MEASURE THEORY

Volume 3

Part I

D.H.Fremlin

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MEASURE THEORY

Volume 3

Measure Algebras

Part I

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to the Publisher

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Contents

General introduction In this treatise I aim to give a comprehensive description of modern abstract measure theory, with some indication of its principal applications. The first two volumes are set at an introductory level; they are intended for students with a solid grounding in the concepts of real analysis, but possibly with rather limited detailed knowledge. As the book proceeds, the level of sophistication and expertise demanded will increase; thus for the volume on topological measure spaces, familiarity with general topology will be assumed. The emphasis throughout is on the mathematical ideas involved, which in this subject are mostly to be found in the details of the proofs.

My intention is that the book should be usable both as a first introduction to the subject and as a reference work. For the sake of the first aim, I try to limit the ideas of the early volumes to those which are really essential to the development of the basic theorems. For the sake of the second aim, I try to express these ideas in their full natural generality, and in particular I take care to avoid suggesting any unnecessary restrictions in their applicability. Of course these principles are to to some extent contradictory. Nevertheless, I find that most of the time they are very nearly reconcilable, *provided* that I indulge in a certain degree of repetition. For instance, right at the beginning, the puzzle arises: should one develop Lebesgue measure first on the real line, and then in spaces of higher dimension, or should one go straight to the multidimensional case? I believe that there is no single correct answer to this question. Most students will find the one-dimensional case easier, and it therefore seems more appropriate for a first introduction, since even in that case the technical problems can be daunting. But certainly every student of measure theory must at a fairly early stage come to terms with Lebesgue area and volume as well as length; and with the correct formulations, the multidimensional case differs from the one-dimensional case only in a definition and a (substantial) lemma. So what I have done is to write them both out (§§114-115). In the same spirit, I have been uninhibited, when setting out exercises, by the fact that many of the results I invite students to look for will appear in later chapters; I believe that throughout mathematics one has a better chance of understanding a theorem if one has previously attempted something similar alone.

The plan of the work is as follows:

Volume 1: The Irreducible Minimum
Volume 2: Broad Foundations
Volume 3: Measure Algebras
Volume 4: Topological Measure Spaces
Volume 5: Set-theoretic Measure Theory.

Volume 1 is intended for those with no prior knowledge of measure theory, but competent in the elementary techniques of real analysis. I hope that it will be found useful by undergraduates meeting Lebesgue measure for the first time. Volume 2 aims to lay out some of the fundamental results of pure measure theory (the Radon-Nikodým theorem, Fubini's theorem), but also gives short introductions to some of the most important applications of measure theory (probability theory, Fourier analysis). While I should like to believe that most of it is written at a level accessible to anyone who has mastered the contents of Volume 1, I should not myself have the courage to try to cover it in an undergraduate course, though I would certainly attempt to include some parts of it. Volumes 3 and 4 are set at a rather higher level, suitable to postgraduate courses; while Volume 5 will assume a wide-ranging competence over large parts of analysis and set theory.

There is a disclaimer which I ought to make in a place where you might see it in time to avoid paying for this book. I make no attempt to describe the history of the subject. This is not because I think the history uninteresting or unimportant; rather, it is because I have no confidence of saying anything which would not be seriously misleading. Indeed I have very little confidence in anything I have ever read concerning the history of ideas. So while I am happy to honour the names of Lebesgue and Kolmogorov and Maharam in more or less appropriate places, and I try to include in the bibliographies the works which I have myself consulted, I leave any consideration of the details to those bolder and better qualified than myself.

For the time being, at least, printing will be in short runs. I hope that readers will be energetic in commenting on errors and omissions, since it should be possible to correct these relatively promptly. An inevitable consequence of this is that paragraph references may go out of date rather quickly. I shall be most flattered if anyone chooses to rely on this book as a source for basic material; and I am willing to attempt to maintain a concordance to such references, indicating where migratory results have come to rest for the moment, if authors will supply me with copies of papers which use them.

I mention some minor points concerning the layout of the material. Most sections conclude with lists of 'basic exercises' and 'further exercises', which I hope will be generally instructive and occasionally entertaining. How many of these you should attempt must be for you and your teacher, if any, to decide, as no two students will have quite the same needs. I mark with a **>** those which seem to me to be particularly important. But while you may not need to write out solutions to all the 'basic exercises', if you are in any doubt as to your capacity to do so you should take this as a warning to slow down a bit. The 'further exercises' are unbounded in difficulty, and are unified only by a presumption that each has at least one solution based on ideas already introduced. Occasionally I add a final 'problem', a question to which I do not know the answer and which seems to arise naturally in the course of the work.

The impulse to write this book is in large part a desire to present a unified account of the subject. Cross-references are correspondingly abundant and wide-ranging. In order to be able to refer freely across the whole text, I have chosen a reference system which gives the same code name to a paragraph wherever it is being called from. Thus 132E is the fifth paragraph in the second section of the third chapter of Volume 1, and is referred to by that name throughout. Let me emphasize that cross-references are supposed to help the reader, not distract her. Do not take the interpolation '(121A)' as an instruction, or even a recommendation, to lift Volume 1 off the shelf and hunt for §121. If you are happy with an argument as it stands, independently of the reference, then carry on. If, however, I seem to have made rather a large jump, or the notation has suddenly become opaque, local cross-references may help you to fill in the gaps.

Each volume will have an appendix of 'useful facts', in which I set out material which is called on somewhere in that volume, and which I do not feel I can take for granted. Typically the arrangement of material in these appendices is directed very narrowly at the particular applications I have in mind, and is unlikely to be a satisfactory substitute for conventional treatments of the topics touched on. Moreover, the ideas may well be needed only on rare and isolated occasions. So as a rule I recommend you to ignore the appendices until you have some direct reason to suppose that a fragment may be useful to you.

During the extended gestation of this project I have been helped by many people, and I hope that my friends and colleagues will be pleased when they recognise their ideas scattered through the pages below. But I am especially grateful to those who have taken the trouble to read through earlier drafts and comment on obscurities and errors.

Introduction to Volume 3

One of the first things one learns, as a student of measure theory, is that sets of measure zero are frequently 'negligible' in the straightforward sense that they can safely be ignored. This is not quite a universal principle, and one of my purposes in writing this treatise is to call attention to the exceptional cases in which negligible sets are important. But very large parts of the theory, including some of the topics already treated in Volume 2, can be expressed in an appropriately abstract language in which negligible sets have been factored out. This is what the present volume is about. A 'measure algebra' is a quotient of an algebra of measurable sets by a null ideal; that is, the elements of the measure algebra are equivalence classes of measurable sets. At the cost of an extra layer of abstraction, we obtain a language which can give concise and elegant expression to a substantial proportion of the ideas of measure theory, and which offers insights almost everywhere in the subject.

It is here that I embark wholeheartedly on 'pure' measure theory. I think it is fair to say that the applications of measure theory to other branches of mathematics are more often through measure *spaces* rather than measure *algebras*. Certainly there will be in this volume many theorems of wide importance outside measure theory; but typically their usefulness will be in forms translated back into the language of the first two volumes. But it is also fair to say that the language of measure algebras is the only reasonable way to discuss large parts of a subject which, as pure mathematics, can bear comparison with any.

In the structure of this volume I can distinguish seven 'working' and two 'accessory' chapters. The 'accessory' chapters are 31 and 35. In these I develop the theories of Boolean algebras and Riesz spaces (= vector lattices) which are needed later. As in Volume 2 you have a certain amount of choice in the order in which you take the material. Everything except Chapter 35 depends on Chapter 31, and everything except Chapters 31 and 35 depends on Chapter 32. Chapters 33, 34 and 36 can be taken in any order, but Chapter 36 relies on Chapter 35. (I do not mean that Chapter 33 is never referred to in Chapter 34, nor even that the later chapters do not rely on results from Chapter 33. What I mean is that their most important ideas are accessible without learning the material of Chapter 33 properly.) Chapter 37 depends on Chapters 35 and 36. Chapter 38 would be difficult to make sense of without some notion of what has been done in Chapter 33. Chapter 39 uses fragments of Chapters 35 and 36.

The first third of the volume follows almost the only line permitted by the structure of the subject. If we are going to study measure algebras at all, we must know the relevant facts about Boolean algebras (Chapter 31) and how to translate what we know about measure spaces into the new language (Chapter 32). Then we must get a proper grip on the two most important theorems: Maharam's theorem on the classification of measure algebras (Chapter 33) and the von Neumann-Maharam lifting theorem (Chapter 34). Since I am now writing for readers who are committed – I hope, happily committed – to learning as much as they can about the subject, I take the space to push these ideas as far as they can easily go, giving a full classification of closed subalgebras of probability algebras, for instance (§333), and investigating special types of lifting (§§345-346). I mention here three sections interpolated into Chapter 34 (§§342-344) which attack a subtle and important question: when can we expect homomorphisms between measure algebras to be realizable in terms of transformations between measure spaces, as discussed briefly in §235 and elsewhere.

Chapters 36 and 37 are devoted to re-working the ideas of Chapter 24 on 'function spaces' in the more abstract context now available, and relating them to the general Riesz spaces of Chapter 35. I am concerned here not to develop new structures, nor even to prove striking new theorems, but rather to offer new ways of looking at the old ones. Only in the Ergodic Theorem (§372) do I come to a really important new result. Chapter 38 looks at two questions, both obvious ones to ask if you have been trained in twentieth-century pure mathematics: what does the automorphism group of a measure algebra look like, and inside such an automorphism group, what do the conjugacy classes look like? (The second question is a fancy way of asking how to decide, given two automorphisms of one of the structures considered in this volume, whether they are really different, or just copies of each other obtained by looking at the structure a different way up.) Finally, in Chapter 39, I discuss what is known about the question of which Boolean algebras can appear as measure algebras.

Concerning the prerequisites for this volume, we certainly do not need everything in Volume 2. The important chapters there are 21, 23, 24, 25 and 27. If you are approaching this volume without having read the earlier parts of this treatise, you will need the Radon-Nikodým theorem and product measures (of arbitrary families of probability spaces), for Maharam's theorem; a simple version of the martingale theorem, for the lifting theorem; and an acquaintance with L^p spaces (particularly, with L^0 spaces) for Chapter 36. But I would recommend the results-only versions of Volumes 1 and 2 in case some reference is totally obscure. Outside measure theory, I call on quite a lot of terms from general topology, but none of the ideas needed are difficult (Baire's and Tychonoff's theorems are the deepest); they are sketched in §§3A3 and 3A4. We do need some functional analysis for Chapters 36 and 39, but very little more than was already used in Volume 2, except that I now call on versions of the Hahn-Banach theorem (§3A5).

In this volume I assume that readers have substantial experience in both real and abstract analysis, and I make few concessions which would not be appropriate when addressing active researchers, except that perhaps I am a little gentler when calling on ideas from set theory and general topology than I should be with my own colleagues, and I continue to include all the easiest exercises I can think of. I do maintain my practice of giving proofs in very full detail, not so much because I am trying to make them easier, but because one of my purposes here is to provide a complete account of the ideas of the subject. I hope that the result will be accessible to most doctoral students who are studying topics in, or depending on, measure theory.

Note on second printing

For the second printing of this volume I have taken the opportunity to strengthen a theorem in §381 concerning automorphisms of Boolean algebras; I am grateful to B.Miller for allowing me to incorporate his ideas into what is now §382. This has forced substantial changes in the rest of the chapter. I have also added a discussion of the order-sequential topology on a Boolean algebra in §393, following Balcar Głowczyński & Jech 98 and Balcar Jech & Pazák 05. A couple of results on the topology of convergence in measure have been added to §367, and a handful of new definitions are included. Besides these changes, there are many minor corrections (once again, I should thank T.D.Austin for his help) and new exercises, and two of the 'problems' (323Z and 381Z) mentioned in the first printing have been solved. Details may be found in `http://www.essex.ac.uk/maths/people/fremlin/mterr3.02.ps`.

Note on second ('Lulu') edition

A good few further errors have been detected and, I hope, dealt with. For the sake of ideas in Volume 5, there is some additional material on homogeneous algebras (§316) and 'reduced products' (§§328 and 377); but the most substantial addition is a new §394, 'Talagrand's example', describing M.Talagrand's solution to the Control Measure Problem. This has naturally lead to a complete re-writing of §393, which is now just an investigation of Maharam algebras.

Chapter 31

Boolean algebras

The theory of measure algebras naturally depends on certain parts of the general theory of Boolean algebras. In this chapter I collect those results which will be useful later. Since many students encounter the formal notion of Boolean algebra for the first time in this context, I start at the beginning; and indeed I include in the Appendix (§3A2) a brief account of the necessary part of the theory of rings, as not everyone will have had time for this bit of abstract algebra in an undergraduate course. But unless you find the algebraic theory of Boolean algebras so interesting that you wish to study it for its own sake – in which case you should perhaps turn to Sikorski 64 or Koppelberg 89 – I do not think it would be very sensible to read the whole of this chapter before proceeding to the main work of the volume in Chapter 32. Probably §311 is necessary to get an idea of what a Boolean algebra looks like, and a glance at the statements of the theorems in §312 and 313A-313B would be useful, but the later sections can wait until you have need of them, on the understanding that apparently innocent formal manipulations may depend on concepts which take some time to master. I hope that the cross-references will be sufficiently well-targeted to make it possible to read this material in parallel with its applications.

As for the actual material covered, §311 introduces Boolean rings and algebras, with M.H.Stone's theorem on their representation as rings and algebras of sets. §312 is devoted to subalgebras, homomorphisms and quotients, following a path parallel to the corresponding ideas in group theory, ring theory and linear algebra. In §313 I come to the special properties of Boolean algebras associated with their lattice structures, with notions of order-preservation, order-continuity and order-closure. §314 continues this with a discussion of order-completeness, and the elaboration of the Stone representation of an arbitrary Boolean algebra into the Loomis-Sikorski representation of a σ-complete Boolean algebra; this brings us to regular open algebras. §315 deals with 'simple' and 'free' products of Boolean algebras, corresponding to 'products' and 'tensor products' of linear spaces, and to projective and inductive limits of families of Boolean algebras. Finally, §316 examines three special topics: the countable chain condition, weak distributivity and homogeneity.

311 Boolean algebras

In this section I try to give a sufficient notion of the character of abstract Boolean algebras to make the calculations which will appear on almost every page of this volume seem both elementary and natural. The principal result is of course M.H.Stone's theorem: every Boolean algebra can be expressed as an algebra of sets (311E). So the section divides naturally into the first part, proving Stone's theorem, and the second, consisting of elementary consequences of the theorem and a little practice in using the insights it offers.

311A Definitions (a) A **Boolean ring** is a ring $(\mathfrak{A},+,.)$ in which $a^2=a$ for every $a\in\mathfrak{A}$.

(b) A **Boolean algebra** is a Boolean ring $\mathfrak{A}$ with a multiplicative identity $1=1_{\mathfrak{A}}$; I allow $1=0$ in this context.

Remark For notes on those parts of the elementary theory of rings which we shall need, see §3A2.

I hope that the rather arbitrary use of the word 'algebra' here will give no difficulties; it gives me the freedom to insist that the ring $\{0\}$ should be accepted as a Boolean algebra.

311B Examples (a) For any set X, $(\mathcal{P}X,\triangle,\cap)$ is a Boolean algebra; its zero is $\emptyset$ and its multiplicative identity is X. **P** We have to check the following, which are all easily established, using Venn diagrams or otherwise:

$A\triangle B\subseteq X$ for all A, $B\subseteq X$,

$(A\triangle B)\triangle C=A\triangle(B\triangle C)$ for all A, B, $C\subseteq X$,

so that $(\mathcal{P}X,\triangle)$ is a semigroup;

$A\triangle\emptyset=\emptyset\triangle A=A$ for every $A\subseteq X$,

so that $\emptyset$ is the identity in $(\mathcal{P}X,\triangle)$;

$A\triangle A=\emptyset$ for every $A\subseteq X$,

so that every element of $\mathcal{P}X$ is its own inverse in $(\mathcal{P}X,\triangle)$, and $(\mathcal{P}X,\triangle)$ is a group;

$A\triangle B=B\triangle A$ for all A, $B\subseteq X$,

so that $(\mathcal{P}X,\triangle)$ is an abelian group;

$A\cap B\subseteq X$ for all A, $B\subseteq X$,

$(A\cap B)\cap C=A\cap(B\cap C)$ for all A, B, $C\subseteq X$, so that $(\mathcal{P}X, \cap)$ is a semigroup;

$A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$, $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$ for all A, B, $C \subseteq X$,

so that $(\mathcal{P}X, \triangle, \cap)$ is a ring;

$A \cap A = A$ for every $A \subseteq X$,

so that $(\mathcal{P}X, \triangle, \cap)$ is a Boolean ring;

$A \cap X = X \cap A = A$ for every $A \subseteq X$,

so that $(\mathcal{P}X, \triangle, \cap)$ is a Boolean algebra and X is its identity. **Q**

(b) Recall that an 'algebra of subsets of X' (136E) is a family $\Sigma \subseteq \mathcal{P}X$ such that $\emptyset \in \Sigma$, $X \setminus E \in \Sigma$ for every $E \in \Sigma$, and $E \cup F \in \Sigma$ for all E, $F \in \Sigma$. In this case $(\Sigma, \triangle, \cap)$ is a Boolean algebra with zero $\emptyset$ and identity X. **P** If E, $F \in \Sigma$, then

$$E \cap F = X \setminus ((X \setminus E) \cup (X \setminus F)) \in \Sigma,$$

$$E \triangle F = (E \cap (X \setminus F)) \cup (F \cap (X \setminus E)) \in \Sigma.$$

Because $\emptyset$ and $X = X \setminus \emptyset$ both belong to Σ, we can work through the identities in (a) above to see that Σ, like $\mathcal{P}X$, is a Boolean algebra. **Q**

(c) Consider the ring $\mathbb{Z}_2 = \{0, 1\}$, with its ring operations $+_2$, $\cdot$ given by setting

$$0 +_2 0 = 1 +_2 1 = 0, \quad 0 +_2 1 = 1 +_2 0 = 1,$$

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1.$$

I leave it to you to check, if you have not seen it before, that this is a ring. Because $0 \cdot 0 = 0$ and $1 \cdot 1 = 1$, it is a Boolean algebra.

311C Proposition Let $\mathfrak{A}$ be a Boolean ring.

(a) $a + a = 0$, that is, $a = -a$, for every $a \in \mathfrak{A}$.

(b) $ab = ba$ for all a, $b \in \mathfrak{A}$.

proof (a) If $a \in \mathfrak{A}$, then

$$a + a = (a + a)(a + a) = a^2 + a^2 + a^2 + a^2 = a + a + a + a,$$

so we must have $0 = a + a$.

(b) Now for any a, $b \in \mathfrak{A}$,

$$a + b = (a + b)(a + b) = a^2 + ab + ba + b^2 = a + ab + ba + b,$$

so

$$0 = ab + ba = ab + ab$$

and $ab = ba$.

311D Lemma Let $\mathfrak{A}$ be a Boolean ring, I an ideal of $\mathfrak{A}$ (3A2E), and $a \in \mathfrak{A} \setminus I$. Then there is a ring homomorphism $\phi : \mathfrak{A} \to \mathbb{Z}_2$ such that $\phi a = 1$ and $\phi d = 0$ for every $d \in I$.

proof (a) Let $\mathcal{I}$ be the family of those ideals J of $\mathfrak{A}$ which include I and do not contain a. Then $\mathcal{I}$ has a maximal element K say. **P** Apply Zorn's lemma. Since $I \in \mathcal{I}$, $\mathcal{I} \neq \emptyset$. If $\mathcal{J}$ is a non-empty totally ordered subset of $\mathcal{I}$, then set $J^* = \bigcup \mathcal{J}$. If b, $c \in J^*$ and $d \in \mathfrak{A}$, then there are J_1, $J_2 \in \mathcal{J}$ such that $b \in J_1$ and $c \in J_2$; now $J = J_1 \cup J_2$ is equal to one of J_1, J_2, so belongs to $\mathcal{J}$, and 0, $b + c$, bd all belong to J, so all belong to J^*. Thus $J^* \triangleleft \mathfrak{A}$; of course $I \subseteq J^*$ and $a \notin J^*$, so $J^* \in \mathcal{I}$ and is an upper bound for $\mathcal{J}$ in $\mathcal{I}$. As $\mathcal{J}$ is arbitrary, the hypotheses of Zorn's lemma are satisfied and $\mathcal{I}$ has a maximal element. **Q**

(b) For $b \in \mathfrak{A}$ set $K_b = \{d : d \in \mathfrak{A}, bd \in K\}$. The following are easy to check:

(i) $K \subseteq K_b$ for every $b \in \mathfrak{A}$, because K is an ideal.

(ii) $K_b \triangleleft \mathfrak{A}$ for every $b \in \mathfrak{A}$. **P** $0 \in K \subseteq K_b$. If d, $d' \in K_b$ and $c \in \mathfrak{A}$ then

$$b(d + d') = bd + bd', \quad b(dc) = (bd)c$$

belong to K, so $d + d'$, $dc \in K_b$. **Q**

(iii) If $b \in \mathfrak{A}$ and $a \notin K_b$, then $K_b \in \mathcal{I}$ so $K_b = K$.
(iv) Now $a^2 = a \notin K$, so $a \notin K_a$ and $K_a = K$.
(v) If $b \in \mathfrak{A} \setminus K$ then $b \notin K_a$, that is, $ba = ab \notin K$, and $a \notin K_b$; consequently $K_b = K$.
(vi) If b, $c \in \mathfrak{A} \setminus K$ then $c \notin K_b$ so $bc \notin K$.
(vii) If b, $c \in \mathfrak{A} \setminus K$ then

$$bc(b+c) = b^2c + bc^2 = bc + bc = 0 \in K,$$

so $b + c \in K_{bc}$. By (vi) and (v), $K_{bc} = K$ so $b + c \in K$.

(c) Now define $\phi : \mathfrak{A} \to \mathbb{Z}_2$ by setting $\phi d = 0$ if $d \in K$, $\phi d = 1$ if $d \in \mathfrak{A} \setminus K$. Then ϕ is a ring homomorphism. **P**
(i) If b, $c \in K$ then $b + c$, $bc \in K$ so

$$\phi(b+c) = 0 = \phi b +_2 \phi c, \quad \phi(bc) = 0 = \phi b\, \phi c.$$

(ii) If $b \in K$, $c \in \mathfrak{A} \setminus K$ then

$$c = (b+b) + c = b + (b+c) \notin K$$

so $b + c \notin K$, while $bc \in K$, so

$$\phi(b+c) = 1 = \phi b +_2 \phi c, \quad \phi(bc) = 0 = \phi b\, \phi c.$$

(iii) Similarly,

$$\phi(b+c) = 1 = \phi b +_2 \phi c, \quad \phi(bc) = 0 = \phi b\, \phi c$$

if $b \in \mathfrak{A} \setminus K$ and $c \in K$.
(iv) If b, $c \in \mathfrak{A} \setminus K$, then by (b-vi) and (b-vii) we have $b + c \in K$, $bc \notin K$ so

$$\phi(b+c) = 0 = \phi b +_2 \phi c, \quad \phi(bc) = 1 = \phi b\, \phi c.$$

Thus ϕ is a ring homomorphism. **Q**

(d) Finally, if $d \in I$ then $d \in K$ so $\phi d = 0$; and $\phi a = 1$ because $a \notin K$.

311E M.H.Stone's theorem: first form Let $\mathfrak{A}$ be any Boolean ring, and let Z be the set of ring homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$. Then we have an injective ring homomorphism $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$, setting $\widehat{a} = \{z : z \in Z,\ z(a) = 1\}$. If $\mathfrak{A}$ is a Boolean algebra, then $\widehat{1_{\mathfrak{A}}} = Z$.

proof (a) If a, $b \in \mathfrak{A}$, then

$$\widehat{a{+}b} = \{z : z(a{+}b) = 1\} = \{z : z(a) +_2 z(b) = 1\} = \{z : \{z(a), z(b)\} = \{0,1\}\} = \widehat{a} \triangle \widehat{b},$$

$$\widehat{ab} = \{z : z(ab) = 1\} = \{z : z(a)z(b) = 1\} = \{z : z(a) = z(b) = 1\} = \widehat{a} \cap \widehat{b}.$$

Thus $a \mapsto \widehat{a}$ is a ring homomorphism.

(b) If $a \in \mathfrak{A}$ and $a \neq 0$, then by 311D, with $I = \{0\}$, there is a $z \in Z$ such that $z(a) = 1$, that is, $z \in \widehat{a}$; so that $\widehat{a} \neq \emptyset$. This shows that the kernel of $a \mapsto \widehat{a}$ is $\{0\}$, so that the homomorphism is injective (3A2Db).

(c) If $\mathfrak{A}$ is a Boolean algebra, and $z \in Z$, then there is some $a \in \mathfrak{A}$ such that $z(a) = 1$, so that $z(1_{\mathfrak{A}})z(a) = z(1_{\mathfrak{A}}a) \neq 0$ and $z(1_{\mathfrak{A}}) \neq 0$; thus $\widehat{1_{\mathfrak{A}}} = Z$.

311F Remarks (a) For any Boolean ring $\mathfrak{A}$, I will say that the **Stone space** of $\mathfrak{A}$ is the set Z of non-zero ring homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$, and the canonical map $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$ is the **Stone representation**.

(b) Because the map $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$ is an injective ring homomorphism, $\mathfrak{A}$ is isomorphic, as Boolean ring, to its image $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$, which is a subring of $\mathcal{P}Z$. Thus the Boolean rings $\mathcal{P}X$ of 311Ba are leading examples in a very strong sense.

(c) I have taken the set Z of the Stone representation to be actually the set of homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$. Of course we could equally well take any set which is in a natural one-to-one correspondence with Z; a popular choice is the set of maximal ideals of $\mathfrak{A}$, since a subset of $\mathfrak{A}$ is a maximal ideal iff it is the kernel of a member of Z, which is then uniquely defined.

311G The operations $\cup$, $\setminus$, $\triangle$ on a Boolean ring Let $\mathfrak{A}$ be a Boolean ring.

(a) Using the Stone representation, we can see that the elementary operations $\cup$, $\cap$, $\setminus$, $\triangle$ of set theory all correspond to operations on $\mathfrak{A}$. If we set

$$a \cup b = a + b + ab, \quad a \cap b = ab, \quad a \setminus b = a + ab, \quad a \triangle b = a + b$$

for a, $b \in \mathfrak{A}$, then we see that

$$\widehat{a \cup b} = \widehat{a} \triangle \widehat{b} \triangle (\widehat{a} \cap \widehat{b}) = \widehat{a} \cup \widehat{b},$$

$$\widehat{a \cap b} = \widehat{a} \cap \widehat{b},$$

$$\widehat{a \setminus b} = \widehat{a} \setminus \widehat{b},$$

$$\widehat{a \triangle b} = \widehat{a} \triangle \widehat{b}.$$

Consequently all the familiar rules for manipulation of $\cap$, $\cup$, etc. will apply also to $\cap$, $\cup$, and we shall have, for instance,

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c), \quad a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$$

for any members a, b, c of any Boolean ring $\mathfrak{A}$.

(b) Still importing terminology from elementary set theory, I will say that a set $A \subseteq \mathfrak{A}$ is **disjoint** if $a \cap b = 0$, that is, $ab = 0$, for all distinct a, $b \in A$; and that an indexed family $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}$ is **disjoint** if $a_i \cap a_j = 0$ for all distinct i, $j \in I$. (Just as I allow $\emptyset$ to be a member of a disjoint family of sets, I allow $0 \in A$ or $a_i = 0$ in the present context.)

(c) A **partition of unity** in $\mathfrak{A}$ will be *either* a disjoint set $C \subseteq \mathfrak{A}$ such that there is no non-zero $a \in \mathfrak{A}$ such that $a \cap c = 0$ for every $c \in C$ *or* a disjoint family $\langle c_i \rangle_{i \in I}$ in $\mathfrak{A}$ such that there is no non-zero $a \in \mathfrak{A}$ such that $a \cap c_i = 0$ for every $i \in I$. (In the first case I allow $0 \in C$, and in the second I allow $c_i = 0$.)

(d) Note that a set $C \subseteq \mathfrak{A}$ is a partition of unity iff $C \cup \{0\}$ is a maximal disjoint set. **P** If C is a partition of unity and $a \in \mathfrak{A} \setminus (C \cup \{0\})$, then there must be a $c \in C$ such that $a \cap c \neq 0$, so that $C \cup \{0, a\}$ is not disjoint; thus $C \cup \{0\}$ is a maximal disjoint set. If $C \cup \{0\}$ is a maximal disjoint set, and $a \in \mathfrak{A} \setminus \{0\}$, then either $a \in C$ and $a \cap a \neq 0$, or $C \cup \{0, a\}$ is not disjoint, so there is a $c \in C$ such that $a \cap c \neq 0$; thus C is a partition of unity. **Q**

If $A \subseteq \mathfrak{A}$ is any disjoint set, there is a partition of unity including A. **P** Apply Zorn's Lemma to $\{C : C$ is a disjoint set including $A\}$. **Q**

(e) If C and D are two partitions of unity, I say that C **refines** D if for every $c \in C$ there is a $d \in D$ such that $cd = c$ (that is, $c \subseteq d$ in the language of 311H below). Note that if C refines D and D refines E then C refines E. **P** If $c \in C$, there is a $d \in D$ such that $cd = c$; now there is an $e \in E$ such that $de = d$; in this case,

$$ce = (cd)e = c(de) = cd = c;$$

as c is arbitrary, C refines E. **Q**

311H The order structure of a Boolean ring Again treating a Boolean ring $\mathfrak{A}$ as an algebra of sets, it has a natural ordering, setting $a \subseteq b$ if $ab = a$, so that $a \subseteq b$ iff $\widehat{a} \subseteq \widehat{b}$. This translation makes it obvious that $\subseteq$ is a partial order on $\mathfrak{A}$, with least element 0, and with greatest element 1 iff $\mathfrak{A}$ is a Boolean algebra. Moreover, $\mathfrak{A}$ is a lattice (definition: 2A1Ad), with $a \cup b = \sup\{a, b\}$ and $a \cap b = \inf\{a, b\}$ for all a, $b \in \mathfrak{A}$. Generally, for $a_0, \dots, a_n \in \mathfrak{A}$,

$$\sup_{i \le n} a_i = a_0 \cup \dots \cup a_n, \quad \inf_{i \le n} a_i = a_0 \cap \dots \cap a_n;$$

suprema and infima of finite subsets of $\mathfrak{A}$ correspond to unions and intersections of the corresponding families in the Stone space. (But suprema and infima of *infinite* subsets of $\mathfrak{A}$ are a very different matter; see §313 below.)

It may be obvious, but it is nevertheless vital to recognise that when $\mathfrak{A}$ is a ring of sets then $\subseteq$ agrees with $\subseteq$.

311I The topology of a Stone space: Theorem Let Z be the Stone space of a Boolean ring $\mathfrak{A}$, and let $\mathfrak{T}$ be

$$\{G : G \subseteq Z \text{ and for every } z \in G \text{ there is an } a \in \mathfrak{A} \text{ such that } z \in \widehat{a} \subseteq G\}.$$

Then $\mathfrak{T}$ is a topology on Z, under which Z is a locally compact zero-dimensional Hausdorff space, and $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$ is precisely the set of compact open subsets of Z. $\mathfrak{A}$ is a Boolean algebra iff Z is compact.

proof (a) Because $\mathcal{E}$ is closed under $\cap$, and $\bigcup \mathcal{E} = Z$ (recall that Z is the set of surjective homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$, so that every $z \in Z$ is somewhere non-zero and belongs to some $\widehat{a}$), $\mathcal{E}$ is a topology base, and $\mathfrak{T}$ is a topology.

(b) $\mathfrak{T}$ is Hausdorff. **P** Take any distinct z, $w \in Z$. Then there is an $a \in \mathfrak{A}$ such that $z(a) \neq w(a)$; let us take it that $z(a) = 1$, $w(a) = 0$. There is also a $b \in \mathfrak{A}$ such that $w(b) = 1$, so that $w(b + ab) = w(b) +_2 w(a)w(b) = 1$ and $w \in (b + ab)\widehat{\ }$; also

$$a(b + ab) = ab + a^2b = ab + ab = 0,$$

so

$$\widehat{a} \cap (b + ab)\widehat{\ } = (a(b + ab))\widehat{\ } = \widehat{0} = \emptyset,$$

and $\widehat{a}$, $(b + ab)\widehat{\ }$ are disjoint members of $\mathfrak{T}$ containing z, w respectively. **Q**

(c) If $a \in \mathfrak{A}$ then $\widehat{a}$ is compact. **P** Let $\mathcal{F}$ be an ultrafilter on Z containing $\widehat{a}$. For each $b \in \mathfrak{A}$, $z_0(b) = \lim_{z\to\mathcal{F}} z(b)$ must be defined in $\mathbb{Z}_2$, since one of the sets $\{z : z(b) = 0\}$, $\{z : z(b) = 1\}$ must belong to $\mathcal{F}$. If b, $c \in \mathfrak{A}$, then the set

$$F = \{z : z(b) = z_0(b),\ z(c) = z_0(c),\ z(b + c) = z_0(b + c),\ z(bc) = z_0(bc)\}$$

belongs to $\mathcal{F}$, so is not empty; take any $z_1 \in F$; then

$$z_0(b + c) = z_1(b + c) = z_1(b) +_2 z_1(c) = z_0(b) +_2 z_0(c),$$

$$z_0(bc) = z_1(bc) = z_1(b)z_1(c) = z_0(b)z_0(c).$$

As b, c are arbitrary, $z_0 : \mathfrak{A} \to \mathbb{Z}_2$ is a ring homomorphism. Also $z_0(a) = 1$, because $\widehat{a} \in \mathcal{F}$, so $z_0 \in \widehat{a}$. Now let G be any open subset of Z containing z_0; then there is a $b \in \mathfrak{A}$ such that $z_0 \subseteq \widehat{b} \subseteq G$; since $\lim_{z\to\mathcal{F}} z(b) = z_0(b) = 1$, we must have $\widehat{b} = \{z : z(b) = 1\} \in \mathcal{F}$ and $G \in \mathcal{F}$. Thus $\mathcal{F}$ converges to z_0. As $\mathcal{F}$ is arbitrary, $\widehat{a}$ is compact (2A3R). **Q**

(d) This shows that $\widehat{a}$ is a compact open set for every $a \in \mathfrak{A}$. Moreover, since every point of Z belongs to some $\widehat{a}$, every point of Z has a compact neighbourhood, and Z is locally compact. Every $\widehat{a}$ is closed (because it is compact, or otherwise), so $\mathcal{E}$ is a base for $\mathfrak{T}$ consisting of open-and-closed sets, and $\mathfrak{T}$ is zero-dimensional.

(e) Now suppose that $E \subseteq Z$ is an open compact set. If $E = \emptyset$ then $E = \widehat{0}$. Otherwise, set

$$\mathcal{G} = \{\widehat{a} : a \in \mathfrak{A},\ \widehat{a} \subseteq E\}.$$

Then $\mathcal{G}$ is a family of open subsets of Z and $\bigcup \mathcal{G} = E$, because E is open. But E is also compact, so there is a finite $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $E = \bigcup \mathcal{G}_0$. Express $\mathcal{G}_0$ as $\{\widehat{a}_0, \dots, \widehat{a}_n\}$. Then

$$E = \widehat{a}_0 \cup \ldots \cup \widehat{a}_n = (a_0 \cup \ldots \cup a_n)\widehat{\ }.$$

This shows that every compact open subset of Z is of the form $\widehat{a}$ for some $a \in \mathfrak{A}$.

(f) Finally, if $\mathfrak{A}$ is a Boolean algebra then $Z = \widehat{1}$ is compact, by (c); while if Z is compact then (e) tells us that $Z = \widehat{a}$ for some $a \in \mathfrak{A}$, and of course this a must be a multiplicative identity for $\mathfrak{A}$, so that $\mathfrak{A}$ is a Boolean algebra.

311J We have a kind of converse of Stone's theorem.

Proposition Let X be a locally compact zero-dimensional Hausdorff space. Then the set $\mathfrak{A}$ of open-and-compact subsets of X is a subring of $\mathcal{P}X$. If Z is the Stone space of $\mathfrak{A}$, there is a unique homeomorphism $\theta : Z \to X$ such that $\widehat{a} = \theta^{-1}[a]$ for every $a \in \mathfrak{A}$.

proof (a) Because X is Hausdorff, all its compact sets are closed, so every member of $\mathfrak{A}$ is closed. Consequently $a \cup b$, $a \setminus b$, $a \cap b$ and $a \triangle b$ belong to $\mathfrak{A}$ for all a, $b \in \mathfrak{A}$, and $\mathfrak{A}$ is a subring of $\mathcal{P}X$.

It will be helpful to know that $\mathfrak{A}$ is a base for the topology of X. **P** If $G \subseteq X$ is open and $x \in G$, then (because X is locally compact) there is a compact set $K \subseteq X$ such that $x \in \operatorname{int} K$; now (because X is zero-dimensional) there is an open-and-closed set $a \subseteq X$ such that $x \in a \subseteq G \cap \operatorname{int} K$; because a is a closed subset of a compact subset of X, it is compact, and belongs to $\mathfrak{A}$, while $x \in a \subseteq G$. **Q**

(b) Let $R \subseteq Z \times X$ be the relation

$$\{(z, x)\colon \text{ for every } a \in \mathfrak{A},\ x \in a \iff z(a) = 1\}.$$

Then R is the graph of a bijective function $\theta : Z \to X$.

P **(i)** If $z \in Z$ and x, $x' \in X$ are distinct, then, because X is Hausdorff, there is an open set $G \subseteq X$ containing x and not containing x'; because $\mathfrak{A}$ is a base for the topology of X, there is an $a \in \mathfrak{A}$ such that $x \in a \subseteq G$, so that $x' \notin a$. Now either $z(a) = 1$ and $(z, x') \notin R$, or $z(a) = 0$ and $(z, x) \notin R$. Thus R is the graph of a function θ with domain included in Z and taking values in X.

(ii) If $z \in Z$, there is an $a_0 \in \mathfrak{A}$ such that $z(a_0) = 1$. Consider $\mathcal{A} = \{a : z(a) = 1\}$. This is a family of closed subsets of X containing the compact set a_0, and $a \cap b \in \mathcal{A}$ for all a, $b \in \mathcal{A}$. So $\bigcap \mathcal{A}$ is not empty (3A3Db); take $x \in \bigcap \mathcal{A}$. Then $x \in a$ whenever $z(a) = 1$. On the other hand, if $z(a) = 0$, then

$$z(a_0 \setminus a) = z(a_0 \triangle (a \cap a_0)) = z(a_0) +_2 z(a_0)z(a) = 1,$$

so $x \in a_0 \setminus a$ and $x \notin a$. Thus $(z, x) \in R$ and $\theta(z) = x$ is defined. As z is arbitrary, the domain of θ is the whole of Z.

(iii) If $x \in X$, define $z : \mathfrak{A} \to \mathbb{Z}_2$ by setting $z(a) = 1$ if $x \in a$, 0 otherwise. It is elementary to check that z is a ring homomorphism form $\mathfrak{A}$ to $\mathbb{Z}_2$. To see that it takes the value 1, note that because $\mathfrak{A}$ is a base for the topology of X there is an $a \in \mathfrak{A}$ such that $x \in a$, so that $z(a) = 1$. So $z \in Z$, and of course $(z, x) \in R$. As x is arbitrary, θ is surjective.

(iv) If z, $z' \in Z$ and $\theta(z) = \theta(z')$, then, for any $a \in \mathfrak{A}$,

$$z(a) = 1 \iff \theta(z) \in a \iff \theta(z') \in a \iff z'(a) = 1,$$

so $z = z'$. Thus θ is injective. **Q**

(c) For any $a \in \mathfrak{A}$,

$$\theta^{-1}[a] = \{z : \theta(z) \in a\} = \{z : z(a) = 1\} = \widehat{a}.$$

It follows that θ is a homeomorphism. **P** (i) If $G \subseteq X$ is open, then (because $\mathfrak{A}$ is a base for the topology of X) $G = \bigcup\{a : a \in \mathfrak{A},\, a \subseteq G\}$ and

$$\theta^{-1}[G] = \bigcup\{\theta^{-1}[a] : a \in \mathfrak{A},\, a \subseteq G\} = \bigcup\{\widehat{a} : a \in \mathfrak{A},\, a \subseteq G\}$$

is an open subset of Z. As G is arbitrary, θ is continuous. (ii) On the other hand, if $G \subseteq X$ and $\theta^{-1}[G]$ is open, then $\theta^{-1}[G]$ is of the form $\bigcup_{a \in \mathcal{A}} \widehat{a}$ for some $\mathcal{A} \subseteq \mathfrak{A}$, so that $G = \bigcup \mathcal{A}$ is an open set in X. Accordingly θ is a homeomorphism. **Q**

(d) Finally, I must check the uniqueness of θ. But of course if $\tilde{\theta} : Z \to X$ is any function such that $\tilde{\theta}^{-1}[a] = \widehat{a}$ for every $a \in \mathfrak{A}$, then the graph of $\tilde{\theta}$ must be R, so $\tilde{\theta} = \theta$.

311K Remark Thus we have a correspondence between Boolean rings and zero-dimensional locally compact Hausdorff spaces which is (up to isomorphism, on the one hand, and homeomorphism, on the other) one-to-one. Every property of Boolean rings which we study will necessarily correspond to some property of zero-dimensional locally compact Hausdorff spaces.

311L Complemented distributive lattices I have introduced Boolean algebras through the theory of rings; this seems to be the quickest route to them from an ordinary undergraduate course in abstract algebra. However there are alternative approaches, taking the order structure rather than the algebraic operations as fundamental, and for the sake of an application in Chapter 35 I give the details of one of these.

Proposition Let $\mathfrak{A}$ be a lattice such that

(i) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, $c \in \mathfrak{A}$;

(ii) there is a permutation $a \mapsto a' : \mathfrak{A} \to \mathfrak{A}$ which is order-reversing, that is, $a \le b$ iff $b' \le a'$, and such that $a'' = a$ for every a;

(iii) $\mathfrak{A}$ has a least element 0 and $a \wedge a' = 0$ for every $a \in \mathfrak{A}$.

Then $\mathfrak{A}$ has a Boolean algebra structure for which $a \subseteq b$ iff $a \le b$.

proof (a) Write 1 for $0'$; if $a \in \mathfrak{A}$, then $a' \ge 0$ so $a = a'' \le 0' = 1$, and 1 is the greatest element of $\mathfrak{A}$.

If a, $b \in \mathfrak{A}$ then, because $'$ is an order-reversing permutation, $a' \vee b' = (a \wedge b)'$. **P** For $c \in \mathfrak{A}$,

$$\begin{aligned} a' \vee b' \le c &\iff a' \le c \;\&\; b' \le c \iff c' \le a \;\&\; c' \le b \\ &\iff c' \le a \wedge b \iff (a \wedge b)' \le c. \; \mathbf{Q} \end{aligned}$$

Similarly, $a' \wedge b' = (a \vee b)'$. If a, b, $c \in \mathfrak{A}$ then

$$(a \wedge b) \vee c = ((a' \vee b') \wedge c')' = ((a' \wedge c') \vee (b' \wedge c'))' = (a \vee c) \wedge (b \vee c).$$

(b) Define addition and multiplication on $\mathfrak{A}$ by setting

$$a + b = (a \wedge b') \vee (a' \wedge b), \quad ab = a \wedge b$$

for a, $b \in \mathfrak{A}$.

(c)(i) If a, $b \in \mathfrak{A}$ then

$$\begin{aligned} (a+b)' &= (a' \vee b) \wedge (a \vee b') = (a' \wedge a) \vee (a' \wedge b') \vee (b \wedge a) \vee (b \wedge b') \\ &= 0 \vee (a' \wedge b') \vee (b \wedge a) = (a' \wedge b') \vee (a \wedge b). \end{aligned}$$

So if a, b, $c \in \mathfrak{A}$ then

$$\begin{aligned} (a+b)+c &= ((a+b) \wedge c') \vee ((a+b)' \wedge c) \\ &= (((a \wedge b') \vee (a' \wedge b)) \wedge c') \vee (((a' \wedge b') \vee (a \wedge b)) \wedge c) \\ &= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \vee (a' \wedge b' \wedge c) \vee (a \wedge b \wedge c); \end{aligned}$$

as this last formula is symmetric in a, b and c, it is also equal to $a + (b + c)$. Thus addition is associative.

(ii) For any $a \in \mathfrak{A}$,

$$a + 0 = 0 + a = (a' \wedge 0) \vee (a \wedge 0') = 0 \vee (a \wedge 1) = a,$$

so 0 is the additive identity of $\mathfrak{A}$. Also

$$a + a = (a \wedge a') \vee (a' \wedge a) = 0 \vee 0 = 0$$

so each element of $\mathfrak{A}$ is its own additive inverse, and $(\mathfrak{A}, +)$ is a group. It is abelian because $\vee$ and $\wedge$ are commutative.

(d) Because $\wedge$ is associative and commutative, $(\mathfrak{A}, \cdot)$ is a commutative semigroup; also 1 is its identity, because $a \wedge 1 = a$ for every $a \in \mathfrak{A}$. As for the distributive law in $\mathfrak{A}$,

$$\begin{aligned} ab + ac &= (a \wedge b \wedge (a \wedge c)') \vee ((a \wedge b)' \wedge a \wedge c) \\ &= (a \wedge b \wedge (a' \vee c')) \vee ((a' \vee b') \wedge a \wedge c) \\ &= (a \wedge b \wedge a') \vee (a \wedge b \wedge c') \vee (a' \wedge a \wedge c) \vee (b' \wedge a \wedge c) \\ &= (a \wedge b \wedge c') \vee (b' \wedge a \wedge c) \\ &= a \wedge ((b \wedge c') \vee (b' \wedge c)) = a(b + c) \end{aligned}$$

for all a, b, $c \in \mathfrak{A}$. Thus $(\mathfrak{A}, +, \cdot)$ is a ring; because $a \wedge a = a$ for every a, it is a Boolean ring.

(e) For a, $b \in \mathfrak{A}$,

$$a \subseteq b \iff ab = a \iff a \wedge b = a \iff a \le b,$$

so the order relations of $\mathfrak{A}$ coincide.

Remark It is the case that the Boolean algebra structure of $\mathfrak{A}$ is uniquely determined by its order structure, but I delay the proof to the next section (312M).

311X Basic exercises (a) Let $A_0, \dots, A_n$ be sets. Show that

$$A_0 \triangle \dots \triangle A_n = \{x : \#(\{i : i \le n,\ x \in A_i\}) \text{ is odd}\}.$$

(b) Let X be a set, and $\Sigma \subseteq \mathcal{P}X$. Show that the following are equiveridical: (i) Σ is an algebra of subsets of X; (ii) Σ is a subring of $\mathcal{P}X$ (that is, contains $\emptyset$ and is closed under $\triangle$ and $\cap$) and contains X; (iii) $\emptyset \in \Sigma$, $X \setminus E \in \Sigma$ for every $E \in \Sigma$, and $E \cap F \in \Sigma$ for all E, $F \in \Sigma$.

(c) Let $\mathfrak{A}$ be any Boolean ring. Let $a \mapsto a'$ be any bijection between $\mathfrak{A}$ and a set B disjoint from $\mathfrak{A}$. Set $\mathfrak{B} = \mathfrak{A} \cup B$, and extend the addition and multiplication of $\mathfrak{A}$ to form binary operations on $\mathfrak{B}$ by using the formulae

$$a + b' = a' + b = (a + b)', \quad a' + b' = a + b,$$

$$a'b = b + ab, \quad ab' = a + ab, \, a'b' = (a + b + ab)'.$$

Show that $\mathfrak{B}$ is a Boolean algebra and that $\mathfrak{A}$ is an ideal in $\mathfrak{B}$.

>(d) Let $\mathfrak{A}$ be a Boolean ring, and K a finite subset of $\mathfrak{A}$. Show that the subring of $\mathfrak{A}$ generated by K has at most $2^{2^{\#(K)}-1}$ members. (*Hint*: count its minimal non-zero elements.)

>(e) Show that any finite Boolean ring is isomorphic to $\mathcal{P}X$ for some finite set X (and, in particular, is a Boolean algebra).

(f) Let $\mathfrak{A}$ be any Boolean ring. Show that

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c), \quad a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$$

for all a, b, $c \in \mathfrak{A}$ directly from the definitions in 311G, without using Stone's theorem.

>(g) Let $\mathfrak{A}$ be any Boolean ring. Show that if we regard the Stone space Z of $\mathfrak{A}$ as a subset of $\{0,1\}^{\mathfrak{A}}$, then the topology of Z (311I) is just the subspace topology induced by the ordinary product topology of $\{0,1\}^{\mathfrak{A}}$.

(h) Let I be any set, and set $X = \{0,1\}^I$ with its usual topology (3A3K). Show that for a subset E of X the following are equiveridical: (i) E is open-and-compact; (ii) E is determined by coordinates in a finite subset of I (definition: 254M); (iii) E belongs to the algebra of subsets of X generated by $\{E_i : i \in I\}$, where $E_i = \{x : x(i) = 1\}$ for each i.

(i) Let $(\mathfrak{A}, \le)$ be a lattice such that (α) $\mathfrak{A}$ has a least element 0 and a greatest element 1 (β) for every a, b, $c \in \mathfrak{A}$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (γ) for every $a \in \mathfrak{A}$ there is an $a' \in \mathfrak{A}$ such that $a \vee a' = 1$ and $a \wedge a' = 0$. Show that there is a Boolean algebra structure on $\mathfrak{A}$ for which $\le$ agrees with $\subseteq$.

311Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean ring, and $\mathfrak{B}$ the Boolean algebra constructed by the method of 311Xc. Show that the Stone space of $\mathfrak{B}$ can be identified with the one-point compactification (3A3O) of the Stone space of $\mathfrak{A}$.

(b) Let $(\mathfrak{A}, \vee, \wedge, 0, 1)$ be such that (i) $(\mathfrak{A}, \vee)$ is a commutative semigroup with identity 0 (ii) $(\mathfrak{A}, \wedge)$ is a commutative semigroup with identity 1 (iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all a, b, $c \in \mathfrak{A}$ (iv) $a \vee a = a \wedge a = a$ for every $a \in \mathfrak{A}$ (v) for every $a \in \mathfrak{A}$ there is an $a' \in \mathfrak{A}$ such that $a \vee a' = 1$ and $a \wedge a' = 0$. Show that there is a Boolean algebra structure on $\mathfrak{A}$ for which $\vee = \cup$, $\wedge = \cap$.

(c) Let $(\mathfrak{A}, \vee, ')$ be such that (i) $(\mathfrak{A}, \vee)$ is a non-empty commutative semigroup (ii) $' : \mathfrak{A} \to \mathfrak{A}$ is a function (iii) $((a \vee b)' \vee (a \vee b')')' = a$ for all a, $b \in \mathfrak{A}$. Show that there is a Boolean algebra structure on $\mathfrak{A}$ for which $\vee = \cup$ and $'$ is complementation. (*Hint*: McCune 97.)

(d) Let P be a distributive lattice, and Z the set of surjective lattice homomorphisms from P to $\{0,1\}$. Show that there is a sublattice of $\mathcal{P}Z$ isomorphic to P.

311 Notes and comments My aim in this section has been to get as quickly as possible to Stone's theorem, since this is surely the best route to a picture of general Boolean algebras; they are isomorphic to algebras of sets. This means that all their elementary algebraic properties – indeed, all their first-order properties – can be effectively studied in the context of elementary set theory. In 311G-311H I describe a few of the ways in which the Stone representation suggests algebraic properties of Boolean algebras.

You should not, however, come too readily to the conclusion that Boolean algebras will never be able to surprise you. In this book, in particular, we shall need to work a good deal with suprema and infima of infinite sets in Boolean algebras, for the ordering of 311H; and even though this corresponds to the ordering $\subseteq$ of ordinary sets, we find that $(\sup A)\widehat{}$ is sufficiently different from $\bigcup_{a \in A} \widehat{a}$ to need new kinds of intuition. (The point is that $\bigcup_{a \in A} \widehat{a}$ is an open set in the Stone space, but need not be compact if A is infinite, so may well be smaller than $(\sup A)\widehat{}$, even when $\sup A$ is defined in $\mathfrak{A}$.) There is also the fact that Stone's theorem depends crucially on a fairly strong form of the axiom of choice (employed through Zorn's Lemma in the argument of 311D). Of course I shall be using the axiom of choice without scruple throughout this volume. But it should be clear that such results as 312B-312C in the next section cannot possibly need the axiom of choice for their proofs, and that to use Stone's theorem in such a context is slightly misleading.

Nevertheless, it is so useful to be able to regard a Boolean algebra as an algebra of sets – especially when dealing with only finitely many elements of the algebra at a time – that henceforth I will almost always use the symbols $\triangle$, $\cap$ for the addition and multiplication of a Boolean ring, and will use $\cup$, $\setminus$, $\subseteq$ without further comment, just as if I were considering $\cup$, $\setminus$ and $\subseteq$ in the Stone space. (In 311Gb I have given a definition of 'disjointness' in a Boolean algebra based on the same idea.) Even without the axiom of choice this approach can be justified, once we have observed that finitely-generated Boolean algebras are finite (311Xd), since relatively elementary methods show that any finite Boolean algebra is isomorphic to $\mathcal{P}X$ for some finite set X.

I have taken a Boolean algebra to be a particular kind of commutative ring with identity. Of course there are other approaches. If we wish to think of the order relation as primary, then 311L and 311Xi are reasonably natural. Other descriptions can be based on a list of the properties of the binary operations $\cup$, $\cap$ and the complementation operation $a \mapsto a' = 1 \setminus a$, as in 311Yb. (The hardest I know of is in 311Yc.) I give extra space to 311L only because this is well adapted to an application in 352Q below.

312 Homomorphisms

I continue the theory of Boolean algebras with a section on subalgebras, ideals and homomorphisms. From now on, I will relegate Boolean rings which are not algebras to the exercises; I think there is no need to set out descriptions of the (mostly trifling) modifications necessary to deal with the extra generality. The first part of the section (312A-312L) concerns the translation of the basic concepts of ring theory into the language which I propose to use for Boolean algebras. 312M shows that the order relation on a Boolean algebra defines the algebraic structure, and in 312N-312O I give a fundamental result on the extension of homomorphisms. I end the section with results relating the previous ideas to the Stone representation of a Boolean algebra (312P-312T).

312A Subalgebras Let $\mathfrak{A}$ be a Boolean algebra. I will use the phrase **subalgebra of** $\mathfrak{A}$ to mean a subring of $\mathfrak{A}$ containing its multiplicative identity $1 = 1_{\mathfrak{A}}$.

312B Proposition Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subset of $\mathfrak{A}$. Then the following are equiveridical, that is, if one is true so are the others:

(i) $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$;

(ii) $0 \in \mathfrak{B}$, $a \cup b \in \mathfrak{B}$ for all $a, b \in \mathfrak{B}$, and $1 \setminus a \in \mathfrak{B}$ for all $a \in \mathfrak{B}$;

(iii) $\mathfrak{B} \neq \emptyset$, $a \cap b \in \mathfrak{B}$ for all $a, b \in \mathfrak{B}$, and $1 \setminus a \in \mathfrak{B}$ for all $a \in \mathfrak{B}$.

proof (i)⇒(iii) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, and $a, b \in \mathfrak{B}$, then of course we shall have

$$0,\ 1 \in \mathfrak{B}, \text{ so } \mathfrak{B} \neq \emptyset,$$

$$a \cap b \in \mathfrak{B}, \quad 1 \setminus a = 1 \triangle a \in \mathfrak{B}.$$

(iii)⇒(ii) If (iii) is true, then there is some $b_0 \in \mathfrak{B}$; now $1 \setminus b_0 \in \mathfrak{B}$, so

$$0 = b_0 \cap (1 \setminus b_0) \in \mathfrak{B}.$$

If $a, b \in \mathfrak{B}$, then

$$a \cup b = 1 \setminus ((1 \setminus a) \cap (1 \setminus b)) \in \mathfrak{B}.$$

So (ii) is true.

(ii)⇒(i) If (ii) is true, then for any $a, b \in \mathfrak{B}$,

$$a \cap b = 1 \setminus ((1 \setminus a) \cup (1 \setminus b)) \in \mathfrak{B},$$

$$a \triangle b = (a \cap (1 \setminus b)) \cup (b \cap (1 \setminus a)) \in \mathfrak{B},$$

so (because also $0 \in \mathfrak{B}$) $\mathfrak{B}$ is a subring of $\mathfrak{A}$, and

$$1 = 1 \setminus 0 \in \mathfrak{B},$$

so $\mathfrak{B}$ is a subalgebra.

Remark Thus an algebra of subsets of a set X, as defined in 136E or 311Bb, is just a subalgebra of the Boolean algebra $\mathcal{P}X$.

312C Ideals in Boolean algebras: Proposition If $\mathfrak{A}$ is a Boolean algebra, a set $I \subseteq \mathfrak{A}$ is an ideal of $\mathfrak{A}$ iff $0 \in I$, $a \cup b \in I$ for all a, $b \in I$, and $a \in I$ whenever $b \in I$ and $a \subseteq b$.

proof (a) Suppose that I is an ideal. Then of course $0 \in I$. If a, $b \in I$ then $a \cap b \in I$ so $a \cup b = (a \triangle b) \triangle (a \cap b) \in I$. If $b \in I$ and $a \subseteq b$ then $a = a \cap b \in I$.

(b) Now suppose that I satisfies the conditions proposed. If a, $b \in I$ then

$$a \triangle b \subseteq a \cup b \in I$$

so $a \triangle b \in I$, while of course $-a = a \in I$, and also $0 \in I$, by hypothesis; thus I is a subgroup of $(\mathfrak{A}, \triangle)$. Finally, if $a \in I$ and $b \in \mathfrak{A}$ then

$$a \cap b \subseteq a \in I,$$

so $b \cap a = a \cap b \in I$; thus I is an ideal.

312D Principal ideals Of course, while an ideal I in a Boolean algebra $\mathfrak{A}$ is necessarily a subring, it is not as a rule a subalgebra, except in the special case $I = \mathfrak{A}$. But if we say that a **principal ideal** of $\mathfrak{A}$ is the ideal $\mathfrak{A}_a$ generated by a single element a of $\mathfrak{A}$, we have a special phenomenon.

312E Proposition Let $\mathfrak{A}$ be a Boolean algebra, and a any element of $\mathfrak{A}$. Then the principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$ generated by a is just $\{b : b \in \mathfrak{A},\, b \subseteq a\}$, and (with the inherited operations $\cap \restriction \mathfrak{A}_a \times \mathfrak{A}_a$, $\triangle \restriction \mathfrak{A}_a \times \mathfrak{A}_a$) is a Boolean algebra in its own right, with multiplicative identity a.

proof $b \subseteq a$ iff $b \cap a = b$, so that

$$\mathfrak{A}_a = \{b : b \subseteq a\} = \{b \cap a : b \in \mathfrak{A}\}$$

is an ideal of $\mathfrak{A}$, and of course it is the smallest ideal of $\mathfrak{A}$ containing a. Being an ideal, it is a subring; the idempotent relation $b \cap b = b$ is inherited from $\mathfrak{A}$, so it is a Boolean ring; and a is plainly its multiplicative identity.

312F Boolean homomorphisms Now suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two Boolean algebras. I will use the phrase **Boolean homomorphism** to mean a function $\pi : \mathfrak{A} \to \mathfrak{B}$ which is a ring homomorphism (that is, $\pi(a \triangle b) = \pi a \triangle \pi b$, $\pi(a \cap b) = \pi a \cap \pi b$ for all a, $b \in \mathfrak{A}$) and is uniferent, that is, $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

312G Proposition Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be Boolean algebras.
(a) If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean homomorphism, then $\pi[\mathfrak{A}]$ is a subalgebra of $\mathfrak{B}$.
(b) If $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\theta : \mathfrak{B} \to \mathfrak{C}$ are Boolean homomorphisms, then $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ is a Boolean homomorphism.
(c) If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a bijective Boolean homomorphism, then $\pi^{-1} : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism.

proof These are all immediate consequences of the corresponding results for ring homomorphisms (3A2D).

312H Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a function. Then the following are equiveridical:
(i) π is a Boolean homomorphism;
(ii) $\pi(a \cap b) = \pi a \cap \pi b$ and $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$ for all a, $b \in \mathfrak{A}$;
(iii) $\pi(a \cup b) = \pi a \cup \pi b$ and $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$ for all a, $b \in \mathfrak{A}$;
(iv) $\pi(a \cup b) = \pi a \cup \pi b$ and $\pi a \cap \pi b = 0_{\mathfrak{B}}$ whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0_{\mathfrak{A}}$, and $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

proof (i)⇒(iv) If π is a Boolean homomorphism then of course $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$; also, given that $a \cap b = 0$ in $\mathfrak{A}$,

$$\pi a \cap \pi b = \pi(a \cap b) = \pi(0_{\mathfrak{A}}) = 0_{\mathfrak{B}},$$

$$\pi(a \cup b) = \pi(a \triangle b) = \pi a \triangle \pi b = \pi a \cup \pi b.$$

(iv)⇒(iii) Assume (iv), and take a, $b \in \mathfrak{A}$. Then

$$\pi a = \pi(a \cap b) \cup \pi(a \setminus b), \quad \pi b = \pi(a \cap b) \cup \pi(b \setminus a),$$

so

$$\pi(a \cup b) = \pi a \cup \pi(b \setminus a) = \pi(a \cap b) \cup \pi(a \setminus b) \cup \pi(b \setminus a) = \pi a \cup \pi b.$$

Taking $b = 1 \setminus a$, we must have

$$1_{\mathfrak{B}} = \pi(1_{\mathfrak{A}}) = \pi a \cup \pi(1_{\mathfrak{A}} \setminus a), \quad 0_{\mathfrak{B}} = \pi a \cap \pi(1_{\mathfrak{A}} \setminus a),$$

so $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$. Thus (iii) is true.

(iii)⇒(ii) If (iii) is true and a, $b \in \mathfrak{A}$, then

$$\begin{aligned}\pi(a \cap b) &= \pi(1_{\mathfrak{A}} \setminus ((1_{\mathfrak{A}} \setminus a) \cup (1_{\mathfrak{A}} \setminus b))) \\ &= 1_{\mathfrak{B}} \setminus ((1_{\mathfrak{B}} \setminus \pi a) \cup (1_{\mathfrak{B}} \setminus \pi b))) = \pi a \cap \pi b.\end{aligned}$$

So (ii) is true.

(ii)⇒(i) If (ii) is true, then

$$\begin{aligned}\pi(a \triangle b) &= \pi((1_{\mathfrak{A}} \setminus ((1_{\mathfrak{A}} \setminus a) \cap (1_{\mathfrak{A}} \setminus b))) \cap (1_{\mathfrak{A}} \setminus (a \cap b))) \\ &= (1_{\mathfrak{B}} \setminus ((1_{\mathfrak{B}} \setminus \pi a) \cap (1_{\mathfrak{B}} \setminus \pi b)) \cap (1_{\mathfrak{B}} \setminus (\pi a \cap \pi b))) = \pi a \triangle \pi b\end{aligned}$$

for all a, $b \in \mathfrak{A}$, so π is a ring homomorphism; and now

$$\pi(1_{\mathfrak{A}}) = \pi(1_{\mathfrak{A}} \setminus 0_{\mathfrak{A}}) = 1_{\mathfrak{B}} \setminus \pi(0_{\mathfrak{A}}) = 1_{\mathfrak{B}} \setminus 0_{\mathfrak{B}} = 1_{\mathfrak{B}},$$

so that π is a Boolean homomorphism.

312I Proposition If $\mathfrak{A}$, $\mathfrak{B}$ are Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean homomorphism, then $\pi a \subseteq \pi b$ whenever $a \subseteq b$ in $\mathfrak{A}$.

proof

$$a \subseteq b \Longrightarrow a \cap b = a \Longrightarrow \pi a \cap \pi b = \pi a \Longrightarrow \pi a \subseteq \pi b.$$

312J Proposition Let $\mathfrak{A}$ be a Boolean algebra, and a any member of $\mathfrak{A}$. Then the map $b \mapsto a \cap b$ is a surjective Boolean homomorphism from $\mathfrak{A}$ onto the principal ideal $\mathfrak{A}_a$ generated by a.

proof This is an elementary verification.

***312K Fixed-point subalgebras** For future reference I introduce the following idea. If $\mathfrak{A}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism, then $\{a : a \in \mathfrak{A}, \pi a = a\}$ is a subalgebra of $\mathfrak{A}$ (put the definitions 312A and 312F together); I will call it the **fixed-point subalgebra** of π.

312L Quotient algebras: Proposition Let $\mathfrak{A}$ be a Boolean algebra and I an ideal of $\mathfrak{A}$. Then the quotient ring $\mathfrak{A}/I$ (3A2F) is a Boolean algebra, and the canonical map $a \mapsto a^\bullet : \mathfrak{A} \to \mathfrak{A}/I$ is a Boolean homomorphism, so that

$$(a \triangle b)^\bullet = a^\bullet \triangle b^\bullet, \quad (a \cup b)^\bullet = a^\bullet \cup b^\bullet, \quad (a \cap b)^\bullet = a^\bullet \cap b^\bullet, \quad (a \setminus b)^\bullet = a^\bullet \setminus b^\bullet$$

for all a, $b \in \mathfrak{A}$.

(b) The order relation on $\mathfrak{A}/I$ is defined by the formula

$$a^\bullet \subseteq b^\bullet \iff a \setminus b \in I.$$

For any $a \in \mathfrak{A}$,

$$\{u : u \subseteq a^\bullet\} = \{b^\bullet : b \subseteq a\}.$$

proof (a) Of course the map $a \mapsto a^\bullet = \{a \triangle b : b \in I\}$ is a ring homomorphism (3A2Fd). Because

$$(a^\bullet)^2 = (a^2)^\bullet = a^\bullet$$

for every $a \in \mathfrak{A}$, $\mathfrak{A}/I$ is a Boolean ring; because $1^\bullet$ is a multiplicative identity, it is a Boolean algebra, and $a \mapsto a^\bullet$ is a Boolean homomorphism. The formulae given are now elementary.

(b) We have

$$a^\bullet \subseteq b^\bullet \iff a^\bullet \setminus b^\bullet = 0 \iff a \setminus b \in I.$$

Now

$$\{u : u \subseteq a^\bullet\} = \{u \cap a^\bullet : u \in \mathfrak{A}/I\} = \{(b \cap a)^\bullet : b \in \mathfrak{A}\} = \{b^\bullet : b \subseteq a\}.$$

312M The above results are both repetitive and nearly trivial. Now I come to something with a little more meat to it.

Proposition If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a bijection such that $\pi a \subseteq \pi b$ whenever $a \subseteq b$, then π is a Boolean algebra isomorphism.

proof (a) Because π is surjective, there must be c_0, $c_1 \in \mathfrak{A}$ such that $\pi c_0 = 0_{\mathfrak{B}}$, $\pi c_1 = 1_{\mathfrak{B}}$; now $\pi(0_{\mathfrak{A}}) \subseteq \pi c_0$ and $\pi c_1 \subseteq \pi(1_{\mathfrak{A}})$, so we must have $\pi(0_{\mathfrak{A}}) = 0_{\mathfrak{B}}$ and $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

(b) If $a \in \mathfrak{A}$, then $\pi a \cup \pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}}$. **P** There is a $c \in \mathfrak{A}$ such that $\pi c = 1_{\mathfrak{B}} \setminus (\pi a \cup \pi(1_{\mathfrak{A}} \setminus a))$. Now

$$\pi(c \cap a) \subseteq \pi c \cap \pi a = 0_{\mathfrak{B}}, \quad \pi(c \setminus a) \subseteq \pi c \cap \pi(1_{\mathfrak{A}} \setminus a) = 0_{\mathfrak{B}};$$

as π is injective, $c \cap a = c \setminus a = 0_{\mathfrak{A}}$ and $c = 0_{\mathfrak{A}}$, $\pi c = 0_{\mathfrak{B}}$, $\pi a \cup \pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}}$. **Q**

(c) If $a \in \mathfrak{A}$, then $\pi a \cap \pi(1_{\mathfrak{A}} \setminus a) = 0_{\mathfrak{B}}$. **P** It may be clear to you that this is just a dual form of (b). If not, I repeat the argument in the form now appropriate. There is a $c \in \mathfrak{A}$ such that $\pi c = 1_{\mathfrak{B}} \setminus (\pi a \cap \pi(1_{\mathfrak{A}} \setminus a))$. Now

$$\pi(c \cup a) \supseteq \pi c \cup \pi a = 1_{\mathfrak{B}}, \quad \pi(c \cup (1_{\mathfrak{A}} \setminus a)) \supseteq \pi c \cup \pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}};$$

as π is injective, $c \cup a = c \cup (1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{A}}$ and $c = 1_{\mathfrak{A}}$, $\pi c = 1_{\mathfrak{B}}$, $\pi a \cap \pi(1_{\mathfrak{A}} \setminus a) = 0_{\mathfrak{B}}$. **Q**

(d) Putting (b) and (c) together, we have $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$ for every $a \in \mathfrak{A}$. Now $\pi(a \cup b) = \pi a \cup \pi b$ for every a, $b \in \mathfrak{A}$. **P** Surely $\pi a \cup \pi b \subseteq \pi(a \cup b)$. Let $c \in \mathfrak{A}$ be such that $\pi c = \pi(a \cup b) \setminus (\pi a \cup \pi b)$. Then

$$\pi(c \cap a) \subseteq \pi c \cap \pi a = 0_{\mathfrak{B}}, \quad \pi(c \cap b) \subseteq \pi c \cap \pi b = 0_{\mathfrak{B}},$$

so $c \cap a = c \cap b = 0$ and $c \subseteq 1_{\mathfrak{A}} \setminus (a \cup b)$; accordingly

$$\pi c \subseteq \pi(1_{\mathfrak{A}} \setminus (a \cup b)) = 1_{\mathfrak{B}} \setminus \pi(a \cup b);$$

as also $\pi c \subseteq \pi(a \cup b)$, $\pi c = 0_{\mathfrak{B}}$ and $\pi(a \cup b) = \pi a \cup \pi b$. **Q**

(e) So the conditions of 312H(iii) are satisfied and π is a Boolean homomorphism; being bijective, it is an isomorphism.

312N I turn next to a fundamental lemma on the construction of homomorphisms. We need to start with a proper description of a certain type of subalgebra.

Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$; let c be any member of $\mathfrak{A}$. Then

$$\mathfrak{A}_1 = \{(a \cap c) \cup (b \setminus c) : a,\, b \in \mathfrak{A}_0\}$$

is a subalgebra of $\mathfrak{A}$; it is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$.

proof We have to check the following:

$$a = (a \cap c) \cup (a \setminus c) \in \mathfrak{A}_1$$

for every $a \in \mathfrak{A}_0$, so $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$; in particular, $0 \in \mathfrak{A}_1$.

$$1 \setminus ((a \cap c) \cup (b \setminus c)) = ((1 \setminus a) \cap c) \cup ((1 \setminus b) \setminus c) \in \mathfrak{A}_1$$

for all a, $b \in \mathfrak{A}_0$, so $1 \setminus d \in \mathfrak{A}_1$ for every $d \in \mathfrak{A}_1$.

$$(a \cap c) \cup (b \setminus c) \cup (a' \cap c) \cup (b' \setminus c) = ((a \cup a') \cap c) \cup ((b \cup b') \setminus c) \in \mathfrak{A}_1$$

for all a, b, a', $b' \in \mathfrak{A}_0$, so $d \cup d' \in \mathfrak{A}_1$ for all d, $d' \in \mathfrak{A}_1$. Thus $\mathfrak{A}_1$ is a subalgebra of $\mathfrak{A}$ (using 312B).

$$c = (1 \cap c) \cup (0 \setminus c) \in \mathfrak{A}_1,$$

so $\mathfrak{A}_1$ includes $\mathfrak{A}_0 \cup \{c\}$; and finally it is clear that any subalgebra of $\mathfrak{A}$ including $\mathfrak{A}_0 \cup \{c\}$, being closed under $\cap$, $\cup$ and complementation, must include $\mathfrak{A}_1$, so that $\mathfrak{A}_1$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$.

312O Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$, $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism, and $c \in \mathfrak{A}$. If $v \in \mathfrak{B}$ is such that $\pi a \subseteq v \subseteq \pi b$ whenever a, $b \in \mathfrak{A}_0$ and $a \subseteq c \subseteq b$, then there is a unique Boolean homomorphism π_1 from the subalgebra $\mathfrak{A}_1$ of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$ such that π_1 extends π and $\pi_1 c = v$.

proof (a) The basic fact we need to know is that if a, a', b, $b' \in \mathfrak{A}_0$ and

$$(a \cap c) \cup (b \setminus c) = d = (a' \cap c) \cup (b' \setminus c),$$

then

$$(\pi a \cap v) \cup (\pi b \setminus v) = (\pi a' \cap v) \cup (\pi b' \setminus v).$$

P We have

$$a \cap c = d \cap c = a' \cap c.$$

Accordingly $(a \triangle a') \cap c = 0$ and $c \subseteq 1 \setminus (a \triangle a')$. Consequently (since $a \triangle a'$ surely belongs to $\mathfrak{A}_0$)

$$v \subseteq \pi(1 \setminus (a \triangle a')) = 1 \setminus (\pi a \triangle \pi a'),$$

and

$$\pi a \cap v = \pi a' \cap v.$$

Similarly,

$$b \setminus c = d \setminus c = b' \setminus c,$$

so

$$(b \triangle b') \setminus c = 0, \quad b \triangle b' \subseteq c, \quad \pi b \triangle \pi b' = \pi(b \triangle b') \subseteq v$$

and

$$\pi b \setminus v = \pi b' \setminus v.$$

Putting these together, we have the result. **Q**

(b) Consequently, we have a function π_1 defined by writing

$$\pi_1((a \cap c) \cup (b \setminus c)) = (\pi a \cap v) \cup (\pi b \setminus v)$$

for all a, $b \in \mathfrak{A}_0$; and 312N tells us that the domain of π_1 is just $\mathfrak{A}_1$. Now π_1 is a Boolean homomorphism. **P** This amounts to running through the proof of 312N again.

(i) If a, $b \in \mathfrak{A}_0$, then

$$\begin{aligned}\pi_1(1 \setminus ((a \cap c) \cup (b \setminus c))) &= \pi_1(((1 \setminus a) \cap c) \cup ((1 \setminus b) \setminus c)) \\ &= (\pi(1 \setminus a) \cap v) \cup (\pi(1 \setminus b) \setminus v) \\ &= ((1 \setminus \pi a) \cap v) \cup ((1 \setminus \pi b) \setminus v) \\ &= 1 \setminus ((\pi a \cap v) \cup (\pi b \setminus v)) = 1 \setminus \pi_1((a \cap c) \cup (b \setminus c)).\end{aligned}$$

So $\pi_1(1 \setminus d) = 1 \setminus \pi_1 d$ for every $d \in \mathfrak{A}_1$.

(ii) If a, b, a', $b' \in \mathfrak{A}_0$, then

$$\begin{aligned}\pi_1((a \cap c) \cup (b \setminus c) \cup (a' \cap c) \cup (b' \setminus c)) &= \pi_1(((a \cup a') \cap c) \cup ((b \cup b') \setminus c)) \\ &= (\pi(a \cup a') \cap v) \cup (\pi(b \cup b') \setminus v) \\ &= ((\pi a \cup \pi a') \cap v) \cup ((\pi b \cup \pi b') \setminus v) \\ &= (\pi a \cap v) \cup (\pi b \setminus v) \cup (\pi a' \cap v) \cup (\pi b' \setminus v) \\ &= \pi_1((a \cap c) \cup (b \setminus c)) \cup \pi_1((a' \cap c) \cup (b' \setminus c)).\end{aligned}$$

So $\pi_1(d \cup d') = \pi_1 d \cup \pi_1 d'$ for all d, $d' \in \mathfrak{A}_1$.

By 312H(iii), π_1 is a Boolean homomorphism. **Q**

(c) If $a \in \mathfrak{A}_0$, then

$$\pi_1 a = \pi_1((a \cap c) \cup (a \setminus c)) = (\pi a \cap v) \cup (\pi a \setminus v) = \pi a,$$

so π_1 extends π. As for the action of π_1 on c,

$$\pi_1 c = \pi_1((1 \cap c) \cup (0 \setminus c)) = (\pi 1 \cap v) \cup (\pi 0 \setminus v) = (1 \cap v) \cup (0 \setminus v) = v,$$

as required.

(d) Finally, the formula of (b) is the only possible definition for any Boolean homomorphism from $\mathfrak{A}_1$ to $\mathfrak{B}$ which will extend π and take c to v. So π_1 is unique.

312P Homomorphisms and Stone spaces Because the Stone space Z of a Boolean algebra $\mathfrak{A}$ (311E) can be constructed explicitly from the algebraic structure of $\mathfrak{A}$, it must in principle be possible to describe any feature of the Boolean structure of $\mathfrak{A}$ in terms of Z. In the next few paragraphs I work through the most important identifications.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; write $\widehat{a} \subseteq Z$ for the open-and-closed set corresponding to $a \in \mathfrak{A}$. Then there is a one-to-one correspondence between ideals I of $\mathfrak{A}$ and open sets $G \subseteq Z$, given by the formulae

$$G = \bigcup_{a\in I} \widehat{a}, \quad I = \{a : \widehat{a} \subseteq G\}.$$

proof (a) For any ideal $I \lhd \mathfrak{A}$, set $H(I) = \bigcup_{a\in I} \widehat{a}$; then $H(I)$ is a union of open subsets of Z, so is open. For any open set $G \subseteq Z$, set $J(G) = \{a : a \in \mathfrak{A},\ \widehat{a} \subseteq G\}$; then $J(G)$ satisfies the conditions of 312C, so is an ideal of $\mathfrak{A}$.

(b) If $I \lhd \mathfrak{A}$, then $J(H(I)) = I$. **P** (i) If $a \in I$, then $\widehat{a} \subseteq H(I)$ so $a \in J(H(I))$. (ii) If $a \in J(H(I))$, then $\widehat{a} \subseteq H(I) = \bigcup_{b\in I} \widehat{b}$. Because $\widehat{a}$ is compact and all the $\widehat{b}$ are open, there must be finitely many $b_0, \dots , b_n \in I$ such that $\widehat{a} \subseteq \widehat{b}_0 \cup \ldots \cup \widehat{b}_n$. But now $a \subseteq b_0 \cup \ldots \cup b_n \in I$, so $a \in I$. **Q**

(c) If $G \subseteq Z$ is open, then $H(J(G)) = G$. **P** (i) If $z \in G$, then (because $\{\widehat{a} : a \in \mathfrak{A}\}$ is a base for the topology of Z) there is an $a \in \mathfrak{A}$ such that $z \in \widehat{a} \subseteq G$; now $a \in J(G)$ and $z \in H(J(G))$. (ii) If $z \in H(J(G))$, there is an $a \in J(G)$ such that $z \in \widehat{a}$; now $\widehat{a} \subseteq G$, so $z \in G$. **Q**

This shows that the maps $G \mapsto J(G)$, $I \mapsto H(I)$ are two halves of a one-to-one correspondence, as required.

312Q Theorem Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z, W; write $\widehat{a} \subseteq Z$, $\widehat{b} \subseteq W$ for the open-and-closed sets corresponding to $a \in \mathfrak{A}$, $b \in \mathfrak{B}$. Then we have a one-to-one correspondence between Boolean homomorphisms $\pi : \mathfrak{A} \to \mathfrak{B}$ and continuous functions $\phi : W \to Z$, given by the formula

$$\pi a = b \iff \phi^{-1}[\widehat{a}] = \widehat{b},$$

that is, $\phi^{-1}[\widehat{a}] = \widehat{\pi a}$.

proof (a) Recall that I have constructed Z and W as the sets of Boolean homomorphisms from $\mathfrak{A}$ and $\mathfrak{B}$ to $\mathbb{Z}_2$ (311F). So if $\pi : \mathfrak{A} \to \mathfrak{B}$ is any Boolean homomorphism, and $w \in W$, $\psi_\pi(w) = w\pi$ is a Boolean homomorphism from $\mathfrak{A}$ to $\mathbb{Z}_2$ (312Gb), and belongs to Z. Now $\psi_\pi^{-1}[\widehat{a}] = \widehat{\pi a}$ for every $a \in \mathfrak{A}$. **P**

$$\psi_\pi^{-1}[\widehat{a}] = \{w : \psi_\pi(w) \in \widehat{a}\} = \{w : w\pi \in \widehat{a}\} = \{w : w\pi(a) = 1\} = \{w : w \in \widehat{\pi a}\}. \textbf{ Q}$$

Consequently ψ_π is continuous. **P** Let G be any open subset of Z. Then $G = \bigcup\{\widehat{a} : \widehat{a} \subseteq G\}$, so

$$\psi_\pi^{-1}[G] = \bigcup\{\psi_\pi^{-1}[\widehat{a}] : \widehat{a} \subseteq G\} = \bigcup\{\widehat{\pi a} : \widehat{a} \subseteq G\}$$

is open. As G is arbitrary, ψ_π is continuous. **Q**

(b) If $\phi : W \to Z$ is continuous, then for any $a \in \mathfrak{A}$ the set $\phi^{-1}[\widehat{a}]$ must be an open-and-closed set in W; consequently there is a unique member of $\mathfrak{B}$, call it $\theta_\phi a$, such that $\phi^{-1}[\widehat{a}] = \widehat{\theta_\phi a}$. Observe that, for any $w \in W$ and $a \in \mathfrak{A}$,

$$w(\theta_\phi a) = 1 \iff w \in \widehat{\theta_\phi a} \iff \phi(w) \in \widehat{a} \iff (\phi(w))(a) = 1,$$

so $\phi(w) = w\theta_\phi$.

Now θ_ϕ is a Boolean homomorphism. **P** (i) If a, $b \in \mathfrak{A}$ then

$$\theta_\phi(a \cup b)\widehat{\ } = \phi^{-1}[(a \cup b)\widehat{\ }] = \phi^{-1}[\widehat{a} \cup \widehat{b}] = \phi^{-1}[\widehat{a}] \cup \phi^{-1}[\widehat{b}] = \widehat{\theta_\phi a} \cup \widehat{\theta_\phi b} = (\theta_\phi a \cup \theta_\phi b)\widehat{\ },$$

so $\theta_\phi(a \cup b) = \theta_\phi a \cup \theta_\phi b$. (ii) If $a \in \mathfrak{A}$, then

$$\theta_\phi(1 \setminus a)\widehat{\ } = \phi^{-1}[(1 \setminus a)\widehat{\ }] = \phi^{-1}[Z \setminus \widehat{a}] = W \setminus \phi^{-1}[\widehat{a}] = W \setminus \widehat{\theta_\phi a} = (1 \setminus \theta_\phi a)\widehat{\ },$$

so $\theta_\phi(1\setminus a)=1\setminus\theta_\phi a$. (iii) By 312H, θ_ϕ is a Boolean homomorphism. **Q**

(c) For any Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$, $\pi=\theta_{\psi_\pi}$. **P** For $a\in\mathfrak{A}$,

$$(\theta_{\psi_\pi}a)\widehat{\ }=\psi_\pi^{-1}[\widehat{a}]=\widehat{\pi a},$$

so $\theta_{\psi_\pi}a=a$. **Q**

(d) For any continuous function $\phi:W\to Z$, $\phi=\psi_{\theta_\phi}$. **P** For any $w\in W$,

$$\psi_{\theta_\phi}(w)=w\theta_\phi=\phi(w).\ \mathbf{Q}$$

(e) Thus $\pi\mapsto\psi_\pi$, $\phi\mapsto\theta_\phi$ are the two halves of a one-to-one correspondence, as required.

312R Theorem Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be Boolean algebras, with Stone spaces Z, W and V. Let $\pi:\mathfrak{A}\to\mathfrak{B}$ and $\theta:\mathfrak{B}\to\mathfrak{C}$ be Boolean homomorphisms, with corresponding continuous functions $\phi:W\to Z$ and $\psi:V\to W$. Then the Boolean homomorphism $\theta\pi:\mathfrak{A}\to\mathfrak{C}$ corresponds to the continuous function $\phi\psi:V\to Z$.

proof For any $a\in\mathfrak{A}$,

$$\widehat{\theta\pi a}=(\theta(\pi a))\widehat{\ }=\psi^{-1}[\widehat{\pi a}]=\psi^{-1}[\phi^{-1}[\widehat{a}]]=(\phi\psi)^{-1}[\widehat{a}].$$

312S Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z and W, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a Boolean homomorphism, with associated continuous function $\phi:W\to Z$. Then

(a) π is injective iff ϕ is surjective;

(b) π is surjective iff ϕ is injective.

proof (a) If $a\in\mathfrak{A}$, then

$$\widehat{a}\cap\phi[W]=\emptyset\iff\phi^{-1}[\widehat{a}]=\emptyset\iff\widehat{\pi a}=\emptyset\iff\pi a=0.$$

Now W is compact, so $\phi[W]$ also is compact, therefore closed, and

$$\begin{aligned}\phi\text{ is not surjective}&\iff Z\setminus\phi[W]\ne\emptyset\\&\iff\text{ there is a non-zero }a\in\mathfrak{A}\text{ such that }\widehat{a}\subseteq Z\setminus\phi[W]\\&\iff\text{ there is a non-zero }a\in\mathfrak{A}\text{ such that }\pi a=0\\&\iff\pi\text{ is not injective}\end{aligned}$$

(3A2Db).

(b)(i) If π is surjective and w, w' are distinct members of W, then there is a $b\in\mathfrak{B}$ such that $w\in\widehat{b}$ and $w'\notin\widehat{b}$. Now $b=\pi a$ for some $a\in\mathfrak{A}$, so $\phi(w)\in\widehat{a}$ and $\phi(w')\notin\widehat{a}$, and $\phi(w)\ne\phi(w')$. As w and w' are arbitrary, ϕ is injective.

(ii) If ϕ is injective and $b\in\mathfrak{B}$, then $K=\phi[\widehat{b}]$, $L=\phi[W\setminus\widehat{b}]$ are disjoint compact subsets of Z. Consider $I=\{a:a\in\mathfrak{A},\,L\cap\widehat{a}=\emptyset\}$. Then $\bigcup_{a\in I}\widehat{a}=Z\setminus L\supseteq K$. Because K is compact and every $\widehat{a}$ is open, there is a finite family $a_0,\dots,a_n\in I$ such that $K\subseteq\widehat{a}_0\cup\ldots\cup\widehat{a}_n$. Set $a=a_0\cup\ldots\cup a_n$. Then $\widehat{a}=\widehat{a}_0\cup\ldots\cup\widehat{a}_n$ includes K and is disjoint from L. So $\widehat{\pi a}=\phi^{-1}[\widehat{a}]$ includes $\widehat{b}$ and is disjoint from $W\setminus\widehat{b}$; that is, $\widehat{\pi a}=\widehat{b}$ and $\pi a=b$. As b is arbitrary, π is surjective.

312T Principal ideals If $\mathfrak{A}$ is a Boolean algebra and $a\in\mathfrak{A}$, we have a natural surjective Boolean homomorphism $b\mapsto b\cap a:\mathfrak{A}\to\mathfrak{A}_a$, the principal ideal generated by a (312J). Writing Z for the Stone space of $\mathfrak{A}$ and Z_a for the Stone space of $\mathfrak{A}_a$, this homomorphism must correspond to an injective continuous function $\phi:Z_a\to Z$ (312Sb). Because Z_a is compact and Z is Hausdorff, ϕ must be a homeomorphism between Z_a and its image $\phi[Z_a]\subseteq Z$ (3A3Dd). To identify $\phi[Z_a]$, note that it is compact, therefore closed, and

$$\begin{aligned}Z\setminus\phi[Z_a]&=\bigcup\{\widehat{b}:b\in\mathfrak{A},\,\widehat{b}\cap\phi[Z_a]=\emptyset\}\\&=\bigcup\{\widehat{b}:\phi^{-1}[\widehat{b}]=\emptyset\}=\bigcup\{\widehat{b}:b\cap a=0\}=Z\setminus\widehat{a},\end{aligned}$$

so that $\phi[Z_a]=\widehat{a}$. It is therefore natural to identify Z_a with the open-and-closed set $\widehat{a}\subseteq Z$.

312X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean ring, and $\mathfrak{B}$ a subset of $\mathfrak{A}$. Show that $\mathfrak{B}$ is a subring of $\mathfrak{A}$ iff $0 \in \mathfrak{B}$ and $a \cup b$, $a \setminus b \in \mathfrak{B}$ for all $a, b \in \mathfrak{B}$.

(b) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a subset of $\mathfrak{A}$. Show that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ iff $1 \in \mathfrak{B}$ and $a \setminus b \in \mathfrak{B}$ for all $a, b \in \mathfrak{B}$.

(c) Let $\mathfrak{A}$ be a Boolean algebra. Suppose that $I \subseteq A \subseteq \mathfrak{A}$ are such that $1 \in A$, $a \cap b \in I$ for all $a, b \in I$ and $a \setminus b \in A$ whenever $a, b \in A$ and $b \subseteq a$. Show that A includes the subalgebra of $\mathfrak{A}$ generated by I. (*Hint*: 136Xf.)

(d) Show that if $\mathfrak{A}$ is a Boolean ring, a set $I \subseteq \mathfrak{A}$ is an ideal of $\mathfrak{A}$ iff $0 \in I$, $a \cup b \in I$ for all $a, b \in I$, and $a \in I$ whenever $b \in I$ and $a \subseteq b$.

(e) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a function such that (i) $\pi(a) \subseteq \pi(b)$ whenever $a \subseteq b$ (ii) $\pi(a) \cap \pi(b) = 0_{\mathfrak{B}}$ whenever $a \cap b = 0_{\mathfrak{A}}$ (iii) $\pi(a) \cup \pi(b) \cup \pi(c) = 1_{\mathfrak{B}}$ whenever $a \cup b \cup c = 1_{\mathfrak{A}}$. Show that π is a Boolean homomorphism.

(f) Let $\mathfrak{A}$ be a Boolean ring, and a any member of $\mathfrak{A}$. Show that the map $b \mapsto a \cap b$ is a ring homomorphism from $\mathfrak{A}$ onto the principal ideal $\mathfrak{A}_a$ generated by a.

(g) Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Boolean rings, and let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be the Boolean algebras constructed from them by the method of 311Xc. Show that any ring homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$ has a unique extension to a Boolean homomorphism from $\mathfrak{B}_1$ to $\mathfrak{B}_2$.

(h) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings, $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$, $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a ring homomorphism, and $c \in \mathfrak{A}$. Show that if $v \in \mathfrak{B}$ is such that $\pi a \setminus v = \pi b \cap v = 0$ whenever $a, b \in \mathfrak{A}_0$ and $a \setminus c = b \cap c = 0$, then there is a unique ring homomorphism π_1 from the subring $\mathfrak{A}_1$ of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$ such that π_1 extends π_0 and $\pi_1 c = v$.

(i) Let $\mathfrak{A}$ be a Boolean ring, and Z its Stone space. Show that there is a one-to-one correspondence between ideals I of $\mathfrak{A}$ and open sets $G \subseteq Z$, given by the formulae $G = \bigcup_{a \in I} \widehat{a}$, $I = \{a : \widehat{a} \subseteq G\}$.

(j) Let $\mathfrak{A}$ be a Boolean algebra, and suppose that $\mathfrak{A}$ is the subalgebra of itself generated by $\mathfrak{A}_0 \cup \{c\}$, where $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}$ and $c \in \mathfrak{A}$. Let Z be the Stone space of $\mathfrak{A}$ and Z_0 the Stone space of $\mathfrak{A}_0$. Let $\psi : Z \to Z_0$ be the continuous surjection corresponding to the embedding of $\mathfrak{A}_0$ in $\mathfrak{A}$. Show that $\psi\restriction\widehat{c}$ and $\psi\restriction Z \setminus \widehat{c}$ are injective.

Now let $\mathfrak{B}$ be another Boolean algebra, with Stone space W, and $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism, with corresponding function $\phi : W \to Z_0$. Show that there is a continuous function $\phi_1 : W \to Z$ such that $\psi\phi_1 = \phi$ iff there is an open-and-closed set $V \subseteq W$ such that $\phi[V] \subseteq \psi[\widehat{c}]$ and $\phi[W \setminus V] \subseteq \psi[Z \setminus \widehat{c}]$.

(k) Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z, and I an ideal of $\mathfrak{A}$, corresponding to an open set $G \subseteq Z$. Show that the Stone space of the quotient algebra $\mathfrak{A}/I$ may be identified with $Z \setminus G$.

312Y Further exercises (a) Find a function $\phi : \mathcal{P}\{0,1,2\} \to \mathbb{Z}_2$ such that $\phi(1 \setminus a) = 1 \setminus \phi a$ for every $a \in \mathcal{P}\{0,1,2\}$ and $\phi(a) \subseteq \phi(b)$ whenever $a \subseteq b$, but ϕ is not a Boolean homomorphism.

(b) Let $\mathfrak{A}$ be the Boolean ring of finite subsets of $\mathbb{N}$. Show that there is a permutation $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi a \subseteq \pi b$ whenever $a \subseteq b$ but π is not a ring homomorphism.

(c) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean rings, with Stone spaces Z, W. Show that we have a one-to-one correspondence between ring homomorphisms $\pi : \mathfrak{A} \to \mathfrak{B}$ and continuous functions $\phi : H \to Z$, where $H \subseteq W$ is an open set, such that $\phi^{-1}[K]$ is compact for every compact set $K \subseteq Z$, given by the formula $\pi a = b \iff \phi^{-1}[\widehat{a}] = \widehat{b}$.

(d) Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be Boolean rings, with Stone spaces Z, W and V. Let $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\theta : \mathfrak{B} \to \mathfrak{C}$ be ring homomorphisms, with corresponding continuous functions $\phi : H \to Z$ and $\psi : G \to W$. Show that the ring homomorphism $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ corresponds to the continuous function $\phi\psi : \psi^{-1}[H] \to Z$.

(e) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings, with Stone spaces Z and W, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a ring homomorphism, with associated continuous function $\phi : H \to Z$. Show that π is injective iff $\phi[H]$ is dense in Z, and that π is surjective iff ϕ is injective and $H = W$.

(f) Let $\mathfrak{A}$ be a Boolean ring and $a \in \mathfrak{A}$. Show that the Stone space of the principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$ generated by a can be identified with the compact open set $\widehat{a}$ in the Stone space of $\mathfrak{A}$. Show that the identity map is a ring homomorphism from $\mathfrak{A}_a$ to $\mathfrak{A}$, and corresponds to the identity function on $\widehat{a}$.

312 Notes and comments The definitions of 'subalgebra' and 'Boolean homomorphism' (312A, 312F), like that of 'Boolean algebra', are a trifle arbitrary, but will be a convenient way of mandating appropriate treatment of multiplicative identities. I run through the work of 312A-312J essentially for completeness; once you are familiar with Boolean algebras, they should all seem obvious. 312M has a little bit more to it. It shows that the order structure of a Boolean algebra defines the ring structure, in a fairly strong sense.

I call 312O a 'lemma', but actually it is the most important result in this section; it is the basic tool we have for extending a homomorphism from a subalgebra to a slightly larger one, and with Zorn's Lemma (another 'lemma' which deserves a capital L) will provide us with general methods of constructing homomorphisms.

In 312P-312T I describe the basic relationships between the Boolean homomorphisms and continuous functions on Stone spaces. 312Q-312R show that, in the language of category theory, the Stone representation provides a 'contravariant functor' from the category of Boolean algebras with Boolean homomorphisms to the category of topological spaces with continuous functions. Using 311I-311J, we know exactly which topological spaces appear, the zero-dimensional compact Hausdorff spaces; and we know also that the functor is faithful, that is, that we can recover Boolean algebras and homomorphisms from the corresponding topological spaces and continuous functions. There is an agreeable duality in 312S. All of this can be done for Boolean rings, but there are some extra complications (312Yc-312Yf).

To my mind, the very essence of the theory of Boolean algebras is the fact that they are abstract rings, but at the same time can be thought of 'locally' as algebras of sets. Consequently we can bring two quite separate kinds of intuition to bear. 312O gives an example of a ring-theoretic problem, concerning the extension of homomorphisms, which has a resolution in terms of the order relation, a concept most naturally described in terms of algebras-of-sets. It is very much a matter of taste and habit, but I myself find that a Boolean homomorphism is easiest to think of in terms of its action on finite subalgebras, which are directly representable as $\mathcal{P}X$ for some finite X (311Xe); the corresponding continuous map between Stone spaces is less helpful. I offer 312Xj, the Stone-space version of 312O, for you to test your own intuitions on.

313 Order-continuous homomorphisms

Because a Boolean algebra has a natural partial order (311H), we have corresponding notions of upper bounds, lower bounds, suprema and infima. These are particularly important in the Boolean algebras arising in measure theory, and the infinitary operations 'sup' and 'inf' require rather more care than the basic binary operations '$\cup$', '$\cap$', because intuitions from elementary set theory are sometimes misleading. I therefore take a section to work through the most important properties of these operations, together with the homomorphisms which preserve them.

313A Relative complementation: Proposition Let $\mathfrak{A}$ be a Boolean algebra, e a member of $\mathfrak{A}$, and A a non-empty subset of $\mathfrak{A}$.

(a) If $\sup A$ is defined in $\mathfrak{A}$, then $\inf\{e \setminus a : a \in A\}$ is defined and equal to $e \setminus \sup A$.

(b) If $\inf A$ is defined in $\mathfrak{A}$, then $\sup\{e \setminus a : a \in A\}$ is defined and equal to $e \setminus \inf A$.

proof (a) Writing a_0 for $\sup A$, we have $e \setminus a_0 \subseteq e \setminus a$ for every $a \in A$, so $e \setminus a_0$ is a lower bound for $C = \{e \setminus a : a \in A\}$. Now suppose that c is any lower bound for C. Then (because A is not empty) $c \subseteq e$, and

$$a = (a \setminus e) \cup (e \setminus (e \setminus a)) \subseteq (a_0 \setminus e) \cup (e \setminus c)$$

for every $a \in A$. Consequently $a_0 \subseteq (a_0 \setminus e) \cup (e \setminus c)$ is disjoint from c and

$$c = c \cap e \subseteq e \setminus a_0.$$

Accordingly $e \setminus a_0$ is the greatest lower bound of C, as claimed.

(b) This time set $a_0 = \inf A$, $C = \{e \setminus a : a \in A\}$. As before, $e \setminus a_0$ is surely an upper bound for C. If c is any upper bound for C, then

$$e \setminus c \subseteq e \setminus (e \setminus a) = e \cap a \subseteq a$$

for every $a \in A$, so $e \setminus c \subseteq a_0$ and $e \setminus a_0 \subseteq c$. As c is arbitrary, $e \setminus a_0$ is indeed the least upper bound of C.

Remark In the arguments above I repeatedly encourage you to treat $\cap$, $\cup$, $\setminus$, $\subseteq$ as if they were the corresponding operations and relation of basic set theory. This is perfectly safe so long as we take care that every manipulation so justified has only finitely many elements of the Boolean algebra in hand at once.

313B General distributive laws: Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $e \in \mathfrak{A}$ and $A \subseteq \mathfrak{A}$ is a non-empty set such that $\sup A$ is defined in $\mathfrak{A}$, then $\sup\{e \cap a : a \in A\}$ is defined and equal to $e \cap \sup A$.

(b) If $e \in \mathfrak{A}$ and $A \subseteq \mathfrak{A}$ is a non-empty set such that $\inf A$ is defined in $\mathfrak{A}$, then $\inf\{e \cup a : a \in A\}$ is defined and equal to $e \cup \inf A$.

(c) Suppose that A, $B \subseteq \mathfrak{A}$ are non-empty and $\sup A$, $\sup B$ are defined in $\mathfrak{A}$. Then $\sup\{a \cap b : a \in A, b \in B\}$ is defined and is equal to $\sup A \cap \sup B$.

(d) Suppose that A, $B \subseteq \mathfrak{A}$ are non-empty and $\inf A$, $\inf B$ are defined in $\mathfrak{A}$. Then $\inf\{a \cup b : a \in A, b \in B\}$ is defined and is equal to $\inf A \cup \inf B$.

proof (a) Set

$$B = \{e \setminus a : a \in A\}, \quad C = \{e \setminus b : b \in B\} = \{e \cap a : a \in A\}.$$

Using 313A, we have

$$\inf B = e \setminus \sup A, \qquad \sup C = e \setminus \inf B = e \cap \sup A,$$

as required.

(b) Set $a_0 = \inf A$, $B = \{e \cup a : a \in A\}$. Then $e \cup a_0 \subseteq e \cup a$ for every $a \in A$, so $e \cup a_0$ is a lower bound for B. If c is any lower bound for B, then $c \setminus e \subseteq a$ for every $a \in A$, so $c \setminus e \subseteq a_0$ and $c \subseteq e \cup a_0$; thus $e \cup a_0$ is the greatest lower bound for B, as claimed.

(c) By (a), we have

$$a \cap \sup B = \sup_{b \in B} a \cap b$$

for every $a \in A$, so

$$\sup_{a \in A, b \in B} a \cap b = \sup_{a \in A}(a \cap \sup B) = \sup A \cap \sup B,$$

using (a) again.

(d) Similarly, using (b) twice,

$$\inf_{a \in A, b \in B} a \cup b = \inf_{a \in A}(a \cup \inf B) = \inf A \cup \inf B.$$

313C As always, it is worth developing a representation of the concepts of sup and inf in terms of Stone spaces.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; for $a \in \mathfrak{A}$ write $\widehat{a}$ for the corresponding open-and-closed subset of Z.

(a) If $A \subseteq \mathfrak{A}$ and $a_0 \in \mathfrak{A}$ then $a_0 = \sup A$ in $\mathfrak{A}$ iff $\widehat{a}_0 = \overline{\bigcup_{a \in A} \widehat{a}}$.

(b) If $A \subseteq \mathfrak{A}$ is non-empty and $a_0 \in \mathfrak{A}$ then $a_0 = \inf A$ in $\mathfrak{A}$ iff $\widehat{a}_0 = \operatorname{int} \bigcap_{a \in A} \widehat{a}$.

(c) If $A \subseteq \mathfrak{A}$ is non-empty then $\inf A = 0$ in $\mathfrak{A}$ iff $\bigcap_{a \in A} \widehat{a}$ is nowhere dense in Z.

proof (a) For any $b \in \mathfrak{A}$,

$$\begin{aligned} b \text{ is an upper bound for } A &\iff \widehat{a} \subseteq \widehat{b} \text{ for every } a \in A \\ &\iff \bigcup_{a \in A} \widehat{a} \subseteq \widehat{b} \iff \overline{\bigcup_{a \in A} \widehat{a}} \subseteq \widehat{b} \end{aligned}$$

because $\widehat{b}$ is certainly closed in Z. It follows at once that if $\widehat{a}_0$ is actually equal to $\overline{\bigcup_{a \in A} \widehat{a}}$ then a_0 must be the least upper bound of A in $\mathfrak{A}$. On the other hand, if $a_0 = \sup A$, then $\overline{\bigcup_{a \in A} \widehat{a}} \subseteq \widehat{a}_0$. **?** If $\widehat{a}_0 \ne \overline{\bigcup_{a \in A} \widehat{a}}$, then $\widehat{a}_0 \setminus \overline{\bigcup_{a \in A} \widehat{a}}$ is a non-empty open set in Z, so includes $\widehat{b}$ for some non-zero $b \in \mathfrak{A}$; now $\widehat{a} \subseteq \widehat{a}_0 \setminus \widehat{b}$, so $a \subseteq a_0 \setminus b$ for every $a \in A$, and $a_0 \setminus b$ is an upper bound for A strictly less than a_0. **X** Thus $\widehat{a}_0$ must be exactly $\overline{\bigcup_{a \in A} \widehat{a}}$.

(b) Take complements: setting $a_1 = 1 \setminus a_0$, we have

$$a_0 = \inf A \iff a_1 = \sup_{a\in A} 1 \setminus a$$

(by 313A)

$$\iff \widehat{a}_1 = \overline{\bigcup_{a\in A} Z \setminus \widehat{a}}$$

$$\iff \widehat{a}_0 = Z \setminus \overline{\bigcup_{a\in A} Z \setminus \widehat{a}} = \operatorname{int} \bigcap_{a\in A} \widehat{a}.$$

(c) Since $\bigcap_{a\in A} \widehat{a}$ is surely a closed set, it is nowhere dense iff it has empty interior, that is, iff $0 = \inf A$.

313D I started the section with the results above because they are easily stated and of great importance. But I must now turn to some new definitions, and I think it may help to clarify the ideas involved if I give them in their own natural context, even though this is far more general than we have any immediate need for here.

Definitions Let P be a partially ordered set and C a subset of P.

(a) C is **order-closed** if $\sup A \in C$ whenever A is a non-empty upwards-directed subset of C such that $\sup A$ is defined in P, and $\inf A \in C$ whenever A is a non-empty downwards-directed subset of C such that $\inf A$ is defined in P.

(b) C is **sequentially order-closed** if $\sup_{n\in\mathbb{N}} p_n \in C$ whenever $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in C such that $\sup_{n\in\mathbb{N}} p_n$ is defined in P, and $\inf_{n\in\mathbb{N}} p_n \in C$ whenever $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in C such that $\inf_{n\in\mathbb{N}} p_n$ is defined in P.

Remark I hope it is obvious that an order-closed set is sequentially order-closed.

313E Order-closed subalgebras and ideals Of course, in the very special cases of a subalgebra or ideal of a Boolean algebra, the concepts 'order-closed' and 'sequentially order-closed' have expressions simpler than those in 313D. I spell them out.

(a) Let $\mathfrak{B}$ be a subalgebra of a Boolean algebra $\mathfrak{A}$.

(i) The following are equiveridical:
(α) $\mathfrak{B}$ is order-closed in $\mathfrak{A}$;
(β) $\sup B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ and $\sup B$ is defined in $\mathfrak{A}$;
(β') $\inf B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ and $\inf B$ is defined in $\mathfrak{A}$;
(γ) $\sup B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ is non-empty and upwards-directed and $\sup B$ is defined in $\mathfrak{A}$;
(γ') $\inf B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ is non-empty and downwards-directed and $\inf B$ is defined in $\mathfrak{A}$.

P Of course (β) $\Rightarrow$ (γ). If (γ) is true and $B \subseteq \mathfrak{B}$ is any set with a supremum in $\mathfrak{A}$, then $B' = \{0\} \cup \{b_0 \cup \ldots \cup b_n : b_0, \ldots, b_n \in B\}$ is a non-empty upwards-directed set with the same upper bounds as B, so $\sup B = \sup B' \in \mathfrak{B}$. Thus ($\gamma$) $\Rightarrow$ (β) and (β), (γ) are equiveridical. Next, if (β) is true and $B \subseteq \mathfrak{B}$ is a set with an infimum in $\mathfrak{A}$, then $B' = \{1 \setminus b : b \in \mathfrak{B}\} \subseteq \mathfrak{B}$ and $\sup B' = 1 \setminus \inf B$ is defined, so $\sup B'$ and $\inf B$ belong to $\mathfrak{B}$. Thus (β) $\Rightarrow$ (β'). In the same way, (γ') $\iff$ (β') $\Rightarrow$ (β) and (β), (β'), (γ), (γ') are all equiveridical. But since we also have (α) $\iff$ (γ)&(γ'), (α) is equiveridical with the others. **Q**

Replacing the sets B above by sequences, the same arguments provide conditions for $\mathfrak{B}$ to be sequentially order-closed, as follows.

(ii) The following are equiveridical:
(α) $\mathfrak{B}$ is sequentially order-closed in $\mathfrak{A}$;
(β) $\sup_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}$ and $\sup_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;
(β') $\inf_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}$ and $\inf_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;
(γ) $\sup_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{B}$ and $\sup_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;
(γ') $\inf_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{B}$ and $\inf_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$.

(b) Now suppose that I is an ideal of $\mathfrak{A}$. Then if $A \subseteq I$ is non-empty all lower bounds of A necessarily belong to I; so that

I is order-closed iff $\sup A \in I$ whenever $A \subseteq I$ is non-empty, upwards-directed and has a supremum in $\mathfrak{A}$;

I is sequentially order-closed iff $\sup_{n\in\mathbb{N}} a_n \in I$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in I with a supremum in $\mathfrak{A}$.

Moreover, because I is closed under $\cup$,

I is order-closed iff $\sup A \in I$ whenever $A \subseteq I$ has a supremum in $\mathfrak{A}$;

I is sequentially order-closed iff $\sup_{n\in\mathbb{N}} a_n \in I$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in I with a supremum in $\mathfrak{A}$.

(c) If $\mathfrak{A} = \mathcal{P}X$ is a power set, then a sequentially order-closed subalgebra of $\mathfrak{A}$ is just a σ-algebra of sets, while a sequentially order-closed ideal of $\mathfrak{A}$ is a what I have called a σ-ideal of sets (112Db). If $\mathfrak{A}$ is itself a σ-algebra of sets, then a sequentially order-closed subalgebra of $\mathfrak{A}$ is a 'σ-subalgebra' in the sense of 233A.

Accordingly I will normally use the phrases **σ-subalgebra**, **σ-ideal** for sequentially order-closed subalgebras and ideals of Boolean algebras.

313F Order-closures and generated sets (a) It is an immediate consequence of the definitions that

(i) if $\mathcal{S}$ is any non-empty family of subalgebras of a Boolean algebra $\mathfrak{A}$, then $\bigcap\mathcal{S}$ is a subalgebra of $\mathfrak{A}$;

(ii) if $\mathcal{F}$ is any non-empty family of order-closed subsets of a partially ordered set P, then $\bigcap\mathcal{F}$ is an order-closed subset of P;

(iii) if $\mathcal{F}$ is any non-empty family of sequentially order-closed subsets of a partially ordered set P, then $\bigcap\mathcal{F}$ is a sequentially order-closed subset of P.

(b) Consequently, given any Boolean algebra $\mathfrak{A}$ and a subset B of $\mathfrak{A}$, we have a smallest subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ including B, being the intersection of all the subalgebras of $\mathfrak{A}$ which include B; a smallest σ-subalgebra $\mathfrak{B}_\sigma$ of $\mathfrak{A}$ including B, being the intersection of all the σ-subalgebras of $\mathfrak{A}$ which include B; and a smallest order-closed subalgebra $\mathfrak{B}_\tau$ of $\mathfrak{A}$ including B, being the intersection of all the order-closed subalgebras of $\mathfrak{A}$ which include B. We call $\mathfrak{B}$, $\mathfrak{B}_\sigma$ and $\mathfrak{B}_\tau$ the subalgebra, σ-subalgebra and order-closed subalgebra **generated** by B. (I will return to this in 331E.)

(c) If $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{B}$ any subalgebra of $\mathfrak{A}$, then the smallest order-closed subset $\overline{\mathfrak{B}}$ of $\mathfrak{A}$ which includes $\mathfrak{B}$ is again a subalgebra of $\mathfrak{A}$ (so is the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$). **P** (i) $0 \in \mathfrak{B} \subseteq \overline{\mathfrak{B}}$. (ii) The set $\{b : 1 \setminus b \in \overline{\mathfrak{B}}\}$ is order-closed (use 313A) and includes $\mathfrak{B}$, so includes $\overline{\mathfrak{B}}$; thus $1 \setminus b \in \overline{\mathfrak{B}}$ for every $b \in \overline{\mathfrak{B}}$. (iii) If $c \in \mathfrak{B}$, the set $\{b : b \cup c \in \overline{\mathfrak{B}}\}$ is order-closed (use 313Bb) and includes $\mathfrak{B}$, so includes $\overline{\mathfrak{B}}$; thus $b \cup c \in \overline{\mathfrak{B}}$ whenever $b \in \overline{\mathfrak{B}}$ and $c \in \mathfrak{B}$. (iv) If $c \in \overline{\mathfrak{B}}$, the set $\{b : b \cup c \in \overline{\mathfrak{B}}\}$ is order-closed and includes $\mathfrak{B}$ (by (iii)), so includes $\overline{\mathfrak{B}}$; thus $b \cup c \in \overline{\mathfrak{B}}$ whenever b, $c \in \overline{\mathfrak{B}}$. (v) By 312B(ii), $\overline{\mathfrak{B}}$ is a subalgebra of $\mathfrak{A}$. **Q**

313G This is a convenient moment at which to spell out an abstract version of the Monotone Class Theorem (136B).

Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) Suppose that $1 \in I \subseteq A \subseteq \mathfrak{A}$ and that

$$a \cap b \in I \text{ for all } a,\, b \in I,$$

$$b \setminus a \in A \text{ whenever } a,\, b \in A \text{ and } a \subseteq b.$$

Then A includes the subalgebra of $\mathfrak{A}$ generated by I.

(b) If moreover $\sup_{n\in\mathbb{N}} a_n \in A$ for every non-decreasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A with a supremum in $\mathfrak{A}$, then A includes the σ-subalgebra of $\mathfrak{A}$ generated by I.

(c) And if $\sup C \in A$ whenever $C \subseteq A$ is an upwards-directed set with a supremum in $\mathfrak{A}$, then A includes the order-closed subalgebra of $\mathfrak{A}$ generated by I.

proof (a)(i) Let $\mathfrak{P}$ be the family of all sets J such that $I \subseteq J \subseteq A$ and $a \cap b \in J$ for all a, $b \in J$. Then $I \in \mathfrak{P}$ and if $\mathfrak{Q} \subseteq \mathfrak{P}$ is upwards-directed and not empty, $\bigcup\mathfrak{Q} \in \mathfrak{P}$. By Zorn's Lemma, $\mathfrak{P}$ has a maximal element $\mathfrak{B}$.

(ii) Now

$$\mathfrak{B} = \{c : c \in \mathfrak{A},\, c \cap b \in A \text{ for every } b \in \mathfrak{B}\}.$$

P If $c \in \mathfrak{B}$, then of course $c \cap b \in \mathfrak{B} \subseteq A$ for every $b \in \mathfrak{B}$, because $\mathfrak{B} \in \mathfrak{P}$. If $c \in \mathfrak{A} \setminus \mathfrak{B}$, consider

$$J = \mathfrak{B} \cup \{c \cap b : b \in \mathfrak{B}\}.$$

Then $c = c \cap 1 \in J$ so J properly includes $\mathfrak{B}$ and cannot belong to $\mathfrak{P}$. On the other hand, if b_1, $b_2 \in \mathfrak{B}$,

$$b_1 \cap b_2 \in \mathfrak{B} \subseteq J, \quad (c \cap b_1) \cap b_2 = b_1 \cap (c \cap b_2) = (c \cap b_1) \cap (c \cap b_2) = c \cap (b_1 \cap b_2) \in J,$$

so $c_1 \cap c_2 \in J$ for all c_1, $c_2 \in J$; and of course $I \subseteq \mathfrak{B} \subseteq J$. So J cannot be a subset of A, and there must be a $b \in \mathfrak{B}$ such that $c \cap b \notin A$. **Q**

(iii) Consequently $c \setminus b \in \mathfrak{B}$ whenever b, $c \in \mathfrak{B}$ and $b \subseteq c$. **P** If $a \in \mathfrak{B}$, then $b \cap a$, $c \cap a \in \mathfrak{B} \subseteq A$ and $b \cap a \subseteq c \cap a$, so

$$(c \setminus b) \cap a = (c \cap a) \setminus (b \cap a) \in A$$

by the hypothesis on A. By (ii), $c \setminus b \in \mathfrak{B}$. **Q**

(iv) It follows that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. **P** If $b \in \mathfrak{B}$, then

$$b \subseteq 1 \in I \subseteq \mathfrak{B},$$

so $1 \setminus b \in \mathfrak{B}$. If a, $b \in \mathfrak{B}$, then

$$a \cup b = 1 \setminus ((1 \setminus a) \cap (1 \setminus b)) \in \mathfrak{B}.$$

$0 = 1 \setminus 1 \in \mathfrak{B}$, so that the conditions of 312B(ii) are satisfied. **Q**

Now the subalgebra of $\mathfrak{A}$ generated by I is included in $\mathfrak{B}$ and therefore in A, as required.

(b) Now suppose that $\sup_{n \in \mathbb{N}} a_n$ belongs to A whenever $\langle a_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in A with a supremum in $\mathfrak{A}$. Then $\mathfrak{B}$, as defined in part (a) of the proof, is a σ-subalgebra of $\mathfrak{A}$. **P** Let $\langle b_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{B}$ with a supremum c in $\mathfrak{A}$. Then for any $b \in \mathfrak{B}$, $\langle b_n \cap b \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in A with a supremum $c \cap b$ in $\mathfrak{A}$ (313Ba). So $c \cap b \in A$. As b is arbitrary, $c \in \mathfrak{B}$, by the criterion in (a-ii) above. As $\langle b_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\mathfrak{B}$ is a σ-subalgebra, by 313Ea. **Q**

Accordingly the σ-subalgebra of $\mathfrak{A}$ generated by I is included in $\mathfrak{B}$ and therefore in $\mathfrak{A}$.

(c) Finally, if $\sup C \in A$ whenever C is a non-empty upwards-directed subset of A with a least upper bound in $\mathfrak{A}$, $\mathfrak{B}$ is order-closed. **P** Let $C \subseteq \mathfrak{B}$ be a non-empty upwards-directed set with a supremum c in $\mathfrak{A}$. Then for any $b \in \mathfrak{B}$, $\{c \cap b : c \in C\}$ is a non-empty upwards-directed set in A with supremum $c \cap b$ in $\mathfrak{A}$. So $c \cap b \in A$. As b is arbitrary, $c \in \mathfrak{B}$. As C is arbitrary, $\mathfrak{B}$ is order-closed in $\mathfrak{A}$ (313Ea(i-α)). **Q**

Accordingly the order-closed subalgebra of $\mathfrak{A}$ generated by I is included in $\mathfrak{B}$ and therefore in $\mathfrak{A}$.

313H Definitions It is worth distinguishing various types of supremum- and infimum-preserving function. Once again, I do this in almost the widest possible context. Let P and Q be two partially ordered sets, and $\phi : P \to Q$ an **order-preserving** function, that is, a function such that $\phi(p) \le \phi(q)$ in Q whenever $p \le q$ in P.

(a) I say that ϕ is **order-continuous** if (i) $\phi(\sup A) = \sup_{p \in A} \phi(p)$ whenever A is a non-empty upwards-directed subset of P and $\sup A$ is defined in P (ii) $\phi(\inf A) = \inf_{p \in A} \phi(p)$ whenever A is a non-empty downwards-directed subset of P and $\inf A$ is defined in P.

(b) I say that ϕ is **sequentially order-continuous** or **σ-order-continuous** if (i) $\phi(p) = \sup_{n \in \mathbb{N}} \phi(p_n)$ whenever $\langle p_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in P and $p = \sup_{n \in \mathbb{N}} p_n$ in P (ii) $\phi(p) = \inf_{n \in \mathbb{N}} \phi(p_n)$ whenever $\langle p_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in P and $p = \inf_{n \in \mathbb{N}} p_n$ in P.

Remark You may feel that one of the equivalent formulations in Proposition 313Lb gives a clearer idea of what is really being demanded of ϕ in the ordinary cases we shall be looking at.

313I Proposition Let P, Q and R be partially ordered sets, and $\phi : P \to Q$, $\psi : Q \to R$ order-preserving functions.

(a) $\psi\phi : P \to R$ is order-preserving.

(b) If ϕ and ψ are order-continuous, so is $\psi\phi$.

(c) If ϕ and ψ are sequentially order-continuous, so is $\psi\phi$.

(d) ϕ is order-continuous iff $\phi^{-1}[B]$ is order-closed for every order-closed $B \subseteq Q$.

proof (a)-(c) I think the only point that needs remarking is that if $A \subseteq P$ is upwards-directed, then $\phi[A] \subseteq Q$ is upwards-directed, because ϕ is order-preserving. So if $\sup A$ is defined in P and ϕ, ψ are order-continuous, we shall have

$$\psi(\phi(\sup A)) = \psi(\sup \phi[A]) = \sup \psi[\phi[A]].$$

(d)(i) Suppose that ϕ is order-continuous and that $B \subseteq Q$ is order-closed. Let $A \subseteq \phi^{-1}[B]$ be a non-empty upwards-directed set with supremum $p \in P$. Then $\phi[A] \subseteq B$ is non-empty and upwards-directed, because ϕ is order-preserving, and $\phi(p) = \sup \phi[A]$ because ϕ is order-continuous. Because B is order-closed, $\phi(p) \in B$ and $p \in \phi^{-1}[B]$. Similarly, if $A \subseteq \phi^{-1}[B]$ is non-empty and downwards-directed, and $\inf A$ is defined in P, then $\phi(\inf A) = \inf \phi[A] \in B$ and $\inf A \in \phi^{-1}[B]$. Thus $\phi^{-1}[B]$ is order-closed; as B is arbitrary, ϕ satisfies the condition.

(ii) Now suppose that $\phi^{-1}[B]$ is order-closed in P whenever $B \subseteq Q$ is order-closed in Q. Let $A \subseteq P$ be a non-empty upwards-directed subset of P with a supremum $p \in P$. Then $\phi(p)$ is an upper bound of $\phi[A]$. Let q be any upper bound of $\phi[A]$ in Q. Consider $B = \{r : r \le q\}$; then $B \subseteq Q$ is upwards-directed and order-closed, so $\phi^{-1}[B]$ is order-closed. Also $A \subseteq \phi^{-1}[B]$ is non-empty and upwards-directed and has supremum p, so $p \in \phi^{-1}[B]$ and $\phi(p) \in B$, that is, $\phi(p) \le q$. As q is arbitrary, $\phi(p) = \sup \phi[A]$. Similarly, $\phi(\inf A) = \inf \phi[A]$ whenever $A \subseteq P$ is non-empty, downwards-directed and has an infimum in P; so ϕ is order-continuous.

313J It is useful to introduce here the following notion.

Definition Let $\mathfrak{A}$ be a Boolean algebra. A set $D \subseteq \mathfrak{A}$ is **order-dense** if for every non-zero $a \in \mathfrak{A}$ there is a non-zero $d \in D$ such that $d \subseteq a$.

Remark Many authors use the simple word 'dense' where I have insisted on the phrase 'order-dense'. In the work of this treatise it will be important to distinguish clearly between this concept of 'dense' set and the topological concept (2A3U).

313K Lemma If $\mathfrak{A}$ is a Boolean algebra and $D \subseteq \mathfrak{A}$ is order-dense, then for any $a \in \mathfrak{A}$ there is a disjoint $C \subseteq D$ such that $\sup C = a$; in particular, $a = \sup\{d : d \in D, d \subseteq a\}$ and there is a partition of unity $C \subseteq D$.

proof Set $D_a = \{d : d \in D, d \subseteq a\}$. Applying Zorn's lemma to the family $\mathcal{C}$ of disjoint sets $C \subseteq D_a$, we have a maximal $C \in \mathcal{C}$. Now if $b \in \mathfrak{A}$ and $b \not\supseteq a$, there is a $d \in D$ such that $0 \ne d \subseteq a \setminus b$. Because C is maximal, there must be a $c \in C$ such that $c \cap d \ne 0$, so that $c \not\subseteq b$. Turning this round, any upper bound of C must include a, so that $a = \sup C$. It follows at once that $a = \sup D_a$.

Taking $a = 1$ we obtain a partition of unity included in D.

313L Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a) π is order-preserving.

(b) The following are equiveridical:

(i) π is order-continuous;

(ii) whenever $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and $\inf A = 0$ in $\mathfrak{A}$, then $\inf \pi[A] = 0$ in $\mathfrak{B}$;

(iii) whenever $A \subseteq \mathfrak{A}$ is non-empty and upwards-directed and $\sup A = 1$ in $\mathfrak{A}$, then $\sup \pi[A] = 1$ in $\mathfrak{B}$;

(iv) whenever $A \subseteq \mathfrak{A}$ and $\sup A$ is defined in $\mathfrak{A}$, then $\pi(\sup A) = \sup \pi[A]$ in $\mathfrak{B}$;

(v) whenever $A \subseteq \mathfrak{A}$ and $\inf A$ is defined in $\mathfrak{A}$, then $\pi(\inf A) = \inf \pi[A]$ in $\mathfrak{B}$;

(vi) whenever $C \subseteq \mathfrak{A}$ is a partition of unity, then $\pi[C]$ is a partition of unity in $\mathfrak{B}$.

(c) The following are equiveridical:

(i) π is sequentially order-continuous;

(ii) whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ and $\inf_{n\in\mathbb{N}} a_n = 0$ in $\mathfrak{A}$, then $\inf_{n\in\mathbb{N}} \pi a_n = 0$ in $\mathfrak{B}$;

(iii) whenever $A \subseteq \mathfrak{A}$ is countable and $\sup A$ is defined in $\mathfrak{A}$, then $\pi(\sup A) = \sup \pi[A]$ in $\mathfrak{B}$;

(iv) whenever $A \subseteq \mathfrak{A}$ is countable and $\inf A$ is defined in $\mathfrak{A}$, then $\pi(\inf A) = \inf \pi[A]$ in $\mathfrak{B}$;

(v) whenever $C \subseteq \mathfrak{A}$ is a countable partition of unity, then $\pi[C]$ is a partition of unity in $\mathfrak{B}$.

(d) If π is bijective, it is order-continuous.

proof (a) This is 312I.

(b)(i)⇒(ii) is trivial, as $\pi 0 = 0$.

(ii)⇒(iv) Assume (ii), and let A be any subset of $\mathfrak{A}$ such that $c = \sup A$ is defined in $\mathfrak{A}$. If $A = \emptyset$, then $c = 0$ and $\sup \pi[A] = 0 = \pi c$. Otherwise, set

$$A' = \{a_0 \cup \ldots \cup a_n : a_0, \ldots , a_n \in A\}, \quad C = \{c \setminus a : a \in A'\}.$$

Then A' is upwards-directed and has the same upper bounds as A, so $c = \sup A'$ and $0 = \inf C$, by 313Aa. Also C is downwards-directed, so $\inf \pi[C] = 0$ in $\mathfrak{B}$. But now

$$\pi[C] = \{\pi c \setminus \pi a : a \in A'\} = \{\pi c \setminus b : b \in \pi[A']\},$$

$$\pi[A'] = \{\pi a_0 \cup \ldots \cup \pi a_n : a_0, \ldots , a_n \in A\} = \{b_0 \cup \ldots \cup b_n : b_0, \ldots , b_n \in \pi[A]\},$$

because π is a Boolean homomorphism. Again using 313Aa and the fact that $b \subseteq \pi c$ for every $b \in \pi[A']$, we get

$$\pi c = \sup \pi[A'] = \sup \pi[A].$$

As A is arbitrary, (iv) is satisfied.

(iv)⇒(v) If $A \subseteq \mathfrak{A}$ and $c = \inf A$ is defined in $\mathfrak{A}$, then $1 \setminus c = \sup_{a\in A} 1 \setminus a$, so

$$\pi c = 1 \setminus \pi(1 \setminus c) = 1 \setminus \sup_{a\in A} \pi(1 \setminus a) = \inf_{a\in A} 1 \setminus \pi(1 \setminus a) = \inf_{a\in A} \pi a.$$

(v)⇒(ii) is trivial, because $\pi 0 = 0$.

(iv)⇒(iii) is similarly trivial.

(iii)⇒(vi) Assume (iii), and let C be a partition of unity in $\mathfrak{A}$. Then $C' = \{c_0 \cup \ldots \cup c_n : c_0, \ldots , c_n \in C\}$ is upwards-directed and has supremum 1, so $\sup \pi[C'] = 1$. But (because π is a Boolean homomorphism) $\pi[C]$ and $\pi[C']$ have the same upper bounds, so $\sup \pi[C] = 1$, as required.

(vi)⇒(ii) Assume (vi), and let $A \subseteq \mathfrak{A}$ be a set with infimum 0. Set

$$D = \{d : d \in \mathfrak{A}, \exists\, a \in A,\ d \cap a = 0\}.$$

Then D is order-dense in $\mathfrak{A}$. **P** If $e \in \mathfrak{A} \setminus \{0\}$, then there is an $a \in A$ such that $e \not\subseteq a$, so that $e \setminus a$ is a non-zero member of D included in e. **Q** Consequently there is a partition of unity $C \subseteq D$, by 313K. But now if b is any lower bound for $\pi[A]$ in $\mathfrak{B}$, we must have $b \cap \pi d = 0$ for every $d \in D$, so $\pi c \subseteq 1 \setminus b$ for every $c \in C$, and $1 \setminus b = 1$, $b = 0$. Thus $\inf \pi[A] = 0$. As A is arbitrary, (ii) is satisfied.

(v)&(iv)⇒(i) is trivial.

(c) We can use nearly identical arguments, remembering only to interpolate the word 'countable' from time to time. I spell out the new version of (ii)⇒(iv), even though it requires no more than an adaptation of the language. Assume (ii), and let A be a countable subset of $\mathfrak{A}$ with a supremum $c \in \mathfrak{A}$. If $A = \emptyset$, then $c = 0$ so $\pi c = 0 = \sup \pi[A]$. Otherwise, let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence running over A; set $a'_n = a_0 \cup \ldots \cup a_n$ and $c_n = c \setminus a'_n$ for each n. Then $\langle a'_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, with supremum c, and $\langle c_n\rangle_{n\in\mathbb{N}}$ is non-increasing, with infimum 0; so $\inf_{n\in\mathbb{N}} \pi c_n = 0$ and

$$\sup_{n\in\mathbb{N}} \pi a_n = \sup_{n\in\mathbb{N}} \pi a'_n = \pi c.$$

For (v)⇒(ii), however, a different idea is involved. Assume (v), and suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0. Set $c_0 = 1 \setminus a_0$, $c_n = a_{n-1} \setminus a_n$ for $n \ge 1$; then $C = \{c_n : n \in \mathbb{N}\}$ is a partition of unity in $\mathfrak{A}$ (because if $c \cap c_n = 0$ for every n, then $c \subseteq a_n$ for every n), so $\pi[C]$ is a partition of unity in $\mathfrak{B}$. Now if $b \subseteq \pi a_n$ for every n, $b \cap \pi c_n$ for every n, so $b = 0$; thus $\inf_{n\in\mathbb{N}} \pi a_n = 0$. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, (ii) is satisfied.

(d) Suppose that $A \subseteq \mathfrak{A}$ is non-empty and $\inf A = 0$ in $\mathfrak{A}$. Let $b \in \mathfrak{B}$ be a lower bound for $\pi[A]$. Because π is surjective, there is a $c \in \mathfrak{A}$ such that $\pi c = b$. If $a \in A$, then

$$\pi(a \cap c) = \pi a \cap \pi c = \pi a \cap b = b = \pi c;$$

because π is injective, $a \cap c = c$ and $c \subseteq a$. As a is arbitrary, c is a lower bound of A and must be 0; so $b_0 = \pi 0 = 0$. As b is arbitrary, $\inf \pi[A] = 0$; as A is arbitrary, π is order-continuous, by (b)(ii)⇒(i).

313M The following result is perfectly elementary, but it will save a moment later on to have it spelt out.

Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism.

(a) If $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{B}$, then $\pi^{-1}[\mathfrak{D}]$ is an order-closed subalgebra of $\mathfrak{A}$.

(b) If $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}$ generated by $C \subseteq \mathfrak{A}$, then the order-closed subalgebra $\mathfrak{D}$ of $\mathfrak{B}$ generated by $\pi[C]$ includes $\pi[\mathfrak{C}]$.

(c) Now suppose that π is surjective and that $C \subseteq \mathfrak{A}$ is such that the order-closed subalgebra of $\mathfrak{A}$ generated by C is $\mathfrak{A}$ itself. Then the order-closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$ is $\mathfrak{B}$.

proof (a) Setting $\mathfrak{C} = \pi^{-1}[\mathfrak{D}]$: if $a, a' \in \mathfrak{C}$ then $\pi(a \cap b) = \pi a \cap \pi b$, $\pi(a \triangle a') = \pi a \triangle \pi a' \in \mathfrak{D}$, so $a \cap a'$, $a \triangle a' \in \mathfrak{C}$; $\pi 1 = 1 \in \mathfrak{D}$ so $1 \in \mathfrak{C}$; thus $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$. By 313Id, $\mathfrak{C}$ is order-closed.

(b) By (a), $\pi^{-1}[\mathfrak{D}]$ is an order-closed subalgebra of $\mathfrak{A}$. It includes C so includes $\mathfrak{C}$, and $\pi[\mathfrak{C}] \subseteq \mathfrak{D}$.

(c) In the language of (b), we have $\mathfrak{C} = \mathfrak{A}$, so $\mathfrak{D}$ must be $\mathfrak{B}$.

313N Definition The phrase **regular embedding** is sometimes used to mean an injective order-continuous Boolean homomorphism; a subalgebra $\mathfrak{B}$ of a Boolean algebra $\mathfrak{A}$ is said to be **regularly embedded** in $\mathfrak{A}$ if the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is order-continuous, that is, if whenever $b \in \mathfrak{B}$ is the supremum (in $\mathfrak{B}$) of $B \subseteq \mathfrak{B}$, then b is also the supremum in $\mathfrak{A}$ of B; and similarly for infima. One important case is when $\mathfrak{B}$ is order-dense (313O); another is in 314Ga below.

It will be useful to be able to say '$\mathfrak{B}$ can be regularly embedded in $\mathfrak{A}$' to mean that there is an injective order-continuous Boolean homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$; that is, that $\mathfrak{B}$ is isomorphic to a regularly embedded subalgebra of $\mathfrak{A}$. In this form it is obvious from 313Ib that if $\mathfrak{C}$ can be regularly embedded in $\mathfrak{B}$, and $\mathfrak{B}$ can be regularly embedded in $\mathfrak{A}$, then $\mathfrak{C}$ can be regularly embedded in $\mathfrak{A}$.

313O Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ an order-dense subalgebra of $\mathfrak{A}$. Then $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$. In particular, if $B \subseteq \mathfrak{B}$ and $c \in \mathfrak{B}$ then $c = \sup B$ in $\mathfrak{B}$ iff $c = \sup B$ in $\mathfrak{A}$.

proof I have to show that the identity homomorphism $\iota : \mathfrak{B} \to \mathfrak{A}$ is order-continuous. **?** Suppose, if possible, otherwise. By 313L(b-ii), there is a non-empty set $B \subseteq \mathfrak{B}$ such that $\inf B = 0$ in $\mathfrak{B}$ but $B = \iota[B]$ has a non-zero lower bound $a \in \mathfrak{A}$. In this case, however (because $\mathfrak{B}$ is order-dense) there is a non-zero $d \in \mathfrak{B}$ with $d \subseteq a$, in which case d is a non-zero lower bound for B in $\mathfrak{B}$. **X**

313P The most important use of these ideas to us concerns quotient algebras (313Q); I approach by means of a superficially more general result.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism with kernel I.

(a)(i) If π is order-continuous then I is order-closed.

(ii) If $\pi[\mathfrak{A}]$ is regularly embedded in $\mathfrak{B}$ and I is order-closed then π is order-continuous.

(b)(i) If π is sequentially order-continuous then I is a σ-ideal.

(ii) If $\pi[\mathfrak{A}]$ is regularly embedded in $\mathfrak{B}$ and I is a σ-ideal then π is sequentially order-continuous.

proof (a)(i) If $A \subseteq I$ is upwards-directed and has a supremum $c \in \mathfrak{A}$, then $\pi c = \sup \pi[A] = 0$, so $c \in I$. As remarked in 313Eb, this shows that I is order-closed.

(ii) We are supposing that the identity map from $\pi[\mathfrak{A}]$ to $\mathfrak{B}$ is order-continuous, so it will be enough to show that π is order-continuous when regarded as a map from $\mathfrak{A}$ to $\pi[\mathfrak{A}]$. Suppose that $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and that $\inf A = 0$. **?** Suppose, if possible, that 0 is not the greatest lower bound of $\pi[A]$ in $\pi[\mathfrak{A}]$. Then there is a $c \in \mathfrak{A}$ such that $0 \neq \pi c \subseteq \pi a$ for every $a \in A$. Now

$$\pi(c \setminus a) = \pi c \setminus \pi a = 0$$

for every $a \in A$, so $c \setminus a \in I$ for every $a \in A$. The set $C = \{c \setminus a : a \in A\}$ is upwards-directed and has supremum c; because I is order-closed, $c = \sup C \in I$, and $\pi c = 0$, contradicting the specification of c. **X** Thus $\inf \pi[A] = 0$ in either $\pi[\mathfrak{A}]$ or $\mathfrak{B}$. As A is arbitrary, π is order-continuous, by the criterion (ii) of 313Lb.

(b) Argue in the same way, replacing each set A by a sequence.

313Q Corollary Let $\mathfrak{A}$ be a Boolean algebra and I an ideal of $\mathfrak{A}$; write π for the canonical map from $\mathfrak{A}$ to $\mathfrak{A}/I$.

(a) π is order-continuous iff I is order-closed.

(b) π is sequentially order-continuous iff I is a σ-ideal.

proof $\pi[\mathfrak{A}] = \mathfrak{A}/I$ is surely regularly embedded in $\mathfrak{A}/I$.

313R For order-continuous homomorphisms, at least, there is an elegant characterization in terms of Stone spaces.

Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism. Let Z and W be their Stone spaces, and $\phi : W \to Z$ the corresponding continuous function (312Q). Then the following are equiveridical:

(i) π is order-continuous;
(ii) $\phi^{-1}[M]$ is nowhere dense in W for every nowhere dense set $M \subseteq Z$;
(iii) $\operatorname{int}\phi[H] \neq \emptyset$ for every non-empty open set $H \subseteq W$.

proof (a)(i)⇒(iii) Suppose that π is order-continuous. **?** Suppose, if possible, that $H \subseteq W$ is a non-empty open set and $\operatorname{int}\phi[H] = \emptyset$. Let $b \in \mathfrak{B} \setminus \{0\}$ be such that $\widehat{b} \subseteq H$. Then $\phi[\widehat{b}]$ has empty interior; but also it is a closed set, so its complement is dense. Set $A = \{a : a \in \mathfrak{A},\, \widehat{a} \cap \phi[\widehat{b}] = \emptyset\}$. Then $\bigcup_{a\in A} \widehat{a} = Z \setminus \phi[\widehat{b}]$ is a dense open set, so $\sup A = 1$ in $\mathfrak{A}$ (313Ca). Because π is order-continuous, $\sup \pi[A] = 1$ in $\mathfrak{B}$ (313L(b-iii)), and there is an $a \in A$ such that $\pi a \cap b \neq 0$. But this means that $\widehat{b} \cap \phi^{-1}[\widehat{a}] \neq \emptyset$ and $\phi[\widehat{b}] \cap \widehat{a} \neq \emptyset$, contrary to the definition of A. **X**

Thus there is no such set H, and (iii) is true.

(b)(iii)⇒(ii) Now assume (iii). If $M \subseteq Z$ is nowhere dense, set $N = \phi^{-1}[\overline{M}]$, so that $N \subseteq W$ is a closed set. If $H = \operatorname{int} N$, then $\operatorname{int}\phi[H] \subseteq \operatorname{int}\overline{M} = \emptyset$, so (iii) tells us that H is empty; thus N and $\phi^{-1}[M]$ are nowhere dense, as required by (ii).

(c)(ii)⇒(i) Assume (ii), and let $A \subseteq \mathfrak{A}$ be a non-empty set such that $\inf A = 0$ in $\mathfrak{A}$. Then $M = \bigcap_{a\in A} \widehat{a}$ has empty interior in Z (313Cb), so (being closed) is nowhere dense, and $\phi^{-1}[M]$ also is nowhere dense. If $b \in \mathfrak{B} \setminus \{0\}$, then

$$\widehat{b} \not\subseteq \phi^{-1}[M] = \textstyle\bigcap_{a\in A} \phi^{-1}[\widehat{a}] = \bigcap_{a\in A} \widehat{\pi a},$$

so b is not a lower bound for $\pi[A]$. This shows that $\inf \pi[A] = 0$ in $\mathfrak{B}$. As A is arbitrary, π is order-continuous (313L(b-ii)).

313S Upper envelopes (a) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. For $a \in \mathfrak{A}$, the **upper envelope** of a in $\mathfrak{C}$, or **projection** of a on $\mathfrak{C}$, is

$$\operatorname{upr}(a, \mathfrak{C}) = \inf\{c : c \in \mathfrak{C},\, a \subseteq c\}$$

if the infimum is defined in $\mathfrak{C}$.

Remark Note that the infima here are to be taken in the subalgebra, so that $\operatorname{upr}(a, \mathfrak{C})$ will always belong to $\mathfrak{C}$. In the great majority of elementary applications, $\mathfrak{C}$ will be order-closed in $\mathfrak{A}$, so that we do not need to distinguish between infima in $\mathfrak{C}$ and infima in $\mathfrak{A}$. But see 313Yh.

(b) If $A \subseteq \mathfrak{A}$ is such that $\operatorname{upr}(a, \mathfrak{C})$ is defined for every $a \in A$, $a_0 = \sup A$ is defined in $\mathfrak{A}$ and $c_0 = \sup_{a\in A} \operatorname{upr}(a, \mathfrak{C})$ is defined in $\mathfrak{C}$, then $c_0 = \operatorname{upr}(a_0, \mathfrak{C})$. **P** If $c \in \mathfrak{C}$ then

$$\begin{aligned} c_0 \subseteq c &\iff \operatorname{upr}(a, \mathfrak{C}) \subseteq c \text{ for every } a \in A \\ &\iff a \subseteq c \text{ for every } a \in A \iff a_0 \subseteq c. \textbf{ Q} \end{aligned}$$

In particular, $\operatorname{upr}(a \cup a', \mathfrak{C}) = \operatorname{upr}(a, \mathfrak{C}) \cup \operatorname{upr}(a', \mathfrak{C})$ whenever the right-hand side is defined.

(c) If $a \in \mathfrak{A}$ is such that $\operatorname{upr}(a, \mathfrak{C})$ is defined, then $\operatorname{upr}(a \cap c, \mathfrak{C}) = c \cap \operatorname{upr}(a, \mathfrak{C})$ for every $c \in \mathfrak{C}$. **P** For $c' \in \mathfrak{C}$,

$$\begin{aligned} a \cap c \subseteq c' &\iff a \subseteq c' \cup (1 \setminus c) \\ &\iff \operatorname{upr}(a, \mathfrak{C}) \subseteq c' \cup (1 \setminus c) \iff c \cap \operatorname{upr}(a, \mathfrak{C}) \subseteq c'. \textbf{ Q} \end{aligned}$$

313X Basic exercises (a) Use 313C to give alternative proofs of 313A and 313B.

(b) Let P be a partially ordered set. Show that there is a topology on P for which the closed sets are just the order-closed sets.

(c) Let P be a partially ordered set, $Q \subseteq P$ an order-closed set, and R a subset of Q which is order-closed in Q when Q is given the partial ordering induced by that of P. Show that R is order-closed in P.

>(d) Let $\mathfrak{A}$ be a Boolean algebra. Suppose that $1 \in I \subseteq \mathfrak{A}$ and that $a \cap b \in I$ for all a, $b \in I$. (i) Let $\mathfrak{B}$ be the intersection of all those subsets A of $\mathfrak{A}$ such that $I \subseteq A$ and $b \setminus a \in A$ whenever a, $b \in A$ and $a \subseteq b$. Show that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. (ii) Let $\mathfrak{B}_\sigma$ be the intersection of all those subsets A of $\mathfrak{A}$ such that $I \subseteq A$, $b \setminus a \in A$ whenever a, $b \in A$ and $a \subseteq b$ and $\sup_{n\in\mathbb{N}} b_n \in A$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in A with a supremum in $\mathfrak{A}$.

