algebra, and ν and ν' two strictly positive Maharam submeasures on $\mathfrak{A}$. Then the metrics they induce on $\mathfrak{A}$ are uniformly equivalent, so we have a topology and uniformity on $\mathfrak{A}$ which we may call the **Maharam-algebra topology** and the **Maharam-algebra uniformity**.

proof By 393F, ν and ν' are mutually absolutely continuous; translating this with the formula of 392Ha, we see that the metrics are uniformly equivalent, so induce the same topology and uniformity. As $\mathfrak{A}$ does have a strictly positive Maharam submeasure, we may use it to define the Maharam-algebra topology and uniformity of $\mathfrak{A}$.

393H Proposition Let $\mathfrak{A}$ be a Boolean algebra, and ν an exhaustive strictly positive totally finite submeasure on $\mathfrak{A}$. Let $\widehat{\mathfrak{A}}$ be the metric completion of $\mathfrak{A}$, as described in 392H, and $\hat{\nu}$ the continuous extension of ν to $\widehat{\mathfrak{A}}$. Then $\hat{\nu}$ is a Maharam submeasure, so $\widehat{\mathfrak{A}}$ is a Maharam algebra.

proof (Compare 392Hd.)

(a) The point is that any non-increasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\widehat{\mathfrak{A}}$ is a Cauchy sequence for the metric $\hat{\rho}$. **P** Let $\epsilon>0$. For each $n\in\mathbb{N}$, choose $b_n\in\mathfrak{A}$ such that $\hat{\rho}(a_n,b_n)\le 2^{-n}\epsilon$, and set $c_n=\inf_{i\le n}b_i$. Then

$$\hat{\rho}(a_n,c_n)=\hat{\rho}(\inf_{i\le n}a_i,\inf_{i\le n}b_i)\le\sum_{i=0}^n\hat{\rho}(a_i,b_i)\le 2\epsilon$$

for every n. Choose $\langle n(k)\rangle_{k\in\mathbb{N}}$ inductively so that, for each $k\in\mathbb{N}$, $n(k+1)\ge n(k)$ and

$$\nu(c_{n(k)}\setminus c_{n(k+1)})\ge\sup_{i\ge n(k)}\nu(c_{n(k)}\setminus c_i)-\epsilon.$$

Then $\langle c_{n(k)}\setminus c_{n(k+1)}\rangle_{k\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, so

$$\begin{aligned}\limsup_{k\to\infty}\sup_{i\ge n(k)}\hat{\rho}(a_{n(k)},a_i)&\le 4\epsilon+\limsup_{k\to\infty}\sup_{i\ge n(k)}\hat{\rho}(c_{n(k)},c_i)\\&=4\epsilon+\limsup_{k\to\infty}\sup_{i\ge n(k)}\nu(c_{n(k)}\setminus c_i)\\&\le 4\epsilon+\limsup_{k\to\infty}\nu(c_{n(k)}\setminus c_{n(k+1)})+\epsilon=5\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\langle a_n\rangle_{n\in\mathbb{N}}$ is Cauchy. **Q**

(b) It follows that $\widehat{\mathfrak{A}}$ is Dedekind σ-complete. **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\widehat{\mathfrak{A}}$, $\langle b_n\rangle_{n\in\mathbb{N}}=\langle\inf_{i\le n}a_i\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence with a limit $b\in\widehat{\mathfrak{A}}$. For any $k\in\mathbb{N}$,

$$\hat{\nu}(b\setminus a_k)=\lim_{n\to\infty}\hat{\nu}(b_n\setminus a_k)=0,$$

so $b\subseteq a_k$, because $\hat{\nu}$ is strictly positive. While if $c\in\widehat{\mathfrak{A}}$ is a lower bound for $\{a_n:n\in\mathbb{N}\}$, we have $c\subseteq b_n$ for every n, so

$$\hat{\nu}(c\setminus b)=\lim_{n\to\infty}\hat{\nu}(c\setminus b_n)=0$$

and $c\subseteq b$. Thus $b=\inf_{n\in\mathbb{N}}a_n$; as $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\widehat{\mathfrak{A}}$ is Dedekind σ-complete (314Bc). **Q**

(c) We find also that $\hat{\nu}$ is a Maharam submeasure, because if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\widehat{\mathfrak{A}}$ with infimum 0, it must have a limit a which (as in (b) above) must be its infimum, that is, $a=0$; consequently

$$\lim_{n\to\infty}\hat{\nu}a_n=\hat{\nu}a=0.$$

(d) It follows at once that $\hat{\nu}$ is exhaustive (393Bc), so that $\widehat{\mathfrak{A}}$ is ccc (392Ca) and Dedekind complete (316Fa).

393I Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν an atomless Maharam submeasure on $\mathfrak{A}$. Then for every $\epsilon>0$ there is a finite partition C of unity in $\mathfrak{A}$ such that $\nu c\le\epsilon$ for every $c\in C$.

proof Let $A\subseteq\mathfrak{A}$ be a maximal disjoint set such that $0<\nu a\le\epsilon$ for every $a\in A$. As ν is exhaustive (393Bc), A is countable. Set $c=1\setminus\sup A$. **?** If $\nu c>0$, then (because ν is atomless) we can choose inductively a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ such that $b_0=c$, $b_{n+1}\subseteq b_n$, $\nu b_{n+1}>0$ and $\nu(b_n\setminus b_{n+1})>0$ for every $n\in\mathbb{N}$. But now $\langle b_n\setminus b_{n+1}\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of elements of non-zero submeasure, so one of them has submeasure in $]0,\epsilon]$ and ought to have been added to A. **X**

If A is finite, we can set $C=A\cup\{c\}$ and stop. Otherwise, enumerate A as $\langle a_n\rangle_{n\in\mathbb{N}}$ and set $c_n=\sup_{i\ge n}a_i$ for each n; then $\lim_{n\to\infty}\nu c_n=0$, so there is an n such that $\nu c_n\le\epsilon$, and we can set $C=\{a_i:i<n\}\cup\{c_n\cup c\}$.

393J Lemma (Maharam 47) Let $\mathfrak{A}$ be a ccc Boolean algebra with a T_1 topology $\mathfrak{T}$ such that (i) $\cup:\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$ is continuous at $(0,0)$ (ii) whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then $\langle a_n\rangle_{n\in\mathbb{N}}\to 0$ for $\mathfrak{T}$. Then $\mathfrak{A}$ has a strictly positive Maharam submeasure.

proof (a) For any $e\in\mathfrak{A}\setminus\{0\}$, there is a Maharam submeasure ν on $\mathfrak{A}$ such that $\nu e>0$.

P(i) Choose a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0, as follows. Because $\mathfrak{T}$ is T_1, $G_0=\mathfrak{A}\setminus\{0\}$ is a neighbourhood of 0 not containing e. Given G_n, choose a neighbourhood G_{n+1} of 0 such that $G_{n+1}\subseteq G_n$ and

$a \cup b \cup c \in G_n$ whenever a, b, $c \in G_{n+1}$. (Take neighbourhoods H, H' of 0 such that $a \cup b \in G_n$ for a, $b \in H$, $b \cup c \in H$ for b, $c \in H'$ and set $G_{n+1} = H \cap H' \cap G_n$.) Define $\nu_0 : \mathfrak{A} \to [0,1]$ by setting

$$\begin{aligned}\nu_0 a &= 1 \text{ if } a \notin G_0,\\ &= 2^{-n} \text{ if } a \in G_n \setminus G_{n+1},\\ &= 0 \text{ if } a \in \bigcap_{n\in\mathbb{N}} G_n.\end{aligned}$$

Then whenever $a_0, \dots , a_r \in \mathfrak{A}$, $n \in \mathbb{N}$ and $\sum_{i=0}^{r} \nu_0 a_i < 2^{-n}$, $\sup_{i\le r} a_i \in G_n$. To see this, induce on r. If $r = 0$ then we have $\nu_0 a_0 < 2^{-n}$ so $a_0 \in G_{n+1} \subseteq G_n$. For the inductive step to $r \ge 1$, there must be a $k \le r$ such that $\sum_{i<k} \nu_0 a_i < 2^{-n-1}$ and $\sum_{k<i\le n} \nu_0 a_i < 2^{-n-1}$ (allowing $k = 0$ or $k = n$, in which case one of the sums will be zero). (If $\sum_{i=0}^{r} \nu_0 a_i < 2^{-n-1}$, take $k = n$; otherwise, take k to be the least number such that $\sum_{i=0}^{k} \nu_0 a_i \ge 2^{-n-1}$.) By the inductive hypothesis, and because 0 certainly belongs to G_{n+1}, $b = \sup_{i<k} a_i$ and $c = \sup_{k<i\le r} a_i$ both belong to G_{n+1}; but also $\nu_0 a_k < 2^{-n}$ so $a_k \in G_{n+1}$. Accordingly, by the choice of G_{n+1},

$$\sup_{i\le r} a_i = b \cup a_k \cup c$$

belongs to G_n, and the induction continues.

(ii) Set

$$\nu_1 a = \inf\{\textstyle\sum_{i=0}^{r} \nu_0 a_i : a_0, \dots , a_r \in \mathfrak{A},\ a = \sup_{i\le r} a_i\}$$

for every $a \in \mathfrak{A}$. It is easy to see that $\nu_1(a \cup b) \le \nu_1 a + \nu_1 b$ for all a, $b \in \mathfrak{A}$; also $a \in G_n$ whenever $\nu_1 a < 2^{-n}$, so, in particular, $\nu_1 e \ge 1$, because $e \notin G_0$.

Set

$$\nu a = \inf\{\nu_1 b : a \cap e \subseteq b \subseteq e\}$$

for every $a \in \mathfrak{A}$. Then of course $0 \le \nu a \le \nu b$ whenever $a \subseteq b$, and

$$\nu 0 \le \nu_1 0 \le \nu_0 0 = 0,$$

so $\nu 0 = 0$. If a, $b \in \mathfrak{A}$ and $\epsilon > 0$, there are a', b' such that $a \cap e \subseteq a' \subseteq e$, $b \cap e \subseteq b' \subseteq e$, $\nu_1 a' \le \nu a + \epsilon$ and $\nu_1 b' \le \nu b + \epsilon$; so that $(a \cup b) \cap e \subseteq a' \cup b' \subseteq e$ and

$$\nu(a \cup b) \le \nu_1(a' \cup b') \le \nu_1 a' + \nu_1 b' \le \nu a + \nu b + 2\epsilon.$$

As ϵ, a and b are arbitrary, ν is a submeasure. Next, if $\langle a_i\rangle_{i\in\mathbb{N}}$ is any non-increasing sequence in $\mathfrak{A}$ with infimum 0, $\langle a_i \cap e\rangle_{i\in\mathbb{N}}$ is another, so converges to 0 for $\mathfrak{T}$. If $n \in \mathbb{N}$ there is an m such that $a_i \cap e \in G_n$ for every $i \ge m$, so that

$$\nu a_i \le \nu_1(a_i \cap e) \le \nu_0(a_i \cap e) \le 2^{-n}$$

for every $i \ge m$. As n is arbitrary, $\lim_{i\to\infty} \nu a_i = 0$; as $\langle a_i\rangle_{i\in\mathbb{N}}$ is arbitrary, ν is a Maharam submeasure. Finally,

$$\nu e = \nu_1 e \ge 1,$$

so $\nu e \ne 0$. **Q**

(b) Write C for the set of those $c \in \mathfrak{A}$ such that there is a strictly positive Maharam submeasure on the principal ideal $\mathfrak{A}_c$. Then C is order-dense in $\mathfrak{A}$. **P** Take any $e \in \mathfrak{A} \setminus \{0\}$. By (a), there is a Maharam submeasure ν such that $\nu e > 0$. Set $A = \{e \setminus a : \nu a = 0\}$. Because ν is a submeasure, A is downwards-directed. **?** If $\inf A = 0$ then, because $\mathfrak{A}$ is ccc, there is a non-increasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A with infimum 0; because ν is a Maharam submeasure,

$$\nu e \le \inf_{n\in\mathbb{N}} \nu a_n + \nu(e \setminus a_n) = \inf_{n\in\mathbb{N}} \nu a_n = 0. \text{ **X**}$$

Thus A has a non-zero lower bound c, and $\nu\restriction\mathfrak{A}_c$ is a strictly positive Maharam submeasure, while $c \subseteq e$. **Q**

(c) Because $\mathfrak{A}$ is ccc, there is a sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in C with supremum 1. For each n, let ν_n be a strictly positive Maharam submeasure on $\mathfrak{A}_{c_n}$; multiplying by a scalar if necessary, we may suppose that $\nu_n c_n \le 2^{-n}$. We can therefore define $\nu : \mathfrak{A} \to [0, 2]$ by setting $\nu a = \sum_{n=0}^{\infty} \nu_n(a \cap c_n)$ for every $a \in \mathfrak{A}$, and it is easy to check that ν is a strictly positive Maharam submeasure on $\mathfrak{A}$.

***393K Theorem** Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra. Then $\mathfrak{A}$ is a Maharam algebra iff there is a Hausdorff linear space topology $\mathfrak{T}$ on $L^0(\mathfrak{A})$ such that for every neighbourhood G of 0 there is a neighbourhood H of 0 such that $u \in G$ whenever $v \in H$ and $|u| \le |v|$.

proof (a) Suppose that $\mathfrak{A}$ is a Maharam algebra; let ν be a strictly positive Maharam submeasure on $\mathfrak{A}$.

(i) For $u \in L^0 = L^0(\mathfrak{A})$ set

$$\tau(u) = \inf\{\alpha : \alpha \geq 0,\, \nu[\![|u| > \alpha]\!] \leq \alpha\}.$$

Then

$$\tau(u+v) \leq \tau(u) + \tau(v), \quad \tau(\alpha u) \leq \tau(u) \text{ if } |\alpha| \leq 1, \quad \lim_{\alpha\to 0} \tau(\alpha u) = 0$$

for every u, $v \in L^0$. **P** (i) It will save a moment if we observe that whenever $\beta > \tau(u)$ there is an $\alpha \leq \beta$ such that $\nu[\![|u| > \alpha]\!] \leq \alpha$, so that

$$\nu[\![|u| > \beta]\!] \leq \nu[\![|u| > \alpha]\!] \leq \alpha \leq \beta.$$

Also, because ν is sequentially order-continuous,

$$\nu[\![|u| > \tau(u)]\!] = \lim_{n\to\infty} \nu[\![|u| > \tau(u) + 2^{-n}]\!] \leq \lim_{n\to\infty} \tau(u) + 2^{-n} = \tau(u).$$

(ii) So

$$\begin{aligned}\nu[\![|u+v| > \tau(u)+\tau(v)]\!] &\leq \nu[\![|u|+|v| > \tau(u)+\tau(v)]\!])\\ &\leq \nu([\![|u| > \tau(u)]\!] \cup [\![|v| > \tau(v)]\!])\end{aligned}$$

(364Ea)

$$\leq \nu[\![|u| > \tau(u)]\!] + \nu[\![|v| > \tau(v)]\!] \leq \tau(u) + \tau(v),$$

and $\tau(u+v) \leq \tau(u) + \tau(v)$. (iii) If $|\alpha| \leq 1$ then

$$\nu[\![|\alpha u| > \tau(u)]\!] \leq \nu[\![|u| > \tau(u)]\!] \leq \tau(u),$$

and $\tau(\alpha u) \leq \tau(u)$. (iv) $\lim_{n\to\infty} \nu[\![|u| > n]\!] = 0$ because $\langle [\![|u| > n]\!] \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0. So if $\epsilon > 0$, there is an $n \geq 1$ such that $\nu[\![|u| > n\epsilon]\!] \leq \epsilon$, in which case $\nu[\![|\alpha u| > \epsilon]\!] \leq \epsilon$ whenever $|\alpha| \leq \frac{1}{n}$, so that $\tau(\alpha u) \leq \epsilon$ whenever $|\alpha| \leq \frac{1}{n}$. As ϵ is arbitrary, $\lim_{\alpha\to 0} \tau(\alpha u) = 0$. **Q**

(ii) Accordingly we have a metric $(u, v) \mapsto \tau(u-v)$ which defines a linear space topology $\mathfrak{T}$ on L^0 (2A5B). Now let G be an open set containing 0. Then there is an $\epsilon > 0$ such that $H = \{u : \tau(u) < \epsilon\}$ is included in G. If $v \in H$ and $|u| \leq |v|$, then

$$\nu[\![|u| > \tau(v)]\!] \leq \nu[\![|v| > \tau(v)]\!] \leq \tau(v),$$

so $\tau(u) \leq \tau(v)$ and $u \in H \subseteq G$. So $\mathfrak{T}$ satisfies all the conditions.

(b) Given such a topology $\mathfrak{T}$ on L^0, let $\mathfrak{S}$ be the topology on $\mathfrak{A}$ induced by $\mathfrak{T}$ and the function $\chi : \mathfrak{A} \to L^0$; that is, $\mathfrak{S} = \{\chi^{-1}[G] : G \in \mathfrak{T}\}$. Then $\mathfrak{S}$ satisfies the conditions of 393J. **P** (i) Because $\mathfrak{T}$ is Hausdorff and χ is injective, $\mathfrak{S}$ is Hausdorff, therefore T_1. (ii) If $0 \in G \in \mathfrak{S}$, there is an $H \in \mathfrak{T}$ such that $G = \chi^{-1}[H]$. Now 0 (the zero of L^0) belongs to H, so there is an open set H_1 containing 0 such that $u \in H$ whenever $v \in H_1$ and $|u| \leq |v|$. Next, addition on L^0 is continuous for $\mathfrak{T}$, so there is an open set H_2 containing 0 such that $u+v \in H_1$ whenever u, $v \in H_2$. Consider $G' = \chi^{-1}[H_2]$. This is an open set in $\mathfrak{A}$ containing $0_{\mathfrak{A}}$, and if a, $b \in G'$ then

$$|\chi(a \cup b)| \leq \chi a + \chi b \in H_2 + H_2 \subseteq H_1,$$

so $\chi(a \cup b) \in H$ and $a \cup b \in G$. As G is arbitrary, $\cup$ is continuous at $(0, 0)$. (iii) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, $u_0 = \sup_{n\in\mathbb{N}} n\chi a_n$ is defined in L^0 (use the criterion of 364L(a-i):

$$\inf_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} [\![n\chi a_n > m]\!] = \inf_{m\in\mathbb{N}} a_{m+1} = 0.)$$

If $0 \in G \in \mathfrak{S}$, take $H \in \mathfrak{T}$ such that $G = \chi^{-1}[H]$, and $H_1 \in \mathfrak{T}$ such that $0 \in H_1$ and $u \in H$ whenever $v \in H_1$ and $|u| \leq |v|$. Because scalar multiplication is continuous for $\mathfrak{T}$, there is a $k \geq 1$ such that $\frac{1}{k}u_0 \in H_1$. For any $n \geq k$, $\chi a_n \leq \frac{1}{k}u_0$ so $\chi a_n \in H$ and $a_n \in G$. As G is arbitrary, $\langle a_n \rangle_{n\in\mathbb{N}} \to 0$ for $\mathfrak{S}$. As $\langle a_n \rangle_{n\in\mathbb{N}}$ is arbitrary, condition (ii) in the statement of 393J is satisfied. **Q**

So 393J tells us that $\mathfrak{A}$ has a strictly positive Maharam submeasure, and is a Maharam algebra.

393L I now turn to some very remarkable ideas relating the order*-convergence of §367 to the questions here.

Definition Let P be a lattice, and consider the relation '$\langle p_n \rangle_{n\in\mathbb{N}}$ order*-converges to p' as a relation between $P^{\mathbb{N}}$ and P. By 367Bc, this satisfies the hypothesis of 3A3Pa, so there is a unique topology on P for which a set $F \subseteq P$

is closed iff $a \in F$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in F which order*-converges to a in P. I will call this topology the **order-sequential topology** of P.

Warning! For the next few paragraphs I shall be closely following the papers Balcar Głowczyński & Jech 98 and Balcar Jech & Pazák 05. I should therefore note explicitly that if $\mathfrak{A}$ is a Boolean algebra which is neither Dedekind σ-complete nor ccc, my 'order-sequential topology' on $\mathfrak{A}$ may not be identical to theirs.

393M Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) A sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ order*-converges to $a \in \mathfrak{A}$ iff there is a partition B of unity in $\mathfrak{A}$ such that $\{n : n \in \mathbb{N}, (a_n \triangle a) \cap b \neq 0\}$ is finite for every $b \in B$.

(b) If $\langle a_n \rangle_{n\in\mathbb{N}}$ order*-converges to a and $c \in \mathfrak{A}$, then $\langle a_n \cup c \rangle_{n\in\mathbb{N}}$, $\langle a_n \cap c \rangle_{n\in\mathbb{N}}$ and $\langle a_n \triangle c \rangle_{n\in\mathbb{N}}$ order*-converge to $a \cup c$, $a \cap c$ and $a \triangle c$ respectively.

(c) The operations $\cap$, $\cup$ and $\triangle$ are separately continuous for the order-sequential topology.

(d) Every disjoint sequence in $\mathfrak{A}$ is order*-convergent to 0.

proof (a) Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ and $a \in \mathfrak{A}$; set

$$C = \{c : \exists\, n \in \mathbb{N},\ c \subseteq a_i \text{ for every } i \geq n\},$$

$$D = \{d : \exists\, n \in \mathbb{N},\ a_i \subseteq d \text{ for every } i \geq n\}.$$

(i) If $\langle a_n \rangle_{n\in\mathbb{N}}$ order*-converges to a, then $a = \sup C = \inf D$ (367Be). Since

$$\inf\{d \setminus a : d \in D\} = \inf\{a \setminus c : c \in C\} = 0,$$

$$E = \{(d \setminus a) \cup (a \setminus c) : c \in C,\ d \in D\}$$

also has infimum 0 (313A, 313B). So there is a partition B of unity such that for every $b \in B$ there is an $e \in E$ such that $b \cap e = 0$. Now, given $b \in B$, there are $c \in C$ and $d \in D$ such that $b \cap (d \setminus c) = 0$; there are n_1, $n_2 \in \mathbb{N}$ such that $c \subseteq a_n$ for $n \geq n_1$ and $a_n \subseteq d$ for $n \geq n_2$; so that $\{n : (a_n \triangle a) \cap b \neq 0\}$ is bounded above by $\max(n_1, n_2)$ and is finite. So B witnesses that the condition is satisfied.

(ii) Now suppose that B is a partition of unity such that $\{n : (a_n \triangle a) \cap b \neq 0\}$ is finite for every $b \in B$. Then $a \cup (1 \setminus b) \in D$ for every $b \in B$, because $\{n : a_n \not\subseteq a \cup (1 \setminus b)\} \subseteq \{n : (a_n \triangle a) \cap b \neq 0\}$ is finite. So any lower bound for D is also a lower bound for $\{a \cup (1 \setminus b) : b \in B\}$ and is included in a. Similarly, any upper bound for C includes a; as $c \subseteq d$ whenever $c \in C$ and $d \in D$, $a = \sup C = \inf D$ and $\langle a_n \rangle_{n\in\mathbb{N}}$ order*-converges to a.

(b) These are all immediate from (a), because

$$(a_n \cup c) \triangle (a \cup c) \subseteq a_n \triangle a, \quad (a_n \cap c) \triangle (a \cap c) \subseteq a_n \triangle a,$$

$$(a_n \triangle c) \triangle (a \triangle c) = a_n \triangle a$$

for every n.

(c) By (b), we can apply 3A3Pb to each of the functions $a \mapsto a \cap b = b \cap a$, $a \mapsto a \cup b = b \cup a$ and $a \mapsto a \triangle b = b \triangle a$ to see that these are all continuous for every $b \in \mathfrak{A}$.

(d) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, there is a partition B of unity in $\mathfrak{A}$ containing every a_n (311Gd); now B witnesses that the condition of (a) is satisfied.

393N Proposition Let $\mathfrak{A}$ be a Maharam algebra. Then the Maharam-algebra topology on $\mathfrak{A}$ is the order-sequential topology.

proof Let $\mathfrak{T}_o$ be the order-sequential topology on $\mathfrak{A}$, ν a strictly positive Maharam submeasure on $\mathfrak{A}$, ρ the metric defined from ν (392H) and $\mathfrak{T}_M$ the Maharam-algebra topology induced by ρ (393G).

(a) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\langle a_n \rangle_{n\in\mathbb{N}} \to^* a$ in $\mathfrak{A}$, then $\lim_{n\to\infty} \rho(a_n, a) = 0$. **P** By 393Mb, $\langle a_n \triangle a \rangle_{n\in\mathbb{N}} \to^* 0$; by 367Bf, $0 = \inf_{n\in\mathbb{N}} \sup_{i\geq n}(a_i \triangle a)$, so

$$\rho(a_n, a) \leq \nu(\sup_{i\geq n} a_i \triangle a) \to 0$$

as $n \to \infty$. **Q**

It follows that every $\mathfrak{T}_M$-closed set is $\mathfrak{T}_o$-closed, and $\mathfrak{T}_o \subseteq \mathfrak{T}_M$.

(b) Conversely, suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ converging for $\mathfrak{T}_M$ to $a\in\mathfrak{A}$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ has a subsequence $\langle a'_n\rangle_{n\in\mathbb{N}}$ such that $\rho(a'_n,a)\le 2^{-n}$ for every $n\in\mathbb{N}$. In this case, setting $b_m=\sup_{n\ge m}a'_n\triangle a$ for each m, $\nu b_m\le 2^{-m+1}$ for every m (393Bb), so $\inf_{m\in\mathbb{N}}b_m=0$, and $\langle a'_n\triangle a\rangle_{n\in\mathbb{N}}\to^* 0$, that is, $\langle a'_n\rangle_{n\in\mathbb{N}}\to^* a$.

Thus every $\mathfrak{T}_M$-convergent sequence has an order*-convergent subsequence with the same limit; it follows that every $\mathfrak{T}_o$-closed set is $\mathfrak{T}_M$-closed, that is, $\mathfrak{T}_M\subseteq\mathfrak{T}_o$.

393O Proposition Let $\mathfrak{A}$ be a ccc Dedekind σ-complete Boolean algebra, with its order-sequential topology, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Then the topological closure of $\mathfrak{B}$ is the smallest order-closed set including $\mathfrak{B}$; in particular, $\mathfrak{B}$ is order-closed iff it is topologically closed.

proof (a) Let $\overline{\mathfrak{B}}$ be the topological closure of $\mathfrak{B}$, and $\mathfrak{B}^\sim$ the smallest order-closed set including $\mathfrak{B}$.

(i) Suppose that $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\overline{\mathfrak{B}}$ with supremum b in $\mathfrak{A}$; then $\langle b_n\rangle_{n\in\mathbb{N}}\to^* b$, by 367Bf or 367Xa. So $b\in\overline{\mathfrak{B}}$. Similarly, $\inf_{n\in\mathbb{N}}b_n\in\overline{\mathfrak{B}}$ for every non-increasing sequence in $\overline{\mathfrak{B}}$. Thus $\overline{\mathfrak{B}}$ is sequentially order-closed. But this means that it is order-closed, by 316Fb. So $\overline{\mathfrak{B}}\supseteq\mathfrak{B}^\sim$.

(ii) By 313Fc, $\mathfrak{B}^\sim$ is a subalgebra of $\mathfrak{A}$. Now suppose that $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}^\sim$ which order*-converges to $a\in\mathfrak{A}$. Then $c_{mn}=\sup_{m\le i\le n}b_i$ belongs to $\mathfrak{B}^\sim$ whenever $m\le n$; as $\langle c_{mn}\rangle_{n\ge m}$ is non-decreasing, $c_m=\sup_{i\ge m}b_i=\sup_{n\ge m}c_{mn}$ belongs to $\mathfrak{B}^\sim$ for every $m\in\mathbb{N}$; as $\langle c_m\rangle_{m\in\mathbb{N}}$ is non-increasing, $\inf_{m\in\mathbb{N}}c_m\in\mathfrak{B}^\sim$. But $c=b$ (367Bf). As $\langle b_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{B}^\sim$ is closed for the order-sequential topology, and must include $\overline{\mathfrak{B}}$.

Thus $\overline{\mathfrak{B}}=\mathfrak{B}^\sim$, as claimed.

(b) Now

$$\mathfrak{B}\text{ is order-closed}\iff\mathfrak{B}=\mathfrak{B}^\sim\iff\mathfrak{B}=\overline{\mathfrak{B}}\iff\mathfrak{B}\text{ is topologically closed.}$$

393P Lemma Let $\mathfrak{A}$ be a ccc weakly (σ,∞)-distributive Boolean algebra, endowed with its order-sequential topology.

(a) If $\langle a_{mn}\rangle_{m,n\in\mathbb{N}}$, $\langle a_m\rangle_{m\in\mathbb{N}}$ and a are such that $\langle a_{mn}\rangle_{n\in\mathbb{N}}$ order*-converges to a_m for each m, while $\langle a_m\rangle_{m\in\mathbb{N}}$ order*-converges to a, then there is a sequence $\langle k(m)\rangle_{m\in\mathbb{N}}$ in $\mathbb{N}$ such that $\langle a_{m,k(m)}\rangle_{m\in\mathbb{N}}$ order*-converges to a.

(b) If $A\subseteq\mathfrak{A}$ and $a\in\overline{A}$, there is a sequence in A which order*-converges to a.

(c) If G is a neighbourhood of 0 in $\mathfrak{A}$ then there is an open neighbourhood H of 0, included in G, such that $[0,a]\subseteq H$ for every $a\in H$.

(d) For $A\subseteq\mathfrak{A}$, set $\bigvee_0(A)=\{0\}$ and $\bigvee_{n+1}(A)=\{a\cup b:a\in\bigvee_n(A),\ b\in A\}$ for $n\in\mathbb{N}$.

(i) If $A\subseteq\mathfrak{A}$ is such that $[0,a]\subseteq A$ for every $a\in A$, and $n\in\mathbb{N}$, then $[0,a]\subseteq\bigvee_n(A)$ for every $a\in\bigvee_n(A)$.

(ii) If $H\subseteq\mathfrak{A}$ is an open set containing 0 such that $[0,a]\subseteq H$ for every $a\in H$, then $\bigvee_{n+1}(H)$ is open and $\overline{\bigvee_n(H)}\subseteq\bigvee_{n+1}(H)$ for every $n\in\mathbb{N}$.

(e) Suppose that $\mathfrak{A}$ is Dedekind σ-complete. Then for every open set G containing 0 there is an open set H containing 0 such that $\bigvee_3(H)\subseteq\bigvee_2(G)$.

proof (a) Let C_m, for $m\in\mathbb{N}$, be partitions of unity in $\mathfrak{A}$ such that

$$\{m:(a_m\triangle a)\cap c\ne 0\}\text{ is finite for every }c\in C_0,$$

$$\{n:(a_{mn}\triangle a_m)\cap c\ne 0\}\text{ is finite whenever }m\in\mathbb{N}\text{ and }c\in C_{m+1}$$

(393Ma). Because $\mathfrak{A}$ is weakly (σ,∞)-distributive, there is a partition B of unity such that $\{c:c\in C_m,\ c\cap b\ne 0\}$ is finite whenever $m\in\mathbb{N}$ and $b\in B$ (316H(ii)). Because $\mathfrak{A}$ is ccc, there is a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ running over $B\cup\{0\}$. Now, for each m, any sufficiently large $k(m)$ will be such that $(a_{m,k(m)}\triangle a_m)\cap b_i=0$ for every $i\le m$. In this case, for any i,

$$\{m:(a_{m,k(m)}\triangle a)\cap b_i\ne 0\}\subseteq\{m:m<i\}\cup\{m:(a_m\triangle a)\cap b_i\ne 0\}$$

is finite, so B witnesses that $\langle a_{m,k(m)}\rangle_{m\in\mathbb{N}}\to^* a$ (393Ma, in the other direction).

(b) Let $A^\sim$ be the set of order*-limits of sequences in A. Of course $A^\sim$ must be included in $\overline{A}$. But from (a) we see that the limit of any order*-convergent sequence in $A^\sim$ belongs to $A^\sim$. So $A^\sim$ is closed and is equal to $\overline{A}$. Turning this round, we see that $\overline{A}$ is just the set of order*-limits of sequences in A, as claimed.

(c) Set $D=\{d:d\in\mathfrak{A},\ [0,d]\not\subseteq G\}$, $H=\mathfrak{A}\setminus\overline{D}$. Since $D\supseteq\mathfrak{A}\setminus G$, H is an open subset of G.

? If $0 \in \overline{D}$, then (b) tells us that there is a sequence $\langle d_n \rangle_{n\in\mathbb{N}}$ in D order*-converging to 0. Now there is for each $n \in \mathbb{N}$ a $c_n \subseteq d_n$ such that $c_n \notin G$. By 367Be or 393Ma, $\langle c_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0, and $0 \in \overline{\mathfrak{A} \setminus G}$; but G is supposed to be a neighbourhood of 0. **X** Thus $0 \in H$ and H is a neighbourhood of 0.

? If $a \in H$ and $b \in [0,a] \setminus H$, then $b \in \overline{D}$, so there is a sequence $\langle d_n \rangle_{n\in\mathbb{N}}$ in D order*-converging to b. In this case, $\langle d_n \cup a \rangle_{n\in\mathbb{N}}$ order*-converges to $b \cup a = a$, by 393Mb. But also $[0, d_n \cup a] \supseteq [0, d_n]$ is not included in G, so $d_n \cup a \in D$ for each n, and $a \in \overline{D}$; which is impossible. **X** Thus $[0,a] \subseteq H$ for every $a \in H$, and H has the properties declared.

(d)(i) This is an elementary induction on n.

(ii) The point is that $\bigvee_{n+1}(H) = \{a \triangle b : a \in \bigvee_n(H),\, b \in H\}$. **P** If $a \in \bigvee_n(H)$ and $b \in H$, then $a \setminus b \in \bigvee_n(H)$, by (i), and $b \setminus a \in H$, so $a \triangle b \in \bigvee_{n+1}(H)$. On the other hand, if $c \in \bigvee_{n+1}(H)$, it is expressible as $a \cup b = a \triangle (b \setminus a)$ where $a \in \bigvee_n(H)$ and b and $b \setminus a$ belong to H. **Q**

Since $\triangle$ is separately continuous, it follows at once that

$$\bigvee_{n+1}(H) = \bigcup_{a\in\bigvee_n(H)}\{a \triangle b : b \in H\} = \bigcup_{a\in\bigvee_n(H)}\{b : a \triangle b \in H\}$$

is open, because $\triangle$ is separately continuous (393Mc). Next, if $d \in \overline{\bigvee_n(H)}$, then there is a sequence $\langle d_n \rangle_{n\in\mathbb{N}}$ in $\bigvee_n(H)$ order*-converging to d, by (b). Now $\langle d_n \triangle d \rangle_{n\in\mathbb{N}} \to^* 0$, by 393Mb, so $\langle d_n \triangle d \rangle_{n\in\mathbb{N}}$ converges topologically to 0, by 3A3Pa, and there is an $n \in \mathbb{N}$ such that $d_n \triangle d \in H$; in which case $d = d_n \triangle (d_n \triangle d)$ belongs to $\bigvee_{n+1}(H)$. As d is arbitrary, $\overline{\bigvee_n(H)} \subseteq \bigvee_{n+1}(H)$.

(e) ? Suppose, if possible, otherwise.

(i) Choose H_n, a_n, b_n and c_n inductively, as follows. $H_0 \subseteq G$ is to be an open neighbourhood of 0 such that $[0,a] \subseteq H_0$ whenever $a \in H_0$ ((c) above). Given that H_n is an open set containing 0 and including $[0,a]$ whenever it contains a, we are supposing that $\bigvee_3(H_n) \not\subseteq \bigvee_2(G)$; choose $a_n, b_n, c_n \in H_n$ such that $a_n \cup b_n \cup c_n \notin \bigvee_2(G)$, and set

$$H_{n+1} = \{a : a,\ a \cup a_n,\ a \cup b_n \text{ and } a \cup c_n \text{ all belong to } H_n\},$$

so that H_{n+1} is an open set containing 0, and $[0,a] \subseteq H_{n+1}$ for every $a \in H_{n+1}$. Continue.

(ii) At the end of the induction, set $F = \bigcap_{n\in\mathbb{N}} \overline{H}_n$ and $a^* = \inf_{n\in\mathbb{N}} \sup_{i\ge n} a_i$. Then $a^* \cup d \in F$ for every $d \in F$. **P** For $m \le n \in \mathbb{N}$, $\sup_{m\le i\le n} a_i \cup d \in H_m$ for every $d \in H_{n+1}$ (induce downwards on m). Because $\cup$ is separately continuous, $\sup_{m\le i\le n} a_i \cup d \in \overline{H}_m$ for every $d \in F$. Letting $n \to \infty$, $d \cup \sup_{i\ge m} a_i \in \overline{H}_m$ whenever $d \in F$ and $m \in \mathbb{N}$. Next, for any $b \in \mathfrak{A}$, $\{a : a \cap b \in \overline{H}_m\}$ is a closed set including H_m, so $a \cap b \in \overline{H}_m$ for every $a \in \overline{H}_m$; that is, $[0,a] \subseteq \overline{H}_m$ for every $a \in \overline{H}_m$. As $a^* \subseteq \sup_{i\ge m} a_i$, $d \cup a^* \in \overline{H}_m$ for every $d \in F$. As m is arbitrary, $d \cup a^* \in F$ for every $d \in F$. **Q**

Similarly, setting $b^* = \inf_{n\in\mathbb{N}} \sup_{i\ge n} b_i$ and $c^* = \inf_{n\in\mathbb{N}} \sup_{i\ge n} c_i$, $d \cup b^*$ and $d \cup c^*$ belong to F for every $d \in F$; and of course $0 \in F$. So $e = a^* \cup b^* \cup c^*$ belongs to F. For each $n \in \mathbb{N}$, $a_n \cup b_n \cup c_n \notin \bigvee_2(H_0)$; but $[0,a] \subseteq \bigvee_2(H_0)$ for every $a \in \bigvee_2(H_0)$, by (d-i), so $\sup_{i\ge n} a_i \cup b_i \cup c_i \notin \bigvee_2(H_0)$. Accordingly $e = \inf_{n\in\mathbb{N}} \sup_{i\ge n} a_i \cup b_i \cup c_i$ does not belong to the open set $\bigvee_2(H_0)$, and $e \notin \overline{H}_0$, by (d-ii). So $e \in F \setminus \overline{H}_0$; which is impossible. **X**

393Q Theorem (Balcar Głowczyński & Jech 98, Balcar Jech & Pazák 05) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is a Maharam algebra;

(ii) $\mathfrak{A}$ is ccc and the order-sequential topology is Hausdorff;

(iii) $\mathfrak{A}$ is weakly (σ,∞)-distributive and $\{0\}$ is a G_δ set for the order-sequential topology of $\mathfrak{A}$;

(iv) $\mathfrak{A}$ is ccc and there is a T_1 topology on $\mathfrak{A}$ such that (α) $\cup : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is continuous at $(0,0)$ (β) whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with infimum 0, then $\langle a_n \rangle_{n\in\mathbb{N}} \to 0$.

proof (a)(i)⇒(ii) By 393Eb, $\mathfrak{A}$ is ccc. By 393N, the order-sequential topology is metrizable, therefore Hausdorff.