Show that $\mathfrak{B}_\sigma$ is a σ-subalgebra of $\mathfrak{A}$. (iii) Let $\mathfrak{B}_\tau$ be the intersection of all those subsets A of $\mathfrak{A}$ such that $I \subseteq A$, $b \setminus a \in A$ whenever a, $b \in A$ and $a \subseteq b$ and $\sup B \in A$ whenever B is a non-empty upwards-directed subset of A with a supremum in $\mathfrak{A}$. Show that $\mathfrak{B}_\tau$ is an order-closed subalgebra of $\mathfrak{A}$. (iv) Hence give a proof of 313G not relying on Zorn's Lemma or any other use of the axiom of choice.

(e) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Let $\mathfrak{B}_\sigma$ be the smallest sequentially order-closed subset of $\mathfrak{A}$ including $\mathfrak{B}$. Show that $\mathfrak{B}_\sigma$ is a subalgebra of $\mathfrak{A}$.

>(f) Let X be a set, and $\mathcal{A}$ a subset of $\mathcal{P}X$. Show that $\mathcal{A}$ is an order-closed subalgebra of $\mathcal{P}X$ iff it is of the form $\{f^{-1}[F] : F \subseteq Y\}$ for some set Y and function $f : X \to Y$.

(g) Let P and Q be partially ordered sets, and $\phi : P \to Q$ an order-preserving function. Show that ϕ is sequentially order-continuous iff $\phi^{-1}[C]$ is sequentially order-closed in $\mathfrak{A}$ for every sequentially order-closed $C \subseteq \mathfrak{B}$.

(h) For partially ordered sets P and Q, let us call a function $\phi : P \to Q$ **monotonic** if it is *either* order-preserving *or* order-reversing. State and prove definitions and results corresponding to 313H, 313I and 313Xg for general monotonic functions.

>(i) Let $\mathfrak{A}$ be a Boolean algebra. Show that the operations $(a,b) \mapsto a \cup b$ and $(a,b) \mapsto a \cap b$ are order-continuous operations from $\mathfrak{A} \times \mathfrak{A}$ to $\mathfrak{A}$, if we give $\mathfrak{A} \times \mathfrak{A}$ the product partial order, saying that $(a,b) \le (a',b')$ iff $a \subseteq a'$ and $b \subseteq b'$.

(j) Let $\mathfrak{A}$ be a Boolean algebra. Show that if a subalgebra of $\mathfrak{A}$ is order-dense then it is dense in the topology of 313Xb.

>(k) Let $\mathfrak{A}$ be a Boolean algebra and $A \subseteq \mathfrak{A}$ any disjoint set. Show that there is a partition of unity in $\mathfrak{A}$ including A.

>(l) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and π_1, $\pi_2 : \mathfrak{A} \to \mathfrak{B}$ two order-continuous Boolean homomorphisms. Show that $\{a : \pi_1 a = \pi_2 a\}$ is an order-closed subalgebra of $\mathfrak{A}$.

(m) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and π_1, $\pi_2 : \mathfrak{A} \to \mathfrak{B}$ two Boolean homomorphisms. Suppose that π_1 and π_2 agree on some order-dense subset of $\mathfrak{A}$, and that one of them is order-continuous. Show that they are equal. (*Hint*: if π_1 is order-continuous, $\pi_2 a \supseteq \pi_1 a$ for every a.)

(n) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, $\mathfrak{A}_0$ an order-dense subalgebra of $\mathfrak{A}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism. Show that π is order-continuous iff $\pi \restriction \mathfrak{A}_0 : \mathfrak{A}_0 \to \mathfrak{B}$ is order-continuous.

(o) Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism with fixed-point subalgebra $\mathfrak{C}$ (312K). (i) Show that if π is sequentially order-continuous then $\mathfrak{C}$ is a σ-subalgebra of $\mathfrak{A}$. (ii) Show that if π is order-continuous then $\mathfrak{C}$ is order-closed.

>(p) Let $\mathfrak{A}$ be a Boolean algebra. For $A \subseteq \mathfrak{A}$ set $A^\perp = \{b : a \cap b = 0 \ \forall \ a \in A\}$. (i) Show that $A^\perp$ is an order-closed ideal of $\mathfrak{A}$. (ii) Show that a set $A \subseteq \mathfrak{A}$ is an order-closed ideal of $\mathfrak{A}$ iff $A = A^{\perp\perp}$. (iii) Show that if $I \subseteq \mathfrak{A}$ is an order-closed ideal then $\{a^\bullet : a \in I^\perp\}$ is an order-dense ideal in the quotient algebra $\mathfrak{A}/I$.

(q) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z and W; let $\pi : \mathfrak{A} \to \mathfrak{B}$ be a Boolean homomorphism, and $\phi : W \to Z$ the corresponding continuous function. Show that the following are equiveridical: (i) π is order-continuous; (ii) $\operatorname{int} \phi^{-1}[F] = \phi^{-1}[\operatorname{int} F]$ for every closed $F \subseteq Z$ (iii) $\overline{\phi^{-1}[G]} = \phi^{-1}[\overline{G}]$ for every open $G \subseteq Z$.

(r) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, $\pi : \mathfrak{A} \to \mathfrak{B}$ an injective Boolean homomorphism and $\mathfrak{C}$ a Boolean subalgebra of $\mathfrak{A}$. Suppose that $a \in \mathfrak{A}$ is such that $c = \operatorname{upr}(a, \mathfrak{C})$ is defined. Show that $\operatorname{upr}(\pi a, \pi[\mathfrak{C}])$ is defined and equal to πc.

313Y Further exercises (a) Prove 313A-313C for general Boolean rings.

(b) Let P be any partially ordered set, and let $\mathfrak{T}$ be the topology of 313Xb. (i) Show that a sequence $\langle p_n \rangle_{n\in\mathbb{N}}$ in P is $\mathfrak{T}$-convergent to $p \in P$ iff every subsequence of $\langle p_n \rangle_{n\in\mathbb{N}}$ has a monotonic sub-subsequence with supremum or infimum equal to p. (ii) Show that a subset A of P is sequentially order-closed, in the sense of 313Db, iff the $\mathfrak{T}$-limit of any $\mathfrak{T}$-convergent sequence in A belongs to A. (iii) Suppose that A is an upwards-directed subset of P with supremum $p_0 \in P$. For $a \in A$ set $F_a = \{p : a \le p \in A\}$, and let $\mathcal{F}$ be the filter on P generated by $\{F_a : a \in A\}$. Show that $\mathcal{F} \to p_0$ for $\mathfrak{T}$. (iv) Show that if Q is another partially ordered set, endowed with a topology $\mathfrak{S}$ in the same way, then a monotonic function $\phi : P \to Q$ is order-continuous iff it is continuous for the topologies $\mathfrak{T}$ and $\mathfrak{S}$, and is sequentially order-continuous iff it is sequentially continuous for these topologies.

(c) Let U be a Banach lattice (242G, 354Ab). Show that its norm is order-continuous in the sense of 242Yg and 354Dc iff its restriction to $\{u : u \geq 0\}$ is order-continuous in the sense of 313Ha.

(d) Let P and Q be lattices, and $f : P \to Q$ a bijective lattice homomorphism. Show that f is order-continuous.

(e) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z and W, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism, with associated continuous function $\phi : W \to Z$. Show that π is sequentially order-continuous iff $\phi^{-1}[M]$ is nowhere dense for every nowhere dense zero set $M \subseteq Z$.

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with Stone spaces Z and W respectively, $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism and $\phi : W \to Z$ the corresponding continuous function. Show that $\pi[\mathfrak{A}]$ is order-dense in $\mathfrak{B}$ iff ϕ is **irreducible**, that is, $\phi[F] \neq \phi[W]$ for any proper closed subset F of W.

(g) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with Stone spaces Z and W respectively, $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism and $\phi : W \to Z$ the corresponding continuous function. Show that the following are equiveridical: (i) π is injective and order-continuous; (ii) for $M \subseteq Z$, M is nowhere dense iff $\phi^{-1}[M]$ is nowhere dense.

(h) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a Boolean subalgebra of $\mathfrak{A}$. Let $\mathcal{I}$ be the set of those $a \in \mathfrak{A}$ such that the upper envelope $\operatorname{upr}(a, \mathfrak{C})$ is zero. (i) Show that $\mathcal{I}$ is an ideal in $\mathfrak{A}$. (ii) Show that $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$ iff $\mathcal{I} = \{0\}$. (iii) Let $\pi : \mathfrak{A} \to \mathfrak{A}/\mathcal{I}$ be the canonical map. Show that $\pi\restriction\mathfrak{C}$ is injective and order-continuous.

313 Notes and comments I give 'elementary' proofs of 313A-313B because I believe that they help to exhibit the relevant aspects of the structure of Boolean algebras; but various abbreviations are possible, notably if we allow ourselves to use the Stone representation (313Xa). 313A and 313Ba-b can be expressed by saying that the Boolean operations $\cup$, $\cap$ and $\setminus$ are (separately) order-continuous. Of course, $\setminus$ is order-reversing, rather than order-preserving, in the second variable; but the natural symmetry in the concept of partial order means that the ideas behind 313H-313I can be applied equally well to order-reversing functions (313Xh). In fact, $\cup$ and $\cap$ can be regarded as order-continuous functions on the product space (313Bc-d, 313Xi). Clearly 313Bc-d can be extended into forms valid for any finite sequence $A_0, \dots, A_n$ of subsets of $\mathfrak{A}$ in place of A, B. But if we seek to go to infinitely many subsets of $\mathfrak{A}$ we find ourselves saying something new; see 316G-316J below.

Proposition 313C, and its companions 313R, 313Xq and 313Ye, are worth studying not only as a useful technique, but also in order to understand the difference between $\sup A$, where A is a set in a Boolean algebra, and $\bigcup \mathcal{A}$, where $\mathcal{A}$ is a family of sets. Somehow $\sup A$ can be larger, and $\inf A$ smaller, than one's first intuition might suggest, corresponding to the fact that not every subset of the Stone space corresponds to an element of the Boolean algebra.

I should like to use the words 'order-closed' and 'sequentially order-closed' to mean closed, or sequentially closed, for some more or less canonical topology. The difficulty is that while a great many topologies can be defined from a partial order (one is described in 313Xb and 313Yb, and another in 367Yb and 393L), none of them has such pre-eminence that it can be called 'the' order-topology. Accordingly there is a degree of arbitrariness in the language I use here. Nevertheless (sequentially) order-closed subalgebras and ideals are of such importance that they seem to deserve a concise denotation. The same remarks apply to (sequential) order-continuity. Concerning the term 'order-dense' in 313J, this has little to do with density in any topological sense, but the word 'dense', at least, is established in this context.

With all these definitions, there is a good deal of scope for possible interrelations. The most important to us is 313Q, which will be used repeatedly (typically, with $\mathfrak{A}$ an algebra of sets), but I think it is worth having the expanded version in 313P available.

I take the opportunity to present an abstract form of an important lemma on σ-algebras generated by families closed under $\cap$ (136B, 313Gb). This time round I use the Zorn's Lemma argument in the text and suggest the alternative, 'elementary' method in the exercises (313Xd). The two methods are opposing extremes in the sense that the Zorn's Lemma argument looks for maximal subalgebras included in A (which are not unique, and have to be picked out using the axiom of choice) and the other approach seeks minimal subalgebras including I (which are uniquely defined, and can be described without the axiom of choice).

Note that the concept of 'order-closed algebra of sets' is not particularly useful; there are too few order-closed subalgebras of $\mathcal{P}X$ and they are of too simple a form (313Xf). It is in abstract Boolean algebras that the idea becomes important. In many of the most important partially ordered sets of measure theory, the sequentially order-closed sets are the same as the order-closed sets (see, for instance, 316Fb below), and most of the important order-closed subalgebras dealt with in this chapter can be thought of as σ-subalgebras which are order-closed because they happen to lie in the right kind of algebra.

314 Order-completeness

The results of §313 are valid in all Boolean algebras, but of course are of most value when many suprema and infima exist. I now set out the most useful definitions which guarantee the existence of suprema and infima (314A) and work through their elementary relationships with the concepts introduced so far (314C-314J). I then embark on the principal theorems concerning order-complete Boolean algebras: the extension theorem for homomorphisms to a Dedekind complete algebra (314K), the Loomis-Sikorski representation of a Dedekind σ-complete algebra as a quotient of a σ-algebra of sets (314M), the characterization of Dedekind complete algebras in terms of their Stone spaces (314S), and the idea of 'Dedekind completion' of a Boolean algebra (314T-314U). On the way I describe 'regular open algebras' (314O-314Q).

314A Definitions Let P be a partially ordered set.

(a) P is **Dedekind complete**, or **order-complete**, or **conditionally complete** if every non-empty subset of P with an upper bound has a least upper bound.

(b) P is **Dedekind σ-complete**, or **σ-order-complete**, if (i) every countable non-empty subset of P with an upper bound has a least upper bound (ii) every countable non-empty subset of P with a lower bound has a greatest lower bound.

314B Remarks (a) I give these definitions in the widest possible generality because they are in fact of great interest for general partially ordered sets, even though for the moment we shall be concerned only with Boolean algebras. Indeed I have already presented the same idea in the context of Riesz spaces (241F).

(b) You will observe that the definition in (a) of 314A is asymmetric, unlike that in (b). This is because the inverted form of the definition is equivalent to that given; that is, P is Dedekind complete (on the definition 314Aa) iff every non-empty subset of P with a lower bound has a greatest lower bound. **P** (i) Suppose that P is Dedekind complete, and that $B \subseteq P$ is non-empty and bounded below. Let A be the set of lower bounds for B. Then A has at least one upper bound (since any member of B is an upper bound for A) and is not empty; so $a_0 = \sup A$ is defined. Now if $b \in B$, b is an upper bound for A, so $a_0 \le b$; thus $a_0 \in A$ and must be the greatest member of A, that is, the greatest lower bound of B. (ii) Similarly, if every non-empty subset of P with a lower bound has a greatest lower bound, P is Dedekind complete. **Q**

(c) In the special case of Boolean algebras, we do not need both halves of the definition 314Ab; in fact we have, for any Boolean algebra $\mathfrak{A}$,

$$\begin{aligned} &A \text{ is Dedekind } \sigma\text{-complete}\\ &\quad\iff \text{every non-empty countable subset of } \mathfrak{A} \text{ has a least upper bound}\\ &\quad\iff \text{every non-empty countable subset of } \mathfrak{A} \text{ has a greatest lower bound.}\end{aligned}$$

P Because $\mathfrak{A}$ has a least element 0 and a greatest element 1, every subset of $\mathfrak{A}$ has upper and lower bounds; so the two one-sided conditions together are equivalent to saying that $\mathfrak{A}$ is Dedekind σ-complete. I therefore have to show that they are equiveridical. Now if $A \subseteq \mathfrak{A}$ is a non-empty countable set, so is $B = \{1 \setminus a : a \in A\}$, and

$$\inf A = 1 \setminus \sup B, \quad \sup A = 1 \setminus \inf B$$

whenever the right-hand-sides are defined (313A). So if the existence of a supremum (resp. infimum) of B is guaranteed, so is the existence of an infimum (resp. supremum) of A. **Q**

The real point here is of course that $(\mathfrak{A}, \subseteq)$ is isomorphic to $(\mathfrak{A}, \supseteq)$.

(d) Most specialists in Boolean algebra speak of 'complete', or 'σ-complete', Boolean algebras. I prefer the longer phrases 'Dedekind complete' and 'Dedekind σ-complete' because we shall be studying metrics on Boolean algebras and shall need the notion of metric completeness as well as that of order-completeness.

(e) I have had to make some rather arbitrary choices in the definition here. The principal examples of partially ordered set to which we shall apply these definitions are Boolean algebras and Riesz spaces, which are all lattices. Consequently it is not possible to distinguish in these contexts between the property of Dedekind completeness, as defined above, and the weaker property, which we might call 'monotone order-completeness',

> (i) whenever $A \subseteq P$ is non-empty, upwards-directed and bounded above then A has a least upper bound in P (ii) whenever $A \subseteq P$ is non-empty, downwards-directed and bounded below then A has a greatest lower bound in P.

(See 314Xa below. 'Monotone order-completeness' is the property involved in 314Ya, for instance.) Nevertheless I am prepared to say, on the basis of my own experience of working with other partially ordered sets, that 'Dedekind completeness', as I have defined it, is at least of sufficient importance to deserve a name. Note that it does not imply that P is a lattice, since it allows two elements of P to have no common upper bound.

(f) The phrase **complete lattice** is sometimes used to mean a Dedekind complete lattice with greatest and least elements; equivalently, a Dedekind complete partially ordered set with greatest and least elements. Thus a Dedekind complete Boolean algebra is a complete lattice in this sense, but $\mathbb{R}$ is not.

(g) The most important Dedekind complete Boolean algebras (at least from the point of view of measure theory) are the 'measure algebras' of the next chapter. I shall not pause here to give other examples, but will proceed directly with the general theory.

314C Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and I a σ-ideal of $\mathfrak{A}$. Then the quotient Boolean algebra $\mathfrak{A}/I$ is Dedekind σ-complete.

proof I use the description in 314Bc. Let $B \subseteq \mathfrak{A}/I$ be a non-empty countable set. For each $u \in B$, choose an $a_u \in \mathfrak{A}$ such that $u = a_u^\bullet$. Then $c = \sup_{u\in B} a_u$ is defined in $\mathfrak{A}$; consider $v = c^\bullet$ in $\mathfrak{A}/I$. Because the map $a \mapsto a^\bullet$ is sequentially order-continuous (313Qb), $v = \sup B$. As B is arbitrary, $\mathfrak{A}/I$ is Dedekind σ-complete.

314D Corollary Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X. Then $\Sigma \cap \mathcal{I}$ is a σ-ideal of the Boolean algebra Σ, and $\Sigma/\Sigma \cap \mathcal{I}$ is Dedekind σ-complete.

proof Of course Σ is Dedekind σ-complete, because if $\langle E_n\rangle_{n\in\mathbb{N}}$ is any sequence in Σ then $\bigcup_{n\in\mathbb{N}} E_n$ is the least upper bound of $\{E_n : n \in \mathbb{N}\}$ in Σ. It is also easy to see that $\Sigma \cap \mathcal{I}$ is a σ-ideal of Σ, since $F \cap \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{I}$ whenever $F \in \Sigma$ and $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\Sigma \cap \mathcal{I}$. So 314C gives the result.

314E Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{A}$ is Dedekind complete, then all its order-closed subalgebras and principal ideals are Dedekind complete.

(b) If $\mathfrak{A}$ is Dedekind σ-complete, then all its σ-subalgebras and principal ideals are Dedekind σ-complete.

proof All we need to note is that if $\mathfrak{C}$ is either an order-closed subalgebra or a principal ideal of $\mathfrak{A}$, and $B \subseteq \mathfrak{C}$ is such that $b = \sup B$ is defined in $\mathfrak{A}$, then $b \in \mathfrak{C}$ (see 313E(a-i-β)), so b is still the supremum of B in $\mathfrak{C}$; while the same is true if $\mathfrak{C}$ is a σ-subalgebra and $B \subseteq \mathfrak{C}$ is countable, using 313E(a-ii-β).

314F I spell out some further connexions between the concepts 'order-closed set', 'order-continuous function' and 'Dedekind complete Boolean algebra' which are elementary without being quite transparent.

Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a)(i) If $\mathfrak{A}$ is Dedekind complete and π is order-continuous, then $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$.

(ii) If $\mathfrak{B}$ is Dedekind complete and π is injective and $\pi[\mathfrak{A}]$ is order-closed then π is order-continuous.

(b)(i) If $\mathfrak{A}$ is Dedekind σ-complete and π is sequentially order-continuous, then $\pi[\mathfrak{A}]$ is a σ-subalgebra of $\mathfrak{B}$.

(ii) If $\mathfrak{B}$ is Dedekind σ-complete and π is injective and $\pi[\mathfrak{A}]$ is a σ-subalgebra of $\mathfrak{B}$ then π is sequentially order-continuous.

proof (a)(i) If $B \subseteq \pi[\mathfrak{A}]$, then $a_0 = \sup(\pi^{-1}[B])$ is defined in $\mathfrak{A}$; now

$$\pi a_0 = \sup(\pi[\pi^{-1}[B]]) = \sup B$$

in $\mathfrak{B}$ (313L(b-iv)), and of course $\pi a_0 \in \pi[\mathfrak{A}]$. By 313E(a-i-$\beta$) again, this is enough to show that $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$.

(ii) Suppose that $A \subseteq \mathfrak{A}$ and $\inf A = 0$ in $\mathfrak{A}$. Then $\pi[A]$ has an infimum b_0 in $\mathfrak{B}$, which belongs to $\pi[\mathfrak{A}]$ because $\pi[\mathfrak{A}]$ is an order-closed subalgebra of $\mathfrak{B}$ (313E(a-i-β')). Now if $a_0 \in \mathfrak{A}$ is such that $\pi a_0 = b_0$, we have

$$\pi(a \cap a_0) = \pi a \cap \pi a_0 = \pi a_0$$

for every $a \in A$, so (because π is injective) $a \cap a_0 = a_0$ and $a_0 \subseteq a$ for every $a \in A$. But this means that $a_0 = 0$ and $b_0 = \pi 0 = 0$. As A is arbitrary, π is order-continuous (313L(b-ii)).

(b) Use the same arguments, but with sequences in place of the sets B, A above.

314G Corollary Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$.

(a) If $\mathfrak{A}$ is Dedekind complete, then $\mathfrak{B}$ is order-closed iff it is Dedekind complete in itself and is regularly embedded in $\mathfrak{A}$.

(b) If $\mathfrak{A}$ is Dedekind σ-complete, then $\mathfrak{B}$ is a σ-subalgebra iff it is Dedekind σ-complete in itself and the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is sequentially order-continuous.

proof (a) Let $\iota : \mathfrak{B} \to \mathfrak{A}$ be the identity map; then it is an injective Boolean homomorphism.

(i) If $\mathfrak{B}$ is order-closed, then it is Dedekind complete in itself by 314Ea. By 314F(a-ii), $\iota : \mathfrak{B} \to \mathfrak{A}$ is order-continuous, that is, $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$.

(ii) If $\mathfrak{B}$ is Dedekind complete in itself and ι is order-continuous, then $\mathfrak{B} = \iota[\mathfrak{B}]$ is order-closed in $\mathfrak{A}$ by 314F(a-i).

(b) Use the same arguments, but with 314Eb and 314Fb in place of 314Ea and 314Fa.

314H Corollary Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism. If $C \subseteq \mathfrak{A}$ and $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}$ generated by C, then $\pi[\mathfrak{C}]$ is the order-closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$.

proof Let $\mathfrak{D}$ be the order-closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$. By 313Mb, $\pi[\mathfrak{C}] \subseteq \mathfrak{D}$. On the other hand, the identity homomorphism $\iota : \mathfrak{C} \to \mathfrak{A}$ is order-continuous, by 314Ga, so $\pi\iota : \mathfrak{C} \to \mathfrak{B}$ is order-continuous, and $\pi[\mathfrak{C}] = \pi\iota[\mathfrak{C}]$ is order-closed in $\mathfrak{B}$, by 314F(a-i). But since $\pi[C]$ is surely included in $\pi[\mathfrak{C}]$, $\mathfrak{D}$ also is included in $\pi[\mathfrak{C}]$. Accordingly $\pi[\mathfrak{C}] = \mathfrak{D}$, as claimed.

314I Corollary (a) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra, $\mathfrak{B}$ is a Boolean algebra, $\pi : \mathfrak{A} \to \mathfrak{B}$ is an injective Boolean homomorphism and $\pi[\mathfrak{A}]$ is order-dense in $\mathfrak{B}$, then π is an isomorphism.

(b) If $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$ which is Dedekind complete in itself, then $\mathfrak{B} = \mathfrak{A}$.

proof (a) Because $\pi[\mathfrak{A}]$ is order-dense, it is regularly embedded in $\mathfrak{B}$ (313O); also, the kernel of π is $\{0\}$, which is surely order-closed in $\mathfrak{A}$, so 313P(a-ii) tells us that π is order-continuous. By 314F(a-i), $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$; being order-dense, it must be the whole of $\mathfrak{B}$ (313K). Thus π is surjective; being injective, it is an isomorphism.

(b) Apply (a) to the identity map from $\mathfrak{B}$ to $\mathfrak{A}$.

314J When we come to applications of the extension procedure in 312O, the following will sometimes be needed.

Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$. Take any $c \in \mathfrak{A}$, and set

$$\mathfrak{A}_1 = \{(a \cap c) \cup (b \setminus c) : a,\, b \in \mathfrak{A}_0\},$$

the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$ (312N).

(a) Suppose that $\mathfrak{A}$ is Dedekind complete. If $\mathfrak{A}_0$ is order-closed in $\mathfrak{A}$, so is $\mathfrak{A}_1$.

(b) Suppose that $\mathfrak{A}$ is Dedekind σ-complete. If $\mathfrak{A}_0$ is a σ-subalgebra of $\mathfrak{A}$, so is $\mathfrak{A}_1$.

proof (a) Let D be any subset of $\mathfrak{A}_1$. Set

$$E = \{e : e \in \mathfrak{A}, \text{ there is some } d \in D \text{ such that } e \subseteq d\},$$

$$A = \{a : a \in \mathfrak{A}_0,\, a \cap c \in E\}, \quad B = \{b : b \in \mathfrak{A}_0,\, b \setminus c \in E\}.$$

Because $\mathfrak{A}$ is Dedekind complete, $a^* = \sup A$ and $b^* = \sup B$ are defined in $\mathfrak{A}$; because $\mathfrak{A}_0$ is order-closed, both belong to $\mathfrak{A}_0$, so $d^* = (a^* \cap c) \cup (b^* \setminus c)$ belongs to $\mathfrak{A}_1$.

Now if $d \in D$, it is expressible as $(a \cap c) \cup (b \setminus c)$ for some a, $b \in \mathfrak{A}_0$; since $a \in A$ and $b \in B$, we have $a \subseteq a^*$ and $b \subseteq b^*$, so $d \subseteq d^*$. Thus d^* is an upper bound for D. On the other hand, if d' is any other upper bound for D in $\mathfrak{A}$, it is also an upper bound for E, so we must have

$$a^* \cap c = \sup_{a \in A} a \cap c \subseteq d', \quad b^* \setminus c = \sup_{b \in B} b \setminus c \subseteq d',$$

and $d^* \subseteq d'$. Thus $d^* = \sup D$. This shows that the supremum of any subset of $\mathfrak{A}_1$ belongs to $\mathfrak{A}_1$, so that $\mathfrak{A}_1$ is order-closed.

(b) The argument is the same, except that we replace D by a sequence $\langle d_n \rangle_{n \in \mathbb{N}}$, and A, B by sequences $\langle a_n \rangle_{n \in \mathbb{N}}$, $\langle b_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}_0$ such that $d_n = (a_n \cap c) \cup (b_n \setminus c)$ for every n.

314K Extension of homomorphisms The following is one of the most striking properties of Dedekind complete Boolean algebras.

Theorem Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a Dedekind complete Boolean algebra. Let $\mathfrak{A}_0$ be a Boolean subalgebra of $\mathfrak{A}$ and $\pi_0 : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism. Then there is a Boolean homomorphism $\pi_1 : \mathfrak{A} \to \mathfrak{B}$ extending π_0.

proof (a) Let P be the set of all Boolean homomorphisms π such that $\operatorname{dom}\pi$ is a Boolean subalgebra of $\mathfrak{A}$ including $\mathfrak{A}_0$ and π extends π_0. Identify each member of P with its graph, which is a subset of $\mathfrak{A}\times\mathfrak{B}$, and order P by inclusion, so that $\pi \subseteq \theta$ means just that θ extends π. Then any non-empty totally ordered subset Q of P has an upper bound in P. **P** Let π^* be the simple union of these graphs. (i) If (a,b) and (a,b') both belong to π^*, then there are π, $\pi' \in Q$ such that $\pi a = b$, $\pi' a = b'$; now either $\pi \subseteq \pi'$ or $\pi' \subseteq \pi$; in either case, $\theta = \pi \cup \pi' \in Q$, so that

$$b = \pi a = \theta a = \pi' a = b'.$$

This shows that π^* is a function. (ii) Because $Q \neq \emptyset$,

$$\operatorname{dom}\pi_0 \subseteq \operatorname{dom}\pi \subseteq \operatorname{dom}\pi^*$$

for some $\pi \in Q$; thus π^* extends π_0 (and, in particular, $0 \in \operatorname{dom}\pi^*$). (iii) Now suppose that a, $a' \in \operatorname{dom}(\pi^*)$. Then there are π, $\pi' \in Q$ such that $a \in \operatorname{dom}\pi$, $a' \in \operatorname{dom}\pi'$; once again, $\theta = \pi \cup \pi' \in Q$, so that a, $a' \in \operatorname{dom}\theta$, and

$$a \cap a' \in \operatorname{dom}\theta \subseteq \operatorname{dom}\pi^*, \quad 1 \setminus a \in \operatorname{dom}\theta \subseteq \operatorname{dom}\pi^*,$$

$$\pi^*(a \cap a') = \theta(a \cap a') = \theta a \cap \theta a' = \pi^* a \cap \pi^* a',$$

$$\pi^*(1 \setminus a) = \theta(1 \setminus a) = 1 \setminus \theta a = 1 \setminus \pi^* a.$$

(iv) This shows that $\operatorname{dom}\pi^*$ is a subalgebra of $\mathfrak{A}$ and that π^* is a Boolean homomorphism, that is, that $\pi^* \in P$; and of course π^* is an upper bound for Q in P. **Q**

(b) By Zorn's Lemma, P has a maximal element π_1 say.

? Suppose, if possible, that $\mathfrak{A}_1 = \operatorname{dom}\pi_1$ is not the whole of $\mathfrak{A}$; take $c \in \mathfrak{A} \setminus \mathfrak{A}_1$. Set $A = \{a : a \in \mathfrak{A}_1,\, a \subseteq c\}$. Because $\mathfrak{B}$ is Dedekind complete, $d = \sup \pi_1[A]$ is defined in $\mathfrak{B}$. If $a' \in \mathfrak{A}_1$ and $c \subseteq a'$, then of course $a \subseteq a'$ and $\pi_1 a \subseteq \pi_1 a'$ whenever $a \in A$, so that $\pi_1 a'$ is an upper bound for $\pi_1[A]$, and $d \subseteq \pi_1 a'$.

But this means that there is an extension of π_1 to a Boolean homomorphism π on the Boolean subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_1 \cup \{c\}$ (312O). And this π must be a member of P properly extending π_1, which is supposed to be maximal. **X**

Thus $\operatorname{dom}\pi_1 = \mathfrak{A}$ and π_1 is an extension of π_0 to $\mathfrak{A}$, as required.

314L The Loomis-Sikorski representation of a Dedekind σ-complete Boolean algebra The construction in 314D is not only the commonest way in which new Dedekind σ-complete Boolean algebras appear, but is adequate to describe them all. I start with an elementary general fact.

Lemma Let X be any topological space, and write $\mathcal{M}$ for the family of meager subsets of X. Then $\mathcal{M}$ is a σ-ideal of subsets of X.

proof The point is that if $A \subseteq X$ is nowhere dense, so is every subset of A; this is obvious, since if $B \subseteq A$ then $\overline{B} \subseteq \overline{A}$ so $\operatorname{int}\overline{B} \subseteq \operatorname{int}\overline{A} = \emptyset$. So if $B \subseteq A \in \mathcal{M}$, let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of nowhere dense sets with union A; then $\langle B \cap A_n\rangle_{n\in\mathbb{N}}$ is a sequence of nowhere dense sets with union B, so $B \in \mathcal{M}$. If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}$ with union A, then for each n we may choose a sequence $\langle A_{nm}\rangle_{m\in\mathbb{N}}$ of nowhere dense sets with union A_n; then the countable family $\langle A_{nm}\rangle_{n,m\in\mathbb{N}}$ may be re-indexed as a sequence of nowhere dense sets with union A, so $A \in \mathcal{M}$. Finally, $\emptyset$ is nowhere dense, so belongs to $\mathcal{M}$.

314M Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and Z its Stone space. Let $\mathcal{E}$ be the algebra of open-and-closed subsets of Z, and $\mathcal{M}$ the σ-ideal of meager subsets of Z. Then $\Sigma = \{E \triangle A : E \in \mathcal{E},\, A \in \mathcal{M}\}$ is a σ-algebra of subsets of Z, $\mathcal{M}$ is a σ-ideal of Σ, and $\mathfrak{A}$ is isomorphic, as Boolean algebra, to $\Sigma/\mathcal{M}$.

proof (a) I start by showing that Σ is a σ-algebra. **P** Of course $\emptyset = \emptyset \triangle \emptyset \in \Sigma$. If $F \in \Sigma$, express it as $E \triangle A$ where $E \in \mathcal{E}$, $A \in \mathcal{M}$; then $Z \setminus F = (Z \setminus E) \triangle A \in \Sigma$.

If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, express each F_n as $E_n \triangle A_n$, where $E_n \in \mathcal{E}$ and $A_n \in \mathcal{M}$. Now each E_n is expressible as $\widehat{a}_n$, where $a_n \in \mathfrak{A}$. Because $\mathfrak{A}$ is Dedekind σ-complete, $a = \sup_{n\in\mathbb{N}} a_n$ is defined in $\mathfrak{A}$. Set $E = \widehat{a} \in \mathcal{E}$.

By 313Ca, $E = \overline{\bigcup_{n\in\mathbb{N}} E_n}$, so the closed set $E \setminus \bigcup_{n\in\mathbb{N}} E_n$ has empty interior and is nowhere dense. Accordingly, setting $A = E \triangle \bigcup_{n\in\mathbb{N}} F_n$, we have

$$A \subseteq (E \setminus \textstyle\bigcup_{n\in\mathbb{N}} E_n) \cup \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M},$$

so that $\bigcup_{n\in\mathbb{N}} F_n = E \triangle A \in \Sigma$. Thus Σ is closed under countable unions and is a σ-algebra. **Q**

Evidently $\mathcal{M} \subseteq \Sigma$, because $\emptyset \in \mathcal{E}$.

(b) For each $F \in \Sigma$, there is exactly one $E \in \mathcal{E}$ such that $F \triangle E \in \mathcal{M}$. **P** There is surely some $E \in \mathcal{E}$ such that F is expressible as $E \triangle A$ where $A \in \mathcal{M}$, so that $F \triangle E = A \in \mathcal{M}$. If E' is any other member of $\mathcal{E}$, then $E' \triangle E$ is a non-empty open set in X, while $E' \triangle E \subseteq A \cup (F \triangle E')$; by Baire's theorem for compact Hausdorff spaces (3A3G), $A \cup (F \triangle E') \notin \mathcal{M}$ and $F \triangle E' \notin \mathcal{M}$. Thus E is unique. **Q**

(c) Consequently the maps $E \mapsto E^\bullet : \mathcal{E} \to \Sigma/\mathcal{M}$ is a bijection. But since it is also a Boolean homomorphism, it is an isomorphism, and $\mathfrak{A} \cong \mathcal{E} \cong \Sigma/\mathcal{M}$, as claimed.

314N Corollary A Boolean algebra $\mathfrak{A}$ is Dedekind σ-complete iff it is isomorphic to a quotient $\Sigma/\mathcal{I}$ where Σ is a σ-algebra of sets and $\mathcal{I}$ is a σ-ideal of Σ.

proof Put 314D and 314M together.

314O Regular open algebras For Boolean algebras which are Dedekind complete in the full sense, there is another general method of representing them, which leads to further very interesting ideas.

Definition Let X be a topological space. A **regular open set** in X is an open set $G \subseteq X$ such that $G = \operatorname{int} \overline{G}$.

Note that if $F \subseteq X$ is any closed set, then $G = \operatorname{int} F$ is a regular open set, because $G \subseteq \overline{G} \subseteq F$ so

$$G \subseteq \operatorname{int} \overline{G} \subseteq \operatorname{int} F = G$$

and $G = \operatorname{int} \overline{G}$.

314P Theorem Let X be any topological space, and write $\mathrm{RO}(X)$ for the set of regular open sets in X. Then $\mathrm{RO}(X)$ is a Dedekind complete Boolean algebra, with $1_{\mathrm{RO}(X)} = X$ and $0_{\mathrm{RO}(X)} = \emptyset$, and with Boolean operations given by

$$G \cap_{\mathrm{RO}} H = G \cap H, \quad G \triangle_{\mathrm{RO}} H = \operatorname{int} \overline{G \triangle H},$$

$$G \cup_{\mathrm{RO}} H = \operatorname{int} \overline{G \cup H}, \quad G \setminus_{\mathrm{RO}} H = G \setminus \overline{H},$$

with Boolean ordering given by

$$G \subseteq_{\mathrm{RO}} H \iff G \subseteq H,$$

and with suprema and infima given by

$$\sup \mathcal{H} = \operatorname{int} \overline{\textstyle\bigcup \mathcal{H}}, \quad \inf \mathcal{H} = \operatorname{int} \textstyle\bigcap \mathcal{H} = \operatorname{int} \overline{\textstyle\bigcap \mathcal{H}}$$

for all non-empty $\mathcal{H} \subseteq \mathrm{RO}(X)$.

Remark I use the expressions

$$\cap_{\mathrm{RO}} \quad \cup_{\mathrm{RO}} \quad \triangle_{\mathrm{RO}} \quad \setminus_{\mathrm{RO}} \quad \subseteq_{\mathrm{RO}}$$

in case the distinction between

$$\cap \quad \cup \quad \triangle \quad \setminus \quad \subseteq$$

and

$$\cap \quad \cup \quad \triangle \quad \setminus \quad \subseteq$$

is insufficiently marked.

proof I base the proof on the study of an auxiliary algebra of sets which involves some of the ideas already used in 314M.

(a) Let $\mathcal{I}$ be the family of nowhere dense subsets of X. Then $\mathcal{I}$ is an ideal of subsets of X. **P** Of course $\emptyset \in \mathcal{I}$. If $A \subseteq B \in \mathcal{I}$ then $\operatorname{int} \overline{A} \subseteq \operatorname{int} \overline{B} = \emptyset$. If A, $B \in \mathcal{I}$ and G is a non-empty open set, then $G \setminus \overline{A}$ is a non-empty open set and $(G \setminus \overline{A}) \setminus \overline{B}$ is non-empty; accordingly G cannot be a subset of $\overline{A} \cup \overline{B} = \overline{A \cup B}$. This shows that $\operatorname{int} \overline{A \cup B} = \emptyset$, so that $A \cup B \in \mathcal{I}$. **Q**

(b) For any set $A \subseteq X$, write ∂A for the boundary of A, that is, $\overline{A} \setminus \operatorname{int} A$. Set

$$\Sigma = \{E : E \subseteq X,\, \partial E \in \mathcal{I}\}.$$

The Σ is an algebra of subsets of X. **P** (i) $\partial\emptyset = \emptyset \in \mathcal{I}$ so $\emptyset \in \Sigma$. (ii) If A, $B \subseteq X$, then $\overline{A \cup B} = \overline{A} \cup \overline{B}$, while $\operatorname{int}(A \cup B) \supseteq \operatorname{int} A \cup \operatorname{int} B$; so $\partial(A \cup B) \subseteq \partial A \cup \partial B$. So if E, $F \in \Sigma$, $\partial(E \cup F) \subseteq \partial E \cup \partial F \in \mathcal{I}$ and $E \cup F \in \Sigma$. (iii) If $A \subseteq X$, then

$$\partial(X \setminus A) = \overline{X \setminus A} \setminus \operatorname{int}(X \setminus A) = (X \setminus \operatorname{int} A) \setminus (X \setminus \overline{A}) = \overline{A} \setminus \operatorname{int} A = \partial A.$$

So if $E \in \Sigma$, $\partial(X \setminus E) = \partial E \in \mathcal{I}$ and $X \setminus E \in \Sigma$. **Q**

If $A \in \mathcal{I}$, then of course $\partial A = \overline{A} \in \mathcal{I}$, so $A \in \Sigma$; accordingly $\mathcal{I}$ is an ideal in the Boolean algebra Σ, and we can form the quotient $\Sigma/\mathcal{I}$.

It will be helpful to note that every open set belongs to Σ, since if G is open then $\partial G = \overline{G} \setminus G$ cannot include any non-empty open set (since any open set meeting $\overline{G}$ must meet G).

(c) For each $E \in \Sigma$, set $V_E = \operatorname{int} \overline{E}$; then V_E is the unique member of $\mathrm{RO}(X)$ such that $E \triangle V_E \in \mathcal{I}$. **P** (i) Being the interior of a closed set, $V_E \in \mathrm{RO}(X)$. Since $\operatorname{int} E \subseteq V_E \subseteq \overline{E}$, $E \triangle V_E \subseteq \partial E \in \mathcal{I}$. (ii) If $G \in \mathrm{RO}(X)$ is such that $E \triangle G \in \mathcal{I}$, then

$$G \setminus \overline{V_E} \subseteq G \setminus V_E \subseteq (G \triangle E) \cup (V_E \triangle E) \in \mathcal{I},$$

so $G \setminus \overline{V_E}$, being open, must be actually empty, and $G \subseteq \overline{V_E}$; but this means that $G \subseteq \operatorname{int} \overline{V_E} = V_E$. Similarly, $V_E \subseteq G$ and $V_E = G$. This shows that V_E is unique. **Q**

(d) It follows that the map $G \mapsto G^\bullet : \mathrm{RO}(X) \to \Sigma/\mathcal{I}$ is a bijection, and we have a Boolean algebra structure on $\mathrm{RO}(X)$ defined by the Boolean algebra structure of $\Sigma/\mathcal{I}$. What this means is that for each of the binary Boolean operations $\cap_{\mathrm{RO}}$, $\triangle_{\mathrm{RO}}$, $\cup_{\mathrm{RO}}$, $\setminus_{\mathrm{RO}}$ and for G, $H \in \mathrm{RO}(X)$ we must have $G *_{\mathrm{RO}} H = \operatorname{int} \overline{G * H}$, writing $*_{\mathrm{RO}}$ for the operation on the algebra $\mathrm{RO}(X)$ and $*$ for the corresponding operation on Σ or $\mathcal{P}X$.

(e) Before working through the identifications, it will be helpful to observe that if $\mathcal{H}$ is any non-empty subset of $\mathrm{RO}(X)$, then $\operatorname{int} \bigcap \mathcal{H} = \operatorname{int} \overline{\bigcap \mathcal{H}}$. **P** Set $G = \operatorname{int} \overline{\bigcap \mathcal{H}}$. For every $H \in \mathcal{H}$, $G \subseteq \overline{H}$ so $G \subseteq \operatorname{int} \overline{H} = H$; thus

$$G \subseteq \operatorname{int} \bigcap \mathcal{H} \subseteq \operatorname{int} \overline{\bigcap \mathcal{H}} = G,$$

so $G = \operatorname{int} \bigcap \mathcal{H}$. **Q** Consequently $\operatorname{int} \bigcap \mathcal{H}$, being the interior of a closed set, belongs to $\mathrm{RO}(X)$.

(f)(i) If G, $H \in \mathrm{RO}(X)$ then their intersection in the algebra $\mathrm{RO}(X)$ is

$$G \cap_{\mathrm{RO}} H = \operatorname{int} \overline{G \cap H} = \operatorname{int}(G \cap H) = G \cap H,$$

using (d) for the first equality and (e) for the second.

(ii) Of course $X \in \mathrm{RO}(X)$ and $X^\bullet = 1_{\Sigma/\mathcal{I}}$, so $X = 1_{\mathrm{RO}(X)}$.

(iii) If $G \in \mathrm{RO}(X)$ then its complement $1_{\mathrm{RO}(X)} \setminus_{\mathrm{RO}} G$ in $\mathrm{RO}(X)$ is

$$\operatorname{int} \overline{X \setminus G} = \operatorname{int}(X \setminus G) = X \setminus \overline{G}.$$

(iv) If G, $H \in \mathrm{RO}(X)$, then the relative complement in $\mathrm{RO}(X)$ is

$$G \setminus_{\mathrm{RO}} H = G \cap_{\mathrm{RO}} (1_{\mathrm{RO}(X)} \setminus_{\mathrm{RO}} H) = G \cap (X \setminus \overline{H}) = G \setminus \overline{H} = \operatorname{int}(G \setminus H).$$

(v) If G, $H \in \mathrm{RO}(X)$, then $G \cup_{\mathrm{RO}} H = \operatorname{int} \overline{G \cup H}$ and $G \triangle_{\mathrm{RO}} H = \operatorname{int} \overline{G \triangle H}$, by the remarks in (d).

(g) We must note that for G, $H \in \mathrm{RO}(X)$,

$$G \subseteq_{\mathrm{RO}} H \iff G \cap_{\mathrm{RO}} H = G \iff G \cap H = G \iff G \subseteq H;$$

that is, the ordering of the Boolean algebra $\mathrm{RO}(X)$ is just the partial ordering induced on $\mathrm{RO}(X)$ by the Boolean ordering $\subseteq$ of $\mathcal{P}X$ or Σ.

(h) If $\mathcal{H}$ is any non-empty subset of $\mathrm{RO}(X)$, consider $G_0 = \operatorname{int} \bigcap \mathcal{H}$ and $G_1 = \operatorname{int} \overline{\bigcup \mathcal{H}}$.

$G_0 = \inf \mathcal{H}$ in $\mathrm{RO}(X)$. **P** By (e), $G_0 \in \mathrm{RO}(X)$. Of course $G_0 \subseteq H$ for every $H \in \mathcal{H}$, so G_0 is a lower bound for $\mathcal{H}$. If G is any lower bound for $\mathcal{H}$ in $\mathrm{RO}(X)$, then $G \subseteq H$ for every $H \in \mathcal{H}$, so $G \subseteq \bigcap \mathcal{H}$; but also G is open, so $G \subseteq \operatorname{int} \bigcap \mathcal{H} = G_0$. Thus G_0 is the greatest lower bound for $\mathcal{H}$. **Q**

$G_1 = \sup \mathcal{H}$ in $\mathrm{RO}(X)$. **P** Being the interior of a closed set, $G_1 \in \mathrm{RO}(X)$, and of course

$$H = \operatorname{int} \overline{H} \subseteq \operatorname{int} \overline{\bigcup \mathcal{H}} = G_1$$

for every $H \in \mathcal{H}$, so G_1 is an upper bound for $\mathcal{H}$ in $\mathrm{RO}(X)$. If G is any upper bound for $\mathcal{H}$ in $\mathrm{RO}(X)$, then

$$G = \operatorname{int} \overline{G} \supseteq \operatorname{int} \overline{\bigcup \mathcal{H}} = G_1;$$

thus G_1 is the least upper bound for $\mathcal{H}$ in $\mathrm{RO}(X)$. **Q**

This shows that every non-empty $\mathcal{H} \subseteq \mathrm{RO}(X)$ has a supremum and an infimum in $\mathrm{RO}(X)$; consequently $\mathrm{RO}(X)$ is Dedekind complete, and the proof is finished.

314Q Remarks (a) $\mathrm{RO}(X)$ is called the **regular open algebra** of the topological space X.

(b) Note that the map $E \mapsto V_E : \Sigma \to \mathrm{RO}(X)$ of part (c) of the proof above is a Boolean homomorphism, if $\mathrm{RO}(X)$ is given its Boolean algebra structure. Its kernel is of course $\mathcal{I}$; the induced map $E^\bullet \mapsto V_E : \Sigma/\mathcal{I} \to \mathrm{RO}(X)$ is just the inverse of the isomorphism $G \mapsto G^\bullet : \mathrm{RO}(X) \to \Sigma/\mathcal{I}$.

***314R** I interpolate a lemma corresponding to 313R, with a couple of other occasionally useful facts.

Lemma (a) Let X and Y be topological spaces, and $f : X \to Y$ a continuous function such that $f^{-1}[M]$ is nowhere dense in X for every nowhere dense $M \subseteq Y$. Then we have an order-continuous Boolean homomorphism π from the regular open algebra $\mathrm{RO}(Y)$ of Y to the regular open algebra $\mathrm{RO}(X)$ of X defined by setting $\pi H = \operatorname{int} \overline{f^{-1}[H]}$ for every $H \in \mathrm{RO}(Y)$.

(b) Let X be a topological space.

(i) If $U \subseteq X$ is open, then $G \mapsto G \cap U$ is a surjective order-continuous Boolean homomorphism from $\mathrm{RO}(X)$ onto $\mathrm{RO}(U)$.

(ii) If $U \in \mathrm{RO}(X)$ then $\mathrm{RO}(U)$ is the principal ideal of $\mathrm{RO}(X)$ generated by U.

proof (a)(i) By the remark in 314O, the formula for πH always defines a member of $\mathrm{RO}(X)$; and of course π is order-preserving.

Observe that if $H \in \mathrm{RO}(Y)$, then $f^{-1}[H]$ is open, so $f^{-1}[H] \subseteq \pi H$. It will be convenient to note straight away that if $V \subseteq Y$ is a dense open set then $f^{-1}[V]$ is dense in X. **P** $M = Y \setminus V$ is nowhere dense, so $f^{-1}[M]$ is nowhere dense and its complement $f^{-1}[V]$ is dense. **Q**

(ii) If H_1, $H_2 \in \mathrm{RO}(Y)$ then $\pi(H_1 \cap H_2) = \pi H_1 \cap \pi H_2$. **P** Because π is order-preserving, $\pi(H_1 \cap H_2) \subseteq \pi H_1 \cap \pi H_2$. **?** Suppose, if possible, that they are not equal. Then (because $\pi(H_1 \cap H_2)$ is a regular open set) $G = \pi H_1 \cap \pi H_2 \setminus \overline{\pi(H_1 \cap H_2)}$ is non-empty. Set $M = \overline{f[G]}$. Then $f^{-1}[M] \supseteq G$ is not nowhere dense, so $H = \operatorname{int} M$ must be non-empty. Now $G \subseteq \pi H_1 \subseteq \overline{f^{-1}[H_1]}$, so

$$f[G] \subseteq f[\overline{f^{-1}[H_1]}] \subseteq \overline{f[f^{-1}[H_1]]} \subseteq \overline{H}_1,$$

so $M \subseteq \overline{H}_1$ and $H \subseteq \operatorname{int} \overline{H}_1 = H_1$. Similarly, $H \subseteq H_2$, and $f^{-1}[H] \subseteq f^{-1}[H_1 \cap H_2] \subseteq \pi(H_1 \cap H_2)$. But also $H \cap f[G]$ is not empty, so

$$\emptyset \ne G \cap f^{-1}[H] \subseteq G \cap \pi(H_1 \cap H_2),$$

which is impossible. **XQ**

(iii) If $H \in \mathrm{RO}(Y)$ and $H' = Y \setminus \overline{H}$ is its complement in $\mathrm{RO}(Y)$ then $\pi H' = X \setminus \overline{\pi H}$ is the complement of πH in $\mathrm{RO}(X)$. **P** By (b), πH and $\pi H'$ are disjoint. Now $H \cup H'$ is a dense open subset of Y, so

$$\pi H \cup \pi H' \supseteq f^{-1}[H] \cup f^{-1}[H'] = f^{-1}[H \cup H']$$

is dense in X, and the regular open set $\pi H'$ must include the complement of πH in $\mathrm{RO}(X)$. **Q**

Putting this together with (b), we see that the conditions of 312H(ii) are satisfied, so that π is a Boolean homomorphism.

(iv) To see that it is order-continuous, let $\mathcal{H} \subseteq \mathrm{RO}(Y)$ be a non-empty set with supremum Y. Then $H_0 = \bigcup \mathcal{H}$ is a dense open subset of Y (see the formula in 314P). So

$$\bigcup_{H\in\mathcal{H}}\pi H\supseteq\bigcup_{H\in\mathcal{H}}f^{-1}[H]=f^{-1}[H_0]$$

is dense in X, and $\sup_{H\in\mathcal{H}}\pi H=X$ in $\mathrm{RO}(X)$. By 313L(b-iii), π is order-continuous.

(b)(i) The idea is to apply (a) to the identity function $f:U\to X$. If $M\subseteq X$ is nowhere dense, then any non-empty open subset of U has a non-empty open subset disjoint from M, so $f^{-1}[M]=M\cap U$ is nowhere dense in U; thus the condition is satisfied, and we have an order-continuous Boolean homomorphism $\pi:\mathrm{RO}(X)\to\mathrm{RO}(U)$ defined by setting $\pi H=\mathrm{int}_U\,\overline{H\cap U}^{(U)}$ for every $H\in\mathrm{RO}(X)$. (I write int_U, $\overline{}^{(U)}$ to indicate interior and closure in the subspace topology.) Now for any open set $G\subseteq X$,

$$U\cap\overline{G}=U\cap(\overline{G\cap U}\cup\overline{G\setminus U})=U\cap\overline{G\cap U}=\overline{G\cap U}^{(U)}.$$

So if $H\in\mathrm{RO}(X)$, then

$$\pi H=\mathrm{int}_U\,\overline{H\cap U}^{(U)}=\mathrm{int}_U(U\cap\overline{H})=U\cap\mathrm{int}\,\overline{H}=U\cap G.$$

So π takes the required form. To see that it is surjective, take any $V\in\mathrm{RO}(U)$. Then $\mathrm{int}\,\overline{V}\in\mathrm{RO}(X)$, and

$$V=\mathrm{int}_U\,\overline{V}^{(U)}=\mathrm{int}_U(U\cap\overline{V})=U\cap\mathrm{int}\,\overline{V}=\pi(\mathrm{int}\,\overline{V})$$

is a value of π.

(ii) If $G\in\mathrm{RO}(X)$ and $G\subseteq U$, then $G=G\cap U\in\mathrm{RO}(U)$. Conversely, if $V\in\mathrm{RO}(U)$, there is a $G\in\mathrm{RO}(X)$ such that $V=G\cap U$; but $G\cap U\in\mathrm{RO}(X)$, by 314P, so $V\in\mathrm{RO}(X)$.

314S It is now easy to characterize the Stone spaces of Dedekind complete Boolean algebras.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; write $\mathcal{E}$ for the algebra of open-and-closed subsets of Z, and $\mathrm{RO}(Z)$ for the regular open algebra of Z. Then the following are equiveridical:
(i) $\mathfrak{A}$ is Dedekind complete;
(ii) Z is extremally disconnected (definition: 3A3Af);
(iii) $\mathcal{E}=\mathrm{RO}(Z)$.

proof For $a\in\mathfrak{A}$, let $\widehat{a}$ be the corresponding member of $\mathcal{E}$.

(i)⇒(ii) If $\mathfrak{A}$ is Dedekind complete, let G be any open set in Z. Set $A=\{a:a\in\mathfrak{A},\,\widehat{a}\subseteq G\}$, $a_0=\sup A$. Then $G=\bigcup\{\widehat{a}:a\in A\}$, because $\mathcal{E}$ is a base for the topology of Z, so $\widehat{a}_0=\overline{G}$, by 313Ca. Consequently $\overline{G}$ is open. As G is arbitrary, Z is extremally disconnected.

(ii)⇒(iii) If $E\in\mathcal{E}$, then of course $E=\overline{E}=\mathrm{int}\,\overline{E}$, so E is a regular open set. Thus $\mathcal{E}\subseteq\mathrm{RO}(Z)$. On the other hand, suppose that $G\subseteq Z$ is a regular open set. Because Z is extremally disconnected, $\overline{G}$ is open; so $G=\mathrm{int}\,\overline{G}=\overline{G}$ is open-and-closed, and belongs to $\mathcal{E}$. Thus $\mathcal{E}=\mathrm{RO}(Z)$.

(iii)⇒(i) Since $\mathrm{RO}(Z)$ is Dedekind complete (314P), $\mathcal{E}$ and $\mathfrak{A}$ are also Dedekind complete Boolean algebras.

Remark Note that if the conditions above are satisfied, either 312M or the formulae in 314P show that the Boolean structures of $\mathcal{E}$ and $\mathrm{RO}(Z)$ are identical.

314T I come now to a construction of great importance, both as a foundation for further constructions and as a source of insight into the nature of Dedekind completeness.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z; for $a\in\mathfrak{A}$ let $\widehat{a}$ be the corresponding open-and-closed subset of Z. Let $\widehat{\mathfrak{A}}$ be the regular open algebra of Z (314P).
(a) The map $a\mapsto\widehat{a}$ is an injective order-continuous Boolean homomorphism from $\mathfrak{A}$ onto an order-dense subalgebra of $\widehat{\mathfrak{A}}$.
(b) If $\mathfrak{B}$ is any Dedekind complete Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{B}$ is an order-continuous Boolean homomorphism, there is a unique order-continuous Boolean homomorphism $\pi_1:\widehat{\mathfrak{A}}\to\mathfrak{B}$ such that $\pi_1\widehat{a}=\pi a$ for every $a\in\mathfrak{A}$.

proof (a)(i) Setting $\mathcal{E}=\{\widehat{a}:a\in\mathfrak{A}\}$, every member of $\mathcal{E}$ is open-and-closed, so is surely equal to the interior of its closure, and is a regular open set; thus $\widehat{a}\in\widehat{\mathfrak{A}}$ for every $a\in\mathfrak{A}$. The formulae in 314P tell us that if a, $b\in\mathfrak{A}$, then $\widehat{a}\cap\widehat{b}$, taken in $\widehat{\mathfrak{A}}$, is just the set-theoretic intersection $\widehat{a}\cap\widehat{b}=(a\cap b)\widehat{\ }$; while $1\setminus\widehat{a}$, taken in $\widehat{\mathfrak{A}}$, is

$$Z\setminus\overline{\widehat{a}}=Z\setminus\widehat{a}=(1\setminus a)\widehat{\ }.$$

And of course $\widehat{0} = \emptyset$ is the zero of $\widehat{\mathfrak{A}}$. Thus the map $a \mapsto \widehat{a} : \mathfrak{A} \to \widehat{\mathfrak{A}}$ preserves $\cap$ and complementation, so is a Boolean homomorphism (312H). Of course it is injective.

(ii) If $A \subseteq \mathfrak{A}$ is non-empty and $\inf A = 0$, then $\bigcap_{a\in A} \widehat{a}$ is nowhere dense in Z (313Cc), so

$$\inf\{\widehat{a} : a \in A\} = \operatorname{int}(\bigcap_{a\in A} \widehat{a}) = \emptyset$$

(314P again). As A is arbitrary, the map $a \mapsto \widehat{a} : \mathfrak{A} \to \widehat{\mathfrak{A}}$ is order-continuous.

(iii) If $G \in \widehat{\mathfrak{A}}$ is not empty, then there is a non-empty member of $\mathcal{E}$ included in it, by the definition of the topology of Z (311I). So $\mathcal{E}$ is an order-dense subalgebra of $\widehat{\mathfrak{A}}$.

(b) Now suppose that $\mathfrak{B}$ is a Dedekind complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism. Write $\iota a = \widehat{a}$ for $a \in \mathfrak{A}$, so that $\iota : \mathfrak{A} \to \widehat{\mathfrak{A}}$ is an isomorphism between $\mathfrak{A}$ and the order-dense subalgebra $\mathcal{E}$ of $\widehat{\mathfrak{A}}$. Accordingly $\pi\iota^{-1} : \mathcal{E} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, being the composition of the order-continuous Boolean homomorphisms π and ι^{-1}. By 314K, it has an extension to a Boolean homomorphism $\pi_1 : \widehat{\mathfrak{A}} \to \mathfrak{B}$, and $\pi_1\iota = \pi$, that is, $\pi_1\widehat{a} = \pi a$ for every $a \in \mathfrak{A}$. Now π_1 is order-continuous. **P** Suppose that $\mathcal{H} \subseteq \widehat{\mathfrak{A}}$ has supremum 1 in $\widehat{\mathfrak{A}}$. Set

$$\mathcal{H}' = \{E : E \in \mathcal{E},\, E \subseteq H \text{ for some } H \in \mathcal{H}\}.$$

Because $\mathcal{E}$ is order-dense in $\widehat{\mathfrak{A}}$,

$$H = \sup_{E\in\mathcal{E}, E\subseteq H} E = \sup_{E\in\mathcal{H}', E\subseteq H} E$$

for every $H \in \mathcal{H}$ (313K), and $\sup \mathcal{H}' = 1$ in $\widehat{\mathfrak{A}}$. It follows at once that $\sup \mathcal{H}' = 1$ in $\mathcal{E}$, so $\sup \pi_1[\mathcal{H}'] = \sup(\pi\iota^{-1})[\mathcal{H}'] = 1$. Since any upper bound for $\pi_1[\mathcal{H}]$ must also be an upper bound for $\pi_1[\mathcal{H}']$, $\sup \pi_1[\mathcal{H}] = 1$ in $\mathfrak{B}$. As $\mathcal{H}$ is arbitrary, π_1 is order-continuous (313L(b-iii)). **Q**

If $\pi_1' : \widehat{\mathfrak{A}} \to \mathfrak{B}$ is any other Boolean homomorphism such that $\pi_1'\widehat{a} = \pi a$ for every $a \in \mathfrak{A}$, then π_1 and π_1' agree on $\mathcal{E}$, and the argument just above shows that π_1' is also order-continuous. But if $G \in \widehat{\mathfrak{A}}$, G is the supremum (in $\widehat{\mathfrak{A}}$) of $\mathcal{F} = \{E : E \in \mathcal{E},\, E \subseteq G\}$, so

$$\pi_1'G = \sup_{E\in\mathcal{F}} \pi_1'E = \sup_{E\in\mathcal{F}} \pi_1 E = \pi_1 G.$$

As G is arbitrary, $\pi_1' = \pi_1$. Thus π_1 is unique.

314U The Dedekind completion of a Boolean algebra (a) For any Boolean algebra $\mathfrak{A}$, I will say that the Boolean algebra $\widehat{\mathfrak{A}}$ constructed in 314T is the **Dedekind completion** of $\mathfrak{A}$.

When using this concept I shall frequently suppress the distinction between $a \in \mathfrak{A}$ and $\widehat{a} \in \widehat{\mathfrak{A}}$, and treat $\mathfrak{A}$ as itself an order-dense subalgebra of $\widehat{\mathfrak{A}}$.

(b) The universal mapping theorem in 314Tb assures us that the Dedekind completion is essentially unique. The commonest way in which this fact appears is the following. If $\mathfrak{C}$ is a Dedekind complete Boolean algebra and $\mathfrak{A}$ is an order-dense subalgebra of $\mathfrak{C}$, then the embedding $\mathfrak{A} \subsetneq_{\to} \mathfrak{C}$ induces an isomorphism from $\widehat{\mathfrak{A}}$ to $\mathfrak{C}$. **P** Write $\pi a = a$ for $a \in \mathfrak{A}$. Because $\mathfrak{A}$ is order-dense, π is order-continuous (313O), so extends to an order-continuous Boolean homomorphism $\pi_1 : \widehat{\mathfrak{A}} \to \mathfrak{C}$. If $b \in \widehat{\mathfrak{A}}$ is non-zero, there is a non-zero $a \in \mathfrak{A}$ such that $a \subseteq b$; now

$$0 \neq a = \pi a = \pi_1 a \subseteq \pi_1 b.$$

As b is arbitrary, π_1 is injective. Next, $\pi_1[\widehat{\mathfrak{A}}]$ must be order-closed in $\mathfrak{C}$, by 314F(a-i); since it includes $\mathfrak{A}$ and $\mathfrak{A}$ is order-dense in $\mathfrak{C}$, $\pi_1[\widehat{\mathfrak{A}}] = \mathfrak{C}$ and π_1 is an isomorphism. **Q**

314X Basic exercises >(a) Let $\mathfrak{A}$ be a Boolean algebra. (i) Show that the following are equiveridical: (α) $\mathfrak{A}$ is Dedekind complete (β) every upwards-directed subset of $\mathfrak{A}$ has a least upper bound (γ) every downwards-directed subset of $\mathfrak{A}$ has a greatest lower bound (δ) every disjoint subset of $\mathfrak{A}$ has a least upper bound. (ii) Show that the following are equiveridical: (α) $\mathfrak{A}$ is Dedekind σ-complete (β) every non-decreasing sequence in $\mathfrak{A}$ has a least upper bound (γ) every non-increasing sequence in $\mathfrak{A}$ has a greatest lower bound (δ) every disjoint sequence in $\mathfrak{A}$ has a least upper bound.

(b) Let $\mathfrak{A}$ be a Boolean algebra. Show that any principal ideal of $\mathfrak{A}$ is order-closed. Show that $\mathfrak{A}$ is Dedekind complete iff every order-closed ideal is principal.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$, and $a \in \mathfrak{A}$; let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Show that $\{a \cap b : b \in \mathfrak{B}\}$ is an order-closed subalgebra of $\mathfrak{A}_a$.

>(d) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ a surjective order-continuous Boolean homomorphism. (i) Show that the kernel of π is a principal ideal in $\mathfrak{A}$. (ii) Show that $\mathfrak{B}$ is isomorphic to the complementary principal ideal in $\mathfrak{A}$, and in particular is Dedekind complete.

(e) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $\mathfrak{C}$ an order-closed subalgebra of $\mathfrak{A}$. Show that an element a of $\mathfrak{A}$ belongs to $\mathfrak{C}$ iff $\mathrm{upr}(1 \setminus a, \mathfrak{C}) = 1 \setminus \mathrm{upr}(a, \mathfrak{C})$ iff $\mathrm{upr}(1 \setminus a, \mathfrak{C}) \cap \mathrm{upr}(a, \mathfrak{C}) = 0$, writing $\mathrm{upr}(a, \mathfrak{C})$ for the upper envelope of a in $\mathfrak{C}$, as in 313S.

>(f) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{C}$ an order-closed subalgebra of $\mathfrak{A}$, $a_0 \in \mathfrak{A}$ and $c_0 \in \mathfrak{C}$. Show that the following are equiveridical: (i) there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\pi c = c$ for every $c \in \mathfrak{C}$ and $\pi a_0 = c_0$ (ii) $1 \setminus \mathrm{upr}(1 \setminus a_0, \mathfrak{C}) \subseteq c_0 \subseteq \mathrm{upr}(a_0, \mathfrak{C})$.

>(g) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. If $C \subseteq \mathfrak{A}$ and $\mathfrak{C}$ is the σ-subalgebra of $\mathfrak{A}$ generated by C, show that $\pi[\mathfrak{C}]$ is the σ-subalgebra of $\mathfrak{B}$ generated by $\pi[C]$.

(h) Let X and Y be extremally disconnected compact Hausdorff spaces, $\mathrm{RO}(X)$ and $\mathrm{RO}(Y)$ their regular open algebras, and $\phi : X \to Y$ a continuous surjection. Show that the following are equiveridical: (i) the Boolean homomorphism $V \mapsto \phi^{-1}[V]$ from $\mathrm{RO}(Y)$ to $\mathrm{RO}(X)$ (312Q, 314S) is order-continuous; (ii) $\phi[U]$ is open-and-closed in Y for every open-and-closed set $U \subseteq X$; (iii) $\phi[G]$ is open in Y for every open set $G \subseteq X$.

(i) Find a proof of 314Tb which does not appeal to 314K.

(j) Let $\mathfrak{B}$ be a Dedekind complete Boolean algebra, and $\mathfrak{A}$ a Boolean algebra which can be regularly embedded in $\mathfrak{B}$. Show that the Dedekind completion of $\mathfrak{A}$ can be regularly embedded in $\mathfrak{B}$.

(k) Let X be a topological space and Y a dense subset of X. Show that $G \mapsto G \cap Y$ is a Boolean isomorphism from $\mathrm{RO}(X)$ to $\mathrm{RO}(Y)$.

314Y Further exercises (a) Let P be a Dedekind complete partially ordered set. Show that a set $Q \subseteq P$ is order-closed iff $\sup R$, $\inf R$ belong to Q whenever $R \subseteq Q$ is a totally ordered subset of Q with upper and lower bounds in P. (*Hint*: show by induction on κ that if $A \subseteq Q$ is upwards-directed and bounded above and $\#(A) \le \kappa$ then $\sup A \in Q$.)

(b) Let P be a lattice. Show that P is Dedekind complete iff every non-empty totally ordered subset of P with an upper bound in P has a least upper bound in P. (*Hint*: if $A \subseteq P$ is non-empty and bounded below in P, let B be the set of lower bounds of A and use Zorn's Lemma to find a maximal element of B.)

(c) Give an example of a Boolean algebra $\mathfrak{A}$ with an order-closed subalgebra $\mathfrak{A}_0$ and an element c such that the subalgebra generated by $\mathfrak{A}_0 \cup \{c\}$ is not order-closed.

(d) Let X be any topological space. Let $\mathcal{M}$ be the σ-ideal of meager subsets of X, and set

$$\widehat{\mathcal{B}} = \{G \triangle A : G \subseteq X \text{ is open, } A \in \mathcal{M}\}.$$

(i) Show that $\widehat{\mathcal{B}}$ is a σ-algebra of subsets of X, and that $\widehat{\mathcal{B}}/\mathcal{M}$ is Dedekind complete. (Members of $\widehat{\mathcal{B}}$ are said to be the subsets of X **with the Baire property**; $\widehat{\mathcal{B}}$ is the **Baire-property algebra** of X.) (ii) Show that if $A \subseteq X$ and $\bigcup\{G : G \subseteq X \text{ is open, } A \cap G \in \widehat{\mathcal{B}}\}$ is dense, then $A \in \widehat{\mathcal{B}}$. (iii) Show that there is a largest open set $V \in \mathcal{M}$. (iv) Let $\mathrm{RO}(X)$ be the regular open algebra of X. Show that the map $G \mapsto G^\bullet$ is an order-continuous Boolean homomorphism from $\mathrm{RO}(X)$ onto $\widehat{\mathcal{B}}/\mathcal{M}$, so induces a Boolean isomorphism between the principal ideal of $\mathrm{RO}(X)$ generated by $X \setminus \overline{V}$ and $\widehat{\mathcal{B}}/\mathcal{M}$. ($\widehat{\mathcal{B}}/\mathcal{M}$ is the **category algebra** of X; it is a Dedekind complete Boolean algebra. X is called a **Baire space** if $V = \emptyset$; in this case $\mathrm{RO}(X) \cong \widehat{\mathcal{B}}/\mathcal{M}$. See 4A3R in Volume 4.)

(e) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle a_n \rangle_{n \in \mathbb{N}}$ any sequence in $\mathfrak{A}$. For $n \in \mathbb{N}$ set $E_n = \{x : x \in \{0,1\}^{\mathbb{N}}, x(n) = 1\}$, and let $\mathcal{B}$ be the σ-algebra of subsets of $\{0,1\}^{\mathbb{N}}$ generated by $\{E_n : n \in \mathbb{N}\}$. ($\mathcal{B}$ is the 'Borel σ-algebra' of $\{0,1\}^{\mathbb{N}}$; see 4A3E in Volume 4.) Show that there is a unique sequentially order-continuous Boolean homomorphism $\theta : \mathcal{B} \to \mathfrak{A}$ such that $\theta(E_n) = a_n$ for every $n \in \mathbb{N}$. (*Hint*: define a suitable function ϕ from the Stone space Z of $\mathfrak{A}$ to $\{0,1\}^{\mathbb{N}}$, and consider $\{E : E \subseteq \{0,1\}^{\mathbb{N}}, \phi^{-1}[E] \text{ has the Baire property in } Z\}$.) Show that $\theta[\mathcal{B}]$ is the σ-subalgebra of $\mathfrak{A}$ generated by $\{a_n : n \in \mathbb{N}\}$.

(f) Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space. Show that $\mathfrak{A}$ is Dedekind σ-complete iff $\overline{G}$ is open whenever G is a cozero set in Z. (Such spaces are called **basically disconnected** or **quasi-Stonian**.)

(g) Let $\mathfrak{A}$, $\mathfrak{B}$ be Dedekind complete Boolean algebras and $D \subseteq \mathfrak{A}$ an order-dense set. Suppose that $\phi : D \to \mathfrak{B}$ is such that (i) $\phi[D]$ is order-dense in $\mathfrak{B}$ (ii) for all d, $d' \in D$, $d \cap d' = 0$ iff $\phi d \cap \phi d' = 0$. Show that ϕ has a unique extension to a Boolean isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(h) Let $\mathfrak{A}$ be any Boolean algebra. Let $\mathcal{J}$ be the family of order-closed ideals in $\mathfrak{A}$. Show that (i) $\mathcal{J}$ is a Dedekind complete Boolean algebra with operations defined by the formulae $I \cap J = I \cap J$, $1 \setminus J = \{a : a \cap b = 0 \text{ for every } b \in J\}$ (ii) the map $a \mapsto \mathfrak{A}_a$, the principal ideal generated by a, is an injective order-continuous Boolean homomorphism from $\mathfrak{A}$ onto an order-dense subalgebra of $\mathcal{J}$ (iii) $\mathcal{J}$ is isomorphic to the Dedekind completion of $\mathfrak{A}$.

314 Notes and comments At the risk of being tiresomely long-winded, I have taken the trouble to spell out a large proportion of the results in this section and the last in their 'sequential' as well as their 'unrestricted' forms. The point is that while (in my view) the underlying ideas are most clearly and dramatically expressed in terms of order-closed sets, order-continuous functions and Dedekind complete algebras, a large proportion of the applications in measure theory deal with sequentially order-closed sets, sequentially order-continuous functions and Dedekind σ-complete algebras. As a matter of simple technique, therefore, it is necessary to master both, and for the sake of later reference I generally give the statements of both versions in full. Perhaps the points to look at most keenly are just those where there is a difference in the ideas involved, as in 314Bb, or in which there is only one version given, as in 314M and 314T.

If you have seen the Hahn-Banach theorem (3A5A), it may have been recalled to your mind by Theorem 314K; in both cases we use an order relation and a bit of algebra to make a single step towards an extension of a function, and Zorn's lemma to turn this into the extension we seek. A good part of this section has turned out to be on the borderland between the theory of Boolean algebra and general topology; naturally enough, since (as always with the general theory of Boolean algebra) one of our first concerns is to establish connexions between algebras and their Stone spaces.

I think 314T is the first substantial 'universal mapping theorem' in this volume; it is by no means the last. The idea of the construction $\widehat{\mathfrak{A}}$ is not just that we obtain a Dedekind complete Boolean algebra in which $\mathfrak{A}$ is embedded as an order-dense subalgebra, but that we simultaneously obtain a theorem on the canonical extension to $\widehat{\mathfrak{A}}$ of order-continuous Boolean homomorphisms defined on $\mathfrak{A}$. This characterization is enough to define the pair $(\widehat{\mathfrak{A}}, a \mapsto \widehat{a})$ up to isomorphism, so the exact method of construction of $\widehat{\mathfrak{A}}$ becomes of secondary importance. The one used in 314T is very natural (at least, if we believe in Stone spaces), but there are others (see 314Yh), with different virtues.

314K and 314T both describe circumstances in which we can find extensions of Boolean homomorphisms. Clearly such results are fundamental in the theory of Boolean algebras, but I shall not attempt any systematic presentation here. 314Ye can also be regarded as belonging to this family of ideas.

315 Products and free products

I describe here two algebraic constructions of fundamental importance. They are very different in character, indeed may be regarded as opposites, despite the common use of the word 'product'. The first part of the section (315A-315H) deals with the easier construction, the 'simple product'; the second part (315I-315Q) with the 'free product'. These constructions lead to descriptions of projective and inductive limits (315R-315S).

315A Products of Boolean algebras (a) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras. Set $\mathfrak{A} = \prod_{i\in I} \mathfrak{A}_i$, with the natural ring structure

$$a \triangle b = \langle a(i) \triangle b(i)\rangle_{i\in I},$$

$$a \cap b = \langle a(i) \cap b(i)\rangle_{i\in I}$$

for a, $b \in \mathfrak{A}$. Then $\mathfrak{A}$ is a ring (3A2H); it is a Boolean ring because

$$a \cap a = \langle a(i) \cap a(i)\rangle_{i\in I} = a$$

for every $a \in \mathfrak{A}$; and it is a Boolean algebra because if we set $1_{\mathfrak{A}} = \langle 1_{\mathfrak{A}_i}\rangle_{i\in I}$, then $1_{\mathfrak{A}} \cap a = a$ for every $a \in \mathfrak{A}$. I will call $\mathfrak{A}$ the **simple product** of the family $\langle\mathfrak{A}_i\rangle_{i\in I}$.

I should perhaps remark that when $I = \emptyset$ then $\mathfrak{A}$ becomes $\{\emptyset\}$, to be interpreted as the singleton Boolean algebra.

(b) The Boolean operations on $\mathfrak{A}$ are now defined by the formulae

$$a \cup b = \langle a(i) \cup b(i)\rangle_{i\in I}, \quad a \setminus b = \langle a(i) \setminus b(i)\rangle_{i\in I}$$

for all a, $b \in \mathfrak{A}$.

315B Theorem Let $\langle \mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and $\mathfrak{A}$ their simple product.

(a) The maps $a \mapsto \pi_i(a) = a(i) : \mathfrak{A} \to \mathfrak{A}_i$ are all Boolean homomorphisms.

(b) If $\mathfrak{B}$ is any other Boolean algebra, then a map $\phi : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism iff $\pi_i\phi : \mathfrak{B} \to \mathfrak{A}_i$ is a Boolean homomorphism for every $i \in I$.

proof Verification of these facts amounts just to applying the definitions with attention.

315C Products of partially ordered sets (a) It is perhaps worth spelling out the following elementary definition. If $\langle P_i\rangle_{i\in I}$ is any family of partially ordered sets, its **product** is the set $P = \prod_{i\in I} P_i$ ordered by saying that $p \le q$ iff $p(i) \le q(i)$ for every $i \in I$; it is easy to check that P is now a partially ordered set.

(b) The point is that if $\mathfrak{A}$ is the simple product of a family $\langle \mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras, then the ordering of $\mathfrak{A}$ is just the product partial order:

$$a \subseteq b \iff a \cap b = a \iff a(i) \cap b(i) = a(i)\ \forall\ i \in I \iff a(i) \subseteq b(i)\ \forall\ i \in I.$$

Now we have the following elementary, but extremely useful, general facts about products of partially ordered sets.

315D Proposition Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets with product P.

(a) For any non-empty set $A \subseteq P$ and $q \in P$,

(i) $\sup A = q$ in P iff $\sup_{p\in A} p(i) = q(i)$ in P_i for every $i \in I$,

(ii) $\inf A = q$ in P iff $\inf_{p\in A} p(i) = q(i)$ in P_i for every $i \in I$.

(b) The coordinate maps $p \mapsto \pi_i(p) = p(i) : P \to P_i$ are all order-preserving and order-continuous.

(c) For any partially ordered set Q and function $\phi : Q \to P$, ϕ is order-preserving iff $\pi_i\phi$ is order-preserving for every $i \in I$.

(d) For any partially ordered set Q and order-preserving function $\phi : Q \to P$,

(i) ϕ is order-continuous iff $\pi_i\phi$ is order-continuous for every i,

(ii) ϕ is sequentially order-continuous iff $\pi_i\phi$ is sequentially order-continuous for every i.

(e)(i) P is Dedekind complete iff every P_i is Dedekind complete.

(ii) P is Dedekind σ-complete iff every P_i is Dedekind σ-complete.

proof All these are elementary verifications. Of course parts (b), (d) and (e) rely on (a).

315E Factor algebras as principal ideals Because Boolean algebras have least elements, we have a second type of canonical homomorphism associated with their products. If $\langle \mathfrak{A}_i\rangle_{i\in I}$ is a family of Boolean algebras with simple product $\mathfrak{A}$, define $\theta_i : \mathfrak{A}_i \to \mathfrak{A}$ by setting $(\theta_i a)(i) = a$, $(\theta_i a)(j) = 0_{\mathfrak{A}_j}$ if $i \in I$, $a \in \mathfrak{A}_i$ and $j \in I \setminus \{i\}$. Each θ_i is a ring homomorphism, and is a Boolean isomorphism between $\mathfrak{A}_i$ and the principal ideal of $\mathfrak{A}$ generated by $\theta_i(1_{\mathfrak{A}_i})$. The family $\langle \theta_i(1_{\mathfrak{A}_i})\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$.

Associated with these embeddings is the following important result.

315F Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\langle e_i\rangle_{i\in I}$ a partition of unity in $\mathfrak{A}$. Suppose

either (i) that I is finite

or (ii) that I is countable and $\mathfrak{A}$ is Dedekind σ-complete

or (iii) that $\mathfrak{A}$ is Dedekind complete.

Then the map $a \mapsto \langle a \cap e_i\rangle_{i\in I}$ is a Boolean isomorphism between $\mathfrak{A}$ and $\prod_{i\in I} \mathfrak{A}_{e_i}$, writing $\mathfrak{A}_{e_i}$ for the principal ideal of $\mathfrak{A}$ generated by e_i for each i.

proof The given map is a Boolean homomorphism because each of the maps $a \mapsto a \cap e_i : \mathfrak{A} \to \mathfrak{A}_{e_i}$ is (312J). It is injective because $\sup_{i\in I} e_i = 1$, so if $a \in \mathfrak{A} \setminus \{0\}$ there is an i such that $a \cap e_i \ne 0$. It is surjective because $\langle e_i\rangle_{i\in I}$ is disjoint and if $c \in \prod_{i\in I} \mathfrak{A}_{e_i}$ then $a = \sup_{i\in I} c(i)$ is defined in $\mathfrak{A}$ and

$$a \cap e_j = \sup_{i\in I} c(i) \cap e_j = c(j)$$

for every $j \in I$ (using 313Ba). The three alternative versions of the hypotheses of this proposition are designed to ensure that the supremum is always well-defined in $\mathfrak{A}$.

315G Algebras of sets and their quotients The Boolean algebras of measure theory are mostly presented as algebras of sets or quotients of algebras of sets, so it is perhaps worth spelling out the ways in which the product construction applies to such algebras.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i an algebra of subsets of X_i for each i.

(a) The simple product $\prod_{i\in I}\Sigma_i$ may be identified with the algebra

$$\Sigma=\{E:E\subseteq X,\,\{x:(x,i)\in E\}\in\Sigma_i\text{ for every }i\in I\}$$

of subsets of $X=\{(x,i):i\in I,\,x\in X_i\}$, with the canonical homomorphisms $\pi_i:\Sigma\to\Sigma_i$ being given by

$$\pi_iE=\{x:(x,i)\in E\}$$

for each $E\in\Sigma$.

(b) Now suppose that $\mathcal{J}_i$ is an ideal of Σ_i for each i. Then $\prod_{i\in I}\Sigma_i/\mathcal{J}_i$ may be identified with $\Sigma/\mathcal{J}$, where

$$\mathcal{J}=\{E:E\in\Sigma,\,\{x:(x,i)\in E\}\in\mathcal{J}_i\text{ for every }i\in I\},$$

and the canonical homomorphisms $\tilde\pi_i:\Sigma/\mathcal{J}\to\Sigma_i/\mathcal{J}_i$ are given by the formula $\tilde\pi_i(E^\bullet)=(\pi_iE)^\bullet$ for every $E\in\Sigma$.

proof (a) It is easy to check that Σ is a subalgebra of $\mathcal{P}X$, and that the map $E\mapsto\langle\pi_iE\rangle_{i\in I}:\Sigma\to\prod_{i\in I}\Sigma_i$ is a Boolean isomorphism.

(b) Again, it is easy to check that $\mathcal{J}$ is an ideal of Σ, that the proposed formula for $\tilde\pi_i$ does indeed define a map from $\Sigma/\mathcal{J}$ to $\Sigma_i/\mathcal{J}_i$, and that $E^\bullet\mapsto\langle\tilde\pi_iE^\bullet\rangle_{i\in I}$ is an isomorphism between $\Sigma/\mathcal{J}$ and $\prod_{i\in I}\Sigma_i/\mathcal{J}_i$.

***315H** There is a particular kind of simple product which arises naturally when we look at regular open algebras.

Proposition Let X be a topological space, and $\mathcal{U}$ a disjoint family of open subsets of X with union dense in X. Then the regular open algebra $\mathrm{RO}(X)$ is isomorphic to the simple product $\prod_{U\in\mathcal{U}}\mathrm{RO}(U)$.

proof By 314R(b-i), $G\mapsto G\cap U$ is a Boolean homomorphism from $\mathrm{RO}(X)$ onto $\mathrm{RO}(U)$, for any $U\in\mathcal{U}$. By 315B, we have a Boolean homomorphism $G\mapsto\pi G=\langle G\cap U\rangle_{U\in\mathcal{U}}:\mathrm{RO}(X)\to\prod_{U\in\mathcal{U}}\mathrm{RO}(U)$. If $G\in\mathrm{RO}(X)\setminus\{\emptyset\}$, then $G\cap\bigcup\mathcal{U}\ne\emptyset$, because $\bigcup\mathcal{U}$ is dense; now there is a $U\in\mathcal{U}$ such that $G\cap U\ne\emptyset$, so πG is non-zero in the Boolean algebra $\prod_{U\in\mathcal{U}}\mathrm{RO}(U)$. As G is arbitrary, π is injective (3A2Db).

To see that π is surjective, suppose that we are given a family $\langle V_U\rangle_{U\in\mathcal{U}}$ with $V_U\in\mathrm{RO}(U)$ for every $U\in\mathcal{U}$. Set $H=\bigcup_{U\in\mathcal{U}}V_U$, $G=\operatorname{int}\overline{H}\in\mathrm{RO}(X)$. Then, for any $U\in\mathcal{U}$, (writing int_U and $\overline{}^{(U)}$ for interior and closure in the subspace topology on U, as in part (b) of the proof of 314R)

$$G\cap U=U\cap\operatorname{int}\overline{H}=\operatorname{int}_U\overline{H\cap U}^{(U)}=\operatorname{int}_U\overline{V}_U^{(U)}=V_U,$$

so $\pi G=\langle V_U\rangle_{U\in\mathcal{U}}$. Thus π is bijective and is a Boolean isomorphism.

315I Free products I come now to the second construction of this section.

(a) Definition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras. For each $i\in I$, let Z_i be the Stone space of $\mathfrak{A}_i$. Set $Z=\prod_{i\in I}Z_i$, with the product topology. Then the **free product** of $\langle\mathfrak{A}_i\rangle_{i\in I}$ is the algebra $\mathfrak{A}$ of open-and-closed sets in Z; I will denote it by $\bigotimes_{i\in I}\mathfrak{A}_i$.

(b) For $i\in I$ and $a\in\mathfrak{A}_i$, the set $\widehat{a}\subseteq Z_i$ representing a is an open-and-closed subset of Z_i; because $z\mapsto z(i):Z\to Z_i$ is continuous, $\varepsilon_i(a)=\{z:z(i)\in\widehat{a}\}$ is open-and-closed, so belongs to $\mathfrak{A}$. In this context I will call $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ the **canonical map**.

(c) The topological space Z may be identified with the Stone space of the Boolean algebra $\mathfrak{A}$. **P** By Tychonoff's theorem (3A3J), Z is compact. If $z\in Z$ and G is an open subset of Z containing z, then there are J, $\langle G_j\rangle_{i\in J}$ such that J is a finite subset of I, G_j is an open subset of Z_j for each $j\in J$, and

$$z\in\{w:w\in Z,\,w(j)\in G_j\text{ for every }j\in J\}\subseteq G.$$

Because each Z_j is zero-dimensional, we can find an open-and-closed set $E_j\subseteq Z_j$ such that $z(j)\in E_j\subseteq G_j$. Now

$$H=Z\cap\bigcap_{j\in J}\{w:w(j)\in E_j\}$$

is a finite intersection of open-and-closed subsets of Z, so is open-and-closed; and $z\in H\subseteq G$. As z and G are arbitrary, Z is zero-dimensional. Finally, Z, being the product of Hausdorff spaces, is Hausdorff. So the result follows from 311J. **Q**

315J Theorem Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, with free product $\mathfrak{A}$.

(a) The canonical map $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ is a Boolean homomorphism for every $i\in I$.

(b) For any Boolean algebra $\mathfrak{B}$ and any family $\langle\phi_i\rangle_{i\in I}$ such that ϕ_i is a Boolean homomorphism from $\mathfrak{A}_i$ to $\mathfrak{B}$ for every i, there is a unique Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{B}$ such that $\phi_i=\phi\varepsilon_i$ for each i.

proof These are both consequences of 312Q-312R. As in 315I, write Z_i for the Stone space of $\mathfrak{A}$, and Z for $\prod_{i\in I}Z_i$, identified with the Stone space of $\mathfrak{A}$, as observed in 315Ic. The maps $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ are defined as the homomorphisms corresponding to the continuous maps $z\mapsto\tilde\varepsilon_i(z)=z(i):Z\to Z_i$, so (a) is surely true.

Now suppose that we are given a Boolean homomorphism $\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ for each $i\in I$. Let W be the Stone space of $\mathfrak{B}$, and let $\tilde\phi_i:W\to Z_i$ be the continuous function corresponding to ϕ_i. By 3A3Ib, the map $w\mapsto\tilde\phi(w)=\langle\tilde\phi_i(w)\rangle_{i\in I}:W\to Z$ is continuous, and corresponds to a Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{B}$; because $\tilde\phi_i=\tilde\varepsilon_i\tilde\phi$, $\phi\varepsilon_i=\phi_i$ for each i. Moreover, ϕ is the only Boolean homomorphism with this property, because if $\psi:\mathfrak{A}\to\mathfrak{B}$ is a Boolean homomorphism such that $\psi\varepsilon_i=\phi_i$ for every i, then ψ corresponds to a continuous function $\tilde\psi:W\to Z$, and we must have $\tilde\varepsilon_i\tilde\psi=\tilde\phi_i$ for each i, so that $\tilde\psi=\tilde\phi$ and $\psi=\phi$. This proves (b).

315K Of course 315J is the defining property of the free product (see 315Xi below). I list a few further basic facts.

Proposition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and $\mathfrak{A}$ their free product; write $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ for the canonical homomorphisms.

(a) $\mathfrak{A}$ is the subalgebra of itself generated by $\bigcup_{i\in I}\varepsilon_i[\mathfrak{A}_i]$.

(b) Write C for the set of those members of $\mathfrak{A}$ expressible in the form $\inf_{j\in J}\varepsilon_j(a_j)$, where $J\subseteq I$ is finite and $a_j\in\mathfrak{A}_j$ for every j. Then every member of $\mathfrak{A}$ is expressible as the supremum of a disjoint finite subset of C. In particular, C is order-dense in $\mathfrak{A}$.

(c) Every ε_i is order-continuous.

(d) $\mathfrak{A}=\{0_{\mathfrak{A}}\}$ iff there is some $i\in I$ such that $\mathfrak{A}_i=\{0_{\mathfrak{A}_i}\}$.

(e) Now suppose that $\mathfrak{A}_i\ne\{0_{\mathfrak{A}_i}\}$ for every $i\in I$.

(i) ε_i is injective for every $i\in I$.

(ii) If $J\subseteq I$ is finite and a_j is a non-zero member of $\mathfrak{A}_j$ for each $j\in J$, then $\inf_{j\in J}\varepsilon_j(a_j)\ne0$.

(iii) If i, j are distinct members of I, $a\in\mathfrak{A}_i$ and $b\in\mathfrak{A}_j$, then $\varepsilon_i(a)=\varepsilon_j(b)$ iff either $a=0_{\mathfrak{A}_i}$ and $b=0_{\mathfrak{A}_j}$ or $a=1_{\mathfrak{A}_i}$ and $b=1_{\mathfrak{A}_j}$.

proof As usual, write Z_i for the Stone space of $\mathfrak{A}_i$, and $Z=\prod_{i\in I}Z_i$, identified with the Stone space of $\mathfrak{A}$ (315Ic).

(a) Write $\mathfrak{A}'$ for the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\varepsilon_i[\mathfrak{A}_i]$. Then $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}'$ is a Boolean homomorphism for each i, so by 315Jb there is a Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{A}'$ such that $\phi\varepsilon_i=\varepsilon_i$ for each i. Now, regarding ϕ as a Boolean homomorphism from $\mathfrak{A}$ to itself, the uniqueness assertion of 315Jb (with $\mathfrak{B}=\mathfrak{A}$) shows that ϕ must be the identity, so that $\mathfrak{A}'=\mathfrak{A}$.

(b) Write $\mathcal{D}$ for the set of finite partitions of unity in $\mathfrak{A}$ consisting of members of C, and A for the set of members of $\mathfrak{A}$ expressible in the form $\sup D'$ where D' is a subset of a member of $\mathcal{D}$. Then A is a subalgebra of $\mathfrak{A}$. **P** (i) $1_{\mathfrak{A}}\in C$ (set $J=\emptyset$ in the definition of members of C) so $\{1_{\mathfrak{A}}\}\in\mathcal{D}$ and $0_{\mathfrak{A}}$, $1_{\mathfrak{A}}\in A$. (ii) Note that if c, $d\in C$ then $c\cap d\in C$. (iii) If a, $b\in A$, express them as $\sup D'$, $\sup E'$ where $D'\subseteq D\in\mathcal{D}$, $E'\subseteq E\in\mathcal{D}$. Then

$$F=\{d\cap e:d\in D,\,e\in E\}\in\mathcal{D},$$

so

$$1_{\mathfrak{A}}\setminus a=\sup(D\setminus D')\in A,$$

$$a\cup b=\sup\{f:f\in F,\,f\subseteq a\cup b\}\in A.\ \mathbf{Q}$$

Also, $\varepsilon_i[\mathfrak{A}_i]\subseteq A$ for each $i\in I$. **P** If $a\in\mathfrak{A}_i$, then $\{\varepsilon_i(a),\varepsilon_i(1_{\mathfrak{A}_i}\setminus a)\}\in\mathcal{D}$, so $\varepsilon_i(a)\in A$. **Q**

So (a) tells us that $A=\mathfrak{A}$, and every member of $\mathfrak{A}$ is a finite disjoint union of members of C.

(c) If $i\in I$ and $A\subseteq\mathfrak{A}_i$ and $\inf A=0$ in $\mathfrak{A}_i$, take any non-zero $c\in\mathfrak{A}$. By (b), we can find a finite $J\subseteq I$ and a family $\langle a_j\rangle_{j\in J}$ such that $c'=\inf_{j\in J}\varepsilon_j(a_j)\subseteq c$ and $c'\ne0$. Regarding c' as a subset of Z, we have a point $z\in c'$. Adding i to J and setting $a_i=1_{\mathfrak{A}_i}$ if necessary, we may suppose that $i\in J$. Now $c'\ne0_{\mathfrak{A}}$ so $a_i\ne0_{\mathfrak{A}_i}$ and there is an $a\in A$ such that $a_i\not\subseteq a$, so there is a $t\in\widehat{a}_i\setminus\widehat{a}$. In this case, setting $w(i)=t$, $w(j)=z(j)$ for $j\ne i$, we have $w\in c'\setminus\varepsilon_i(a)$, and c', c are not included in $\varepsilon_i(a)$. As c is arbitrary, this shows that $\inf\varepsilon_i[A]=0$. As A is arbitrary, ε_i is order-continuous.

(d) The point is that $\mathfrak{A} = \{0_{\mathfrak{A}}\}$ iff $Z = \emptyset$, which is so iff some Z_i is empty.

(e)(i) Because no Z_i is empty, all the coordinate maps from Z to Z_i are surjective, so the corresponding homomorphisms ε_i are injective (312Sa).

(ii) Because J is finite,

$$\inf_{j\in J} \varepsilon_j(a_j) = \{z : z \in Z,\ z(j) \in \widehat{a}_j \text{ for every } j \in J\}$$

is not empty.

(iii) If $\varepsilon_i(a) = \varepsilon_j(b) = 0_{\mathfrak{A}}$ then (using (i)) $a = 0_{\mathfrak{A}_i}$ and $b = 0_{\mathfrak{A}_j}$; if $\varepsilon_i(a) = \varepsilon_j(b) = 1_{\mathfrak{A}}$ then $a = 1_{\mathfrak{A}_i}$ and $b = 1_{\mathfrak{A}_j}$. **?** If $\varepsilon_i(a) = \varepsilon_j(b) \in \mathfrak{A} \setminus \{0_{\mathfrak{A}}, 1_{\mathfrak{A}}\}$, then there are $t \in \widehat{a}$ and $u \in Z_j \setminus \widehat{b}$. Now there is a $z \in Z$ such that $z(i) = t$ and $z(j) = u$, so that $z \in \varepsilon_i(a) \setminus \varepsilon_j(b)$. **X**

315L Proposition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, and $\langle J_k\rangle_{k\in K}$ any partition (that is, disjoint cover) of I. Then the free product $\mathfrak{A}$ of $\langle\mathfrak{A}_i\rangle_{i\in I}$ is isomorphic to the free product $\mathfrak{B}$ of $\langle\mathfrak{B}_k\rangle_{k\in K}$, where each $\mathfrak{B}_k$ is the free product of $\langle\mathfrak{A}_i\rangle_{i\in J_k}$.

proof Write $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$, $\varepsilon'_i : \mathfrak{A}_i \to \mathfrak{B}_k$ and $\delta_k : \mathfrak{B}_k \to \mathfrak{B}$ for the canonical homomorphisms when $k \in K$ and $i \in J_k$. Then the homomorphisms $\delta_k\varepsilon'_i : \mathfrak{A}_i \to \mathfrak{B}$ correspond to a homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $\phi\varepsilon_i = \delta_k\varepsilon'_i$ whenever $i \in J_k$. Next, for each k, the homomorphisms $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$, for $i \in J_k$, correspond to a homomorphism $\psi_k : \mathfrak{B}_k \to \mathfrak{A}$ such that $\psi_k\varepsilon'_i = \varepsilon_i$ for $i \in J_k$; and the family $\langle\psi_k\rangle_{k\in K}$ corresponds to a homomorphism $\psi : \mathfrak{B} \to \mathfrak{A}$ such that $\psi\delta_k = \psi_k$ for $k \in K$. Consequently

$$\psi\phi\varepsilon_i = \psi\delta_k\varepsilon'_i = \psi_k\varepsilon'_i = \varepsilon_i$$

whenever $k \in K$, $i \in J_k$. Once again using the uniqueness assertion in 315Jb, $\psi\phi$ is the identity homomorphism on $\mathfrak{A}$. On the other hand, if we look at $\phi\psi : \mathfrak{B} \to \mathfrak{B}$, then we see that

$$\phi\psi\delta_k\varepsilon'_i = \phi\psi_k\varepsilon'_i = \phi\varepsilon_i = \delta_k\varepsilon'_i$$

whenever $k \in K$, $i \in J_k$. Now, for given k, $\{b : b \in \mathfrak{B}_k,\ \phi\psi\delta_k b = \delta_k b\}$ is a subalgebra of $\mathfrak{B}_k$ including $\bigcup_{i\in J_k} \varepsilon'_i[\mathfrak{A}_i]$, and must be the whole of $\mathfrak{B}_k$, by 315Ka. So $\{b : b \in \mathfrak{B},\ \phi\psi b = b\}$ is a subalgebra of $\mathfrak{B}$ including $\bigcup_{k\in K} \delta_k[\mathfrak{B}_k]$, and is the whole of $\mathfrak{B}$. Thus $\phi\psi$ is the identity on $\mathfrak{B}$ and ϕ, ψ are the two halves of an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

315M Algebras of sets and their quotients Once again I devote a paragraph to spelling out the application of the construction to the algebras most important to us.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i an algebra of subsets of X_i for each i.

(a) The free product $\bigotimes_{i\in I} \Sigma_i$ may be identified with the algebra Σ of subsets of $X = \prod_{i\in I} X_i$ generated by the set $\{\varepsilon_i(E) : i \in I,\ E \in \Sigma_i\}$, where $\varepsilon_i(E) = \{x : x \in X,\ x(i) \in E\}$.

(b) Now suppose that $\mathcal{J}_i$ is an ideal of Σ_i for each i. Then $\bigotimes_{i\in I} \Sigma_i/\mathcal{J}_i$ may be identified with $\Sigma/\mathcal{J}$, where $\mathcal{J}$ is the ideal of Σ generated by $\{\varepsilon_i(E) : i \in I,\ E \in \mathcal{J}_i\}$; the corresponding canonical maps $\tilde{\varepsilon}_i : \Sigma_i/\mathcal{J}_i \to \Sigma/\mathcal{J}$ being defined by the formula $\tilde{\varepsilon}_i(E^\bullet) = (\varepsilon_i(E))^\bullet$ for $i \in I$, $E \in \Sigma_i$.

proof I start by proving (b) in detail; the argument for (a) is then easy to extract. Write $\mathfrak{A}_i = \Sigma_i/\mathcal{J}_i$, $\mathfrak{A} = \Sigma/\mathcal{J}$.

(i) Fix $i \in I$ for the moment. By the definition of Σ, $\varepsilon_i(E) \in \Sigma$ for $E \in \Sigma_i$, and it is easy to check that $\varepsilon_i : \Sigma_i \to \Sigma$ is a Boolean homomorphism. Again, because $\varepsilon_i(E) \in \mathcal{J}$ whenever $E \in \mathcal{J}_i$, the kernel of the homomorphism $E \mapsto (\varepsilon_i(E))^\bullet : \Sigma_i \to \mathfrak{A}$ includes $\mathcal{J}_i$, so the formula for $\tilde{\varepsilon}_i$ defines a homomorphism from $\mathfrak{A}_i$ to $\mathfrak{A}$.

Now let $\mathfrak{C} = \bigotimes_{i\in I} \mathfrak{A}_i$ be the free product, and write $\varepsilon'_i : \mathfrak{A}_i \to \mathfrak{C}$ for the canonical homomorphisms. By 315J, there is a Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{A}$ such that $\phi\varepsilon'_i = \tilde{\varepsilon}_i$ for each i. The set

$$\{H : H \in \Sigma,\ H^\bullet \in \phi[\mathfrak{C}]\}$$

is a subalgebra of Σ including $\varepsilon_i[\Sigma_i]$ for every i, so is Σ itself, and ϕ is surjective.

(ii) We need a simple description of the ideal $\mathcal{J}$, as follows: a set $H \in \Sigma$ belongs to $\mathcal{J}$ iff there are a finite $K \subseteq I$ and a family $\langle F_k\rangle_{k\in K}$ such that $F_k \in \mathcal{J}_k$ for each k and $H \subseteq \bigcup_{k\in K} \varepsilon_k(F_k)$. For evidently such sets have to belong to $\mathcal{J}$, since the $\varepsilon_k(F_k)$ will be in $\mathcal{J}$, while the family of all these sets is an ideal containing $\varepsilon_i(F)$ whenever $i \in I$ and $F \in \mathcal{J}_i$.

(iii) Now we can see that $\phi:\mathfrak{C}\to\mathfrak{A}$ is injective. **P** Take any non-zero $c\in\mathfrak{C}$. By 315Kb, we can find a finite $J\subseteq I$ and a family $\langle a_j\rangle_{j\in J}$ in $\prod_{j\in J}\mathfrak{A}_j$ such that $0\ne\inf_{j\in J}\varepsilon'_j a_j\subseteq c$. Express each a_j as $E_j^\bullet$, where $E_j\in\Sigma_j$, and consider $H=X\cap\bigcap_{j\in J}\varepsilon_j(E_j)\in\Sigma$. Then

$$H^\bullet=\inf_{j\in J}\tilde\varepsilon_j a_j=\phi(\inf_{j\in J}\varepsilon'_j a_j)\subseteq\phi(c).$$

Also, because $\varepsilon'_j a_j\ne 0$, $E_j\notin\mathcal{J}_j$ for each j. But it follows that $H\notin\mathcal{J}$, because if $K\subseteq I$ is finite and $F_k\in\mathcal{J}_k$ for each $k\in K$, set $E_i=X_i$ for $i\in I\setminus J$, $F_i=\emptyset$ for $i\in I\setminus K$; then there is an $x\in X$ such that $x(i)\in E_i\setminus F_i$ for each $i\in I$, so that $x\in H\setminus\bigcup_{k\in K}\varepsilon_k(F_k)$. By the criterion of (ii), $H\notin\mathcal{J}$. So

$$0\ne E^\bullet\subseteq\phi(c).$$

As c is arbitrary, the kernel of ϕ is $\{0\}$, and ϕ is injective. **Q**

So $\phi:\mathfrak{C}\to\mathfrak{A}$ is the required isomorphism.

(iv) This proves (b). Reading through the arguments above, it is easy to see the simplifications which compose a proof of (a), reading Σ_i for $\mathfrak{A}_i$ and $\{\emptyset\}$ for $\mathcal{J}_i$.

315N Notation Free products are sufficiently surprising that I think it worth taking a moment to look at a pair of examples relevant to the kinds of application I wish to make of the concept in the next chapter. First let me introduce a somewhat more direct notation which seems appropriate for the free product of finitely many factors. If $\mathfrak{A}$ and $\mathfrak{B}$ are two Boolean algebras, I write $\mathfrak{A}\otimes\mathfrak{B}$ for their free product, and for $a\in\mathfrak{A}$, $b\in\mathfrak{B}$ I write $a\otimes b$ for $\varepsilon_1(a)\cap\varepsilon_2(b)$, where $\varepsilon_1:\mathfrak{A}\to\mathfrak{A}\otimes\mathfrak{B}$, $\varepsilon_2:\mathfrak{B}\to\mathfrak{A}\otimes\mathfrak{B}$ are the canonical maps. Observe that $(a_1\otimes b_1)\cap(a_2\otimes b_2)=(a_1\cap a_2)\otimes(b_1\cap b_2)$, and that the maps $a\mapsto a\otimes b_0$, $b\mapsto a_0\otimes b$ are always ring homomorphisms. Now 315K(e-ii) tells us that $a\otimes b=0$ only when one of a, b is 0. In the context of 315M, we can identify $E\otimes F$ with $E\times F$ for $E\in\Sigma_1$ and $F\in\Sigma_2$, and $E^\bullet\otimes F^\bullet$ with $(E\times F)^\bullet$.

315O Lemma Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras.

(a) Any element of $\mathfrak{A}\otimes\mathfrak{B}$ is expressible as $\sup_{i\in I}a_i\otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$.

(b) If $c\in\mathfrak{A}\otimes\mathfrak{B}$ is non-zero there are non-zero $a\in\mathfrak{A}$, $b\in\mathfrak{B}$ such that $a\otimes b\subseteq c$.

proof (a) Let C be the set of elements of $\mathfrak{A}\otimes\mathfrak{B}$ representable in this form. Then C is a subalgebra of $\mathfrak{A}\otimes\mathfrak{B}$. **P** (i) If $\langle a_i\rangle_{i\in I}$, $\langle a'_j\rangle_{j\in J}$ are finite partitions of unity in $\mathfrak{A}$, and b_i, b'_j members of $\mathfrak{B}$ for $i\in I$ and $j\in J$, then $\langle a_i\cap a'_j\rangle_{i\in I,j\in J}$ is a partition of unity in $\mathfrak{A}$, and

$$\begin{aligned}(\sup_{i\in I}a_i\otimes b_i)\cap(\sup_{j\in J}a'_j\otimes b'_j)&=\sup_{i\in I,j\in J}(a_i\otimes b_i)\cap(a'_j\otimes b'_j)\\&=\sup_{i\in I,j\in J}(a_i\cap a'_j)\otimes(b_i\cap b'_j)\in C.\end{aligned}$$

So $c\cap c'\in C$ for all c, $c'\in C$. (ii) If $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$ and $b_i\in\mathfrak{B}$ for each i, then

$$1\setminus\sup_{i\in I}a_i\otimes b_i=(\sup_{i\in I}a_i\otimes 1)\setminus(\sup_{i\in I}a_i\otimes b_i)=\sup_{i\in I}a_i\otimes(1\setminus b_i)\in C.$$

Thus $1\setminus c\in C$ for every $c\in C$. **Q**

Since $a\otimes 1=(a\otimes 1)\cup((1\setminus a)\otimes 0)$ and $1\otimes b$ belong to C for every $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$, C must be the whole of $\mathfrak{A}\otimes\mathfrak{B}$, by 315Ka.

(b) Now this follows at once, as well as being a special case of 315Kb.

315P Example $\mathfrak{A}=\mathcal{P}\mathbb{N}\otimes\mathcal{P}\mathbb{N}$ is not Dedekind σ-complete. **P** Consider $A=\{\{n\}\otimes\{n\}:n\in\mathbb{N}\}\subseteq\mathfrak{A}$. **?** If A has a least upper bound c in $\mathfrak{A}$, then c is expressible as a supremum $\sup_{j\le k}a_j\otimes b_j$, by 315Kb. Because k is finite, there must be distinct m, n such that $\{j:m\in a_j\}=\{j:n\in a_j\}$. Now $\{n\}\times\{n\}\subseteq c$, so there is a $j\le k$ such that

$$(a_j\cap\{n\})\otimes(b_j\cap\{n\})=(\{n\}\otimes\{n\})\cap(a_j\otimes b_j)\ne 0,$$

so that neither $a_j\cap\{n\}$ nor $b_j\cap\{n\}$ is empty, that is, $n\in a_j\cap b_j$. But this means that $m\in a_j$, so that

$$(a_j\otimes b_j)\cap(\{m\}\otimes\{n\})=(a_j\cap\{m\})\otimes(b_j\cap\{n\})\ne 0,$$

and $c\cap(\{m\}\otimes\{n\})\ne 0$, even though $a\cap(\{m\}\otimes\{n\})=0$ for every $a\in A$. **X** Thus we have found a countable subset of $\mathfrak{A}$ with no supremum in $\mathfrak{A}$, and $\mathfrak{A}$ is not Dedekind σ-complete. **Q**

315Q Example Now let $\mathfrak{A}$ be any non-trivial atomless Boolean algebra, and $\mathfrak{B}$ the free product $\mathfrak{A}\otimes\mathfrak{A}$. Then the identity homomorphism from $\mathfrak{A}$ to itself induces a homomorphism $\phi:\mathfrak{B}\to\mathfrak{A}$ given by setting $\phi(a\otimes b)=a\cap b$ for every a, $b\in\mathfrak{A}$. The point I wish to make is that ϕ is not order-continuous. **P** Let C be the set $\{a\otimes b: a, b\in\mathfrak{A}, a\cap b=0\}$. Then $\phi(c)=0_{\mathfrak{A}}$ for every $c\in C$. If $d\in\mathfrak{B}$ is non-zero, then by 315Ob there are non-zero a, $b\in\mathfrak{A}$ such that $a\otimes b\subseteq d$; now, because $\mathfrak{A}$ is atomless, there is a non-zero $a'\subseteq a$ such that $a\setminus a'\ne 0$. At least one of $b\setminus a'$, $b\setminus(a\setminus a')$ is non-zero; suppose the former. Then $a'\otimes(b\setminus a')$ is a non-zero member of C included in d. As d is arbitrary, this shows that $\sup C=1_{\mathfrak{B}}$. So

$$\sup_{c\in C}\phi(c)=0_{\mathfrak{A}}\ne 1_{\mathfrak{A}}=\phi(\sup C),$$

and ϕ is not order-continuous. **Q**

Thus the free product (unlike the product, see 315Dd) does not respect order-continuity.

***315R Projective and inductive limits: Proposition** Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and R a subset of $I\times I$; suppose that $\pi_{ji}:\mathfrak{A}_i\to\mathfrak{A}_j$ is a Boolean homomorphism for each $(i,j)\in R$.

(a) There are a Boolean algebra $\mathfrak{C}$ and a family $\langle\pi_i\rangle_{i\in I}$ such that

$\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ is a Boolean homomorphism for each $i\in I$,

$\pi_j=\pi_{ji}\pi_i$ whenever $(i,j)\in R$,

and whenever $\mathfrak{B}$, $\langle\phi_i\rangle_{i\in I}$ are such that

$\mathfrak{B}$ is a Boolean algebra,

$\phi_i:\mathfrak{B}\to\mathfrak{A}_i$ is a Boolean homomorphism for each $i\in I$,

$\phi_j=\pi_{ji}\phi_i$ whenever $(i,j)\in R$,

then there is a unique Boolean homomorphism $\phi:\mathfrak{B}\to\mathfrak{C}$ such that $\pi_i\phi=\phi_i$ for every $i\in I$.

(b) There are a Boolean algebra $\mathfrak{C}$ and a family $\langle\pi_i\rangle_{i\in I}$ such that

$\pi_i:\mathfrak{A}_i\to\mathfrak{C}$ is a Boolean homomorphism for each $i\in I$,

$\pi_i=\pi_j\pi_{ji}$ whenever $(i,j)\in R$,

and whenever $\mathfrak{B}$, $\langle\phi_i\rangle_{i\in I}$ are such that

$\mathfrak{B}$ is a Boolean algebra,

$\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ is a Boolean homomorphism for each $i\in I$,

$\phi_i=\phi_j\pi_{ji}$ whenever $(i,j)\in R$,

then there is a unique Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\pi_i=\phi_i$ for every $i\in I$.

proof (a) Let $\mathfrak{A}$ be the simple product $\prod_{i\in I}\mathfrak{A}_i$, and set

$$\mathfrak{C}=\{a: a\in\mathfrak{A},\ a(j)=\pi_{ji}a(i)\text{ whenever }(i,j)\in R\}.$$

Because every π_{ji} is a Boolean homomorphism, $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$. Set $\pi_i(a)=a(i)$ for $i\in I$ and $a\in\mathfrak{C}$; then $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ is a Boolean homomorphism for every i, and $\pi_j=\pi_{ji}\pi_i$ whenever $(i,j)\in R$.

Now suppose that $\mathfrak{B}$ and $\langle\phi_i\rangle_{i\in I}$ have the declared properties. For $b\in\mathfrak{B}$, set $\phi b=\langle\phi_i b\rangle_{i\in I}\in\mathfrak{A}$; because $\phi_j=\pi_{ji}\phi_i$ whenever $(i,j)\in R$, $\phi b\in\mathfrak{C}$. Of course ϕb is the unique member of $\mathfrak{C}$ such that $\pi_i\phi b=\phi_i b$ for every $i\in I$. And $\phi:\mathfrak{B}\to\mathfrak{A}$ is a Boolean homomorphism by 315Bb, so $\phi:\mathfrak{B}\to\mathfrak{C}$ is a Boolean homomorphism.

(b) This time, let $\mathfrak{A}=\bigotimes_{i\in I}\mathfrak{A}_i$ be the free product of $\langle\mathfrak{A}_i\rangle_{i\in I}$; for each $i\in I$, let $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ be the canonical map. Let J be the ideal of $\mathfrak{A}$ generated by elements of the form $\varepsilon_i a\,\triangle\,\varepsilon_j\pi_{ji}a$ where $(i,j)\in R$ and $a\in\mathfrak{A}_i$; let $\mathfrak{C}$ be the quotient algebra $\mathfrak{A}/J$, and set $\pi_i a=(\varepsilon_i a)^\bullet\in\mathfrak{C}$ for $i\in I$ and $a\in\mathfrak{A}_i$. Then every π_i is a Boolean homomorphism, and if $(i,j)\in R$ and $a\in\mathfrak{A}_i$, then

$$\pi_i a=(\varepsilon_i a)^\bullet=(\varepsilon_j\pi_{ji}a)^\bullet=\pi_j\pi_{ji}a$$

because $\varepsilon_i a\,\triangle\,\varepsilon_j\pi_{ji}a$ belongs to J.

Once again, suppose that $\mathfrak{B}$ and $\langle\phi_i\rangle_{i\in I}$ have the properties declared in this part of the proposition. By 315Jb, there is a Boolean homomorphism $\tilde\phi:\mathfrak{A}\to\mathfrak{B}$ such that $\tilde\phi\varepsilon_i=\phi_i$ for every $i\in I$. Now the kernel of $\tilde\phi$ includes J. **P** The kernel of $\tilde\phi$ is an ideal of $\mathfrak{A}$, so all we have to check is that it contains $\varepsilon_i a\,\triangle\,\varepsilon_j\pi_{ji}a$ whenever $(i,j)\in R$ and $a\in\mathfrak{A}_i$; but in this case

$$\tilde\phi(\varepsilon_i a\,\triangle\,\varepsilon_j\pi_{ji}a)=\tilde\phi\varepsilon_i a\,\triangle\,\tilde\phi\varepsilon_j\pi_{ji}a=\phi_i a\,\triangle\,\phi_j\pi_{ji}a=\phi_i a\,\triangle\,\phi_i a=0.\ \mathbf{Q}$$

Accordingly there is a unique ring homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ defined by saying that $\phi c^\bullet=\tilde\phi c$ for every $c\in\mathfrak{A}$ (3A2G). As

$$\phi 1_{\mathfrak{C}}=\phi(1_{\mathfrak{A}}^\bullet)=\tilde\phi 1_{\mathfrak{A}}=1_{\mathfrak{B}},$$

ϕ is a Boolean homomorphism. Now, of course,

$$\phi\pi_i a = \phi(\varepsilon_i a)^\bullet = \tilde{\phi}\varepsilon_i a = \phi_i a$$

whenever $i \in I$ and $a \in \mathfrak{A}_i$.

To see that ϕ is unique, observe that if $\phi' : \mathfrak{C} \to \mathfrak{B}$ has the same property, then we have a Boolean homomorphism $\tilde{\phi}' : \mathfrak{A} \to \mathfrak{B}$ defined by setting $\tilde{\phi}'c = \phi'c^\bullet$ for every $c \in \mathfrak{A}$; in which case

$$\tilde{\phi}'\varepsilon_i a = \phi'(\varepsilon_i a)^\bullet = \phi'\pi_i a = \phi_i a$$

whenever $i \in I$ and $a \in \mathfrak{A}_i$, so that $\tilde{\phi}' = \tilde{\phi}$ and $\phi' = \phi$.

***315S Definitions** In 315Ra, we call $\mathfrak{A}$, together with $\langle\pi_i\rangle_{i\in I}$, 'the' **projective limit** of $(\langle\mathfrak{A}_i\rangle_{i\in I}, \langle\pi_{ji}\rangle_{(i,j)\in R})$; in 315Rb, we call $\mathfrak{A}$, together with $\langle\pi_i\rangle_{i\in I}$, 'the' **inductive limit** of $(\langle\mathfrak{A}_i\rangle_{i\in I}, \langle\pi_{ji}\rangle_{(i,j)\in R})$.

315X Basic exercises (a) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, with simple product $\mathfrak{A}$, and $\pi_i : \mathfrak{A} \to \mathfrak{A}_i$ the coordinate homomorphisms. Suppose we have another Boolean algebra $\mathfrak{A}'$, with homomorphisms $\pi'_i : \mathfrak{A}' \to \mathfrak{A}_i$, such that for every Boolean algebra $\mathfrak{B}$ and every family $\langle\phi_i\rangle_{i\in I}$ of homomorphisms from $\mathfrak{B}$ to the $\mathfrak{A}_i$ there is a unique homomorphism $\phi : \mathfrak{B} \to \mathfrak{A}'$ such that $\phi_i = \pi'_i\phi$ for every i. Show that there is a unique isomorphism $\psi : \mathfrak{A} \to \mathfrak{A}'$ such that $\pi'_i\psi = \pi_i$ for every $i \in I$.

(b) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras with simple product $\mathfrak{A} = \prod_{i\in I}\mathfrak{A}_i$. (i) Show that $\mathfrak{A}$ is Dedekind complete iff every $\mathfrak{A}_i$ is Dedekind complete. (ii) Show that $\mathfrak{A}$ is Dedekind σ-complete iff every $\mathfrak{A}_i$ is Dedekind σ-complete.

(c) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras with simple product $\mathfrak{A} = \prod_{i\in I}\mathfrak{A}_i$. Suppose that for every $i \in I$ we are given a subalgebra $\mathfrak{B}_i$ of $\mathfrak{A}_i$. (i) Show that the simple product $\mathfrak{B} = \prod_{i\in I}\mathfrak{B}_i$ is a subalgebra of $\mathfrak{A}$. (ii) Show that $\mathfrak{B}$ is order-closed in $\mathfrak{A}$ iff $\mathfrak{B}_i$ is order-closed in $\mathfrak{A}_i$ for every $i \in I$.

(d) Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets, with product partially ordered set P. Show that P is a lattice iff every P_i is a lattice, and that in this case it is the product lattice in the sense that $p\vee q = \langle p(i)\vee q(i)\rangle_{i\in I}$, $p \wedge q = \langle p(i) \wedge q(i)\rangle_{i\in I}$ for all p, $q \in P$.

(e) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras with simple product $\mathfrak{A}$. For each $i \in I$ let Z_i be the Stone space of $\mathfrak{A}_i$, and let Z be the Stone space of $\mathfrak{A}$. (i) Show that the coordinate maps from $\mathfrak{A}$ onto $\mathfrak{A}_i$ induce homeomorphisms between the Z_i and open-and-closed subsets Z_i^* of Z. (ii) Show that $\langle Z_i^*\rangle_{i\in I}$ is disjoint. (iii) Show that $\bigcup_{i\in I} Z_i^*$ is dense in Z, and is equal to Z iff $\{i : \mathfrak{A}_i \neq \{0\}\}$ is finite.

(f) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, with simple product $\mathfrak{A}$. Suppose that for each $i \in I$ we are given an ideal I_i of $\mathfrak{A}_i$. Show that $I = \prod_{i\in I} I_i$ is an ideal of $\mathfrak{A}$, and that $\mathfrak{A}/I$ may be identified, as Boolean algebra, with $\prod_{i\in I}\mathfrak{A}_i/I_i$.

(g) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces. Let X be their disjoint union $\{(x,i) : i \in I,\, x \in X_i\}$, with the disjoint union topology; that is, a set $G \subseteq X$ is open in X iff $\{x : (x,i) \in G\}$ is open in X_i for every $i \in I$. Show that the algebra of open-and-closed subsets of X can be identified, as Boolean algebra, with the simple product of the algebras of open-and-closed sets of the X_i.

(h) Show that the topological product of any family of zero-dimensional spaces is zero-dimensional.

(i) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, with free product $\mathfrak{A}$, and $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ the canonical homomorphisms. Suppose we have another Boolean algebra $\mathfrak{A}'$, with homomorphisms $\varepsilon'_i : \mathfrak{A}_i \to \mathfrak{A}'$, such that for every Boolean algebra $\mathfrak{B}$ and every family $\langle\phi_i\rangle_{i\in I}$ of homomorphisms from the $\mathfrak{A}_i$ to $\mathfrak{B}$ there is a unique homomorphism $\phi : \mathfrak{A}' \to \mathfrak{B}$ such that $\phi_i = \phi\varepsilon'_i$ for every i. Show that there is a unique isomorphism $\psi : \mathfrak{A} \to \mathfrak{A}'$ such that $\varepsilon'_i = \psi\varepsilon_i$ for every $i \in I$.

(j) Let I be any set, and let $\mathfrak{A}$ be the algebra of open-and-closed sets of $\{0,1\}^I$; for each $i \in I$ set $a_i = \{x : x \in \{0,1\}^I,\, x(i) = 1\} \in \mathfrak{A}$. Show that for any Boolean algebra $\mathfrak{B}$ and any family $\langle b_i\rangle_{i\in I}$ in $\mathfrak{B}$ there is a unique Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $\phi(a_i) = b_i$ for every $i \in I$.

(k) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$, $\langle\mathfrak{B}_j\rangle_{j\in J}$ be two families of Boolean algebras. Show that there is a natural injective homomorphism $\phi:\prod_{i\in I}\mathfrak{A}_i\otimes\prod_{j\in J}\mathfrak{B}_j\to\prod_{i\in I,j\in J}\mathfrak{A}_i\otimes\mathfrak{B}_j$ defined by saying that

$$\phi(a\otimes b)=\langle a(i)\otimes b(j)\rangle_{i\in I,j\in J}$$

for $a\in\prod_{i\in I}\mathfrak{A}_i$, $b\in\prod_{j\in J}\mathfrak{B}_j$. Show that ϕ is surjective if I and J are finite.

(l) Let $\langle J(i)\rangle_{i\in I}$ be a family of sets, with product $Q=\prod_{i\in I}J(i)$. Let $\langle\mathfrak{A}_{ij}\rangle_{i\in I,j\in J(i)}$ be a family of Boolean algebras. Describe a natural injective homomorphism $\phi:\bigotimes_{i\in I}\prod_{j\in J(i)}\mathfrak{A}_{ij}\to\prod_{q\in Q}\bigotimes_{i\in I}\mathfrak{A}_{i,q(i)}$.

(m) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with partitions of unity $\langle a_i\rangle_{i\in I}$, $\langle b_j\rangle_{j\in J}$. Show that $\langle a_i\otimes b_j\rangle_{i\in I,j\in J}$ is a partition of unity in $\mathfrak{A}\otimes\mathfrak{B}$.

(n) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $a\in\mathfrak{A}$, $b\in\mathfrak{B}$. Write $\mathfrak{A}_a$, $\mathfrak{B}_b$ for the corresponding principal ideals. Show that there is a canonical isomorphism between $\mathfrak{A}_a\otimes\mathfrak{B}_b$ and the principal ideal of $\mathfrak{A}\otimes\mathfrak{B}$ generated by $a\otimes b$.

(o) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, with free product $\bigotimes_{i\in I}\mathfrak{A}_i$, and $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ the canonical maps. Show that $\varepsilon_i[\mathfrak{A}_i]$ is an order-closed subalgebra of $\mathfrak{A}$ for every i.

(p) Let $\mathfrak{A}$ be a Boolean algebra. Let us say that a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is **Boolean-independent** if $\inf_{j\in J}a_j\ne 0$ whenever $J\subseteq I$ is finite and $a_j\in\mathfrak{A}_j\setminus\{0\}$ for every $j\in J$. Show that in this case the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{A}_i$ is isomorphic to the free product $\bigotimes_{i\in I}\mathfrak{A}_i$.

(q) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ and $\langle\mathfrak{B}_i\rangle_{i\in I}$ be two families of Boolean algebras, and suppose that for each $i\in I$ we are given a Boolean homomorphism $\phi_i:\mathfrak{A}_i\to\mathfrak{B}_i$ with kernel $K_i\lhd\mathfrak{A}_i$. Show that the ϕ_i induce a Boolean homomorphism $\phi:\bigotimes_{i\in I}\mathfrak{A}_i\to\bigotimes_{i\in I}\mathfrak{B}_i$ with kernel generated by $\bigcup_{i\in I}\varepsilon[K_i]$, where $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ is the canonical homomorphism. Show that if every ϕ_i is surjective, so is ϕ.

(r) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of non-trivial Boolean algebras. Show that if $J\subseteq I$ and $\mathfrak{B}_j$ is a subalgebra of $\mathfrak{A}_j$ for each $j\in J$, then $\bigotimes_{j\in J}\mathfrak{B}_j$ is canonically embedded as a subalgebra of $\bigotimes_{i\in I}\mathfrak{A}_i$.

(s) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, neither $\{0\}$. Show that any element of $\mathfrak{A}\otimes\mathfrak{B}$ is uniquely expressible as $\sup_{i\in I}a_i\otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, with no a_i equal to 0, and $b_i\ne b_j$ in $\mathfrak{B}$ for $i\ne j$.

315Y Further exercises (a) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ and $\langle\mathfrak{B}_i\rangle_{i\in I}$ be two families of Boolean algebras, and suppose that we are given Boolean homomorphisms $\phi_i:\mathfrak{A}_i\to\mathfrak{B}_i$ for each i; let $\phi:\bigotimes_{i\in I}\mathfrak{A}_i\to\bigotimes_{i\in I}\mathfrak{B}_i$ be the induced homomorphism. (i) Show that if every ϕ_i is order-continuous, so is ϕ. (ii) Show that if every ϕ_i is sequentially order-continuous, so is ϕ.

(b) Let $\langle Z_i\rangle_{i\in I}$ be any family of topological spaces with product Z. For $i\in I$, $z\in Z$ set $\tilde\varepsilon_i(z)=z(i)$. Show that if $M\subseteq Z_i$ is nowhere dense in Z_i then $\tilde\varepsilon_i^{-1}[M]$ is nowhere dense in Z. Use this to prove 315Kc.

(c) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and suppose that we are given subalgebras $\mathfrak{B}_i$ of $\mathfrak{A}_i$ for each i; set $\mathfrak{A}=\bigotimes_{i\in I}\mathfrak{A}_i$ and $\mathfrak{B}=\bigotimes_{i\in I}\mathfrak{B}_i$, and let $\phi:\mathfrak{B}\to\mathfrak{A}$ be the homomorphism induced by the embeddings $\mathfrak{B}_i\subsetneq\mathfrak{A}_i$. (i) Show that if every $\mathfrak{B}_i$ is order-closed in $\mathfrak{A}_i$, then $\phi[\mathfrak{B}]$ is order-closed in $\mathfrak{A}$. (ii) Show that if every $\mathfrak{B}_i$ is a σ-subalgebra of $\mathfrak{A}_i$, then $\phi[\mathfrak{B}]$ is a σ-subalgebra in $\mathfrak{A}$.

(d) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, with product X. Let $\mathrm{RO}(X_i)$, $\mathrm{RO}(X)$ be the corresponding regular open algebras. Show that $\mathrm{RO}(X)$ can be identified with the Dedekind completion of $\bigotimes_{i\in I}\mathrm{RO}(X_i)$.

(e) Use the ideas of 315Xj and 315M to give an alternative construction of 'free product', for which 315J and 315K(e-ii) are true, and which does not depend on the concept of Stone space nor on any other use of the axiom of choice. (*Hint*: show that for any Boolean algebra $\mathfrak{A}$ there is a canonical surjection from the algebra $\mathcal{E}_{\mathfrak{A}}$ onto $\mathfrak{A}$, where $\mathcal{E}_J$ is the algebra of subsets of $\{0,1\}^J$ generated by sets of the form $\{x:x(j)=1\}$; show that for such algebras $\mathcal{E}_J$, at least, the method of 315I-315J can be used; now apply the method of 315M to describe $\bigotimes_{i\in I}\mathfrak{A}_i$ as a quotient of $\mathcal{E}_J$ where $J=\{(a,i):i\in I,\,a\in\mathfrak{A}_i\}$. Finally check 315K(e-ii).)

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. Show that $\mathfrak{A}\otimes\mathfrak{B}$ is Dedekind complete iff *either* $\mathfrak{A}=\{0\}$ *or* $\mathfrak{B}=\{0\}$ *or* $\mathfrak{A}$ is finite and $\mathfrak{B}$ is Dedekind complete *or* $\mathfrak{B}$ is finite and $\mathfrak{A}$ is Dedekind complete.

(g) Let $\langle P_i\rangle_{i\in I}$ be any family of partially ordered spaces. (i) Give a construction of a partially ordered space P, together with a family of order-preserving maps $\varepsilon_i : P_i \to P$, such that whenever Q is a partially ordered set and $\phi_i : P_i \to Q$ is order-preserving for every $i \in I$, there is a unique order-preserving map $\phi : P \to Q$ such that $\phi_i = \phi\varepsilon_i$ for every i. (ii) Show that ϕ will be order-continuous iff every ϕ_i is. (iii) Show that P will be Dedekind complete iff every P_i is, but (except in trivial cases) is not a lattice.

(h) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and R a subset of $I\times I$; suppose that $\pi_{ji} : \mathfrak{A}_i \to \mathfrak{A}_j$ is a Boolean homomorphism for each $(i,j)\in R$. For each $i\in I$, let Z_i be the Stone space of $\mathfrak{A}_i$; for $(i,j)\in R$, let $f_{ji} : Z_j \to Z_i$ be the continuous function corresponding to π_{ji}. Show that the Stone space of the inductive limit of the system $(\langle\mathfrak{A}_i\rangle_{i\in I}, \langle\pi_{ji}\rangle_{(i,j)\in R})$ can be identified with $\{z : z \in \prod_{i\in I} Z_i,\ f_{ji}(z(j)) = z(i) \text{ whenever } (i,j)\in R\}$.

315 Notes and comments In this section I find myself asking for slightly more sophisticated algebra than seems necessary elsewhere. The point is that simple products and free products are best regarded as defined by the properties described in 315B and 315J. That is, it is sometimes right to think of *a* simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras as being a structure $(\mathfrak{A}, \langle\pi_i\rangle_{i\in I})$ where $\mathfrak{A}$ is a Boolean algebra, $\pi_i : \mathfrak{A} \to \mathfrak{A}_i$ is a homomorphism for every $i \in I$, and every family of homomorphisms from a Boolean algebra $\mathfrak{B}$ to the $\mathfrak{A}_i$ can be uniquely represented by a single homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$. Similarly, reversing the direction of the homomorphisms, we can speak of *a* free product (it would be natural to say 'coproduct') $(\mathfrak{A}, \langle\varepsilon_i\rangle_{i\in I})$ of $\langle\mathfrak{A}_i\rangle_{i\in I}$. On such definitions, it is elementary that any two simple products, or free products, are isomorphic in the obvious sense (315Xa, 315Xi), and very general arguments from abstract algebra, not restricted to Boolean algebras (see Bourbaki 68, IV.3.2), show that they exist. (But in order to prove such basic facts as that the π_i are surjective, or that the ε_i are, except when the construction collapses altogether, injective, we do of course have to look at the special properties of Boolean algebras.) Now in the case of simple products, the Cartesian product construction is so direct and so familiar that there seems no need to trouble our imaginations with any other. But in the case of free products, things are more complicated. I have given primacy to the construction in terms of Stone spaces because I believe that this is the fastest route to effective mental pictures. But in some ways this approach seems to be inappropriate. If you take what in my view is a tenable position, and say that a Boolean algebra is best regarded as the limit of its finite subalgebras, then you might prefer a construction of a free product as a limit of free products of finitely many finite subalgebras. Or you might feel that it is wrong to rely on the axiom of choice to prove a result which certainly does not need it (see 315Ye).

Because I believe that the universal mapping theorem 315J is the right basis for the study of free products, I am naturally led to use it as the starting point for proofs of theorems about free products, as in 315L. But 315K(e-ii) seems to lie deeper. (Note, for instance, that in 315M we *do* need the axiom of choice, in part (a-iii) of the proof, since without it the product $\prod_{i\in I} X_i$ could be empty.)

Both 'simple product' and 'free product' are essentially algebraic constructions involving the category of Boolean algebras and Boolean homomorphisms, and any relationships with such concepts as order-continuity can be regarded as accidental, in so far as there are accidents in mathematics. 315Cb and 315D show that simple products behave very straightforwardly when the homomorphisms involved are order-continuous. 315Q, 315Xo and 315Ya-315Yc show that free products are much more complex and subtle.

For finite products, we have a kind of distributivity; $(\mathfrak{A}\times\mathfrak{B})\otimes\mathfrak{C}$ can be identified with $(\mathfrak{A}\otimes\mathfrak{C})\times(\mathfrak{B}\otimes\mathfrak{C})$ (315Xk, 315Xl). There are contexts in which this makes it seem more natural to write $\mathfrak{A}\oplus\mathfrak{B}$ in place of $\mathfrak{A}\times\mathfrak{B}$, and indeed I have already spoken of a 'direct sum' of measure spaces (214L) in terms which correspond closely to the simple product of algebras of sets described in 315Ga. Generally, the simple product corresponds to disjoint unions of Stone spaces (315Xe) and the free product to products of Stone spaces. But the simple product is indeed the product Boolean algebra, in the ordinary category sense; the universal mapping theorem 315B is exactly of the type we expect from products of topological spaces (3A3Ib) or partially ordered sets (315Dc), etc. It is the 'free product' which is special to Boolean algebras. The nearest analogy that I know of elsewhere is with the concept of 'tensor product' of linear spaces (cf. §253).

It is perhaps worth noting that projective limits of systems of Boolean algebras have a straightforward description in terms of the algebras themselves (315Ra), while inductive limits have a similarly direct description in terms of Stone spaces (315Yh).

316 Further topics

I introduce three special properties of Boolean algebras which will be of great importance in the rest of this volume: the countable chain condition (316A-316F), weak (σ,∞)-distributivity (316G-316J) and homogeneity (316N-316Q). I add some brief notes on atoms in Boolean algebras (316K-316L), with a characterization of the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ (316M).

316A Definitions (a) A Boolean algebra $\mathfrak{A}$ is **ccc**, or satisfies the **countable chain condition**, if every disjoint subset of $\mathfrak{A}$ is countable.

(b) A topological space X is **ccc**, or satisfies the **countable chain condition**, or has **Souslin's property**, if every disjoint collection of open sets in X is countable.

316B Theorem A Boolean algebra $\mathfrak{A}$ is ccc iff its Stone space Z is ccc.

proof (a) If $\mathfrak{A}$ is ccc and $\mathcal{G}$ is a disjoint family of open sets in Z, then for each $G \in \mathcal{G}' = \mathcal{G} \setminus \{\emptyset\}$ we can find a non-zero $a_G \in \mathfrak{A}$ such that the corresponding open-and-closed set $\widehat{a}_G$ is included in G. Now $\{a_G : G \in \mathcal{G}'\}$ is a disjoint family in $\mathfrak{A}$, so is countable; since $a_G \neq a_H$ for distinct G, $H \in \mathcal{G}'$, $\mathcal{G}'$ and $\mathcal{G}$ must be countable. As $\mathcal{G}$ is arbitrary, Z is ccc.

(b) If Z is ccc and $A \subseteq \mathfrak{A}$ is disjoint, then $\{\widehat{a} : a \in A\}$ is a disjoint family of open subsets of Z, so must be countable, and A is countable. As A is arbitrary, $\mathfrak{A}$ is ccc.

316C Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\mathcal{I}$ a σ-ideal of $\mathfrak{A}$. Then the quotient algebra $\mathfrak{B} = \mathfrak{A}/\mathcal{I}$ is ccc iff every disjoint family in $\mathfrak{A} \setminus \mathcal{I}$ is countable.

proof (a) Suppose that $\mathfrak{B}$ is ccc and that A is a disjoint family in $\mathfrak{A} \setminus \mathcal{I}$. Then $\{a^\bullet : a \in A\}$ is a disjoint family in $\mathfrak{B}$, therefore countable, and $a^\bullet \neq b^\bullet$ when a, b are distinct members of A; so A is countable.

(b) Now suppose that $\mathfrak{B}$ is not ccc. Then there is an uncountable disjoint set $B \subseteq \mathfrak{B}$. Of course $B \setminus \{0\}$ is still uncountable, so may be enumerated as $\langle b_\xi \rangle_{\xi<\kappa}$, where κ is an uncountable cardinal (2A1K), so that $\omega_1 \leq \kappa$. For each $\xi < \omega_1$, choose $a_\xi \in \mathfrak{A}$ such that $a_\xi^\bullet = b_\xi$. Of course $a_\xi \notin \mathcal{I}$. If $\eta < \xi < \omega_1$, then $b_\eta \cap b_\xi = 0$, so $a_\xi \cap a_\eta \in \mathcal{I}$. Because $\xi < \omega_1$, it is countable; because $\mathcal{I}$ is a σ-ideal, and $\mathfrak{A}$ is Dedekind σ-complete,

$$d_\xi = \sup_{\eta<\xi} a_\xi \cap a_\eta,$$

belongs to $\mathcal{I}$, and

$$c_\xi = a_\xi \setminus d_\xi \in \mathfrak{A} \setminus \mathcal{I}.$$

But now of course

$$c_\xi \cap c_\eta \subseteq c_\xi \cap a_\eta \subseteq c_\xi \cap d_\xi = 0$$

whenever $\eta < \xi < \omega_1$, so $\{c_\xi : \xi < \omega_1\}$ is an uncountable disjoint family in $\mathfrak{A} \setminus \mathcal{I}$.

Remark An ideal $\mathcal{I}$ satisfying the conditions of this proposition is said to be ω_1**-saturated** in $\mathfrak{A}$.

316D Corollary Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ. Then the quotient algebra $\Sigma/\mathcal{I}$ is ccc iff every disjoint family in $\Sigma \setminus \mathcal{I}$ is countable.

316E Proposition Let $\mathfrak{A}$ be a ccc Boolean algebra. Then for any $A \subseteq \mathfrak{A}$ there is a countable $B \subseteq A$ such that B has the same upper and lower bounds as A.

proof (a) Set

$$D = \bigcup_{a \in A} \{d : d \subseteq a\}.$$

Applying Zorn's lemma to the family $\mathcal{C}$ of disjoint subsets of D, we have a maximal $C_0 \in \mathcal{C}$. For each $c \in C_0$ choose a $b_c \in A$ such that $c \subseteq b_c$, and set $B_0 = \{b_c : c \in C_0\}$. Because $\mathfrak{A}$ is ccc, C_0 is countable, so B_0 also is countable. **?** If there is an upper bound e for B_0 which is not an upper bound for A, take $a \in A$ such that $c' = a \setminus e \neq 0$; then $c' \in D$ and $c' \cap c = c' \cap b_c = 0$ for every $c \in C_0$, so $C_0 \cup \{c'\} \in \mathcal{C}$; but C_0 was supposed to be maximal in $\mathcal{C}$. **X** Thus every upper bound for B_0 is also an upper bound for A.

(b) Similarly, there is a countable set $B_1' \subseteq A' = \{1 \setminus a : a \in A\}$ such that every upper bound for B_1' is an upper bound for A'. Set $B_1 = \{1 \setminus b : b \in B_1'\}$; then B_1 is a countable subset of A and every lower bound for B_1 is a lower bound for A. Try $B = B_0 \cup B_1$. Then B is a countable subset of A and every upper (resp. lower) bound for B is an upper (resp. lower) bound for A; so that B must have exactly the same upper and lower bounds as A has.

316F Corollary Let $\mathfrak{A}$ be a ccc Boolean algebra.

(a) If $\mathfrak{A}$ is Dedekind σ-complete it is Dedekind complete.

(b) If $A \subseteq \mathfrak{A}$ is sequentially order-closed it is order-closed.

(c) If Q is any partially ordered set and $\phi : \mathfrak{A} \to Q$ is a sequentially order-continuous order-preserving function, it is order-continuous.

(d) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a sequentially order-continuous Boolean homomorphism, it is order-continuous.

proof (a) If A is any subset of $\mathfrak{A}$, let $B \subseteq A$ be a countable set with the same upper bounds as A; then $\sup B$ is defined in $\mathfrak{A}$ and must be $\sup A$.

(b) Suppose that $B \subseteq A$ is non-empty and upwards-directed and has a supremum a in $\mathfrak{A}$. Then there is a non-empty countable set $C \subseteq B$ with the same upper bounds as B. Let $\langle c_n \rangle_{n\in\mathbb{N}}$ be a sequence running over C. Because B is upwards-directed, we can choose $\langle b_n \rangle_{n\in\mathbb{N}}$ inductively such that

$$b_0 = c_0, \quad b_{n+1} \in B,\, b_{n+1} \supseteq b_n \cup c_{n+1} \text{ for every } n \in \mathbb{N}.$$

Now any upper bound for $\{b_n : n \in \mathbb{N}\}$ must also be an upper bound for $\{c_n : n \in \mathbb{N}\} = C$, so is an upper bound for the whole set B. But this means that $a = \sup_{n\in\mathbb{N}} b_n$. As $\langle b_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in A, and A is sequentially order-closed, $a \in A$.

In the same way, if $B \subseteq A$ is downwards-directed and has an infimum in $\mathfrak{A}$, this is also the infimum of some non-increasing sequence in B, so must belong to A. Thus A is order-closed.

(c)(i) Suppose that $A \subseteq \mathfrak{A}$ is a non-empty upwards-directed set with supremum $a_0 \in \mathfrak{A}$. As in (b), there is a non-decreasing sequence $\langle c_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with supremum a_0. Because ϕ is sequentially order-continuous, $\phi a_0 = \sup_{n\in\mathbb{N}} \phi c_n$ in Q. But this means that ϕa_0 must be the least upper bound of $\phi[A]$.