(b)(ii)⇒(iii) Suppose that the conditions of (ii) are satisfied. In the following argument, all topological terms will refer to the order-sequential topology on $\mathfrak{A}$.

(α) $\mathfrak{A}$ is weakly (σ,∞)-distributive. **P** Let $\langle A_n \rangle_{n\in\mathbb{N}}$ be a sequence of partitions of unity in $\mathfrak{A}$, and set

$$D = \{d : d \in \mathfrak{A},\ \{a : a \in A_n,\ a \cap d \ne 0\} \text{ is finite for every } n \in \mathbb{N}\}.$$

Take any $c \in \mathfrak{A}^+$. Let G, H be disjoint open sets containing 0, c respectively. Choose $\langle c_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. $c_0 = c$. Given $c_n \in H$, let $\langle a_{ni} \rangle_{i\in\mathbb{N}}$ be a sequence running over A_n, and set $c_{nj} = \sup_{i\le j} c_n \cap a_{ni}$; then $\langle c_{nj} \rangle_{j\in\mathbb{N}}$ order*-converges to c_n (367Bf/367Xa), so there is a j_n such that $c_{nj_n} \in H$; set $c_{n+1} = c_{nj_n}$, and continue.

This gives us a non-increasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in H. Set $d=\inf_{n\in\mathbb{N}}c_n$; then $d\notin G$ so $d\ne 0$, while $d\subseteq\sup_{i\le j_n}a_{ni}$ for each n, so $d\in D$.

As c is arbitrary, D is order-dense in $\mathfrak{A}$ and includes a partition of unity. As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive (316H). **Q**

(β) For any $a\in\mathfrak{A}^+$ there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0 such that $a\not\subseteq\sup(\bigcap_{n\in\mathbb{N}}H_n)$. **P** For $A\subseteq\mathfrak{A}$ and $n\in\mathbb{N}$, define $\bigvee_n(A)$ as in 393Pd. Let G, G' be disjoint neighbourhoods of 0 and a respectively, and set $G_0=G\cap\{a\triangle b:b\in G'\}$; then G_0 is a neighbourhood of 0 (393Mc). By 393Pc, we can find a neighbourhood H_0 of 0 such that $H_0\subseteq G_0$ and $[0,b]\subseteq H_0$ for every $b\in H_0$, in which case $[0,b]\subseteq\bigvee_2(H_0)$ for every $b\in\bigvee_2(H_0)$, while $a\notin\bigvee_2(H_0)$. By 393Pe, we can choose neighbourhoods H_n of 0 such that $H_n\subseteq H_{n-1}$ and $\bigvee_3(H_n)\subseteq\bigvee_2(H_{n-1})$ for every $n\ge 1$; by 393Pc, we can suppose that $[0,b]\subseteq H_n$ whenever $b\in H_n$. But this will ensure that $\bigvee_4(H_{n+2})\subseteq\bigvee_2(H_n)$ for every n, so that $\bigvee_{2^k}(H_{2k})\subseteq\bigvee_2(H_2)$ for every $k\ge 1$. Set $F=\bigcap_{n\in\mathbb{N}}H_n$. Then

$$\bigvee_{2^k}(F)\subseteq\bigvee_{2^k}(H_{2k})\subseteq\bigvee_2(H_2)$$

for every $k\ge 1$. Since $\sup F$ is the limit of a sequence in $\bigcup_{k\ge 1}\bigvee_{2^k}(F)$,

$$\sup F\in\overline{\bigvee_2(H_2)}\subseteq\bigvee_3(H_2)\subseteq\bigvee_2(H_0)$$

(using 393P(d-ii) for the first inequality) and cannot include a. **Q**

(γ) Now consider the set D of those $d\in\mathfrak{A}$ such that there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0 such that $d\cap\sup(\bigcap_{n\in\mathbb{N}}H_n)=0$. By ($\beta$), D is order-dense, so includes a partition of unity A. A is countable, so there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0 such that $d\cap\sup(\bigcap_{n\in\mathbb{N}}H_n)=0$ for every $d\in A$; but this means that $\bigcap_{n\in\mathbb{N}}H_n=\{0\}$. So (iii) is true.

(c)(iii)⇒(iv) Now suppose that the conditions in (iii) are satisfied. As in (b), all topological terms will refer to the order-sequential topology on $\mathfrak{A}$.

(α) There is a non-increasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open neighbourhoods of 0 such that $\bigcap_{n\in\mathbb{N}}\overline{G}_n=\{0\}$. **P** Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence of open sets with intersection $\{0\}$. Set $G_0=\mathfrak{A}$, and for $n\in\mathbb{N}$ choose an open neighbourhood G_{n+1} of 0, included in $U_n\cap G_n$, such that $[0,a]\subseteq G_{n+1}$ for every $a\in G_{n+1}$ (393P). **?** If $0\ne d\in\bigcap_{n\in\mathbb{N}}\overline{G}_n$, then for each $n\in\mathbb{N}$ we can find a sequence $\langle a_{ni}\rangle_{i\in\mathbb{N}}$ in G_n order*-converging to d (393Pc). By 393Pa, there is a sequence $\langle k(n)\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $\langle a_{n,k(n)}\rangle_{n\in\mathbb{N}}$ order*-converges to d. Now $d=\sup_{n\in\mathbb{N}}\inf_{i\ge n}a_{i,k(i)}$ (367Bf), so there is an $n\in\mathbb{N}$ such that $c=\inf_{i\ge n}a_{i,k(i)}$ is non-zero. But in this case we must have $c\le a_{i,k(i)}\in G_i$ and $c\in G_i\subseteq U_j$ whenever $i\ge\max(n,j+1)$, so $c=0$. **X** Thus $\bigcap_{n\in\mathbb{N}}\overline{G}_n=\{0\}$, as required. **Q**

(β) For every neighbourhood G of 0 there is a neighbourhood H of 0 such that $a\cup b\in G$ for all a, $b\in H$. **P** **?** Otherwise, choose $\langle H_n\rangle_{n\in\mathbb{N}}$, $\langle a_n\rangle_{n\in\mathbb{N}}$ and $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Start with an open neighbourhood H_0 of 0 such that $H_0\subseteq G$ and $[0,a]\subseteq H_0$ for every $a\in H_0$. Given that H_n is an open neighbourhood of 0, let a_n, $b_n\in H_n$ be such that $a_n\cup b_n\notin G$. Because the maps $a\mapsto a\cup a_n$ and $a\mapsto a\cup b_n$ are continuous, there is an open neighbourhood H_{n+1} of 0 such that $a\cup a_n$ and $a\cup b_n$ belong to H_n for every $a\in H_{n+1}$; and we may suppose that $H_{n+1}\subseteq G_n$. Continue.

An easy induction on k shows that $a\cup\sup_{n\le i\le n+k}a_i$ and $a\cup\sup_{n\le i\le n+k}b_i$ belong to H_n whenever $k\in\mathbb{N}$ and $a\in H_{n+k+1}$. In particular, $\sup_{n\le i\le n+k}a_i\in H_n$ for every k; since $\langle\sup_{n\le i\le n+k}a_i\rangle_{k\in\mathbb{N}}$ is order*-convergent to $\sup_{i\ge n}a_i$, $\sup_{i\ge n}a_i\in\overline{H}_n\subseteq\overline{G}_n$ for every n. Set $a^*=\inf_{n\in\mathbb{N}}\sup_{i\ge n}a_i$. Then $\langle\sup_{i\ge n}a_i\rangle_{n\in\mathbb{N}}\to^* a^*$, and $\sup_{i\ge n}a_i\in\overline{G}_m$ whenever $n\ge m$, so $a^*\in\overline{G}_m$ for every m, and $a^*=0$.

In the same way, $\inf_{n\in\mathbb{N}}\sup_{i\ge n}b_i=0$. It follows that $\inf_{n\in\mathbb{N}}c_n=0$, where $c_n=\sup_{i\ge n}a_i\cup\sup_{i\ge n}b_i$ for each n. But now $\langle c_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0, so order*-converges to 0, and there must be an n such that $c_n\in H_0$. Since $a_n\cup b_n\subseteq c_n$, $a_n\cup b_n\in H_0\subseteq G$, contrary to the choice of a_n and b_n. **XQ**

(γ) $\mathfrak{A}$ is ccc. **P** Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence of open sets with intersection $\{0\}$, and $A\subseteq\mathfrak{A}\setminus\{0\}$ a partition of unity. If $\langle a_i\rangle_{i\in\mathbb{N}}$ is a sequence of distinct elements of A, then $\langle a_i\rangle_{i\in\mathbb{N}}\to^* 0$ (393Md); so $A\setminus U_n$ is finite for every n, and A is countable. **Q**

(δ) (β) means just that $\cup$ is continuous at $(0,0)$. Also a non-increasing sequence with infimum 0 order*-converges to 0, so converges topologically to 0 (3A3Pa); and the topology is certainly T_1. So all the conditions of (iv) are satisfied by the order-sequential topology.

(d)(iv)⇒(i) By 393J, there is a strictly positive Maharam submeasure on $\mathfrak{A}$; as $\mathfrak{A}$ is Dedekind σ-complete, it is a Maharam algebra.

393R Definition Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is **σ-finite-cc** if $\mathfrak{A}$ can be expressed as $\bigcup_{n\in\mathbb{N}} A_n$ where no A_n includes any infinite disjoint set.

393S Theorem (TODORČEVIĆ 04) Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is a Maharam algebra iff it is σ-finite-cc, weakly (σ,∞)-distributive and Dedekind σ-complete.

proof (B.Balcar)**(a)** If $\mathfrak{A}$ is a Maharam algebra, then of course it is Dedekind σ-complete, and we have known since 393C that it is weakly (σ,∞)-distributive. Also it carries a strictly positive exhaustive submeasure, so is σ-finite-cc.

Of course $\{0\}$ is a Maharam algebra. For the rest of the proof, therefore, I suppose that $\mathfrak{A}$ is a non-trivial algebra satisfying the conditions, and seek to show that it is a Maharam algebra.

(b)(i) Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets, with union $\mathfrak{A}^+$, such that no A_n includes any infinite disjoint set. For each n, set $B_n=\bigcup_{m\le n}\bigcup_{a\in A_m}[a,1]$, so that B_n includes no infinite disjoint subset. Now there is an n such that 1 is in the interior of B_n for the order-sequential topology. **P?** Otherwise, of course $\mathfrak{A}$ is ccc, so there is for each $n\in\mathbb{N}$ a sequence $\langle b_{ni}\rangle_{i\in\mathbb{N}}$ in $\mathfrak{A}\setminus B_n$ which is order*-convergent to 1 (393Pb). By 393Pa, there is a sequence $\langle k(n)\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $\langle b_{n,k(n)}\rangle_{n\in\mathbb{N}}$ order*-converges to 1. As $1\ne 0$, there must be an $m\in\mathbb{N}$ such that $c=\inf_{i\ge m} b_{i,k(i)}\ne 0$. There is an n such that $c\in A_n$, in which case $b_{i,k(i)}\in B_n\subseteq B_i$ for every $i\ge\max(m,n)$. **XQ**

(ii) Set $H=\operatorname{int} B_n$. Then there is a $c\in H$ such that for every $d\in\mathfrak{A}$ one of $c\cap d$, $c\setminus d\notin H$. **P?** Otherwise, we can choose a sequence $\langle c_i\rangle_{i\in\mathbb{N}}$ in H such that $c_0=1$ and, for each $i\in\mathbb{N}$, $c_{i+1}\subseteq c_i$ and $c_i\setminus c_{i+1}\in H$. But in this case $\langle c_i\setminus c_{i+1}\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in B_n, which is impossible. **XQ**

(iii) 0 and 1 can be separated by open sets. **P** Take H and c from (ii). Then $G_0=\{d: c\setminus d\in H\}$ and $G_1=\{d: c\cap d\in H\}$ are disjoint open sets containing 0 and 1 respectively. **Q**

(b) It follows that $\mathfrak{A}$ is actually Hausdorff in the order-sequential topology. **P** Let a_0, $a_1\in\mathfrak{A}$ be such that $b=a_1\setminus a_0$ is non-zero. Consider the principal ideal $\mathfrak{A}_b$. Like $\mathfrak{A}$, this is σ-finite-cc, weakly (σ,∞)-distributive and Dedekind σ-complete. By (a), there are disjoint subsets U, V of $\mathfrak{A}_b$, open for the order-sequential topology of $\mathfrak{A}_b$, such that $0\in U$ and $b\in V$. The function $a\mapsto a\cap b:\mathfrak{A}\to\mathfrak{A}_b$ is continuous for the order-sequential topologies (3A3Pb), so $G=\{a: a\cap b\in U\}$ and $H=\{a: a\cap b\in V\}$ are open. Now G and H are disjoint open sets in $\mathfrak{A}$ containing a_0, a_1 respectively. As a_0 and a_1 are arbitrary, $\mathfrak{A}$ is Hausdorff. **Q**

By 393Q, $\mathfrak{A}$ is a Maharam algebra.

393X Basic exercises >(a) Let $\mathfrak{A}$ be the finite-cofinite algebra on an uncountable set (316Yl). (i) Set $\nu_1 0=0$, $\nu_1 a=1$ for $a\in\mathfrak{A}\setminus\{0\}$. Show that ν_1 is a strictly positive Maharam submeasure but is not exhaustive. (ii) Set $\nu_2 a=0$ for finite a, 1 for cofinite a. Show that ν_2 is a uniformly exhaustive Maharam submeasure but is not order-continuous.

>(b) Let $\mathfrak{A}$ be a Boolean algebra and ν a submeasure on $\mathfrak{A}$. Set $I=\{a:\nu a=0\}$. Show that (i) I is an ideal of $\mathfrak{A}$ (ii) there is a submeasure $\bar\nu$ on $\mathfrak{A}/I$ defined by setting $\bar\nu a^\bullet=\nu a$ for every $a\in\mathfrak{A}$ (iii) if ν is exhaustive, so is $\bar\nu$ (iv) if ν is uniformly exhaustive, so is $\bar\nu$ (v) if ν is a Maharam submeasure, I is a σ-ideal (vi) if ν is a Maharam submeasure and $\mathfrak{A}$ is Dedekind σ-complete, $\bar\nu$ is a Maharam submeasure.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and ν an order-continuous submeasure on $\mathfrak{A}$. Show that ν has a unique support $a\in\mathfrak{A}$ such that $\nu\restriction\mathfrak{A}_a$ is strictly positive and $\nu\restriction\mathfrak{A}_{1\setminus a}$ is identically zero.

(d) Let $\mathfrak{A}$ be a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$ such that $\nu a=\lim_{n\to\infty}\nu a_n$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum a. Show that ν is a Maharam submeasure.

(e) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν a uniformly exhaustive Maharam submeasure on $\mathfrak{A}$. Show that there is a non-negative countably additive functional μ on $\mathfrak{A}$ such that $\{a:\mu a=0\}=\{a:\nu a=0\}$. (*Hint*: 393Xb(vi).)

(f) Let $\mathfrak{A}$ be a Maharam algebra with its Maharam-algebra topology and uniformity. (i) Let $B\subseteq\mathfrak{A}$ be a non-empty upwards-directed set. For $b\in B$ set $F_b=\{c: b\subseteq c\in B\}$. Show that $\{F_b: b\in B\}$ generates a Cauchy filter $\mathcal{F}(B\uparrow)$ on $\mathfrak{A}$ which converges to $\sup B$. (ii) Show that closed subsets of $\mathfrak{A}$ are order-closed. (iii) Show that an order-dense subalgebra of $\mathfrak{A}$ must be dense in the topological sense.

(g) Let $\mathfrak{A}$ be a Maharam algebra. Show that it is a measurable algebra iff for every $A \subseteq \mathfrak{A}$ including antichains of all finite sizes there is a sequence in A which is order*-convergent to 0.

(h) Let $\mathfrak{A}$ be a Boolean algebra. Suppose that $\langle a_n \rangle_{n\in\mathbb{N}}$, $\langle b_n \rangle_{n\in\mathbb{N}}$ are sequences in $\mathfrak{A}$ order*-converging to a, b respectively. Show that $\langle a_n \odot b_n \rangle_{n\in\mathbb{N}} \to^* a \odot b$ when $\odot$ is any of the operations $\cup$, $\cap$, $\triangle$ or $\setminus$.

(i) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra. Write $\mathfrak{T}_{\text{os}}$ for the order-sequential topology on $\mathfrak{A}$ and $\mathfrak{T}_{\text{ma}}$ for the measure-algebra topology. Show that $\mathfrak{T}_{\text{os}} \supseteq \mathfrak{T}_{\text{ma}}$, with equality iff $(\mathfrak{A}, \bar\mu)$ is σ-finite.

(j) (Jech 08) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle A_n \rangle_{n\in\mathbb{N}}$ a sequence of subsets of $\mathfrak{A}$ such that (α) for every $n \in \mathbb{N}$, any antichain in A_n has at most n elements (β) a sequence $\langle a_k \rangle_{k\in\mathbb{N}}$ in $\mathfrak{A}$ is order*-convergent to 0 iff $\{k : a_k \in A_n\}$ is finite for every $n \in \mathbb{N}$. (i) Show that $\mathfrak{A}$ is ccc. (ii) Show that $\mathfrak{A}$ is weakly (σ,∞)-distributive. (*Hint*: if C_n is non-empty and downwards-directed with infimum 0 for each n, show that there is a sequence $\langle a_n \rangle_{n\in\mathbb{N}} \to^* 0$ such that $a_n \in C_n$ for every n.) (iii) Show that $\mathfrak{A}$ is a Maharam algebra. (*Hint*: 393S.) (iv) Show that any Maharam submeasure on $\mathfrak{A}$ is uniformly exhaustive. (v) Show that $\mathfrak{A}$ is a measurable algebra.

393Y Further exercises (a) Let $\mathfrak{A}$ be any Boolean algebra with a strictly positive Maharam submeasure. Show that $\mathfrak{A}$ is weakly σ-distributive.

(b) Let U be a Riesz space, with its order-sequential topology. (i) Show that addition and subtraction are separately continuous. (ii) Show that U is Archimedean iff scalar multiplication is separately continuous as a function from $\mathbb{R} \times U$ to U, and that in this case scalar multiplication is actually continuous.

(c) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and give $L^0 = L^0(\mathfrak{A})$ its order-sequential topology. Suppose $h : \mathbb{R} \to \mathbb{R}$ is continuous, and let $\bar h : L^0 \to L^0$ be the corresponding function as defined in 364H. Show that $\bar h$ is continuous.

(d) Let $\mathfrak{A}$ be a Maharam algebra. Show that a topology $\mathfrak{T}$ on $L^0(\mathfrak{A})$ defined by the method of 393K must be the order-sequential topology on $L^0(\mathfrak{A})$.

(e) Let U be a weakly (σ,∞)-distributive Riesz space with the countable sup property, with its order-sequential topology, and A a subset of U. Show that $\overline{A}$ is the set of order*-limits of sequences in A.

(f) Let U be a weakly (σ,∞)-distributive Dedekind complete Riesz space with the countable sup property, endowed with its order-sequential topology, and $\mathfrak{A}$ its band algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is a Maharam algebra; (ii) U is Hausdorff; (iii) addition on U is continuous at $(0,0)$; (iv) $\vee : U \times U \to U$ is continuous at $(0,0)$.

(g) Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$, with its order-sequential topology. (i) Show that if U, V are open sets in $\mathfrak{G}$ containing $0_{\mathfrak{G}} = \emptyset$ and $1_{\mathfrak{G}} = \mathbb{R}$ respectively, then $U \cap V \ne \emptyset$. (ii) Show that if U is an open set in $\mathfrak{G}$ containing $\emptyset$ then there are G, $H \in U$ such that $H = \mathbb{R} \setminus \overline{G}$. (iii) Show that $\{\emptyset\}$ is a G_δ set in $\mathfrak{G}$. (iv) Show that there is no non-zero Maharam submeasure on $\mathfrak{G}$. (v) Show that there is no non-zero countably additive functional on $\mathfrak{G}$.

(h) In 393Xj, show that each of the sets A_n must have non-zero intersection number.

(i) Let $\mathfrak{A}$ be an atomless Boolean algebra with countable Maharam type. Show that there is a submeasure μ on $\mathfrak{A}$, order-continuous on the left, such that whenever $a \in \mathfrak{A} \setminus \{0\}$ there is a $b \subseteq a$ such that $\mu b < \mu a$.

393 Notes and comments For many years it was not known whether there were any Maharam algebras which were not measurable algebras; this was the famous 'control measure problem', eventually solved by M.Talagrand. I will present his example in the next section. We now know that we have a larger class, but it remains very poorly understood, and the material presented here must be regarded as work in progress. As in §§391-392, the stimulus for these ideas has been the attempt to characterize measurable algebras in more or less algebraic terms. If we are prepared to allow order*-convergence of sequences to be an 'algebraic' notion, then 393Xj is such a characterization; but it shares with Kelley's criterion 391K the need for a sequence $\langle A_n \rangle_{n\in\mathbb{N}}$, covering $\mathfrak{A}^+$, with defined properties. The advance, if any, is that the properties (α) and (β) of 393Xj are a good deal farther from any formula for a measure.

The first few results of this section, down to 393G, are concerned with checking that Maharam algebras share properties with measurable algebras, and the proofs use the same ideas, with occasional minor modifications. In 393H we have to think a little, since exhaustivity is less familiar, and harder to apply, than additivity. From this proposition we see that exhaustive submeasures are to uniformly exhaustive submeasures something like what Maharam algebras are to measurable algebras. 393K is a further example of a well-known construction – this time, convergence in measure – which has a version based on Maharam algebras.

In §367 I examined order*-convergence in Riesz spaces, without explicitly discussing the associated topology, and in 393L-393Q here I look at Boolean algebras. In both cases the usefulness of the idea starts with the fact that the algebraic operations are separately continuous (367C, 393M), which is itself a consequence of the strong distributive laws in 313A-313B and 352E. It is easy to see that in a Maharam algebra the order-sequential topology is the Maharam-algebra topology (393N). What is remarkable is that natural questions about the order-sequential topology lead to characterizations of Maharam algebras (393Q). This leads directly to an astonishing algebraic characterization of Maharam algebras (393S). (But once again we need to hypothesize the existence of a suitable sequence of sets covering $\mathfrak{A}^+$.)

394 Talagrand's example

I rewrite the construction in TALAGRAND 08 of an exhaustive submeasure which is not uniformly exhaustive.

394A I begin with two elementary combinatorial facts.

Lemma Suppose that $\mathcal{K}$ is a non-empty finite family of subsets of $\mathbb{N}$ and $r \in \mathbb{N}$ is such that $\#(K) \ge r\#(\mathcal{K})$ for every $K \in \mathcal{K}$. Then we have an enumeration $\langle K_i\rangle_{i<s}$ of $\mathcal{K}$ and a non-decreasing family $\langle n_i\rangle_{i\le s}$ such that $\#(K_i \cap n_{i+1} \setminus n_i) = r$ for every $i < s$.

proof Set $s = \#(\mathcal{K})$. Choose n_i, K_i inductively, as follows. Start with $n_0 = 0$. Given $j < s$, $n_j \in \mathbb{N}$ and $\langle K_i\rangle_{i<j}$ such that $\#(K \setminus n_j) \ge r(s-j)$ for every $K \in \mathcal{K}_j = \mathcal{K} \setminus \{K_i : i < j\}$, set

$$n_{j+1} = \min\{n : \#(K \cap n \setminus n_j) \ge r \text{ for some } K \in \mathcal{K}_j\}$$

and choose $K_j \in \mathcal{K}_j$ such that $\#(K_j \cap n_{j+1} \setminus n_j) \ge r$. Observe that $\#(K \cap n_{j+1} \setminus n_j) \le r$ for every $K \in \mathcal{K}_j$, so that $\#(K_j \cap n_{j+1} \setminus n_j)$ must in fact be equal to r and $\#(K \setminus n_{j+1}) \ge r(s-j-1)$ for every $K \in \mathcal{K}_j$; thus the induction proceeds.

394B Lemma Suppose that $\langle K_i\rangle_{i<s}$ is a family of finite subsets of $\mathbb{N}$, all of size at least $r \ge 2$, such that $\max K_i < \min K_{i+1}$ for $i \le s-2$. Let $\mathcal{J}$ be a finite subset of $([\mathbb{N}]^{<\omega} \setminus \{\emptyset\}) \times [0,\infty[$, and set $\gamma = \sum_{(I,w)\in\mathcal{J}} w$. Then we can find $\langle u_i\rangle_{i<s}$ and $\langle v_i\rangle_{i<s}$ such that u_i, v_i are successive members of K_i for each $i < s$ and, setting $W = \bigcup_{i<s} v_i \setminus u_i$,

$$\sum_{\substack{(J,w)\in\mathcal{J}\\ \#(J\cap W)\ge\frac{1}{2}\#(J)}} w \le \frac{2\gamma}{r-1}.$$

proof Set $K = \bigcup_{i<s} K_i$ and for $n \in K \setminus \{\max K\}$ let n^+ be the next member of K above n. For $z \in Z = \prod_{i<s}(K_i \setminus \{\max K_i\})$ set

$$W(z) = \bigcup_{i<s} z(i)^+ \setminus z(i),$$

$$S(z) = \sum_{(J,w)\in\mathcal{J},\#(J\cap W(z))\ge\frac{1}{2}\#(J)} w.$$

Consider the uniform probability measure λ on Z. For any set $J \subseteq \mathbb{N}$,

$$\begin{aligned}\int \#(J\cap W(z))\lambda(dz) &= \sum_{i=0}^{s-1}\int \#(J\cap z(i)^+\setminus z(i))\lambda(dz)\\ &= \sum_{i=0}^{s-1}\sum_{n\in K_i\setminus\{\max K_i\}}\frac{1}{\#(K_i)-1}\#(J\cap n^+\setminus n)\\ &= \sum_{i=0}^{s-1}\frac{\#(J\cap\max K_i\setminus\min K_i)}{\#(K_i)-1}\le\frac{\#(J)}{r-1}.\end{aligned}$$

So, writing $V_J=\{z:\#(J\cap W(z))\ge\frac{1}{2}\#(J)\}$, $\lambda V_J\le\frac{2}{r-1}$ for any non-empty J. Now

$$\int S(z)\lambda(dz)=\sum_{(J,w)\in\mathcal{J}}w\lambda V_J\le\frac{2}{r-1}\sum_{(J,w)\in\mathcal{J}}w=\frac{2\gamma}{r-1},$$

so there must be a $z\in Z$ such that $S(z)\le\frac{2\gamma}{r-1}$, and we can take $u_i=z(i)$, $v_i=z(i)^+$ for each i.

394C Definitions We are now ready to begin work. The construction is complex and demands a large volume of special notation.

(a) I shall work throughout with $X=\prod_{n\in\mathbb{N}}T_n$ where $\langle T_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty finite sets and $\sup_{n\in\mathbb{N}}\#(T_n)$ is infinite. X may be regarded as a compact Hausdorff space with the product of the discrete topologies on the T_n. For each $n\in\mathbb{N}$, $\mathfrak{B}_n$ will be the algebra of subsets of X determined by coordinates less than n and $\mathcal{A}_n$ the set of its atoms, that is, the family of sets of the form $\{x:z\subseteq x\in X\}$ for some $z\in\prod_{i<n}T_i$. $\mathfrak{B}=\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$ will be the algebra of open-and-closed subsets of X. For $n\in\mathbb{N}$ and $t\in T_n$, Y_{nt} will be $\{x:x\in X,\,x(n)=t\}$.

(b) We shall need a sequence $\langle\alpha_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{R}$ and a sequence $\langle N_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$. It is easy enough to give appropriate formulae but perhaps the ideas will be clearer if instead I declare the properties they must have.

(i) $\alpha_k>0$ and $(2^{k+4})^{\alpha_k}\le 2$ for every $k\in\mathbb{N}$, $\langle\alpha_k\rangle_{k\in\mathbb{N}}$ is non-increasing, and $\sum_{k=0}^{\infty}\alpha_k\le\frac{1}{2}$.

(ii) $N_k\in\mathbb{N}$ and $2^{-k}(2^{-2k-12}N_k)^{\alpha_k}\ge 2^4$ for every $k\in\mathbb{N}$.

(c) Now we come to some of the key ideas. For a set $\mathcal{I}\subseteq\mathcal{P}X\times\mathcal{P}\mathbb{N}\times[0,\infty[$, define its 'spread' $\operatorname{spr}\mathcal{I}$ to be $\bigcup_{(E,I,w)\in\mathcal{I}}E$ and its 'weight' $\operatorname{wt}\mathcal{I}$ to be $\sum_{(E,I,w)\in\mathcal{I}}w$.

(d) For any family $\mathcal{E}\subseteq\mathcal{P}X\times\mathcal{P}\mathbb{N}\times[0,\infty[$ define $\phi_{\mathcal{E}}:\mathfrak{B}\to[0,\infty]$ by setting

$$\phi_{\mathcal{E}}E=\inf\{\operatorname{wt}\mathcal{I}:\mathcal{I}\subseteq\mathcal{E}\text{ is finite, }E\subseteq\operatorname{spr}\mathcal{I}\},$$

counting $\inf\emptyset$ as ∞. So $\phi_\emptyset\emptyset=0$ and $\phi_\emptyset E=\infty$ for $E\in\mathfrak{B}\setminus\{\emptyset\}$.

(e) For $D\subseteq X$ and $I\subseteq\mathbb{N}$ set

$$\theta_I(D)=\{y:y\in X,\,y\restriction I=x\restriction I\text{ for some }x\in D\}.$$

(f)(i) If $m<n$ in $\mathbb{N}$, $\phi:\mathfrak{B}\to[0,\infty]$ is a function and $E\in\mathfrak{B}$, then E is **ϕ-thin between m and n** if $\phi(X\setminus\theta_{n\setminus m}(A\cap E))\ge 1$ for every $A\in\mathcal{A}_m$.

(ii) If $I\subseteq\mathbb{N}$, $\phi:\mathfrak{B}\to[0,\infty]$ is a function and $E\in\mathfrak{B}$, then E is **ϕ-thin along I** if it is ϕ-thin between m and n whenever m, $n\in I$ and $m<n$.

(g) For $k\le p\in\mathbb{N}$ define $\mathcal{C}_{kp}$ and ν_{kp} by downwards induction on k, as follows. Start with $\mathcal{C}_{pp}=\emptyset$ for every p. Given $\mathcal{C}_{kp}$, set $\nu_{kp}=\phi_{\mathcal{C}_{kp}}$. Given that $k<p$ and $\mathcal{C}_{k+1,p}$ and $\nu_{k+1,p}=\phi_{\mathcal{C}_{k+1,p}}$ have been defined, set

$$\begin{aligned}\mathcal{E}_{kp}=\{(E,I,w):E\in\mathfrak{B},\,I\subseteq\mathbb{N},\,1\le\#(I)\le N_k,&\\ w\ge 2^{-k}\Bigl(\frac{N_k}{\#(I)}\Bigr)^{\alpha_k},\,E\text{ is }\nu_{k+1,p}\text{-thin along }I\},&\\ \mathcal{C}_{kp}=\mathcal{E}_{kp}\cup\mathcal{C}_{k+1,p}&\end{aligned}$$

and continue.

(h) Define $\langle c_k\rangle_{k\in\mathbb{N}}$ by setting $c_0 = 8$, $c_{k+1} = 2^{2\alpha_k}c_k$ for every k.

394D Very elementary facts In the hope of aiding digestion of the definitions here, of which 394Cf and 394Cg are likely to be wholly obscure to anyone who has not worked through this proof before, I run over some obvious facts which will be used below.

(a) $\phi_{\mathcal{E}} : \mathfrak{B} \to [0,\infty]$ is a submeasure for any $\mathcal{E} \subseteq \mathcal{P}X \times \mathcal{P}\mathbb{N} \times [0,\infty[$. (Subadditivity and monotonicity are written into the definition.)

(b) If I, $J \subseteq \mathbb{N}$ then $\theta_I\theta_J = \theta_{I\cap J}$. If $I \subseteq J \subseteq \mathbb{N}$ then $\theta_I(D) = \theta_I\theta_J(D) \supseteq \theta_J(D)$ for all $D \subseteq X$. If $I \subseteq \mathbb{N}$ then $\theta_I(D \cap \theta_I(E)) = \theta_I(E \cap \theta_I(D))$ for all D, $E \subseteq X$. For any $I \subseteq \mathbb{N}$ and any family $\mathcal{D}$ of subsets of X, $\theta_I(\bigcup\mathcal{D}) = \bigcup_{D\in\mathcal{D}}\theta_I(D)$.

For $n \in \mathbb{N}$ and $D \subseteq X$, $D \in \mathfrak{B}_n$ iff $\theta_n(D) = D$. If $E \in \mathfrak{B}$ and $I \subseteq \mathbb{N}$, $\theta_I(E) \in \mathfrak{B}$. If $m \le n$ in $\mathbb{N}$, $A \in \mathcal{A}_m$ and $A_1 \in \mathcal{A}_n$, then $A \cap \theta_{n\setminus m}(A_1) \in \mathcal{A}_n$. If $m \in \mathbb{N}$ and $A \in \mathcal{A}_m$ then $E \mapsto \theta_{\mathbb{N}\setminus m}(A \cap E) : \mathfrak{B} \to \mathfrak{B}$ is a Boolean homomorphism.

(c) If $m < n$, $\phi : \mathfrak{B} \to [0,\infty]$ is a non-decreasing function, $E \in \mathfrak{B}$ is ϕ-thin between m and n and $E' \in \mathfrak{B}$ is included in E, then E' is ϕ-thin between m and n' for every $n' \ge n$.

(d) All the classes $\mathcal{E}_{kp}$, $\mathcal{C}_{kp}$ are closed under increases in the scalar variable and decreases in the first variable, that is,

—— if $k < p$, $(E, I, w) \in \mathcal{E}_{kp}$, $E' \in \mathfrak{B}$, $E' \subseteq E$ and $w' \ge w$ then $(E', I, w') \in \mathcal{E}_{kp}$,

—— if $k \le p$, $(E, I, w) \in \mathcal{C}_{kp}$, $E' \in \mathfrak{B}$, $E' \subseteq E$ and $w' \ge w$ then $(E', I, w') \in \mathcal{C}_{kp}$.

(e) If $k \le p$ in $\mathbb{N}$, $\mathcal{C}_{kp} = \bigcup_{k\le l<p}\mathcal{E}_{lp}$.

(f) If $k < p$ in $\mathbb{N}$, $\nu_{kp} \le \nu_{k+1,p}$, because $\mathcal{C}_{kp} \supseteq \mathcal{C}_{k+1,p}$.

(g) $8 \le c_k \le 16$ for every $k \in \mathbb{N}$, because $\sum_{k=0}^{\infty} 2\alpha_k \le 1$.

(h) If $k < p$ in $\mathbb{N}$, then $(X, \{0\}, 2^{-k}N_k^{\alpha_k}) \in \mathcal{E}_{kp}$ so $\nu_{kp}X \le 2^{-k}N_k^{\alpha_k}$ and ν_{kp} is totally finite.

394E Lemma Suppose that $k \le p$, $m < n$, $A \in \mathcal{A}_m$, $(E, I, w) \in \mathcal{C}_{kp}$ and $I' = I \cap n \setminus m$ is non-empty. If $E' = \theta_{n\setminus m}(E \cap A)$ and $w' \ge \bigl(\frac{\#(I)}{\#(I')}\bigr)^{\alpha_k} w$, then $(E', I', w') \in \mathcal{C}_{kp}$.

proof There is an l such that $k \le l < p$ and $(E, I, w) \in \mathcal{E}_{lp}$. Now E' is $\nu_{l+1,p}$-thin along I'. **P** Suppose that i, $j \in I'$ and $i < j$, so that $m \le i < j < n$. Take any $A_1 \in \mathcal{A}_i$, and set $A_2 = A \cap \theta_{n\setminus m}(A_1)$, so that A_2 also belongs to $\mathcal{A}_i$. Then, using the list in 394Db,

$$\begin{aligned}
\theta_{j\setminus i}(E' \cap A_1) &= \theta_{j\setminus i}(A_1 \cap \theta_{n\setminus m}(E \cap A)) \\
&= \theta_{j\setminus i}(\theta_{n\setminus m}(A_1 \cap \theta_{n\setminus m}(E \cap A))) \\
&= \theta_{j\setminus i}(\theta_{n\setminus m}(E \cap A \cap \theta_{n\setminus m}(A_1))) \\
&= \theta_{j\setminus i}(\theta_{n\setminus m}(E \cap A_2)) = \theta_{j\setminus i}(E \cap A_2).
\end{aligned}$$

So

$$\nu_{l+1,p}(X \setminus \theta_{j\setminus i}(E' \cap A_1)) = \nu_{l+1,p}(X \setminus \theta_{j\setminus i}(E \cap A_2)) \ge 1$$

because E is $\nu_{l+1,p}$-thin between i and j. As i, j and A_1 are arbitrary, E' is $\nu_{l+1,p}$-thin along I'. **Q**

Of course $\#(I') \le \#(I) \le N_l$. Finally, because $\alpha_l \le \alpha_k$, we have

$$w' \ge \bigl(\frac{\#(I)}{\#(I')}\bigr)^{\alpha_k} w \ge \bigl(\frac{\#(I)}{\#(I')}\bigr)^{\alpha_l} \cdot 2^{-l}\bigl(\frac{N_l}{\#(I)}\bigr)^{\alpha_l} = 2^{-l}\bigl(\frac{N_l}{\#(I')}\bigr)^{\alpha_l},$$

and $(E', I', w') \in \mathcal{E}_{lp} \subseteq \mathcal{C}_{kp}$.

394F Corollary (a) Suppose that $n \in \mathbb{N}$ and $k \le p$ and that $\mathcal{I} \subseteq \mathcal{C}_{kp}$ is a finite set such that $\#(I \cap n) \ge \frac{1}{4}\#(I)$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_n(\operatorname{spr}\mathcal{I})) \le 2\operatorname{wt}\mathcal{I}$.

(b) Suppose that $m \in \mathbb{N}$, $k \le p$ and $A \in \mathcal{A}_m$. Let $\mathcal{I}$ be a finite subset of $\mathcal{C}_{kp}$ such that $\#(I \setminus m) \ge \frac{1}{4}\#(I)$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_{\mathbb{N}\setminus m}(A \cap \operatorname{spr}\mathcal{I})) \le 2\,\mathrm{wt}\,\mathcal{I}$.

(c) Suppose that $m < n$ in $\mathbb{N}$, $k \le p$ and $A \in \mathcal{A}_m$. Let $\mathcal{I}$ be a finite subset of $\mathcal{C}_{kp}$ such that $\#(I \cap n \setminus m) \ge 2^{-k-4}\#(I)$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_{n\setminus m}(A \cap \operatorname{spr}\mathcal{I})) \le 2\,\mathrm{wt}\,\mathcal{I}$.

proof (a) For each $(E, I, w) \in \mathcal{I}$ set $E' = \theta_n(E) \in \mathcal{B}_n$, $I' = I \cap n$ and

$$w' = \bigl(\frac{\#(I)}{\#(I')}\bigr)^{\alpha_k} w \le 4^{\alpha_k} w \le 2w.$$

By 394E, with $m = 0$ and $A = X$, $(E', I', w') \in \mathcal{C}_{kp}$. Set $\mathcal{J} = \{(E', I', w') : (E, I, w) \in \mathcal{I}\}$ and $B = \operatorname{spr}\mathcal{J}$. Then

$$B = \bigcup_{(E,I,w)\in\mathcal{I}} \theta_n(E) = \theta_n(\operatorname{spr}\mathcal{I})$$

and

$$\nu_{kp}B \le \mathrm{wt}\,\mathcal{J} = \sum_{(E,I,w)\in\mathcal{I}} w' \le 2\,\mathrm{wt}\,\mathcal{I},$$

as required.

(b) This time, take $n > m$ so large that $I \subseteq n$ whenever $(E, I, w) \in \mathcal{I}$. For $(E, I, w) \in \mathcal{I}$, set

$$E' = \theta_{n\setminus m}(A \cap E), \quad I' = I \setminus m = I \cap n \setminus m, \quad w' = \bigl(\frac{\#(I)}{\#(I')}\bigr)^{\alpha_k} w \le 2w.$$

Then 394E tells us that $(E', I', w') \in \mathcal{C}_{kp}$. Setting $\mathcal{J} = \{(E', I', w') : (E, I, w) \in \mathcal{I}\}$,

$$\theta_{\mathbb{N}\setminus m}(A \cap \operatorname{spr}\mathcal{I}) = \bigcup_{(E,I,w)\in\mathcal{I}} \theta_{\mathbb{N}\setminus m}(A \cap E) \subseteq \bigcup_{(E,I,w)\in\mathcal{I}} E' = \operatorname{spr}\mathcal{J},$$

so

$$\nu_{kp}(\theta_{N\setminus m}(A \cap \operatorname{spr}\mathcal{I})) \le \mathrm{wt}\,\mathcal{J} \le 2\,\mathrm{wt}\,\mathcal{I}.$$

(c) For $(E, I, w) \in \mathcal{I}$ set

$$E' = \theta_{n\setminus m}(A \cap E), \quad I' = I \cap n \setminus m, \quad w' = \bigl(\frac{\#(I)}{\#(I')}\bigr)^{\alpha_k} w \le (2^{k+4})^{\alpha_k} w \le 2w.$$

Then $(E', I', w') \in \mathcal{C}_{kp}$. Setting $\mathcal{J} = \{(E', I', w') : (E, I, w) \in \mathcal{I}\}$,

$$\theta_{n\setminus m}(A \cap \operatorname{spr}\mathcal{I}) = \bigcup_{(E,I,w)\in\mathcal{I}} \theta_{n\setminus m}(A \cap E) = \bigcup_{(E,I,w)\in\mathcal{I}} E' = \operatorname{spr}\mathcal{J},$$

so

$$\nu_{kp}(\theta_{n\setminus m}(A \cap \operatorname{spr}\mathcal{I})) \le \mathrm{wt}\,\mathcal{J} \le 2\,\mathrm{wt}\,\mathcal{I}.$$

394G We are at the centre of the argument.

Lemma Suppose that $r \in \mathbb{N}$ and $t \in T_r$. Then $\nu_{kp}Y_{rt} \ge c_k$ whenever $k \le p$ in $\mathbb{N}$.

proof Induce on $p - k$.

(a) If $k = p$ then

$$\mathcal{C}_{pp} = \emptyset, \quad \nu_{pp}Y_{rt} = \infty.$$

For the downwards step to $k < p$, given that $\nu_{k+1,p}Y_{rt} \ge c_{k+1}$, take a finite set $\mathcal{I} \subseteq \mathcal{C}_{kp}$ such that $\mathrm{wt}\,\mathcal{I} < c_k$. The rest of the proof is devoted to showing that $Y_{rt} \not\subseteq \operatorname{spr}\mathcal{I}$.

(b) It will help to get a trivial case out of the way. If $\mathcal{I} \subseteq \mathcal{C}_{k+1,p}$, then we have

$$\mathrm{wt}\,\mathcal{I} < c_k \le c_{k+1} \le \nu_{k+1,p}Y_{rt},$$

by the inductive hypothesis, so certainly $Y_{rt} \not\subseteq \operatorname{spr}\mathcal{I}$. Accordingly we may suppose henceforth that $\mathcal{I} \not\subseteq \mathcal{C}_{k+1,p}$.

A second elementary point is that $\#(I) \ge 2^{2k+12}$ whenever $(E, I, w) \in \mathcal{I}$. **P** We have an l such that $k \le l < p$ and $(E, I, w) \in \mathcal{E}_{kp}$, so

$$2^{-l}\bigl(\frac{N_l}{\#(I)}\bigr)^{\alpha_l} \le w \le c_k \le 2^4$$

and $\#(I) \ge 2^{2l+12} \ge 2^{2k+12}$, by the choice of N_l. **Q**

(c) Express $\mathcal{I}$ as $\mathcal{J} \cup \mathcal{K}$ where $\mathcal{J} \subseteq \mathcal{C}_{k+1,p}$ and $\mathcal{K} \subseteq \mathcal{E}_{kp}$. Set $s = \#(\mathcal{K}) > 0$. For $(E, I, w) \in \mathcal{K}$ we have $w \ge 2^{-k}$, so $s \le 2^k c_k \le 2^{k+4}$ (394Dg). Consequently $\#(I) \ge 2^{2k+12} \ge 2^{k+8}s$ whenever $(E, I, w) \in \mathcal{K}$. By 394A, we can find $m_0 < m_1 < \ldots < m_s$ and an enumeration $\langle(E_i, K_i, w_i)\rangle_{i<s}$ of $\mathcal{K}$ such that $\#(K_i \cap m_{i+1} \setminus m_i) = 2^{k+8}$ for $i < s$.

For each $i < s$, set $u'_i = \sup(K_i \cap (r+1))$ and $K'_i = (K_i \setminus \{u'_i\}) \cap m_{i+1} \setminus m_i$, so that $\#(K'_i) \ge 2^{k+8} - 1$. By 394B we can find successive members u_i, v_i of K'_i, for $i < s$, such that, setting $W = \bigcup_{i<s} v_i \setminus u_i$,

$$S = \sum_{\substack{(E,I,w)\in\mathcal{J} \\ \#(I\cap W)\ge\frac{1}{2}\#(I)}} w \le \frac{2}{2^{k+8}-2} \sum_{(E,I,w)\in\mathcal{J}} w \le \frac{2}{2^{k+7}} c_k \le 2^{-k-2}.$$

Also $r \notin W$. **P** If $i < s$ and $u_i \le r$, then $u_i \le u'_i \le r$; but $u_i \ne u'_i$, so $u_i < u'_i$ and $v_i \le u'_i \le r$. **Q**

(d) Set

$$\mathcal{J}_1 = \{(E, I, w) : (E, I, w) \in \mathcal{J},\ \#(I \cap W) \ge \frac{1}{2}\#(I)\}, \quad \mathcal{J}_2 = \mathcal{J} \setminus \mathcal{J}_1;$$

then

$$\mathrm{wt}\,\mathcal{J}_1 = S \le 2^{-k-2} \le \frac{1}{4}.$$

For $i < s$ set

$$\mathcal{J}_{1i} = \{(E, I, w) : (E, I, w) \in \mathcal{J}_1,\ \#(I \cap v_i \setminus u_i) \ge 2^{-k-5}\#(I)\}.$$

Since $s \le 2^{k+4}$, $\mathcal{J}_1 = \bigcup_{i<s} \mathcal{J}_{1i}$.

(e) Suppose that $i < s$ and $A \in \mathcal{A}_{u_i}$. Then there is an $A' \in \mathcal{A}_{v_i}$ such that $A' \subseteq A \setminus (E_i \cup \mathrm{spr}\,\mathcal{J}_{1i})$. **P** Set $C = \theta_{v_i \setminus u_i}(A \cap \mathrm{spr}\,\mathcal{J}_{1i}) \in \mathfrak{B}_{v_i}$. By 394Fc, applied to $\mathcal{C}_{k+1,p}$,

$$\nu_{k+1,p}C \le 2\,\mathrm{wt}\,\mathcal{J}_{1i} \le 2\,\mathrm{wt}\,\mathcal{J}_1 < 1.$$

As $(E_i, K_i, w_i) \in \mathcal{E}_{kp}$, E_i is $\nu_{k+1,p}$-thin between u_i and v_i, $\nu_{k+1,p}(X \setminus \theta_{v_i \setminus u_i}(A \cap E_i)) \ge 1$ and C does not include $X \setminus \theta_{v_i \setminus u_i}(A \cap E_i)$. Since these sets both belong to $\mathfrak{B}_{v_i}$ there is an $A_1 \in \mathcal{A}_{v_i}$ disjoint from both C and $\theta_{v_i \setminus u_i}(A \cap E_i)$, that is, disjoint from $\theta_{v_i \setminus u_i}(A \cap (E_i \cup \mathrm{spr}\,\mathcal{J}_{1i}))$. Now $A' = A \cap \theta_{v_i \setminus u_i}(A_1)$ belongs to $\mathcal{A}_{v_i}$, is included in A and is disjoint from $E_i \cup \mathrm{spr}\,\mathcal{J}_{1i}$. **Q**

(f) We can therefore find a function $\Gamma : X \to X$ such that $\Gamma[X]$ is disjoint from $\mathrm{spr}(\mathcal{K} \cup \mathcal{J}_1)$, while $\Gamma(x){\restriction}m$ is determined by $x{\restriction}m$ for every $m \in \mathbb{N}$. **P** By (e) just above, we have for each $i < s$ a function $q_i : \mathcal{A}_{u_i} \to \mathcal{A}_{v_i}$ such that $q_i(A) \subseteq A \setminus (E_i \cup \mathrm{spr}\,\mathcal{J}_{1i})$ for every $A \in \mathcal{A}_{u_i}$. We can re-interpret q_i as a function $h_i : \prod_{n<u_i} T_n \to \prod_{n<v_i} T_n$ defined by saying that if $A = \{x : x{\restriction}u_i = z\}$ then $q_i(A) = \{x : x{\restriction}v_i = h_i(z)\}$; note that $z = h_i(z){\restriction}u_i$ for every $z \in \prod_{n<u_i} T_n$. Now, for $x \in X$, define $\Gamma(x)(n)$ inductively by saying that

$$\begin{aligned}\Gamma(x)(n) &= x(n) \text{ if } n \in \mathbb{N} \setminus W, \\ &= h_i(\Gamma(x){\restriction}u_i)(n) \text{ if } i < s \text{ and } u_i \le n < v_i.\end{aligned}$$

Of course this ensures that $\Gamma(x){\restriction}m$ is determined by $x{\restriction}m$ for every m. If $i < s$, $x \in X$, and $A \in \mathcal{A}_{u_i}$ is such that $\Gamma(x) \in A$, then $\Gamma(x) \in q_i(A)$, which is disjoint from $E_i \cup \mathrm{spr}\,\mathcal{J}_{1i}$. Thus $\Gamma[X]$ is disjoint from $\bigcup_{i<s} E_i \cup \mathrm{spr}\,\mathcal{J}_{1i} = \mathrm{spr}(\mathcal{K} \cup \mathcal{J}_1)$. **Q**

(g) Take $(E, I, w) \in \mathcal{J}_2$ and consider $\nu_{k+1,p}(\Gamma^{-1}[E])$.

(i) There is an l such that $k < l < p$ and $(E, I, w) \in \mathcal{E}_{lp}$. Now if m, $n \in I$ are such that $m < n$ and $n \setminus m$ is disjoint from W, $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin between m and n. **P** Take any $A \in \mathcal{A}_m$. Because $\Gamma(x){\restriction}m$ is determined by $x{\restriction}m$, we can find an $A' \in \mathcal{A}_m$ such that $\Gamma[A] \subseteq A'$. In this case,

$$A \cap \Gamma^{-1}[E] \subseteq \Gamma^{-1}[\Gamma[A] \cap E] \subseteq \Gamma^{-1}[A' \cap E] \subseteq \theta_{n\setminus m}(A' \cap E)$$

because $\Gamma(x)(i) = x(i)$ whenever $x \in X$ and $i \in n \setminus m$. So $\theta_{n\setminus m}(A \cap \Gamma^{-1}[E]) \subseteq \theta_{n\setminus m}(A' \cap E)$ and

$$\nu_{l+1,p}(X \setminus \theta_{n\setminus m}(A \cap \Gamma^{-1}[E])) \ge \nu_{l+1,p}(X \setminus \theta_{n\setminus m}(A' \cap E)) \ge 1$$

because E is $\nu_{l+1,p}$-thin between m and n. **Q**

(ii) As noted in (b), $\#(I) \ge 2^{2k+12} \ge 4s$. For each $i < s$ such that $\min I \le u_i$, let u_i^- be the largest element of I which is less than or equal to u_i. Set $I' = I \setminus (W \cup \{u_i^- : i < s,\ \min I \le u_i\})$. Then

$$\#(I') \geq \frac{\#(I)}{2} - s \geq \frac{\#(I)}{4}.$$

Now $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin along I'. **P** Suppose that m, $n \in I'$ and $m < n$. Let m^+ be the least element of I such that $m < m^+$. Then $m^+ \leq n$. **?** If $W \cap m^+ \setminus m \neq \emptyset$, there is an $i < s$ such that $m^+ \setminus m$ meets $v_i \setminus u_i$, that is, $m < v_i$ and $u_i < m^+$. Since $m \in I' \subseteq \mathbb{N} \setminus W$, $m \leq u_i$ and u_i^- is defined; now $m \neq u_i^-$ so $m < u_i^- \in I$ and $m^+ \leq u_i^- \leq u_i$. **X** Thus $W \cap m^+ \setminus m$ is empty and (i) tells us that $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin between m and m^+, therefore $\nu_{l+1,p}$-thin between m and n (394Dc). As m and n are arbitrary, $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin along I'. **Q**

(iii) If we now set $w' = 4^{\alpha_l} w$, we see that

$$1 \leq \#(I') \leq \#(I) \leq N_l, \quad w' \geq 2^{-l} 4^{\alpha_l} \bigl(\frac{N_l}{\#(I)}\bigr)^{\alpha_l} \geq 2^{-l} \bigl(\frac{N_l}{\#(I')}\bigr)^{\alpha_l},$$

so

$$(\Gamma^{-1}[E], I', w') \in \mathcal{E}_{lp} \subseteq \mathcal{C}_{k+1,p}$$

and

$$\nu_{k+1,p}(\Gamma^{-1}[E]) \leq w' \leq 4^{\alpha_l} w \leq 4^{\alpha_k} w.$$

(h) We are nearly done. Applying (g) to each member of $\mathcal{J}_2$,

$$\nu_{k+1,p}(\Gamma^{-1}[\operatorname{spr} \mathcal{J}_2]) \leq 4^{\alpha_k} \operatorname{wt} \mathcal{J}_2 \leq 4^{\alpha_k} \operatorname{wt} \mathcal{I} < 4^{\alpha_k} c_k = c_{k+1} \leq \nu_{k+1,p} Y_{rt}$$

by the inductive hypothesis in its full strength. So there is a $y \in Y_{rt} \setminus \Gamma^{-1}[\operatorname{spr} \mathcal{J}_2]$. With (f), this means that $\Gamma(y)$ does not belong to

$$\operatorname{spr}(\mathcal{K} \cup \mathcal{J}_1) \cup \operatorname{spr}(\mathcal{J}_2) = \operatorname{spr} \mathcal{I}.$$

On the other hand, $\Gamma(y) \in Y_{rt}$ because $r \notin W$. As $\mathcal{I}$ was arbitrary, $\nu_{kp} Y_{rt}$ must be at least c_k, which is what we need to know to proceed with the induction.

394H Definitions I present the last two definitions required. Fix on a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. For $k \in \mathbb{N}$, set

$$\nu_k E = \lim_{p \to \mathcal{F}} \nu_{kp} E \in [0, \infty]$$

for every $E \in \mathfrak{B}$; finally, write ν for ν_0.

394I Proposition (a) For every $k \in \mathbb{N}$, ν_k is a totally finite submeasure and $\nu_k X \geq 8$.
(b) ν is not uniformly exhaustive.

proof (a) It follows directly from the definition in 392A that ν_k, being a limit of submeasures, is a submeasure. By 394Dh, $\nu_k X \leq 2^{-k} N_k^{\alpha_k}$ is finite. By 394G and 394Dg,

$$\nu_k X = \lim_{p \to \mathcal{F}} \nu_{kp} X \geq c_k \geq 8.$$

(b) For any $n \in \mathbb{N}$ and $t \in T_n$,

$$\nu Y_{nt} = \lim_{p \to \mathcal{F}} \nu_{0p} Y_{nt} \geq 8$$

by 394G. As $\sup_{n \in \mathbb{N}} \#(T_n)$ is infinite, and $\langle Y_{nt} \rangle_{t \in T_n}$ is disjoint for every n, ν is not uniformly exhaustive.

394J Lemma Suppose that $k \in \mathbb{N}$, $E \in \mathfrak{B}$, $I \in [\mathbb{N}]^{<\omega}$ and E is $\frac{1}{2}\nu_k$-thin along I. Then

$$\{p : p \geq k,\, E \text{ is } \nu_{kp}\text{-thin along } I\} \in \mathcal{F}.$$

If $k \geq 1$ and $\#(I) = N_{k-1}$, then $\nu_{k-1} E \leq 2^{-k+1}$.

proof If m, $n \in I$, $m < n$ and $A \in \mathcal{A}_m$, then $\nu_k(X \setminus \theta_{n \setminus m}(A \cap E)) \geq 2$. So

$$U_{An} = \{p : p \geq k,\, \nu_{kp}(X \setminus \theta_{n \setminus m}(A \cap E)) \geq 1\}$$

belongs to $\mathcal{F}$. Setting $U = \bigcap_{m < n \text{ in } I, A \in \mathcal{A}_m} U_{An}$, $U \in \mathcal{F}$ and E is ν_{kp}-thin along I for every $p \in U$.

If $k \geq 1$ and $\#(I) = N_{k-1}$, then $(E, I, 2^{-k+1}) \in \mathcal{E}_{k-1,p}$ for every $p \in U$, so $\nu_{k-1,p} E \leq 2^{-k+1}$ for every $p \in U$ and $\nu_{k-1} E \leq 2^{-k+1}$.

394K Lemma Let m, $k \in \mathbb{N}$ and let $\langle E_i \rangle_{i\in\mathbb{N}}$ be a sequence in $\mathfrak{B}$ such that

every E_i is determined by coordinates in $\mathbb{N}\setminus m$,

$\nu_k(\bigcup_{i\le n} E_i) < 2$ for every $n \in \mathbb{N}$.

Then for every $\eta > 0$ there is a $C \in \mathfrak{B}$, determined by coordinates in $\mathbb{N}\setminus m$, such that $\nu_k C \le 4$ and $\nu_k(E_i \setminus C) \le \eta$ for each i.

proof (a) For each $n > m$, set

$$\tilde{E}_n = \bigcup\{E_i : i \le n,\, E_i \in \mathfrak{B}_n\},$$

so that $\tilde{E}_n$ is determined by coordinates in $n\setminus m$ and $\nu_k\tilde{E}_n < 2$. Set

$$U_n = \{p : p \ge k,\, \nu_{kp}\tilde{E}_n < 2\} \in \mathcal{F}.$$

For $p \in U_n$ we can find a finite $\mathcal{I}_{np} \subseteq \mathcal{C}_{kp}$ such that $\tilde{E}_n \subseteq \operatorname{spr}\mathcal{I}_{np}$ and $\operatorname{wt}\mathcal{I}_{np} \le 2$. For $r > m$ set

$$\begin{aligned}\mathcal{I}_{npr} = \{(E, I, w) : {} & (E, I, w) \in \mathcal{I}_{np},\\ & \#(I \cap (r-1)\setminus m) < \tfrac{1}{2}\#(I) \le \#(I \cap r \setminus m)\},\end{aligned}$$

and set

$$\mathcal{I}'_{np} = \{(E, I, w) : (E, I, w) \in \mathcal{I}_{np},\, \#(I \cap m) \ge \tfrac{1}{4}\#(I)\}.$$

Set $B_{np} = \theta_m(\operatorname{spr}\mathcal{I}'_{np})$; then

$$\nu_{kp}B_{np} \le 2\operatorname{wt}\mathcal{I}'_{np} \le 4,$$

by 394Fa. Since $\nu_{kp}X \ge c_k \ge 8$ (394G, 394Dg again), $B_{np} \ne X$ and there is an $A_{np} \in \mathcal{A}_m$ disjoint from $\operatorname{spr}\mathcal{I}'_{np}$. Next, for $m < r \le n$ and $p \in U_n$ set

$$\mathcal{J}_{npr} = \{(\theta_{r\setminus m}(A_{np} \cap E), I \cap r\setminus m, 2w) : (E, I, w) \in \mathcal{I}_{npr}\}, \quad F_{npr} = \operatorname{spr}\mathcal{J}_{npr}.$$

By 394E, $\mathcal{J}_{npr} \subseteq \mathcal{C}_{kp}$, so $\nu_{kp}F_{npr} \le \operatorname{wt}\mathcal{J}_{npr} \le 2\operatorname{wt}\mathcal{I}_{npr}$. Note that F_{npr} is determined by coordinates in $r\setminus m$ and includes $A_{np} \cap \operatorname{spr}\mathcal{I}_{npr}$. Now if $m < j \le n$ and $p \in U_n$, $\tilde{E}_j \subseteq \bigcup_{m<r\le j} F_{npr}$. **P?** Otherwise, since both sets are determined by coordinates in $j\setminus m$, and since $A_{np} \in \mathcal{A}_m$, there is an $A \in \mathcal{A}_j$ with

$$A \subseteq A_{np} \cap \tilde{E}_j \setminus \bigcup_{m<r\le j} F_{npr} \subseteq A_{np} \cap \tilde{E}_j \setminus \bigcup_{m<r\le j} \operatorname{spr}\mathcal{I}_{npr}.$$

Since A is also disjoint from $\operatorname{spr}\mathcal{I}'_{np}$ and $A \subseteq \tilde{E}_j \subseteq \operatorname{spr}\mathcal{I}_{np}$, $A \subseteq \operatorname{spr}\mathcal{I}$, where

$$\begin{aligned}\mathcal{I} &= \mathcal{I}_{np} \setminus (\mathcal{I}'_{np} \cup \bigcup_{m<r\le j} \mathcal{I}_{npr})\\ &\subseteq \{(E, I, w) : (E, I, w) \in \mathcal{I}_{np},\, \#(I\setminus j) \ge \tfrac{1}{4}\#(I)\}.\end{aligned}$$

Since $\mathcal{I} \subseteq \mathcal{C}_{kp}$,

$$\begin{aligned}8 \le \nu_{kp}X = \nu_{kp}(\theta_{\mathbb{N}\setminus j}(A)) &= \nu_{kp}(\theta_{\mathbb{N}\setminus j}(A \cap \operatorname{spr}\mathcal{I})) \le 2\operatorname{wt}\mathcal{I}\\ &\qquad (394\text{Fb})\\ &\le 4. \ \mathbf{XQ}\end{aligned}$$

(b) For $r > m$ we can find $F_r \in \mathfrak{B}$ such that

$\sum_{r=m+1}^{\infty} \nu_k F_r \le 4$,

$\tilde{E}_j \subseteq \bigcup_{m<r\le j} F_r$ for every $j > m$,

F_r is determined by coordinates in $r\setminus m$.

P If $n \ge r > m$, then, because $\mathfrak{B}_r$ is finite, there is a set $F_{nr} \in \mathfrak{B}_r$ such that $\{p : p \in U_n,\, F_{npr} = F_{nr}\}$ belongs to $\mathcal{F}$. Next, if $r > m$ there is an $F_r \in \mathfrak{B}_r$ such that $\{n : n \ge r,\, F_{nr} = F_r\}$ belongs to $\mathcal{F}$. Now

$$\nu_k F_r = \lim_{n\to\mathcal{F}} \nu_k F_{nr}$$

(because $\{n : \nu_k F_{nr} = \nu_k F_r\} \supseteq \{n : F_{nr} = F_r\} \in \mathcal{F}$)

$$= \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \nu_{kp} F_{nr} = \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \nu_{kp} F_{npr} \le 2 \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \mathrm{wt}\,\mathcal{I}_{npr}.$$

So, for $s > m$,

$$\sum_{r=m+1}^{s} \nu_k F_r \le 2 \sum_{r=m+1}^{s} \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \mathrm{wt}\,\mathcal{I}_{npr}$$

$$= 2 \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \sum_{r=m+1}^{s} \mathrm{wt}\,\mathcal{I}_{npr} \le 2 \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \mathrm{wt}\,\mathcal{I}_{np} \le 4.$$

As s is arbitrary, $\sum_{r=m+1}^{\infty} \nu_k F_r \le 4$.

If $n \ge j > m$, then we saw in (a) that $\tilde{E}_j \subseteq \bigcup_{m<r\le j} F_{npr}$ for every $p \in U_n$. Since there are many p such that $F_{nr} = F_{npr}$ whenever $m < r \le j$, $\tilde{E}_j \subseteq \bigcup_{m<r\le j} F_{nr}$. Now, given $j > m$, there are many n such that $F_{nr} = F_r$ whenever $m < r \le j$, so $\tilde{E}_j \subseteq \bigcup_{m<r\le j} F_r$.

Finally, take any $r > m$. Since F_{npr} is determined by coordinates in $r \setminus m$ whenever $n \ge r$ and $p \in U_n$, F_{nr} is determined by coordinates in $r \setminus m$ whenever $n \ge r$, and F_r also is determined by coordinates in $r \setminus m$. **Q**

(c) Let $r_0 \ge m$ be such that $\sum_{r=r_0+1}^{\infty} \nu_k F_r \le \eta$. Set $C = \bigcup_{m<r\le r_0} F_r$. Then C is determined by coordinates in $\mathbb{N} \setminus m$ and

$$\nu_k C \le \sum_{r=m+1}^{r_0} \nu_k F_r \le 4.$$

For any $i \in \mathbb{N}$, there is some $j > r_0$ such that $E_i \subseteq \tilde{E}_j$, in which case

$$E_i \setminus C \subseteq \bigcup_{r_0<r\le j} F_r$$

and

$$\nu_k(E_i \setminus C) \le \sum_{r=r_0+1}^{j} \nu_k F_r \le \eta,$$

as required.

394L Lemma Suppose that $k \in \mathbb{N}$, $\epsilon > 0$, $m \in \mathbb{N}$, $B \in \mathfrak{B}_m$ and that $\langle E_i \rangle_{i\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$. Then there are $n > m$ and $B' \in \mathfrak{B}_n$ such that $B' \subseteq B$, B' is $\frac{1}{2}\nu_k$-thin between m and n and $\limsup_{i\to\infty} \nu_k(E_i \cap B \setminus B') \le \epsilon$.

proof Set $\eta = \dfrac{\epsilon}{\#(\mathcal{A}_m)}$. For those $A \in \mathcal{A}_m$ included in B define $C'_A \subseteq A$ as follows.

case 1 If there is some r such that $\nu_k(\theta_{\mathbb{N}\setminus m}(A \cap \bigcup_{i\le r} E_i)) \ge 2$, set $C'_A = A \setminus \bigcup_{i\le r} E_i$, so that $\frac{1}{2}\nu_k(\theta_{\mathbb{N}\setminus m}(A \setminus C'_A)) \ge 1$ and $E_i \cap A \setminus C'_A = \emptyset$ for $i > r$.

case 2 If $\nu_k(\theta_{\mathbb{N}\setminus m}(A \cap \bigcup_{i\le r} E_i)) < 2$ for every r, then by 394K, applied to the sequence $\langle \theta_{\mathbb{N}\setminus m}(A \cap E_i) \rangle_{i\in\mathbb{N}}$, we can find a $C \in \mathfrak{B}$, determined by coordinates in $\mathbb{N} \setminus m$, such that $\nu_k C \le 4$ and $\nu_k(\theta_{\mathbb{N}\setminus m}(A \cap E_i) \setminus C) \le \eta$ for every i. Set $C'_A = C \cap A$. Because C is determined by coordinates in $\mathbb{N} \setminus m$ and $A \in \mathcal{A}_m$, $\nu_k(\theta_{\mathbb{N}\setminus m}(C'_A)) = \nu_k C \le 4$. Also $E_i \cap A \setminus C'_A \subseteq \theta_{\mathbb{N}\setminus m}(A \cap E_i) \setminus C$ so $\nu_k(E_i \cap A \setminus C'_A) \le \eta$ for every i.

Set

$$B' = \bigcup\{C'_A : A \in \mathcal{A}_m,\ A \subseteq B\}.$$

Then $B' \in \mathfrak{B}$, $B' \subseteq B$ and

$$\limsup_{i\to\infty} \nu_k(E_i \cap B \setminus B') \le \sum_{A\in\mathcal{A}_m, A\subseteq B} \limsup_{i\to\infty} \nu_k(E_i \cap A \setminus C'_A)$$

$$\le \sum_{A\in\mathcal{A}_m, A\subseteq B} \eta \le \epsilon.$$

Let $n > m$ be such that $C'_A \in \mathfrak{B}_n$ whenever $A \in \mathcal{A}_m$ and $A \subseteq B$. Then B' is $\frac{1}{2}\nu_k$-thin between m and n. **P** Take any $A \in \mathcal{A}_m$ and set $\tilde{C} = \theta_{n\setminus m}(A \cap B')$. If $A \not\subseteq B$ then $A \cap B'$ and $\tilde{C}$ are empty and $\nu_k(X \setminus \tilde{C}) \ge 8$ (394Ia). Otherwise, $A \cap B' = C'_A \in \mathfrak{B}_n$ so $\tilde{C} = \theta_{\mathbb{N}\setminus m}(C'_A)$ is disjoint from $\theta_{\mathbb{N}\setminus m}(A \setminus C'_A)$ (see the last remark in 394Db). If C'_A was chosen as in case 1 above,

$$\nu_k(X \setminus \tilde{C}) \ge \nu_k(\theta_{\mathbb{N}\setminus m}(A \setminus C'_A)) \ge 2.$$

If C'_A was chosen as in case 2,

$$\nu_k(X \setminus \tilde{C}) = \nu_k(X \setminus \theta_{\mathbb{N}\setminus m}(C'_A)) \ge \nu_k X - \nu_k(\theta_{\mathbb{N}\setminus m}(C'_A)) \ge 8 - 4.$$

So in all three cases we have $\frac{1}{2}\nu_k(X \setminus \tilde{C}) \ge 1$, as required. **Q**

Thus we have an appropriate B'.

394M Theorem ν is exhaustive.

proof Let $\langle E_i\rangle_{i\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{B}$. Take any $k \in \mathbb{N}$ and $\epsilon > 0$, and choose $\langle B_j\rangle_{j\in\mathbb{N}}$ and $\langle n_j\rangle_{j\in\mathbb{N}}$ inductively, as follows. $B_0 = X$ and $n_0 = 0$. Given that $B_j \in \mathfrak{B}_{n_j}$, take $n_{j+1} > n_j$ and $B_{j+1} \in \mathfrak{B}_{n_{j+1}}$ such that $B_{j+1} \subseteq B_j$, B_{j+1} is $\frac{1}{2}\nu_{k+1}$-thin between n_j and n_{j+1}, and $\limsup_{i\to\infty} \nu_{k+1}(E_i \cap B_j \setminus B_{j+1}) \le \epsilon$ (394L). Continue. Note that $\limsup_{i\to\infty} \nu_{k+1}(E_i \setminus B_j) \le j\epsilon$ for every j.

Set $I = \{n_j : j < N_k\}$ and $B = B_{N_k-1}$. Then B is $\frac{1}{2}\nu_{k+1}$-thin along I (use 394Dc). By 394J, $\nu_k B \le 2^{-k}$.

Of course $\nu \le \nu_k \le \nu_{k+1}$ (394Df). So

$$\limsup_{i\to\infty} \nu E_i \le \nu_k B + \limsup_{i\to\infty} \nu_{k+1}(E_i \setminus B) \le 2^{-k} + (N_k - 1)\epsilon.$$

As k, ϵ and $\langle E_i\rangle_{i\in\mathbb{N}}$ are arbitrary, ν is exhaustive.

394N Remarks (a) Note that the whole construction is invariant under the action of the group $\prod_{n\in\mathbb{N}} G_n$ where G_n is the group of all permutations of T_n for each n. In particular, if we give each T_n a group structure and X the product group structure, then ν is translation-invariant.

(b) It follows that ν is strictly positive. **P** For each $n \in \mathbb{N}$, ν is constant on $\mathcal{A}_n$, so $\nu E \ge \nu X/\#(\mathcal{A}_n) > 0$ for every non-empty $E \in \mathfrak{B}_n$. **Q**

(c) We can therefore form the metric completion $\widehat{\mathfrak{B}}$ of $\mathfrak{B}$, as in 392H, and $\widehat{\mathfrak{B}}$ will be a Maharam algebra, with a strictly positive Maharam submeasure $\hat{\nu}$ continuously extending ν (393H). Now $\widehat{\mathfrak{B}}$ is not measurable. **P?** Otherwise, let $\bar{\mu}$ be such that $(\widehat{\mathfrak{B}}, \bar{\mu})$ is a probability algebra. Then $\bar{\mu}$ and $\hat{\nu}$ are strictly positive Maharam submeasures on $\widehat{\mathfrak{B}}$, so $\hat{\nu}$ is absolutely continuous with respect to $\bar{\mu}$ (393F). Let $n \ge 1$ be such that $\hat{\nu}b < 8$ whenever $\bar{\mu}b \le 1/\#(T_n)$. Then there must be a $t \in T_n$ such that $\bar{\mu}Y_{nt} \le 1/\#(T_n)$; but $\hat{\nu}Y_{nt} = \nu Y_{nt} \ge 8$ (see the proof of 394Ib). **XQ**

In fact, $\widehat{\mathfrak{B}}$ is nowhere measurable (394Ya).

***394O Control measures** One of the original reasons for studying Maharam submeasures was their connexion with the following notion. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and U a Hausdorff linear topological space. (The idea is intended to apply, in particular, when $\mathfrak{A}$ is a σ-algebra of subsets of a set.) A function $\theta : \mathfrak{A} \to U$ is a **vector measure** if $\sum_{n=0}^{\infty} \theta a_n = \lim_{n\to\infty} \sum_{i=0}^{n} \theta a_i$ is defined in U and equal to $\theta(\sup_{n\in\mathbb{N}} a_n)$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. In this case, a non-negative countably additive functional $\mu : \mathfrak{A} \to [0,\infty[$ is a **control measure** for θ if $\theta a = 0$ whenever $\mu a = 0$.