(ii) Similarly, if $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum a_0, there is a non-increasing sequence $\langle c_n \rangle_{n\in\mathbb{N}}$ in A with infimum a_0, so that

$$\inf \phi[A] = \inf_{n\in\mathbb{N}} \phi c_n = \phi a_0.$$

Putting this together with (i), we see that ϕ is order-continuous, as claimed.

(d) This is a special case of (c).

316G Definition Let $\mathfrak{A}$ be a Boolean algebra. I will say that $\mathfrak{A}$ is **weakly (σ,∞)-distributive** if whenever $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of downwards-directed subsets of $\mathfrak{A}$ and $\inf A_n = 0$ for every n, then $\inf B = 0$, where

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in A_n \text{ such that } b \supseteq a\}.$$

316H Proposition Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is weakly (σ,∞)-distributive;

(ii) whenever $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of partitions of unity in $\mathfrak{A}$, there is a partition of unity B in $\mathfrak{A}$ such that $\{a : a \in A_n,\, a \cap b \neq 0\}$ is finite for every $n \in \mathbb{N}$ and $b \in B$;

(iii) whenever $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of upwards-directed subsets of $\mathfrak{A}$, each with a supremum $c_n = \sup A_n$, and

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in A_n \text{ such that } b \subseteq a\},$$

then $\inf\{c_n \setminus b : n \in \mathbb{N},\, b \in B\} = 0$;

(iv) whenever $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of upwards-directed subsets of $\mathfrak{A}$, each with a supremum $c_n = \sup A_n$, and $\inf_{n\in\mathbb{N}} c_n = c$ is defined, then $c = \sup B$, where

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in A_n \text{ such that } b \subseteq a\}.$$

proof (i)⇒(ii) Suppose that $\mathfrak{A}$ is weakly (σ,∞)-distributive and that $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of partitions of unity in $\mathfrak{A}$. For each $n \in \mathbb{N}$, set

$$C_n = \{1 \setminus \sup D : D \in A_n \text{ is finite}\},$$

so that C_n is downwards-directed and has infimum 0. Set $E = \{e : \text{for every } n \in \mathbb{N} \text{ there is a } c \in C_n \text{ such that } c \subseteq e\}$; then $\inf E = 0$. So $B_0 = \{b : b \cap e = 0 \text{ for some } e \in E\}$ is order-dense in $\mathfrak{A}$ and includes a partition B of unity. If $n \in \mathbb{N}$ and $b \in B$, take $e \in E$ such that $b \cap e = 0$, $c \in C_n$ such that $c \subseteq e$, and a finite set $D \subseteq A_n$ such that $c = 1 \setminus \sup D$; then

$$b \subseteq 1 \setminus e \subseteq 1 \setminus c \subseteq \sup D$$

and $\{a : a \in A_n,\ a \cap b \neq 0\} \subseteq D$ is finite. As $\langle A_n \rangle_{n \in \mathbb{N}}$ is arbitrary, (ii) is true.

(ii)⇒(iii) Suppose that (ii) is true, and that $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of upwards-directed subsets of $\mathfrak{A}$, each with a supremum $c_n = \sup A_n$. For each $n \in \mathbb{N}$,

$$D_n = \{d : d \subseteq 1 \setminus c_n\} \cup \bigcup_{a \in A_n} \{d : d \subseteq a\}$$

is order-dense in $\mathfrak{A}$, so there is a partition of unity $D'_n \subseteq D_n$ (313K). By (ii), there is a partition of unity E such that $\{d : d \in D'_n,\ d \cap e \neq 0\}$ is finite for every $n \in \mathbb{N}$ and $e \in E$. **?** Suppose, if possible, that $\{c_n \setminus b : n \in \mathbb{N},\ b \in B\}$ has a non-zero lower bound c. Let $e \in E$ be such that $c \cap e \neq 0$. For each $n \in \mathbb{N}$, set $D''_n = \{d : d \in D'_n,\ c \cap e \cap d \neq 0\}$. Then D''_n is finite so $d_n = \sup D''_n$ is defined and $c \cap e \subseteq d_n$. Also, because $c \subseteq c_n$, each element of D''_n is included in a member of A_n; as A_n is upwards-directed, so are d_n and $c \cap e$. As n is arbitrary, $c \cap e \in B$; and c was supposed to be disjoint from every member of B. **X**

Thus $\inf\{c_n \setminus b : n \in \mathbb{N},\ b \in B\} = 0$; as $\langle A_n \rangle_{n \in \mathbb{N}}$ is arbitrary, (iii) is true.

(iii)⇒(iv) Suppose that (iii) is true and that A_n, c_n, c and B are as in the statement of (iv). Then

$$\inf\{c \setminus b : b \in B\} = \inf\{c_n \setminus b : n \in \mathbb{N},\ b \in B\} = 0;$$

as $b \subseteq c_n$ whenever $b \in B$ and $n \in \mathbb{N}$, we have $b \subseteq c$ for every $b \in B$, and $\sup B = c$, by 313Ab. Thus (iv) is true.

(iv)⇒(i) Suppose that (iv) is true and that A_n and B are as in 316G. Set $A'_n = \{1 \setminus a : a \in A_n\}$, so that A'_n is an upwards-directed set with supremum 1 for each n, and

$$B' = \{b : \text{for every } n \in \mathbb{N} \text{ there is an } a \in A'_n \text{ such that } b \subseteq a\} = \{1 \setminus b : b \in B\};$$

then

$$\inf B = 1 \setminus \sup B' = 1 \setminus \inf_{n \in \mathbb{N}} \sup A'_n = 0,$$

as required.

316I As usual, a characterization of the property in terms of the Stone spaces is extremely valuable.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space. Then $\mathfrak{A}$ is weakly (σ, ∞)-distributive iff every meager set in Z is nowhere dense.

proof (a) The point is that if $M \subseteq Z$ then M is nowhere dense iff there is a partition of unity A in $\mathfrak{A}$ such that $M \cap \widehat{a} = \emptyset$ for every $a \in A$. **P** If M is nowhere dense, then $\{a : M \cap \widehat{a} = \emptyset\}$ is order-dense in $\mathfrak{A}$, so includes a partition of unity. In the other direction, if A is a partition of unity such that M is disjoint from $\widehat{a}$ for every $a \in A$, then $\sup A = 1$ so $G = \bigcup_{a \in A} \widehat{a}$ is dense (313Ca); now G is a dense open set disjoint from M, so M is nowhere dense. **Q**

(b) Suppose that $\mathfrak{A}$ is weakly (σ, ∞)-distributive and that M is a meager subset of Z. Then M can be expressed as $\bigcup_{n \in \mathbb{N}} M_n$ where each M_n is nowhere dense. For each $n \in \mathbb{N}$, let A_n be a partition of unity such that $M_n \cap \widehat{a} = \emptyset$ for every $a \in A_n$. By 316H(i)⇒(ii), there is a partition of unity B such that $\{a : a \in A_n,\ a \cap b \neq 0\}$ is finite for every $n \in \mathbb{N}$ and $b \in B$. Now $M_n \cap \widehat{b} = \emptyset$ for every $n \in \mathbb{N}$ and $b \in B$. **P** $C = \{a : a \in A_n,\ b \cap a \neq 0\}$ is finite. So $F = \bigcup_{a \in C} \widehat{a}$ is closed and $G = \widehat{b} \setminus F$ is open. But $G \cap \widehat{a} = \emptyset$ for every $a \in A$, so G is empty and $\widehat{b} \subseteq F \subseteq Z \setminus M_n$. **Q** Accordingly $M \cap \widehat{b} = \emptyset$ for every $b \in B$ and M is nowhere dense.

(c) Suppose that every meager set in Z is nowhere dense, and that $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of partitions of unity in $\mathfrak{A}$. Then $M_n = Z \setminus \bigcup_{a \in A_n} \widehat{a}$ is nowhere dense for each n (313Cc), so $M = \bigcup_{n \in \mathbb{N}} M_n$ is meager, therefore nowhere dense. Let B be a partition of unity in $\mathfrak{A}$ such that $M \cap \widehat{b} = \emptyset$ for every $b \in B$. If $n \in \mathbb{N}$ and $b \in B$, then

$$\widehat{b} \subseteq Z \setminus M \subseteq Z \setminus M_n = \bigcup_{a \in A_n} \widehat{a}.$$

As $\widehat{b}$ is compact, there is some finite $C \subseteq A$ such that $\widehat{b} \subseteq \bigcup_{a \in C} \widehat{a}$ and $b \subseteq \sup C$; but this means that $\{a : a \in A_n,\ a \cap b \neq 0\} \subseteq C$ is finite. As $\langle A_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ, ∞)-distributive, by 316H(ii)⇒(i).

316J The regular open algebra of $\mathbb{R}$ For examples of weakly (σ,∞)-distributive algebras, I think we can wait for Chapter 32 (see also 393C). But the standard example of an algebra which is *not* weakly (σ,∞)-distributive is of such importance that (even though it has nothing to do with measure theory, narrowly defined) I think it right to describe it here.

Proposition The algebra $\mathrm{RO}(\mathbb{R})$ of regular open subsets of $\mathbb{R}$ (314O) is not weakly (σ,∞)-distributive.

proof Enumerate $\mathbb{Q}$ as $\langle q_n\rangle_{n\in\mathbb{N}}$. For each $n\in\mathbb{N}$, set

$$A_n=\{G:G\in\mathrm{RO}(\mathbb{R}),\ q_i\in G\text{ for every }i\le n\}.$$

Then A_n is downwards-directed, and

$$\inf A_n=\operatorname{int}\bigcap A_n=\operatorname{int}\{q_i:i\le n\}=\emptyset.$$

But if $A\subseteq\mathrm{RO}(\mathbb{R})$ is such that

for every $n\in\mathbb{N}$, $G\in A$ there is an $H\in A_n$ such that $H\subseteq G$,

then we must have $\mathbb{Q}\subseteq G$ for every $G\in A$, so that

$$\mathbb{R}=\operatorname{int}\overline{\mathbb{Q}}\subseteq\operatorname{int}\overline{G}=G$$

for every $G\in A$, and $A\subseteq\{\mathbb{R}\}$; which means that $\inf A\ne\emptyset$ in $\mathrm{RO}(\mathbb{R})$, and 316G cannot be satisfied.

316K Atoms in Boolean algebras (a) If $\mathfrak{A}$ is a Boolean algebra, an **atom** in $\mathfrak{A}$ is a non-zero $a\in\mathfrak{A}$ such that the only elements included in a are 0 and a.

(b) A Boolean algebra is **atomless** if it has no atoms.

(c) A Boolean algebra is **purely atomic** if every non-zero element includes an atom.

316L Proposition Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z.

(a) There is a one-to-one correspondence between atoms a of $\mathfrak{A}$ and isolated points $z\in Z$, given by the formula $\widehat{a}=\{z\}$.

(b) $\mathfrak{A}$ is atomless iff Z has no isolated points.

(c) $\mathfrak{A}$ is purely atomic iff the isolated points of Z form a dense subset of Z.

proof (a)(i) If z is an isolated point in Z, then $\{z\}$ is an open-and-closed subset of Z, so is of the form $\widehat{a}$ for some $a\in\mathfrak{A}$; now if $b\subseteq a$, $\widehat{b}$ must be either $\emptyset$ or $\{z\}$, so b must be either a or 0, and a is an atom.

(ii) If $a\in\mathfrak{A}$ and $\widehat{a}$ has two points z and w, then (because Z is Hausdorff, 311I) there is an open set G containing z but not w. Now there is a $c\in\mathfrak{A}$ such that $z\in\widehat{c}\subseteq G$, so that $a\cap c$ must be different from both 0 and a, and a is not an atom.

(b) This follows immediately from (a).

(c) From (a) we see that $\mathfrak{A}$ is purely atomic iff $\widehat{a}$ contains an isolated point for every non-zero $a\in\mathfrak{A}$; since every non-empty open set in Z includes a non-empty set of the form $\widehat{a}$, this happens iff every non-empty open set in Z contains an isolated point, that is, iff the set of isolated points is dense.

316M Proposition Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. Then a Boolean algebra $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ iff it is atomless, countable and not $\{0\}$.

proof (a) I must check that $\mathfrak{B}$ has the declared properties. The point is that it is the subalgebra $\mathfrak{B}'$ of $\mathcal{P}X$ generated by $\{b_i:i\in\mathbb{N}\}$, where I write $X=\{0,1\}^{\mathbb{N}}$, $b_i=\{x:x\in X,\ x(i)=1\}$. **P** Of course b_i and its complement $\{x:x(i)=0\}$ are open, so $b_i\in\mathfrak{B}$ for each i, and $\mathfrak{B}'\subseteq\mathfrak{B}$. In the other direction, the open cylinder sets of X are all of the form $c_z=\{x:x(i)=z(i)\text{ for every }i\in J\}$, where $J\subseteq I$ is finite and $z\in\{0,1\}^J$; now

$$c_z=X\cap\bigcap\nolimits_{z(i)=1}b_i\setminus\bigcup\nolimits_{z(i)=0}b_i\in\mathfrak{B}'.$$

If $b\in\mathfrak{B}$ then b is expressible as a union of such cylinder sets, because it is open; but also it is compact, so is the union of finitely many of them, and must belong to $\mathfrak{B}'$. Thus $\mathfrak{B}=\mathfrak{B}'$, as claimed. **Q**

For each $n\in\mathbb{N}$ let $\mathfrak{B}_n$ be the finite subalgebra of $\mathfrak{B}$ generated by $\{b_i:i<n\}$ (so that $\mathfrak{B}_0=\{0,1\}$). Then $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ is an increasing sequence of subalgebras of $\mathfrak{B}$ with union $\mathfrak{B}$; so $\mathfrak{B}$ is countable. Also $b\cap b_n$, $b\setminus b_n$ are non-zero for every $n\in\mathbb{N}$ and non-zero $b\in\mathfrak{B}_n$, so no member of any $\mathfrak{B}_n$ can be an atom in $\mathfrak{B}$, and $\mathfrak{B}$ is atomless.

(b) Now suppose that $\mathfrak{A}$ is another algebra with the same properties. Enumerate $\mathfrak{A}$ as $\langle a_n\rangle_{n\in\mathbb{N}}$. Choose finite subalgebras $\mathfrak{A}_n \subseteq \mathfrak{A}$ and isomorphisms $\pi_n : \mathfrak{A}_n \to \mathfrak{B}_n$ as follows. $\mathfrak{A}_0 = \{0, 1\}$, $\pi_0 0 = 0$, $\pi_0 1 = 1$. Given $\mathfrak{A}_n$ and π_n, let A_n be the set of atoms of $\mathfrak{A}_n$. For $a \in A_n$, choose $a' \in \mathfrak{A}$ such that

if $a_n \cap a$, $a_n \setminus a$ are both non-zero, then $a' = a_n \cap a$;

otherwise, $a' \subseteq a$ is any element such that a', $a \setminus a'$ are both non-zero.

(This is where I use the hypothesis that $\mathfrak{A}$ is atomless.) Set $a'_n = \sup_{a\in A_n} a'$. Then we see that $a \cap a'_n$, $a \setminus a'_n$ are non-zero for every $a \in A_n$ and therefore for every non-zero $a \in \mathfrak{A}_n$, that is, that

$$\sup\{a : a \in \mathfrak{A}_n,\, a \subseteq a'_n\} = 0, \quad \inf\{a : a \in \mathfrak{A}_n,\, a \supseteq a'_n\} = 1.$$

Let $\mathfrak{A}_{n+1}$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_n \cup \{a'_n\}$. Since we have

$$\sup\{b : b \in \mathfrak{B}_n,\, b \subseteq b_n\} = 0, \quad \inf\{b : b \in \mathfrak{B}_n,\, b \supseteq b_n\} = 1,$$

there is a (unique) extension of $\pi_n : \mathfrak{A}_n \to \mathfrak{B}_n$ to a homomorphism $\pi_{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{B}_{n+1}$ such that $\pi_{n+1}a'_n = b_n$ (312O). Since we similarly have an extension ϕ of π_n^{-1} to a homomorphism from $\mathfrak{B}_{n+1}$ to $\mathfrak{A}_{n+1}$ with $\phi b_n = a'_n$, and since $\phi\pi_{n+1}$, $\pi_{n+1}\phi$ must be the respective identity homomorphisms, π_{n+1} is an isomorphism, and the induction continues.

Since π_{n+1} extends π_n for each n, these isomorphisms join together to give us an isomorphism

$$\pi : \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n \to \bigcup_{n\in\mathbb{N}} \mathfrak{B}_n = \mathfrak{B}.$$

Observe next that the construction ensures that $a_n \in \mathfrak{A}_{n+1}$ for each n, since $a_n \cap a$ is either 0 or a or $a'_n \cap a$ for every $a \in A_n$, and in all cases belongs to $\mathfrak{A}_{n+1}$. So $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ contains every a_n and (by the choice of $\langle a_n\rangle_{n\in\mathbb{N}}$) must be the whole of $\mathfrak{A}$. Thus $\pi : \mathfrak{A} \to \mathfrak{B}$ witnesses that $\mathfrak{A} \cong \mathfrak{B}$.

316N Definition A Boolean algebra $\mathfrak{A}$ is **homogeneous** if every non-trivial principal ideal of $\mathfrak{A}$ is isomorphic to $\mathfrak{A}$.

***316O Lemma** Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that

$$D = \{d : d \in \mathfrak{A},\, \mathfrak{A} \text{ is isomorphic to the principal ideal } \mathfrak{A}_d\}$$

is order-dense in $\mathfrak{A}$. Then $\mathfrak{A}$ is homogeneous.

proof (a) If $\mathfrak{A} = \{0\}$ then it has no non-trivial principal ideals, so is homogeneous. If $\mathfrak{A}$ is not atomless, let $a \in \mathfrak{A}$ be an atom; then there is a non-zero $d \in D$ such that $d \subseteq a$ and $d = a$; so $\mathfrak{A} \cong \mathfrak{A}_d = \{0, d\}$ and $\mathfrak{A} = \{0, 1\}$ is homogeneous because its only non-trivial principal ideal is itself. So suppose henceforth that $\mathfrak{A}$ is atomless and not $\{0\}$.

(b) Take any $a \in \mathfrak{A} \setminus \{0\}$. Choose $\langle a_n\rangle_{n\in\mathbb{N}}$ inductively in $\mathfrak{A}$ in such a way that $a_0 = a$ and that $a_{n+1} \subseteq a_n$ is neither 0 nor a_n for any n. Let D' be

$$\{d : d \in D, \text{ either } d \subseteq \inf_{n\in\mathbb{N}} a_n \text{ or } d \subseteq a_n \setminus a_{n+1} \text{ for some } n \text{ or } d \subseteq 1 \setminus a_0\}.$$

Then $D' \subseteq D$ is still order-dense. Let $C \subseteq D'$ be a partition of unity (313K); then C is infinite. We have

$$\mathfrak{A} \cong \prod_{c\in C} \mathfrak{A}_c \cong \mathfrak{A}^C$$

(315F). Moreover, every member of C is either included in a or disjoint from it, so setting $C' = \{c : c \in C,\, c \subseteq a\}$ we see that C' is a partition of unity in $\mathfrak{A}_a$; as $\mathfrak{A}_a$ is Dedekind complete (314Ea),

$$\mathfrak{A}_a \cong \prod_{c\in C'} \mathfrak{A}_c \cong \mathfrak{A}^{C'} \cong (\mathfrak{A}^C)^{C'} \cong \mathfrak{A}^{C\times C'} \cong \mathfrak{A}^C \cong \mathfrak{A}$$

(because C is infinite and C' is not empty, so $\#(C \times C') = \#(C)$). As a is arbitrary, $\mathfrak{A}$ is homogeneous.

***316P Proposition** Let $\mathfrak{A}$ be a homogeneous Boolean algebra. Then its Dedekind completion $\widehat{\mathfrak{A}}$ is homogeneous.

proof Regarding $\mathfrak{A}$ as a subset of $\widehat{\mathfrak{A}}$, it is order-dense. Next, if $a \in \mathfrak{A}$, then the principal ideal $\widehat{\mathfrak{A}}_a$ which it generates in $\widehat{\mathfrak{A}}$ can be identified with the Dedekind completion of the principal ideal $\mathfrak{A}_a$ which it generates in $\mathfrak{A}$. **P** $\mathfrak{A}_a$ is order-dense in $\widehat{\mathfrak{A}}_a$ and $\widehat{\mathfrak{A}}_a$ is Dedekind complete, so we can use 314Ub. **Q** But this means that

$$\widehat{\mathfrak{A}}_a \cong \widehat{\mathfrak{A}_a} \cong \widehat{\mathfrak{A}}$$

for every $a \in \mathfrak{A} \setminus \{0\}$. As $\mathfrak{A} \setminus \{0\}$ is order-dense in $\widehat{\mathfrak{A}}$, 316O tells us that $\widehat{\mathfrak{A}}$ is homogeneous.

***316Q Proposition** The free product of any family of homogeneous Boolean algebras is homogeneous.

proof (a) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of homogeneous Boolean algebras and $\mathfrak{A} = \bigotimes_{i\in I}\mathfrak{A}_i$ their free product; let $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ be the canonical homomorphisms. If any of the $\mathfrak{A}_i$ is $\{0\}$, so is $\mathfrak{A}$ (315Kd), and the result is trivial; so let us suppose that every $\mathfrak{A}_i$ has at least two elements. If $\mathfrak{A}_i = \{0,1\}$ for every $i \in I$, then $\mathfrak{A} = \{0,1\}$ is homogeneous; so we may suppose that at least one $\mathfrak{A}_i$ is infinite.

(b) If we have a family $\langle a_i\rangle_{i\in I}$ such that $a_i \in \mathfrak{A}_i$ for every i and $J = \{i : a_i \neq 1\}$ is finite, consider $a = \inf_{i\in J}\varepsilon_i(a_i)$ in $\mathfrak{A}$. Then $\mathfrak{A}_a \cong \bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}$. **P** For $j \in I$, let ε'_j be the canonical homomorphism from $(\mathfrak{A}_j)_{a_j}$ to $\bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}$. Set $\phi_i(c) = a \cap \varepsilon_i(c)$ for $i \in I$ and $c \in (\mathfrak{A}_i)_{a_i}$. Then $\phi_i : (\mathfrak{A}_i)_{a_i} \to \mathfrak{A}_a$ is always a Boolean homomorphism, so we have a Boolean homomorphism $\phi : \bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i} \to \mathfrak{A}_a$ such that $\phi_i = \phi\varepsilon'_i$ for each i (315J).

If $K \subseteq I$ is finite, $J \subseteq K$, $b_k \in (\mathfrak{A}_k)_{a_k}$ for each $k \in K$ and b is the infimum $\inf_{k\in K}\varepsilon'_k(b_k)$ taken in $\bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}$, then

$$\phi(b) = \inf_{k\in K}\phi\varepsilon'_k(b_k)$$

(here taking the infimum in $\mathfrak{A}_a$)

$$= \inf_{k\in K}\phi_k(b_k) = \inf_{k\in K} a \cap \varepsilon_k(b_k) = a \cap \inf_{k\in K}\varepsilon_k(b_k)$$

(here taking the infimum in $\mathfrak{A}$)

$$= \inf_{k\in K}\varepsilon_k(b_k)$$

because $K \supseteq J$.

Now suppose that $b \in \bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}$ is non-zero. Then there are a finite $K \subseteq I$ and a family $\langle b_k\rangle_{k\in K}$ such that $b_k \in (\mathfrak{A}_k)_{a_k} \setminus \{0\}$ for each k and b includes $\inf_{k\in K}\varepsilon'_k b_k$ in $\bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}$ (315Kb). Set $K' = J \cup K$ and $b_k = a_k$ for $k \in J \setminus K$. Then

$$\phi(b) \supseteq \phi(\inf_{k\in K}\varepsilon'_k(b_k)) \supseteq \inf_{k\in K'}\phi\varepsilon'_k(b_k) = \inf_{k\in K'}\varepsilon_k(b_k) \neq 0$$

(315K(e-ii)). As b is arbitrary, ϕ is injective.

To see that ϕ is surjective, use 315Kb; every element of $\mathfrak{A}_a$ is expressible as a finite union of elements of the form $c = \inf_{k\in K}\varepsilon_k(c_k)$ where $K \subseteq I$ is finite and $c_k \in \mathfrak{A}_k$ for each $k \in K$. Again set $K' = J \cup K$; this time, take $c_k = 1$ for any $k \in J \setminus K$. Then

$$\begin{aligned} c = c \cap a &= \inf_{k\in K'}\varepsilon_k(c_k) \cap \inf_{k\in K'}\varepsilon_k(a_k) \\ &= \inf_{k\in K'}(\varepsilon_k(c_k) \cap \varepsilon_k(a_k)) = \inf_{k\in K'}(\varepsilon_k(c_k \cap a_k)) \\ &= \phi(\inf_{k\in K'}\varepsilon'_k(c_k \cap a_k) \in \phi[\bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}]. \end{aligned}$$

So $\mathfrak{A}_a \subseteq \phi[\bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}]$. **Q**

(c) Let A be the set of those $a \in \mathfrak{A}$ expressible in the form considered in (b), with every a_i non-zero. If $a \in A$, then

$$\mathfrak{A}_a \cong \bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i} \cong \bigotimes_{i\in I}\mathfrak{A}_i = \mathfrak{A}$$

because every $\mathfrak{A}_i$ is homogeneous.

(d) We need to know that $\mathfrak{A}$ is isomorphic to the simple power $\mathfrak{A}^n$ for every $n \geq 1$. **P** We are supposing that there is a $k \in I$ such that $\mathfrak{A}_k$ is infinite. In this case there must be a partition of unity $(d_1, \dots, d_n)$ in $\mathfrak{A}_k \setminus \{0\}$. (Induce on n, noting at the inductive step that if $\{d_1, \dots, d_n\}$ is a partition of unity then not all the d_j can be atoms, because $\#(\mathfrak{A}_k) > 2^n$.) Now, setting $a^{(j)} = \varepsilon_k(d_j)$ for each j, $(a^{(1)}, \dots, a^{(n)})$ is a partition of unity in $\mathfrak{A}$ and (by (c)) all the principal ideals $\mathfrak{A}_{a^{(j)}}$ are isomorphic to $\mathfrak{A}$, so

$$\mathfrak{A} \cong \prod_{j\leq n}\mathfrak{A}_{a^{(j)}} \cong \mathfrak{A}^n$$

by 315F(i). **Q**

(e) Now take any $a \in \mathfrak{A} \setminus \{0\}$. Then a is expressible as $\sup_{1\leq j\leq n} a^{(j)}$ where $a^{(1)}, \dots, a^{(n)}$ are disjoint members of A (315Kb). So, putting (c) and (d) together,

$$\mathfrak{A}_a \cong \prod_{1\le j\le n} \mathfrak{A}_{a(j)} \cong \mathfrak{A}^n \cong \mathfrak{A}.$$

As a is arbitrary, $\mathfrak{A}$ is homogeneous.

316X Basic exercises (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that it is ccc iff there is no family $\langle a_\xi\rangle_{\xi<\omega_1}$ in $\mathfrak{A}$ such that $a_\xi \subset a_\eta$ whenever $\xi < \eta < \omega_1$.

(b) Let $\mathfrak{A}$ be a ccc Boolean algebra. Show that if $\mathcal{I}$ is a σ-ideal of $\mathfrak{A}$, then it is order-closed, and $\mathfrak{A}/\mathcal{I}$ is ccc.

(c) Let $\mathfrak{A}$ be a Boolean algebra and $\mathcal{I}$ an order-closed ideal of $\mathfrak{A}$. Show that $\mathfrak{A}/\mathcal{I}$ is ccc iff there is no uncountable disjoint family in $\mathfrak{A}\setminus\mathcal{I}$.

(d) Let $\mathfrak{A}$ be a Boolean algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is ccc; (ii) every σ-ideal of $\mathfrak{A}$ is order-closed; (iii) every σ-subalgebra of $\mathfrak{A}$ is order-closed; (iv) every sequentially order-continuous Boolean homomorphism from $\mathfrak{A}$ to another Boolean algebra is order-continuous. (*Hint*: 313Q.)

(e) Show that any purely atomic Boolean algebra is weakly (σ,∞)-distributive.

>(f) Let $\mathfrak{A}$ be a Dedekind complete purely atomic Boolean algebra. Show that it is isomorphic to $\mathcal{P}A$, where A is the set of atoms of $\mathfrak{A}$.

(g) Show that a homogeneous Boolean algebra is either atomless or $\{0,1\}$.

(h) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Show that if $\mathfrak{A}$ is ccc, then $\mathfrak{B}$ is ccc.

(i) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ which is regularly embedded in $\mathfrak{A}$. (i) Show that if $\mathfrak{A}$ is weakly (σ,∞)-distributive, then $\mathfrak{B}$ is weakly (σ,∞)-distributive. (ii) Show that every atom of $\mathfrak{A}$ is included in an atom of $\mathfrak{B}$. (iii) Show that if $\mathfrak{A}$ is purely atomic, so is $\mathfrak{B}$. >(iv) Show that if $\mathfrak{B}$ is atomless, so is $\mathfrak{A}$.

(j) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ an order-dense subalgebra of $\mathfrak{A}$. (i) Show that $\mathfrak{A}$ is ccc iff $\mathfrak{B}$ is ccc. (ii) Show that $\mathfrak{A}$ is weakly (σ,∞)-distributive iff $\mathfrak{B}$ is weakly (σ,∞)-distributive. (iii) Show that $\mathfrak{A}$ and $\mathfrak{B}$ have the same atoms, so that $\mathfrak{A}$ is atomless, or purely atomic, iff $\mathfrak{B}$ is.

(k) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a surjective order-continuous Boolean homomorphism. (i) Show that if $\mathfrak{A}$ is ccc, then $\mathfrak{B}$ is ccc. (ii) Show that if $\mathfrak{A}$ is weakly (σ,∞)-distributive, then $\mathfrak{B}$ is weakly (σ,∞)-distributive. (iii) Show that if $\mathfrak{A}$ is atomless, then $\mathfrak{B}$ is atomless. (iv) Show that if $\mathfrak{A}$ is purely atomic, then $\mathfrak{B}$ is purely atomic.

(l) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, neither $\{0\}$, and $\mathfrak{A}\otimes\mathfrak{B}$ their free product. (i) Show that if $\mathfrak{A}\otimes\mathfrak{B}$ is ccc, then $\mathfrak{A}$ and $\mathfrak{B}$ are both ccc. (ii) Show that if $\mathfrak{A}\otimes\mathfrak{B}$ is weakly (σ,∞)-distributive, then $\mathfrak{A}$ and $\mathfrak{B}$ are both weakly (σ,∞)-distributive. (iii) Show that $\mathfrak{A}\otimes\mathfrak{B}$ is atomless iff either $\mathfrak{A}$ or $\mathfrak{B}$ is atomless. (iv) Show that $\mathfrak{A}\otimes\mathfrak{B}$ is purely atomic iff $\mathfrak{A}$ and $\mathfrak{B}$ are both purely atomic.

(m) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{A}_a$ a principal ideal of $\mathfrak{A}$. (i) Show that of $\mathfrak{A}$ is ccc, then $\mathfrak{A}_a$ is ccc. (ii) Show that if $\mathfrak{A}$ is weakly (σ,∞)-distributive, then $\mathfrak{A}_a$ is weakly (σ,∞)-distributive. (iii) Show that if $\mathfrak{A}$ is atomless, then $\mathfrak{A}_a$ is atomless. (iv) Show that if $\mathfrak{A}$ is purely atomic, then $\mathfrak{A}_a$ is purely atomic. (v) Show that if $\mathfrak{A}$ is homogeneous, then $\mathfrak{A}_a$ is homogeneous.

(n) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, with simple product $\mathfrak{A}$. (i) Show that $\mathfrak{A}$ is ccc iff every $\mathfrak{A}_i$ is ccc and $\{i:\mathfrak{A}_i\ne\{0\}\}$ is countable. (ii) Show that $\mathfrak{A}$ is weakly (σ,∞)-distributive iff every $\mathfrak{A}_i$ is weakly (σ,∞)-distributive. (iii) Show that $\mathfrak{A}$ is atomless iff every $\mathfrak{A}_i$ is atomless. (iv) Show that $\mathfrak{A}$ is purely atomic iff every $\mathfrak{A}_i$ is purely atomic.

>(o) Let X be a separable topological space. Show that X is ccc.

(p) Let X be a topological space, and $\mathrm{RO}(X)$ its regular open algebra. Show that X is ccc iff $\mathrm{RO}(X)$ is ccc.

(q) Let X be a zero-dimensional compact Hausdorff space. Show that the regular open algebra of X is weakly (σ,∞)-distributive iff the algebra of open-and-closed subsets of X is weakly (σ,∞)-distributive.

(r) Show that the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$, with its usual topology, is ccc, homogeneous and not weakly (σ,∞)-distributive.

(s) Show that the regular open algebra $\mathrm{RO}(\mathbb{R})$ is ccc and homogeneous.

316Y Further exercises (a) Let I be any set. Show that $\{0,1\}^I$, with its usual topology, is ccc. (*Hint*: show that if $E\subseteq\{0,1\}^I$ is a non-empty open-and-closed set, then $\nu_I E>0$, where ν_I is the usual measure on $\{0,1\}^I$.)

(b) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. Show that $\mathfrak{A}$ is ccc iff every nowhere dense subset of Z is included in a nowhere dense zero set.

(c) Let X be a zero-dimensional topological space. Show that X is ccc iff the regular open algebra of X is ccc iff the algebra of open-and-closed subsets of X is ccc.

(d) Set $X=\{0,1\}^{\omega_1}$, and for $\xi<\omega_1$ set $E_\xi=\{x:x\in X,\,x(\xi)=1\}$. Let Σ be the algebra of subsets of X generated by $\{E_\xi:\xi<\omega_1\}\cup\{\{x\}:x\in X\}$, and $\mathcal{I}$ the σ-ideal of Σ generated by $\{E_\xi\cap E_\eta:\xi<\eta<\omega_1\}\cup\{\{x\}:x\in X\}$. Show that $\Sigma/\mathcal{I}$ is not ccc, but that there is no uncountable disjoint family in $\Sigma\setminus\mathcal{I}$.

(e) Let $\mathfrak{A}$ be a Boolean algebra. $\mathfrak{A}$ is **weakly σ-distributive** if whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of countable partitions of unity in $\mathfrak{A}$ then there is a partition B of unity such that $\{a:a\in A_n,\,a\cap b\neq 0\}$ is finite for every $b\in B$ and $n\in\mathbb{N}$. (Dedekind complete weakly σ-distributive algebras are also called **ω^ω-bounding**.) $\mathfrak{A}$ has the **Egorov property** if whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of countable partitions of unity in $\mathfrak{A}$ then there is a *countable* partition B of unity such that $\{a:a\in A_n,\,a\cap b\neq 0\}$ is finite for every $b\in B$ and $n\in\mathbb{N}$. (i) Show that if $\mathfrak{A}$ is ccc then it is weakly (σ,∞)-distributive iff it has the Egorov property iff it is weakly σ-distributive. (ii) Show that $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ does not have the Egorov property, even though it is weakly (σ,∞)-distributive. (*Hint*: try $a_{nm}=\{f:f(n)=m\}$.)

(f) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with the Egorov property and I a σ-ideal of $\mathfrak{A}$. Show that $\mathfrak{A}/I$ has the Egorov property.

(g) Let X be a regular topological space and $\mathrm{RO}(X)$ its regular open algebra. Show that $\mathrm{RO}(X)$ is weakly (σ,∞)-distributive iff every meager set in X is nowhere dense.

(h) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. (i) Show that $\mathfrak{A}$ is weakly σ-distributive iff the union of any sequence of nowhere dense zero sets in Z is nowhere dense. (ii) Show that $\mathfrak{A}$ has the Egorov property iff the union of any sequence of nowhere dense zero sets in Z is included in a nowhere dense zero set.

(i) Let $\mathfrak{A}$ be a Dedekind σ-complete weakly (σ,∞)-distributive Boolean algebra, Z its Stone space, $\mathcal{E}$ the algebra of open-and-closed subsets of Z, $\mathcal{M}$ the σ-ideal of meager subsets of Z, and Σ the algebra $\{E\triangle M:E\in\mathcal{E},\,M\in\mathcal{M}\}$, as in 314M. (i) Let $f:Z\to\mathbb{R}$ be a function. Show that f is Σ-measurable iff there is a dense open set $G\subseteq Z$ such that $f\restriction G$ is continuous. (ii) Now suppose that $\mathfrak{A}$ is Dedekind complete and that $f:Z\to\mathbb{R}$ is a bounded function. Show that f is Σ-measurable iff there is a continuous function $g:Z\to\mathbb{R}$ such that $\{z:f(z)\neq g(z)\}$ is meager. (*Hint*: if G is a dense open set and $f\restriction G$ is continuous, the closure in $Z\times\mathbb{R}$ of the graph of $f\restriction G$ is a function, because Z is extremally disconnected.)

(j) Show that the Stone space of $\mathrm{RO}(\mathbb{R})$ is separable. More generally, show that if a topological space X is separable so is the Stone space of its regular open algebra.

(k)(i) Let X be a non-empty separable Hausdorff space without isolated points. Show that its regular open algebra is not weakly (σ,∞)-distributive. (ii) Let (X,ρ) be a non-empty metric space without isolated points. Show that its regular open algebra is not weakly (σ,∞)-distributive. (iii) Let I be any infinite set. Show that the algebra of open-and-closed subsets of $\{0,1\}^I$ is not weakly (σ,∞)-distributive. Show that the regular open algebra of $\{0,1\}^I$ is not weakly (σ,∞)-distributive.

(l) For any set X, write

$$\mathcal{C}_X=\{I:I\subseteq X\text{ is finite}\}\cup\{X\setminus I:I\subseteq X\text{ is finite}\}.$$

(i) Show that $\mathcal{C}_X$ is an algebra of subsets of X (the **finite-cofinite algebra**). (ii) Show that a Boolean algebra is purely atomic iff it has an order-dense subalgebra isomorphic to the finite-cofinite algebra of some set. (iii) Show that a Dedekind σ-complete Boolean algebra is purely atomic iff it has an order-dense subalgebra isomorphic to the countable-cocountable algebra of some set.

(m) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, none of them $\{0\}$, with free product $\mathfrak{A}$. (i) Show that $\mathfrak{A}$ is purely atomic iff every $\mathfrak{A}_i$ is purely atomic and $\{i:\mathfrak{A}_i\ne\{0,1\}\}$ is finite. (ii) Show that $\mathfrak{A}$ is atomless iff either some $\mathfrak{A}_i$ is atomless or $\{i:\mathfrak{A}_i\ne\{0,1\}\}$ is infinite.

(n) Let X be a Hausdorff space and $\mathrm{RO}(X)$ its regular open algebra. (i) Show that the atoms of $\mathrm{RO}(X)$ are precisely the sets $\{x\}$ where x is an isolated point in X. (ii) Show that $\mathrm{RO}(X)$ is atomless iff X has no isolated points. (iii) Show that $\mathrm{RO}(X)$ is purely atomic iff the set of isolated points in X is dense in X.

(o) Show that a Boolean algebra is isomorphic to $\mathrm{RO}(\mathbb{R})$ iff it is atomless, Dedekind complete, has a countable order-dense subalgebra and is not $\{0\}$.

(p) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, none of them $\{0\}$, and $\mathfrak{A}$ their simple product. Show that $\mathfrak{A}$ is homogeneous iff (i) $\mathfrak{A}_i$ is isomorphic to $\mathfrak{A}_j$ for all i, $j\in I$ (ii) for every $i\in I$ there is a partition of unity $A\subseteq\mathfrak{A}_i\setminus\{0\}$ with $\#(A)=\#(I)$.

(q) Let $\mathfrak{A}$ be a Boolean algebra such that $\{d:d\in\mathfrak{A},\,\mathfrak{A}_d\cong\mathfrak{A}\}$ is order-dense in $\mathfrak{A}$. Show that the Dedekind completion $\widehat{\mathfrak{A}}$ is homogeneous.

(r) Write $[\mathbb{N}]^{<\omega}$ for the ideal of $\mathcal{P}\mathbb{N}$ consisting of the finite subsets of $\mathbb{N}$. Show that $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ is atomless, weakly (σ,∞)-distributive and not ccc, and that its Dedekind completion is homogeneous.

(s) Show that the regular open algebra of $\{0,1\}^I$ is homogeneous for any infinite set I.

316 Notes and comments The phrase 'countable chain condition' is perhaps unfortunate, since the disjoint sets to which the definition 316A refers could more naturally be called 'antichains'; but there is in fact a connexion between countable chains and countable antichains (316Xa). While some authors speak of the 'countable antichain condition' or 'cac', the term 'ccc' has become solidly established. In the Boolean algebra context, it could equally well be called the 'countable sup property' (316E).

The countable chain condition can be thought of as a restriction on the 'width' of a Boolean algebra; it means that the algebra cannot spread too far laterally (see 316Xn(i)), though it may be indefinitely complex in other ways. Generally it means that in a wide variety of contexts we need look only at countable families and monotonic sequences, rather than arbitrary families and directed sets (316E, 316F, 316Ye). Many of the ideas of 316C-316E have already appeared in 215B; see 322G below.

I remarked in the notes to §313 that the distributive laws described in 313B have important generalizations, of which 'weak (σ,∞)-distributivity' and its cousin 'weak σ-distributivity' (316Ye) are two. They are characteristic of the measure algebras which are the chief subject of this volume. The 'Egorov property' (316Ye again) is an alternative formulation applicable to ccc spaces.

Of course every property of Boolean algebras has a reflection in a topological property of their Stone spaces; happily, most of the concepts of this section correspond to reasonably natural topological expressions (316B, 316I, 316L, 316Yh). 'Homogeneity' is the odd one out. In fact only the definition of 'homogeneous' Boolean algebra is particularly worth noting at this stage. The homogeneous algebras we are primarily interested in will appear in §331, and they are too special for any general theory to be very helpful.

With five new properties (ccc, weakly (σ,∞)-distributive, atomless, purely atomic, homogeneous) to incorporate into the constructions of the last few sections, a very large number of questions can be asked; most are elementary. In 316Xh-316Xn I list the properties which are inherited by subalgebras, order-continuous homomorphic images, free products, principal ideals and simple products. The countable chain condition is so important that it is worth noting that a sequentially order-continuous image of a ccc algebra is ccc (316Xb), and that there is a useful necessary and sufficient condition for a sequentially order-continuous image of a σ-complete algebra to be ccc (316C, 316D, 316Xc; but see also 316Yd). To see that sequentially order-continuous images do not inherit weak (σ,∞)-distributivity, recall that the regular open algebra of $\mathbb{R}$ is isomorphic to the quotient of the Baire-property algebra $\widehat{\mathcal{B}}$ of $\mathbb{R}$ by the meager ideal $\mathcal{M}$ (314Yd); but that $\widehat{\mathcal{B}}$ is purely atomic (since it contains all singletons), therefore weakly (σ,∞)-distributive (316Xe). Similarly, $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ is a non-ccc image of a ccc algebra (316Yr). Free products of weakly (σ,∞)-distributive algebras need not be weakly (σ,∞)-distributive (325Ye). There are important cases in which the simple product of homogeneous algebras is homogeneous (316Yp).

The definitions here provide a language in which a remarkably interesting question can be asked: is the free product of ccc Boolean algebras always ccc? equivalently, is the product of ccc topological spaces always ccc? What is special about this question is that it cannot be answered within the ordinary rules of mathematics (even including the axiom of choice); it is undecidable, in the same way that the continuum hypothesis is. I will deal with a variety of undecidable questions in Volume 5; this particular one is mentioned in 516U, 517Xe and 553J.

I have taken the opportunity to mention three of the most important of all Boolean algebras: the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ (316M, 316Xr), the regular open algebra of $\mathbb{R}$ (316J, 316Xs, 316Yo) and the quotient $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ (316Yr). A fourth algebra which belongs in this company is the Lebesgue measure algebra, which is atomless, ccc, weakly (σ,∞)-distributive and homogeneous (so that every countable subset of its Stone space Z is nowhere dense, and Z is a non-separable ccc space); but for this I wait for the next chapter.

Chapter 32

Measure algebras

I now come to the real work of this volume, the study of the Boolean algebras of equivalence classes of measurable sets. In this chapter I work through the 'elementary' theory, defining this to consist of the parts which do not depend on Maharam's theorem or the lifting theorem or non-trivial set theory.

§321 gives the definition of 'measure algebra', and relates this idea to its origin as the quotient of a σ-algebra of measurable sets by a σ-ideal of negligible sets, both in its elementary properties (following those of measure spaces treated in §112) and in an appropriate version of the Stone representation. §322 deals with the classification of measure algebras according to the scheme already developed in §211 for measure spaces. §323 discusses the standard topology and uniformity of a measure algebra. §324 contains results concerning Boolean homomorphisms between measure algebras, with the relationships between topological continuity, order-continuity and preservation of measure. §325 is devoted to the measure algebras of product measures, and their abstract characterization as completed free products. §§326-327 address the properties of additive functionals on Boolean algebras, generalizing the ideas of Chapter 23. Finally, §328 looks at 'reduced products' of probability algebras and some related constructions, including inductive limits.

321 Measure algebras

I begin by defining 'measure algebra' and relating this concept to the work of Chapter 31 and to the elementary properties of measure spaces.

321A Definition A **measure algebra** is a pair $(\mathfrak{A}, \bar{\mu})$, where $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\bar{\mu} : \mathfrak{A} \to [0, \infty]$ is a function such that

$\bar{\mu}0 = 0$;

whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, $\bar{\mu}(\sup_{n\in\mathbb{N}} a_n) = \sum_{n=0}^{\infty} \bar{\mu}a_n$;

$\bar{\mu}a > 0$ whenever $a \in \mathfrak{A}$ and $a \neq 0$.

321B Elementary properties of measure algebras Corresponding to the most elementary properties of measure spaces (112C in Volume 1), we have the following basic properties of measure algebras. Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) If a, $b \in \mathfrak{A}$ and $a \cap b = 0$ then $\bar{\mu}(a \cup b) = \bar{\mu}a + \bar{\mu}b$. **P** Set $a_0 = a$, $a_1 = b$, $a_n = 0$ for $n \geq 2$; then

$$\bar{\mu}(a \cup b) = \bar{\mu}(\sup_{n\in\mathbb{N}} a_n) = \sum_{n=0}^{\infty} \bar{\mu}a_n = \bar{\mu}a + \bar{\mu}b. \text{ **Q**}$$

(b) If a, $b \in \mathfrak{A}$ and $a \subseteq b$ then $\bar{\mu}a \leq \bar{\mu}b$. **P**

$$\bar{\mu}a \leq \bar{\mu}a + \bar{\mu}(b \setminus a) = \bar{\mu}b. \text{ **Q**}$$

(c) For any a, $b \in \mathfrak{A}$, $\bar{\mu}(a \cup b) \leq \bar{\mu}a + \bar{\mu}b$. **P**

$$\bar{\mu}(a \cup b) = \bar{\mu}a + \bar{\mu}(b \setminus a) \leq \bar{\mu}a + \bar{\mu}b. \text{ **Q**}$$

(d) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{A}$, then $\bar{\mu}(\sup_{n\in\mathbb{N}} a_n) \leq \sum_{n=0}^{\infty} \bar{\mu}a_n$. **P** For each n, set $b_n = a_n \setminus \sup_{i<n} a_i$. Inducing on n, we see that $\sup_{i\leq n} a_i = \sup_{i\leq n} b_i$ for each n, so $\sup_{n\in\mathbb{N}} a_n = \sup_{n\in\mathbb{N}} b_n$ and

$$\bar{\mu}(\sup_{n\in\mathbb{N}} a_n) = \bar{\mu}(\sup_{n\in\mathbb{N}} b_n) = \sum_{n=0}^{\infty} \bar{\mu}b_n \leq \sum_{n=0}^{\infty} \bar{\mu}a_n$$

because $\langle b_n \rangle_{n\in\mathbb{N}}$ is disjoint. **Q**

(e) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$, then $\bar{\mu}(\sup_{n\in\mathbb{N}} a_n) = \lim_{n\to\infty} \bar{\mu}a_n$. **P** Set $b_0 = a_0$, $b_n = a_n \setminus a_{n-1}$ for $n \geq 1$. Then

$$\begin{aligned}\bar{\mu}(\sup_{n\in\mathbb{N}} a_n) &= \bar{\mu}(\sup_{n\in\mathbb{N}} b_n) = \sum_{n=0}^{\infty} \bar{\mu}b_n \\ &= \lim_{n\to\infty} \sum_{i=0}^{n} \bar{\mu}b_i = \lim_{n\to\infty} \bar{\mu}(\sup_{i\leq n} b_i) = \lim_{n\to\infty} \bar{\mu}a_n. \text{ **Q**}\end{aligned}$$

(f) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ and $\inf_{n\in\mathbb{N}}\bar\mu a_n<\infty$, then $\bar\mu(\inf_{n\in\mathbb{N}}a_n)=\lim_{n\to\infty}\bar\mu a_n$. **P** (Cf. 112Cf.) Set $a=\inf_{n\in\mathbb{N}}a_n$. Take $k\in\mathbb{N}$ such that $\bar\mu a_k<\infty$. Set $b_n=a_k\setminus a_n$ for $n\in\mathbb{N}$; then $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and $\sup_{n\in\mathbb{N}}b_n=a_k\setminus a$ (313Ab). Because $\bar\mu a_k$ is finite,

$$\bar\mu a=\bar\mu a_k-\bar\mu(a_k\setminus a)=\bar\mu a_k-\lim_{n\to\infty}\bar\mu b_n$$

(by (e) above)

$$=\lim_{n\to\infty}\bar\mu(a_k\setminus b_n)=\lim_{n\to\infty}\bar\mu a_n.\ \mathbf{Q}$$

321C Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $A\subseteq\mathfrak{A}$ a non-empty upwards-directed set. If $\sup_{a\in A}\bar\mu a<\infty$, then $\sup A$ is defined in $\mathfrak{A}$ and $\bar\mu(\sup A)=\sup_{a\in A}\bar\mu a$.

proof (Compare 215A.) Set $\gamma=\sup_{a\in A}\bar\mu a$, and for each $n\in\mathbb{N}$ choose $a_n\in A$ such that $\bar\mu a_n\ge\gamma-2^{-n}$. Next, choose $\langle b_n\rangle_{n\in\mathbb{N}}$ in A such that $b_{n+1}\supseteq b_n\cup a_n$ for each n, and set $b=\sup_{n\in\mathbb{N}}b_n$. Then

$$\bar\mu b=\lim_{n\to\infty}\bar\mu b_n\le\gamma,\quad \bar\mu a_n\le\bar\mu b\text{ for every }n\in\mathbb{N},$$

so $\bar\mu b=\gamma$.

If $a\in A$, then for every $n\in\mathbb{N}$ there is an $a'_n\in A$ such that $a\cup a_n\subseteq a'_n$, so that

$$\bar\mu(a\setminus b)\le\bar\mu(a\setminus a_n)\le\bar\mu(a'_n\setminus a_n)=\bar\mu a'_n-\bar\mu a_n\le\gamma-\bar\mu a_n\le 2^{-n}.$$

This means that $\bar\mu(a\setminus b)=0$, so $a\setminus b=0$ and $a\subseteq b$. Accordingly b is an upper bound of A, and is therefore $\sup A$; since we already know that $\bar\mu b=\gamma$, the proof is complete.

321D Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $A\subseteq\mathfrak{A}$ a non-empty upwards-directed set. If $\sup A$ is defined in $\mathfrak{A}$, then $\bar\mu(\sup A)=\sup_{a\in A}\bar\mu a$.

proof If $\sup_{a\in A}\bar\mu a=\infty$, this is trivial; otherwise it follows from 321C.

321E Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $A\subseteq\mathfrak{A}$ a disjoint set. If $\sup A$ is defined in $\mathfrak{A}$, then $\bar\mu(\sup A)=\sum_{a\in A}\bar\mu a$.

proof If $A=\emptyset$ then $\sup A=0$ and the result is trivial. Otherwise, set $B=\{a_0\cup\ldots\cup a_n : a_0,\ldots,a_n\in A$ are distinct$\}$. Then B is upwards-directed, and $\sup_{b\in B}\bar\mu b=\sum_{a\in A}\bar\mu a$ because A is disjoint. Also B has the same upper bounds as A, so $\sup B=\sup A$ and

$$\bar\mu(\sup A)=\bar\mu(\sup B)=\sup_{b\in B}\bar\mu b=\sum_{a\in A}\bar\mu a.$$

321F Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $A\subseteq\mathfrak{A}$ a non-empty downwards-directed set. If $\inf_{a\in A}\bar\mu a<\infty$, then $\inf A$ is defined in $\mathfrak{A}$ and $\bar\mu(\inf A)=\inf_{a\in A}\bar\mu a$.

proof Take $a_0\in A$ with $\bar\mu a_0<\infty$, and set $B=\{a_0\setminus a : a\in A\}$. Then B is upwards-directed, and $\sup_{b\in B}\bar\mu b\le\bar\mu a_0<\infty$, so $\sup B$ is defined. Accordingly $\inf A=a_0\setminus\sup B$ is defined (313Aa), and

$$\bar\mu(\inf A)=\bar\mu a_0-\bar\mu(\sup B)=\bar\mu a_0-\sup_{b\in B}\bar\mu b$$

$$=\inf_{b\in B}\bar\mu(a_0\setminus b)=\inf_{a\in A}\bar\mu(a_0\cap a)=\inf_{a\in A}\bar\mu a.$$

321G Subalgebras If $(\mathfrak{A},\bar\mu)$ is a measure algebra, and $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$, then $(\mathfrak{B},\bar\mu\restriction\mathfrak{B})$ is a measure algebra. **P** As remarked in 314Eb, $\mathfrak{B}$ is Dedekind σ-complete. If $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$, then the supremum $b=\sup_{n\in\mathbb{N}}b_n$ is the same whether taken in $\mathfrak{B}$ or $\mathfrak{A}$, so that we have $\bar\mu b=\sum_{n=0}^\infty\bar\mu b_n$. **Q**

321H The measure algebra of a measure space I introduce the abstract notion of 'measure algebra' because I believe that this is the right language in which to formulate the questions addressed in this volume. However it is very directly linked with the idea of 'measure space', as the next two results show.

Theorem Let (X,Σ,μ) be a measure space, and $\mathcal{N}$ the null ideal of μ. Let $\mathfrak{A}$ be the Boolean algebra quotient $\Sigma/\Sigma\cap\mathcal{N}$. Then we have a functional $\bar\mu:\mathfrak{A}\to[0,\infty]$ defined by setting

$$\bar\mu E^\bullet=\mu E \text{ for every } E\in\Sigma,$$

and $(\mathfrak{A},\bar\mu)$ is a measure algebra. The canonical map $E\mapsto E^\bullet:\Sigma\to\mathfrak{A}$ is sequentially order-continuous.

proof (a) By 314C, $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra. By 313Qb, $E\mapsto E^\bullet$ is sequentially order-continuous, because $\Sigma\cap\mathcal{N}$ is a σ-ideal of Σ.

(b) If E, $F\in\Sigma$ and $E^\bullet=F^\bullet$ in $\mathfrak{A}$, then $E\triangle F\in\mathcal{N}$, so

$$\mu E\le\mu F+\mu(E\setminus F)=\mu F\le\mu E+\mu(F\setminus E)=\mu E$$

and $\mu E=\mu F$. Accordingly the given formula does indeed define a function $\bar\mu:\mathfrak{A}\to[0,\infty]$.

(c) Now

$$\bar\mu 0=\bar\mu\emptyset^\bullet=\mu\emptyset=0.$$

If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, choose for each $n\in\mathbb{N}$ an $E_n\in\Sigma$ such that $E_n^\bullet=a_n$. Set $F_n=E_n\setminus\bigcup_{i<n}E_i$; then

$$F_n^\bullet=E_n^\bullet\setminus\sup_{i<n}E_i^\bullet=a_n\setminus\sup_{i<n}a_i=a_n$$

for each n, so $\bar\mu a_n=\mu F_n$ for each n. Now set $E=\bigcup_{n\in\mathbb{N}}E_n=\bigcup_{n\in\mathbb{N}}F_n$; then $E^\bullet=\sup_{n\in\mathbb{N}}F_n^\bullet=\sup_{n\in\mathbb{N}}a_n$. So

$$\bar\mu(\sup_{n\in\mathbb{N}}a_n)=\mu E=\sum_{n=0}^\infty\mu F_n=\sum_{n=0}^\infty\bar\mu a_n.$$

Finally, if $a\ne0$, then there is an $E\in\Sigma$ such that $E^\bullet=a$, and $E\notin\mathcal{N}$, so $\bar\mu a=\mu E>0$. Thus $(\mathfrak{A},\bar\mu)$ is a measure algebra.

321I Definition For any measure space (X,Σ,μ) I will call $(\mathfrak{A},\bar\mu)$, as constructed above, the **measure algebra of** (X,Σ,μ).

321J The Stone representation of a measure algebra Just as with Dedekind σ-complete Boolean algebras (314N), every measure algebra is obtainable from the construction above.

Theorem Let $(\mathfrak{A},\bar\mu)$ be any measure algebra. Then it is isomorphic, as measure algebra, to the measure algebra of some measure space.

proof (a) We know from 314M that $\mathfrak{A}$ is isomorphic, as Boolean algebra, to a quotient algebra $\Sigma/\mathcal{M}$ where Σ is a σ-algebra of subsets of the Stone space Z of $\mathfrak{A}$, and $\mathcal{M}$ is the ideal of meager subsets of Z. Let $\pi:\Sigma/\mathcal{M}\to\mathfrak{A}$ be the canonical isomorphism, and set $\theta E=\pi E^\bullet$ for each $E\in\Sigma$; then $\theta:\Sigma\to\mathfrak{A}$ is a sequentially order-continuous surjective Boolean homomorphism with kernel $\mathcal{M}$.

(b) For $E\in\Sigma$, set

$$\nu E=\bar\mu(\theta E).$$

Then (Z,Σ,ν) is a measure space. **P** (i) We know already that Σ is a σ-algebra of subsets of Z. (ii)

$$\nu\emptyset=\bar\mu(\theta\emptyset)=\bar\mu 0=0.$$

(iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ, then (because θ is a Boolean homomorphism) $\langle\theta E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ and (because θ is sequentially order-continuous) $\theta(\bigcup_{n\in\mathbb{N}}E_n)=\sup_{n\in\mathbb{N}}\theta E_n$; so

$$\nu(\bigcup_{n\in\mathbb{N}}E_n)=\bar\mu(\sup_{n\in\mathbb{N}}\theta E_n)=\sum_{n=0}^\infty\bar\mu(\theta E_n)=\sum_{n=0}^\infty\nu E_n. \text{ } \mathbf{Q}$$

(c) For $E\in\Sigma$,

$$\nu E=0\iff\bar\mu(\theta E)=0\iff\theta E=0\iff E\in\mathcal{M}.$$

So the measure algebra of (Z,Σ,ν) is just $\Sigma/\mathcal{M}$, with

$$\bar\nu E^\bullet=\nu E=\bar\mu(\theta E)=\bar\mu(\pi E^\bullet)$$

for every $E\in\Sigma$. Thus the Boolean algebra isomorphism π is also an isomorphism between the measure algebras $(\Sigma/\mathcal{M},\bar\nu)$ and $(\mathfrak{A},\bar\mu)$, and $(\mathfrak{A},\bar\mu)$ is represented in the required form.

321K Definition I will call the measure space (Z, Σ, ν) constructed in the proof of 321J the **Stone space** of the measure algebra $(\mathfrak{A}, \bar{\mu})$.

For later reference, I repeat the description of this space as developed in 311E, 311I, 314M and 321J. Z is a compact Hausdorff space, being the Stone space of $\mathfrak{A}$. $\mathfrak{A}$ can be identified with the algebra of open-and-closed sets in Z. The null ideal of ν coincides with the ideal of meager subsets of Z; in particular, ν is complete. The measurable sets are precisely those expressible in the form $E = \widehat{a} \triangle M$ where $a \in \mathfrak{A}$, $\widehat{a} \subseteq Z$ is the corresponding open-and-closed set, and M is meager; in this case $\nu E = \bar{\mu} a$ and $a = \theta E$ is the member of $\mathfrak{A}$ corresponding to E.

For the most important classes of measure algebras, more can be said; see 322O *et seq.* below.

321X Basic exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $a \in \mathfrak{A}$; write $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a. Show that $(\mathfrak{A}_a, \bar{\mu}\restriction\mathfrak{A}_a)$ is a measure algebra.

(b) Let $(X, \Sigma, \bar{\mu})$ be a measure space, and $\mathfrak{A}$ its measure algebra. (i) Show that if T is a σ-subalgebra of Σ, then $\{E^\bullet : E \in \mathrm{T}\}$ is a σ-subalgebra of $\mathfrak{A}$. (ii) Show that if $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$ then $\{E : E \in \Sigma,\ E^\bullet \in \mathfrak{B}\}$ is a σ-subalgebra of Σ.

321Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $I \triangleleft \mathfrak{A}$ a σ-ideal. For $u \in \mathfrak{A}/I$ set $\bar{\nu} u = \inf\{\bar{\mu} a : a \in \mathfrak{A},\ a^\bullet = u\}$. (i) Show that the infimum is always attained. (ii) Show that $(\mathfrak{A}/I, \bar{\nu})$ is a measure algebra.

321 Notes and comments The idea behind taking the quotient $\Sigma/\mathcal{N}$, where Σ is the algebra of measurable sets and $\mathcal{N}$ is the null ideal, is just that if negligible sets can be ignored – as is the case for a very large proportion of the results of measure theory – then two measurable sets can be counted as virtually the same if they differ by a negligible set, that is, if they represent the same member of the measure algebra. The definition in 321A is designed to be an exact characterization of these quotient algebras, taking into account the measures with which they are endowed. In the course of the present chapter I will work through many of the basic ideas dealt with in Volumes 1 and 2 to show how they can be translated into theorems about measure algebras, as I have done in 321B-321F. It is worth checking these correspondences carefully, because some of the ideas mutate significantly in translation. In measure algebras, it becomes sensible to take seriously the suprema and infima of uncountable sets (see 321C-321F).

I should perhaps remark that while the Stone representation (321J-321K) is significant, it is not the most important method of representing measure algebras, which is surely Maharam's theorem, to be dealt with in the next chapter. Nevertheless, the Stone representation is a canonical one, and will appear at each point that we meet a new construction involving measure algebras, just as the ordinary Stone representation of Boolean algebras can be expected to throw light on any aspect of Boolean algebra.

322 Taxonomy of measure algebras

Before going farther with the general theory of measure algebras, I run through those parts of the classification of measure spaces in §211 which have expressions in terms of measure algebras. The most important concepts at this stage are those of 'semi-finite', 'localizable' and 'σ-finite' measure algebra (322Ac-322Ae); these correspond exactly to the same terms applied to measure spaces (322B). I briefly investigate the Boolean-algebra properties of semi-finite and σ-finite measure algebras (322F, 322G), with mentions of completions and c.l.d. versions (322D), subspace measures (322I-322J), indefinite-integral measures (322K), direct sums of measure spaces (322L, 322M) and subalgebras of measure algebras (322N). It turns out that localizability of a measure algebra is connected in striking ways to the properties of the canonical measure on its Stone space (322O). I end the section with a description of the 'localization' of a semi-finite measure algebra (322P-322Q) and with some further properties of Stone spaces (322R).

322A Definitions Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) I will say that $(\mathfrak{A}, \bar{\mu})$ is a **probability algebra** if $\bar{\mu} 1 = 1$.

(b) $(\mathfrak{A}, \bar{\mu})$ is **totally finite** if $\bar{\mu} 1 < \infty$.

(c) $(\mathfrak{A}, \bar{\mu})$ is **σ-finite** if there is a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\bar{\mu} a_n < \infty$ for every $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} a_n = 1$. Note that in this case $\langle a_n \rangle_{n\in\mathbb{N}}$ can be taken *either* to be non-decreasing (consider $a_n' = \sup_{i<n} a_i$) *or* to be disjoint (consider $a_n'' = a_n \setminus a_n'$).

(d) $(\mathfrak{A}, \bar{\mu})$ is **semi-finite** if whenever $a \in \mathfrak{A}$ and $\bar{\mu}a = \infty$ there is a non-zero $b \subseteq a$ such that $\bar{\mu}b < \infty$.

(e) $(\mathfrak{A}, \bar{\mu})$ is **localizable** if it is semi-finite and the Boolean algebra $\mathfrak{A}$ is Dedekind complete.

322B The first step is to relate these concepts to the corresponding ones for measure spaces.

Theorem Let (X, Σ, μ) be a measure space, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Then
(a) (X, Σ, μ) is a probability space iff $(\mathfrak{A}, \bar{\mu})$ is a probability algebra;
(b) (X, Σ, μ) is totally finite iff $(\mathfrak{A}, \bar{\mu})$ is;
(c) (X, Σ, μ) is σ-finite iff $(\mathfrak{A}, \bar{\mu})$ is;
(d) (X, Σ, μ) is semi-finite iff $(\mathfrak{A}, \bar{\mu})$ is;
(e) (X, Σ, μ) is localizable iff $(\mathfrak{A}, \bar{\mu})$ is;
(f) if $E \in \Sigma$, then E is an atom for μ iff $E^\bullet$ is an atom in $\mathfrak{A}$;
(g) (X, Σ, μ) is atomless iff $\mathfrak{A}$ is;
(h) (X, Σ, μ) is purely atomic iff $\mathfrak{A}$ is.

proof (a), (b) are trivial, since $\bar{\mu}1 = \mu X$.

(c)(i) If μ is σ-finite, let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a sequence of sets of finite measure covering X; then $\bar{\mu}E_n^\bullet < \infty$ for every n, and

$$\sup_{n\in\mathbb{N}} E_n^\bullet = (\bigcup_{n\in\mathbb{N}} E_n)^\bullet = 1,$$

so $(\mathfrak{A}, \bar{\mu})$ is σ-finite.

(ii) If $(\mathfrak{A}, \bar{\mu})$ is σ-finite, let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\bar{\mu}a_n < \infty$ for every n and $\sup_{n\in\mathbb{N}} a_n = 1$. For each n, choose $E_n \in \Sigma$ such that $E_n^\bullet = a_n$. Set $E = \bigcup_{n\in\mathbb{N}} E_n$; then $E^\bullet = \sup_{n\in\mathbb{N}} a_n = 1$, so E is conegligible. Now $(X \setminus E, E_0, E_1, \dots)$ is a sequence of sets of finite measure covering X, so μ is σ-finite.

(d)(i) Suppose that μ is semi-finite and that $a \in \mathfrak{A}$, $\bar{\mu}a = \infty$. Then there is an $E \in \Sigma$ such that $E^\bullet = a$, so that $\mu E = \bar{\mu}a = \infty$. As μ is semi-finite, there is an $F \in \Sigma$ such that $F \subseteq E$ and $0 < \mu F < \infty$. Set $b = F^\bullet$; then $b \subseteq a$ and $0 < \bar{\mu}b < \infty$.

(ii) Suppose that $(\mathfrak{A}, \bar{\mu})$ is semi-finite and that $E \in \Sigma$, $\mu E = \infty$. Then $\bar{\mu}E^\bullet = \infty$, so there is a $b \subseteq E^\bullet$ such that $0 < \bar{\mu}b < \infty$. Let $F \in \Sigma$ be such that $F^\bullet = b$. Then $F \cap E \in \Sigma$, $F \cap E \subseteq E$ and $(F \cap E)^\bullet = E^\bullet \cap b = b$, so that $\mu(F \cap E) = \bar{\mu}b \in]0, \infty[$.

(e)(i) Note first that if $\mathcal{E} \subseteq \Sigma$ and $F \in \Sigma$, then

$$\begin{aligned} &E \setminus F \text{ is negligible for every } E \in \mathcal{E} \\ &\iff E^\bullet \setminus F^\bullet = 0 \text{ for every } E \in \mathcal{E} \\ &\iff F^\bullet \text{ is an upper bound for } \{E^\bullet : E \in \mathcal{E}\}. \end{aligned}$$

So if $\mathcal{E} \subseteq \Sigma$ and $H \in \Sigma$, then H is an essential supremum of $\mathcal{E}$ in Σ, in the sense of 211G, iff $H^\bullet$ is the supremum of $A = \{E^\bullet : E \in \mathcal{E}\}$ in $\mathfrak{A}$. **P** Writing $\mathcal{F}$ for

$$\{F : F \in \Sigma,\ E \setminus F \text{ is negligible for every } E \in \mathcal{E}\},$$

we see that $B = \{F^\bullet : F \in \mathcal{F}\}$ is just the set of upper bounds of A, and that H is an essential supremum of $\mathcal{E}$ iff $H \in \mathcal{F}$ and $H^\bullet$ is a lower bound for B; that is, iff $H^\bullet = \sup A$. **Q**

(ii) Thus $\mathfrak{A}$ is Dedekind complete iff every family in Σ has an essential supremum in Σ. Since we already know that $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff μ is, we see that $(\mathfrak{A}, \bar{\mu})$ is localizable iff μ is.

(f) This is immediate from the definitions in 211I and 316K, if we remember always that $\{b : b \subseteq E^\bullet\} = \{F^\bullet : F \in \Sigma,\ F \subseteq E\}$ (312Lb).

(g), (h) follow at once from (f).

322C I copy out the relevant parts of Theorem 211L in the new context.

Theorem (a) A probability algebra is totally finite.
(b) A totally finite measure algebra is σ-finite.
(c) A σ-finite measure algebra is localizable.

(d) A localizable measure algebra is semi-finite.

proof All except (c) are trivial; and (c) may be deduced from 211Lc-211Ld, 322Bc, 322Be and 321J, or from 316Fa and 322G below.

322D Of course not all the definitions in §211 are directly relevant to measure algebras. The concepts of 'complete', 'locally determined' and 'strictly localizable' measure space do not correspond in any direct way to properties of the measure algebras. Indeed, completeness is just irrelevant, as the next proposition shows.

Proposition Let (X,Σ,μ) be a measure space, with completion $(X,\hat{\Sigma},\hat{\mu})$ and c.l.d. version $(X,\tilde{\Sigma},\tilde{\mu})$ (213E). Write $(\mathfrak{A},\bar{\mu})$, $(\mathfrak{A}_1,\bar{\mu}_1)$ and $(\mathfrak{A}_2,\bar{\mu}_2)$ for the measure algebras of μ, $\hat{\mu}$ and $\tilde{\mu}$ respectively.

(a) The embedding $\Sigma \subsetneqq \hat{\Sigma}$ corresponds to an isomorphism between $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{A}_1,\bar{\mu}_1)$.

(b)(i) The embedding $\Sigma \subsetneqq \tilde{\Sigma}$ defines an order-continuous Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}_2$. Setting $\mathfrak{A}^f=\{a: a\in\mathfrak{A},\ \bar{\mu}a<\infty\}$, $\pi\restriction\mathfrak{A}^f$ is a measure-preserving bijection between $\mathfrak{A}^f$ and $\mathfrak{A}_2^f=\{c:c\in\mathfrak{A}_2,\ \bar{\mu}_2c<\infty\}$.

(ii) π is injective iff μ is semi-finite, and in this case $\bar{\mu}_2(\pi a)=\bar{\mu}a$ for every $a\in\mathfrak{A}$.

(iii) If μ is localizable, π is a bijection.

proof For $E\in\Sigma$, I write E° for its image in $\mathfrak{A}$; for $F\in\hat{\Sigma}$, I write F^* for its image in $\mathfrak{A}_1$; and for $G\in\tilde{\Sigma}$, I write $G^\bullet$ for its image in $\mathfrak{A}_2$.

(a) This is nearly trivial. The map $E\mapsto E^*:\Sigma\to\mathfrak{A}_1$ is a Boolean homomorphism, being the composition of the Boolean homomorphisms $E\mapsto E:\Sigma\to\hat{\Sigma}$ and $F\mapsto F^*:\hat{\Sigma}\to\mathfrak{A}_1$. Its kernel is $\{E:E\in\Sigma,\ \hat{\mu}E=0\}=\{E:E\in\Sigma,\ \mu E=0\}$, so it induces an injective Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{A}_1$ given by the formula $\phi(E^\circ)=E^*$ for every $E\in\Sigma$ (312F, 3A2G). To see that ϕ is surjective, take any $b\in\mathfrak{A}_1$. There is an $F\in\hat{\Sigma}$ such that $F^*=b$, and there is an $E\in\Sigma$ such that $E\subseteq F$ and $\hat{\mu}(F\setminus E)=0$, so that

$$\pi(E^\circ)=E^*=F^*=b.$$

Thus π is a Boolean algebra isomorphism. It is a measure algebra isomorphism because for any $E\in\Sigma$

$$\bar{\mu}_1\phi(E^\circ)=\bar{\mu}_1E^*=\hat{\mu}E=\mu E=\bar{\mu}E^\circ.$$

(b)(i) The map $E\mapsto E^\bullet:\Sigma\to\mathfrak{A}_2$ is a Boolean homomorphism with kernel $\{E:E\in\Sigma,\ \tilde{\mu}E=0\}\supseteq\{E:E\in\Sigma,\ \mu E=0\}$, so induces a Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}_2$, defined by saying that $\pi E^\circ=E^\bullet$ for every $E\in\Sigma$.

If $a\in\mathfrak{A}^f$, it is expressible as E° where $\mu E<\infty$. Then $\tilde{\mu}E=\mu E$ (213Fa), so $\pi a=E^\bullet$ belongs to $\mathfrak{A}_2^f$, and $\bar{\mu}_2(\pi a)=\bar{\mu}a$. If a, a' are distinct members of $\mathfrak{A}^f$, then

$$\bar{\mu}_2(\pi a\,\triangle\,\pi a')=\bar{\mu}_2\pi(a\,\triangle\,a')=\bar{\mu}(a\,\triangle\,a')>0,$$

so $\pi a\ne\pi a'$; thus $\pi\restriction\mathfrak{A}^f$ is an injective map from $\mathfrak{A}^f$ to $\mathfrak{A}_2^f$. If $c\in\mathfrak{A}_2^f$, then $c=G^\bullet$ where $\tilde{\mu}G<\infty$; by 213Fc, there is an $E\in\Sigma$ such that $E\subseteq G$, $\mu E=\tilde{\mu}G$ and $\tilde{\mu}(G\setminus E)=0$, so that $E^\circ\in\mathfrak{A}^f$ and

$$\pi E^\circ=E^\bullet=G^\bullet=c.$$

As c is arbitrary, $\phi[\mathfrak{A}^f]=\mathfrak{A}_2^f$.

Finally, π is order-continuous. **P** Let $A\subseteq\mathfrak{A}$ be a non-empty downwards-directed set with infimum 0, and $b\in\mathfrak{A}_2$ a lower bound for $\pi[A]$. **?** If $b\ne0$, then (because $(\mathfrak{A}_2,\bar{\mu}_2)$ is semi-finite) there is a $b_0\in\mathfrak{A}_2^f$ such that $0\ne b_0\subseteq b$. Let $a_0\in\mathfrak{A}$ be such that $\pi a_0=b_0$. Then $a_0\ne0$, so there is an $a\in A$ such that $a\not\supseteq a_0$, that is, $a\cap a_0\ne a_0$. But now, because $\pi\restriction\mathfrak{A}^f$ is injective,

$$b_0=\pi a_0\ne\pi(a\cap a_0)=\pi a\cap\pi a_0=\pi a\cap b_0,$$

and $b_0\not\subseteq\pi a$, which is impossible. **X** Thus $b=0$, and 0 is the only lower bound of $\pi[A]$. As A is arbitrary, π is order-continuous (313L(b-ii)). **Q**

(ii) (α) If μ is semi-finite, then $\tilde{\mu}E=\mu E$ for every $E\in\Sigma$ (213Hc), so

$$\bar{\mu}_2(\pi E^\circ)=\bar{\mu}_2E^\bullet=\tilde{\mu}E=\mu E=\bar{\mu}E^\circ$$

for every $E\in\Sigma$. In particular,

$$\pi a=0\Longrightarrow 0=\bar{\mu}_2(\pi a)=\bar{\mu}a\Longrightarrow a=0,$$

so π is injective. (β) If μ is not semi-finite, there is an $E\in\Sigma$ such that $\mu E=\infty$ but $\mu H=0$ whenever $H\in\Sigma$, $H\subseteq E$ and $\mu H<\infty$; so that $\tilde{\mu}E=0$ and

$$E^\circ \neq 0, \quad \pi E^\circ = E^\bullet = 0.$$

So in this case π is not injective.

(iii) Now suppose that μ is localizable. Then for every $G \in \tilde{\Sigma}$ there is an $E \in \Sigma$ such that $\tilde{\mu}(E \triangle G) = 0$, by 213Hb; accordingly $\pi E^\circ = E^\bullet = G^\bullet$. As G is arbitrary, π is surjective; and we know from (ii) that π is injective, so it is a bijection, as claimed.

322E Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff it has a partition of unity consisting of elements of finite measure.

(b) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite, $a = \sup\{b : b \subseteq a,\, \bar{\mu}b < \infty\}$ and $\bar{\mu}a = \sup\{\bar{\mu}b : b \subseteq a,\, \bar{\mu}b < \infty\}$ for every $a \in \mathfrak{A}$.

proof Set $\mathfrak{A}^f = \{b : b \in \mathfrak{A},\, \bar{\mu}b < \infty\}$.

(a)(i) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite, then $\mathfrak{A}^f$ is order-dense in $\mathfrak{A}$, so there is a partition of unity consisting of members of $\mathfrak{A}^f$ (313K).

(ii) If there is a partition of unity $C \subseteq \mathfrak{A}^f$, and $\bar{\mu}a = \infty$, then there is a $c \in C$ such that $a \cap c \neq 0$, and now $a \cap c \subseteq a$ and $0 < \bar{\mu}(a \cap c) < \infty$; as a is arbitrary, $(\mathfrak{A}, \bar{\mu})$ is semi-finite.

(b) Of course $\mathfrak{A}^f$ is upwards-directed, by 321Bc, and we are supposing that its supremum is 1. If $a \in \mathfrak{A}$, then

$$B = \{b : b \in \mathfrak{A}^f,\, b \subseteq a\} = \{a \cap b : b \in \mathfrak{A}^f\}$$

is upwards-directed and has supremum a (313Ba), so $\bar{\mu}a = \sup_{b \in B} \bar{\mu}b$, by 321D.

Remark Compare 213A.

322F Proposition If $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra, then $\mathfrak{A}$ is a weakly (σ, ∞)-distributive Boolean algebra.

proof Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence of non-empty downwards-directed subsets of $\mathfrak{A}$, all with infimum 0. Set

$$B = \{b : \text{for every } n \in \mathbb{N} \text{ there is an } a \in A_n \text{ such that } b \supseteq a\}.$$

If $c \in \mathfrak{A} \setminus \{0\}$, let $c' \subseteq c$ be such that $0 < \bar{\mu}c' < \infty$. For each $n \in \mathbb{N}$, $\inf_{a \in A_n} \bar{\mu}(c' \cap a) = 0$, by 321F; so we may choose $a_n \in A_n$ such that $\bar{\mu}(c' \cap a_n) \leq 2^{-n-2}\bar{\mu}b$. Set $b = \sup_{n \in \mathbb{N}} a_n \in B$. Then

$$\bar{\mu}(c' \cap b) \leq \textstyle\sum_{n=0}^\infty \bar{\mu}(c' \cap a_n) < \bar{\mu}c',$$

so $c' \not\subseteq b$ and $c \not\subseteq b$. As c is arbitrary, $\inf B = 0$; as $\langle A_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ, ∞)-distributive (316G).

322G Corresponding to 215B, we have the following description of σ-finite algebras.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra. Then the following are equiveridical:

(i) $(\mathfrak{A}, \bar{\mu})$ is σ-finite;

(ii) $\mathfrak{A}$ is ccc;

(iii) *either* $\mathfrak{A} = \{0\}$ *or* there is a functional $\bar{\nu} : \mathfrak{A} \to [0, 1]$ such that $(\mathfrak{A}, \bar{\nu})$ is a probability algebra.

proof (i)$\Leftrightarrow$(ii) By 321J, it is enough to consider the case in which $(\mathfrak{A}, \bar{\mu})$ is the measure algebra of a measure space (X, Σ, μ), and μ is semi-finite, by 322Bd. We know that $\mathfrak{A}$ is ccc iff there is no uncountable disjoint set in $\Sigma \setminus \mathcal{N}$, where $\mathcal{N}$ is the null ideal of μ (316D). But 215B(iii) shows that this is equivalent to μ being σ-finite, which is equivalent to $(\mathfrak{A}, \bar{\mu})$ being σ-finite, by 322Bc.

(i)$\Rightarrow$(iii) If $(\mathfrak{A}, \bar{\mu})$ is σ-finite, and $\mathfrak{A} \neq \{0\}$, let $\langle a_n \rangle_{n \in \mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$ such that $\bar{\mu}a_n < \infty$ for every n and $\sup_{n \in \mathbb{N}} a_n = 1$. Then $\bar{\mu}a_n > 0$ for some n, so there are $\gamma_n > 0$ such that $\sum_{n=0}^\infty \gamma_n \bar{\mu}a_n = 1$. (Set $\gamma'_n = 2^{-n}/(1 + \bar{\mu}a_n)$, $\gamma_n = \gamma'_n / (\sum_{i=0}^\infty \gamma'_i \bar{\mu}a_i)$.) Set $\bar{\nu}a = \sum_{n=0}^\infty \gamma_n \bar{\mu}(a \cap a_n)$ for every $a \in \mathfrak{A}$; it is easy to check that $(\mathfrak{A}, \bar{\nu})$ is a probability algebra.

(iii)$\Rightarrow$(i) is a consequence of (i)$\Leftrightarrow$(ii).

322H Principal ideals If $(\mathfrak{A}, \bar{\mu})$ is a measure algebra and $a \in \mathfrak{A}$, then it is easy to see (using 314Eb) that $(\mathfrak{A}_a, \bar{\mu} \restriction \mathfrak{A}_a)$ is a measure algebra, where $\mathfrak{A}_a$ is the principal ideal of $\mathfrak{A}$ generated by a.

322I Subspace measures General subspace measures give rise to complications in the measure algebra (see 322Xf, 322Yd). But subspaces with measurable envelopes (132D, 213L) are manageable.

Proposition Let (X,Σ,μ) be a measure space, and $A\subseteq X$ a set with a measurable envelope E. Let μ_A be the subspace measure on A, and Σ_A its domain; let $(\mathfrak{A},\bar\mu)$ be the measure algebra of (X,Σ,μ) and $(\mathfrak{A}_A,\bar\mu_A)$ the measure algebra of (A,Σ_A,μ_A). Set $a=E^\bullet$ and let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Then we have an isomorphism between $(\mathfrak{A}_a,\bar\mu\restriction\mathfrak{A}_a)$ and $(\mathfrak{A}_A,\bar\mu_A)$ given by the formula

$$F^\bullet\mapsto(F\cap A)^\circ$$

whenever $F\in\Sigma$ and $F\subseteq E$, writing $F^\bullet$ for the equivalence class of F in $\mathfrak{A}$ and $(F\cap A)^\circ$ for the equivalence class of $F\cap A$ in $\mathfrak{A}_A$.

proof Set $\Sigma_E=\{E\cap F:F\in\Sigma\}$. For F, $G\in\Sigma_E$,

$$F^\bullet=G^\bullet\iff\mu(F\triangle G)=0\iff\mu_A(A\cap(F\triangle G))=0\iff(F\cap A)^\circ=(G\cap A)^\circ,$$

because E is a measurable envelope of A. Accordingly the given formula defines an injective function from the image $\{F^\bullet:F\in\Sigma_E\}$ of Σ_E in $\mathfrak{A}$ to $\mathfrak{A}_A$; but this image is just the principal ideal $\mathfrak{A}_a$. It is easy to check that the map is a Boolean homomorphism from $\mathfrak{A}_a$ to $\mathfrak{A}_A$, and it is a Boolean isomorphism because $\Sigma_A=\{F\cap A:F\in\Sigma_E\}$. Finally, it is measure-preserving because

$$\bar\mu F^\bullet=\mu F=\mu^*(F\cap A)=\mu_A(F\cap A)=\bar\mu_A(F\cap A)^\circ$$

for every $F\in\Sigma_E$, again using the fact that E is a measurable envelope of A.

322J Corollary Let (X,Σ,μ) be a measure space, with measure algebra $(\mathfrak{A},\bar\mu)$.

(a) If $E\in\Sigma$, then the measure algebra of the subspace measure μ_E can be identified with the principal ideal $\mathfrak{A}_{E^\bullet}$ of $\mathfrak{A}$.

(b) If $A\subseteq X$ is a set of full outer measure (in particular, if $\mu^*A=\mu X<\infty$), then the measure algebra of the subspace measure μ_A can be identified with $\mathfrak{A}$.

322K Indefinite-integral measures: Proposition Let (X,Σ,μ) be a measure space and ν an indefinite-integral measure over μ (234J). Then the measure algebra of ν can be identified, as Boolean algebra, with a principal ideal of the measure algebra of μ.

proof Taking $(X,\hat\Sigma,\hat\mu)$ to be the completion of (X,Σ,μ), then we can identify the measure algebras of μ and $\hat\mu$, by 322Da; and ν is still an indefinite-integral measure over $\hat\mu$, just because μ and $\hat\mu$ give rise to the same theory of integration (212Fb). Now there is a $G\in\hat\Sigma$ such that the domain T of ν is $\{E:E\subseteq X,\ E\cap G\in\hat\Sigma\}$ and the null ideal $\mathcal{N}_\nu$ of ν is $\{A:A\subseteq X,\ A\cap G\in\mathcal{N}_\mu\}$, where $\mathcal{N}_\mu$ is the null ideal of μ or $\hat\mu$ (234Lc[1], 212Eb). Writing $\mathfrak{A}$ for the measure algebra of $\hat\mu$, $c=G^\bullet\in\mathfrak{A}$, and $\mathfrak{A}_c$ for the principal ideal of $\mathfrak{A}$ generated by c, we have a Boolean homomorphism $E\mapsto(E\cap G)^\bullet:\mathrm{T}\to\mathfrak{A}_c$ with kernel $\mathcal{N}_\nu$. So, writing $E^\circ\in\mathfrak{B}$ for the equivalence class of $E\in\mathrm{T}$, we have an injective Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}_c$ defined by setting $\pi E^\circ=(E\cap G)^\bullet$ for every $E\in\mathrm{T}$. Of course

$$\pi[\mathfrak{B}]\supseteq\{(E\cap G)^\bullet:E\in\hat\Sigma\}=\{a\cap c:a\in\mathfrak{A}\}=\mathfrak{A}_c,$$

so π is actually an isomorphism, as required.

322L Simple products (a) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be an indexed family of measure algebras. Let $\mathfrak{A}$ be the simple product Boolean algebra $\prod_{i\in I}\mathfrak{A}_i$ (315A), and for $a\in\mathfrak{A}$ set $\bar\mu a=\sum_{i\in I}\bar\mu_i a(i)$. Then it is easy to check (using 315D(e-ii)) that $(\mathfrak{A},\bar\mu)$ is a measure algebra; I will call it the **simple product** of the family $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$. Each of the $\mathfrak{A}_i$ corresponds to a principal ideal $\mathfrak{A}_{e_i}$ say in $\mathfrak{A}$, where $e_i\in\mathfrak{A}$ corresponds to $1_{\mathfrak{A}_i}\in\mathfrak{A}_i$ (315E), and the Boolean isomorphism between $\mathfrak{A}_i$ and $\mathfrak{A}_{e_i}$ is a measure algebra isomorphism between $(\mathfrak{A}_i,\bar\mu_i)$ and $(\mathfrak{A}_{e_i},\bar\mu\restriction\mathfrak{A}_{e_i})$.

(b) If $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ is a family of measure spaces, with direct sum (X,Σ,μ) (214L), then the measure algebra $(\mathfrak{A},\bar\mu)$ of (X,Σ,μ) can be identified with the simple product of the measure algebras $(\mathfrak{A}_i,\bar\mu_i)$ of the (X_i,Σ_i,μ_i). **P** If, as in 214L, we set $X=\{(x,i):i\in I,\ x\in X_i\}$, and for $E\subseteq X$, $i\in I$ we set $E_i=\{x:(x,i)\in E\}$, then the Boolean isomorphism $E\mapsto\langle E_i\rangle_{i\in I}:\Sigma\to\prod_{i\in I}\Sigma_i$ induces a Boolean isomorphism from $\mathfrak{A}$ to $\prod_{i\in I}\mathfrak{A}_i$, which is also a measure algebra isomorphism, because

$$\bar\mu E^\bullet=\mu E=\sum_{i\in I}\mu_iE_i=\sum_{i\in I}\bar\mu_iE_i^\bullet$$

for every $E\in\Sigma$. **Q**

[1] Formerly 234D.

(c) A simple product of measure algebras is semi-finite, or localizable, or atomless, or purely atomic, iff every factor is. (Compare 214Kb.)

(d) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra.

(i) If $\langle e_i\rangle_{i\in I}$ is any partition of unity in $\mathfrak{A}$, then $(\mathfrak{A},\bar{\mu})$ is isomorphic to the product $\prod_{i\in I}(\mathfrak{A}_{e_i},\bar{\mu}\restriction\mathfrak{A}_{e_i})$ of the corresponding principal ideals. **P** By 315F(iii), the map $a\mapsto\langle a\cap e_i\rangle_{i\in I}$ is a Boolean isomorphism between $\mathfrak{A}$ and $\prod_{i\in I}\mathfrak{A}_i$. Because $\langle e_i\rangle_{i\in I}$ is disjoint and $a=\sup_{i\in I}a\cap e_i$, $\bar{\mu}a=\sum_{i\in I}\bar{\mu}(a\cap e_i)$ (321E), for every $a\in\mathfrak{A}$. So $a\mapsto\langle a\cap e_i\rangle_{i\in I}$ is a measure algebra isomorphism between $(\mathfrak{A},\bar{\mu})$ and $\prod_{i\in I}(\mathfrak{A}_i,\bar{\mu}\restriction\mathfrak{A}_{e_i})$. **Q**

(ii) In particular, since $\mathfrak{A}$ has a partition of unity consisting of elements of finite measure (322Ea), $(\mathfrak{A},\bar{\mu})$ is isomorphic to a simple product of totally finite measure algebras. Each of these is isomorphic to the measure algebra of a totally finite measure space, so $(\mathfrak{A},\bar{\mu})$ is isomorphic to the measure algebra of a direct sum of totally finite measure spaces, which is strictly localizable.

Thus every localizable measure algebra is isomorphic to the measure algebra of a strictly localizable measure space. (See also 322O below.)

***322M Strictly localizable spaces** The following fact is occasionally useful.

Proposition Let (X,Σ,μ) be a strictly localizable measure space with $\mu X>0$, and $(\mathfrak{A},\bar{\mu})$ its measure algebra. If $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, there is a partition $\langle X_i\rangle_{i\in I}$ of X into members of Σ such that $X_i^\bullet=a_i$ for every $i\in I$ and

$$\Sigma=\{E:E\subseteq X,\,E\cap X_i\in\Sigma\;\forall\,i\in I\},$$

$$\mu E=\textstyle\sum_{i\in I}\mu(E\cap X_i)\text{ for every }E\in\Sigma;$$

that is, the isomorphism between $\mathfrak{A}$ and the simple product $\prod_{i\in I}\mathfrak{A}_{a_i}$ of its principal ideals (315F) corresponds to an isomorphism between (X,Σ,μ) and the direct sum of the subspace measures on X_i.

proof (a) Suppose to begin with that $\mu X<\infty$. In this case $J=\{i:a_i\ne 0\}$ must be countable (322G). For each $i\in J$, choose $E_i\in\Sigma$ such that $E_i^\bullet=a_i$, and set $F_i=E_i\setminus\bigcup_{j\in J,j\ne i}E_j$; then $F_i^\bullet=a_i$ for each $i\in J$, and $\langle F_i\rangle_{i\in J}$ is disjoint. Because $\mu X>0$, J is non-empty; fix some $j_0\in J$ and set

$$\begin{aligned}X_i&=F_{j_0}\cup(X\setminus\bigcup_{j\in J}F_j)\text{ if }i=j_0,\\&=F_i\text{ for }i\in J\setminus\{j_0\},\\&=\emptyset\text{ for }i\in I\setminus J.\end{aligned}$$

Then $\langle X_i\rangle_{i\in I}$ is a disjoint family in Σ, $\bigcup_{i\in I}X_i=X$ and $X_i^\bullet=a_i$ for every i. Moreover, because only countably many of the X_i are non-empty, we certainly have

$$\Sigma=\{E:E\subseteq X,\,E\cap X_i\in\Sigma\;\forall\,i\in I\},$$

$$\mu E=\textstyle\sum_{i\in I}\mu(E\cap X_i)\text{ for every }E\in\Sigma.$$

(b) For the general case, start by taking a decomposition $\langle Y_j\rangle_{j\in J}$ of X. We can suppose that no Y_j is negligible, because there is certainly some j_0 such that $\mu Y_{j_0}>0$, and we can if necessary replace Y_{j_0} by $Y_{j_0}\cup\bigcup\{Y_j:\mu Y_j=0\}$. For each j, we can identify the measure algebra of the subspace measure on Y_j with the principal ideal $\mathfrak{A}_{b_j}$ generated by $b_j=Y_j^\bullet$ (322I). Now $\langle a_i\cap b_j\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}_{b_j}$, so by (a) just above we can find a disjoint family $\langle X_{ji}\rangle_{i\in I}$ in Σ such that $\bigcup_{i\in I}X_{ji}=Y_j$, $X_{ji}^\bullet=a_i\cap b_j$ for every i and

$$\Sigma\cap\mathcal{P}Y_j=\{E:E\subseteq Y_j,\,E\cap X_{ji}\in\Sigma\;\forall\,i\in I\},$$

$$\mu E=\textstyle\sum_{i\in I}\mu(E\cap X_{ji})\text{ for every }E\in\Sigma\cap\mathcal{P}Y_j.$$

Set $X_i=\bigcup_{j\in I}X_{ji}$ for every $i\in I$. Then $\langle X_i\rangle_{i\in I}$ is a partition of X. Because $X_i\cap Y_j=X_{ji}$ is measurable for every j, $X_i\in\Sigma$. Because $X_i^\bullet\supseteq a_i\cap b_j$ for every j, and $\langle b_j\rangle_{j\in J}$ is a partition of unity in $\mathfrak{A}$ (322Lb), $X_i^\bullet\supseteq a_i$ for each i; because $\langle X_i^\bullet\rangle_{i\in I}$ is disjoint and $\sup_{i\in I}a_i=1$, $X_i^\bullet=a_i$ for every i. If $E\subseteq X$ is such that $E\cap X_i\in\Sigma$ for every i, then $E\cap X_{ji}\in\Sigma$ for all $i\in I$ and $j\in J$, so $E\cap Y_j\in\Sigma$ for every $j\in J$ and $E\in\Sigma$. If $E\in\Sigma$, then

$$\mu E = \sum_{j\in J}\mu(E\cap Y_j) = \sum_{j\in J}\sum_{i\in I}\mu(E\cap X_{ji})$$
$$= \sum_{i\in I}\sum_{j\in J}\mu(E\cap X_i\cap Y_j) = \sum_{i\in I}\mu(E\cap X_i).$$

Thus $\langle X_i\rangle_{i\in I}$ is a suitable family.

322N Subalgebras: Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Set $\bar\nu = \bar\mu\restriction\mathfrak{B}$.
(a) $(\mathfrak{B},\bar\nu)$ is a measure algebra.
(b) If $(\mathfrak{A},\bar\mu)$ is totally finite, or a probability algebra, so is $(\mathfrak{B},\bar\nu)$.
(c) If $(\mathfrak{A},\bar\mu)$ is σ-finite and $(\mathfrak{B},\bar\nu)$ is semi-finite, then $(\mathfrak{B},\bar\nu)$ is σ-finite.
(d) If $(\mathfrak{A},\bar\mu)$ is localizable and $\mathfrak{B}$ is order-closed and $(\mathfrak{B},\bar\nu)$ is semi-finite, then $(\mathfrak{B},\bar\nu)$ is localizable.
(e) If $(\mathfrak{B},\bar\nu)$ is a probability algebra, or totally finite, or σ-finite, so is $(\mathfrak{A},\bar\mu)$.

proof (a) By 314Eb, $\mathfrak{B}$ is Dedekind σ-complete, and the identity map $\pi:\mathfrak{B}\to\mathfrak{A}$ is sequentially order-continuous; so that $\bar\nu = \bar\mu\pi$ will be countably additive and $(\mathfrak{B},\bar\nu)$ will be a measure algebra.

(b) This is trivial.

(c) Use 322G. Every disjoint subset of $\mathfrak{B}$ is disjoint in $\mathfrak{A}$, therefore countable, because $\mathfrak{A}$ is ccc; so $\mathfrak{B}$ also is ccc and $(\mathfrak{B},\bar\nu)$ (being semi-finite) is σ-finite.

(d) By 314Ea, $\mathfrak{B}$ is Dedekind complete; we are supposing that $(\mathfrak{B},\bar\nu)$ is semi-finite, so it is localizable.

(e) This is elementary.

322O The Stone space of a localizable measure algebra I said above that the concepts 'strictly localizable' and 'locally determined' measure space have no equivalents in the theory of measure algebras. But when we look at the canonical measure on the Stone space of a measure algebra, we can of course hope that properties of the measure algebra will be reflected in the properties of this measure, as happens in the next theorem.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, Z the Stone space of $\mathfrak{A}$, and ν the standard measure on Z constructed by the method of 321J-321K. Then the following are equiveridical:
(i) $(\mathfrak{A},\bar\mu)$ is localizable;
(ii) ν is localizable;
(iii) ν is locally determined;
(iv) ν is strictly localizable.

proof Write Σ for the domain of ν, that is,

$$\{E\triangle A : E\subseteq Z \text{ is open-and-closed, } A\subseteq Z \text{ is meager}\},$$

and $\mathcal{M}$ for the ideal of meager subsets of Z, that is, the null ideal of ν (314M, 321K). Then $a\mapsto\widehat{a}^\bullet:\mathfrak{A}\to\Sigma/\mathcal{M}$ is an isomorphism between $(\mathfrak{A},\bar\mu)$ and the measure algebra of (Z,Σ,ν) (314M). Note that because any subset of a meager set is meager, ν is surely complete.

(a)(i)$\Leftrightarrow$(ii) is a consequence of 322Be.

(b)(ii)$\Rightarrow$(iii) Suppose that ν is localizable. Of course it is semi-finite. Let $V\subseteq Z$ be a set such that $V\cap E\in\Sigma$ whenever $E\in\Sigma$ and $\nu E<\infty$. Because ν is localizable, there is a $W\in\Sigma$ which is an essential supremum in Σ of $\{V\cap E : E\in\Sigma, \nu E<\infty\}$, that is, $W^\bullet = \sup\{(V\cap E)^\bullet : \nu E<\infty\}$ in $\Sigma/\mathcal{M}$. I claim that $W\triangle V$ is nowhere dense. **P** Let $G\subseteq Z$ be a non-empty open set. Then there is a non-zero $a\in\mathfrak{A}$ such that $\widehat{a}\subseteq G$. Because $(\mathfrak{A},\bar\mu)$ is semi-finite, we may suppose that $\bar\mu a<\infty$. Now

$$(W\cap\widehat{a})^\bullet = W^\bullet\cap\widehat{a}^\bullet = \sup\nolimits_{\nu E<\infty}(V\cap E)^\bullet\cap\widehat{a}^\bullet = \sup\nolimits_{\nu E<\infty}(V\cap E\cap\widehat{a})^\bullet = (V\cap\widehat{a})^\bullet,$$

so $(W\triangle V)\cap\widehat{a}$ is negligible, therefore meager. But we know that $\mathfrak{A}$ is weakly (σ,∞)-distributive (322F), so that meager sets in Z are nowhere dense (316I), and there is a non-empty open set $H\subseteq\widehat{a}\setminus(W\triangle V)$. Now $H\subseteq G\setminus\overline{W\triangle V}$. As G is arbitrary, $\operatorname{int}\overline{W\triangle V}=\emptyset$ and $W\triangle V$ is nowhere dense. **Q**

But this means that $W\triangle V\in\mathcal{M}\subseteq\Sigma$ and $V = W\triangle(W\triangle V)\in\Sigma$. As V is arbitrary, ν is locally determined.

(c)(iii)⇒(iv) Assume that ν is locally determined. Because $(\mathfrak{A}, \bar{\mu})$ is semi-finite, there is a partition of unity $C \subseteq \mathfrak{A}$ consisting of elements of finite measure (322Ea). Set $\mathcal{C} = \{\widehat{c} : c \in C\}$. This is a disjoint family of sets of finite measure for ν. Now suppose that $F \in \Sigma$ and $\nu F > 0$. Then there is an open-and-closed set $E \subseteq Z$ such that $F \triangle E$ is meager, and E is of the form $\widehat{a}$ for some $a \in \mathfrak{A}$. Since

$$\bar{\mu}a = \nu\widehat{a} = \nu F > 0,$$

there is some $c \in C$ such that $a \cap c \neq 0$, and now

$$\nu(F \cap \widehat{c}) = \bar{\mu}(a \cap c) > 0.$$

This means that ν satisfies the conditions of 213O and must be strictly localizable.

(d)(iv)⇒(ii) This is just 211Ld.

322P Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and let $\widehat{\mathfrak{A}}$ be the Dedekind completion of $\mathfrak{A}$ (314U). Then there is a unique extension of $\bar{\mu}$ to a functional $\tilde{\mu}$ on $\widehat{\mathfrak{A}}$ such that $(\widehat{\mathfrak{A}}, \tilde{\mu})$ is a localizable measure algebra. The embedding $\mathfrak{A} \subsetneq\!\!\!\to \widehat{\mathfrak{A}}$ identifies the ideals $\{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\}$ and $\{a : a \in \widehat{\mathfrak{A}},\ \tilde{\mu}a < \infty\}$.

proof (I write the argument out as if $\mathfrak{A}$ were actually a subalgebra of $\widehat{\mathfrak{A}}$.) For $c \in \widehat{\mathfrak{A}}$, set

$$\tilde{\mu}c = \sup\{\bar{\mu}a : a \in \mathfrak{A},\ a \subseteq c\}.$$

Evidently $\tilde{\mu}$ is a function from $\widehat{\mathfrak{A}}$ to $[0, \infty]$ extending $\bar{\mu}$, so $\tilde{\mu}0 = 0$. Because $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$, $\tilde{\mu}c > 0$ whenever $c \neq 0$, because any such c includes a non-zero member of $\mathfrak{A}$. If $\langle c_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\widehat{\mathfrak{A}}$ with supremum c, then $\tilde{\mu}c = \sum_{n=0}^{\infty} \tilde{\mu}c_n$. **P** Let A be the set of all members of $\mathfrak{A}$ expressible as $a = \sup_{n \in \mathbb{N}} a_n$ where $a_n \in \mathfrak{A}$ and $a_n \subseteq c_n$ for every $n \in \mathbb{N}$. Now

$$\begin{aligned}\sup_{a \in A} \bar{\mu}a &= \sup\{\sum_{n=0}^{\infty} \bar{\mu}a_n : a_n \in \mathfrak{A},\ a_n \subseteq c_n \text{ for every } n \in \mathbb{N}\} \\ &= \sum_{n=0}^{\infty} \sup\{\bar{\mu}a_n : a_n \subseteq c_n\} = \sum_{n=0}^{\infty} \tilde{\mu}c_n.\end{aligned}$$

Also, because $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$, $c_n = \sup\{a : a \in \mathfrak{A},\ a \subseteq c_n\}$ for each n, and $\sup A$, taken in $\widehat{\mathfrak{A}}$, must be c. But this means that if $a' \in \mathfrak{A}$ and $a' \subseteq c$ then $a' = \sup_{a \in A} a' \cap a$ in $\widehat{\mathfrak{A}}$ and therefore also in $\mathfrak{A}$; so that

$$\bar{\mu}a' = \sup_{a \in A} \bar{\mu}(a' \cap a) \le \sup_{a \in A} \bar{\mu}a.$$

Accordingly

$$\tilde{\mu}c = \sup_{a \in A} \bar{\mu}a = \sum_{n=0}^{\infty} \tilde{\mu}c_n. \quad \mathbf{Q}$$

This shows that $(\widehat{\mathfrak{A}}, \tilde{\mu})$ is a measure algebra. It is semi-finite because $(\mathfrak{A}, \bar{\mu})$ is and every non-zero element of $\widehat{\mathfrak{A}}$ includes a non-zero element of $\mathfrak{A}$, which in turn includes a non-zero element of finite measure. Since $\widehat{\mathfrak{A}}$ is Dedekind complete, $(\widehat{\mathfrak{A}}, \bar{\mu})$ is localizable.

If $\bar{\mu}a$ is finite, then surely $\tilde{\mu}a = \bar{\mu}a$ is finite. If $\tilde{\mu}c$ is finite, then $A = \{a : a \in \mathfrak{A},\ a \subseteq c\}$ is upwards-directed and $\sup_{a \in A} \bar{\mu}a = \tilde{\mu}c$ is finite, so $b = \sup A$ is defined in $\mathfrak{A}$ and $\bar{\mu}b = \tilde{\mu}c$. Because $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$, $b = c$ (313K, 313O) and $c \in \mathfrak{A}$, with $\bar{\mu}c = \tilde{\mu}c$.

322Q Definition Let $(\mathfrak{A}, \bar{\mu})$ be any semi-finite measure algebra. I will call $(\widehat{\mathfrak{A}}, \tilde{\mu})$, as constructed above, the **localization** of $(\mathfrak{A}, \bar{\mu})$. Of course it is unique just in so far as the Dedekind completion of $\mathfrak{A}$ is.

322R Further properties of Stone spaces: Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra and (Z, Σ, ν) its Stone space.

(a) Meager sets in Z are nowhere dense; every $E \in \Sigma$ is uniquely expressible as $G \triangle M$ where $G \subseteq Z$ is open-and-closed and M is nowhere dense, and $\nu E = \sup\{\nu H : H \subseteq E \text{ is open-and-closed}\}$.

(b) The c.l.d. version $\tilde{\nu}$ of ν is strictly localizable, and has the same negligible sets as ν.

(c) If $(\mathfrak{A}, \bar{\mu})$ is totally finite then $\nu E = \inf\{\nu H : H \supseteq E \text{ is open-and-closed}\}$ for every $E \in \Sigma$.

proof (a) I have already remarked (in the proof of 322O) that $\mathfrak{A}$ is weakly (σ, ∞)-distributive, so that meager sets in Z are nowhere dense. But we know that every member of Σ is expressible as $G \triangle M$ where G is open-and-closed and M is meager, therefore nowhere dense. Moreover, the expression is unique, because if $G\triangle M = G'\triangle M'$ then $G\triangle G' \subseteq M \cup M'$ is open and nowhere dense, therefore empty, so $G = G'$ and $M = M'$.

Now let $a \in \mathfrak{A}$ be such that $\widehat{a} = G$, and consider $B = \{b : b \in \mathfrak{A}, \widehat{b} \subseteq E\}$. Then $\sup B = a$ in $\mathfrak{A}$. **P** If $b \in B$, then $\widehat{b} \setminus \widehat{a} \subseteq M$ is nowhere dense, therefore empty; so a is an upper bound for B. **?** If a is not the supremum of B, then there is a non-zero $c \subseteq a$ such that $b \subseteq a \setminus c$ for every $b \in B$. But now $\widehat{c}$ cannot be empty, so $\widehat{c} \setminus \overline{M}$ is non-empty, and there is a non-zero $d \in \mathfrak{A}$ such that $\widehat{d} \subseteq \widehat{c} \setminus \overline{M}$. In this case $d \in B$ and $d \not\subseteq a \setminus c$. **X** Thus $a = \sup B$. **Q**

It follows that

$$\begin{aligned}\nu E = \nu G = \bar{\mu} a &= \sup_{b\in B} \bar{\mu} b \\ &= \sup_{b\in B} \nu\widehat{b} \le \sup\{\nu H : H \subseteq E \text{ is open-and-closed}\} \le \nu E\end{aligned}$$

and $\nu E = \sup\{\nu H : H \subseteq E \text{ is open-and-closed}\}$.

(b) This is the same as part (c) of the proof of 322O. We have a disjoint family $\mathcal{C}$ of sets of finite measure for ν such that whenever $E \in \Sigma$ and $\nu E > 0$ there is a $C \in \mathcal{C}$ such that $\mu(C \cap E) > 0$. Now if $\tilde{\nu} F$ is defined and not 0, there is an $E \in \Sigma$ such that $E \subseteq F$ and $\nu E > 0$ (213Fc), so that there is a $C \in \mathcal{C}$ such that $\nu(E \cap C) > 0$; since $\nu C < \infty$, we have

$$\tilde{\nu}(F \cap C) \ge \tilde{\nu}(E \cap C) = \nu(E \cap C) > 0.$$

And of course $\tilde{\nu} C < \infty$ for every $C \in \mathcal{C}$. This means that $\mathcal{C}$ witnesses that $\tilde{\nu}$ satisfies the conditions of 213O, so that $\tilde{\nu}$ is strictly localizable.

Any ν-negligible set is surely $\tilde{\nu}$-negligible. If M is $\tilde{\nu}$-negligible then it is nowhere dense. **P** If $G \subseteq Z$ is open and not empty then there is a non-empty open-and-closed set $H_1 \subseteq G$, and now $H_1 \in \Sigma$, so there is a non-empty open-and-closed set $H \subseteq H_1$ such that νH is finite (because ν is semi-finite). In this case $H \cap M$ is ν-negligible, therefore nowhere dense, and $H \not\subseteq \overline{M}$. But this means that $G \not\subseteq \overline{M}$; as G is arbitrary, M is nowhere dense. **Q** Accordingly $M \in \mathcal{M}$ and is ν-negligible.

Thus ν and $\tilde{\nu}$ have the same negligible sets.

(c) Because $\nu Z < \infty$,

$$\begin{aligned}\nu E = \nu Z - \nu(Z \setminus E) &= \nu Z - \sup\{\nu H : H \subseteq Z \setminus E \text{ is open-and-closed}\} \\ &= \inf\{\nu(Z \setminus H) : H \subseteq Z \setminus E \text{ is open-and-closed}\} \\ &= \inf\{\nu H : H \supseteq E \text{ is open-and-closed}\}.\end{aligned}$$

322X Basic exercises >(a) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Let I_∞ be the set of those $a \in \mathfrak{A}$ which are either 0 or 'purely infinite', that is, $\bar{\mu} b = \infty$ for every non-zero $b \subseteq a$. Show that I_∞ is a σ-ideal of $\mathfrak{A}$. Show that there is a function $\bar{\mu}_{sf} : \mathfrak{A}/I_\infty \to [0, \infty]$ defined by setting $\bar{\mu}_{sf} a^\bullet = \sup\{\bar{\mu} b : b \subseteq a, \bar{\mu} b < \infty\}$ for every $a \in \mathfrak{A}$. Show that $(\mathfrak{A}/I_\infty, \bar{\mu}_{sf})$ is a semi-finite measure algebra.

(b) Let (X, Σ, μ) be a measure space and let μ_{sf} be the 'semi-finite version' of μ, as defined in 213Xc. Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of (X, Σ, μ). Show that the measure algebra of (X, Σ, μ_{sf}) is isomorphic to the measure algebra $(\mathfrak{A}/I_\infty, \bar{\mu}_{sf})$ of (a) above.

(c) Let (X, Σ, μ) be a measure space and $(X, \tilde{\Sigma}, \tilde{\mu})$ its c.l.d. version. Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be the corresponding measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{A}_2$ the canonical homomorphism, as in 322Db. Show that the kernel of π is the ideal I_∞, as described in 322Xa, so that $\mathfrak{A}/I_\infty$ is isomorphic, as Boolean algebra, to $\pi[\mathfrak{A}] \subseteq \mathfrak{A}_2$. Show that this isomorphism identifies $\bar{\mu}_{sf}$, as described in 322Xa, with $\bar{\mu}_2 \restriction \pi[\mathfrak{A}]$.

(d) Give a direct proof of 322G, not relying on 215B and 321J.

>(e) Let $(\mathfrak{A}, \bar{\mu})$ be any measure algebra, A a non-empty subset of $\mathfrak{A}$, and $c \in \mathfrak{A}$ such that $\bar{\mu} c < \infty$. Show that (i) $c_0 = \sup\{a \cap c : a \in A\}$ is defined in $\mathfrak{A}$ (ii) there is a countable set $B \subseteq A$ such that $c_0 = \sup\{a \cap c : a \in B\}$.

(f) Let (X, Σ, μ) be a measure space and A any subset of X; let μ_A be the subspace measure on A and Σ_A its domain. Write $(\mathfrak{A}, \bar{\mu})$ for the measure algebra of (X, Σ, μ) and $(\mathfrak{A}_A, \bar{\mu}_A)$ for the measure algebra of (A, Σ_A, μ_A). Show that the formula $F^\bullet \mapsto (F \cap A)^\bullet$ defines a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}_A$ which has kernel $I = \{F^\bullet : F \in \Sigma, F \cap A = \emptyset\}$. Show that for any $a \in \mathfrak{A}$, $\bar{\mu}_A(\pi a) = \min\{\bar{\mu} b : b \in \mathfrak{A}, a \setminus b \in I\}$.

(g) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and $\mathfrak{B}$ a regularly embedded σ-subalgebra of $\mathfrak{A}$. Suppose that $(\mathfrak{B},\bar{\mu}\restriction\mathfrak{B})$ is semi-finite. Show that $(\mathfrak{A},\bar{\mu})$ is semi-finite.

(h) Let $(\mathfrak{A},\bar{\mu})$ be any measure algebra and (Z,Σ,ν) its Stone space. Show that the c.l.d. version of ν is strictly localizable.

322Y Further exercises (a) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ. Set $\mathcal{N}=\{N:\exists F\in\mathcal{I},\,N\subseteq F\}$. Show that $\mathcal{N}$ is a σ-ideal of subsets of X. Set $\hat{\Sigma}=\{E\triangle N:E\in\Sigma,\,N\in\mathcal{N}\}$. Show that $\hat{\Sigma}$ is a σ-algebra of subsets of X and that $\hat{\Sigma}/\mathcal{N}$ is isomorphic to $\Sigma/\mathcal{I}$.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and (Z,Σ,ν) its Stone space. Let $\tilde{\nu}$ be the c.l.d. version of ν, and $\tilde{\Sigma}$ its domain. Show that $\tilde{\Sigma}$ is precisely the Baire-property algebra $\{G\triangle A:G\subseteq Z$ is open, $A\subseteq Z$ is meager$\}$, so that $\tilde{\Sigma}/\mathcal{M}$ can be identified with the regular open algebra of Z (314Yd) and the measure algebra of $\tilde{\nu}$ can be identified with the localization of $\mathfrak{A}$.

(c) Give an example of a localizable measure algebra $(\mathfrak{A},\bar{\mu})$ with a σ-subalgebra $\mathfrak{B}$ such that $(\mathfrak{B},\bar{\mu}\restriction\mathfrak{B})$ is semi-finite and atomless, but $\mathfrak{A}$ has an atom.

(d) Let (X,Σ,μ) be a measure space and $A\subseteq X$ a subset; let μ_A be the subspace measure on A, $\mathfrak{A}$ and $\mathfrak{A}_A$ the measure algebras of μ and μ_A, and $\pi:\mathfrak{A}\to\mathfrak{A}_A$ the canonical homomorphism, as described in 322Xf. (i) Show that if μ_A is semi-finite, then π is order-continuous. (ii) Show that if μ is semi-finite but μ_A is not, then π is not order-continuous.

(e) Show that if $(\mathfrak{A},\bar{\mu})$ is a semi-finite measure algebra, with Stone space (Z,Σ,ν), then ν has locally determined negligible sets in the sense of 213I.

(f) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra and (Z,Σ,ν) its Stone space. (i) Show that a function $f:Z\to\mathbb{R}$ is Σ-measurable iff there is a conegligible set $G\subseteq X$ such that $f\restriction G$ is continuous. (*Hint*: 316Yi.) (ii) Show that $f:Z\to[0,1]$ is Σ-measurable iff there is a continuous function $g:Z\to[0,1]$ such that $f=g$ ν-a.e.

322 Notes and comments I have taken this leisurely tour through the concepts of Chapter 21 partly to recall them (or persuade you to look them up) and partly to give you practice in the elementary manipulations of measure algebras. The really vital result here is the correspondence between 'localizability' in measure spaces and measure algebras. Part of the object of this volume (particularly in Chapter 36) is to try to make sense of the properties of localizable measure spaces, as discussed in Chapter 24 and elsewhere, in terms of their measure algebras. I hope that 322Be has already persuaded you that the concept really belongs to measure algebras, and that the formulation in terms of 'essential suprema' is a dispensable expedient.

I have given proofs of 322C and 322G depending on the realization of an arbitrary measure algebra as the measure algebra of a measure space, and the corresponding theorems for measure spaces, because this seems the natural approach from where we presently stand; but I am sympathetic to the view that such proofs must be inappropriate, and that it is in some sense better style to look for arguments which speak only of measure algebras (322Xd).

For any measure algebra $(\mathfrak{A},\bar{\mu})$, the set $\mathfrak{A}^f$ of elements of finite measure is an ideal of $\mathfrak{A}$; consequently it is order-dense iff it includes a partition of unity (322E). In 322F we have something deeper: any semi-finite measure algebra must be weakly (σ,∞)-distributive when regarded as a Boolean algebra, and this has significant consequences in its Stone space, which are used in the proofs of 322O and 322R. Of course a result of this kind must depend on the semi-finiteness of the measure algebra, since any Dedekind σ-complete Boolean algebra becomes a measure algebra if we give every non-zero element the measure ∞. It is natural to look for algebraic conditions on a Boolean algebra sufficient to make it 'measurable', in the sense that it should carry a semi-finite measure; this is an unresolved problem to which I will return in Chapter 39.

Subspace measures, indefinite-integral measures, simple products, direct sums, principal ideals and order-closed subalgebras give no real surprises; I spell out the details in 322H-322N and 322Xf-322Xg. It is worth noting that completing a measure space has no effect on its measure algebra (322D, 322Ya). We see also that from the point of view of measure algebras there is no distinction to be made between 'localizable' and 'strictly localizable', since every localizable measure algebra is representable as the measure algebra of a strictly localizable measure space (322Ld). (But strict localizability does have implications for some processes starting in the measure algebra; see 322M.) It is nevertheless remarkable that the canonical measure on the Stone space of a semi-finite measure algebra is localizable iff it is strictly localizable (322O). This canonical measure has many other interesting properties, which I skim over in 322R, 322Xh, 322Yb and 322Yf. In Chapter 21 I discussed a number of methods of improving measure spaces, notably 'completions' (212C) and 'c.l.d. versions' (213E). Neither of these is applicable in any general way to measure algebras. But in fact we have a more effective construction, at least for semi-finite measure algebras, that of 'localization' (322P-322Q); I say that it is more effective just because localizability is more important than completeness or local determinedness, being of vital importance in the behaviour of function spaces (241Gb, 243Gb, 245Ec, 363M, 364M, 365M, 367M, 369A, 369C). Note that the localization of a semi-finite measure algebra does in fact correspond to the c.l.d. version of a certain measure (322Yb). But of course $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ do *not* have the same Stone spaces, even when $\widehat{\mathfrak{A}}$ can be effectively represented as the measure algebra of a measure on the Stone space of $\mathfrak{A}$. What is happening in 322Yb is that we are using all the open sets of Z to represent members of $\widehat{\mathfrak{A}}$, not just the open-and-closed sets, which correspond to members of $\mathfrak{A}$.

323 The topology of a measure algebra

I take a short section to discuss one of the fundamental tools for studying totally finite measure algebras, the natural metric that each carries. The same ideas, suitably adapted, can be applied to an arbitrary measure algebra, where we have a topology corresponding closely to the topology of convergence in measure on the function space L^0. Most of the section consists of an analysis of the relations between this topology and the order structure of the measure algebra.

323A The pseudometrics ρ_a **(a)** Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Write $\mathfrak{A}^f=\{a:a\in\mathfrak{A},\,\bar\mu a<\infty\}$. For $a\in\mathfrak{A}^f$ and b, $c\in\mathfrak{A}$, write $\rho_a(b,c)=\bar\mu(a\cap(b\triangle c))$. Then ρ_a is a pseudometric on $\mathfrak{A}$. **P** (i) Because $\bar\mu a<\infty$, ρ_a takes values in $[0,\infty[$. (ii) If b, c, $d\in\mathfrak{A}$ then $b\triangle d\subseteq(b\triangle c)\cup(c\triangle d)$, so

$$\begin{aligned}\rho_a(b,d)=\bar\mu(a\cap(b\triangle d))&\le\bar\mu((a\cap(b\triangle c))\cup(a\cap(c\triangle d)))\\&\le\bar\mu(a\cap(b\triangle c))+\bar\mu(a\cap(c\triangle d))=\rho_a(b,c)+\rho_a(c,d).\end{aligned}$$

(iii) If b, $c\in\mathfrak{A}$ then

$$\rho_a(b,c)=\bar\mu(a\cap(b\triangle c))=\bar\mu(a\cap(c\triangle b))=\rho_a(c,b).\ \mathbf{Q}$$

(b) Now the **measure-algebra topology** of the measure algebra $(\mathfrak{A},\bar\mu)$ is that generated by the family $\mathrm{P}=\{\rho_a:a\in\mathfrak{A}^f\}$ of pseudometrics on $\mathfrak{A}$. Similarly the **measure-algebra uniformity** on $\mathfrak{A}$ is that generated by P. For the rest of this section I will take it that every measure algebra is endowed with its measure-algebra topology and uniformity.

(For a general discussion of topologies defined by pseudometrics, see 2A3F *et seq.* For the associated uniformities see §3A4.)

(c) Note that P is upwards-directed, since $\rho_{a\cup a'}\ge\max(\rho_a,\rho_{a'})$ for all a, $a'\in\mathfrak{A}^f$.

(d) On the ideal $\mathfrak{A}^f$ we have an actual metric ρ defined by saying that $\rho(a,b)=\bar\mu(a\triangle b)$ for a, $b\in\mathfrak{A}^f$ (to see that ρ is a metric, repeat the formulae of (a) above); this is the **measure metric** or **Fréchet-Nikodým metric**. I will call the topology it generates the **strong measure-algebra topology** on $\mathfrak{A}^f$.

When $\bar\mu$ is totally finite, that is, $\mathfrak{A}^f=\mathfrak{A}$, $\rho=\rho_1$ defines the measure-algebra topology and uniformity of $\mathfrak{A}$.

323B Proposition Let $(\mathfrak{A},\bar\mu)$ be any measure algebra, and give $\mathfrak{A}$ its measure-algebra topology.

(a) The operations $\cup$, $\cap$, $\setminus$ and $\triangle$ are all uniformly continuous.

(b) $\mathfrak{A}^f$ is dense in $\mathfrak{A}$.

proof (a) The point is that for any b, c, b', $c'\in\mathfrak{A}$ we have

$$(b*c)\triangle(b'*c')\subseteq(b\triangle b')\cup(c\triangle c')$$

for any of the operations $*=\cup,\ \cap$ etc.; so that if $a\in\mathfrak{A}^f$ then

$$\rho_a(b*c,b'*c')\le\rho_a(b,b')+\rho_a(c,c').$$

Consequently the operation $*$ must be uniformly continuous.

(b) Given $b\in\mathfrak{A}$, $a\in\mathfrak{A}^f$ and $\epsilon>0$, then $a\cap b\in\mathfrak{A}^f$ and $\rho_a(b,a\cap b)=0$. Because the family $\{\rho_a:a\in\mathfrak{A}^f\}$ is upwards-directed, this is enough to show that every neighbourhood of b meets $\mathfrak{A}^f$; as b is arbitrary, $\mathfrak{A}^f$ is dense.

323C Proposition (a) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra. Then $\bar\mu:\mathfrak{A}\to[0,\infty[$ is uniformly continuous.
(b) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Then $\bar\mu:\mathfrak{A}\to[0,\infty]$ is lower semi-continuous.
(c) Let $(\mathfrak{A},\bar\mu)$ be any measure algebra. If $a\in\mathfrak{A}$ and $\bar\mu a<\infty$, then $b\mapsto\bar\mu(b\cap a):\mathfrak{A}\to\mathbb{R}$ is uniformly continuous.

proof (a) For any a, $b\in\mathfrak{A}$,

$$|\bar\mu a-\bar\mu b|\le\bar\mu(a\triangle b)=\rho_1(a,b).$$

(b) Suppose that $b\in\mathfrak{A}$ and $\bar\mu b>\alpha\in\mathbb{R}$. Then there is an $a\subseteq b$ such that $\alpha<\bar\mu a<\infty$ (322Eb). If $c\in\mathfrak{A}$ is such that $\rho_a(b,c)<\bar\mu a-\alpha$, then

$$\bar\mu c\ge\bar\mu(a\cap c)=\bar\mu a-\bar\mu(a\cap(b\setminus c))>\alpha.$$

Thus $\{b:\bar\mu b>\alpha\}$ is open; as α is arbitrary, $\bar\mu$ is lower semi-continuous.

(c) $|\bar\mu(a\cap b)-\bar\mu(a\cap c)|\le\rho_a(b,c)$ for all b, $c\in\mathfrak{A}$.

323D The following facts are basic to any understanding of the relationship between the order structure and topology of a measure algebra.

Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.
(a) Let $B\subseteq\mathfrak{A}$ be a non-empty upwards-directed set. For $b\in B$ set $F_b=\{c:b\subseteq c\in B\}$.
(i) $\{F_b:b\in B\}$ generates a Cauchy filter $\mathcal{F}(B\uparrow)$ on $\mathfrak{A}$.
(ii) If $\sup B$ is defined in $\mathfrak{A}$, then it is a topological limit of $\mathcal{F}(B\uparrow)$; in particular, it belongs to the topological closure of B.
(b) Let $B\subseteq\mathfrak{A}$ be a non-empty downwards-directed set. For $b\in B$ set $F'_b=\{c:b\supseteq c\in B\}$.
(i) $\{F'_b:b\in B\}$ generates a Cauchy filter $\mathcal{F}(B\downarrow)$ on $\mathfrak{A}$.
(ii) If $\inf B$ is defined in $\mathfrak{A}$, then it is a topological limit of $\mathcal{F}(B\downarrow)$; in particular, it belongs to the topological closure of B.
(c)(i) Closed subsets of $\mathfrak{A}$ are order-closed in the sense of 313Da.
(ii) An order-dense subalgebra of $\mathfrak{A}$ must be dense in the topological sense.
(d) Now suppose that $(\mathfrak{A},\bar\mu)$ is semi-finite.
(i) The sets $\{b:b\subseteq c\}$, $\{b:b\supseteq c\}$ are closed for every $c\in\mathfrak{A}$.
(ii) If $B\subseteq\mathfrak{A}$ is non-empty and upwards-directed and e is a cluster point of $\mathcal{F}(B\uparrow)$, then $e=\sup B$.
(iii) If $B\subseteq\mathfrak{A}$ is non-empty and downwards-directed and e is a cluster point of $\mathcal{F}(B\downarrow)$, then $e=\inf B$.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A.

(a)(i) (α) If b, $c\in B$ then there is a $d\in B$ such that $b\cup c\subseteq d$, so that $F_d\subseteq F_b\cap F_c$; consequently

$$\mathcal{F}(B\uparrow)=\{F:F\subseteq\mathfrak{A},\ \exists\,b\in B,\ F_b\subseteq F\}$$

is a filter on $\mathfrak{A}$. (β) Let $a\in\mathfrak{A}^f$, $\epsilon>0$. Then there is a $b\in B$ such that $\bar\mu(a\cap c)\le\bar\mu(a\cap b)+\frac{1}{2}\epsilon$ for every $c\in B$, and $F_b\in\mathcal{F}(B\uparrow)$. If now c, $c'\in F_b$, $c\triangle c'\subseteq(c\setminus b)\cup(c'\setminus b)$, so

$$\rho_a(c,c')\le\bar\mu(a\cap c\setminus b)+\bar\mu(a\cap c'\setminus b)=\bar\mu(a\cap c)+\bar\mu(a\cap c')-2\bar\mu(a\cap b)\le\epsilon.$$

As a and ϵ are arbitrary, $\mathcal{F}(B\uparrow)$ is Cauchy.

(ii) Suppose that $e=\sup B$ is defined in $\mathfrak{A}$. Let $a\in\mathfrak{A}^f$, $\epsilon>0$. By 313Ba, $a\cap e=\sup_{b\in B}a\cap b$; but $\{a\cap b:b\in B\}$ is upwards-directed, so $\bar\mu(a\cap e)=\sup_{b\in B}\bar\mu(a\cap b)$, by 321D. Let $b\in B$ be such that $\bar\mu(a\cap b)\ge\bar\mu(a\cap e)-\epsilon$. Then for any $c\in F_b$, $e\triangle c\subseteq e\setminus b$, so

$$\rho_a(e,c)=\bar\mu(a\cap(e\triangle c))\le\bar\mu(a\cap(e\setminus b))=\bar\mu(a\cap e)-\bar\mu(a\cap b)\le\epsilon.$$

As a and ϵ are arbitrary, $\mathcal{F}(B\uparrow)\to e$.

Because $B\in\mathcal{F}(B\uparrow)$, e surely belongs to the topological closure of B.

(b) Either repeat the arguments above, with appropriate inversions, using 321F in place of 321D, or apply (a) to the set $\{1\setminus b:b\in B\}$.

(c)(i) This follows at once from (a) and (b) and the definition in 313Da.

(ii) If $\mathfrak{B}\subseteq\mathfrak{A}$ is an order-dense subalgebra and $a\in\mathfrak{A}$, then $B=\{b:b\in\mathfrak{B},\ b\subseteq a\}$ is upwards-directed and has supremum a (313K); by (a-ii), $a\in\overline{B}\subseteq\overline{\mathfrak{B}}$. As a is arbitrary, $\mathfrak{B}$ is topologically dense.

(d)(i) Set $F = \{b : b \subseteq c\}$. If $d \in \mathfrak{A} \setminus F$, then (because $(\mathfrak{A}, \bar{\mu})$ is semi-finite) there is an $a \in \mathfrak{A}^f$ such that $\delta = \bar{\mu}(a \cap d \setminus c) > 0$; now if $b \in F$,

$$\rho_a(d, b) \ge \bar{\mu}(a \cap d \setminus b) \ge \delta,$$

so that d cannot belong to the closure of F. As d is arbitrary, F is closed. Similarly, $\{b : b \supseteq c\}$ is closed.

(ii) (α) If $b \in B$, then $e \in \overline{F_b}$, because $F_b \in \mathcal{F}(B\uparrow)$; but $\{c : b \subseteq c\}$ is a closed set including F_b, so contains e, and $b \subseteq e$. As b is arbitrary, e is an upper bound for B. (β) If d is an upper bound of B, then $\{c : c \subseteq d\}$ is a closed set belonging to $\mathcal{F}(B\uparrow)$, so contains e. As d is arbitrary, this shows that e is the supremum of B, as claimed.

(iii) Use the same arguments as in (ii), but inverted.

323E Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) If $\langle b_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum b, then $\langle b_n \rangle_{n \in \mathbb{N}}$ converges topologically to b.

(b) If $\langle b_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum b, then $\langle b_n \rangle_{n \in \mathbb{N}}$ converges topologically to b.

proof I call this a 'corollary' because it is the special case of 323Da-323Db in which B is the set of terms of a monotonic sequence; but it is probably easier to work directly from the definition in 323A, and use 321Be or 321Bf to see that $\lim_{n\to\infty} \rho_a(b_n, b) = 0$ whenever $\bar{\mu}a < \infty$.

323F The following is a useful calculation.

Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $\langle c_n \rangle_{n \in \mathbb{N}}$ a sequence in $\mathfrak{A}$ such that the sum $\sum_{n=0}^{\infty} \bar{\mu}(c_n \triangle c_{n+1})$ is finite. Set $d_0 = \sup_{n \in \mathbb{N}} \inf_{m \ge n} c_m$, $d_1 = \inf_{n \in \mathbb{N}} \sup_{m \ge n} c_m$. Then $d_0 = d_1$ and, writing d for their common value, $\lim_{n\to\infty} \bar{\mu}(c_n \triangle d) = 0$.

proof Write $\alpha_n = \bar{\mu}(c_n \triangle c_{n+1})$, $\beta_n = \sum_{k=n}^{\infty} \alpha_k$ for $n \in \mathbb{N}$; we are supposing that $\lim_{n\to\infty} \beta_n = 0$. Set $b_n = \sup_{m \ge n} c_m \triangle c_{m+1}$; then

$$\bar{\mu}b_n \le \sum_{m=n}^{\infty} \bar{\mu}(c_m \triangle c_{m+1}) = \beta_n$$

for each n. If $m \ge n$, then

$$c_m \triangle c_n \subseteq \sup_{n \le k < m} c_k \triangle c_{k+1} \subseteq b_n,$$

so

$$c_n \setminus b_n \subseteq c_m \subseteq c_n \cup b_n.$$

Consequently

$$c_n \setminus b_n \subseteq \inf_{k \ge m} c_k \subseteq \sup_{k \ge m} c_k \subseteq c_n \cup b_n$$

for every $m \ge n$, and

$$c_n \setminus b_n \subseteq d_0 \subseteq d_1 \subseteq c_n \cup b_n,$$

so that

$$c_n \triangle d_0 \subseteq b_n, \quad c_n \triangle d_1 \subseteq b_n, \quad d_1 \setminus d_0 \subseteq b_n.$$

As this is true for every n,

$$\lim_{n\to\infty} \bar{\mu}(c_n \triangle d_i) \le \lim_{n\to\infty} \bar{\mu}b_n = 0$$

for both i, and

$$\bar{\mu}(d_1 \triangle d_0) \le \inf_{n \in \mathbb{N}} \bar{\mu}b_n = 0,$$

so that $d_1 = d_0$.

323G The classification of measure algebras: Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, $\mathfrak{T}$ its measure-algebra topology and $\mathcal{U}$ its measure-algebra uniformity.

(a) $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff $\mathfrak{T}$ is Hausdorff.

(b) $(\mathfrak{A}, \bar{\mu})$ is σ-finite iff $\mathfrak{T}$ is metrizable, and in this case $\mathcal{U}$ also is metrizable.

(c) $(\mathfrak{A}, \bar{\mu})$ is localizable iff $\mathfrak{T}$ is Hausdorff and $\mathfrak{A}$ is complete under $\mathcal{U}$.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A.

(a)(i) Suppose that $(\mathfrak{A}, \bar{\mu})$ is semi-finite and that b, c are distinct members of $\mathfrak{A}$. Then there is an $a \subseteq b \mathbin{\triangle} c$ such that $0 < \bar{\mu}a < \infty$, and now $\rho_a(b,c) > 0$. As b and c are arbitrary, $\mathfrak{T}$ is Hausdorff (2A3L).

(ii) Suppose that $\mathfrak{T}$ is Hausdorff and that $b \in \mathfrak{A}$ has $\bar{\mu}b = \infty$. Then $b \neq 0$ so there must be an $a \in \mathfrak{A}^f$ such that $\bar{\mu}(a \cap b) = \rho_a(0,b) > 0$; in which case $a \cap b \subseteq b$ and $0 < \bar{\mu}(a \cap b) < \infty$. As b is arbitrary, $\bar{\mu}$ is semi-finite.

(b)(i) Suppose that $\bar{\mu}$ is σ-finite. Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}^f$ with supremum 1. Set

$$\rho(b,c) = \sum_{n=0}^{\infty} \frac{\rho_{a_n}(b,c)}{1 + 2^n \bar{\mu}a_n}$$

for b, $c \in \mathfrak{A}$. Then ρ is a metric on $\mathfrak{A}$, because if $\rho(b,c) = 0$ then $a_n \cap (b \mathbin{\triangle} c) = 0$ for every n, so $b \mathbin{\triangle} c = 0$ and $b = c$.

If $a \in \mathfrak{A}^f$ and $\epsilon > 0$, take n such that $\bar{\mu}(a \setminus a_n) \le \frac{1}{2}\epsilon$. If b, $c \in \mathfrak{A}$ and $\rho(b,c) \le \epsilon/2(1 + 2^n\bar{\mu}a_n)$, then

$$\begin{aligned}\rho_a(b,c) &= \rho_{a \setminus a_n}(b,c) + \rho_{a \cap a_n}(b,c) \le \bar{\mu}(a \setminus a_n) + \rho_{a_n}(b,c)\\ &\le \frac{1}{2}\epsilon + (1 + 2^n\bar{\mu}a_n)\rho(b,c) \le \epsilon.\end{aligned}$$

In the other direction, given $\epsilon > 0$, take $n \in \mathbb{N}$ such that $2^{-n} \le \frac{1}{2}\epsilon$; then $\rho(b,c) \le \epsilon$ whenever $\rho_{a_n}(b,c) \le \epsilon/2(n+1)$.

This shows that $\mathcal{U}$ is the same as the metrizable uniformity defined by $\{\rho\}$; accordingly $\mathfrak{T}$ also is defined by ρ.

(ii) Now suppose that $\mathfrak{T}$ is metrizable, and let ρ be a metric defining $\mathfrak{T}$. For each $n \in \mathbb{N}$ there must be $a_{n0}, \dots, a_{nk_n} \in \mathfrak{A}^f$ and $\delta_n > 0$ such that

$$\rho_{a_{ni}}(b,1) \le \delta_n \text{ for every } i \le k_n \Longrightarrow \rho(b,1) \le 2^{-n}.$$

Set $d = \sup_{n\in\mathbb{N}, i\le k_n} a_{ni}$. Then $\rho_{a_{ni}}(d,1) = 0$ for every n and i, so $\rho(d,1) \le 2^{-n}$ for every n and $d = 1$. Thus 1 is the supremum of countably many elements of finite measure and $(\mathfrak{A}, \bar{\mu})$ is σ-finite.

(c)(i) Suppose that $(\mathfrak{A}, \bar{\mu})$ is localizable. Then $\mathfrak{T}$ is Hausdorff, by (a). Let $\mathcal{F}$ be a Cauchy filter on $\mathfrak{A}$. For each $a \in \mathfrak{A}^f$, choose a sequence $\langle F_n(a) \rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\rho_a(b,c) \le 2^{-n}$ whenever b, $c \in F_n(a)$ and $n \in \mathbb{N}$. Choose $c_{an} \in \bigcap_{k\le n} F_k(a)$ for each n; then $\rho_a(c_{an}, c_{a,n+1}) \le 2^{-n}$ for each n. Set $d_a = \sup_{n\in\mathbb{N}} \inf_{k\ge n} a \cap c_{ak}$. Then

$$\lim_{n\to\infty} \rho_a(d_a, c_{an}) = \lim_{n\to\infty} \bar{\mu}(d_a \mathbin{\triangle} (a \cap c_{an})) = 0,$$

by 323F.

If a, $b \in \mathfrak{A}^f$ and $a \subseteq b$, then $d_a = a \cap d_b$. **P** For each $n \in \mathbb{N}$, $F_n(a)$ and $F_n(b)$ both belong to $\mathcal{F}$, so must have a point e in common; now

$$\begin{aligned}\rho_a(d_a, d_b) &\le \rho_a(d_a, c_{an}) + \rho_a(c_{an}, e) + \rho_a(e, c_{bn}) + \rho_a(c_{bn}, d_b)\\ &\le \rho_a(d_a, c_{an}) + \rho_a(c_{an}, e) + \rho_b(e, c_{bn}) + \rho_b(c_{bn}, d_b)\\ &\le \rho_a(d_a, c_{an}) + 2^{-n} + 2^{-n} + \rho_b(c_{bn}, d_b)\\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Consequently $\rho_a(d_a, d_b) = 0$, that is,

$$d_a = a \cap d_a = a \cap d_b. \textbf{ Q}$$

Set $d = \sup\{d_b : b \in \mathfrak{A}^f\}$; this is defined because $\mathfrak{A}$ is Dedekind complete. Then $\mathcal{F} \to d$. **P** If $a \in \mathfrak{A}^f$ and $\epsilon > 0$, then

$$a \cap d = \sup_{b\in\mathfrak{A}^f} a \cap d_b = \sup_{b\in\mathfrak{A}^f} a \cap b \cap d_{a\cup b} = \sup_{b\in\mathfrak{A}^f} a \cap b \cap d_a = a \cap d_a.$$

So if we choose $n \in \mathbb{N}$ such that $2^{-n} + \rho_a(c_{an}, d_a) \le \epsilon$, then for any $e \in F_n(a)$ we shall have

$$\rho_a(e,d) \le \rho_a(e, c_{an}) + \rho_a(c_{an}, d) \le 2^{-n} + \rho_a(c_{an}, d_a) \le \epsilon.$$

Thus

$$\{e : \rho_a(d,e) \le \epsilon\} \supseteq F_n(a) \in \mathcal{F}.$$

As a, ϵ are arbitrary, $\mathcal{F}$ converges to d. **Q** As $\mathcal{F}$ is arbitrary, $\mathfrak{A}$ is complete.

(ii) Now suppose that $\mathfrak{T}$ is Hausdorff and that $\mathfrak{A}$ is complete under $\mathcal{U}$. By (a), $(\mathfrak{A}, \bar{\mu})$ is semi-finite. Let B be any non-empty subset of $\mathfrak{A}$, and set $B' = \{b_0 \cup \dots \cup b_n : b_0, \dots, b_n \in B\}$, so that B' is upwards-directed and has the same upper bounds as B. By 323Da, we have a Cauchy filter $\mathcal{F}(B'\uparrow)$; because $\mathfrak{A}$ is complete, this is convergent; and because $(\mathfrak{A},\bar\mu)$ is semi-finite, its limit must be $\sup B'=\sup B$, by 323Dd. As B is arbitrary, $\mathfrak{A}$ is Dedekind complete, so $(\mathfrak{A},\bar\mu)$ is localizable.

323H Closed subalgebras The ideas used in the proof of (c) above have many other applications, of which one of the most important is the following. You may find it helpful to read the next theorem first on the assumption that $(\mathfrak{A},\bar\mu)$ is a probability algebra.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Then it is topologically closed iff it is order-closed.

proof (a) If $\mathfrak{B}$ is closed, it must be order-closed, by 323Dc.

(b) Now suppose that $\mathfrak{B}$ is order-closed. I repeat the ideas of part (c-i) of the proof of 323G. Let e be any member of the closure of $\mathfrak{B}$ in $\mathfrak{A}$. For each $a\in\mathfrak{A}^f$ and $n\in\mathbb{N}$ choose $c_{an}\in\mathfrak{B}$ such that $\rho_a(c_{an},e)\le 2^{-n}$. Then

$$\begin{aligned}\sum_{n=0}^\infty\bar\mu((a\cap c_{an})\mathbin{\triangle}(a\cap c_{a,n+1}))&=\sum_{n=0}^\infty\rho_a(c_{an},c_{a,n+1})\\&\le\sum_{n=0}^\infty\rho_a(c_{an},e)+\rho_a(e,c_{a,n+1})<\infty.\end{aligned}$$

So if we set $e_a=\sup_{n\in\mathbb{N}}\inf_{k\ge n}c_{ak}$, then

$$\rho_a(e_a,c_{an})=\rho_a(a\cap e_a,a\cap c_{an})\to 0$$

as $n\to\infty$, by 323F, and $\rho_a(e,e_a)=0$, that is, $a\cap e_a=a\cap e$. Also, because $\mathfrak{B}$ is order-closed, $\inf_{k\ge n}c_{ak}\in\mathfrak{B}$ for every n, and $e_a\in\mathfrak{B}$.