***394P Example** There are a metrizable linear topological space U and a vector measure $\theta : \Sigma \to U$, where Σ is a σ-algebra of sets, such that θ has no control measure.

proof As in 394Nc, let $\widehat{\mathfrak{B}}$ be the metric completion of $\mathfrak{B}$, and $\hat{\nu}$ the continuous extension of ν to $\widehat{\mathfrak{B}}$. Give $L^0 = L^0(\widehat{\mathfrak{B}})$ the topology defined from $\hat{\nu}$ as in 393K, so that L^0 is a metrizable linear topological space. By 314M, we can identify $\widehat{\mathfrak{B}}$ with a quotient algebra $\Sigma/\mathcal{N}$ where Σ is a σ-algebra of subsets of a set Ω and $\mathcal{N}$ is a σ-ideal in Σ. Set $\theta E = \chi E^\bullet \in L^0$ for $E \in \Sigma$. Then θ is a vector measure. **P** If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ with union E, set $F_n = \bigcup_{i\le n} E_i$, so that $\chi F_n^\bullet = \sum_{i=0}^{n} \chi E_i^\bullet$ for each n. We have $\hat{\nu}(E^\bullet \setminus F_n^\bullet) \to 0$, so that

$$\tau(\theta E - \theta F_n) = \tau(\chi E^\bullet - \chi F_n^\bullet)) = \min(1, \hat{\nu}(E^\bullet \setminus F_n^\bullet)) \to 0,$$

where τ is the functional of the proof of 393K, and $\theta E = \sum_{i=0}^{\infty} \theta E_i$ in L^0. **Q**

If μ is a totally finite measure with domain Σ, set

$$\lambda a = \inf\{\mu E : E \in \Sigma,\ E^\bullet = a\}$$

for every $a \in \widehat{\mathfrak{B}}$. Note that the infimum is always attained. **P** If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ such that $E_n^\bullet = a$ for every $n \in \mathbb{N}$ and $\lambda a = \lim_{n\to\infty} \mu E_n$, set $E = \bigcap_{n\in\mathbb{N}} E_n$; then $E^\bullet = a$ and $\mu E = \lambda a$. **Q** Next, λ is countably additive. **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\widehat{\mathfrak{B}}$ with supremum a, take $E_n \in \Sigma$ such that $E_n^\bullet = a_n$ and $\mu E_n = \lambda a_n$ for each n, and $E \in \Sigma$ such that $E^\bullet = a$ and $\mu E = \lambda a$. Set $F_n = E \cap E_n \setminus \bigcup_{i<n} E_i$ for each n, and $F = \bigcup_{n\in\mathbb{N}} F_n$. Then $F_n^\bullet = a_n$ and $F_n \subseteq E_n$, so $\mu F_n = \lambda a_n$ for each n; similarly, $F^\bullet = a$ and $F \subseteq E$, so $\mu F = \lambda a$. Also $\langle F_n\rangle_{n\in\mathbb{N}}$ is disjoint and has union F. Accordingly

$$\lambda a = \mu F = \sum_{n=0}^\infty \mu F_n = \sum_{n=0}^\infty \lambda a_n. \textbf{ Q}$$

Since $\widehat{\mathfrak{B}}$ is not a measurable algebra, λ cannot be strictly positive, and there is a non-zero $a \in \widehat{\mathfrak{B}}$ such that $\lambda a = 0$. Let $E \in \Sigma$ be such that $E^\bullet = a$ and $\mu E = 0$; then $\theta E = \chi a \ne 0$. So μ is not a control measure for θ.

***394Q** This is not a book about vector measures, but having gone so far I ought to note that the generality of the phrase 'metrizable linear topological space' in 394P is essential. If we look only at normed spaces the situation is very different.

Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, U a normed space and $\theta : \mathfrak{A} \to U$ a vector measure. Then θ has a control measure.

proof (a) Since U can certainly be embedded in a Banach space $\hat U$ (3A5Jb), and as θ will still be a vector measure when regarded as a map from $\mathfrak{A}$ to $\hat U$, we may assume from the beginning that U itself is complete.

(b) θ is bounded (that is, $\sup_{a\in\mathfrak{A}} \|\theta a\|$ is finite). **P?** (Cf. 326M.) Suppose, if possible, otherwise. Choose $\langle a_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $a_0 = 1$. Given that $\sup_{a\subseteq a_n} \|\theta a\| = \infty$, choose $b \subseteq a_n$ such that $\|\theta b\| \ge \|\theta a_n\| + 1$. Then $\|\theta(a_n \setminus b)\| \ge 1$. Also

$$\sup_{a\subseteq a_n} \|\theta a\| \le \sup_{a\subseteq a_n} \|\theta(a\cap b)\| + \|\theta(a\setminus b)\|,$$

so at least one of $\sup_{a\subseteq b} \|\theta a\|$, $\sup_{a\subseteq a_n\setminus b} \|\theta a\|$ must be infinite. We may therefore take a_{n+1} to be either b or $a_n \setminus b$ and such that $\sup_{a\subseteq a_{n+1}} \|\theta a\| = \infty$. Observe that in either case we shall have $\|\theta(a_n \setminus a_{n+1})\| \ge 1$. Continue.

At the end of the induction we shall have a disjoint sequence $\langle a_n \setminus a_{n+1}\rangle_{n\in\mathbb{N}}$ such that $\|\theta(a_n \setminus a_{n+1})\| \ge 1$ for every n, so that $\sum_{n=0}^\infty \theta(a_n \setminus a_{n+1})$ cannot be defined in U; which is impossible. **XQ**

(c) Accordingly we have a bounded linear operator $T : L^\infty \to U$, where $L^\infty = L^\infty(\mathfrak{A})$, such that $T\chi = \theta$ (363Ea).

Now the key to the proof is the following fact: if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a disjoint order-bounded sequence in $(L^\infty)^+$, $\langle Tu_n\rangle_{n\in\mathbb{N}} \to 0$ in U. **P** Let γ be such that $u_n \le \gamma\chi 1$ for every n. Let $\epsilon > 0$, and let k be the integer part of γ/ϵ. For $n \in \mathbb{N}$, $i \le k$ set $a_{ni} = [\![u_n > \epsilon(i+1)]\!]$; then $\langle a_{ni}\rangle_{n\in\mathbb{N}}$ is disjoint for each i, and if we set $v_n = \epsilon\sum_{i=0}^k \chi a_{ni}$, we get $v_n \le u_n \le v_n + \epsilon\chi 1$, so $\|u_n - v_n\|_\infty \le \epsilon$.

Because $\langle a_{ni}\rangle_{n\in\mathbb{N}}$ is disjoint, $\sum_{n=0}^\infty \theta a_{ni}$ is defined in U, and $\langle\theta a_{ni}\rangle_{n\in\mathbb{N}} \to 0$, for each $i \le k$. Consequently

$$Tv_n = \epsilon\sum_{i=0}^k \theta a_{ni} \to 0$$

as $n \to \infty$. But

$$\|Tu_n - Tv_n\| \le \|T\|\|u_n - v_n\|_\infty \le \epsilon\|T\|$$

for each n, so $\limsup_{n\to\infty} \|Tu_n\| \le \epsilon\|T\|$. As ϵ is arbitrary, $\lim_{n\to\infty} \|Tu_n\| = 0$. **Q**

(d) Consider the adjoint operator $T' : U^* \to (L^\infty)^*$. Recall that L^∞ is an M-space (363Ba) so that its dual is an L-space (356N). Write

$$A = \{T'g : g \in U^*,\ \|g\| \le 1\} \subseteq (L^\infty)^* = (L^\infty)^\sim.$$

If $u \in L^\infty$, then

$$\sup_{f\in A} |f(u)| = \sup_{\|g\|\le 1} |(T^*g)(u)| = \sup_{\|g\|\le 1} |g(Tu)| = \|Tu\|.$$

Now A is uniformly integrable. **P** I use the criterion of 356O. Of course $\|f\| \le \|T'\|$ for every $f \in A$, so A is norm-bounded. If $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded disjoint sequence in $(L^\infty)^+$, then

$$\sup_{f\in A} |f(u_n)| = \|Tu_n\| \to 0$$

as $n \to \infty$. So A is uniformly integrable. **Q**

(e) Next, $A \subseteq (L^\infty)^\sim_c$. **P** If $f \in A$, it is of the form $T'g$ for some $g \in U^*$, that is,

$$f(\chi a) = (T'g)(\chi a) = gT(\chi a) = g(\theta a)$$

for every $a \in \mathfrak{A}$. If now $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a,

$$f(\chi a) = g(\theta(\sup_{n\in\mathbb{N}} a_n)) = g(\sum_{n=0}^\infty \theta a_n) = \sum_{n=0}^\infty g(\theta a_n) = \sum_{n=0}^\infty f(\chi a_n).$$

So $f\chi$ is countably additive. By 363K, $f \in (L^\infty)^\sim_c$. **Q**

(f) Because A is uniformly integrable, there is for each $m \in \mathbb{N}$ an $f_m \ge 0$ in $(L^\infty)^*$ such that $\|(|f| - f_m)^+\| \le 2^{-m}$ for every $f \in A$; moreover, we can suppose that f_m is of the form $\sup_{i\le k_m} |f_{mi}|$ where every f_{mi} belongs to A (354R(b-iii)), so that $f_m \in (L^\infty)^\sim_c$ and $\mu_m = f_m\chi$ is countably additive. Set

$$\mu = \sum_{m=0}^\infty \frac{1}{2^m(1+\mu_m 1)}\mu_m;$$

then $\mu : \mathfrak{A} \to [0, \infty[$ is a non-negative countably additive functional.

Now μ is a control measure for θ. **P** If $\mu a = 0$, then $\mu_m a = 0$, that is, $f_m(\chi a) = 0$, for every $m \in \mathbb{N}$. But this means that if $g \in U^*$ and $\|g\| \le 1$,

$$|g(\theta a)| = |(T'g)(\chi a)| \le f_m(\chi a) + \|(|T'g| - f_m)^+\| \le 2^{-m}$$

for every m, by the choice of f_m; so that $g(\theta a) = 0$. As g is arbitrary, $\theta a = 0$; as a is arbitrary, μ is a control measure for θ. **Q**

394X Basic exercises (a) Show that the metric completion $\widehat{\mathfrak{B}}$ of $\mathfrak{B}$, as defined in 394N, has many involutions (definition: 382O).

394Y Further exercises (a)(i) Show that if $r \in \mathbb{N}$, $k \le p$ and $E \in \mathfrak{B}_{r+1}$ are such that $\nu_{kp}E < c_k$, then $\nu_{kp}(\theta_r(E)) \le \frac{32}{c_k}\nu_{kp}E$. (ii) Show that if $E \in \mathcal{B}_r$ then $\nu(E \cap Y_{rt}) \ge \min(8, \frac{1}{4}\nu E)$ for every $t \in T_r$. (iii) Let $\widehat{\mathfrak{B}}$ be the metric completion of $\mathfrak{B}$ and $\hat\nu$ the continuous extension of ν to $\widehat{\mathfrak{B}}$. Show that for every $a \in \mathfrak{B}$ and $n \in \mathbb{N}$ there is a disjoint family $\langle c_i \rangle_{i\le n}$ such that $c_i \subseteq a$ and $\hat\nu c_i \ge \min(7, \frac{1}{5}\hat\nu a)$ for every $i \le n$. (iv) Show that the only countably additive real-valued functional on $\widehat{\mathfrak{B}}$ is the zero functional. (v) Show that $\widehat{\mathfrak{B}}$ is nowhere measurable. (vi) Show that if ν' is a uniformly exhaustive submeasure on $\mathfrak{B}$ which is absolutely continuous with respect to ν, then $\nu' = 0$.

394Z Problems (a) Does $\widehat{\mathfrak{B}}$ have an order-closed subalgebra isomorphic to the measure algebra of Lebesgue measure? In particular, if we take $\mathfrak{C} \subseteq \mathfrak{B}$ to be the algebra of sets generated by sets of the form $\{x : x \in X, x(n) = 0\}$ for $n \in \mathbb{N}$, is $\nu\restriction\mathfrak{C}$ uniformly exhaustive?

(b) Suppose that instead of taking large sets T_n, we simply set $T_n = \{0, 1\}$ for every n, but otherwise used the same construction. Should we then find that ν was uniformly exhaustive? (This might be relevant to (a) above.)

(c) Is the Boolean algebra $\widehat{\mathfrak{B}}$ homogeneous?

394 Notes and comments 'Maharam's problem', or the 'control measure problem', was for fifty years one of the most vexing questions in abstract measure theory. To begin with, there were reasonable hopes that there was a positive answer – in the language of this book, that every Maharam algebra was a measurable algebra. If this had been the case, there would have been consequences all over the theories of topological Boolean algebras, topological Riesz spaces and vector measures. In the 1970s, it began to seem too much to ask for. In 1983 the Kalton-Roberts theorem gave new life to the conjecture for a moment, but Roberts 93 demonstrated a major obstacle, which Talagrand (building on some further ideas of I.Farah) eventually developed into the construction above. The ideas which for a generation were collected together by their association with the control measure problem no longer have this as a unifying principle, and (as after any successful revolution) are now more naturally grouped in other ways. In the 2004 edition of this volume, maintained in `http://www.essex.ac.uk/maths/people/fremlin/mt3.2004`, you can find a list of formulations of the control measure problem as it then appeared to be. There is a relic of this era in 394P.

Now that we know for sure that there are non-measurable Maharam algebras, it becomes possible to ask questions about their structure. Frustratingly, practically none of these questions has yet been answered even for the examples constructed by Talagrand's method. (I should of course note that while I speak of 'the' submeasure ν and 'the' Maharam algebra $\widehat{\mathfrak{B}}$, they depend on the sequences $\langle\#(T_n)\rangle_{n\in\mathbb{N}}$, $\langle\alpha_k\rangle_{k\in\mathbb{N}}$ and $\langle N_k\rangle_{k\in\mathbb{N}}$ and the filter $\mathcal{F}$, and there is every reason to suppose that $\mathfrak{c}$ non-isomorphic examples can be constructed by the formulae set out above, without considering elementary variations.) I will return briefly to such questions in Volumes 4 and 5, as I come to further properties of measure algebras which can be interpreted in Maharam algebras.

395 Kawada's theorem

I now describe a completely different characterization of (homogeneous) measurable algebras, based on the special nature of their automorphism groups. The argument depends on the notion of 'non-paradoxical' group of automorphisms; this is an idea of great importance in other contexts, and I therefore aim at a fairly thorough development, with proofs which are adaptable to other circumstances.

395A Definitions Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. For a, $b\in\mathfrak{A}$ I will say that an isomorphism $\phi:\mathfrak{A}_a\to\mathfrak{A}_b$ between the corresponding principal ideals belongs to the **full local semigroup generated by** G if there are a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle\pi_i\rangle_{i\in I}$ in G such that $\phi c=\pi_i c$ whenever $i\in I$ and $c\subseteq a_i$. If such an isomorphism exists I will say that a and b are **G-τ-equidecomposable**.

I will write $a\preccurlyeq^{\tau}_{G}b$ to mean that there is a $b'\subseteq b$ such that a and b' are G-τ-equidecomposable.

For any function f with domain $\mathfrak{A}$, I will say that f is **G-invariant** if $f(\pi a)=f(a)$ whenever $a\in\mathfrak{A}$ and $\pi\in G$.

395B The notion of 'full local semigroup' is of course an extension of the idea of 'full subgroup' (381Be; see also 381Yb). The word 'semigroup' is justified by (c) of the following lemma, and the word 'full' by (e).

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_τ for the full local semigroup generated by G.

(a) Suppose that a, $b\in\mathfrak{A}$ and that $\phi:\mathfrak{A}_a\to\mathfrak{A}_b$ is an isomorphism. Then the following are equiveridical:

(i) $\phi\in G^*_\tau$;

(ii) for every non-zero $c_0\subseteq a$ there are a non-zero $c_1\subseteq c_0$ and a $\pi\in G$ such that $\phi c=\pi c$ for every $c\subseteq c_1$;

(iii) for every non-zero $c_0\subseteq a$ there are a non-zero $c_1\subseteq c_0$ and a $\psi\in G^*_\tau$ such that $\phi c=\psi c$ for every $c\subseteq c_1$.

(b) If a, $b\in\mathfrak{A}$ and $\phi:\mathfrak{A}_a\to\mathfrak{A}_b$ belongs to G^*_τ, then $\phi^{-1}:\mathfrak{A}_b\to\mathfrak{A}_a$ also belongs to G^*_τ.

(c) Suppose that a, b, a', $b'\in\mathfrak{A}$ and that $\phi:\mathfrak{A}_a\to\mathfrak{A}_{a'}$, $\psi:\mathfrak{A}_b\to\mathfrak{A}_{b'}$ belong to G^*_τ. Then $\psi\phi\in G^*_\tau$; its domain is $\mathfrak{A}_c$ where $c=\phi^{-1}(b\cap a')$, and its set of values is $\mathfrak{A}_{c'}$ where $c'=\psi(b\cap a')$.

(d) If a, $b\in\mathfrak{A}$ and $\phi:\mathfrak{A}_a\to\mathfrak{A}_b$ belongs to G^*_τ, then $\phi\restriction\mathfrak{A}_c\in G^*_\tau$ for any $c\subseteq a$.

(e) Suppose that a, $b\in\mathfrak{A}$ and that $\psi:\mathfrak{A}_a\to\mathfrak{A}_b$ is an isomorphism such that there are a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle\phi_i\rangle_{i\in I}$ in G^*_τ such that $\psi c=\phi_i c$ whenever $i\in I$ and $c\subseteq a_i$. Then $\psi\in G^*_\tau$.

proof (a) (Compare 381I.)

(i)⇒(iii) is trivial, since of course $G\subseteq G^*_\tau$.

(iii)⇒(ii) Suppose that ϕ satisfies (iii), and that $0\ne c_0\subseteq a$. Then we can find a $\psi\in G^*_\tau$ and a non-zero $c_1\subseteq c_0$ such that ϕ agrees with ψ on $\mathfrak{A}_{c_1}$. Suppose that $\operatorname{dom}\psi=\mathfrak{A}_d$, where necessarily $d\supseteq c_1$. Then there are a partition of unity $\langle d_i\rangle_{i\in I}$ in $\mathfrak{A}_d$ and a family $\langle\pi_i\rangle_{i\in I}$ such that $\psi c=\pi_i c$ whenever $c\subseteq d_i$. There is some $i\in I$ such that $c_2=c_1\cap d_i\ne 0$, and we see that $\phi c=\psi c=\pi_i c$ for every $c\subseteq c_2$. As c_0 is arbitrary, ϕ satisfies (ii).

(ii)⇒(i) If ϕ satisfies (ii), set

$$D=\{d:d\subseteq a,\text{ there is some }\pi\in G\text{ such that }\pi c=\phi c\text{ for every }c\subseteq d\}.$$

The hypothesis is that D is order-dense in $\mathfrak{A}$, so there is a partition of unity $\langle a_i\rangle_{i\in I}$ of $\mathfrak{A}_a$ lying within D (313K); for each $i\in I$ take $\pi_i\in G$ such that $\phi c=\pi_i c$ for $c\subseteq a_i$; then $\langle a_i\rangle_{i\in I}$ and $\langle\pi_i\rangle_{i\in I}$ witness that $\phi\in G^*_\tau$.

(b) This is elementary; if $\langle a_i\rangle_{i\in I}$, $\langle\pi_i\rangle_{i\in I}$ witness that $\phi\in G^*_\tau$, then $\langle\phi a_i\rangle_{i\in I}=\langle\pi_i a_i\rangle_{i\in I}$, $\langle\pi_i^{-1}\rangle_{i\in I}$ witness that $\phi^{-1}\in G^*_\tau$.

(c) I ought to start by computing the domain of $\psi\phi$:

$$\begin{aligned} d \in \operatorname{dom}(\psi\phi) &\iff d \in \operatorname{dom}\phi,\ \phi d \in \operatorname{dom}\psi \\ &\iff d \subseteq a,\ \phi d \subseteq b \iff d \subseteq \phi^{-1}(a' \cap b) = c. \end{aligned}$$

So the domain of $\psi\phi$ is indeed $\mathfrak{A}_c$; now $\phi\restriction\mathfrak{A}_c$ is an isomorphism between $\mathfrak{A}_c$ and $\mathfrak{A}_{\phi c}$, where $\phi c = a' \cap b \in \mathfrak{A}_b$, so $\psi\phi$ is an isomorphism between $\mathfrak{A}_c$ and $\mathfrak{A}_{\psi\phi c} = \mathfrak{A}_{c'}$. Let $\langle a_i\rangle_{i\in I}$, $\langle b_j\rangle_{j\in J}$ be partitions of unity in $\mathfrak{A}_a$, $\mathfrak{A}_b$ respectively, and $\langle \pi_i\rangle_{i\in I}$, $\langle \theta_j\rangle_{j\in J}$ families in G such that $\phi d = \pi_i d$ for $d \subseteq a_i$, $\psi e = \theta_j e$ for $e \subseteq b_j$. Set $c_{ij} = a_i \cap \pi_i^{-1} b_j$; then $\langle c_{ij}\rangle_{i\in I, j\in J}$ is a partition of unity in $\mathfrak{A}_c$ and $\psi\phi d = \theta_j\pi_i d$ for $d \subseteq c_{ij}$, so $\psi\phi \in G^*_\tau$ (because all the $\theta_j\pi_i$ belong to G).

(d) This is nearly trivial; use the definition of G^*_τ or the criteria of (a), or apply (c) with the identity map on $\mathfrak{A}_c$ as one of the factors.

(e) This follows at once from the criterion (a-iii) above, or otherwise.

395C Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_τ for the full local semigroup generated by G.

(a) For a, $b \in \mathfrak{A}$, $a \preccurlyeq^\tau_G b$ iff there is a $\phi \in G^*_\tau$ such that $a \in \operatorname{dom}\phi$ and $\phi a \subseteq b$.

(b)(i) $\preccurlyeq^\tau_G$ is transitive and reflexive;

(ii) if $a \preccurlyeq^\tau_G b$ and $b \preccurlyeq^\tau_G a$ then a and b are G-τ-equidecomposable.

(c) G-τ-equidecomposability is an equivalence relation on $\mathfrak{A}$.

(d) If $\langle a_i\rangle_{i\in I}$ and $\langle b_i\rangle_{i\in I}$ are families in $\mathfrak{A}$, of which $\langle b_i\rangle_{i\in I}$ is disjoint, and $a_i \preccurlyeq^\tau_G b_i$ for every $i \in I$, then $\sup_{i\in I} a_i \preccurlyeq^\tau_G \sup_{i\in I} b_i$.

proof (a) This is immediate from the definition of 'G-τ-equidecomposable' and 395Bd.

(b)(i) $a \preccurlyeq^\tau_G a$ because the identity homomorphism belongs to G^*_τ. If $a \preccurlyeq^\tau_G b \preccurlyeq^\tau_G c$ there are ϕ, $\psi \in G^*_\tau$ such that $\phi a \subseteq b$, $\psi b \subseteq c$ so that $\psi\phi a \subseteq c$; as $\psi\phi \in G^*_\tau$ (395Bc), $a \preccurlyeq^\tau_G c$.

(ii) (This is of course a Schröder-Bernstein theorem, and the proof is the usual one.) Take ϕ, $\psi \in G^*_\tau$ such that $\phi a \subseteq b$, $\psi b \subseteq a$. Set $a_0 = a$, $b_0 = b$, $a_{n+1} = \psi b_n$ and $b_{n+1} = \phi a_n$ for each n. Then $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ are non-increasing sequences; set $a_\infty = \inf_{n\in\mathbb{N}} a_n$, $b_\infty = \inf_{n\in\mathbb{N}} b_n$. For each n,

$$\phi\restriction\mathfrak{A}_{a_{2n}\setminus a_{2n+1}} : \mathfrak{A}_{a_{2n}\setminus a_{2n+1}} \to \mathfrak{A}_{b_{2n+1}\setminus b_{2n+2}},$$

$$\psi\restriction\mathfrak{A}_{b_{2n}\setminus b_{2n+1}} : \mathfrak{A}_{b_{2n}\setminus b_{2n+1}} \to \mathfrak{A}_{a_{2n+1}\setminus a_{2n+2}}$$

are isomorphisms, while

$$\phi\restriction\mathfrak{A}_{a_\infty} : \mathfrak{A}_{a_\infty} \to \mathfrak{A}_{b_\infty}$$

is another. So we can define an isomorphism $\theta : \mathfrak{A}_a \to \mathfrak{A}_b$ by setting

$$\begin{aligned} \theta c &= \phi c \text{ if } c \subseteq a_\infty \cup \sup_{n\in\mathbb{N}} a_{2n} \setminus a_{2n+1}, \\ &= \psi^{-1} c \text{ if } c \subseteq \sup_{n\in\mathbb{N}} a_{2n+1} \setminus a_{2n+2}. \end{aligned}$$

By 395Be, $\theta \in G^*_\tau$, so a and b are G-τ-equidecomposable.

(c) This is easy to prove directly from the results in 395B, but also follows at once from (b); any transitive reflexive relation gives rise to an equivalence relation.

(d) We may suppose that I is well-ordered by a relation $\le$. For $i \in I$, set $a'_i = a_i \setminus \sup_{j<i} a_j$. Set $a = \sup_{i\in I} a_i = \sup_{i\in I} a'_i$, $b = \sup_{i\in I} b_i$. For each $i \in I$, we have a $b'_i \subseteq b_i$ and a $\phi_i \in G^*_\tau$ such that $\phi_i a'_i = b'_i$. Set $b' = \sup_{i\in I} b'_i \subseteq b$; then we have an isomorphism $\psi : \mathfrak{A}_a \to \mathfrak{A}_{b'}$ defined by setting $\psi d = \phi_i d$ if $d \subseteq a'_i$, and $\psi \in G^*_\tau$, so a and b' are G-τ-equidecomposable and $a \preccurlyeq^\tau_G b$.

395D Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is an $a \ne 1$ such that a is G-τ-equidecomposable with 1;

(ii) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-τ-equidecomposable;

(iii) there are non-zero G-τ-equidecomposable a, b, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) there are G-τ-equidecomposable a, $b \in \mathfrak{A}$ such that $a \subset b$.

proof Write G^*_τ for the full local semigroup generated by G.

(i)⇒(ii) Assume (i). There is a $\phi \in G^*_\tau$ such that $\phi 1 = a$. Set $a_n = \phi^n(1 \setminus a)$ for each $n \in \mathbb{N}$; because every ϕ^n belongs to G^*_τ (counting ϕ^0 as the identity operator on $\mathfrak{A}$, and using 395Bc), with $\operatorname{dom}\phi^n = \mathfrak{A}$, a_n is G-τ-equidecomposable with $a_0 = 1 \setminus a$ for every n. Also $a_n = \phi^n 1 \setminus \phi^{n+1} 1$ for each n, while $\langle \phi^n 1 \rangle_{n\in\mathbb{N}}$ is non-increasing, so $\langle a_n \rangle_{n\in\mathbb{N}}$ is disjoint. Thus (ii) is true.

(ii)⇒(iii) Assume (ii). Set $a = \sup_{n\in\mathbb{N}} a_{2n}$, $b = \sup_{n\in\mathbb{N}} a_{2n+1}$, $c = \sup_{n\in\mathbb{N}} a_n$, so that $a \cap b = 0$ and $a \cup b = c$. For each n we have a $\phi_n \in G^*_\tau$ such that $\phi_n a_0 = a_n$. So if we set

$$\psi d = \sup_{n\in\mathbb{N}} \phi_n \phi_{2n}^{-1}(d \cap a_{2n}) \text{ for } d \subseteq a,$$

ψ belongs to G^*_τ (using 395B) and witnesses that a and c are G-τ-equidecomposable. Similarly, b and c are G-τ-equidecomposable, so (iii) is true.

(iii)⇒(iv) is trivial.

(iv)⇒(i) Take $\phi \in G^*_\tau$ such that $\phi b = a$. Set

$$\psi d = \phi(d \cap b) \cup (d \setminus b)$$

for $d \in \mathfrak{A}$; then $\psi \in G^*_\tau$ witnesses that 1 is G-τ-equidecomposable with $a \cup (1 \setminus b) \neq 1$.

395E Definition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. I will say that G is **fully non-paradoxical** if the statements of 395D are false; that is, if one of the following equiveridical statements is true:

(i) if a is G-τ-equidecomposable with 1 then $a = 1$;

(ii) there is no disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-τ-equidecomposable;

(iii) there are no non-zero G-τ-equidecomposable a, b, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) if $a \subseteq b \in \mathfrak{A}$ and a, b are G-τ-equidecomposable then $a = b$.

Note that if G is fully non-paradoxical, and H is a subgroup of G, then H also is fully non-paradoxical, because if $a \preccurlyeq^\tau_H b$ then $a \preccurlyeq^\tau_G b$, so that a and b are G-τ-equidecomposable whenever they are H-τ-equidecomposable.

395F Proposition Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, and $G = \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ the group of all measure-preserving automorphisms of $\mathfrak{A}$. Then G is fully non-paradoxical.

proof If $\phi : \mathfrak{A} \to \mathfrak{A}_a$ belongs to the full local semigroup generated by G, then we have a partition of unity $\langle a_i \rangle_{i\in I}$ and a family $\langle \pi_i \rangle_{i\in I}$ in G such that $\phi a_i = \pi_i a_i$ for every i; but this means that

$$\bar\mu a = \sum_{i\in I} \bar\mu \phi_i a_i = \sum_{i\in I} \bar\mu \pi_i a_i = \sum_{i\in I} \bar\mu a_i = \bar\mu 1.$$

As $\bar\mu 1 < \infty$, we can conclude that $a = 1$, so that G satisfies the condition (i) of 395E.

395G The fixed-point subalgebra of a group Let $\mathfrak{A}$ be a Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$.

(a) By the **fixed-point subalgebra** of G I mean

$$\mathfrak{C} = \{c : c \in \mathfrak{A},\ \pi c = c \text{ for every } \pi \in G\}.$$

(I looked briefly at this construction in 333R, and in the special case of a group generated by a single element it appeared at various points in Chapter 38.) This is a subalgebra of $\mathfrak{A}$, and is order-closed, because every $\pi \in G$ is order-continuous.

(b) Now suppose that $\mathfrak{A}$ is Dedekind complete. In this case $\mathfrak{C}$ is Dedekind complete (314Ea), and we have, for any $a \in \mathfrak{A}$, an upper envelope $\operatorname{upr}(a, \mathfrak{C})$ of $\mathfrak{C}$, defined by setting

$$\operatorname{upr}(a, \mathfrak{C}) = \inf\{c : a \subseteq c \in \mathfrak{C}\}$$

(313S). Now $\operatorname{upr}(a, \mathfrak{C}) = \sup\{\pi a : \pi \in G\}$. **P** Set $c_1 = \operatorname{upr}(a, \mathfrak{C})$, $c_2 = \sup\{\pi a : \pi \in G\}$. (i) Because $a \subseteq c_1 \in \mathfrak{C}$, $\pi a \subseteq \pi c_1 = c_1$ for every $\pi \in G$, and $c_2 \subseteq c_1$. (ii) For any $\phi \in G$,

$$\phi c_2 = \sup_{\pi\in G} \phi\pi a = \sup_{\pi\in G} \pi a = c_2$$

because $G = \{\phi\pi : \pi \in G\}$. So $c_2 \in \mathfrak{C}$; since also $a \subseteq c_2$, $c_1 \subseteq c_2$, and $c_1 = c_2$, as claimed. **Q**

(c) Again supposing that $\mathfrak{A}$ is Dedekind complete, write G^*_τ for the full local semigroup generated by G. Then $\phi(a \cap c) = \phi a \cap c$ whenever $\phi \in G^*_\tau$, $a \in \operatorname{dom}\phi$ and $c \in \mathfrak{C}$. **P** We have $\phi a = \sup_{i\in I} \pi_i a_i$, where $a = \sup_{i\in I} a_i$ and $\pi_i \in G$ for every i. Now

$$\phi(a \cap c) = \sup_{i\in I} \pi_i(a_i \cap c) = \sup_{i\in I} \pi_i a_i \cap c = \phi a \cap c. \textbf{ Q}$$

Consequently $\operatorname{upr}(\phi a, \mathfrak{C}) = \operatorname{upr}(a, \mathfrak{C})$ whenever $\phi \in G^*_\tau$ and $a \in \operatorname{dom}\phi$. **P** For $c \in \mathfrak{C}$,

$$a \subseteq c \iff a \cap c = a \iff \phi(a \cap c) = \phi a \iff \phi a \cap c = \phi a \iff \phi a \subseteq c. \textbf{ Q}$$

It follows that $\operatorname{upr}(a, \mathfrak{C}) \subseteq \operatorname{upr}(b, \mathfrak{C})$ whenever $a \preccurlyeq^\tau_G b$.

(d) Still supposing that $\mathfrak{A}$ is Dedekind complete, we also find that if $a \preccurlyeq^\tau_G b$ and $c \in \mathfrak{C}$ then $a \cap c \preccurlyeq^\tau_G b \cap c$. **P** There is a $\phi \in G^*_\tau$ such that $\phi a \subseteq b$; now $\phi(a \cap c) = \phi a \cap c \subseteq b \cap c$. **Q** Hence, or otherwise, $a \cap c$ and $b \cap c$ are G-τ-equidecomposable whenever a and b are G-τ-equidecomposable and $c \in \mathfrak{C}$.

(e) By analogy with the notion of 'ergodic automorphism', I will say that G is **ergodic** if $\sup_{\pi\in G} \pi a = 1$ for every non-zero $a \in \mathfrak{A}$. Thus an automorphism π is ergodic in the sense of 372Oa iff the group $\{\pi^n : n \in \mathbb{Z}\}$ it generates is ergodic (372Pb).

(f) If G is ergodic, then $\mathfrak{C} = \{0, 1\}$. (If $c \in \mathfrak{C} \setminus \{0\}$, then $1 = \sup_{\pi\in G} \pi c = c$.) If $\mathfrak{A}$ is Dedekind complete and $\mathfrak{C} = \{0, 1\}$ then G is ergodic. (If $a \in \mathfrak{A} \setminus \{0\}$, then $1 = \operatorname{upr}(a, \mathfrak{C}) = \sup_{\pi\in G} \pi a$, by (b) above.) (Cf. 392Sa, 392Sc.)

395H I now embark on a series of lemmas leading to the main theorem (395N).

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Write $\mathfrak{C}$ for the fixed-point subalgebra of G. Take any a, $b \in \mathfrak{A}$. Set $c_0 = \sup\{c : c \in \mathfrak{C},\, a \cap c \preccurlyeq^\tau_G b\}$; then $a \cap c_0 \preccurlyeq^\tau_G b$ and $b \setminus c_0 \preccurlyeq^\tau_G a$.

proof Enumerate G as $\langle \pi_\xi \rangle_{\xi<\kappa}$, where $\kappa = \#(G)$. Define $\langle a_\xi \rangle_{\xi<\kappa}$, $\langle b_\xi \rangle_{\xi<\kappa}$ inductively, setting

$$a_\xi = (a \setminus \sup_{\eta<\xi} a_\eta) \cap \pi_\xi^{-1}(b \setminus \sup_{\eta<\xi} b_\eta), \qquad b_\xi = \pi_\xi a_\xi.$$

Then $\langle a_\xi \rangle_{\xi<\kappa}$ is a disjoint family in $\mathfrak{A}_a$ and $\langle b_\xi \rangle_{\xi<\kappa}$ is a disjoint family in $\mathfrak{A}_b$, and $\sup_{\xi<\kappa} a_\xi$ is G-τ-equidecomposable with $\sup_{\xi<\kappa} b_\xi$. Set $a' = a \setminus \sup_{\xi<\kappa} a_\xi$, $b' = b \setminus \sup_{\xi<\kappa} b_\xi$,

$$\tilde{c}_0 = 1 \setminus \operatorname{upr}(a', \mathfrak{C}) = \sup\{c : c \in \mathfrak{C},\, c \cap a' = 0\}.$$

Then

$$a \cap \tilde{c}_0 \subseteq \sup_{\xi<\kappa} a_\xi \preccurlyeq^\tau_G b,$$

so $\tilde{c}_0 \subseteq c_0$.

Now $b' \subseteq \tilde{c}_0$. **P?** Otherwise, because $\tilde{c}_0 = 1 \setminus \sup_{\xi<\kappa} \pi_\xi a'$ (395Gb), there must be a $\xi < \kappa$ such that $\pi_\xi a' \cap b' \neq 0$. But in this case $d = a' \cap \pi_\xi^{-1} b' \neq 0$, and we have

$$d \subseteq (a \setminus \sup_{\eta<\xi} a_\eta) \cap \pi_\xi^{-1}(b \setminus \sup_{\eta<\xi} b_\eta),$$

so that $d \subseteq a_\xi$, which is absurd. **XQ** Consequently

$$b \setminus \tilde{c}_0 \subseteq \sup_{\xi<\kappa} b_\xi \preccurlyeq^\tau_G a.$$

Now take any $c \in \mathfrak{C}$ such that $a \cap c \preccurlyeq^\tau_G b$, and consider $c' = c \setminus \tilde{c}_0$. Then $b' \cap c' = 0$, that is, $b \cap c' = \sup_{\xi<\kappa} b_\xi \cap c'$, which is G-τ-equidecomposable with $\sup_{\xi<\kappa} a_\xi \cap c' = (a \setminus a') \cap c'$ (395Gd). But now

$$a \cap c' = a \cap c \cap c' \preccurlyeq^\tau_G b \cap c' \preccurlyeq^\tau_G (a \cap c') \setminus (a' \cap c');$$

because G is fully non-paradoxical, $a' \cap c'$ must be 0, that is, $c' \subseteq \tilde{c}_0$ and $c' = 0$. As c' is arbitrary, $c_0 \subseteq \tilde{c}_0$ and $c_0 = \tilde{c}_0$. So c_0 has the required properties.

Remark By analogy with the notation I used in discussing the Hahn decomposition of countably additive functionals (326S-326T), we might denote c_0 as '$[\![a \preccurlyeq^\tau_G b]\!]$', or perhaps '$[\![a \preccurlyeq^\tau_G b]\!]_{\mathfrak{C}}$', 'the region (in $\mathfrak{C}$) where $a \preccurlyeq^\tau_G b$'. The same notation would write $\operatorname{upr}(a, \mathfrak{C})$ as '$[\![a \neq 0]\!]_{\mathfrak{C}}$'.

395I The construction I wish to use depends essentially on L^0 spaces as described in §364. The next step is the following.

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of G. Suppose that a, $b \in \mathfrak{A}$ and that $\operatorname{upr}(a,\mathfrak{C}) = 1$. Then there are u, $v \in L^0 = L^0(\mathfrak{C})$ such that

$$\begin{aligned}[\![u \ge n]\!] = \max\{c : c \in \mathfrak{C},\ &\text{there is a disjoint family } \langle d_i\rangle_{i<n} \\ &\text{such that } c \cap a \preccurlyeq^\tau_G d_i \subseteq b \text{ for every } i < n\}, \\ [\![v \le n]\!] = \max\{c : c \in \mathfrak{C},\ &\text{there is a family } \langle d_i\rangle_{i<n} \\ &\text{such that } d_i \preccurlyeq^\tau_G a \text{ for every } i < n \text{ and } b \cap c \subseteq \sup_{i<n} d_i\}\end{aligned}$$

for every $n \in \mathbb{N}$. Moreover, we can arrange that

(i) $[\![u \in \mathbb{N}]\!] = [\![v \in \mathbb{N}]\!] = 1$,

(ii) $[\![v > 0]\!] = \operatorname{upr}(b,\mathfrak{C})$,

(iii) $u \le v \le u + \chi 1$.

Remark By writing 'max' in the formulae above, I mean to imply that the elements $[\![u \ge n]\!]$, $[\![v \le n]\!]$ belong to the sets described.

proof (a) Choose $\langle c_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ as follows. Given $\langle b_i\rangle_{i<n}$, set $b'_n = b \setminus \sup_{i<n} b_i$,

$$c_n = \sup\{c : c \in \mathfrak{C},\ a \cap c \preccurlyeq^\tau_G b'_n\},$$

so that $a \cap c_n \preccurlyeq^\tau_G b'_n$ (395H); choose $b_n \subseteq b'_n$ such that $a \cap c_n$ is G-τ-equidecomposable with b_n, and continue. Then $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_b$ and $\langle c_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{C}$.

For each n, we have $b'_n \setminus c_n \preccurlyeq^\tau_G a$, by 395H; while $a \cap c \not\preccurlyeq^\tau_G b'_n$ whenever $c \in \mathfrak{C}$ and $c \not\subseteq c_n$. Note also that, because $\operatorname{upr}(a,\mathfrak{C}) = 1$,

$$c_n = \operatorname{upr}(a \cap c_n,\mathfrak{C}) = \operatorname{upr}(b_n,\mathfrak{C}) \subseteq \operatorname{upr}(b'_n,\mathfrak{C})$$

(using 395Gc for the second equality).

(b) Now $\inf_{n\in\mathbb{N}} c_n = 0$. **P** Setting $c_\infty = \inf_{n\in\mathbb{N}} c_n$, $\langle b_n \cap c_\infty\rangle_{n\in\mathbb{N}}$ is a disjoint sequence, all G-τ-equidecomposable with $a \cap c_\infty$, so $a \cap c_\infty = 0$, because G is fully non-paradoxical; because $\operatorname{upr}(a,\mathfrak{C}) = 1$, it follows that $c_\infty = 0$. **Q** Accordingly, if we set $u = \sup_{n\in\mathbb{N}}(n+1)\chi c_n$, $u \in L^0$ and $[\![u \ge n]\!] = c_{n-1}$ for $n \ge 1$. The construction ensures that $[\![u \in \mathbb{N}]\!]$, as defined in 364G, is equal to 1.

(c) Consider next $c'_0 = \operatorname{upr}(b,\mathfrak{C})$, $c'_n = c_{n-1} \cap \operatorname{upr}(b'_n,\mathfrak{C})$ for $n \ge 1$. Then $\langle c'_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with zero infimum, so again we can define $v \in L^0$ by setting $v = \sup_{n\in\mathbb{N}}(n+1)\chi c'_n$. Once again, $[\![v \in \mathbb{N}]\!] = 1$, and $[\![v \le n]\!] = 1 \setminus c'_n$ for each n.

Of course $[\![v > 0]\!] = c'_0 = \operatorname{upr}(b,\mathfrak{C})$. Because $c_n \subseteq c'_n \subseteq c_{n-1}$,

$$(n+1)\chi c_n \le (n+1)\chi c'_n \le n\chi c_{n-1} + \chi 1$$

for each $n \ge 1$, and $u \le v \le u + \chi 1$.

(d) Now set

$$\begin{aligned}C_n = \{c : c \in \mathfrak{C},\ &\text{there is a disjoint family } \langle d_i\rangle_{i<n} \\ &\text{such that } c \cap a \preccurlyeq^\tau_G d_i \subseteq b \text{ for every } i < n\}.\end{aligned}$$

Then $c_n = \max C_{n+1}$.

P$(\boldsymbol{\alpha})$ Because $c_n \subseteq c_{n-1} \subseteq \ldots \subseteq c_0$, $a \cap c_n \preccurlyeq^\tau_G b_i$ for every $i \le n$, so that $\langle b_i\rangle_{i\le n}$ witnesses that $c_n \in C_{n+1}$.

$(\boldsymbol{\beta})$ Suppose that $c \in C_{n+1}$; let $\langle d_i\rangle_{i\le n}$ be a disjoint family such that $c \cap a \preccurlyeq^\tau_G d_i \subseteq b$ for every i. Set $c' = c \setminus c_n$. For each $i < n$, $b_i \preccurlyeq^\tau_G a$, so

$$b_i \cap c' \preccurlyeq^\tau_G a \cap c' \preccurlyeq^\tau_G d_i \cap c',$$

while also

$$b'_n \cap c' \preccurlyeq^\tau_G a \cap c' \preccurlyeq^\tau_G d_n \cap c'.$$

Take $d \subseteq d_n \cap c'$ such that $b'_n \cap c'$ is G-τ-equidecomposable with d. Then

$$b \cap c' = (b'_n \cap c') \cup \sup_{i<n}(b_i \cap c') \preccurlyeq^\tau_G d \cup \sup_{i<n}(d_i \cap c') \subseteq b \cap c'.$$

Because G is fully non-paradoxical, $d \cup \sup_{i<n}(d_i \cap c')$ must be exactly $b \cap c'$, so d must be the whole of $d_n \cap c'$, and

$$a \cap c' \preccurlyeq^\tau_G d_n \cap c' = d \preccurlyeq^\tau_G b'_n.$$

But this means that $c' \subseteq c_n$. Thus $c' = 0$ and $c \subseteq c_n$. So $c_n = \sup C_{n+1} = \max C_{n+1}$. **Q**

Accordingly

$$[\![u \geq n]\!] = c_{n-1} = \max C_n$$

for $n \geq 1$. For $n = 0$ we have $[\![u \geq 0]\!] = 1 = \max C_0$. So $[\![u \geq n]\!] = \max C_n$ for every n, as required.

(e) Similarly, if we set

$$C'_n = \{c : c \in \mathfrak{C}, \text{ there is a family } \langle d_i \rangle_{i<n} \text{ such that } d_i \preccurlyeq^\tau_G a \text{ for every } i < n \text{ and } b \cap c \subseteq \sup_{i<n} d_i\}$$

then $1 \setminus c'_n = \max C'_n$ for every n.

P (α) If $n = 0$, then of course (interpreting $\sup \emptyset$ as 0) $1 \setminus c'_0 \in C'_0$ because $b \subseteq c'_0$. For each $n \in \mathbb{N}$, set

$$\tilde{b}_n = b_n \cup (b'_n \setminus c_n) = (b_n \cap c_n) \cup (b'_n \setminus c_n).$$

Because $b_n \preccurlyeq^\tau_G a$ and $b'_n \setminus c_n \preccurlyeq^\tau_G a$, we have $b_n \cap c_n \preccurlyeq^\tau_G a \cap c_n$ and $b'_n \setminus c_n \preccurlyeq^\tau_G a \setminus c_n$, so $\tilde{b}_n \preccurlyeq^\tau_G a$ (395Cd). If we look at

$$\sup_{i<n} \tilde{b}_i \supseteq \sup_{i<n} b_i \cup (b'_{n-1} \setminus c_{n-1}),$$

we see that, for $n \geq 1$,

$$b \setminus \sup_{i<n} \tilde{b}_i \subseteq b'_n \cap c_{n-1} \subseteq c'_n,$$

so that $b \setminus c'_n \subseteq \sup_{i<n} \tilde{b}_i$ and $\{\tilde{b}_i : i < n\}$ witnesses that $1 \setminus c'_n \in C'_n$.

(β) Now take any $c \in C'_n$ and a corresponding family $\langle d_i \rangle_{i<n}$ such that $d_i \preccurlyeq^\tau_G a$ for every $i < n$ and $b \cap c \subseteq \sup_{i<n} d_i$.

Set $c' = c \cap c'_n$. For each $i < n$,

$$c' \cap d_i \preccurlyeq^\tau_G c' \cap a \preccurlyeq^\tau_G b_i$$

because $c' \subseteq c_i$. So (by 395Cd, as usual)

$$c' \cap b \preccurlyeq^\tau_G c' \cap \sup_{i<n} b_i \subseteq c' \cap b,$$

and (again because G is fully non-paradoxical) $c' \cap b = c' \cap \sup_{i<n} b_i$, that is, $c' \cap b'_n = 0$. But $c' \subseteq c'_n \subseteq \operatorname{upr}(b'_n, \mathfrak{C})$, so c' must be 0, which means that $c \subseteq 1 \setminus c'_n$. As c is arbitrary, $1 \setminus c'_n = \sup C'_n = \max C'_n$. **Q**

Thus $[\![v \leq n]\!] = \max C'_n$, as declared.

395J Notation Observe that the specification of $[\![u \geq n]\!]$ and $[\![v \leq n]\!]$, together with the declaration that $[\![u \in \mathbb{N}]\!] = [\![v \in \mathbb{N}]\!] = 1$, determine u and v uniquely, because $\langle [\![u = n]\!] \rangle_{n\in\mathbb{N}}$ and $\langle [\![v = n]\!] \rangle_{n\in\mathbb{N}}$ must be partitions of unity. So, in the context of 395I, we can write $\lfloor b : a \rfloor$ for u and $\lceil b : a \rceil$ for v.

395K Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that a, b, b_1, $b_2 \in \mathfrak{A}$ and that $\operatorname{upr}(a, \mathfrak{C}) = 1$.

(a) $\lfloor 0 : a \rfloor = \lceil 0 : a \rceil = 0$, $\lfloor 1 : a \rfloor \geq \chi 1$ and $\lfloor 1 : 1 \rfloor = \chi 1$.

(b) If $b_1 \preccurlyeq^\tau_G b_2$ then $\lfloor b_1 : a \rfloor \leq \lfloor b_2 : a \rfloor$ and $\lceil b_1 : a \rceil \leq \lceil b_2 : a \rceil$.

(c) $\lceil b_1 \cup b_2 : a \rceil \leq \lceil b_1 : a \rceil + \lceil b_2 : a \rceil$.

(d) If $b_1 \cap b_2 = 0$, $\lfloor b_1 : a \rfloor + \lfloor b_2 : a \rfloor \leq \lfloor b_1 \cup b_2 : a \rfloor$.

(e) If $c \in \mathfrak{C}$ is such that $a \cap c$ is a relative atom over $\mathfrak{C}$ (definition: 331A), then $c \subseteq [\![\lceil b : a \rceil - \lfloor b : a \rfloor = 0]\!]$.

proof (a)-(b) are immediate from the definitions and the basic properties of $\preccurlyeq^\tau_G$, $\lceil \dots \rceil$ and $\lfloor \dots \rfloor$, as listed in 395C and 395I.

(c) For j, $k \in \mathbb{N}$, set $c_{jk} = [\![\lceil b_1 : a \rceil = j]\!] \cap [\![\lceil b_2 : a \rceil = k]\!]$. Then

$$c_{jk} \subseteq [\![\lceil b_1 \cup b_2 : a \rceil \leq j + k]\!] \cap [\![\lceil b_1 : a \rceil + \lceil b_2 : a \rceil = j + k]\!].$$

P We may suppose that $c_{jk} \neq 0$. Of course

$$c_{jk} \subseteq [\![\lceil b_1 : a\rceil + \lceil b_2 : a\rceil = j+k]\!].$$

Next, there are sets J, $J' \subseteq \mathfrak{A}$ such that $d \preccurlyeq^\tau_G a$ for every $d \in J \cup J'$, $\#(J) \le j$, $\#(J') \le k$, $\sup J \supseteq b_1 \cap c_{jk}$ and $\sup J' \supseteq b_2 \cap c_{jk}$. So $\sup(J \cup J') \supseteq (b_1 \cup b_2) \cap c_{jk}$ and $J \cup J'$ witnesses that $c_{jk} \subseteq [\![\lceil b_1 \cup b_2 : a\rceil \le j+k]\!]$. **Q**

Accordingly

$$c_{jk} \subseteq [\![\lceil b_1 : a\rceil + \lceil b_2 : a\rceil - \lceil b_1 \cup b_2 : a\rceil \ge 0]\!].$$

Now as $\sup_{j,k\in\mathbb{N}} c_{jk} = 1$, we must have $\lceil b_1 \cup b_2 : a\rceil \le \lceil b_1 : a\rceil + \lceil b_2 : a\rceil$.

(d) This time, set $c_{jk} = [\![\lfloor b_1 : a\rfloor = j]\!] \cap [\![\lfloor b_2 : a\rfloor = k]\!]$ for j, $k \in \mathbb{N}$. Then

$$c_{jk} \subseteq [\![\lfloor b_1 \cup b_2 : a\rfloor \ge j+k]\!] \cap [\![\lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor = j+k]\!]$$

for every j, $k \in \mathbb{N}$. **P** Once again, we surely have

$$c_{jk} \subseteq [\![\lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor = j+k]\!].$$

Next, we can find a family $\langle d_i\rangle_{i<j+k}$ such that

$$\langle d_i\rangle_{i<j} \text{ is disjoint, } a \cap c_{jk} \preccurlyeq^\tau_G d_i \subseteq b_1 \text{ for every } i<k,$$

$$\langle d_i\rangle_{j\le i<j+k} \text{ is disjoint, } a \cap c_{jk} \preccurlyeq^\tau_G d_i \subseteq b_2 \text{ for } j \le i < j+k.$$

As $b_1 \cap b_2 = 0$, the whole family $\langle d_i\rangle_{i<j+k}$ is disjoint and witnesses that $c_{jk} \subseteq [\![\lfloor b_1 \cup b_2 : a\rfloor \ge j+k]\!]$. **Q**

So

$$c_{jk} \subseteq [\![\lfloor b_1 \cup b_2 : a\rfloor - \lfloor b_1 : a\rfloor - \lfloor b_2 : a\rfloor \ge 0]\!]$$

Since $\sup_{j,k\in\mathbb{N}} c_{jk} = 1$, as before, we must have $\lfloor b_1 \cup b_2 : a\rfloor \ge \lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor$.

(e) ? Otherwise, there must be some $k \in \mathbb{N}$ such that

$$c_0 = c \cap [\![\lfloor b : a\rfloor = k]\!] \cap [\![\lceil b : a\rceil > k]\!] \ne 0.$$

Let $\langle d_i\rangle_{i<k}$ be a disjoint family in $\mathfrak{A}_b$ such that $a \cap c_0 \preccurlyeq^\tau_G d_i$ for each i; cutting the d_i down if necessary, we may suppose that $a \cap c_0$ is G-τ-equidecomposable with d_i for each i. As $c_0 \not\subseteq [\![\lceil b : a\rceil \le k]\!]$, $b \cap c_0 \not\subseteq \sup_{i<k} d_i$; set $d = b \cap c_0 \setminus \sup_{i<k} d_i \ne 0$. By 395H, there is a $c_1 \in \mathfrak{C}$ such that $d \cap c_1 \preccurlyeq^\tau_G a$ and $a \setminus c_1 \preccurlyeq^\tau_G d$. Setting $d_k = d$, $\langle d_i\rangle_{i\le k}$ witnesses that $c_0 \subseteq c_1 \subseteq [\![\lfloor b : a\rfloor \ge k+1]\!]$, so $c_0 \subseteq c_1$ must be 0 and $d \cap c_0 \preccurlyeq^\tau_G a$. There is therefore a non-zero $\tilde a \subseteq a \cap c_0$ such that $\tilde a \preccurlyeq^\tau_G d$. But now remember that $a \cap c$ is supposed to be a relative atom over $\mathfrak{C}$, so $\tilde a = a \cap \tilde c$ for some $\tilde c \in \mathfrak{C}$ such that $\tilde c \subseteq c_0$. In this case, $a \cap \tilde c \preccurlyeq^\tau_G d_i$ for every $i<k$ and also $a \cap \tilde c \preccurlyeq^\tau_G d$, so $0 \ne \tilde c \subseteq [\![\lfloor b : a\rfloor \ge k+1]\!]$, which is absurd. **X**

395L Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that a_1, a_2, $b \in \mathfrak{A}$ and that $\operatorname{upr}(a_1,\mathfrak{C}) = \operatorname{upr}(a_2,\mathfrak{C}) = 1$. Then

$$\lfloor b : a_2\rfloor \ge \lfloor b : a_1\rfloor \times \lfloor a_1 : a_2\rfloor, \quad \lceil b : a_2\rceil \le \lceil b : a_1\rceil \times \lceil a_1 : a_2\rceil.$$

proof I use the same method as in 395K. As usual, write G^*_τ for the full local semigroup generated by G.

(a) For j, $k \in \mathbb{N}$ set

$$c_{j,k} = [\![\lfloor b : a_1\rfloor = j]\!] \cap [\![\lfloor a_1 : a_2\rfloor = k]\!].$$

Then

$$c_{j,k} \subseteq [\![\lfloor b : a_1\rfloor \times \lfloor a_1 : a_2\rfloor = jk]\!] \cap [\![\lfloor b : a_2\rfloor \ge jk]\!].$$

P Write c for $c_{j,k}$. As in parts (c) and (d) of the proof of 395K, the fact that $c \subseteq [\![\lfloor b : a_1\rfloor \times \lfloor a_1 : a_2\rfloor = jk]\!]$ is elementary; what we need to check is that $c \subseteq [\![\lfloor b : a_2\rfloor \ge jk]\!]$. Again, we may suppose that $c \ne 0$. There are families $\langle d_i\rangle_{i<j}$, $\langle d^*_l\rangle_{l<k}$ such that

$$\langle d_i\rangle_{i<j} \text{ is disjoint, } a_1 \cap c \preccurlyeq^\tau_G d_i \subseteq b \text{ for every } i<j,$$

$$\langle d^*_l\rangle_{l<k} \text{ is disjoint, } a_2 \cap c \preccurlyeq^\tau_G d^*_l \subseteq a_1 \text{ for every } l<k.$$

For each $i<j$, let $\phi_i \in G^*_\tau$ be such that $\phi_i(a_1 \cap c) \subseteq d_i$. If $i<j$ and $l<k$, then

$$a_2 \cap c \preccurlyeq^\tau_G d^*_l \cap c \preccurlyeq^\tau_G \phi_i(d^*_l \cap c) \subseteq \phi_i(a_1 \cap c) \subseteq d_i \subseteq b.$$

Also $\langle\phi_i(d_l^*\cap c)\rangle_{i<j,l<k}$ is disjoint because $\langle\phi_i(a_1\cap c)\rangle_{i<j}$ and $\langle d_l^*\rangle_{l<k}$ are, so witnesses that $c\subseteq[\![\lfloor b:a_2\rfloor\ge jk]\!]$. **Q**

Now, just as in 395K, it follows from the fact that $\sup_{j,k\in\mathbb{N}}c_{j,k}=1$ that $\lfloor b:a_1\rfloor\times\lfloor a_1:a_2\rfloor\le\lfloor b:a_2\rfloor$.

(b) For j, $k\in\mathbb{N}$ set

$$c_{j,k}=[\![\lceil b:a_1\rceil=j]\!]\cap[\![\lceil a_1:a_2\rceil=k]\!].$$

Then

$$c_{j,k}\subseteq[\![\lceil b:a_1\rceil\times\lceil a_1:a_2\rceil=jk]\!]\cap[\![\lceil b:a_2\rceil\le jk]\!].$$

P Write c for $c_{j,k}$. Then $c\subseteq[\![\lceil b:a_1\rceil\times\lceil a_1:a_2\rceil=jk]\!]$. There are families $\langle d_i\rangle_{i<j}$, $\langle d_l^*\rangle_{l<k}$ such that $d_i\preccurlyeq_G^\tau a_1$ for every $i<j$, $d_l^*\preccurlyeq_G^\tau a_2$ for every $l<k$, $b\cap c\subseteq\sup_{i<j}d_i$ and $a_1\cap c\subseteq\sup_{l<k}d_l^*$. For each $i<j$, let $d_i'\subseteq a_1$ be G-τ-equidecomposable with d_i, and take $\phi_i\in G_\tau^*$ such that $\phi_i d_i'=d_i$. Then

$$\phi_i(d_i'\cap d_l^*)\preccurlyeq_G^\tau d_l^*\preccurlyeq_G^\tau a_2 \text{ for every } i<j,\ l<k,$$

$$\begin{aligned}\sup_{i<j,l<k}\phi_i(d_i'\cap d_l^*)&=\sup_{i<j}\phi_i(d_i'\cap\sup_{l<k}d_l^*)\supseteq\sup_{i<j}\phi_i(d_i'\cap c)\\&=\sup_{i<j}d_i\cap c\supseteq b\cap c.\end{aligned}$$

So $\langle\phi_i(d_i'\cap d_l^*)\rangle_{i<j,l<k}$ witnesses that $c\subseteq[\![\lceil b:a_2\rceil\le jk]\!]$. **Q**

Once again, it follows easily that $\lceil b:a_1\rceil\times\lceil a_1:a_2\rceil\ge\lceil b:a_2\rceil$.

395M Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$.

(a) For any $a\in\mathfrak{A}$, there is a $b\subseteq a$ such that $b\preccurlyeq_G^\tau a\setminus b$ and $a\setminus\operatorname{upr}(b,\mathfrak{C})$ is a either 0 or a relative atom over $\mathfrak{C}$.

(b) Now suppose that G is fully non-paradoxical. Then for any $\epsilon>0$ there is an $a\in\mathfrak{A}$ such that $\operatorname{upr}(a,\mathfrak{C})=1$ and $\lceil b:a\rceil\le\lfloor b:a\rfloor+\epsilon\lfloor 1:a\rfloor$ for every $b\in\mathfrak{A}$.

proof (a) Set $B=\{d:d\subseteq a,\ d\preccurlyeq_G^\tau a\setminus d\}$ and let $D\subseteq B$ be a maximal subset such that $\operatorname{upr}(d,\mathfrak{C})\cap\operatorname{upr}(d',\mathfrak{C})=0$ for all distinct d, $d'\in\mathfrak{D}$. Set $b=\sup D$. For any $d\in D$, $d\preccurlyeq_G^\tau a\setminus d$, so

$$\begin{aligned}b\cap\operatorname{upr}(d,\mathfrak{C})&=\sup_{d'\in D}d'\cap\operatorname{upr}(d,\mathfrak{C})=\sup_{d'\in D}d'\cap\operatorname{upr}(d',\mathfrak{C})\cap\operatorname{upr}(d,\mathfrak{C})=d\cap\operatorname{upr}(d,\mathfrak{C})\\&\preccurlyeq_G^\tau(a\setminus d)\cap\operatorname{upr}(d,\mathfrak{C})=(a\setminus b)\cap\operatorname{upr}(d,\mathfrak{C})\subseteq a\setminus b\end{aligned}$$

by 395Gc. By 395H,

$$b=\sup\nolimits_{d\in D}b\cap\operatorname{upr}(d,\mathfrak{C})\preccurlyeq_G^\tau a\setminus b.$$

? Suppose, if possible, that $a'=a\setminus\operatorname{upr}(b,\mathfrak{C})$ is neither 0 nor a relative atom over $\mathfrak{C}$. Let $d_0\subseteq a'$ be an element not expressible as $a'\cap c$ for any $c\in\mathfrak{C}$; then $d_0\ne a\cap\operatorname{upr}(d_0,\mathfrak{C})$ and there must be a $\pi\in G$ such that $d_1=\pi d_0\cap a\setminus d_0$ is non-zero (395Gb). In this case

$$d_1\preccurlyeq_G^\tau\pi^{-1}d_1\subseteq d_0\subseteq a\setminus d_1,$$

so $d_1\in B$; but also

$$d_1\cap\operatorname{upr}(d,\mathfrak{C})\subseteq d_1\cap\operatorname{upr}(b,\mathfrak{C})=0,$$

so $\operatorname{upr}(d_1,\mathfrak{C})\cap\operatorname{upr}(d,\mathfrak{C})=0$, for every $d\in D$, and we ought to have put d_1 into D. **X**

Thus b has the required properties.

(b)(i) For every $n\in\mathbb{N}$ we can find $a_n\in\mathfrak{A}$ and $c_n\in\mathfrak{C}$ such that $\operatorname{upr}(a_n,\mathfrak{C})=1$, $a_n\setminus c_n$ is either 0 or a relative atom over $\mathfrak{C}$, and $\lfloor 1:a_n\rfloor\ge 2^n\chi c_n$. **P** Induce on n. The induction starts with $a_0=c_0=1$, because $\lfloor 1:1\rfloor=\chi 1$. For the inductive step, having found a_n and c_n, let $d\subseteq a_n\cap c_n$ be such that $d\preccurlyeq_G^\tau a_n\cap c_n\setminus d$ and $a_n\cap c_n\setminus\operatorname{upr}(d,\mathfrak{C})$ is either 0 or a relative atom over $\mathfrak{C}$, as in (a). Set $c_{n+1}=\operatorname{upr}(d,\mathfrak{C})$, $a_{n+1}=(a_n\setminus c_{n+1})\cup d$; then

$$\begin{aligned}\operatorname{upr}(a_{n+1},\mathfrak{C})&=\operatorname{upr}(a_n\setminus c_{n+1},\mathfrak{C})\cup\operatorname{upr}(d,\mathfrak{C})\\&=(\operatorname{upr}(a_n,\mathfrak{C})\setminus c_{n+1})\cup c_{n+1}=(1\setminus c_{n+1})\cup c_{n+1}=1\end{aligned}$$

by 313Sb-313Sc and the inductive hypothesis.

We have $c_{n+1}\cap d\preccurlyeq_G^\tau c_{n+1}\cap a_n\setminus d$, so

$$c_{n+1} \cap a_{n+1} = d \subseteq a_n, \quad c_{n+1} \cap a_{n+1} \preccurlyeq_G^\tau a_n \setminus d,$$

and $\lfloor a_n : a_{n+1} \rfloor \geq 2\chi c_{n+1}$; by 395L,

$$\lfloor 1 : a_{n+1} \rfloor \geq \lfloor 1 : a_n \rfloor \times \lfloor a_n : a_{n+1} \rfloor \geq 2^n \chi c_n \times 2\chi c_{n+1} = 2^{n+1}\chi c_{n+1}.$$

If

$$b \subseteq a_{n+1} \setminus c_{n+1} = (a_n \setminus c_n) \cup (a_n \cap c_n \setminus c_{n+1}),$$

then, because both terms on the right are either 0 or relative atoms over $\mathfrak{C}$, there are c', $c'' \in \mathfrak{C}$ such that

$$\begin{aligned} b &= (b \cap a_n \setminus c_n) \cup (b \cap a_n \cap c_n \setminus c_{n+1}) \\ &= (c' \cap a_n \setminus c_n) \cup (c'' \cap a_n \cap c_n \setminus c_{n+1}) = c \cap a_{n+1} \setminus c_{n+1} \end{aligned}$$

where $c = (c' \setminus c_n) \cup (c'' \cap c_n)$ belongs to $\mathfrak{C}$. So $a_{n+1} \setminus c_{n+1}$ is either 0 or a relative atom over $\mathfrak{C}$.
Thus the induction continues. **Q**

(ii) Now suppose that $\epsilon > 0$. Take n such that $2^{-n} \leq \epsilon$, and consider a_n, c_n taken from (i) above.
Let $b \in \mathfrak{A}$. Set

$$c = [\![\lceil b : a_n \rceil - \lfloor b : a_n \rfloor - \epsilon \lfloor 1 : a_n \rfloor > 0]\!] \in \mathfrak{C}.$$

Since we know that

$$\epsilon\lfloor 1 : a_n \rfloor \geq 2^{-n}2^n\chi c_n = \chi c_n, \quad \lceil b : a_n \rceil \leq \lfloor b : a_n \rfloor + \chi 1,$$

we must have $c \cap c_n = 0$. But this means that $a_n \cap c$ is either 0 or a relative atom over $\mathfrak{C}$. By 395Ke, c is included in $[\![\lceil b : a_n \rceil - \lfloor b : a_n \rfloor = 0]\!]$; as also $\lfloor 1 : a_n \rfloor \geq \chi 1$ (395Ka), c must be zero, that is, $\lceil b : a_n \rceil \leq \lfloor b : a_n \rfloor + \epsilon \lfloor 1 : a_n \rfloor$.

395N We are at last ready for the theorem.

Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Then there is a unique function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{C})$ such that
(i) θ is additive, non-negative and order-continuous;
(ii) $[\![\theta a > 0]\!] = \operatorname{upr}(a, \mathfrak{C})$ for every $a \in \mathfrak{A}$; in particular, $\theta a = 0$ iff $a = 0$;
(iii) $\theta 1 = \chi 1$;
(iv) $\theta(a \cap c) = \theta a \times \chi c$ for every $a \in \mathfrak{A}$, $c \in \mathfrak{C}$; in particular, $\theta c = \chi c$ for every $c \in \mathfrak{C}$;
(v) If a, $b \in \mathfrak{A}$ are G-τ-equidecomposable, then $\theta a = \theta b$; in particular, θ is G-invariant.

proof If $\mathfrak{A} = \{0\}$ this is trivial; so I suppose henceforth that $\mathfrak{A} \neq \{0\}$.

(a) Set $A^* = \{a : a \in \mathfrak{A}, \operatorname{upr}(a, \mathfrak{C}) = 1\}$ and for $a \in A^*$, $b \in \mathfrak{A}$ set

$$\theta_a(b) = \frac{\lceil b : a \rceil}{\lfloor 1 : a \rfloor} \in L^0 = L^0(\mathfrak{C});$$

the first thing to note is that because $\lfloor 1 : a \rfloor \geq \chi 1$, we can always do the divisions to obtain elements $\theta_a(b)$ of $L^0(\mathfrak{A})$ (364N). Set

$$\theta b = \inf_{a \in A^*} \theta_a b$$

for $b \in \mathfrak{A}$. (Note that $L^0(\mathfrak{C})$ is Dedekind complete, by 364M, so the infimum is defined.)

(b) The formulae of 395K tell us that, for $a \in A^*$ and b_1, $b_2 \in \mathfrak{A}$,

$$\theta_a 0 = 0, \quad \theta_a b_1 \leq \theta_a b_2 \text{ if } b_1 \subseteq b_2,$$

$$\theta_a(b_1 \cup b_2) \leq \theta_a b_1 + \theta_a b_2,$$

$$\theta_a 1 \geq \chi 1.$$

It follows at once that

$$\theta 0 = 0, \quad \theta b_1 \leq \theta b_2 \text{ if } b_1 \subseteq b_2,$$

$$\theta 1 \geq \chi 1.$$

(c) For each $n\in\mathbb{N}$ there is an $e_n\in A^*$ such that $\lceil b:e_n\rceil\le\lfloor b:e_n\rfloor+2^{-n}\lfloor 1:e_n\rfloor$ for every $b\in\mathfrak{A}$ (395Mb). Now $\theta_{e_n}b\le\theta_a b+2^{-n}\lceil b:a\rceil$ for every $a\in A^*$, $b\in\mathfrak{A}$. **P** $\lceil a:e_n\rceil\le\lfloor a:e_n\rfloor+2^{-n}\lfloor 1:e_n\rfloor$, so

$$\begin{aligned}\lceil a:e_n\rceil\times\lfloor 1:a\rfloor&\le\lfloor a:e_n\rfloor\times\lfloor 1:a\rfloor+2^{-n}\lfloor 1:e_n\rfloor\times\lfloor 1:a\rfloor\\&\le\lfloor 1:e_n\rfloor+2^{-n}\lfloor 1:e_n\rfloor\times\lfloor 1:a\rfloor\end{aligned}$$

(by 395L); accordingly

$$\lceil b:e_n\rceil\times\lfloor 1:a\rfloor\le\lceil b:a\rceil\times\lceil a:e_n\rceil\times\lfloor 1:a\rfloor$$

(by the other half of 395L)

$$\le\lceil b:a\rceil\times\lfloor 1:e_n\rfloor+2^{-n}\lceil b:a\rceil\times\lfloor 1:e_n\rfloor\times\lfloor 1:a\rfloor$$

and, dividing by $\lfloor 1:a\rfloor\times\lfloor 1:e_n\rfloor$, we get $\theta_{e_n}b\le\theta_a b+2^{-n}\lceil b:a\rceil$. **Q**

(d) Now θ is additive. **P** Taking $\langle e_n\rangle_{n\in\mathbb{N}}$ from (c), observe first that

$$\inf_{n\in\mathbb{N}}\theta_{e_n}b\le\theta_a b+\inf_{n\in\mathbb{N}}2^{-n}\lceil b:a\rceil=\theta_a b$$

for every $a\in A^*$, $b\in\mathfrak{A}$, so that $\theta b=\inf_{n\in\mathbb{N}}\theta_{e_n}b$ for every b. Now suppose that b_1, $b_2\in\mathfrak{A}$ and $b_1\cap b_2=0$. Then, for any $n\in\mathbb{N}$,

$$\begin{aligned}\lceil b_1:e_n\rceil+\lceil b_2:e_n\rceil&\le\lfloor b_1:e_n\rfloor+\lfloor b_2:e_n\rfloor+2^{-n+1}\lfloor 1:e_n\rfloor\\&\le\lfloor b_1\cup b_2:e_n\rfloor+2^{-n+1}\lfloor 1:e_n\rfloor\end{aligned}$$

(by 395Kd)

$$\le\lceil b_1\cup b_2:e_n\rceil+2^{-n+1}\lfloor 1:e_n\rfloor.$$

Dividing by $\lfloor 1:e_n\rfloor$, we have

$$\theta b_1+\theta b_2\le\theta_{e_n}b_1+\theta_{e_n}b_2\le\theta_{e_n}(b_1\cup b_2)+2^{-n+1}\chi 1.$$

Taking the infimum over n, we get

$$\theta b_1+\theta b_2\le\theta(b_1\cup b_2).$$

In the other direction, if a, $a'\in A^*$ and $n\in\mathbb{N}$,

$$\begin{aligned}\theta(b_1\cup b_2)\le\theta_{e_n}(b_1\cup b_2)&\le\theta_{e_n}(b_1)+\theta_{e_n}(b_2)\\&\le\theta_a(b_1)+2^{-n}\lceil b_1:a\rceil+\theta_{a'}(b_2)+2^{-n}\lceil b_2:a'\rceil.\end{aligned}$$

As n is arbitrary, $\theta(b_1\cup b_2)\le\theta_a(b_1)+\theta_{a'}(b_2)$; as a and a' are arbitrary, $\theta(b_1\cup b_2)\le\theta b_1+\theta b_2$ (using 351Dc).

As b_1 and b_2 are arbitrary, θ is additive. **Q**

We see also that $\lceil 1:e_n\rceil\le(1+2^{-n})\lfloor 1:e_n\rfloor$, so that $\theta_{e_n}1\le(1+2^{-n})\chi 1$ for each n; since we already know that $\theta 1\ge\chi 1$, we have $\theta 1=\chi 1$ exactly.

(e) If $c\in\mathfrak{C}$ then

$$[\![\theta c>0]\!]\subseteq[\![\theta_1 c>0]\!]\subseteq[\![\lceil c:1\rceil>0]\!]=\operatorname{upr}(c,\mathfrak{C})=c$$

(395I(ii)). It follows that

$$\theta(b\cap c)\le\theta b\wedge\theta c\le\theta b\times\chi c$$

for any $b\in\mathfrak{A}$, $c\in\mathfrak{C}$. Similarly, $\theta(b\setminus c)\le\theta b\times\chi(1\setminus c)$; adding, we must have equality in both, and $\theta(b\cap c)=\theta b\times\chi c$.

Rather late, I point out that

$$0\le\theta a\le\theta 1=\chi 1\in L^\infty=L^\infty(\mathfrak{C})$$

for every $a\in\mathfrak{A}$, so that $\theta a\in L^\infty$ for every a.