Because $\mathfrak{A}$ is Dedekind complete, we can set

$$e'_a=\inf\{e_b:b\in\mathfrak{A}^f,\ a\subseteq b\};$$

then $e'_a\in\mathfrak{B}$ and

$$e'_a\cap a=\inf\nolimits_{b\supseteq a}e_b\cap a=\inf\nolimits_{b\supseteq a}e_b\cap b\cap a=\inf\nolimits_{b\supseteq a}e\cap b\cap a=e\cap a.$$

Now $e'_a\subseteq e'_b$ whenever $a\subseteq b$, so $B=\{e'_a:a\in\mathfrak{A}^f\}$ is upwards-directed, and

$$\sup B=\sup\{e'_a\cap a:a\in\mathfrak{A}^f\}=\sup\{e\cap a:a\in\mathfrak{A}^f\}=e$$

because $(\mathfrak{A},\bar\mu)$ is semi-finite. Accordingly $e\in\mathfrak{B}$. As e is arbitrary, $\mathfrak{B}$ is closed, as claimed.

323I Notation In the context of 323H, I will say simply that $\mathfrak{B}$ is a **closed subalgebra** of $\mathfrak{A}$.

323J Proposition If $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then the topological closure $\overline{\mathfrak{B}}$ of $\mathfrak{B}$ in $\mathfrak{A}$ is precisely the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$.

proof Write $\mathfrak{B}_\tau$ for the smallest order-closed subset of $\mathfrak{A}$ including $\mathfrak{B}$. By 313Gc, $\mathfrak{B}_\tau$ is a subalgebra of $\mathfrak{A}$, and is the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$. Being an order-closed subalgebra of $\mathfrak{A}$, it is topologically closed, by 323H, and must include $\overline{\mathfrak{B}}$. On the other hand, $\overline{\mathfrak{B}}$, being topologically closed, is order-closed (323D(c-i)), so includes $\mathfrak{B}_\tau$. Thus $\overline{\mathfrak{B}}=\mathfrak{B}_\tau$ is the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$.

323K I note some simple results for future reference.

Lemma If $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra and $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, then for any $a\in\mathfrak{A}$ the subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\mathfrak{B}\cup\{a\}$ is closed.

proof By 314Ja, $\mathfrak{C}$ is order-closed.

323L Proposition Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of measure algebras with simple product $(\mathfrak{A},\bar\mu)$ (322K). Then the measure-algebra topology on $\mathfrak{A}=\prod_{i\in I}\mathfrak{A}_i$ defined by $\bar\mu$ is just the product of the measure-algebra topologies of the $\mathfrak{A}_i$.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A. Write $\mathfrak{T}$ for the topology of $\mathfrak{A}$ and $\mathfrak{S}$ for the product topology. For $i \in I$ and $d \in \mathfrak{A}_i^f$ define a pseudometric $\tilde{\rho}_{di}$ on $\mathfrak{A}$ by setting

$$\tilde{\rho}_{di}(b, c) = \rho_d(b(i), c(i))$$

whenever b, $c \in \mathfrak{A}$; then $\mathfrak{S}$ is defined by $\mathrm{P} = \{\tilde{\rho}_{di} : i \in I,\, a \in \mathfrak{A}_i^f\}$ (3A3Ig). Now each $\tilde{\rho}_{di}$ is one of the defining pseudometrics for $\mathfrak{T}$, since

$$\tilde{\rho}_{di}(b, c) = \bar{\mu}(\tilde{d} \cap (b \triangle c))$$

where $\tilde{d}(i) = d$, $\tilde{d}(j) = 0$ for $j \neq i$. So $\mathfrak{S} \subseteq \mathfrak{T}$.

Now suppose that $a \in \mathfrak{A}^f$ and $\epsilon > 0$. Then $\sum_{i \in I} \bar{\mu}_i a(i) = \bar{\mu} a$ is finite, so there is a finite set $J \subseteq I$ such that $\sum_{i \in I \setminus J} \bar{\mu}_i a(i) \leq \frac{1}{2}\epsilon$. For each $j \in J$, $\tau_j = \tilde{\rho}_{a(j),j}$ belongs to P, and

$$\begin{aligned} \rho_a(b, c) &= \sum_{i \in I} \bar{\mu}_i(a(i) \cap (b(i) \triangle c(i))) \\ &\leq \sum_{j \in J} \bar{\mu}_j(a(j) \cap (b(j) \triangle c(j))) + \frac{1}{2}\epsilon = \sum_{j \in J} \tau_j(b, c) + \frac{1}{2}\epsilon \leq \epsilon \end{aligned}$$

whenever b, c are such that $\tau_j(b, c) \leq \epsilon/(1 + 2\#(J))$ for every $j \in J$. By 2A3H, the identity map from $(\mathfrak{A}, \mathfrak{S})$ to $(\mathfrak{A}, \mathfrak{T})$ is continuous, that is, $\mathfrak{T} \subseteq \mathfrak{S}$.

Putting these together, we see that $\mathfrak{S} = \mathfrak{T}$, as claimed.

***323M** In this volume we shall have little need to consider the measure metric on $\mathfrak{A}^f$, but the following facts are sometimes useful.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and give $\mathfrak{A}^f$ its measure metric.

(a) The Boolean operations $\triangle$, $\cap$, $\cup$ and $\setminus$ on $\mathfrak{A}^f$ are uniformly continuous.

(b) $\bar{\mu}{\restriction}\mathfrak{A}^f : \mathfrak{A}^f \to [0, \infty[$ is 1-Lipschitz, therefore uniformly continuous.

(c) $\mathfrak{A}^f$ is complete.

proof (a) Writing ρ for the measure metric on $\mathfrak{A}^f$, then, just as in the proof of 323Ba,

$$\rho(b * c, b' * c') \leq \rho(b, b') + \rho(c, c')$$

for all b, c, b', $c' \in \mathfrak{A}^f$ and any of the Boolean operations $* = \triangle$, $\cap$, $\cup$ and $\setminus$.

(b) If a, $b \in \mathfrak{A}^f$ then

$$|\bar{\mu}a - \bar{\mu}b| \leq |\bar{\mu}a - \bar{\mu}(a \cap b)| + |\bar{\mu}b - \bar{\mu}(a \cap b)| = \bar{\mu}(a \setminus b) + \bar{\mu}(b \setminus a) = \rho(a, b).$$

(c) If $\langle a_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathfrak{A}^f$ such that $\sum_{n=0}^{\infty} \rho(a_n, a_{n+1}) < \infty$, set $d = \sup_{n \in \mathbb{N}} \inf_{m \geq n} a_m$. By 323F, $\lim_{n \to \infty} \bar{\mu}(d \triangle a_n) = 0$. In particular, there is some $n \in \mathbb{N}$ such that $\bar{\mu}(d \setminus a_n)$ is finite, so $d \in \mathfrak{A}^f$ and $\lim_{n \to \infty} \rho(d, a_n) = 0$. As in 2A4E, this is enough to show that $\mathfrak{A}^f$ is complete.

323X Basic exercises >(a) Let (X, Σ, μ) be a measure space, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. (i) Show that we have an injection $\chi : \mathfrak{A} \to L^0(\mu)$ (see §241) given by setting $\chi(E^\bullet) = (\chi E)^\bullet$ for every $E \in \Sigma$. (ii) Show that χ is a homeomorphism between $\mathfrak{A}$ and its image if $\mathfrak{A}$ is given its measure-algebra topology and $L^0(\mu)$ is given its topology of convergence in measure (245A).

>(b) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and ρ the measure metric on the ideal $\mathfrak{A}^f$ of elements of finite measure. (i) Show that the embedding $\mathfrak{A}^f \subseteq \mathfrak{A}$ is uniformly continuous for the measure-algebra uniformity on $\mathfrak{A}$. (ii) In the context of 323Xa, show that $\chi : \mathfrak{A}^f \to L^0(\mu)$ is an isometry between $\mathfrak{A}^f$ and a subset of $L^1(\mu)$.

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra. Show that the set $\{(a, b) : a \subseteq b\}$ is a closed set in $\mathfrak{A} \times \mathfrak{A}$.

>(d) Let (X, Σ, μ) be a σ-finite measure space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. (i) Show that if T is a σ-subalgebra of Σ, then $\{F^\bullet : F \in \mathrm{T}\}$ is a closed subalgebra of $\mathfrak{A}$. (ii) Show that if $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, then $\{F : F \in \Sigma,\, F^\bullet \in \mathfrak{B}\}$ is a σ-subalgebra of Σ.

(e) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra, and $C \subseteq \mathfrak{A}$ a set such that $\sup A$, $\inf A$ belong to C for all non-empty subsets A of C. Show that C is topologically closed.

(f) Show that if $(\mathfrak{A},\bar{\mu})$ is any measure algebra and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then its topological closure $\overline{\mathfrak{B}}$ is again a subalgebra.

(g) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $e\in\mathfrak{A}$; let $\mathfrak{A}_e$ be the principal ideal of $\mathfrak{A}$ generated by e, and $\bar{\mu}_e$ its measure (322H). (i) Show that the measure-algebra topology on $\mathfrak{A}_e$ defined by $\bar{\mu}_e$ is just the subspace topology induced by the measure-algebra topology of $\mathfrak{A}$. (ii) Show that the measure-algebra uniformity on $\mathfrak{A}_e$ is the subspace uniformity induced by the measure-algebra uniformity of $\mathfrak{A}$. (iii) Show that the strong measure-algebra topology on $\mathfrak{A}_e^f$ is the subspace topology induced by the strong measure-algebra topology of $\mathfrak{A}^f$.

(h) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Show that its localization (322P) can be identified with its completion under its measure-algebra uniformity.

323Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a σ-finite measure algebra. Show that a set $F\subseteq\mathfrak{A}$ is topologically closed iff $e\in F$ whenever there are non-empty sets B, $C\subseteq\mathfrak{A}$ such that B is upwards-directed, C is downwards-directed, $\sup B=\inf C=e$ and $[b,c]\cap F\ne\emptyset$ for every $b\in B$, $c\in C$, writing $[b,c]=\{d:b\subseteq d\subseteq c\}$.

(b) Give an example to show that (a) is false for general localizable measure algebras.

(c) Give an example of a semi-finite measure algebra $(\mathfrak{A},\bar{\mu})$ with an order-closed subalgebra which is not topologically closed.

(d) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and write $\mathbb{B}$ for the family of closed subalgebras of $\mathfrak{A}$. For $\mathfrak{B}$, $\mathfrak{C}\in\mathbb{B}$ set $\rho(\mathfrak{B},\mathfrak{C})=\max(\sup_{b\in\mathfrak{B}}\inf_{c\in\mathfrak{C}}\bar{\mu}(b\triangle c),\sup_{c\in\mathfrak{C}}\inf_{b\in\mathfrak{B}}\bar{\mu}(b\triangle c))$. Show that $(\mathbb{B},\rho)$ is a complete metric space. (Cf. 246Yb, 4A2T.)

(e) Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of Lebesgue measure on $\mathbb{R}$. Show that it is separable in its measure-algebra topology. (*Hint*: 245Yj.)

323 Notes and comments The message of this section is that the topology of a measure algebra is essentially defined by its order and algebraic structure; see also 324F-324H below. Of course the results are really about semi-finite measure algebras, and indeed this whole volume, like the rest of measure theory, has little of interest to say about others; they are included only because they arise occasionally and it is not absolutely essential to exclude them. We therefore expect to be able to describe such things as closed subalgebras and continuous homomorphisms in terms of the ordering, as in 323H and 324G. For σ-finite algebras, indeed, there is an easy description of the topology in terms of the order (323Ya). I think the result of this section on which I shall most often depend is 323H: in most contexts, there is no need to distinguish between 'topologically closed subalgebra' and 'order-closed subalgebra'. However a σ-subalgebra of a localizable measure algebra need not be topologically sequentially closed; there is an example in Fremlin Pagter & Ricker 05.

It is also the case that the topology of a measure algebra corresponds very closely indeed to the topology of convergence in measure. A description of this correspondence is in 323Xa. Indeed all the results of this section have analogues in the theory of topological Riesz spaces. I will enlarge on the idea here in §367. For the moment, however, if you look back to Chapter 24, you will see that 323B and 323G are closely paralleled by 245D and 245E, while 323Ya is related to 245L.

It is I think natural to ask whether there are any other topological Boolean algebras with the properties 323B-323D. In fact there are; see 393G and 393Xf below.

324 Homomorphisms

In the course of Volume 2, I had occasion to remark that elementary measure theory was unusual among abstract topics in pure mathematics in not being dominated by any particular class of structure-preserving operators. We now come to what I think is one of the reasons for the gap: the most important operators of the theory are not between measure spaces at all, but between their measure algebras. In this section I run through the most elementary facts about Boolean homomorphisms between measure algebras. I start with results on the construction of such homomorphisms from functions between measure spaces (324A-324E), then investigate continuity and order-continuity of homomorphisms (324F-324H) before turning to measure-preserving homomorphisms (324I-324O).

324A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ their measure algebras. Write $\hat\Sigma$ for the domain of the completion $\hat\mu$ of μ. Let $D\subseteq X$ be a set of full outer measure (definition: 132F), and let $\hat\Sigma_D$ be the subspace σ-algebra on D induced by $\hat\Sigma$. Let $\phi:D\to Y$ be a function such that $\phi^{-1}[F]\in\hat\Sigma_D$ for every $F\in\mathrm{T}$ and $\phi^{-1}[F]$ is μ-negligible whenever $\nu F=0$. Then there is a sequentially order-continuous Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by the formula

$$\pi F^\bullet=E^\bullet \text{ whenever } F\in\mathrm{T},\ E\in\Sigma \text{ and } (E\cap D)\triangle\phi^{-1}[F] \text{ is negligible.}$$

proof Let $F\in\mathrm{T}$. Then there is an $H\in\hat\Sigma$ such that $H\cap D=\phi^{-1}[F]$; now there is an $E\in\Sigma$ such that $E\triangle H$ is negligible, so that $(E\cap D)\triangle\phi^{-1}[F]$ is negligible. If E_1 is another member of Σ such that $(E_1\cap D)\triangle\phi^{-1}[F]$ is negligible, then $(E\triangle E_1)\cap D$ is negligible, so is included in a negligible member G of Σ. Since $(E\triangle E_1)\setminus G$ belongs to Σ and is disjoint from D, it is negligible; accordingly $E\triangle E_1$ is negligible and $E^\bullet=E_1^\bullet$ in $\mathfrak{A}$.

What this means is that the formula offered defines a map $\pi:\mathfrak{B}\to\mathfrak{A}$. It is now easy to check that π is a Boolean homomorphism, because if

$$(E\cap D)\triangle\phi^{-1}[F],\quad (E'\cap D)\triangle\phi^{-1}[F']$$

are negligible, so are

$$((X\setminus E)\cap D)\triangle\phi^{-1}[Y\setminus F],\quad ((E\cup E')\cap D)\triangle\phi^{-1}[F\cup F'].$$

To see that π is sequentially order-continuous, let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{B}$. For each n we may choose an $F_n\in\mathrm{T}$ such that $F_n^\bullet=b_n$, and $E_n\in\Sigma$ such that $(E_n\cap D)\triangle\phi^{-1}[F_n]$ is negligible; now, setting $F=\bigcup_{n\in\mathbb{N}}F_n$, $E=\bigcup_{n\in\mathbb{N}}E_n$,

$$(E\cap D)\triangle\phi^{-1}[F]\subseteq\bigcup_{n\in\mathbb{N}}(E_n\cap D)\triangle\phi^{-1}[F_n]$$

is negligible, so

$$\pi(\sup_{n\in\mathbb{N}}b_n)=\pi(F^\bullet)=E^\bullet=\sup_{n\in\mathbb{N}}E_n^\bullet=\sup_{n\in\mathbb{N}}\pi b_n.$$

(Recall that the maps $E\mapsto E^\bullet$, $F\mapsto F^\bullet$ are sequentially order-continuous, by 321H.) So π is sequentially order-continuous (313L(c-iii)).

324B Corollary Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ their measure algebras. Let $\phi:X\to Y$ be a function such that $\phi^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$ and $\mu\phi^{-1}[F]=0$ whenever $\nu F=0$. Then there is a sequentially order-continuous Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by the formula

$$\pi F^\bullet=(\phi^{-1}[F])^\bullet \text{ for every } F\in\mathrm{T}.$$

324C Remarks (a) In §235 and elsewhere in Volume 2 I spent a good deal of time on functions between measure spaces which satisfy the conditions of 324A. Indeed, I take the trouble to spell 324A out in such generality just in order to catch these applications. Some of the results of the present chapter (322D, 322Jb) can also be regarded as special cases of 324A.

(b) The question of which homomorphisms between the measure algebras of measure spaces (X,Σ,μ), (Y,T,ν) can be realized by functions between X and Y is important and deep; I will return to it in §§343-344.

(c) In the simplified context of 324B, I have actually defined a contravariant functor; the relevant facts are the following.

324D Proposition Let (X,Σ,μ), (Y,T,ν) and (Z,Λ,λ) be measure spaces, with measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$, $(\mathfrak{C},\bar\lambda)$. Suppose that $\phi:X\to Y$ and $\psi:Y\to Z$ satisfy the conditions of 324B, that is,

$$\phi^{-1}[F]\in\Sigma \text{ if } F\in\mathrm{T},\quad \mu\phi^{-1}[F]=0 \text{ if } \nu F=0,$$

$$\psi^{-1}[G]\in\mathrm{T} \text{ if } G\in\Lambda,\quad \mu\psi^{-1}[G]=0 \text{ if } \lambda G=0.$$

Let $\pi_\phi:\mathfrak{B}\to\mathfrak{A}$, $\pi_\psi:\mathfrak{C}\to\mathfrak{B}$ be the corresponding homomorphisms. Then $\psi\phi:X\to Z$ is another map of the same type, and $\pi_{\psi\phi}=\pi_\phi\pi_\psi:\mathfrak{C}\to\mathfrak{A}$.

proof The necessary checks are all elementary.

324E Stone spaces While in the context of general measure spaces the question of realizing homomorphisms is difficult, in the case of the Stone representation it is relatively straightforward.

Proposition Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, with Stone spaces Z and W; let μ, ν be the corresponding measures on Z and W, as described in 321J-321K, and Σ, T their domains. If $\pi:\mathfrak{B}\to\mathfrak{A}$ is any order-continuous Boolean homomorphism, let $\phi:Z\to W$ be the corresponding continuous function, as described in 312Q. Then $\phi^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$, $\mu\phi^{-1}[F]=0$ whenever $\nu F=0$, and (writing E^* for the member of $\mathfrak{A}$ corresponding to $E\in\Sigma$) $\pi F^*=(\phi^{-1}[F])^*$ for every $F\in\mathrm{T}$.

proof Recall that $E^*=a$ iff $E\triangle\widehat{a}$ is meager, where $\widehat{a}$ is the open-and-closed subset of Z corresponding to $a\in\mathfrak{A}$. In particular, $\mu E=0$ iff E is meager. Now the point is that $\phi^{-1}[F]$ is nowhere dense in Z whenever F is a nowhere dense subset of W, by 313R. Consequently $\phi^{-1}[F]$ is meager whenever F is meager in W, since F is then just a countable union of nowhere dense sets. Thus we see already that $\mu\phi^{-1}[F]=0$ whenever $\nu F=0$. If F is any member of T, there is an open-and-closed set F_0 such that $F\triangle F_0$ is meager; now $\phi^{-1}[F_0]$ is open-and-closed, so $\phi^{-1}[F]=\phi^{-1}[F_0]\triangle\phi^{-1}[F\triangle F_0]$ belongs to Σ. Moreover, if $b\in\mathfrak{B}$ is such that $\widehat{b}=F_0$, and $a=\pi b$, then $\widehat{a}=\phi^{-1}[F_0]$, so

$$\pi F^*=\pi b=a=(\phi^{-1}[F_0])^*=(\phi^{-1}[F])^*,$$

as required.

324F I turn now to the behaviour of order-continuous homomorphisms between measure algebras.

Theorem Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras and $\pi:\mathfrak{A}\to\mathfrak{B}$ a Boolean homomorphism. Give $\mathfrak{A}$ and $\mathfrak{B}$ their measure-algebra topologies and uniformities (323Ab).

(a) π is continuous iff it is continuous at 0 iff it is uniformly continuous.

(b) If $(\mathfrak{B},\bar{\nu})$ is semi-finite and π is continuous, then it is order-continuous.

(c) If $(\mathfrak{A},\bar{\mu})$ is semi-finite and π is order-continuous, then it is continuous.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A.

(a) Suppose that π is continuous at 0; I seek to show that it is uniformly continuous. Take $b\in\mathfrak{B}^f$ and $\epsilon>0$. Then there are $a_0,\dots,a_n\in\mathfrak{A}^f$ and $\delta>0$ such that

$$\bar{\nu}(b\cap\pi c)=\rho_b(\pi c,0)\le\epsilon\text{ whenever }\max_{i\le n}\rho_{a_i}(c,0)\le\delta;$$

setting $a=\sup_{i\le n}a_i$,

$$\bar{\nu}(b\cap\pi c)\le\epsilon\text{ whenever }\bar{\mu}(a\cap c)\le\delta.$$

Now suppose that $\rho_a(c,c')\le\delta$. Then $\bar{\mu}(a\cap(c\mathbin{\triangle}c'))\le\delta$, so

$$\rho_b(\pi c,\pi c')=\bar{\nu}(b\cap(\pi c\mathbin{\triangle}\pi c'))=\bar{\nu}(b\cap\pi(c\mathbin{\triangle}c'))\le\epsilon.$$

As b, ϵ are arbitrary, π is uniformly continuous. The rest of the implications are elementary.

(b) Let A be a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0. Then $0\in\overline{A}$ (323D(b-ii)); because π is continuous, $0\in\overline{\pi[A]}$. **?** If b is a non-zero lower bound for $\pi[A]$ in $\mathfrak{B}$, then (because $(\mathfrak{B},\bar{\nu})$ is semi-finite) there is a $c\subseteq b$ with $0<\bar{\nu}c<\infty$; now

$$\rho_c(\pi a,0)=\bar{\nu}(c\cap\pi a)=\bar{\nu}c>0$$

for every $a\in A$, so $0\notin\overline{\pi[A]}$. **X**

Thus $\inf\pi[A]=0$ in $\mathfrak{B}$; as A is arbitrary, π is order-continuous (313L(b-ii)).

(c) By (a), it will be enough to show that π is continuous at 0. Let $b\in\mathfrak{B}^f$, $\epsilon>0$. **?** Suppose, if possible, that for every $a\in\mathfrak{A}^f$, $\delta>0$ there is a $c\in\mathfrak{A}$ such that $\bar{\mu}(a\cap c)\le\delta$ but $\bar{\nu}(b\cap\pi c)\ge\epsilon$. For each $a\in\mathfrak{A}^f$, $n\in\mathbb{N}$ choose c_{an} such that $\bar{\mu}(a\cap c_{an})\le 2^{-n}$ but $\bar{\nu}(b\cap\pi c_{an})\ge\epsilon$. Set $c_a=\inf_{n\in\mathbb{N}}\sup_{m\ge n}c_{am}$; then

$$\bar{\mu}(a\cap c_a)\le\inf_{n\in\mathbb{N}}\sum_{m=n}^\infty\bar{\mu}(a\cap c_{an})=0,$$

so $c_a\cap a=0$. On the other hand, because π is order-continuous, $\pi c_a=\inf_{n\in\mathbb{N}}\sup_{m\ge n}\pi c_{am}$, so that

$$\bar{\nu}(b\cap\pi c_a)=\lim_{n\to\infty}\bar{\nu}(b\cap\sup_{m\ge n}\pi c_{am})\ge\epsilon.$$

This shows that

$$\rho_b(\pi(1\setminus a),0)=\bar{\nu}(b\cap\pi(1\setminus a))\ge\bar{\nu}(b\cap\pi c_a)\ge\epsilon.$$

But now observe that $A = \{1 \setminus a : a \in \mathfrak{A}^f\}$ is a downwards-directed subset of $\mathfrak{A}$ with infimum 0, because $(\mathfrak{A}, \bar{\mu})$ is semi-finite. So $\pi[A]$ is downwards-directed and has infimum 0, and 0 must be in the closure of $\pi[A]$, by 323D(b-ii); while we have just seen that $\rho_b(d, 0) \geq \epsilon$ for every $d \in \pi[A]$. **X**

Thus there must be $a \in \mathfrak{A}^f$, $\delta > 0$ such that

$$\rho_b(\pi c, 0) = \bar{\nu}(b \cap \pi c) \leq \epsilon$$

whenever

$$\rho_a(c, 0) = \bar{\mu}(a \cap c) \leq \delta.$$

As b, ϵ are arbitrary, π is continuous at 0 and therefore continuous.

324G Corollary If $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are semi-finite measure algebras, a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is continuous iff it is order-continuous.

324H Corollary If $\mathfrak{A}$ is a Boolean algebra and $\bar{\mu}$, $\bar{\nu}$ are two measures both rendering $\mathfrak{A}$ a semi-finite measure algebra, then they endow $\mathfrak{A}$ with the same uniformity (and, of course, the same topology).

proof By 324G, the identity map from $\mathfrak{A}$ to itself is continuous whichever of the topologies we place on $\mathfrak{A}$; and by 324F it is therefore uniformly continuous.

324I Definition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras. A Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is **measure-preserving** if $\bar{\nu}(\pi a) = \bar{\mu} a$ for every $a \in \mathfrak{A}$.

324J Proposition Let $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{B}, \bar{\nu})$ and $(\mathfrak{C}, \bar{\lambda})$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$, $\theta : \mathfrak{B} \to \mathfrak{C}$ measure-preserving Boolean homomorphisms. Then $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ is a measure-preserving Boolean homomorphism.

proof Elementary.

324K Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving Boolean homomorphism.

(a) π is injective.

(b) $(\mathfrak{A}, \bar{\mu})$ is totally finite iff $(\mathfrak{B}, \bar{\nu})$ is, and in this case π is order-continuous, therefore continuous, and $\pi[\mathfrak{A}]$ is a closed subalgebra of $\mathfrak{B}$.

(c) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite and $(\mathfrak{B}, \bar{\nu})$ is σ-finite, then $(\mathfrak{A}, \bar{\mu})$ is σ-finite.

(d) If $(\mathfrak{A}, \bar{\mu})$ is σ-finite and π is sequentially order-continuous, then $(\mathfrak{B}, \bar{\nu})$ is σ-finite.

(e) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite and π is order-continuous, then $(\mathfrak{B}, \bar{\nu})$ is semi-finite.

(f) If $(\mathfrak{A}, \bar{\mu})$ is atomless and semi-finite, and π is order-continuous, then $\mathfrak{B}$ is atomless.

(g) If $\mathfrak{B}$ is purely atomic and $(\mathfrak{A}, \bar{\mu})$ is semi-finite, then $\mathfrak{A}$ is purely atomic.

proof (a) If $a \neq 0$ in $\mathfrak{A}$, then $\bar{\nu}\pi a = \bar{\mu} a > 0$ so $\pi a \neq 0$. By 3A2Db, π is injective.

(b) Because

$$\bar{\nu} 1_{\mathfrak{B}} = \bar{\nu}\pi 1_{\mathfrak{A}} = \bar{\mu} 1_{\mathfrak{A}},$$

$(\mathfrak{A}, \bar{\mu})$ is totally finite iff $(\mathfrak{B}, \bar{\nu})$ is. Now suppose that $A \subseteq \mathfrak{A}$ is downwards-directed and non-empty and that $\inf A = 0$. Then

$$\inf_{a \in A} \bar{\nu}\pi a = \inf_{a \in A} \bar{\mu} a = 0$$

by 321F. So $\bar{\nu} b = 0$ for any lower bound b of $\pi[A]$, and $\inf \pi[A] = 0$. As A is arbitrary, π is order-continuous.

By 324Fc, π is continuous. By 314Fa, $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$, that is, 'closed' in the sense of 323I.

(c) I appeal to 322G. If C is a disjoint family in $\mathfrak{A} \setminus \{0\}$, then $\langle \pi c \rangle_{c \in C}$ is a disjoint family in $\mathfrak{B} \setminus \{0\}$, so is countable, and C must be countable, because π is injective. Thus $\mathfrak{A}$ is ccc and (being semi-finite) $(\mathfrak{A}, \bar{\mu})$ is σ-finite.

(d) Let $\langle a_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\bar{\mu} a_n < \infty$ for every n and $\sup_{n \in \mathbb{N}} a_n = 1$. Then $\bar{\nu}\pi a_n < \infty$ for every n and (because π is sequentially order-continuous) $\sup_{n \in \mathbb{N}} \pi a_n = 1$, so $(\mathfrak{B}, \bar{\nu})$ is σ-finite.

(e) Setting $\mathfrak{A}^f = \{a : \bar{\mu} a < \infty\}$, $\sup \mathfrak{A}^f = 1$; because π is order-continuous, $\sup \pi[\mathfrak{A}^f] = 1$ in $\mathfrak{B}$. So if $\bar{\nu} b = \infty$, there is an $a \in \mathfrak{A}^f$ such that $\pi a \cap b \neq 0$, and now $0 < \bar{\nu}(b \cap \pi a) < \infty$.

(f) Take any non-zero $b \in \mathfrak{B}$. As in (e), there is an $a \in \mathfrak{A}$ such that $\bar{\mu}a < \infty$ and $\pi a \cap b \neq 0$. If $\pi a \cap b \neq b$, then surely b is not an atom. Otherwise, set

$$C = \{c : c \in \mathfrak{A},\, c \subseteq a,\, b \subseteq \pi c\}.$$

Then C is downwards-directed and contains a, so $c_0 = \inf C$ is defined in $\mathfrak{A}$ (321F), and

$$\bar{\mu}c_0 = \inf_{c\in C} \bar{\mu}c \geq \bar{\nu}b > 0,$$

so $c_0 \neq 0$. Because $\mathfrak{A}$ is atomless, there is a $d \subseteq c_0$ such that neither d nor $c_0 \setminus d$ is zero, so that neither $c_0 \setminus d$ nor d can belong to C. But this means that $b \cap \pi d$ and $b \cap \pi(c_0 \setminus d)$ are both non-zero, so that again b is not an atom. As b is arbitrary, $\mathfrak{B}$ is atomless.

(g) Take any non-zero $a \in \mathfrak{A}$. Then there is an $a' \subseteq a$ such that $0 < \bar{\mu}a' < \infty$. Because $\mathfrak{B}$ is purely atomic, there is an atom b of $\mathfrak{B}$ with $b \subseteq \pi a'$. Set

$$C = \{c : c \in \mathfrak{A},\, c \subseteq a',\, b \subseteq \pi c\}.$$

Then C is downwards-directed and contains a', so $c_0 = \inf C$ is defined in $\mathfrak{A}$, and

$$\bar{\mu}c_0 = \inf_{c\in C} \bar{\mu}c \geq \bar{\nu}b > 0,$$

so $c_0 \neq 0$. If $d \subseteq c_0$, then $b \cap \pi d$ must be either b or 0. If $b \cap \pi d = b$, then $d \in C$ and $d = c_0$. If $b \cap \pi d = 0$, then $c_0 \setminus d \in C$ and $d = 0$. Thus c_0 is an atom in $\mathfrak{A}$. As a is arbitrary, $\mathfrak{A}$ is purely atomic.

324L Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $(\mathfrak{B}, \bar{\nu})$ a measure algebra, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving homomorphism. If $C \subseteq \mathfrak{A}$ and $\mathfrak{C}$ is the closed subalgebra of $\mathfrak{A}$ generated by C, then $\pi[\mathfrak{C}]$ is the closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$.

proof This is a special case of 314H.

324M Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with measure algebras $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$. Let $\phi : X \to Y$ be inverse-measure-preserving. Then we have a sequentially order-continuous measure-preserving Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ defined by setting $\pi F^\bullet = \phi^{-1}[F]^\bullet$ for every $F \in \mathrm{T}$.

proof This is immediate from 324B.

324N Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras, with Stone spaces Z and W; let μ, ν be the corresponding measures on Z and W. If $\pi : \mathfrak{B} \to \mathfrak{A}$ is an order-continuous measure-preserving Boolean homomorphism, and $\phi : Z \to W$ the corresponding continuous function, then ϕ is inverse-measure-preserving.

proof Use 324E. In the notation there, if $F \in \mathrm{T}$, then

$$\nu F = \bar{\nu}F^* = \bar{\mu}\pi F^* = \bar{\mu}\phi^{-1}[F]^* = \mu\phi^{-1}[F].$$

324O Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras, $\mathfrak{A}_0$ a topologically dense subalgebra of $\mathfrak{A}$, and $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism such that $\bar{\nu}\pi a = \bar{\mu}a$ for every $a \in \mathfrak{A}_0$. Then π has a unique extension to a measure-preserving homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

proof Let ρ, σ be the measure metrics on $\mathfrak{A}$, $\mathfrak{B}$ respectively, as in 323Ad. Then for any a, $a' \in \mathfrak{A}_0$

$$\sigma(\pi a, \pi a') = \bar{\nu}(\pi a \triangle \pi a') = \bar{\nu}\pi(a \triangle a') = \bar{\mu}(a \triangle a') = \rho(a, a');$$

that is, $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ is an isometry. Because $\mathfrak{A}_0$ is dense in the metric space $(\mathfrak{A}, \rho)$, while $\mathfrak{B}$ is complete under σ (323Gc), there is a unique continuous function $\hat{\pi} : \mathfrak{A} \to \mathfrak{B}$ extending π (3A4G). Now the operations

$$(a, a') \mapsto \hat{\pi}(a \cup a'), \quad (a, a') \mapsto \hat{\pi}a \cup \hat{\pi}a' : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{B},$$

are continuous and agree on the dense subset $\mathfrak{A}_0 \times \mathfrak{A}_0$ of $\mathfrak{A} \times \mathfrak{A}$; because the topology of $\mathfrak{B}$ is Hausdorff, they agree on $\mathfrak{A} \times \mathfrak{A}$, that is, $\hat{\pi}(a \cup a') = \hat{\pi}a \cup \hat{\pi}a'$ for all a, $a' \in \mathfrak{A}$ (2A3Uc). Similarly, the operations

$$a \mapsto \hat{\pi}(1 \setminus a), \quad a \mapsto 1 \setminus \hat{\pi}a : \mathfrak{A} \to \mathfrak{B}$$

are continuous and agree on the dense subset $\mathfrak{A}_0$ of $\mathfrak{A}$, so they agree on $\mathfrak{A}$, that is, $\hat{\pi}(1 \setminus a) = 1 \setminus a$ for every $a \in \mathfrak{A}$. Thus $\hat{\pi}$ is a Boolean homomorphism. To see that it is measure-preserving, observe that

$$a \mapsto \bar{\mu}a = \rho(a, 0), \quad a \mapsto \bar{\nu}(\hat{\pi}a) = \sigma(\hat{\pi}a, 0) : \mathfrak{A} \to \mathbb{R}$$

are continuous and agree on $\mathfrak{A}_0$, so agree on $\mathfrak{A}$. Finally, $\hat{\pi}$ is the only measure-preserving Boolean homomorphism extending π, because any such map must be continuous (324Kb), and $\hat{\pi}$ is the only continuous extension of π.

***324P** The following fact will be applied in §387, by which time it will seem perfectly elementary; for the moment, it may be a useful exercise.

Proposition Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be totally finite measure algebras such that $\bar{\mu}1=\bar{\nu}1$. Suppose that $A\subseteq\mathfrak{A}$ and $\phi:A\to\mathfrak{B}$ are such that $\bar{\nu}(\inf_{i\le n}\phi a_i)=\bar{\mu}(\inf_{i\le n}a_i)$ for all $a_0,\ldots,a_n\in A$. Let $\mathfrak{C}$ be the smallest closed subalgebra of $\mathfrak{A}$ including A. Then ϕ has a unique extension to a measure-preserving Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$.

proof (a) Let Ψ be the family of all functions ψ extending ϕ and having the same properties; that is, ψ is a function from a subset of $\mathfrak{A}$ to $\mathfrak{B}$, and $\bar{\nu}(\inf_{i\le n}\psi a_i)=\bar{\mu}(\inf_{i\le n}a_i)$ for all $a_0,\ldots,a_n\in\operatorname{dom}\psi$. By Zorn's Lemma, Ψ has a maximal member θ. Write D for the domain of θ.

(b)(i) If c, $d\in D$ then $c\cap d\in D$. **P?** Otherwise, set $D'=D\cup\{c\cap d\}$ and extend θ to $\theta':D'\to\mathfrak{B}$ by writing $\theta'(c\cap d)=\theta c\cap\theta d$. It is easy to check that $\theta'\in\Psi$, which is supposed to be impossible. **XQ**

Now

$$\bar{\nu}(\theta c\cap\theta d\cap\theta(c\cap d))=\bar{\mu}(c\cap d)=\bar{\nu}(\theta c\cap\theta d)=\bar{\nu}\theta(c\cap d),$$

so $\theta(c\cap d)=\theta c\cap\theta d$.

(ii) If $d\in D$ then $1\setminus d\in D$. **P?** Otherwise, set $D'=D\cup\{1\setminus d\}$ and extend θ to D' by writing $\theta'(1\setminus d)=1\setminus\theta d$. Once again, it is easy to check that $\theta'\in\Psi$, which is impossible. **XQ**

Consequently (since D is certainly not empty, even if C is), D is a subalgebra of $\mathfrak{A}$ (312B(iii)).

(iii) Since

$$\bar{\nu}\theta1=\bar{\mu}1=\bar{\nu}1,$$

$\theta1=1$. If $d\in D$ then

$$\bar{\nu}\theta(1\setminus d)=\bar{\mu}(1\setminus d)=\bar{\mu}1-\bar{\mu}d=\bar{\nu}1-\bar{\nu}\theta d=\bar{\nu}(1\setminus\theta d),$$

while

$$\bar{\nu}(\theta d\cap\theta(1\setminus d))=\bar{\mu}(d\cap(1\setminus d))=0,$$

so $\theta d\cap\theta(1\setminus d))=0$, $\theta(1\setminus d)\subseteq 1\setminus\theta d$ and $\theta(1\setminus d)$ must be equal to $1\setminus\theta d$.

By 312H(ii), $\theta:D\to\mathfrak{B}$ is a Boolean homomorphism.

(iv) Let $\mathfrak{D}$ be the topological closure of D in $\mathfrak{A}$. Then it is an order-closed subalgebra of $\mathfrak{A}$ (323J), so, with $\bar{\mu}\restriction\mathfrak{D}$, is a totally finite measure algebra in which D is a topologically dense subalgebra. By 324O, there is an extension of θ to a measure-preserving Boolean homomorphism from $\mathfrak{D}$ to $\mathfrak{B}$; of course this extension belongs to Ψ, so in fact $D=\mathfrak{D}$ is a closed subalgebra of $\mathfrak{A}$.

(c) Since $A\subseteq D$, $\mathfrak{C}\subseteq\mathfrak{D}$ and $\phi_1=\theta\restriction\mathfrak{C}$ is a suitable extension of ϕ.

To see that ϕ_1 is unique, let $\phi_2:\mathfrak{C}\to\mathfrak{B}$ be any other measure-preserving Boolean homomorphism extending ϕ. Set $C=\{a:\phi_1a=\phi_2a\}$; then C is a topologically closed subalgebra of $\mathfrak{A}$ including A, so is the whole of $\mathfrak{C}$, and $\phi_2=\phi_1$.

324X Basic exercises (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, of which $\mathfrak{A}$ is Dedekind σ-complete, and $\phi:\mathfrak{A}\to\mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let I be an ideal of $\mathfrak{A}$ included in the kernel of ϕ. Show that we have a sequentially order-continuous Boolean homomorphism $\pi:\mathfrak{A}/I\to\mathfrak{B}$ given by setting $\phi(a^\bullet)=\phi a$ for every $a\in\mathfrak{A}$.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Show that *provided that* $(\mathfrak{B},\bar{\mu}\restriction\mathfrak{B})$ *is semi-finite*, then the topology of $\mathfrak{B}$ induced by $\bar{\mu}\restriction\mathfrak{B}$ is just the subspace topology induced by the topology of $\mathfrak{A}$. (*Hint*: apply 324Fc to the embedding $\mathfrak{B}\hookrightarrow\mathfrak{A}$.)

(c) Let (X,Σ,μ) be a measure space and $(X,\tilde{\Sigma},\tilde{\mu})$ its c.l.d. version. Let $\mathfrak{A}$, $\mathfrak{A}_2$ be the corresponding measure algebras and $\pi:\mathfrak{A}\to\mathfrak{A}_2$ the canonical homomorphism (see 322Db). Show that π is topologically continuous.

(d) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a bijective measure-preserving Boolean homomorphism. Show that $\pi^{-1} : \mathfrak{B} \to \mathfrak{A}$ is a measure-preserving homomorphism.

(e) Let $\bar{\mu}$ be counting measure on $\mathcal{P}\mathbb{N}$. Show that $(\mathcal{P}\mathbb{N}, \bar{\mu})$ is a σ-finite measure algebra. Find a measure-preserving Boolean homomorphism from $\mathcal{P}\mathbb{N}$ to itself which is not sequentially order-continuous.

324Y Further exercises (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, of which $\mathfrak{A}$ is Dedekind complete, and $\phi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism. Let I be an ideal of $\mathfrak{A}$ included in the kernel of ϕ. Show that we have an order-continuous Boolean homomorphism $\pi : \mathfrak{A}/I \to \mathfrak{B}$ given by setting $\phi(a^\bullet) = \phi a$ for every $a \in \mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and Z its Stone space. Write $\mathcal{E}$ for the algebra of open-and-closed subsets of Z, and $\mathcal{Z}$ for the family of nowhere dense zero sets of Z; let $\mathcal{Z}_\sigma$ be the σ-ideal of subsets of Z generated by $\mathcal{Z}$. Show that $\Sigma = \{E \triangle U : E \in \mathcal{E}, U \in \mathcal{Z}_\sigma\}$ is a σ-algebra of subsets of Z, and describe a canonical isomorphism between $\Sigma/\mathcal{Z}_\sigma$ and $\mathfrak{A}$.

(c) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, with Stone spaces Z and W. Construct $\mathcal{Z}_\sigma \subseteq \Sigma \subseteq \mathcal{P}Z$ as in 324Yb, and let $\mathcal{W}_\sigma \subseteq \mathrm{T} \subseteq \mathcal{P}W$ be the corresponding structure defined from $\mathfrak{B}$. Let $\pi : \mathfrak{B} \to \mathfrak{A}$ be a sequentially order-continuous Boolean homomorphism, and $\phi : Z \to W$ the corresponding continuous map. Show that if $E^* \in \mathfrak{A}$ corresponds to $E \in \Sigma$, then $\pi F^* = \phi^{-1}[F]^*$ for every $F \in \mathrm{T}$. (*Hint*: 313Ye.)

(d) Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a ccc Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ an injective Boolean homomorphism. Show that $\mathfrak{A}$ is ccc.

(e) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra, and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism. Show that for every atom $b \in \mathfrak{B}$ there is an atom $a \in \mathfrak{A}$ such that $\pi a \supseteq b$. Hence show that if $\mathfrak{A}$ is atomless so is $\mathfrak{B}$, and that if $\mathfrak{B}$ is purely atomic and π is injective then $\mathfrak{A}$ is purely atomic.

(f) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be localizable measure algebras and $\mathfrak{A}_0$ an order-dense subalgebra of $\mathfrak{A}$. Suppose that $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ is an order-continuous Boolean homomorphism such that $\bar{\nu}\pi a = \bar{\mu} a$ for every $a \in \mathfrak{A}_0$. Show that π has a unique extension to a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(g) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $[0, 1]$. (i) Show that there is an injective order-preserving function $f : \mathfrak{A} \to \mathcal{P}\mathbb{N}$. (*Hint*: take a countable topologically dense subset D of $\mathfrak{A}$, and define $f : \mathfrak{A} \to \mathcal{P}(D \times \mathbb{Q})$ by setting $f(a) = \{(d, q) : \bar{\mu}(a \cap d) \ge q\}$.) (ii) Show that there is an order-preserving function $h : \mathcal{P}\mathbb{N} \to \mathfrak{A}$ such that $h(f(a)) = a$ for every $a \in \mathfrak{A}$. (*Hint*: set $h(I) = \sup\{a : f(a) \subseteq I\}$.)

(h) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be probability algebras, and $f : \mathfrak{A} \to \mathfrak{B}$ an isometry for the measure metrics. Show that $a \mapsto f(a) \triangle f(0)$ is a measure-preserving Boolean homomorphism.

324 Notes and comments If you examine the arguments of this section carefully, you will see that rather little depends on the measures named. Really this material deals with structures $(X, \Sigma, \mathcal{I})$ where X is a set, Σ is a σ-ideal of subsets of X, and $\mathcal{I}$ is a σ-ideal of Σ, corresponding to the family of measurable negligible sets. In this abstract form it is natural to think in terms of sequentially order-continuous homomorphisms, as in 324Yc. I have stated 324E in terms of order-continuous homomorphisms just for a slight gain in simplicity. But in fact, when there is a difference, it is likely that order-continuity, rather than sequential order-continuity, will be the more significant condition. Note that when the domain algebra is σ-finite, it is ccc (322G), so the two concepts coincide (316Fd).

Of course I need to refer to measures when looking at such concepts as σ-finite measure algebra or measure-preserving homomorphism, but even here the real ideas involved are such notions as order-continuity and the countable chain condition, as you will see if you work through 324K. It is instructive to look at the translations of these facts into the context of inverse-measure-preserving functions; see 234B.

324H shows that we may speak of 'the' topology and uniformity of a Dedekind σ-complete Boolean algebra which carries any semi-finite measure; the topology of such an algebra is determined by its algebraic structure. Contrast this with the theory of normed spaces: two Banach spaces (e.g., ℓ^1 and ℓ^2) can be isomorphic as linear spaces, both being of algebraic dimension $\mathfrak{c}$, while they are not isomorphic as topological linear spaces. When we come to the theory of ordered linear topological spaces, however, we shall again find ourselves with operators whose algebraic properties guarantee continuity (355C, 367O).

325 Free products and product measures

In this section I aim to describe the measure algebras of product measures as defined in Chapter 25. This will involve the concept of 'free product' set out in §315. It turns out that we cannot determine the measure algebra of a product measure from the measure algebras of the factors (325B), unless we are told that the product measure is localizable; but that there is nevertheless a general construction of 'localizable measure algebra free product', applicable to any pair of semi-finite measure algebras (325D), which represents the measure algebra of the product measure in the most important cases (325Eb). In the second part of the section (325I-325M) I deal with measure algebra free products of probability algebras, corresponding to the products of probability spaces treated in §254.

325A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with measure algebras $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$. Let λ be the c.l.d. product measure on $X\times Y$, and Λ its domain; let $(\mathfrak{C},\bar\lambda)$ be the corresponding measure algebra.

(a)(i) The map $E\mapsto E\times Y:\Sigma\to\Lambda$ induces an order-continuous Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{C}$.

(ii) The map $F\mapsto X\times F:\mathrm{T}\to\Lambda$ induces an order-continuous Boolean homomorphism from $\mathfrak{B}$ to $\mathfrak{C}$.

(b) The map $(E,F)\mapsto E\times F:\Sigma\times\mathrm{T}\to\Lambda$ induces a Boolean homomorphism $\psi:\mathfrak{A}\otimes\mathfrak{B}\to\mathfrak{C}$.

(c) $\psi[\mathfrak{A}\otimes\mathfrak{B}]$ is topologically dense in $\mathfrak{C}$ for the measure-algebra topology of $\mathfrak{C}$.

(d) For every $c\in\mathfrak{C}$,

$$\bar\lambda c=\sup\{\bar\lambda(c\cap\psi(a\otimes b)):a\in\mathfrak{A},\,b\in\mathfrak{B},\,\bar\mu a<\infty,\,\bar\nu b<\infty\}.$$

(e) If μ and ν are semi-finite, ψ is injective and $\bar\lambda\psi(a\otimes b)=\bar\mu a\cdot\bar\mu b$ for every $a\in\mathfrak{A}$, $b\in\mathfrak{B}$.

proof (a) $E\times Y\in\Lambda$ for every $E\in\Sigma$ (251E), and $\lambda(E\times Y)=0$ whenever $\mu E=0$ (251Ia). Thus $E\mapsto(E\times Y)^\bullet:\Sigma\to\mathfrak{C}$ is a Boolean homomorphism with kernel including $\{E:\mu E=0\}$, so descends to a Boolean homomorphism $\varepsilon_1:\mathfrak{A}\to\mathfrak{C}$.

To see that ε_1 is order-continuous, let $A\subseteq\mathfrak{A}$ be a non-empty downwards-directed set with infimum 0. **?** If there is a non-zero lower bound c of $\varepsilon_1[A]$, express c as $W^\bullet$ where $W\in\Lambda$. We have $\lambda(W)>0$; by the definition of λ (251F), there are $G\in\Sigma$, $H\in\mathrm{T}$ such that $\mu G<\infty$, $\nu H<\infty$ and $\lambda(W\cap(G\times H))>0$. Of course $\inf_{a\in A}a\cap G^\bullet=0$ in $\mathfrak{A}$, so $\inf_{a\in A}\bar\mu(a\cap G^\bullet)=0$, by 321F; let $a\in A$ be such that $\bar\mu(a\cap G^\bullet)\cdot\nu H<\lambda(W\cap(G\times H))$. Express a as $E^\bullet$, where $E\in\Sigma$. Then $\lambda(W\setminus(E\times Y))=0$. But this means that

$$\lambda(W\cap(G\times H))\le\lambda((E\cap G)\times H)=\mu(E\cap G)\cdot\nu H=\bar\mu(a\cap G^\bullet)\cdot\nu H,$$

contradicting the choice of a. **X** Thus $\inf\varepsilon_1[A]=0$ in $\mathfrak{C}$; as A is arbitrary, ε_1 is order-continuous.

Similarly $\varepsilon_2:\mathfrak{B}\to\mathfrak{C}$, induced by $F\mapsto X\times F:\mathrm{T}\to\Lambda$, is order-continuous.

(b) Now there must be a corresponding Boolean homomorphism $\psi:\mathfrak{A}\otimes\mathfrak{B}\to\mathfrak{C}$ such that $\psi(a\otimes b)=\varepsilon_1a\cap\varepsilon_2b$ for every $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$, that is,

$$\psi(E^\bullet\otimes F^\bullet)=(E\times Y)^\bullet\cap(X\times F)^\bullet=(E\times F)^\bullet$$

for every $E\in\Sigma$, $F\in\mathrm{T}$ (315Jb).

(c) Suppose that c, $e\in\mathfrak{C}$, $\bar\lambda e<\infty$ and $\epsilon>0$. Express c, e as $U^\bullet$, $W^\bullet$ where U, $W\in\Lambda$. By 251Ie, there are $E_0,\dots,E_n\in\Sigma$, $F_0,\dots,F_n\in\mathrm{T}$, all of finite measure, such that $\lambda((U\cap W)\triangle\bigcup_{i\le n}E_i\times F_i)\le\epsilon$. Set

$$c_1=(\bigcup_{i\le n}E_i\times F_i)^\bullet\in\psi[\mathfrak{A}\otimes\mathfrak{B}];$$

then

$$\bar\lambda(e\cap(c\,\triangle\,c_1))=\lambda(W\cap(U\triangle\bigcup_{i\le n}E_i\times F_i))\le\epsilon.$$

As c, e and ϵ are arbitrary, $\psi[\mathfrak{A}\otimes\mathfrak{B}]$ is topologically dense in $\mathfrak{C}$.

(d) By the definition of λ, we have

$$\lambda W=\sup\{\lambda(W\cap(E\times F)):E\in\Sigma,\,F\in\mathrm{T},\,\mu E<\infty,\,\nu F<\infty\}$$

for every $W\in\Lambda$; so all we have to do is express c as $W^\bullet$.

(e) Now suppose that μ and ν are semi-finite. Then $\lambda(E\times F)=\mu E\cdot\nu F$ for any $E\in\Sigma$, $F\in\mathrm{T}$ (251J), so $\bar\lambda\psi(a\otimes b)=\bar\mu a\cdot\bar\nu b$ for every $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$.

To see that ψ is injective, take any non-zero $c\in\mathfrak{A}\otimes\mathfrak{B}$; then there must be non-zero $a\in\mathfrak{A}$, $b\in\mathfrak{B}$ such that $a\otimes b\subseteq c$ (315Kb), so that

$$\bar\lambda\psi c\ge\bar\lambda\psi(a\otimes b)=\bar\mu a\cdot\bar\nu b>0$$

and $\psi c\ne 0$.

325B Characterizing the measure algebra of a product space A very natural question to ask is, whether it is possible to define a 'measure algebra free product' of two abstract measure algebras in a way which will correspond to one of the constructions above. I give an example to show the difficulties involved.

Example There are complete locally determined localizable measure spaces (X,μ), (X',μ'), with isomorphic measure algebras, and a probability space (Y,ν) such that the measure algebras of the c.l.d. product measures on $X\times Y$, $X'\times Y$ are not isomorphic.

proof Let (X,Σ,μ) be the complete locally determined localizable not-strictly-localizable measure space described in 216E. Recall that, for $E\in\Sigma$, $\mu E=\#(\{\gamma:\gamma\in C,\,f_\gamma\in E\})$ if this is finite, ∞ otherwise (216Eb), where C is a set of cardinal greater than $\mathfrak{c}$. The map $E\mapsto\{\gamma:f_\gamma\in E\}:\Sigma\to\mathcal{P}C$ is surjective (216Ec), so descends to an isomorphism between $\mathfrak{A}$, the measure algebra of μ, and $\mathcal{P}C$. Let (X',Σ',μ') be C with counting measure, so that its measure algebra $(\mathfrak{A}',\bar\mu')$ is isomorphic to $(\mathfrak{A},\bar\mu)$, while μ' is of course strictly localizable.

Let (Y,T,ν) be $\{0,1\}^C$ with its usual measure. Let λ, λ' be the c.l.d. product measures on $X\times Y$, $X'\times Y$ respectively, and $(\mathfrak{C},\bar\lambda)$, $(\mathfrak{C}',\bar\lambda')$ the corresponding measure algebras. Then λ is not localizable (254U), so $(\mathfrak{C},\bar\lambda)$ is not localizable (322Be). On the other hand, λ', being the c.l.d. product of strictly localizable measures, is strictly localizable (251O), therefore localizable, so $(\mathfrak{C}',\bar\lambda')$ is localizable, and is not isomorphic to $(\mathfrak{C},\bar\lambda)$.

325C Thus there can be no universally applicable method of identifying the measure algebra of a product measure from the measure algebras of the factors. However, you have no doubt observed that the example above involves non-σ-finite spaces, and conjectured that this is not an accident. In contexts in which we know that the algebras involved are localizable, there are positive results available, such as the following.

Theorem Let (X_1,Σ_1,μ_1) and (X_2,Σ_2,μ_2) be semi-finite measure spaces, with measure algebras $(\mathfrak{A}_1,\bar\mu_1)$ and $(\mathfrak{A}_2,\bar\mu_2)$. Let λ be the c.l.d. product measure on $X_1\times X_2$, and $(\mathfrak{C},\bar\lambda)$ the corresponding measure algebra. Let $(\mathfrak{B},\bar\nu)$ be a localizable measure algebra, and $\phi_1:\mathfrak{A}_1\to\mathfrak{B}$, $\phi_2:\mathfrak{A}_2\to\mathfrak{B}$ order-continuous Boolean homomorphisms such that $\bar\nu(\phi_1(a_1)\cap\phi_2(a_2))=\bar\mu_1a_1\cdot\bar\mu_2a_2$ for all $a_1\in\mathfrak{A}_1$, $a_2\in\mathfrak{A}_2$. Then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi(\psi(a_1\otimes a_2))=\phi_1(a_1)\cap\phi_2(a_2)$ for all $a_1\in\mathfrak{A}_1$, $a_2\in\mathfrak{A}_2$, writing $\psi:\mathfrak{A}_1\otimes\mathfrak{A}_2\to\mathfrak{C}$ for the canonical map described in 325A.

proof (a) Because ψ is injective, it is an isomorphism between $\mathfrak{A}_1\otimes\mathfrak{A}_2$ and its image in $\mathfrak{C}$. I trust it will cause no confusion if I abuse notation slightly and treat $\mathfrak{A}_1\otimes\mathfrak{A}_2$ as actually a subalgebra of $\mathfrak{C}$. Now the Boolean homomorphisms ϕ_1, ϕ_2 correspond to a Boolean homomorphism $\theta:\mathfrak{A}_1\otimes\mathfrak{A}_2\to\mathfrak{B}$. The point is that $\bar\nu\theta c=\bar\lambda c$ for every $c\in\mathfrak{A}_1\otimes\mathfrak{A}_2$. **P** By 315Kb, every member of $\mathfrak{A}_1\otimes\mathfrak{A}_2$ is expressible as $\sup_{i\le n}a_i\otimes a_i'$, where $a_i\in\mathfrak{A}_1$, $a_i'\in\mathfrak{A}_2$ for each i and $\langle a_i\otimes a_i'\rangle_{i\le n}$ is disjoint. Now for each i we have

$$\bar\nu\theta(a_i\otimes a_i')=\bar\nu(\phi_1(a_i)\cap\phi_2(a_i'))=\bar\mu_1a_i\cdot\bar\mu_2a_i'=\bar\lambda(a_i\otimes a_i'),$$

by 325Ae. So

$$\bar\nu\theta(c)=\sum_{i=0}^n\bar\nu\theta(a_i\otimes a_i')=\sum_{i=0}^n\bar\lambda(a_i\otimes a_i')=\bar\lambda c.\ \textbf{Q}$$

(b) The following fact will underlie many of the arguments below. If $e\in\mathfrak{B}$, $\bar\nu e<\infty$ and $\epsilon>0$, there are $e_1\in\mathfrak{A}_1^f$, $e_2\in\mathfrak{A}_2^f$ such that $\bar\nu(e\setminus\theta(e_1\otimes e_2))\le\epsilon$, writing $\mathfrak{A}_i^f$ for $\{a:\bar\mu_ia<\infty\}$. **P** Because $(\mathfrak{A}_1,\bar\mu_1)$ is semi-finite, $\mathfrak{A}_1^f$ has supremum 1 in $\mathfrak{A}_1$; because ϕ_1 is order-continuous, $\sup\{\phi_1(a):a\in\mathfrak{A}_1^f\}=1$ in $\mathfrak{B}$, and $\inf\{e\setminus\phi_1(a):a\in\mathfrak{A}_1^f\}=0$ (313Aa). Because $\mathfrak{A}_1^f$ is upwards-directed, $\{e\setminus\phi_1(a):a\in\mathfrak{A}_1^f\}$ is downwards-directed, so $\inf\{\bar\nu(e\setminus\phi(a)):a\in\mathfrak{A}_1^f\}=0$ (321F again). Let $e_1\in\mathfrak{A}_1^f$ be such that $\bar\nu(e\setminus\phi_1(e_1))\le\frac12\epsilon$.

In the same way, there is an $e_2\in\mathfrak{A}_2^f$ such that $\bar\nu(e\setminus\phi_2(e_2))\le\frac12\epsilon$. Consider $e'=e_1\otimes e_2\in\mathfrak{C}$. Then

$$\bar\nu(e\setminus\theta e')=\bar\nu(e\setminus(\phi_1(e_1)\cap\phi_2(e_2)))\le\bar\nu(e\setminus\phi_1(e_1))+\bar\nu(e\setminus\phi_2(e_2))\le\epsilon.\ \textbf{Q}$$

(c) The next step is to check that θ is uniformly continuous for the measure-algebra uniformities defined by $\bar\nu$ and $\bar\lambda$. **P** Take any $e\in\mathfrak{B}^f$ and $\epsilon>0$. By (b), there are e_1, e_2 such that $\bar\lambda(e_1\otimes e_2)<\infty$ and $\bar\nu(e\setminus\theta(e_1\otimes e_2))\le\frac12\epsilon$. Set $e'=e_1\otimes e_2$. Now suppose that c, $c'\in\mathfrak{A}_1\otimes\mathfrak{A}_2$ and $\bar\lambda((c\mathbin{\triangle}c')\cap e')\le\frac12\epsilon$. Then

$$\bar\nu((\theta(c)\mathbin{\triangle}\theta(c'))\cap e)\le\bar\nu\theta((c\mathbin{\triangle}c')\cap e')+\bar\nu(e\setminus\theta e')\le\bar\lambda((c\mathbin{\triangle}c')\cap e')+\frac{1}{2}\epsilon\le\epsilon.$$

By 3A4Cc, θ is uniformly continuous for the subspace uniformity on $\mathfrak{A}_1\otimes\mathfrak{A}_2$. **Q**

(d) Recall that $\mathfrak{A}_1\otimes\mathfrak{A}_2$ is topologically dense in $\mathfrak{C}$ (325Ac), while $\mathfrak{B}$ is complete for its uniformity (323Gc). So there is a uniformly continuous function $\phi:\mathfrak{C}\to\mathfrak{B}$ extending θ (3A4G).

(e) Because θ is a Boolean homomorphism, so is ϕ. **P** (i) The functions $c \mapsto \phi(1 \setminus c)$, $c \mapsto 1 \setminus \phi(c)$ are continuous and the topology of $\mathfrak{B}$ is Hausdorff, so $\{c : \phi(1 \setminus c) = 1 \setminus \phi(c)\}$ is closed; as it includes $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, it must be the whole of $\mathfrak{C}$. (ii) The functions $(c, c') \mapsto \phi(c \cup c')$, $(c, c') \mapsto \phi(c) \cup \phi(c')$ are continuous, so $\{(c, c') : \phi(c \cup c') = \phi(c) \cup \phi(c')\}$ is closed in $\mathfrak{C} \times \mathfrak{C}$; as it includes $(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \times (\mathfrak{A}_1 \otimes \mathfrak{A}_2)$, it must be the whole of $\mathfrak{C} \times \mathfrak{C}$. **Q**

(f) Because θ is measure-preserving, so is ϕ. **P** Take any $e_1 \in \mathfrak{A}_1^f$, $e_2 \in \mathfrak{A}_2^f$. Then the functions $c \mapsto \bar{\lambda}(c \cap (e_1 \otimes e_2))$, $c \mapsto \bar{\nu}\phi(c \cap (e_1 \otimes e_2))$ are continuous and equal on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, so are equal on $\mathfrak{C}$. The argument of (b) shows that for any $b \in \mathfrak{B}$,

$$\begin{aligned}\bar{\nu}b &= \sup\{\bar{\nu}(b \cap e) : e \in \mathfrak{B}^f\}\\ &= \sup\{\bar{\nu}(b \cap \phi(e_1 \otimes e_2)) : e_1 \in \mathfrak{A}_1^f,\ e_2 \in \mathfrak{A}_2^f\},\end{aligned}$$

so that

$$\begin{aligned}\bar{\nu}\phi(c) &= \sup\{\bar{\nu}\phi(c \cap (e_1 \otimes e_2)) : e_1 \in \mathfrak{A}_1^f,\ e_2 \in \mathfrak{A}_2^f\}\\ &= \sup\{\bar{\lambda}(c \cap (e_1 \otimes e_2)) : e_1 \in \mathfrak{A}_1^f,\ e_2 \in \mathfrak{A}_2^f\} = \bar{\lambda}c\end{aligned}$$

for every $c \in \mathfrak{C}$. **Q**

(g) To see that ϕ is order-continuous, take any non-empty downwards-directed set $C \subseteq \mathfrak{C}$ with infimum 0. **?** If $\phi[C]$ has a non-zero lower bound b in $\mathfrak{B}$, let $e \subseteq b$ be such that $0 < \bar{\nu}e < \infty$. Let $e' \in \mathfrak{C}$ be such that $\bar{\lambda}e' < \infty$ and $\bar{\nu}(e \setminus \phi(e')) < \bar{\nu}e$, as in (b) above, so that $\bar{\nu}(e \cap \phi(e')) > 0$. Now, because $\inf C = 0$, there is a $c \in C$ such that $\bar{\lambda}(c \cap e') < \bar{\nu}(e \cap \phi(e'))$. But this means that

$$\bar{\nu}(b \cap \phi(e')) \le \bar{\nu}\phi(c \cap e') = \bar{\lambda}(c \cap e') < \bar{\nu}(e \cap \phi(e')) \le \bar{\nu}(b \cap \phi(e')),$$

which is absurd. **X** Thus $\inf \phi[C] = 0$ in $\mathfrak{B}$. As C is arbitrary, ϕ is order-continuous.

(h) Finally, to see that ϕ is unique, observe that any order-continuous Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$ must be continuous (324Fc); so that if it agrees with ϕ on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ it must agree with ϕ on $\mathfrak{C}$.

325D Theorem Let $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be semi-finite measure algebras.

(a) There is a localizable measure algebra $(\mathfrak{C}, \bar{\lambda})$, together with order-continuous Boolean homomorphisms $\varepsilon_1 : \mathfrak{A}_1 \to \mathfrak{C}$ and $\varepsilon_2 : \mathfrak{A}_2 \to \mathfrak{C}$, such that whenever $(\mathfrak{B}, \bar{\nu})$ is a localizable measure algebra, and $\phi_1 : \mathfrak{A}_1 \to \mathfrak{B}$, $\phi_2 : \mathfrak{A}_2 \to \mathfrak{B}$ are order-continuous Boolean homomorphisms and $\bar{\nu}(\phi_1(a_1) \cap \phi_2(a_2)) = \bar{\mu}_1 a_1 \cdot \bar{\mu}_2 a_2$ for all $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$, then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi\varepsilon_j = \phi_j$ for both j.

(b) The structure $(\mathfrak{C}, \bar{\lambda}, \varepsilon_1, \varepsilon_2)$ is determined up to isomorphism by this property.

(c)(i) The Boolean homomorphism $\psi : \mathfrak{A}_1 \otimes \mathfrak{A}_2 \to \mathfrak{C}$ defined from ε_1 and ε_2 is injective, and $\psi[\mathfrak{A}_1 \otimes \mathfrak{A}_2]$ is topologically dense in $\mathfrak{C}$.

(ii) The closed subalgebra of $\mathfrak{C}$ generated by $\psi[\mathfrak{A}_1 \otimes \mathfrak{A}_2]$ is the whole of $\mathfrak{C}$.

(d) If $j \in \{1, 2\}$ and $(\mathfrak{A}_j, \bar{\mu}_j)$ is localizable, then $\varepsilon_j[\mathfrak{A}_j]$ is a closed subalgebra of $(\mathfrak{C}, \bar{\lambda})$.

proof (a)(i) We may regard $(\mathfrak{A}_1, \bar{\mu}_1)$ as the measure algebra of (Z_1, Σ_1, μ_1) where Z_1 is the Stone space of $\mathfrak{A}_1$, Σ_1 is the algebra of subsets of Z_1 differing from an open-and-closed set by a meager set, and μ_1 is an appropriate measure (321K). Note that in this representation, each $a \in \mathfrak{A}_1$ becomes identified with $\widehat{a}^\bullet$, where $\widehat{a}$ is the open-and-closed subset of Z_1 corresponding to a. Similarly, we may think of $(\mathfrak{A}_2, \bar{\mu}_2)$ as the measure algebra of (Z_2, Σ_2, μ_2), where Z_2 is the Stone space of $\mathfrak{A}_2$.

(ii) Let λ be the c.l.d. product measure on $Z_1 \times Z_2$. The point is that λ is strictly localizable. **P** By 322Ea, both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have partitions of unity consisting of elements of finite measure; let $\langle c_i \rangle_{i \in I}$, $\langle d_j \rangle_{j \in J}$ be such partitions. Then $\langle \widehat{c}_i \times \widehat{d}_j \rangle_{i \in I, j \in J}$ is a disjoint family of sets of finite measure in $Z_1 \times Z_2$. If $W \subseteq Z_1 \times Z_2$ is such that $\lambda W > 0$, there must be sets E_1, E_2 of finite measure such that $\lambda(W \cap (E_1 \times E_2)) > 0$. Because $E_1^\bullet = \sup_{i \in I} E_1^\bullet \cap c_i$, we must have

$$\mu_1 E_1 = \bar{\mu}_1 E_1^\bullet = \sum_{i \in I} \bar{\mu}_1(E_1^\bullet \cap c_i) = \sum_{i \in I} \mu_1(E_1 \cap \widehat{c}_i).$$

Similarly, $\mu_2 E_2 = \sum_{i \in J} \mu_2(E_2 \cap \widehat{d}_j)$. But this means that there must be finite $I' \subseteq I$, $J' \subseteq J$ such that

$$\sum_{i \in I', j \in J'} \mu_1(E_1 \cap \widehat{c}_i)\mu_2(E_2 \cap \widehat{d}_j) > \mu_1 E_1 \cdot \mu_2 E_2 - \lambda(W \cap (E_1 \times E_2)),$$

so that there have to be $i \in I'$, $j \in J'$ such that $\lambda(W \cap (\widehat{c}_i \times \widehat{d}_j)) > 0$.

Now this means that $\langle \widehat{c}_i \times \widehat{d}_j \rangle_{i\in I, j\in J}$ satisfies the conditions of 213O. Because λ is surely complete and locally determined, it is strictly localizable. **Q**

(iii) We may therefore take $(\mathfrak{C}, \bar\lambda)$ to be just the measure algebra of λ. The maps ε_1, ε_2 will be the canonical maps described in 325Aa, inducing the map $\psi : \mathfrak{A}_1 \otimes \mathfrak{A}_2 \to \mathfrak{C}$ referred to in 325C; and 325C now gives the result.

(b) This is nearly obvious. Suppose we had an alternative structure $(\mathfrak{C}', \bar\lambda', \varepsilon_1', \varepsilon_2')$ with the same property. Then we must have an order-continuous measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{C}'$ such that $\phi\varepsilon_j = \varepsilon_j'$ for both j; and similarly we have an order-continuous measure-preserving Boolean homomorphism $\phi' : \mathfrak{C}' \to \mathfrak{C}$ such that $\phi'\varepsilon_j' = \varepsilon_j$ for both j. Now $\phi'\phi : \mathfrak{C} \to \mathfrak{C}$ is an order-continuous measure-preserving Boolean homomorphism such that $\phi'\varepsilon_j = \varepsilon_j$ for both j. By the uniqueness assertion in (a), applied with $\mathfrak{B} = \mathfrak{C}$, $\phi'\phi$ must be the identity on $\mathfrak{C}$. In the same way, $\phi\phi'$ is the identity on $\mathfrak{C}'$. So ϕ and ϕ' are the two halves of the required isomorphism.

(c) In view of the construction for $\mathfrak{C}$ offered in part (a) of the proof, (i) is just a consequence of 325Ac and 325Ae. Now (ii) follows by 323J.

(d) If $\mathfrak{A}_j$ is Dedekind complete then $\varepsilon_j[\mathfrak{A}_j]$ is order-closed in $\mathfrak{C}$ because ε_j is order-continuous (314F(a-i)).

325E Remarks (a) We could say that a measure algebra $(\mathfrak{C}, \bar\lambda)$, together with embeddings ε_1 and ε_2, as described in 325D, is a **localizable measure algebra free product** of $(\mathfrak{A}_1, \bar\mu_1)$ and $(\mathfrak{A}_2, \bar\mu_2)$; and its uniqueness up to isomorphism makes it safe, most of the time, to call it 'the' localizable measure algebra free product. Observe that it can equally well be regarded as the uniform space completion of the algebraic free product; see 325Yc.

(b) As the example in 325B shows, the localizable measure algebra free product of the measure algebras of given measure spaces need not appear directly as the measure algebra of their product. But there is one context in which it must so appear: if the product measure is localizable, 325C tells us at once that it has the right measure algebra. For σ-finite measure algebras, of course, any corresponding measure spaces have to be strictly localizable, so again we can use the product measure directly.

325F I ought not to proceed to the next topic without giving another pair of examples to show the subtlety of the concept of 'measure algebra free product'.

Example Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of Lebesgue measure μ on $[0,1]$, and $(\mathfrak{C}, \bar\lambda)$ the measure algebra of Lebesgue measure λ on $[0,1]^2$. Then $(\mathfrak{C}, \bar\lambda)$ can be regarded as the localizable measure algebra free product of $(\mathfrak{A}, \bar\mu)$ with itself, by 251N and 325Eb. Let $\psi : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{C}$ be the canonical map, as described in 325A. Then $\psi[\mathfrak{A} \otimes \mathfrak{A}]$ is not order-dense in $\mathfrak{C}$, and ψ is not order-continuous.

proof (a) Let $\langle \epsilon_n \rangle_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ such that $\sum_{n=0}^\infty \epsilon_n = \infty$, but $\sum_{n=0}^\infty \epsilon_n^2 < 1$; for instance, we could take $\epsilon_n = \frac{1}{n+2}$. Let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a stochastically independent sequence of measurable subsets of $[0,1]$ such that $\mu E_n = \epsilon_n$ for each n. In $\mathfrak{A}$ set $a_n = E_n^\bullet$, and consider $c_n = \sup_{i\le n} a_i \otimes a_i \in \mathfrak{A} \otimes \mathfrak{A}$ for each n.

(b) We have $\sup_{n\in\mathbb{N}} c_n = 1$ in $\mathfrak{A} \otimes \mathfrak{A}$. **P?** Otherwise, there is a non-zero $a \in \mathfrak{A} \otimes \mathfrak{A}$ such that $a \cap (a_n \otimes a_n) = 0$ for every n, and now there are non-zero b, $b' \in \mathfrak{A}$ such that $b \otimes b' \subseteq a$. Set $I = \{n : a_n \cap b = 0\}$, $J = \{n : a_n \cap b'\} = 0$. Then $\langle E_n \rangle_{n\in I}$ is an independent family and $\mu(\bigcup_{n\in I} E_i) \le 1 - \bar\mu b < 1$, so $\sum_{n\in I} \mu E_n < \infty$, by the Borel-Cantelli lemma (273K). Similarly $\sum_{n\in J} \mu E_n < \infty$. Because $\sum_{n\in\mathbb{N}} \mu E_n = \infty$, there must be some $n \in \mathbb{N} \setminus (I \cup J)$. Now $a_n \cap b$ and $a_n \cap b'$ are both non-zero, so

$$0 \ne (a_n \cap b) \otimes (a_n \cap b') = (a_n \otimes a_n) \cap (b \otimes b') = 0,$$

which is absurd. **XQ**

(c) On the other hand,

$$\sum_{n=0}^\infty \bar\lambda\psi(c_n) \le \sum_{n=0}^\infty (\bar\mu a_n)^2 = \sum_{n=0}^\infty \epsilon_n^2 < 1,$$

by the choice of the ϵ_n. So $\sup_{n\in\mathbb{N}} \psi(c_n)$ cannot be 1 in $\mathfrak{C}$.

Thus ψ is not order-continuous.

(d) By 313P(a-ii) and 313O, $\psi[\mathfrak{A} \otimes \mathfrak{A}]$ cannot be order-dense in $\mathfrak{C}$; alternatively, (b) shows that there can be no non-zero member of $\psi[\mathfrak{A} \otimes \mathfrak{A}]$ included in $1 \setminus \sup_{n\in\mathbb{N}} \psi(c_n)$. (Both these arguments rely tacitly on the fact that ψ is injective, as noted in 325Ae.)

325G Since 325F shows that the free product and the localizable measure algebra free product are very different constructions, I had better repeat an idea from §315 in the new context.

Example Again, let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1]$, and $(\mathfrak{C},\bar{\lambda})$ the measure algebra of Lebesgue measure on $[0,1]^2$. Then there is no order-continuous Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{A}$ such that $\phi(a\otimes b)=a\cap b$ for all a, $b\in\mathfrak{A}$. **P** Let $\phi:\mathfrak{C}\to\mathfrak{A}$ be a Boolean homomorphism such that $\phi(a\otimes b)=a\cap b$ for all a, $b\in\mathfrak{A}$. For $i<2^n$ let a_{ni} be the equivalence class in $\mathfrak{A}$ of the interval $[2^{-n}i,2^{-n}(i+1)]$, and set $c_n=\sup_{i<2^n}a_{ni}\otimes a_{ni}$. Then $\phi c_n=1$ for every n, but $\bar{\lambda}c_n=2^{-n}$ for each n, so $\inf_{n\in\mathbb{N}}c_n=0$ in $\mathfrak{C}$; thus ϕ cannot be order-continuous. **Q** (Compare 315Q.)

***325H Products of more than two factors** We can of course extend the ideas of 325A, 325C and 325D to products of any finite number of factors. No new ideas are needed, so I spell the results out without proofs.

(a) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a non-empty finite family of semi-finite measure algebras. Then there is a localizable measure algebra $(\mathfrak{C},\bar{\lambda})$, together with order-continuous Boolean homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ for $i\in I$, such that whenever $(\mathfrak{B},\bar{\nu})$ is a localizable measure algebra, and $\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ are order-continuous Boolean homomorphisms such that $\bar{\nu}(\inf_{i\in I}\phi_i(a_i))=\prod_{i\in I}\bar{\mu}_ia_i$ whenever $a_i\in\mathfrak{A}_i$ for each i, then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\varepsilon_i=\phi_i$ for every i.

(b) The structure $(\mathfrak{C},\bar{\lambda},\langle\varepsilon_i\rangle_{i\in I})$ is determined up to isomorphism by this property.

(c) The Boolean homomorphism $\psi:\bigotimes_{i\in I}\mathfrak{A}_i\to\mathfrak{C}$ defined from the ε_i is injective, and $\psi[\bigotimes_{i\in I}\mathfrak{A}_i]$ is topologically dense in $\mathfrak{C}$.

(d) Write $\widehat{\bigotimes}^{\text{loc}}_{i\in I}(\mathfrak{A}_i,\bar{\mu}_i)$ for (a particular version of) the localizable measure algebra free product described in (a). If $\langle(A_i,\bar{\mu}_i)\rangle_{i\in I}$ is a finite family of semi-finite measure algebras and $\langle I_k\rangle_{k\in K}$ is a partition of I into non-empty sets, then $\widehat{\bigotimes}^{\text{loc}}_{i\in I}(\mathfrak{A}_i,\bar{\mu}_i)$ is isomorphic, in a canonical way, to $\widehat{\bigotimes}^{\text{loc}}_{k\in K}\bigl(\widehat{\bigotimes}^{\text{loc}}_{i\in I_k}(\mathfrak{A}_i,\bar{\mu}_i)\bigr)$.

(e) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a finite family of semi-finite measure spaces, and write $(\mathfrak{A}_i,\bar{\mu}_i)$ for the measure algebra of (X_i,Σ_i,μ_i). Let λ be the c.l.d. product measure on $\prod_{i\in I}X_i$ (251W), and $(\mathfrak{C},\bar{\lambda})$ the corresponding measure algebra. Then there is a canonical order-continuous measure-preserving embedding of $(\mathfrak{C},\bar{\lambda})$ into the localizable measure algebra free product of the $(\mathfrak{A}_i,\bar{\mu}_i)$. If each μ_i is strictly localizable, this embedding is an isomorphism.

325I Infinite products Just as in §254, we can now turn to products of infinite families of probability algebras.

Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any family of probability spaces, with measure algebras $(\mathfrak{A}_i,\bar{\mu}_i)$. Let λ be the product measure on $X=\prod_{i\in I}X_i$, and $(\mathfrak{C},\bar{\lambda})$ the corresponding measure algebra. For each $i\in I$, we have a measure-preserving homomorphism $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ corresponding to the inverse-measure-preserving function $x\mapsto x(i):X\to X_i$. Let $(\mathfrak{B},\bar{\nu})$ be a probability algebra, and $\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ Boolean homomorphisms such that $\bar{\nu}(\inf_{i\in J}\phi_i(a_i))=\prod_{i\in J}\bar{\mu}_ia_i$ whenever $J\subseteq I$ is a finite set and $a_i\in\mathfrak{A}_i$ for every i. Then there is a unique measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\varepsilon_i=\phi_i$ for every $i\in I$.

proof (a) As remarked in 254Fb, all the maps $x\mapsto x(i)$ are inverse-measure-preserving, so correspond to measure-preserving homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ (324M). It will be helpful to use some notation from §254. Write $\mathcal{C}$ for the family of measurable cylinders in X expressible in the form

$$E=\{x:x\in X,\,x(i)\in E_i\text{ for every }i\in J\},$$

where $J\subseteq I$ is finite and $E_i\in\Sigma_i$ for every $i\in J$. Note that in this case

$$E^\bullet=\inf_{i\in J}\varepsilon_i(E_i^\bullet).$$

Set

$$C=\{E^\bullet:E\in\mathcal{C}\}\subseteq\mathfrak{C},$$

so that C is precisely the family of elements of $\mathfrak{C}$ expressible in the form $\inf_{i\in J}\phi_i(a_i)$ where $J\subseteq I$ is finite and $a_i\in\mathfrak{A}_i$ for each i.

The homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ define a Boolean homomorphism $\psi:\bigotimes_{i\in I}\mathfrak{A}_i\to\mathfrak{C}$ (315J), which is injective. **P** If $c\in\bigotimes_{i\in I}\mathfrak{A}_i$ is non-zero, there must be a finite set $J\subseteq I$ and a family $\langle a_i\rangle_{i\in J}$ such that $a_i\in\mathfrak{A}_i\setminus\{0\}$ for each i and $c\supseteq\inf_{i\in J}\tilde{\varepsilon}_i(a_i)$, where for the moment I write $\tilde{\varepsilon}_i$ for the canonical map from $\mathfrak{A}_i$ to $\bigotimes_{i\in I}\mathfrak{A}_i$ (315Kb). Express each a_i as $E_i^\bullet$, where $E_i\in\Sigma_i$. Then

$$E = \{x : x \in X,\, x(i) \in E_i \text{ for each } i \in J\}$$

has measure

$$\lambda E = \prod_{i\in J} \mu E_i = \prod_{i\in J} \bar\mu a_i \neq 0,$$

while

$$E^\bullet = \psi(\inf_{i\in J} \tilde\varepsilon_i(a_i)) \subseteq \psi(c),$$

so $\psi(c) \neq 0$. As c is arbitrary, ψ is injective. **Q**

(b) Because ψ is injective, it is an isomorphism between $\bigotimes_{i\in I} \mathfrak{A}_i$ and its image in $\mathfrak{C}$. I trust it will cause no confusion if I abuse notation slightly and treat $\bigotimes_{i\in I} \mathfrak{A}_i$ as actually a subalgebra of $\mathfrak{C}$, so that $\varepsilon_j : \mathfrak{A}_j \to \mathfrak{C}$ becomes identified with $\tilde\varepsilon_j : \mathfrak{A}_j \to \bigotimes_{i\in I} \mathfrak{A}_i$. Now the Boolean homomorphisms $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ correspond to a Boolean homomorphism $\theta : \bigotimes_{i\in I} \mathfrak{A}_i \to \mathfrak{B}$. The point is that $\bar\nu\theta(c) = \bar\lambda c$ for every $c \in \bigotimes_{i\in I} \mathfrak{A}_i$. **P** Suppose to begin with that $c \in C$. Then we have $c = E^\bullet$, where $E = \{x : x(i) \in E_i\ \forall\ i \in J\}$ and $E_i \in \Sigma_i$ for each $i \in J$. So

$$\begin{aligned}\bar\lambda c = \lambda E = \prod_{i\in J} \mu E_i = \prod_{i\in J} \bar\mu_i E_i^\bullet = \bar\nu(\inf_{i\in J} \phi a_i)\\ = \bar\nu(\inf_{i\in J} \theta\varepsilon_i(a_i)) = \bar\nu\theta(\inf_{i\in J} \varepsilon_i(a_i)) = \bar\nu\theta(c).\end{aligned}$$

Next, any $c \in \bigotimes_{i\in I} \mathfrak{A}_i$ is expressible as the supremum of a finite disjoint family $\langle c_k\rangle_{k\in K}$ in C (315Kb), so

$$\bar\nu\theta(c) = \sum_{k\in K} \bar\nu\theta(c_k) = \sum_{k\in K} \bar\lambda(c_k) = \bar\lambda c. \ \mathbf{Q}$$

(c) It follows that θ is uniformly continuous for the measure metrics defined by $\bar\nu$ and $\bar\lambda$, since

$$\bar\nu(\theta(c) \triangle \theta(c')) = \bar\nu\theta(c \triangle c') = \bar\lambda(c \triangle c')$$

for all c, $c' \in \bigotimes_{i\in I} \mathfrak{A}_i$.

(d) Next, $\bigotimes_{i\in I} \mathfrak{A}_i$ is topologically dense in $\mathfrak{C}$. **P** Let $c \in \mathfrak{C}$, $\epsilon > 0$. Express c as $W^\bullet$. Then by 254Fe there are $H_0, \dots , H_k \in \mathcal{C}$ such that $\lambda(W \triangle \bigcup_{j\le k} H_j) \le \epsilon$. Now $c_j = H_j^\bullet \in C$ for each j, so

$$c' = \sup_{j\le k} c_j = (\bigcup_{j\le k} H_j)^\bullet \in \bigotimes_{i\in I} \mathfrak{A}_i,$$

and $\bar\lambda(c \triangle c') \le \epsilon$. **Q**

Since $\mathfrak{B}$ is complete for its uniformity (323Gc), there is a uniformly continuous function $\phi : \mathfrak{C} \to \mathfrak{B}$ extending θ (3A4G).

(e) Because θ is a Boolean homomorphism, so is ϕ. **P** (i) The functions $c \mapsto \phi(1 \setminus c)$, $1 \setminus \phi(c)$ are continuous and the topology of $\mathfrak{B}$ is Hausdorff, so $\{c : \phi(1 \setminus c) = 1 \setminus \phi(c)\}$ is closed; as it includes $\bigotimes_{i\in I} \mathfrak{A}_i$, it must be the whole of $\mathfrak{C}$. (ii) The functions $(c, c') \mapsto \phi(c \cup c')$, $(c, c') \mapsto \phi(c) \cup \phi(c')$ are continuous, so $\{(c, c') : \phi(c \cup c') = \phi(c) \cup \phi(c')\}$ is closed in $\mathfrak{C} \times \mathfrak{C}$; as it includes $\bigotimes_{i\in I} \mathfrak{A}_I \times \bigotimes_{i\in I} \mathfrak{A}_i$, it must be the whole of $\mathfrak{C} \times \mathfrak{C}$. **Q**

(f) Because θ is measure-preserving, so is ϕ. **P** The functions $c \mapsto \bar\lambda c$, $c \mapsto \bar\nu\phi(c)$ are continuous and equal on $\bigotimes_{i\in I} \mathfrak{A}_i$, so are equal on $\mathfrak{C}$. **Q**

(g) Finally, to see that ϕ is unique, observe that any measure-preserving Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$ must be continuous, so that if it agrees with ϕ on $\bigotimes_{i\in I} \mathfrak{A}_i$ it must agree with ϕ on $\mathfrak{C}$.

325J Of course this leads at once to a result corresponding to 325D.

Theorem Let $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras.

(a) There is a probability algebra $(\mathfrak{C}, \bar\lambda)$, together with measure-preserving Boolean homomorphisms $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{C}$ for $i \in I$, such that whenever $(\mathfrak{B}, \bar\nu)$ is a probability algebra, and $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ are Boolean homomorphisms such that $\bar\nu(\inf_{i\in J} \phi_i(a_i)) = \prod_{i\in J} \bar\mu_i a_i$ whenever $J \subseteq I$ is finite and $a_i \in \mathfrak{A}_i$ for each $i \in J$, then there is a unique measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi\varepsilon_i = \phi_i$ for every $i \in I$.

(b) The structure $(\mathfrak{C}, \bar\lambda, \langle\varepsilon_i\rangle_{i\in I})$ is determined up to isomorphism by this property.

(c) The Boolean homomorphism $\psi : \bigotimes_{i\in I} \mathfrak{A}_i \to \mathfrak{C}$ defined from the ε_i is injective, and $\psi[\bigotimes_{i\in I} \mathfrak{A}_i]$ is topologically dense in $\mathfrak{C}$.

proof For (a) and (c), all we have to do is represent each $(\mathfrak{A}_i, \bar\mu_i)$ as the measure algebra of a probability space, and apply 325I. The uniqueness of $\mathfrak{C}$ and the ε_i follows from the uniqueness of the homomorphisms ϕ, as in 325Db.

325K Definition As in 325Ea, we can say that $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$ is a, or the, **probability algebra free product** of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$.

325L Independent subalgebras If $(\mathfrak{A},\bar\mu)$ is a probability algebra, we say that a family $\langle\mathfrak{B}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is **stochastically independent** if $\bar\mu(\inf_{i\in J}b_i)=\prod_{i\in J}\bar\mu b_i$ whenever $J\subseteq I$ is finite and $b_i\in\mathfrak{B}_i$ for each i. (Compare 272Ab.) If every $\mathfrak{B}_i$ is closed, so that $(\mathfrak{B}_i,\bar\mu\restriction\mathfrak{B}_i)$ is a probability algebra, the identity maps $\iota_i:\mathfrak{B}_i\to\mathfrak{A}$ satisfy the conditions of the universal mapping theorem 325Ja, so we have a probability algebra free product $(\mathfrak{C},\bar\mu\restriction\mathfrak{C},\langle\iota_i\rangle_{i\in I})$ of $\langle(\mathfrak{B}_i,\bar\mu\restriction\mathfrak{B}_i)\rangle_{i\in I}$, where $\mathfrak{C}=\bigvee_{i\in I}\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{B}_i$.

Conversely, if $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ is any family of probability algebras with probability algebra free product $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$, then $\langle\varepsilon_i[\mathfrak{A}_i]\rangle_{i\in I}$ is an independent family of closed subalgebras of $\mathfrak{C}$. (Compare 272J, 315Xp.)

325M We can now make a general trawl through Chapters 25 and 27 seeking results which can be expressed in the language of this section. I give some in 325Xf-325Xi. Some ideas from §254 which are thrown into sharper relief by a reformulation are in the following theorem.

Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras and $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$ their probability algebra free product. For $J\subseteq I$ let $\mathfrak{C}_J=\bigvee_{i\in J}\varepsilon_i[\mathfrak{A}_i]$ be the closed subalgebra of $\mathfrak{C}$ generated by $\bigcup_{i\in J}\varepsilon_i[\mathfrak{A}_i]$.

(a) For any $J\subseteq I$, $(\mathfrak{C}_J,\bar\lambda\restriction\mathfrak{C}_J,\langle\varepsilon_i\rangle_{i\in J})$ is a probability algebra free product of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in J}$.

(b)(i) For any $c\in\mathfrak{C}$, there is a unique smallest $J_c\subseteq I$ such that $c\in\mathfrak{C}_{J_c}$, and this J_c is countable.

(ii) If c, $d\in\mathfrak{C}$ and $c\subseteq d$, then there is an $e\in\mathfrak{C}_{J_c\cap J_d}$ such that $c\subseteq e\subseteq d$.

(c) For any non-empty family $\mathcal{J}\subseteq\mathcal{P}I$, $\bigcap_{J\in\mathcal{J}}\mathfrak{C}_J=\mathfrak{C}_{\bigcap\mathcal{J}}$.

proof (a) If $(\mathfrak{B},\bar\nu,\langle\phi_i\rangle_{i\in J})$ is any probability algebra free product of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in J}$, then we have a measure-preserving homomorphism $\psi:\mathfrak{B}\to\mathfrak{C}$ such that $\psi\phi_i=\varepsilon_i$ for every $i\in J$. Because the subalgebra $\mathfrak{B}_0$ of $\mathfrak{B}$ generated by $\bigcup_{i\in J}\phi_i[\mathfrak{A}_i]$ is topologically dense in $\mathfrak{B}$ (325Jc), and ψ is continuous (324Kb), $\bigcup_{i\in J}\varepsilon_i[\mathfrak{A}_i]$ is topologically dense in $\psi[\mathfrak{B}]$; also $\psi[\mathfrak{B}]$ is closed in $\mathfrak{C}$ (324Kb again). But this means that $\psi[\mathfrak{B}]$ is just the topological closure of $\bigcup_{i\in I}\varepsilon_i[\mathfrak{A}_i]$ and must be $\mathfrak{C}_J$. Thus ψ is an isomorphism, and

$$(\mathfrak{C}_J,\bar\lambda\restriction\mathfrak{C}_J,\langle\varepsilon_i\rangle_{i\in J})=(\psi[\mathfrak{B}],\bar\nu\psi^{-1},\langle\psi\phi_i\rangle_{i\in J})$$

also is a probability algebra free product of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in J}$.

(b) As in 325J, we may suppose that each $(\mathfrak{A}_i,\bar\mu_i)$ is the measure algebra of a probability space (X_i,Σ_i,μ_i), and that $\mathfrak{C}$ is the measure algebra of their product (X,Λ,λ). For $J\subseteq I$ let Λ_J be the set of members of Λ which are determined by coordinates in J. Then $\{x:x(i)\in E\}\in\Lambda_J$ for every $i\in J$ and $E\in\Sigma_i$; so $\{U^\bullet:U\in\Lambda_J\}$ is a closed subalgebra of $\mathfrak{C}$ including $\varepsilon_i[\mathfrak{A}_i]$ for every $i\in J$, and therefore including $\mathfrak{C}_J$. On the other hand, as observed in 254Ob, any member of Λ_J is approximated, in measure, by sets in the σ-algebra T_J generated by sets of the form $\{x:x(i)\in E\}$ where $i\in J$ and $E\in\Sigma_i$. Of course $\mathrm{T}_J\subseteq\Lambda_J$, so $\{W^\bullet:W\in\Lambda_J\}=\{W^\bullet:W\in\mathrm{T}_J\}$ is the closed subalgebra of $\mathfrak{C}$ generated by $\bigcup_{i\in J}\varepsilon_i[\mathfrak{A}_i]$, which is $\mathfrak{C}_J$.

(i) Let $W\in\Lambda$ be such that $c=W^\bullet$. By 254Rd, there is a smallest $J_c\subseteq I$ such that $W\triangle U$ is negligible for some $U\in\Lambda_{J_c}$, and J_c is countable. By the remarks above, J_c is also the unique smallest subset of I such that $c\in\mathfrak{C}_{J_c}$.

(ii) Let $U\in\Lambda_{J_c}$, $V\in\Lambda_{J_d}$ be such that $c=U^\bullet$ and $d=V^\bullet$. We can think of λ as a product $\lambda'\times\lambda''$ where λ' is the product measure on $X'=\prod_{i\in J_d}X_i$ and λ'' is the product measure on $X''=\prod_{i\in I\setminus J_d}X_i$ (254N). Express V as $V_0\times X''$ where $V_0\subseteq X'$ belongs to the domain of λ' (254Ob). Consider

$$W_0=\{z:z\in X',\,\{w:w\in X'',\,(z,w)\in U\}\text{ is not }\lambda''\text{-negligible}\};$$

then W_0 is measured by λ', by Fubini's theorem (252B or 252D). Because $c\subseteq d$, $U\setminus V$ is λ-negligible and $W_0\setminus V_0$ is λ'-negligible, while W_0 is determined by coordinates in $J_c\cap J_d$. So $W=W_0\times X''$ also is determined by coordinates in $J_c\cap J_d$, while $U\setminus W$ and $W\setminus V$ are λ-negligible. We can therefore take $e=W^\bullet$.

(c) Of course $\mathfrak{C}_K\subseteq\mathfrak{C}_J$ whenever $K\subseteq J\subseteq I$, so $\bigcap_{J\in\mathcal{J}}\mathfrak{C}_J\supseteq\mathfrak{C}_{\bigcap\mathcal{J}}$. On the other hand, suppose that $c\in\bigcap_{J\in\mathcal{J}}\mathfrak{C}_J$; then by (b-i) there is some $K\subseteq\bigcap\mathcal{J}$ such that $c\in\mathfrak{C}_K\subseteq\mathfrak{C}_{\bigcap\mathcal{J}}$. As c is arbitrary, $\bigcap_{J\in\mathcal{J}}\mathfrak{C}_J=\mathfrak{C}_{\bigcap\mathcal{J}}$.

***325N Notation** In this context, I will say that an element c of $\mathfrak{C}$ is **determined by coordinates in** J if $c\in\mathfrak{C}_J$.

325X Basic exercises (a) Let $(\mathfrak{A}_1,\bar{\mu}_1)$, $(\mathfrak{A}_2,\bar{\mu}_2)$ be two semi-finite measure algebras, and suppose that for each j we are given a closed subalgebra $\mathfrak{B}_j$ of $\mathfrak{A}_j$ such that $(\mathfrak{B}_j,\bar{\nu}_j)$ also is semi-finite, where $\bar{\nu}_j=\bar{\mu}_j\restriction\mathfrak{B}_j$. Show that the localizable measure algebra free product $(\mathfrak{B}_1,\bar{\nu}_1)\widehat{\otimes}_{\text{loc}}(\mathfrak{B}_2,\bar{\nu}_2)$ can be thought of as a closed subalgebra of $(\mathfrak{A}_1,\bar{\mu}_1)\widehat{\otimes}_{\text{loc}}(\mathfrak{A}_2,\bar{\mu}_2)$.

(b) Let $(\mathfrak{A}_1,\bar{\mu}_1)$ and $(\mathfrak{A}_2,\bar{\mu}_2)$ be two semi-finite measure algebras, and suppose that for each j we are given a principal ideal $\mathfrak{B}_j$ of $\mathfrak{A}_j$. Set $\bar{\nu}_j=\bar{\mu}_j\restriction\mathfrak{B}_j$. Show that the localizable measure algebra free product $(\mathfrak{B}_1,\bar{\nu}_1)\widehat{\otimes}_{\text{loc}}(\mathfrak{B}_2,\bar{\nu}_2)$ can be thought of as a principal ideal of $(\mathfrak{A}_1,\bar{\mu}_1)\widehat{\otimes}_{\text{loc}}(\mathfrak{A}_2,\bar{\mu}_2)$.

(c) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras with localizations $(\widehat{\mathfrak{A}},\tilde{\mu})$ and $(\widehat{\mathfrak{B}},\tilde{\nu})$. Show that the localizable measure algebra free products $(\mathfrak{A},\bar{\mu})\widehat{\otimes}_{\text{loc}}(\mathfrak{B},\bar{\nu})$ and $(\widehat{\mathfrak{A}},\tilde{\mu})\widehat{\otimes}_{\text{loc}}(\widehat{\mathfrak{B}},\tilde{\nu})$ are isomorphic.

>(d) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ and $\langle(\mathfrak{B}_j,\bar{\nu}_j)\rangle_{j\in J}$ be families of semi-finite measure algebras, with simple products $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ (322L). Show that the localizable measure algebra free product $(\mathfrak{A},\bar{\mu})\widehat{\otimes}_{\text{loc}}(\mathfrak{B},\bar{\nu})$ can be identified with the simple product of the family $\langle(\mathfrak{A}_i,\bar{\mu}_i)\widehat{\otimes}_{\text{loc}}(\mathfrak{B}_j,\bar{\nu}_j)\rangle_{i\in I,j\in J}$.

>(e) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ and $\langle(\mathfrak{A}'_i,\bar{\mu}'_i)\rangle_{i\in I}$ be two families of probability algebras, and $(\mathfrak{C},\bar{\lambda},\langle\varepsilon_i\rangle_{i\in I})$, $(\mathfrak{C}',\bar{\lambda}',\langle\varepsilon'_i\rangle_{i\in I})$ their probability algebra free products. Suppose that for each $i\in I$ we are given a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_i\to\mathfrak{A}'_i$. Show that there is a unique measure-preserving Boolean homomorphism $\pi:\mathfrak{C}\to\mathfrak{C}'$ such that $\pi\varepsilon_i=\varepsilon'_i\pi_i$ for every $i\in I$.

>(f) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra. We say that a family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ is **stochastically independent** if $\bar{\mu}(\inf_{i\in J}a_i)=\prod_{i\in J}\bar{\mu}a_i$ for every non-empty finite $J\subseteq I$. Show that this is so iff $\langle\mathfrak{A}_i\rangle_{i\in I}$ is stochastically independent, where $\mathfrak{A}_i=\{0,a_i,1\setminus a_i,1\}$ for each i. (Compare 272F.)

>(g) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\langle\mathfrak{A}_i\rangle_{i\in I}$ a stochastically independent family of closed subalgebras of $\mathfrak{A}$. Let $\langle J(k)\rangle_{k\in K}$ be a disjoint family of subsets of I, and for each $k\in K$ let $\mathfrak{B}_k=\bigvee_{i\in J(k)}\mathfrak{A}_i$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in J(k)}\mathfrak{A}_i$. Show that $\langle\mathfrak{B}_k\rangle_{k\in K}$ is stochastically independent. (Compare 272K.)

(h) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\langle\mathfrak{A}_i\rangle_{i\in I}$ a stochastically independent family of closed subalgebras of $\mathfrak{A}$. For $J\subseteq I$ set $\mathfrak{B}_J=\bigvee_{i\in J}\mathfrak{A}_i$. Show that $\bigcap\{\mathfrak{B}_{I\setminus J}:J$ is a finite subset of $I\}=\{0,1\}$. (*Hint*: For $J\subseteq I$, show that $\bar{\mu}(b\cap c)=\bar{\mu}b\cdot\bar{\mu}c$ for every $b\in\mathfrak{B}_{I\setminus J}$ and $c\in\mathfrak{B}_J$. Compare 272O, 325M.)

(i) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras with probability algebra free product $(\mathfrak{C},\bar{\lambda},\langle\varepsilon_i\rangle_{i\in I})$. For $J\subseteq I$ set $\mathfrak{C}_J=\bigvee_{i\in J}\varepsilon_i[\mathfrak{A}_i]$. Show that for any J, $K\subseteq I$ and $c\in\mathfrak{C}$, $\mathfrak{C}_J\cap\mathfrak{C}_K=\mathfrak{C}_{J\cap K}$ and the upper envelope $\text{upr}(c,\mathfrak{C}_{J\cap K})$ is equal to $\text{upr}(\text{upr}(c,\mathfrak{C}_J),\mathfrak{C}_K)$.

325Y Further exercises (a) Let μ be counting measure on $X=\{0\}$, μ' the countable-cocountable measure on $X'=\omega_1$, and ν counting measure on $Y=\omega_1$. Show that the measure algebras of the primitive product measures on $X\times Y$, $X'\times Y$ are not isomorphic.

(b) Let $(\mathfrak{A}_1,\bar{\mu}_1)$, $(\mathfrak{A}_2,\bar{\mu}_2)$, $(\mathfrak{A}'_1,\bar{\mu}'_1)$ and $(\mathfrak{A}'_2,\bar{\mu}'_2)$ be semi-finite measure algebras with localizable measure algebra free products $(\mathfrak{C},\bar{\lambda},\varepsilon_1,\varepsilon_2)$ and $(\mathfrak{C}',\bar{\lambda}',\varepsilon'_1,\varepsilon'_2)$. Suppose that $\pi_1:\mathfrak{A}_1\to\mathfrak{A}'_1$ and $\pi_2:\mathfrak{A}_2\to\mathfrak{A}'_2$ are measure-preserving Boolean homomorphisms. Show that there is a measure-preserving Boolean homomorphism $\pi:\mathfrak{C}\to\mathfrak{C}'$ such that $\pi\varepsilon_i=\varepsilon'_i\pi_i$ for both i, but that π is not necessarily unique.

(c) Let $\mathfrak{A}$ be a Boolean algebra, and $\mu:\mathfrak{A}\to[0,\infty]$ a functional such that $\mu0=0$, $\mu a>0$ for every $a\neq 0$, and $\mu(a\cup b)=\mu a+\mu b$ whenever a, $b\in\mathfrak{A}$ and $a\cap b=0$; suppose that $\mathfrak{A}^f=\{a:\mu a<\infty\}$ is order-dense in $\mathfrak{A}$. For $e\in\mathfrak{A}^f$, a, $b\in\mathfrak{A}$ set $\rho_e(a,b)=\mu(e\cap(a\,\triangle\,b))$. Give $\mathfrak{A}$ the uniformity defined by $\{\rho_e:\mu e<\infty\}$. (i) Show that the completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ under this uniformity has a measure $\hat{\mu}$, extending μ, under which it is a localizable measure algebra. (ii) Show that if $a\in\widehat{\mathfrak{A}}$, $\hat{\mu}a<\infty$ and $\epsilon>0$, there is a $b\in\mathfrak{A}$ such that $\hat{\mu}(a\,\triangle\,b)\le\epsilon$. (iii) Show that for every $a\in\widehat{\mathfrak{A}}$ there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $a\supseteq\sup_{n\in\mathbb{N}}\inf_{m\ge n}a_m$ and $\hat{\mu}a=\hat{\mu}(\sup_{n\in\mathbb{N}}\inf_{m\ge n}a_m)$. (iv) In particular, the set of infima in $\widehat{\mathfrak{A}}$ of sequences in $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$. (v) Explain the relevance of this construction to the embedding $\mathfrak{A}_1\otimes\mathfrak{A}_2\subsetneqq\mathfrak{C}$ in 325D.

(d) In 325F, set $W=\bigcup_{n\in\mathbb{N}}E_n\times E_n$. Show that if A, B are any non-negligible subsets of $[0,1]$, then $W\cap(A\times B)$ is not negligible.

(e) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$. Show that $\mathfrak{A}\otimes\mathfrak{A}$ is ccc but not weakly (σ,∞)-distributive. (*Hint*: (i) $\mathfrak{A}\otimes\mathfrak{A}$ is embeddable as a subalgebra of a probability algebra (ii) in the notation of 325F, look at $c_{mn}=\sup_{m\le i\le n}e_i\otimes e_i$.)

(f) Repeat 325F-325G and 325Yd-325Ye with an arbitrary atomless probability space in place of $[0,1]$.

(g) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\langle a_i\rangle_{i\in I}$ a stochastically independent family in $\mathfrak{A}$. Show that for any $a\in\mathfrak{A}$ and $\epsilon>0$ the set $\{i:i\in I,\,|\bar\mu(a\cap a_i)-\bar\mu a\cdot\bar\mu a_i|\ge\epsilon\}$ is finite, so that $\{i:\bar\mu(a\cap a_i)\ne\bar\mu a\cdot\bar\mu a_i\}$ is countable. (*Hint*: 272Ye[2].)

325 Notes and comments 325B shows that the measure algebra of a product measure may be irregular if we have factor measures which are not strictly localizable. But two facts lead the way to the 'localizable measure algebra free product' in 325D-325E. The first is that every semi-finite measure algebra is embeddable, in a canonical way, in a localizable measure algebra (322P); and the second is that the Stone representation of a localizable measure algebra is strictly localizable (322O). It is a happy coincidence that we can collapse these two facts together in the construction of 325D. Another way of looking at the localizable measure algebra free product of two localizable measure algebras is to express it as the simple product of measure algebra free products of totally finite measure algebras, using 325Xd and the fact that for σ-finite measure algebras there is only one reasonable measure algebra free product, being that provided by any representation of them as measure algebras of measure spaces (325Eb).

Yet a third way of approaching measure algebra free products is as the uniform space completions of algebraic free products, using 325Yc. This gives the same result as the construction of 325D because the algebraic free product appears as a topologically dense subalgebra of the localizable measure algebra free product, which is complete as uniform space (325Dc). (I have to repeat such phrases as 'topologically dense' because the algebraic free product is emphatically *not* order-dense in the measure algebra free product (325F).) The results in 251I on approximating measurable sets for a c.l.d. product measure by combinations of measurable rectangles correspond to general facts about completions of finitely-additive measures (325Yc(ii), 325Yc(iii)). It is worth noting that the completion process can be regarded as made up of two steps; first take infima of sequences of sets of finite measure, and then take arbitrary suprema (325Yc(iv)).

The idea of 325F appears in many guises, and this is only the first time that I shall wish to call on it. The point of the set $W=\bigcup_{n\in\mathbb{N}}E_n\times E_n$ is that it is a measurable subset of the square (indeed, by taking the E_n to be open sets we can arrange that W should be open), of measure strictly less than 1 (in fact, as small as we wish), such that its complement does not include any non-negligible 'measurable rectangle' $G\times H$; indeed, $W\cap(A\times B)$ is non-negligible for any non-negligible sets A, $B\subseteq[0,1]$ (325Yd). I believe that the first published example of such a set was by Erdős & Oxtoby 55 (a version of which is in 532N in Volume 5); I learnt the method of 325F from R.O.Davies.

I include 325G as a kind of guard-rail. The relationship between preservation of measure and order-continuity is a subtle one, as I have already tried to show in 324K, and it is often worth considering the possibility that a result involving order-continuous measure-preserving homomorphisms has a form applying to all order-continuous homomorphisms. However, there is no simple expression of such an idea in the present context.

In the context of infinite free products of probability algebras, there is a degree of simplification, since there is only one algebra which can plausibly be called the probability algebra free product, and this is produced by any realization of the algebras as measure algebras of probability spaces (325I-325K). The examples 325F-325G apply equally, of course, to this context. At this point I mention the concept of 'stochastically independent' family (325L, 325Xf) because we have the machinery to translate several results from §272 into the language of measure algebras (325Xf-325Xh). I feel that I have to use the phrase 'stochastically independent' here because there is the much weaker alternative concept of 'Boolean independence' (315Xp) also present. But I leave most of this as exercises, because the language of measure algebras offers few ideas to the probability theory already covered in Chapter 27. All it can do is formalise the ever-present principle that negligible sets often can and should be ignored.

[2] Formerly 272Yd.

326 Additive functionals on Boolean algebras

I devote two sections to the general theory of additive functionals on measure algebras. As many readers will rightly be in a hurry to get on to the next two chapters, I remark that the only significant result needed for §§331-332 is the Hahn decomposition of a countably additive functional (326M), and that this is no more than a translation into the language of measure algebras of a theorem already given in Chapter 23. The concept of 'standard extension' of a countably additive functional from a subalgebra (327F-327G) will be used for a theorem in §333, and as preparation for Chapter 36.

I begin with notes on the space of additive functionals on an arbitrary Boolean algebra (326A-326D), corresponding to 231A-231B, but adding a more general form of the Jordan decomposition of a bounded additive functional into positive and negative parts (326D). The next four paragraphs are starred, because they will not be needed in this volume; 326E is essential if you want to look at additive functionals on free products, 326F is a basic classification criterion, and 326H is an important extension of a fundamental fact about atomless measures noted in 215D, but all can be passed over on first reading. The next subsection (326I-326M) deals with countably additive functionals, corresponding to 231C-231F. In 326N-326T I develop a new idea, that of 'completely additive' functional, which does not match anything in the previous treatment.

326A Additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **finitely additive**, or just **additive**, if $\nu(a \cup b) = \nu a + \nu b$ whenever $a, b \in \mathfrak{A}$ and $a \cap b = 0$.

A non-negative additive functional is sometimes called a **finitely additive measure** or **charge**.

326B Elementary facts Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. The following will I hope be obvious.

(a) $\nu 0 = 0$ (because $\nu 0 = \nu 0 + \nu 0$).

(b) If $c \in \mathfrak{A}$, then $a \mapsto \nu(a \cap c)$ is additive (because $(a \cap c) \cup (b \cap c) = (a \cup b) \cap c$).

(c) $\alpha\nu$ is an additive functional for any $\alpha \in \mathbb{R}$. If ν' is another finitely additive functional on $\mathfrak{A}$, then $\nu + \nu'$ is additive.

(d) If $\langle \nu_i \rangle_{i \in I}$ is any family of finitely additive functionals such that $\nu' a = \sum_{i \in I} \nu_i a$ is defined in $\mathbb{R}$ for every $a \in \mathfrak{A}$, then ν' is additive.

(e) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism, then $\nu\pi : \mathfrak{B} \to \mathbb{R}$ is additive. In particular, if $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then $\nu\restriction\mathfrak{B} : \mathfrak{B} \to \mathbb{R}$ is additive.

(f) ν is non-negative iff it is order-preserving – that is,

$$\nu a \ge 0 \text{ for every } a \in \mathfrak{A} \iff \nu b \le \nu c \text{ whenever } b \subseteq c$$

(because $\nu c = \nu b + \nu(c \setminus b)$ if $b \subseteq c$).

326C The space of additive functionals Let $\mathfrak{A}$ be any Boolean algebra. From 326Bc we see that the set M of all finitely additive real-valued functionals on $\mathfrak{A}$ is a linear space (a linear subspace of $\mathbb{R}^{\mathfrak{A}}$). We give it the ordering induced by that of $\mathbb{R}^{\mathfrak{A}}$, so that $\nu \le \nu'$ iff $\nu a \le \nu' a$ for every $a \in \mathfrak{A}$. This renders it a partially ordered linear space (because $\mathbb{R}^{\mathfrak{A}}$ is).

326D The Jordan decomposition (I): Proposition Let $\mathfrak{A}$ be a Boolean algebra, and ν a finitely additive real-valued functional on $\mathfrak{A}$. Then the following are equiveridical:

(i) ν is bounded;

(ii) $\sup_{n \in \mathbb{N}} |\nu a_n| < \infty$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;

(iii) $\lim_{n \to \infty} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;

(iv) $\sum_{n=0}^{\infty} |\nu a_n| < \infty$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;

(v) ν is expressible as the difference of two non-negative additive functionals.

proof (a)(i)$\Rightarrow$(v) Assume that ν is bounded. For each $a \in \mathfrak{A}$, set

$$\nu^+ a = \sup\{\nu b : b \subseteq a\}.$$

Because ν is bounded, ν^+ is real-valued. Now ν^+ is additive. **P** If $a, b \in \mathfrak{A}$ and $a \cap b = 0$, then

$$\nu^+(a \cup b) = \sup_{c \subseteq a \cup b} \nu c = \sup_{d \subseteq a, e \subseteq b} \nu(d \cup e) = \sup_{d \subseteq a, e \subseteq b} \nu d + \nu e$$

(because $d \cap e \subseteq a \cap b = 0$ whenever $d \subseteq a$, $e \subseteq b$)

$$= \sup_{d \subseteq a} \nu d + \sup_{e \subseteq b} \nu e = \nu^+ a + \nu^+ b. \text{ **Q**}$$

Consequently $\nu^- = \nu^+ - \nu$ also is additive (326Bc).

Since

$$0 = \nu 0 \le \nu^+ a, \quad \nu a \le \nu^+ a$$

for every $a \in \mathfrak{A}$, $\nu^+ \ge 0$ and $\nu^- \ge 0$. Thus $\nu = \nu^+ - \nu^-$ is the difference of two non-negative additive functionals.

(b)(v)⇒(iv) If ν is expressible as $\nu_1 - \nu_2$, where ν_1 and ν_2 are non-negative additive functionals, and $\langle a_n \rangle_{n \in \mathbb{N}}$ is disjoint, then

$$\sum_{i=0}^{n} \nu_j a_i = \nu_j(\sup_{i \le n} a_i) \le \nu_j 1$$

for every n, both j, so that

$$\sum_{i=0}^{\infty} |\nu a_i| \le \sum_{i=0}^{\infty} \nu_1 a_i + \sum_{i=0}^{\infty} \nu_2 a_i \le \nu_1 1 + \nu_2 1 < \infty.$$

(c)(iv)⇒(iii)⇒(ii) are trivial.

(d) not-(i)⇒not-(ii) Suppose that ν is unbounded. Choose sequences $\langle a_n \rangle_{n \in \mathbb{N}}$, $\langle b_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. $b_0 = 1$. Given that $\sup_{a \subseteq b_n} |\nu a| = \infty$, choose $c_n \subseteq b_n$ such that $|\nu c_n| \ge |\nu b_n| + n$; then $|\nu c_n| \ge n$ and

$$|\nu(b_n \setminus c_n)| = |\nu b_n - \nu c_n| \ge |\nu c_n| - |\nu b_n| \ge n.$$

We have

$$\begin{aligned} \infty = \sup_{a \subseteq b_n} |\nu a| &= \sup_{a \subseteq b_n} |\nu(a \cap c_n) + \nu(a \setminus c_n)| \\ &\le \sup_{a \subseteq b_n} |\nu(a \cap c_n)| + |\nu(a \setminus c_n)| \le \sup_{a \subseteq b_n \cap c_n} |\nu a| + \sup_{a \subseteq b_n \setminus c_n} |\nu a|, \end{aligned}$$

so at least one of $\sup_{a \subseteq b_n \cap c_n} |\nu a|$, $\sup_{a \subseteq b_n \setminus c_n} |\nu a|$ must be infinite; take b_{n+1} to be one of c_n, $b_n \setminus c_n$ such that $\sup_{a \subseteq b_{n+1}} |\nu a| = \infty$, and set $a_n = b_n \setminus b_{n+1}$, so that $|\nu a_n| \ge n$. Continue.

On completing the induction, we have a disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ such that $|\nu a_n| \ge n$ for every n, so that (ii) is false.

Remark I hope that this reminds you of the decomposition of a function of bounded variation as the difference of monotonic functions (224D).

***326E Additive functionals on free products** In Volume 4, when we return to the construction of measures on product spaces, the following fundamental fact will be useful.

Theorem Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a non-empty family of Boolean algebras, with free product $\mathfrak{A}$; write $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ for the canonical maps, and

$$C = \{\inf_{j \in J} \varepsilon_j(a_j) : J \subseteq I \text{ is finite, } a_j \in \mathfrak{A}_j \text{ for every } j \in J\}.$$

Suppose that $\theta : C \to \mathbb{R}$ is such that

$$\theta c = \theta(c \cap \varepsilon_i(a)) + \theta(c \cap \varepsilon_i(1 \setminus a))$$

whenever $c \in C$, $i \in I$ and $a \in \mathfrak{A}_i$. Then there is a unique finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ extending θ.

proof (a) It will help if I note at once that $\theta 0 = 0$. **P**

$$\theta 0 = \theta(0 \cap \varepsilon_i(0)) + \theta(0 \cap \varepsilon_i(1)) = 2\theta 0$$

for any $i \in I$. **Q**

(b) The key is of course the following fact: if $\langle c_r\rangle_{r\le m}$ and $\langle d_s\rangle_{s\le n}$ are two disjoint families in C with the same supremum in $\mathfrak{A}$, then $\sum_{r=0}^m \theta c_r = \sum_{s=0}^n \theta d_s$. **P** Let $J \subseteq I$ be a finite set and $\mathfrak{B}_i \subseteq \mathfrak{A}_i$ a finite subalgebra, for each $i \in J$, such that every c_r and every d_s belongs to the subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ generated by $\{\varepsilon_j(b) : j \in J,\, b \in \mathfrak{B}_j\}$. Next, if $j \in J$ and $b \in \mathfrak{B}_j$, then

$$\sum_{r=0}^m \theta c_r = \sum_{r=0}^m \theta(c_r \cap \varepsilon_j(b)) + \sum_{r=0}^m \theta(c_r \setminus \varepsilon_j(b)).$$

We can therefore find a disjoint family $\langle c'_r\rangle_{r\le m'}$ in $C \cap \mathfrak{A}_0$ such that

$$\sup_{r\le m'} c'_r = \sup_{r\le m} c_r, \quad \sum_{r=0}^{m'} \theta c'_r = \sum_{r=0}^m \theta c_r,$$

and whenever $r \le m'$, $j \in J$ and $b \in \mathfrak{B}_j$ then either $c'_r \subseteq \varepsilon_j(b)$ or $c'_r \cap \varepsilon_j(b) = 0$; that is, every c'_r is either 0 or of the form $\inf_{j\in J} \varepsilon_j(b_j)$ where b_j is an atom of $\mathfrak{B}_j$ for every j. Similarly, we can find $\langle d'_s\rangle_{s\le n'}$ such that

$$\sup_{s\le n'} d'_s = \sup_{s\le n} d_s, \quad \sum_{s=0}^{n'} \theta d'_s = \sum_{s=0}^n \theta d_s,$$

and whenever $s \le n'$ and $j \in J$ then d'_s is either 0 or of the form $\inf_{j\in J} \varepsilon_j(b_j)$ where b_j is an atom of $\mathfrak{B}_j$ for every j. But we now have $\sup_{r\le m'} c'_r = \sup_{s\le n'} d'_s$ while for any $r \le m'$, $s \le n'$ either $c'_r = d'_s$ or $c'_r \cap d'_s = 0$. It follows that the non-zero terms in the finite sequence $\langle c'_r\rangle_{r\le m'}$ are just a rearrangement of the non-zero terms in $\langle d'_s\rangle_{s\le n'}$, so that

$$\sum_{r=0}^m \theta c_r = \sum_{r=0}^{m'} \theta c'_r = \sum_{s=0}^{n'} \theta d'_s = \sum_{s=0}^n \theta d_s,$$

as required. **Q**

(c) By 315Kb, this means that we have a functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu(\sup_{r\le m} c_r) = \sum_{r=0}^m \theta c_r$ whenever $\langle c_r\rangle_{r\le m}$ is a disjoint family in C. It is now elementary to check that ν is additive, and it is clearly the only additive functional on $\mathfrak{A}$ extending θ.

***326F** I give a couple of pages to an interesting property of additive functionals on Dedekind σ-complete Boolean algebras. I do not think it will be used in this book, and it really belongs to the theory of vector measures, which is hardly considered here, but the ideas are important, and the following definition has other uses.

Definition Let $\mathfrak{A}$ be a Boolean algebra, and ν a finitely additive functional on $\mathfrak{A}$. I will say that ν is **properly atomless** if for every $\epsilon > 0$ there is a finite partition $\langle a_i\rangle_{i\in I}$ of unity in $\mathfrak{A}$ such that $|\nu a| \le \epsilon$ whenever $i \in I$ and $a \subseteq a_i$.

***326G Lemma** Let $\mathfrak{A}$ be a Boolean algebra.

(a)(i) If ν, $\nu' : \mathfrak{A} \to \mathbb{R}$ are properly atomless finitely additive functionals and $\alpha \in \mathbb{R}$, then $\alpha\nu$ and $\nu + \nu'$ are properly atomless additive functionals.

(ii) If $\nu : \mathfrak{A} \to \mathbb{R}$ is a properly atomless finitely additive functional, then ν is bounded and ν can be expressed as the difference of two non-negative properly atomless additive functionals.

(b) Suppose that $\mathfrak{A}$ is Dedekind σ-complete and that $\langle \nu_i\rangle_{i\in I}$ is a family of non-negative additive functionals on $\mathfrak{A}$ such that for every $a \in \mathfrak{A}$ there are an $\alpha \in [\frac{1}{3}, \frac{2}{3}]$ and an $a' \subseteq a$ such that $\nu_i a' = \alpha \nu_i a$ for every $i \in I$. Then for any $a \in \mathfrak{A}$ there is a non-decreasing family $\langle a_t\rangle_{t\in[0,1]}$ in $\mathfrak{A}$ such that $a_0 = 0$, $a_1 = a$ and $\nu_i a_t = t\nu_i a$ for every $t \in [0,1]$ and $i \in I$.

(c) Suppose that $\mathfrak{A}$ is Dedekind σ-complete and that $\nu_0, \dots, \nu_n : \mathfrak{A} \to [0,\infty[$ are properly atomless additive functionals such that $\nu_i a \le \nu_0 a$ for every $i \le n$ and $a \in \mathfrak{A}$. Then for any $a \in \mathfrak{A}$ there is a non-decreasing family $\langle a_t\rangle_{t\in[0,1]}$ in $\mathfrak{A}$ such that $a_0 = 0$, $a_1 = a$ and $\nu_i a_t = t\nu_i a$ for every $t \in [0,1]$ and $i \le n$.

proof (a)(i) Let $\epsilon > 0$. Then there are finite partitions $\langle a_i\rangle_{i\in I}$, $\langle b_j\rangle_{j\in J}$ of unity in $\mathfrak{A}$ such that $|\nu a| \le \frac{\epsilon}{2+|\alpha}$ whenever $i \in I$ and $a \subseteq a_i$, while $|\nu' a| \le \frac{\epsilon}{2}$ whenever $j \in J$ and $a \subseteq b_j$. Now $|(\alpha\nu)(a)| \le \epsilon$ whenever $i \in I$ and $a \subseteq a_i$. Moreover, $\langle a_i \cap b_j\rangle_{(i,j)\in I\times J}$ is a finite partition of unity in $\mathfrak{A}$, and $|(\nu + \nu')(a)| \le \epsilon$ whenever $i \in I$, $j \in J$ and $a \subseteq a_i \cap b_j$.

(ii)(α) There is a finite partition $\langle c_j\rangle_{j\in J}$ of unity in $\mathfrak{A}$ such that $|\nu a| \le 1$ whenever $i \in J$ and $a \subseteq c_j$; now $|\nu a| \le \sum_{j\in J} |\nu(a \cap c_j)| \le \#(J)$ for every $a \in \mathfrak{A}$, so ν is bounded.

(β) Define ν^+ as in part (a) of the proof of 326D, so that $\nu^+ : \mathfrak{A} \to [0,\infty[$ is additive. Now ν^+ is properly atomless. **P** Given $\epsilon > 0$, there is a finite partition $\langle a_i\rangle_{i\in I}$ of unity in $\mathfrak{A}$ such that $|\nu a| \le \epsilon$ whenever $i \in I$ and $a \subseteq a_i$; in which case $\nu^+ a = \sup_{b \subseteq a} \nu b \le \epsilon$ whenever $i \in I$ and $a \subseteq a_i$. **Q** As in 326D, $\nu^- = \nu^+ - \nu$ is non-negative, and by (i) just above (or otherwise) it is properly atomless, so $\nu=\nu^+-\nu^-$ is the difference of non-negative properly atomless functionals.

(b) If $\nu_i a=0$ for every $i\in I$, we can take $a_t=0$ for $0\le t<1$ and $a_1=a$. So suppose that $k\in I$ is such that $\nu_k a>0$. For $i\in I$, set $\gamma_i=\dfrac{\nu_i a}{\nu_k a}$. Choose $\langle D_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $D_0=\{0,a\}$. Given that D_n is a finite totally ordered subset of $\{b:b\subseteq a\}$ containing 0 and a and $\nu_i d=\gamma_i\nu_k d$ for every $d\in D_n$ and $i\in I$, then for each $d\in D_n\setminus\{a\}$ let d' be the next member of D_n strictly including d, and take $b_d\subseteq d'\setminus d$, $\alpha_d\in[\frac13,\frac23]$ such that $\nu_i b_d=\alpha_d\nu_i(d'\setminus d)$ for every $i\in I$. Then

$$\nu_i(d\cup b_d)=(1-\alpha_d)\nu_i d+\alpha_d\nu_i d'=\gamma_i((1-\alpha_d)\nu_k d+\alpha_d\nu_k d')=\gamma_i\nu_k(d\cup b_d)$$

for every i. Set $D_{n+1}=D_n\cup\{d\cup b_d:d\in D_n\}$; observe that D_{n+1} is still totally ordered, and continue. At the end of the induction, it is easy to see that $\nu_k(d'\setminus d)\le(\frac23)^n\nu_i a$ whenever $n\in\mathbb{N}$ and $d\subset d'$ are successive members of D_n.

Set $D=\bigcup_{n\in\mathbb{N}}D_n$. Then D is a countable totally ordered set with least element 0 and greatest element a, and $\{\nu_k d:d\in D\}$ is dense in $[0,\nu_k a]$. For $t\in\,]0,1]$, set $a_t=\sup\{d:d\in D,\ \nu_k d\le t\nu_k a\}$; this is where we need to know that $\mathfrak{A}$ is Dedekind σ-complete. Set $a_0=0$. Then $\langle a_t\rangle_{t\in[0,1]}$ is a non-decreasing family with $a_0=0$ and $a_1=a$. If $0<t<1$, $i\in I$ and $\epsilon>0$, there are d, $d'\in D$ such that

$$t\nu_k a-\epsilon\le\nu_k d\le t\nu_k a<\nu_k d'\le t\nu_k a+\epsilon,$$

$$t\nu_i a-\gamma_i\epsilon\le\nu_i d\le t\nu_i a<\nu_i d'\le t\nu_i a+\gamma_i\epsilon;$$

in this case $d\subseteq a_t\subseteq d'$, so

$$t\nu_i a-\gamma_i\epsilon\le\nu_i a_t\le t\nu_i a+\gamma_i\epsilon;$$

as ϵ is arbitrary, $\nu_i a_t=t\nu_i a$. Thus we have a suitable family $\langle a_t\rangle_{t\ge0}$.

(c) Induce on n.

(i) The induction starts with a single non-negative properly atomless functional ν_0. Now for any $a\in\mathfrak{A}$ there is an $a'\subseteq a$ such that $\frac13\nu_0 a\le\nu_0 a'\le\frac23\nu_0 a$. **P** This is trivial if $\nu_0 a=0$. Otherwise, let C be a finite partition of unity in $\mathfrak{A}$ such that $\nu_0 c\le\frac13\nu_0 a$ for every $c\in C$. Enumerate C as $\langle c_i\rangle_{i<m}$ and for $i\le m$ set $b_i=a\cap\sup_{j<i}c_j$. Then $b_0=0$, $b_m=a$ and $\nu_0 b_{i+1}-\nu_0 b_i\le\nu_0 c_i\le\frac13\nu_0 a$ for each i. So there must be an $i\le m$ such that $\frac13\nu_0 a\le\nu_0 b_i\le\frac23\nu_0 a$, and we can set $a'=b_i$. **Q**

Now (b), with $I=\{0\}$, gives the result.

(ii) For the inductive step to $n\ge1$, I show first that if $a\in\mathfrak{A}$ there is an $a'\subseteq a$ such that $\nu_i a'=\frac12\nu_i a$ for every $i\le n$. **P** By the inductive hypothesis, we have a non-decreasing family $\langle a_t\rangle_{t\in[0,1]}$ such that $a_0=0$, $a_1=a$ and $\nu_i a_t=t\nu_i a$ whenever $t\in[0,1]$ and $i<n$. Now observe that for $0\le s\le t\le1$,

$$|\nu_n a_t-\nu_n a_s|=\nu_n(a_t\setminus a_s)\le\nu_0(a_t\setminus a_s)=(t-s)\nu_0 a.$$

So the functions $t\mapsto\nu_n a_t:[0,1]\to[0,\infty[$ and $f:[0,\frac12]\to[0,\infty[$ are continuous, where $f(t)=\nu_n a_{t+\frac12}-\nu_n a_t$ for $0\le t\le\frac12$. However, $f(0)+f(\frac12)=\nu_n a$, so $\frac12\nu_n a$ lies between $f(0)$ and $f(\frac12)$ and there is a $t\in[0,\frac12]$ such that $f(t)=\frac12\nu_n a$. Set $a'=a_{t+\frac12}\setminus a_t$; then $\nu_i a'=\frac12\nu_i a$ for every $i\le n$, as required. **Q**

Once again (b), with $I=\{0,\dots,n\}$, shows that for any $a\in\mathfrak{A}$ we have a non-decreasing family $\langle a_t\rangle_{t\in[0,1]}$ such that $a_0=0$, $a_1=1$ and $\nu_i a_t=t\nu_i a$ whenever $t\in[0,1]$ and $i\le n$.

***326H Liapounoff's convexity theorem** (Liapounoff 40) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $r\ge1$ an integer. Suppose that $\nu:\mathfrak{A}\to\mathbb{R}^r$ is additive in the sense that $\nu(a\cup b)=\nu a+\nu b$ whenever $a\cap b=0$ (see 361B), and properly atomless in the sense that for every $\epsilon>0$ there is a finite partition $\langle a_j\rangle_{j\in J}$ of unity in $\mathfrak{A}$ such that $\|\nu a\|\le\epsilon$ whenever $j\in J$ and $a\subseteq a_j$. Then $\{\nu a:a\in\mathfrak{A}\}$ is a convex set in $\mathbb{R}^r$.([3])

proof For $1\le i\le r$, let ν_i be the ith component of ν, so that $\nu a=\langle\nu_i a\rangle_{1\le i\le r}$ for each $a\in\mathfrak{A}$. Then every ν_i is additive. Moroever, it is properly atomless. **P** Given $\epsilon>0$, there is a finite partition $\langle a_j\rangle_{j\in J}$ of unity in $\mathfrak{A}$ such that $|\nu_i a|\le\|\nu a\|\le\epsilon$ whenever $j\in J$ and $a\subseteq a_j$. **Q** So we can express ν_i as $\nu_i^+-\nu_i^-1$ where ν_i^+ and ν_i^{-1} are non-negative properly atomless non-negative functionals (326G(a-ii)). Set $\tilde\nu a=\sum_{i=1}^r\nu_i^+a+\nu_i^-a$ for $a\in\mathfrak{A}$. Then $\tilde\nu$ is again properly atomless (326G(a-i)).

[3] I learnt this version of the theorem from K.P.S.Bhaskara Rao.

Suppose that x, $y \in \nu[\mathfrak{A}]$ and $\alpha \in [0,1]$. Let a, $b \in \mathfrak{A}$ be such that $\nu a = x$ and $\nu b = y$. By 326Gc, applied to $\tilde{\nu}, \nu_1^+, \nu_1^-, \dots, \nu_r^+, \nu_r^-$, there is an $c \subseteq a \setminus b$ such that

$$\nu_i^+ c = \alpha \nu_i^+(a \setminus b), \quad \nu_i^- c = \alpha \nu_i^-(a \setminus b),$$

for every $i \le r$, so that $\nu_i c = \alpha \nu_i(a \setminus b)$ for every $i \le r$. Similarly, there is a $d \subseteq b \setminus a$ such that $\nu d = (1-\alpha)\nu(b \setminus d)$. Now $e = c \cup (a \cap b) \cup d$,

$$\begin{aligned}\alpha x + (1-\alpha)y &= \alpha \nu a + (1-\alpha)\nu b \\ &= \alpha\nu(a \setminus b) + \alpha\nu(a \cap b) + (1-\alpha)\nu(a \cap b) + (1-\alpha)\nu(b \setminus a) \\ &= \nu c + \nu(a \cap b) + \nu d = \nu(c \cup (a \cap b) \cup d) \in \nu[\mathfrak{A}].\end{aligned}$$

As x, y and α are arbitrary, $\nu[\mathfrak{A}]$ is convex.

326I Countably additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **countably additive** or **σ-additive** if $\sum_{n=0}^\infty \nu a_n$ is defined and equal to $\nu(\sup_{n\in\mathbb{N}} a_n)$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ and $\sup_{n\in\mathbb{N}} a_n$ is defined in $\mathfrak{A}$.

A warning is perhaps in order. It can happen that $\mathfrak{A}$ is presented to us as a subalgebra of a larger algebra $\mathfrak{B}$; for instance, $\mathfrak{A}$ might be an algebra of sets, a subalgebra of some σ-algebra $\Sigma \subseteq \mathcal{P}X$. In this case, there may be sequences in $\mathfrak{A}$ which have a supremum in $\mathfrak{A}$ which is not a supremum in $\mathfrak{B}$ (indeed, this will happen just when the embedding is not sequentially order-continuous). So we can have a countably additive functional $\nu : \mathfrak{B} \to \mathbb{R}$ such that $\nu{\restriction}\mathfrak{A}$ is not countably additive in the sense used here. A similar phenomenon will arise when we come to the Daniell integral in Volume 4 (§436).

326J Elementary facts Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional.

(a) ν is finitely additive. (Setting $a_n = 0$ for every n, we see from the definition in 326I that $\nu 0 = 0$. Now, given $a \cap b = 0$, set $a_0 = a$, $a_1 = b$, $a_n = 0$ for $n \ge 2$ to see that $\nu(a \cup b) = \nu a + \nu b$.)

(b) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with a supremum $a \in \mathfrak{A}$, then

$$\nu a = \nu a_0 + \sum_{n=0}^\infty \nu(a_{n+1} \setminus a_n) = \lim_{n\to\infty} \nu a_n.$$

(c) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with an infimum $a \in \mathfrak{A}$, then $\langle a_0 \setminus a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum $a_0 \setminus a$, so

$$\nu a = \nu a_0 - \nu(a_0 \setminus a) = \nu a_0 - \lim_{n\to\infty} \nu(a_0 \setminus a_n) = \lim_{n\to\infty} \nu a_n.$$

(d) If $c \in \mathfrak{A}$, then $a \mapsto \nu(a \cap c)$ is countably additive. (For $\sup_{n\in\mathbb{N}} a_n \cap c = c \cap \sup_{n\in\mathbb{N}} a_n$ whenever the right-hand-side is defined, by 313Ba.)

(e) $\alpha\nu$ is a countably additive functional for any $\alpha \in \mathbb{R}$. If ν' is another countably additive functional on $\mathfrak{A}$, then $\nu + \nu'$ is countably additive.

(f) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism, then $\nu\pi$ is a countably additive functional on $\mathfrak{B}$. (For if $\langle b_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$ with supremum b, then $\langle \pi b_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence with supremum πb.)

(g) If $\mathfrak{A}$ is Dedekind σ-complete and $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$, then $\nu{\restriction}\mathfrak{B} : \mathfrak{B} \to \mathbb{R}$ is countably additive. (For the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is sequentially order-continuous, by 314Gb.)

326K Corollary Let $\mathfrak{A}$ be a Boolean algebra and ν a finitely additive real-valued functional on $\mathfrak{A}$.

(a) ν is countably additive iff $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0 in $\mathfrak{A}$.

(b) If ν' is an additive functional on $\mathfrak{A}$ and $|\nu' a| \le \nu a$ for every $a \in \mathfrak{A}$, and ν is countably additive, then ν' is countably additive.

(c) If ν is non-negative, then ν is countably additive iff it is sequentially order-continuous.

proof (a)(i) If ν is countably additive and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then $\lim_{n\to\infty}\nu a_n=0$ by 326Jc. **(ii)** If ν satisfies the condition, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, set $b_n=a\setminus\sup_{i\le n}a_i$ for each $n\in\mathbb{N}$; then $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0, so

$$\nu a-\sum_{i=0}^{n}\nu a_i=\nu a-\nu(\sup_{i\le n}a_i)=\nu b_n\to 0$$

as $n\to\infty$, and $\nu a=\sum_{n=0}^{\infty}\nu a_n$; thus ν is countably additive.

(b) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, set $b_n=\sup_{i\le n}a_i$ for each n; then $\nu a=\lim_{n\to\infty}\nu b_n$, so

$$\lim_{n\to\infty}|\nu' a-\nu' b_n|=\lim_{n\to\infty}|\nu'(a\setminus b_n)|\le\lim_{n\to\infty}\nu(a\setminus b_n)=0,$$

and

$$\sum_{n=0}^{\infty}\nu' a_n=\lim_{n\to\infty}\nu' b_n=\nu' a.$$

(c) If ν is countably additive, then it is sequentially order-continuous by 326Jb-326Jc. If ν is sequentially order-continuous, then of course it satisfies the condition of (a), so is countably additive.

326L The Jordan decomposition (II): Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν a bounded countably additive real-valued functional on $\mathfrak{A}$. Then ν is expressible as the difference of two non-negative countably additive functionals.

proof Consider the functional $\nu^+a=\sup_{b\subseteq a}\nu b$ defined in the proof of 326D. If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, and $b\subseteq a$, then

$$\nu b=\sum_{n=0}^{\infty}\nu(b\cap a_n)\le\sum_{n=0}^{\infty}\nu^+a_n.$$

As b is arbitrary, $\nu^+a\le\sum_{n=0}^{\infty}\nu^+a_n$. But of course

$$\nu^+a\ge\nu^+(\sup_{i\le n}a_i)=\sum_{i=0}^{n}\nu^+a_i$$

for every $n\in\mathbb{N}$, so $\nu^+a=\sum_{n=0}^{\infty}\nu^+a_n$. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ν^+ is countably additive.

Now $\nu^-=\nu^+-\nu$ also is countably additive, and $\nu=\nu^+-\nu^-$ is the difference of non-negative countably additive functionals.

326M The Hahn decomposition: Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu:\mathfrak{A}\to\mathbb{R}$ a countably additive functional. Then ν is bounded and there is a $c\in\mathfrak{A}$ such that $\nu a\ge 0$ whenever $a\subseteq c$, while $\nu a\le 0$ whenever $a\cap c=0$.

first proof By 314M, there are a set X and a σ-algebra Σ of subsets of X and a sequentially order-continuous Boolean homomorphism π from Σ onto $\mathfrak{A}$. Set $\nu_1=\nu\pi:\Sigma\to\mathbb{R}$. Then ν_1 is countably additive (326Jf). So ν_1 is bounded and there is a set $H\in\Sigma$ such that $\nu_1F\ge 0$ whenever $F\in\Sigma$ and $F\subseteq H$ and $\nu_1F\le 0$ whenever $F\in\Sigma$ and $F\cap H=\emptyset$ (231Eb). Set $c=\pi H\in\mathfrak{A}$. If $a\subseteq c$, then there is an $F\in\Sigma$ such that $\pi F=a$; now $\pi(F\cap H)=a\cap c=a$, so $\nu a=\nu_1(F\cap H)\ge 0$. If $a\cap c=0$, then there is an $F\in\Sigma$ such that $\pi F=a$; now $\pi(F\setminus H)=a\setminus c=a$, so $\nu a=\nu_1(F\setminus H)\le 0$.

second proof (a) Note first that ν is bounded. **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, then $\sum_{n=0}^{\infty}\nu a_n$ must exist and be equal to $\nu(\sup_{n\in\mathbb{N}}a_n)$; in particular, $\lim_{n\to\infty}\nu a_n=0$. By 326D, ν is bounded. **Q**

(b)(i) We know that $\gamma=\sup\{\nu a:a\in\mathfrak{A}\}<\infty$. Choose a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\nu a_n\ge\gamma-2^{-n}$ for every $n\in\mathbb{N}$. For $m\le n\in\mathbb{N}$, set $b_{mn}=\inf_{m\le i\le n}a_i$. Then $\nu b_{mn}\ge\gamma-2\cdot 2^{-m}+2^{-n}$ for every $n\ge m$. **P** Induce on n. For $n=m$, this is due to the choice of $a_m=b_{mm}$. For the inductive step, we have $b_{m,n+1}=b_{mn}\cap a_{n+1}$, while surely $\gamma\ge\nu(a_{n+1}\cup b_{mn})$, so

$$\begin{aligned}\gamma+\nu b_{m,n+1}&\ge\nu(a_{n+1}\cup b_{mn})+\nu(a_{n+1}\cap b_{mn})\\&=\nu a_{n+1}+\nu b_{mn}\ge\gamma-2^{-n-1}+\gamma-2\cdot 2^{-m}+2^{-n}\end{aligned}$$

(by the choice of a_{n+1} and the inductive hypothesis)

$$=2\gamma-2\cdot 2^{-m}+2^{-n-1}.$$

Subtracting γ from both sides, $\nu b_{m,n+1}\ge\gamma-2\cdot 2^{-m}+2^{-n-1}$ and the induction proceeds. **Q**

(ii) Set

$$b_m = \inf_{n\ge m} b_{mn} = \inf_{n\ge m} a_n.$$

Then

$$\nu b_m = \lim_{n\to\infty} \nu b_{mn} \ge \gamma - 2\cdot 2^{-m},$$

by 326Jc. Next, $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, so setting $c = \sup_{n\in\mathbb{N}} b_n$ we have

$$\nu c = \lim_{n\to\infty} \nu b_n \ge \gamma;$$

since νc is surely less than or equal to γ, $\nu c = \gamma$.