(f) If $b\in\mathfrak{A}\setminus\{0\}$, then

$$[\![\theta b>0]\!]\subseteq[\![\theta_1 b>0]\!]\subseteq[\![\lceil b:1\rceil>0]\!]=\operatorname{upr}(b,\mathfrak{C})$$

by 395I(ii) again. **?** Suppose, if possible, that $[\![\theta b > 0]\!] \ne \operatorname{upr}(b, \mathfrak{C})$. Set $c_0 = \operatorname{upr}(b, \mathfrak{C}) \setminus [\![\theta b > 0]\!]$, $a_0 = b \cup (1 \setminus \operatorname{upr}(b, \mathfrak{C})) \in A^*$. Let $k \ge 1$ be such that $c_1 = c_0 \cap [\![\lceil 1 : a_0 \rceil \le k]\!] \ne 0$. Then $a_0 \cap c_1 = b \cap c_1$, so

$$\theta a_0 \times \chi c_1 = \theta(a_0 \cap c_1) = \theta(b \cap c_1) = \theta b \times \chi c_1 = 0.$$

By 364L(b-ii), there is an $a \in A^*$ such that $c_1 \not\subseteq [\![\theta_a a_0 \times \chi c_1 \ge \frac{1}{k}]\!]$, that is, $c_2 = c_1 \cap [\![\theta_a a_0 < \frac{1}{k}]\!] \ne 0$. Now

$$c_2 \subseteq [\![\lfloor 1 : a \rfloor - k\lceil a_0 : a \rceil > 0]\!] \subseteq [\![\lceil 1 : a_0 \rceil \times \lceil a_0 : a \rceil - k\lceil a_0 : a \rceil > 0]\!] \subseteq [\![\lceil 1 : a_0 \rceil > k]\!],$$

which is impossible, as $c_2 \subseteq c_1$. **X**

Thus $[\![\theta b > 0]\!] = \operatorname{upr}(b, \mathfrak{C})$. In particular, $\theta b = 0$ iff $b = 0$.

(g) If $b, b' \in \mathfrak{A}$ and $b \preccurlyeq^\tau_G b'$, then $\theta b \le \theta b'$. **P** For every $a \in A^*$, $\lceil b : a \rceil \le \lceil b' : a \rceil$ (395Kb) so $\theta_a b \le \theta_a b'$. **Q** So if b, $b' \in \mathfrak{A}$ and $c = [\![\theta b - \theta b' > 0]\!]$, $b' \cap c \preccurlyeq^\tau_G b$. **P?** Otherwise, by 395H, there is a non-zero $c' \subseteq c$ such that $b \cap c' \preccurlyeq^\tau_G b'$. But in this case $\theta b \times \chi c' = \theta(b \cap c') \le \theta b'$ and $c' \subseteq [\![\theta b' - \theta b \ge 0]\!]$. **XQ**

(h) If $\langle a_i \rangle_{i \in I}$ is any disjoint family in $\mathfrak{A}$ with supremum a, $\theta a = \sum_{i \in I} \theta a_i$, where the sum is to be interpreted as $\sup_{J \subseteq I \text{ is finite}} \sum_{i \in J} \theta a_i$. **P** Induce on $\#(I)$. If $\#(I)$ is finite, this is just finite additivity ((d) above). For the inductive step to $\#(I) = \kappa \ge \omega$, we may suppose that I is actually equal to the cardinal κ. Of course

$$\theta a \ge \theta(\sup_{\xi \in J} a_\xi) = \sum_{\xi \in J} \theta a_\xi$$

for every finite $J \subseteq \kappa$, so (because $L^\infty(\mathfrak{C})$ is Dedekind complete) $u = \sum_{\xi < \kappa} \theta a_\xi$ is defined, and $u \le \theta a$.

For $\zeta < \kappa$, set $b_\zeta = \sup_{\xi < \zeta} a_\xi$. By the inductive hypothesis,

$$\theta b_\zeta = \sum_{\xi < \zeta} \theta a_\xi = \sup_{J \subseteq \zeta \text{ is finite}} \sum_{\xi \in J} \theta a_\xi \le u.$$

At the same time, if $J \subseteq \kappa$ is finite, there is some $\zeta < \kappa$ such that $J \subseteq \zeta$, so that $\sum_{\xi \in J} \theta a_\xi \le \theta b_\zeta$; accordingly $\sup_{\zeta < \kappa} \theta b_\zeta = u$.

? Suppose, if possible, that $u < \theta a$; set $v = \theta a - u$. Take $\delta > 0$ such that $c_0 = [\![v > \delta]\!] \ne 0$. Let $\zeta < \kappa$ be such that $c_1 = c_0 \setminus [\![u - \theta b_\zeta > \delta]\!]$ is non-zero (cf. 364L(b-ii)). Now $v = \theta a - u \le \theta(a \setminus b_\zeta)$, so

$$c_1 \subseteq [\![v > \delta]\!] \subseteq [\![\theta(a \setminus b_\zeta) > 0]\!] = \operatorname{upr}(a \setminus b_\zeta, \mathfrak{C}),$$

and $c_1 \cap (a \setminus b_\zeta) \ne 0$; there is therefore an $\eta' \ge \zeta$ such that $d = c_1 \cap a_{\eta'} \ne 0$. Since $\theta d \le u - \theta b_\zeta$ and c_1 is included in $[\![u - \theta b_\zeta \le \delta]\!] \cap [\![v > \delta]\!]$, $[\![v - \theta d > 0]\!] \supseteq c_1$.

Choose $\langle d_\xi \rangle_{\xi < \kappa}$ inductively, as follows. Given that $\langle d_\eta \rangle_{\eta < \xi}$ is a disjoint family in $\mathfrak{A}_{a \setminus d}$ such that d_η is G-τ-equidecomposable with $a_\eta \cap c_1$ for every $\eta < \xi$, then $e_\xi = \sup_{\eta < \xi} d_\eta$ is G-τ-equidecomposable with $b_\xi \cap c_1$, so that $\theta e_\xi \le \theta b_\xi$, and

$$\begin{aligned} [\![\theta(a \setminus (d \cup e_\xi)) - \theta a_\xi > 0]\!] &= [\![\theta a - \theta d - \theta e_\xi - \theta a_\xi > 0]\!] \supseteq [\![\theta a - \theta d - \theta b_\xi - \theta a_\xi > 0]\!] \\ &= [\![\theta a - \theta d - \theta b_{\xi+1} > 0]\!] \supseteq [\![v - \theta d > 0]\!] \supseteq c_1. \end{aligned}$$

By (g), $a_\xi \cap c_1 \preccurlyeq^\tau_G a \setminus (d \cup e_\xi)$; take $d_\xi \subseteq a \setminus (d \cup e_\xi)$ G-τ-equidecomposable with $a_\xi \cap c_1$, and continue.

At the end of this induction, we have a disjoint family $\langle d_\xi \rangle_{\xi < \kappa}$ in $\mathfrak{A}_{a \setminus d}$ such that d_ξ is G-τ-equidecomposable with $a_\xi \cap c_1$ for every ξ. But this means that $a' = \sup_{\xi < \kappa} d_\xi$ is G-τ-equidecomposable with $a \cap c_1$, while $a' \subseteq (a \setminus d) \cap c_1$; since $d \cap a \cap c_1 \ne 0$, G cannot be fully non-paradoxical. **X**

Thus $\theta a = u = \sum_{\xi < \kappa} \theta a_\xi$ and the induction continues. **Q**

(i) It follows that θ is order-continuous. **P** (α) If $B \subseteq \mathfrak{A}$ is non-empty and upwards-directed and has supremum e, then $\bigcup_{b \in B} \mathfrak{A}_b$ is order-dense in $\mathfrak{A}_e$, so includes a partition of unity A of $\mathfrak{A}_e$; now (h) tells us that

$$\theta e = \sum_{a \in A} \theta a \le \sup_{b \in B} \theta b.$$

Since of course $\theta b \le \theta e$ for every $b \in B$, $\theta e = \sup_{b \in B} \theta b$. (β) If $B \subseteq \mathfrak{A}$ is non-empty and downwards-directed and has infimum e, then, using (α), we see that

$$\theta 1 - \theta e = \theta(1 \setminus e) = \sup_{b \in B} \theta(1 \setminus b) = \sup_{b \in B} \theta 1 - \theta b,$$

so that $\theta e = \inf_{b \in B} \theta b$. **Q**

(j) I still have to show that θ is unique. Let $\theta' : \mathfrak{A} \to L^\infty$ be any non-negative order-continuous G-invariant additive function such that $\theta' c = \chi c$ for every $c \in \mathfrak{C}$.

(i) Just as in (e) of this proof, but more easily, we see that $\theta'(b \cap c) = \theta' b \times \chi c$ whenever $b \in \mathfrak{A}$ and $c \in \mathfrak{C}$.

(ii) If $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ with supremum a, then $\langle \sup_{i\in J} a_i\rangle_{J\subseteq I \text{ is finite}}$ is an upwards-directed family with supremum a, so that

$$\theta' a = \sup_{J\subseteq I \text{ is finite}} \theta'(\sup_{i\in J} a_i) = \sup_{J\subseteq I \text{ is finite}} \sum_{i\in J} \theta' a_i = \sum_{i\in I} \theta' a_i.$$

(iii) $\theta' a = \theta' b$ whenever a and b are G-τ-equidecomposable. **P** Take a partition $\langle a_i\rangle_{i\in I}$ of a and a family $\langle \pi_i\rangle_{i\in I}$ in G such that $\langle \pi_i a_i\rangle_{i\in I}$ is a partition of b. Then

$$\theta' a = \sum_{i\in I} \theta' a_i = \sum_{i\in I} \theta' \pi_i a_i = \theta' b. \text{ } \mathbf{Q}$$

Consequently $\theta' a \le \theta' b$ whenever $a \preccurlyeq_G^\tau b$.

(iv) Take $a \in A^*$, $b \in \mathfrak{A}$ and for j, $k \in \mathbb{N}$ set $c_{jk} = [\![\lfloor 1 : a\rfloor = j]\!] \cap [\![\lceil b : a\rceil = k]\!]$. Then

$$\lceil b : a\rceil \times \chi c_{jk} \ge \theta' b \times \lfloor 1 : a\rfloor \times \chi c_{jk}.$$

P If $c_{jk} = 0$ this is trivial; suppose $c_{jk} \ne 0$. Now we have sets I, J such that $\#(I) = j$, $\#(J) \le k$, $a \cap c_{jk} \preccurlyeq_G^\tau d$ for every $d \in I$, $e \preccurlyeq_G^\tau a$ for every $e \in J$, I is disjoint, and $b \cap c_{jk} \subseteq \sup J$. So

$$\begin{aligned}\theta' b \times \lfloor 1 : a\rfloor \times \chi c_{jk} &= j\theta' b \times \chi c_{jk} = j\theta'(b \cap c_{jk}) \le j \sum_{e\in J} \theta'(e \cap c_{jk}) \\ &\le jk\theta'(a \cap c_{jk}) \le k \sum_{d\in I} \theta'(d \cap c_{jk}) \le k\theta' c_{jk} \\ &= k\chi c_{jk} = \lceil b : a\rceil \times \chi c_{jk}. \text{ } \mathbf{Q}\end{aligned}$$

Summing over j and k, $\lceil b : a\rceil \ge \theta' b \times \lfloor 1 : a\rfloor$, that is, $\theta_a b \ge \theta' b$. Taking the infimum over a, $\theta b \ge \theta' b$. But also

$$\theta b = \chi 1 - \theta(1 \setminus b) \le \chi 1 - \theta'(1 \setminus b) = \theta' b,$$

so $\theta b = \theta' b$. As b is arbitrary, $\theta = \theta'$. This completes the proof.

395O We have reached the summit. The rest of the section is a list of easy corollaries.

Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Then there is a G-invariant additive functional $\nu : \mathfrak{A} \to [0, 1]$ such that $\nu 1 = 1$.

proof Let $\mathfrak{C}$ be the fixed-point subalgebra of G, and $\theta : \mathfrak{A} \to L^\infty(\mathfrak{C})$ the function of 395N. By 311D, there is a ring homomorphism $\nu_0 : \mathfrak{C} \to \{0, 1\}$ such that $\nu_0 1 = 1$; now ν_0 can also be regarded as an additive functional from $\mathfrak{C}$ to $\mathbb{R}$. Let $f_0 : L^\infty(\mathfrak{C}) \to \mathbb{R}$ be the corresponding positive linear functional (363K). Set $\nu = f_0\theta$. Then ν is order-preserving and additive because f_0 and θ are, $\nu 1 = f_0(\chi 1) = \nu_0 1 > 0$, and ν is G-invariant because θ is.

395P Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Then the following are equiveridical:

(i) $\mathfrak{A}$ is a measurable algebra;
(ii) $\mathfrak{C}$ is a measurable algebra;
(iii) there is a strictly positive G-invariant countably additive real-valued functional on $\mathfrak{A}$.

proof (iii)$\Rightarrow$(i)$\Rightarrow$(ii) are trivial. For (ii)$\Rightarrow$(iii), let $\theta : \mathfrak{A} \to L^\infty(\mathfrak{C})$ be the function of 395N, and $\bar\nu : \mathfrak{C} \to \mathbb{R}$ a strictly positive countably additive functional. Let $f : L^\infty(\mathfrak{C}) \to \mathbb{R}$ be the corresponding linear operator; then f is sequentially order-continuous (363K again). Set $\bar\mu = f\theta$. Then $\bar\mu$ is additive and order-preserving and sequentially order-continuous because f and θ are. It is also strictly positive, because if $a \in \mathfrak{A} \setminus \{0\}$ then $\theta a > 0$ (395N(ii)), that is, there is some $\delta > 0$ such that $[\![\theta a > \delta]\!] \ne 0$, so that

$$\bar\mu a \ge \delta\bar\nu[\![\theta a > \delta]\!] > 0.$$

Finally, $\bar\mu$ is G-invariant because θ is.

395Q Corollary: Kawada's theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that $\operatorname{Aut}\mathfrak{A}$ has a subgroup which is ergodic and fully non-paradoxical. Then $\mathfrak{A}$ is measurable.

proof By 395Gf, this is the case $\mathfrak{C} = \{0, 1\}$ of 395P.

395R Thus the existence of an ergodic fully non-paradoxical subgroup is a sufficient condition for a Dedekind complete Boolean algebra to be measurable. It is not quite necessary, because if a measure algebra $\mathfrak{A}$ is not homogeneous then its automorphism group is not ergodic. But for homogeneous algebras the condition is necessary as well as sufficient, by the following result.

Proposition If $(\mathfrak{A},\bar{\mu})$ is a homogeneous totally finite measure algebra, $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is ergodic.

proof If $\mathfrak{A}=\{0,1\}$ this is trivial. Otherwise, $\mathfrak{A}$ is atomless. If a, $b\in\mathfrak{A}\setminus\{0,1\}$, set $\gamma=\min(\bar{\mu}a,\bar{\mu}b)$; then there are $a'\subseteq a$ and $b'\subseteq b$ such that $\bar{\mu}a'=\bar{\mu}b'=\gamma$. By 383Fb, there is a $\pi\in G$ such that $\pi a'=b'$, so that $\pi a\cap a\ne 0$. As b is arbitrary, $\sup_{\pi\in G}\pi a=1$; as a is arbitrary, G is ergodic.

395X Basic exercises (a) Re-write the section on the assumption that every group G is ergodic, so that $L^0(\mathfrak{C})$ may be identified with $\mathbb{R}$, the functions $\lceil\ldots\rceil$ and $\lfloor\ldots\rfloor$ become real-valued, the functionals θ_a (395N) become submeasures and θ becomes a measure.

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that $\langle c_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{C}$ and that a, $b\in\mathfrak{A}$ are such that $a\cap c_i\preccurlyeq_G^\tau b$ for every $i\in I$. Show that $a\preccurlyeq_G^\tau b$.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Show that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$ iff the full subgroup generated by G has many involutions (definition: 382O).

(d) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Show that the following are equiveridical: (i) $\mathfrak{A}$ is chargeable (definition: 391Bb); (ii) $\mathfrak{C}$ is chargeable; (iii) there is a strictly positive G-invariant real-valued additive functional on $\mathfrak{A}$.

(e) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Show that the following are equiveridical: (i) there is a non-zero completely additive functional on $\mathfrak{A}$; (ii) there is a non-zero completely additive functional on $\mathfrak{C}$; (iii) there is a non-zero G-invariant completely additive functional on $\mathfrak{A}$.

(f) Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra. Show that it is a measurable algebra iff there is a fully non-paradoxical subgroup G of $\operatorname{Aut}\mathfrak{A}$ such that the fixed-point subalgebra of G is purely atomic.

(g) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra. Show that the following are equiveridical: (i) $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is ergodic; (ii) $\mathfrak{A}$ is quasi-homogeneous in the sense of 374G.

(h) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra. Show that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is fully non-paradoxical iff (i) for every infinite cardinal κ, the Maharam-type-κ component of $\mathfrak{A}$ (definition: 332Gb) has finite measure (ii) for every $\gamma\in\,]0,\infty[$ there are only finitely many atoms of measure γ.

(i) Let $\mathfrak{A}$ be a Boolean algebra, G a subgroup of $\operatorname{Aut}\mathfrak{A}$, and G^* the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by G. Show that G^* is ergodic iff G is ergodic.

395Y Further exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, G a subgroup of $\operatorname{Aut}\mathfrak{A}$, and G^*_τ the full local semigroup generated by G. For ϕ, $\psi\in G^*_\tau$, say that $\phi\le\psi$ if ψ extends ϕ. (i) Show that every member of G^*_τ can be extended to a maximal member of G^*_τ. (ii) Show that G is fully non-paradoxical iff every maximal member of G^*_τ is actually a Boolean automorphism of $\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Show that G is fully non-paradoxical iff $\langle\pi_n a_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ and $\langle\pi_n\rangle_{n\in\mathbb{N}}$ is a sequence in G.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. (i) Show that $\mathfrak{A}$ is ccc iff $\mathfrak{C}$ is ccc. (*Hint*: if $\mathfrak{C}$ is ccc, $L^\infty(\mathfrak{C})$ has the countable sup property.) (ii) Show that $\mathfrak{A}$ is weakly (σ,∞)-distributive iff $\mathfrak{C}$ is. (iii) Show that $\mathfrak{A}$ is a Maharam algebra iff $\mathfrak{C}$ is.

(d) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, G an ergodic subgroup of $\operatorname{Aut}\mathfrak{A}$, and G^*_τ the full local semigroup generated by G. Suppose that there is a non-zero $a\in\mathfrak{A}$ for which there is no $\phi\in G^*_\tau$ such that $\phi a\subset a$. Show that there is a measure $\bar{\mu}$ such that $(\mathfrak{A},\bar{\mu})$ is a localizable measure algebra. (*Hint*: show that $\mathfrak{A}_a$ is a measurable algebra.)

(e) Show that there are a semi-finite measure algebra $(\mathfrak{A}, \bar\mu)$ and a subgroup G of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ such that G is not ergodic but has fixed-point algebra $\{0,1\}$.

395Z Problem Suppose that $\mathfrak{A}$ is a Dedekind complete Boolean algebra, not $\{0\}$, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ such that whenever $\langle a_i\rangle_{i\le n}$ is a finite partition of unity in $\mathfrak{A}$ and we are given π_i, $\pi'_i \in G$ for every $i \le n$, then the elements $\pi_0 a_0$, $\pi'_0 a_0$, $\pi_1 a_1$, $\pi'_1 a_1, \dots, \pi'_n a_n$ are not all disjoint. Must there be a non-zero non-negative G-invariant finitely additive functional θ on $\mathfrak{A}$?

(See 'Tarski's theorem' in the notes below.)

395 Notes and comments Regarded as a sufficient condition for measurability, Kawada's theorem suffers from the obvious defect that it is going to be rather rarely that we can verify the existence of an ergodic fully non-paradoxical group of automorphisms without having some quite different reason for supposing that our algebra is measurable. If we think of it as a criterion for the existence of a G-invariant measure, rather than as a criterion for measurability in the abstract, it seems to make better sense. But if we know from the start that the algebra $\mathfrak{A}$ is measurable, the argument short-circuits, as we shall see in §396.

I take the trouble to include the 'τ' in every 'G-τ-equidecomposable', 'G^*_τ' and '$\preccurlyeq^\tau_G$' because there are important variations on the concept, in which the partitions $\langle a_i\rangle_{i\in I}$ of 395A are required to be finite or countable. Indeed **Tarski's theorem** relies on one of these. I spell it out because it is close to Kawada's in spirit, though there are significant differences in the ideas needed in the proof:

> Let X be a set and G a subgroup of $\operatorname{Aut}\mathcal{P}X$. Then the following are equiveridical: (i) there is a G-invariant additive functional $\theta : \mathcal{P}X \to [0,1]$ such that $\theta A = 1$; (ii) there are no $A_0, \dots, A_n$, $\pi_0, \dots, \pi_n$, $\pi'_0, \dots, \pi'_n$ such that $A_0, \dots, A_n$ are subsets of X covering X, $\pi_0, \dots, \pi'_n$ all belong to G, and $\pi_0[A_0]$, $\pi'_0[A_0]$, $\pi_1[A_1]$, $\pi'_1[A_1], \dots, \pi'_n[A_n]$ are all disjoint.

For a proof, see 449L in Volume 4; for an illuminating discussion of this theorem, see Wagon 85, Chapter 9. But it seems to be unknown whether the natural translation of this result is valid in all Dedekind complete Boolean algebras (395Z). Note that we are looking for theorems which do not depend on any special properties of the group G or the Boolean algebra $\mathfrak{A}$. For abelian or 'amenable' groups, or weakly (σ,∞)-distributive algebras, for instance, much more can be done, as described in 396Ya and §449.

The methods of this section can, however, be used to prove similar results for *countable* groups of automorphisms on Dedekind σ-complete Boolean algebras; I will return to such questions in §448. The presentation here owes a good deal to Nadkarni 90 and something to Becker & Kechris 96.

As noted, Kawada (Kawada 44) treated the case in which the group G of automorphisms is ergodic, that is, the fixed-point subalgebra $\mathfrak{C}$ is trivial. Under this hypothesis the proof is of course very much simpler. (You may find it useful to reconstruct the original version, as suggested in 395Xa.) I give the more general argument partly for the sake of 395O, partly to separate out the steps which really need ergodicity from those which depend only on non-paradoxicality, partly to prepare the ground for the countable version in the next volume, partly to show off the power of the construction in §364, and partly to get you used to 'Boolean-valued' arguments. A bolder use of language could indeed simplify some formulae slightly by writing (for instance) $[\![k\lceil a_0 : a\rceil < \lfloor 1 : a\rfloor]\!]$ in place of $[\![\lfloor 1 : a\rfloor - k\lceil a_0 : a\rceil > 0]\!]$ (see part (f) of the proof of 395N). As in §388, the differences involved in the extension to non-ergodic groups are, in a sense, just a matter of technique; but this time the technique is more obtrusive. In §556 of Volume 5 I will try to explain a general approach to questions of this kind, using metamathematical ideas.

396 The Hajian-Ito theorem

In the notes to the last section, I said that the argument there short-circuits if we are told that we are dealing with a measurable algebra. The point is that in this case there is a much simpler criterion for the existence of a G-invariant measure (396B(ii)), with a proof which is independent of §395 in all its non-trivial parts, which makes it easy to prove that non-paradoxicality is sufficient as well as necessary.

396A Lemma Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra.

(a) Let $\pi \in \operatorname{Aut}\mathfrak{A}$ be a Boolean automorphism (not necessarily measure-preserving), and T_π the corresponding Riesz homomorphism from $L^0 = L^0(\mathfrak{A})$ to itself (364P). Then there is a unique $w_\pi \in (L^0)^+$ such that $\int w_\pi \times v = \int T_\pi v$ for every $v \in (L^0)^+$.

(b) If ϕ, $\pi \in \operatorname{Aut}\mathfrak{A}$ then $w_{\pi\phi} = w_\phi \times T_{\phi^{-1}} w_\pi$.

(c) For each $\pi \in \operatorname{Aut}\mathfrak{A}$ we have a norm-preserving isomorphism U_π from $L^2 = L^2(\mathfrak{A}, \bar\mu)$ to itself defined by setting

$$U_\pi v = T_\pi v \times \sqrt{w_{\pi^{-1}}}$$

for every $v \in L^2$, and $U_{\pi\phi} = U_\pi U_\phi$ for all π, $\phi \in \operatorname{Aut}\mathfrak{A}$.

proof (a) Set $\bar\nu a = \bar\mu(\pi a)$ for $a \in \mathfrak{A}$. Then $(\mathfrak{A}, \bar\nu)$ is a semi-finite measure algebra. **P** $\bar\nu 0 = \bar\mu 0 = 0$; if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, then $\langle \pi a_n\rangle_{n\in\mathbb{N}}$ is disjoint and (because π is sequentially order-continuous) $a = \sup_{n\in\mathbb{N}} \pi a_n$, so $\bar\nu a = \sum_{n=0}^\infty \bar\nu a_n$; if $a \ne 0$ then $\pi a \ne 0$ so $\bar\nu a > 0$. Thus $(\mathfrak{A}, \bar\nu)$ is a measure algebra. If $a \ne 0$ there is a $b \subseteq \pi a$ such that $0 < \bar\mu b < \infty$, and now $\pi^{-1} b \subseteq a$ and $0 < \bar\nu(\pi^{-1} b) < \infty$; thus $\bar\nu$ is semi-finite. **Q**

By 365T, there is a unique $w_\pi \in (L^0)^+$ such that $\int_a w_\pi = \bar\mu(\pi a)$ for every $a \in \mathfrak{A}$. If we look at

$$W = \{v : v \in (L^0)^+,\ \int v \times w_\pi = \int T_\pi v\},$$

we see that W contains χa for every $a \in \mathfrak{A}$, that $v + v' \in W$ and $\alpha v \in W$ whenever v, $v' \in W$ and $\alpha \ge 0$, and that $\sup_{n\in\mathbb{N}} v_n \in W$ whenever $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in W which is bounded above in L^0. By 364Jd, $W = (L^0)^+$, as required.

(b) For any $v \in (L^0)^+$,

$$\int w_{\pi\phi} \times v = \int T_{\pi\phi} v = \int T_\pi T_\phi v$$

(364Pe)

$$= \int w_\pi \times T_\phi v = \int T_\phi(T_{\phi^{-1}} w_\pi \times v)$$

(recalling that T_ϕ is multiplicative)

$$= \int w_\phi \times T_{\phi^{-1}} w_\pi \times v.$$

As v is arbitrary (and $(\mathfrak{A}, \bar\mu)$ is semi-finite), $w_{\pi\phi} = w_\phi \times T_{\phi^{-1}} w_\pi$.

(c)(i) For any $v \in L^0$,

$$\int (T_\pi v \times \sqrt{w_{\pi^{-1}}})^2 = \int T_\pi v^2 \times w_{\pi^{-1}} = \int T_{\pi^{-1}} T_\pi v^2 = \int v^2.$$

So $U_\pi v \in L^2$ and $\|U_\pi v\|_2 = \|v\|_2$ whenever $v \in L^2$, and U_π is a norm-preserving operator on L^2.

(ii) Now consider $U_{\pi\phi}$. For any $v \in L^2$, we have

$$U_\pi U_\phi v = T_\pi(T_\phi v \times \sqrt{w_{\phi^{-1}}}) \times \sqrt{w_{\pi^{-1}}}$$
$$= T_\pi T_\phi v \times \sqrt{T_\pi w_{\phi^{-1}} \times w_{\pi^{-1}}}$$

(using 364Pd)

$$= T_{\pi\phi} v \times \sqrt{w_{\phi^{-1}\pi^{-1}}}$$

(by (b) above)

$$= U_{\pi\phi} v.$$

So $U_{\pi\phi} = U_\pi U_\phi$.

(iii) Writing ι for the identity operator on $\mathfrak{A}$, we see that T_ι is the identity operator on L^0, $w_\iota = \chi 1$ and U_ι is the identity operator on L^2. Since $U_{\pi^{-1}} U_\pi = U_\pi U_{\pi^{-1}} = U_\iota$, $U_\pi : L^2 \to L^2$ is an isomorphism, with inverse $U_{\pi^{-1}}$, for every $\pi \in \operatorname{Aut}\mathfrak{A}$.

396B Theorem (Hajian & Ito 69) Let $\mathfrak{A}$ be a measurable algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is a G-invariant functional $\bar\nu$ such that $(\mathfrak{A}, \bar\nu)$ is a totally finite measure algebra;

(ii) whenever $a \in \mathfrak{A} \setminus \{0\}$ and $\langle \pi_n\rangle_{n\in\mathbb{N}}$ is a sequence in G, $\langle \pi_n a\rangle_{n\in\mathbb{N}}$ is not disjoint;

(iii) G is fully non-paradoxical (definition: 395E).

proof (a) (i)$\Rightarrow$(iii) by the argument of 395F, and (iii)$\Rightarrow$(ii) by the criterion (ii) of 395E. So for the rest of the proof I assume that (ii) is true and seek to prove (i).

(b) Let $\bar{\mu}$ be such that $(\mathfrak{A},\bar{\mu})$ is a totally finite measure algebra. If $a\in\mathfrak{A}\setminus\{0\}$, then $\inf_{\pi\in G}\bar{\mu}(\pi a)>0$. **P?** Otherwise, let $\langle\pi_n\rangle_{n\in\mathbb{N}}$ be a sequence in G such that $\bar{\mu}\pi_n a\le 2^{-n}$ for each $n\in\mathbb{N}$. Set $b_n=\sup_{k\ge n}\pi_k a$ for each n; then $\inf_{n\in\mathbb{N}}b_n=0$, so that

$$\inf_{n\in\mathbb{N}}\pi b_n=0,\quad \lim_{n\to\infty}\bar{\mu}(\pi\pi_n a)=0$$

for every $\pi\in\operatorname{Aut}\mathfrak{A}$. Choose $\langle n_i\rangle_{i\in\mathbb{N}}$ inductively so that

$$\bar{\mu}(\pi_{n_i}^{-1}\pi_{n_j}a)\le 2^{-j-2}\bar{\mu}a$$

whenever $i<j$. Set

$$c=a\setminus\sup_{i<j}\pi_{n_i}^{-1}\pi_{n_j}a.$$

Because

$$\sum_{j=1}^{\infty}\sum_{i=0}^{j-1}\bar{\mu}(\pi_{n_i}^{-1}\pi_{n_j}a)<\bar{\mu}a,$$

$c\ne 0$, while $\pi_{n_i}c\cap\pi_{n_j}c=0$ whenever $i<j$, contrary to the hypothesis (ii). **XQ**

(c) For each $\pi\in G$, define $w_\pi\in L^0=L^0(\mathfrak{A})$ and $U_\pi:L^2\to L^2$ as in 396A, where $L^2=L^2(\mathfrak{A},\bar{\mu})$. If $a\in\mathfrak{A}\setminus\{0\}$, then $\inf_{\pi\in G}\int_a\sqrt{w_\pi}>0$. **P?** Otherwise, there is a sequence $\langle\pi_n\rangle_{n\in\mathbb{N}}$ in G such that $\int_a v_n\le 4^{-n-2}\bar{\mu}a$ for every n, where $v_n=\sqrt{w_{\pi_n}}$. In this case, $\bar{\mu}(a\cap[\![v_n\ge 2^{-n}]\!])\le 2^{-n-2}\bar{\mu}a$ for every n, so that $b=a\setminus\sup_{n\in\mathbb{N}}[\![v_n\ge 2^{-n}]\!]$ is non-zero. But now

$$\bar{\mu}(\pi_n b)=\int_b w_{\pi_n}=\int_b v_n^2\le 4^{-n}\bar{\mu}b\to 0$$

as $n\to\infty$, contradicting (b) above. **XQ**

(d) Write $e=\chi 1$ for the standard weak order unit of L^0 or L^2. Let $C\subseteq L^2$ be the convex hull of $\{U_\pi e:\pi\in G\}$. Then C and its norm closure $\overline{C}$ are G-invariant in the sense that $U_\pi v\in C$, $U_\pi v'\in\overline{C}$ whenever $v\in C$, $v'\in\overline{C}$ and $\pi\in G$. By 3A5Md, there is a unique $u_0\in\overline{C}$ such that $\|u_0\|_2\le\|u\|_2$ for every $u\in\overline{C}$. Now if $\pi\in G$, $U_\pi u_0\in\overline{C}$, while $\|U_\pi u_0\|_2=\|u_0\|_2$; so $U_\pi u_0=u_0$. Also, if $a\in\mathfrak{A}\setminus\{0\}$,

$$\int_a u_0\ge\inf_{u\in\overline{C}}\int_a u=\inf_{u\in C}\int_a u$$

(because $u\mapsto\int_a u$ is $\|\,\|_2$-continuous)

$$=\inf_{\pi\in G}\int_a U_\pi e=\inf_{\pi\in G}\int_a T_\pi e\times\sqrt{w_{\pi^{-1}}}$$
$$=\inf_{\pi\in G}\int_a\sqrt{w_{\pi^{-1}}}=\inf_{\pi\in G}\int_a\sqrt{w_\pi}>0$$

by (c). So $[\![u_0>0]\!]=1$.

(e) For $a\in\mathfrak{A}$, set $\bar{\nu}a=\int_a u_0^2$. Because $u_0\in L^2$, $\bar{\nu}$ is a non-negative countably additive functional on $\mathfrak{A}$; because $[\![u_0^2>0]\!]=[\![u_0>0]\!]=1$, $\bar{\nu}$ is strictly positive, and $(\mathfrak{A},\bar{\nu})$ is a totally finite measure algebra. Finally, $\bar{\nu}$ is G-invariant. **P** If $a\in\mathfrak{A}$ and $\pi\in G$, then

$$\bar{\nu}(\pi a)=\int_{\pi a}u_0^2=\int u_0^2\times\chi(\pi a)=\int T_\pi(T_{\pi^{-1}}u_0^2\times\chi a)$$
$$=\int w_\pi\times T_{\pi^{-1}}u_0^2\times\chi a=\int_a(T_{\pi^{-1}}u_0\times\sqrt{w_\pi})^2$$
$$=\int_a(U_{\pi^{-1}}u_0)^2=\int_a u_0^2=\bar{\nu}a.\ \mathbf{Q}$$

So (i) is true.

396C Remark If $\mathfrak{A}$ is a Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$, a non-zero element a of $\mathfrak{A}$ is called **weakly wandering** if there is a sequence $\langle\pi_n\rangle_{n\in\mathbb{N}}$ in G such that $\langle\pi_n a\rangle_{n\in\mathbb{N}}$ is disjoint. Thus condition (ii) of 396B may be read as 'there is no weakly wandering element of $\mathfrak{A}$'.

396X Basic exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ an order-continuous Boolean homomorphism. Let $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ be the corresponding Riesz homomorphism. Show that there is a unique $w_\pi \in L^1(\mathfrak{A}, \bar{\mu})$ such that $\int T_\pi v = \int v \times w_\pi$ for every $v \in L^0(\mathfrak{A})^+$.

(b) In 396A, show that the map $\pi \mapsto U_\pi : \operatorname{Aut}\mathfrak{A} \to \mathsf{B}(L^2; L^2)$ is injective.

(c) Let $\mathfrak{A}$ be a measurable algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Suppose that there is a strictly positive G-invariant finitely additive functional on $\mathfrak{A}$. Show that there is a G-invariant $\bar{\mu}$ such that $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra.

396Y Further exercises (a) Let $\mathfrak{A}$ be a weakly (σ, ∞)-distributive Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. For a, $b \in \mathfrak{A}$, say that a and b are G**-equidecomposable** if there are *finite* partitions of unity $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}_a$ and $\langle b_i \rangle_{i \in I}$ in $\mathfrak{A}_b$, and a family $\langle \pi_i \rangle_{i \in I}$ in G, such that $\pi_i a_i = b_i$ for every $i \in I$. Show that the following are equiveridical: (i) G is fully non-paradoxical in the sense of 395E; (ii) if $\langle a_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence of mutually G-equidecomposable elements of $\mathfrak{A}$, they must all be 0.

396 Notes and comments I have separated these few pages from §395 partly because §395 was already up to full weight and partly in order that the ideas here should not be entirely overshadowed by those of the earlier section. It will be evident that the construction of the U_π in 396A, providing us with a faithful representation, acting on a Hilbert space, of the whole group $\operatorname{Aut}\mathfrak{A}$, is a basic tool for the study of that group.

Appendix to Volume 3

Useful Facts

This volume assumes a fairly wide-ranging competence in analysis, a solid understanding of elementary set theory and some straightforward Boolean algebra. As in previous volumes, I start with a few pages of revision in set theory, but the absolutely essential material is in §3A2, on commutative rings, which is the basis of the treatment of Boolean rings in §311. I then give three sections of results in analysis: topological spaces (§3A3), uniform spaces (§3A4) and normed spaces (§3A5). Finally, I add six sentences on group theory (§3A6).

3A1 Set Theory

3A1A The axioms of set theory This treatise is based on arguments within, or in principle reducible to, 'ZFC', meaning 'Zermelo-Fraenkel set theory, including the Axiom of Choice'. For discussions of this system, see, for instance, Krivine 71, Jech 03 or Kunen 80. As I remarked in §2A1, I believe that it is helpful, as a matter of general principle, to distinguish between results dependent on the axiom of choice and those which can be proved without it, or with some relatively weak axiom such as 'countable choice'. (See 134C. I will go much more deeply into this in Chapter 56 of Volume 5.) In Volumes 1 and 2, such a distinction is useful in appreciating the special features of different ideas. In the present volume, however, most of the principal theorems require something close to the full axiom of choice, and there are few areas where it seems at present appropriate to work with anything weaker. Indeed, at many points we shall approach questions which are, or may be, undecidable in ZFC; but with very few exceptions I postpone discussion of these to Volume 5. In particular, I specifically exclude, for the time being, results dependent on such axioms as the continuum hypothesis.

3A1B Definition Let X be a set. By an **enumeration** of X I mean a bijection $f:\kappa\to X$ where $\kappa=\#(X)$is the initial ordinal equipollent with X (2A1Kb); more often than not I shall express such a function in the form $\langle x_\xi\rangle_{\xi<\kappa}$. In this case I say that the function f, or the family $\langle x_\xi\rangle_{\xi<\kappa}$, **enumerates** X. You will see that I am tacitly assuming that $\#(X)$ is always defined, that is, that the axiom of choice is true.