If $b \in \mathfrak{A}$ and $b \subseteq c$, then

$$\nu c - \nu b = \nu(c\setminus b) \le \gamma = \nu c,$$

so $\nu b \ge 0$. If $b\in\mathfrak{A}$ and $b\cap c = 0$ then

$$\nu c + \nu b = \nu(c\cup b) \le \gamma = \nu c$$

so $\nu b \le 0$. This completes the proof.

326N Completely additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **completely additive** or **τ-additive** if it is finitely additive and $\inf_{a\in A} |\nu a| = 0$ whenever A is a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0.

326O Basic facts Let $\mathfrak{A}$ be a Boolean algebra and ν a completely additive real-valued functional on $\mathfrak{A}$.

(a) ν is countably additive. **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then for any infinite $I\subseteq\mathbb{N}$ the set $\{a_i : i \in I\}$ is downwards-directed and has infimum 0, so $\inf_{i\in I} |\nu a_i| = 0$; which means that $\lim_{n\to\infty} \nu a_n$ must be zero. By 326Ka, ν is countably additive. **Q**

(b) Let A be a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0. Then for every $\epsilon > 0$ there is an $a \in A$ such that $|\nu b| \le \epsilon$ whenever $b \subseteq a$. **P?** Suppose, if possible, otherwise. Set

$$B = \{b : |\nu b| \ge \epsilon,\ \exists\, a \in A,\ b \supseteq a\}.$$

If $a\in A$ there is a $b' \subseteq a$ such that $|\nu b'| > \epsilon$. Now $\{a'\setminus b' : a' \in A,\ a' \subseteq a\}$ is downwards-directed and has infimum 0, so there is an $a' \in A$ such that $a' \subseteq a$ and $|\nu(a'\setminus b')| \le |\nu b'| - \epsilon$. Set $b = b'\cup a'$; then $a' \subseteq b$ and

$$|\nu b| = |\nu b' + \nu(a'\setminus b')| \ge |\nu b'| - |\nu(a'\setminus b')| \ge \epsilon,$$

so $b \in B$. But also $b \subseteq a$. Thus every member of A includes some member of B. Since every member of B includes a member of A, B is downwards-directed and has infimum 0; but this is impossible, since $\inf_{b\in B} |\nu b| \ge \epsilon$. **XQ**

(c) If ν is non-negative, it is order-continuous. **P** (i) If A is a non-empty upwards-directed set with supremum a_0, then $\{a_0\setminus a : a \in A\}$ is a non-empty downwards-directed set with infimum 0, so

$$\sup_{a\in A} \nu a = \nu a_0 - \inf_{a\in A} \nu(a_0\setminus a) = \nu a_0.$$

(ii) If A is a non-empty downwards-directed set with infimum a_0, then $\{a\setminus a_0 : a\in A\}$ is a non-empty downwards-directed set with infimum 0, so

$$\inf_{a\in A} \nu a = \nu a_0 + \inf_{a\in A} \nu(a\setminus a_0) = \nu a_0. \quad \textbf{Q}$$

(d) If $c \in \mathfrak{A}$, then $a \mapsto \nu(a\cap c)$ is completely additive. **P** If A is a non-empty downwards-directed set with infimum 0, so is $\{a\cap c : a\in A\}$, and $\inf_{a\in A} |\nu(a\cap c)| = 0$. **Q**

(e) $\alpha\nu$ is a completely additive functional for any $\alpha\in\mathbb{R}$. If ν' is another completely additive functional on $\mathfrak{A}$, then $\nu+\nu'$ is completely additive. **P** We know from 326Bc that $\nu+\nu'$ is additive. Let A be a non-empty downwards-directed set with infimum 0. For any $\epsilon > 0$, (b) tells us that there are a, $a' \in A$ such that $|\nu b| \le \epsilon$ whenever $b\subseteq a$ and $|\nu' b| \le \epsilon$ whenever $b \subseteq a'$. But now, because A is downwards-directed, there is a $b\in A$ such that $b \subseteq a\cap a'$, which means that $|\nu b + \nu' b| \le |\nu b| + |\nu' b|$ is at most 2ϵ. As ϵ is arbitrary, $\inf_{a\in A} |(\nu+\nu')(a)| = 0$, and $\nu+\nu'$ is completely additive. **Q**

(f) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is an order-continuous Boolean homomorphism, then $\nu\pi$ is a completely additive functional on $\mathfrak{B}$. **P** By 326Be, $\nu\pi$ is additive. If $B \subseteq \mathfrak{B}$ is a non-empty downwards-directed set with infimum 0 in $\mathfrak{B}$, then $\pi[B]$ is a non-empty downwards-directed set with infimum 0 in $\mathfrak{A}$, because π is order-continuous, so $\inf_{b\in B} |\nu\pi b| = 0$. **Q** In particular, if $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then $\nu\restriction\mathfrak{B}$ is completely additive.

(g) If ν' is another additive functional on $\mathfrak{A}$ and $|\nu' a| \le \nu a$ for every $a \in \mathfrak{A}$, then ν' is completely additive. **P** If $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and $\inf A = 0$, then $\inf_{a\in A} |\nu' a| \le \inf_{a\in A} \nu a = 0$. **Q**

326P I squeeze a useful fact in here.

Proposition If $\mathfrak{A}$ is a ccc Boolean algebra, a functional $\nu : \mathfrak{A} \to \mathbb{R}$ is countably additive iff it is completely additive.

proof If ν is completely additive it is countably additive, by 326Oa. If ν is countably additive and A is a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0, then there is a (non-empty) countable subset B of A also with infimum 0 (316E). Let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a sequence running over B, and choose $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that $a_0 = b_0$, $a_{n+1} \subseteq a_n \cap b_n$ for every $n \in \mathbb{N}$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0, so $\lim_{n\to\infty} \nu a_n = 0$ (326Jc) and $\inf_{a\in A} |\nu a| = 0$. As A is arbitrary, ν is completely additive.

326Q The Jordan decomposition (III): Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν a completely additive real-valued functional on $\mathfrak{A}$. Then ν is bounded and expressible as the difference of two non-negative completely additive functionals.

proof (a) I must first check that ν is bounded. **P** Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$. Set

$$A = \{a : a \in \mathfrak{A}, \text{ there is an } n \in \mathbb{N} \text{ such that } a_i \subseteq a \text{ for every } i \ge n\}.$$

Then A is closed under $\cap$, and if b is any lower bound for A then $b \subseteq 1 \setminus a_n \in A$, so $b \cap a_n = 0$, for every $n \in \mathbb{N}$; but this means that $1 \setminus b \in A$, so that $b \subseteq 1 \setminus b$ and $b = 0$. Thus $\inf A = 0$. By 326Ob, there is an $a \in A$ such that $|\nu b| \le 1$ whenever $b \subseteq a$. By the definition of A, there must be an $n \in \mathbb{N}$ such that $|\nu a_i| \le 1$ for every $i \ge n$. But this means that $\sup_{n\in\mathbb{N}} |\nu a_n|$ is finite. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ν is bounded, by 326D(ii). **Q**

(b) As in 326D and 326L, set $\nu^+ a = \sup_{b\subseteq a} \nu b$ for every $a \in \mathfrak{A}$. Then ν^+ is completely additive. **P** We know that ν^+ is additive. If A is a non-empty downwards-directed subset of $\mathfrak{A}$ with infimum 0, then for every $\epsilon > 0$ there is an $a \in A$ such that $|\nu b| \le \epsilon$ whenever $b \subseteq a$; in particular, $\nu^+ a \le \epsilon$. As ϵ is arbitrary, $\inf_{a\in A} \nu^+ a = 0$; as A is arbitrary, ν^+ is completely additive. **Q**

Consequently $\nu^- = \nu^+ - \nu$ is completely additive (326Oe) and $\nu = \nu^+ - \nu^-$ is the difference of non-negative completely additive functionals.

326R I give an alternative definition of 'completely additive' which you may feel clarifies the concept.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, and $\nu : \mathfrak{A} \to \mathbb{R}$ a function. Then the following are equiveridical:

(i) ν is completely additive;
(ii) $\nu 1 = \sum_{i\in I} \nu a_i$ whenever $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$;
(iii) $\nu a = \sum_{i\in I} \nu a_i$ whenever $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ with supremum a.

proof (For notes on sums $\sum_{i\in I}$, see 226A.)

(a)(i)⇒(ii) If ν is completely additive and $\langle a_i\rangle_{i\in I}$ is a partition of unity in A, then (inducing on $\#(J)$) $\nu(\sup_{i\in J} a_i) = \sum_{i\in J} \nu a_i$ for every finite $J \subseteq I$. Consider

$$A = \{1 \setminus \sup_{i\in J} a_i : J \subseteq I \text{ is finite}\}.$$

Then A is non-empty and downwards-directed and has infimum 0, so for every $\epsilon > 0$ there is an $a \in A$ such that $|\nu b| \le \epsilon$ whenever $b \subseteq a$ (326Ob again). Express a as $1 \setminus \sup_{i\in J} a_i$ where $J \subseteq I$ is finite. If now K is another finite subset of I including J,

$$|\nu 1 - \textstyle\sum_{i\in K} a_i| = |\nu(1 \setminus \sup_{i\in K} a_i)| \le \epsilon.$$

As remarked in 226Ad, this means that $\nu 1 = \sum_{i\in I} \nu a_i$, as claimed.

(b)(ii)⇒(iii) Suppose that ν satisfies the condition (ii), and that $\langle a_i\rangle_{i\in I}$ is a disjoint family with supremum a. Take any $j \notin I$, set $J = I \cup \{j\}$ and $a_j = 1 \setminus a$; then $\langle a_i\rangle_{i\in J}$, $(a, 1 \setminus a)$ are both partitions of unity, so

$$\nu(1\setminus a)+\nu a=\nu 1=\sum_{i\in J}\nu a_i=\nu(1\setminus a)+\sum_{i\in I}\nu a_i,$$

and $\nu a=\sum_{i\in I}\nu a_i$.

(c)(iii)⇒(i) Suppose that ν satisfies (iii). Then ν is additive.

(α) ν is bounded. **P** Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$. Applying Zorn's Lemma to the set $\mathcal{C}$ of all disjoint families $C\subseteq\mathfrak{A}$ including $\{a_n:n\in\mathbb{N}\}$, we find a partition of unity $C\supseteq\{a_n:n\in\mathbb{N}\}$. Now $\sum_{c\in C}\nu c$ is defined in $\mathbb{R}$, so $\sup_{n\in\mathbb{N}}|\nu a_n|\le\sup_{c\in C}|\nu c|$ is finite. By 326D, ν is bounded. **Q**

(β) Define ν^+ from ν as in 326D. Then ν^+ satisfies the same condition as ν. **P** Let $\langle a_i\rangle_{i\in I}$ be a disjoint family in $\mathfrak{A}$ with supremum a. Then for any $b\subseteq a$, we have $b=\sup_{i\in I}b\cap a_i$, so

$$\nu b=\sum_{i\in I}\nu(b\cap a_i)\le\sum_{i\in I}\nu^+a_i.$$

Thus $\nu^+a\le\sum_{i\in I}\nu^+a_i$. But of course

$$\begin{aligned}\sum_{i\in I}\nu^+a_i&=\sup\{\sum_{i\in J}\nu^+a_i:J\subseteq I\text{ is finite}\}\\&=\sup\{\nu^+(\sup_{i\in J}a_i):J\subseteq I\text{ is finite}\}\le\nu^+a,\end{aligned}$$

so $\nu^+a=\sum_{i\in I}\nu^+a_i$. **Q**

(γ) It follows that ν^+ is completely additive. **P** If A is a non-empty downwards-directed set with infimum 0, then $B=\{b:\exists\,a\in A,\,b\cap a=0\}$ is order-dense in $\mathfrak{A}$, so there is a partition of unity $\langle b_i\rangle_{i\in I}$ lying in B (313K). Now if $J\subseteq I$ is finite, there is an $a\in A$ such that $a\cap\sup_{i\in J}b_i=0$ (because A is downwards-directed), and

$$\nu^+a+\sum_{i\in J}\nu^+b_i\le\nu^+1.$$

Since $\nu^+1=\sup_{J\subseteq I\text{ is finite}}\sum_{i\in J}\nu^+b_i$, $\inf_{a\in A}\nu^+a=0$. As A is arbitrary, ν^+ is completely additive. **Q**

(δ) Now consider $\nu^-=\nu^+-\nu$. Of course

$$\nu^-a=\nu^+a-\nu a=\sum_{i\in I}\nu^+a_i-\sum_{i\in I}\nu a_i=\sum_{i\in I}\nu^-a_i$$

whenever $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ with supremum a. Because ν^- is non-negative, the argument of (γ) shows that $\nu^-=(\nu^-)^+$ is completely additive. So $\nu=\nu^+-\nu^-$ is completely additive, as required.

326S For completely additive functionals, we have a useful refinement of the Hahn decomposition. I give it in a form adapted to the applications I have in mind.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu:\mathfrak{A}\to\mathbb{R}$ a completely additive functional. Then there is a unique element of $\mathfrak{A}$, which I will denote $[\![\nu>0]\!]$, 'the region where $\nu>0$', such that $\nu a>0$ whenever $0\ne a\subseteq[\![\nu>0]\!]$, while $\nu a\le 0$ whenever $a\cap[\![\nu>0]\!]=0$.

proof Set

$$C_1=\{c:c\in\mathfrak{A}\setminus\{0\},\,\nu a>0\text{ whenever }0\ne a\subseteq c\},$$

$$C_2=\{c:c\in\mathfrak{A},\,\nu a\le 0\text{ whenever }a\subseteq c\}.$$

Then $C_1\cup C_2$ is order-dense in $\mathfrak{A}$. **P** There is a $c_0\in\mathfrak{A}$ such that $\nu a\ge 0$ for every $a\subseteq c_0$ and $\nu a\le 0$ whenever $a\cap c_0=0$ (326M). Given $b\in\mathfrak{A}\setminus\{0\}$, then $b\setminus c_0\in C_2$, so if $b\setminus c_0\ne 0$ we can stop. Otherwise, $b\subseteq c_0$. If $b\in C_1$ we can stop. Otherwise, there is a non-zero $c\subseteq b$ such that $\nu c\le 0$; but in this case $\nu a\ge 0$ and $\nu(c\setminus a)\ge 0$ so $\nu a=0$ for every $a\subseteq c$, and $c\in C_2$. **Q**

There is therefore a partition of unity $D\subseteq C_1\cup C_2$. Now $D\cap C_1$ is countable. **P** If $d\in D\cap C_1$, $\nu d>0$. Also

$$\#(\{d:d\in D,\,\nu d\ge 2^{-n}\})\le 2^n\sup_{a\in\mathfrak{A}}\nu a$$

is finite for each n, so $D\cap C_1$ is the union of a sequence of finite sets, and is countable. **Q**

Accordingly $D\cap C_1$ has a supremum e. If $0\ne a\subseteq e$ then

$$\nu a=\sum_{c\in D}\nu(a\cap c)=\sum_{c\in D\cap C_1}\nu(a\cap c)\ge 0$$

by 326R. Also there must be some $c\in D\cap C_1$ such that $a\cap c\ne 0$, in which case $\nu(a\cap c)>0$, so that $\nu a>0$. If $a\cap e=0$, then

$$\nu a = \sum_{c \in D} \nu(a \cap c) = \sum_{c \in D \cap C_2} \nu(a \cap c) \le 0.$$

Thus e has the properties demanded of $[\![\nu > 0]\!]$. To see that e is unique, we need observe only that if e' has the same properties then $\nu(e \setminus e') \le 0$ (because $(e \setminus e') \cap e' = 0$), so $e \setminus e' = 0$ (because $e \setminus e' \subseteq e$). Similarly, $e' \setminus e = 0$ and $e = e'$. Thus we may properly denote e by the formula $[\![\nu > 0]\!]$.

326T Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and μ, ν two completely additive functionals on $\mathfrak{A}$. Then there is a unique element of $\mathfrak{A}$, which I will denote $[\![\mu > \nu]\!]$, 'the region where $\mu > \nu$', such that

$$\mu a > \nu a \text{ whenever } 0 \ne a \subseteq [\![\mu > \nu]\!],$$

$$\mu a \le \nu a \text{ whenever } a \cap [\![\mu > \nu]\!] = 0.$$

proof Apply 326S to the functional $\mu - \nu$, and set $[\![\mu > \nu]\!] = [\![\mu - \nu > 0]\!]$.

326X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Show that (i) $\nu(a \cup b) = \nu a + \nu b - \nu(a \cap b)$ (ii) $\nu(a \cup b \cup c) = \nu a + \nu b + \nu c - \nu(a \cap b) - \nu(a \cap c) - \nu(b \cap c) + \nu(a \cap b \cap c)$ for all a, b, $c \in \mathfrak{A}$. Generalize these results to longer sequences in $\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Boolean algebra. (i) Show that a finitely additive functional ν is properly atomless iff there is a properly atomless additive functional ν' such that $|\nu a| \le \nu' a$ for every $a \in \mathfrak{A}$. (ii) Show that a non-negative finitely additive functional ν on $\mathfrak{A}$ is properly atomless iff whenever ν' is a non-zero finitely additive functional such that $0 \le \nu' a \le \nu a$ for every $a \in \mathfrak{A}$ there is an $a \in \mathfrak{A}$ such that $\nu' a$ and $\nu'(1 \setminus a)$ are both non-zero.

(c) (i) Suppose that $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ is countably additive. Show that $\mathcal{I} = \{a : \nu b = 0 \text{ for every } b \subseteq a\}$ is an ideal of $\mathfrak{A}$. Show that the following are equiveridical: (α) ν is properly atomless; (β) whenever $\nu a \ne 0$ there is a $b \subseteq a$ such that $\nu b \notin \{0, \nu a\}$; (γ) the quotient algebra $\mathfrak{A}/\mathcal{I}$ is atomless. (ii) Find an atomless Dedekind complete Boolean algebra $\mathfrak{A}$ and a finitely additive $\nu : \mathfrak{A} \to [0,1]$ such that $\nu a > 0$ for every non-zero $a \in \mathfrak{A}$ but ν is not properly atomless.

(d) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Show that the following are equiveridical: (i) ν is countably additive; (ii) $\lim_{n\to\infty} \nu a_n = \nu a$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum a.

(e) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Show that the following are equiveridical: (i) ν is countably additive; (ii) $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ and $\inf_{n\in\mathbb{N}} \sup_{m\ge n} a_m = 0$; (iii) $\lim_{n\to\infty} \nu a_n = \nu a$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ and $a = \inf_{n\in\mathbb{N}} \sup_{m\ge n} a_m = \sup_{n\in\mathbb{N}} \inf_{m\ge n} a_m$. (*Hint*: for (i)$\Rightarrow$(iii), consider non-negative ν first.)

(f) Let X be an uncountable set, and J an infinite subset of X. Let $\mathfrak{A}$ be the finite-cofinite algebra of X (316Yl), and for $a \in A$ set $\nu a = \#(a \cap J)$ if a is finite, $-\#(J \setminus a)$ if a is cofinite. Show that ν is countably additive and unbounded.

>(g) Let $\mathfrak{A}$ be the algebra of subsets of $[0,1]$ generated by the family of (closed) intervals. Show that there is a unique additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu[\alpha, \beta] = \beta - \alpha$ whenever $0 \le \alpha \le \beta \le 1$. Show that ν is countably additive but not completely additive.

(h)(i) Let (X, Σ, μ) be any atomless probability space. Show that $\mu : \Sigma \to \mathbb{R}$ is a countably additive functional which is not completely additive. (ii) Let X be any uncountable set and μ the countable-cocountable measure on X (211R). Show that μ is countably additive but not completely additive.

(i) Let $\mathfrak{A}$ be an atomless Boolean algebra. Show that every completely additive functional on $\mathfrak{A}$ is properly atomless.

(j) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a function. (i) Show that ν is finitely additive iff $\sum_{i\in I} \nu a_i = \nu 1$ for every finite partition of unity $\langle a_i \rangle_{i\in I}$. (ii) Show that ν is countably additive iff $\sum_{i\in I} \nu a_i = \nu 1$ for every countable partition of unity $\langle a_i \rangle_{i\in I}$.

(k) Show that 326S can fail if ν is only countably additive, rather than completely additive. (*Hint*: 326Xh.)

(l) Let $\mathfrak{A}$ be a Boolean algebra and ν a finitely additive real-valued functional on $\mathfrak{A}$. Let us say that $a \in \mathfrak{A}$ is a **support** of ν if (α) $\nu b = 0$ whenever $b \cap a = 0$ (β) for every non-zero $b \subseteq a$ there is a $c \subseteq b$ such that $\nu c \neq 0$. (i) Check that ν can have at most one support. (ii) Show that if a is a support for ν and ν is bounded, then the principal ideal $\mathfrak{A}_a$ generated by a is ccc. (iii) Show that if $\mathfrak{A}$ is Dedekind σ-complete and ν is countably additive, then ν is completely additive iff it has a support, and that in the language of 326S this is $[\![\nu > 0]\!] \cup [\![-\nu > 0]\!]$. (iv) Taking $J = X$ in 326Xf, show that X is the support of the functional ν there.

326Y Further exercises (a) Show that there is a finitely additive functional $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ such that $\nu\{n\} = 1$ for every $n \in \mathbb{N}$, so that ν is not bounded. (*Hint*: Use Zorn's Lemma to construct a maximal linearly independent subset of ℓ^∞ including $\{\chi\{n\} : n \in \mathbb{N}\}$, and hence to construct a linear map $f : \ell^\infty \to \mathbb{R}$ such that $f(\chi\{n\}) = 1$ for every n.)

(b) Let $\mathfrak{A}$ be any infinite Boolean algebra. Show that there is an unbounded finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$. (*Hint*: let $\langle t_n \rangle_{n\in\mathbb{N}}$ be a sequence of distinct points in the Stone space of $\mathfrak{A}$, and set $\nu a = \nu'\{n : t_n \in \widehat{a}\}$ for a suitable ν'.)

(c) Let $\mathfrak{A}$ be a Boolean algebra, and give $\mathbb{R}^{\mathfrak{A}}$ its product topology. Show that the space of finitely additive functionals on $\mathfrak{A}$ is a closed subset of $\mathbb{R}^{\mathfrak{A}}$, but that the space of bounded finitely additive functionals is closed only when $\mathfrak{A}$ is finite.

(d) Let $\mathfrak{A}$ be a Boolean algebra, and M the linear space of all bounded finitely additive real-valued functionals on $\mathfrak{A}$. For ν, $\nu' \in M$ say that $\nu \le \nu'$ if $\nu a \le \nu' a$ for every $a \in \mathfrak{A}$. Show that

(i) ν^+, as defined in the proof of 326D, is just $\sup\{0, \nu\}$ in M;

(ii) M is a Dedekind complete Riesz space (241E-241F, 353G);

(iii) for ν, $\nu' \in M$, $|\nu| = \nu \vee (-\nu)$, $\nu \vee \nu'$ and $\nu \wedge \nu'$ are given by the formulae

$$|\nu|(a) = \sup_{b \subseteq a} \nu b - \nu(a \setminus b), \quad (\nu \vee \nu')(a) = \sup_{b \subseteq a} \nu b + \nu'(a \setminus b),$$

$$(\nu \wedge \nu')(a) = \inf_{b \subseteq a} \nu b + \nu'(a \setminus b);$$

(iv) for any non-empty $A \subseteq M$, A is bounded above in M iff

$$\sup\{\textstyle\sum_{i=0}^n \nu_i a_i : \nu_i \in A \text{ for each } i \le n,\ \langle a_i \rangle_{i \le n} \text{ is disjoint}\}$$

is finite, and then $\sup A$ is defined by the formula

$$(\sup A)(a) = \sup\{\textstyle\sum_{i=0}^n \nu_i a_i : \nu_i \in A \text{ for each } i \le n,\ \langle a_i \rangle_{i \le n} \text{ is disjoint}, \sup_{i \le n} a_i = a\}$$

for every $a \in \mathfrak{A}$;

(v) setting $\|\nu\| = |\nu|(1)$, $\|\ \|$ is an order-continuous norm (definition: 354Dc) on M under which M is a Banach lattice.

(e) Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{C}$ is **finitely additive** if its real and imaginary parts are. Show that the space of bounded finitely additive functionals from $\mathfrak{A}$ to $\mathbb{C}$ is a Banach space under the total variation norm $\|\nu\| = \sup\{\sum_{i=0}^n |\nu a_i| : \langle a_i \rangle_{i \le n}$ is a partition of unity in $\mathfrak{A}\}$.

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and μ, ν finitely additive functionals on $\mathfrak{A}$, $\mathfrak{B}$ respectively. Show that there is a unique finitely additive functional λ on the free product $\mathfrak{A} \otimes \mathfrak{B} \to \mathbb{R}$ such that $\lambda(a \otimes b) = \mu a \cdot \nu b$ for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$.

(g) Let $\langle \mathfrak{A}_i \rangle_{i\in I}$ be a family of Boolean algebras, with free product $(\bigotimes_{i\in I} \mathfrak{A}_i, \langle \varepsilon_i \rangle_{i\in I})$, and for each $i \in I$ let ν_i be a finitely additive functional on $\mathfrak{A}_i$ such that $\nu_i 1 = 1$. Show that there is a unique finitely additive functional $\nu : \bigotimes_{i\in I} \mathfrak{A}_i \to \mathbb{R}$ such that $\nu(\inf_{i\in J} \varepsilon_i(a_i)) = \prod_{i\in J} \nu_i a_i$ whenever $J \subseteq I$ is non-empty and finite and $a_i \in \mathfrak{A}_i$ for each $i \in J$.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to [0, \infty[$ a countably additive functional. Show that ν is properly atomless iff whenever $a \in \mathfrak{A}$ and $\nu a \neq 0$ there is a $b \subseteq a$ such that $0 < \nu b < \nu a$.

(i) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional. Show that $\nu[\mathfrak{A}]$ is a compact subset of $\mathbb{R}$.

(j) Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$ (314P). Find a properly atomless finitely additive $\nu : \mathfrak{G} \to \mathbb{R}$ such that $\nu[\mathfrak{G}]$.

(k) (Halmos 48) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $r \geq 1$ an integer. (i) Let $C \subseteq \mathbb{R}^r$ be a non-empty bounded convex set, and for $z \in \mathbb{R}^r$ set $H_z = \{x : x \mathbin{.} z = \sup_{y \in C} y \mathbin{.} z\}$. Suppose that $H_z \cap \overline{C} \subseteq C$ for every $z \in \mathbb{R}^r \setminus \{0\}$. Show that C is closed. (ii) Suppose that $\nu : \mathfrak{A} \to \mathbb{R}^r$ is countably additive in the sense that all its coordinates are countably additive functionals. Show that $\nu[\mathfrak{A}]$ is compact.

(l) Let $\mathfrak{A}$ be a Boolean algebra, and give it the topology $\mathfrak{T}_\sigma$ for which the closed sets are the sequentially order-closed sets. Show that a finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is countably additive iff it is continuous for $\mathfrak{T}_\sigma$.

(m) Let $\mathfrak{A}$ be a Boolean algebra, and M_σ the set of all bounded countably additive real-valued functionals on $\mathfrak{A}$. Show that M_σ is a closed and order-closed linear subspace of the normed space M of all additive functionals on $\mathfrak{A}$ (326Yd), and that $|\nu| \in M_\sigma$ whenever $\nu \in M_\sigma$.

(n) Let $\mathfrak{A}$ be a Boolean algebra and ν a non-negative finitely additive functional on $\mathfrak{A}$. Set

$$\nu_\sigma a = \inf\{\sup_{n \in \mathbb{N}} \nu a_n : \langle a_n \rangle_{n \in \mathbb{N}} \text{ is a non-decreasing sequence with supremum } a\}$$

for every $a \in \mathfrak{A}$. Show that ν_σ is countably additive, and is $\sup\{\nu' : \nu' \leq \nu \text{ is countably additive}\}$.

(o) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle \nu_n \rangle_{n \in \mathbb{N}}$ a sequence of countably additive real-valued functionals on $\mathfrak{A}$ such that $\nu a = \lim_{n \to \infty} \nu_n a$ is defined in $\mathbb{R}$ for every $a \in \mathfrak{A}$. Show that ν is countably additive. (*Hint*: use arguments from part (a) of the proof of 247C to see that $\lim_{n \to \infty} \sup_{k \in \mathbb{N}} |\nu_k a_n| = 0$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$, and therefore that $\lim_{n \to \infty} \sup_{k \in \mathbb{N}} |\nu_k a_n| = 0$ whenever $\langle a_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence with infimum 0.)

(p) Let $\mathfrak{A}$ be a Boolean algebra, and M_τ the set of all completely additive real-valued functionals on $\mathfrak{A}$. Show that M_τ is a closed and order-closed linear subspace of the normed space M of all additive functionals, and that $|\nu| \in M_\tau$ whenever $\nu \in M_\tau$.

(q) Let $\mathfrak{A}$ be a Boolean algebra and ν a non-negative finitely additive functional on $\mathfrak{A}$. Set

$$\nu_\tau b = \inf\{\sup_{a \in A} \nu a : A \text{ is a non-empty upwards-directed set with supremum } b\}$$

for every $b \in \mathfrak{A}$. Show that ν_τ is completely additive, and is $\sup\{\nu' : \nu' \leq \nu \text{ is completely additive}\}$.

(r) Let $\mathfrak{A}$ be a Boolean algebra, and give it the topology $\mathfrak{T}$ for which the closed sets are the order-closed sets (313Xb). Show that a finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is completely additive iff it is continuous for $\mathfrak{T}$.

(s) Let X be a set, Σ any σ-algebra of subsets of X, and $\nu : \Sigma \to \mathbb{R}$ a functional. Show that ν is completely additive iff there are sequences $\langle x_n \rangle_{n \in \mathbb{N}}$, $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ such that $\sum_{n=0}^\infty |\alpha_n| < \infty$ and $\nu E = \sum_{n=0}^\infty \alpha_n \chi E(x_n)$ for every $E \in \Sigma$.

326 Notes and comments I have not mentioned the phrase 'measure algebra' anywhere in this section, and in principle this material could have been part of Chapter 31; but countably additive functionals are kissing cousins of measures, and most of the ideas here surely belong to 'measure theory' rather than to 'Boolean algebra', in so far as such divisions are meaningful at all. I have given as much as possible of the theory in a general form because the simplifications which are possible when we look only at measure algebras are seriously confusing if they are allowed too much prominence. In particular, it is important to understand that the principal properties of completely additive functionals do not depend on Dedekind completeness of the algebra, provided we take care over the definitions. Similarly, the definition of 'countably additive' functional for algebras which are not Dedekind σ-complete needs a moment's attention to the phrase 'and $\sup_{n \in \mathbb{N}} a_n$ is defined in $\mathfrak{A}$'. It can happen that a functional is countably additive mostly because there are too few such sequences (326Xf).

The formulations I have chosen as principal definitions (326A, 326I, 326N) are those which I find closest to my own intuitions of the concepts, but you may feel that 326K(i), 326Xe(iii) and 326R, or 326Yl and 326Yr, provide useful alternative patterns. The point is that countable additivity corresponds to sequential order-continuity (326Jb, 326Jc, 326Jf), while complete additivity corresponds to order-continuity (326Oc, 326Of); the difficulty is that we must consider functionals which are not order-preserving, so that the simple definitions in 313H cannot be applied directly. It is fair to say that all the additive functionals ν we need to understand are bounded, and therefore may be studied in terms of their positive and negative parts ν^+, ν^-, which are order-preserving (326Bf); but many of the most important applications of these ideas depend precisely on using facts about ν to deduce facts about ν^+ and ν^-.

It is in 326D that we seem to start getting more out of the theory than we have put in. The ideas here have vast ramifications. What it amounts to is that we can discover much more than we might expect by looking at disjoint sequences. To begin with, the conditions here lead directly to 326M and 326Q: every completely additive functional is bounded, and every countably additive functional on a Dedekind σ-complete Boolean algebra is bounded. (But note 326Ya-326Yb.)

I have expressed 326H in terms of an additive function from a Boolean algebra to a finite-dimensional space (it is already non-trivial in the two-dimensional case, which would correspond to an additive complex-valued functional, as in 326Ye). It is usually regarded as a theorem about countably additive functions, or 'vector measures' (see 394O below), but rather remarkably we do not in fact need countable additivity. Of course it can also be regarded as a kind of ham-sandwich theorem for measures; we can simultaneously bisect an element of a Dedekind σ-complete Boolean algebra with respect to finitely many additive functionals. If you like, the dimensionality requirement of the ordinary ham-sandwich theorems of topology is met by the requirement of atomlessness here. A companion result, also due to Liapounoff, which requires countable additivity but allows atoms, is in 362Yx.

Naturally enough, the theory of countably additive functionals on general Boolean algebras corresponds closely to the special case of countably additive functionals on σ-algebras of sets, already treated in §§231-232 for the sake of the Radon-Nikodým theorem. This should make 326I-326M very straightforward. When we come to completely additive functionals, however, there is room for many surprises. The natural map from a σ-algebra of measurable sets to the corresponding measure algebra is sequentially order-continuous but rarely order-continuous, so that there can be completely additive functionals on the measure algebra which do not correspond to completely additive functionals on the σ-algebra. Indeed there are very few completely additive functionals on σ-algebras of sets (326Ys). Of course these surprises can arise only when there is a difference between completely additive and countably additive functionals, that is, when the algebra involved is not ccc (326P). But I think that neither 326Q nor 326R is obvious.

I find myself generally using the phrase 'countably additive' in preference to 'completely additive' in the context of ccc algebras, where there is no difference between them. This is an attempt at user-friendliness; the phrase 'countably additive' is the commoner one in ordinary use. But I must say that my personal inclination is to the other side. The reason why so many theorems apply to countably additive functionals in these contexts is just that they are completely additive.

I have given two proofs of 326M. I certainly assume that if you have got this far you are acquainted with the Radon-Nikodým theorem and the associated basic facts about countably additive functionals on σ-algebras of sets; so that the 'first proof' should be easy and natural. On the other hand, there are purist objections on two fronts. First, it relies on the Stone representation, which involves a much stronger form of the axiom of choice than is actually necessary. Second, the classical Hahn decomposition in 231E is evidently a special case of 326M, and if we need both (as we certainly do) then one expects the ideas to stand out more clearly if they are applied directly to the general case. In fact the two versions of the argument are so nearly identical that (as you will observe, if you have Volume 2 to hand) they can share nearly every word. You can take the 'second proof', therefore, as a worked example in the translation of ideas from the context of σ-algebras of sets to the context of Dedekind σ-complete Boolean algebras. What makes it possible is the fact that the only limit operations referred to involve countable families.

Arguments not involving limit operations can generally, of course, be applied to all Boolean algebras; I have lifted some exercises (326Yd, 326Yn) from §231 to give you some practice in such generalizations.

Almost any non-trivial measure provides an example of a countably additive functional on a Dedekind σ-complete algebra which is not completely additive (326Xh). The question of whether such a functional can exist on a Dedekind complete algebra is the 'Banach-Ulam problem', to which I will return in 363S.

In this section I have looked only at questions which can be adequately treated in terms of the underlying algebras $\mathfrak{A}$, without using any auxiliary structure. To go much farther we shall need to study the 'function spaces' $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ of Chapter 36. In particular, the ideas of 326Ya, 326Yd-326Ye and 326Ym-326Yq will make better sense when redeveloped in §362.

327 Additive functionals on measure algebras

When we turn to measure algebras, we have a simplification, relative to the general context of §326, because the algebras are always Dedekind σ-complete; but there are also elaborations, because we can ask how the additive functionals we examine are related to the measure. In 327A-327C I work through the relationships between the concepts of 'absolute continuity', '(true) continuity' and 'countable additivity', following 232A-232B, and adding 'complete additivity' from §326. These ideas provide a new interpretation of the Radon-Nikodým theorem (327D). I then use this theorem to develop some machinery (the 'standard extension' of an additive functional from a closed subalgebra to the whole algebra, 327F-327G) which will be used in §333.

327A I start with the following definition and theorem corresponding to 232A-232B.

Definition Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and $\nu:\mathfrak{A}\to\mathbb{R}$ a finitely additive functional. Then ν is **absolutely continuous** with respect to $\bar{\mu}$ if for every $\epsilon>0$ there is a $\delta>0$ such that $|\nu a|\le\epsilon$ whenever $\bar{\mu}a\le\delta$.

327B Theorem Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\nu:\mathfrak{A}\to\mathbb{R}$ a finitely additive functional. Give $\mathfrak{A}$ its measure-algebra topology and uniformity (§323).

(a) If ν is continuous at 0, it is completely additive.

(b) If ν is countably additive, it is absolutely continuous with respect to $\bar{\mu}$.

(c) The following are equiveridical:

(i) ν is continuous at 0;

(ii) ν is countably additive and whenever $a\in\mathfrak{A}$ and $\nu a\ne 0$ there is a $b\in\mathfrak{A}$ such that $\bar{\mu}b<\infty$ and $\nu(a\cap b)\ne 0$;

(iii) ν is continuous everywhere on $\mathfrak{A}$;

(iv) ν is uniformly continuous.

(d) If $(\mathfrak{A},\bar{\mu})$ is semi-finite, then ν is continuous iff it is completely additive.

(e) If $(\mathfrak{A},\bar{\mu})$ is σ-finite, then ν is continuous iff it is countably additive iff it is completely additive.

(f) If $(\mathfrak{A},\bar{\mu})$ is totally finite, then ν is continuous iff it is absolutely continuous with respect to $\bar{\mu}$ iff it is countably additive iff it is completely additive.

proof (a) If ν is continuous, and $A\subseteq\mathfrak{A}$ is non-empty, downwards-directed and has infimum 0, then $0\in\overline{A}$ (323D(b-ii)), so $\inf_{a\in A}|\nu a|=0$.

(b) ? Suppose, if possible, that ν is countably additive but not absolutely continuous. Then there is an $\epsilon>0$ such that for every $\delta>0$ there is an $a\in\mathfrak{A}$ such that $\bar{\mu}a\le\delta$ but $|\nu a|\ge\epsilon$. For each $n\in\mathbb{N}$ we may choose a $b_n\in\mathfrak{A}$ such that $\bar{\mu}b_n\le 2^{-n}$ and $|\nu b_n|\ge\epsilon$. Consider $b_n^*=\sup_{k\ge n}b_k$, $b=\inf_{n\in\mathbb{N}}b_n^*$. Then we have

$$\bar{\mu}b\le\inf_{n\in\mathbb{N}}\bar{\mu}(\sup_{k\ge n}b_k)\le\inf_{n\in\mathbb{N}}\sum_{k=n}^{\infty}2^{-k}=0,$$

so $\bar{\mu}b=0$ and $b=0$. On the other hand, ν is expressible as a difference $\nu^+-\nu^-$ of non-negative countably additive functionals (326L), each of which is sequentially order-continuous (326Kc), and

$$0=\lim_{n\to\infty}(\nu^++\nu^-)b_n^*\ge\inf_{n\in\mathbb{N}}(\nu^++\nu^-)b_n\ge\inf_{n\in\mathbb{N}}|\nu b_n|\ge\epsilon,$$

which is absurd. **X**

(c)(i)⇒(ii) Suppose that ν is continuous at 0. Then it is completely additive, by (a), therefore countably additive. If $\nu a\ne 0$, there must be a b of finite measure such that $|\nu d|<|\nu a|$ whenever $d\cap b=0$, so that $|\nu(a\setminus b)|<|\nu a|$ and $\nu(a\cap b)\ne 0$. Thus the conditions are satisfied.

(ii)⇒(iv) Now suppose that ν satisfies the two conditions in (ii). Because $\mathfrak{A}$ is Dedekind σ-complete, ν must be bounded (326M), therefore expressible as the difference $\nu^+-\nu^-$ of countably additive functionals. Set $\nu_1=\nu^++\nu^-$. Set

$$\gamma=\sup\{\nu_1 b:b\in\mathfrak{A},\ \bar{\mu}b<\infty\},$$

and choose a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ of elements of $\mathfrak{A}$ of finite measure such that $\lim_{n\to\infty}\nu_1 b_n=\gamma$; set $b^*=\sup_{n\in\mathbb{N}}b_n$. If $d\in\mathfrak{A}$ and $d\cap b^*=0$ then $\nu d=0$. **P** If $b\in\mathfrak{A}$ and $\bar{\mu}b<\infty$, then

$$|\nu(d\cap b)|\le\nu_1(d\cap b)\le\nu_1(b\setminus b_n)=\nu_1(b\cup b_n)-\nu_1 b_n\le\gamma-\nu_1 b_n$$

for every $n\in\mathbb{N}$, so $\nu(d\cap b)=0$. As b is arbitrary, the second condition in (ii) tells us that $\nu d=0$. **Q**

Setting $b_n^*=\sup_{k\le n}b_k$ for each n, we have $\lim_{n\to\infty}\nu_1(b^*\setminus b_n^*)=0$. Take any $\epsilon>0$, and (using (b) above) let $\delta>0$ be such that $|\nu a|\le\epsilon$ whenever $\bar{\mu}a\le\delta$. Let n be such that $\nu_1(b^*\setminus b_n^*)\le\epsilon$. Then

$$\begin{aligned}|\nu a| &\le |\nu(a\cap b_n^*)| + |\nu(a\cap(b^*\setminus b_n^*))| + |\nu(a\setminus b^*)|\\ &\le |\nu(a\cap b_n^*)| + \nu_1(b^*\setminus b_n^*) \le |\nu(a\cap b_n^*)| + \epsilon\end{aligned}$$

for any $a\in\mathfrak{A}$.

Now if b, $c\in\mathfrak{A}$ and $\bar\mu((b\triangle c)\cap b_n^*)\le\delta$ then

$$\begin{aligned}|\nu b-\nu c| &\le |\nu(b\setminus c)| + |\nu(c\setminus b)|\\ &\le |\nu((b\setminus c)\cap b^*)| + |\nu((c\setminus b)\cap b^*)| + 2\epsilon \le \epsilon+\epsilon+2\epsilon = 4\epsilon\end{aligned}$$

because $\bar\mu((b\setminus c)\cap b_n^*)$, $\bar\mu((c\setminus b)\cap b_n^*)$ are both less than or equal to δ. As ϵ is arbitrary, ν is uniformly continuous.

(iv)⇒(iii)⇒(i) are trivial.

(d) One implication is covered by (a). For the other, suppose that ν is completely additive. Then it is countably additive. On the other hand, if $\nu a\ne 0$, consider $B=\{b : b\subseteq a, \bar\mu b<\infty\}$. Then B is upwards-directed and $\sup B=a$, because $\bar\mu$ is semi-finite (322Eb), so $\{a\setminus b : b\in B\}$ is downwards-directed and has infimum 0. Accordingly $\inf_{b\in B}|\nu(a\setminus b)|=0$, and there must be a $b\in B$ such that $\nu b\ne 0$. But this means that condition (ii) of (c) is satisfied, so that ν is continuous.

(e) Now suppose that $(\mathfrak{A},\bar\mu)$ is σ-finite. In this case $\mathfrak{A}$ is ccc (322G) so complete additivity and countable additivity are the same (326P) and we have a special case of (d).

(f) Finally, suppose that $\bar\mu 1<\infty$ and that ν is absolutely continuous with respect to $\bar\mu$. If $A\subseteq\mathfrak{A}$ is non-empty and downwards-directed and has infimum 0, then $\inf_{a\in A}\bar\mu a=0$ (321F), so $\inf_{a\in A}|\nu a|$ must be 0; thus ν is completely additive. With (b) and (e) this shows that all four conditions are equiveridical.

327C Proposition Let (X,Σ,μ) be a measure space and $(\mathfrak{A},\bar\mu)$ its measure algebra.

(a) There is a one-to-one correspondence between finitely additive functionals $\bar\nu$ on $\mathfrak{A}$ and finitely additive functionals ν on Σ such that $\nu E=0$ whenever $\mu E=0$, given by the formula $\bar\nu E^\bullet=\nu E$ for every $E\in\Sigma$.

(b) In (a), $\bar\nu$ is absolutely continuous with respect to $\bar\mu$ iff ν is absolutely continuous with respect to μ.

(c) In (a), $\bar\nu$ is countably additive iff ν is countably additive; so that we have a one-to-one correspondence between the countably additive functionals on $\mathfrak{A}$ and the absolutely continuous countably additive functionals on Σ.

(d) In (a), $\bar\nu$ is continuous for the measure-algebra topology on $\mathfrak{A}$ iff ν is truly continuous in the sense of 232Ab.

(e) Suppose that μ is semi-finite. Then, in (a), $\bar\nu$ is completely additive iff ν is truly continuous.

proof (a) This should be nearly obvious. If $\bar\nu:\mathfrak{A}\to\mathbb{R}$ is additive, then the formula defines a functional $\nu:\Sigma\to\mathbb{R}$ which is additive by 326Be. Also, of course,

$$\mu E=0 \implies E^\bullet=0 \implies \nu E=0.$$

On the other hand, if ν is an additive functional on Σ which is zero on negligible sets, then, for E, $F\in\Sigma$,

$$\begin{aligned}E^\bullet=F^\bullet &\implies \mu(E\setminus F)=\mu(F\setminus E)=0\\ &\implies \nu(E\setminus F)=\nu(F\setminus E)=0\\ &\implies \nu F=\nu E-\nu(E\setminus F)+\nu(F\setminus E)=\nu E,\end{aligned}$$

so we have a function $\bar\nu:\mathfrak{A}\to\mathbb{R}$ defined by the given formula. If E, $F\in\Sigma$ and $E^\bullet\cap F^\bullet=0$, then

$$\begin{aligned}\bar\nu(E^\bullet\cup F^\bullet) &= \bar\nu(E\cup F)^\bullet = \nu(E\cup F)\\ &= \nu(E\setminus F)+\nu F = \bar\nu E^\bullet + \bar\nu F^\bullet\end{aligned}$$

because $(E\setminus F)^\bullet=E^\bullet\setminus F^\bullet=E^\bullet$. Thus $\bar\nu$ is additive, and the correspondence is complete.

(b) This is immediate from the definitions.

(c)(i) If ν is countably additive, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, we can express it as $\langle E_n\rangle_{n\in\mathbb{N}}$ where $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ. Setting $F_n=E_n\setminus\bigcup_{i<n}E_i$, $\langle F_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ and

$$F_n^\bullet = a_n\setminus\sup_{i<n}a_i = a_n$$

for each n. So

$$\bar{\nu}(\sup_{n\in\mathbb{N}} a_n) = \nu(\bigcup_{n\in\mathbb{N}} F_n) = \sum_{n=0}^{\infty} \nu F_n = \sum_{n=0}^{\infty} \bar{\nu} a_n.$$

As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\bar{\nu}$ is countably additive.

(ii) If $\bar{\nu}$ is countably additive, then ν is countably additive by 326Jf.

(iii) For the last remark, note that by 232Ba a countably additive functional on Σ is absolutely continuous with respect to μ iff it is zero on the μ-negligible sets.

(d) The definition of 'truly continuous' functional translates directly to continuity at 0 in the measure algebra. But by 327Bc this is the same thing as continuity.

(e) Put (d) and 327Bd together.

327D The Radon-Nikodým theorem We are now ready for another look at this theorem.

Theorem Let (X, Σ, μ) be a semi-finite measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Let L^1 be the space of equivalence classes of real-valued integrable functions on X (§242), and write M_τ for the set of completely additive real-valued functionals on $\mathfrak{A}$. Then there is an ordered linear space bijection between M_τ and L^1 defined by saying that $\bar{\nu} \in M_\tau$ corresponds to $u \in L^1$ if

$$\bar{\nu} a = \int_E f \text{ whenever } a = E^\bullet \text{ in } \mathfrak{A} \text{ and } f^\bullet = u \text{ in } L^1.$$

proof (a) Given $\bar{\nu} \in M_\tau$, we have a truly continuous $\nu : \Sigma \to \mathbb{R}$ given by setting $\nu E = \bar{\nu} E^\bullet$ for every $E \in \Sigma$ (327Ce). Now there is an integrable function f such that $\nu E = \int_E f$ for every $E \in \Sigma$ (232E). There is likely to be more than one such function, but any two must be equal almost everywhere (232Hd), so the corresponding equivalence class $u_{\bar{\nu}} = f^\bullet$ is uniquely defined.

(b) Conversely, given $u \in L^1$, we have a well-defined functional ν_u on Σ given by setting

$$\nu_u E = \int_E u = \int_E f \text{ whenever } f^\bullet = u$$

for every $E \in \Sigma$ (242Ac). By 232E, ν_u is additive and truly continuous, and of course it is zero when μ is zero, so corresponds to a completely additive functional $\bar{\nu}_u$ on $\mathfrak{A}$ (327Ce).

(c) Clearly the maps $u \mapsto \bar{\nu}_u$ and $\bar{\nu} \mapsto u_{\bar{\nu}}$ are now the two halves of a one-to-one correspondence. To see that it is linear, we need note only that

$$(\bar{\nu}_u + \bar{\nu}_v)E^\bullet = \bar{\nu}_u E^\bullet + \bar{\nu}_v E^\bullet = \int_E u + \int_E v = \int_E u + v = \bar{\nu}_{u+v} E^\bullet$$

for every $E \in \Sigma$, so $\bar{\nu}_u + \bar{\nu}_v = \bar{\nu}_{u+v}$ for all u, $v \in L^1$; and similarly $\bar{\nu}_{\alpha u} = \alpha \bar{\nu}_u$ for $u \in L^1$ and $\alpha \in \mathbb{R}$. As for the ordering, given u and $v \in L^1$, take integrable f, g such that $u = f^\bullet$ and $v = g^\bullet$; then

$$\begin{aligned}
\bar{\nu}_u \le \bar{\nu}_v &\iff \bar{\nu}_u E^\bullet \le \bar{\nu}_v E^\bullet \text{ for every } E \in \Sigma \\
&\iff \int_E u \le \int_E v \text{ for every } E \in \Sigma \\
&\iff \int_E f \le \int_E g \text{ for every } E \in \Sigma \\
&\iff f \le_{\text{a.e.}} g \iff u \le v,
\end{aligned}$$

using 131Ha.

327E I slip in an elementary fact.

Proposition If $(\mathfrak{A}, \bar{\mu})$ is a measure algebra, then the functional $a \mapsto \mu_c a = \bar{\mu}(a \cap c)$ is completely additive whenever $c \in \mathfrak{A}$ and $\bar{\mu} c < \infty$.

proof μ_c is additive because $\bar{\mu}$ is additive, and by 321F again $\inf_{a\in A} \mu_c a = 0$ whenever A is non-empty, downwards-directed and has infimum 0.

327F Standard extensions The machinery of 327D provides the basis of a canonical method for extending countably additive functionals from closed subalgebras, which we shall need in §333.

Lemma Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra and $\mathfrak{C}\subseteq\mathfrak{A}$ a closed subalgebra. Write $M_\sigma(\mathfrak{A})$, $M_\sigma(\mathfrak{C})$ for the spaces of countably additive real-valued functionals on $\mathfrak{A}$, $\mathfrak{C}$ respectively.

(a) There is an operator $R:M_\sigma(\mathfrak{C})\to M_\sigma(\mathfrak{A})$ defined by saying that, for every $\nu\in M_\sigma(\mathfrak{C})$, $R\nu$ is the unique member of $M_\sigma(\mathfrak{A})$ such that $[\![R\nu>\alpha\bar{\mu}]\!]=[\![\nu>\alpha\bar{\mu}\restriction\mathfrak{C}]\!]$ for every $\alpha\in\mathbb{R}$.

(b)(i) $R\nu$ extends ν for every $\nu\in M_\sigma(\mathfrak{C})$.

(ii) R is linear and order-preserving.

(iii) $R(\bar{\mu}\restriction\mathfrak{C})=\bar{\mu}$.

(iv) If $\langle\nu_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-negative functionals in $M_\sigma(\mathfrak{C})$ such that $\sum_{n=0}^\infty\nu_nc=\bar{\mu}c$ for every $c\in\mathfrak{C}$, then $\sum_{n=0}^\infty(R\nu_n)(a)=\bar{\mu}a$ for every $a\in\mathfrak{A}$.

Remarks When saying that $\mathfrak{C}$ is 'closed', I mean, indifferently, 'topologically closed' or 'order-closed'; see 323H-323I.

For the notation '$[\![\nu>\alpha\bar{\mu}]\!]$' see 326S-326T.

proof (a)(i) By 321J-321K, we may represent $(\mathfrak{A},\bar{\mu})$ as the measure algebra of a measure space (X,Σ,μ); write π for the canonical map from Σ to $\mathfrak{A}$. Write T for $\{E:E\in\Sigma,\,\pi E\in\mathfrak{C}\}$. Because $\mathfrak{C}$ is a σ-subalgebra of $\mathfrak{C}$ and π is a sequentially order-continuous Boolean homomorphism, T is a σ-subalgebra of Σ.

(ii) For each $\nu\in M_\sigma(\mathfrak{C})$, $\nu\pi:\mathrm{T}\to\mathbb{R}$ is countably additive and zero on $\{F:F\in\mathrm{T},\,\mu F=0\}$, so we can choose a T-measurable function $f_\nu:X\to\mathbb{R}$ such that $\int_F f_\nu d(\mu\restriction\mathrm{T})=\nu\pi F$ for every $F\in\mathrm{T}$. Of course we can now think of f_ν as a μ-integrable function (233B), so we get a corresponding countably additive functional $R\nu:\mathfrak{A}\to\mathbb{R}$ defined by setting $(R\nu)(\pi E)=\int_E f_\nu$ for every $E\in\Sigma$ (327D). (In this context, of course, countably additive functionals are completely additive, by 327Bf.) Note that if $c\in\mathfrak{C}$ there is an $F\in\mathrm{T}$ such that $F^\bullet=c$, so that

$$(R\nu)(c)=\int_F f_\nu=\nu c.$$

For $\alpha\in\mathbb{R}$, set $H_\alpha=\{x:f_\nu(x)>\alpha\}\in\mathrm{T}$. Then for any $E\in\Sigma$,

$$E\subseteq H_\alpha,\,\mu E>0\Longrightarrow\int_E f_\nu>\alpha\mu E,$$

$$E\cap H_\alpha=\emptyset\Longrightarrow\int_E f_\nu\le\alpha\mu E.$$

Translating into terms of elements of $\mathfrak{A}$, and setting $c_\alpha=\pi H_\alpha\in\mathfrak{C}$, we have

$$0\ne a\subseteq c_\alpha\Longrightarrow(R\nu)(a)>\alpha\bar{\mu}a,$$

$$a\cap c_\alpha=0\Longrightarrow(R\nu)(a)\le\alpha\bar{\mu}a.$$

So $[\![R\nu>\alpha\bar{\mu}]\!]=c_\alpha\in\mathfrak{C}$. Of course we now have

$$\nu c=(R\nu)(c)>\alpha\bar{\mu}c\text{ when }c\in\mathfrak{C},\,0\ne c\subseteq c_\alpha,$$

$$\nu c\le\alpha\bar{\mu}c\text{ when }c\in\mathfrak{C},\,c\cap c_\alpha=0,$$

so that c_α is also equal to $[\![\nu>\alpha\bar{\mu}\restriction\mathfrak{C}]\!]$.

Thus the functional $R\nu$ satisfies the declared formula.

(iii) To see that $R\nu$ is uniquely defined, observe that if $\lambda\in M_\sigma(\mathfrak{A})$ and $[\![\lambda>\alpha\bar{\mu}]\!]=[\![R\nu>\alpha\bar{\mu}]\!]$ for every α, then there is a Σ-measurable function $g:X\to\mathbb{R}$ such that $\int_E g\,d\mu=\lambda\pi E$ for every $E\in\Sigma$; but in this case (just as in (ii)) $[\![\lambda>\alpha\bar{\mu}]\!]=\pi G_\alpha$, where $G_\alpha=\{x:g(x)>\alpha\}$, for each α. So we must have $\pi G_\alpha=\pi H_\alpha$, that is, $\mu(G_\alpha\triangle H_\alpha)=0$, for every α. Accordingly

$$\{x:f_\nu(x)\ne g(x)\}=\bigcup_{q\in\mathbb{Q}}G_q\triangle H_q$$

is negligible; $f_\nu=_{\text{a.e.}}g$, $\int_E f_\nu d\mu=\int_E g\,d\mu$ for every $E\in\Sigma$ and $\lambda=R\nu$.

(b)(i) I have already noted that $(R\nu)c=\nu c$ for every $\nu\in M_\sigma(\mathfrak{C})$ and $c\in\mathfrak{C}$.

(ii) If $\nu=\nu_1+\nu_2$, we must have, in the language of (a) above,

$$\int_F f_\nu=\nu\pi F=\nu_1\pi F+\nu_2\pi F=\int_F f_{\nu_1}+\int_F f_{\nu_2}=\int_F f_{\nu_1}+f_{\nu_2}$$

for every $F\in\mathrm{T}$, so $f_\nu=_{\text{a.e.}}f_{\nu_1}+f_{\nu_2}$, and we can repeat the formulae

$$(R\nu)(\pi E)=\int_E f_\nu=\int_E f_{\nu_1}+f_{\nu_2}=\int_E f_{\nu_1}+\int_E f_{\nu_2}=(R\nu_1)(\pi E)+(R\nu_2)(\pi E),$$

in a different order, for every $E\in\Sigma$, to see that $R\nu=R\nu_1+R\nu_2$. Similarly, if $\nu\in M_\sigma(\mathfrak{C})$ and $\gamma\in\mathbb{R}$, $f_{\gamma\nu}=_{\text{a.e.}}\gamma f_\nu$ and $R(\gamma\nu)=\gamma R\nu$. If $\nu_1\le\nu_2$ in $M_\sigma(\mathfrak{C})$, then

$$\int_F f_{\nu_1} = \nu_1\pi F \le \nu_2\pi F = \int_F f_{\nu_2}$$

for every $F \in \mathrm{T}$, so $f_{\nu_1} \le_{\text{a.e.}} f_{\nu_2}$ (131Ha again), and $R\nu_1 \le R\nu_2$.

Thus R is linear and order-preserving.

(iii) If $\nu = \bar{\mu}\restriction\mathfrak{C}$ then

$$\int_F f_\nu = \nu\pi F = \mu F = \int_F \chi X$$

for every $F \in \mathrm{T}$, so $f_\nu =_{\text{a.e.}} \chi X$ and $R\nu = \bar{\mu}$.

(iv) Now suppose that $\langle \nu_n\rangle_{n\in\mathbb{N}}$ is a sequence in $M_\sigma(\mathfrak{C})$ such that, for every $c \in \mathfrak{C}$, $\nu_n c \ge 0$ for every n and $\sum_{n=0}^\infty \nu_n c = \bar{\mu}c$. Set $g_n = \sum_{i=0}^n f_{\nu_i}$ for each n; then $0 \le_{\text{a.e.}} g_n \le_{\text{a.e.}} g_{n+1} \le_{\text{a.e.}} \chi X$ for every n, and

$$\lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \sum_{i=0}^n \nu_i 1 = \bar{\mu}1.$$

But this means that, setting $g = \lim_{n\to\infty} g_n$, $g \le_{\text{a.e.}} \chi X$ and $\int g = \int \chi X$, so that $g =_{\text{a.e.}} \chi X$ and

$$\sum_{n=0}^\infty (R\nu_i)(\pi E) = \lim_{n\to\infty} \int_E g_n = \mu E$$

for every $E \in \Sigma$. Thus $\sum_{n=0}^\infty (R\nu_i)(a) = \bar{\mu}a$ for every $a \in \mathfrak{A}$.

327G Definition In the context of 327F, I will call $R\nu$ the **standard extension** of ν to $\mathfrak{A}$.

Remark The point of my insistence on the uniqueness of R, and on the formula in 327Fa, is that $R\nu$ really is defined by the abstract structure $(\mathfrak{A}, \bar{\mu}, \mathfrak{C}, \nu)$, even though I have used a proof which runs through the representation of $(\mathfrak{A}, \bar{\mu})$ as the measure algebra of a measure space (X, Σ, μ).

327X Basic exercises (a) Let (X, Σ, μ) be a probability space, and T a σ-subalgebra of Σ. Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of (X, Σ, μ). Show that $\mathfrak{C} = \{F^\bullet : F \in \mathrm{T}\}$ is a closed subalgebra of $\mathfrak{A}$. Identify the spaces $M_\sigma(\mathfrak{A})$, $M_\sigma(\mathfrak{C})$ of countably additive functionals with $L^1(\mu)$, $L^1(\mu\restriction\mathrm{T})$, as in 327D. Show that the conditional expectation operator $P : L^1(\mu) \to L^1(\mu\restriction\mathrm{T})$ (242Jd) corresponds to the map $\nu \mapsto \nu\restriction\mathfrak{C} : M_\sigma(\mathfrak{A}) \to M_\sigma(\mathfrak{C})$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional. Show that, for any $a \in \mathfrak{A}$,

$$\nu a = \int_0^\infty \bar{\mu}(a \cap [\![\nu > \alpha\bar{\mu}]\!])d\alpha - \int_{-\infty}^0 \bar{\mu}(a \setminus [\![\nu > \alpha\bar{\mu}]\!])d\alpha,$$

the integrals being taken with respect to Lebesgue measure. (*Hint*: take $(\mathfrak{A}, \bar{\mu})$ to be the measure algebra of (X, Σ, μ); represent ν by a μ-integrable function f; apply Fubini's theorem to the sets $\{(x,t) : x \in E,\ 0 \le t < f(x)\}$, $\{(x,t) : x \in E,\ f(x) \le t \le 0\}$ in $X \times \mathbb{R}$, where $a = E^\bullet$.)

(c) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\mu}')$ be totally finite measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving Boolean homomorphism. Let $\mathfrak{C}$ be a closed subalgebra of $\mathfrak{A}$, and ν a countably additive functional on the closed subalgebra $\pi[\mathfrak{C}]$ of $\mathfrak{B}$ (324L). (i) Show that $\nu\pi$ is a countably additive functional on $\mathfrak{C}$. (ii) Show that if $\tilde{\nu}$ is the standard extension of ν to $\mathfrak{B}$, then $\tilde{\nu}\pi$ is the standard extension of $\nu\pi$ to $\mathfrak{A}$. (*Hint*: take $\alpha \in \mathbb{R}$ and set $e_0 = [\![\tilde{\nu} > \alpha\bar{\mu}']\!] = [\![\nu > \alpha\bar{\mu}'\restriction\pi[\mathfrak{C}]]\!]$; there is a $c_0 \in \mathfrak{C}$ such that $\pi c_0 = e_0$; check that $c_0 = [\![\tilde{\nu}\pi > \alpha\bar{\mu}]\!] = [\![\nu\pi > \alpha\bar{\mu}\restriction\mathfrak{C}]\!]$.)

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$ and $\nu : \mathfrak{C} \to \mathbb{R}$ a countably additive functional with standard extension $\tilde{\nu} : \mathfrak{A} \to \mathbb{R}$. Show that, for any $a \in \mathfrak{A}$,

$$\tilde{\nu}a = \int_0^\infty \bar{\mu}(a \cap [\![\nu > \alpha\bar{\mu}\restriction\mathfrak{C}]\!])d\alpha - \int_{-\infty}^0 \bar{\mu}(a \setminus [\![\nu > \alpha\bar{\mu}\restriction\mathfrak{C}]\!])d\alpha.$$

(e) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\mathfrak{B}$, $\mathfrak{C}$ stochastically independent closed subalgebras of $\mathfrak{A}$ (definition: 325L). Let ν be a countably additive functional on $\mathfrak{C}$, and $\tilde{\nu}$ its standard extension to $\mathfrak{A}$. Show that $\tilde{\nu}(b \cap c) = \bar{\mu}b \cdot \nu c$ for every $b \in \mathfrak{B}$, $c \in \mathfrak{C}$.

(f) Let (X, Σ, μ) be a probability space, and T a σ-subalgebra of Σ. Let ν be a probability measure with domain T such that $\nu E = 0$ whenever $E \in \mathrm{T}$ and $\mu E = 0$. Show that there is a probability measure λ with domain Σ which extends ν.

327Y Further exercises (a) Let $(\mathfrak{A}_1,\bar{\mu}_1)$ and $(\mathfrak{A}_2,\bar{\mu}_2)$ be localizable measure algebras with localizable measure algebra free product $(\mathfrak{C},\bar{\lambda})$. Show that if ν_1, ν_2 are completely additive functionals on $\mathfrak{A}_1$, $\mathfrak{A}_2$ respectively, there is a unique completely additive functional $\nu:\mathfrak{C}\to\mathbb{R}$ such that $\nu(a_1\otimes a_2)=\nu_1a_1\cdot\nu_2a_2$ for every $a_1\in\mathfrak{A}_1$, $a_2\in\mathfrak{A}_2$. (*Hint*: 253D.)

(b) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra; let $R:M_\sigma(\mathfrak{C})\to M_\sigma(\mathfrak{A})$ be the standard extension operator (327G). Show (i) that R is order-continuous (ii) that $R(\nu^+)=(R\nu)^+$, $\|R\nu\|=\|\nu\|$ for every $\nu\in M_\sigma(\mathfrak{C})$, defining ν^+ and $\|\nu\|$ as in 326Yd.

(c) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. For a countably additive functional ν on $\mathfrak{C}$ write $\tilde{\nu}$ for its standard extension to $\mathfrak{A}$. Show that if ν, $\langle\nu_n\rangle_{n\in\mathbb{N}}$ are countably additive functionals on $\mathfrak{C}$ and $\lim_{n\to\infty}\nu_nc=\nu c$ for every $c\in\mathfrak{C}$, then $\lim_{n\to\infty}\tilde{\nu}_na=\tilde{\nu}a$ for every $a\in\mathfrak{A}$. (*Hint*: use ideas from §§246-247, as well as from 327F and 326Yo.)

327 Notes and comments When we come to measure algebras, it is the completely additive functionals which fit most naturally into the topological theory (327Bd); they correspond to the 'truly continuous' functionals which I discussed in §232 (327Cd), and therefore to the Radon-Nikodým theorem (327D). I will return to some of these questions in Chapter 36. I myself regard the form here as the best expression of the essence of the Radon-Nikodým theorem, if not the one most commonly applied.

The concept of 'standard extension' of a countably additive functional (or, as we could equally well say, of a completely additive functional, since in the context of 327F the two coincide) is in a sense dual to the concept of 'conditional expectation'. If (X,Σ,μ) is a probability space and T is a σ-subalgebra of Σ, then we have a corresponding closed subalgebra $\mathfrak{C}$ of the measure algebra $(\mathfrak{A},\bar{\mu})$ of μ, and identifications between the spaces $M_\sigma(\mathfrak{A})$, $M_\sigma(\mathfrak{C})$ of countably additive functionals and the spaces $L^1(\mu)$, $L^1(\mu\restriction\mathrm{T})$. Now we have a natural embedding S of $L^1(\mu\restriction\mathrm{T})$ as a subspace of $L^1(\mu)$ (242Jb), and a natural restriction map from $M_\sigma(\mathfrak{A})$ to $M_\sigma(\mathfrak{C})$. These give rise to corresponding operators between the opposite members of each pair; the standard extension operator R of 327F-327G, and the conditional expectation operator P of 242Jd. (See 327Xa.) The fundamental fact

$$PSv=v\text{ for every }v\in L^1(\mu\restriction\mathrm{T})$$

(242Jg) is matched by the fact that

$$R\nu\restriction\mathfrak{C}=\nu\text{ for every }\nu\in M_\sigma(\mathfrak{C}).$$

The further identification of $R\nu$ in terms of integrals $\int\bar{\mu}(a\cap[\![\nu>\alpha\bar{\mu}]\!])d\alpha$ (327Xd) is relatively inessential, but is striking, and perhaps makes it easier to believe that R is truly 'standard' in the abstract contexts which will arise in §333 below. It is also useful in such calculations as 327Xe.

The isomorphisms between M_τ spaces and L^1 spaces described here mean that any of the concepts involving L^1 spaces discussed in Chapter 24 can be applied to M_τ spaces, at least in the case of measure algebras. In fact, as I will show in Chapter 36, there is much more to be said here; the space of bounded additive functionals on a Boolean algebra is already an L^1 space in an abstract sense, and ideas such as 'uniform integrability' are relevant and significant there, as well as in the spaces of countably additive and completely additive functionals. I hope that 326Yd, 326Ym-326Yn, 326Yp-326Yq and 327Yb will provide some hints to be going on with for the moment.

*328 Reduced products and other constructions

I devote a section to some related constructions. At the end of §315 I mentioned projective and inductive limits of systems of Boolean algebras with linking homomorphisms. In the context of the present chapter, we naturally ask whether similar constructions can be found for probability algebras. For projective limits there is no difficulty (328I). For inductive limits the situation is more complex (328H). Some ideas in Volume 5 will depend on what I call 'reduced products' (328A-328F), which also provide a route to 328H. The same methods give a route to a useful result relating measure-preserving Boolean homomorphisms on a probability algebra to measure-preserving automorphisms on a larger probability algebra (328J).

328A Construction Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, and $\mathcal{F}$ an ultrafilter on I.

(a) Set

$$\mathcal{J}=\{\langle a_i\rangle_{i\in I}:\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i,\ \lim_{i\to\mathcal{F}}\bar{\mu}_ia_i=0\}.$$

Then $\mathcal{J}$ is an ideal in the simple product Boolean algebra $\prod_{i\in I}\mathfrak{A}_i$. **P** If $\langle a_i\rangle_{i\in I}$ and $\langle b_i\rangle_{i\in I}$ belong to $\mathcal{J}$, and $\langle c_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ is such that $\langle c_i\rangle_{i\in I}\subseteq\langle a_i\rangle_{i\in I}\cup\langle b_i\rangle_{i\in I}$, then $c_i\subseteq a_i\cup b_i$ for every i, so

$$\lim_{i\to\mathcal{F}}\bar\mu_i c_i\le\lim_{i\to\mathcal{F}}\bar\mu_i a_i+\bar\mu_i b_i=\lim_{i\to\mathcal{F}}\bar\mu_i a_i+\lim_{i\to\mathcal{F}}\bar\mu_i b_i=0$$

and $\langle c_i\rangle_{i\in I}\in\mathcal{J}$. Of course $\langle 0_{\mathfrak{A}_i}\rangle_{i\in I}$ belongs to $\mathcal{J}$, so $\mathcal{J}\triangleleft\prod_{i\in I}\mathfrak{A}_i$. **Q**

(b) Let $\mathfrak{A}$ be the quotient Boolean algebra $\prod_{i\in I}\mathfrak{A}_i/\mathcal{J}$. Then we have a functional $\bar\mu:\mathfrak{A}\to[0,1]$ defined by saying that

$$\bar\mu(\langle a_i\rangle^\bullet_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu_i a_i$$

whenever $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$. **P** If $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ and $\langle a_i\rangle^\bullet_{i\in I}=\langle b_i\rangle^\bullet_{i\in I}$, then $\langle a_i\triangle b_i\rangle_{i\in I}\in\mathcal{J}$, so

$$|\lim_{i\to\mathcal{F}}\bar\mu_i a_i-\lim_{i\to\mathcal{F}}\bar\mu_i b_i|=\lim_{i\to\mathcal{F}}|\bar\mu_i a_i-\bar\mu_i b_i|\le\lim_{i\to\mathcal{F}}\bar\mu_i(a_i\triangle b_i)=0. \text{ **Q**}$$

328B Proposition Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras and $\mathcal{F}$ an ultrafilter on I, and construct $\mathfrak{A}$ and $\bar\mu$ as in 328A. Then $(\mathfrak{A},\bar\mu)$ is a probability algebra.

proof (a) If $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ and $\langle a_i\rangle^\bullet_{i\in I}\cap\langle b_i\rangle^\bullet_{i\in I}=0$, then $\langle a_i\cap b_i\rangle_{i\in I}\in\mathcal{J}$, so

$$\begin{aligned}\bar\mu(\langle a_i\rangle^\bullet_{i\in I}\cup\langle b_i\rangle^\bullet_{i\in I})&=\bar\mu(\langle a_i\cup b_i\rangle^\bullet_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu_i(a_i\cup b_i)\\&=\lim_{i\to\mathcal{F}}\bar\mu_i a_i+\bar\mu_i b_i-\bar\mu_i(a_i\cap b_i)\\&=\lim_{i\to\mathcal{F}}\bar\mu_i a_i+\lim_{i\to\mathcal{F}}\bar\mu_i b_i-\lim_{i\to\mathcal{F}}\bar\mu_i(a_i\cap b_i)\\&=\lim_{i\to\mathcal{F}}\bar\mu_i a_i+\lim_{i\to\mathcal{F}}\bar\mu_i b_i=\bar\mu(\langle a_i\rangle^\bullet_{i\in I})+\bar\mu(\langle b_i\rangle^\bullet_{i\in I}).\end{aligned}$$

So $\bar\mu$ is additive.

(b) $1_{\mathfrak{A}}=\langle 1_{\mathfrak{A}_i}\rangle^\bullet_{i\in I}$ so

$$\bar\mu 1_{\mathfrak{A}}=\lim_{i\to\mathcal{F}}\bar\mu_i 1_{\mathfrak{A}_i}=1.$$

(c) If $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ and $\bar\mu(\langle a_i\rangle^\bullet_{i\in I})=0$, then $\langle a_i\rangle_{i\in I}\in\mathcal{J}$ and $\langle a_i\rangle^\bullet_{i\in I}=0$; thus $\bar\mu$ is strictly positive.

(d) Suppose that $\langle\tilde a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$. Express each $\tilde a_n$ as $\langle a_{ni}\rangle^\bullet_{i\in I}$ where $a_{ni}\in\mathfrak{A}_i$ for each i. Set $b_{ni}=\sup_{m\le n}a_{mi}$ for $n\in\mathbb{N}$ and $i\in I$; then $\langle b_{ni}\rangle^\bullet_{i\in I}=\sup_{m\le n}\tilde a_m$ in $\mathfrak{A}$. Set

$$\gamma=\sum_{n=0}^\infty\bar\mu\tilde a_n=\sup_{n\in\mathbb{N}}\bar\mu(\langle b_{ni}\rangle^\bullet_{i\in I})=\sup_{n\in\mathbb{N}}\lim_{i\to\mathcal{F}}\bar\mu_i b_{ni}.$$

Set $A_n=\{i:i\in I,\ \bar\mu_i b_{ni}\le\gamma+2^{-n}\}$ for each $n\in\mathbb{N}$; then $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{F}$, and $A_0=I$. For $i\in I$ set

$$\begin{aligned}b_i&=b_{ni}\text{ if }i\in A_n\setminus A_{n+1},\\&=\sup_{n\in\mathbb{N}}b_{ni}\text{ if }i\in\bigcap_{n\in\mathbb{N}}A_n.\end{aligned}$$

Consider $\tilde b=\langle b_i\rangle^\bullet_{i\in I}\in\mathfrak{A}$. For each $n\in\mathbb{N}$, $\{i:a_{ni}\subseteq b_i,\ \bar\mu b_i\le\gamma+2^{-n}\}$ includes $A_n\in\mathcal{F}$, so $\tilde a_n\subseteq\tilde b$ for every n and $\bar\mu\tilde b\le\gamma$.

If $\tilde c\in\mathfrak{A}$ is another upper bound for $\{\tilde a_n:n\in\mathbb{N}\}$, then, using (a),

$$\gamma=\sup_{n\in\mathbb{N}}\bar\mu(\sup_{m\le n}\tilde a_m)\le\bar\mu(\tilde b\cap\tilde c)\le\bar\mu\tilde b\le\gamma;$$

so $\bar\mu(\tilde b\setminus\tilde c)=0$ and $\tilde b\setminus\tilde c=0$, by (c). Thus $\tilde b=\sup_{n\in\mathbb{N}}\tilde a_n$ in $\mathfrak{A}$, while $\bar\mu\tilde b=\sum_{n=0}^\infty\bar\mu\tilde a_n$.

(e) If $\langle\tilde a_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{A}$, then (iv) tells us that $\{\tilde a_n\setminus\sup_{m<n}\tilde a_m:n\in\mathbb{N}\}$ has a supremum in $\mathfrak{A}$, which is also the supremum of $\{\tilde a_n:n\in\mathbb{N}\}$. So $\mathfrak{A}$ is Dedekind σ-complete. Now (d) tells us also that $\bar\mu$ is countably additive, so that $(\mathfrak{A},\bar\mu)$ is a probability algebra.

328C Definition In the context of 328A/328B, I will call $(\mathfrak{A},\bar\mu)$ the **probability algebra reduced product** of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ modulo $\mathcal{F}$; I will sometimes write it as $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$. (There are dangers in this notation. In 351M I will speak of 'reduced powers' $\mathbb{R}^I|\mathcal{F}$, and the rules will be significantly different there.)

If all the $(\mathfrak{A}_i,\bar\mu_i)$ are the same, with common value $(\mathfrak{B},\bar\nu)$, I will write $(\mathfrak{B},\bar\nu)^I|\mathcal{F}$ for $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$, and call it the **probability algebra reduced power**.

328D Proposition Let I be a set, $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$, $\langle(\mathfrak{B}_i,\bar\nu_i)\rangle_{i\in I}$ and $\langle(\mathfrak{C}_i,\bar\lambda_i)\rangle_{i\in I}$ three families of probability algebras, and $\mathcal{F}$ an ultrafilter on I; let $(\mathfrak{A},\bar\mu)=\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$, $(\mathfrak{B},\bar\nu)=\prod_{i\in I}(\mathfrak{B}_i,\bar\nu_i)|\mathcal{F}$ and $(\mathfrak{C},\bar\lambda)=\prod_{i\in I}(\mathfrak{C}_i,\bar\lambda_i)|\mathcal{F}$ be the corresponding reduced products.

(a) If $\pi_i:\mathfrak{A}_i\to\mathfrak{B}_i$ is a measure-preserving Boolean homomorphism for each $i\in I$, we have a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ given by saying that

$$\pi(\langle a_i\rangle^\bullet_{i\in I})=\langle\pi_i a_i\rangle^\bullet_{i\in I}$$

whenever $a_i\in\mathfrak{A}_i$ for every $i\in I$.

(b) If, in addition, $\phi_i:\mathfrak{B}_i\to\mathfrak{C}_i$ is a measure-preserving Boolean homomorphism for each $i\in I$, and $\phi:\mathfrak{B}\to\mathfrak{C}$ is constructed as in (a), then $\phi\pi:\mathfrak{A}\to\mathfrak{C}$ corresponds to the family $\langle\phi_i\pi_i\rangle_{i\in I}$.

proof (a) Following through the construction in 328A, we have ideals

$$\mathcal{J}=\{\langle a_i\rangle_{i\in I}:\lim_{i\to\mathcal{F}}\bar\mu_i a_i=0\}\lhd\prod_{i\in I}\mathfrak{A}_i,$$

$$\mathcal{K}=\{\langle b_i\rangle_{i\in I}:\lim_{i\to\mathcal{F}}\bar\nu_i b_i=0\}\lhd\prod_{i\in I}\mathfrak{B}_i,$$

and a Boolean homomorphism $\hat\pi:\prod_{i\in I}\mathfrak{A}_i\to\prod_{i\in I}\mathfrak{B}_i$ given by the formula $\hat\pi\langle a_i\rangle_{i\in I}=\langle\pi_i a_i\rangle_{i\in I}$ (use 315Bb). Because the homomorphisms π_i are measure-preserving, $\hat\pi\boldsymbol{a}\in\mathcal{K}$ whenever $\boldsymbol{a}\in\mathcal{J}$. Consequently we have a Boolean homomorphism $\pi:\prod_{i\in I}\mathfrak{A}_i/\mathcal{J}\to\prod_{i\in I}\mathfrak{B}_i/\mathcal{K}$ given by setting $\pi\boldsymbol{a}^\bullet=(\hat\pi\boldsymbol{a})^\bullet$ whenever $\boldsymbol{a}\in\prod_{i\in I}\mathfrak{A}_i$ (3A2G). And

$$\bar\nu\pi(\langle a_i\rangle^\bullet_{i\in I})=\bar\nu(\langle\pi_i a_i\rangle^\bullet_{i\in I})=\lim_{i\to\mathcal{F}}\bar\nu_i\pi_i a_i=\lim_{i\to\mathcal{F}}\bar\mu_i a_i=\bar\mu(\langle a_i\rangle^\bullet_{i\in I})$$

whenever $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$, so π is measure-preserving.

(b) is now just a matter of writing the defining formulae out.

328E Proposition Let I be a non-empty set, $\le$ a reflexive transitive relation on I, and $\mathcal{F}$ an ultrafilter on I such that $\{j:j\in I,\,j\ge i\}$ belongs to $\mathcal{F}$ for every $i\in I$. Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, and suppose that we are given a family $\langle\pi_{ji}\rangle_{i\le j}$ such that

π_{ji} is a measure-preserving Boolean homomorphism from $\mathfrak{A}_i$ to $\mathfrak{A}_j$ whenever $i\le j$ in I,

$\pi_{ki}=\pi_{kj}\pi_{ji}$ whenever $i\le j\le k$ in I.

Let $(\mathfrak{A},\bar\mu)$ be the probability algebra reduced product $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$.

(a) For each $i\in I$ we have a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_i\to\mathfrak{A}$ defined by saying that $\pi_i a=\langle a_j\rangle^\bullet_{j\in I}$ whenever $a_j=\pi_{ji}a$ for every $j\ge i$, and $\pi_i=\pi_j\pi_{ji}$ whenever $i\le j$ in I.

(b) $\langle a_i\rangle^\bullet_{i\in I}\subseteq\sup_{j\in A}\pi_j a_j$ whenever $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ and $A\in\mathcal{F}$.

proof (a) π_i is well-defined because $\{j:j\ge i\}\in\mathcal{F}$. It is a measure-preserving Boolean homomorphism because every π_{ji} is. If $i\le j$ in I, $a\in\mathfrak{A}_i$ and $a_k=\pi_{ki}a$ for every $k\ge i$, then $a_k=\pi_{kj}\pi_{ji}a$ for every $k\ge j$, so $\pi_j\pi_{ji}a=\langle a_k\rangle^\bullet_{k\in I}=\pi_i a$; as a is arbitrary, $\pi_j\pi_{ji}=\pi_i$.

(b) Set $c=\sup_{j\in A}\pi_j a_j$ in $\mathfrak{A}$. For any $\epsilon>0$, there is a finite $K\subseteq A$ such that $\bar\mu c\le\epsilon+\bar\mu(\sup_{j\in K}\pi_j a_j)$ (321C). The set $B=\{k:k\in I,\,j\le k\text{ for every }j\in K\}$ belongs to $\mathcal{F}$, so is not empty; fix $k\in B$, and set $b=\sup_{j\in K}\pi_{kj}a_j\in\mathfrak{A}_k$,

$$\begin{aligned}b_i&=\pi_{ik}b\text{ if }i\ge k,\\&=0\text{ for other }i\in I.\end{aligned}$$

Then

$$\langle b_i\rangle^\bullet_{i\in I}=\pi_k b=\pi_k(\sup_{j\in K}\pi_{kj}a_j)=\sup_{j\in K}\pi_k\pi_{kj}a_j=\sup_{j\in K}\pi_j a_j\subseteq c.$$

If $i\in A$ and $i\ge k$, then

$$\begin{aligned}\bar\mu_i(a_i\setminus b_i)&=\bar\mu(\pi_i a_i\setminus\pi_i b_i)=\bar\mu(\pi_i a_i\setminus\pi_i\pi_{ik}b)\\&=\bar\mu(\pi_i a_i\setminus\pi_k b)=\bar\mu(\pi_i a_i\setminus\sup_{j\in K}\pi_j a_j)\le\bar\mu(c\setminus\sup_{j\in K}\pi_j a_j)\le\epsilon\end{aligned}$$

by the choice of K. So

$$\begin{aligned}\bar\mu(\langle a_i\rangle^\bullet_{i\in I}\setminus c)&\le\bar\mu(\langle a_i\rangle^\bullet_{i\in I}\setminus\langle b_i\rangle^\bullet_{i\in I})=\bar\mu(\langle a_i\setminus b_i\rangle^\bullet_{i\in I})\\&=\lim_{i\to\mathcal{F}}\bar\mu_i(a_i\setminus b_i)\le\sup_{i\in A,i\ge k}\bar\mu_i(a_i\setminus b_i)\le\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\langle a_i\rangle^\bullet_{i\in I}\subseteq c$.

328F Corollary Suppose that $\langle(\mathfrak{A}_n,\bar\mu_n)\rangle_{n\in\mathbb{N}}$ is a sequence of probability algebras, $\phi_n:\mathfrak{A}_n\to\mathfrak{A}_{n+1}$ is a measure-preserving Boolean homomorphism for each n and $\mathcal{F}$ is a non-principal ultrafilter on $\mathbb{N}$. Let $(\mathfrak{A},\bar\mu)$ be the probability algebra reduced product $\prod_{n\in\mathbb{N}}(\mathfrak{A}_n,\bar\mu_n)|\mathcal{F}$. Then we have canonical measure-preserving Boolean homomorphisms $\pi_n:\mathfrak{A}_n\to\mathfrak{A}$ such that $\langle a_n\rangle^\bullet_{n\in\mathbb{N}}\subseteq\sup_{n\in A}\pi_n a_n$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}\in\prod_{n\in\mathbb{N}}\mathfrak{A}_n$ and $A\in\mathcal{F}$, and $\pi_{n+1}\phi_n=\pi_n$ for every $n\in\mathbb{N}$.

proof Apply 328E with $\pi_{ji}=\phi_{j-1}\ldots\phi_{i+1}\phi_i$ whenever $i<j$.

328G Corollary Let $(\mathfrak{B},\bar\nu)$ be a probability algebra, I a non-empty set, and $\mathcal{F}$ an ultrafilter on I. Let $(\mathfrak{A},\bar\mu)$ be the probability algebra reduced power $(\mathfrak{B},\bar\nu)^I|\mathcal{F}$.

(a) We have a measure-preserving Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by saying that $\pi b=\langle b\rangle^\bullet_{i\in I}$ for $b\in\mathfrak{B}$.

(b)

$$\langle b_i\rangle^\bullet_{i\in I}\subseteq\sup_{j\in A}\pi b_j=\pi(\sup_{j\in A}b_j)$$

whenever $A\in\mathcal{F}$ and $\langle b_i\rangle_{i\in I}\in\mathfrak{B}^I$.

proof Apply 328E with $\le\,=I\times I$ and π_{ji} the identity operator on $\mathfrak{B}$ for all i, $j\in I$.

328H Proposition Let $(I,\le)$ be an upwards-directed partially ordered set, and $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ a family of probability algebras; suppose that $\pi_{ji}:\mathfrak{A}_i\to\mathfrak{A}_j$ is a measure-preserving Boolean homomorphism whenever $i\le j$, and that $\pi_{ki}=\pi_{kj}\pi_{ji}$ whenever $i\le j\le k$. Then there are a probability algebra $(\mathfrak{C},\bar\lambda)$ and a family $\langle\pi_i\rangle_{i\in I}$ such that

$\pi_i:\mathfrak{A}_i\to\mathfrak{C}$ is a measure-preserving Boolean homomorphism for each $i\in I$,

$\pi_i=\pi_j\pi_{ji}$ whenever $i\le j$,

$\{0,1\}\cup\bigcup_{i\in I}\pi_i[\mathfrak{A}_i]$ is topologically dense in $\mathfrak{C}$,

and whenever $(\mathfrak{B},\bar\nu)$, $\langle\phi_i\rangle_{i\in I}$ are such that

$(\mathfrak{B},\bar\nu)$ is a probability algebra,

$\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ is a measure-preserving Boolean homomorphism for each $i\in I$,

$\phi_i=\phi_j\pi_{ji}$ whenever $i\le j$,

then there is a unique measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\pi_i=\phi_i$ for every $i\in I$.

proof (a) If I is empty the result is trivial (take $\mathfrak{C}=\{0,1\}$); so let us suppose henceforth that $I\ne\emptyset$. In this case,

$$\{A:A\subseteq I,\text{ there is some }i\in I\text{ such that }j\in A\text{ whenever }i\le j\}$$

is a filter on I, and is included in an ultrafilter $\mathcal{F}$ say (2A1O). Let $(\mathfrak{A},\bar\mu)$ be the reduced product $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$. Then we have for each $i\in I$ a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_i\to\mathfrak{A}$ such that $\pi_i=\pi_j\pi_{ji}$ whenever $i\le j$ (328E). If $i\le j$ in I, then $\pi_i[\mathfrak{A}_i]\subseteq\pi_j[\mathfrak{A}_j]$; because $(I,\le)$ is upwards-directed, $\langle\pi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is an upwards-directed family of subalgebras of $\mathfrak{A}$, and $\mathfrak{D}=\bigcup_{i\in I}\pi_i[\mathfrak{A}_i]$ is a subalgebra of $\mathfrak{A}$; let $\mathfrak{C}$ be its closure (323J). Set $\bar\lambda=\bar\mu\restriction\mathfrak{C}$, so that $(\mathfrak{C},\bar\lambda)$ is a probability algebra, and $\pi_i:\mathfrak{A}_i\to\mathfrak{C}$ is a measure-preserving Boolean homomorphism for each $i\in I$, with $\pi_i=\pi_j\pi_{ji}$ whenever $i\le j$.

(b) Now suppose that $\mathfrak{B}$ and $\langle\phi_i\rangle_{i\in I}$ are as declared.

(i) Set

$$\phi'=\{(\pi_i a,\phi_i a):i\in I,\,a\in\mathfrak{A}_i\}\subseteq\mathfrak{D}\times\mathfrak{B}.$$

Then ϕ' is (the graph of) a function from $\mathfrak{D}$ to $\mathfrak{B}$. **P** If $c\in\mathfrak{D}$, there is surely an $i\in I$ such that $c\in\pi_i[\mathfrak{A}_i]$, so that $(c,\phi_i a)\in\phi'$ for some $a\in\mathfrak{A}_i$. If (c,b) and (c,b') belong to ϕ', there are i, $j\in I$ and $a\in\mathfrak{A}_i$, $a'\in\mathfrak{A}_j$ such that

$$\pi_i a=\pi_j a'=c,\quad\phi_i a=b,\quad\phi_j a'=b'.$$

Let $k\in I$ be such that $i\le k$ and $j\le k$; then

$$\pi_k\pi_{ki}a=\pi_i a=c=\pi_j a'=\pi_k\pi_{kj}a'.$$

As π_k is measure-preserving, therefore injective, $\pi_{ki}a=\pi_{kj}a'$, and

$$b=\phi_i a=\phi_k\pi_{ki}a=\phi_k\pi_{kj}a'=\phi_j a'=b'.$$

So each element of $\mathfrak{D}$ is the first member of exactly one element of ϕ', and ϕ' is the graph of a function. **Q** Of course the defining formula for ϕ' guarantees that $\phi'\pi_i=\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ for every $i\in I$.

(ii) Next, $\phi' : \mathfrak{D} \to \mathfrak{B}$ is a measure-preserving Boolean homomorphism. **P** If c, $c' \in \mathfrak{D}$ then there are i, $j \in I$ and $a \in \mathfrak{A}_i$, $a' \in \mathfrak{A}_j$ such that $c = \pi_i a$ and $c' = \pi_j a'$. Again take $k \in I$ such that $i \le k$ and $j \le k$; then

$$c = \pi_k \pi_{ki} a, \quad c' = \pi_k \pi_{kj} a', \quad \phi' c = \phi_k \pi_{ki} a, \quad \phi' c' = \phi_k \pi_{kj} a'.$$

In this case, for either of the Boolean operations $* = \triangle$ or $* = \cap$, we have

$$\begin{aligned}\phi' c * \phi' c' &= \phi_k \pi_{ki} a * \phi_k \pi_{kj} a' = \phi_k(\pi_{ki} a * \pi_{kj} a') \\ &= \phi' \pi_k(\pi_{ki} a * \pi_{kj} a') = \phi'(\pi_k \pi_{ki} a * \pi_k \pi_{kj} a') = \phi'(c * c').\end{aligned}$$

As c, c' and $*$ are arbitrary, ϕ' is a ring homomorphism. Moreover, in the same context,

$$\bar\nu \phi' c = \bar\nu \phi_i a = \bar\mu_i a = \bar\mu \pi_i a = \bar\lambda c,$$

so ϕ' is measure-preserving. It follows that $\phi' 1_{\mathfrak{C}} = 1_{\mathfrak{B}}$, and ϕ' is a Boolean homomorphism. **Q**

(iii) By 324O, there is a unique extension of ϕ' to a measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$; and of course we still have $\phi \pi_i = \phi_i$ for every $i \in I$.

(iv) To see that ϕ is unique, take any measure-preserving Boolean homomorphism $\tilde\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\tilde\phi \pi_i = \phi_i$ for every i. Then $\tilde\phi$ must agree with ϕ on $\pi_i[\mathfrak{A}_i]$ for every i, so $\tilde\phi \restriction \mathfrak{D} = \phi \restriction \mathfrak{D}$; as $\mathfrak{D}$ is topologically dense in $\mathfrak{C}$, $\tilde\phi = \phi$ (324O again).

328I For completeness, I spell out the relatively elementary construction for projective limits.

Proposition Let $(I, \le)$ be a non-empty upwards-directed set, and $\langle (\mathfrak{A}_i, \bar\mu_i) \rangle_{i \in I}$ a family of probability algebras; suppose that $\pi_{ij} : \mathfrak{A}_j \to \mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for $i \le j$ in I, and that $\pi_{ij} \pi_{jk} = \pi_{ik}$ whenever $i \le j \le k$. Then there are a probability algebra $(\mathfrak{C}, \bar\lambda)$ and a family $\langle \pi_i \rangle_{i \in I}$ such that

$\pi_i : \mathfrak{C} \to \mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for each $i \in I$,

$\pi_i = \pi_{ij} \pi_j$ whenever $i \le j$,

and whenever $(\mathfrak{B}, \bar\nu)$, $\langle \phi_i \rangle_{i \in I}$ are such that

$(\mathfrak{B}, \bar\nu)$ is a probability algebra,

$\phi_i : \mathfrak{B} \to \mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for each $i \in I$,

$\phi_i = \pi_{ij} \phi_j$ whenever $i \le j$,

then there is a unique measure-preserving Boolean homomorphism $\phi : \mathfrak{B} \to \mathfrak{C}$ such that $\pi_i \phi = \phi_i$ for every $i \in I$.

proof (a) Let $\mathfrak{C} \subseteq \prod_{i \in I} \mathfrak{A}_i$ be the set

$$\{\langle a_i \rangle_{i \in I} : \pi_{ij} a(j) = a(i) \text{ whenever } i \le j \text{ in } I\}.$$

Because every π_{ij} is a Boolean homomorphism, $\mathfrak{C}$ is a subalgebra of $\prod_{i \in I} \mathfrak{A}_i$; taking $\pi_j(\langle a_i \rangle_{i \in I}) = a_j$ whenever $\langle a_i \rangle_{i \in I} \in \mathfrak{C}$, $\pi_j : \mathfrak{C} \to \mathfrak{A}_j$ is a Boolean homomorphism for every $j \in I$, and $\pi_i = \pi_{ij} \pi_j$ whenever $i \le j$.

Because every π_{ij} is order-continuous, $\mathfrak{C}$ is an order-closed subalgebra of $\prod_{i \in I} \mathfrak{A}_i$, so is Dedekind complete.

(b) If $c = \langle a_i \rangle_{i \in I} \in \mathfrak{C}$, then

$$\bar\mu_i \pi_i c = \bar\mu_i a_i = \bar\mu_i \pi_{ij} a_j = \bar\mu_j a_j = \bar\mu_j \pi_j c$$

whenever $i \le j$ in I; because I is upwards-directed, $\bar\mu_i \pi_i c = \bar\mu_j \pi_j c$ for all i, $j \in I$. So we have a functional $\bar\lambda : \mathfrak{C} \to [0,1]$ defined by setting $\bar\lambda c = \bar\mu_i \pi_i c$ whenever $c \in \mathfrak{C}$ and $i \in I$. Note that $1_{\mathfrak{C}} = \langle 1_{\mathfrak{A}_i} \rangle_{i \in I}$, so $\bar\lambda 1_{\mathfrak{C}} = \bar\mu_i 1_{\mathfrak{A}_i} = 1$, for any $i \in I$.

If $\langle c_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\mathfrak{C}$ with supremum c, then express each c_n as $\langle a_{ni} \rangle_{i \in I}$; we must have $c = \langle \sup_{n \in \mathbb{N}} a_{ni} \rangle_{i \in I}$, so

$$\bar\lambda c = \bar\mu_i(\sup_{n \in \mathbb{N}} a_{ni}) = \sum_{n=0}^\infty \bar\mu_i a_{ni} = \sum_{n=0}^\infty \bar\lambda c_n$$

for any $i \in I$. Thus $\bar\lambda$ is countably additive. If $c \in \mathfrak{C}$ is non-zero, express it as $\langle a_i \rangle_{i \in I}$; there must be an $i \in I$ such that $a_i \ne 0$, so that $\bar\lambda c = \bar\mu_i a_i > 0$. Thus $\bar\lambda$ is strictly positive, and $(\mathfrak{C}, \bar\lambda)$ is a probability algebra.

(c) If $(\mathfrak{B}, \bar\nu)$ is a probability algebra and $\langle \phi_i \rangle_{i \in I}$ is a family such that $\phi_i : \mathfrak{B} \to \mathfrak{A}_i$ is a measure-preserving Boolean homomorphism and $\phi_i = \pi_{ij} \phi_j$ whenever $i \le j$ in I, set $\phi b = \langle \phi_i b \rangle_{i \in I}$ for $b \in \mathfrak{B}$. Then $\phi : \mathfrak{B} \to \prod_{i \in I} \mathfrak{A}_i$ is a Boolean homomorphism; also

$$\pi_{ij}(\phi b)(j) = \pi_{ij} \phi_j b = \phi_i b = (\phi b)(i)$$

whenever $i \le j$ and $b \in \mathfrak{B}$, so $\phi[\mathfrak{B}] \subseteq \mathfrak{C}$, while $\pi_i\phi = \phi_i$ for every $i \in I$. And of course this uniquely determines ϕ. To see that ϕ is measure-preserving, we have only to check that

$$\bar{\lambda}\phi b = \bar{\mu}_i\pi_i\phi b = \bar{\mu}_i\phi_i b = \bar{\nu}b$$

whenever $b \in \mathfrak{B}$ and $i \in I$.

328J A different application of the method in 328A yields the following result on commuting families of Boolean homomorphisms.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and Φ a family of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself such that $\phi\psi = \psi\phi$ for all ϕ, $\psi \in \Phi$. Then there are a probability algebra $(\mathfrak{C}, \bar{\lambda})$, a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ and a family $\langle\tilde{\phi}\rangle_{\phi\in\Phi}$ such that

(i) $\tilde{\phi} : \mathfrak{C} \to \mathfrak{C}$ is a measure-preserving Boolean automorphism and $\tilde{\phi}\pi = \pi\phi$ for every $\phi \in \Phi$;

(ii) $(\phi\psi)^{\sim} = \tilde{\phi}\tilde{\psi}$ for all ϕ, $\psi \in \Phi$.

proof (a) Let Ψ be the set of all products $\phi_0\phi_1 \ldots \phi_n$ where $\phi_i \in \Phi \cup \{\iota\}$ for every $i \le n$, ι here being the identity map from $\mathfrak{A}$ to itself. Then Ψ is a family of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself, and $\phi\psi = \psi\phi \in \Psi$ for all ϕ, $\psi \in \Psi$.

(b) For ϕ, $\psi \in \Psi$, say that $\phi \le \psi$ if there is a $\theta \in \Psi$ such that $\phi\theta = \psi$. Then $\le$ is a reflexive transitive relation on Ψ. Note that if $\phi \le \psi$ in Ψ then there is exactly one $\theta \in \Psi$ such that $\phi\theta = \psi$, because ϕ is injective. So we may define $\pi_{\psi,\phi} \in \Psi$ by saying that $\phi\pi_{\psi,\phi} = \psi$ whenever $\phi \le \psi$ in Ψ; that is, $\pi_{\phi\psi,\phi} = \psi$ whenever ϕ, $\psi \in \Psi$. Observe that if $\phi \le \psi \le \theta$ in Ψ, then

$$\phi\pi_{\psi,\phi}\pi_{\theta,\psi} = \psi\pi_{\theta,\psi} = \theta = \phi\pi_{\theta,\phi},$$

so

$$\pi_{\theta,\phi} = \pi_{\psi,\phi}\pi_{\theta,\psi} = \pi_{\theta,\psi}\pi_{\psi,\phi}.$$

Of course $\iota \le \phi$ for every $\phi \in \Psi$.

(c) If ϕ_1, $\phi_2 \in \Psi$ then $\phi_1 \le \phi_1\phi_2$ and $\phi_2 \le \phi_2\phi_1 = \phi_1\phi_2$; generally, if $D \subseteq \Psi$ is finite, there is a $\psi \in \Psi$ such that $\phi \le \psi$ for every $\phi \in D$. Consequently

$$\{A : A \subseteq \Psi, \text{ there is some } \phi \in \Psi \text{ such that } \psi \in A \text{ whenever } \phi \le \psi\}$$

is a filter on Ψ, and is included in an ultrafilter $\mathcal{F}$ say. Let $(\mathfrak{C}_0, \bar{\lambda}_0)$ be the probability algebra reduced power $(\mathfrak{A}, \bar{\mu})^\Psi|\mathcal{F}$. By 328E, we have for each $\phi \in \Psi$ a measure-preserving Boolean homomorphism $\pi_\phi : \mathfrak{A} \to \mathfrak{C}_0$ defined by saying that $\pi_\phi a = \langle a_\psi\rangle^\bullet_{\psi\in\Psi}$ if $a_\psi = \pi_{\psi,\phi}a$ whenever $\phi \le \psi$ in Ψ, and $\pi_\phi = \pi_\psi\pi_{\psi,\phi}$ whenever $\phi \le \psi$. Re-interpreting this in terms of the definitions of $\le$ and $\pi_{\psi,\phi}$, we have $\pi_\phi = \pi_{\phi\psi}\psi$ whenever ϕ, $\psi \in \Psi$.

(d) If ϕ, ψ in Ψ, then

$$\pi_\phi[\mathfrak{A}] \cup \pi_\psi[\mathfrak{A}] = \pi_{\phi\psi}[\psi[\mathfrak{A}]] \cup \pi_{\psi\phi}[\phi[\mathfrak{A}]] \subseteq \pi_{\phi\psi}[\mathfrak{A}] \cup \pi_{\psi\phi}[\mathfrak{A}] = \pi_{\phi\psi}[\mathfrak{A}],$$

which is a subalgebra of $\mathfrak{C}_0$. So $\mathfrak{D} = \bigcup_{\phi\in\Psi} \pi_\phi[\mathfrak{A}]$ is a subalgebra of $\mathfrak{C}_0$, and its closure $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{C}_0$; set $\bar{\lambda} = \bar{\lambda}_0{\restriction}\mathfrak{C}$. Then $\pi = \pi_\iota : \mathfrak{A} \to \mathfrak{C}$ is a measure-preserving Boolean homomorphism.

(e) If $\theta \in \Psi$, we have a measure-preserving Boolean homomorphism $\hat{\theta} : \mathfrak{C} \to \mathfrak{C}$ defined by the formula

$$\hat{\theta}(\langle a_\psi\rangle^\bullet_{\psi\in\Psi}) = \langle\theta a_\psi\rangle^\bullet_{\psi\in\Psi}$$

for every family $\langle a_\psi\rangle_{\psi\in\Psi}$ in $\mathfrak{A}$ (328Da); and $\widehat{\theta\phi} = \hat{\theta}\hat{\phi}$ for all θ, $\phi \in \Psi$ (328Db). Also $\hat{\theta}\pi_\phi = \pi_\phi\theta$ for every ϕ, $\theta \in \Psi$. **P** Let $a \in \mathfrak{A}$. Define $\langle a_\psi\rangle_{\psi\in\Psi}$, $\langle a'_\psi\rangle_{\psi\in\Psi}$ by setting

$$\begin{aligned} a_\psi &= \pi_{\psi,\phi}a \text{ when } \phi \le \psi, \\ &= 0 \text{ otherwise}, \\ a'_\psi &= \pi_{\psi,\phi}\theta a = \theta\pi_{\psi,\phi}a \text{ when } \phi \le \psi, \\ &= 0 \text{ otherwise}. \end{aligned}$$

Then

$$\pi_\phi a = \langle a_\psi\rangle^\bullet_{\psi\in\Psi},$$

$$\hat{\theta}\pi_\phi a = \langle\theta a_\psi\rangle^\bullet_{\psi\in\Psi} = \langle a'_\psi\rangle^\bullet_{\psi\in\Psi} = \pi_\phi\theta a. \textbf{ Q}$$

(f) It follows that, for $\theta \in \Psi$,

$$\hat{\theta}[\mathfrak{D}] = \bigcup_{\phi\in\Psi} \hat{\theta}[\pi_\phi[\mathfrak{A}]] = \bigcup_{\phi\in\Psi} \pi_\phi[\theta[\mathfrak{A}]] \subseteq \mathfrak{D}.$$

But in fact $\hat{\theta}[\mathfrak{D}] = \mathfrak{D}$. **P** If $d \in \mathfrak{D}$, there are $\phi \in \Psi$ and $a \in \mathfrak{A}$ such that $\pi_\phi a = d$. Now define

$$\begin{aligned} a_\psi &= \pi_{\psi,\phi} a \text{ if } \phi \le \psi, \\ &= 0 \text{ for other } \psi \in \Psi, \\ a'_\psi &= \pi_{\psi,\phi\theta} a \text{ if } \phi\theta \le \psi, \\ &= 0 \text{ for other } \psi \in \Psi, \\ d' &= \pi_{\phi\theta} a = \langle a'_\psi\rangle^\bullet_{\psi\in\Psi}. \end{aligned}$$

In this case, if $\phi\theta \le \psi$,

$$\phi\theta a'_\psi = \psi a, \quad \phi a_\psi = \psi a$$

so $\theta a'_\psi = a_\psi$. Consequently

$$\hat{\theta} d' = \hat{\theta}(\langle a'_\psi\rangle^\bullet_{\psi\in\Psi}) = \langle\theta a'_\psi\rangle^\bullet_{\psi\in\Psi} = \langle a_\psi\rangle^\bullet_{\psi\in\Psi}$$

(because $\{\psi : \phi\theta \le \psi\} \in \mathcal{F}$)

$$= d,$$

and $d = \hat{\theta}\pi_{\phi\theta} a \in \hat{\theta}[\mathfrak{D}]$. **Q**

(g) Since $\hat{\theta}[\mathfrak{C}]$ is a closed subalgebra of $\mathfrak{C}_0$ (324Kb) in which $\hat{\theta}[\mathfrak{D}] = \mathfrak{D}$ is topologically dense (3A3Eb), $\hat{\theta}[\mathfrak{C}] = \mathfrak{C}$. Setting $\tilde{\theta} = \hat{\theta}\restriction\mathfrak{C}$, we see that $\tilde{\theta} : \mathfrak{C} \to \mathfrak{C}$ is a surjective measure-preserving Boolean homomorphism, so is a Boolean automorphism. Since $\widehat{\phi\theta} = \hat{\phi}\hat{\theta}$, we have $(\phi\theta)^\sim = \tilde{\phi}\tilde{\theta}$ for all ϕ, $\theta \in \Psi$.

(h) Finally, as observed at the beginning of (e),

$$\tilde{\theta}\pi = \tilde{\theta}\pi_\iota = \hat{\theta}\pi_\iota = \pi_\iota\theta = \pi\theta$$

for every $\theta \in \Psi$. So $(\mathfrak{C}, \bar{\lambda}, \pi, \langle\tilde{\theta}\rangle_{\theta\in\Phi})$ has the required properties.

328X Basic exercises (a) Write out a version of the proof of 328J adapted to the case in which $\Phi = \{\phi\}$ is a singleton. (This is an abstract version of a construction known as the 'natural extension' of an inverse-measure-preserving function; see PETERSEN 83, 1.3G.)

(b) Let $\nu_\mathbb{N}$ be the usual measure on $X = \{0,1\}^\mathbb{N}$, and $(\mathfrak{B}_\mathbb{N}, \bar{\nu}_\mathbb{N})$ its measure algebra. (i) Find inverse-measure-preserving functions f, $g : X \to X$ such that $gf = g$ but $f(x) \ne x$ for every $x \in X$. (*Hint*: try $g(x)(n) = x(n+1)$.) (ii) Find measure-preserving Boolean homomorphisms ϕ, $\psi : \mathfrak{B}_\mathbb{N} \to \mathfrak{B}_\mathbb{N}$ such that $\phi\psi = \psi$ but ϕ is not the identity. (iii) In 328J, show that the hypothesis that members of Φ commute cannot be omitted.

328 Notes and comments I have starred this section because it is far from the main line of argument of the volume, and most readers should be moving on to Maharam's theorem and the Lifting Theorem. However the results here, while natural enough, have some features which demand a little attention, and it will be useful to be able to call on exact formulations of the ideas.

The proof of 328H begins by taking an ultrafilter on I. This ought to ring bells. It should be clear from the statement of the proposition that $(\mathfrak{C}, \bar{\lambda}, \langle\pi_i\rangle_{i\in I})$ is determined up to isomorphism by the properties declared here. It cannot therefore depend on which ultrafilter we pick, and there ought to be a construction not relying on this approach (and, we can hope, not demanding any application of the axiom of choice). This is indeed the case, and in 392Yd below I will sketch a method which can be adapted to give such a proof. Yet another proof of 328H is proposed in 418Yp in Volume 4.

The same remarks apply to the proof of 328J. In the result as stated, I have not imposed conditions on the structure $(\mathfrak{C}, \bar{\lambda}, \pi, \langle\tilde{\phi}\rangle_{\phi\in\Phi})$ sufficient to define it uniquely, but once again it is not necessary to employ an ultrafilter, and in fact the filter

$$\{A : A \subseteq \Psi, \text{ there is some } \phi \in \Psi \text{ such that } \psi \in A \text{ whenever } \phi \le \psi\}$$

is already enough, if we take the trouble to move to the right subalgebra of $\mathfrak{A}^\Psi$ before taking the quotient algebra.

Chapter 33

Maharam's theorem

We are now ready for the astonishing central fact about measure algebras: there are very few of them. Any localizable measure algebra has a canonical expression as a simple product of measure algebras of easily described types. This complete classification necessarily dominates all further discussion of measure algebras; to the point that all the results of Chapter 32 have to be regarded as 'elementary', since however complex their formulation they have been proved by techniques not involving, nor providing, any particular insight into the special nature of measure algebras. The proof depends, of course, on developing methods of defining measure-preserving homomorphisms and isomorphisms; I give a number of results, progressively more elaborate, but all based on the same idea. These techniques are of great power, leading, for instance, to an effective classification of closed subalgebras and their embeddings.

'Maharam's theorem' itself, the classification of localizable measure algebras, is in §332. I devote §331 to the definition and description of 'homogeneous' probability algebras. In §333 I turn to the problem of describing pairs $(\mathfrak{A},\mathfrak{C})$ where $\mathfrak{A}$ is a probability algebra and $\mathfrak{C}$ is a closed subalgebra. Finally, in §334, I give some straightforward results on the classification of free products of probability algebras.

331 Maharam types and homogeneous measure algebras

I embark directly on the principal theorem of this chapter (331I), split between 331B, 331D and 331I; 331B and 331D will be the basis of many of the results in later sections of this chapter. In 331E-331H I introduce the concepts of 'Maharam type' and 'Maharam-type-homogeneity'. I discuss the measure algebras of products $\{0,1\}^\kappa$, showing that these provide a complete set of examples of Maharam-type-homogeneous probability algebras (331J-331L).

331A Definition The following idea is almost the key to the whole chapter. Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$. A non-zero element a of $\mathfrak{A}$ is a **relative atom** over $\mathfrak{B}$ if every $c\subseteq a$ is of the form $a\cap b$ for some $b\in\mathfrak{B}$; that is, $\{a\cap b: b\in\mathfrak{B}\}$ is the principal ideal generated by a. We say that $\mathfrak{A}$ is **relatively atomless** over $\mathfrak{B}$ if there are no relative atoms in $\mathfrak{A}$ over $\mathfrak{B}$.

(I'm afraid the phrases 'relative atom', 'relatively atomless' are bound to seem opaque at this stage. I hope that after the structure theory of §333 they will seem more natural. For the moment, note only that a is an atom in $\mathfrak{A}$ iff it is a relative atom over the smallest subalgebra $\{0,1\}$, and every element of $\mathfrak{A}$ is a relative atom over the largest subalgebra $\mathfrak{A}$. In a way, a is a relative atom over $\mathfrak{B}$ if its image is an atom in a kind of quotient $\mathfrak{A}/\mathfrak{B}$. But we are two volumes away from any prospect of making sense of this kind of quotient.)

331B The first lemma is the heart of Maharam's theorem.

Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$, and $a_0\in\mathfrak{A}$. Let $\nu:\mathfrak{B}\to\mathbb{R}$ be an additive functional such that $0\le\nu b\le\bar\mu(b\cap a_0)$ for every $b\in\mathfrak{B}$. Then there is a $c\in\mathfrak{A}$ such that $c\subseteq a_0$ and $\nu b=\bar\mu(b\cap c)$ for every $b\in\mathfrak{B}$.

Remark Recall that by 323H we need not distinguish between 'order-closed' and 'topologically closed' subalgebras.

proof (a) It is worth noting straight away that ν is necessarily countably additive. This is easy to check from first principles, but if you want to trace the underlying ideas they are in 313O (the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is order-continuous), 326Jf (so $\mu\restriction\mathfrak{B}:\mathfrak{B}\to\mathbb{R}$ is countably additive) and 326Kb (therefore ν is countably additive).

(b) For each $a\in\mathfrak{A}$ set $\nu_a b=\bar\mu(b\cap a)$ for every $b\in\mathfrak{B}$; then $\nu_a:\mathfrak{B}\to\mathbb{R}$ is countably additive (326Jd). Note that $\nu_{c\cup d}=\nu_c+\nu_d$ whenever $c.\ d\in\mathfrak{A}$ are disjoint. The key idea is the following fact: for every non-zero $a\in\mathfrak{A}$ there is a non-zero $d\subseteq a$ such that $\nu_d\le\frac{1}{2}\nu_a$. **P** Because $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$, there is an $e\subseteq a$ such that $e\ne a\cap b$ for any $b\in\mathfrak{B}$. Consider the countably additive functional $\lambda=\nu_a-2\nu_e:\mathfrak{B}\to\mathbb{R}$. By 326M, there is a $b_0\in\mathfrak{B}$ such that $\lambda b\ge0$ whenever $b\in\mathfrak{B}$ and $b\subseteq b_0$, while $\lambda b\le0$ whenever $b\in\mathfrak{B}$ and $b\cap b_0=0$.

If $e\cap b_0\ne0$, try $d=e\cap b_0$. Then $0\ne d\subseteq a$, and for every $b\in\mathfrak{B}$

$$\nu_d b=\nu_e(b\cap b_0)=\frac{1}{2}(\nu_a(b\cap b_0)-\lambda(b\cap b_0))\le\frac{1}{2}\nu_a b$$

(because $\lambda(b\cap b_0)\ge0$) so $\nu_d\le\frac{1}{2}\nu_a$.

If $e\cap b_0=0$, then (by the choice of e) $e\ne a\cap(1\setminus b_0)$, so $d=a\setminus(e\cup b_0)\ne0$, and of course $d\subseteq a$. In this case, for every $b\in\mathfrak{B}$,

$$\nu_d b = \nu_a(b \setminus b_0) - \nu_e(b \setminus b_0) = \frac{1}{2}(\lambda(b \setminus b_0) + \nu_a(b \setminus b_0)) \le \frac{1}{2}\nu_a b$$

(because $\lambda(b \setminus b_0) \le 0$), so once again $\nu_d \le \frac{1}{2}\nu_a$.

Thus in either case we have a suitable d. **Q**

(c) It follows at once, by induction on n, that if a is any non-zero element of $\mathfrak{A}$ and $n \in \mathbb{N}$ then there is a non-zero $d \subseteq a$ such that $\nu_d \le 2^{-n}\nu_a$.

(d) Now suppose that $a \in \mathfrak{A}$ and that $\lambda : \mathfrak{B} \to [0, \infty[$ is a non-zero countably additive functional such that $\lambda \le \nu_a$. Then there is a non-zero $d \subseteq a$ such that $\nu_d \le \lambda$. **P** Let $b^* \in \mathfrak{B}$ be such that $\lambda b^* > 0$; then

$$\lambda(b^* \setminus a) \le \nu_a(b^* \setminus a) = 0,$$

so $\lambda b^* = \lambda(b^* \cap a)$. Take $n \in \mathbb{N}$ such that $2^{-n}\nu_a b^* < \lambda b^*$; set $\lambda_1 = \lambda - 2^{-n}\nu_a$. By 326M (for the second time), there is a $b_1 \in \mathfrak{B}$ such that $\lambda_1 b \ge 0$ if $b \subseteq b_1$ and $\lambda_1 b \le 0$ if $b \cap b_1 = 0$. Set $c = a \cap b_1$. Now

$$2^{-n}\nu_a(a \cap b^*) = 2^{-n}\nu_a b^* < \lambda b^* = \lambda(a \cap b^*),$$

so $\lambda_1(a \cap b^*) > 0$ and $a \cap b^* \cap b_1 \ne 0$ and $c \ne 0$. By (c), we have a non-zero $d \subseteq c$ such that $\nu_d \le 2^{-n}\nu_c$. If $b \in \mathfrak{B}$ then

$$\nu_d b \le 2^{-n}\nu_c b = 2^{-n}\nu_a(b \cap b_1) = \lambda(b \cap b_1) - \lambda_1(b \cap b_1) \le \lambda(b \cap b_1) \le \lambda b,$$

so $\nu_d \le \lambda$, as required. **Q**

(e) Let C be the set

$$\{a : a \in \mathfrak{A},\ a \subseteq a_0,\ \nu_a \le \nu\}.$$

Then $0 \in C$, so $C \ne \emptyset$. If $D \subseteq C$ is upwards-directed and not empty, then $a = \sup D$ is defined in $\mathfrak{A}$ and included in a_0, and

$$\nu_{\sup D} b = \bar\mu(b \cap \sup D) = \bar\mu(\sup_{d \in D} b \cap d) = \sup_{d \in D} \bar\mu(b \cap d) = \sup_{d \in D} \nu_d b \le \nu b$$

using 313Ba and 321C. So $a \in C$ and is an upper bound for D in C. In particular, any non-empty totally ordered subset of C has an upper bound in C. By Zorn's Lemma, C has a maximal element c say. Of course $c \subseteq a_0$.

(e) **?** Suppose, if possible, that $\nu_c \ne \nu$. Set $\lambda = \nu - \nu_c$; note that

$$\lambda b = \nu b - \nu_c b \le \bar\mu(b \cap a_0) - \bar\mu(b \cap c) = \nu_{a_0 \setminus c} b$$

for every $b \in \mathfrak{B}$, so $\lambda \le \nu_{a_0 \setminus c}$. Because $c \in C$, $\lambda \ge 0$, and we are supposing that $\lambda \ne 0$. By (d), there is a non-zero $d \subseteq a_0 \setminus c$ such that $\nu_d \le \lambda$. But now $c \cup d \subseteq a_0$,

$$\nu_{c \cup d} = \nu_c + \nu_d \le \nu_c + \lambda = \nu$$

and $c \cup d \in C$, so c is not maximal in C. **X**

Thus c is an element of $\mathfrak{A}$, included in a_0, giving a representation of ν.

331C Corollary Let $(\mathfrak{A}, \bar\mu)$ be an atomless semi-finite measure algebra, and $a \in \mathfrak{A}$. Suppose that $0 \le \gamma \le \bar\mu a$. Then there is a $c \in \mathfrak{A}$ such that $c \subseteq a$ and $\bar\mu c = \gamma$.

proof If $\gamma = \bar\mu a$, take $c = a$. If $\gamma < \bar\mu a$, there is a $d \in \mathfrak{A}$ such that $d \subseteq a$ and $\gamma \le \bar\mu d < \infty$ (322Eb). Apply 331B to the principal ideal $\mathfrak{A}_d$ generated by d, with $a_0 = d$, $\mathfrak{B} = \{0, d\}$ and $\nu d = \gamma$. (The point is that because $\mathfrak{A}$ is atomless, no non-trivial principal ideal of $\mathfrak{A}_d$ can be of the form $\{c \cap b : b \in \mathfrak{B}\} = \{0, c\}$, so $\mathfrak{A}_d$ is relatively atomless over $\{0, d\}$.)

Remark Of course this is also an easy consequence of either 215D or the one-dimensional case of 326H.

331D Lemma Let $(\mathfrak{A}, \bar\mu)$, $(\mathfrak{B}, \bar\nu)$ be totally finite measure algebras and $\mathfrak{C} \subseteq \mathfrak{A}$ a closed subalgebra. Suppose that $\pi : \mathfrak{C} \to \mathfrak{B}$ is a measure-preserving Boolean homomorphism such that $\mathfrak{B}$ is relatively atomless over $\pi[\mathfrak{C}]$. Take any $a \in \mathfrak{A}$, and let $\mathfrak{C}_1$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{a\}$. Then there is a measure-preserving homomorphism from $\mathfrak{C}_1$ to $\mathfrak{B}$ extending π.

proof We know that $\pi[\mathfrak{C}]$ is a closed subalgebra of $\mathfrak{B}$ (324Kb), and that π is a Boolean isomorphism between $\mathfrak{C}$ and $\pi[\mathfrak{C}]$. Consequently the countably additive functional $c \mapsto \bar\mu(c \cap a) : \mathfrak{C} \to \mathbb{R}$ is transferred to a countably additive functional $\lambda : \pi[\mathfrak{C}] \to \mathbb{R}$, writing $\lambda(\pi c) = \bar{\mu}(c \cap a)$ for every $c \in \mathfrak{C}$. Of course $\lambda(\pi c) \le \bar{\mu}c = \bar{\nu}(\pi c)$ for every $c \in \mathfrak{C}$. So by 331B there is a $b \in \mathfrak{B}$ such that $\lambda(\pi c) = \bar{\nu}(b \cap \pi c)$ for every $c \in \mathfrak{C}$.

If $c \in \mathfrak{C}$ and $c \subseteq a$ then

$$\bar{\nu}(b \cap \pi c) = \lambda(\pi c) = \bar{\mu}(a \cap c) = \bar{\mu}c = \bar{\nu}(\pi c),$$

so $\pi c \subseteq b$. Similarly, if $a \subseteq c \in \mathfrak{C}$, then

$$\bar{\nu}(b \cap \pi c) = \bar{\mu}(a \cap c) = \bar{\mu}(a \cap 1) = \bar{\nu}(b \cap \pi 1) = \bar{\nu}b,$$

so $b \subseteq \pi c$. It follows from 312O that there is a Boolean homomorphism $\pi_1 : \mathfrak{C}_1 \to \mathfrak{B}$, extending π, such that $\pi_1 a = b$.

To see that π_1 is measure-preserving, take any member of $\mathfrak{C}_1$. By 312N, this is expressible as $e = (c_1 \cap a) \cup (c_2 \setminus a)$, where c_1, $c_2 \in \mathfrak{C}$. Now

$$\begin{aligned}\bar{\nu}(\pi_1 e) &= \bar{\nu}((\pi c_1 \cap b) \cup (\pi c_2 \setminus b)) = \bar{\nu}(\pi c_1 \cap b) + \bar{\nu}(\pi c_2) - \bar{\nu}(\pi c_2 \cap b)\\ &= \bar{\mu}(c_1 \cap a) + \bar{\mu}c_2 - \bar{\mu}(c_2 \cap a) = \bar{\mu}e.\end{aligned}$$

As e is arbitrary, π_1 is measure-preserving.

331E Generating sets For the sake of the next definition, we need a language a little more precise than I have felt the need to use so far. The point is that if $\mathfrak{A}$ is a Boolean algebra and B is a subset of $\mathfrak{A}$, there is more than one subalgebra of $\mathfrak{A}$ which can be said to be 'generated' by B, because we can look at any of the three algebras

– $\mathfrak{B}$, the smallest subalgebra of $\mathfrak{A}$ including B;

– $\mathfrak{B}_\sigma$, the smallest σ-subalgebra of $\mathfrak{A}$ including B;

– $\mathfrak{B}_\tau$, the smallest order-closed subalgebra of $\mathfrak{A}$ including B.

(See 313Fb.) Now I will say henceforth, in this context, that

– $\mathfrak{B}$ is the subalgebra of $\mathfrak{A}$ generated by B, and B **generates** $\mathfrak{A}$ if $\mathfrak{A} = \mathfrak{B}$;

– $\mathfrak{B}_\sigma$ is the σ-subalgebra of $\mathfrak{A}$ generated by B, and B **σ-generates** $\mathfrak{A}$ if $\mathfrak{A} = \mathfrak{B}_\sigma$;

– $\mathfrak{B}_\tau$ is the order-closed subalgebra of $\mathfrak{A}$ generated by B, and B **τ-generates** or **completely generates** $\mathfrak{A}$ if $\mathfrak{A} = \mathfrak{B}_\tau$.

There is a danger inherent in these phrases, because if we have $B \subseteq \mathfrak{A}'$, where $\mathfrak{A}'$ is a subalgebra of $\mathfrak{A}$, it is possible that the smallest order-closed subalgebra of $\mathfrak{A}'$ including B might not be recoverable from the smallest order-closed subalgebra of $\mathfrak{A}$ including B. (See 331Yb-331Yc.) This problem will not seriously interfere with the ideas below; but for definiteness let me say that the phrases 'B σ-generates $\mathfrak{A}$', 'B τ-generates $\mathfrak{A}$' will always refer to suprema and infima taken in $\mathfrak{A}$ itself, not in any larger algebra in which it may be embedded.

331F Maharam types (a) With the language of 331E established, I can now define the **Maharam type** or **complete generation number** $\tau(\mathfrak{A})$ of any Boolean algebra $\mathfrak{A}$; it is the smallest cardinal of any subset of $\mathfrak{A}$ which τ-generates $\mathfrak{A}$.

(I think that this is the first 'cardinal function' which I have mentioned in this treatise. All you need to know, to confirm that the definition is well-conceived, is that there is *some* set which τ-generates $\mathfrak{A}$; and obviously $\mathfrak{A}$ τ-generates itself. For this means that the set $A = \{\#(B) : B \subseteq \mathfrak{A}\ \tau\text{-generates } \mathfrak{A}\}$ is a non-empty class of cardinals, and therefore, assuming the axiom of choice, has a least member (2A1Lf). In 331Ye-331Yf I mention a further function, the 'density' of a topological space, which is closely related to Maharam type.)

(b) A Boolean algebra $\mathfrak{A}$ is **Maharam-type-homogeneous** if $\tau(\mathfrak{A}_a) = \tau(\mathfrak{A})$ for every non-zero $a \in \mathfrak{A}$, writing $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a.

(c) Let (X, Σ, μ) be a measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Then the **Maharam type** of (X, Σ, μ), or of μ, is the Maharam type of $\mathfrak{A}$; and (X, Σ, μ), or μ, is **Maharam-type-homogeneous** if $\mathfrak{A}$ is.

Remark I should perhaps remark that the phrases 'Maharam type' and 'Maharam-type-homogeneous', while well established in the context of probability algebras, are not in common use for general Boolean algebras. But the cardinal $\tau(\mathfrak{A})$ is important in the general context, and is such an obvious extension of Maharam's idea (Maharam 42) that I am happy to propose this extension of terminology.

331G For the sake of those who have not mixed set theory and algebra before, I had better spell out some basic facts.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, B a subset of $\mathfrak{A}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by B, $\mathfrak{B}_\sigma$ the σ-subalgebra of $\mathfrak{A}$ generated by B, and $\mathfrak{B}_\tau$ the order-closed subalgebra of $\mathfrak{A}$ generated by B.

(a) $\mathfrak{B} \subseteq \mathfrak{B}_\sigma \subseteq \mathfrak{B}_\tau$.

(b) If B is finite, so is $\mathfrak{B}$, and in this case $\mathfrak{B} = \mathfrak{B}_\sigma = \mathfrak{B}_\tau$.

(c) For every $a \in \mathfrak{B}$, there is a finite $B' \subseteq B$ such that a belongs to the subalgebra of $\mathfrak{A}$ generated by B'. Consequently $\#(\mathfrak{B}) \le \max(\omega, \#(B))$.

(d) For every $a \in \mathfrak{B}_\sigma$, there is a countable $B' \subseteq B$ such that a belongs to the σ-subalgebra of $\mathfrak{A}$ generated by B'.

(e) If $\mathfrak{A}$ is ccc, then $\mathfrak{B}_\sigma = \mathfrak{B}_\tau$.

proof (a) All we need to know is that $\mathfrak{B}_\sigma$ is a subalgebra of $\mathfrak{A}$ including B, and that $\mathfrak{B}_\tau$ is a σ-subalgebra of $\mathfrak{A}$ including B.

(b) Induce on $\#(B)$, using 312N for the inductive step, to see that $\mathfrak{B}$ is finite. In this case it must be order-closed, so is equal to $\mathfrak{B}_\tau$.

(c)(i) For $I \subseteq B$, let $\mathfrak{C}_I$ be the subalgebra of $\mathfrak{A}$ generated by I. If I, $J \subseteq B$ then $\mathfrak{C}_I \cup \mathfrak{C}_J \subseteq \mathfrak{C}_{I\cup J}$. So $\bigcup\{\mathfrak{C}_I : I \subseteq B$ is finite$\}$ is a subalgebra of $\mathfrak{A}$, and must be equal to $\mathfrak{B}$, as claimed.

(ii) To estimate the size of $\mathfrak{B}$, recall that the set $[B]^{<\omega}$ of all finite subsets of B has cardinal at most $\max(\omega, \#(B))$ (3A1Cd). For each $I \in [B]^{<\omega}$, $\mathfrak{C}_I$ is finite, so

$$\#(\mathfrak{B}) = \#(\bigcup_{I\in[B]^{<\omega}} \mathfrak{C}_I) \le \max(\omega, \#(I), \sup_{I\in[B]^{<\omega}} \#(\mathfrak{C}_I)) \le \max(\omega, \#(B))$$

by 3A1Cc.

(d) For $I \subseteq B$, let $\mathfrak{D}_I \subseteq \mathfrak{B}_\sigma$ be the σ-subalgebra of $\mathfrak{A}$ generated by I. If I, $J \subseteq B$ then $\mathfrak{D}_I \cup \mathfrak{D}_J \subseteq \mathfrak{D}_{I\cup J}$, so $\mathfrak{B}'_\sigma = \bigcup\{\mathfrak{D}_I : I \subseteq B$ is countable$\}$ is a subalgebra of $\mathfrak{A}$. But also it is sequentially order-closed in $\mathfrak{A}$. **P** Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{B}'_\sigma$ with supremum a in $\mathfrak{A}$. For each $n \in \mathbb{N}$ there is a countable $I(n) \subseteq B$ such that $a_n \in \mathfrak{C}_{I(n)}$. Set $K = \bigcup_{n\in\mathbb{N}} I(n)$; then K is a countable subset of B and every a_n belongs to $\mathfrak{D}_K$, so $a \in \mathfrak{D}_K \subseteq \mathfrak{B}'_\sigma$. **Q** So $\mathfrak{B}'_\sigma$ is a σ-subalgebra of $\mathfrak{A}$ including B and must be the whole of $\mathfrak{B}_\sigma$.

(e) By 316Fb, $\mathfrak{B}_\sigma$ is order-closed in $\mathfrak{A}$, so must be equal to $\mathfrak{B}_\tau$.

331H Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a)(i) $\tau(\mathfrak{A}) = 0$ iff $\mathfrak{A}$ is either $\{0\}$ or $\{0, 1\}$.

(ii) $\tau(\mathfrak{A})$ is finite iff $\mathfrak{A}$ is finite.

(b) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then $\tau(\mathfrak{B}) \le \tau(\mathfrak{A})$.

(c) If $a \in \mathfrak{A}$ then $\tau(\mathfrak{A}_a) \le \tau(\mathfrak{A})$, where $\mathfrak{A}_a$ is the principal ideal of $\mathfrak{A}$ generated by a.

(d) If $\mathfrak{A}$ has an atom and is Maharam-type-homogeneous, then $\mathfrak{A} = \{0, 1\}$.

proof (a)(i) $\tau(\mathfrak{A}) = 0$ iff $\mathfrak{A}$ has no proper subalgebras.

(ii) If $\mathfrak{A}$ is finite, then $\tau(\mathfrak{A}) \le \#(\mathfrak{A})$ is finite. If $\tau(\mathfrak{A})$ is finite, then there is a finite set $B \subseteq \mathfrak{A}$ which τ-generates $\mathfrak{A}$; by 331Gb, $\mathfrak{A}$ is finite.

(b) We know that there is a set $A \subseteq \mathfrak{A}$, τ-generating $\mathfrak{A}$, with $\#(A) = \tau(\mathfrak{A})$. Now $\pi[A]$ τ-generates $\pi[\mathfrak{A}] = \mathfrak{B}$ (313Mb), so

$$\tau(\mathfrak{B}) \le \#(\pi[A]) \le \#(A) = \tau(\mathfrak{A}).$$

(c) Apply (b) to the map $b \mapsto a \cap b : \mathfrak{A} \to \mathfrak{A}_a$.

(d) If $a \in \mathfrak{A}$ is an atom, then $\tau(\mathfrak{A}_a) = 0$, so if $\mathfrak{A}$ is Maharam-type-homogeneous then $\tau(\mathfrak{A}) = 0$ and $\mathfrak{A} = \{0, a\} = \{0, 1\}$.

331I We are now ready for the theorem.

Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be Maharam-type-homogeneous measure algebras of the same Maharam type, with $\bar\mu 1 = \bar\nu 1 < \infty$. Then they are isomorphic as measure algebras.

proof (a) If $\tau(\mathfrak{A}) = \tau(\mathfrak{B}) = 0$, this is trivial. So let us take $\kappa = \tau(\mathfrak{A}) = \tau(\mathfrak{B}) > 0$. In this case, because $\mathfrak{A}$ and $\mathfrak{B}$ are Maharam-type-homogeneous, they can have no atoms and must be infinite, so κ is infinite (331H). Let $\langle a_\xi\rangle_{\xi<\kappa}$ and $\langle b_\xi\rangle_{\xi<\kappa}$ enumerate τ-generating subsets of $\mathfrak{A}$, $\mathfrak{B}$ respectively.

The strategy of the proof is to define a measure-preserving isomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ as the last of an increasing family $\langle\pi_\xi\rangle_{\xi\le\kappa}$ of isomorphisms between closed subalgebras $\mathfrak{C}_\xi$, $\mathfrak{D}_\xi$ of $\mathfrak{A}$ and $\mathfrak{B}$. The inductive hypothesis will be that, for some families $\langle a'_\xi\rangle_{\xi<\kappa}$, $\langle b'_\xi\rangle_{\xi<\kappa}$ to be determined,

$\mathfrak{C}_\xi$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta:\eta<\xi\}\cup\{a'_\eta:\eta<\xi\}$,

$\mathfrak{D}_\xi$ is the closed subalgebra of $\mathfrak{B}$ generated by $\{b_\eta:\eta<\xi\}\cup\{b'_\eta:\eta<\xi\}$,

$\pi_\xi:\mathfrak{C}_\xi\to\mathfrak{D}_\xi$ is a measure-preserving isomorphism,

π_ξ extends π_η whenever $\eta<\xi$.

(Formally speaking, this will be a transfinite recursion, defining a function $\xi\mapsto f(\xi)=(\mathfrak{C}_\xi,\mathfrak{D}_\xi,\pi_\xi,a'_\xi,b'_\xi)$ on the ordinal $\kappa+1$ by a rule which chooses $f(\xi)$ in terms of $f\restriction\xi$, as described in 2A1B. The construction of an actual function F for which $f(\xi)=F(f\restriction\xi)$ will necessitate the axiom of choice.)

(b) The induction starts with $\mathfrak{C}_0=\{0,1\}$, $\mathfrak{D}_0=\{0,1\}$, $\pi_0(0)=0$, $\pi_0(1)=1$. (The hypothesis $\bar\mu1=\bar\nu1$ is what we need to ensure that π_0 is measure-preserving.)

(c) For the inductive step to a successor ordinal $\xi+1$, where $\xi<\kappa$, suppose that $\mathfrak{C}_\xi$, $\mathfrak{D}_\xi$ and π_ξ have been defined.

(i) For any non-zero $b\in\mathfrak{B}$, the principal ideal $\mathfrak{B}_b$ of $\mathfrak{B}$ generated by b has Maharam type κ, because $\mathfrak{B}$ is Maharam-type-homogeneous. On the other hand, the Maharam type of $\mathfrak{D}_\xi$ is at most

$$\#(\{b_\eta:\eta<\xi\}\cup\{b'_\eta:\eta<\xi\})\le\#(\xi\times\{0,1\})<\kappa,$$

because if ξ is finite so is $\xi\times\{0,1\}$, while if ξ is infinite then $\#(\xi\times\{0,1\})=\#(\xi)\le\xi<\kappa$. Consequently $\mathfrak{B}_b$ cannot be an order-continuous image of $\mathfrak{D}_\xi$ (331Hb). Now the map $c\mapsto c\cap b:\mathfrak{D}_\xi\to\mathfrak{B}_b$ is order-continuous, because $\mathfrak{D}_\xi$ is closed, so that the embedding $\mathfrak{D}_\xi\subsetneqq\mathfrak{B}$ is order-continuous. It therefore cannot be surjective, and

$$\{b\cap\pi_\xi a:a\in\mathfrak{C}_\xi\}=\{b\cap d:d\in\mathfrak{D}_\xi\}\ne\mathfrak{B}_b.$$

This means that $\pi_\xi:\mathfrak{C}_\xi\to\mathfrak{D}_\xi$ satisfies the conditions of 331D, and must have an extension ϕ_ξ to a measure-preserving homomorphism from the subalgebra $\mathfrak{C}'_\xi$ of $\mathfrak{A}$ generated by $\mathfrak{C}_\xi\cup\{a_\xi\}$ to $\mathfrak{B}$. We know that $\mathfrak{C}'_\xi$ is a closed subalgebra of $\mathfrak{A}$ (314Ja), so it must be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta:\eta\le\xi\}\cup\{a'_\eta:\eta<\xi\}$. Also $\mathfrak{D}'_\xi=\phi_\xi[\mathfrak{C}'_\xi]$ will be the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D}_\xi\cup\{b'_\xi\}$, where $b'_\xi=\phi_\xi(a_\xi)$, so is closed in $\mathfrak{B}$, and is the closed subalgebra of $\mathfrak{B}$ generated by $\{b_\eta:\eta<\xi\}\cup\{b'_\eta:\eta\le\xi\}$.

(ii) The next step is to repeat the whole of the argument above, but applying it to $\phi_\xi^{-1}:\mathfrak{D}'_\xi\to\mathfrak{C}_\xi$, b_ξ in place of $\pi_\xi:\mathfrak{C}_\xi\to\mathfrak{D}_\xi$ and a_ξ. Once again, we have $\tau(\mathfrak{D}'_\xi)<\kappa=\tau(\mathfrak{A}_a)$ for every $a\in\mathfrak{A}$, so we can use Lemma 331D to find a measure-preserving isomorphism $\psi_\xi:\mathfrak{D}_{\xi+1}\to\mathfrak{C}_{\xi+1}$ extending ϕ_ξ^{-1}, where $\mathfrak{D}_{\xi+1}$ is the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D}'_\xi\cup\{b_\xi\}$, and $\mathfrak{C}_{\xi+1}$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}'_\xi\cup\{a'_\xi\}$, setting $a'_\xi=\psi_\xi(b_\xi)$. As in (i), we find that $\mathfrak{C}_{\xi+1}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta:\eta\le\xi\}\cup\{a'_\eta:\eta\le\xi\}$, while $\mathfrak{D}_{\xi+1}$ is the closed subalgebra of $\mathfrak{B}$ generated by $\{b_\eta:\eta\le\xi\}\cup\{b'_\eta:\eta\le\xi\}$.

(iii) We can therefore take $\pi_{\xi+1}=\psi_\xi^{-1}:\mathfrak{C}_{\xi+1}\to\mathfrak{D}_{\xi+1}$, and see that $\pi_{\xi+1}$ is a measure-preserving isomorphism, extending π_ξ, such that $\pi_{\xi+1}(a_\xi)=b'_\xi$ and $\pi_{\xi+1}(a'_\xi)=b_\xi$. Evidently $\pi_{\xi+1}$ extends π_η for every $\eta\le\xi$ because it extends π_ξ and (by the inductive hypothesis) π_ξ extends π_η for every $\eta<\xi$.

(d) For the inductive step to a limit ordinal ξ, where $0<\xi\le\kappa$, suppose that $\mathfrak{C}_\eta$, $\mathfrak{D}_\eta$, a'_η, b'_η, π_η have been defined for $\eta<\xi$. Set $\mathfrak{C}^*_\xi=\bigcup_{\eta<\xi}\mathfrak{C}_\xi$. Then $\mathfrak{C}^*_\xi$ is a subalgebra of $\mathfrak{A}$, because it is the union of an upwards-directed family of subalgebras; similarly, $\mathfrak{D}^*_\xi=\bigcup_{\eta<\xi}\mathfrak{D}_\xi$ is a subalgebra of $\mathfrak{B}$. Next, we have a function $\pi^*_\xi:\mathfrak{C}^*_\xi\to\mathfrak{D}^*_\xi$ defined by setting $\pi^*_\xi a=\pi_\eta a$ whenever $\eta<\xi$ and $a\in\mathfrak{C}_\eta$; for if η, $\zeta<\xi$ and $a\in\mathfrak{C}_\eta\cap\mathfrak{C}_\zeta$, then $\pi_\eta a=\pi_{\max(\eta,\zeta)}a=\pi_\zeta a$. Clearly

$$\pi^*_\xi[\mathfrak{C}^*_\xi]=\bigcup_{\eta<\xi}\pi_\eta[\mathfrak{C}_\eta]=\mathfrak{D}^*_\xi.$$

Moreover, $\bar\nu(\pi^*_\xi a)=\bar\mu a$ for every $a\in\mathfrak{C}^*_\xi$, since $\bar\nu(\pi_\eta a)=\bar\mu a$ whenever $\eta<\xi$ and $a\in\mathfrak{C}_\eta$.

Now let $\mathfrak{C}_\xi$ be the smallest closed subalgebra of $\mathfrak{A}$ including $\mathfrak{C}^*_\xi$, that is, the topological closure of $\mathfrak{C}^*_\xi$ in $\mathfrak{A}$ (323J). Since $\mathfrak{C}_\xi$ is the smallest closed subalgebra of $\mathfrak{A}$ including $\mathfrak{C}_\eta$ for every $\eta<\xi$, it must be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta:\eta<\xi\}\cup\{a'_\eta:\eta<\xi\}$. By 324O, π^*_ξ has an extension to a measure-preserving homomorphism $\pi_\xi:\mathfrak{C}_\xi\to\mathfrak{B}$. Set $\mathfrak{D}_\xi=\pi_\xi[\mathfrak{C}_\xi]$; by 324Kb, $\mathfrak{D}_\xi$ is a closed subalgebra of $\mathfrak{B}$. Because $\pi_\xi:\mathfrak{C}_\xi\to\mathfrak{B}$ is continuous (also noted in 324Kb),

$$\mathfrak{D}^*_\xi=\pi^*_\xi[\mathfrak{C}^*_\xi]=\pi_\xi[\mathfrak{C}^*_\xi]$$

is topologically dense in $\mathfrak{D}_\xi$ (3A3Eb), and $\mathfrak{D}_\xi=\overline{\mathfrak{D}^*_\xi}$ is the closed subalgebra of $\mathfrak{B}$-generated by $\{b_\eta:\eta<\xi\}\cup\{b'_\eta:\eta<\xi\}$. Finally, if $\eta<\xi$, π_ξ extends π_η because π^*_ξ extends π_η. Thus the induction continues.