3A1C Calculation of cardinalities The following formulae are basic.

(a) For any sets X and Y, $\#(X\times Y)\le\max(\omega,\#(X),\#(Y))$. (Enderton 77, p. 64; Jech 03, p. 51; Krivine 71, p. 33; Kunen 80, 10.13.)

(b) For any $r\in\mathbb{N}$ and any family $\langle X_i\rangle_{i\le r}$ of sets, $\#(\prod_{i=0}^r X_i)\le\max(\omega,\max_{i\le r}\#(X_i))$. (Induce on r.)

(c) For any family $\langle X_i\rangle_{i\in I}$ of sets, $\#(\bigcup_{i\in I}X_i)\le\max(\omega,\#(I),\sup_{i\in I}\#(X_i))$. (Jech 03, p. 52; Krivine 71, p. 33; Kunen 80, 10.21.)

(d) For any set X, the set $[X]^{<\omega}$ of finite subsets of X has cardinal at most $\max(\omega,\#(X))$. (There is a surjection from $\bigcup_{r\in\mathbb{N}}X^r$ onto $[X]^{<\omega}$. For the notation $[X]^{<\omega}$ see 3A1J below.)

3A1D Cardinal exponentiation For a cardinal κ, I write 2^κ for $\#(\mathcal{P}\kappa)$. So $2^\omega=\mathfrak{c}$, and $\kappa^+\le 2^\kappa$ for every κ. (Enderton 77, p. 132; Lipschutz 64, p. 139; Jech 03, p. 29; Krivine 71, p. 25; Halmos 60, p. 93.)

3A1E Definition The class of infinite initial ordinals, or cardinals, is a subclass of the class On of all ordinals, so is itself well-ordered; being unbounded, it is a proper class; consequently there is a unique increasing enumeration of it as $\langle\omega_\xi\rangle_{\xi\in\mathrm{On}}$. We have $\omega_0=\omega$, $\omega_{\xi+1}=\omega_\xi^+$ for every ξ (compare 2A1Fc), $\omega_\xi=\bigcup_{\eta<\xi}\omega_\eta$ for non-zero limit ordinals ξ. (Enderton 77, pp. 213-214; Jech 03, p. 30; Krivine 71, p. 31.)

3A1F Cofinal sets (a) If P is any partially ordered set (definition: 2A1Aa), a subset Q of P is **cofinal** with P if for every $p\in P$ there is a $q\in Q$ such that $p\le q$.

(b) If P is any partially ordered set, the **cofinality** of P, $\operatorname{cf}P$, is the least cardinal of any cofinal subset of P. Note that $\operatorname{cf}P=0$ iff $P=\emptyset$, and that $\operatorname{cf}P=1$ iff P has a greatest element.

(c) Observe that if P is upwards-directed and $\operatorname{cf} P$ is finite, then $\operatorname{cf} P$ is either 0 or 1; for if Q is a finite, non-empty cofinal set then it has an upper bound, which must be the greatest element of P.

(d) If P is a totally ordered set of cofinality κ, then there is a strictly increasing family $\langle p_\xi\rangle_{\xi<\kappa}$ in P such that $\{p_\xi : \xi < \kappa\}$ is cofinal with P. **P** If $\kappa = 0$ then $P = \emptyset$ and this is trivial. Otherwise, let Q be a cofinal subset of P of cardinal κ, and $\{q_\xi : \xi < \kappa\}$ an enumeration of Q. Define $\langle p_\xi\rangle_{\xi<\kappa}$ inductively, as follows. Start with $p_0 = q_0$. Given $\langle p_\eta\rangle_{\eta<\xi}$, where $\xi < \kappa$, then if $p_\eta < q_\xi$ for every $\eta < \xi$, take $p_\xi = q_\xi$; otherwise, because $\#(\xi) \le \xi < \kappa$, $\{p_\eta : \eta < \xi\}$ cannot be cofinal with P, so there is a $p_\xi \in P$ such that $p_\xi \not\le p_\eta$ for every $\eta < \xi$, that is, $p_\eta < p_\xi$ for every $\eta < \xi$. Note that there is some $\eta < \xi$ such that $q_\xi \le p_\eta$, so that $q_\xi \le p_\xi$. Continue.

Now $\langle p_\xi\rangle_{\xi<\kappa}$ is a strictly increasing family in P such that $q_\xi \le p_\xi$ for every ξ; it follows at once that $\{p_\xi : \xi < \kappa\}$ is cofinal with P. **Q**

(e) In particular, for a totally ordered set P, $\operatorname{cf} P = \omega$ iff there is a cofinal strictly increasing sequence in P.

3A1G Zorn's Lemma In Volume 2 I used Zorn's Lemma only once or twice, giving the arguments in detail. In the present volume I feel that continuing in such a manner would often be tedious; but nevertheless the arguments are not always quite obvious, at least until you have gained a good deal of experience. I therefore take a paragraph to comment on some of the standard forms in which they appear.

The statement of Zorn's Lemma, as quoted in 2A1M, refers to arbitrary partially ordered sets P. A large proportion of the applications can in fact be represented more or less naturally by taking P to be a family $\mathfrak{P}$ of sets ordered by $\subseteq$; in such a case, it will be sufficient to check that (i) $\mathfrak{P}$ is not empty (ii) $\bigcup\mathfrak{Q} \in \mathfrak{P}$ for every non-empty totally ordered $\mathfrak{Q} \subseteq \mathfrak{P}$. More often than not, this will in fact be true for all non-empty upwards-directed sets $\mathfrak{Q} \subseteq \mathfrak{P}$, and the line of the argument is sometimes clearer if phrased in this form.

Within this class of partially ordered sets, we can distinguish a special subclass. If A is any set and $\perp$ any relation on A, we can consider the collection $\mathfrak{P}$ of sets $I \subseteq A$ such that $a \perp b$ for all distinct a, $b \in I$. In this case we need look no farther before declaring '$\mathfrak{P}$ has a maximal element'; for $\emptyset$ necessarily belongs to $\mathfrak{P}$, and if $\mathfrak{Q}$ is any upwards-directed subset of $\mathfrak{P}$, then $\bigcup\mathfrak{Q} \in \mathfrak{P}$. **P** If a, b are distinct elements of $\bigcup\mathfrak{Q}$, there are I_1, $I_2 \in \mathfrak{Q}$ such that $a \in I_1$, $b \in I_2$; because $\mathfrak{Q}$ is upwards-directed, there is an $I \in \mathfrak{Q}$ such that $I_1 \cup I_2 \subseteq I$, so that a, b are distinct members of $I \in \mathfrak{P}$, and $a \perp b$. **Q** So $\bigcup\mathfrak{Q}$ is an upper bound of $\mathfrak{Q}$ in $\mathfrak{P}$; as $\mathfrak{Q}$ is arbitrary, $\mathfrak{P}$ satisfies the conditions of Zorn's Lemma, and must have a maximal element.

Another important type of partially ordered set in this context is a family Φ of functions, ordered by saying that $f \le g$ if g is an extension of f. In this case, for any non-empty upwards-directed $\Psi \subseteq \Phi$, we shall have a function h defined by saying that

$$\operatorname{dom} h = \bigcup_{f\in\Psi} \operatorname{dom} f, \quad h(x) = f(x) \text{ whenever } f \in \Psi,\, x \in \operatorname{dom} f,$$

and the usual attack is to seek to prove that any such h belongs to Φ.

I find that at least once I wish to use Zorn's Lemma 'upside down': that is, I have a non-empty partially ordered set P in which every non-empty totally ordered subset has a *lower* bound. In this case, of course, P has a *minimal* element. The point is that the definition of 'partial order' is symmetric, so that $(P, \ge)$ is a partially ordered set whenever $(P, \le)$ is; and we can seek to apply Zorn's Lemma to either.

3A1H Natural numbers and finite ordinals I remarked in 2A1De that the first few ordinals

$$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \quad \ldots$$

may be identified with the natural numbers $0, 1, 2, 3, \ldots$; the idea being that $n = \{0, 1, \ldots, n-1\}$ is a set with n elements. If we do this, then the set $\mathbb{N}$ of natural numbers becomes identified with the first infinite ordinal ω. This convention makes it possible to present a number of arguments in a particularly elegant form. A typical example is in 344H. There I wish to describe an inductive construction for a family $\langle K_\sigma\rangle_{\sigma\in S^*}$ where $S^* = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$. If we think of n as the set of its predecessors, then $\sigma \in \{0,1\}^n$ becomes a function from n to $\{0,1\}$; since the set n has just n members, this corresponds well to the idea of σ as the list of its n coordinates, except that it would now be natural to list them as $\sigma(0), \ldots, \sigma(n-1)$ rather than as $x_1, \ldots, x_n$, which was the language I favoured in Volume 2. An extension of σ to a member of $\{0,1\}^{n+1}$ is of the form $\tau = \sigma^\frown{<}i{>}$ where $\tau(k) = \sigma(k)$ for $k < n$ ('$\tau\restriction n = \sigma$') and $\tau(n) = i$. If $w \in \{0,1\}^{\mathbb{N}}$, then we can identify the initial segment $(w(0), w(1), \ldots, w(n-1))$ of its first n coordinates with the restriction $w\restriction n$ of w to the set $n = \{0, \ldots, n-1\}$.

3A1I Definitions (a) If P and Q are lattices (2A1Ad), a **lattice homomorphism** from P to Q is a function $f:P\to Q$ such that $f(p\wedge p')=f(p)\wedge f(p')$ and $f(p\vee p')=f(p)\vee f(p')$ for all p, $p'\in P$. Such a homomorphism is surely order-preserving (313H), for if $p\le p'$ in P then $f(p')=f(p\vee p')=f(p)\vee f(p')$ and $f(p)\le f(p')$.

(b) If P is a lattice, a **sublattice** of P is a set $Q\subseteq P$ such that $p\vee q$ and $p\wedge q$ belong to Q for all p, $q\in Q$.

(c)(i) A lattice P is **distributive** if

$$(p\wedge q)\vee r=(p\vee r)\wedge(q\vee r),\quad (p\vee q)\wedge r=(p\wedge r)\vee(q\wedge r)$$

for all p, q, $r\in P$.

(ii) In a distributive lattice we have a **median function** of three variables

$$\begin{aligned}\operatorname{med}(p,q,r)&=(p\wedge q)\vee(p\wedge r)\vee(q\wedge r)\\&=((p\wedge(q\vee r))\vee(q\wedge r)=(p\vee q)\wedge(p\vee r)\wedge(q\vee r).\end{aligned}$$

If P and Q are distributive lattices and $f:P\to Q$ is a lattice homomorphism, $f(\operatorname{med}(p,q,r))=\operatorname{med}(f(p),f(q),f(r))$ for all p, q, $r\in P$.

(iii) If P is a distributive lattice and $I\subseteq P$ is finite, then the sublattice of P generated by I is finite. **P** If $J=\{\sup I_0:\emptyset\ne I_0\subseteq I\}$ and $K=\{\inf J_0:\emptyset\ne J_0\subseteq J\}$ then K is a sublattice of P. **Q**

3A1J Subsets of given size The following concepts are used often enough for a special notation to be helpful. If X is a set and κ is a cardinal, write

$$[X]^\kappa=\{A:A\subseteq X,\ \#(A)=\kappa\},$$

$$[X]^{\le\kappa}=\{A:A\subseteq X,\ \#(A)\le\kappa\},$$

$$[X]^{<\kappa}=\{A:A\subseteq X,\ \#(A)<\kappa\}.$$

Thus

$$[X]^0=[X]^{\le 0}=[X]^{<1}=\{\emptyset\},$$

$[X]^2$ is the set of doubleton subsets of X, $[X]^{<\omega}$ is the set of finite subsets of X, $[X]^{\le\omega}$ is the set of countable subsets of X, and so on.

3A1K The next result is one of the fundamental theorems of combinatorics. In this volume it is used in the proofs of Ornstein's theorem (§387) and the Kalton-Roberts theorem (§392).

Hall's Marriage Lemma Suppose that X and Y are finite sets and $R\subseteq X\times Y$ is a relation such that $\#(R[I])\ge\#(I)$ for every $I\subseteq X$. Then there is an injective function $f:X\to Y$ such that $(x,f(x))\in R$ for every $x\in X$.

Remark Recall that $R[I]$ is the set $\{y:\exists\,x\in I,\ (x,y)\in R\}$ (1A1Bc). If we identify a function with its graph, then '$(x,f(x))\in R$ for every $x\in X$' becomes '$f\subseteq R$'.

proof Bollobás 79, p. 54, Theorem 7; Anderson 87, 2.2.1; Bose & Manvel 84, §10.2.

3A2 Rings

I give a very brief outline of the indispensable parts of the elementary theory of (commutative) rings. I assume that you have seen at least a little group theory.

3A2A Definition A **ring** is a triple $(R,+,.)$ such that

$(R,+)$ is an abelian group; its identity will always be denoted 0 or 0_R;

$(R,.)$ is a semigroup, that is, $ab\in R$ for all a, $b\in R$ and $a(bc)=(ab)c$ for all a, b, $c\in R$;

$a(b+c)=ab+ac$, $(a+b)c=ac+bc$ for all a, b, $c\in R$.

A **commutative ring** is one in which multiplication is commutative, that is, $ab=ba$ for all a, $b\in R$.

3A2B Elementary facts Let R be a ring.

(a) $a0 = 0a = 0$ for every $a \in R$. **P**

$$a0 = a(0+0) = a0 + a0, \quad 0a = (0+0)a = 0a + 0a;$$

because $(R, +)$ is a group, we may subtract $a0$ or $0a$ from each side of the appropriate equation to see that $0 = a0$, $0 = 0a$. **Q**

(b) $(-a)b = a(-b) = -(ab)$ for all a, $b \in R$. **P**

$$ab + ((-a)b) = (a + (-a))b = 0b = 0 = a0 = a(b + (-b)) = ab + a(-b);$$

subtracting ab from each term, we get $(-a)b = -(ab) = a(-b)$. **Q**

3A2C Subrings If R is a ring, a **subring** of R is a set $S \subseteq R$ such that $0 \in S$ and $a + b$, ab, $-a$ belong to S for all a, $b \in S$. In this case S, together with the addition and multiplication induced by those of R, is a ring in its own right.

3A2D Homomorphisms (a) Let R, S be two rings. A function $\phi : R \to S$ is a **ring homomorphism** if $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all a, $b \in R$. The **kernel** of ϕ is $\{a : a \in R, \phi(a) = 0_S\}$.

(b) Note that if $\phi : R \to S$ is a ring homomorphism, then it is also a group homomorphism from $(R, +)$ to $(S, +)$, so that $\phi(0_R) = 0_S$ and $\phi(-a) = -\phi(a)$ for every $a \in R$; moreover, $\phi[R]$ is a subring of S, and ϕ is injective iff its kernel is $\{0_R\}$.

(c) If R, S and T are rings, and $\phi : R \to S$, $\psi : S \to T$ are ring homomorphisms, then $\psi\phi : R \to T$ is a ring homomorphism, because

$$(\psi\phi)(a * b) = \psi(\phi(a * b)) = \psi(\phi(a) * \phi(b)) = \psi(\phi(a)) * \psi(\phi(b))$$

for all a, $b \in R$, taking $*$ to be either addition or multiplication. If ϕ is bijective, then $\phi^{-1} : S \to R$ is a ring homomorphism, because

$$\phi^{-1}(c * d) = \phi^{-1}(\phi(\phi^{-1}(c)) * \phi(\phi^{-1}(d))) = \phi^{-1}\phi(\phi^{-1}(c) * \phi^{-1}(d)) = \phi^{-1}(c) * \phi^{-1}(d)$$

for all c, $d \in S$, again taking $*$ to be either addition or multiplication.

3A2E Ideals (a) Let R be a ring. An **ideal** of R is a subring I of R such that $ab \in I$ and $ba \in I$ whenever $a \in I$ and $b \in R$. In this case we write $I \lhd R$.

Note that R and $\{0\}$ are always ideals of R.

(b) If R and S are rings and $\phi : R \to S$ is a ring homomorphism, then the kernel I of ϕ is an ideal of R. **P** (i) Because ϕ is a group homomorphism, I is a subgroup of $(R, +)$. (ii) If $a \in I$, $b \in R$ then

$$\phi(ab) = \phi(a)\phi(b) = 0_S\phi(b) = 0_S, \quad \phi(ba) = \phi(b)\phi(a) = \phi(b)0_S = 0_S$$

so ab, $ba \in I$. **Q**

3A2F Quotient rings (a) Let R be a ring and I an ideal of R. A **coset** of I is a set of the form $a + I = \{a + x : x \in I\}$ where $a \in R$. (Because $+$ is commutative, we do not need to distinguish between 'left cosets' $a + I$ and 'right cosets' $I + a$.) Let R/I be the set of cosets of I in R.

(b) For A, $B \in R/I$, set

$$A + B = \{x + y : x \in A, y \in B\}, \quad A \cdot B = \{xy + z : x \in A, y \in B, z \in I\}.$$

Then $A + B$, $A \cdot B$ both belong to R/I; moreover, if $A = a + I$ and $B = b + I$, then $A + B = (a + b) + I$ and $A \cdot B = ab + I$. **P** (i)

$$\begin{aligned} A + B &= (a + I) + (b + I) \\ &= \{(a + x) + (b + y) : x, y \in I\} \\ &= \{(a + b) + (x + y) : x, y \in I\} \end{aligned}$$

(because addition is associative and commutative)

$$\subseteq \{(a+b)+z : z \in I\} = (a+b)+I$$

(because $I+I \subseteq I$)

$$= \{(a+b)+(z+0) : z \in I\}$$
$$\subseteq (a+I)+(b+I) = A+B$$

because $0 \in I$. (ii)

$$\begin{aligned} A \cdot B &= \{(a+x)(b+y)+z : x,\, y,\, z \in I\} \\ &= \{ab+(ay+xb+z) : x,\, y,\, z \in I\} \\ &\subseteq \{ab+w : w \in I\} = ab+I \end{aligned}$$

(because ay, $xb \in I$ for all x, $y \in I$, and I is closed under addition)

$$\begin{aligned} &= \{(a+0)(b+0)+w : w \in I\} \\ &\subseteq A \cdot B. \textbf{ Q} \end{aligned}$$

(c) It is now an elementary exercise to check that $(R/I, +, \cdot)$ is a ring, with zero $0+I = I$ and additive inverses $-(a+I) = (-a)+I$.

(d) Moreover, the map $a \mapsto a+I : R \to R/I$ is a ring homomorphism.

(e) Note that for a, $b \in R$, the following are equiveridical: (i) $a \in b+I$; (ii) $b \in a+I$; (iii) $(a+I)\cap(b+I) \neq \emptyset$; (iv) $a+I = b+I$; (v) $a-b \in I$. Thus the cosets of I are just the equivalence classes in R under the equivalence relation $a \sim b \iff a+I = b+I$; accordingly I shall generally write $a^\bullet$ for $a+I$, if there seems no room for confusion. In particular, the kernel of the canonical map from R to R/I is just $\{a : a+I = I\} = I = 0^\bullet$.

(f) If R is commutative so is R/I, since

$$a^\bullet b^\bullet = (ab)^\bullet = (ba)^\bullet = b^\bullet a^\bullet$$

for all a, $b \in R$.

3A2G Factoring homomorphisms through quotient rings: Proposition Let R and S be rings, I an ideal of R, and $\phi : R \to S$ a homomorphism such that I is included in the kernel of ϕ. Then we have a ring homomorphism $\pi : R/I \to S$ such that $\pi(a^\bullet) = \phi(a)$ for every $a \in R$. π is injective iff I is precisely the kernel of ϕ.

proof If a, $b \in R$ and $a^\bullet = b^\bullet$ in R/I, then $a-b \in I$ (3A2Fe), so $\phi(a)-\phi(b) = \phi(a-b) = 0$, and $\phi(a) = \phi(b)$. This means that the formula offered does indeed define a function π from R/I to S. Now if a, $b \in R$ and $*$ is either multiplication or addition,

$$\pi(a^\bullet * b^\bullet) = \pi((a*b)^\bullet) = \phi(a*b) = \phi(a)*\phi(b) = \pi(a^\bullet)*\pi(b^\bullet).$$

So π is a ring homomorphism.

The kernel of π is $\{a^\bullet : \phi(a) = 0\}$, which is $\{0\}$ iff $\phi(a) = 0 \iff a^\bullet = 0 \iff a \in I$.

3A2H Product rings (a) Let $\langle R_i \rangle_{i \in I}$ be any family of rings. Set $R = \prod_{i \in I} R_i$ and for a, $b \in R$ define $a+b$, $ab \in R$ by setting

$$(a+b)(i) = a(i)+b(i), \quad (ab)(i) = a(i)b(i)$$

for every $i \in I$. It is easy to check from the definition in 3A2A that R is a ring; its zero is given by the formula

$$0_R(i) = 0_{R_i} \text{ for every } i \in I,$$

and its additive inverses by the formula

$$(-a)(i) = -a(i) \text{ for every } i \in I.$$

(b) Now let S be any other ring. Then it is easy to see that a function $\phi : S \to R$ is a ring homomorphism iff $s \mapsto \phi(s)(i) : S \to R_i$ is a ring homomorphism for every $i \in I$.

(c) Note that R is commutative iff R_i is commutative for every i.

3A3 General topology

In §2A3, I looked at a selection of topics in general topology in some detail, giving proofs; the point was that an ordinary elementary course in the subject would surely go far beyond what we needed there, and at the same time might omit some of the results I wished to quote. It seemed therefore worth taking a bit of space to cover the requisite material, giving readers the option of delaying a proper study of the subject until a convenient opportunity arose. In the context of the present volume, this approach is probably no longer appropriate, since we need a much greater proportion of the fundamental ideas, and by the time you have reached familiarity with the topics here you will be well able to find your way about one of the many excellent textbooks on the subject. This time round, therefore, I give most of the results without proofs (as in §§2A1 and 3A1), hoping that some of the references I offer will be accessible in all senses. I do, however, give a full set of definitions, partly to avoid ambiguity (since even in this relatively mature subject, there are some awkward divergences remaining in the usage of different authors), and partly because many of the proofs are easy enough for even a novice to fill in with a bit of thought, once the meaning of the words is clear. In fact this happens so often that I will mark with a $*$ those points where a proof needs an idea not implicit in the preceding work.

3A3A Taxonomy of topological spaces I begin with the handful of definitions we need in order to classify the different types of topological space used in this volume. A couple have already been introduced in Volume 2, but I repeat them because the list would look so odd without them.

Definitions Let $(X, \mathfrak{T})$ be a topological space.

(a) X is $\mathbf{T_1}$ if singleton subsets of X are closed.

(b) X is **Hausdorff** if for any distinct points x, $y \in X$ there are disjoint open sets G, $H \subseteq X$ such that $x \in G$ and $y \in H$.

(c) X is **regular** if whenever $F \subseteq X$ is closed and $x \in X \setminus F$ there are disjoint open sets G, $H \subseteq X$ such that $x \in G$ and $F \subseteq H$. (Note that in this definition I do not require X to be Hausdorff, following JAMES 87 but not ENGELKING 89, BOURBAKI 66, DUGUNDJI 66, SCHUBERT 68 or GAAL 64.)

(d) X is **completely regular** if whenever $F \subseteq X$ is closed and $x \in X \setminus F$ there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0$ for every $y \in F$. (Note that many authors restrict the phrase 'completely regular' to Hausdorff spaces.)

(e) X is **zero-dimensional** if whenever $G \subseteq X$ is an open set and $x \in G$ then there is an open-and-closed set H such that $x \in H \subseteq G$.

(f) X is **extremally disconnected** if the closure of every open set in X is open.

(g) X is **compact** if every open cover of X has a finite subcover.

(h) X is **locally compact** if for every $x \in X$ there is a set $K \subseteq X$ such that $x \in \operatorname{int} K$ and K is compact (in its subspace topology, as defined in 2A3C).

(i) If every subset of X is open, we call $\mathfrak{T}$ the **discrete topology** on X.

3A3B Elementary relationships (a) A completely regular space is regular. (ENGELKING 89, p. 39; DUGUNDJI 66, p. 154; SCHUBERT 68, p. 104.)

(b) A locally compact Hausdorff space is completely regular, therefore regular. $*$ (ENGELKING 89, 3.3.1; DUGUNDJI 66, p. 238; GAAL 64, p. 149.)

(c) A compact Hausdorff space is locally compact, therefore completely regular and regular.

(d) A regular extremally disconnected space is zero-dimensional. (Engelking 89, 6.2.25.)

(e) Any topology defined by pseudometrics (2A3F), in particular the weak topology of a normed space (2A5I), is completely regular, therefore regular. (Bourbaki 66, IX.1.5; Dugundji 66, p. 200.)

(f) If X is a completely regular Hausdorff space (in particular, if X is (locally) compact and Hausdorff), and x, y are distinct points in X, then there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) \ne f(y)$. (Apply 3A3Ad with $F = \{y\}$, which is closed because X is Hausdorff.)

(g) An open set in a locally compact Hausdorff space is locally compact in its subspace topology. (Engelking 89, 3.3.8; Bourbaki 66, I.9.7.)

3A3C Continuous functions Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological spaces.

(a) If $f : X \to Y$ is a function and $x \in X$, we say that f is **continuous at** x if $x \in \operatorname{int} f^{-1}[H]$ whenever $H \subseteq Y$ is an open set containing $f(x)$.

(b) Now a function from X to Y is continuous iff it is continuous at every point of X. (Bourbaki 66, I.2.1; Dugundji 66, p. 80; Schubert 68, p. 24; Gaal 64, p. 183; James 87, p. 26.)

(c) If $f : X \to Y$ is continuous at $x \in X$, and $A \subseteq X$ is such that $x \in \overline{A}$, then $f(x) \in \overline{f[A]}$. (Bourbaki 66, I.2.1; Schubert 68, p. 23.)

(d) If $f : X \to Y$ is continuous, then $f[\overline{A}] \subseteq \overline{f[A]}$ for every $A \subseteq X$. (Engelking 89, 1.4.1; Bourbaki 66, I.2.1; Dugundji 66, p. 80; Schubert 68, p. 24; Gaal 64, p. 184; James 87, p. 27.)

(e) A function $f : X \to Y$ is a **homeomorphism** if it is a continuous bijection and its inverse is also continuous; that is, if $\mathfrak{S} = \{f[G] : G \in \mathfrak{T}\}$ and $\mathfrak{T} = \{f^{-1}[H] : H \in \mathfrak{S}\}$.

(f) A function $f : X \to [-\infty, \infty]$ is **lower semi-continuous** if $\{x : x \in X,\ f(x) > \alpha\}$ is open for every $\alpha \in \mathbb{R}$. (Cf. 225H.)

3A3D Compact spaces Any extended series of applications of general topology is likely to involve some new features of compactness. I start with the easy bits, continuing from 2A3Nb.

(a) The first is just a definition of compactness in terms of closed sets instead of open sets. A family $\mathcal{F}$ of sets has the **finite intersection property** if $\bigcap \mathcal{F}_0$ is non-empty for every finite $\mathcal{F}_0 \subseteq \mathcal{F}$. Now a topological space X is compact iff $\bigcap \mathcal{F} \ne \emptyset$ whenever $\mathcal{F}$ is a family of closed subsets of X with the finite intersection property. (Engelking 89, 3.1.1; Bourbaki 66, I.9.1; Dugundji 66, p., 223; Schubert 68, p. 68; Gaal 64, p. 127.)

(b) A marginal generalization of this is the following. Let X be a topological space and $\mathcal{F}$ a family of closed subsets of X with the finite intersection property. If $\mathcal{F}$ contains a compact set then $\bigcap \mathcal{F} \ne \emptyset$. (Apply (a) to $\{K \cap F : F \in \mathcal{F}\}$ where $K \in \mathcal{F}$ is compact.)

(c) In a Hausdorff space, compact subsets are closed. (Engelking 89, 3.1.8; Bourbaki 66, I.9.4; Dugundji 66, p. 226; Schubert 68, p. 70; Gaal 64, p. 138; James 87, p. 77.)

(d) If X is compact, Y is Hausdorff and $\phi : X \to Y$ is continuous and injective, then ϕ is a homeomorphism between X and $\phi[X]$ (where $\phi[X]$ is given the subspace topology). (Engelking 89, 3.1.12; Bourbaki 66, I.9.4; Dugundji 66, p. 226; Schubert 68, p. 71; Gaal 64, p. 207.)

(e) Let X be a regular topological space and A a subset of X. Then the following are equiveridical: (i) A is relatively compact in X (that is, A is included in some compact subset of X, as in 2A3Na); (ii) $\overline{A}$ is compact; (iii) every ultrafilter on X which contains A has a limit in X. **P**(ii)⇒(i) is trivial, and (i)⇒(iii) is a consequence of 2A3R; neither of these requires X to be regular. Now assume (iii) and let $\mathcal{F}$ be an ultrafilter on X containing $\overline{A}$. Set

$$\mathcal{H} = \{B : B \subseteq X, \text{ there is an open set } G \in \mathcal{F} \text{ such that } A \cap G \subseteq B\}.$$

Then $\mathcal{H}$ does not contain $\emptyset$ and $B_1 \cap B_2 \in \mathcal{H}$ whenever B_1, $B_2 \in \mathcal{H}$, so $\mathcal{H}$ is a filter on X, and it contains A. Let $\mathcal{H}^* \supseteq \mathcal{H}$ be an ultrafilter (2A1O). By hypothesis, $\mathcal{H}^*$ has a limit x say. Because $A \in \mathcal{H}^*$, $X \setminus \overline{A}$ is an open set not belonging to $\mathcal{H}^*$, and cannot be a neighbourhood of x; thus x must belong to $\overline{A}$. Let G be an open set containing x. Then there is an open set H such that $x \in H \subseteq \overline{H} \subseteq G$ (this is where I use the hypothesis that X is regular). Because $\mathcal{H}^* \to x$, $H \in \mathcal{H}^*$ so $X \setminus \overline{H}$ does not belong to $\mathcal{H}^*$ and therefore does not belong to $\mathcal{H}$. But $X \setminus \overline{H}$ is open, so by the definition of $\mathcal{H}$ it cannot belong to $\mathcal{F}$. As $\mathcal{F}$ is an ultrafilter, $\overline{H} \in \mathcal{F}$ and $G \in \mathcal{F}$. As G is arbitrary, $\mathcal{F} \to x$. As $\mathcal{F}$ is arbitrary, $\overline{A}$ is compact (2A3R). Thus (iii)$\Rightarrow$(ii). **Q**

3A3E Dense sets Recall that a set D in a topological space X is **dense** if $\overline{D} = X$, and that X is **separable** if it has a countable dense subset (2A3Ud).

(a) If X is a topological space, $D \subseteq X$ is dense and $G \subseteq X$ is dense and open, then $G \cap D$ is dense. (Engelking 89, 1.3.6.) Consequently the intersection of finitely many dense open sets is always dense.

(b) If X and Y are topological spaces, $D \subseteq A \subseteq X$, D is dense in A and $f : X \to Y$ is a continuous function, then $f[D]$ is dense in $f[A]$. (Use 3A3Cd.)

3A3F Meager sets Let X be a topological space.

(a) A set $A \subseteq X$ is **nowhere dense** if $\operatorname{int} \overline{A} = \emptyset$, that is, $\operatorname{int}(X \setminus A) = X \setminus \overline{A}$ is dense, that is, for every non-empty open set G there is a non-empty open set $H \subseteq G \setminus A$.

(b) A set $M \subseteq X$ is **meager** if it is expressible as the union of a sequence of nowhere dense sets. A subset of X is **comeager** if its complement is meager.

(c) Any subset of a nowhere dense set is nowhere dense; the union of finitely many nowhere dense sets is nowhere dense. (3A3Ea.)

(d) Any subset of a meager set is meager; the union of countably many meager sets is meager. (314L.)

3A3G Baire's theorem for locally compact Hausdorff spaces Let X be a locally compact Hausdorff space and $\langle G_n \rangle_{n \in \mathbb{N}}$ a sequence of dense open subsets of X. Then $\bigcap_{n \in \mathbb{N}} G_n$ is dense. $*$ (Engelking 89, 3.9.4; Bourbaki 66, IX.5.3; Dugundji 66, p. 249; Schubert 68, p. 148.) Consequently every comeager subset of X is dense.

3A3H Corollary (a) Let X be a compact Hausdorff space. Then a non-empty open subset of X cannot be meager. (Dugundji 66, p. 250; Schubert 68, p. 147.)

(b) Let X be a non-empty locally compact Hausdorff space. If $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of sets covering X, then there is some $n \in \mathbb{N}$ such that $\operatorname{int} \overline{A}_n$ is non-empty. (Dugundji 66, p. 250.)

3A3I Product spaces (a) Definition Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces, and $X = \prod_{i \in I} X_i$ their Cartesian product. We say that a set $G \subseteq X$ is open for the **product topology** if for every $x \in G$ there are a finite $J \subseteq I$ and a family $\langle G_j \rangle_{j \in J}$ such that every G_j is an open set in the corresponding X_j and

$$\{y : y \in X,\ y(j) \in G_j \text{ for every } j \in J\}$$

contains x and is included in G.

(Of course we must check that this does indeed define a topology; see Engelking 89, 2.3.1; Bourbaki 66, I.4.1; Schubert 68, p. 38; Gaal 64, p. 144.)

(b) If $\langle X_i \rangle_{i \in I}$ is a family of topological spaces, with product X, and Y another topological space, a function $\phi : Y \to X$ is continuous iff $\pi_i \phi$ is continuous for every $i \in I$, where $\pi_i(x) = x(i)$ for $x \in X$ and $i \in I$. (Engelking 89, 2.3.6; Bourbaki 66, I.4.1; Dugundji 66, p. 101; Schubert 68, p. 62; James 87, p. 31.)

(c) Let $\langle X_i \rangle_{i \in I}$ be any family of non-empty topological spaces, with product X. If $\mathcal{F}$ is a filter on X and $x \in X$, then $\mathcal{F} \to x$ iff $\pi_i[[\mathcal{F}]] \to x(i)$ for every i, where $\pi_i(y) = y(i)$ for $y \in X$, and $\pi_i[[\mathcal{F}]]$ is the image filter on X_i (2A1Ib). (Bourbaki 66, I.7.6; Schubert 68, p. 61; James 87, p. 32.

(d) The product of any family of Hausdorff spaces is Hausdorff. (Engelking 89, 2.3.11; Bourbaki 66, I.8.2; Schubert 68, p. 62; James 87, p. 87.)

(e) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces. If D_i is a dense subset of X_i for each i, then $\prod_{i\in I} D_i$ is dense in $\prod_{i\in I} X_i$. (Engelking 89, 2.3.5.).

(f) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces. If F_i is a closed subset of X_i for each i, then $\prod_{i\in I} F_i$ is closed in $\prod_{i\in I} X_i$. (Engelking 89, 2.3.4; Bourbaki 66, I.4.3.)

(g) Let $\langle(X_i,\mathfrak{T}_i)\rangle_{i\in I}$ be a family of topological spaces with product $(X,\mathfrak{T})$. Suppose that each $\mathfrak{T}_i$ is defined by a family P_i of pseudometrics on X_i (2A3F). Then $\mathfrak{T}$ is defined by the family $\mathrm{P}=\{\tilde\rho_i : i\in I,\rho\in\mathrm{P}_i\}$ of pseudometrics on X, where I write $\tilde\rho_i(x,y)=\rho(\pi_i(x),\pi_i(y))$ whenever $i\in I$, $\rho\in\mathrm{P}_i$ and x, $y\in X$, taking π_i to be the coordinate map from X to X_i, as in (b)-(c). **P** (Compare 2A3Tb). (i) It is easy to check that every $\tilde\rho_i$ is a pseudometric on X. Write $\mathfrak{T}_{\mathrm{P}}$ for the topology generated by P. (ii) If $x\in G\in\mathfrak{T}_{\mathrm{P}}$, let $\mathrm{P}'\subseteq\mathrm{P}$ and $\delta>0$ be such that P' is finite and $\{y:\tau(y,x)\le\delta$ for every $\tau\in\mathrm{P}'\}$ is included in G. Express P' as $\{\tilde\rho_j : j\in J,\,\rho\in\mathrm{P}'_j\}$ where $J\subseteq I$ is finite and $\mathrm{P}'_j\subseteq\mathrm{P}_j$ is finite for each $j\in J$. Set

$$G_j=\{t:t\in X_j,\ \rho(t,\pi_j(x))<\delta \text{ for every } \rho\in\mathrm{P}'_j\}$$

for every $j\in J$. Then $G'=\{y:\pi_j(y)\in G_j$ for every $j\in J\}$ contains x, belongs to $\mathfrak{T}$ and is included in G. As x is arbitrary, $G\in\mathfrak{T}$; as G is arbitrary, $\mathfrak{T}_{\mathrm{P}}\subseteq\mathfrak{T}$. (iii) Every π_i is $(\mathfrak{T}_{\mathrm{P}},\mathfrak{T}_i)$-continuous, by 2A3H; by (b) above, the identity map from X to itself is $(\mathfrak{T}_{\mathrm{P}},\mathfrak{T})$-continuous, that is, $\mathfrak{T}_{\mathrm{P}}\subseteq\mathfrak{T}$ and $\mathfrak{T}_{\mathrm{P}}=\mathfrak{T}$, as claimed. **Q**

(h) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces with product X, and Y another topological space. Then a function $f:X\to Y$ is **separately continuous** if for every $j\in I$ and $z\in\prod_{i\in I\setminus\{j\}} X_i$ the function $t\mapsto f(z^\frown{<}t{>}):X_j\to Y$ is continuous, where $z^\frown{<}t{>}$ is the member of X extending z and such that $(z^\frown{<}t{>})(j)=t$.