(e) The induction ends with $\xi=\kappa$, $\mathfrak{C}_\kappa=\mathfrak{A}$, $\mathfrak{D}_\kappa=\mathfrak{B}$ and $\pi=\pi_\kappa:\mathfrak{A}\to\mathfrak{B}$ the required measure algebra isomorphism.

331J Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and κ an infinite cardinal.

(a) If there is a family $\langle e_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}$ such that $\inf_{\xi\in I}a_\xi=0$ and $\sup_{\xi\in I}a_\xi=1$ for every infinite $I\subseteq\kappa$, then $\tau(\mathfrak{A}_d)\ge\kappa$ for every non-zero $d\in\mathfrak{A}$.

(b) Let ν_κ be the usual measure on $\{0,1\}^\kappa$ (254J) and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ its measure algebra. If there is an order-continuous Boolean homomorphism from $\mathfrak{B}_\kappa$ to $\mathfrak{A}$, $\tau(\mathfrak{A}_d)\ge\kappa$ for every non-zero $d\in\mathfrak{A}$.

proof (a)(i) To begin with (down to the end of (iii)), let us take it that $d=1$. For $a\in\mathfrak{A}$, $\delta>0$ set $U(a,\delta)=\{a':\bar\mu(a'\triangle a)<\delta\}$, the ordinary open δ-neighbourhood of a. If $a\in\mathfrak{A}$, then there is a $\delta>0$ such that $\{\xi:\xi<\kappa,\ a_\xi\in U(a,\delta)\}$ is finite. **P?** Suppose, if possible, otherwise. Then there is a sequence $\langle\xi_n\rangle_{n\in\mathbb{N}}$ of distinct elements of κ such that $\bar\mu(a\triangle a_{\xi_n})\le 2^{-n-2}\bar\mu 1$ for every $n\in\mathbb{N}$. Now $\inf_{n\in\mathbb{N}}a_{\xi_n}=0$, so

$$\bar\mu a=\bar\mu(a\setminus\inf_{n\in\mathbb{N}}a_{\xi_n})\le\textstyle\sum_{n=0}^\infty\bar\mu(a\setminus a_{\xi_n}).$$

Similarly

$$\bar\mu(1\setminus a)=\bar\mu(\sup_{n\in\mathbb{N}}a_{\xi_n}\setminus a)\le\textstyle\sum_{n=0}^\infty\bar\mu(a_{\xi_n}\setminus a).$$

Putting these together,

$$\begin{aligned}\bar\mu 1=\bar\mu a+\bar\mu(1\setminus a)&\le\sum_{n=0}^\infty\bar\mu(a\setminus a_{\xi_n})+\sum_{n=0}^\infty\bar\mu(a_{\xi_n}\setminus a)\\&=\sum_{n=0}^\infty\bar\mu(a\triangle a_{\xi_n})\le\sum_{n=0}^\infty 2^{-n-2}\bar\mu 1<\bar\mu 1,\end{aligned}$$

which is impossible. **XQ**

(ii) Note that $\mathfrak{A}$ is infinite; for if $a\in\mathfrak{A}$ the set $\{\xi:a_\xi=a\}$ must be finite, and κ is supposed to be infinite. So $\tau(\mathfrak{A})$ must be infinite.

(iii) Now take a set $C\subseteq\mathfrak{A}$, of cardinal $\tau(\mathfrak{A})$, which τ-generates $\mathfrak{A}$. By (ii), C is infinite. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by C; then $\#(\mathfrak{C})=\#(C)=\tau(\mathfrak{A})$, by 331Gc, and $\mathfrak{C}$ is topologically dense in $\mathfrak{A}$ (323J again). If $a\in\mathfrak{A}$, there are $c\in\mathfrak{C}$ and $k\in\mathbb{N}$ such that $a\in U(c,2^{-k})$ and $\{\xi:a_\xi\in U(c,2^{-k})\}$ is finite. **P** By (b), there is a $\delta>0$ such that $\{\xi:a_\xi\in U(a,\delta)\}$ is finite. Take $k\in\mathbb{N}$ such that $2\cdot 2^{-k}\le\delta$, and $c\in\mathfrak{C}\cap U(a,2^{-k})$; then $U(c,2^{-k})\subseteq U(a,\delta)$ can contain only finitely many a_ξ, so these c, k serve. **Q**

Consider

$$\mathcal{U}=\{U(c,2^{-k}):c\in\mathfrak{C},\ k\in\mathbb{N},\ \{\xi:a_\xi\in U(c,2^{-k})\}\text{ is finite}\}.$$

Then $\#(\mathcal{U})\le\max(\#(\mathfrak{C}),\omega)=\tau(\mathfrak{A})$. Also $\mathcal{U}$ is a cover of $\mathfrak{A}$. In particular, $\kappa=\bigcup_{U\in\mathcal{U}}J_U$, where $J_U=\{\xi:a_\xi\in U\}$. But this means that

$$\kappa=\#(\kappa)\le\max(\omega,\#(\mathcal{U}),\sup_{U\in\mathcal{U}}\#(J_U))=\tau(\mathfrak{A}).$$

This proves the result when $d=1$.

(iv) For the general case, given $d\in\mathfrak{A}\setminus\{0\}$, set $a'_\xi=a_\xi\cap d$ for each ξ. Since $\inf_{\xi\in I}a_\xi\cap d=0$ and $\sup_{\xi\in I}a_\xi\cap d=d$ for every infinite $I\subseteq\kappa$, we can apply (i)-(iii) to $(\mathfrak{A}_d,\bar\mu\restriction\mathfrak{A}_d,\langle a'_\xi\rangle_{\xi<\kappa})$ to see that $\tau(\mathfrak{A}_d)\ge\kappa$, as required.

(b) Let $\pi:\mathfrak{B}_\kappa\to\mathfrak{A}$ be an order-continuous Boolean homomorphism. Set $E_\xi=\{x:x\in\{0,1\}^\kappa,\ x(\xi)=1\}$, $e_\xi=E_\xi^\bullet\in\mathfrak{B}_\kappa$ and $a_\xi=\pi e_\xi\in\mathfrak{A}$ for each $\xi<\kappa$. If $\langle\xi_n\rangle_{n\in\mathbb{N}}$ is any sequence of distinct elements of κ,

$$\nu_\kappa(\textstyle\bigcap_{n\in\mathbb{N}}E_{\xi_n})=\lim_{n\to\infty}\nu_\kappa(\bigcap_{i\le n}E_{\xi_n})=\lim_{n\to\infty}2^{-n-1}=0,$$

so that $\bar\nu_\kappa(\inf_{n\in\mathbb{N}}e_{\xi_n})=0$ and $\inf_{n\in\mathbb{N}}e_{\xi_n}=0$. Because π is order-continuous, $\inf_{n\in\mathbb{N}}a_{\xi_n}=0$ in $\mathfrak{A}$. Similarly, $\nu_\kappa(\bigcup_{n\in\mathbb{N}}E_{\xi_n})=1$ and $\sup_{n\in\mathbb{N}}a_{\xi_n}=1$. As $\langle\xi_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\inf_{\xi\in I}a_\xi=0$ and $\sup_{\xi\in I}a_\xi=1$ for every infinite $I\subseteq\kappa$. So we can apply (a) to get the result.

331K Theorem Let κ be any infinite cardinal. Let ν_κ be the usual measure on $\{0,1\}^\kappa$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ its measure algebra. Then $\mathfrak{B}_\kappa$ is Maharam-type-homogeneous, with Maharam type κ.

proof Set $X=\{0,1\}^\kappa$ and write Σ for the domain of ν_κ.

(a) To see that $\tau(\mathfrak{B}_\kappa) \le \kappa$, set $E_\xi = \{x : x \in X,\ x(\xi) = 1\}$ and $e_\xi = E_\xi^\bullet$ for each $\xi < \kappa$. Writing $\mathcal{E}$ for the algebra of subsets of X generated by $\{E_\xi : \xi < \kappa\}$, we see that every measurable cylinder in X, as defined in 254Aa, belongs to $\mathcal{E}$, so that every member of Σ is approximated, in measure, by members of $\mathcal{E}$ (254Fe), that is, $\{E^\bullet : E \in \mathcal{E}\}$ is topologically dense in $\mathfrak{A}$. But this means just that the subalgebra $\mathfrak{C}$ of $\mathfrak{B}_\kappa$ generated by $\{e_\xi : \xi < \kappa\}$ is topologically dense in $\mathfrak{B}_\kappa$, so that $\{e_\xi : \xi < \kappa\}$ τ-generates $\mathfrak{B}_\kappa$, and $\tau(\mathfrak{B}_\kappa) \le \kappa$.

(b) Next, if $c \in \mathfrak{B}_\kappa \setminus \{0\}$ and $(\mathfrak{B}_\kappa)_c$ is the principal ideal of $\mathfrak{B}_\kappa$ generated by c, the map $b \mapsto b \cap c$ is an order-continuous Boolean homomorphism from $\mathfrak{B}_\kappa$ to $(\mathfrak{B}_\kappa)_c$, so by 331Jb we must have $\tau((\mathfrak{B}_\kappa)_c) \ge \kappa$. Thus

$$\kappa \le \tau((\mathfrak{B}_\kappa)_c) \le \tau(\mathfrak{B}_\kappa) \le \kappa.$$

As c is arbitrary, $\mathfrak{B}_\kappa$ is Maharam-type-homogeneous of Maharam type κ.

331L Theorem Let $(\mathfrak{A}, \bar\mu)$ be a Maharam-type-homogeneous probability algebra. Then there is exactly one κ, either 0 or an infinite cardinal, such that $(\mathfrak{A}, \bar\mu)$ is isomorphic, as measure algebra, to the measure algebra $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ of the usual measure on $\{0,1\}^\kappa$.

proof If $\tau(\mathfrak{A})$ is finite, it is zero, and $\mathfrak{A} = \{0,1\}$ (331Ha, 331Hd) so that (interpreting $\{0,1\}^0$ as $\{\emptyset\}$) we have the case $\kappa = 0$. If $\kappa = \tau(\mathfrak{A})$ is infinite, then by 331K we know that $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ also is Maharam-type-homogeneous with Maharam type κ, so 331I gives the required isomorphism. Of course κ is uniquely defined by $\mathfrak{A}$.

331M Homogeneous Boolean algebras Of course a homogeneous Boolean algebra (definition: 316N) must be Maharam-type-homogeneous, since $\tau(\mathfrak{A}) = \tau(\mathfrak{A}_c)$ whenever $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_c$. In general, a Boolean algebra can be Maharam-type-homogeneous without being homogeneous (331Xj, 331Yg). But for σ-finite measure algebras this doesn't happen.

331N Proposition Let $(\mathfrak{A}, \bar\mu)$ be a Maharam-type-homogeneous σ-finite measure algebra. Then it is homogeneous as a Boolean algebra.

proof If $\mathfrak{A} = \{0\}$ this is trivial; so suppose that $\mathfrak{A} \ne \{0\}$. By 322G, there is a measure $\bar\nu$ on $\mathfrak{A}$ such that $(\mathfrak{A}, \bar\nu)$ is a probability algebra. Now let c be any non-zero member of $\mathfrak{A}$, and set $\gamma = \bar\nu c$, $\bar\nu_c' = \gamma^{-1}\bar\nu_c$, where $\bar\nu_c$ is the restriction of $\bar\nu$ to the principal ideal $\mathfrak{A}_c$ of $\mathfrak{A}$ generated by c. Then $(\mathfrak{A}, \bar\nu)$ and $(\mathfrak{A}_c, \bar\nu_c')$ are Maharam-type-homogeneous probability algebras of the same Maharam type, so are isomorphic as measure algebras, and *a fortiori* as Boolean algebras.

331O I will wait until Chapter 52 of Volume 5 for a systematic discussion of properties of measure algebras which depend on their Maharam types. There is however a result which is easy, useful and expressible in terms already introduced.

Proposition Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra with countable Maharam type. Then $\mathfrak{A}$ is separable in its measure-algebra topology.

proof Let $B \subseteq \mathfrak{A}$ be a countable set which τ-generates $\mathfrak{A}$. Then the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by B is countable (331Gc). Now $\mathfrak{B}$ is dense for the measure-algebra topology. **P** Let G be a non-empty open subset of $\mathfrak{A}$, and c any element of G. Let $\mathrm{P} = \{\rho_a : a \in \mathfrak{A}^f\}$ be the upwards-directed family of pseudometrics defining the topology of $\mathfrak{A}$, as described in 323A. Then there must be an $a \in \mathfrak{A}^f$ and an $\epsilon > 0$ such that $\{b : \rho_a(b, c) \le \epsilon\} \subseteq G$. Let $\mathfrak{C}$ be the order-closed subalgebra of the principal ideal $\mathfrak{A}_a$ generated by $\mathfrak{B}_a = \{b \cap a : b \in \mathfrak{B}\}$. Because $b \mapsto b \cap a : \mathfrak{A} \to \mathfrak{A}_a$ is an order-continuous Boolean homomorphism, $\{b : b \in \mathfrak{A},\ b \cap a \in \mathfrak{C}\}$ is an order-closed subalgebra of $\mathfrak{A}$, and must be the whole of $\mathfrak{A}$, because it includes B. So $\mathfrak{C} = \mathfrak{A}_a$. By 323J, $\mathfrak{C}$ is the topological closure of $\mathfrak{B}_a$ in $\mathfrak{A}_a$, and there must be a $b \in \mathfrak{B}_a$ such that $\bar\mu(b \mathbin{\triangle} (c \cap a)) \le \epsilon$; that is, there is a $b \in \mathfrak{B}$ such that $\bar\mu(a \cap (b \mathbin{\triangle} c)) \le \epsilon$ and $b \in G$. Thus $\mathfrak{B}$ meets G; as G is arbitrary, $\mathfrak{B}$ is dense. **Q**

So $\mathfrak{B}$ is a countable dense subset of $\mathfrak{A}$ and $\mathfrak{A}$ is separable.

331X Basic exercises (a) Let (X, Σ, μ) be a probability space and T a σ-subalgebra of Σ such that for any non-negligible $E \in \Sigma$ there is an $F \in \Sigma$ such that $F \subseteq E$ and $\mu(F \triangle (E \cap H)) > 0$ for every $H \in \mathrm{T}$. Suppose that $f : X \to [0,1]$ is a measurable function. Show that there is an $F \in \Sigma$ such that $\int_H f = \mu(H \cap F)$ for every $H \in \mathrm{T}$.

>(b) Write out a direct proof of 331C not relying on 331B or 321J.

(c) Let $\mathfrak{A}$ be a finite Boolean algebra with n atoms. Show that $\tau(\mathfrak{A})$ is the least k such that $n \le 2^k$.

>(d) Show that the measure algebra of Lebesgue measure on $\mathbb{R}$ is Maharam-type-homogeneous with Maharam type ω. (*Hint*: show that it is τ-generated by $\{]-\infty, q]^\bullet : q \in \mathbb{Q}\}$.)

(e) Show that the measure algebra of Lebesgue measure on $\mathbb{R}^r$ is Maharam-type-homogeneous with Maharam type ω, for any $r \ge 1$. (*Hint*: show that it is τ-generated by $\{]-\infty, q]^\bullet : q \in \mathbb{Q}^r\}$.)

(f) Show that the measure algebra of any Radon measure on $\mathbb{R}^r$ (256Ad) has countable Maharam type. (*Hint*: show that it is τ-generated by $\{]-\infty, q]^\bullet : q \in \mathbb{Q}^r\}$.)

>(g) Show that $\mathcal{P}\mathbb{R}$ has Maharam type ω. (*Hint*: show that it is τ-generated by $\{]-\infty, q] : q \in \mathbb{Q}\}$.)

>(h) Show that the regular open algebra of $\mathbb{R}$ is Maharam-type-homogeneous with Maharam type ω. (*Hint*: show that it is τ-generated by $\{]-\infty, q]^\bullet : q \in \mathbb{Q}\}$.)

(i) Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, and κ an infinite cardinal. Suppose that there is a family $\langle a_\xi \rangle_{\xi<\kappa}$ in $\mathfrak{A}$ such that $\inf_{\xi \in I} a_\xi = 0$, $\sup_{\xi \in I} a_\xi = 1$ for every infinite $I \subseteq \kappa$. Show that $\tau(\mathfrak{A}_a) \ge \kappa$ for every non-zero principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$.

(j) Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure on $\mathbb{R}$, and $\mathfrak{G}$ the regular open algebra of $\mathbb{R}$. Show that the simple product $\mathfrak{A} \times \mathfrak{G}$ is Maharam-type-homogeneous of Maharam type ω, but is not homogeneous. (*Hint*: $\mathfrak{A}$ is weakly (σ, ∞)-distributive, but $\mathfrak{G}$ is not, so they are not isomorphic.)

(k) Show that a homogeneous semi-finite measure algebra is σ-finite.

(l) Let (X, Σ, μ) be a measure space, and A a subset of X which has a measurable envelope. Show that the Maharam type of the subspace measure on A is less than or equal to the Maharam type of μ.

(m) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ an order-dense subalgebra of $\mathfrak{A}$. Show that $\tau(\mathfrak{A}) \le \tau(\mathfrak{B})$.

(n) Let μ be a semi-finite measure, and $\tilde\mu$ the c.l.d. version of μ. Show that the Maharam type of $\tilde\mu$ is at most the Maharam type of μ. (*Hint*: 322Db.)

(o) Let $\langle \mu_i \rangle_{i \in I}$ be a non-empty countable family of σ-finite measures all with the same domain; let μ be the sum measure $\sum_{i \in I} \mu_i$. Writing $\tau(\mu)$, $\tau(\mu_i)$ for the Maharam types of the measures, show that $\sup_{i \in I} \tau(\mu_i) \le \tau(\mu) \le \max(\omega, \sup_{i \in I} \tau(\mu_i))$.

331Y Further exercises (a) Suppose that $\mathfrak{A}$ is a Dedekind complete Boolean algebra, $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ and $\mathfrak{C}$ is an order-closed subalgebra of $\mathfrak{B}$. Show that if $a \in \mathfrak{A}$ is a relative atom in $\mathfrak{A}$ over $\mathfrak{C}$, then $\mathrm{upr}(a, \mathfrak{B})$ is a relative atom in $\mathfrak{B}$ over $\mathfrak{C}$. So if $\mathfrak{B}$ is relatively atomless over $\mathfrak{C}$, then $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$.

(b) Give an example of a Boolean algebra $\mathfrak{A}$ with a subalgebra $\mathfrak{A}'$ and a proper subalgebra $\mathfrak{B}$ of $\mathfrak{A}'$ which is order-closed in $\mathfrak{A}'$, but τ-generates $\mathfrak{A}$. (*Hint*: take $\mathfrak{A}$ to be the measure algebra $\mathfrak{A}_L$ of Lebesgue measure on $\mathbb{R}$ and $\mathfrak{B}$ the subalgebra $\mathfrak{B}_{\mathbb{Q}}$ of $\mathfrak{A}$ generated by $\{[a, b]^\bullet : a, b \in \mathbb{Q}\}$. Take $E \subseteq \mathbb{R}$ such that $I \cap E$, $I \setminus E$ have non-zero measure for every non-trivial interval $I \subseteq \mathbb{R}$ (134Jb), and let $\mathfrak{A}'$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{E^\bullet\}$.)

(c) Give an example of a Boolean algebra $\mathfrak{A}$ with a subalgebra $\mathfrak{A}'$ and a proper subalgebra $\mathfrak{B}$ of $\mathfrak{A}'$ which is order-closed in $\mathfrak{A}$, but τ-generates $\mathfrak{A}'$. (*Hint*: in the notation of 331Yb, take Z to be the Stone space of $\mathfrak{A}_L$, and set $\mathfrak{A}' = \{\widehat{a} : a \in \mathfrak{A}_L\}$, $\mathfrak{B} = \{\widehat{a} : a \in \mathfrak{B}_{\mathbb{Q}}\}$; let $\mathfrak{A}$ be the subalgebra of $\mathcal{P}Z$ generated by $\mathfrak{A}' \cup \{\{z\} : z \in Z\}$.)

(d) Let $\mathfrak{A}$ be a Dedekind complete purely atomic Boolean algebra, and A the set of its atoms. Show that $\tau(\mathfrak{A})$ is the least cardinal κ such that $\#(A) \le 2^\kappa$.

(e) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Write $d(\mathfrak{A})$ for the smallest cardinal of any subset of $\mathfrak{A}$ which is dense for the measure-algebra topology. Show that $d(\mathfrak{A}) \le \max(\omega, \tau(\mathfrak{A}))$. Show that if $(\mathfrak{A}, \bar\mu)$ is semi-finite, then $\tau(\mathfrak{A}) \le d(\mathfrak{A})$.

(f) Let (X, ρ) be a metric space. Write $d(X)$ for the **density** of X, the smallest cardinal of any dense subset of X. (i) Show that if $\mathcal{G}$ is any family of open subsets of X, there is a family $\mathcal{H} \subseteq \mathcal{G}$ such that $\bigcup \mathcal{H} = \bigcup \mathcal{G}$ and $\#(\mathcal{H}) \le \max(\omega, d(X))$. (ii) Show that if $\kappa > \max(\omega, d(X))$ and $\langle x_\xi \rangle_{\xi<\kappa}$ is any family in X, then there is an $x \in X$ such that $\#(\{\xi : x_\xi \in G\}) > \max(\omega, d(X))$ for every open set G containing x, and that there is a strictly increasing sequence $\langle \xi_n \rangle_{n \in \mathbb{N}}$ in κ such that $x = \lim_{n\to\infty} x_{\xi_n}$.

(g) Let $(\mathfrak{A},\bar{\mu})$ be the simple product (322L) of ω_1 copies of the measure algebra of the usual measure on $\{0,1\}^{\omega_1}$. Show that $\mathfrak{A}$ is Maharam-type-homogeneous but not homogeneous.

(h) Let κ be an infinite cardinal, ν_κ the usual measure on $\{0,1\}^\kappa$ and $(\mathfrak{B}_\kappa,\bar{\nu}_\kappa)$ its measure algebra. Suppose that $(\mathfrak{A},\bar{\mu})$ is a totally finite measure algebra and such that $\tau(\mathfrak{A})<\kappa$, and $\pi:\mathfrak{B}_\kappa\to\mathfrak{A}$ a Boolean homomorphism. Show that (i) for every $\epsilon>0$ there is a $b\in\mathfrak{B}_\kappa$ such that $\bar{\nu}_\kappa b\ge 1-\epsilon$ and $\bar{\mu}(\pi b)\le\epsilon$ (ii) π is not injective.

(i) Give an example of a semi-finite measure space (X,Σ,μ) such that the Maharam type of μ is greater than the Maharam type of its c.l.d. version.

(j) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra which is separable when given its measure-algebra topology. Show that it has countable Maharam type.

331 Notes and comments Maharam's theorem belongs with the Radon-Nikodým theorem, Fubini's theorem and the strong law of large numbers as one of the theorems which make measure theory what it is. Once you have this theorem and its consequences in the next section properly absorbed, you will never again look at a measure space without trying to classify its measure algebra in terms of the Maharam types of its homogeneous principal ideals. As one might expect, a very large proportion of the important measure algebras of analysis are homogeneous, and indeed a great many are homogeneous with Maharam type ω.

In this section I have contented myself with the basic statement of Theorem 331I on the isomorphism of Maharam-type-homogeneous measure algebras and the identification of representative homogeneous probability algebras (331K). The same techniques lead to an enormous number of further facts, some of which I will describe in the rest of the chapter. For the moment, it gives us a complete description of Maharam-type-homogeneous probability algebras (331L). There is the atomic algebra $\{0,1\}$, with Maharam type 0, and for each infinite cardinal κ there is the measure algebra of $\{0,1\}^\kappa$, with Maharam type κ; these are all non-isomorphic, and every Maharam-type-homogeneous probability algebra is isomorphic to exactly one of them. The isomorphisms here are not unique; indeed, it is characteristic of measure algebras that they have very large automorphism groups (see Chapter 38 below), and there are correspondingly large numbers of isomorphisms between any isomorphic pair. The proof of 331I already suggests this, since we have such a vast amount of choice concerning the lists $\langle a_\xi\rangle_{\xi<\kappa}$ and $\langle b_\xi\rangle_{\xi<\kappa}$, and even with these fixed there remains a good deal of scope in the choice of $\langle a'_\xi\rangle_{\xi<\kappa}$ and $\langle b'_\xi\rangle_{\xi<\kappa}$.

The isomorphisms described in Theorem 331I are measure algebra isomorphisms, that is, measure-preserving Boolean isomorphisms. Obvious questions arise concerning Boolean isomorphisms which are not necessarily measure-preserving; the theorem also helps us to settle many of these (see 331N). But we can observe straight away the remarkable fact that two homogeneous probability algebras which are isomorphic as Boolean algebras are also isomorphic as probability algebras, since they must have the same Maharam type.

I have already mentioned certain measure space isomorphisms (254K, 255A). Of course any isomorphism between measure spaces must induce an isomorphism between their measure algebras (see 324M), and any isomorphism between measure algebras corresponds to an isomorphism between their Stone spaces (see 324N). But there are many important examples of isomorphisms between measure algebras which do not correspond to isomorphisms between the measure spaces most naturally involved. (I describe one in 343J.) Maharam's theorem really is a theorem about measure *algebras* rather than measure *spaces*.

The particular method I use to show that the measure algebra of the usual measure on $\{0,1\}^\kappa$ is homogeneous for infinite κ (331J-331K) is chosen with a view to a question in the next section (332O). There are other ways of doing it. But I recommend study of this particular one because of the way in which it involves the topological, algebraic and order properties of the algebra $\mathfrak{B}$. I have extracted some of the elements of the argument in 331Xi and 331Ye-331Yf. These use the concept of 'density' of a topological space. This does not seem the moment to go farther along this road, but I hope you can see that there are likely to be many further 'cardinal functions' to provide useful measures of complexity in both algebraic and topological structures.

332 Classification of localizable measure algebras

In this section I present what I call 'Maharam's theorem', that every localizable measure algebra is expressible as a weighted simple product of measure algebras of spaces of the form $\{0,1\}^\kappa$ (332B). Among its many consequences is a complete description of the isomorphism classes of localizable measure algebras (332J). This description needs the concepts of 'cellularity' of a Boolean algebra (332D) and its refinement, the 'magnitude' of a measure algebra (332G). I end this section with a discussion of those pairs of measure algebras for which there is a measure-preserving homomorphism from one to the other (332P-332Q), and a general formula for the Maharam type of a localizable measure algebra (332S).

332A Lemma Let $\mathfrak{A}$ be any Boolean algebra. Writing $\mathfrak{A}_a$ for the principal ideal generated by $a \in \mathfrak{A}$, the set $\{a : a \in \mathfrak{A},\, \mathfrak{A}_a \text{ is Maharam-type-homogeneous}\}$ is order-dense in $\mathfrak{A}$.

proof Take any $a \in \mathfrak{A} \setminus \{0\}$. Then $A = \{\tau(\mathfrak{A}_b) : 0 \neq b \subseteq a\}$ has a least member; take $c \subseteq a$ such that $c \neq 0$ and $\tau(\mathfrak{A}_c) = \min A$. If $0 \neq b \subseteq c$, then $\tau(\mathfrak{A}_b) \leq \tau(\mathfrak{A}_c)$, by 331Hc, while $\tau(\mathfrak{A}_b) \in A$, so $\tau(\mathfrak{A}_c) \leq \tau(\mathfrak{A}_b)$. Thus $\tau(\mathfrak{A}_b) = \tau(\mathfrak{A}_c)$ for every non-zero $b \subseteq c$, and $\mathfrak{A}_c$ is Maharam-type-homogeneous.

332B Maharam's theorem Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra. Then it is isomorphic to the simple product of a family $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$ of measure algebras, where for each $i \in I$ $(\mathfrak{A}_i, \bar\mu_i)$ is isomorphic, up to a re-normalization of the measure, to the measure algebra of the usual measure on $\{0,1\}^{\kappa_i}$, where κ_i is either 0 or an infinite cardinal.

proof (a) For $a \in \mathfrak{A}$, let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Then

$$D = \{a : a \in \mathfrak{A},\, 0 < \bar\mu a < \infty,\, \mathfrak{A}_a \text{ is Maharam-type-homogeneous}\}$$

is order-dense in $\mathfrak{A}$. **P** If $a \in \mathfrak{A} \setminus \{0\}$, then (because $(\mathfrak{A}, \bar\mu)$ is semi-finite) there is a $b \subseteq a$ such that $0 < \bar\mu b < \infty$; now by 332A there is a non-zero $d \subseteq b$ such that $\mathfrak{A}_d$ is Maharam-type-homogeneous. **Q**

(b) By 313K, there is a partition of unity $\langle e_i\rangle_{i\in I}$ consisting of members of D; by 322L(d-i), $(\mathfrak{A}, \bar\mu)$ is isomorphic, as measure algebra, to the simple product of the principal ideals $\mathfrak{A}_i = \mathfrak{A}_{e_i}$.

(c) For each $i \in I$, $(\mathfrak{A}_i, \bar\mu_i)$ is a non-trivial totally finite Maharam-type-homogeneous measure algebra, writing $\bar\mu_i = \bar\mu\restriction\mathfrak{A}_i$. Take $\gamma_i = \bar\mu_i(1_{\mathfrak{A}_i}) = \bar\mu e_i$, and set $\bar\mu_i' = \gamma_i^{-1}\bar\mu_i$. Then $(\mathfrak{A}_i, \bar\mu_i')$ is a Maharam-type-homogeneous probability algebra, so by 331L is isomorphic to the measure algebra $(\mathfrak{B}_{\kappa_i}, \bar\nu_{\kappa_i})$ of the usual measure on $\{0,1\}^{\kappa_i}$, where κ_i is either 0 or an infinite cardinal. Thus $(\mathfrak{A}_i, \bar\mu_i)$ is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_{\kappa_i}, \bar\nu_{\kappa_i})$.

Remark For the case of totally finite measure algebras, this is Theorem 2 of Maharam 42.

332C Corollary Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra. For any cardinal κ, write ν_κ for the usual measure on $\{0,1\}^\kappa$, and T_κ for its domain. Then we can find families $\langle\kappa_i\rangle_{i\in I}$, $\langle\gamma_i\rangle_{i\in I}$ such that every κ_i is either 0 or an infinite cardinal, every γ_i is a strictly positive real number, and $(\mathfrak{A}, \bar\mu)$ is isomorphic to the measure algebra of (X, Σ, ν), where

$$X = \{(x, i) : i \in I,\, x \in \{0,1\}^{\kappa_i}\},$$

$$\Sigma = \{E : E \subseteq X,\, \{x : (x, i) \in E\} \in \mathrm{T}_{\kappa_i} \text{ for every } i \in I\},$$

$$\nu E = \sum_{i\in I} \gamma_i \nu_{\kappa_i}\{x : (x, i) \in E\}$$

for every $E \in \Sigma$.

proof Take the family $\langle\kappa_i\rangle_{i\in I}$ from the last theorem, take the $\gamma_i = \bar\mu e_i$ to be the normalizing factors of the proof there, and apply 322Lb to identify the simple product of the measure algebras of $(\{0,1\}^{\kappa_i}, \mathrm{T}_{\kappa_i}, \gamma_i\nu_{\kappa_i})$ with the measure algebra of their direct sum (X, Σ, ν).

332D The cellularity of a Boolean algebra In order to properly describe non-sigma-finite measure algebras, we need the following concept. If $\mathfrak{A}$ is any Boolean algebra, write

$$c(\mathfrak{A}) = \sup\{\#(C) : C \subseteq \mathfrak{A} \setminus \{0\} \text{ is disjoint}\},$$

the **cellularity** of $\mathfrak{A}$. (If $\mathfrak{A} = \{0\}$, take $c(\mathfrak{A}) = 0$.) Thus $\mathfrak{A}$ is ccc (316A) iff $c(\mathfrak{A}) \leq \omega$.

332E Proposition Let $(\mathfrak{A},\bar{\mu})$ be any semi-finite measure algebra, and C any partition of unity in $\mathfrak{A}$ consisting of elements of finite measure. Then $\max(\omega,\#(C))=\max(\omega,c(\mathfrak{A}))$.

proof Of course $\#(C\setminus\{0\})\le c(\mathfrak{A})$, because $C\setminus\{0\}$ is disjoint, so

$$\max(\omega,\#(C))=\max(\omega,\#(C\setminus\{0\})\le\max(\omega,c(\mathfrak{A})).$$

Now suppose that D is any disjoint set in $\mathfrak{A}\setminus\{0\}$. For $c\in C$, $\{d\cap c:d\in D\}$ is a disjoint set in the principal ideal $\mathfrak{A}_c$ generated by c. But $\mathfrak{A}_c$ is ccc (322G), so $\{d\cap c:d\in D\}$ must be countable, and $D_c=\{d:d\in D,\ d\cap c\ne 0\}$ is countable. Because $\sup C=1$, $D=\bigcup_{c\in C}D_c$, so

$$\#(D)\le\max(\omega,\#(C),\sup_{c\in C}\#(D_c))=\max(\omega,\#(C)).$$

As D is arbitrary, $c(\mathfrak{A})\le\max(\omega,\#(C))$ and $\max(\omega,c(\mathfrak{A}))=\max(\omega,\#(C))$.

332F Corollary Let $(\mathfrak{A},\bar{\mu})$ be any semi-finite measure algebra. Then there is a disjoint set in $\mathfrak{A}\setminus\{0\}$ of cardinal $c(\mathfrak{A})$.

proof Start by taking any partition of unity C consisting of non-zero elements of finite measure. If $\#(C)=c(\mathfrak{A})$ we can stop, because C is a disjoint set in $\mathfrak{A}\setminus\{0\}$. Otherwise, by 332E, we must have C finite and $c(\mathfrak{A})=\omega$. Let A be the set of atoms in $\mathfrak{A}$. If A is infinite, it is a disjoint set of cardinal ω, so we can stop. Otherwise, since there is certainly a disjoint set $D\subseteq\mathfrak{A}\setminus\{0\}$ of cardinal greater than $\#(A)$, and since each member of A can meet at most one member of D, there must be a member d of D which does not include any atom. Accordingly we can choose inductively a sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ such that $d_0=d$, $0\ne d_{n+1}\subset d_n$ for every n. Now $\{d_n\setminus d_{n+1}:n\in\mathbb{N}\}$ is a disjoint set in $\mathfrak{A}\setminus\{0\}$ of cardinal $\omega=c(\mathfrak{A})$.

332G Definitions For the next theorem, it will be convenient to have some special terminology.

(a) The first word I wish to introduce is a variant of the idea of 'cellularity', adapted to measure algebras. If $(\mathfrak{A},\bar{\mu})$ is a semi-finite measure algebra, let us say that the **magnitude** of an $a\in\mathfrak{A}$ is $\bar{\mu}a$ if $\bar{\mu}a$ is finite, and otherwise is the cellularity of the principal ideal $\mathfrak{A}_a$ generated by a. (This is necessarily infinite, since any partition of a into sets of finite measure must be infinite.) If we take it that any real number is less than any infinite cardinal, then the class of possible magnitudes is totally ordered.

I shall sometimes speak of the **magnitude** of the measure algebra $(\mathfrak{A},\bar{\mu})$ itself, meaning the magnitude of $1_{\mathfrak{A}}$. Similarly, if (X,Σ,μ) is a semi-finite measure space, the **magnitude** of (X,Σ,μ), or of μ, is the magnitude of its measure algebra.

(b) Next, for any Dedekind complete Boolean algebra $\mathfrak{A}$, and any cardinal κ, we can look at the element

$$e_\kappa=\sup\{a:a\in\mathfrak{A}\setminus\{0\},\ \mathfrak{A}_a\text{ is Maharam-type-homogeneous with Maharam type }\kappa\},$$

writing $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a, as usual. I will call this the **Maharam-type-κ component** of $\mathfrak{A}$. Of course $e_\kappa\cap e_\lambda=0$ whenever λ, κ are distinct cardinals. **P** $a\cap b=0$ whenever $\mathfrak{A}_a$, $\mathfrak{A}_b$ are Maharam-type-homogeneous of different Maharam types, since $\tau(\mathfrak{A}_{a\cap b})$ cannot be equal simultaneously to $\tau(\mathfrak{A}_a)$ and $\tau(\mathfrak{A}_b)$. **Q**

Also $\{e_\kappa:\kappa\text{ is a cardinal}\}$ is a partition of unity in $\mathfrak{A}$, because

$$\sup\{e_\kappa:\kappa\text{ is a cardinal}\}=\sup\{a:\mathfrak{A}_a\text{ is Maharam-type-homogeneous}\}=1$$

by 332A. Note that there is no claim that $\mathfrak{A}_{e_\kappa}$ itself is homogeneous; but we do have a useful result in this direction.

332H Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and κ an infinite cardinal. Let e be the Maharam-type-κ component of $\mathfrak{A}$. If $0\ne d\subseteq e$ and the principal ideal $\mathfrak{A}_d$ generated by d is ccc, then it is Maharam-type-homogeneous with Maharam type κ.

proof (a) The point is that $\tau(\mathfrak{A}_d)\le\kappa$. **P** Set

$$A=\{a:a\in\mathfrak{A}\setminus\{0\},\ \mathfrak{A}_a\text{ is Maharam-type-homogeneous of Maharam type }\kappa\}.$$

Then $d=\sup\{a\cap d:a\in A\}$. Because $\mathfrak{A}_d$ is ccc, there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that $d=\sup_{n\in\mathbb{N}}d\cap a_n$ (316E); set $b_n=d\cap a_n$. We have $\tau(\mathfrak{A}_{b_n})\le\tau(\mathfrak{A}_{a_n})=\kappa$ for each n; let D_n be a subset of $\mathfrak{A}_{b_n}$, of cardinal at most κ, which τ-generates $\mathfrak{A}_{b_n}$. Set

$$D=\bigcup_{n\in\mathbb{N}}D_n\cup\{b_n:n\in\mathbb{N}\}\subseteq\mathfrak{A}_d.$$

If $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}_d$ generated by D, then $\mathfrak{C}\cap\mathfrak{A}_{b_n}$ is an order-closed subalgebra of $\mathfrak{A}_{b_n}$ including D_n, so is equal to $\mathfrak{A}_{b_n}$, for every n. But $a = \sup_{n\in\mathbb{N}} a\cap b_n$ for every $a \in \mathfrak{A}_d$, so $\mathfrak{C} = \mathfrak{A}_d$. Thus D τ-generates $\mathfrak{A}_d$, and

$$\tau(\mathfrak{A}_d) \le \#(D) \le \max(\omega, \sup_{n\in\mathbb{N}} \#(D_n)) = \kappa. \mathbf{Q}$$

(b) If now b is any non-zero member of $\mathfrak{A}_d$, there is some $a \in A$ such that $b \cap a \neq 0$, so that

$$\kappa = \tau(\mathfrak{A}_{b\cap a}) \le \tau(\mathfrak{A}_b) \le \tau(\mathfrak{A}_d) \le \kappa.$$

Thus we must have $\tau(\mathfrak{A}_b) = \kappa$ for every non-zero $b \in \mathfrak{A}_d$, and $\mathfrak{A}_d$ is Maharam-type-homogeneous with type κ, as claimed.

332I Lemma Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra which is not totally finite. Then it has a partition of unity consisting of elements of measure 1.

proof Let A be the set $\{a : \bar\mu a = 1\}$, and $\mathcal{C}$ the family of disjoint subsets of A. By Zorn's lemma, $\mathcal{C}$ has a maximal member C_0 (compare the proof of 313K). Set $D = \{d : d \in \mathfrak{A},\ d\cap c = 0$ for every $c \in C_0\}$. Then D is upwards-directed. If $d \in D$, then $\bar\mu a \neq 1$ for every $a \subseteq d$, so $\bar\mu d < 1$, by 331C. So $d_0 = \sup D$ is defined in $\mathfrak{A}$ (321C); of course $d_0 \in D$, so $\bar\mu d_0 < 1$. Observe that $\sup C_0 = 1 \setminus d_0$.

Because $\bar\mu 1 = \infty$, C_0 must be infinite; let $\langle a_n\rangle_{n\in\mathbb{N}}$ be any sequence of distinct elements of C_0. For each $n \in \mathbb{N}$, use 331C again to choose an $a'_n \subseteq a_n$ such that $\bar\mu a'_n = \bar\mu d_0$. Set

$$b_0 = d_0 \cup (a_0 \setminus a'_0), \quad b_n = a'_{n-1} \cup (a_n \setminus a'_n)$$

for every $n \ge 1$. Then $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of elements of measure 1 and $\sup_{n\in\mathbb{N}} b_n = \sup_{n\in\mathbb{N}} a_n \cup d_0$. Now

$$(C_0 \setminus \{a_n : n \in \mathbb{N}\}) \cup \{b_n : n \in \mathbb{N}\}$$

is a partition of unity consisting of elements of measure 1.

332J Now I can formulate a complete classification theorem for localizable measure algebras, refining the expression in 332B.

Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras. For each cardinal κ, let e_κ, f_κ be the Maharam-type-κ components of $\mathfrak{A}$, $\mathfrak{B}$ respectively. Then $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are isomorphic, as measure algebras, iff (i) e_κ and f_κ have the same magnitude for every infinite cardinal κ (ii) for every $\gamma \in \,]0,\infty[$, $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ have the same number of atoms of measure γ.

proof Throughout the proof, write $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a, and $\bar\mu_a$ for the restriction of $\bar\mu$ to $\mathfrak{A}_a$; and define $\mathfrak{B}_b$, $\bar\nu_b$ similarly for $b \in \mathfrak{B}$.

(a) If $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are isomorphic, then of course the isomorphism matches their Maharam-type components together and retains their magnitudes, and matches atoms of the same measure together; so the conditions are surely satisfied.

(b) Now suppose that the conditions are satisfied. Set

$$K = \{\kappa : \kappa \text{ is an infinite cardinal}, e_\kappa \neq 0\} = \{\kappa : \kappa \text{ is an infinite cardinal}, f_\kappa \neq 0\}.$$

For $\gamma \in \,]0,\infty[$, let A_γ be the set of atoms of measure γ in $\mathfrak{A}$, and set $e_\gamma = \sup A_\gamma$. Write $I = K \cup \,]0,\infty[$. Then $\langle e_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, so $(\mathfrak{A},\bar\mu)$ is isomorphic to the simple product of $\langle(\mathfrak{A}_{e_i},\bar\mu_{e_i})\rangle_{i\in I}$, writing $\mathfrak{A}_{e_i}$ for the principal ideal generated by e_i and $\bar\mu_{e_i}$ for the restriction $\bar\mu\restriction\mathfrak{A}_{e_i}$.

In the same way, writing B_γ for the set of atoms of measure γ in $\mathfrak{B}$, f_γ for $\sup B_\gamma$, $\mathfrak{B}_{f_i}$ for the principal ideal generated by f_i and $\bar\nu_{f_i}$ for the restriction of $\bar\nu$ fo $\mathfrak{B}_{f_i}$, we have $(\mathfrak{B},\bar\nu)$ isomorphic to the simple product of $\langle(\mathfrak{B}_{f_i},\bar\nu_{f_i})\rangle_{i\in I}$.

(c) It will therefore be enough if I can show that $(\mathfrak{A}_{e_i},\bar\mu_{e_i}) \cong (\mathfrak{B}_{f_i},\bar\nu_{f_i})$ for every $i \in I$.

(i) For $\kappa \in K$, the hypothesis is that e_κ and f_κ have the same magnitude. If they are both of finite magnitude, that is, $\bar\mu e_\kappa = \bar\nu f_\kappa < \infty$, then both $(\mathfrak{A}_{e_\kappa},\bar\mu_{e_\kappa})$ and $(\mathfrak{B}_{f_\kappa},\bar\nu_{f_\kappa})$ are homogeneous and of Maharam type κ, by 332H. So 331I tells us that they are isomorphic. If they are both of infinite magnitude λ, then 332I tells us that both $\mathfrak{A}_{e_\kappa}$, $\mathfrak{B}_{f_\kappa}$ have partitions of unity C, D consisting of sets of measure 1. So $(\mathfrak{A}_{e_\kappa},\bar\mu_{e_\kappa})$ is isomorphic to the simple product of $\langle(\mathfrak{A}_c,\bar\mu_c)\rangle_{c\in C}$, while $(\mathfrak{B}_{f_\kappa},\bar\nu_{f_\kappa})$ is isomorphic to the simple product of $\langle(\mathfrak{B}_d,\bar\nu_d)\rangle_{d\in D}$. But we know also that every $(\mathfrak{A}_c,\bar\mu_c)$, $(\mathfrak{B}_d,\bar\nu_d)$ is a homogeneous probability algebra with Maharam type κ, by 332H again, so by Maharam's theorem again they are all isomorphic. Since C, D and λ are all infinite,

$$\#(C) = c(\mathfrak{A}_{e_\kappa}) = \lambda = c(\mathfrak{B}_{f_\kappa}) = \#(D)$$

by 332E. So we are taking the same number of factors in each product and $(\mathfrak{A}_{e_\kappa}, \bar\mu_{e_\kappa})$ must be isomorphic to $(\mathfrak{B}_{f_\kappa}, \bar\nu_{f_\kappa})$.

(ii) For $\gamma \in]0,\infty[$, our hypothesis is that $\#(A_\gamma) = \#(B_\gamma)$. Now A_γ is a partition of unity in $\mathfrak{A}_{e_\gamma}$, so $(\mathfrak{A}_{e_\gamma}, \bar\mu_{e_\gamma})$ is isomorphic to the simple product of $\langle(\mathfrak{A}_a, \bar\mu_a)\rangle_{a\in A_\gamma}$. Similarly, $(\mathfrak{B}_{f_\gamma}, \bar\nu_{f_\gamma})$ is isomorphic to the simple product of $\langle(\mathfrak{B}_b, \bar\nu_b)\rangle_{b\in B_\gamma}$. Since every $(\mathfrak{A}_a, \bar\mu_a)$, $(\mathfrak{B}_b, \bar\nu_b)$ is just a simple atom of measure γ, these are all isomorphic; since we are taking the same number of factors in each product, $(\mathfrak{A}_{e_\gamma}, \bar\mu_{e_\gamma})$ must be isomorphic to $(\mathfrak{B}_{f_\gamma}, \bar\nu_{f_\gamma})$.

(iii) Thus we have the full set of required isomorphisms, and $(\mathfrak{A}, \bar\mu)$ is isomorphic to $(\mathfrak{B}, \bar\nu)$.

332K Remarks (a) The partition of unity $\{e_i : i \in I\}$ of $\mathfrak{A}$ used in the above theorem is in some sense canonical. (You might feel it more economical to replace I by $K \cup \{\gamma : A_\gamma \neq \emptyset\}$.) The further partition of the atomic part into individual atoms (part (c-ii) of the proof) is also canonical. But of course the partition of the e_κ of infinite magnitude into elements of measure 1 requires a degree of arbitrary choice.

The value of the expressions in 332C is that the parameters κ_i, γ_i there are sufficient to identify the measure algebra up to isomorphism. For, amalgamating the language of 332C and 332J, we see that the magnitude of e_κ in 332J is just $\sum_{\kappa_i=\kappa} \gamma_i$ if this is finite, $\#(\{i : \kappa_i = \kappa\})$ otherwise (using 332E, as usual); while the number of atoms of measure γ is $\#(\{i : \kappa_i = 0,\ \gamma_i = \gamma\})$.

(b) The classification which Maharam's theorem gives us is not merely a listing. It involves a real insight into the nature of the algebras, enabling us to answer a very wide variety of natural questions. I give the next couple of results as a sample of what we can expect these methods to do for us.

332L Proposition Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and a, $b \in \mathfrak{A}$ two elements of finite measure. Suppose that $\pi : \mathfrak{A}_a \to \mathfrak{A}_b$ is a measure-preserving isomorphism, where $\mathfrak{A}_a$, $\mathfrak{A}_b$ are the principal ideals generated by a and b. Then there is a measure-preserving automorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ which extends π.

proof The point is that $\mathfrak{A}_{b\setminus a}$ is isomorphic, as measure algebra, to $\mathfrak{A}_{a\setminus b}$. **P** Set $c = a \cup b$. For each infinite cardinal κ, let e_κ be the Maharam-type-κ component of $\mathfrak{A}_c$. Then $e_\kappa \cap a$ is the Maharam-type-κ component of $\mathfrak{A}_a$, because if $d \subseteq c$ and $\mathfrak{A}_d$ is Maharam homogeneous with Maharam type κ, then $\mathfrak{A}_{d\cap a}$ is either $\{0\}$ or again Maharam-type-homogeneous with Maharam type κ. Similarly, $e_\kappa \setminus a$ is the Maharam-type-κ component of $\mathfrak{A}_{c\setminus a} = \mathfrak{A}_{b\setminus a}$, $e_\kappa \cap b$ is the Maharam-type-κ component of $\mathfrak{A}_b$ and $e_\kappa \setminus b$ is the Maharam-type-κ component of $\mathfrak{A}_{a\setminus b}$. Now $\pi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism, so $\pi(e_\kappa \cap a)$ must be $e_\kappa \cap b$, and

$$\begin{aligned}\bar\mu(e_\kappa \setminus a) &= \bar\mu e_\kappa - \bar\mu(e_\kappa \cap a) = \bar\mu e_\kappa - \bar\mu\pi(e_\kappa \cap a)\\ &= \bar\mu e_\kappa - \bar\mu(e_\kappa \cap b) = \bar\mu(e_\kappa \setminus b).\end{aligned}$$

In the same way, if we write $n_\gamma(d)$ for the number of atoms of measure γ in $\mathfrak{A}_d$, then

$$n_\gamma(b \setminus a) = n_\gamma(c) - n_\gamma(a) = n_\gamma(c) - n_\gamma(b) = n_\gamma(a \setminus b)$$

for every $\gamma \in]0,\infty[$. By 332J, there is a measure-preserving isomorphism $\pi_1 : \mathfrak{A}_{b\setminus a} \to \mathfrak{A}_{a\setminus b}$. **Q**

If we now set

$$\phi d = \pi(d \cap a) \cup \pi_1(d \cap b \setminus a) \cup (d \setminus c)$$

for every $d \in \mathfrak{A}$, $\phi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving isomorphism which agrees with π on $\mathfrak{A}_a$.

332M Lemma Suppose that $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ are homogeneous measure algebras, with $\tau(\mathfrak{A}) \leq \tau(\mathfrak{B})$ and $\bar\mu 1 = \bar\nu 1 < \infty$. Then there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

proof The case $\tau(\mathfrak{A}) = 0$ is trivial. Otherwise, considering normalized versions of the measures, we are reduced to the case $\bar\mu 1 = \bar\nu 1 = 1$, $\tau(\mathfrak{A}) = \kappa \geq \omega$, $\tau(\mathfrak{B}) = \lambda \geq \kappa$, so that $(\mathfrak{A}, \bar\mu)$ is isomorphic to the measure algebra $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ of the usual measure $\bar\nu_\kappa$ on $\{0,1\}^\kappa$; and similarly $(\mathfrak{B}, \bar\nu)$ is isomorphic to the measure algebra $(\mathfrak{B}_\lambda, \bar\nu_\lambda)$ of the usual measure on $\{0,1\}^\lambda$. Now (identifying the cardinals κ, λ with von Neumann ordinals, as usual), $\kappa \subseteq \lambda$, so we have an inverse-measure-preserving map $x \mapsto x{\restriction}\kappa : \{0,1\}^\lambda \to \{0,1\}^\kappa$ (254Oa), which induces a measure-preserving Boolean homomorphism from $\mathfrak{B}_\kappa$ to $\mathfrak{B}_\lambda$ (324M), and hence a measure-preserving homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

332N Lemma If $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\kappa\ge\max(\omega,\tau(\mathfrak{A}))$, then there is a measure-preserving Boolean homomorphism from $(\mathfrak{A},\bar\mu)$ to the measure algebra $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ of the usual measure ν on $\{0,1\}^\kappa$; that is, $(\mathfrak{A},\bar\mu)$ is isomorphic to a closed subalgebra of $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$.

proof Let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ such that every principal ideal $\mathfrak{A}_{c_i}$ is homogeneous and no c_i is zero. Then I is countable and $\sum_{i\in I}\bar\mu c_i=1$. Accordingly there is a partition of unity $\langle d_i\rangle_{i\in I}$ in $\mathfrak{B}_\kappa$ such that $\bar\nu d_i=\bar\mu c_i$ for every i. **P** Because I is countable, we may suppose that it is either $\mathbb{N}$ or an initial segment of $\mathbb{N}$. In this case, choose $\langle d_i\rangle_{i\in I}$ inductively such that $d_i\subseteq 1\setminus\sup_{j<i}d_j$ and $\bar\nu d_i=\bar\mu d_i$ for each $i\in I$, using 331C. **Q**

If $i\in I$, then $\tau(\mathfrak{A}_{c_i})\le\kappa=\tau((\mathfrak{B}_\kappa)_{d_i})$, so there is a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_{c_i}\to(\mathfrak{B}_\kappa)_{d_i}$. Setting $\pi a=\sup_{i\in I}\pi_i(a\cap c_i)$ for $a\in\mathfrak{A}$, we have a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}_\kappa$. By 324Kb, $\pi[\mathfrak{A}]$ is a closed subalgebra of $\mathfrak{B}_\kappa$, and of course $(\pi[\mathfrak{A}],\bar\nu_\kappa\restriction\pi[\mathfrak{A}])$ is isomorphic to $(\mathfrak{A},\bar\mu)$.

332O Lemma Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras. For each infinite cardinal κ let e_κ, f_κ be their Maharam-type-κ components, and for $\gamma\in\,]0,\infty[$ let e_γ, f_γ be the suprema of the atoms of measure γ in $\mathfrak{A}$, $\mathfrak{B}$ respectively. If there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then the magnitude of $\sup_{\kappa\ge\lambda}e_\kappa$ is not greater than the magnitude of $\sup_{\kappa\ge\lambda}f_\kappa$ whenever λ is an infinite cardinal, while the magnitude of $\sup_{\kappa\ge\omega}e_\kappa\cup\sup_{\gamma\le\delta}e_\gamma$ is not greater than the magnitude of $\sup_{\kappa\ge\omega}f_\kappa\cup\sup_{\gamma\le\delta}f_\gamma$ for any $\delta\in\,]0,\infty[$.

proof Suppose that $\pi:\mathfrak{A}\to\mathfrak{B}$ is a measure-preserving Boolean homomorphism. For infinite cardinals λ, set $e^*_\lambda=\sup_{\kappa\ge\lambda}e_\kappa$, $f^*_\lambda=\sup_{\kappa\ge\lambda}f_\kappa$, while for $\delta\in\,]0,\infty[$ set $e^*_\delta=\sup_{\kappa\ge\omega}e_\kappa\cup\sup_{\gamma\le\delta}e_\gamma$, $f^*_\delta=\sup_{\kappa\ge\omega}f_\kappa\cup\sup_{\gamma\le\delta}f_\gamma$. Let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ such that all the principal ideals $\mathfrak{A}_{c_i}$ are totally finite and homogeneous, as in 332B. Then $c_i\subseteq e_\kappa$ whenever $\kappa=\tau(\mathfrak{A}_{c_i})$ is infinite, and $c_i\subseteq e_\gamma$ if c_i is an atom of measure γ. Take v to be either an infinite cardinal or a strictly positive real number. Set

$$J=\{i:i\in I,\,c_i\subseteq e^*_v\};$$

then $e^*_v=\sup_{i\in J}c_i$.

Now the point is that if $i\in J$ then $\pi c_i\subseteq f^*_v$. **P** We need to consider two cases. (i) If c_i is an atom, then $v\in\,]0,\infty[$ and $\bar\mu c_i\le v$. So we need only observe that $1\setminus f^*_v$ is just the supremum in $\mathfrak{B}$ of the atoms of measure greater than v, none of which can meet πc_i, since this has measure at most v. (ii) Now suppose that $\mathfrak{A}_{c_i}$ is atomless, with $\tau(\mathfrak{A}_{c_i})=\kappa\ge v$. If $0\ne b\subseteq\pi c_i$, then $a\mapsto b\cap\pi a:\mathfrak{A}_{c_i}\to\mathfrak{B}_b$ is an order-continuous Boolean homomorphism, while $\mathfrak{A}_{c_i}$ is isomorphic (as Boolean algebra) to the measure algebra of $\{0,1\}^\kappa$, so 331J tells us that $\tau(\mathfrak{B}_b)\ge\kappa$. This means, first, that b cannot be an atom, so that πc_i cannot meet $\sup_{\gamma\in]0,\infty[}f_\gamma$; and also that b cannot be included in $f_{\kappa'}$ for any infinite $\kappa'<\kappa$, so that πc_i cannot meet $\sup_{\omega\le\kappa'<\kappa}f_\kappa$. Thus πc_i must be included in $\sup_{\kappa'\ge\kappa}f_\kappa\subseteq f^*_v$. **Q**

Of course $\langle\pi c_i\rangle_{i\in J}$ is disjoint. So if e^*_v has finite magnitude, the magnitude of f^*_v is at least

$$\sum_{i\in J}\bar\nu\pi c_i=\sum_{i\in J}\bar\mu c_i=\bar\mu e^*_v,$$

the magnitude of e^*_v. While if e^*_v has infinite magnitude, this is $\#(J)$, by 332E, which is not greater than the magnitude of f^*_v.

332P Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be atomless totally finite measure algebras. For each infinite cardinal κ let e_κ, f_κ be their Maharam-type-κ components. Then the following are equiveridical:

(i) $(\mathfrak{A},\bar\mu)$ is isomorphic to a closed subalgebra of a principal ideal of $(\mathfrak{B},\bar\nu)$;

(ii) for every cardinal λ,

$$\bar\mu(\sup_{\kappa\ge\lambda}e_\kappa)\le\bar\nu(\sup_{\kappa\ge\lambda}f_\kappa).$$

proof (a)(i)⇒(ii) Suppose that $\pi:\mathfrak{A}\to\mathfrak{B}_d$ is a measure-preserving isomorphism between $\mathfrak{A}$ and a closed subalgebra of a principal ideal $\mathfrak{B}_d$ of $\mathfrak{B}$. The Maharam-type-κ component of $\mathfrak{B}_d$ is just $d\cap f_\kappa$, so 332O tells us that

$$\bar\mu(\sup_{\kappa\ge\lambda}e_\kappa)\le\bar\nu(\sup_{\kappa\ge\lambda}d\cap f_\kappa)\le\bar\nu(\sup_{\kappa\ge\lambda}f_\kappa)$$

for every λ.

(b)(ii)⇒(i) Now suppose that the condition is satisfied.

(α) Let P be the set of all measure-preserving Boolean homomorphisms π from principal ideals $\mathfrak{A}_{c_\pi}$ of $\mathfrak{A}$ to principal ideals $\mathfrak{B}_{d_\pi}$ of $\mathfrak{B}$ such that

$$\bar\mu(\sup_{\kappa\ge\lambda}e_\kappa\setminus c_\pi)\le\bar\nu(\sup_{\kappa\ge\lambda}\bar\nu f_\kappa\setminus d_\pi)$$

for every cardinal $\lambda \ge \omega$. Then the trivial homomorphism from $\mathfrak{A}_0$ to $\mathfrak{B}_0$ belongs to P, so P is not empty. Order P by saying that $\pi \le \pi'$ if π' extends π, that is, if $c_\pi \subseteq c_{\pi'}$ and $\pi' a = \pi a$ for every $a \in \mathfrak{A}_{c_\pi}$. Then P is a partially ordered set.

(β) If $Q \subseteq P$ is non-empty and totally ordered, it is bounded above in P. **P** Set $c^* = \sup_{\pi\in Q} c_\pi$, $d^* = \sup_{\pi\in Q} d_\pi$. For $a \subseteq c^*$ set $\pi^* a = \sup_{\pi\in Q}\pi(a\cap c_\pi)$. Because Q is totally ordered, π^* extends all the functions in Q. It is also easy to check that $\pi^*0 = 0$, $\pi^*(a\cap a') = \pi^*a\cap\pi^*a'$ and $\pi^*(a\cup a') = \pi^*a\cup\pi^*a'$ for all a, $a' \in \mathfrak{A}_{c^*}$, $\pi^*c^* = d^*$ and that $\bar\nu\pi^*a = \bar\mu a$ for every $a \in \mathfrak{A}_{c^*}$; so that π^* is a measure-preserving Boolean homomorphism from $\mathfrak{A}_{c^*}$ to $\mathfrak{B}_{d^*}$.

Now suppose that λ is any cardinal; then

$$\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa \setminus c^*) = \inf_{\pi\in Q}\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa\setminus c_\pi) \le \inf_{\pi\in Q}\bar\nu(\sup_{\kappa\ge\lambda} f_\kappa\setminus d_\pi) = \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa\setminus d^*).$$

So $\pi^* \in P$ and is the required upper bound of Q. **Q**

(γ) By Zorn's Lemma, P has a maximal element $\tilde\pi$ say. Now $c_{\tilde\pi} = 1$. **P?** If not, then let κ_0 be the least cardinal such that $e_{\kappa_0}\setminus c_{\tilde\pi} \ne 0$. Then

$$0 < \bar\mu(\sup_{\kappa\ge\kappa_0} e_\kappa\setminus c_{\tilde\pi}) \le \bar\nu(\sup_{\kappa\ge\kappa_0}\bar\nu f_\kappa\setminus d_{\tilde\pi}),$$

so there is a least $\kappa_1 \ge \kappa_0$ such that $f_{\kappa_1}\setminus d_{\tilde\pi} \ne 0$. Set $\delta = \min(\bar\mu(e_{\kappa_0}\setminus c_{\tilde\pi}), \bar\nu(f_{\kappa_1}\setminus d_{\tilde\pi})) > 0$. Because $\mathfrak{A}$ and $\mathfrak{B}$ are atomless, there are $a \subseteq e_{\kappa_0}\setminus c_{\tilde\pi}$ and $b \subseteq f_{\kappa_1}\setminus d_{\tilde\pi}$ such that $\bar\mu a = \bar\nu b = \delta$ (331C). Now $\mathfrak{A}_a$ is homogeneous with Maharam type κ_0, while $\mathfrak{B}_b$ is homogeneous with Maharam type κ_1 (332H), so there is a measure-preserving Boolean homomorphism $\phi : \mathfrak{A}_a \to \mathfrak{B}_b$ (332M). Set

$$c^* = c_{\tilde\pi}\cup a, \quad d^* = d_{\tilde\pi}\cup b,$$

and define $\pi^* : \mathfrak{A}_{c^*} \to \mathfrak{B}_{d^*}$ by setting $\pi^*(g) = \tilde\pi(g\cap c_{\tilde\pi})\cup\phi(g\cap a)$ for every $g \subseteq c^*$. It is easy to check that π^* is a measure-preserving Boolean homomorphism.

If λ is a cardinal and $\lambda \le \kappa_0$,

$$\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa\setminus c^*) = \bar\mu(\sup_{\kappa\ge\lambda} e_\kappa\setminus c_{\tilde\pi}) - \delta \le \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa\setminus d_{\tilde\pi}) - \delta = \bar\nu(\sup_{\kappa\ge\lambda}\bar\nu f_\kappa\setminus d^*).$$

If $\kappa_0 < \lambda \le \kappa_1$,

$$\begin{aligned}\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa\setminus c^*) &= \bar\mu(\sup_{\kappa\ge\lambda} e_\kappa\setminus c_{\tilde\pi}) \le \bar\mu(\sup_{\kappa\ge\kappa_0} e_\kappa\setminus c_{\tilde\pi}) - \bar\mu(e_{\kappa_0}\setminus c_{\tilde\pi})\\ &\le \bar\mu(\sup_{\kappa\ge\kappa_0} e_\kappa\setminus c_{\tilde\pi}) - \delta \le \bar\nu(\sup_{\kappa\ge\kappa_0} f_\kappa\setminus d_{\tilde\pi}) - \delta\\ &= \bar\nu(\sup_{\kappa\ge\kappa_1} f_\kappa\setminus d_{\tilde\pi}) - \delta\end{aligned}$$

(by the choice of κ_1)

$$= \bar\nu(\sup_{\kappa\ge\kappa_1} f_\kappa\setminus d^*) \le \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa\setminus d^*).$$

If $\lambda > \kappa_1$,

$$\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa\setminus c^*) = \bar\mu(\sup_{\kappa\ge\lambda} e_\kappa\setminus c_{\tilde\pi}) \le \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa\setminus d_{\tilde\pi}) = \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa\setminus d^*).$$

But this means that $\pi^* \in P$, and evidently it is a proper extension of $\tilde\pi$, which is supposed to be impossible. **XQ**

(δ) Thus $\tilde\pi$ has domain $\mathfrak{A}$ and is the required measure-preserving homomorphism from $\mathfrak{A}$ to the principal ideal $\mathfrak{B}_{d_{\tilde\pi}}$ of $\mathfrak{B}$.

332Q Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be totally finite measure algebras, and suppose that there are measure-preserving Boolean homomorphisms $\pi_1 : \mathfrak{A}\to\mathfrak{B}$ and $\pi_2 : \mathfrak{B}\to\mathfrak{A}$. Then $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are isomorphic.

proof Writing e_κ, f_κ for their Maharam-type-κ components, 332O (applied to both π_1 and π_2) tells us that

$$\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa) = \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa)$$

for every λ. Because all these measures are finite,

$$\begin{aligned}\bar\mu e_\lambda &= \bar\mu(\sup_{\kappa\ge\lambda} e_\kappa) - \bar\mu(\sup_{\kappa>\lambda} e_\kappa)\\ &= \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa) - \bar\nu(\sup_{\kappa>\lambda} f_\kappa) = \bar\nu f_\lambda\end{aligned}$$

for every λ.

Similarly, writing e_γ, f_γ for the suprema in $\mathfrak{A}$, $\mathfrak{B}$ of the atoms of measure γ, 332O tells us that

$$\bar\mu(\sup_{\gamma\le\delta} e_\gamma) = \bar\nu(\sup_{\gamma\le\delta} f_\gamma)$$

for every $\delta \in \,]0,\infty[$, and hence that $\bar\mu e_\gamma = \bar\nu f_\gamma$ for every γ, that is, that $\mathfrak{A}$ and $\mathfrak{B}$ have the same number of atoms of measure γ.

So $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are isomorphic, by 332J.

332R 332J tells us that if we know the magnitudes of the Maharam-type-κ components of a localizable measure algebra, we shall have specified the algebra completely, so that all its properties are determined. The calculation of its Maharam type is straightforward and useful, so I give the details.

Lemma Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Then $c(\mathfrak{A}) \le 2^{\tau(\mathfrak{A})}$.

proof Let $C \subseteq \mathfrak{A}\setminus\{0\}$ be a disjoint set, and $B \subseteq \mathfrak{A}$ a τ-generating set of size $\tau(\mathfrak{A})$.

(a) If $\mathfrak{A}$ is purely atomic, then for each $c \in C$ choose an atom $c' \subseteq c$, and set $f(c) = \{b : b \in B,\, c' \subseteq b\}$. If c_1, c_2 are distinct members of C, the set

$$\{a : a \in \mathfrak{A},\, c_1' \subseteq a \iff c_2' \subseteq a\}$$

is an order-closed subalgebra of $\mathfrak{A}$ not containing either c_1' or c_2', so cannot include B, and $f(c_1) \ne f(c_2)$. Thus f is injective, and

$$\#(C) \le \#(\mathcal{P}B) = 2^{\tau(\mathfrak{A})}.$$

(b) Now suppose that $\mathfrak{A}$ is not purely atomic; in this case $\tau(\mathfrak{A})$ is infinite. For each $c \in C$ choose an element $c' \subseteq c$ of non-zero finite measure. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by B. Then the topological closure of $\mathfrak{B}$ is $\mathfrak{A}$ itself (323J), and $\#(\mathfrak{B}) = \tau(\mathfrak{A})$ (331Gc). For $c \in C$ set

$$f(c) = \{b : b \in \mathfrak{B},\, \bar\mu(b \cap c') \ge \tfrac{1}{2}\bar\mu c'\}.$$

Then $f : C \to \mathcal{P}\mathfrak{B}$ is injective. **P** If c_1, c_2 are distinct members of C, then (because $\mathfrak{B}$ is topologically dense in $\mathfrak{A}$) there is a $b \in \mathfrak{B}$ such that

$$\bar\mu((c_1' \cup c_2') \cap (c_1' \triangle b)) \le \frac{1}{3}\min(\bar\mu c_1', \bar\mu c_2').$$

But in this case

$$\bar\mu(c_1' \setminus b) \le \frac{1}{3}\bar\mu c_1', \quad \bar\mu(c_2' \cap b) \le \frac{1}{3}\bar\mu c_2',$$

and $b \in f(c_1) \triangle f(c_2)$, so $f(c_1) \ne f(c_2)$. **Q** Accordingly $\#(C) \le 2^{\#(\mathfrak{B})} = 2^{\tau(\mathfrak{A})}$ in this case also.

As C is arbitrary, $c(\mathfrak{A}) \le 2^{\tau(\mathfrak{A})}$.

332S Theorem Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Then $\tau(\mathfrak{A})$ is the least cardinal λ such that (α) $c(\mathfrak{A}) \le 2^\lambda$ (β) $\tau(\mathfrak{A}_a) \le \lambda$ for every Maharam-type-homogeneous principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$.

proof Fix λ as the least cardinal satisfying (α) and (β).

(a) By 331Hc, $\tau(\mathfrak{A}_a) \le \tau(\mathfrak{A})$ for every $a \in \mathfrak{A}$, while $c(\mathfrak{A}) \le 2^{\tau(\mathfrak{A})}$ by 332R; so $\lambda \le \tau(\mathfrak{A})$.

(b) Let C be a partition of unity in $\mathfrak{A}$ consisting of elements of non-zero finite measure generating Maharam-type-homogeneous principal ideals (as in the proof of 332B); then $\#(C) \le c(\mathfrak{A}) \le 2^\lambda$, and there is an injective function $f : C \to \mathcal{P}\lambda$. For each $c \in C$, let $B_c \subseteq \mathfrak{A}_c$ be a τ-generating set of cardinal $\tau(\mathfrak{A}_c)$, and $f_c : B_c \to \lambda$ an injection. Set

$$b_\xi = \sup\{c : c \in C,\, \xi \in f(c)\},$$

$$b_\xi' = \sup\{b : \text{there is some } c \in C \text{ such that } b \in B_c \text{ and } f_c(b) = \xi\}$$

for $\xi < \lambda$. Set $B = \{b_\xi : \xi < \lambda\} \cup \{b'_\xi : \xi < \lambda\}$ if λ is infinite, $\{b_\xi : \xi < \lambda\}$ if λ is finite; then $\#(B) \le \lambda$. Note that if $c \in C$ and $b \in B_c$ there is a $b' \in B$ such that $b = b' \cap c$. **P** Since $B_c \ne \emptyset$, $\tau(\mathfrak{A}_c) > 0$; but this means that $\tau(\mathfrak{A}_c)$ is infinite (see 331H) so λ is infinite and $b'_\xi \in B$, where $\xi = f_c(b)$; now $b = b'_\xi \cap c$. **Q**

Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by B. Then $C \subseteq \mathfrak{B}$. **P** For $c \in C$, we surely have $c \subseteq b_\xi$ if $\xi \in f(c)$; but also, because C is disjoint, $c \cap b_\xi = 0$ if $\xi \in \lambda \setminus f(c)$. Consequently

$$c^* = \inf_{\xi\in f(c)} b_\xi \cap \inf_{\xi\in\lambda\setminus f(c)}(1 \setminus b_\xi)$$

includes c. On the other hand, if d is any other member of C, there is some $\xi \in f(c)\triangle f(d)$, so that

$$d^* \cap c^* \subseteq b_\xi \cap (1 \setminus b_\xi) = 0.$$

Since $\sup C = 1$, it follows that $c = c^*$; but $c^* \in \mathfrak{B}$, so $c \in \mathfrak{B}$. **Q**

For any $c \in C$, look at $\{b \cap c : b \in \mathfrak{B}\} \subseteq \mathfrak{B}$. This is a closed subalgebra of $\mathfrak{A}_c$ (314F(a-i)) including B_c, so must be the whole of $\mathfrak{A}_c$. Thus $\mathfrak{A}_c \subseteq \mathfrak{B}$ for every $c \in C$. But $\sup C = 1$, so $a = \sup_{c\in C} a \cap c \in \mathfrak{B}$ for every $a \in \mathfrak{A}$, and $\mathfrak{A} = \mathfrak{B}$. Consequently $\tau(\mathfrak{A}) \le \#(B) \le \lambda$, and $\tau(\mathfrak{A}) = \lambda$.

332T Proposition Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra and $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$. Then
(a) there is a function $\bar\nu : \mathfrak{B} \to [0,\infty]$ such that $(\mathfrak{B}, \bar\nu)$ is a localizable measure algebra;
(b) $\tau(\mathfrak{B}) \le \tau(\mathfrak{A})$.

proof (a) Let D be the set of those $b \in \mathfrak{B}$ such that the principal ideal $\mathfrak{B}_b$ has Maharam type at most $\tau(\mathfrak{A})$ and is a totally finite measure algebra when endowed with an appropriate measure. Then D is order-dense in $\mathfrak{B}$. **P** Take any non-zero $b_0 \in \mathfrak{B}$. Then there is an $a \in \mathfrak{A}$ such that $a \subseteq b_0$ and $0 < \bar\mu a < \infty$. Set $c = \mathrm{upr}(a, \mathfrak{B}) = \min\{b : b \in \mathfrak{B}, a \subseteq b\}$; then $c \in \mathfrak{B}$ and $a \subseteq c \subseteq b_0$. If $0 \ne b \in \mathfrak{B}_c$, then $c \setminus b$ belongs to $\mathfrak{B}$ and is properly included in c, so cannot include a; accordingly $a \cap b \ne 0$. For $b \in \mathfrak{B}_c$, set $\bar\nu b = \bar\mu(a \cap b)$. Because the map $b \mapsto a \cap b$ is an injective order-continuous Boolean homomorphism, $\bar\nu$ is countably additive and strictly positive, that is, $(\mathfrak{B}_c, \bar\nu)$ is a measure algebra. It is totally finite because $\bar\nu c = \bar\mu a < \infty$.

Let $d \in \mathfrak{B}_c \setminus \{0\}$ be such that $\mathfrak{B}_d$ is Maharam-type-homogeneous; suppose that its Maharam type is κ. The map $b \mapsto b \cap a$ is a measure-preserving Boolean homomorphism from $\mathfrak{B}_d$ to $\mathfrak{A}_{a\cap d}$, so by 332O $\mathfrak{A}_{a\cap d}$ must have a non-zero Maharam-type-κ' component for some $\kappa' \ge \kappa$; but this means that

$$\tau(\mathfrak{B}_d) \le \kappa \le \kappa' \le \tau(\mathfrak{A}_{a\cap d}) \le \tau(\mathfrak{A}).$$

Thus $d \in D$, while $0 \ne d \subseteq c \subseteq b_0$. As b_0 is arbitrary, D is order-dense. **Q**

Accordingly there is a partition of unity C in $\mathfrak{B}$ such that $C \subseteq D$. For each $c \in C$ we have a functional $\bar\nu_c$ such that $(\mathfrak{B}_c, \bar\nu_c)$ is a totally finite measure algebra with Maharam type at most $\tau(\mathfrak{A})$; define $\bar\nu : \mathfrak{B} \to [0,\infty]$ by setting $\bar\nu b = \sum_{c\in C} \bar\nu_c(b \cap c)$ for every $b \in \mathfrak{B}$. It is easy to check that $(\mathfrak{B}, \bar\nu)$ is a measure algebra (compare 322La); it is localizable because $\mathfrak{B}$ (being order-closed in a Dedekind complete partially ordered set) is Dedekind complete.

(b) The construction above ensures that every homogeneous principal ideal of $\mathfrak{B}$ can have Maharam type at most $\tau(\mathfrak{A})$, since it must share a principal ideal with some $\mathfrak{B}_c$ for $c \in C$. Moreover, any disjoint set in $\mathfrak{B}$ is also a disjoint set in $\mathfrak{A}$, so $c(\mathfrak{B}) \le c(\mathfrak{A})$. So 332S tells us that $\tau(\mathfrak{B}) \le \tau(\mathfrak{A})$.

Remark I think the only direct appeal I shall make to this result will be when $(\mathfrak{A}, \bar\mu)$ is a probability algebra, in which case (a) above becomes trivial, and the proof of (b) can be shortened to some extent, though I think we still need some of the ideas of 332S.

332X Basic exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. Show that it is isomorphic to a simple product of Maharam-type-homogeneous Boolean algebras.

(b) Let $\mathfrak{A}$ be a Boolean algebra of finite cellularity. Show that $\mathfrak{A}$ is purely atomic.

(c) Let $\mathfrak{A}$ be a purely atomic Boolean algebra. Show that $c(\mathfrak{A})$ is the number of atoms in $\mathfrak{A}$.

(d) Let $\mathfrak{A}$ be any Boolean algebra, and Z its Stone space. Show that $c(\mathfrak{A})$ is equal to

$$c(Z) = \sup\{\#(\mathcal{G}) : G \text{ is a disjoint family of non-empty open subsets of } Z\},$$

the **cellularity** of the topological space Z.

(e) Let X be a topological space, and $\mathrm{RO}(X)$ its regular open algebra. Show that $c(\mathrm{RO}(X)) = c(X)$ as defined in 332Xd.

(f) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ any subalgebra of $\mathfrak{A}$. Show that $c(\mathfrak{B}) \le c(\mathfrak{A})$.

(g) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, with simple product $\mathfrak{A}$. Show that the cellularity of $\mathfrak{A}$ is at most $\max(\omega, \#(I), \sup_{i\in I} c(\mathfrak{A}_i))$. Devise an elegant expression of a necessary and sufficient condition for equality.

(h) Let $\mathfrak{A}$ be any Boolean algebra, and $a \in \mathfrak{A}$; let $\mathfrak{A}_a$ be the principal ideal generated by a. Show that $c(\mathfrak{A}_a) \le c(\mathfrak{A})$.

(i) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra. Show that it has a partition of unity of cardinal $c(\mathfrak{A})$.

(j) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be localizable measure algebras. For each cardinal κ let e_κ, f_κ be their Maharam-type-κ components, and $\mathfrak{A}_{e_\kappa}$, $\mathfrak{B}_{f_\kappa}$ the corresponding principal ideals. Show that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, as Boolean algebras, iff $c(\mathfrak{A}_{e_\kappa}) = c(\mathfrak{B}_{f_\kappa})$ for every κ.

(k) Let ζ be an ordinal, and $\langle\alpha_\xi\rangle_{\xi<\zeta}$, $\langle\beta_\xi\rangle_{\xi<\zeta}$ two families of non-negative real numbers such that $\sum_{\theta\le\xi<\zeta}\alpha_\xi \le \sum_{\theta\le\eta<\zeta}\beta_\eta < \infty$ for every $\theta \le \zeta$. Show that there is a family $\langle\gamma_{\xi\eta}\rangle_{\xi\le\eta<\zeta}$ of non-negative real numbers such that $\alpha_\xi = \sum_{\xi\le\eta<\zeta}\gamma_{\xi\eta}$ for every $\xi < \zeta$ and $\beta_\eta \ge \sum_{\xi\le\eta}\gamma_{\xi\eta}$ for every $\eta < \zeta$. (If only finitely many of the α_ξ, β_ξ are non-zero, this is an easy special case of the max-flow min-cut theorem; see Bollobás 79, §III.1 or Anderson 87, 12.3.1; there is a statement of the theorem in 4A4N in the next volume.) Show that the $\gamma_{\xi\eta}$ can be chosen in such a way that if $\xi < \xi' \le \eta' < \eta$ then at least one of $\gamma_{\xi\eta}$, $\gamma_{\xi'\eta'}$ is zero.

(l) Use 332Xk and 332M to give another proof of 332P.

(m) For each cardinal κ, write $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$. Let $(\mathfrak{A}, \bar\mu)$ be the simple product of $\langle(\mathfrak{B}_{\omega_n}, \bar\nu_{\omega_n})\rangle_{n\in\mathbb{N}}$ and $(\mathfrak{B}, \bar\nu)$ the simple product of $(\mathfrak{A}, \bar\mu)$ with $(\mathfrak{B}_{\omega_\omega}, \bar\nu_{\omega_\omega})$. (See 3A1E if you are puzzled by the names ω_n, ω_ω.) Show that there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, but that no such homomorphism can be order-continuous.

(n) For each cardinal κ, write $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$. Let $(\mathfrak{A}, \bar\mu)$ be the simple product of $\langle(\mathfrak{B}_{\kappa_n}, \bar\nu_{\kappa_n})\rangle_{n\in\mathbb{N}}$ and $(\mathfrak{B}, \bar\nu)$ the simple product of $\langle(\mathfrak{B}_{\lambda_n}, \bar\nu_{\lambda_n})\rangle_{n\in\mathbb{N}}$, where $\kappa_n = \omega$ for even n, ω_n for odd n, while $\lambda_n = \omega$ for odd n, ω_n for even n. Show that there are order-continuous measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ and from $\mathfrak{B}$ to $\mathfrak{A}$, but that these two measure algebras are not isomorphic.

(o) Let $\mathfrak{C}$ be a Boolean algebra. Show that the following are equiveridical: (i) $\mathfrak{C}$ is isomorphic (as Boolean algebra) to a closed subalgebra of a localizable measure algebra; (ii) there is a $\bar\mu$ such that $(\mathfrak{C}, \bar\mu)$ is itself a localizable measure algebra; (iii) $\mathfrak{C}$ is Dedekind complete and for every non-zero $c \in \mathfrak{C}$ there is a completely additive real-valued functional ν on $\mathfrak{C}$ such that $\nu c \ne 0$. (*Hint for (iii)⇒(ii)*: show that the set of supports of non-negative completely additive functionals is order-dense in $\mathfrak{C}$, so includes a partition of unity.)

332Y Further exercises (a) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be atomless localizable measure algebras. For each infinite cardinal κ let e_κ, f_κ be their Maharam-type-κ components. Show that the following are equiveridical: (i) $(\mathfrak{A}, \bar\mu)$ is isomorphic to a closed subalgebra of a principal ideal of $(\mathfrak{B}, \bar\nu)$; (ii) for every cardinal λ, the magnitude of $\sup_{\kappa\ge\lambda} e_\kappa$ is not greater than the magnitude of $\sup_{\kappa\ge\lambda} f_\kappa$.

(b) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be any semi-finite measure algebras, and $(\widehat{\mathfrak{A}}, \hat\mu)$, $(\widehat{\mathfrak{B}}, \hat\nu)$ their localizations (322P-322Q). Let $\langle e_i\rangle_{i\in I}$, $\langle f_j\rangle_{j\in J}$ be partitions of unity in $\mathfrak{A}$, $\mathfrak{B}$ respectively into elements of finite measure generating homogeneous principal ideals $\mathfrak{A}_{e_i}$, $\mathfrak{B}_{f_j}$. For each infinite cardinal κ set $I_\kappa = \{i : \tau(\mathfrak{A}_{e_i}) = \kappa\}$, $J_\kappa = \{j : \tau(\mathfrak{B}_{f_j}) = \kappa\}$; for $\gamma \in]0,\infty[$, set $I_\gamma = \{i : e_i \text{ is an atom}, \bar\mu e_i = \gamma\}$, $J_\gamma = \{j : f_j \text{ is an atom}, \bar\nu f_j = \gamma\}$. Show that $(\widehat{\mathfrak{A}}, \hat\mu)$ and $(\widehat{\mathfrak{B}}, \hat\nu)$ are isomorphic iff for each u, either $\sum_{i\in I_u}\bar\mu e_i = \sum_{j\in J_u}\bar\nu f_j < \infty$ or $\sum_{i\in I_u}\bar\mu e_i = \sum_{j\in J_u}\bar\nu f_j = \infty$ and $\#(I_u) = \#(J_u)$.

(c) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be non-zero localizable measure algebras; let e_κ, f_κ be their Maharam-type-κ components. Show that the following are equiveridical: (i) $\mathfrak{A}$ is isomorphic, as Boolean algebra, to an order-closed subalgebra of a principal ideal of $\mathfrak{B}$; (ii) $c(\mathfrak{A}^*_\lambda) \le c(\mathfrak{B}^*_\lambda)$ for every cardinal λ, where $\mathfrak{A}^*_\lambda$, $\mathfrak{B}^*_\lambda$ are the principal ideals generated by $\sup_{\kappa\ge\lambda} e_\kappa$ and $\sup_{\kappa\ge\lambda} f_\kappa$ respectively.

332 Notes and comments Maharam's theorem tells us that all localizable measure algebras – in particular, all σ-finite measure algebras – can be obtained from the basic algebra $\mathfrak{A} = \{0, a, 1 \setminus a, 1\}$, with $\bar{\mu}a = \bar{\mu}(1 \setminus a) = \frac{1}{2}$, by combining the constructions of probability algebra free products, scalar multiples of measures and simple products. But what is much more important is the fact that we get a description of our measure algebras in terms sufficiently explicit to make a very wide variety of questions resolvable. The description I offer in 332J hinges on the complementary concepts of 'Maharam type' and 'magnitude'. If you like, the magnitude of a measure algebra is a measure of its width, while its Maharam type is a measure of its depth. The latter is more important just because, for localizable algebras, we have such a simple decomposition into algebras of finite magnitude. Of course there is a good deal of scope for further complications if we seek to consider non-localizable semi-finite algebras. For these, the natural starting point is a proper description of their localizations, which is not difficult (332Yb).

Observe that 332C gives a representation of a localizable measure algebra as the measure algebra of a measure space which is completely different from the Stone representation in 321K. It is less canonical (since there is a degree of choice about the partition $\langle e_i \rangle_{i\in I}$) but very much more informative, since the κ_i, γ_i carry enough information to identify the measure algebra up to isomorphism (332K).

'Cellularity' is the second cardinal function I have introduced in this chapter. It refers as much to topological spaces as to Boolean algebras (see 332Xd-332Xe). There is an interesting question in this context. If $\mathfrak{A}$ is an arbitrary Boolean algebra, is there necessarily a disjoint set in $\mathfrak{A}$ of cardinal $c(\mathfrak{A})$? This is believed to be undecidable from the ordinary axioms of set theory (including the axiom of choice); see the 'Erdős-Tarski theorem' in Volume 5. But for semi-finite measure algebras we have a definite answer (332F).

Maharam's classification not only describes the isomorphism classes of localizable measure algebras, but also tells us when to expect Boolean homomorphisms between them (332P, 332Yc). I have given 332P only for atomless totally finite measure algebras because the non-totally-finite case (332Ya, 332Yc) seems to require a new idea, while atoms introduce combinatorial complications.

I offer 332T as an example of the kind of result which these methods make very simple. It fails for general Boolean algebras; in fact, there is for any κ a countably τ-generated Dedekind complete Boolean algebra $\mathfrak{A}$ with cellularity κ (514Yb in Volume 5, or Koppelberg 89, 13.1), so that $\mathcal{P}\kappa$ is isomorphic to an order-closed subalgebra of $\mathfrak{A}$, and if $\kappa > \mathfrak{c}$ then $\tau(\mathcal{P}\kappa) > \omega$ (332R).

For totally finite measure algebras we have a kind of weak Schröder-Bernstein theorem: if we have two of them, each isomorphic to a closed subalgebra of the other, they are isomorphic (332Q). This fails for σ-finite algebras (332Xn). I call it a 'weak' Schröder-Bernstein theorem because it is not clear how to build the isomorphism from the two injections; 'strong' Schröder-Bernstein theorems include definite recipes for constructing the isomorphisms declared to exist (see, for instance, 344D below).

333 Closed subalgebras

Proposition 332P tells us, in effect, which totally finite measure algebras can be embedded as closed subalgebras of each other. Similar techniques make it possible to describe the possible forms of such embeddings. In this section I give the fundamental theorems on extension of measure-preserving homomorphisms from closed subalgebras (333C, 333D); these rely on the concept of 'relative Maharam type' (333A). I go on to describe possible canonical forms for structures $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$, where $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ (333K, 333N). I end the section with a description of fixed-point subalgebras (333R).

333A Definitions (a) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. The **relative Maharam type of $\mathfrak{A}$ over $\mathfrak{C}$**, $\tau_{\mathfrak{C}}(\mathfrak{A})$, is the smallest cardinal of any set $A \subseteq \mathfrak{A}$ such that $A \cup \mathfrak{C}$ τ-generates $\mathfrak{A}$.

(b) In this section, I will regularly use the following notation: if $\mathfrak{A}$ is a Boolean algebra, $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, and $a \in \mathfrak{A}$, then I will write $\mathfrak{C}_a$ for $\{c \cap a : c \in \mathfrak{C}\}$. Observe that $\mathfrak{C}_a$ is a subalgebra of the principal ideal $\mathfrak{A}_a$ (because $c \mapsto c \cap a : \mathfrak{C} \to \mathfrak{A}_a$ is a Boolean homomorphism); it is included in $\mathfrak{C}$ iff $a \in \mathfrak{C}$.

(c) Still taking $\mathfrak{A}$ to be a Boolean algebra and $\mathfrak{C}$ to be a subalgebra of $\mathfrak{A}$, I will say that an element a of $\mathfrak{A}$ is **relatively Maharam-type-homogeneous over $\mathfrak{C}$** if $\tau_{\mathfrak{C}_b}(\mathfrak{A}_b) = \tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ for every non-zero $b \subseteq a$.

(d) If κ is a cardinal which is either infinite or zero, I will write $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ for the measure algebra of the usual measure ν_κ on $\{0,1\}^\kappa$. I hope that there will be no confusion between this notation and the use, in 333C-333F, of the formula $\mathfrak{B}_b$ for the principal ideal generated by b in an arbitrary Boolean algebra $\mathfrak{B}$.

333B Evidently this is a generalization of the ordinary concept of Maharam type as used in §§331-332; if $\mathfrak{C} = \{0,1\}$ then $\tau_{\mathfrak{C}}(\mathfrak{A}) = \tau(\mathfrak{A})$. The first step is naturally to check the results corresponding to 331H.

Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$.

(a) If $a \subseteq b$ in $\mathfrak{A}$, then $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \le \tau_{\mathfrak{C}_b}(\mathfrak{A}_b)$. In particular, $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \le \tau_{\mathfrak{C}}(\mathfrak{A})$ for every $a \in \mathfrak{A}$.

(b) The set $\{a : a \in \mathfrak{A}$ is relatively Maharam-type-homogeneous over $\mathfrak{C}\}$ is order-dense in $\mathfrak{A}$.

(c) If $\mathfrak{A}$ is Dedekind complete and $\mathfrak{C}$ is order-closed in $\mathfrak{A}$, then $\mathfrak{C}_a$ is order-closed in $\mathfrak{A}_a$.

(d) If $a \in \mathfrak{A}$ is relatively Maharam-type-homogeneous over $\mathfrak{C}$ then either $\mathfrak{A}_a = \mathfrak{C}_a$, so that $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) = 0$ and a is a relative atom of $\mathfrak{A}$ over $\mathfrak{C}$ (definition: 331A), or $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \ge \omega$.

(e) If $\mathfrak{D}$ is another subalgebra of $\mathfrak{A}$ and $\mathfrak{D} \subseteq \mathfrak{C}$, then

$$\tau(\mathfrak{A}_a) = \tau_{\{0,a\}}(\mathfrak{A}_a) \ge \tau_{\mathfrak{D}_a}(\mathfrak{A}_a) \ge \tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \ge \tau_{\mathfrak{A}_a}(\mathfrak{A}_a) = 0$$

for every $a \in \mathfrak{A}$.

proof (a) Let $D \subseteq \mathfrak{A}_b$ be a set of cardinal $\tau_{\mathfrak{C}_b}(\mathfrak{A}_b)$ such that $D \cup \mathfrak{C}_b$ τ-generates $\mathfrak{A}_b$. Set $D' = \{d \cap a : d \in D\}$. Then $D' \cup \mathfrak{C}_a$ τ-generates $\mathfrak{A}_a$. **P** Apply 313Mc to the map $d \mapsto d \cap a : \mathfrak{A}_b \to \mathfrak{A}_a$, as in 331Hc. **Q** Consequently

$$\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \le \#(D') \le \#(D) = \tau_{\mathfrak{C}_b}(\mathfrak{A}_b),$$

as claimed. Setting $b = 1$ we get $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \le \tau_{\mathfrak{C}}(\mathfrak{A})$.

(b) Just as in the proof of 332A, given $b \in \mathfrak{A} \setminus \{0\}$, there is an $a \in \mathfrak{A}_b \setminus \{0\}$ minimising $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$, and this a must be relatively Maharam-type-homogeneous over $\mathfrak{C}$.

(c) $\mathfrak{C}_a$ is the image of the Dedekind complete Boolean algebra $\mathfrak{C}$ under the order-continuous Boolean homomorphism $c \mapsto c \cap a$, so must be order-closed (314F(a-i)).

(d) Suppose that $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ is finite. Let $D \subseteq \mathfrak{A}_a$ be a finite set such that $D \cup \mathfrak{C}_a$ τ-generates $\mathfrak{A}_a$. Then there is a non-zero $b \in \mathfrak{A}_a$ such that $b \cap d$ is either 0 or b for every $d \in D$. But this means that $\mathfrak{C}_b = \{d \cap b : d \in D \cup \mathfrak{C}_a\}$, which τ-generates $\mathfrak{A}_b$; so that $\tau_{\mathfrak{C}_b}(\mathfrak{A}_b) = 0$. Since a is relatively Maharam-type-homogeneous over $\mathfrak{C}$, $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ must be zero, that is, $\mathfrak{A}_a = \mathfrak{C}_a$.

(e) The middle inequality is true just because $\mathfrak{A}_a$ will be τ-generated by $D \cup \mathfrak{C}_a$ whenever it is τ-generated by $D \cup \mathfrak{D}_a$. The neighbouring inequalities are special cases of the middle one, and the outer equalities are elementary.

333C Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras, and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Let $\phi : \mathfrak{C} \to \mathfrak{B}$ be a measure-preserving Boolean homomorphism.

(a) If, in the notation of 333A, $\tau_{\mathfrak{C}}(\mathfrak{A}) \le \tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)$ for every non-zero $b \in \mathfrak{B}$, there is a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ extending ϕ.

(b) If $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) = \tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)$ for every non-zero $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, then there is a measure algebra isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ extending ϕ.

proof In both parts, the idea is to use the technique of the proof of 331I to construct π as the last of an increasing family $\langle \pi_\xi \rangle_{\xi \le \kappa}$ of measure-preserving homomorphisms from closed subalgebras $\mathfrak{C}_\xi$ of $\mathfrak{A}$, where $\kappa = \tau_{\mathfrak{C}}(\mathfrak{A})$. Let $\langle a_\xi \rangle_{\xi < \kappa}$ be a family in $\mathfrak{A}$ such that $\mathfrak{C} \cup \{a_\xi : \xi < \kappa\}$ τ-generates $\mathfrak{A}$. Write $\mathfrak{D}$ for $\phi[\mathfrak{C}]$; remember that $\mathfrak{D}$ is a closed subalgebra of $\mathfrak{B}$ (324L).

(a)(i) In this case, we can describe the $\mathfrak{C}_\xi$ immediately; $\mathfrak{C}_\xi$ will be the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{a_\eta : \eta < \xi\}$. The induction starts with $\mathfrak{C}_0 = \mathfrak{C}$, $\pi_0 = \phi$.

(ii) For the inductive step to a successor ordinal $\xi + 1$, where $\xi < \kappa$, suppose that $\mathfrak{C}_\xi$ and π_ξ have been defined. Take any non-zero $b \in \mathfrak{B}$. We are supposing that $\tau_{\mathfrak{D}_b}(\mathfrak{B}_b) \ge \kappa > \#(\xi)$, so $\mathfrak{B}_b$ cannot be τ-generated by

$$D = \mathfrak{D}_b \cup \{b \cap \pi_\xi a_\eta : \eta < \xi\} = \pi_\xi[\mathfrak{C}]_b \cup \{b \cap \pi_\xi a_\eta : \eta < \xi\} = \psi[\mathfrak{C} \cup \{a_\eta : \eta < \xi\}],$$

writing $\psi c = b \cap \pi_\xi c$ for $c \in \mathfrak{C}_\xi$. As ψ is order-continuous, $\psi[\mathfrak{C}_\xi]$ is precisely the closed subalgebra of $\mathfrak{B}_b$ generated by D (314H), and is therefore not the whole of $\mathfrak{B}_b$.

But this means that $\mathfrak{B}_b \ne \{b \cap \pi_\xi c : c \in \mathfrak{C}_\xi\}$. As b is arbitrary, π_ξ satisfies the conditions of 331D, and has an extension to a measure-preserving Boolean homomorphism $\pi_{\xi+1} : \mathfrak{C}_{\xi+1} \to \mathfrak{B}$, since $\mathfrak{C}_{\xi+1}$ is just the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{a_\xi\}$.

(iii) For the inductive step to a non-zero limit ordinal $\xi \le \kappa$, we can argue exactly as in part (d) of the proof of 331I; $\mathfrak{C}_\xi$ will be the metric closure of $\mathfrak{C}^*_\xi = \bigcup_{\eta < \xi} \mathfrak{C}_\eta$, so we can take $\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{B}$ to be the unique measure-preserving homomorphism extending $\pi^*_\xi = \bigcup_{\eta < \xi} \pi_\eta$.

Thus the induction proceeds, and evidently $\pi = \pi_\kappa$ will be a measure-preserving homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ extending ϕ.

(b) (This is rather closer to the proof of 331I, being indeed a direct generalization of it.) Observe that the hypothesis (b) implies that $1_{\mathfrak{A}}$ is relatively Maharam-type-homogeneous over $\mathfrak{C}$; so either $\kappa = 0$, in which case $\mathfrak{A} = \mathfrak{C}$, $\mathfrak{B} = \phi[\mathfrak{C}]$ and the result is trivial, or $\kappa \ge \omega$, by 333Bd. Let us therefore take it that κ is infinite.

We are supposing, among other things, that $\tau_{\mathfrak{D}}(\mathfrak{B}) = \kappa$; let $\langle b_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}$ such that $\mathfrak{B}$ is τ-generated by $\mathfrak{D} \cup \{b_\xi : \xi < \kappa\}$. This time, as in 331I, we shall have to choose further families $\langle a'_\xi\rangle_{\xi<\kappa}$ and $\langle b'_\xi\rangle_{\xi<\kappa}$, and

$\mathfrak{C}_\xi$ will be the closed subalgebra of $\mathfrak{A}$ generated by

$$\mathfrak{C} \cup \{a_\eta : \eta < \xi\} \cup \{a'_\eta : \eta < \xi\},$$

$\mathfrak{D}_\xi$ will be the closed subalgebra of $\mathfrak{B}$ generated by

$$\mathfrak{D} \cup \{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta < \xi\},$$

$\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{D}_\xi$ will be a measure-preserving homomorphism.

The induction will start with $\mathfrak{C}_0 = \mathfrak{C}$, $\mathfrak{D}_0 = \mathfrak{D}$ and $\pi_0 = \phi$, as in (a).

(i) For the inductive step to a successor ordinal $\xi + 1$, where $\xi < \kappa$, suppose that $\mathfrak{C}_\xi$, $\mathfrak{D}_\xi$ and π_ξ have been defined.

(α) Let $b \in \mathfrak{B} \setminus \{0\}$. Because

$$\tau_{\mathfrak{D}_b}(\mathfrak{B}_b) = \kappa > \#(\{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta < \xi\}),$$

$\mathfrak{B}_b$ cannot be τ-generated by $\mathfrak{D}_b \cup \{b \cap b_\eta : \eta < \xi\} \cup \{b \cap b'_\eta : \eta < \xi\}$, and cannot be equal to $\{b \cap d : d \in \mathfrak{D}_\xi\}$. As b is arbitrary, there is an extension of π_ξ to a measure-preserving homomorphism ϕ_ξ from $\mathfrak{C}'_\xi$ to $\mathfrak{B}$, where $\mathfrak{C}'_\xi$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{a_\eta : \eta \le \xi\} \cup \{a'_\eta : \eta < \xi\}$. Setting $b'_\xi = \phi_\xi(a_\xi)$, the image $\mathfrak{D}'_\xi = \phi_\xi[\mathfrak{C}'_\xi]$ will be the closed subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D} \cup \{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta \le \xi\}$.

(β) Next, as in 331I, we must repeat the argument of (α), applying it now to $\phi_\xi^{-1} : \mathfrak{D}_\xi \to \mathfrak{A}$. If $a \in \mathfrak{A} \setminus \{0\}$,

$$\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) = \kappa > \#(\{a_\eta : \eta \le \xi\} \cup \{a'_\eta : \eta < \xi\}),$$

so that $\mathfrak{A}_a$ cannot be $\{a \cap c : c \in \mathfrak{C}'_\xi\}$. As a is arbitrary, ϕ_ξ^{-1} has an extension to a measure-preserving homomorphism $\psi_\xi : \mathfrak{D}_{\xi+1} \to \mathfrak{C}_{\xi+1}$, where $\mathfrak{D}_{\xi+1}$ is the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D}'_\xi \cup \{b_\xi\}$, that is, the closed subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D} \cup \{b_\eta : \eta \le \xi\} \cup \{b'_\eta : \eta \le \xi\}$, and $\mathfrak{C}_{\xi+1}$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}'_\xi \cup \{a'_\xi\}$, setting $a'_\xi = \psi_\xi(b_\xi)$.

We can therefore take $\pi_{\xi+1} = \psi_\xi^{-1} : \mathfrak{C}_{\xi+1} \to \mathfrak{D}_{\xi+1}$, as in 331I.

(ii) The inductive step to a non-zero limit ordinal $\xi \le \kappa$ is exactly the same as in (a) above or in 331I; $\mathfrak{C}_\xi$ is the metric closure of $\mathfrak{C}^*_\xi = \bigcup_{\eta<\xi} \mathfrak{C}_\eta$, $\mathfrak{D}_\xi$ is the metric closure of $\mathfrak{D}^*_\xi = \bigcup_{\eta<\xi} \mathfrak{D}_\eta$, and π_ξ is the unique measure-preserving homomorphism from $\mathfrak{C}_\xi$ to $\mathfrak{D}_\xi$ extending every π_η for $\eta < \xi$.

(iii) The induction stops, as before, with $\pi = \pi_\kappa : \mathfrak{C}_\kappa \to \mathfrak{D}_\kappa$, where $\mathfrak{C}_\kappa = \mathfrak{A}$, $\mathfrak{D}_\kappa = \mathfrak{B}$.

333D Corollary Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be totally finite measure algebras and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Suppose that

$$\tau(\mathfrak{C}) < \max(\omega, \tau(\mathfrak{A})) \le \min\{\tau(\mathfrak{B}_b) : b \in \mathfrak{B} \setminus \{0\}\}.$$

Then any measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ can be extended to a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$.

proof Set $\kappa = \min\{\tau(\mathfrak{B}_b) : b \in \mathfrak{B} \setminus \{0\}\}$. Then for any non-zero $b \in \mathfrak{B}$,

$$\tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b) \ge \kappa.$$

P There is a set $C \subseteq \mathfrak{C}$, of cardinal $\tau(\mathfrak{C})$, which τ-generates $\mathfrak{C}$, so that $C' = \{b \cap \phi c : c \in C\}$ τ-generates $\phi[\mathfrak{C}]_b$. Now there is a set $D \subseteq \mathfrak{B}_b$, of cardinal $\tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)$, such that $\phi[\mathfrak{C}]_b \cup D$ τ-generates $\mathfrak{B}_b$. In this case $C' \cup D$ must τ-generate $\mathfrak{B}_b$, so $\kappa \le \#(C' \cup D)$. But $\#(C') \le \#(C) < \kappa$ and κ is infinite, so we must have $\#(D) \ge \kappa$, as claimed. **Q**

On the other hand, $\tau_{\mathfrak{C}}(\mathfrak{A}) \le \tau(\mathfrak{A}) \le \kappa$. So we can apply 333Ca to give the result.

333E Theorem Let $(\mathfrak{C},\bar{\mu})$ be a totally finite measure algebra and κ an infinite cardinal. Let $(\mathfrak{A},\bar{\lambda})$ be the localizable measure algebra free product of $(\mathfrak{C},\bar{\mu})$ and $(\mathfrak{B}_\kappa,\bar{\nu}_\kappa)$ (notation: 333Ad), and $\varepsilon:\mathfrak{C}\to\mathfrak{A}$ the corresponding homomorphism. Then for any non-zero $a\in\mathfrak{A}$,

$$\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a)=\kappa,$$

in the notation of 333A above.

proof Recall from 325Dd that $\varepsilon[\mathfrak{C}]$ is a closed subalgebra of $\mathfrak{A}$.

(a) Let $\langle e_\xi\rangle_{\xi<\kappa}$ be the standard generating family in $\mathfrak{B}_\kappa$, corresponding to the sets $\{x:x\in\{0,1\}^\kappa,\ x(\xi)=1\}$. Let $\varepsilon':\mathfrak{B}_\kappa\to\mathfrak{A}$ be the canonical map, and set $e'_\xi=\varepsilon' e_\xi$ for each ξ.

We know that $\{e_\xi:\xi<\kappa\}$ τ-generates $\mathfrak{B}_\kappa$ (see part (a) of the proof of 331K). Consequently $\varepsilon[\mathfrak{C}]\cup\{e'_\xi:\xi<\kappa\}$ τ-generates $\mathfrak{A}$. **P** Let $\mathfrak{A}_1$ be the closed subalgebra of $\mathfrak{A}$ generated by $\varepsilon[\mathfrak{C}]\cup\{e'_\xi:\xi<\kappa\}$. Because $\varepsilon':\mathfrak{B}_\kappa\to\mathfrak{A}$ is order-continuous (325Da), $\varepsilon'[\mathfrak{B}_\kappa]\subseteq\mathfrak{A}_1$ (313Mb). But this means that $\mathfrak{A}_1$ includes $\varepsilon[\mathfrak{C}]\cup\varepsilon'[\mathfrak{B}_\kappa]$ and therefore includes the image of $\mathfrak{C}\otimes\mathfrak{B}_\kappa$ in $\mathfrak{A}$; because this is topologically dense in $\mathfrak{A}$ (325Dc), $\mathfrak{A}_1=\mathfrak{A}$, as claimed. **Q**

(b) It follows that

$$\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a)\le\tau_{\varepsilon[\mathfrak{C}]}(\mathfrak{A})\le\kappa$$

(333Ba).

(c) We need to know that if $\xi<\kappa$ and a belongs to the closed subalgebra $\mathfrak{C}_\xi$ of $\mathfrak{A}$ generated by $\varepsilon[\mathfrak{C}]\cup\{e'_\eta:\eta\ne\xi\}$, then $\bar{\lambda}(a\cap e'_\xi)=\frac{1}{2}\bar{\lambda}a$. **P** Set

$$E=\varepsilon[\mathfrak{C}]\cup\{e'_\eta:\eta\ne\xi\},\quad F=\{a_0\cap\ldots\cap a_n:a_0,\ldots,a_n\in E\}.$$

Then every member of F is expressible in the form

$$a=\varepsilon c\cap\inf_{\eta\in J}e'_\eta,$$

where $c\in\mathfrak{C}$ and $J\subseteq\kappa\setminus\{\xi\}$ is finite. Now

$$\bar{\lambda}a=\bar{\mu}c\cdot\bar{\nu}(\inf_{\eta\in J}e_\eta)=2^{-\#(J)}\bar{\mu}c,$$

$$\bar{\lambda}(e'_\xi\cap a)=\bar{\mu}c\cdot\bar{\nu}(e_\xi\cap\inf_{\eta\in J}e_\eta)=2^{-\#(J\cup\{\xi\})}\bar{\mu}c=\frac{1}{2}\bar{\lambda}a.$$

Now consider the set

$$G=\{a:a\in\mathfrak{A},\ \bar{\lambda}(e_\xi\cap a)=\frac{1}{2}\bar{\lambda}a\}.$$

We have $1_{\mathfrak{A}}\in F\subseteq G$, and F is closed under $\cap$. Secondly, if $a,\ a'\in G$ and $a\subseteq a'$, then

$$\bar{\lambda}(e_\xi\cap(a'\setminus a))=\bar{\lambda}(e_\xi\cap a')-\bar{\lambda}(e_\xi\cap a)=\frac{1}{2}\bar{\lambda}a'-\frac{1}{2}\bar{\lambda}a=\frac{1}{2}\bar{\lambda}(a'\setminus a),$$

so $a'\setminus a\in G$. Also, if $H\subseteq G$ is non-empty and upwards-directed,

$$\bar{\lambda}(e_\xi\cap\sup H)=\bar{\lambda}(\sup_{a\in H}e_\xi\cap a)=\sup_{a\in H}\bar{\lambda}(e_\xi\cap a)=\sup_{a\in H}\frac{1}{2}\bar{\lambda}a=\frac{1}{2}\bar{\lambda}(\sup H),$$

so $\sup H\in G$. By the Monotone Class Theorem (313Gc), G includes the order-closed subalgebra of $\mathfrak{D}$ generated by F. But this is just $\mathfrak{C}_\xi$. **Q**

(d) The next step is to see that $\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a)>0$. **P** By (a) and 323J, $\mathfrak{A}$ is the metric closure of the subalgebra $\mathfrak{A}_0$ generated by $\varepsilon[\mathfrak{C}]\cup\{e'_\eta:\eta<\kappa\}$, so there must be an $a_0\in\mathfrak{A}_0$ such that $\bar{\lambda}(a_0\triangle a)\le\frac{1}{4}\bar{\lambda}a$. Now there is a finite $J\subseteq\kappa$ such that a_0 belongs to the subalgebra $\mathfrak{A}_1$ generated by $\varepsilon[\mathfrak{C}]\cup\{e'_\eta:\eta\in J\}$. Take any $\xi\in\kappa\setminus J$ (this is where I use the hypothesis that κ is infinite). If $c\in\mathfrak{C}$, then by (c) we have

$$\begin{aligned}\bar{\lambda}((a\cap\varepsilon c)\triangle(a\cap e'_\xi))&=\bar{\lambda}(a\cap(\varepsilon c\triangle e'_\xi))\ge\bar{\lambda}(a_0\cap(\varepsilon c\triangle e'_\xi))-\bar{\lambda}(a\triangle a_0)\\&=\bar{\lambda}(a_0\cap e'_\xi)+\bar{\lambda}(a_0\cap\varepsilon c)-2\bar{\lambda}(a_0\cap\varepsilon c\cap e'_\xi)-\bar{\lambda}(a\triangle a_0)\\&=\frac{1}{2}\bar{\lambda}a_0-\bar{\lambda}(a\triangle a_0)\end{aligned}$$

(because both a_0 and $a_0\cap\varepsilon c$ belong to $\mathfrak{C}_\xi$)

$$\geq \frac{1}{2}\bar{\lambda}a - \frac{3}{2}\bar{\lambda}(a \triangle a_0) > 0.$$

Thus $a \cap e'_\xi$ is not of the form $a \cap \varepsilon c$ for any $c \in \mathfrak{C}$, and $\mathfrak{A}_a \neq \varepsilon[\mathfrak{C}]_a$, so that $\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a) > 0$. **Q**

(e) It follows that $\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a)$ is infinite. **P** There is a non-zero $d \subseteq a$ which is relatively Maharam-type-homogeneous over $\varepsilon[\mathfrak{C}]$. By (d), applied to d, $\tau_{\varepsilon[\mathfrak{C}]_d}(\mathfrak{A}_d) > 0$; but now 333Bd tells us that $\tau_{\varepsilon[\mathfrak{C}]_d}(\mathfrak{A}_d)$ must be infinite, so $\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a)$ is infinite. **Q**

(f) If $\kappa = \omega$, we can stop here. If $\kappa > \omega$, we continue, as follows. Let $D \subseteq \mathfrak{A}_a$ be any set of cardinal less than κ. Each $d \in D \cup \{a\}$ belongs to the closed subalgebra of $\mathfrak{A}$ generated by $C = \varepsilon[\mathfrak{C}] \cup \{e'_\xi : \xi < \kappa\}$. But because $\mathfrak{A}$ is ccc, this is just the σ-subalgebra of $\mathfrak{A}$ generated by C (331Ge). So d belongs to the closed subalgebra of $\mathfrak{A}$ generated by some countable subset C_d of C, by 331Gd. Now $J_d = \{\eta : e'_\eta \in C_d\}$ is countable. Set $J = \bigcup_{d \in D \cup \{a\}} J_d$; then

$$\#(J) \leq \max(\omega, \#(D \cup \{a\})) = \max(\omega, \#(D)) < \kappa,$$

so $J \neq \kappa$, and there is a $\xi \in \kappa \setminus J$. Accordingly $\varepsilon[\mathfrak{C}] \cup D \cup \{a\}$ is included in $\mathfrak{E}_\xi$, as defined in (c) above, and $\varepsilon[\mathfrak{C}]_a \cup D \subseteq \mathfrak{E}_\xi$. As $\mathfrak{A}_a \cap \mathfrak{E}_\xi$ is a closed subalgebra of $\mathfrak{A}_a$, it includes the closed subalgebra generated by $\varepsilon[\mathfrak{C}]_a \cup D$. But $a \cap e'_\xi$ surely does not belong to $\mathfrak{E}_\xi$, since

$$\bar{\lambda}(a \cap e'_\xi \cap e'_\xi) = \bar{\lambda}(a \cap e'_\xi) = \frac{1}{2}\bar{\lambda}a > 0,$$

and $\bar{\lambda}(a \cap e'_\xi \cap e'_\xi) \neq \frac{1}{2}\bar{\lambda}(a \cap e'_\xi)$. Thus $a \cap e'_\xi$ cannot belong to the closed subalgebra of $\mathfrak{A}_a$ generated by $\varepsilon[\mathfrak{C}]_a \cup D$, and $\varepsilon[\mathfrak{C}]_a \cup D$ does not τ-generate $\mathfrak{A}_a$. As D is arbitrary, $\tau_{\phi[\mathfrak{C}]_a}(\mathfrak{A}_a) \geq \kappa$.

This completes the proof.

333F Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$ and κ an infinite cardinal.

(a) Suppose that $\kappa \geq \tau_{\mathfrak{C}}(\mathfrak{A})$. Let $(\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa, \bar{\lambda})$ be the localizable measure algebra free product of $(\mathfrak{C}, \bar{\mu}\restriction\mathfrak{C})$ and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$, and $\varepsilon : \mathfrak{C} \to \mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ the corresponding homomorphism. Then there is a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ extending ε.

(b) Suppose further that $\kappa = \tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ for every non-zero $a \in \mathfrak{A}$. Then π can be taken to be an isomorphism.

proof All we have to do is apply 333C with $\mathfrak{B} = \mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$, using 333E to see that the hypothesis

$$\tau_{\varepsilon[\mathfrak{C}]_b}(\mathfrak{B}_b) = \kappa \text{ for every non-zero } b \in \mathfrak{B}$$

is satisfied.

333G Corollary Let $(\mathfrak{C}, \bar{\mu})$ be a totally finite measure algebra. Suppose that $\kappa \geq \max(\omega, \tau(\mathfrak{C}))$ is a cardinal. Let $(\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa, \bar{\lambda})$ be the localizable measure algebra free product of $(\mathfrak{C}, \bar{\mu})$ and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$. Then

(a) $\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ is Maharam-type-homogeneous, with Maharam type κ if $\mathfrak{C} \neq \{0\}$;

(b) for every measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{C}$ there is a measure-preserving automorphism $\pi : \mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa \to \mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ such that $\pi(c \otimes 1) = \phi c \otimes 1$ for every $c \in \mathfrak{C}$, writing $c \otimes 1$ for the canonical image in $\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ of any $c \in \mathfrak{C}$.

proof Write $\mathfrak{A}$ for $\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$, as in 333E, and $\mathfrak{D}$ for $\{c \otimes 1 : c \in \mathfrak{C}\} \subseteq \mathfrak{A}$.

(a) If $C \subseteq \mathfrak{C}$ is a set of cardinal $\tau(\mathfrak{C})$ which τ-generates $\mathfrak{C}$, and $B \subseteq \mathfrak{B}_\kappa$ a set of cardinal κ which τ-generates $\mathfrak{B}_\kappa$ (331K), then $\{c \otimes b : c \in C, b \in B\}$ is a set of cardinal at most $\max(\omega, \tau(\mathfrak{C}), \kappa) = \kappa$ which τ-generates $\mathfrak{A}$ (because the subalgebra it generates is topologically dense in $\mathfrak{A}$, by 325Dc). So $\tau(\mathfrak{A}) \leq \kappa$. On the other hand, if $a \in \mathfrak{A}$ is non-zero, then $\tau(\mathfrak{A}_a) \geq \tau_{\mathfrak{D}_a}(\mathfrak{A}_a) \geq \kappa$, by 333E; so $\mathfrak{A}$ is Maharam-type-homogeneous, with Maharam type κ unless $\mathfrak{C} = \{0\}$.

(b) We have a measure-preserving automorphism $\phi_1 : \mathfrak{D} \to \mathfrak{D}$ defined by setting $\phi_1(c \otimes 1) = \phi c \otimes 1$ for every $c \in \mathfrak{C}$. Because $\phi_1[\mathfrak{D}] \subseteq \mathfrak{D}$, 333Be and 333E tell us that

$$\kappa = \tau(\mathfrak{A}_a) \geq \tau_{\phi_1[\mathfrak{D}]_a}(\mathfrak{A}_a) \geq \tau_{\mathfrak{D}_a}(\mathfrak{A}_a) = \kappa$$

for every non-zero $a \in \mathfrak{A}$, so we can use 333Cb, with $\mathfrak{B} = \mathfrak{A}$, to see that ϕ_1 can be extended to a measure-preserving automorphism on $\mathfrak{A}$.

333H I turn now to the classification of closed subalgebras.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Then there are $\langle\mu_i\rangle_{i\in I}$, $\langle c_i\rangle_{i\in I}$, $\langle\kappa_i\rangle_{i\in I}$ such that

for each $i\in I$, μ_i is a non-negative completely additive functional on $\mathfrak{C}$,

$c_i=[\![\mu_i>0]\!]\in\mathfrak{C}$,

κ_i is 0 or an infinite cardinal,

$(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ is a totally finite measure algebra, writing $\mathfrak{C}_{c_i}$ for the principal ideal of $\mathfrak{C}$ generated by c_i,

$\sum_{i\in I}\mu_i c=\bar\mu c$ for every $c\in\mathfrak{C}$,

there is a measure-preserving isomorphism π from $\mathfrak{A}$ to the simple product $\prod_{i\in I}\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ of the localizable measure algebra free products $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ of $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ and $(\mathfrak{B}_{\kappa_i},\bar\nu_{\kappa_i})$.

Moreover, π may be taken such that

for every $c\in\mathfrak{C}$, $\pi c=\langle(c\cap c_i)\otimes 1\rangle_{i\in I}$, writing $c\otimes 1$ for the image in $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ of $c\in\mathfrak{C}_{c_i}$.

Remark Recall that $[\![\mu_i>0]\!]$ is that element of $\mathfrak{C}$ such that $\mu_i c>0$ whenever $c\in\mathfrak{C}$ and $0\ne c\subseteq[\![\mu_i>0]\!]$, $\mu_i c\le 0$ whenever $c\in\mathfrak{C}$ and $c\cap[\![\mu_i>0]\!]=0$ (326S).

proof (a) Let A be the set of those elements of $\mathfrak{A}$ which are relatively Maharam-type-homogeneous over $\mathfrak{C}$ (see 333Ac). By 333Bb, A is order-dense in $\mathfrak{A}$ (compare part (a) of the proof of 332B), and consequently $A'=\{a:a\in A,\ \bar\mu a<\infty\}$ is order-dense in $\mathfrak{A}$. So there is a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ consisting of members of A' (313K). For each $i\in I$, set $\mu_i c=\bar\mu(a_i\cap c)$ for every $c\in\mathfrak{C}$; then μ_i is non-negative, and it is completely additive by 327E. Because $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$,

$$\bar\mu c=\sum_{i\in I}\bar\mu(c\cap a_i)=\sum_{i\in I}\mu_i c$$

for every $c\in\mathfrak{C}$. Next, $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ is a totally finite measure algebra. **P** $\mathfrak{C}_{c_i}$ is a Dedekind σ-complete Boolean algebra because $\mathfrak{C}$ is. $\mu_i\restriction\mathfrak{C}_{c_i}$ is a non-negative countably additive functional because μ_i is. If $c\in\mathfrak{C}_{c_i}$ and $\mu_i c=0$, then $c=0$ by the choice of c_i. **Q** Note also that

$$\bar\mu(a_i\setminus c_i)=\mu_i(1\setminus c_i)=0,$$

so that $a_i\subseteq c_i$.

(b) By 333Bd, any finite κ_i must actually be zero. The next element we need is the fact that, for each $i\in I$, we have a measure-preserving isomorphism $c\mapsto c\cap a_i$ from $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ to $(\mathfrak{C}_{a_i},\bar\mu\restriction\mathfrak{C}_{a_i})$. **P** Of course this is a ring homomorphism. Because $a_i\subseteq c_i$, it is a surjective Boolean homomorphism. It is measure-preserving by the definition of μ_i, and therefore injective. **Q**

(c) Still focusing on a particular $i\in I$, let $\mathfrak{A}_{a_i}$ be the principal ideal of $\mathfrak{A}$ generated by a_i. Then we have a measure-preserving isomorphism $\tilde\pi_i:\mathfrak{A}_{a_i}\to\mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$, extending the canonical homomorphism $c\mapsto c\otimes 1:\mathfrak{C}_{a_i}\to\mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$. **P** When κ_i is infinite, this is just 333Fb. But the only other case is when $\kappa_i=0$, that is, $\mathfrak{C}_{a_i}=\mathfrak{A}_{a_i}$, while $\mathfrak{B}_{\kappa_i}=\{0,1\}$ and $\mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}\cong\mathfrak{C}_{c_i}$. **Q**

The isomorphism between $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ and $(\mathfrak{C}_{a_i},\bar\mu\restriction\mathfrak{C}_{a_i})$ induces an isomorphism between $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ and $\mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$. So we have a measure-preserving isomorphism $\pi_i:\mathfrak{A}_{a_i}\to\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ such that $\pi_i(c\cap a_i)=c\otimes 1$ for every $c\in\mathfrak{C}_{c_i}$.

(d) By 322Ld, we have a measure-preserving isomorphism $a\mapsto\langle a\cap a_i\rangle_{i\in I}:\mathfrak{A}\to\prod_{i\in I}\mathfrak{A}_{a_i}$. Putting this together with the isomorphisms of (c), we have a measure-preserving isomorphism π from $\mathfrak{A}$ to $\prod_{i\in I}\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$, setting $\pi a=\langle\pi_i(a\cap a_i)\rangle_{i\in I}$ for $a\in\mathfrak{A}$. Observe that, for $c\in\mathfrak{C}$,

$$\pi c=\langle\pi_i(c\cap a_i)\rangle_{i\in I}=\langle(c\cap c_i)\otimes 1\rangle_{i\in I},$$

as required.

333I Remarks (a) I hope it is clear that whenever $(\mathfrak{C},\bar\mu)$ is a Dedekind complete measure algebra, $\langle\mu_i\rangle_{i\in I}$ is a family of non-negative completely additive functionals on $\mathfrak{C}$ such that $\sum_{i\in I}\mu_i=\bar\mu$, and $\langle\kappa_i\rangle_{i\in I}$ is a family of cardinals all infinite or zero, then the construction above can be applied to give a measure algebra $(\mathfrak{A},\bar\lambda)$, the product of the family $\langle\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}\rangle_{i\in I}$, together with an order-continuous measure-preserving homomorphism $\pi:\mathfrak{C}\to\mathfrak{A}$; and that the partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ corresponding to this product (315E) has $\mu_i c=\bar\lambda(a_i\cap\pi c)$ for every $c\in\mathfrak{C}$ and $i\in I$, while each principal ideal $\mathfrak{A}_{a_i}$ can be identified with $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$, so that a_i is relatively Maharam-type-homogeneous over $\pi[\mathfrak{C}]$. Thus any structure $(\mathfrak{C}, \bar{\mu}, \langle\mu_i\rangle_{i\in I}, \langle\kappa_i\rangle_{i\in I})$ of the type described here corresponds to an embedding of $\mathfrak{C}$ as a closed subalgebra of a localizable measure algebra.

(b) The obvious next step is to seek a complete classification of objects $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$, where $(\mathfrak{A}, \bar{\mu})$ is a localizable measure algebra and $\mathfrak{C}$ is a closed subalgebra, corresponding to the classification of localizable measure algebras in terms of the magnitudes of their Maharam-type-κ components in 332J. The general case seems to be complex. But I can deal with the special case in which $(\mathfrak{A}, \bar{\mu})$ is totally finite. In this case, we have the following facts.

333J Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, and $\mathfrak{C}$ a closed subalgebra. Let A be the set of relative atoms of $\mathfrak{A}$ over $\mathfrak{C}$. Then there is a unique sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ of additive functionals on $\mathfrak{C}$ such that (i) $\mu_{n+1} \le \mu_n$ for every n (ii) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that $\sup_{n\in\mathbb{N}} a_n = \sup A$ and $\mu_n c = \bar{\mu}(a_n \cap c)$ for every $n \in \mathbb{N}$ and $c \in \mathfrak{C}$.

Remark I hope it is plain from my wording that it is the μ_n which are unique, not the a_n.

proof (a) For each $a \in \mathfrak{A}$ set $\theta_a(c) = \bar{\mu}(c \cap a)$ for $c \in \mathfrak{C}$. Then θ_a is a non-negative completely additive real-valued functional on $\mathfrak{C}$ (see 326Od).

The key step is I suppose in (c) below; I approach by a two-stage argument. For each $b \in \mathfrak{A}$ write $A_b^\perp$ for $\{a : a \in A,\ a \cap b = 0\}$.

(b) For every $b \in \mathfrak{A}$ and non-zero $c \in \mathfrak{C}$ there are $a \in A_b^\perp$, $c' \in \mathfrak{C}$ such that $0 \ne c' \subseteq c$ and $\theta_a(d) \ge \theta_e(d)$ whenever $d \in \mathfrak{C}$, $e \in A_b^\perp$ and $d \subseteq c'$. **P?** Otherwise, choose $\langle a_n\rangle_{n\in\mathbb{N}}$ and $\langle c_n\rangle_{n\in\mathbb{N}}$ as follows. Since 0, c won't serve for a, c', there must be an $a_0 \in A_b^\perp$ such that $\theta_{a_0}(c) > 0$. Let $\delta > 0$ be such that $\theta_{a_0}(c) > \delta\bar{\mu}c$ and set $c_0 = c \cap [\![\theta_{a_0} > \delta\bar{\mu}\restriction\mathfrak{C}]\!]$; then $c_0 \in \mathfrak{C}$ and $0 \ne c_0 \subseteq c$. Given that $a_n \in A_b^\perp$, $c_n \in \mathfrak{C}$ and $0 \ne c_n \subseteq c$, then there must be $a_{n+1} \in A_b^\perp$, $d_n \in \mathfrak{C}$ such that $d_n \subseteq c_n$ and $\theta_{a_{n+1}}(d_n) > \theta_{a_n}(d_n)$. Set $c_{n+1} = d_n \cap [\![\theta_{a_{n+1}} > \theta_{a_n}]\!]$, so that $c_{n+1} \in \mathfrak{C}$ and $0 \ne c_{n+1} \subseteq c_n$, and continue.

There is some $n \in \mathbb{N}$ such that $n\delta \ge 1$. For any $i < n$, the construction ensures that

$$0 \ne c_{n+1} \subseteq c_{i+1} \subseteq [\![\theta_{a_{i+1}} > \theta_{a_i}]\!],$$

so $\theta_{a_i}(c_{n+1}) < \theta_{a_{i+1}}(c_{n+1})$; also $c_{n+1} \subseteq c_0$ so

$$\bar{\mu}(a_i \cap c_{n+1}) = \theta_{a_i}(c_{n+1}) \ge \theta_{a_0}(c_{n+1}) > \delta\bar{\mu}c_{n+1}.$$

But this means that $\sum_{i=0}^{n-1} \bar{\mu}(a_i \cap c_{n+1}) > \bar{\mu}c_{n+1}$ and there must be distinct j, $k < n$ such that $a_j \cap a_k \cap c_{n+1}$ is non-zero. Because a_j, $a_k \in A$ there are d', $d'' \in \mathfrak{C}$ such that $a_j \cap a_k = a_j \cap d' = a_k \cap d''$; set $d = c_{n+1} \cap d' \cap d''$, so that $d \in \mathfrak{C}$ and

$$a_j \cap d = a_j \cap a_k \cap c_{n+1} = a_k \cap d, \quad \theta_{a_j}(d) = \bar{\mu}(a_j \cap a_k \cap c_{n+1}) = \theta_{a_k}(d).$$

But as $0 \ne d \subseteq [\![\theta_{a_{i+1}} > \theta_{a_i}]\!]$ for every $i < n$, $\theta_{a_0}(d) < \theta_{a_1}(d) < \ldots < \theta_{a_n}(d)$, so this is impossible. **XQ**

(c) Now for a global, rather than local, version of the same idea. For every $b \in \mathfrak{A}$ there is an $a \in A_b^\perp$ such that and $\theta_a \ge \theta_e$ whenever $e \in A_b^\perp$. **P** (i) By (b), the set C of those $c \in \mathfrak{C}$ such that there is an $a \in A_b^\perp$ such that $\theta_a\restriction\mathfrak{C}_c \ge \theta_e\restriction\mathfrak{C}_c$ for every $e \in A_b^\perp$ is order-dense in $\mathfrak{C}$. Let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{C}$ consisting of members of C, and for each $i \in I$ choose $a_i \in A_b^\perp$ such that $\theta_{a_i}\restriction\mathfrak{C}_{c_i} \ge \theta_e\restriction\mathfrak{C}_{c_i}$ for every $e \in A_b^\perp$. Consider $a = \sup_{i\in I} a_i \cap c_i$. (ii) If $a' \in \mathfrak{A}$ and $a' \subseteq a$, then for each $i \in I$ there is a $d_i \in \mathfrak{C}$ such that $a_i \cap a' = a_i \cap d_i$. Set $d' = \sup_{i\in I} c_i \cap d_i$; then (because $\langle c_i\rangle_{i\in I}$ is disjoint)

$$a \cap d' = \sup_{i\in I} a_i \cap c_i \cap d_i = \sup_{i\in I} a_i \cap c_i \cap a' = a \cap a' = a'.$$

As a' is arbitrary, this shows that $a \in A$. (iii) Of course $a \cap b = 0$, so $a \in A_b^\perp$. Now take any $e \in A_b^\perp$ and $d \in \mathfrak{C}$. Then

$$\theta_a(d) = \sum_{i\in I} \theta_{a_i}(c_i \cap d) \ge \sum_{i\in I} \theta_e(c_i \cap d) = \theta_e(d).$$

So this a has the required property. **Q**

(d) Choose $\langle a_n\rangle_{n\in\mathbb{N}}$ inductively in A so that, for each n, $a_n \cap \sup_{i<n} a_i = 0$ and $\theta_{a_n} \ge \theta_e$ whenever $e \in A$ and $e \cap \sup_{i<n} a_i = 0$. Set $\mu_n = \theta_{a_n}$. Because $a_{n+1} \cap \sup_{i<n} a_i = 0$, $\mu_{n+1} \le \mu_n$ for each n. Also $\sup_{n\in\mathbb{N}} a_n = \sup A$. **P** Take any $a \in A$ and set $e = a \setminus \sup_{n\in\mathbb{N}} a_n$. Then $e \in A$ and, for any $n \in \mathbb{N}$, $e \cap \sup_{i<n} a_i = 0$, so $\theta_e \le \theta_{a_n}$ and

$$\bar{\mu}e = \theta_e(1) \le \theta_{a_n}(1) = \bar{\mu}a_n.$$

But as $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint, this means that $e = 0$, that is, $a \subseteq \sup_{n\in\mathbb{N}} a_n$. As a is arbitrary, $\sup A \subseteq \sup_{n\in\mathbb{N}} a_n$. **Q**

(e) Thus we have a sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ of the required type, witnessed by $\langle a_n\rangle_{n\in\mathbb{N}}$. To see that it is unique, suppose that $\langle\mu'_n\rangle_{n\in\mathbb{N}}$, $\langle a'_n\rangle_{n\in\mathbb{N}}$ are another pair of sequences with the same properties. Note first that if $c\in\mathfrak{C}$ and $0\ne c\subseteq[\![\mu'_i>0]\!]$ there is some $k\in\mathbb{N}$ such that $c\cap a'_i\cap a_k\ne 0$; this is because $\bar\mu(a'_i\cap c)=\mu'_i(c)>0$, so that $a'_i\cap c\ne 0$, while $a'_i\subseteq\sup A=\sup_{k\in\mathbb{N}}a_k$. **?** Suppose, if possible, that there is some n such that $\mu_n\ne\mu'_n$; since the situation is symmetric, there is no loss of generality in supposing that $\mu'_n\not\le\mu_n$, that is, that $c=[\![\mu'_n>\mu_n]\!]\ne 0$. For any $i\le n$, $\mu'_i\ge\mu'_n$ so $c\subseteq[\![\mu'_i>0]\!]$. We may therefore choose $c_0,\dots,c_{n+1}\in\mathfrak{C}_c\setminus\{0\}$ and $k(0),\dots,k(n)\in\mathbb{N}$ such that $c_0=c$ and, for $i\le n$,

$$c_i\cap a'_i\cap a_{k(i)}\ne 0$$

(choosing $k(i)$, recalling that $0\ne c_i\subseteq c\subseteq[\![\mu'_i>0]\!]$),

$$c_{i+1}\in\mathfrak{C},\quad c_{i+1}\subseteq c_i,\quad c_{i+1}\cap a'_i=c_{i+1}\cap a_{k(i)}=c_i\cap a'_i\cap a_{k(i)}$$

(choosing c_{i+1}, using the fact that a'_i and $a_{k(i)}$ both belong to A – see the penultimate sentence in part (b) of the proof). On reaching c_{n+1}, we have $0\ne c_{n+1}\subseteq c$ so $\mu_n(c_{n+1})<\mu'_n(c_{n+1})$. On the other hand, for each $i\le n$,

$$c_{n+1}\cap a'_i\cap a_{k(i)}=c_{n+1}\cap c_{i+1}\cap a'_i\cap a_{k(i)}=c_{n+1}\cap a'_i=c_{n+1}\cap a_{k(i)},$$

so

$$\mu_n(c_{n+1})<\mu'_n(c_{n+1})\le\mu'_i(c_{n+1})=\bar\mu(c_{n+1}\cap a'_i)=\bar\mu(c_{n+1}\cap a_{k(i)})=\mu_{k(i)}(c_{n+1}),$$

and $k(i)$ must be less than n. There are therefore distinct i, $j\le n$ such that $k(i)=k(j)$. But in this case

$$c_{n+1}\cap a'_i=c_{n+1}\cap a_{k(i)}=c_{n+1}\cap a_{k(j)}=c_{n+1}\cap a'_j\ne 0$$

because $0\ne c_{n+1}\subseteq[\![\mu'_j>0]\!]$. So a'_i, a'_j cannot be disjoint, breaking one of the rules of the construction. **X** Thus $\mu_n=\mu'_n$ for every $n\in\mathbb{N}$.

This completes the proof.

333K Theorem Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Then there are unique families $\langle\mu_n\rangle_{n\in\mathbb{N}}$, $\langle\mu_\kappa\rangle_{\kappa\in K}$ such that

K is a countable set of infinite cardinals,

for $i\in\mathbb{N}\cup K$, μ_i is a non-negative countably additive functional on $\mathfrak{C}$, and $\sum_{i\in\mathbb{N}\cup K}\mu_i c=\bar\mu c$ for every $c\in\mathfrak{C}$,

$\mu_{n+1}\le\mu_n$ for every $n\in\mathbb{N}$, and $\mu_\kappa\ne 0$ for $\kappa\in K$,

setting $e_i=[\![\mu_i>0]\!]\in\mathfrak{C}$, and giving the principal ideal $\mathfrak{C}_{e_i}$ generated by e_i the measure $\mu_i\restriction\mathfrak{C}_{e_i}$ for each $i\in\mathbb{N}\cup K$, we have a measure algebra isomorphism

$$\pi:\mathfrak{A}\to\prod_{n\in\mathbb{N}}\mathfrak{C}_{e_n}\times\prod_{\kappa\in K}\mathfrak{C}_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$$

such that

$$\pi c=(\langle c\cap e_n\rangle_{n\in\mathbb{N}},\langle(c\cap e_\kappa)\otimes 1\rangle_{\kappa\in K})$$

for each $c\in C$, writing $c\otimes 1$ for the canonical image in $C_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$ of $c\in\mathfrak{C}_{e_\kappa}$.

proof (a) I aim to use the construction of 333H, but taking much more care over the choice of $\langle a_i\rangle_{i\in I}$ in part (a) of the proof there. We start by taking $\langle a_n\rangle_{n\in\mathbb{N}}$ as in 333J, and setting $\mu_n c=\bar\mu(a_n\cap c)$ for every $n\in\mathbb{N}$, $c\in\mathfrak{C}$; then these a_n will deal with the relative atoms over $\mathfrak{C}$.

(b) The further idea required here concerns the treatment of infinite κ. Let $\langle b_i\rangle_{i\in I}$ be any partition of unity in $\mathfrak{A}$ consisting of non-zero members of $\mathfrak{A}$ which are relatively Maharam-type-homogeneous over $\mathfrak{C}$, and $\langle\kappa_i\rangle_{i\in I}$ the corresponding cardinals, so that $\kappa_i=0$ iff b_i is a relative atom. Set $I_1=\{i:i\in I,\ \kappa_i\ge\omega\}$. Set $K=\{\kappa_i:i\in I_1\}$, so that K is a countable set of infinite cardinals, and for $\kappa\in K$ set $J_\kappa=\{i:\kappa_i=\kappa\}$, $a_\kappa=\sup_{i\in J_\kappa}b_i$ for $\kappa\in K$. Now every a_κ is relatively Maharam-type-homogeneous over $\mathfrak{C}$. **P** (Compare 332H.) J_κ must be countable, because $\mathfrak{A}$ is ccc. If $0\ne a\subseteq a_\kappa$, there is some $i\in J_\kappa$ such that $a\cap b_i\ne 0$; now

$$\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)\ge\tau_{\mathfrak{C}_{a\cap b_i}}(\mathfrak{A}_{a\cap b_i})=\kappa_i=\kappa$$

(333Ba). At the same time, for each $i\in J_\kappa$, there is a set $D_i\subseteq\mathfrak{A}_{b_i}$ such that $\#(D_i)=\kappa$ and $\mathfrak{C}_{b_i}\cup D_i$ τ-generates $\mathfrak{A}_{b_i}$. Set $D=\bigcup_{i\in J_\kappa}D_i\cup\{b_i:i\in J_\kappa\}$; then

$$\#(D)\le\max(\omega,\#(J_\kappa),\sup_{i\in K}\#(D_i))=\kappa.$$

Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}_{a_\kappa}$ generated by $\mathfrak{C}_{a_\kappa} \cup D$. Then

$$\mathfrak{C}_{b_i} \cup D_i \subseteq \{b \cap b_i : b \in \mathfrak{B}\} = \mathfrak{B} \cap \mathfrak{A}_{b_i}$$

so $\mathfrak{B} \supseteq \mathfrak{A}_{b_i}$ for each $i \in J_\kappa$, and $\mathfrak{B} = \mathfrak{A}_{a_\kappa}$. Thus $\mathfrak{C}_{a_\kappa} \cup D$ τ-generates $\mathfrak{A}_{a_\kappa}$, and

$$\tau_{\mathfrak{C}_{a_\kappa}}(\mathfrak{A}_{a_\kappa}) \le \kappa \le \min_{0 \ne a \subseteq a_\kappa} \tau_{\mathfrak{C}_a}(\mathfrak{A}_a).$$

This shows that a_κ is relatively Maharam-type-homogeneous over $\mathfrak{C}$, with $\tau_{\mathfrak{C}_{a_\kappa}}(\mathfrak{A}_{a_\kappa}) = \kappa$. **Q**

Since evidently $\langle J_\kappa \rangle_{\kappa \in K}$ and $\langle a_\kappa \rangle_{\kappa \in K}$ are disjoint, and $\sup_{\kappa \in K} a_\kappa = \sup_{i \in I_1} b_i$, this process yields a partition $\langle a_i \rangle_{i \in \mathbb{N} \cup K}$ of unity in $\mathfrak{A}$. Now the arguments of 333H show that we get an isomorphism π of the kind described.

(c) To see that the families $\langle \mu_n \rangle_{n \in \mathbb{N}}$, $\langle \mu_\kappa \rangle_{\kappa \in K}$ (and therefore the e_i and the $(\mathfrak{C}_{e_i}, \mu_i \restriction \mathfrak{C}_{e_i})$, but not π) are uniquely defined, argue as follows. Let A be the set of those $a \in \mathfrak{A}$ which are relatively Maharam-type-homogeneous over $\mathfrak{C}$. Take families $\langle \tilde{\mu}_n \rangle_{n \in \mathbb{N}}$, $\langle \tilde{\mu}_\kappa \rangle_{\kappa \in \tilde{K}}$ which correspond to an isomorphism

$$\tilde{\pi} : \mathfrak{A} \to \mathfrak{D} = \prod_{n \in \mathbb{N}} \mathfrak{C}_{\tilde{e}_n} \times \prod_{\kappa \in \tilde{K}} \mathfrak{C}_{\tilde{e}_\kappa} \widehat{\otimes} \mathfrak{B}_\kappa,$$

writing $\tilde{e}_i = [\![\tilde{\mu}_i > 0]\!]$ for $i \in \mathbb{N} \cup \tilde{K}$. In the simple product $\prod_{n \in \mathbb{N}} \mathfrak{C}_{\tilde{e}_n} \times \prod_{\kappa \in \tilde{K}} \mathfrak{C}_{\tilde{e}_\kappa} \widehat{\otimes} \mathfrak{B}_\kappa$, we have a partition of unity $\langle e_i^* \rangle_{i \in \mathbb{N} \cup \tilde{K}}$ corresponding to the product structure. Now for $d \subseteq e_i^*$, we have

$$\begin{aligned} \tau_{\tilde{\pi}[\mathfrak{C}]_d}(\mathfrak{D}_d) &= 0 \text{ if } i \in \mathbb{N}, \\ &= \kappa \text{ if } i = \kappa \in \tilde{K}. \end{aligned}$$

So $\tilde{K}$ must be

$$\{\kappa : \kappa \ge \omega, \exists\, a \in A, \tau_{\mathfrak{C}_a}(\mathfrak{A}_a) = \kappa\} = K,$$

and for $\kappa \in \tilde{K}$,

$$\tilde{\pi}^{-1} e_\kappa^* = \sup\{a : a \in A, \tau_{\mathfrak{C}_a}(\mathfrak{A}_a) = \kappa\} = a_\kappa,$$

so that $\tilde{\mu}_\kappa = \mu_\kappa$. On the other hand, $\langle \tilde{\pi}^{-1} e_n^* \rangle_{n \in \mathbb{N}}$ must be a disjoint sequence with supremum $\sup A$, and the corresponding functionals $\tilde{\mu}_n$ are supposed to form a non-increasing sequence, so must be equal to the μ_n by 333J.