3A3J Tychonoff's theorem The product of any family of compact topological spaces is compact.

proof Engelking 89, 3.2.4; Bourbaki 66, I.9.5; Dugundji 66, p. 224; Schubert 68, p. 72; Gaal 64, p. 146 and p. 272; James 87, p. 67.

3A3K The spaces $\{0,1\}^I$, $\mathbb{R}^I$ For any set I, we can think of $\{0,1\}^I$ as the product $\prod_{i\in I} X_i$ where $X_i=\{0,1\}$ for each i. If we endow each X_i with its discrete topology, the product topology is the **usual topology** on $\{0,1\}^I$. Being a product of Hausdorff spaces, it is Hausdorff; by Tychonoff's theorem, it is compact. A subset G of $\{0,1\}^I$ is open iff for every $x\in G$ there is a finite $J\subseteq I$ such that $\{y:y\in\{0,1\}^I,\ y{\restriction}J=x{\restriction}J\}\subseteq G$.

Similarly, the 'usual topology' of $\mathbb{R}^I$ is the product topology when each factor is given its Euclidean topology (cf. 2A3Tc).

3A3L Cluster points of filters (a) Let X be a topological space and $\mathcal{F}$ a filter on X. A point x of X is a **cluster point** of $\mathcal{F}$ if $x\in\overline{A}$ for every $A\in\mathcal{F}$.

(b) For any topological space X, filter $\mathcal{F}$ on X and $x\in X$, x is a cluster point of $\mathcal{F}$ iff there is a filter $\mathcal{G}\supseteq\mathcal{F}$ such that $\mathcal{G}\to x$. (Engelking 89, 1.6.8; Bourbaki 66, I.7.2; Gaal 64, p. 260; James 87, p. 22.)

(c) If $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$, $\alpha\in\mathbb{R}$ and $\lim_{n\to\mathcal{H}}\alpha_n=\alpha$ for every non-principal ultrafilter $\mathcal{H}$ on $\mathbb{N}$ (definition: 2A3Sb), then $\lim_{n\to\infty}\alpha_n=\alpha$. **P?** If $\langle\alpha_n\rangle_{n\in\mathbb{N}}\not\to\alpha$, there is some $\epsilon>0$ such that $I=\{n:|\alpha_n-\alpha|\ge\epsilon\}$ is infinite. Now $\mathcal{F}_0=\{F:F\subseteq\mathbb{N},\,I\setminus F$ is finite$\}$ is a filter on $\mathbb{N}$, so there is an ultrafilter $\mathcal{F}\supseteq\mathcal{F}_0$. But now α cannot be $\lim_{n\to\mathcal{F}}\alpha_n$. **XQ**

3A3M Topology bases (a) If X is a set and $\mathbb{T}$ is any non-empty family of topologies on X, $\bigcap\mathbb{T}$ is a topology on X. So if $\mathcal{A}$ is any family of subsets of X, the intersection of all the topologies on X including $\mathcal{A}$ is a topology on X; this is the **topology generated by** $\mathcal{A}$.

(b) If X is a set and $\mathfrak{T}$ is a topology on X, a **base** for $\mathfrak{T}$ is a set $\mathcal{U}\subseteq\mathfrak{T}$ such that whenever $x\in G\in\mathfrak{T}$ there is a $U\in\mathcal{U}$ such that $x\in U\subseteq G$; that is, such that $\mathfrak{T}=\{\bigcup\mathcal{G}:\mathcal{G}\subseteq\mathcal{U}\}$. In this case, of course, $\mathcal{U}$ generates $\mathfrak{T}$.

(c) If X is a set and $\mathcal{E}$ is a family of subsets of X, then $\mathcal{E}$ is a base for a topology on X iff (i) whenever E_1, $E_2 \in \mathcal{E}$ and $x \in E_1 \cap E_2$ then there is an $E \in \mathcal{E}$ such that $x \in E \subseteq E_1 \cap E_2$ (ii) $\bigcup \mathcal{E} = X$. (Engelking 89, p. 12.)

3A3N Uniform convergence (a) Let X be a set, (Y, ρ) a metric space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of functions from X to Y. We say that $\langle f_n \rangle_{n\in\mathbb{N}}$ converges **uniformly** to a function $f : X \to Y$ if for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $\rho(f_n(x), f(x)) \le \epsilon$ whenever $n \ge n_0$ and $x \in X$.

(b) Let X be a topological space and (Y, ρ) a metric space. Suppose that $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence of continuous functions from X to Y converging uniformly to $f : X \to Y$. Then f is continuous. $*$ (Engelking 89, 1.4.7/4.2.19; Gaal 64, p. 202.)

3A3O One-point compactifications Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space. Take any object x_∞ not belonging to X and set $X^* = X \cup \{x_\infty\}$. Let $\mathfrak{T}^*$ be the family of those sets $H \subseteq X^*$ such that $H \cap X \in \mathfrak{T}$ and either $x_\infty \notin H$ or $X \setminus H$ is compact (for $\mathfrak{T}$). Then $\mathfrak{T}^*$ is the unique compact Hausdorff topology on X^* inducing $\mathfrak{T}$ as the subspace topology on X; $(X^*, \mathfrak{T}^*)$ is the **one-point compactification** of $(X, \mathfrak{T})$. (Engelking 89, 3.5.11; Bourbaki 66, I.9.8; Dugundji 66, p. 246.)

3A3P Topologies defined from a sequential convergence: Proposition (a) Let X be a set and $\to^*$ a relation between $X^{\mathbb{N}}$ and X such that whenever $\langle x_n \rangle_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, $x \in X$, $\langle x_n \rangle_{n\in\mathbb{N}} \to^* x$ and $\langle x'_n \rangle_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ is a subsequence of $\langle x_n \rangle_{n\in\mathbb{N}}$ then $\langle x'_n \rangle_{n\in\mathbb{N}} \to^* x$. Then there is a unique topology on X for which a set $F \subseteq X$ is closed iff $x \in F$ whenever $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in F and $\langle x_n \rangle_{n\in\mathbb{N}} \to^* x$. Moreover, if $\langle x_n \rangle_{n\in\mathbb{N}} \to^* x$ then $\langle x_n \rangle_{n\in\mathbb{N}}$ converges to x for this topology.

(b) Let X and Y be sets, and suppose that $\to^*_X \subseteq X^{\mathbb{N}} \times X$, $\to^*_Y \subseteq Y^{\mathbb{N}} \times Y$ are relations with the subsequence property described in (a). Give X and Y the corresponding topologies. If $f : X \to Y$ is a function such that $\langle f(x_n) \rangle_{n\in\mathbb{N}} \to^*_Y f(x)$ whenever $\langle x_n \rangle_{n\in\mathbb{N}} \to^*_X x$, then f is continuous.

proof (a)(i) Let $\mathcal{F}$ be the family of those $F \subseteq X$ which are closed under *-convergence, that is, such that $x \in F$ whenever $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in F and $\langle x_n \rangle_{n\in\mathbb{N}} \to^* x$. Of course $\emptyset$ and X belong to $\mathcal{F}$; also the intersection of any non-empty subset of $\mathcal{F}$ belongs to $\mathcal{F}$. The point is that the union of two members of $\mathcal{F}$ belongs to $\mathcal{F}$. **P** Suppose that F_1, $F_2 \in \mathcal{F}$ and that $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in $F_1 \cup F_2$ *-converging to x. Then there is a subsequence $\langle x'_n \rangle_{n\in\mathbb{N}}$ of $\langle x_n \rangle_{n\in\mathbb{N}}$ which lies entirely in one of the sets; say $x'_n \in F_j$ for every n. By hypothesis, $\langle x'_n \rangle_{n\in\mathbb{N}} \to^* x$, so $x \in F_j \subseteq F_1 \cup F_2$. As $\langle x_n \rangle_{n\in\mathbb{N}}$ and x are arbitrary, $F_1 \cup F_2 \in \mathcal{F}$. **Q** Taking complements, we see that $\{X \setminus F : F \in \mathcal{F}\}$ is a topology on X for which $\mathcal{F}$ is the family of closed sets; and of course there can be only one such topology.

(ii) Now suppose that $\langle x_n \rangle_{n\in\mathbb{N}} \to^* x$. **?** If $\langle x_n \rangle_{n\in\mathbb{N}}$ does not converge topologically to x, then there is an open set G containing x such that $\{n : x_n \notin G\}$ is infinite, that is, there is a subsequence $\langle x'_n \rangle_{n\in\mathbb{N}}$ of $\langle x_n \rangle_{n\in\mathbb{N}}$ such that $x'_n \notin G$ for every n. Now $\langle x'_n \rangle_{n\in\mathbb{N}} \to^* x$; but $X \setminus G$ is supposed to be closed under *-convergence, so this is impossible. **X**

(b) Let $H \subseteq Y$ be open; set $F = Y \setminus H$ and let $\langle x_n \rangle_{n\in\mathbb{N}}$ be a sequence in $f^{-1}[F]$ such that $\langle x_n \rangle_{n\in\mathbb{N}} \to^*_X x$. Then $\langle f(x_n) \rangle_{n\in\mathbb{N}} \to^*_Y f(x)$, so $f(x) \in F$ and $x \in f^{-1}[F]$, As $\langle x_n \rangle_{n\in\mathbb{N}}$ and x are arbitrary, $f^{-1}[F]$ is closed and $f^{-1}[H]$ is open; as H is arbitrary, f is continuous.

3A3Q Miscellaneous definitions Let X be a topological space.

(a) A subset of X is a **zero set** if it is of the form $f^{-1}[\{0\}]$ for some continuous function $f : X \to \mathbb{R}$. A subset of X is a **cozero set** if its complement is a zero set. A subset of X is a **$\mathbf{G}_\delta$ set** if it is expressible as the intersection of a sequence of open sets.

(b) An **isolated point** of X is a point $x \in X$ such that the singleton set $\{x\}$ is open.

3A4 Uniformities

I continue the work of §3A3 with some notes on uniformities, so as to be able to discuss completeness and the extension of uniformly continuous functions in non-metrizable contexts (3A4F-3A4H). As in §3A3, most of the individual steps are elementary; I mark exceptions with a $*$.

3A4A Uniformities (a) Let X be a set. A **uniformity** on X is a filter $\mathcal{W}$ on $X \times X$ such that

(i) $(x,x) \in W$ for every $x \in X$, $W \in \mathcal{W}$;

(ii) for every $W \in \mathcal{W}$, $W^{-1} = \{(y,x) : (x,y) \in W\} \in \mathcal{W}$;

(iii) for every $W \in \mathcal{W}$, there is a $V \in \mathcal{W}$ such that

$$V \circ V = \{(x,z) : \exists\, y,\, (x,y) \in V \,\&\, (y,z) \in V\} \subseteq W.$$

It is convenient to allow the special case $X = \emptyset$, $\mathcal{W} = \{\emptyset\}$, even though this is not properly speaking a filter.

The pair $(X, \mathcal{W})$ is now a **uniform space**.

(b) If $\mathcal{W}$ is a uniformity on a set X, the associated topology $\mathfrak{T}$ is the set of sets $G \subseteq X$ such that for every $x \in G$ there is a $W \in \mathcal{W}$ such that $W[\{x\}] = \{y : (x,y) \in W\}$ is included in G. (Engelking 89, 8.1.1; Bourbaki 66, II.1.2; Gaal 64, p. 48; Schubert 68, p. 115; James 87, p. 101.)

(c) We say that a uniformity is **Hausdorff** if the associated topology is Hausdorff.

(d) If U is a linear topological space, then it has an associated uniformity

$$\begin{aligned}\mathcal{W} = \{W : W \subseteq U \times U, \text{ there is an open set } G \text{ containing } 0 \\ \text{such that } (u,v) \in W \text{ whenever } u - v \in G\},\end{aligned}$$

and $\mathcal{W}$ defines the topology of U in the sense of (b) above (Schaefer 66, I.1.4).

3A4B Uniformities and pseudometrics (a) If P is a family of pseudometrics on a set X, then the associated uniformity is the smallest uniformity on X containing all the sets $W(\rho;\epsilon) = \{(x,y) : \rho(x,y) < \epsilon\}$ as ρ runs over P, ϵ over $]0,\infty[$. (Engelking 89, 8.1.18; Bourbaki 66, IX.1.2.)

(b) If $\mathcal{W}$ is the uniformity defined by a family P of pseudometrics, then the topology associated with $\mathcal{W}$ is the topology defined from P (2A3F). (Dugundji 66, p. 203.)

(c) A uniformity $\mathcal{W}$ is **metrizable** if it can be defined by a single metric.

(d) If U is a linear space with a topology defined from a family of functionals $\tau : U \to [0,\infty[$ such that $\tau(u+v) \le \tau(u) + \tau(v)$, $\tau(\alpha u) \le \tau(u)$ when $|\alpha| \le 1$, and $\lim_{\alpha\to 0} \tau(\alpha u) = 0$ (2A5B), the uniformity defined from the topology (3A4Ad) coincides with the uniformity defined from the pseudometrics $\rho_\tau(u,v) = \tau(u-v)$. (Immediate from the definitions.)

3A4C Uniform continuity (a) If $(X,\mathcal{W})$ and $(Y,\mathcal{V})$ are uniform spaces, a function $\phi : X \to Y$ is **uniformly continuous** if $\{(x,y) : (\phi(x),\phi(y)) \in V\}$ belongs to $\mathcal{W}$ for every $V \in \mathcal{V}$.

(b) The composition of uniformly continuous functions is uniformly continuous. (Bourbaki 66, II.2.1; Schubert 68, p. 118.)

(c) If uniformities $\mathcal{W}$, $\mathcal{V}$ on sets X, Y are defined by non-empty families P, Θ of pseudometrics, then a function $\phi : X \to Y$ is uniformly continuous iff for every $\theta \in \Theta$, $\epsilon > 0$ there are $\rho_0, \dots, \rho_n \in$ P and $\delta > 0$ such that $\theta(\phi(x),\phi(y)) \le \epsilon$ whenever x, $y \in X$ and $\max_{i\le n} \rho_i(x,y) \le \delta$. (Elementary verification.)

(d) A uniformly continuous function is continuous for the associated topologies. (Bourbaki 66, II.2.1; Schubert 68, p. 118; James 87, p. 102.)

(e) Two metrics ρ, σ on a set X are **uniformly equivalent** if they give rise to the same uniformity; that is, if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\rho(x,y) \le \delta \Rightarrow \sigma(x,y) \le \epsilon, \quad \sigma(x,y) \le \delta \Rightarrow \rho(x,y) \le \epsilon.$$

3A4D Subspaces (a) If $(X,\mathcal{W})$ is a uniform space and Y is any subset of X, then $\mathcal{W}_Y=\{W\cap(Y\times Y):W\in\mathcal{W}\}$ is a uniformity on Y; it is the **subspace uniformity**. (Bourbaki 66, II.2.4; Schubert 68, p. 122.)

(b) If $\mathcal{W}$ defines a topology $\mathfrak{T}$ on X, then the topology defined by $\mathcal{W}_Y$ is the subspace topology on Y, as defined in 2A3C. (Schubert 68, p. 122; James 87, p. 103.)

(c) If $\mathcal{W}$ is defined by a family P of pseudometrics on X, then $\mathcal{W}_Y$ is defined by $\{\rho\restriction Y\times Y:\rho\in\mathrm{P}\}$. (Elementary verification.)

3A4E Product uniformities (a) If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are uniform spaces, the **product uniformity** is the smallest uniformity $\mathcal{W}$ on $X\times Y$ containing all sets of the form

$$\{((x,y),(x',y')):(x,x')\in U,\ (y,y')\in V\}$$

as U runs over $\mathcal{U}$ and V over $\mathcal{V}$. (Engelking 89, §8.2; Bourbaki 66, II.2.6; Schubert 68, p. 124; James 87, p. 93.)

(b) If $\mathcal{U}$, $\mathcal{V}$ are defined from families P, Θ of pseudometrics, then $\mathcal{W}$ will be defined by the family $\{\tilde\rho:\rho\in\mathrm{P}\}\cup\{\bar\theta:\theta\in\Theta\}$, writing

$$\tilde\rho((x,y),(x',y'))=\rho(x,x'),\quad \bar\theta((x,y),(x',y'))=\theta(y,y')$$

as in 2A3Tb. (Elementary verification.)

(c) If $(X,\mathcal{U})$, $(Y,\mathcal{V})$ and $(Z,\mathcal{W})$ are uniform spaces, a map $\phi:Z\to X\times Y$ is uniformly continuous iff the coordinate maps $\phi_1:Z\to X$ and $\phi_2:Z\to Y$ are uniformly continuous. (Engelking 89, 8.2.1; Bourbaki 66, II.2.6; Schubert 68, p. 125; James 87, p. 93.)

3A4F Completeness (a) If $\mathcal{W}$ is a uniformity on a set X, a filter $\mathcal{F}$ on X is **Cauchy** if for every $W\in\mathcal{W}$ there is an $F\in\mathcal{F}$ such that $F\times F\subseteq W$.

Any convergent filter in a uniform space is Cauchy. (Bourbaki 66, II.3.1; Gaal 64, p. 276; Schubert 68, p. 134; James 87, p. 109.)

(b) A uniform space is **complete** if every Cauchy filter is convergent.

(c) If $\mathcal{W}$ is defined from a family P of pseudometrics, then a filter $\mathcal{F}$ on X is Cauchy iff for every $\rho\in\mathrm{P}$ and $\epsilon>0$ there is an $F\in\mathcal{F}$ such that $\rho(x,y)\le\epsilon$ for all x, $y\in F$; equivalently, for every $\rho\in\mathrm{P}$, $\epsilon>0$ there is an $x\in X$ such that $U(x;\rho;\epsilon)\in\mathcal{F}$. (Elementary verification.)

(d) A complete subspace of a Hausdorff uniform space is closed. (Engelking 89, 8.3.6; Bourbaki 66, II.3.4; Schubert 68, p. 135; James 87, p. 148.)

(e) A metric space is complete iff every Cauchy sequence converges (cf. 2A4Db). (Schubert 68, p. 141; Gaal 64, p. 276; James 87, p. 150.)

(f) If (X,ρ) is a complete metric space, $D\subseteq X$ a dense subset, (Y,σ) a metric space and $f:X\to Y$ is an **isometry** (that is, $\sigma(f(x),f(x'))=\rho(x,x')$ for all x, $x'\in X$), then $f[X]$ is precisely the closure of $f[D]$ in Y. (For $f[X]$ must be complete, and we can use (d).)

(g) If U is a linear space with a linear space topology and the associated uniformity (3A4Ad), then a filter $\mathcal{F}$ on U is Cauchy iff for every open set G containing 0 there is an $F\in\mathcal{F}$ such that $F-F\subseteq G$ (cf. 2A5F). (Immediate from the definitions.)

3A4G Extension of uniformly continuous functions: Theorem If $(X,\mathcal{W})$ is a uniform space, $(Y,\mathcal{V})$ is a complete uniform space, $D\subseteq X$ is a dense subset of X, and $\phi:D\to Y$ is uniformly continuous (for the subspace uniformity of D), then there is a uniformly continuous $\hat\phi:X\to Y$ extending ϕ. If Y is Hausdorff, the extension is unique. $*$ (Engelking 89, 8.3.10; Bourbaki 66, II.3.6; Gaal 64, p. 300; Schubert 68, p. 137; James 87, p. 152.)

In particular, if (X,ρ) is a metric space, (Y,σ) is a complete metric space, $D\subseteq X$ is a dense subset, and $\phi:D\to Y$ is an isometry, then there is a unique isometry $\hat\phi:X\to Y$ extending ϕ.

3A4H Completions (a) Theorem If $(X, \mathcal{W})$ is any Hausdorff uniform space, then we can find a complete Hausdorff uniform space $(\hat{X}, \hat{\mathcal{W}})$ in which X is embedded as a dense subspace; moreover, any two such spaces are essentially unique. $*$ (Engelking 89, 8.3.12; Bourbaki 66, II.3.7; Gaal 64, p. 297 & p. 300; Schubert 68, p. 139; James 87, p. 156.)

(b) Such a space $(\hat{X}, \hat{\mathcal{W}})$ is called a **completion** of $(X, \mathcal{W})$. Because it is unique up to isomorphism as a uniform space, we may call it 'the' completion.

(c) If $\mathcal{W}$ is the uniformity defined by a metric ρ on a set X, then there is a unique extension of ρ to a metric $\hat{\rho}$ on $\hat{X}$ defining the uniformity $\hat{\mathcal{W}}$. (Bourbaki 66, IX.1.3.)

3A4I A note on metric spaces I mention some elementary facts. Let (X, ρ) be a metric space. If $x \in X$ and $A \subseteq X$ is non-empty, set

$$\rho(x, A) = \inf_{y \in A} \rho(x, y).$$

Then $\rho(x, A) = 0$ iff $x \in \overline{A}$ (2A3Kb). If $B \subseteq X$ is another non-empty set, then

$$\rho(x, B) \le \rho(x, A) + \sup_{y \in A} \rho(y, B).$$

In particular, $\rho(x, \overline{A}) = \rho(x, A)$. If $\langle A_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of non-empty sets with union A, then

$$\rho(x, A) = \lim_{n \to \infty} \rho(x, A_n).$$

3A5 Normed spaces

I run as quickly as possible over the results, nearly all of them standard elements of any introductory course in functional analysis, which I find myself calling on in this volume. As in the corresponding section of Volume 2 (§2A4), a large proportion of these are valid for both real and complex normed spaces, but as the present volume is almost exclusively concerned with real linear spaces I leave this unsaid, except in 3A5M, and if in doubt you may suppose for the time being that scalars belong to the field $\mathbb{R}$. A couple of the most basic results will be used in their complex forms in Volume 4.

3A5A The Hahn-Banach theorem: analytic forms The Hahn-Banach theorem is one of the central ideas of functional analysis, both finite- and infinite-dimensional, and appears in a remarkable variety of forms. I list those formulations which I wish to quote, starting with those which are more or less 'analytic', according to the classification of Bourbaki 87. Recall that if U is a normed space I write U^* for the Banach space of bounded linear functionals on U.

(a) Let U be a linear space and $p : U \to [0, \infty[$ a functional such that $p(u + v) \le p(u) + p(v)$ and $p(\alpha u) = \alpha p(u)$ whenever u, $v \in U$ and $\alpha \ge 0$. Then for any $u_0 \in U$ there is a linear functional $f : U \to \mathbb{R}$ such that $f(u_0) = p(u_0)$ and $f(u) \le p(u)$ for every $u \in U$. (Rudin 91, 3.2; Dunford & Schwartz 57, II.3.10.)

(b) Let U be a normed space and V a linear subspace of U. Then for any $f \in V^*$ there is a $g \in U^*$, extending f, with $\|g\| = \|f\|$. (363R; Bourbaki 87, II.3.2; Rudin 91, 3.3; Dunford & Schwartz 57, II.3.11; Lang 93, p. 69; Wilansky 64, p. 66; Taylor 64, 3.7-B & 4.3-A.)

(c) If U is a normed space and $u \in U$ there is an $f \in U^*$ such that $\|f\| \le 1$ and $f(u) = \|u\|$. (Bourbaki 87, II.3.2; Rudin 91, 3.3; Dunford & Schwartz 57, II.3.14; Wilansky 64, p. 67; Taylor 64, 3.7-C & 4.3-B.)

(d) If U is a normed space and $V \subseteq U$ is a linear subspace which is not dense, then there is a non-zero $f \in U^*$ such that $f(v) = 0$ for every $v \in V$. (Rudin 91, 3.5; Dunford & Schwartz 57, II.3.12; Taylor 64, 4.3-D.)

(e) If U is a normed space, U^* separates the points of U. (Rudin 91, 3.4; Lang 93, p. 70; Dunford & Schwartz 57, II.3.14.)

3A5B Cones (a) Let U be a linear space. A **convex cone** (with apex 0) is a set $C \subseteq U$ such that $\alpha u + \beta v \in C$ whenever u, $v \in C$ and α, $\beta \ge 0$. The intersection of any family of convex cones is a convex cone, so for every subset A of U there is a smallest convex cone including A.

(b) Let U be a normed space. Then the closure of a convex cone is a convex cone. (Bourbaki 87, II.2.6; Dunford & Schwartz 57, V.2.1.)

3A5C Hahn-Banach theorem: geometric forms (a) Let U be a normed space and $C \subseteq U$ a convex set such that $\|u\| \geq 1$ for every $u \in C$. Then there is an $f \in U^*$ such that $\|f\| \leq 1$ and $f(u) \geq 1$ for every $u \in C$. (Dunford & Schwartz 57, V.1.12.)

(b) Let U be a normed space and $C \subseteq U$ a non-empty convex set such that $0 \notin \overline{C}$. Then there is an $f \in U^*$ such that $\inf_{u \in C} f(u) > 0$. (Bourbaki 87, II.4.1; Rudin 91, 3.4; Lang 93, p. 70; Dunford & Schwartz 57, V.2.12.)

(c) Let U be a normed space, C a closed convex subset of U containing 0, and u a point of $U \setminus C$. Then there is an $f \in U^*$ such that $f(u) > 1$ and $f(v) \leq 1$ for every $v \in C$. (Apply (b) to $C - u$ to find a $g \in U^*$ such that $g(u) < \inf_{v \in C} g(v)$ and now set $f = -\frac{1}{\alpha}g$ where $g(u) < \alpha < \inf_{v \in C} g(v)$).

3A5D Separation from finitely-generated cones Let U be a linear space over $\mathbb{R}$ and u, $v_0, \dots, v_n$ points of U such that u does not belong to the convex cone generated by $\{v_0, \dots, v_n\}$. Then there is a linear functional $f : U \to \mathbb{R}$ such that $f(v_i) \geq 0$ for every i and $f(u) < 0$.

proof (a) If U is finite-dimensional this is covered by Gale 60, p. 56.

(b) For the general case, let V be the linear subspace of U generated by u, $v_0, \dots, v_n$. Then there is a linear functional $f_0 : V \to \mathbb{R}$ such that $f_0(u) < 0 \leq f_0(v_i)$ for every i. By Zorn's Lemma, there is a maximal linear subspace $W \subseteq U$ such that $W \cap V = \{0\}$. Now $W + V = U$ (for if $u \notin W + V$, the linear subspace W' generated by $W \cup \{u\}$ still has trivial intersection with V), so we have an extension of f_0 to a linear functional $f : U \to \mathbb{R}$ defined by setting $f(v + w) = f_0(v)$ whenever $v \in V$ and $w \in W$. Now $f(u) < 0 \leq \min_{i \leq n} f(v_i)$, as required.

3A5E Weak topologies (a) Let U be any linear space over $\mathbb{R}$ and W a subset of the space U' of all linear functionals from U to $\mathbb{R}$. Then I write $\mathfrak{T}_s(U, W)$ for the linear space topology defined by the method of 2A5B from the functionals $u \mapsto |f(u)|$ as f runs over W. (Bourbaki 87, II.6.2; Rudin 91, 3.10; Dunford & Schwartz 57, V.3.2; Taylor 64, 3.81.)

(b) I note that the weak topology of a normed space U (2A5Ia) is $\mathfrak{T}_s(U, U^*)$, while the weak* topology of U^* (2A5Ig) is $\mathfrak{T}_s(U^*, W)$ where W is the canonical image of U in U^{**}. (Rudin 91, 3.14.)

(c) Let U and V be linear spaces over $\mathbb{R}$ and $T : U \to V$ a linear operator. If $W \subseteq U'$ and $Z \subseteq V'$ are such that $gT \in W$ for every $g \in Z$, then T is continuous for $\mathfrak{T}_s(U, W)$ and $\mathfrak{T}_s(V, Z)$. (Bourbaki 87, II.6.4.)

(d) If U and V are normed spaces and $T : U \to V$ is a bounded linear operator then we have an **adjoint** operator $T' : V^* \to U^*$ defined by saying that $T'g = gT$ for every $g \in V^*$. T' is linear and is continuous for the weak* topologies of U^* and V^*. (Bourbaki 87, II.6.4; Dunford & Schwartz 57, §VI.2; Taylor 64, 4.5.)

(e) If U is a normed space and $A \subseteq U$ is convex, then the closure of A for the norm topology is the same as the closure of A for the weak topology of U. In particular, norm-closed convex subsets (for instance, norm-closed linear subspaces) of U are closed for the weak topology. (Rudin 91, 3.12; Lang 93, p. 88; Dunford & Schwartz 57, V.3.13.)

3A5F Weak* topologies: Theorem If U is a normed space, the unit ball of U^* is compact and Hausdorff for the weak* topology. (Rudin 91, 3.15; Lang 93, p. 71; Dunford & Schwartz 57, V.4.2; Taylor 64, 4.61-A.)

3A5G Reflexive spaces (a) A normed space U is **reflexive** if every member of U^{**} is of the form $f \mapsto f(u)$ for some $u \in U$.

(b) A normed space is reflexive iff bounded sets are relatively weakly compact. (Dunford & Schwartz 57, V.4.8; Taylor 64, 4.61-C.)

(c) If U is a reflexive space, $\langle u_n \rangle_{n \in \mathbb{N}}$ is a bounded sequence in U and $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$, then $\lim_{n \to \mathcal{F}} u_n$ is defined in U for the weak topology. (Use (b) and 2A3Se.)

3A5H (a) Uniform Boundedness Theorem Let U be a Banach space, V a normed space, and $A \subseteq \mathsf{B}(U;V)$ a set such that $\{Tu : T \in A\}$ is bounded in V for every $u \in U$. Then A is bounded in $\mathsf{B}(U;V)$. (Rudin 91, 2.6; Dunford & Schwartz 57, II.3.21; Taylor 64, 4.4-E.)

(b) Corollary If U is a normed space and $A \subseteq U$ is such that $f[A]$ is bounded for every $f \in U^*$, then A is bounded. (Wilansky 64, p. 117; Taylor 64, 4.4-AS.) Consequently any relatively weakly compact set in U is bounded. (Rudin 91, 3.18.)

***3A5I Strong operator topologies** If U and V are normed spaces, the **strong operator topology** on $\mathsf{B}(U;V)$ is that defined by the seminorms $T \mapsto \|Tu\|$ as u runs over U. If U is a Banach space, V is a normed space and $A \subseteq \mathsf{B}(U;V)$, then A is relatively compact for the strong operator topology iff $\{Tu : T \in A\}$ is relatively compact in V for every $u \in U$. (Put 3A5Ha and 2A3R together.)

3A5J Completions Let U be a normed space.

(a) Recall that U has a metric ρ associated with the norm (2A4Bb), and that the topology defined by ρ is a linear space topology (2A5D, 2A5B). This topology defines a uniformity $\mathcal{W}$ (3A4Ad) which is also the uniformity defined by ρ (3A4Bd). The norm itself is a uniformly continuous function from U to $\mathbb{R}$ (because $|\|u\| - \|v\|| \le \|u - v\|$ for all u, $v \in U$).

(b) Let $(\hat{U}, \hat{\mathcal{W}})$ be the uniform space completion of $(U, \mathcal{W})$ (3A4H). Then addition and scalar multiplication and the norm extend uniquely to make $\hat{U}$ a Banach space. (Schaefer 66, I.1.5; Lang 93, p. 78.)

(c) If U and V are Banach spaces with dense linear subspaces U_0 and V_0, then any norm-preserving isomorphism between U_0 and V_0 extends uniquely to a norm-preserving isomorphism between U and V (use 3A4G).

3A5K Normed algebras If U is a normed algebra (2A4J), its multiplication, regarded as a function from $U \times U$ to U, is continuous. (Wilansky 64, p. 259.)

3A5L Compact operators Let U and V be Banach spaces.

(a) A linear operator $T : U \to V$ is **compact** if $\{Tu : \|u\| \le 1\}$ is relatively compact in V for the topology defined by the norm of V.

(b) A linear operator $T : U \to V$ is **weakly compact** if $\{Tu : \|u\| \le 1\}$ is relatively weakly compact in V. Of course compact operators are weakly compact; because weakly compact sets must be norm-bounded (3A5Hb), weakly compact operators are bounded.

3A5M Hilbert spaces I mentioned the phrases 'inner product space', 'Hilbert space' briefly in 244N and 244P, without explanation, as I did not there rely on any of the abstract theory of these spaces. For the main result of §396 we need one of their fundamental properties, so I now skim over the definitions.

(a) An **inner product space** is a linear space U over $\substack{\mathbb{R}\\\mathbb{C}}$ together with an operator $(\ |\ ) : U \times U \to \substack{\mathbb{R}\\\mathbb{C}}$ such that

$$(u_1 + u_2|v) = (u_1|v) + (u_2|v), \quad (\alpha u|v) = \alpha(u|v), \quad (u|v) = \overline{(v|u)}$$

(the complex conjugate of $(v|u)$),

$$(u|u) \ge 0, \quad u = 0 \text{ whenever } (u|u) = 0$$

for all u, u_1, u_2, $v \in U$ and $\alpha \in \substack{\mathbb{R}\\\mathbb{C}}$.

(b) If U is any inner product space, we have a norm on U defined by setting $\|u\| = \sqrt{(u|u)}$ for every $u \in U$. (Taylor 64, 3.2-B.)

(c) A **Hilbert space** is an inner product space which is a Banach space under the norm of (b) above, that is, is complete in the metric defined from its norm.

(d) If U is a Hilbert space, $C \subseteq U$ is a non-empty closed convex set, and $u \in U$, then there is a unique $v \in C$ such that $\|u - v\| = \inf_{w \in C} \|u - w\|$. (Taylor 64, 4.81-A; compare 244Yn.)

***3A5N Bounded sets in linear topological spaces** There is a point in §377 where a concept from the general theory of linear topological spaces helps an idea to flow more freely. Let U be a linear topological space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$.

(a) A set $A \subseteq U$ is **bounded** if for every neighbourhood G of 0 there is an $n \in \mathbb{N}$ such that $A \subseteq nG$.

(b) If $A \subseteq U$ is bounded, then
(i) every subset of A is bounded;
(ii) the closure of A is bounded;
(iii) αA is bounded for every $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$;
(iv) $A \cup B$ and $A + B$ are bounded for every bounded $B \subseteq U$;
(v) if V is another linear topological space, and $T : U \to V$ is a continuous linear operator, then $T[A]$ is bounded.

(c) If $A \subseteq U$ is relatively compact, it is bounded.

(d) If U is a normed space, and $A \subseteq U$, then the following are equiveridical:
(i) A is bounded in the sense of (a) above for the norm topology of U;
(ii) A is bounded in the sense of 2A4Bc, that is, $\{\|u\| : u \in A\}$ is bounded above in $\mathbb{R}$;
(iii) A is bounded for the weak topology of U.

proof Schaefer 66, §I.5; Köthe 69, §15.6. For (d-iii), use 3A5Hb.

3A6 Group Theory

For Chapter 38 we need four definitions and two results from elementary abstract group theory.

3A6A Definition If G is a group, I will say that an element g of G is an **involution** if its order is 2, that is, $g^2 = e$, the identity of G, but $g \neq e$.

3A6B Definition If G is a group, the set $\operatorname{Aut} G$ of **automorphisms** of G (that is, bijective homomorphisms from G to itself) is a group. For $g \in G$ define $\hat{g} : G \to G$ by writing $\hat{g}(h) = ghg^{-1}$ for every $h \in G$; then $\hat{g} \in \operatorname{Aut} G$, and the map $g \mapsto \hat{g}$ is a homomorphism from G onto a normal subgroup J of $\operatorname{Aut} G$ (Rotman 84, p. 130). We call J the group of **inner automorphisms** of G. Members of $(\operatorname{Aut} G) \setminus J$ are called **outer automorphisms**.

3A6C Normal subgroups For any group G, the family of normal subgroups of G, ordered by $\subseteq$, is a Dedekind complete lattice, with $H \vee K = HK$ and $H \wedge K = H \cap K$. (Davey & Priestley 90, 2.8 & 2.19.)

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this volume, and which have since been changed.

354Yk Complexifications of normed Riesz spaces This exercise, referred to in the 2003 edition of Volume 4, is now 354Yl.

372Xm The tent map, referred to in the 2003 and 2006 editions of Volume 4, is now in 372Xp.

393B The association of a metric with a strictly positive submeasure, used in the 2003 and 2006 editions of Volume 4, is now in 392H and 393H.

393C The result that a non-negative additive functional on a Boolean algebra can be factored through a measure algebra, used in the 2003 and 2006 editions of Volume 4, is now in 392I.

393O The note on control measures for vector measures, referred to in the 2003 and 2006 editions of Volume 4, is now in 394Q.

§**394** Kawada's theorem, referred to in the 2003 and 2006 editions of Volume 4, is now in §395.

3A3P Definitions This paragraph, referred to in the 2003 edition of Volume 4, is now 3A3Q.

3A5K Compact operators This paragraph, referred to in the 2003 and 2006 editions of Volume 4, is now 3A5L.

3A5L Inner product spaces This paragraph, referred to in the 2003 and 2006 editions of Volume 4, is now 3A5M.

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Index to volumes 1, 2 and 3

Principal topics and results

The general index below is intended to be comprehensive. Inevitably the entries are voluminous to the point that they are often unhelpful. I have therefore prepared a shorter, better-annotated, index which will, I hope, help readers to focus on particular areas. It does not mention definitions, as the bold-type entries in the main index are supposed to lead efficiently to these; and if you draw blank here you should always, of course, try again in the main index. Entries in the form of mathematical assertions frequently omit essential hypotheses and should be checked against the formal statements in the body of the work.

General index

References in **bold** type refer to definitions; references in *italics* are passing references. Definitions marked with **>** are those in which my usage is dangerously at variance with that of some other author or authors.

[1] A.M.Lyapunov, 1857-1918

[2] A.A.Lyapunov, 1911-1973