333L Remark Thus for the classification of structures $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$, where $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra, it will be enough to classify objects $(\mathfrak{C}, \bar{\mu}, \langle \mu_n \rangle_{n \in \mathbb{N}}, \langle \mu_\kappa \rangle_{\kappa \in K})$, where

$(\mathfrak{C}, \bar{\mu})$ is a totally finite measure algebra,

$\langle \mu_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of non-negative countably additive functionals on $\mathfrak{C}$,

K is a countable set of infinite cardinals (possibly empty),

$\langle \mu_\kappa \rangle_{\kappa \in K}$ is a family of non-zero non-negative countably additive functionals on $\mathfrak{C}$,

$\sum_{n=0}^\infty \mu_n + \sum_{\kappa \in K} \mu_\kappa = \bar{\mu}$.

To do this we need the concept of 'standard extension' of a countably additive functional on a closed subalgebra of a measure algebra, treated in 327F-327G, together with the following idea.

333M Lemma Let $(\mathfrak{C}, \bar{\mu})$ be a totally finite measure algebra and $\langle \mu_i \rangle_{i \in I}$ a family of countably additive functionals on $\mathfrak{C}$. For $i \in I$, $\alpha \in \mathbb{R}$ set $e_{i\alpha} = [\![\mu_i > \alpha\bar{\mu}]\!]$ (326T), and let $\mathfrak{C}_0$ be the closed subalgebra of $\mathfrak{C}$ generated by $\{e_{i\alpha} : i \in I, \alpha \in \mathbb{R}\}$. Write Σ for the σ-algebra of subsets of $\mathbb{R}^I$ generated by sets of the form $E_{i\alpha} = \{x : x(i) > \alpha\}$ as i runs over I and α runs over $\mathbb{R}$. Then

(a) there is a measure μ, with domain Σ, such that there is a measure-preserving isomorphism $\pi : \Sigma/\mathcal{N}_\mu \to \mathfrak{C}_0$ for which $\pi E_{i\alpha}^\bullet = e_{i\alpha}$ for every $i \in I$ and $\alpha \in \mathbb{R}$, writing $\mathcal{N}_\mu$ for $\mu^{-1}[\{0\}]$;

(b) this formula determines both μ and π;

(c) for every $E \in \Sigma$ and $i \in I$, we have

$$\mu_i \pi E^\bullet = \int_E x(i) \mu(dx);$$

(d) for every $i \in I$, μ_i is the standard extension of $\mu_i \restriction \mathfrak{C}_0$ to $\mathfrak{C}$;

(e) for every $i \in I$, $\mu_i \ge 0$ iff $x(i) \ge 0$ for μ-almost every x;

(f) for every $i, j \in I$, $\mu_i \ge \mu_j$ iff $x(i) \ge x(j)$ for μ-almost every x;

(g) for every $i \in I$, $\mu_i = 0$ iff $x(i) = 0$ for μ-almost every x.

proof (a) Express $(\mathfrak{C},\bar\mu)$ as the measure algebra of a measure space (Y,T,ν); write $\phi:\mathrm{T}\to\mathfrak{C}$ for the corresponding homomorphism. For each $i\in I$ let $f_i:Y\to\mathbb{R}$ be a T-measurable, ν-integrable function such that $\int_H f_i=\mu_i\phi H$ for every $H\in\mathrm{T}$. Define $\psi:Y\to\mathbb{R}^I$ by setting $\psi(y)=\langle f_i(y)\rangle_{i\in I}$; then $\psi^{-1}[E_{i\alpha}]\in\Sigma$, and $e_{i\alpha}=\phi(\psi^{-1}[E_{i\alpha}])$ for every $i\in I$ and $\alpha\in\mathbb{R}$. (See part (a) of the proof of 327F.) So $\{E:E\subseteq\mathbb{R}^I,\,\psi^{-1}[E]\in\mathrm{T}\}$, which is a σ-algebra of subsets of $\mathbb{R}^I$, contains every $E_{i\alpha}$, and therefore includes Σ; that is, $\psi^{-1}[E]\in\mathrm{T}$ for every $E\in\Sigma$. Accordingly we may define μ by setting $\mu E=\nu\psi^{-1}[E]$ for every $E\in\Sigma$, and μ will be a measure on $\mathbb{R}^I$ with domain Σ. The Boolean homomorphism $E\mapsto\phi\psi^{-1}[E]:\Sigma\to\mathfrak{C}$ has kernel $\mathcal{N}_\mu$, so descends to a homomorphism $\pi:\Sigma/\mathcal{N}_\mu\to\mathfrak{C}$, which is measure-preserving. To see that $\pi[\Sigma/\mathcal{N}_\mu]=\mathfrak{C}_0$, observe that because Σ is the σ-algebra generated by $\{E_{i\alpha}:i\in I,\,\alpha\in\mathbb{R}\}$, $\pi[\Sigma/\mathcal{N}_\mu]$ must be the closed subalgebra of $\mathfrak{C}$ generated by $\{\pi E^\bullet_{i\alpha}:i\in I,\,\alpha\in\mathbb{R}\}=\{e_{i\alpha}:i\in I,\,\alpha\in\mathbb{R}\}$, which is $\mathfrak{C}_0$.

(b) Now suppose that μ', π' have the same properties. Consider

$$\mathcal{A}=\{E:E\in\Sigma,\,\pi E^\bullet=\pi' E^\circ\},$$

where I write $E^\bullet$ for the equivalence class of E in $\Sigma/\mathcal{N}_\mu$, and E° for the equivalence class of E in $\Sigma/\mathcal{N}_{\mu'}$. Then $\mathcal{A}$ is a σ-subalgebra of Σ, because $E\mapsto\pi E^\bullet$, $E\mapsto\pi'E^\circ$ are both sequentially order-continuous Boolean homomorphisms, and contains every $E_{i\alpha}$, so must be the whole of Σ. Consequently

$$\mu E=\bar\mu\pi E^\bullet=\bar\mu\pi' E^\circ=\mu' E$$

for every $E\in\Sigma$, and $\mu'=\mu$; it follows at once that $\pi'=\pi$. So μ and π are uniquely determined.

(c) If $E\in\Sigma$ and $i\in I$,

$$\int_E x(i)\mu(dx)=\int x(i)\chi E(x)\mu(dx)=\int\psi(y)(i)\chi E(\psi(y))\nu(dy)$$

(applying 235G[1] to the inverse-measure-preserving function $\psi:Y\to\mathbb{R}^I$)

$$=\int_{\psi^{-1}[E]}f_i(y)\nu(dy)$$

(by the definition of ψ)

$$=\mu_i\phi(\psi^{-1}[E])$$

(by the choice of f_i)

$$=\mu_i\pi E^\bullet$$

by the definition of π.

(d) For every $\alpha\in\mathbb{R}$, $[\![\mu_i>\alpha\bar\mu]\!]$ belongs to $\mathfrak{C}_0$, so must be equal to $[\![\mu_i{\restriction}\mathfrak{C}_0>\alpha\bar\mu{\restriction}\mathfrak{C}_0]\!]$. Thus μ_i is the standard extension of $\mu_i{\restriction}\mathfrak{C}_0$ (327G).

(e)-(g) The point is that, because the standard-extension operator is order-preserving (327F(b-ii)),

$$\begin{aligned}
\mu_i\ge0&\iff\mu_i{\restriction}\mathfrak{C}_0\ge0\\
&\iff\int_E x(i)\mu(dx)\ge0\text{ for every }E\in\Sigma\\
&\iff x(i)\ge0\ \mu\text{-a.e.},\\
\mu_i\ge\mu_j&\iff\mu_i{\restriction}\mathfrak{C}_0\ge\mu_j{\restriction}\mathfrak{C}_0\\
&\iff\int_E x(i)\mu(dx)\ge\int_E x(j)\mu(dx)\text{ for every }E\in\Sigma\\
&\iff x(i)\ge x(j)\ \mu\text{-a.e.},\\
\mu_i=0&\iff\mu_i{\restriction}\mathfrak{C}_0=0\\
&\iff\int_E x(i)\mu(dx)=0\text{ for every }E\in\Sigma\\
&\iff x(i)=0\ \mu\text{-a.e.}.
\end{aligned}$$

[1] Formerly 235I.

333N A canonical form for closed subalgebras We now have all the elements required to describe a canonical form for structures

$$(\mathfrak{A}, \bar{\mu}, \mathfrak{C}),$$

where $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$. The first step is the matching of such structures with structures

$$(\mathfrak{C}, \bar{\mu}, \langle \mu_n \rangle_{n\in\mathbb{N}}, \langle \mu_\kappa \rangle_{\kappa\in K}),$$

where $(\mathfrak{C}, \bar{\mu})$ is a totally finite measure algebra, $\langle \mu_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-negative countably additive functionals on $\mathfrak{C}$, K is a countable set of infinite cardinals, $\langle \mu_\kappa \rangle_{\kappa\in K}$ is a family of non-zero non-negative countably additive functionals on $\mathfrak{C}$, and $\sum_{n=0}^{\infty} \mu_n + \sum_{\kappa\in K} \mu_\kappa = \bar{\mu}$; this is the burden of 333K.

Next, given any structure of this second kind, we have a corresponding closed subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$, a measure μ on $\mathbb{R}^I$, where $I = \mathbb{N} \cup K$, and an isomorphism π from the measure algebra $\mathfrak{C}_0^*$ of μ to $\mathfrak{C}_0$, all uniquely defined from the family $\langle \mu_i \rangle_{i\in I}$ by the process of 333M. For any E belonging to the domain Σ of μ, and $i \in I$, we have

$$\mu_i \pi E^\bullet = \int_E x(i) \mu(dx)$$

(333Mc), so that $\mu_i \restriction \mathfrak{C}_0$ is fixed by π and μ. Moreover, the functionals μ_i can be recovered from their restrictions to $\mathfrak{C}_0$ by the formulae of 327F (333Md). Thus from $(\mathfrak{C}, \bar{\mu}, \langle \mu_i \rangle_{i\in I})$ we are led, by a canonical and reversible process, to the structure

$$(\mathfrak{C}, \bar{\mu}, \mathfrak{C}_0, I, \mu, \pi).$$

But the extension $\mathfrak{C}$ of $\mathfrak{C}_0 = \pi[\mathfrak{C}_0^*]$ can be described, up to isomorphism, by the same process as before; that is, it corresponds to a sequence $\langle \theta'_n \rangle_{n\in\mathbb{N}}$ and a family $\langle \theta'_\kappa \rangle_{\kappa\in L}$ of countably additive functionals on $\mathfrak{C}_0$ satisfying the conditions of 333K. We can transfer these to $\mathfrak{C}_0^*$, where they correspond to families $\langle \theta_n \rangle_{n\in\mathbb{N}}$, $\langle \theta_\kappa \rangle_{\kappa\in L}$ of absolutely continuous countably additive functionals defined on Σ, setting

$$\theta_j E = \theta'_j \pi E^\bullet$$

for $E \in \Sigma$, $j \in \mathbb{N} \cup L$. This process too is reversible; every absolutely continuous countably additive functional ν on Σ corresponds to countably additive functionals on $\mathfrak{C}_0^*$ and $\mathfrak{C}_0$. Let me repeat that the results of 327F mean that the whole structure $(\mathfrak{C}, \bar{\mu}, \langle \mu_i \rangle_{i\in I})$ can be recovered from $(\mathfrak{C}_0, \bar{\mu}\restriction\mathfrak{C}_0, \langle \mu_i \restriction \mathfrak{C}_0 \rangle_{i\in I})$ if we can get the description of $(\mathfrak{C}, \bar{\mu})$ right, and that the requirements $\mu_i \ge 0$, $\mu_n \ge \mu_{n+1}$, $\mu_\kappa \ne 0$, $\sum_{i\in I} \mu_i = \bar{\mu}$ imposed in 333K will survive the process (327F(b-iv)).

Putting all this together, a structure $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$ leads, in a canonical and (up to isomorphism) reversible way, to a structure

$$(K, \mu, L, \langle \theta_j \rangle_{j\in\mathbb{N}\cup L})$$

such that

K and L are countable sets of infinite cardinals,

μ is a totally finite measure on $\mathbb{R}^I$, where $I = \mathbb{N} \cup K$, and its domain Σ is precisely the σ-algebra of subsets of $\mathbb{R}^I$ defined by the coordinate functionals,

for μ-almost every $x \in \mathbb{R}^I$ we have $x(i) \ge 0$ for every $i \in I$, $x(n) \ge x(n+1)$ for every $n \in \mathbb{N}$ and $\sum_{i\in I} x(i) = 1$,

for $\kappa \in K$, $\mu\{x : x(\kappa) > 0\} > 0$,

(these two clauses corresponding to the requirements $\mu_i \ge 0$, $\mu_n \ge \mu_{n+1}$, $\sum_{i\in I} \mu_i = \bar{\mu}$, $\mu_\kappa \ne 0$ – see 333M(e)-(g))

for $j \in J = \mathbb{N} \cup L$, θ_j is a non-negative countably additive functional on Σ,

$\theta_n \ge \theta_{n+1}$ for every $n \in \mathbb{N}$, $\theta_\kappa \ne 0$ for every $\kappa \in L$, $\sum_{j\in J} \theta_j = \mu$.

333O Remark I do not envisage quoting the result above very often. Indeed I do not claim that its final form adds anything to the constituent results 333K, 327F and 333M. I have taken the trouble to spell it out, however, because it does not seem to me obvious that the trail is going to end quite as quickly as it does. We need to use 333K twice, but only twice. The most important use of the ideas expressed here, I suppose, is in constructing examples to strengthen our intuition for the structures $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$ under consideration, and I hope that you will experiment in this direction.

333P At the risk of trespassing on the province of Chapter 38, I turn now to a special type of closed subalgebra, in which there is a particularly elegant alternative form for a canonical description. The first step is an important result concerning automorphisms of homogeneous probability algebras.

Proposition Let $(\mathfrak{B},\bar\nu)$ be a homogeneous probability algebra. Then there is a measure-preserving automorphism $\phi:\mathfrak{B}\to\mathfrak{B}$ such that

$$\lim_{n\to\infty}\bar\nu(c\cap\phi^n(b))=\bar\nu c\cdot\bar\nu b$$

for all b, $c\in\mathfrak{B}$.

proof (a) The case $\mathfrak{B}=\{0,1\}$ is trivial (ϕ is, and must be, the identity map) so we may take it that $(\mathfrak{B},\bar\nu)=(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ for some infinite cardinal κ is an infinite cardinal. Because $\#(\kappa\times\mathbb{Z})=\max(\omega,\kappa)=\kappa$, there must be a permutation $\theta:\kappa\to\kappa$ such that every orbit of θ in κ is infinite (take θ to correspond to the bijection $(\xi,n)\mapsto(\xi,n+1):\kappa\times\mathbb{Z}\to\kappa\times\mathbb{Z}$). This induces a permutation $\hat\theta:\{0,1\}^\kappa\to\{0,1\}^\kappa$ through the formula $\hat\theta(x)=x\theta$ for every $x\in\{0,1\}^\kappa$, and of course $\hat\theta$ is an automorphism of the measure space $(\{0,1\}^\kappa,\nu_\kappa)$. We therefore have a corresponding automorphism ϕ of $\mathfrak{B}$, setting $\phi E^\bullet=(\hat\theta^{-1}[E])^\bullet$ for every E in the domain T_κ of ν_κ.

(b) Let $\mathcal{E}$ be the family of subsets E of $\{0,1\}^\kappa$ which are determined by coordinates in finite sets, that is, are expressible in the form $E=\{x:x\restriction J\in\tilde E\}$ for some finite set $J\subseteq\kappa$ and some $\tilde E\subseteq\{0,1\}^J$; equivalently, expressible as a finite union of basic cylinder sets $\{x:x\restriction J=y\}$. Then $\mathcal{E}$ is a subalgebra of T_κ, so $\mathfrak{C}=\{E^\bullet:E\in\mathcal{E}\}$ is a subalgebra of $\mathfrak{B}$.

(c) Now if b, $c\in\mathfrak{C}$, there is an $n\in\mathbb{N}$ such that $\bar\nu(c\cap\phi^m(b))=\bar\nu c\cdot\bar\nu b$ for every $m\ge n$. **P** Express b, c as $E^\bullet$, $F^\bullet$ where $E=\{x:x\restriction J\in\tilde E\}$, $F=\{x:x\restriction K\in\tilde F\}$ and J, K are finite subsets of κ. For $\xi\in K$, all the $\theta^n(\xi)$ are distinct, so only finitely many of them can belong to J; as K is also finite, there is an n such that $\theta^m[J]\cap K=\emptyset$ for every $m\ge n$. Fix $m\ge n$. Then $\phi^m(b)=H^\bullet$ where

$$H=\{x:x\theta^m\in E\}=\{x:x\theta^m\restriction J\in\tilde E\}=\{x:x\restriction L\in\tilde H\},$$

where $L=\theta^m[J]$ and $\tilde H=\{z\theta^{-m}:z\in\tilde E\}$. So $\bar\nu(c\cap\phi^m(b))=\nu(F\cap H)$. But L and K are disjoint, because $m\ge n$, so F and H must be independent (cf. 272K), and

$$\bar\nu(c\cap\phi^m(b))=\nu F\cdot\nu H=\nu F\cdot\nu E=\bar\nu c\cdot\bar\nu b,$$

as claimed. **Q**

(d) Now recall that for every $E\in\mathrm{T}_\kappa$ and $\epsilon>0$ there is an $E'\in\mathcal{E}$ such that $\nu(E\triangle E')\le\epsilon$ (254Fe). So, given b, $c\in\mathfrak{B}$ and $\epsilon>0$, we can find b', $c'\in\mathfrak{C}$ such that $\bar\nu(b\triangle b')\le\epsilon$ and $\bar\nu(c\triangle c')\le\epsilon$, and in this case

$$\begin{aligned}
\limsup_{n\to\infty}&|\bar\nu(c\cap\phi^n(b))-\bar\nu c\cdot\bar\nu b|\\
&\le\limsup_{n\to\infty}|\bar\nu(c\cap\phi^n(b))-\bar\nu(c'\cap\phi^n(b'))|\\
&\qquad\qquad+|\bar\nu(c'\cap\phi^n(b'))-\bar\nu c'\cdot\bar\nu b'|+|\bar\nu c\cdot\bar\nu b-\bar\nu c'\cdot\bar\nu b'|\\
&=\limsup_{n\to\infty}|\bar\nu(c\cap\phi^n(b))-\bar\nu(c'\cap\phi^n(b'))|+|\bar\nu c\cdot\bar\nu b-\bar\nu c'\cdot\bar\nu b'|\\
&\le\limsup_{n\to\infty}\bar\nu(c\triangle c')+\bar\nu(\phi^n(b)\triangle\phi^n(b'))\\
&\qquad\qquad+\bar\nu c|\bar\nu b-\bar\nu b'|+|\bar\nu c-\bar\nu c'|\bar\nu b'\\
&\le\bar\nu(c\triangle c')+\bar\nu(b\triangle b')+\bar\nu c\cdot\bar\nu(b\triangle b')+\bar\nu(c\triangle c')\cdot\bar\nu b'\le4\epsilon.
\end{aligned}$$

As ϵ is arbitrary,

$$\lim_{n\to\infty}\bar\nu(c\cap\phi^n(b))=\bar\nu c\cdot\bar\nu b,$$

as required.

Remark Automorphisms of this type are called **mixing** (see 372O below).

333Q Corollary Let $(\mathfrak{C},\bar\mu_0)$ be a totally finite measure algebra and $(\mathfrak{B},\bar\nu)$ a probability algebra which is *either* homogeneous *or* purely atomic with finitely many atoms all of the same measure. Let $(\mathfrak{A},\bar\mu)$ be the localizable measure algebra free product of $(\mathfrak{C},\bar\mu_0)$ and $(\mathfrak{B},\bar\nu)$. Then there is a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that

$$\{a : a \in \mathfrak{A},\, \pi a = a\} = \{c \otimes 1 : c \in \mathfrak{C}\}.$$

Remark I am following 315N in using the notation $c \otimes b$ for the intersection in $\mathfrak{A}$ of the canonical images of $c \in \mathfrak{C}$ and $b \in \mathfrak{B}$. By 325D(c-i) I need not distinguish between the free product $\mathfrak{C} \otimes \mathfrak{B}$ and its image in $\mathfrak{A}$.

proof Set $\gamma = \bar{\mu}1 = \bar{\mu}_0 1$.

(a) Let me deal with the case of atomic $\mathfrak{B}$ first. In this case, if $\mathfrak{B}$ has $n+1$ atoms $b_0, \dots, b_n$, let $\phi : \mathfrak{B} \to \mathfrak{B}$ be the measure-preserving homomorphism cyclically permuting these atoms, so that $\phi b_0 = b_1, \dots, \phi b_n = b_0$. Because ϕ is an automorphism of $(\mathfrak{B}, \bar{\nu})$, it induces an automorphism π of $(\mathfrak{A}, \bar{\mu})$; any member of $\mathfrak{A}$ is uniquely expressible as $a = \sup_{i \le n} c_i \otimes b_i$, and now $\pi a = \sup_{i \le n} c_i \otimes b_{i+1}$, if we set $b_{n+1} = b_0$. So $\pi a = a$ iff $c_i = c_{i+1}$ for $i < n$ and $c_n = c_0$, that is, iff all the c_i are the same and $a = \sup_{i \le n} c \otimes b_i = c \otimes 1$ for some $c \in \mathfrak{C}$.

(b) If $\mathfrak{B}$ is homogeneous, then take a mixing measure-preserving automorphism $\phi : \mathfrak{B} \to \mathfrak{B}$ as described in 333P. As in (a), this corresponds to an automorphism π of $\mathfrak{A}$, defined by saying that $\pi(c \otimes b) = c \otimes \phi(b)$ for every $c \in \mathfrak{C}$, $b \in \mathfrak{A}$. Of course $\pi(c \otimes 1) = c \otimes 1$ for every $c \in \mathfrak{C}$.

Now suppose that $a \in \mathfrak{A}$ and $\pi a = a$; I need to show that $a \in \mathfrak{C}_1 = \{c \otimes 1 : c \in \mathfrak{C}\}$. Take any $\epsilon \in \left]0, \frac{1}{4}\right]$. We know that $\mathfrak{C} \otimes \mathfrak{B}$ is topologically dense in $\mathfrak{A}$ (325Dc), so there is an $a' \in \mathfrak{C} \otimes \mathfrak{B}$ such that $\bar{\mu}(a \triangle a') \le \epsilon^2$. Express a' as $\sup_{i \in I} c_i \otimes b_i$, where $\langle c_i \rangle_{i \in I}$ is a finite partition of unity in $\mathfrak{C}$ (315Oa). Then

$$\pi a' = \sup_{i \in I} c_i \otimes \phi(b_i), \quad \pi^n(a') = \sup_{i \in I} c_i \otimes \phi^n(b_i) \text{ for every } n \in \mathbb{N}.$$

So we can get a formula for

$$\begin{aligned}\lim_{n \to \infty} \bar{\mu}(a' \cap \pi^n(a')) &= \lim_{n \to \infty} \bar{\mu}(\sup_{i \in I} c_i \otimes (b_i \cap \phi^n(b_i))) \\ &= \lim_{n \to \infty} \sum_{i \in I} \bar{\mu}_0 c_i \cdot \bar{\nu}(b_i \cap \phi^n(b_i)) = \sum_{i \in I} \bar{\mu}_0 c_i \cdot (\bar{\nu} b_i)^2.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{i \in I} \bar{\mu}_0 c_i \cdot (\bar{\nu} b_i)^2 &= \lim_{n \to \infty} \bar{\mu}(a' \cap \pi^n(a')) \\ &\ge \limsup_{n \to \infty} \bar{\mu}(a \cap \pi^n(a)) - \bar{\mu}(a \triangle a') - \bar{\mu}(\pi^n(a) \triangle \pi^n(a')) \\ &= \bar{\mu} a - 2\bar{\mu}(a \triangle a') \ge \bar{\mu} a' - 3\bar{\mu}(a \triangle a') \ge \sum_{i \in I} \bar{\mu}_0 c_i \cdot \bar{\nu} b_i - 3\epsilon^2,\end{aligned}$$

that is,

$$\textstyle\sum_{i \in I} \bar{\mu}_0 c_i \cdot \bar{\nu} b_i \cdot (1 - \bar{\nu} b_i) \le 3\epsilon^2.$$

But this means that, setting $J = \{i : i \in I,\, \bar{\nu} b_i \cdot (1 - \bar{\nu} b_i) \ge \epsilon\}$, we must have $\sum_{i \in J} \bar{\mu}_0 c_i \le 3\epsilon$. Set

$$K = \{i : i \in I,\, \bar{\nu} b_i \ge 1 - 2\epsilon\}, \quad L = \{i : i \in I \setminus K,\, \bar{\nu} b_i \le 2\epsilon\}, \quad c = \sup_{i \in K} c_i.$$

Then $I \setminus (K \cup L) \subseteq J$, so

$$\begin{aligned}\bar{\mu}(a' \triangle (c \otimes 1)) &= \sum_{i \in I \setminus K} \bar{\mu}_0 c_i \cdot \bar{\nu} b_i + \sum_{i \in K} \bar{\mu}_0 c_i \cdot (1 - \bar{\nu} b_i) \\ &\le \sum_{i \in J} \bar{\mu}_0 c_i \cdot \bar{\nu} b_i + \sum_{i \in L} \bar{\mu}_0 c_i \cdot \bar{\nu} b_i + \sum_{i \in K} \bar{\mu}_0 c_i \cdot (1 - \bar{\nu} b_i) \\ &\le \sum_{i \in J} \bar{\mu}_0 c_i + 2\epsilon \sum_{i \in L} \bar{\mu}_0 c_i + 2\epsilon \sum_{i \in K} \bar{\mu}_0 c_i \le 3\epsilon + 2\epsilon\gamma,\end{aligned}$$

and

$$\bar{\mu}(a \triangle (c \otimes 1)) \le \epsilon^2 + 3\epsilon + 2\epsilon\gamma.$$

As ϵ is arbitrary, a belongs to the topological closure of $\mathfrak{C}_1$. But of course $\mathfrak{C}_1$ is a closed subalgebra of $\mathfrak{A}$ (325Dd), so must actually contain a.

As a is arbitrary, π has the required property.

333R Now for the promised special type of closed subalgebra. It will be convenient to have the following temporary notation. For an integer $n \ge 1$, I will (for this paragraph only) write $\mathfrak{B}_n$ for the power set of $\{0, \dots, n\}$ and set $\bar\nu_n b = \frac{1}{n+1}\#(b)$ for $b \in \mathfrak{B}_n$. (The natural interpretation of $(\mathfrak{B}_n, \bar\nu_n)$ as defined in 333Ad corresponds to $(\mathfrak{B}_{2^{n+1}-1}, \bar\nu_{2^{n+1}-1})$ here, so we have a match if $n = 0$.)

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\mathfrak{C}$ a subset of $\mathfrak{A}$. Then the following are equiveridical:

(i) there is some set G of measure-preserving automorphisms of $\mathfrak{A}$ such that

$$\mathfrak{C} = \{c : c \in \mathfrak{A},\ \pi c = c \text{ for every } \pi \in G\};$$

(ii) $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ and there is a partition of unity $\langle e_i\rangle_{i\in I}$ in $\mathfrak{C}$, where I is a countable set of cardinals, such that $\mathfrak{A}$ is isomorphic to $\prod_{i\in I}\mathfrak{C}_{e_i}\widehat{\otimes}\mathfrak{B}_i$, writing $\mathfrak{C}_{e_i}$ for the principal ideal of $\mathfrak{C}$ generated by e_i and endowed with $\bar\mu\restriction\mathfrak{C}_{e_i}$, and $\mathfrak{C}_{e_i}\widehat{\otimes}\mathfrak{B}_i$ for the localizable measure algebra free product of $\mathfrak{C}_{e_i}$ and $\mathfrak{B}_i$ – the isomorphism being one which takes any $c \in \mathfrak{C}$ to $\langle (c \cap e_i)\otimes 1\rangle_{i\in I}$, as in 333H and 333K;

(iii) there is a single measure-preserving automorphism π of $\mathfrak{A}$ such that

$$\mathfrak{C} = \{c : c \in \mathfrak{A},\ \pi c = c\}.$$

proof (a)(i)⇒(ii)(α) $\mathfrak{C}$ is a subalgebra because every $\pi \in G$ is a Boolean homomorphism, and it is order-closed because every π is order-continuous (324Kb). (Or, if you prefer, $\mathfrak{C}$ is topologically closed because every π is continuous.)

(β) Because $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$, its embedding can be described in terms of families $\langle\mu_n\rangle_{n\in\mathbb{N}}$, $\langle\mu_\kappa\rangle_{\kappa\in K}$ as in Theorem 333K. Set $I = K \cup \mathbb{N}$. Recall that each μ_i is defined by setting $\mu_i c = \bar\mu(c \cap a_i)$, where $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ (see the proofs of 333H and 333K). For $\kappa \in K$, a_κ is the maximal element of $\mathfrak{A}$ which is relatively Maharam-type-homogeneous over $\mathfrak{C}$ with relative Maharam type κ (part (b) of the proof of 333K). Consequently we must have $\pi a_\kappa = a_\kappa$ for any measure algebra automorphism of $(\mathfrak{A}, \bar\mu)$ which leaves $\mathfrak{C}$ invariant; in particular, for every $\pi \in G$. Thus $a_\kappa \in \mathfrak{C}$ for every $\kappa \in K$.

(γ) Now consider the relatively atomic part of $\mathfrak{A}$. The elements a_n, for $n \in \mathbb{N}$, are not uniquely defined. However, the functionals μ_n and their supports $e'_n = [\![\mu_n > 0]\!]$ are uniquely defined from the structure $(\mathfrak{A}, \bar\mu, \mathfrak{C})$ and therefore invariant under G. Since

$$\begin{aligned} e'_n &= 1 \setminus \sup\{c : c \in \mathfrak{C},\ \mu_n c \le 0\} \\ &= 1 \setminus \sup\{c : c \in \mathfrak{C},\ c \cap a_n = 0\} = \inf\{c : c \in \mathfrak{C},\ c \supseteq a_n\}, \end{aligned}$$

and $\sup_{n\in\mathbb{N}} a_n = 1 \setminus \sup_{\kappa\in K} a_\kappa$ belongs to $\mathfrak{C}$, while $e'_n \supseteq e'_{n+1}$ for every n, we must have $e'_0 = \sup_{n\in\mathbb{N}} a_n$.

Let G^* be the set of all those automorphisms π of the measure algebra $(\mathfrak{A}, \bar\mu)$ such that $\pi c = c$ for every $c \in \mathfrak{C}$. Then of course G^* is a group including G. Now $\sup_{\pi\in G^*}\pi a_n$ must be invariant under every member of G^*, so belongs to $\mathfrak{C}$; it includes a_n and is included in any member of $\mathfrak{C}$ including a_n, so must be e'_n.

(δ) I claim now that if $n \in \mathbb{N}$ then $e'_n \cap [\![\mu_0 > \mu_n]\!] = 0$. **P?** Otherwise, set $c = [\![\mu_0 > \mu_n]\!] \cap e'_n$. Then $\mu_0 c > 0$ so $c \cap a_0 \ne 0$. By the last remark in (γ), there is a $\pi \in G^*$ such that $c \cap a_0 \cap \pi a_n \ne 0$. Now there is a $c' \in \mathfrak{C}$ such that $c \cap a_0 \cap \pi a_n = c' \cap a_0$, and of course we may suppose that $c' \subseteq c$. But this means that

$$\pi(c' \cap a_n) = c' \cap \pi a_n \supseteq c' \cap a_0 \cap \pi a_n = c' \cap a_0,$$

so that

$$\mu_n c' = \bar\mu(c' \cap a_n) = \bar\mu\pi(c' \cap a_n) \ge \bar\mu(c' \cap a_0) = \mu_0 c',$$

which is impossible, because $0 \ne c' \subseteq [\![\mu_0 > \mu_n]\!]$. **XQ**

So $\mu_0 c \le \mu_n c$ whenever $c \in \mathfrak{C}$ and $c \subseteq e'_n$. Because the μ_k have been chosen to be a non-increasing sequence, we must have $\mu_0 c = \mu_1 c = \ldots = \mu_n c$ for every $c \subseteq e'_n$.

(ϵ) Recalling now that $\sum_{i\in I}\mu_i = \bar\mu\restriction\mathfrak{C}$, we see that $\mu_0 c \le \frac{1}{n+1}\bar\mu c$ for every $c \subseteq e'_n$. It follows that if $e^* = \inf_{n\in\mathbb{N}} e'_n$, $\mu_0 e^* = 0$; but this must mean that $e^* = 0$. Consequently, setting $e_n = e'_n \setminus e'_{n+1}$ for $n \in \mathbb{N}$, $e_\kappa = a_\kappa$ for $\kappa \in K$, we find that $\langle e_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{C}$.

Moreover, for $n \in \mathbb{N}$ and $c \subseteq e_n$, we must have $\mu_{n+1}c = 0$,

$$\bar\mu c = \sum_{i\in I}\mu_i c = \sum_{k=0}^{n}\mu_k c = (n+1)\mu_0 c,$$

so that $\mu_k c = \frac{1}{n+1}\mu c$ for every $k \le n$. But this means that we have a measure-preserving homomorphism $\psi_n : \mathfrak{A}_{e_n} \to \mathfrak{C}_{e_n}\widehat{\otimes}\mathfrak{B}_n$ given by setting

$$\psi_n(a_k \cap c) = c \otimes \{k\}$$

whenever $c \in \mathfrak{C}_{e_n}$ and $k \leq n$; this is well-defined because $e_n \subseteq e'_k$, so that $a_k \cap c \neq a_k \cap c'$ if c, c' are distinct members of $\mathfrak{C}_{e_n}$, and it is measure-preserving because

$$\bar{\mu}(a_k \cap c) = \mu_k c = \frac{1}{n+1}\bar{\mu}c = \bar{\mu}c \cdot \bar{\nu}_n\{k\}$$

for all relevant k and c. Because $\mathfrak{B}_n$ is finite, ψ_n is surjective.

(ζ) Just as in 333H, we now see that because $\langle e_i \rangle_{i \in I}$ is a partition of unity in $\mathfrak{A}$ as well as in $\mathfrak{C}$, we can identify $\mathfrak{A}$ with $\prod_{i \in I} \mathfrak{A}_{e_i}$ and therefore with $\prod_{i \in I} \mathfrak{C}_{e_i} \widehat{\otimes} \mathfrak{B}_i$.

(b)(ii)$\Rightarrow$(iii) Let us work in $\mathfrak{D} = \prod_{i \in I} \mathfrak{C}_{e_i} \widehat{\otimes} \mathfrak{B}_i$, writing $\psi : \mathfrak{A} \to \mathfrak{D}$ for the given isomorphism. For each $i \in I$, we have a measure-preserving automorphism π_i of $\mathfrak{C}_{e_i} \widehat{\otimes} \mathfrak{B}_i$ with fixed-point subalgebra $\{c \otimes 1 : c \in \mathfrak{C}_{e_i}\}$ (333Q). For $d = \langle d_i \rangle_{i \in I} \in \mathfrak{D}$, set

$$\pi d = \langle \pi_i d_i \rangle_{i \in I}.$$

Then π is a measure-preserving automorphism because every π_i is. If $\pi d = d$, then for every $i \in I$ there must be a $c_i \subseteq e_i$ such that $d_i = c_i \otimes 1$. But this means that $d = \psi c$, where $c = \sup_{i \in I} c_i \in \mathfrak{C}$. Thus the fixed-point subalgebra of π is just $\psi[\mathfrak{C}]$. Transferring the structure $(\mathfrak{D}, \psi[\mathfrak{C}], \pi)$ back to $\mathfrak{A}$, we obtain a measure-preserving automorphism $\psi^{-1}\pi\psi$ of $\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$, as required.

(c)(iii)$\Rightarrow$(i) is trivial.

333X Basic exercises >(a) Show that, in the proof of 333H, $c_i = \operatorname{upr}(a_i, \mathfrak{C})$ (definition: 313S) for every $i \in I$.

(b) In Lemma 333J, show that every relative atom in $\mathfrak{A}$ over $\mathfrak{C}$ belongs to the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{a_n : n \in \mathbb{N}\}$.

(c) In the context of Lemma 333M, show that if I is countable we have a one-to-one correspondence between atoms c of $\mathfrak{C}_0$ and points x of non-zero mass in $\mathbb{R}^I$, given by the formula $\pi\{x\}^\bullet = c$.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be totally finite measure algebra and G a set of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself such that $\pi\phi \in G$ for all π, $\phi \in G$. (i) Show that $a \subseteq \sup_{\pi \in G} \pi a$ for every $a \in \mathfrak{A}$. (*Hint*: if $\pi c \subseteq c$, where $\pi \in G$ and $c \in \mathfrak{A}$, then $\pi c = c$; apply this to $c = \sup_{\pi \in G} \pi a$.) (ii) Set $\mathfrak{C} = \{c : c \in \mathfrak{A},\ \pi c = c \text{ for every } \pi \in G\}$. Show that $\sup_{\pi \in G} \pi a = \operatorname{upr}(a, \mathfrak{C})$ for every $a \in \mathfrak{A}$.

333Y Further exercises (a) Show that when $I = \mathbb{N}$ the algebra Σ of subsets of $\mathbb{R}^I$, used in 333M, is precisely the Borel σ-algebra as described in 271Ya.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, and $\mathfrak{B}$, $\mathfrak{C}$ two closed subalgebras of $\mathfrak{A}$ with $\mathfrak{C} \subseteq \mathfrak{B}$. Show that $\tau_{\mathfrak{C}}(\mathfrak{B}) \leq \tau_{\mathfrak{C}}(\mathfrak{A})$. (*Hint*: use 333K and the ideas of 332T.)

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. Show that $\mathfrak{A}$ is homogeneous iff there is a measure-preserving automorphism of $\mathfrak{A}$ which is mixing in the sense of 333P.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, and G a set of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself. Set $\mathfrak{C} = \{c : c \in \mathfrak{A},\ \pi c = c \text{ for every } \pi \in G\}$. Show that $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ of the type described in 333R. (*Hint*: in the language of part (a) of the proof of 333R, show that every a_κ still belongs to $\mathfrak{C}$.)

333 Notes and comments I have done my best, in the first part of this section, to follow the lines already laid out in §§331-332, using what should (once you have seen them) be natural generalizations of the former definitions and arguments. Thus the Maharam type $\tau(\mathfrak{A})$ of an algebra is just the relative Maharam type $\tau_{\{0,1\}}(\mathfrak{A})$, and $\mathfrak{A}$ is Maharam-type-homogeneous iff it is relatively Maharam-type-homogeneous over $\{0, 1\}$. To help you trace through the correspondence, I list the code numbers: 331Fa$\to$333Aa, 331Fb$\to$333Ac, 331Hc$\to$333Ba, 331Hd$\to$333Bd, 332A$\to$333Bb, 331I$\to$333Cb, 331K$\to$333E, 331L$\to$333Fb, 332B$\to$333H, 332J$\to$333K. 333D overlaps with 332P. Throughout, the principle is the same: everything can be built up from products and free products.

Theorem 333Ca does not generalize any explicitly stated result, but overlaps with Proposition 332P. In the proof of 333E I have used a new idea; the same method would of course have worked just as well for 331K, but I thought it worth while to give an example of an alternative technique, which displays a different facet of homogeneous algebras, and a different way in which the algebraic, topological and metric properties of homogeneous algebras interact. The argument of 331K-331L relies (without using the term) on the fact that measure algebras of Maharam type κ have topological density at most $\max(\kappa,\omega)$ (see 331Ye), while the the argument of 333E uses the rather more sophisticated concept of stochastic independence.

Corollary 333Fa is cruder than the more complicated results which follow, but I think that it is invaluable as a first step in forming a picture of the possible embeddings of a given (totally finite) measure algebra $\mathfrak{C}$ in a larger algebra $\mathfrak{A}$. If we think of $\mathfrak{C}$ as the measure algebra of a measure space (X,Σ,μ), then we can be sure that $\mathfrak{A}$ is representable as a closed subalgebra of the measure algebra of $X\times\{0,1\}^\kappa$ for some κ, that is, the measure algebra of $\lambda\restriction\mathrm{T}$ where λ is the product measure on $X\times\{0,1\}^\kappa$ and T is some σ-subalgebra of the domain of λ; the embedding of $\mathfrak{C}$ in $\mathfrak{A}$ being defined by the formula $E^\bullet\to(E\times\{0,1\}^\kappa)^\bullet$ for $E\in\Sigma$ (325A, 325D). Identifying, in our imaginations, both X and $\{0,1\}^\kappa$ with the unit interval, we can try to picture everything in the unit square – and these pictures, although necessarily inadequate for algebras of uncountable Maharam type, already give a great deal of scope for invention.

I said above that everything can be constructed from simple products and free products, judiciously combined; of course some further ideas must be mixed with these. The difference between 332B and 333H, for instance, is partly in the need for the functionals μ_i in the latter, whereas in the former the decomposition involves only principal ideals with the induced measures. Because the μ_i are completely additive, they all have supports c_i (326Xl) and we get measure algebras $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ to use in the products. (I note that the c_i can be obtained directly from the a_i, without mentioning the functionals μ_i, by the process of 333Xa.) The fact that the c_i can overlap means that the 'relatively atomic' part of the larger algebra $\mathfrak{A}$ needs a much more careful description than before; this is the burden of 333J, and also the principal complication in the proof of 333R. The 'relatively atomless' part is (comparatively) straightforward, since we can use the same kind of amalgamation as before (part (c-i) of the proof of 332J, part (b) of the proof of 333K), simplified because I am no longer seeking to deal with algebras of infinite magnitude.

Theorem 333K gives a canonical form for superalgebras of a given totally finite measure algebra $(\mathfrak{C},\bar\mu)$, taking the structure $(\mathfrak{C},\bar\mu)$ itself for granted. I hope it is clear that while the μ_i amd e_i and the algebra $\widehat{\mathfrak{A}}=\prod_{n\in\mathbb{N}}\mathfrak{C}_{e_n}\times\prod_{\kappa\in K}\mathfrak{C}_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$ and the embedding of $\mathfrak{C}$ in $\widehat{\mathfrak{A}}$ are uniquely defined, the rest of the isomorphism $\pi:\mathfrak{A}\to\widehat{\mathfrak{A}}$ generally is not. Even when the a_κ are uniquely defined the isomorphisms between $\mathfrak{A}_{a_\kappa}$ and $\mathfrak{C}_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$ depend on choosing generating families in the $\mathfrak{A}_{a_\kappa}$; see the proof of 333Cb.

To understand the possible structures $(\mathfrak{C},\langle\mu_i\rangle_{i\in I})$ of that theorem, we have to go rather deeper. The route I have chosen is to pick out the subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$ determined by $\langle\mu_i\rangle_{i\in I}$ and identify it with the measure algebra of a particular measure on $\mathbb{R}^I$. Perhaps I should apologise for not stating explicitly in the course of 333N that the measure μ there is a 'Borel measure' (see 333Ya); but I am afraid of opening a door to an invasion of ideas which belong in Volume 4. Besides, if I were going to do anything more with these measures than observe that they are uniquely defined by the construction proposed, I would complete them and call them Radon measures. In order to validate this approach, I must show that the μ_i can be recovered from their restrictions to $\mathfrak{C}_0$; this is 333Md, and is the motive for the discussion of 'standard extensions' in §327. No doubt there are other ways of doing it. One temptation which I felt it right to resist was the idea of decomposing $\mathfrak{C}$ into its homogeneous principal ideals; this seemed merely an additional complication. Of course the subalgebra $\mathfrak{C}_0$ has countable Maharam type (being τ-generated by the elements e_{iq}, for $i\in I$ and $q\in\mathbb{Q}$, of 333M), so that its decomposition is relatively simple, being just a matter of picking out the atoms (333Xc).

Another way of looking at the expression in 333N is to observe that $\mathfrak{A}$ is obtained by amalgamating two extensions of the core subalgebra $\mathfrak{C}_0$; one defined by $\langle\mu_i\restriction\mathfrak{C}_0\rangle_{i\in\mathbb{N}\cup K}$, and one by $\langle\theta'_j\rangle_{j\in\mathbb{N}\cup L}$. After using the second family to represent $(\mathfrak{C},\bar\mu\restriction\mathfrak{C},\mathfrak{C}_0)$, we obtain standard extensions μ_i from which we can represent $(\mathfrak{A},\bar\mu,\mathfrak{C})$. Put this way, it seems that the process demands that the steps be performed in the given order. In fact it can be made symmetric; but for that I think we need the theory of 'relative free products', which I will come to in §458 of Volume 4.

In 333P I find myself presenting an important fact about homogeneous measure algebras, rather out of context; but I hope that it will help you to believe that I have by no means finished with the insights which Maharam's theorem provides. I give it here for the sake of 333R. For the moment, I invite you to think of 333R as just a demonstration of the power of the techniques I have developed in this chapter, and of the kind of simplification (in the equivalence of conditions (i) and (iii)) which seems to arise repeatedly in the theory of measure algebras. But you will see that the first step to understanding any automorphism will be a description of its fixed-point subalgebra, so 333R will also be basic to the theory of automorphisms of measure algebras. I note that the hypothesis (i) of 333R can in fact be relaxed (333Yd), but this seems to need an extra idea.

334 Products

I devote a short section to results on the Maharam classification of the measure algebras of product measures, or, if you prefer, of the free products of measure algebras. The complete classification, even for probability algebras, is complex (334Xc, 334Ya), so I content myself with a handful of the most useful results. I start with upper bounds for the Maharam type of the c.l.d. product of two measure spaces (334A) and the localizable measure algebra free product of two semi-finite measure algebras (334B), and go on to the corresponding results for general products of probability spaces and algebras (334C-334D). Finally, I show that any infinite power of a probability space is Maharam-type-homogeneous (334E).

In this section I will write $\tau(\mu)$ for the Maharam type of a measure μ, defined as the Maharam type of its measure algebra (331Fc).

334A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and λ the c.l.d. product measure on $X\times Y$. Then $\tau(\lambda)\le\max(\omega,\tau(\mu),\tau(\nu))$.

proof Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be the measure algebras of μ, ν and λ, respectively. Recall from 325A that we have order-continuous Boolean homomorphisms $\varepsilon_1:\mathfrak{A}\to\mathfrak{C}$ and $\varepsilon_2:\mathfrak{B}\to\mathfrak{C}$ defined by setting $\varepsilon_1(E^\bullet)=(E\times Y)^\bullet$, $\varepsilon_2(F^\bullet)=(X\times F)^\bullet$ for $E\in\Sigma$ and $F\in\mathrm{T}$. Let $A\subseteq\mathfrak{A}$, $B\subseteq\mathfrak{B}$ be τ-generating sets with $\#(A)=\tau(\mathfrak{A})=\tau(\mu)$, $\#(B)=\tau(\nu)$; set $C=\varepsilon_1[A]\cup\varepsilon_2[B]$. Then C τ-generates $\mathfrak{C}$. **P** Let $\mathfrak{C}_1$ be the order-closed subalgebra of $\mathfrak{C}$ generated by C. Because ε_1 is order-continuous, $\varepsilon_1^{-1}[\mathfrak{C}_1]$ is an order-closed subalgebra of $\mathfrak{A}$, and it includes A, so must be the whole of $\mathfrak{A}$; thus $\varepsilon_1 a\in\mathfrak{C}_1$ for every $a\in\mathfrak{A}$. Similarly, $\varepsilon_2 b\in\mathfrak{C}_1$ for every $b\in\mathfrak{B}$.

This means that

$$\Lambda_1=\{W:W\in\Lambda,\,W^\bullet\in\mathfrak{C}_1\}$$

contains $E\times F$ for every $E\in\Sigma$, $F\in\mathrm{T}$. Also Λ_1 is a σ-algebra of subsets of $X\times Y$, because $\mathfrak{C}_1$ is (sequentially) order-closed in $\mathfrak{C}$. So $\Lambda_1\supseteq\Sigma\widehat{\otimes}\mathrm{T}$ (definition: 251D). But this means that if $W\in\Lambda$ there is a $V\in\Lambda_1$ such that $V\subseteq W$ and $\lambda V=\lambda W$ (251Ib); that is, if $c\in\mathfrak{C}$ there is a $d\in\mathfrak{C}_1$ such that $d\subseteq c$ and $\bar\lambda d=\bar\lambda c$. Thus $\mathfrak{C}_1$ is order-dense in $\mathfrak{C}$, and

$$c=\sup\{d:d\in\mathfrak{C}_1,\,d\subseteq c\}\in\mathfrak{C}_1$$

for every $c\in\mathfrak{C}$. So $\mathfrak{C}_1=\mathfrak{C}$ and C τ-generates $\mathfrak{C}$, as claimed. **Q**

Consequently

$$\tau(\lambda)=\tau(\mathfrak{C})\le\#(C)\le\max(\omega,\tau(\mu),\tau(\nu)).$$

334B Corollary Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, with localizable measure algebra free product $(\mathfrak{C},\bar\lambda)$ (325E). Then $\tau(\mathfrak{C})\le\max(\omega,\tau(\mathfrak{A}),\tau(\mathfrak{B}))$.

proof By the construction of part (a) of the proof of 325D, $\mathfrak{C}$ can be regarded as the measure algebra of the c.l.d. product of the Stone representations of $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$; so the result follows at once from 334A.

334C Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X,Λ,λ). Then

$$\tau(\lambda)\le\max(\omega,\#(I),\sup_{i\in I}\tau(\mu_i)).$$

proof For $i\in I$, let $\mathfrak{A}_i$ be the measure algebra of μ_i; let $\mathfrak{C}$ be the measure algebra of λ. Recall from 325I that we have order-continuous Boolean homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ corresponding to the inverse-measure-preserving maps $x\mapsto\pi_i(x)=x(i):X\to X_i$. For each $i\in I$, let $A_i\subseteq\mathfrak{A}_i$ be a set of cardinal $\tau(\mu_i)$ which τ-generates $\mathfrak{A}_i$. Set $C=\bigcup_{i\in I}\varepsilon_i[A_i]$. Then C τ-generates $\mathfrak{C}$. **P** Let $\mathfrak{C}_1$ be the order-closed subalgebra of $\mathfrak{C}$ generated by C. Because ε_i is order-continuous, $\varepsilon_i^{-1}[\mathfrak{C}_1]$ is an order-closed subalgebra of $\mathfrak{A}_i$, and it includes A_i, so must be the whole of $\mathfrak{A}_i$; thus $\varepsilon_i a\in\mathfrak{C}_1$ for every $a\in\mathfrak{A}_i$, $i\in I$.

This means that

$$\Lambda_1=\{W:W\in\Lambda,\,W^\bullet\in\mathfrak{C}_1\}$$

contains $\pi_i^{-1}[E]$ for every $E\in\Sigma_i$, $i\in I$. Also Λ_1 is a σ-algebra of subsets of X, because $\mathfrak{C}_1$ is (sequentially) order-closed in $\mathfrak{C}$. So $\Lambda_1\supseteq\widehat{\bigotimes}_{i\in I}\Sigma_i$. But this means that if $W\in\Lambda$ there is a $V\in\Lambda_1$ such that $V^\bullet=W^\bullet$ (254Ff), that is, that $\mathfrak{C}_1=\mathfrak{C}$, and C τ-generates $\mathfrak{C}$, as claimed. **Q**

Consequently

$$\tau(\lambda) = \tau(\mathfrak{C}) \le \#(C) \le \max(\omega, \#(I), \sup_{i\in I} \tau(\mu_i)).$$

334D Corollary Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, with probability algebra free product $(\mathfrak{C}, \bar{\lambda})$. Then

$$\tau(\mathfrak{C}) \le \max(\omega, \#(I), \sup_{i\in I} \tau(\mathfrak{A}_i)).$$

proof See 325J-325K.

334E I come now to the question of when a product of probability spaces is Maharam-type-homogeneous. I give just one result in detail, leaving others to the exercises.

Theorem Let (X, Σ, μ) be a probability space and I an infinite set; let λ be the product measure on X^I. Then λ is Maharam-type-homogeneous. If $\tau(\mu) = 0$ then $\tau(\lambda) = 0$; otherwise $\tau(\lambda) = \max(\tau(\mu), \#(I))$.

proof (a) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of μ and $(\mathfrak{C}, \bar{\lambda})$ the measure algebra of λ. If $\tau(\mu) = 0$, that is, $\mathfrak{A} = \{0, 1\}$, then $\mathfrak{C} = \{0, 1\}$ (by 254Fe, or 325Jc, or otherwise), and in this case is surely homogeneous, with $\tau(\mathfrak{C}) = 0$; so that λ is Maharam-type-homogeneous and $\tau(\lambda) = 0$. So let us suppose hencforth that $\tau(\mu) > 0$. We have

$$\tau(\mathfrak{C}) \le \max(\omega, \#(I), \tau(\mathfrak{A})) = \max(\#(I), \tau(\mathfrak{A})$$

by 334C.

(b) Fix on $b \in \mathfrak{A} \setminus \{0, 1\}$. For each $i \in I$, let $\varepsilon_i : \mathfrak{A} \to \mathfrak{C}$ be the canonical measure-preserving homomorphism corresponding to the inverse-measure-preserving function $x \mapsto x(i) : X^I \to X$. For each $n \in \mathbb{N}$, there is a set $J \subseteq I$ of cardinal n, and now the finite subalgebra of $\mathfrak{C}$ generated by $\{\varepsilon_i b : i \in J\}$ has atoms of measure at most δ^n, where $\delta = \max(\bar{\mu}b, 1 - \bar{\mu}b) < 1$. Consequently $\mathfrak{C}$ can have no atom of measure greater than δ^n, for any n, and is therefore atomless.

(c) Because I is infinite, there is a bijection between I and $I \times \mathbb{N}$; that is, there is a partition $\langle J_i\rangle_{i\in I}$ of I into countably infinite sets. Now (X^I, λ) can be identified with the product of the family $\langle(X^{J_i}, \lambda_i)\rangle_{i\in I}$, where λ_i is the product measure on X^{J_i} (254N). By (b), every λ_i is atomless, so there are sets $E_i \subseteq X^{J_i}$ of measure $\frac{1}{2}$. The sets $E'_i = \{x : x{\restriction}J_i \in E_i\}$ are now stochastically independent in X. Accordingly we have an inverse-measure-preserving function $f : X \to \{0, 1\}^I$, endowed with its usual measure ν_I, defined by setting $f(x)(i) = 1$ if $x \in E'_i$, 0 otherwise, and therefore a measure-preserving Boolean homomorphism $\pi : \mathfrak{B}_I \to \mathfrak{C}$, writing $\mathfrak{B}_I$ for the measure algebra of ν_I.

Now if $c \in \mathfrak{C} \setminus \{0\}$ and $\mathfrak{C}_c$ is the corresponding ideal, $b \mapsto c \cap \pi b : \mathfrak{B}_I \to \mathfrak{C}_c$ is an order-continuous Boolean homomorphism. It follows that $\tau(\mathfrak{C}_c) \ge \#(I)$ (331Jb).

(d) Again take any non-zero $c \in \mathfrak{C}$. For each $i \in I$, set $a_i = \inf\{a : \varepsilon_i a \supseteq c\}$. Writing $\mathfrak{A}_{a_i}$ for the corresponding principal ideal of $\mathfrak{A}$, we have an order-continuous Boolean homomorphism $\varepsilon'_i : \mathfrak{A}_{a_i} \to \mathfrak{C}_c$, given by the formula

$$\varepsilon'_i a = \varepsilon_i a \cap c \text{ for every } a \in \mathfrak{A}_{a_i}.$$

Now ε'_i is injective, so is a Boolean isomorphism between $\mathfrak{A}_{a_i}$ and its image $\varepsilon'_i[\mathfrak{A}_{a_i}]$, which by 314F(a-i) is a closed subalgebra of $\mathfrak{C}_c$. So

$$\tau(\mathfrak{A}_{a_i}) = \tau(\varepsilon'_i[\mathfrak{A}_{a_i}]) \le \tau(\mathfrak{C}_c)$$

by 332Tb.

For any finite $J \subseteq I$,

$$0 < \bar{\lambda}c \le \bar{\lambda}(\inf_{i\in J} \varepsilon_i a_i) = \prod_{i\in J} \bar{\lambda}(\varepsilon_i a_i) = \prod_{i\in J} \bar{\mu}a_i.$$

So for any $\delta < 1$, $\{i : \bar{\mu}a_i \le \delta\}$ must be finite, and $\sup_{i\in I} \bar{\mu}a_i = 1$. In particular, $\sup_{i\in I} a_i = 1$ in $\mathfrak{A}$. But this means that if ζ is any cardinal such that the Maharam-type-ζ component e_ζ of $\mathfrak{A}$ is non-zero, then $e_\zeta \cap a_i \ne 0$ for some $i \in I$, so that

$$\zeta \le \tau(\mathfrak{A}_{e_\zeta \cap a_i}) \le \tau(\mathfrak{A}_{a_i}) \le \tau(\mathfrak{C}_c).$$

As ζ is arbitrary, $\tau(\mathfrak{A}) \le \max(\omega, \tau(\mathfrak{C}_c))$ (332S).

(e) Putting (a)-(d) together, we have

$$\max(\tau(\mathfrak{A}), \#(I)) \le \max(\omega, \tau(\mathfrak{C}_c)) = \tau(\mathfrak{C}_c) \le \tau(\mathfrak{C}) \le \max(\tau(\mathfrak{A}), \#(I))$$

for every non-zero $c \in \mathfrak{C}$; so $\mathfrak{C}$ is homogeneous, with $\tau(\mathfrak{C}) = \max(\tau(\mathfrak{A}), \#(I))$. Re-stating this in terms of λ and μ, λ is Maharam-type-homogeneous and $\tau(\lambda) = \max(\tau(\mu), \#(I))$.

334X Basic exercises (a) Let (X, Σ, μ) and (Y, T, ν) be complete locally determined measure spaces with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Show that if $\nu Y > 0$ then $\tau(\mu) \le \tau(\lambda)$.

>(b) Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, with probability algebra free product $(\mathfrak{C}, \bar{\lambda})$. Show that $\tau(\mathfrak{A}_i) \le \tau(\mathfrak{C})$ for every i, and that

$$\#(\{i : i \in I,\, \tau(\mathfrak{A}_i) > 0\}) \le \tau(\mathfrak{C}).$$

(c) Let (X, Σ, μ) and (Y, T, ν) be σ-finite measure spaces, and λ the product measure on $X \times Y$. Show that λ is Maharam-type-homogeneous iff one of μ, ν is Maharam-type-homogeneous with Maharam type at least as great as the Maharam type of the other.

(d) Show that the product of any family of Maharam-type-homogeneous probability spaces is again Maharam-type-homogeneous.

>(e) Let (X, Σ, μ) be a probability space of Maharam type κ, and I any set of cardinal at least $\max(\omega, \kappa)$. Show that the product measure on $X \times \{0,1\}^I$ is Maharam-type-homogeneous, with Maharam type $\#(I)$.

334Y Further exercises (a) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be an infinite family of probability spaces, with product (X, Λ, λ). Let κ_i be the Maharam type of μ_i for each i; set $\kappa = \max(\#(I), \sup_{i\in I} \kappa_i)$. Show that *either* λ is Maharam-type-homogeneous, with Maharam type κ, *or* there are $\kappa' < \kappa$, $X_i' \in \Sigma_i$ such that $\sum_{i\in I} \mu_i(X_i \setminus X_i') < \infty$, the Maharam type of the subspace measure on X_i' is at most κ' for every $i \in I$ and $\#(\{i : \kappa_i \ne 0\}) \le \kappa'$.

334 Notes and comments The results above are all very natural ones; I have spelt them out partly for completeness and partly for the sake of an application in §346 below. But note the second alternative in 334Ya; it is possible, even in an infinite product, for a kernel of relatively small Maharam type to be preserved.

Chapter 34

The lifting theorem

Whenever we have a surjective homomorphism $\phi : P \to Q$, where P and Q are mathematical structures, we can ask whether there is a right inverse of ϕ, a homomorphism $\psi : Q \to P$ such that $\phi\psi$ is the identity on Q. As a general rule, we expect a negative answer; those categories in which epimorphisms always have right inverses (e.g., the category of linear spaces) are rather special, and elsewhere the phenomenon is relatively rare and almost always important. So it is notable that we have a case of this at the very heart of the theory of measure algebras: for any complete probability space (X, Σ, μ) (in fact, for any complete strictly localizable space of non-zero measure) the canonical homomorphism from Σ to the measure algebra of μ has a right inverse (341K). This is the von Neumann-Maharam lifting theorem. Its proof, together with some essentially elementary remarks, takes up the whole of of §341.

As a first application of the theorem (there will be others in Volume 4) I apply it to one of the central problems of measure theory: under what circumstances will a homomorphism between measure algebras be representable by a function between measure spaces? Variations on this question are addressed in §343. For a reasonably large proportion of the measure spaces arising naturally in analysis, homomorphisms are representable (343B). New difficulties arise if we ask for isomorphisms of measure algebras to be representable by isomorphisms of measure spaces, and here we have to work rather hard for rather narrowly applicable results; but in the case of Lebesgue measure and its closest relatives, a good deal can be done, as in 344I-344K.

Returning to liftings, there are many difficult questions concerning the extent to which liftings can be required to have special properties, reflecting the natural symmetries of the standard measure spaces. For instance, Lebesgue measure is translation-invariant; if liftings were in any sense canonical, they could be expected to be automatically translation-invariant in some sense. It seems sure that there is no canonical lifting for Lebesgue measure – all constructions of liftings involve radical use of the axiom of choice – but even so we do have many translation-invariant liftings (§345). We have less luck with product spaces; here the construction of liftings which respect the product structure is fraught with difficulties. I give the currently known results in §346.

341 The lifting theorem

I embark directly on the principal theorem of this chapter (341K, 'every non-trivial complete strictly localizable measure space has a lifting'), using the minimum of advance preparation. 341A-341B give the definition of 'lifting'; the main argument is in 341F-341K, using the concept of 'lower density' (341C-341E) and a theorem on martingales from §275. In 341P I describe an alternative way of thinking about liftings in terms of the Stone space of the measure algebra.

341A Definition Let (X, Σ, μ) be a measure space, and $\mathfrak{A}$ its measure algebra. By a **lifting** for $\mathfrak{A}$ (or for (X, Σ, μ), or for μ) I shall mean

either a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$

or a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that (i) $\phi E = \emptyset$ whenever $\mu E = 0$ (ii) $\mu(E \triangle \phi E) = 0$ for every $E \in \Sigma$.

341B Remarks (a) I trust that the ambiguities permitted by this terminology will not cause any confusion. The point is that there is a natural one-to-one correspondence between liftings $\theta : \mathfrak{A} \to \Sigma$ and liftings $\phi : \Sigma \to \Sigma$ given by the formula

$$\theta E^\bullet = \phi E \text{ for every } E \in \Sigma.$$

P (i) Given a lifting $\theta : \mathfrak{A} \to \Sigma$, the formula defines a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that

$$\phi\emptyset = \theta 0 = \emptyset, \quad (E \triangle \phi E)^\bullet = E^\bullet \triangle (\theta E^\bullet)^\bullet = 0 \ \forall \ E \in \Sigma,$$

so that ϕ is a lifting. (ii) Given a lifting $\phi : \Sigma \to \Sigma$, the kernel of ϕ includes $\{E : \mu E = 0\}$, so there is a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $\theta E^\bullet = \phi E$ for every E (3A2G), and now

$$(\theta E^\bullet)^\bullet = (\phi E)^\bullet = E^\bullet$$

for every $E \in \Sigma$, so θ is a lifting. **Q**

I suppose that the word 'lifting' applies most naturally to functions from $\mathfrak{A}$ to Σ; but for applications in measure theory the other type of lifting is used at least equally often.

(b) Note that if $\phi : \Sigma \to \Sigma$ is a lifting then $\phi^2 = \phi$. **P** For any $E \in \Sigma$,

$$\phi^2 E \triangle \phi E = \phi(E \triangle \phi E) = \emptyset. \text{ } \mathbf{Q}$$

If ϕ is associated with $\theta : \mathfrak{A} \to \Sigma$, then $\phi\theta a = \theta a$ for every $a \in \mathfrak{A}$. **P** $\phi\theta a = \theta((\theta a)^\bullet) = \theta a$. **Q**

(c) In the theorems to follow, there will occasionally intrude a hypothesis '$\mu X > 0$'. The point is that if we have a measure space (X, Σ, μ) which is trivial in the sense that $\mu X = 0$, then the only candidate for a 'lifting' $\phi : \Sigma \to \Sigma$ is the constant function with value $\emptyset$; and if $X \neq \emptyset$ this is not a Boolean homomorphism in the sense of this book. The simplest way of dealing with these cases is to rule them out of the discussion.

341C Definition Let (X, Σ, μ) be a measure space, and $\mathfrak{A}$ its measure algebra. By a **lower density** for $\mathfrak{A}$ (or for (X, Σ, μ), or for μ) I shall mean

either a function $\underline{\theta} : \mathfrak{A} \to \Sigma$ such that (i) $(\underline{\theta}a)^\bullet = a$ for every $a \in \mathfrak{A}$ (ii) $\underline{\theta}0 = \emptyset$ (iii) $\underline{\theta}(a \cap b) = \underline{\theta}a \cap \underline{\theta}b$ for all a, $b \in \mathfrak{A}$

or a function $\underline{\phi} : \Sigma \to \Sigma$ such that (i) $\underline{\phi}E = \underline{\phi}F$ whenever E, $F \in \Sigma$ and $\mu(E \triangle F) = 0$ (ii) $\mu(E \triangle \underline{\phi}E) = 0$ for every $E \in \Sigma$ (iii) $\underline{\phi}\emptyset = \emptyset$ (iv) $\underline{\phi}(E \cap F) = \underline{\phi}E \cap \underline{\phi}F$ for all E, $F \in \Sigma$.

341D Remarks (a) As in 341B, there is a natural one-to-one correspondence between lower densities $\underline{\theta} : \mathfrak{A} \to \Sigma$ and lower densities $\underline{\phi} : \Sigma \to \Sigma$ given by the formula

$$\underline{\theta}E^\bullet = \underline{\phi}E \text{ for every } E \in \Sigma.$$

(For the requirement $\underline{\phi}E = \underline{\phi}F$ whenever $E^\bullet = F^\bullet$ in $\mathfrak{A}$ means that every $\underline{\phi}$ corresponds to a function $\underline{\theta}$, and the other clauses match each other directly.)

(b) As before, if $\underline{\phi} : \Sigma \to \Sigma$ is a lower density then $\underline{\phi}^2 = \underline{\phi}$. If $\underline{\phi}$ is associated with $\underline{\theta} : \mathfrak{A} \to \Sigma$, then $\underline{\phi}\underline{\theta} = \underline{\theta}$.

(c) It will be convenient, in the course of the proofs of 341F-341H below, to have the following concept available. If (X, Σ, μ) is a measure space with measure algebra $\mathfrak{A}$, a **partial lower density** of $\mathfrak{A}$ is a function $\underline{\theta} : \mathfrak{B} \to \Sigma$ such that (i) the domain $\mathfrak{B}$ of $\underline{\theta}$ is a subalgebra of $\mathfrak{A}$ (ii) $(\underline{\theta}b)^\bullet = b$ for every $b \in \mathfrak{B}$ (iii) $\underline{\theta}0 = \emptyset$ (iv) $\underline{\theta}(a \cap b) = \underline{\theta}a \cap \underline{\theta}b$ for all a, $b \in \mathfrak{B}$.

Similarly, if T is a subalgebra of Σ, a function $\underline{\phi} : \mathrm{T} \to \Sigma$ is a **partial lower density** if (i) $\underline{\phi}E = \underline{\phi}F$ whenever E, $F \in \mathrm{T}$ and $\mu(E \triangle F) = 0$ (ii) $\mu(E \triangle \underline{\phi}E) = 0$ for every $E \in \mathrm{T}$ (iii) $\underline{\phi}\emptyset = \emptyset$ (iv) $\underline{\phi}(E \cap F) = \underline{\phi}E \cap \underline{\phi}F$ for all E, $F \in \mathrm{T}$.

(d) Note that lower densities and partial lower densities are order-preserving; if $a \subseteq b$ in $\mathfrak{A}$, and $\underline{\theta}$ is a lower density for $\mathfrak{A}$, then

$$\underline{\theta}a = \underline{\theta}(a \cap b) = \underline{\theta}a \cap \underline{\theta}b \subseteq \underline{\theta}b.$$

(e) Of course a Boolean homomorphism from $\mathfrak{A}$ to Σ, or from Σ to itself, is a lifting iff it is a lower density.

341E Example Let μ be Lebesgue measure on $\mathbb{R}^r$, where $r \geq 1$, and Σ its domain. For $E \in \Sigma$ set

$$\mathrm{int}^* E = \{x : x \in \mathbb{R}^r, \lim_{\delta \downarrow 0} \tfrac{\mu(E \cap B(x,\delta))}{\mu B(x,\delta)} = 1\}.$$

(Here $B(x, \delta)$ is the closed ball with centre x and radius δ.) Then int^* is a lower density for μ; we may call it **lower Lebesgue density**. **P** (You may prefer at first to suppose that $r = 1$, so that $B(x, \delta) = [x - \delta, x + \delta]$ and $\mu B(x, \delta) = 2\delta$.) By 261Db (or 223B, for the one-dimensional case) $E \triangle \mathrm{int}^* E$ is negligible for every E; in particular, $\mathrm{int}^* E \in \Sigma$ for every $E \in \Sigma$. If $E \triangle F$ is negligible, then $\mu(E \cap B(x, \delta)) = \mu(F \cap B(x, \delta))$ for every x and δ, so $\mathrm{int}^* E = \mathrm{int}^* F$. If $E \subseteq F$, then $\mu(E \cap B(x, \delta)) \leq \mu(F \cap B(x, \delta))$ for every x, δ, so $\mathrm{int}^* E \subseteq \mathrm{int}^* F$; consequently $\mathrm{int}^*(E \cap F) \subseteq \mathrm{int}^* E \cap \mathrm{int}^* F$ for all E, $F \in \Sigma$. If E, $F \in \Sigma$ and $x \in \mathrm{int}^* E \cap \mathrm{int}^* F$, then

$$\begin{aligned}\mu(E \cap F \cap B(x, \delta)) &= \mu(E \cap B(x, \delta)) + \mu(F \cap B(x, \delta)) - \mu((E \cup F) \cap B(x, \delta)) \\ &\geq \mu(E \cap B(x, \delta)) + \mu(F \cap B(x, \delta)) - \mu(B(x, \delta))\end{aligned}$$

for every δ, so

$$\tfrac{\mu(E \cap F \cap B(x,\delta))}{\mu B(x,\delta)} \geq \tfrac{\mu(E \cap B(x,\delta))}{\mu B(x,\delta)} + \tfrac{\mu(F \cap B(x,\delta))}{\mu B(x,\delta)} - 1 \to 1$$

as $\delta \downarrow 0$, and $x \in \mathrm{int}^*(E \cap F)$. Thus $\mathrm{int}^*(E \cap F) = \mathrm{int}^*E \cap \mathrm{int}^*F$ for all E, $F \in \Sigma$, and int^* is a lower density. **Q**

Remark In Chapter 47 of Volume 4 I will return to the operator int* in a context in which an alternative name, 'essential interior', is more natural.

341F The hard work of this section is in the proof of 341H below. To make it a little more digestible, I extract two parts of the proof as separate lemmas.

Lemma Let (X, Σ, μ) be a probability space and $\mathfrak{A}$ its measure algebra. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ and $\underline{\theta} : \mathfrak{B} \to \Sigma$ a partial lower density. Then for any $e \in \mathfrak{A}$ there is a partial lower density $\underline{\theta}_1$, extending $\underline{\theta}$, defined on the subalgebra $\mathfrak{B}_1$ of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{e\}$.

proof (a) Because $\mathfrak{B}$ is order-closed, therefore Dedekind complete in itself (314Ea),

$$v = \mathrm{upr}(e, \mathfrak{B}) = \inf\{a : a \in \mathfrak{B},\, a \supseteq e\}, \quad w = \mathrm{upr}(1 \setminus e, \mathfrak{B})$$

are defined in $\mathfrak{B}$. Let $E \in \Sigma$ be such that $E^\bullet = e$.

(b) We have a function $\underline{\theta}_1 : \mathfrak{B}_1 \to \Sigma$ defined by writing

$$\underline{\theta}_1((a \cap e) \cup (b \setminus e)) = \big(\underline{\theta}((a \cap v) \cup (b \setminus v)) \cap E\big) \cup \big(\underline{\theta}((a \setminus w) \cup (b \cap w)) \setminus E\big)$$

for a, $b \in \mathfrak{B}$. **P** By 312N, every element of $\mathfrak{B}_1$ is expressible as $(a \cap e) \cup (b \setminus e)$ for some a, $b \in \mathfrak{B}$. If a, a', b, $b' \in \mathfrak{B}$ are such that $(a \cap e) \cup (b \setminus e) = (a' \cap e) \cup (b' \setminus e)$, then $a \cap e = a' \cap e$ and $b \setminus e = b' \setminus e$, that is,

$$a \mathbin{\triangle} a' \subseteq 1 \setminus e \subseteq w, \quad b \mathbin{\triangle} b' \subseteq e \subseteq v.$$

This means that $e \subseteq 1 \setminus (a \mathbin{\triangle} a') \in \mathfrak{B}$ and $1 \setminus e \subseteq 1 \setminus (b \mathbin{\triangle} b') \in \mathfrak{B}$. So we also have $v \subseteq 1 \setminus (a \mathbin{\triangle} a')$ and $w \subseteq 1 \setminus (b \mathbin{\triangle} b')$. Accordingly

$$a \cap v = a' \cap v, \quad b \cap w = b' \cap w. \quad a \setminus w = a' \setminus w, \quad b \setminus v = b' \setminus v.$$

But this means that

$$\begin{aligned}\big(\underline{\theta}((a \cap v) \cup (b \setminus v)) \cap E\big) &\cup \big(\underline{\theta}((a \setminus w) \cup (b \cap w)) \setminus E\big) \\ &= \big(\underline{\theta}((a' \cap v) \cup (b' \setminus v)) \cap E\big) \cup \big(\underline{\theta}((a' \setminus w) \cup (b' \cap w)) \setminus E\big).\end{aligned}$$

Thus the formula given defines $\underline{\theta}_1$ uniquely. **Q**

(c) Now $\underline{\theta}_1$ is a partial lower density.

P(i) If a, $b \in \mathfrak{B}$,

$$\begin{aligned}(\underline{\theta}_1((a \cap e) \cup (b \setminus e)))^\bullet &= \Big(\big(\underline{\theta}((a \cap v) \cup (b \setminus v)) \cap E\big) \cup \big(\underline{\theta}((a \setminus w) \cup (b \cap w)) \setminus E\big)\Big)^\bullet \\ &= \big(((a \cap v) \cup (b \setminus v)) \cap e\big) \cup \big(((a \setminus w) \cup (b \cap w)) \setminus e\big) \\ &= (a \cap e) \cup (b \setminus e).\end{aligned}$$

So $(\underline{\theta}_1 c)^\bullet = c$ for every $c \in \mathfrak{B}_1$.

(ii)

$$\underline{\theta}_1(0) = \big(\underline{\theta}((0 \cap v) \cup (0 \setminus v)) \cap E\big) \cup \big(\underline{\theta}((0 \setminus w) \cup (0 \cap w)) \setminus E\big) = \emptyset.$$

(iii) If a, a', b, $b' \in \mathfrak{B}$, then

$$\begin{aligned}
\underline{\theta}_1(((a\cap e)\cup(b\setminus e))\cap((a'\cap e)\cup(b'\setminus e)))&\\
=\underline{\theta}_1((a\cap a'\cap e)\cup(b\cap b'\setminus e))&\\
=\big(\underline{\theta}((a\cap a'\cap v)\cup(b\cap b'\setminus v))\cap E\big)\cup\big(\underline{\theta}((a\cap a'\setminus w)\cup(b\cap b'\cap w))\setminus E\big)&\\
=\big(\underline{\theta}(((a\cap v)\cup(b\setminus v))\cap((a'\cap v)\cup(b'\setminus v)))\cap E\big)&\\
\cup\big(\underline{\theta}(((a\setminus w)\cup(b\cap w))\cap((a'\setminus w)\cup(b'\cap w)))\setminus E\big)&\\
=\big(\underline{\theta}((a\cap v)\cup(b\setminus v))\cap\underline{\theta}((a'\cap v)\cup(b'\setminus v))\cap E\big)&\\
\cup\big(\underline{\theta}((a\setminus w)\cup(b\cap w))\cap\underline{\theta}((a'\setminus w)\cup(b'\cap w))\setminus E\big)&\\
=\big(\big(\underline{\theta}((a\cap v)\cup(b\setminus v))\cap E\big)\cup\big(\underline{\theta}((a\setminus w)\cup(b\cap w))\setminus E\big)\big)&\\
\cap\big(\big(\underline{\theta}((a'\cap v)\cup(b'\setminus v))\cap E\big)\cup\big(\underline{\theta}((a'\setminus w)\cup(b'\cap w))\setminus E\big)\big)&\\
=\underline{\theta}_1((a\cap e)\cup(b\setminus e))\cap\underline{\theta}_1((a'\cap e)\cup(b'\setminus e)).&
\end{aligned}$$

So $\underline{\theta}_1(c\cap c')=\underline{\theta}_1(c)\cap\underline{\theta}_1(c')$ for all c, $c'\in\mathfrak{B}_1$. **Q**

(d) If $a\in\mathfrak{B}$, then

$$\begin{aligned}
\underline{\theta}_1(a)&=\underline{\theta}_1((a\cap e)\cup(a\setminus e))\\
&=\big(\underline{\theta}((a\cap v)\cup(a\setminus v))\cap E\big)\cup\big(\underline{\theta}((a\setminus w)\cup(a\cap w))\setminus E\big)\\
&=\big(\underline{\theta}(a)\cap E\big)\cup\big(\underline{\theta}(a)\setminus E\big)=\underline{\theta}a.
\end{aligned}$$

Thus $\underline{\theta}_1$ extends $\underline{\theta}$, as required.

341G Lemma Let (X,Σ,μ) be a probability space and $(\mathfrak{A},\bar\mu)$ its measure algebra. Suppose we have a sequence $\langle\underline{\theta}_n\rangle_{n\in\mathbb{N}}$ of partial lower densities such that, for each n, (i) the domain $\mathfrak{B}_n$ of $\underline{\theta}_n$ is a closed subalgebra of $\mathfrak{A}$ (ii) $\mathfrak{B}_n\subseteq\mathfrak{B}_{n+1}$ and $\underline{\theta}_{n+1}$ extends $\underline{\theta}_n$. Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$. Then there is a partial lower density $\underline{\theta}$, with domain $\mathfrak{B}$, extending every $\underline{\theta}_n$.

proof (a) For each n, set

$$\Sigma_n=\{E:E\in\Sigma,\,E^\bullet\in\mathfrak{B}_n\},$$

and set

$$\Sigma_\infty=\{E:E\in\Sigma,\,E^\bullet\in\mathfrak{B}\}.$$

Then (because all the $\mathfrak{B}_n$, $\mathfrak{B}$ are σ-subalgebras of $\mathfrak{A}$, and $E\mapsto E^\bullet$ is sequentially order-continuous) all the Σ_n, Σ_∞ are σ-subalgebras of Σ. We need to know that Σ_∞ is just the σ-algebra Σ_∞^* of subsets of X generated by $\bigcup_{n\in\mathbb{N}}\Sigma_n$. **P** Because Σ_∞ is a σ-algebra including $\bigcup_{n\in\mathbb{N}}\Sigma_n$, $\Sigma_\infty^*\subseteq\Sigma_\infty$. On the other hand, $\mathfrak{B}^*=\{E^\bullet:E\in\Sigma_\infty^*\}$ is a σ-subalgebra of $\mathfrak{A}$ including $\mathfrak{B}_n$ for every $n\in\mathbb{N}$. Because $\mathfrak{A}$ is ccc, $\mathfrak{B}^*$ is (order-)closed (316Fb), so includes $\mathfrak{B}$. This means that if $E\in\Sigma_\infty$ there must be an $F\in\Sigma_\infty^*$ such that $E^\bullet=F^\bullet$. But now $(E\triangle F)^\bullet=0\in\mathfrak{B}_0$, so $E\triangle F\in\Sigma_0\subseteq\Sigma_\infty^*$, and E also belongs to Σ_∞^*. This shows that $\Sigma_\infty\subseteq\Sigma_\infty^*$ and the two algebras are equal. **Q**

(b) For each $n\in\mathbb{N}$, we have the partial lower density $\underline{\theta}_n:\mathfrak{B}_n\to\Sigma$. Since $(\underline{\theta}_na)^\bullet=a\in\mathfrak{B}_n$ for every $a\in\mathfrak{B}_n$, $\underline{\theta}_n$ takes all its values in Σ_n. For $n\in\mathbb{N}$, let $\underline{\phi}_n:\Sigma_n\to\Sigma_n$ be the lower density corresponding to $\underline{\theta}_n$ (341Ba), that is, $\underline{\phi}_nE=\underline{\theta}_nE^\bullet$ for every $E\in\Sigma_n$.

(c) For $a\in\mathfrak{A}$, $n\in\mathbb{N}$ choose $G_a\in\Sigma$, g_{an} such that $G_a^\bullet=a$ and g_{an} is a conditional expectation of χG_a on Σ_n; that is,

$$\int_E g_{an}=\int_E\chi G_a=\mu(E\cap G_a)=\bar\mu(E^\bullet\cap a)$$

for every $E\in\Sigma_n$. As remarked in 233Db, such a function g_{an} can always be found, and moreover we may take it to be Σ_n-measurable and defined everywhere on X. Now if $a\in\mathfrak{B}$, $\lim_{n\to\infty}g_{an}(x)$ exists and is equal to $\chi G_a(x)$ for almost every x. **P** By Lévy's martingale theorem (275I), $\lim_{n\to\infty}g_{an}$ is defined almost everywhere and is a conditional expectation of χG_a on the σ-algebra generated by $\bigcup_{n\in\mathbb{N}}\Sigma_n$. As observed in (a), this is just Σ_∞; and as χG_a is itself Σ_∞-measurable, it is also a conditional expectation of itself on Σ_∞, and must be equal almost everywhere to $\lim_{n\to\infty}g_{an}$. **Q**

(d) For $a\in\mathfrak{B}$, $k\geq 1$, $n\in\mathbb{N}$ set

$$H_{kn}(a)=\{x:x\in X,\,g_{an}(x)\ge 1-2^{-k}\}\in\Sigma_n,\quad \tilde H_{kn}(a)=\underline{\phi}_n(H_{kn}(a)),$$

$$\underline{\theta}a=\bigcap_{k\ge 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde H_{km}(a).$$

The rest of the proof is devoted to showing that $\underline{\theta}:\mathfrak{B}\to\Sigma$ has the required properties.

(e) G_0 is negligible, so every g_{0n} is zero almost everywhere, every $H_{kn}(0)$ is negligible and every $\tilde H_{kn}(0)$ is empty; so $\underline{\theta}0=\emptyset$.

(f) If $a\subseteq b$ in $\mathfrak{B}$, then $\underline{\theta}a\subseteq\underline{\theta}b$. **P** $G_a\setminus G_b$ is negligible, $g_{an}\le g_{bn}$ almost everywhere for every n, every $H_{kn}(a)\setminus H_{kn}(b)$ is negligible, $\tilde H_{kn}(a)\subseteq\tilde H_{kn}(b)$ for every n and k, and $\underline{\theta}a\subseteq\underline{\theta}b$. **Q**

(g) If a, $b\in\mathfrak{B}$ then $\underline{\theta}(a\cap b)=\underline{\theta}a\cap\underline{\theta}b$. **P** $\chi G_{a\cap b}\ge_{\text{a.e.}}\chi G_a+\chi G_b-1$ so $g_{a\cap b,n}\ge_{\text{a.e.}} g_{an}+g_{bn}-1$ for every n. Accordingly

$$H_{k+1,n}(a)\cap H_{k+1,n}(b)\setminus H_{kn}(a\cap b)$$

is negligible, and (because $\underline{\phi}_n$ is a lower density)

$$\tilde H_{kn}(a\cap b)\supseteq\underline{\phi}_n(H_{k+1,n}(a)\cap H_{k+1,n}(b))=\tilde H_{k+1,n}(a)\cap\tilde H_{k+1,n}(b)$$

for all $k\ge 1$, $n\in\mathbb{N}$. Now, if $x\in\underline{\theta}a\cap\underline{\theta}b$, then, for any $k\ge 1$, there are n_1, $n_2\in\mathbb{N}$ such that

$$x\in\bigcap_{m\ge n_1}\tilde H_{k+1,m}(a),\quad x\in\bigcap_{m\ge n_2}\tilde H_{k+1,m}(b).$$

But this means that

$$x\in\bigcap_{m\ge\max(n_1,n_2)}\tilde H_{km}(a\cap b).$$

As k is arbitrary, $x\in\underline{\theta}(a\cap b)$; as x is arbitrary, $\underline{\theta}a\cap\underline{\theta}b\subseteq\underline{\theta}(a\cap b)$. We know already from (f) that $\underline{\theta}(a\cap b)\subseteq\underline{\theta}a\cap\underline{\theta}b$, so $\underline{\theta}(a\cap b)=\underline{\theta}a\cap\underline{\theta}b$. **Q**

(h) If $a\in\mathfrak{B}$, then $\underline{\theta}a^\bullet=a$. **P** $\langle g_{an}\rangle_{n\in\mathbb{N}}\to\chi G_a$ a.e., so setting

$$V_a=\bigcap_{k\ge 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}H_{km}(a)=\{x:\liminf_{n\to\infty}g_{an}(x)\ge 1\},$$

$V_a\triangle G_a$ is negligible, and $V_a^\bullet=a$; but

$$\underline{\theta}a\triangle V_a\subseteq\bigcup_{k\ge 1,n\in\mathbb{N}}H_{kn}(a)\triangle\tilde H_{kn}(a)$$

is negligible, so $\underline{\theta}a^\bullet$ is also equal to a. **Q** Thus $\underline{\theta}$ is a partial lower density with domain $\mathfrak{B}$.

(i) Finally, $\underline{\theta}$ extends $\underline{\theta}_n$ for every $n\in\mathbb{N}$. **P** If $a\in\mathfrak{B}_n$, then $G_a\in\Sigma_m$ for every $m\ge n$, so $g_{am}=_{\text{a.e.}}\chi G_a$ for every $m\ge n$; $H_{km}(a)\triangle G_a$ is negligible for $k\ge 1$, $m\ge n$;

$$\tilde H_{km}=\underline{\phi}_m G_a=\underline{\theta}_m a=\underline{\theta}_n a$$

for $k\ge 1$, $m\ge n$ (this is where I use the hypothesis that $\underline{\theta}_{m+1}$ extends $\underline{\theta}_m$ for every m); and

$$\begin{aligned}\underline{\theta}a&=\bigcap_{k\ge 1}\bigcup_{r\in\mathbb{N}}\bigcap_{m\ge r}\tilde H_{km}(a)\\&=\bigcap_{k\ge 1}\bigcup_{r\ge n}\bigcap_{m\ge r}\tilde H_{km}(a)=\bigcap_{k\ge 1}\bigcup_{r\ge n}\underline{\theta}_n a=\underline{\theta}_n a.\ \mathbf{Q}\end{aligned}$$

The proof is complete.

341H Now for the first main theorem.

Theorem Let (X,Σ,μ) be any strictly localizable measure space. Then it has a lower density $\underline{\phi}:\Sigma\to\Sigma$. If $\mu X>0$ we can take $\underline{\phi}X=X$.

proof : Part A I deal first with the case of probability spaces. Let (X,Σ,μ) be a probability space, and $(\mathfrak{A},\bar\mu)$ its measure algebra.

(a) Set $\kappa=\#(\mathfrak{A})$ and enumerate $\mathfrak{A}$ as $\langle a_\xi\rangle_{\xi<\kappa}$. For $\xi\le\kappa$ let $\mathfrak{A}_\xi$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta:\eta<\xi\}$. I seek to define a lower density $\underline{\theta}:\mathfrak{A}\to\Sigma$ as the last of a family $\langle\underline{\theta}_\xi\rangle_{\xi\le\kappa}$, where $\underline{\theta}_\xi:\mathfrak{A}_\xi\to\Sigma$ is a partial lower density for each ξ. The inductive hypothesis will be that $\underline{\theta}_\xi$ extends $\underline{\theta}_\eta$ whenever $\eta\le\xi\le\kappa$.

To start the induction, we have $\mathfrak{A}_0=\{0,1\}$, $\underline{\theta}_0 0=\emptyset$, $\underline{\theta}_0 1=X$.

(b) *Inductive step to a successor ordinal* ξ Given a successor ordinal $\xi \le \kappa$, express it as $\zeta + 1$; we are supposing that $\underline{\theta}_\zeta : \mathfrak{A}_\zeta \to \Sigma$ has been defined. Now $\mathfrak{A}_\xi$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\zeta \cup \{a_\zeta\}$ (because this is a closed subalgebra, by 323K). So 341F tells us that $\underline{\theta}_\zeta$ can be extended to a partial lower density $\underline{\theta}_\xi$ with domain $\mathfrak{A}_\xi$.

(c) *Inductive step to a non-zero limit ordinal* ξ *of countable cofinality* In this case, there is a strictly increasing sequence $\langle \zeta(n) \rangle_{n\in\mathbb{N}}$ with supremum ξ. Applying 341G with $\mathfrak{B}_n = \mathfrak{A}_{\zeta(n)}$, we see that there is a partial lower density $\underline{\theta}_\xi$, with domain the closed subalgebra $\mathfrak{B}$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_{\zeta(n)}$, extending every $\underline{\theta}_{\zeta(n)}$. Now $\mathfrak{A}_{\zeta(n)} \subseteq \mathfrak{A}_\xi$ for every n, so $\mathfrak{B} \subseteq \mathfrak{A}_\xi$; but also, if $\eta < \xi$, there is an $n \in \mathbb{N}$ such that $\eta < \zeta(n)$, so that $a_\eta \in \mathfrak{A}_{\zeta(n)} \subseteq \mathfrak{B}$; as η is arbitrary, $\mathfrak{A}_\xi \subseteq \mathfrak{B}$ and $\mathfrak{A}_\xi = \mathfrak{B}$. Again, if $\eta < \xi$, there is an n such that $\eta \le \zeta(n)$, so that $\underline{\theta}_{\zeta(n)}$ extends $\underline{\theta}_\eta$ and $\underline{\theta}_\xi$ extends $\underline{\theta}_\eta$. Thus the induction continues.

(d) *Inductive step to a limit ordinal* ξ *of uncountable cofinality* In this case, $\mathfrak{A}_\xi = \bigcup_{\eta<\xi} \mathfrak{A}_\eta$. **P** Because $\mathfrak{A}$ is ccc, every member a of $\mathfrak{A}_\xi$ must be in the closed subalgebra of $\mathfrak{A}$ generated by some countable subset A of $\{a_\eta : \eta < \xi\}$ (331Gd-331Ge). Now A can be expressed as $\{a_\eta : \eta \in I\}$ for some countable $I \subseteq \xi$. As I cannot be cofinal with ξ, there is a $\zeta < \xi$ such that $\eta < \zeta$ for every $\eta \in I$, so that $A \subseteq \mathfrak{A}_\zeta$ and $a \in \mathfrak{A}_\zeta$. **Q**

But now, because $\underline{\theta}_\zeta$ extends $\underline{\theta}_\eta$ whenever $\eta \le \zeta < \xi$, we have a function $\underline{\theta}_\xi : \mathfrak{A}_\xi \to \Sigma$ defined by writing $\underline{\theta}_\xi a = \underline{\theta}_\eta a$ whenever $\eta < \xi$ and $a \in \mathfrak{A}_\eta$. Because the family $\{\mathfrak{A}_\eta : \eta < \xi\}$ is totally ordered and every $\underline{\theta}_\eta$ is a partial lower density, $\underline{\theta}_\xi$ is a partial lower density.

Thus the induction proceeds when ξ is a limit ordinal of uncountable cofinality.

(e) The induction stops when we reach $\underline{\theta}_\kappa : \mathfrak{A} \to \Sigma$, which is a lower density such that $\underline{\theta}_\kappa 1 = X$. Setting $\underline{\phi} E = \underline{\theta}_\kappa E^\bullet$, $\underline{\phi}$ is a lower density such that $\underline{\phi} X = X$.

Part B The general case of a strictly localizable measure space follows easily. First, if $\mu X = 0$, then $\mathfrak{A} = \{0\}$ and we can set $\underline{\phi} 0 = \emptyset$. Second, if μ is totally finite but not zero, we can replace it by ν, where $\nu E = \mu E/\mu X$ for every $E \in \Sigma$; a lower density for ν is also a lower density for μ. Third, if μ is not totally finite, let $\langle X_i \rangle_{i\in I}$ be a decomposition of X (211E). There is surely some j such that $\mu X_j > 0$; replacing X_j by $X_j \cup \bigcup\{X_i : i \in I,\, \mu X_i = 0\}$, we may assume that $\mu X_i > 0$ for every $i \in I$. For each $i \in I$, let $\underline{\phi}_i : \Sigma_i \to \Sigma_i$ be a lower density for μ_i, where $\Sigma_i = \Sigma \cap \mathcal{P} X_i$ and $\mu_i = \mu \restriction \Sigma_i$, such that $\underline{\phi}_i X_i = X_i$. Then it is easy to check that we have a lower density $\underline{\phi} : \Sigma \to \Sigma$ given by setting

$$\underline{\phi} E = \bigcup_{i\in I} \underline{\phi}_i (E \cap X_i)$$

for every $E \in \Sigma$, and that $\underline{\phi} X = X$.

341I The next step is to give a method of moving from lower densities to liftings. I start with an elementary remark on lower densities on complete measure spaces.

Lemma Let (X, Σ, μ) be a complete measure space with measure algebra $\mathfrak{A}$.

(a) Suppose that $\underline{\theta} : \mathfrak{A} \to \Sigma$ is a lower density and $\underline{\theta}_1 : \mathfrak{A} \to \mathcal{P} X$ is a function such that $\underline{\theta}_1 0 = \emptyset$, $\underline{\theta}_1 (a \cap b) = \underline{\theta}_1 a \cap \underline{\theta}_1 b$ for all a, $b \in \mathfrak{A}$ and $\underline{\theta}_1 a \supseteq \underline{\theta} a$ for all $a \in \mathfrak{A}$. Then $\underline{\theta}_1$ is a lower density. If $\underline{\theta}_1$ is a Boolean homomorphism, it is a lifting.

(b) Suppose that $\underline{\phi} : \Sigma \to \Sigma$ is a lower density and $\underline{\phi}_1 : \Sigma \to \mathcal{P} X$ is a function such that $\underline{\phi}_1 E = \underline{\phi}_1 F$ whenever $E \triangle F$ is negligible, $\underline{\phi}_1 \emptyset = \emptyset$, $\underline{\phi}_1 (E \cap F) = \underline{\phi}_1 E \cap \underline{\phi}_1 F$ for all E, $F \in \Sigma$ and $\underline{\phi}_1 E \supseteq \underline{\phi} E$ for all $E \in \Sigma$. Then $\underline{\phi}_1$ is a lower density. If $\underline{\phi}_1$ is a Boolean homomorphism, it is a lifting.

proof (a) All I have to check is that $\underline{\theta}_1 a \in \Sigma$ and $(\underline{\theta}_1 a)^\bullet = a$ for every $a \in \mathfrak{A}$. But

$$\underline{\theta} a \subseteq \underline{\theta}_1 a, \quad \underline{\theta}(1 \setminus a) \subseteq \underline{\theta}_1 (1 \setminus a), \quad \underline{\theta}_1 a \cap \underline{\theta}_1 (1 \setminus a) = \underline{\theta}_1 0 = \emptyset.$$

So

$$\underline{\theta} a \subseteq \underline{\theta}_1 a \subseteq X \setminus \underline{\theta}(1 \setminus a).$$

Since

$$(\underline{\theta} a)^\bullet = a = (X \setminus \underline{\theta}(1 \setminus a))^\bullet,$$

and μ is complete, $\underline{\theta}_1$ is a lower density. If it is a Boolean homomorphism, then it is also a lifting (341De).

(b) This follows by the same argument, or by looking at the functions from $\mathfrak{A}$ to Σ defined by $\underline{\phi}$ and $\underline{\phi}_1$ and using (a).

341J Proposition Let (X,Σ,μ) be a complete measure space such that $\mu X>0$, and $\mathfrak{A}$ its measure algebra.

(a) If $\underline{\theta}:\mathfrak{A}\to\Sigma$ is any lower density, there is a lifting $\theta:\mathfrak{A}\to\Sigma$ such that $\theta a\supseteq\underline{\theta}a$ for every $a\in\mathfrak{A}$.

(b) If $\underline{\phi}:\Sigma\to\Sigma$ is any lower density, there is a lifting $\phi:\Sigma\to\Sigma$ such that $\phi E\supseteq\underline{\phi}E$ for every $E\in\Sigma$.

proof (a) For each $x\in\underline{\theta}1$, set

$$I_x=\{a:a\in\mathfrak{A},\,x\in\underline{\theta}(1\setminus a)\}.$$

Then I_x is a proper ideal of $\mathfrak{A}$. **P** We have

$0\in I_x$, because $x\in\underline{\theta}1$,

if $b\subseteq a\in I_x$ then $b\in I_x$, because $x\in\underline{\theta}(1\setminus a)\subseteq\underline{\theta}(1\setminus b)$,

if $a,\,b\in I_x$ then $a\cup b\in I_x$, because $x\in\underline{\theta}(1\setminus a)\cap\underline{\theta}(1\setminus b)=\underline{\theta}(1\setminus(a\cup b))$,

$1\notin I_x$ because $x\notin\emptyset=\underline{\theta}0$. **Q**

For $x\in X\setminus\underline{\theta}1$, set $I_x=\{0\}$; this is also a proper ideal of $\mathfrak{A}$, because $\mathfrak{A}\ne\{0\}$. By 311D, there is a surjective Boolean homomorphism $\pi_x:\mathfrak{A}\to\{0,1\}$ such that $\pi_x d=0$ for every $d\in I_x$.

Define $\theta:\mathfrak{A}\to\mathcal{P}X$ by setting

$$\theta a=\{x:x\in X,\,\pi_x(a)=1\}$$

for every $a\in\mathfrak{A}$. It is easy to check that, because every π_x is a surjective Boolean homomorphism, θ is a Boolean homomorphism. Now for any $a\in\mathfrak{A}$, $x\in X$,

$$x\in\underline{\theta}a\implies 1\setminus a\in I_x\implies\pi_x(1\setminus a)=0\implies\pi_x a=1\implies x\in\theta a.$$

Thus $\theta a\supseteq\underline{\theta}a$ for every $a\in\mathfrak{A}$. By 341I, θ is a lifting, as required.

(b) Repeat the argument above, or apply it, defining $\underline{\theta}$ by setting $\underline{\theta}(E^\bullet)=\underline{\phi}E$ for every $E\in\Sigma$, and ϕ by setting $\phi E=\theta(E^\bullet)$ for every E.

341K The Lifting Theorem Every complete strictly localizable measure space of non-zero measure has a lifting.

proof By 341H, it has a lower density, so by 341J it has a lifting.

341L Remarks If we count 341F-341K as a single argument, it may be the longest proof, after Carleson's theorem (§286), which I have yet presented in this treatise, and perhaps it will be helpful if I suggest ways of looking at its components.

(a) The first point is that the theorem should be thought of as one about probability spaces. The shift to general strictly localizable spaces (Part B of the proof of 341H) is purely a matter of technique. I would not have presented it if I did not think that it's worth doing, for a variety of reasons, but there is no significant idea needed, and if – for instance – the result were valid only for σ-finite spaces, it would still be one of the great theorems of mathematics. So the rest of these remarks will be directed to the ideas needed in probability spaces.

(b) All the proofs I know of the theorem depend in one way or another on an inductive construction. We do not, of course, need a transfinite induction written out in the way I have presented it in 341H above. Essentially the same proof can be presented as an application of Zorn's Lemma; if we take P to be the set of partial lower densities, then the arguments of 341G and part (A-d) of the proof of 341H can be adapted to prove that any totally ordered subset of P has an upper bound in P, while the argument of 341F shows that any maximal element of P must have domain $\mathfrak{A}$. I think it is purely a matter of taste which form one prefers. I suppose I have used the ordinal-indexed form largely because that seemed appropriate for Maharam's theorem in the last chapter.

(c) There are then three types of inductive step to examine, corresponding to 341F, 341G and (A-d) in 341H. The first and last are easier than the second. Seeking the one-step extension of $\underline{\theta}:\mathfrak{B}\to\Sigma$ to $\underline{\theta}_1:\mathfrak{B}_1\to\Sigma$, the natural model to use is the one-step extension of a Boolean homomorphism presented in 312O. The situation here is rather more complicated, as $\underline{\theta}_1$ is not fully specified by the value of $\underline{\theta}_1 e$, and we do in fact have more freedom at this point than is entirely welcome. The formula used in the proof of 341F is derived from Graf & Weizsäcker 76.

(d) At this point I must call attention to the way in which the whole proof is dominated by the choice of *closed* subalgebras as the domains of our partial liftings. This is what makes the inductive step to a limit ordinal ξ of countable cofinality difficult, because $\mathfrak{A}_\xi$ will ordinarily be larger than $\bigcup_{\eta<\xi}\mathfrak{A}_\eta$. But it is absolutely essential in the one-step extensions as treated here. (I will return to this point in §535 of Volume 5. See also 341Ye.)

Because we are dealing with a ccc algebra $\mathfrak{A}$, the requirement that the $\mathfrak{A}_\xi$ should be closed is not a problem when $\operatorname{cf}\xi$ is uncountable, since in this case $\bigcup_{\eta<\xi}\mathfrak{A}_\eta$ is already a closed subalgebra; this is the only idea needed in (A-d) of 341H.

(e) So we are left with the inductive step to ξ when $\operatorname{cf}\xi=\omega$, which is 341G. Here we actually need some measure theory, and a particularly striking bit. (You will see that the *measure* μ, as opposed to the algebras Σ and $\mathfrak{A}$ and the homomorphism $E\mapsto E^\bullet$ and the ideal of negligible sets, is simply not mentioned anywhere else in the whole argument.)

(i) The central idea is to use the fact that bounded martingales converge to define $\underline{\theta}a$ in terms of a sequence of conditional expectations. Because I have chosen a fairly direct assault on the problem, some of the surrounding facts are not perhaps so clearly visible as they might have been if I had used a more leisurely route. For each $a\in\mathfrak{A}$, I start by choosing a representative $G_a\in\Sigma$; let me emphasize that this is a crude application of the axiom of choice, and that the different sets G_a are in no way coordinated. (The theorem we are proving is that they *can* be coordinated, but we have not reached that point yet.) Next, I choose, arbitrarily, a conditional expectation g_{an} of χG_a on each Σ_n. Once again, the choices are not coordinated; but the martingale theorem assures us that $g_a=\lim_{n\to\infty}g_{an}$ is defined almost everywhere, and is equal almost everywhere to χG_a if $a\in\mathfrak{B}$. Of course I could have gone to the g_{an} directly, without mentioning the G_a; g_{an} is a Radon-Nikodým derivative of the countably additive functional $E\mapsto\bar\mu(E^\bullet\cap a):\Sigma_n\to\mathbb{R}$. Now the g_{an}, like the G_a, are not uniquely defined. But they are defined 'up to a negligible set', so that any alternative functions g'_{an} would have $g'_{an}=_{\text{a.e.}}g_{an}$. This means that the sets $H_{kn}(a)=\{x:g_{an}(x)\ge 1-2^{-k}\}$ are also defined 'up to a negligible set', and consequently the sets $\tilde H_{kn}(a)=\underline{\phi}_n(H_{kn}(a))$ are uniquely defined. I point this out to show that it is not a complete miracle that we have formulae

$$\tilde H_{kn}(a)\subseteq\tilde H_{kn}(b)\text{ if }a\subseteq b,$$

$$\tilde H_{kn}(a\cap b)\supseteq\tilde H_{k+1,n}(a)\cap\tilde H_{k+1,n}(b)\text{ for all }a,\,b\in\mathfrak{A}$$

which do not ask us to turn a blind eye to any negligible sets. I note in passing that I could have defined the $\tilde H_{kn}(a)$ without mentioning the g_{an}; in fact

$$\tilde H_{kn}(a)=\underline{\theta}_n(\sup\{c:c\in\mathfrak{B}_n,\,\bar\mu(a\cap d)\ge 1-2^{-k}\bar\mu d\text{ whenever }d\in\mathfrak{B}_n\text{ and }d\subseteq c\}).$$

(ii) Now, with the sets $\tilde H_{kn}(a)$ in hand, we can look at

$$\tilde V_a=\bigcap_{k\ge 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde H_{kn}(a);$$

because $g_{an}\to\chi G_a$ a.e., $\tilde V_a\triangle G_a$ is negligible and $\tilde V_a^\bullet=a$ for every $a\in\mathfrak{A}_\xi$. The rest of the argument amounts to checking that $a\mapsto\tilde V_a$ will serve for $\underline{\theta}$.

(f) The arguments above apply to all probability spaces, and show that every probability space has a lower density. The next step is to convert a lower density into a lifting. It is here that we need to assume completeness. The point is that we can find a Boolean homomorphism $\theta:\mathfrak{A}\to\mathcal{P}X$ such that $\underline{\theta}a\subseteq\theta a$ for every a; this corresponds just to extending the ideals $I_x=\{a:x\in\underline{\theta}(1\setminus a)\}$ to maximal ideals (and giving a moment's thought to $x\in X\setminus\underline{\theta}1$). In order to ensure that $\theta a\in\Sigma$ and $(\theta a)^\bullet=a$, we have to observe that θa is sandwiched between $\underline{\theta}a$ and $X\setminus\underline{\theta}(1\setminus a)$, which differ by a negligible set; so that if μ is complete all will be well.

(g) The fact that completeness is needed at only one point in the argument makes it natural to wonder whether the theorem might be true for probability spaces in general. (I will come later, in 341M, to non-strictly-localizable spaces.) There is as yet no satisfactory answer to this. For Borel measure on $\mathbb{R}$, the question is known to be undecidable from the ordinary axioms of set theory (including the axiom of choice, but not the continuum hypothesis, as usual); I will give the easy part of the argument in §535; see Burke 93 for the rest. But I conjecture that there is a counter-example under the ordinary axioms (see 341Z below).

(h) Quite apart from whether completeness is needed in the argument, it is not absolutely clear why measure theory is required. The general question of whether a lifting exists can be formulated for any triple $(X,\Sigma,\mathcal{I})$ where X is a set, Σ is a σ-algebra of subsets of X, and $\mathcal{I}$ is a σ-ideal of Σ. (See 341Ya below.) S.Shelah has given an example of such a triple without a lifting in which two of the basic properties of the measure-theoretic case are satisfied: $(X,\Sigma,\mathcal{I})$ is 'complete' in the sense that every subset of any member of $\mathcal{I}$ belongs to Σ (and therefore to $\mathcal{I}$), and $\mathcal{I}$ is ω_1-saturated in Σ in the sense of 316C (see Shelah Sh636). But many other cases are known (e.g., 341Yb) in which liftings do exist.

(i) It is of course possible to prove 341K without mentioning 'lower densities', and there are even some advantages in doing so. The idea is to follow the lines of 341H, but with 'liftings' instead of 'lower densities' throughout. The inductive step to a successor ordinal is actually easier, because we have a Boolean homomorphism θ in 341F to extend, and we can use 312O as it stands if we can choose the pair E, $F = X \setminus E$ correctly. The inductive step to an ordinal of uncountable cofinality remains straightforward. But in the inductive step to an ordinal of countable cofinality, we find that in 341G we get no help from assuming that the $\underline{\theta}_n$ are actually liftings; we are still led to to a lower density $\underline{\theta}$. So at this point we have to interpolate the argument of 341J to convert this lower density into a lifting.

I have chosen the more leisurely exposition, with the extra concept, partly in order to get as far as possible without assuming completeness of the measure and partly because lower densities are an important tool for further work (see §§345-346).

(j) For more light on the argument of 341G see also 363Xe and 363Yf below.

341M I remarked above that the shift from probability spaces to general strictly localizable spaces was simply a matter of technique. The question of which spaces have liftings is also primarily a matter concerning probability spaces, as the next result shows.

Proposition Let (X, Σ, μ) be a complete locally determined space with $\mu X > 0$. Then it has a lifting iff it has a lower density iff it is strictly localizable.

proof If (X, Σ, μ) is strictly localizable then it has a lifting, by 341K. A lifting is already a lower density, and if (X, Σ, μ) has a lower density it has a lifting, by 341J. So we have only to prove that if it has a lifting then it is strictly localizable.

Let $\theta : \mathfrak{A} \to \Sigma$ be a lifting, where $\mathfrak{A}$ is the measure algebra of (X, Σ, μ). Let C be a partition of unity in $\mathfrak{A}$ consisting of elements of finite measure (322Ea). Set $\mathcal{A} = \{\theta c : c \in C\}$. Because C is disjoint, so is $\mathcal{A}$. Because $\sup C = 1$ in $\mathfrak{A}$, every set of positive measure meets some member of $\mathcal{A}$ in a set of positive measure. So the conditions of 213O are satisfied, and (X, Σ, μ) is strictly localizable.

341N Extension of partial liftings The following facts are obvious from the proof of 341H, but it will be useful to have them out in the open.

Proposition Let (X, Σ, μ) be a probability space and T a σ-subalgebra of Σ.

(a) Any partial lower density $\underline{\phi}_0 : \mathrm{T} \to \Sigma$ has an extension to a lower density $\underline{\phi} : \Sigma \to \Sigma$.

(b) Suppose now that μ is complete. If $\underline{\phi}_0$ is a Boolean homomorphism, it has an extension to a lifting ϕ for μ.

proof (a) In Part A of the proof of 341H, let $\mathfrak{A}_\xi$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{E^\bullet : E \in \mathrm{T}\} \cup \{a_\eta : \eta < \xi\}$, and set $\underline{\theta}_0 E^\bullet = \underline{\phi}_0 E$ for every $E \in \mathrm{T}$. Proceed with the induction as before. The only difference is that we no longer have a guarantee that $\underline{\phi} X = X$.

(b) Suppose now that $\underline{\phi}_0$ is a Boolean homomorphism and μ is complete. 341J tells us that there is a lifting $\phi : \Sigma \to \Sigma$ such that $\phi E \supseteq \underline{\phi} E$ for every $E \in \Sigma$. But if $E \in \mathrm{T}$ we must have $\phi E \supseteq \underline{\phi}_0 E$,

$$\phi E \setminus \underline{\phi}_0 E = \phi E \cap \underline{\phi}_0 (X \setminus E) \subseteq \phi E \cap \phi(X \setminus E) = \emptyset,$$

so that $\phi E = \underline{\phi}_0 E$, and ϕ extends $\underline{\phi}_0$.

341O Liftings and Stone spaces The arguments of this section so far involve repeated use of the axiom of choice, and offer no suggestion that any liftings (or lower densities) are in any sense 'canonical'. There is however one context in which we have a distinguished lifting. Suppose that we have the Stone space (Z, T, ν) of a measure algebra $(\mathfrak{A}, \bar{\mu})$; as in 311E, I think of Z as being the set of surjective Boolean homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$, so that each $a \in \mathfrak{A}$ corresponds to the open-and-closed set $\widehat{a} = \{z : z(a) = 1\}$. Then we have a lifting $\theta : \mathfrak{A} \to \mathrm{T}$ defined by setting $\theta a = \widehat{a}$ for each $a \in \mathfrak{A}$. (I am identifying $\mathfrak{A}$ with the measure algebra of ν, as in 321J.) The corresponding lifting $\phi : \mathrm{T} \to \mathrm{T}$ is defined by taking ϕE to be that unique open-and-closed set such that $E \triangle \phi E$ is negligible (or, if you prefer, meager).

Generally, liftings can be described in terms of Stone spaces, as follows.

341P Proposition Let (X, Σ, μ) be a measure space, $(\mathfrak{A}, \bar{\mu})$ its measure algebra, and (Z, T, ν) the Stone space of $(\mathfrak{A}, \bar{\mu})$ with its canonical measure.

(a) There is a one-to-one correspondence between liftings $\theta : \mathfrak{A} \to \Sigma$ and functions $f : X \to Z$ such that $f^{-1}[\widehat{a}] \in \Sigma$ and $(f^{-1}[\widehat{a}])^\bullet = a$ for every $a \in \mathfrak{A}$, defined by the formula

$$\theta a = f^{-1}[\widehat{a}] \text{ for every } a \in \mathfrak{A}.$$

(b) If (X, Σ, μ) is complete and locally determined, then a function $f : X \to Z$ satisfies the conditions of (a) iff (α) it is inverse-measure-preserving (β) the homomorphism it induces between the measure algebras of μ and ν is the canonical isomorphism defined by the construction of Z.

proof Recall that T is just the set $\{\widehat{a} \triangle M : a \in \mathfrak{A},\ M \subseteq Z \text{ is meager}\}$, and that $\nu(\widehat{a} \triangle M) = \bar{\mu} a$ for all such a, M; while the canonical isomorphism π between $\mathfrak{A}$ and the measure algebra of ν is defined by the formula

$$\pi F^\bullet = a \text{ whenever } F \in \mathrm{T},\ a \in \mathfrak{A} \text{ and } F \triangle \widehat{a} \text{ is meager}$$

(341K).

(a) If $\theta : \mathfrak{A} \to \Sigma$ is any Boolean homomorphism, then for every $x \in X$ we have a surjective Boolean homomorphism $f_\theta(x) : \mathfrak{A} \to \mathbb{Z}_2$ defined by saying that $f_\theta(x)(a) = 1$ if $x \in \theta a$, 0 otherwise. f_θ is a function from X to Z. We can recover θ from f_θ by the formula

$$\theta a = \{x : f_\theta(x)(a) = 1\} = \{x : f_\theta(x) \in \widehat{a}\} = f_\theta^{-1}[\widehat{a}].$$

So $f_\theta^{-1}[\widehat{a}] \in \Sigma$ and, if θ is a lifting,

$$(f_\theta^{-1}[\widehat{a}])^\bullet = (\theta a)^\bullet = a.$$

for every $a \in \mathfrak{A}$.

Similarly, given a function $f : X \to Z$ with this property, then we can set $\theta a = f^{-1}[\widehat{a}]$ for every $a \in \mathfrak{A}$ to obtain a lifting $\theta : \mathfrak{A} \to \Sigma$; and of course we now have

$$f(x)(a) = 1 \iff f(x) \in \widehat{a} \iff x \in \theta a,$$

so $f_\theta = f$.

(b) Assume now that (X, Σ, μ) is complete and locally determined.

(i) Let $f : X \to Z$ be the function associated with a lifting θ, as in (a). I show first that f is inverse-measure-preserving. **P** If $F \in \mathrm{T}$, express it as $\widehat{a} \triangle M$, where $a \in \mathfrak{A}$ and $M \subseteq Z$ is meager. By 322F, $\mathfrak{A}$ is weakly (σ, ∞)-distributive, so M is nowhere dense (316I). Consider $f^{-1}[M]$. If $E \subseteq X$ is measurable and of finite measure, then $E \cap f^{-1}[M]$ has a measurable envelope H (132Ee). **?** If $\mu H > 0$, then $b = H^\bullet \neq 0$ and $\widehat{b}$ is a non-empty open set in Z. Because M is nowhere dense, there is a non-zero $a \in \mathfrak{A}$ such that $\widehat{a} \subseteq \widehat{b} \setminus M$. Now $\mu(f^{-1}[\widehat{b}] \triangle H) = 0$, so $f^{-1}[\widehat{a}] \setminus H$ is negligible, and $f^{-1}[\widehat{a}] \cap H$ is a non-negligible measurable set disjoint from $E \cap f^{-1}[M]$ and included in H; which is impossible. **X** Thus H and $E \cap f^{-1}[M]$ are negligible. This is true for every measurable set E of finite measure. Because μ is complete and locally determined, $f^{-1}[M] \in \Sigma$ and $\mu f^{-1}[M] = 0$. So $f^{-1}[F] = f^{-1}[\widehat{a}] \triangle f^{-1}[M]$ is measurable, and

$$\mu f^{-1}[F] = \mu f^{-1}[\widehat{a}] = \mu \theta a = \bar{\mu} a = \nu \widehat{a} = \nu F.$$

As F is arbitrary, f is inverse-measure-preserving. **Q**

It follows at once that for any $F \in \mathrm{T}$,

$$f^{-1}[F]^\bullet = a = \pi F^\bullet$$

where a is that element of $\mathfrak{A}$ such that $M = F \triangle a$ is meager, because in this case $f^{-1}[\widehat{a}]^\bullet = a$, by (a), while $f^{-1}[M]$ is negligible. So π is the homomorphism induced by f.

(ii) Now suppose that $f : X \to Z$ is an inverse-measure-preserving function such that $f^{-1}[F]^\bullet = \pi F^\bullet$ for every $F \in \mathrm{T}$. Then, in particular,

$$f^{-1}[\widehat{a}]^\bullet = \pi \widehat{a}^\bullet = a$$

for every $a \in \mathfrak{A}$, so that f satisfies the conditions of (a).

341Q Corollary Let (X, Σ, μ) be a strictly localizable measure space, $(\mathfrak{A}, \bar{\mu})$ its measure algebra, and Z the Stone space of $\mathfrak{A}$; suppose that $\mu X > 0$. For $E \in \Sigma$ write E^* for the open-and-closed subset of Z corresponding to $E^\bullet \in \mathfrak{A}$. Then there is a function $f : X \to Z$ such that $E \triangle f^{-1}[E^*]$ is negligible for every $E \in \Sigma$. If μ is complete, then f is inverse-measure-preserving.

proof Let $\hat{\mu}$ be the completion of μ, and $\hat{\Sigma}$ its domain. Then we can identify $(\mathfrak{A}, \bar{\mu})$ with the measure algebra of $\hat{\mu}$ (322Da). Let $\theta : \mathfrak{A} \to \hat{\Sigma}$ be a lifting, and $f : X \to Z$ the corresponding function. If $E \in \Sigma$ then $E^* = \widehat{a}$ where $a = E^\bullet$, so $E \triangle f^{-1}[E^*] = E \triangle \theta E^\bullet$ is negligible. If μ is itself complete, so that $\hat{\Sigma} = \Sigma$, then f is inverse-measure-preserving, by 341Pb.

341X Basic exercises (a) Let (X, Σ, μ) be a measure space and $\phi : \Sigma \to \Sigma$ a function. Show that ϕ is a lifting iff it is a lower density and $\phi E \cup \phi(X \setminus E) = X$ for every $E \in \Sigma$.

>(b) Let $\nu_{\mathbb{N}}$ be the usual measure on $X = \{0,1\}^{\mathbb{N}}$, and $\mathrm{T}_{\mathbb{N}}$ its domain. For $x \in X$ and $n \in \mathbb{N}$ set $U_n(x) = \{y : y \in X,\ y \restriction n = x \restriction n\}$. For $E \in \mathrm{T}_{\mathbb{N}}$ set $\underline{\phi} E = \{x : \lim_{n\to\infty} 2^n \mu(E \cap U_n(x)) = 1\}$. Show that $\underline{\phi}$ is a lower density for $\nu_{\mathbb{N}}$.

>(c) Let $\mathfrak{A}$ be a Boolean algebra, I an ideal of $\mathfrak{A}$, and $\mathfrak{B}$ a countable subalgebra of the quotient algebra $\mathfrak{A}/I$. Show that there is a Boolean homomorphism $\theta : \mathfrak{B} \to \mathfrak{A}$ such that $(\theta b)^\bullet = b$ for every $b \in \mathfrak{B}$. (*Hint*: let $\langle b_n \rangle_{n\in\mathbb{N}}$ run over $\mathfrak{B}$; let $\mathfrak{B}_n$ be the subalgebra of $\mathfrak{B}$ generated by $\{b_i : i < n\}$; given $\theta \restriction \mathfrak{B}_n$, show that there is an $a_n \in \mathfrak{A}$ such that $a_n^\bullet = b_n$ and $\theta b' \subseteq a_n \subseteq \theta b''$ whenever $b',\ b'' \in \mathfrak{B}_n$ and $b' \subseteq b_n \subseteq b''$.)

>(d) Let P be the set of all lower densities of a complete measure space (X, Σ, μ), with measure algebra $\mathfrak{A}$, ordered by saying that $\underline{\theta} \le \underline{\theta}'$ if $\underline{\theta} a \subseteq \underline{\theta}' a$ for every $a \in \mathfrak{A}$. Show that any non-empty totally ordered subset of P has an upper bound in P. Show that if $\underline{\theta} \in P$, $a \in \mathfrak{A} \setminus \{0\}$ and $x \in X \setminus (\underline{\theta} a \cup \underline{\theta}(1 \setminus a))$, then $\underline{\theta}' : \mathfrak{A} \to \Sigma$ is a lower density, where $\underline{\theta}' b = \underline{\theta} b \cup \{x\}$ if either $a \subseteq b$ or there is a $c \in \mathfrak{A}$ such that $x \in \underline{\theta} c$ and $a \cap c \subseteq b$, and $\underline{\theta}' b = \underline{\theta} b$ otherwise. Hence prove 341J.

(e) Let (X, Σ, μ) and (Y, T, ν) be measure spaces and suppose that there is an inverse-measure-preserving function $f : X \to Y$ such that the associated homomorphism from the measure algebra of ν to that of μ (324M) is an isomorphism. Show that for every lifting ϕ for (Y, T, ν) we have a corresponding lifting ψ of (X, Σ, μ) defined uniquely by the formula

$$\psi(f^{-1}[F]) = f^{-1}[\phi F] \text{ for every } F \in \mathrm{T}.$$

(f) Let (X, Σ, μ) be a measure space, and write $\mathcal{L}^\infty(\Sigma)$ for the linear space of all bounded Σ-measurable functions from X to $\mathbb{R}$. Show that for any lifting $\phi : \Sigma \to \Sigma$ of μ there is a unique linear operator $T : L^\infty(\mu) \to \mathcal{L}^\infty(\Sigma)$ such that $T(\chi E)^\bullet = \chi(\phi E)$ for every $E \in \Sigma$ and $Tu \ge 0$ in $\mathcal{L}^\infty(\Sigma)$ whenever $u \ge 0$ in $L^\infty(\mu)$. Show that (i) $(Tu)^\bullet = u$ and $\sup_{x\in X} |(Tu)(x)| = \|u\|_\infty$ for every $u \in L^\infty(\mu)$ (ii) $T(u \times v) = Tu \times Tv$ for all $u,\ v \in L^\infty(\mu)$.

(g) Let μ be Lebesgue measure on $[0,1]$. Write $\mathcal{L}^1_{\Sigma_L}$ for the linear space of integrable functions $f : [0,1] \to \mathbb{R}$. Show that there is no operator $T : L^1(\mu) \to \mathcal{L}^1_{\Sigma_L}$ such that (i) $(Tu)^\bullet = u$ for every $u \in L^1(\mu)$ (ii) $Tu \ge Tv$ whenever $u \ge v$ in $L^1(\mu)$. (*Hint*: Let $F \subseteq \mathcal{L}^1_{\Sigma_L}$ be the countable set $\{n\chi[2^{-n}k, 2^{-n}(k+1)] : n \in \mathbb{N},\ k < 2^n\}$. Show that if T satisfies (i) then there is an $x \in \{0,1\}^{\mathbb{N}}$ such that $T(f^\bullet)(x) = f(x)$ for every $f \in F$; find a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in F such that $\{f_n^\bullet : n \in \mathbb{N}\}$ is bounded above in $L^1(\mu)$ but $\sup_{n\in\mathbb{N}} f_n(x) = \infty$.)

341Y Further exercises (a) Let X be a set, Σ an algebra of subsets of X and $\mathcal{I}$ an ideal of Σ; let $\mathfrak{A}$ be the quotient Boolean algebra $\Sigma/\mathcal{I}$. We say that a function $\theta : \mathfrak{A} \to \Sigma$ is a **lifting** if it is a Boolean homomorphism and $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$, and that $\underline{\theta} : \mathfrak{A} \to \Sigma$ is a **lower density** if $\underline{\theta} 0 = \emptyset$, $\underline{\theta}(a \cap b) = \underline{\theta} a \cap \underline{\theta} b$ for all $a,\ b \in \mathfrak{A}$, and $(\underline{\theta} a)^\bullet = a$ for every $a \in \mathfrak{A}$.

Show that if $(X, \Sigma, \mathcal{I})$ is 'complete' in the sense that $F \in \Sigma$ whenever $F \subseteq E \in \mathcal{I}$, and if $X \notin \mathcal{I}$, and $\underline{\theta} : \mathfrak{A} \to \Sigma$ is a lower density, then there is a lifting $\theta : \mathfrak{A} \to \Sigma$ such that $\underline{\theta} a \subseteq \theta a$ for every $a \in \mathfrak{A}$.

(b) Let X be a Baire space, $\widehat{\mathcal{B}}$ Baire-property algebra of X (314Yd) and $\mathcal{M}$ the ideal of meager subsets of X. Show that there is a lifting θ from $\widehat{\mathcal{B}}/\mathcal{M}$ to $\widehat{\mathcal{B}}$ such that $\theta G^\bullet \supseteq G$ for every open $G \subseteq X$. (*Hint*: in 341Ya, set $\underline{\theta}(G^\bullet) = G$ for every regular open set G.)

(c) Let (X, Σ, μ) be a Maharam-type-homogeneous probability space with Maharam type $\kappa \ge \omega$. Let $\mathcal{B}\mathfrak{a}_\kappa$ be the Baire σ-algebra of $Y = \{0,1\}^\kappa$, that is, the σ-algebra of subsets of Y generated by the family $\{\{x : x(\xi) = 1\} : \xi < \kappa\}$, and let ν be the restriction to $\mathcal{B}\mathfrak{a}_\kappa$ of the usual measure on $\{0,1\}^\kappa$. Show that there is an inverse-measure-preserving function $f : X \to Y$ which induces an isomorphism between the measure algebras of μ and ν.

(d) Let (X, Σ, μ) be a complete Maharam-type-homogeneous probability space with Maharam type $\kappa \geq \omega$, and give $Y = \{0,1\}^\kappa$ its usual measure ν_κ. Show that there is an inverse-measure-preserving function $f : X \to Y$ which induces an isomorphism between the measure algebras of μ and ν_κ.

***(e)** Give an example of a complete probability space (X, Σ, μ), a subalgebra T of Σ, and a partial lower density $\underline{\phi} : \mathrm{T} \to \Sigma$ which has no extension to a lower density for μ. (*Hint*: There is a subset of $\{0,1\}^{\mathfrak{c}}$, of cardinal $\mathfrak{c}$, which is non-negligible for the usual measure on $\{0,1\}^{\mathfrak{c}}$.)

341Z Problems (a) Can we construct, using the ordinary axioms of mathematics (including the axiom of choice, but not the continuum hypothesis), a probability space (X, Σ, μ) with no lifting?

(b) Set $\kappa = \omega_3$. (There is a reason for taking ω_3 here; see 535E in Volume 5.) Let $\mathcal{B}\mathfrak{a}_\kappa$ be the Baire σ-algebra of $\{0,1\}^\kappa$ (as in 341Yc), and μ the restriction to $\mathcal{B}\mathfrak{a}_\kappa$ of the usual measure on $\{0,1\}^\kappa$. Can we show that μ has no lifting?

341 Notes and comments Innumerable variations of the proof of 341K have been devised, as each author has struggled with the technical complications. I have discussed the reasons for my own choices in 341L.

The theorem has a curious history. It was originally announced by von Neumann, but he seems never to have written his proof down, and the first published proof is that of Maharam 58. That argument is based on Maharam's theorem, 341Xe and 341Yd, which show that it is enough to find liftings for every $\{0,1\}^\kappa$; this requires most of the ideas presented above, but feels more concrete, and some of the details are slightly simpler. The argument as I have written it owes a great deal to Ionescu Tulcea & Ionescu Tulcea 69.

The lifting theorem and Maharam's theorem are the twin pillars of modern abstract measure theory. But there remains a degree of mystery about the lifting theorem which is absent from the other. The first point is that there is nothing canonical about the liftings we can construct, except in the quite exceptional case of Stone spaces (341O). Even when there is a more or less canonical lower density present (341E, 341Xb), the conversion of this into a lifting requires arbitrary choices, as in 341J. While we can distinguish some liftings as being somewhat more regular than others, I know of no criterion which marks out any particular lifting for Lebesgue measure, for instance, among the rest. Perhaps associated with this arbitrariness is the extreme difficulty of deciding whether liftings of any given type exist. Neither positive nor negative results are easily come by (I will present a few in the later sections of this chapter), and the nature of the obstacles remains quite unclear.

342 Compact measure spaces

The next three sections amount to an extended parenthesis, showing how the Lifting Theorem can be used to attack one of the fundamental problems of measure theory: the representation of Boolean homomorphisms between measure algebras by functions between appropriate measure spaces. This section prepares for the main idea by introducing the class of 'locally compact' measures (342Ad), with the associated concepts of 'compact' and 'perfect' measures (342Ac, 342K). These depend on the notions of 'inner regularity' (342Aa, 342B) and 'compact class' (342Ab, 342D). I list the basic permanence properties for compact and locally compact measures (342G-342I) and mention some of the compact measures which we have already seen (342J). Concerning perfect measures, I content myself with the proof that a locally compact measure is perfect (342L). I end the section with two examples (342M, 342N).

342A Definitions (a) Let (X, Σ, μ) be a measure space. If $\mathcal{K} \subseteq \mathcal{P}X$, I will say that μ is **inner regular** with respect to $\mathcal{K}$ if

$$\mu E = \sup\{\mu K : K \in \mathcal{K} \cap \Sigma,\ K \subseteq E\}$$

for every $E \in \Sigma$.

Of course μ is inner regular with respect to $\mathcal{K}$ iff it is inner regular with respect to $\mathcal{K} \cap \Sigma$.

(b) A family $\mathcal{K}$ of sets is a **compact class** if $\bigcap \mathcal{K}' \neq \emptyset$ whenever $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property.

Note that any subset of a compact class is again a compact class. (In particular, it is convenient to allow the empty set as a compact class.)

(c) A measure space (X,Σ,μ), or a measure μ, is **compact** if μ is inner regular with respect to some compact class of subsets of X.

Allowing $\emptyset$ as a compact class, and interpreting $\sup\emptyset$ as 0 in (a) above, μ is a compact measure whenever $\mu X=0$.

(d) A measure space (X,Σ,μ), or a measure μ, is **locally compact** if the subspace measure μ_E is compact whenever $E\in\Sigma$ and $\mu E<\infty$.

Remark I ought to point out that the original definitions of 'compact class' and 'compact measure' (Marczewski 53) correspond to what I will call 'countably compact class' and 'countably compact measure' in Volume 4. For another variation on the concept of 'compact class' see condition (β) in 343B(ii)-(iii).

For examples of compact measure spaces see 342J and 342Xf.

342B I prepare the ground with some straightforward lemmas.

Lemma Let (X,Σ,μ) be a measure space, and $\mathcal{K}\subseteq\Sigma$ a set such that whenever $E\in\Sigma$ and $\mu E>0$ there is a $K\in\mathcal{K}$ such that $K\subseteq E$ and $\mu K>0$. Let $E\in\Sigma$.

(a) There is a countable disjoint set $\mathcal{K}_1\subseteq\mathcal{K}$ such that $K\subseteq E$ for every $K\in\mathcal{K}_1$ and $\mu(\bigcup\mathcal{K}_1)=\mu E$.

(b) If $\mu E<\infty$ then $\mu(E\setminus\bigcup\mathcal{K}_1)=0$.

(c) In any case, there is for any $\gamma<\mu E$ a finite disjoint $\mathcal{K}_0\subseteq\mathcal{K}$ such that $K\subseteq E$ for every $K\in\mathcal{K}_0$ and $\mu(\bigcup\mathcal{K}_0)\ge\gamma$.

proof Set $\mathcal{K}'=\{K:K\in\mathcal{K},\,K\subseteq E,\,\mu K>0\}$. Let $\mathcal{K}^*$ be a maximal disjoint subfamily of $\mathcal{K}'$. If $\mathcal{K}^*$ is uncountable, then there is some $n\in\mathbb{N}$ such that $\{K:K\in\mathcal{K}^*,\,\mu K\ge 2^{-n}\}$ is infinite, so that there is a countable $\mathcal{K}_1\subseteq\mathcal{K}^*$ such that $\mu(\bigcup\mathcal{K}_1)=\infty=\mu E$.

If $\mathcal{K}^*$ is countable, set $\mathcal{K}_1=\mathcal{K}^*$. Then $F=\bigcup\mathcal{K}_1$ is measurable, and $F\subseteq E$. Moreover, there is no member of $\mathcal{K}'$ disjoint from F; but this means that $E\setminus F$ must be negligible. So $\mu F=\mu E$, and (a) is true. Now (b) and (c) follow at once, because

$$\mu(\bigcup\mathcal{K}_1)=\sup\{\mu(\bigcup\mathcal{K}_0):\mathcal{K}_0\subseteq\mathcal{K}_1\text{ is finite}\}.$$

Remark This lemma can be thought of as more versions of the principle of exhaustion; compare 215A.

342C Corollary Let (X,Σ,μ) be a measure space and $\mathcal{K}\subseteq\mathcal{P}X$ a family of sets such that (α) $K\cup K'\in\mathcal{K}$ whenever $K,\,K'\in\mathcal{K}$ and $K\cap K'=\emptyset$ (β) whenever $E\in\Sigma$ and $\mu E>0$, there is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E$ and $\mu K>0$. Then μ is inner regular with respect to $\mathcal{K}$.

proof Apply 342Bc to $\mathcal{K}\cap\Sigma$.

342D Lemma Let X be a set and $\mathcal{K}$ a family of subsets of X.

(a) The following are equiveridical:

(i) $\mathcal{K}$ is a compact class;

(ii) there is a topology $\mathfrak{T}$ on X such that X is compact and every member of $\mathcal{K}$ is a closed set for $\mathfrak{T}$.

(b) If $\mathcal{K}$ is a compact class, so are the families $\mathcal{K}_1=\{K_0\cup\ldots\cup K_n:K_0,\ldots,K_n\in\mathcal{K}\}$ and $\mathcal{K}_2=\{\bigcap\mathcal{K}':\emptyset\ne\mathcal{K}'\subseteq\mathcal{K}\}$.

proof (a)(i)$\Rightarrow$(ii) Let $\mathfrak{T}$ be the topology generated by $\{X\setminus K:K\in\mathcal{K}\}$. Then of course every member of $\mathcal{K}$ is closed for $\mathfrak{T}$. Let $\mathcal{F}$ be an ultrafilter on X. Then $\mathcal{K}\cap\mathcal{F}$ has the finite intersection property; because $\mathcal{K}$ is a compact class, it has non-empty intersection; take $x\in X\cap\bigcap(\mathcal{K}\cap\mathcal{F})$. The family

$$\{G:G\subseteq X,\text{ either }G\in\mathcal{F}\text{ or }x\notin G\}$$

is easily seen to be a topology on X, and contains $X\setminus K$ for every $K\in\mathcal{K}$ (because if $X\setminus K\notin\mathcal{F}$ then $K\in\mathcal{F}$ and $x\in K$), so includes $\mathfrak{T}$; but this just means that every $\mathfrak{T}$-open set containing x belongs to $\mathcal{F}$, that is, that $\mathcal{F}\to x$. As $\mathcal{F}$ is arbitrary, X is compact for $\mathfrak{T}$ (2A3R).

(ii)$\Rightarrow$(i) Use 3A3Da.

(b) Let $\mathfrak{T}$ be a topology on X such that X is compact and every member of $\mathcal{K}$ is closed for $\mathfrak{T}$; then the same is true of every member of $\mathcal{K}_1$ or $\mathcal{K}_2$.

342E Corollary Suppose that (X,Σ,μ) is a measure space and that $\mathcal{K}$ is a compact class such that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$. Then μ is compact.

proof Set $\mathcal{K}_1 = \{K_0 \cup \ldots \cup K_n : K_0, \ldots, K_n \in \mathcal{K}\}$. By 342Db, $\mathcal{K}_1$ is a compact class, and by 342C μ is inner regular with respect to $\mathcal{K}_1$.

342F Corollary A measure space (X,Σ,μ) is compact iff there is a topology on X such that X is compact and μ is inner regular with respect to the closed sets.

proof (a) If μ is inner regular with respect to a compact class $\mathcal{K}$, then there is a compact topology on X such that every member of $\mathcal{K}$ is closed (342Da); now the family $\mathcal{F}$ of closed sets includes $\mathcal{K}$, so μ is also inner regular with respect to $\mathcal{F}$.

(b) If there is a compact topology on X such that μ is inner regular with respect to the family $\mathcal{K}$ of closed sets, then this is a compact class, so μ is a compact measure.

342G Now I look at the standard questions concerning preservation of the properties of 'compactness' or 'local compactness' under the usual manipulations.

Proposition (a) Any measurable subspace of a compact measure space is compact.
(b) The completion and c.l.d. version of a compact measure space are compact.
(c) A semi-finite measure space is compact iff its completion is compact iff its c.l.d. version is compact.
(d) The direct sum of a family of compact measure spaces is compact.
(e) The c.l.d. product of two compact measure spaces is compact.
(f) The product of any family of compact probability spaces is compact.

proof (a) Let (X,Σ,μ) be a compact measure space, and $E \in \Sigma$. If $\mathcal{K}$ is a compact class such that μ is inner regular with respect to $\mathcal{K}$, then $\mathcal{K}_E = \mathcal{K} \cap \mathcal{P}E$ is a compact class (just because it is a subset of $\mathcal{K}$) and the subspace measure μ_E is inner regular with respect to $\mathcal{K}_E$.

(b) Let (X,Σ,μ) be a compact measure space. Write $(X,\check{\Sigma},\check{\mu})$ for *either* the completion *or* the c.l.d. version of (X,Σ,μ). Let $\mathcal{K} \subseteq \mathcal{P}X$ be a compact class such that μ is inner regular with respect to $\mathcal{K}$. Then $\check{\mu}$ also is inner regular with respect to $\mathcal{K}$. **P** If $E \in \check{\Sigma}$ and $\gamma < \check{\mu}E$ there is an $E' \in \Sigma$ such that $E' \subseteq E$ and $\mu E' > \gamma$; if $\check{\mu}$ is the c.l.d. version of μ, we may take $\mu E'$ to be finite. There is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E'$ and $\mu K \geq \gamma$. Now $\check{\mu}K = \mu K \geq \gamma$ and $K \subseteq E$ and $K \in \mathcal{K} \cap \check{\Sigma}$. **Q**

(c) Now suppose that (X,Σ,μ) is semi-finite; again write $(X,\check{\Sigma},\check{\mu})$ for *either* its completion *or* its c.l.d. version. We already know that if μ is compact, so is $\check{\mu}$. If $\check{\mu}$ is compact, let $\mathcal{K} \subseteq \mathcal{P}X$ be a compact class such that $\check{\mu}$ is inner regular with respect to $\mathcal{K}$. Set $\mathcal{K}^* = \{\bigcap \mathcal{K}' : \emptyset \neq \mathcal{K}' \subseteq \mathcal{K}\}$; then $\mathcal{K}^*$ is a compact class (342Db). Now μ is inner regular with respect to $\mathcal{K}^*$. **P** Take $E \in \Sigma$ and $\gamma < \mu E$. Choose $\langle E_n \rangle_{n\in\mathbb{N}}$, $\langle K_n \rangle_{n\in\mathbb{N}}$ as follows. Because μ is semi-finite, there is an $E_0 \subseteq E$ such that $E_0 \in \Sigma$ and $\gamma < \mu E_0 < \infty$. Given $E_n \in \Sigma$ such that $\mu E_n > \gamma$, there is a $K_n \in \mathcal{K} \cap \check{\Sigma}$ such that $K_n \subseteq E_n$ and $\check{\mu}K_n > \gamma$. Now there is an $E_{n+1} \in \Sigma$ such that $E_{n+1} \subseteq K_n$ and $\mu E_{n+1} > \gamma$. Continue. On completing the induction, set $K = \bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} E_n$, so that $K \in \mathcal{K}^* \cap \Sigma$ and $K \subseteq E$ and $\mu K = \lim_{n\to\infty} \mu E_n \geq \gamma$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}^*$. **Q** As $\mathcal{K}^*$ is a compact class, μ is a compact measure.

(d) Let $\langle (X_i,\Sigma_i,\mu_i) \rangle_{i\in I}$ be a family of compact measure spaces, with direct sum (X,Σ,μ). We may suppose that each X_i is actually a subset of X, with μ_i the subspace measure. For each $i \in I$ let $\mathcal{K}_i \subseteq \mathcal{P}X_i$ be a compact class such that μ_i is inner regular with respect to $\mathcal{K}_i$. Then $\mathcal{K} = \bigcup_{i\in I} \mathcal{K}_i$ is a compact class, for if $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property, then $\mathcal{K}' \subseteq \mathcal{K}_i$ for some i, so has non-empty intersection. Now if $E \in \Sigma$ and $\mu E > 0$ there is some $i \in I$ such that $\mu_i(E \cap X_i) > 0$, and we can find a $K \in \mathcal{K}_i \cap \Sigma_i \subseteq \mathcal{K} \cap \Sigma$ such that $K \subseteq E \cap X_i$ and $\mu_i K > 0$, in which case $\mu K > 0$. By 342E, μ is compact.

(e) Let (X,Σ,μ) and (Y,T,ν) be two compact measure spaces, with c.l.d. product measure $(X \times Y,\Lambda,\lambda)$. Let $\mathfrak{T}$, $\mathfrak{S}$ be topologies on X, Y respectively such that X and Y are compact spaces and μ, ν are inner regular with respect to the closed sets. Then the product topology on $X \times Y$ is compact (3A3J).

The point is that λ is inner regular with respect to the family $\mathcal{K}$ of closed subsets of $X \times Y$. **P** Suppose that $W \in \Lambda$ and $\lambda W > \gamma$. Then there are $E \in \Sigma$, $F \in \mathrm{T}$ such that $\mu E < \infty$, $\nu F < \infty$ and $\lambda(W \cap (E \times F)) > \gamma$ (251F). Now there are sequences $\langle E_n \rangle_{n\in\mathbb{N}}$, $\langle F_n \rangle_{n\in\mathbb{N}}$ in Σ, T respectively such that

$$(E \times F) \setminus W \subseteq \bigcup_{n\in\mathbb{N}} E_n \times F_n,$$

$$\sum_{n=0}^{\infty}\mu E_n\cdot\nu F_n<\lambda((E\times F)\setminus W)+\lambda((E\times F)\cap W)-\gamma=\lambda(E\times F)-\gamma$$

(251C). Set

$$W'=(E\times F)\setminus\bigcup_{n\in\mathbb{N}}E_n\times F_n=\bigcap_{n\in\mathbb{N}}((E\times(F\setminus F_n))\cup((E\setminus E_n)\times F)).$$

Then $W'\subseteq W$, and

$$\lambda((E\times F)\setminus W')\le\lambda(\bigcup_{n\in\mathbb{N}}E_n\times F_n)\le\sum_{n=0}^{\infty}\mu E_n\cdot\nu F_n<\lambda(E\times F)-\gamma,$$

so $\lambda W'>\gamma$.

Set $\epsilon=\frac{1}{4}(\lambda W'-\gamma)/(1+\mu E+\mu F)$. For each n, we can find closed measurable sets K_n, $K'_n\subseteq X$ and L_n, $L'_n\subseteq Y$ such that

$$K_n\subseteq E,\quad \mu(E\setminus K_n)\le 2^{-n}\epsilon,$$

$$L'_n\subseteq F\setminus F_n,\quad \nu((F\setminus F_n)\setminus L'_n)\le 2^{-n}\epsilon,$$

$$K'_n\subseteq E\setminus E_n,\quad \mu((E\setminus E_n)\setminus K'_n)\le 2^{-n}\epsilon,$$

$$L_n\subseteq F,\quad \nu(F\setminus L_n)\le 2^{-n}\epsilon.$$

Set

$$V=\bigcap_{n\in\mathbb{N}}(K_n\times L'_n)\cup(K'_n\times L_n)\subseteq W'\subseteq W.$$

Now

$$\begin{aligned}W'\setminus V\subseteq\bigcup_{n\in\mathbb{N}}&((E\setminus K_n)\times F)\cup(E\times((F\setminus F_n)\setminus L'_n))\\&\cup(((E\setminus E_n)\setminus K'_n)\times F)\cup(E\times(F\setminus L_n)),\end{aligned}$$

so

$$\begin{aligned}\lambda(W'\setminus V)&\le\sum_{n=0}^{\infty}\mu(E\setminus K_n)\cdot\nu F+\mu E\cdot\nu((F\setminus F_n)\setminus L'_n)\\&\qquad+\mu((E\setminus E_n)\setminus K'_n)\cdot\nu F+\mu E\cdot\nu(F\setminus L_n)\\&\le\sum_{n=0}^{\infty}2^{-n}\epsilon(2\mu E+2\mu F)\le\lambda W'-\gamma,\end{aligned}$$

and $\lambda V\ge\gamma$. But V is a countable intersection of finite unions of products of closed measurable sets, so is itself a closed measurable set, and belongs to $\mathcal{K}\cap\Lambda$. **Q**

Accordingly the product topology on $X\times Y$ witnesses that λ is a compact measure.

(f) The same method works. In detail: let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of compact probability spaces, with product (X,Λ,λ). For each i, let $\mathfrak{T}_i$ be a topology on X_i such that X_i is compact and μ_i is inner regular with respect to the closed sets. Give X the product topology; this is compact. If $W\in\Lambda$ and $\epsilon>0$, let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence of measurable cylinders (in the sense of 254A) such that $X\setminus W\subseteq\bigcup_{n\in\mathbb{N}}C_n$ and $\sum_{n=0}^{\infty}\lambda C_n\le\lambda(X\setminus W)+\epsilon$. Express each C_n as $\prod_{i\in I}E_{ni}$ where $E_{ni}\in\Sigma_i$ for each i and $J_n=\{i:E_{ni}\ne X_i\}$ is finite. For $n\in\mathbb{N}$ set $\epsilon_n=2^{-n}\epsilon/(1+\#(J_n))$. Choose closed measurable sets $K_{ni}\subseteq X_i\setminus E_{ni}$ such that $\mu_i((X_i\setminus E_{ni})\setminus K_{ni})\le\epsilon_n$ whenever $n\in\mathbb{N}$ and $i\in J_n$. For each $n\in\mathbb{N}$, set

$$V_n=\bigcup_{i\in J_n}\{x:x\in X,\,x(i)\in K_{ni}\},$$

so that V_n is a closed measurable subset of X. Observe that

$$X\setminus V_n=\{x:x(i)\in X\setminus K_{ni}\text{ for }i\in J_n\}$$

includes C_n, and that

$$\lambda(X\setminus(V_n\cup C_n))\le\sum_{i\in J_n}\lambda\{x:x(i)\in X_i\setminus(K_{ni}\cup E_{ni})\}\le\sum_{i\in J_n}\epsilon_n\le 2^{-n}\epsilon.$$

Now set $V=\bigcap_{n\in\mathbb{N}}V_n$; then V is again a closed measurable set, and

$$X\setminus V\subseteq\bigcup_{n\in\mathbb{N}}C_n\cup(X\setminus(C_n\cup V_n))$$

has measure at most

$$\sum_{n=0}^{\infty} \lambda C_n + 2^{-n}\epsilon \le 1 - \lambda W + \epsilon + 2\epsilon,$$

so $\lambda V \ge \lambda W - 3\epsilon$. As W and ϵ are arbitrary, λ is inner regular with respect to the closed sets, and is a compact measure.

342H Proposition (a) A compact measure space is locally compact.

(b) A strictly localizable locally compact measure space is compact.

(c) Let (X,Σ,μ) be a measure space. Suppose that whenever $E \in \Sigma$ and $\mu E > 0$ there is an $F \in \Sigma$ such that $F \subseteq E$, $\mu F > 0$ and the subspace measure on F is compact. Then μ is locally compact.

proof (a) This is immediate from 342Ga and the definition of 'locally compact' measure space.

(b) Suppose that (X,Σ,μ) is a strictly localizable locally compact measure space. Let $\langle X_i\rangle_{i\in I}$ be a decomposition of X, and for each $i \in I$ let μ_i be the subspace measure on X_i. Then μ_i is compact. Now μ can be identified with the direct sum of the μ_i, so itself is compact, by 342Gd.

(c) Write $\mathcal{F}$ for the set of measurable sets $F \subseteq X$ such that the subspace measures μ_F are compact. Take $E \in \Sigma$ with $\mu E < \infty$. By 342Bb, there is a countable disjoint family $\langle F_i\rangle_{i\in I}$ in $\mathcal{F}$ such that $F_i \subseteq E$ for each i, and $F' = E \setminus \bigcup_{i\in I} F_i$ is negligible; now this means that $F' \in \mathcal{F}$ (342Ac), so we may take it that $E = \bigcup_{i\in I} F_i$. In this case μ_E is isomorphic to the direct sum of the measures μ_{F_i} and is compact. As E is arbitrary, μ is locally compact.

342I Proposition (a) Any measurable subspace of a locally compact measure space is locally compact.

(b) A measure space is locally compact iff its completion is locally compact iff its c.l.d. version is locally compact.

(c) The direct sum of a family of locally compact measure spaces is locally compact.

(d) The c.l.d. product of two locally compact measure spaces is locally compact.

proof (a) Trivial: if (X,Σ,μ) is locally compact, and $E \in \Sigma$, and $F \subseteq E$ is a measurable set of finite measure for the subspace measure on E, then $F \in \Sigma$ and $\mu F < \infty$, so the subspace measure on F is compact.

(b) Let (X,Σ,μ) be a measure space, and write $(X,\check{\Sigma},\check{\mu})$ for *either* its completion *or* its c.l.d. version.

(i) Suppose that μ is locally compact, and that $\check{\mu}F < \infty$. Then there is an $E \in \Sigma$ such that $E \subseteq F$ and $\mu E = \check{\mu}F$. Let μ_E be the subspace measure on E induced by the measure μ; then we are assuming that μ_E is compact. Let $\mathcal{K} \subseteq \mathcal{P}E$ be a compact class such that μ_E is inner regular with respect to $\mathcal{K}$. Then, as in the proof of 342Gb, the subspace measure $\check{\mu}_F$ on F induced by $\check{\mu}$ is also inner regular with respect to $\mathcal{K}$, so $\check{\mu}_F$ is compact; as F is arbitrary, $\check{\mu}$ is locally compact.

(ii) Now suppose that $\check{\mu}$ is locally compact, and that $\mu E < \infty$. Then the subspace measure $\check{\mu}_E$ is compact. But this is just the completion of the subspace measure μ_E, so μ_E is compact, by 342Gc; as E is arbitrary, μ is locally compact.

(c) Put (a) and 342Hc together.

(d) Let (X,Σ,μ) and (Y,T,ν) be locally compact measure spaces, with product $(X\times Y,\Lambda,\lambda)$. If $W \in \Lambda$ and $\lambda W > 0$, there are $E \in \Sigma$, $F \in \mathrm{T}$ such that $\mu E < \infty$, $\nu F < \infty$ and $\lambda(W \cap (E\times F)) > 0$. Now the subspace measure $\lambda_{E\times F}$ induced by λ on $E\times F$ is just the product of the subspace measures (251Q(ii-α)), so is compact, and the subspace measure $\lambda_{W\cap(E\times F)}$ is therefore again compact, by 342Ga. By 342Hc, this is enough to show that λ is locally compact.

342J Examples It is time I listed some examples of compact measure spaces.

(a) Lebesgue measure on $\mathbb{R}^r$ is compact. (Let $\mathcal{K}$ be the family of subsets of $\mathbb{R}^r$ which are compact for the usual topology. By 134Fb, Lebesgue measure is inner regular with respect to $\mathcal{K}$.)

(b) Similarly, any Radon measure on $\mathbb{R}^r$ (256A) is compact.

(c) If $(\mathfrak{A},\bar{\mu})$ is any semi-finite measure algebra, the standard measure ν on its Stone space Z is compact. (By 322Ra, ν is inner regular with respect to the family of open-and-closed subsets of Z, which are all compact for the standard topology of Z, so form a compact class.)

(d) The usual measure on $\{0,1\}^I$ is compact, for any set I. (It is obvious that the usual measure on $\{0,1\}$ is compact; now use 342Gf.)

Remark Actually all these measures are 'Radon' in the sense of Volume 4.

342K One of the most important properties of (locally) compact measure spaces has been studied under the following name.

Definition Let (X,Σ,μ) be a measure space. Then (X,Σ,μ), or μ, is **perfect** if whenever $f:X\to\mathbb{R}$ is measurable, $E\in\Sigma$ and $\mu E>0$, then there is a compact set $K\subseteq f[E]$ such that $\mu f^{-1}[K]>0$.

342L Theorem A semi-finite locally compact measure space is perfect.

proof Let (X,Σ,μ) be a semi-finite locally compact measure space, $f:X\to\mathbb{R}$ a measurable function, and $E\in\Sigma$ a set of non-zero measure. Because μ is semi-finite, there is an $F\in\Sigma$ such that $F\subseteq E$ and $0<\mu F<\infty$. Now the subspace measure μ_F is compact; let $\mathfrak{T}$ be a topology on F such that F is compact and μ_F is inner regular with respect to the family $\mathcal{K}$ of closed sets for $\mathfrak{T}$.

Let $\langle\epsilon_q\rangle_{q\in\mathbb{Q}}$ be a family of strictly positive real numbers such that $\sum_{q\in\mathbb{Q}}\epsilon_q<\frac{1}{2}\mu F$. (For instance, you could set $\epsilon_{q(n)}=2^{-n-3}\mu F$ where $\langle q(n)\rangle_{n\in\mathbb{N}}$ is an enumeration of $\mathbb{Q}$.) For each $q\in\mathbb{Q}$, set $E_q=\{x:x\in F,\,f(x)\le q\}$, $E'_q=\{x:x\in F,\,f(x)>q\}$, and choose K_q, $K'_q\in\mathcal{K}\cap\Sigma$ such that $K_q\subseteq E_q$, $K'_q\subseteq E'_q$, $\mu(E_q\setminus K_q)\le\epsilon_q$ and $\mu(E'_q\setminus K'_q)\le\epsilon_q$. Then $K=\bigcap_{q\in\mathbb{Q}}(K_q\cup K'_q)\in\mathcal{K}\cap\Sigma$, $K\subseteq F$ and

$$\mu(F\setminus K)\le\textstyle\sum_{q\in\mathbb{Q}}\mu(E_q\setminus K_q)+\mu(E'_q\setminus K'_q)<\mu F,$$

so $\mu K>0$.

The point is that $f\restriction K$ is continuous. **P** For any $q\in\mathbb{Q}$, $\{x:x\in K,\,f(x)\le q\}=K\cap K_q$ and $\{x:x\in K,\,f(x)>q\}=K\cap K'_q$. If $H\subseteq\mathbb{R}$ is open and $x\in K\cap f^{-1}[H]$, take q, $q'\in\mathbb{Q}$ such that $f(x)\in\,]q,q']\subseteq H$; then $G=K\setminus(K_q\cup K'_{q'})$ is a relatively open subset of K containing x and included in $f^{-1}[H]$. Thus $K\cap f^{-1}[H]$ is relatively open in K; as H is arbitrary, $f\restriction K$ is continuous. **Q**

Accordingly $f[K]$ is a continuous image of a compact set, therefore compact; it is a subset of $f[E]$, and $\mu f^{-1}[f[K]]\ge\mu K>0$. As f and E are arbitrary, μ is perfect.

342M I ought to give examples to distinguish between the concepts introduced here, partly on general principles, but also because it is not obvious that the concept of 'locally compact' measure space is worth spending time on at all. It is easy to distinguish between 'perfect' and '(locally) compact'; 'locally compact' and 'compact' are harder to separate.

Example Let X be an uncountable set and μ the countable-cocountable measure on X (211R). Then μ is perfect but not compact or locally compact.

proof (a) If $f:X\to\mathbb{R}$ is measurable and $E\subseteq X$ is measurable, with measure greater than 0, set $A=\{\alpha:\alpha\in\mathbb{R},\,\{x:x\in X,\,f(x)\le\alpha\}$ is negligible$\}$. Then $\alpha\in A$ whenever $\alpha\le\beta\in A$. Since $X=\bigcup_{n\in\mathbb{N}}\{x:f(x)\le n\}$, there is some n such that $n\notin A$, in which case A is bounded above by n. Also there is some $m\in\mathbb{N}$ such that $\{x:f(x)>-m\}$ is non-negligible, in which case it must be conegligible, and $-m\in A$, so A is non-empty. Accordingly $\gamma=\sup A$ is defined in $\mathbb{R}$. Now for any $k\in\mathbb{N}$, $\{x:f(x)\le\gamma-2^{-k}\}$ is negligible, so $\{x:f(x)<\gamma\}$ is negligible. Also, for any k, $\{x:f(x)\le\gamma+2^{-k}\}$ is non-negligible, so $\{x:f(x)>\gamma+2^{-k}\}$ must be negligible; accordingly, $\{x:f(x)>\gamma\}$ is negligible. But this means that $\{x:f(x)=\gamma\}$ is conegligible and has measure 1. Thus we have a compact set $K=\{\gamma\}$ such that $\mu f^{-1}[K]=1$, and γ must belong to $f[E]$. As f and E are arbitrary, μ is perfect.

(b) μ is not compact. **P?** Suppose, if possible, that $\mathcal{K}\subseteq\mathcal{P}X$ is a compact class such that μ is inner regular with respect to $\mathcal{K}$. Then for every $x\in X$ there is a measurable set $K_x\in\mathcal{K}$ such that $K_x\subseteq X\setminus\{x\}$ and $\mu K_x>0$, that is, K_x is conegligible. But this means that $\{K_x:x\in X\}$ must have the finite intersection property; as it also has empty intersection, $\mathcal{K}$ cannot be a compact class. **XQ**

(c) Because μ is totally finite, it cannot be locally compact.

Remark See also 342X(n-viii).

***342N Example** There is a complete locally determined localizable locally compact measure space which is not compact.

proof (a) I refer to the example of 216E. In that construction, we have a set I and a family $\langle x_\gamma\rangle_{\gamma\in C}$ in $X=\{0,1\}^I$ such that for every $D\subseteq C$ there is an $i\in I$ such that $D=\{\gamma:x_\gamma(i)=1\}$; moreover, $\#(C)>\mathfrak{c}$. The σ-algebra Σ is the family of sets $E\subseteq X$ such that for every γ there is a countable set $J\subseteq I$ such that $\{x:x\restriction J=x_\gamma\restriction J\}$ is a subset of either E or $X\setminus E$; and for $E\in\Sigma$, μE is $\#(\{\gamma:x_\gamma\in E\})$ if this is finite, ∞ otherwise. Note that any subset of X determined by a countable set of coordinates belongs to Σ.

For each $\gamma\in C$, let $i_\gamma\in I$ be such that $x_\gamma(i_\gamma)=1$, $x_\delta(i_\gamma)=0$ for $\delta\ne\gamma$. (In 216E I took I to be $\mathcal{P}C$, and i_γ would be $\{\gamma\}$.) Set

$$Y=\{x:x\in X,\,\{\gamma:\gamma\in C,\,x(i_\gamma)=1\}\text{ is finite}\}.$$

Give Y its subspace measure μ_Y with domain Σ_Y. Then μ_Y is complete, locally determined and localizable (214Ie). Note that $x_\gamma\in Y$ for every $\gamma\in C$.

(b) μ_Y is locally compact. **P** Suppose that $F\in\Sigma_Y$ and $\mu_YF<\infty$. If $\mu_YF=0$ then surely the subspace measure μ_F is compact. Otherwise, we can express F as $E\cap Y$ where $E\in\Sigma$ and $\mu E=\mu_YF$. Then $D=\{\gamma:x_\gamma\in E\}=\{\gamma:x_\gamma\in F\}$ is finite. For $\gamma\in D$ set

$$G'_\gamma=\{x:x\in X,\,x(i_\gamma)=1,\,x(i_\delta)=0\text{ for every }\delta\in D\setminus\{\gamma\}\}\in\Sigma,$$

$$\mathcal{K}_\gamma=\{K:x_\gamma\in K\subseteq F\cap G'_\gamma\}.$$

Then each $\mathcal{K}_\gamma$ is a compact class, and members of different $\mathcal{K}_\gamma$'s are disjoint, so $\mathcal{K}=\bigcup_{\gamma\in D}\mathcal{K}_\gamma$ is a compact class.

Now suppose that H belongs to the subpsace σ-algebra Σ_F and $\mu_FH>0$. Then there is a $\gamma\in D$ such that $x_\gamma\in H$, so that $H\cap G'_\gamma\in\mathcal{K}\cap\Sigma_F$ and $\mu_F(H\cap G'_\gamma)>0$. By 342E, this is enough to show that μ_F is compact. As F is arbitrary, μ_Y is locally compact. **Q**

(c) μ_Y is not compact. **P?** Suppose, if possible, that μ_Y is inner regular with respect to a compact class $\mathcal{K}\subseteq\mathcal{P}Y$. For each $\gamma\in C$ set $G_\gamma=\{x:x\in X,\,x(i_\gamma)=1\}$, so that $x_\gamma\in G_\gamma\in\Sigma$ and $\mu_Y(G_\gamma\cap Y)=1$. There must therefore be a $K_\gamma\in\mathcal{K}$ such that $K_\gamma\subseteq G_\gamma\cap Y$ and $\mu_YK_\gamma=1$ (since μ_Y takes no value in $]0,1[$). Express K_γ as $Y\cap E_\gamma$, where $E_\gamma\in\Sigma$, and let $J_\gamma\subseteq I$ be a countable set such that

$$E_\gamma\supseteq\{x:x\in X,\,x\restriction J_\gamma=x_\gamma\restriction J_\gamma\}.$$

At this point I call on the full strength of 2A1P. There is a set $B\subseteq C$, of cardinal greater than $\mathfrak{c}$, such that $x_\gamma\restriction J_\gamma\cap J_\delta=x_\delta\restriction J_\gamma\cap J_\delta$ for all γ, $\delta\in B$. But this means that, for any finite set $D\subseteq B$, we can define $x\in X$ by setting

$$\begin{aligned}x(i)&=x_\alpha(i)\text{ if }\alpha\in D,\,i\in J_\alpha,\\&=0\text{ if }i\in I\setminus\bigcup_{\alpha\in D}J_\alpha.\end{aligned}$$

It is easy to check that $\{\gamma:\gamma\in C,\,x(i_\gamma)=1\}=D$, so that $x\in Y$; but now

$$x\in Y\cap\bigcap_{\alpha\in D}E_\alpha=\bigcap_{\alpha\in D}K_\alpha.$$

What this shows is that $\{K_\alpha:\alpha\in B\}$ has the finite intersection property. It must therefore have non-empty intersection; say

$$y\in\bigcap_{\alpha\in B}K_\alpha\subseteq\bigcap_{\alpha\in B}G_\alpha.$$

But now we have a member y of Y such that $\{\gamma:y(i_\gamma)=1\}\supseteq B$ is infinite, contrary to the definition of Y. **XQ**

342X Basic exercises >(a) Show that a measure space (X,Σ,μ) is semi-finite iff μ is inner regular with respect to $\{E:\mu E<\infty\}$.

(b) Find a proof of 342B based on 215A.

(c) Let (X,Σ,μ) be a locally compact semi-finite measure space in which all singleton sets are negligible. Show that it is atomless.

(d) Let (X,Σ,μ) be a measure space, and ν an indefinite-integral measure over μ (234J[1]). Show that ν is compact, or locally compact, if μ is. (*Hint*: if $\mathcal{K}$ satisfies the conditions of 342E with respect to μ, then it satisfies them for ν.)

[1]Formerly 234B.

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be any non-decreasing function, and ν_f the corresponding Lebesgue-Stieltjes measure. Show that ν_f is compact. (*Hint*: 256Xg.)

(f) Let μ be Lebesgue measure on $[0,1]$, ν the countable-cocountable measure on $[0,1]$, and λ their c.l.d. product. Show that λ is a compact measure. (*Hint*: let $\mathcal{K}$ be the family of sets $K \times A$ where $A \subseteq [0,1]$ is cocountable and $K \subseteq A$ is compact.)

(g)(i) Give an example of a compact probability space (X, Σ, μ), a set Y and a function $f : X \to Y$ such that the image measure μf^{-1} is not compact. (ii) Give an example of a compact probability space (X, Σ, μ) and a σ-subalgebra T of Σ such that $(X, \mathrm{T}, \mu{\restriction}\mathrm{T})$ is not compact. (*Hint*: 342Xf.)

(h) Let (X, Σ, μ) be a perfect measure space, and $f : X \to \mathbb{R}$ a measurable function. Show that the image measure μf^{-1} is inner regular with respect to the compact subsets of $\mathbb{R}$, so is a compact measure.

(i) Let (X, Σ, μ) be a σ-finite measure space. Show that it is perfect iff for every measurable $f : X \to \mathbb{R}$ there is a Borel set $H \subseteq f[X]$ such that $f^{-1}[H]$ is conegligible in X. (*Hint*: 342Xh for 'only if', 256C for 'if'.)

(j) Let (X, Σ, μ) be a complete totally finite perfect measure space and $f : X \to \mathbb{R}$ a measurable function. Show that the image measure μf^{-1} is a Radon measure, and is the only Radon measure on $\mathbb{R}$ for which f is inverse-measure-preserving. (*Hint*: 256G.)

(k) Suppose that (X, Σ, μ) is a perfect measure space. (i) Show that if (Y, T, ν) is a measure space, and $f : X \to Y$ is a function such that $f^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$ and $f^{-1}[F]$ is μ-negligible for every ν-negligible set F, then (Y, T, ν) is perfect. (ii) Show that if T is a σ-subalgebra of Σ then $(X, \mathrm{T}, \mu{\restriction}\mathrm{T})$ is perfect.

(l) Let (X, Σ, μ) be a perfect measure space such that Σ is the σ-algebra generated by a sequence of sets. Show that μ is compact. (*Hint*: if Σ is generated by $\{E_n : n \in \mathbb{N}\}$, set $f = \sum_{n=0}^{\infty} 3^{-n}\chi E_n$ and consider $\{f^{-1}[K] : K \subseteq f[X]$ is compact$\}$.)

(m) Let (X, Σ, μ) be a semi-finite measure space. Show that μ is perfect iff $\mu{\restriction}\mathrm{T}$ is compact for every countably generated σ-subalgebra T of Σ.

(n) Show that (i) a measurable subspace of a perfect measure space is perfect (ii) a semi-finite measure space is perfect iff all its totally finite subspaces are perfect (iii) the direct sum of any family of perfect measure spaces is perfect (iv) the c.l.d. product of two perfect measure spaces is perfect (*hint*: put 342Xm and 342Ge together) (v) the product of any family of perfect probability spaces is perfect (vi) a measure space is perfect iff its completion is perfect (vii) the c.l.d. version of a perfect measure space is perfect (viii) any purely atomic measure space is perfect (ix) an indefinite-integral measure over a perfect measure is perfect (x) a sum (234G[2]) of perfect measures is perfect.

(o) Let μ be Lebesgue measure on $\mathbb{R}$, A a subset of $\mathbb{R}$, and μ_A the subspace measure on A. Show that μ_A is compact iff it is perfect iff A is Lebesgue measurable. (*Hint*: if μ_A is perfect, consider the image measure $\mu_A h^{-1}$ on $\mathbb{R}$, where $h(x) = x$ for $x \in A$.)

342Y Further exercises (a) Let U be a Banach space such that there is a linear operator $T : U^{**} \to U$, of norm at most 1, such that $T\hat{u} = u$ for every $u \in U$, writing $\hat{u}$ for the member of U^{**} corresponding to u. Show that the family of closed balls in U is a compact class.

(b) Give an example of a compact class $\mathcal{K}$ of subsets of $\mathbb{N}$ such that there is no compact Hausdorff topology on $\mathbb{N}$ for which every member of $\mathcal{K}$ is closed.

(c) Show that the space (X, Σ, μ) of 216E and 342N is a compact measure space. (*Hint*: use the usual topology on $X = \{0,1\}^I$.)

(d) Give an example of a compact complete locally determined measure space which is not localizable. (*Hint*: in 216D, add a point to each horizontal and vertical section of X, so that all the sections become compact measure spaces.)

[2]Later editions only.

342 Notes and comments The terminology I find myself using in this section – 'compact', 'locally compact', 'perfect' – is not entirely satisfactory, in that it risks collision with the same words applied to topological spaces. For the moment, this is not a serious problem; but when in Volume 4 we come to the systematic analysis of spaces which have both topologies and measures present, it will be necessary to watch our language carefully. Of course there are cases in which a 'compact class' of the sort discussed here can be taken to be the family of compact sets for some familiar topology, as in 342Ja-342Jd, but in others this is not so (see 342Xf); and even when we have a familiar compact class, the topology constructed from it by the method of 342Da need not be one we might expect. (Consider, for instance, the topology on $\mathbb{R}$ for which the closed sets are just the sets which are compact for the usual topology, together with the set $\mathbb{R}$ itself.)

I suppose that 'compact' and 'perfect' measure spaces look reasonably natural objects to study; they offer to illuminate one of the basic properties of Radon measures, the fact that (at least for totally finite Radon measures on Euclidean space) the image measure of a Radon measure under a measurable function is again Radon (256G, 342Xj). Indeed this was the original impetus for the study of perfect measures (Gnedenko & Kolmogorov 54, Sazonov 66). It is not obvious that there is any need to examine 'locally compact' measure spaces, but actually they are the chief purpose of this section, since the main theorem of the next section is an alternative characterization of semi-finite locally compact measure spaces (343B). Of course you may feel that the fact that 'locally compact' and 'compact' coincide for strictly localizable spaces (342Hb) excuses you from troubling about the distinction at first reading.

As with any new classification of measure spaces, it is worth finding out how the classes of 'compact' and 'perfect' measure spaces behave with respect to the standard constructions. I run through the basic facts in 342G-342I, 342Xd, 342Xk and 342Xn. We can also look for relationships between the new properties and those already studied. Here, in fact, there is not much to be said; 342N and 342Yd show that 'compactness' is largely independent of the classification in §211. However there are interactions with the concept of 'atom' (342Xc, 342Xn(viii)).

I give examples to show that perfect measure spaces need not be locally compact, and that locally compact measure spaces need not be compact (342M, 342N). The standard examples of measure spaces which are not perfect are non-measurable subspaces (342Xo); I will return to these in the next section (343L-343M).

Something which is not important to us at the moment, but is perhaps worth taking note of, is the following observation. To determine whether a measure space (X,Σ,μ) is compact, we need only the structure $(X,\Sigma,\mathcal{N})$, where $\mathcal{N}$ is the σ-ideal of negligible sets, since that is all that is referred to in the criterion of 342E. The same is true of local compactness, by 342Hc, and of perfectness, by the definition in 342K. Compare 342Xd, 342Xk and 342Xn(ix).

Much of the material of this section will be repeated in Volume 4 as part of a more systematic analysis of inner regularity.

343 Realization of homomorphisms

We are now in a position to make progress in one of the basic questions of abstract measure theory. In §324 I have already described the way in which a function between two measure spaces can give rise to a homomorphism between their measure algebras. In this section I discuss some conditions under which we can be sure that a homomorphism can be represented by a function.

The principal theorem of the section is 343B. If a measure space (X,Σ,μ) is locally compact, then many homomorphisms from the measure algebra of μ to other measure algebras will be representable by functions into X; moreover, this characterizes locally compact spaces. In general, a homomorphism between measure algebras can be represented by widely different functions (343I, 343J). But in some of the most important cases (e.g., Lebesgue measure) representing functions are 'almost' uniquely defined; I introduce the concept of 'countably separated' measure space to describe these (343D-343H).

343A Preliminary remarks It will be helpful to establish some vocabulary and a couple of elementary facts.

(a) If (X,Σ,μ) and (Y,T,ν) are measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, I will say that a function $f:X\to Y$ **represents** a homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ if $f^{-1}[F]\in\Sigma$ and $(f^{-1}[F])^\bullet=\pi(F^\bullet)$ for every $F\in\mathrm{T}$.

(Perhaps I should emphasize here that some homomorphisms are representable in this sense, and some are not; see 343M below for examples of non-representable homomorphisms.)

(b) If (X,Σ,μ) and (Y,T,ν) are measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, $f:X\to Y$ is a function, and $\pi:\mathfrak{B}\to\mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism, then

$$\{F : F \in \mathrm{T},\ f^{-1}[F] \in \Sigma \text{ and } f^{-1}[F]^\bullet = \pi F^\bullet\}$$

is a σ-subalgebra of T. (The verification is elementary.)

(c) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, and $\pi : \mathfrak{B} \to \mathfrak{A}$ a Boolean homomorphism which is represented by a function $f : X \to Y$. Let $(X, \hat{\Sigma}, \hat{\mu})$, $(Y, \hat{\mathrm{T}}, \hat{\nu})$ be the completions of (X, Σ, μ), (Y, T, ν); then $\mathfrak{A}$ and $\mathfrak{B}$ can be identified with the measure algebras of $\hat{\mu}$ and $\hat{\nu}$ (322Da). Now f still represents π when regarded as a function from $(X, \hat{\Sigma}, \hat{\mu})$ to $(Y, \hat{\mathrm{T}}, \hat{\nu})$. **P** If G is ν-negligible, there is a negligible $F \in \mathrm{T}$ such that $G \subseteq F$; since

$$f^{-1}[F]^\bullet = \pi F^\bullet = 0,$$

$f^{-1}[F]$ is μ-negligible, so $f^{-1}[G]$ is negligible, therefore belongs to $\hat{\Sigma}$. If G is any element of $\hat{\mathrm{T}}$, there is an $F \in \mathrm{T}$ such that $G \triangle F$ is negligible, so that

$$f^{-1}[G] = f^{-1}[F] \triangle f^{-1}[G \triangle F] \in \hat{\Sigma},$$

and

$$f^{-1}[G]^\bullet = f^{-1}[F]^\bullet = \pi F^\bullet = \pi G^\bullet. \ \mathbf{Q}$$

343B Theorem Let (X, Σ, μ) be a non-empty semi-finite measure space, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Let (Z, Λ, λ) be the Stone space of $(\mathfrak{A}, \bar{\mu})$; for $E \in \Sigma$ write E^* for the open-and-closed subset of Z corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Then the following are equiveridical.

(i) (X, Σ, μ) is locally compact in the sense of 342Ad.

(ii) There is a family $\mathcal{K} \subseteq \Sigma$ such that (α) whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$ (β) whenever $\mathcal{K}' \subseteq \mathcal{K}$ is such that $\mu(\bigcap \mathcal{K}_0) > 0$ for every non-empty finite set $\mathcal{K}_0 \subseteq \mathcal{K}'$, then $\bigcap \mathcal{K}' \neq \emptyset$.

(iii) There is a family $\mathcal{K} \subseteq \Sigma$ such that $(\alpha)'$ μ is inner regular with respect to $\mathcal{K}$ (β) whenever $\mathcal{K}' \subseteq \mathcal{K}$ is such that $\mu(\bigcap \mathcal{K}_0) > 0$ for every non-empty finite set $\mathcal{K}_0 \subseteq \mathcal{K}'$, then $\bigcap \mathcal{K}' \neq \emptyset$.

(iv) There is a function $f : Z \to X$ such that $f^{-1}[E] \triangle E^*$ is negligible for every $E \in \Sigma$.

(v) Whenever (Y, T, ν) is a complete strictly localizable measure space, with measure algebra $\mathfrak{B}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, then there is a $g : Y \to X$ representing π.

(vi) Whenever (Y, T, ν) is a complete strictly localizable measure space, with measure algebra $\mathfrak{B}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous measure-preserving Boolean homomorphism, then there is a $g : Y \to X$ representing π.

proof (a)(i)$\Rightarrow$(ii) Because μ is semi-finite, there is a partition of unity $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}$ such that $\bar{\mu} a_i < \infty$ for each i. For each $i \in I$, let $E_i \in \Sigma$ be such that $E_i^\bullet = a_i$. Then the subspace measure μ_{E_i} on E_i is compact; let $\mathcal{K}_i \subseteq \mathcal{P}E_i$ be a compact class such that μ_{E_i} is inner regular with respect to $\mathcal{K}_i$. Set $\mathcal{K} = \bigcup_{i \in I} \mathcal{K}_i$. If $\mathcal{K}' \subseteq \mathcal{K}$ and $\mu(\bigcap \mathcal{K}_0) > 0$ for every non-empty finite $\mathcal{K}_0 \subseteq \mathcal{K}$, then $\mathcal{K}' \subseteq \mathcal{K}_i$ for some i, and surely has the finite intersection property, so $\bigcap \mathcal{K}' \neq \emptyset$; thus $\mathcal{K}'$ satisfies (β) of condition (ii). And if $E \in \Sigma$, $\mu E > 0$ then there must be some $i \in I$ such that $E_i^\bullet \cap a_i \neq 0$, that is, $\mu(E \cap E_i) > 0$, in which case there is a $K \in \mathcal{K}_i \subseteq \mathcal{K}$ such that $K \subseteq E \cap E_i$ and $\mu K > 0$; so that $\mathcal{K}$ satisfies condition (α).

(b)(ii)$\Rightarrow$(iii) Suppose that $\mathcal{K} \subseteq \Sigma$ witnesses that (ii) is true. If $\mu X = 0$ then $\mathcal{K}$ already witnesses that (iii) is true, so we need consider only the case $\mu X > 0$. Set $\mathcal{L} = \{K_0 \cup \ldots \cup K_n : K_0, \ldots, K_n \in \mathcal{K}\}$. Then $\mathcal{L}$ witnesses that (iii) is true. **P** By 342Ba, μ is inner regular with respect to $\mathcal{L}$. Let $\mathcal{L}' \subseteq \mathcal{L}$ be such that $\mu(\bigcap \mathcal{L}_0) > 0$ for every non-empty finite $\mathcal{L}_0 \subseteq \mathcal{L}'$. Then

$$\mathcal{F}_0 = \{A : A \subseteq X, \text{ there is a finite } \mathcal{L}_0 \subseteq \mathcal{L}' \text{ such that } X \cap \bigcap \mathcal{L}_0 \setminus A \text{ is negligible}\}$$

is a filter on X, so there is an ultrafilter $\mathcal{F}$ on X including $\mathcal{F}_0$. Note that every conegligible set belongs to $\mathcal{F}_0$, so no negligible set can belong to $\mathcal{F}$. Set $\mathcal{K}' = \mathcal{K} \cap \mathcal{F}$; then $\bigcap \mathcal{K}_0$ belongs to $\mathcal{F}$, so is not negligible, for every non-empty finite $\mathcal{K}_0 \subseteq \mathcal{K}'$. Accordingly there is some $x \in \bigcap \mathcal{K}'$. But any member of $\mathcal{L}'$ is of the form $L = K_0 \cup \ldots \cup K_n$ where each $K_i \in \mathcal{K}$; because $\mathcal{F}$ is an ultrafilter and $L \in \mathcal{F}$, there must be some $i \leq n$ such that $K_i \in \mathcal{F}$, in which case $x \in K_i \subseteq L$. Thus $x \in \bigcap \mathcal{L}'$. As $\mathcal{L}'$ is arbitrary, $\mathcal{L}$ satisfies the condition (β). **Q**

(c)(iii)$\Rightarrow$(iv) Let $\mathcal{K} \subseteq \Sigma$ witness that (iii) is true. For any $z \in Z$, set $\mathcal{K}_z = \{K : K \in \mathcal{K}, z \in K^*\}$. If $K_0, \ldots, K_n \in \mathcal{K}_z$, then $z \in \bigcap_{i \leq n} K_i^* = (\bigcap_{i \leq n} K_i)^*$, so $(\bigcap_{i \leq n} K_i)^* \neq \emptyset$ and $\mu(\bigcap_{i \leq n} K_i) > 0$. By (β) of condition (iii), $\bigcap \mathcal{K}_z \neq \emptyset$; and even if $\mathcal{K}_z = \emptyset$, $X \cap \bigcap \mathcal{K}_z \neq \emptyset$ because X is non-empty. So we may choose $f(z) \in X \cap \bigcap \mathcal{K}_z$. This defines a function $f : Z \to X$. Observe that, for $K \in \mathcal{K}$ and $z \in Z$,

$$z \in K^* \Longrightarrow K \in \mathcal{K}_z \Longrightarrow f(z) \in K \Longrightarrow z \in f^{-1}[K],$$

so that $K^* \subseteq f^{-1}[K]$.

Now take any $E \in \Sigma$. Consider

$$U_1 = \bigcup\{K^* : K \in \mathcal{K},\, K \subseteq E\} \subseteq \bigcup\{E^* \cap f^{-1}[K] : K \in \mathcal{K},\, K \subseteq E\} \subseteq E^* \cap f^{-1}[E],$$

$$U_2 = \bigcup\{K^* : K \in \mathcal{K},\, K \subseteq X \setminus E\} \subseteq (X \setminus E)^* \cap f^{-1}[X \setminus E] = Z \setminus (f^{-1}[E] \cup E^*),$$

so that $f^{-1}[E] \triangle E^* \subseteq Z \setminus (U_1 \cup U_2)$. Now U_1 and U_2 are open subsets of Z, so $M = Z \setminus (U_1 \cup U_2)$ is closed, and in fact M is nowhere dense. **P?** Otherwise, there is a non-zero $a \in \mathfrak{A}$ such that the corresponding open-and-closed set $\widehat{a}$ is included in M, and an $F \in \Sigma$ of non-zero measure such that $a = F^\bullet$. At least one of $F \cap E$, $F \setminus E$ is non-negligible and therefore includes a non-negligible member K of $\mathcal{K}$. But in this case K^* is a non-empty open subset of M which is included in either U_1 or U_2, which is impossible. **XQ**

By the definition of λ (321J-321K), M is λ-negligible, so $f^{-1}[E] \triangle E^* \subseteq M$ is negligible, as required.

(d)(iv)⇒(v) Now assume that $f : Z \to X$ witnesses (iv), and let (Y, T, ν) be a complete strictly localizable measure space, with measure algebra $\mathfrak{B}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism. If $\nu Y = 0$ then any function from Y to X will represent π, so we may suppose that $\nu Y > 0$. Write W for the Stone space of $\mathfrak{B}$. Then we have a continuous function $\phi : W \to Z$ such that $\phi^{-1}[\widehat{a}] = \widehat{\pi a}$ for every $a \in \mathfrak{A}$ (312Q), and $\phi^{-1}[M]$ is nowhere dense in W for every nowhere dense $M \subseteq Z$ (313R). It follows that $\phi^{-1}[M]$ is meager for every meager $M \subseteq Z$, that is, $\phi^{-1}[M]$ is negligible in W for every negligible $M \subseteq Z$. By 341Q, there is an inverse-measure-preserving function $h : Y \to W$ such that $h^{-1}[\widehat{b}]^\bullet = b$ for every $b \in \mathfrak{B}$. Consider $g = f\phi h : Y \to X$.

If $E \in \Sigma$, set $a = E^\bullet \in \mathfrak{A}$, so that $E^* = \widehat{a} \subseteq Z$, and $M = f^{-1}[E] \triangle E^*$ is λ-negligible; consequently $\phi^{-1}[M]$ is negligible in W. Because h is inverse-measure-preserving,

$$g^{-1}[E] \triangle h^{-1}[\phi^{-1}[E^*]] = h^{-1}[\phi^{-1}[f^{-1}[E]]] \triangle h^{-1}[\phi^{-1}[E^*]] = h^{-1}[\phi^{-1}[M]]$$

is negligible. But $\phi^{-1}[E^*] = \widehat{\pi a}$, so

$$g^{-1}[E]^\bullet = h^{-1}[\phi^{-1}[E^*]]^\bullet = \pi a.$$

As E is arbitrary, g induces the homomorphism π.

(e)(v)⇒(vi) is trivial.

(f)(vi)⇒(iv) Assume (vi). Let ν be the c.l.d. version of λ, T its domain, and $\mathfrak{B}$ its measure algebra; then ν is strictly localizable (322Rb). The embedding $\Lambda \subsetneqq \mathrm{T}$ corresponds to an order-continuous measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ (322Db). By (vi), there is a function $f : Z \to X$ such that $f^{-1}[E] \in \mathrm{T}$ and $f^{-1}[E]^\bullet = (E^*)^\bullet$ in $\mathfrak{B}$ for every $E \in \Sigma$. But as ν and λ have the same negligible sets (322Rb), $f^{-1}[E] \triangle E^*$ is λ-negligible for every $E \in \Sigma$, as required by (iv).

(g)(iv)⇒(i)(α) To begin with (down to the end of (γ) below) I suppose that μ is totally finite. In this case we have a function $g : X \to Z$ such that $E \triangle g^{-1}[E^*]$ is negligible for every $E \in \Sigma$ (341Q again). We are supposing also that there is a function $f : Z \to X$ such that $f^{-1}[E] \triangle E^*$ is negligible for every $E \in \Sigma$. Write $\mathcal{K}$ for the family of sets $K \subseteq X$ such that $K \in \Sigma$ and there is a compact set $L \subseteq Z$ such that $f[L] \subseteq K \subseteq g^{-1}[L]$.

(β) μ is inner regular with respect to $\mathcal{K}$. **P** Take $F \in \Sigma$ and $\gamma < \mu F$. Choose $\langle V_n \rangle_{n\in\mathbb{N}}$, $\langle F_n \rangle_{n\in\mathbb{N}}$ as follows. $F_0 = F$. Given that $\mu F_n > \gamma$, then

$$\lambda(f^{-1}[F_n] \cap F_n^*) = \lambda F_n^* = \mu F_n > \gamma,$$

so there is an open-and-closed set $V_n \subseteq f^{-1}[F_n] \cap F_n^*$ with $\lambda V_n > \gamma$. Express V_n as F_{n+1}^* where $F_{n+1} \in \Sigma$; since $F_n \triangle g^{-1}[F_n^*]$ is negligible, and $V_n \subseteq F_n^*$, we may take it that $F_{n+1} \subseteq g^{-1}[F_n^*]$. Continue.

At the end of the induction, set $K = \bigcap_{n\in\mathbb{N}} F_n \in \Sigma$ and $L = \bigcap_{n\in\mathbb{N}} F_n^*$. Because $F_{n+1} \setminus F_n \subseteq g^{-1}[F_n^*] \setminus F_n$ is negligible for each n, $\mu K = \lim_{n\to\infty} \mu F_n \ge \gamma$, while $K \subseteq F$ and L is surely compact. We have

$$L \subseteq \bigcap_{n\in\mathbb{N}} V_n \subseteq \bigcap_{n\in\mathbb{N}} f^{-1}[F_n] = f^{-1}[K],$$

so $f[L] \subseteq K$. Also

$$K \subseteq \bigcap_{n\in\mathbb{N}} F_{n+1} \subseteq \bigcap_{n\in\mathbb{N}} g^{-1}[F_n^*] = g^{-1}[L].$$

So $K \in \mathcal{K}$. As F and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$. **Q**

(γ) Next, $\mathcal{K}$ is a compact class. **P** Suppose that $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property. If $\mathcal{K}' = \emptyset$, of course $\bigcap \mathcal{K}' \neq \emptyset$; suppose that $\mathcal{K}'$ is non-empty. Let $\mathcal{L}$ be the family of closed sets $L \subseteq Z$ such that $g^{-1}[L]$ includes some member of $\mathcal{K}'$. Then $\mathcal{L}$ has the finite intersection property, and Z is compact, so there is some $z \in \bigcap \mathcal{L}$; also

$Z \in \mathcal{L}$, so $z \in Z$. For any $K \in \mathcal{K}'$, there is some closed set $L \subseteq Z$ such that $f[L] \subseteq K \subseteq g^{-1}[L]$, so that $L \in \mathcal{L}$ and $z \in L$ and $f(z) \in K$. Thus $f(z) \in \bigcap \mathcal{K}'$. As $\mathcal{K}'$ is arbitrary, $\mathcal{K}$ is a compact class. **Q**

So $\mathcal{K}$ witnesses that μ is a compact measure.

(δ) Now consider the general case. Take any $E \in \Sigma$ of finite measure. If $E = \emptyset$ then surely the subspace measure μ_E is compact. Otherwise, we can identify the measure algebra of μ_E with the principal ideal $\mathfrak{A}_{E^\bullet}$ of $\mathfrak{A}$ generated by $E^\bullet$ (322Ja), and $E^* \subseteq Z$ with the Stone space of $\mathfrak{A}_{E^\bullet}$ (312T). Take any $x_0 \in E$ and define $\tilde{f} : E^* \to E$ by setting $\tilde{f}(z) = f(z)$ if $z \in E^* \cap f^{-1}[E]$, x_0 if $z \in E^* \setminus f^{-1}[E]$. Then f and $\tilde{f}$ agree almost everywhere in E^*, so $\tilde{f}^{-1}[F] \triangle F^*$ is negligible for every $F \in \Sigma_E$, that is, $\tilde{f}$ represents the canonical isomorphism between the measure algebras of μ_E and the subspace measure λ_{E^*} on E^*. But this means that condition (iv) is true of μ_E, so μ_E is compact, by (α)-(γ) above. As E is arbitrary, μ is locally compact.

This completes the proof.

343C Examples (a) Let I be any set. We know that the usual measure ν_I on $\{0,1\}^I$ is compact (342Jd). It follows that if (X, Σ, μ) is any complete probability space such that the measure algebra $\mathfrak{B}_I$ of ν_I can be embedded as a subalgebra of the measure algebra $\mathfrak{A}$ of μ, there is an inverse-measure-preserving function from X to $\{0,1\}^I$. For infinite I, this is so iff every non-zero principal ideal of $\mathfrak{A}$ has Maharam type at least κ, by 332P. Of course this does not depend in any way on the results of the present chapter. If $\mathfrak{B}_\kappa$ can be embedded in $\mathfrak{A}$, there must be a stochastically independent family $\langle E_\xi \rangle_{\xi<\kappa}$ of sets of measure $\frac{1}{2}$; now we get a map $h : X \to \{0,1\}^\kappa$ by saying that $h(x)(\xi) = 1$ iff $x \in E_\xi$, which by 254G is inverse-measure-preserving.

(b) In particular, if μ is atomless, there is an inverse-measure-preserving function from X to $\{0,1\}^{\mathbb{N}}$; since this is isomorphic, as measure space, to $[0,1]$ with Lebesgue measure (254K), there is an inverse-measure-preserving function from X to $[0,1]$.

(c) More generally, if (X, Σ, μ) is any complete atomless totally finite measure space, there is an inverse-measure-preserving function from X to the interval $[0, \mu X]$ endowed with Lebesgue measure. (If $\mu X > 0$, apply (b) to the normalized measure $(\mu X)^{-1}\mu$; or argue directly from 343B, using the fact that Lebesgue measure on $[0, \mu X]$ is compact; or use the idea suggested in 343Xd.)

(d) In the other direction, if (X, Σ, μ) is a compact probability space with Maharam type at most $\kappa \geq \omega$, then there is an inverse-measure-preserving function from $\{0,1\}^\kappa$ to X. **P** By 332N, there is a measure-preserving homomorphism from the measure algebra of μ to the measure algebra of ν_κ; by 343B, this is represented by an inverse-measure-preserving function from $\{0,1\}^\kappa$ to X. **Q**

(e) Throughout the work above – in §254 as well as in 343B – I have taken the measures involved to be complete. It does occasionally happen, in this context, that this restriction is inconvenient. Typical results not depending on completeness in the domain space X are in 343Xc-343Xd. Of course these depend not only on the very special nature of the codomain spaces $\{0,1\}^I$ or $[0,1]$, but also on the measures on these spaces being taken to be incomplete.

343D Uniqueness of realizations The results of 342E-342J, together with 343B, give a respectable number of contexts in which homomorphisms between measure algebras can be represented by functions between measure spaces. They say nothing about whether such functions are unique, or whether we can distinguish, among the possible representations of a homomorphism, any canonical one. In fact the proof of 343B, using the Lifting Theorem as it does, strongly suggests that this is like looking for a canonical lifting, and I am sure that (outside a handful of very special cases) any such search is vain. Nevertheless, we do have a weak kind of uniqueness theorem, valid in a useful number of spaces, as follows.

Definition A measure space (X, Σ, μ) is **countably separated** if there is a countable set $\mathcal{A} \subseteq \Sigma$ separating the points of X in the sense that for any distinct x, $y \in X$ there is an $E \in \mathcal{A}$ containing one but not the other. (Of course this is a property of the structure (X, Σ) rather than of (X, Σ, μ).)

343E Lemma A measure space (X, Σ, μ) is countably separated iff there is an injective measurable function from X to $\mathbb{R}$.

proof If (X, Σ, μ) is countably separated, let $\mathcal{A} \subseteq \Sigma$ be a countable set separating the points of X. Let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{A} \cup \{\emptyset\}$. Set

$$f=\sum_{n=0}^{\infty}3^{-n}\chi E_n:X\to\mathbb{R}.$$

Then f is measurable (because every E_n is measurable) and injective (because if $x\neq y$ in X and $n=\min\{i:\#(E_i\cap\{x,y\})=1\}$ and $x\in E_n$, then

$$f(x)\geq 3^{-n}+\sum_{i<n}3^{-i}\chi E_i(x)>\sum_{i>n}3^{-i}+\sum_{i<n}3^{-i}\chi E_i(y)\geq f(y).)$$

On the other hand, if $f:X\to\mathbb{R}$ is measurable and injective, then $\mathcal{A}=\{f^{-1}[\,]-\infty,q]\,]:q\in\mathbb{Q}\}$ is a countable subset of Σ separating the points of X, so (X,Σ,μ) is countably separated.

Remark The construction of the function f from the sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in the proof above is a standard trick; such f are sometimes called **Marczewski functionals**.

343F Proposition Let (X,Σ,μ) be a countably separated measure space and (Y,T,ν) any measure space. Let f, $g:Y\to X$ be two functions such that $f^{-1}[E]$ and $g^{-1}[E]$ both belong to T, and $f^{-1}[E]\triangle g^{-1}[E]$ is ν-negligible, for every $E\in\Sigma$. Then $f=g$ ν-almost everywhere, and $\{y:y\in Y,\ f(y)\neq g(y)\}$ is measurable as well as negligible.

proof Let $\mathcal{A}\subseteq\Sigma$ be a countable set separating the points of X. Then

$$\{y:f(y)\neq g(y)\}=\bigcup_{E\in\mathcal{A}}f^{-1}[E]\triangle g^{-1}[E]$$

is measurable and negligible.

343G Corollary If, in 343B, (X,Σ,μ) is countably separated, then the functions $g:Y\to X$ of 343B(v)-(vi) are almost uniquely defined in the sense that if f, g both represent the same homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ then $f=_{\text{a.e.}}g$.

343H Examples Leading examples of countably separated measure spaces are
(i) $\mathbb{R}$ (take $\mathcal{A}=\{]-\infty,q]:q\in\mathbb{Q}\}$);
(ii) $\{0,1\}^{\mathbb{N}}$ (take $\mathcal{A}=\{E_n:n\in\mathbb{N}\}$, where $E_n=\{x:x(n)=1\}$);
(iii) subspaces (measurable or not) of countably separated spaces;
(iv) finite products of countably separated spaces;
(v) countable products of countably separated probability spaces;
(vi) completions and c.l.d. versions of countably separated spaces.
As soon as we move away from these elementary ideas, however, some interesting difficulties arise.

343I Example Let $\nu_{\mathfrak{c}}$ be the usual measure on $X=\{0,1\}^{\mathfrak{c}}$, where $\mathfrak{c}=\#(\mathbb{R})$, and $\mathrm{T}_{\mathfrak{c}}$ its domain. Then there is a function $f:X\to X$ such that $f(x)\neq x$ for every $x\in X$, but $E\triangle f^{-1}[E]$ is negligible for every $E\in\mathrm{T}_{\mathfrak{c}}$. **P** The set $\mathfrak{c}\setminus\omega$ is still of cardinal $\mathfrak{c}$, so there is an injection $h:\{0,1\}^{\omega}\to\mathfrak{c}\setminus\omega$. (As usual, I am identifying the cardinal number $\mathfrak{c}$ with the corresponding initial ordinal. But if you prefer to argue without the full axiom of choice, you can express all the same ideas with $\mathbb{R}$ in the place of $\mathfrak{c}$ and $\mathbb{N}$ in the place of ω.) For $x\in X$, set

$$\begin{aligned}f(x)(\xi)&=1-x(\xi)\text{ if }\xi=h(x\restriction\omega),\\&=x(\xi)\text{ otherwise}.\end{aligned}$$

Evidently $f(x)\neq x$ for every x. If $E\subseteq X$ is measurable, then we can find a countable set $J\subseteq\mathfrak{c}$ and sets E', E'', both determined by coordinates in J, such that $E'\subseteq E\subseteq E''$ and $E''\setminus E'$ is negligible (254Oc). Now for any particular $\xi\in\mathfrak{c}\setminus\omega$, $\{x:h(x\restriction\omega)=\xi\}$ is negligible, being either empty or of the form $\{x:x(n)=z(n)$ for every $n<\omega\}$ for some $z\in\{0,1\}^{\omega}$. So $H=\{x:h(x\restriction\omega)\in J\}$ is negligible. Now we see that for $x\in X\setminus H$, $f(x)\restriction J=x\restriction J$, so for $x\in X\setminus(H\cup(E''\setminus E'))$,

$$x\in E\Longrightarrow x\in E'\Longrightarrow f(x)\in E'\Longrightarrow f(x)\in E,$$

$$x\notin E\Longrightarrow x\notin E''\Longrightarrow f(x)\notin E''\Longrightarrow f(x)\notin E.$$

Thus $E\triangle f^{-1}[E]\subseteq H\cup(E''\setminus E')$ is negligible. **Q**

343J The split interval I introduce a construction which here will seem essentially elementary, but in other contexts is of great interest, as will appear in Volume 4.

(a) Take $I^{\|}$ to consist of two copies of each point of the unit interval, so that $I^{\|}=\{t^+:t\in[0,1]\}\cup\{t^-:t\in[0,1]\}$. For $A\subseteq I^{\|}$ write $A_l=\{t:t^-\in A\}$, $A_r=\{t:t^+\in A\}$. Let Σ be the set

$$\{E : E \subseteq I^{\parallel},\ E_l \text{ and } E_r \text{ are Lebesgue measurable and } E_l \triangle E_r \text{ is Lebesgue negligible}\}.$$

For $E \in \Sigma$, set

$$\mu E = \mu_L E_l = \mu_L E_r$$

where μ_L is Lebesgue measure on $[0,1]$. It is easy to check that $(I^{\parallel}, \Sigma, \mu)$ is a complete probability space (cf. 234F, 234Ye). Also it is compact. **P** Take $\mathcal{K}$ to be the family of sets $K \subseteq I^{\parallel}$ such that $K_l = K_r$ is a compact subset of $[0,1]$, and check that $\mathcal{K}$ is a compact class and that μ is inner regular with respect to $\mathcal{K}$; or use 343Xa below. **Q** The sets $\{t^- : t \in [0,1]\}$ and $\{t^+ : t \in [0,1]\}$ are non-measurable subsets of $I^{\parallel}$; on both of them the subspace measures correspond exactly to μ_L. We have a canonical inverse-measure-preserving function $h : I^{\parallel} \to [0,1]$ given by setting $h(t^+) = h(t^-) = t$ for every $t \in [0,1]$; h induces an isomorphism between the measure algebras of μ and μ_L.

$I^{\parallel}$ is called the **split interval** or (especially when given its standard topology, as in 343Yc below) the **double arrow space** or **two arrows space**.

Now the relevance to the present discussion is this: we have a map $f : I^{\parallel} \to I^{\parallel}$ given by setting

$$f(t^+) = t^-,\ f(t^-) = t^+ \text{ for every } t \in [0,1]$$

such that $f(x) \neq x$ for every x, but $E \triangle f^{-1}[E]$ is negligible for every $E \in \Sigma$, so that f represents the identity homomorphism on the measure algebra of μ. The function $h : I^{\parallel} \to [0,1]$ is canonical enough, but is two-to-one, and the canonical map from the measure algebra of μ to the measure algebra of μ_L is represented equally by the functions $t \mapsto t^-$ and $t \mapsto t^+$, which are nowhere equal.

(b) Consider the direct sum (Y, ν) of $(I^{\parallel}, \mu)$ and $([0,1], \mu_L)$; for definiteness, take Y to be $(I^{\parallel} \times \{0\}) \cup ([0,1] \times \{1\})$. Setting

$$h_1(t^+, 0) = h_1(t^-, 0) = (t, 1), \quad h_1(t, 1) = (t^+, 0),$$

we see that $h_1 : Y \to Y$ induces a measure-preserving involution of the measure algebra $\mathfrak{B}$ of ν, corresponding to its expression as a simple product of the isomorphic measure algebras of μ and μ_L. But h_1 is not invertible, and indeed there is no invertible function from Y to itself which induces this involution of $\mathfrak{B}$. **P?** Suppose, if possible, that $g : Y \to Y$ were such a function. Looking at the sets

$$E_q = [0, q] \times \{1\}, \quad F_q = \{(t^+, 0) : t \in [0, q]\} \cup \{(t^-, 0) : t \in [0, q]\}$$

for $q \in \mathbb{Q}$, we must have $g^{-1}[E_q] \triangle F_q$ negligible for every q, so that we must have $g(t^+, 0) = g(t^-, 0) = (t, 1)$ for almost every $t \in [0,1]$, and g cannot be injective. **XQ**

(c) Thus even with a compact probability space, and an automorphism ϕ of its measure algebra, we cannot be sure of representing ϕ and ϕ^{-1} by functions which will be inverses of each other.

343K 342L has a partial converse.

Proposition If (X, Σ, μ) is a semi-finite countably separated measure space, it is compact iff it is locally compact iff it is perfect.

proof We already know that compact measure spaces are locally compact and locally compact semi-finite measure spaces are perfect (342Ha, 342L). So suppose that (X, Σ, μ) is a perfect semi-finite countably separated measure space. Let $f : X \to \mathbb{R}$ be an injective measurable function (343E). Consider

$$\mathcal{K} = \{f^{-1}[L] : L \subseteq f[X],\ L \text{ is compact in } \mathbb{R}\}.$$

The definition of 'perfect' measure space states exactly that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$. And $\mathcal{K}$ is a compact class. **P** If $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property, $\mathcal{L} = \{f[K] : K \in \mathcal{K}'\}$ is a family of compact sets in $\mathbb{R}$ with the finite intersection property, and has non-empty intersection; so that $\bigcap \mathcal{K}'$ is also non-empty, because f is injective. **Q** By 342E, (X, Σ, μ) is compact.

343L The time has come to give examples of spaces which are *not* locally compact, so that we can expect to have measure-preserving homomorphisms not representable by inverse-measure-preserving functions. The most commonly arising ones are covered by the following result.

Proposition Let (X, Σ, μ) be a complete locally determined countably separated measure space, and $A \subseteq X$ a set such that the subspace measure μ_A is perfect. Then A is measurable.

proof ? Otherwise, there is a set $E \in \Sigma$ such that $\mu E < \infty$ and $B = A \cap E \notin \Sigma$. Let $f : X \to \mathbb{R}$ be an injective measurable function (343E again). Then $f \restriction B$ is Σ_B-measurable, where Σ_B is the domain of the subspace measure μ_B on B. Set

$$\mathcal{K} = \{f^{-1}[L] : L \subseteq f[B],\, L \text{ is compact in } \mathbb{R}\}.$$

Just as in the proof of 343K, $\mathcal{K}$ is a compact class and μ_B is inner regular with respect to $\mathcal{K}$. By 342Bb, there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $\mu_B(B \setminus \bigcup_{n\in\mathbb{N}} K_n) = 0$. But of course $\mathcal{K} \subseteq \Sigma$, because f is Σ-measurable, so $\bigcup_{n\in\mathbb{N}} K_n \in \Sigma$. Because μ is complete, $B \setminus \bigcup_{n\in\mathbb{N}} K_n \in \Sigma$ and $B \in \Sigma$. **X**

343M Example 343L tells us that any non-measurable set X of $\mathbb{R}^r$, or of $\{0,1\}^{\mathbb{N}}$, with their usual measures, is not perfect, therefore not (locally) compact, when given its subspace measure.

To find a non-representable homomorphism, we do not need to go through the whole apparatus of 343B. Take Y to be a measurable envelope of X (132Ee). Then the identity function from X to Y induces an isomorphism of their measure algebras. But there is no function from Y to X inducing the same isomorphism. **P?** Writing Z for $\mathbb{R}^r$ or $\{0,1\}^{\mathbb{N}}$ and μ for its measure, Z is countably separated; suppose $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable sets in Z separating its points. For each n, $(Y \cap E_n)^\bullet$ in the measure algebra of μ_Y corresponds to $(X \cap E_n)^\bullet$ in the measure algebra of μ_X. So if $f : Y \to X$ were a function representing the isomorphism of the measure algebras, $(Y \cap E_n) \triangle f^{-1}[E_n]$ would have to be negligible for each n, and $A = \bigcup_{n\in\mathbb{N}}(Y \cap E_n) \triangle f^{-1}[E_n]$ would be negligible. But for $y \in Y \setminus A$, $f(y)$ belongs to just the same E_n as y does, so must be equal to y. Accordingly $X \supseteq Y \setminus A$ and X is measurable. **XQ**

343X Basic exercises (a) Let (X, Σ, μ) be a semi-finite measure space. (i) Suppose that there is a set $A \subseteq X$, of full outer measure, such that the subspace measure on A is compact. Show that μ is locally compact. (*Hint*: show that μ satisfies (ii) or (v) of 343B.) (ii) Suppose that for every non-negligible $E \in \Sigma$ there is a non-negligible set $A \subseteq E$ such that the subspace measure on A is compact. Show that μ is locally compact.

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of non-empty sets, with product X; write $\pi_i : X \to X_i$ for the coordinate map. Suppose we are given a σ-algebra Σ_i of subsets of X_i for each i; let $\Sigma = \widehat{\bigotimes}_{i\in I}\Sigma_i$ be the corresponding σ-algebra of subsets of X generated by $\{\pi_i^{-1}[E] : i \in I,\, E \in \Sigma_i\}$. Let μ be a totally finite measure with domain Σ, and for $i \in I$ let μ_i be the image measure $\mu\pi_i^{-1}$. Check that the domain of μ_i is Σ_i. Show that if every (X_i, Σ_i, μ_i) is compact, then so is (X, Σ, μ). (*Hint*: *either* show that μ satisfies (v) of 343B *or* adapt the method of 342Gf.)

(c) Let I be any set. Let $\mathcal{B}\mathfrak{a}$ be the σ-algebra of subsets of $\{0,1\}^I$ generated by the sets $F_i = \{z : z(i) = 1\}$ for $i \in I$, and ν any probability measure with domain $\mathcal{B}\mathfrak{a}$; let $\mathfrak{B}$ be the measure algebra of ν. Let (X, Σ, μ) be a measure space with measure algebra $\mathfrak{A}$, and $\phi : \mathfrak{B} \to \mathfrak{A}$ an order-continuous Boolean homomorphism. Show that there is an inverse-measure-preserving function $f : X \to \{0,1\}^I$ representing ϕ. (*Hint*: for each $i \in I$, take $E_i \in \Sigma$ such that $E_i^\bullet = \phi F_i^\bullet$; set $f(x)(i) = 1$ if $x \in E_i$, and use 343Ab.)

(d) Let (X, Σ, μ) be an atomless probability space. Let $\mu_{\mathcal{B}}$ be the restriction of Lebesgue measure to the σ-algebra of Borel subsets of $[0,1]$. Show that there is a function $g : X \to [0,1]$ which is inverse-measure-preserving for μ and $\mu_{\mathcal{B}}$. (*Hint*: find an $f : X \to \{0,1\}^{\mathbb{N}}$ as in 343Xc, and set $g = hf$ where $h(z) = \sum_{n=0}^{\infty} 2^{-n-1}g(n)$, as in 254K; or choose $E_q \in \Sigma$ such that $\mu E_q = q$, $E_q \subseteq E_{q'}$ whenever $q \le q'$ in $[0,1] \cap \mathbb{Q}$, and set $f(x) = \inf\{q : x \in E_q\}$ for $x \in E_1$.)

(e) Let (X, Σ, μ) be a countably separated measure space, with measure algebra $\mathfrak{A}$. (i) Show that $\{x\} \in \Sigma$ for every $x \in X$. (ii) Show that every atom of $\mathfrak{A}$ is of the form $\{x\}^\bullet$ for some $x \in X$.

(f) Let (X, Σ, μ) be a semi-finite countably separated measure space. (i) Show that μ is point-supported iff it is complete, strictly localizable and purely atomic. (ii) Show that μ is atomless iff $\mu\{x\} = 0$ for every $x \in X$.

(g) Let $I^{\parallel}$ be the split interval, with its usual measure μ described in 343J, and $h : I^{\parallel} \to [0,1]$ the canonical surjection. Show that the canonical isomorphism between the measure algebras of μ and Lebesgue measure on $[0,1]$ is given by the formula '$E^\bullet \mapsto h[E]^\bullet$ for every measurable $E \subseteq I^{\parallel}$'.

(h) Let (X, Σ, μ) and (Y, T, ν) be measure spaces with measure algebras $(\mathfrak{A}, \bar\mu)$, $(\mathfrak{B}, \bar\nu)$. Suppose that $X \cap Y = \emptyset$ and that we have a measure-preserving isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$. Set

$$\Lambda = \{W : W \subseteq X \cup Y,\, W \cap X \in \Sigma,\, W \cap Y \in \mathrm{T},\, \pi(W \cap X)^\bullet = (W \cap Y)^\bullet\},$$

and for $W \in \Lambda$ set $\lambda W = \mu(W \cap X) = \nu(W \cap Y)$. Show that $(X \cup Y, \Lambda, \lambda)$ is a measure space which is locally compact, or perfect, if (X, Σ, μ) is.

>(i) Let (X,Σ,μ) be a complete perfect totally finite measure space, (Y,T,ν) a complete countably separated measure space, and $f:X\to Y$ an inverse-measure-preserving function. Show that $\mathrm{T}=\{F:F\subseteq Y,\,f^{-1}[F]\in\Sigma\}$, so that a function $h:Y\to\mathbb{R}$ is ν-integrable iff hf is μ-integrable. (*Hint*: if $A\subseteq Y$ and $E=f^{-1}[A]\in\Sigma$, $f\restriction E$ is inverse-measure-preserving for the subspace measures μ_E, ν_A; by 342Xk, ν_A is perfect, so by 343L $A\in\mathrm{T}$. Now use 235J.)

(j) Let (X,Σ,μ) be a complete compact measure space, Y a set and $f:Y\to X$ a surjection; set

$$\mathrm{T}=\{F:F\subseteq Y,\,f[F]\in\Sigma,\,\mu(f[F]\cap f[Y\setminus F])=0\},\quad \nu F=\mu f[F]\text{ for }F\in\mathrm{T},$$

so that ν is a measure on Y and f is inverse-measure-preserving (234Ye). Show that ν is a compact measure.

343Y Further exercises (a) Let (X,Σ,μ) be a semi-finite measure space, and suppose that there is a compact class $\mathcal{K}\subseteq\mathcal{P}X$ such that (α) whenever $E\in\Sigma$ and $\mu E>0$ there is a non-negligible $K\in\mathcal{K}$ such that $K\subseteq E$ (β) whenever $K_0,\dots,K_n\in\mathcal{K}$ and $\bigcap_{i\le n}K_i=\emptyset$ then there are measurable sets $E_0,\dots,E_n$ such that $E_i\supseteq K_i$ for every i and $\bigcap_{i\le n}E_i$ is negligible. Show that μ is locally compact.

(b)(i) Show that a countably separated semi-finite measure space has magnitude and Maharam type at most $2^{\mathfrak{c}}$. (ii) Show that the direct sum of $\mathfrak{c}$ or fewer countably separated measure spaces is countably separated. (iii) Show that a countably separated perfect measure space has countable Maharam type.

(c) Let $I^{\parallel}=\{t^+:t\in[0,1]\}\cup\{t^-:t\in[0,1]\}$ be the split interval (343J). (i) Show that the rules

$$s^-\le t^-\iff s^+\le t^+\iff s\le t,\quad s^+\le t^-\iff s<t,$$

$$t^-\le t^+\text{ for all }t\in[0,1]$$

define a Dedekind complete total order on $I^{\parallel}$ with greatest and least elements. (ii) Show that the intervals $[0^-,t^-]$, $[t^+,1^+]$, interpreted for this ordering, generate a compact Hausdorff topology on $I^{\parallel}$ for which the map $h:I^{\parallel}\to[0,1]$ of 343J is continuous. (iii) Show that a subset E of $I^{\parallel}$ is Borel for this topology iff the sets E_r, $E_l\subseteq[0,1]$, as described in 343Ja, are Borel and $E_r\triangle E_l$ is countable. (iv) Show that if $f:[0,1]\to\mathbb{R}$ is of bounded variation then there is a continuous $g:I^{\parallel}\to\mathbb{R}$ such that $g=fh$ except perhaps at countably many points. (v) Show that the measure μ of 343J is inner regular with respect to the compact subsets of $I^{\parallel}$. (vi) Show that we have a lower density ϕ for μ defined by setting

$$\phi E=\{t^-:0<t\le 1,\,\lim_{\delta\downarrow 0}\frac{1}{\delta}\mu(E\cap[(t-\delta)^+,t^-])=1\}$$

$$\cup\{t^+:0\le t<1,\,\lim_{\delta\downarrow 0}\frac{1}{\delta}\mu(E\cap[t^+,(t+\delta)^-])=1\}$$

for measurable sets $E\subseteq I^{\parallel}$.

(d) Set $X=\{0,1\}^{\mathfrak{c}}$, with its usual measure $\nu_{\mathfrak{c}}$. Show that there is an inverse-measure-preserving function $f:X\to X$ such that $f[X]$ is non-measurable but f induces the identity automorphism of the measure algebra of $\nu_{\mathfrak{c}}$. (*Hint*: use the idea of 343I.) Show that under these conditions $f[X]$, with its subspace measure, must be compact. (*Hint*: use 343B(iv).)

(e) Let μ_{Hr} be r-dimensional Hausdorff measure on $\mathbb{R}^s$, where $s\ge 1$ is an integer and $r\ge 0$ (§264). (i) Show that μ_{Hr} is countably separated. (ii) Show that the c.l.d. version of μ_{Hr} is compact. (*Hint*: 264Yi.)

(f) Give an example of a countably separated probability space (X,Σ,μ) and a function f from X to a set Y such that the image measure μf^{-1} is not countably separated. (*Hint*: use 223B to show that if $E\subseteq\mathbb{R}$ is Lebesgue measurable and not negligible, then $E+\mathbb{Q}$ is conegligible; or use the zero-one law to show that if $E\subseteq\mathcal{P}\mathbb{N}$ is measurable and not negligible for the usual measure on $\mathcal{P}\mathbb{N}$, then $\{a\triangle b:a\in E,\,b\in[\mathbb{N}]^{<\omega}\}$ is conegligible.)

343 Notes and comments The points at which the Lifting Theorem impinges on the work of this section are in the proofs of (iv)⇒(i) and (iv)⇒(v) in Theorem 343B. In fact the ideas can be rearranged to give a proof of 343B which does not rely on the Lifting Theorem; I give a hint in Volume 4 (413Yc).

I suppose the significant new ideas of this section are in 343B and 343K. The rest is mostly a matter of being thorough and careful. But I take this material at a slow pace because there are some potentially confusing features, and the underlying question is of the greatest importance: when, given a Boolean homomorphism from one measure algebra to another, can we be sure of representing it by a measurable function between measure spaces? The concept of 'compact' measure puts the burden firmly on the measure space corresponding to the *domain* of the Boolean homomorphism, which will be the *codomain* of the measurable function. So the first step is to try to understand properly which measures are compact, and what other properties they can be expected to have; which accounts for much of the length of §342. But having understood that many of our favourite measures are compact, we have to come to terms with the fact that we still cannot count on a measure algebra isomorphism corresponding to a measure space isomorphism. I introduce the split interval (343J, 343Xg, 343Yc) as a close approximation to Lebesgue measure on $[0,1]$ which is not isomorphic to it. Of course we have already seen a more dramatic example: the Stone space of the Lebesgue measure algebra also has the same measure algebra as Lebesgue measure, while being in almost every other way very much more complex, as will appear in Volumes 4 and 5.

As 343C suggests, elementary cases in which 343B can be applied are often amenable to more primitive methods, avoiding not only the concept of 'compact' measure, but also Stone spaces and the Lifting Theorem. For substantial examples in which we can prove that a measure space (X,μ) is compact, without simultaneously finding direct constructions for inverse-measure-preserving functions into X (as in 343Xc-343Xd), I think we shall have to wait until Volume 4.

The concept of 'countably separated' measure space does not involve the measure at all, nor even the null ideal; it belongs to the theory of σ-algebras of sets. Some simple permanence properties are in 343H and 343Yb(ii). Let us note in passing that 343Xi describes some more situations in which the 'image measure catastrophe', described in 235H, cannot arise.

I include the variants 343B(ii), 343B(iii) and 343Ya of the notion of 'local compactness' because they are not obvious and may illuminate it.

344 Realization of automorphisms

In 343Jb, I gave an example of a 'good' (compact, complete) probability space X with an automorphism ϕ of its measure algebra such that both ϕ and ϕ^{-1} are representable by functions from X to itself, but there is no such representation in which the two functions are inverses of each other. The present section is an attempt to describe the further refinements necessary to ensure that automorphisms of measure algebras can be represented by automorphisms of the measure spaces. It turns out that in the most important contexts in which this can be done, a little extra work yields a significant generalization: the simultaneous realization of countably many homomorphisms by a consistent family of functions.

I will describe three cases in which such simultaneous realizations can be achieved: Stone spaces (344A), perfect complete countably separated spaces (344C) and suitable measures on $\{0,1\}^I$ (344E-344G). The arguments for 344C, suitably refined, give a complete description of perfect complete countably separated strictly localizable spaces which are not purely atomic (344I, 344Xc). At the same time we find that Lebesgue measure, and the usual measure on $\{0,1\}^I$, are 'homogeneous' in the strong sense that two measurable subspaces (of non-zero measure) are isomorphic iff they have the same measure (344J, 344L).

344A Stone spaces The first case is immediate from the work of §§312, 313 and 321, as collected in 324E. If (Z,Σ,μ) is actually the Stone space of a measure algebra $(\mathfrak{A},\bar{\mu})$, then every order-continuous Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{A}$ corresponds to a unique continuous function $f_\phi:Z\to Z$ (312Q) which represents ϕ (324E). The uniqueness of f_ϕ means that we can be sure that $f_{\phi\psi}=f_\psi f_\phi$ for all order-continuous homomorphisms ϕ and ψ; and of course f_ι is the identity map on Z, so that $f_{\phi^{-1}}$ will have to be f_ϕ^{-1} whenever ϕ is invertible. Thus in this special case we can consistently, and canonically, represent all order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself.

Now for two cases where we have to work for the results.

344B Theorem Let (X,Σ,μ) be a countably separated measure space with measure algebra $\mathfrak{A}$, and G a countable semigroup of Boolean homomorphisms from $\mathfrak{A}$ to itself such that every member of G can be represented by some function from X to itself. Then a family $\langle f_\phi\rangle_{\phi\in G}$ of such representatives can be chosen in such a way that $f_{\phi\psi}=f_\psi f_\phi$ for all ϕ, $\psi\in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X.

proof (a) Because $G \cup \{\iota\}$ satisfies the same conditions as G, we may suppose from the beginning that ι belongs to G itself. Let $\mathcal{A} \subseteq \Sigma$ be a countable set separating the points of X. For each $\phi \in G$ take some representing function $g_\phi : X \to X$; take g_ι to be the identity function. If ϕ, $\psi \in G$, then of course

$$((g_\phi g_\psi)^{-1}[E])^\bullet = (g_\psi^{-1}[g_\phi^{-1}[E]])^\bullet = \psi(g_\phi^{-1}[E])^\bullet = \psi\phi E^\bullet = (g_{\psi\phi}^{-1}[E])^\bullet$$

for every $E \in \Sigma$. By 343F, the set

$$H_{\phi\psi} = \{x : g_{\psi\phi}(x) \neq g_\phi g_\psi(x)\}$$

is negligible and belongs to Σ.

(b) Set

$$H = \bigcup_{\phi,\psi \in G} H_{\phi\psi};$$

because G is countable, H also is measurable and negligible. Try defining $f_\phi : X \to X$ by setting $f_\phi(x) = g_\phi(x)$ if $x \in X \setminus H$, $f_\phi(x) = x$ if $x \in H$. Because H is measurable, $f_\phi^{-1}[E] \in \Sigma$ for every $E \in \Sigma$; because H is negligible,

$$(f_\phi^{-1}[E])^\bullet = (g_\phi^{-1}[E])^\bullet = \phi E^\bullet$$

for every $E \in \Sigma$, and f_ϕ represents ϕ, for every $\phi \in G$. Of course $f_\iota = g_\iota$ is the identity function on X.

(c) If $\theta \in G$ then $f_\theta^{-1}[H] = H$. **P** (i) If $x \in H$ then $f_\theta(x) = x \in H$. (ii) If $f_\theta(x) \in H$ and $f_\theta(x) = x$ then of course $x \in H$. (iii) If $f_\theta(x) = g_\theta(x) \in H$ then there are ϕ, $\psi \in G$ such that $g_\phi g_\psi g_\theta(x) \neq g_{\psi\phi} g_\theta(x)$. So either

$$g_\psi g_\theta(x) \neq g_{\theta\psi}(x),$$

or

$$g_\phi g_{\theta\psi}(x) \neq g_{\theta\psi\phi}(x)$$

or

$$g_{\theta\psi\phi}(x) \neq g_{\psi\phi} g_\theta(x),$$

and in any case $x \in H$. **Q**

(d) It follows that $f_\phi f_\psi = f_{\psi\phi}$ for every ϕ, $\psi \in G$. **P** (i) If $x \in H$ then

$$f_\phi f_\psi(x) = x = f_{\psi\phi}(x).$$

(ii) If $x \in X \setminus H$ then $f_\psi(x) \notin H$, by (c), so

$$f_\phi f_\psi(x) = g_\phi g_\psi(x) = g_{\psi\phi}(x) = f_{\psi\phi}(x). \quad \textbf{Q}$$

344C Corollary Let (X, Σ, μ) be a countably separated perfect complete strictly localizable measure space with measure algebra $\mathfrak{A}$, and G a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi \in G$, a function $f_\phi : X \to X$ representing ϕ, in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi \in G$ is invertible, and $\phi^{-1} \in G$, we shall have $f_{\phi^{-1}} = f_\phi^{-1}$; so that if moreover ϕ and ϕ^{-1} are measure-preserving, f_ϕ will be an automorphism of the measure space (X, Σ, μ).

proof By 343K, (X, Σ, μ) is compact. So 343B(v) tells us that every member of G is representable, and we can apply 344B.

Reminder Spaces satisfying the conditions of this corollary include Lebesgue measure on $\mathbb{R}^r$, the usual measure on $\{0,1\}^{\mathbb{N}}$, and their measurable subspaces; see also 342J, 342Xe, 343H and 343Ye.

344D The third case I wish to present requires a more elaborate argument. I start with a kind of Schröder-Bernstein theorem for measurable spaces.

Lemma Let X and Y be sets, and $\Sigma \subseteq \mathcal{P}X$, $\mathrm{T} \subseteq \mathcal{P}Y$ σ-algebras. Suppose that there are $f : X \to Y$, $g : Y \to X$ such that $F = f[X] \in \mathrm{T}$, $E = g[Y] \in \Sigma$, f is an isomorphism between (X, Σ) and (F, T_F) and g is an isomorphism between (Y, T) and (E, Σ_E), writing Σ_E, T_F for the subspace σ-algebras (see 121A). Then (X, Σ) and (Y, T) are isomorphic, and there is an isomorphism $h : X \to Y$ which is covered by f and g in the sense that

$$\{(x, h(x)) : x \in X\} \subseteq \{(x, f(x)) : x \in X\} \cup \{(g(y), y) : y \in Y\}.$$

proof Set $X_0 = X$, $Y_0 = Y$, $X_{n+1} = g[Y_n]$ and $Y_{n+1} = f[X_n]$ for each $n \in \mathbb{N}$; then $\langle X_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ and $\langle Y_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T. Set $X_\infty = \bigcap_{n\in\mathbb{N}} X_n$, $Y_\infty = \bigcap_{n\in\mathbb{N}} Y_n$. Then $f\restriction X_{2k}\setminus X_{2k+1}$ is an isomorphism between $X_{2k}\setminus X_{2k+1}$ and $Y_{2k+1}\setminus Y_{2k+2}$, while $g\restriction Y_{2k}\setminus Y_{2k+1}$ is an isomorphism between $Y_{2k}\setminus Y_{2k+1}$ and $X_{2k+1}\setminus X_{2k+2}$; and $g\restriction Y_\infty$ is an isomorphism between Y_∞ and X_∞. So the formula

$$\begin{aligned} h(x) &= f(x) \text{ if } x \in \bigcup_{k\in\mathbb{N}} X_{2k}\setminus X_{2k+1},\\ &= g^{-1}(x) \text{ for other } x \in X \end{aligned}$$

gives the required isomorphism between X and Y.

Remark You will recognise the ordinary Schröder-Bernstein theorem (2A1G) as the case $\Sigma = \mathcal{P}X$, $\mathrm{T} = \mathcal{P}Y$.

344E Theorem Let I be any set, and let μ be a σ-finite measure on $X = \{0,1\}^I$ with domain the σ-algebra $\mathcal{B}\mathfrak{a}_I$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I; write $\mathfrak{A}$ for the measure algebra of μ. Let G be a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi \in G$, a function $f_\phi : X \to X$ representing ϕ, in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi \in G$ is invertible and $\phi^{-1} \in G$, we shall have $f_{\phi^{-1}} = f_\phi^{-1}$; so that if moreover ϕ is measure-preserving, f_ϕ will be an automorphism of the measure space $(X, \mathcal{B}\mathfrak{a}_I, \mu)$.

proof (a) As in 344C, we may as well suppose from the beginning that $\iota \in G$. The case of finite I is trivial, so I will suppose that I is infinite. For $i \in I$, set $E_i = \{x : x(i) = 1\}$; for $J \subseteq I$, let $\mathcal{B}_J$ be the σ-subalgebra of $\mathcal{B}\mathfrak{a}_I$ generated by $\{E_i : I \in J\}$. For $i \in I$, $\phi \in G$ choose $F_{\phi i} \in \mathcal{B}\mathfrak{a}_I$ such that $F_{\phi i}^\bullet = \phi E_i^\bullet$. (Of course we take $F_{\iota i} = E_i$ for every i.) Let $\mathcal{J}$ be the family of those subsets J of I such that $F_{\phi i} \in \mathcal{B}_J$ for every $i \in J$ and $\phi \in G$.

(b) For the purposes of this proof, I will say that a pair $(J, \langle g_\phi\rangle_{\phi\in G})$ is **consistent** if $J \in \mathcal{J}$ and, for each $\phi \in G$, g_ϕ is a function from X to itself such that

$g_\phi^{-1}[E_i] \in \mathcal{B}_J$ and $(g_\phi^{-1}[E_i])^\bullet = \phi E_i^\bullet$ whenever $i \in J$, $\phi \in G$,

$g_\phi^{-1}[E_i] = E_i$ whenever $i \in I \setminus J$, $\phi \in G$,

$g_\phi g_\psi = g_{\psi\phi}$ whenever ϕ, $\psi \in G$,

$g_\iota(x) = x$ for every $x \in X$.

Now the key to the proof is the following fact: if $(J, \langle g_\phi\rangle_{\phi\in G})$ is consistent, and $\tilde{J}$ is a member of $\mathcal{J}$ such that $\tilde{J}\setminus J$ is countably infinite, then there is a family $\langle tildeg_\phi\rangle_{\phi\in G}$ such that $(\tilde{J}, \langle\tilde{g}_\phi\rangle_{\phi\in G})$ is consistent and $\tilde{g}_\phi^{-1}[E_i] = g_\phi^{-1}[E_i]$ whenever $i \in J$ and $\phi \in G$, that is, $\tilde{g}_\phi(x)\restriction J = g_\phi(x)\restriction J$ whenever $\phi \in G$ and $x \in X$. The construction is as follows.

(i) Start by fixing on any infinite set $K \subseteq \tilde{J}\setminus J$ such that $(\tilde{J}\setminus J)\setminus K$ also is infinite. For $z \in \{0,1\}^K$, set $V_z = \{x : x \in X, x\restriction K = z\}$; then $V_z \in \mathcal{B}_{\tilde{J}}$. All the sets V_z, as z runs over the uncountable set $\{0,1\}^K$, are disjoint, so they cannot all have non-zero measure (because μ is σ-finite), and we can choose z such that V_z is μ-negligible.

(ii) Define $h_\phi : X \to X$, for $\phi \in G$, by setting

$$\begin{aligned} h_\phi(x)(i) &= g_\phi(x)(i) \text{ if } i \in J,\\ &= x(i) \text{ if } i \in I\setminus\tilde{J},\\ &= x(i) \text{ if } i \in \tilde{J}\setminus J \text{ and } x \in V_z,\\ &= 1 \text{ if } i \in \tilde{J}\setminus J \text{ and } x \in F_{\phi i}\setminus V_z,\\ &= 0 \text{ if } i \in \tilde{J}\setminus J \text{ and } x \notin F_{\phi i}\cup V_z. \end{aligned}$$

Because $V_z \in \mathcal{B}_{\tilde{J}}$ and $\mu V_z = 0$, we see that

(α) $h_\phi^{-1}[E_i] = g_\phi^{-1}[E_i] \in \mathcal{B}_J$ if $i \in J$,

(β) $h_\phi^{-1}[E_i] \in \mathcal{B}_{\tilde{J}}$ and $h_\phi^{-1}[E_i]\triangle F_{\phi i}$ is negligible if $i \in \tilde{J}\setminus J$,

and consequently

(γ) $(h_\phi^{-1}[E_i])^\bullet = \phi E_i^\bullet$ for every $i \in \tilde{J}$,

(δ) $(h_\phi^{-1}[E])^\bullet = \phi E^\bullet$ for every $E \in \mathcal{B}_{\tilde{J}}$

(by 343Ab); moreover,

(ϵ) $h_\phi^{-1}[E]=g_\phi^{-1}[E]$ for every $E\in\mathcal{B}_J$,

(ζ) $h_\phi^{-1}[E]\in\mathcal{B}_{\tilde J}$ for every $E\in\mathcal{B}_{\tilde J}$,

(η) $h_\phi^{-1}[E_i]=E_i$ if $i\in I\setminus\tilde J$,

so that

(θ) $h_\phi^{-1}[E]\in\mathcal{B}\mathfrak{a}_I$ for every $E\in\mathcal{B}\mathfrak{a}_I$;

finally, because $F_{\iota i}=E_i$,

(ι) $h_\iota(x)=x$ for every $x\in X$.

(iii) The next step is to note that if ϕ, $\psi\in G$ then

$$H_{\phi,\psi}=\{x:x\in X,\,h_\phi h_\psi(x)\ne h_{\psi\phi}(x)\}$$

belongs to $\mathcal{B}_{\tilde J}$ and is negligible. **P**

$$H_{\phi,\psi}=\bigcup_{i\in I}h_\psi^{-1}[h_\phi^{-1}[E_i]]\triangle h_{\psi\phi}^{-1}[E_i].$$

Now if $i\in J$, then $h_\phi^{-1}[E_i]=g_\phi^{-1}[E_i]\in\mathcal{B}_J$, so

$$h_\psi^{-1}[h_\phi^{-1}[E_i]]=h_\psi^{-1}[g_\phi^{-1}[E_i]]=g_\psi^{-1}[g_\phi^{-1}[E_i]]=g_{\psi\phi}^{-1}[E_i]=h_{\psi\phi}^{-1}[E_i].$$

Next, for $i\in I\setminus\tilde J$,

$$h_\psi^{-1}[h_\phi^{-1}[E_i]]=h_\psi^{-1}[E_i]=E_i=h_{\psi\phi}^{-1}[E_i].$$

So

$$H_{\phi,\psi}=\bigcup_{i\in\tilde J\setminus J}h_\psi^{-1}[h_\phi^{-1}[E_i]]\triangle h_{\psi\phi}^{-1}[E_i].$$

But for any particular $i\in\tilde J\setminus J$, E_i and $h_\phi^{-1}[E_i]$ belong to $\mathcal{B}_{\tilde J}$, so

$$(h_\psi^{-1}[h_\phi^{-1}[E_i]])^\bullet=\psi(h_\phi^{-1}[E_i])^\bullet=\psi\phi E_i^\bullet=(h_{\psi\phi}^{-1}[E_i])^\bullet,$$

and $h_\psi^{-1}[h_\phi^{-1}[E_i]]\triangle h_{\psi\phi}^{-1}[E_i]$ is a negligible set, which by (ii-ζ) belongs to $\mathcal{B}_{\tilde J}$. So $H_{\phi,\psi}$ is a countable union of sets of measure 0 in $\mathcal{B}_{\tilde J}$ and is itself a negligible member of $\mathcal{B}_{\tilde J}$, as claimed. **Q**

(iv) Set

$$H=\bigcup_{\phi,\psi\in G}H_{\phi,\psi}\cup\bigcup_{\phi\in G}h_\phi^{-1}[V_z].$$

Then $H\in\mathcal{B}_{\tilde J}$ and $\mu H=0$. **P** We know that every $H_{\phi,\psi}$ is negligible and belongs to $\mathcal{B}_{\tilde J}$ ((iii) above), that every $h_\phi^{-1}[V_z]$ belongs to $\mathcal{B}_{\tilde J}$ (by (ii-ζ), and that $(h_\phi^{-1}[V_z])^\bullet=\phi V_z^\bullet=0$, so that $h_\phi^{-1}[V_z]$ is negligible, for every $\phi\in G$ (by (ii-δ)). Consequently H is negligible and belongs to $\mathcal{B}_{\tilde J}$. **Q** Also, of course, $V_z=h_\iota^{-1}[V_z]\subseteq H$.

Next, $h_\phi(x)\notin H$ whenever $x\in X\setminus H$ and $\phi\in G$. **P** If ψ, $\theta\in G$ then

$$h_{\theta\psi}h_\phi(x)=h_{\phi\theta\psi}(x)=h_\psi h_{\phi\theta}(x)=h_\psi h_\theta h_\phi(x),$$

$$h_\psi h_\phi(x)=h_{\phi\psi}(x)\notin V_z$$

because

$$x\notin H_{\theta\psi,\phi}\cup H_{\psi,\phi\theta}\cup H_{\theta,\phi}\cup H_{\psi,\phi}\cup h_{\phi\psi}^{-1}[V_z];$$

thus $h_\phi(x)\notin H_{\psi,\theta}\cup h_\psi^{-1}[V_z]$; as ψ and θ are arbitrary, $h_\phi(x)\notin H$. **Q**

(v) The next fact we need is that there is a bijection $q:X\to H$ such that (α) for $E\subseteq H$, $E\in\mathcal{B}_{\tilde J}$ iff $q^{-1}[E]\in\mathcal{B}_{\tilde J}$ (β) $q(x)(i)=x(i)$ for every $i\in I\setminus(\tilde J\setminus J)$ and $x\in X$. **P** Fix any bijection $r:\tilde J\setminus J\to\tilde J\setminus(J\cup K)$. Consider the maps $p_1:X\to H$, $p_2:H\to X$ given by

$$\begin{aligned}p_1(x)(i)&=x(r^{-1}(i))\text{ if }i\in\tilde J\setminus(J\cup K),\\&=z(i)\text{ if }i\in K,\\&=x(i)\text{ if }i\in X\setminus(\tilde J\setminus J),\\p_2(y)&=y\end{aligned}$$

for $x \in X$, $y \in H$. Then p_1 is actually an isomorphism between $(X, \mathcal{B}_{\tilde{J}})$ and $(V_z, \mathcal{B}_{\tilde{J}} \cap \mathcal{P}V_z)$. So p_1, p_2 are isomorphisms between $(X, \mathcal{B}_{\tilde{J}})$, $(H, \mathcal{B}_{\tilde{J}} \cap \mathcal{P}H)$ and measurable subspaces of H, X respectively. By 344D, there is an isomorphism q between X and H such that, for every $x \in X$, either $q(x) = p_1(x)$ or $p_2(q(x)) = x$. Since $p_1(x) \restriction I \setminus (\tilde{J} \setminus J) = x \restriction I \setminus (\tilde{J} \setminus J)$ for every $x \in X$, and $p_2(y) \restriction I \setminus (\tilde{J} \setminus J) = y \restriction I \setminus (\tilde{J} \setminus J)$ for every $y \in H$, $q(x) \restriction I \setminus (\tilde{J} \setminus J) = x \restriction I \setminus (\tilde{J} \setminus J)$ for every $x \in X$. **Q**

(vi) An incidental fact which will be used below is the following: if $i \in \tilde{J}$ and $\phi \in G$ then $g_\phi^{-1}[E_i]$ belongs to $\mathcal{B}_{\tilde{J}}$, because it belongs to $\mathcal{B}_J$ if $i \in J$, and otherwise is equal to E_i. Consequently $g_\phi^{-1}[E] \in \mathcal{B}_{\tilde{J}}$ for every $E \in \mathcal{B}_{\tilde{J}}$.

(vii) I am at last ready to give a formula for $\tilde{g}_\phi$. For $\phi \in G$ set

$$\begin{aligned}\tilde{g}_\phi(x) &= h_\phi(x) \text{ if } x \in X \setminus H,\\ &= q g_\phi q^{-1}(x) \text{ if } x \in H.\end{aligned}$$

Now $(\tilde{J}, \langle \tilde{g}_\phi \rangle_{\phi \in G})$ is consistent. **P**

(α) If $i \in \tilde{J}$ and $\phi \in G$,

$$\tilde{g}_\phi^{-1}[E_i] = (h_\phi^{-1}[E_i] \setminus H) \cup q[g_\phi^{-1}[q^{-1}[E_i \cap H]]] \in \tilde{\mathcal{B}}_J$$

because $H \in \mathcal{B}_{\tilde{J}}$ and $h_\phi^{-1}[E]$, $q^{-1}[H \cap E]$, $g_\phi^{-1}[E]$ and $q[E]$ all belong to $\mathcal{B}_{\tilde{J}}$ for every $E \in \mathcal{B}_{\tilde{J}}$. At the same time, because $\tilde{g}_\phi$ agrees with h_ϕ on the conegligible set $X \setminus H$,

$$(\tilde{g}_\phi^{-1}[E_i])^\bullet = (h_\phi^{-1}[E_i])^\bullet = \phi E_i^\bullet.$$

(β) If $i \in I \setminus \tilde{J}$, $\phi \in G$ and $x \in X$ then

$$g_\phi(x)(i) = h_\phi(x)(i) = q(x)(i) = x(i),$$

and if $x \in H$ then $q^{-1}(x)(i)$ also is equal to $x(i)$; so $\tilde{g}_\phi(x)(i) = x(i)$. But this means that $\tilde{g}_\phi^{-1}[E_i] = E_i$.

(γ) If ϕ, $\psi \in G$ and $x \in X \setminus H$, then

$$\tilde{g}_\psi(x) = h_\psi(x) \in X \setminus H$$

by (iv) above. So

$$\tilde{g}_\phi \tilde{g}_\psi(x) = h_\phi h_\psi(x) = h_{\psi\phi}(x) = \tilde{g}_{\psi\phi}(x)$$

because $x \notin H_{\phi,\psi}$. While if $x \in H$, then

$$\tilde{g}_\psi(x) = q g_\psi q^{-1}(x) \in H,$$

so

$$\tilde{g}_\phi \tilde{g}_\psi(x) = q g_\phi q^{-1} q g_\psi q^{-1}(x) = q g_\phi g_\psi q^{-1}(x) = q g_{\psi\phi} q^{-1}(x) = \tilde{g}_{\psi\phi}(x).$$

Thus $\tilde{g}_\phi \tilde{g}_\psi = \tilde{g}_{\psi\phi}$.

(δ) Because $g_\iota(x) = h_\iota(x) = x$ for every x, $\tilde{g}_\iota(x) = x$ for every x. **Q**

(viii) Finally, if $i \in J$ and $\phi \in G$, $q^{-1}[E_i] = E_i$, so that $q[E_i] = E_i \cap H$. Accordingly $q(x) \restriction J = x \restriction J$ for every $x \in X$, while $q^{-1}(x) \restriction J = x \restriction J$ for $x \in H$. So $g_\phi q^{-1}(x) \restriction J = g_\phi(x) \restriction J$ for $x \in H$, and

$$\begin{aligned}\tilde{g}_\phi(x) \restriction J = h_\phi(x) \restriction J &= g_\phi(x) \restriction J \text{ if } x \in X \setminus H,\\ = q g_\phi q^{-1}(x) \restriction J = g_\phi q^{-1}(x) \restriction J &= g_\phi(x) \restriction J \text{ if } x \in H.\end{aligned}$$

Thus $(\tilde{J}, \langle \tilde{g}_\phi \rangle_{\phi \in G})$ satisfies all the required conditions.

(c) The remaining idea we need is the following: there is a non-decreasing family $\langle J_\xi \rangle_{\xi \le \kappa}$ in $\mathcal{J}$, for some cardinal κ, such that $J_{\xi+1} \setminus J_\xi$ is countably infinite for every $\xi < \kappa$, $J_\xi = \bigcup_{\eta < \xi} J_\eta$ for every limit ordinal $\eta < \kappa$, and $J_\kappa = I$. **P** Recall that I am already supposing that I is infinite. If I is countable, set $\kappa = 1$, $J_0 = \emptyset$, $J_1 = I$. Otherwise, set $\kappa = \#(I)$ and let $\langle i_\xi \rangle_{\xi < \kappa}$ be an enumeration of I. For $i \in I$, $\phi \in G$ let $K_{\phi i} \subseteq I$ be a countable set such that $F_{\phi i} \in \mathcal{B}_{K_{\phi i}}$. Choose the J_ξ inductively, as follows. The inductive hypothesis must include the requirement that $\#(J_\xi) \le \max(\omega, \#(\xi))$ for every ξ. Start by setting $J_0 = \emptyset$. Given $\xi < \kappa$ and $J_\xi \in \mathcal{J}$ with $\#(J_\xi) \le \max(\omega, \#(\xi)) < \kappa$, take an infinite set $L \subseteq \kappa \setminus J_\xi$ and set $J_{\xi+1} = J_\xi \cup \bigcup_{n \in \mathbb{N}} L_n$, where

$$L_0 = L \cup \{i_\xi\},$$

$$L_{n+1} = \bigcup_{i \in L_n, \phi \in G} K_{\phi i}$$

for $n \in \mathbb{N}$, so that every L_n is countable,

$$F_{\phi i} \in \mathcal{B}_{L_{n+1}} \text{ whenever } i \in L_n,\ \phi \in G$$

and $J_{\xi+1} \in \mathcal{J}$; since $L \subseteq J_{\xi+1} \setminus J_\xi \subseteq \bigcup_{n \in \mathbb{N}} L_n$, $J_{\xi+1} \setminus J_\xi$ is countably infinite, and

$$\#(J_{\xi+1}) = \max(\omega, \#(J_\xi)) \le \max(\omega, \#(\xi)) = \max(\omega, \#(\xi+1)).$$

For non-zero limit ordinals $\xi < \kappa$, set $J_\xi = \bigcup_{\eta < \xi} J_\eta$; then

$$\#(J_\xi) \le \max(\omega, \#(\xi), \sup_{\eta<\xi} \#(J_\eta)) \le \max(\omega, \#(\xi)).$$

Thus the induction proceeds. Observing that the construction puts i_ξ into $J_{\xi+1}$ for every ξ, we see that J_κ will be the whole of I, as required. **Q**

(d) Now put (b) and (c) together, as follows. Take $\langle J_\xi \rangle_{\xi \le \kappa}$ from (c). Set $f_{\phi 0}(x) = x$ for every $\phi \in G$, $x \in X$; then, because $J_0 = \emptyset$, $(J_0, \langle f_{\phi 0} \rangle_{\phi \in G})$ is consistent in the sense of (b). Given that $(J_\xi, \langle f_{\phi\xi} \rangle_{\phi \in G})$ is consistent, where $\xi < \kappa$, use the construction of (b) to find a family $\langle f_{\phi,\xi+1} \rangle_{\phi \in G}$ such that $(J_{\xi+1}, \langle f_{\phi,\xi+1} \rangle_{\phi \in G})$ is consistent and $f_{\phi,\xi+1}(x)(i) = f_{\phi\xi}(x)(i)$ for every $i \in J_\xi$ and $x \in X$. At a non-zero limit ordinal $\xi \le \kappa$, set

$$\begin{aligned} f_{\phi\xi}(x)(i) &= f_{\phi\eta}(x)(i) \text{ if } x \in X,\ \eta < \xi,\ i \in J_\eta, \\ &= x(i) \text{ if } i \in I \setminus J_\xi. \end{aligned}$$

(The inductive hypothesis includes the requirement that $f_{\phi\eta}(x) \restriction J_\zeta = f_{\phi\zeta}(x) \restriction J_\zeta$ whenever $\phi \in G$, $x \in X$ and $\zeta \le \eta < \xi$.) To see that $(J_\xi, \langle f_{\phi\xi} \rangle_{\phi \in G})$ is consistent, the only non-trivial point to check is that

$$f_{\phi,\xi} f_{\psi,\xi} = f_{\psi\phi,\xi}$$

for all ϕ, $\psi \in G$. But if $i \in J_\xi$ there is some $\eta < \xi$ such that $i \in J_\eta$, and in this case

$$f_{\psi,\xi}^{-1}[E_i] = f_{\psi,\eta}^{-1}[E_i] \in \mathcal{B}_{J_\eta}$$

is determined by coordinates in J_η, so that (because $f_{\phi,\xi}(x) \restriction J_\eta = f_{\phi,\eta}(x) \restriction J_\eta$ for every x)

$$f_{\phi,\xi}^{-1}[f_{\psi,\xi}^{-1}[E_i]] = f_{\phi,\eta}^{-1}[f_{\psi,\eta}^{-1}[E_i]] = f_{\psi\phi,\eta}^{-1}[E_i] = f_{\psi\phi,\xi}^{-1}[E_i];$$

while if $i \in I \setminus J_\xi$ then

$$f_{\psi\phi,\xi}^{-1}[E_i] = E_i = f_{\phi,\xi}^{-1}[E_i] = f_{\psi,\xi}^{-1}[E_i] = f_{\psi,\xi}^{-1}[f_{\phi,\xi}^{-1}[E_i]].$$

Thus

$$f_{\psi,\xi}^{-1}[f_{\phi,\xi}^{-1}[E_i]] = f_{\psi\phi,\xi}^{-1}[E_i]$$

for every i, and $f_{\phi,\xi} f_{\psi,\xi} = f_{\psi\phi,\xi}$.

On completing the induction, set $f_\phi = f_{\phi\kappa}$ for every $\phi \in G$; it is easy to see that $\langle f_\phi \rangle_{\phi \in G}$ satisfies the conditions of the theorem.

344F Corollary Let I be any set, and let μ be a σ-finite measure on $X = \{0,1\}^I$. Suppose that μ is the completion of its restriction to the σ-algebra $\mathcal{B}\mathfrak{a}_I$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I. Write $\mathfrak{A}$ for the measure algebra of μ. Let G be a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi \in G$, a function $f_\phi : X \to X$ representing ϕ, in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi \in G$ is invertible and $\phi^{-1} \in G$, we shall have $f_{\phi^{-1}} = f_\phi^{-1}$; so that if moreover ϕ is measure-preserving, f_ϕ will be an automorphism of the measure space (X, Σ, μ).

proof Apply 344E to $\mu \restriction \mathcal{B}\mathfrak{a}_I$; of course $\mathfrak{A}$ is canonically isomorphic to the measure algebra of $\mu \restriction \mathcal{B}\mathfrak{a}_I$ (322Da). The functions f_ϕ provided by 344E still represent the homomorphisms ϕ when re-interpreted as functions on the completed measure space $(\{0,1\}^I, \mu)$, by 343Ac.

344G Corollary Let I be any set, ν_I the usual measure on $\{0,1\}^I$, and $\mathfrak{B}_I$ its measure algebra. Then any measure-preserving automorphism of $\mathfrak{B}_I$ is representable by a measure space automorphism of $(\{0,1\}^I, \nu_I)$.

344H Lemma Let (X, Σ, μ) be a perfect semi-finite measure space. If $H \in \Sigma$ is a non-negligible set which includes no atom, there is a negligible subset of H of cardinal $\mathfrak{c}$.

proof (a) Consider first the case in which μ is atomless, compact and totally finite, and $H = X$. Let $\mathcal{K} \subseteq \mathcal{P}X$ be a compact class such that μ is inner regular with respect to $\mathcal{K}$. Set $S^* = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$, and choose $\langle K_\sigma\rangle_{\sigma\in S^*}$ inductively, as follows. $K_\emptyset$ is to be any non-negligible member of $\mathcal{K}\cap\Sigma$. Given that $\mu K_\sigma > 0$, where $\sigma \in \{0,1\}^n$, take F_σ, $F'_\sigma \subseteq K_\sigma$ to be disjoint non-negligible measurable sets both of measure at most 3^{-n}; such exist because μ is atomless (215C). Choose $K_{\sigma^\frown<0>} \subseteq F_\sigma$, $K_{\sigma^\frown<1>} \subseteq F'_\sigma$ to be non-negligible members of $\mathcal{K}\cap\Sigma$.

For each $w \in \{0,1\}^{\mathbb{N}}$, $\langle K_{w\restriction n}\rangle_{n\in\mathbb{N}}$ is a decreasing sequence of members of $\mathcal{K}$ all of non-zero measure, so has non-empty intersection; choose a point $x_w \in \bigcap_{n\in\mathbb{N}} K_{w\restriction n}$. Since $K_{\sigma^\frown<0>} \cap K_{\sigma^\frown<1>} = \emptyset$ for every $\sigma \in S^*$, all the x_w are distinct, and $A = \{x_w : w \in \{0,1\}^{\mathbb{N}}\}$ has cardinal $\mathfrak{c}$. Also

$$A \subseteq \bigcup_{\sigma\in\{0,1\}^n} K_\sigma$$

which has measure at most $2^n 3^{-(n-1)}$ for every $n \ge 1$, so $\mu^* A = 0$ and A is negligible.

(b) Now consider the case in which μ is atomless and totally finite and perfect, but not necessarily compact, while again $H = X$. In this case, by 215C, we can choose $\langle E_n\rangle_{n\in\mathbb{N}}$ inductively so that $\mu(E_n \cap E) = \frac{1}{2}\mu E$ whenever $n \in \mathbb{N}$ and E is an atom of the subalgebra of $\mathcal{P}X$ generated by $\{E_i : i < n\}$. Now define $f : X \to \{0,1\}^{\mathbb{N}}$ by setting $f(x) = \langle\chi E_n(x)\rangle_{n\in\mathbb{N}}$ for $x \in X$. Consider the image measure $\nu = \mu f^{-1}$ on $Y = f[X] \subseteq \{0,1\}^{\mathbb{N}}$. This is perfect. **P** If $g : Y \to \mathbb{R}$ is T-measurable, where $\mathrm{T} = \operatorname{dom}\nu$, and $\nu F > 0$, then $gf : X \to \mathbb{R}$ is Σ-measurable and $\mu f^{-1}[F] > 0$. There is therefore a compact set $K \subseteq gf[f^{-1}[F]]$ such that $\mu(gf)^{-1}[K] > 0$. In this case, $K \subseteq g[F]$ and $\nu g^{-1}[K] > 0$. **Q**

Next, for every $n \in \mathbb{N}$ and $\sigma \in \{0,1\}^n$,

$$\nu\{y : y \in Y,\, y\restriction n = \sigma\} = \mu\{x : \forall\, i < n,\, x \in E_i \iff \sigma(i) = 1\} = 2^{-n}\mu X.$$

So ν can have no atom of measure greater than $2^{-n}\mu X$; as n is arbitrary, ν is atomless. Thirdly, (Y, T, ν) is countably separated, because $\langle\{y : y \in Y,\, y(n) = 1\}\rangle_{n\in\mathbb{N}}$ is a sequence of measurable sets separating the points of Y. By 343K, ν is compact; by (a) here, there is a ν-negligible set $B \subseteq Y$ of cardinal $\mathfrak{c}$. Now $f^{-1}[B]$ is μ-negligible, and because $B \subseteq f[X]$, $\#(f^{-1}[B]) \ge \#(B) = \mathfrak{c}$. We therefore have a set $A \subseteq f^{-1}[B]$ of cardinal $\mathfrak{c}$, and A is μ-negligible.

(c) Finally, for the general case in which μ is just semi-finite and perfect, and H is a non-negligible subset of X not including an atom, let $E \subseteq H$ be a set of non-zero finite measure. Then the subspace measure μ_E is atomless. Also μ_E is perfect. **P** Let $f : E \to \mathbb{R}$ be a measurable function. Define $g : X \to \mathbb{R}$ by setting

$$\begin{aligned} g(x) &= e^{f(x)} \text{ if } x \in E,\\ &= 0 \text{ if } x \in X \setminus E. \end{aligned}$$

Then g is measurable. There is therefore a compact set $K \subseteq g[E]$ such that $\mu g^{-1}[K] > 0$. Now $\ln[K] \subseteq f[E]$ is compact and $\mu_E f^{-1}[\ln[K]] = \mu g^{-1}[K] > 0$. **Q**

By (b), there is a μ_E-negligible set $A \subseteq E$ of cardinal $\mathfrak{c}$, and of course A is also a μ-negligible subset of H.

Remark I see that in this proof I have slipped into a notation which is a touch more sophisticated than what I have used so far. See 3A1H for a note on the interpretations of '$\{0,1\}^n$', '$\{0,1\}^{\mathbb{N}}$' which make sense of the formulae here.

344I Theorem Let (X, Σ, μ) and (Y, T, ν) be atomless, perfect, complete, strictly localizable, countably separated measure spaces of the same non-zero magnitude. Then they are isomorphic.

proof (a) The point is that the measure algebra $(\mathfrak{A}, \bar\mu)$ of μ has Maharam type ω. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ separating the points of X. Let Σ_0 be the σ-subalgebra of Σ generated by $\{E_n : n \in \mathbb{N}\}$, and $\mathfrak{A}_0$ the order-closed subalgebra of $\mathfrak{A}$ generated by $\{E_n^\bullet : n \in \mathbb{N}\}$; then $E^\bullet \in \mathfrak{A}_0$ for every $E \in \Sigma_0$, and $(X, \Sigma_0, \mu\restriction\Sigma_0)$ is countably separated. Let $f : X \to \mathbb{R}$ be Σ_0-measurable and injective (343E). Of course f is also Σ-measurable. If $a \in \mathfrak{A}\setminus\{0\}$, express a as $E^\bullet$ where $E \in \Sigma$. Because (X, Σ, μ) is perfect, there is a compact $K \subseteq \mathbb{R}$ such that $K \subseteq f[E]$ and $\mu f^{-1}[K] > 0$. K is surely a Borel set, so $f^{-1}[K] \in \Sigma_0$ and

$$b = f^{-1}[K]^\bullet \in \mathfrak{A}_0 \setminus \{0\}.$$

But because f is injective, we also have $f^{-1}[K] \subseteq E$ and $b \subseteq a$. As a is arbitrary, $\mathfrak{A}_0$ is order-dense in $\mathfrak{A}$; but $\mathfrak{A}_0$ is order-closed, so must be the whole of $\mathfrak{A}$. Thus $\mathfrak{A}$ is τ-generated by the countable set $\{E_n^\bullet : n \in \mathbb{N}\}$, and $\tau(\mathfrak{A}) \le \omega$. **Q**

On the other hand, because $\mathfrak{A}$ is atomless, and not $\{0\}$, none of its principal ideals can have finite Maharam type, and it is Maharam-type-homogeneous, with type ω.

(b) Writing $(\mathfrak{B},\bar{\nu})$ for the measure algebra of ν, we see that the argument of (a) applies equally to $(\mathfrak{B},\bar{\nu})$, so that $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ are atomless localizable measure algebras, with Maharam type ω and the same magnitude. Consequently they are isomorphic as measure algebras, by 332J. Let $\phi:\mathfrak{A}\to\mathfrak{B}$ be a measure-preserving isomorphism.

By 343K, both μ and ν are (locally) compact. As they are also complete and strictly localizable, 343B tells us that there are functions $g:Y\to X$ and $f:X\to Y$ representing ϕ and ϕ^{-1}. Now $fg:Y\to Y$ and $gf:X\to X$ represent the identity automorphisms on $\mathfrak{B}$, $\mathfrak{A}$, so by 343F are equal almost everywhere to the identity functions on Y, X respectively. Set

$$E=\{x:x\in X,\,gf(x)=x\},\quad F=\{y:y\in Y,\,fg(y)=y\};$$

then both E and F are conegligible. Of course $f[E]\subseteq F$ (since $fgf(x)=f(x)$ for every $x\in E$), and similarly $g[F]\subseteq E$; consequently $f\restriction E$, $g\restriction F$ are the two halves of a one-to-one correspondence between E and F. Because ϕ is measure-preserving, $\mu f^{-1}[H]=\nu H$ and $\nu g^{-1}[G]=\mu G$ for every $G\in\Sigma$, $H\in\mathrm{T}$; accordingly $f\restriction E$ is an isomorphism between the subspace measures on E and F.

(c) By 344H, there is a negligible set $A\subseteq E$ of cardinal $\mathfrak{c}$. Now X and Y, being countably separated, both have cardinal at most $\mathfrak{c}$. (There are injective functions from X and Y to $\mathbb{R}$.) Set

$$B=A\cup(X\setminus E),\quad C=f[A]\cup(Y\setminus F).$$

Then B and C are negligible subsets of X, Y respectively, and both have cardinal $\mathfrak{c}$ precisely, so there is a bijection $h:B\to C$. Set

$$\begin{aligned} f_1(x)&=f(x)\text{ if }x\in X\setminus B=E\setminus A,\\ &=h(x)\text{ if }x\in B. \end{aligned}$$

Then, because μ and ν are complete, f_1 is an isomorphism between the measure spaces (X,Σ,μ) and (Y,T,ν), as required.

344J Corollary Suppose that E, F are two Lebesgue measurable subsets of $\mathbb{R}^r$ of the same non-zero measure. Then the subspace measures on E and F are isomorphic.

344K Corollary (a) A measure space is isomorphic to Lebesgue measure on $[0,1]$ iff it is an atomless countably separated compact (or perfect) complete probability space; in this case it is also isomorphic to the usual measure on $\{0,1\}^{\mathbb{N}}$.

(b) A measure space is isomorphic to Lebesgue measure on $\mathbb{R}$ iff it is an atomless countably separated compact (or perfect) σ-finite measure space which is not totally finite; in this case it is also isomorphic to Lebesgue measure on any Euclidean space $\mathbb{R}^r$.

(c) Let μ be Lebesgue measure on $\mathbb{R}$. If $0<\mu E<\infty$ and we set $\nu F=\dfrac{1}{\mu E}\mu F$ for every measurable $F\subseteq E$, then (E,ν) is isomorphic to Lebesgue measure on $[0,1]$.

344L The homogeneity property of Lebesgue measure described in 344J is repeated in $\{0,1\}^I$ for any infinite I.

Theorem Let I be an infinite set, and ν_I the usual measure on $\{0,1\}^I$. If $E\subseteq\{0,1\}^I$ is a measurable set of non-zero measure, the subspace measure on E is isomorphic to $(\nu_I E)\nu_I$.

proof For $J\subseteq I$ let ν_J be the usual measure on $X_J=\{0,1\}^J$.

(a) If I is countably infinite, then the subspace measure on E is perfect and complete and countably separated, so is isomorphic to Lebesgue measure on the interval $[0,\nu_I E]$, by 344I. But by 344Kc, or otherwise, this is isomorphic, up to a scalar multiple of the measure, to Lebesgue measure on $[0,1]$, which is in turn isomorphic to ν_I.

So henceforth we can suppose that I is uncountable.

(b) By 254Oc there are a countable set $J\subseteq I$ and a set $E'\subseteq E$, determined by coordinates in J, such that $E\setminus E'$ is negligible. Identifying X_I with $X_J\times X_{I\setminus J}$ (254N), we can think of E' as $V\times X_{I\setminus J}$ where V is measured by ν_J (see 254O). Take $v_0\in V$ and set

$$V'=V\setminus\{v_0\},\quad W'=X_J\setminus\{v_0\},\quad E''=V'\times X_{I\setminus J},\quad F''=W'\times X_{I\setminus J}.$$

Then by (a), applied to V' and W' in turn, we have a bijection $g:V'\to W'$ which, up to a scalar multiple of the measure, is an isomorphism between the subspace measures. Now the subspace measure on $V'\times X_{I\setminus J}$ is just the product of the subspace measure on V' with $\nu_{I\setminus J}$ (251Q(ii)), so if we set $f_0(x,z) = (g(x),z)$ for $x \in V'$ and $z \in X_{I\setminus J}$, then $f_0 : E'' \to F''$ is an isomorphism of the subspace measures on E'' and F'', up to a scalar multiple of the measures as always. On the other hand, $E \setminus E''$ and $X_I \setminus F''$ are negligible and both have cardinal $\#(X_{I\setminus J}) = \#(X_I)$, so we have a bijection $f_1 : E \setminus E'' \to X_I \setminus F''$. Putting f_0 and f_1 together, we have a bijection $f : E \to X_I$ which, up to a scalar multiple of the measure, is an isomorphism of the subspace measure on E with ν_I.

344X Basic exercises (a) Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and suppose that there are $E \in \Sigma$, $F \in \mathrm{T}$ such that (X,Σ,μ) is isomorphic to the subspace (F,T_F,ν_F), while (Y,T,ν) is isomorphic to (E,Σ_E,μ_E). Show that (X,Σ,μ) and (Y,T,ν) are isomorphic.

(b) Let (X,Σ,μ) and (Y,T,ν) be perfect countably separated complete strictly localizable measure spaces with isomorphic measure algebras. Show that there are conegligible subsets $X' \subseteq X$, $Y' \subseteq Y$ such that X' and Y', with the subspace measures, are isomorphic.

(c) Let (X,Σ,μ) and (Y,T,ν) be perfect countably separated complete strictly localizable measure spaces with isomorphic measure algebras. Suppose that they are not purely atomic. Show that they are isomorphic.

(d) Give an example of two perfect countably separated complete probability spaces, with isomorphic measure algebras, which are not isomorphic.

(e) Let (Z,Σ,μ) be the Stone space of a homogeneous measure algebra. Show that if E, $F \in \Sigma$ have the same non-zero finite measure, then the subspace measures on E and F are isomorphic.

(f) Let $(I^{\|},\Sigma,\mu)$ be the split interval with its usual measure (343J), and $\mathfrak{A}$ its measure algebra. (i) Show that every measure-preserving automorphism of $\mathfrak{A}$ is represented by a measure space automorphism of $I^{\|}$. (ii) Show that if E, $F \in \Sigma$ and $\mu E = \mu F > 0$ then the subspace measures on E and F are isomorphic.

344Y Further exercises (a) Let X be a set, Σ a σ-algebra of subsets of X, $\mathcal{I}$ a σ-ideal of Σ, and $\mathfrak{A}$ the quotient $\Sigma/\mathcal{I}$. Suppose that there is a countable set $\mathcal{A} \subseteq \Sigma$ separating the points of X. Let G be a countable semigroup of Boolean homomorphisms from $\mathfrak{A}$ to itself such that every member of G can be represented by some function from X to itself. Show that a family $\langle f_\phi \rangle_{\phi\in G}$ of such representatives can be chosen in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X.

(b) Let $\mathfrak{A}$, $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras. Suppose that each is isomorphic to a principal ideal of the other. Show that they are isomorphic.

(c) Let I be an infinite set, and write $\mathcal{B}\mathfrak{a}_I$ for the σ-algebra of subsets of $X = \{0,1\}^I$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I. Let μ and ν be σ-finite measures on X, both with domain $\mathcal{B}\mathfrak{a}_I$, and with measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$. Show that any Boolean isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ is represented by a permutation $f : X \to X$ such that f^{-1} represents $\phi^{-1} : \mathfrak{B} \to \mathfrak{A}$, and hence that $(\mathfrak{A},\bar\mu)$ is isomorphic to $(\mathfrak{B},\bar\nu)$ iff $(X,\mathcal{B}\mathfrak{a}_I,\mu)$ is isomorphic to $(X,\mathcal{B}\mathfrak{a}_I,\nu)$.

(d) Let I be any set, and write $\mathcal{B}\mathfrak{a}_I$ for the σ-algebra of subsets of $X = \{0,1\}^I$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I. Let $\mathcal{I}$ be an ω_1-saturated σ-ideal of $\mathcal{B}\mathfrak{a}_I$, and write $\mathfrak{A}$ for the quotient Boolean algebra $\mathcal{B}/\mathcal{I}$. Let G be a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Show that we can choose simultaneously, for each $\phi \in G$, a function $f_\phi : X \to X$ representing ϕ, in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi \in G$ is invertible and $\phi^{-1} \in G$, f_ϕ will be an automorphism of the structure $(X,\mathcal{B}\mathfrak{a}_I,\mathcal{I})$.

(e) Let I be any set, and write $\mathcal{B}\mathfrak{a}_I$ for the σ-algebra of subsets of $X = \{0,1\}^I$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I. Let $\mathcal{I}$, $\mathcal{J}$ be ω_1-saturated σ-ideals of $\mathcal{B}\mathfrak{a}_I$. Show that if the Boolean algebras $\mathcal{B}\mathfrak{a}_I/\mathcal{I}$ and $\mathcal{B}\mathfrak{a}_I/\mathcal{J}$ are isomorphic, so are the structures $(X,\mathcal{B}\mathfrak{a}_I,\mathcal{I})$ and $(X,\mathcal{B}\mathfrak{a}_I,\mathcal{J})$.

344 Notes and comments In this section and the last, I have allowed myself to drift some distance from the avowed subject of this chapter; but it seemed a suitable place for this material, which is fundamental to abstract measure theory. We find that the concepts of §§342-343 are just what is needed to characterise Lebesgue measure (344K), and the characterization shows that among non-negligible measurable subspaces of $\mathbb{R}^r$ the isomorphism classes are determined by a single parameter, the measure of the subspace. Of course a very large number of other spaces – indeed, most of those appearing in ordinary applications of measure theory to other topics – are perfect and countably separated (for example, those of 342Xe and 343Ye), and therefore covered by this classification. I note that it includes, as a special case, the isomorphism between Lebesgue measure on $[0,1]$ and the usual measure on $\{0,1\}^{\mathbb{N}}$ already described in 254K.

In 344I, the first part of the proof is devoted to showing that a perfect countably separated measure space has countable Maharam type; I ought perhaps to note here that we must resist the temptation to suppose that all countably separated measure spaces have countable Maharam type. In fact there are countably separated probability spaces with Maharam type as high as $2^{\mathfrak{c}}$. The arguments are elementary but seem to fit better into §521 of Volume 5 than here.

I have offered three contexts in which automorphisms of measure algebras are represented by automorphisms of measure spaces (344A, 344C, 344E). In the first case, every automorphism can be represented simultaneously in a consistent way. In the other two cases, there is, I am sure, no such consistent family of representations which can be constructed within ZFC; but the theorems I give offer consistent simultaneous representations of countably many homomorphisms. The question arises, whether 'countably many' is the true natural limit of the arguments. In fact it is possible to extend both results to families of at most ω_1 automorphisms.

Having successfully characterized Lebesgue measure – or, what is very nearly the same thing, the usual measure on $\{0,1\}^{\mathbb{N}}$ – it is natural to seek similar characterizations of the usual measures on $\{0,1\}^{\kappa}$ for uncountable cardinals κ. This seems to be hard. A variety of examples (some touched on in the exercises to §521) show that none of the most natural conjectures can be provable in ZFC.

In fact the principal new ideas of this section do not belong specifically to measure theory; rather, they belong to the general theory of σ-algebras and σ-ideals of sets. In the case of the Schröder-Bernstein-type theorem 344D, this is obvious from the formulation I give. (See also 344Yb.) In the case of 344B and 344E, I offer generalizations in 344Ya-344Ye. Of course the applications of 344B here, in 344C and its corollaries, depend on Maharam's theorem and the concept of 'compact' measure space. The former has no generalization to the wider context, and the value of the latter is based on the equivalences in Theorem 343B, which also do not have simple generalizations.

The property described in 344J – a measure space (X,Σ,μ) in which any two measurable subsets of the same non-zero measure are isomorphic – seems to be a natural concept of 'homogeneity' for measure spaces; it seems unreasonable to ask for all sets of zero measure to be isomorphic, since finite sets of different cardinalities can be expected to be of zero measure. An extra property, shared by Lebesgue measure and the usual measure on $\{0,1\}^I$ and by the measure on the split interval (344Kc, 344L, 344Xf) but not by counting measure, would be the requirement that measurable sets of different non-zero finite measures should be isomorphic up to a scalar multiple of the measure. All these examples have the further property, that all automorphisms of their measure algebras correspond to automorphisms of the measure spaces.

345 Translation-invariant liftings

In this section and the next I complement the work of §341 by describing some important special properties which can, in appropriate circumstances, be engineered into our liftings. I begin with some remarks on translation-invariance. I restrict my attention to measure spaces which we have already seen, delaying a general discussion of translation-invariant measures on groups until Volume 4.

345A Translation-invariant liftings I shall consider two forms of translation-invariance, as follows.

(a) Let μ be Lebesgue measure on $\mathbb{R}^r$, and Σ its domain. A lifting $\phi:\Sigma\to\Sigma$ is **translation-invariant** if $\phi(E+x)=\phi E+x$ for every $E\in\Sigma$, $x\in\mathbb{R}^r$. (Recall from 134A that $E+x=\{y+x:y\in E\}$ belongs to Σ for every $E\in\Sigma$, $x\in\mathbb{R}^r$.)

Similarly, writing $\mathfrak{A}$ for the measure algebra of μ, a lifting $\theta:\mathfrak{A}\to\Sigma$ is **translation-invariant** if $\theta(E+x)^\bullet=\theta E^\bullet+x$ for every $E\in\Sigma$, $x\in\mathbb{R}^r$.

It is easy to see that if θ and ϕ correspond to each other in the manner of 341B, then one is translation-invariant if and only if the other is.

(b) Now let I be any set, and let ν_I be the usual measure on $X=\{0,1\}^I$, with T_I its domain and $\mathfrak{B}_I$ its measure algebra. For x, $y\in X$, define $x+y\in X$ by setting $(x+y)(i)=x(i)+_2 y(i)$ for every $i\in I$; that is, give X the group structure of the product group $\mathbb{Z}_2^I$. This makes X an abelian group (isomorphic to the additive group $(\mathcal{P}I,\triangle)$ of the Boolean algebra $\mathcal{P}I$, if we match $x\in X$ with $\{i:x(i)=1\}\subseteq I$).

Recall that the measure ν_I is a product measure (254J), being the product of copies of the fair-coin probability measure on the two-element set $\{0,1\}$. If $x\in X$, then for each $i\in I$ the map $\epsilon\mapsto\epsilon+_2 x(i):\{0,1\}\to\{0,1\}$ is a measure space automorphism of $\{0,1\}$, since the two singleton sets $\{0\}$ and $\{1\}$ have the same measure $\frac{1}{2}$. It follows at once that the map $y\mapsto y+x:X\to X$ is a measure space automorphism.

Accordingly we can again say that a lifting $\theta:\mathfrak{B}_I\to\mathrm{T}_I$, or $\phi:\mathrm{T}_I\to\mathrm{T}_I$, is **translation-invariant** if

$$\theta(E+x)^\bullet=\theta E^\bullet+x,\quad \phi(E+x)=\phi E+x$$

whenever $E\in\Sigma$ and $x\in X$.

345B Theorem For any $r\ge 1$, there is a translation-invariant lifting for Lebesgue measure on $\mathbb{R}^r$.

proof (a) Write μ for Lebesgue measure on $\mathbb{R}^r$, Σ for its domain. Let $\underline{\phi}:\Sigma\to\Sigma$ be lower Lebesgue density (341E). Then $\underline{\phi}$ is translation-invariant in the sense that $\underline{\phi}(E+x)=\underline{\phi}E+x$ for every $E\in\Sigma$, $x\in\mathbb{R}^r$. **P**

$$\begin{aligned}\underline{\phi}(E+x)&=\{y:y\in\mathbb{R}^r,\ \lim_{\delta\downarrow 0}\tfrac{\mu((E+x)\cap B(y,\delta))}{\mu(B(y,\delta))}=1\}\\&=\{y:y\in\mathbb{R}^r,\ \lim_{\delta\downarrow 0}\tfrac{\mu(E\cap B(y-x,\delta))}{\mu(B(y-x,\delta))}=1\}\end{aligned}$$

(because μ is translation-invariant)

$$\begin{aligned}&=\{y+x:y\in\mathbb{R}^r,\ \lim_{\delta\downarrow 0}\tfrac{\mu(E\cap B(y,\delta))}{\mu(B(y,\delta))}=1\}\\&=\underline{\phi}E+x.\ \mathbf{Q}\end{aligned}$$

(b) Let ϕ_0 be any lifting for μ such that $\phi_0E\supseteq\underline{\phi}E$ for every $E\in\Sigma$ (341Jb). Consider

$$\phi E=\{y:\mathbf{0}\in\phi_0(E-y)\}$$

for $E\in\Sigma$. It is easy to check that $\phi:\Sigma\to\Sigma$ is a Boolean homomorphism because ϕ_0 is, so that, for instance,

$$\begin{aligned}y\in\phi E\triangle\phi F&\iff\mathbf{0}\in\phi_0(E-y)\triangle\phi_0(F-y)\\&\iff\mathbf{0}\in\phi_0((E-y)\triangle(F-y))=\phi_0((E\triangle F)-y)\\&\iff y\in\phi(E\triangle F).\end{aligned}$$

(c) If $\mu E=0$, then $E-y$ is negligible for every $y\in\mathbb{R}^r$, so $\phi_0(E-y)$ is always empty and $\phi E=\emptyset$.

(d) Next, $\underline{\phi}E\subseteq\phi E$ for every $E\in\Sigma$. **P** If $y\in\underline{\phi}E$, then

$$\mathbf{0}=y-y\in\underline{\phi}E-y=\underline{\phi}(E-y)\subseteq\phi_0(E-y),$$

so $y\in\phi E$. **Q** By 341Ib, ϕ is a lifting for μ.

(e) Finally, ϕ is translation-invariant, because if $E\in\Sigma$ and x, $y\in\mathbb{R}^r$ then

$$\begin{aligned}y\in\phi(E+x)&\iff\mathbf{0}\in\phi_0(E+x-y)=\phi_0(E-(y-x))\\&\iff y-x\in\phi E\\&\iff y\in\phi E+x.\end{aligned}$$

345C Theorem For any set I, there is a translation-invariant lifting for the usual measure on $\{0,1\}^I$.

proof I base the argument on the same programme as in 345B. This time we have to work rather harder, as we have no simple formula for a translation-invariant lower density. However, the ideas already used in 341F-341H are in fact adequate, if we take care, to produce one.

(a) Since there is certainly a bijection between I and its cardinal $\kappa=\#(I)$, it is enough to consider the case $I=\kappa$. Write ν_κ for the usual measure on $X=\{0,1\}^I=\{0,1\}^\kappa$ and T_κ for its domain. For each $\xi<\kappa$ set $E_\xi=\{x:x\in X,\ x(\xi)=1\}$, and let Σ_ξ be the σ-algebra generated by $\{E_\eta:\eta<\xi\}$. Because $x+E_\eta$ is either E_η

or $X \setminus E_\eta$, and in either case belongs to Σ_ξ, for every $\eta < \xi$ and $x \in X$, Σ_ξ is translation-invariant. (Consider the algebra

$$\Sigma'_\xi = \{E : E + x \in \Sigma_\xi \text{ for every } x \in X\};$$

this must be Σ_ξ.) Let Φ_ξ be the set of partial lower densities $\underline{\phi} : \Sigma_\xi \to \mathrm{T}_\kappa$ which are translation-invariant in the sense that $\underline{\phi}(E + x) = \underline{\phi}E + x$ for any $E \in \Sigma_\xi$, $x \in X$.

(b)(i) For $\xi < \kappa$, $\Sigma_{\xi+1}$ is just the algebra of subsets of X generated by $\Sigma_\xi \cup \{E_\xi\}$, that is, sets of the form $(F \cap E_\xi) \cup (G \setminus E_\xi)$ where F, $G \in \Sigma_\xi$ (312N). Moreover, the expression is unique. **P** Define $x_\xi \in X$ by setting $x_\xi(\xi) = 1$, $x_\xi(\eta) = 0$ if $\eta \neq \xi$. Then $x_\xi + E_\eta = E_\eta$ for every $\eta < \xi$, so $x_\xi + F = F$ for every $F \in \Sigma_\xi$. If $H = (F \cap E_\xi) \cup (G \setminus E_\xi)$ where F, $G \in \Sigma_\xi$, then

$$x_\xi + H = ((x_\xi + F) \cap (x_\xi + E_\xi)) \cup ((x_\xi + G) \setminus (x_\xi + E_\xi)) = (F \setminus E_\xi) \cup (G \cap E_\xi),$$

so

$$F = (H \cap E_\xi) \cup ((x_\xi + H) \setminus E_\xi) = F_H,$$

$$G = (H \setminus E_\xi) \cup ((x_\xi + H) \cap E_\xi) = G_H$$

are determined by H. **Q**

(ii) The functions $H \mapsto F_H$, $H \mapsto G_H : \Sigma_{\xi+1} \to \Sigma_\xi$ defined above are clearly Boolean homomorphisms; moreover, if H, $H' \in \Sigma_{\xi+1}$ and $H \triangle H'$ is negligible, then

$$(F_H \triangle F_{H'}) \cup (G_H \triangle G_{H'}) \subseteq (H \triangle H') \cup (x_\xi + (H \triangle H'))$$

is negligible. It follows at once that if $\xi < \kappa$ and $\underline{\phi} \in \Phi_\xi$, we can define $\underline{\phi}_1 : \Sigma_{\xi+1} \to \mathrm{T}_\kappa$ by setting

$$\underline{\phi}_1 H = (\underline{\phi} F_H \cap E_\xi) \cup (\underline{\phi} G_H \setminus E_\xi),$$

and $\underline{\phi}_1$ will be a lower density. If $H \in \Sigma_\xi$ then $F_H = G_H = H$, so $\underline{\phi}_1 H = \underline{\phi} H$. Generally, if H, $H' \in \Sigma_\xi$ then

$$\underline{\phi}_1((H \cap E_\xi) \cup (H' \setminus E_\xi)) = (\underline{\phi} F_H \cap E_\xi) \cup (\underline{\phi} G_{H'} \setminus E_\xi) = (\underline{\phi} H \cap E_\xi) \cup (\underline{\phi} H' \setminus E_\xi).$$

(iii) To see that $\underline{\phi}_1$ is translation-invariant, observe that if $x \in X$ and $x(\xi) = 0$ then $x + E_\xi = E_\xi$, so, for any F, $G \in \Sigma_\xi$,

$$\begin{aligned}
\underline{\phi}_1(x + ((F \cap E_\xi) \cup (G \setminus E_\xi))) &= \underline{\phi}_1(((F + x) \cap E_\xi) \cup ((G + x) \setminus E_\xi)) \\
&= (\underline{\phi}(F + x) \cap E_\xi) \cup (\underline{\phi}(G + x) \setminus E_\xi) \\
&= ((\underline{\phi} F + x) \cap E_\xi) \cup ((\underline{\phi} G + x) \setminus E_\xi) \\
&= x + (\underline{\phi} F \cap E_\xi) \cup (\underline{\phi} G \setminus E_\xi) \\
&= x + \underline{\phi}_1((F \cap E_\xi) \cup (G \setminus E_\xi)).
\end{aligned}$$

While if $x(\xi) = 1$ then $x + E_\xi = X \setminus E_\xi$, so

$$\begin{aligned}
\underline{\phi}_1(x + ((F \cap E_\xi) \cup (G \setminus E_\xi))) &= \underline{\phi}_1(((F + x) \setminus E_\xi) \cup ((G + x) \cap E_\xi)) \\
&= (\underline{\phi}(F + x) \setminus E_\xi) \cup (\underline{\phi}(G + x) \cap E_\xi) \\
&= ((\underline{\phi} F + x) \setminus E_\xi) \cup ((\underline{\phi} G + x) \cap E_\xi) \\
&= x + (\underline{\phi} F \cap E_\xi) \cup (\underline{\phi} G \setminus E_\xi) \\
&= x + \underline{\phi}_1((F \cap E_\xi) \cup (G \setminus E_\xi)).
\end{aligned}$$

So $\underline{\phi}_1 \in \Phi_{\xi+1}$.

(iv) Thus every member of Φ_ξ has an extension to a member of $\Phi_{\xi+1}$.

(c) Now suppose that $\langle \zeta(n) \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in κ with supremum $\xi < \kappa$. Then Σ_ξ is just the σ-algebra generated by $\bigcup_{n \in \mathbb{N}} \Sigma_{\zeta(n)}$. If we have a sequence $\langle \underline{\phi}_n \rangle_{n \in \mathbb{N}}$ such that $\underline{\phi}_n \in \Phi_{\zeta(n)}$ and $\underline{\phi}_{n+1}$ extends $\underline{\phi}_n$ for every n, then there is a $\underline{\phi} \in \Phi_\xi$ extending every $\underline{\phi}_n$. **P** I repeat the ideas of 341G.

(i) For $E \in \Sigma_\xi$, $n \in \mathbb{N}$ choose g_{En} such that g_{En} is a conditional expectation of χE on $\Sigma_{\zeta(n)}$; that is,

$$\int_F g_{En} = \int_F \chi E = \nu_\kappa(F\cap E)$$

for every $E\in\Sigma_{\zeta(n)}$. Moreover, make these choices in such a way that (α) every g_{En} is $\Sigma_{\zeta(n)}$-measurable and defined everywhere on X (β) $g_{En}=g_{E'n}$ for every n if $E\triangle E'$ is negligible. Now $\lim_{n\to\infty}g_{En}$ exists and is equal to χE almost everywhere, by Lévy's martingale theorem (275I).

(ii) For $E\in\Sigma_\xi$, $k\ge 1$, $n\in\mathbb{N}$ set

$$H_{kn}(E)=\{x:x\in X,\,g_{En}(x)\ge 1-2^{-k}\}\in\Sigma_{\zeta(n)},\qquad \tilde H_{kn}(E)=\underline{\phi}_n(H_{kn}(E)),$$

$$\underline{\phi}E=\bigcap_{k\ge 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde H_{km}(E).$$

(iii) Every $g_{\emptyset n}$ is zero almost everywhere, every $H_{kn}(\emptyset)$ is negligible and every $\tilde H_{kn}(\emptyset)$ is empty; so $\underline{\phi}\emptyset=\emptyset$. If E, $E'\in\Sigma_\xi$ and $E\triangle E'$ is negligible, $g_{En}=g_{E'n}$ for every n, $H_{nk}(E)=H_{nk}(E')$ and $\tilde H_{nk}(E)=\tilde H_{nk}(E')$ for all n, k, and $\underline{\phi}E=\underline{\phi}E'$.

(iv) If $E\subseteq F$ in Σ_ξ, then $g_{En}\le g_{Fn}$ almost everywhere for every n, every $H_{kn}(E)\setminus H_{kn}(F)$ is negligible, $\tilde H_{kn}(E)\subseteq\tilde H_{kn}(F)$ for every n, k, and $\underline{\phi}E\subseteq\underline{\phi}F$.

(v) If E, $F\in\Sigma_\xi$ then $\chi(E\cap F)\ge_{\text{a.e.}}\chi E+\chi F-1$ so $g_{E\cap F,n}\ge_{\text{a.e.}}g_{En}+g_{Fn}-1$ for every n. Accordingly

$$H_{k+1,n}(E)\cap H_{k+1,n}(F)\setminus H_{kn}(E\cap F)$$

is negligible, and (because $\underline{\phi}_n$ is a lower density)

$$\tilde H_{kn}(E\cap F)\supseteq\underline{\phi}_n(H_{k+1,n}(E)\cap H_{k+1,n}(F))=\tilde H_{k+1,n}(E)\cap\tilde H_{k+1,n}(F)$$

for all $k\ge 1$, $n\in\mathbb{N}$. Now, if $x\in\underline{\phi}E\cap\underline{\phi}F$, then, for any $k\ge 1$, there are n_1, $n_2\in\mathbb{N}$ such that

$$x\in\bigcap_{m\ge n_1}\tilde H_{k+1,m}(E),\qquad x\in\bigcap_{m\ge n_2}\tilde H_{k+1,m}(F).$$

But this means that

$$x\in\bigcap_{m\ge\max(n_1,n_2)}\tilde H_{km}(E\cap F).$$

As k is arbitrary, $x\in\underline{\phi}(E\cap F)$; as x is arbitrary, $\underline{\phi}E\cap\underline{\phi}F\subseteq\underline{\phi}(E\cap F)$. We know already from (iv) that $\underline{\phi}(E\cap F)\subseteq\underline{\phi}E\cap\underline{\phi}F$, so $\underline{\phi}(E\cap F)=\underline{\phi}E\cap\underline{\phi}F$.

(vi) If $E\in\Sigma_\xi$, then $g_{En}\to\chi E$ a.e., so setting

$$V=\bigcap_{k\ge 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}H_{km}(E)=\{x:\limsup_{n\to\infty}g_{En}(x)\ge 1\},$$

$V\triangle E$ is negligible; but

$$\underline{\phi}E\triangle V\subseteq\bigcup_{k\ge 1,n\in\mathbb{N}}H_{kn}(E)\triangle\tilde H_{kn}(E)$$

is also negligible, so $\underline{\phi}E\triangle E$ is negligible. Thus $\underline{\phi}$ is a partial lower density with domain Σ_ξ.

(vii) If $E\in\Sigma_{\zeta(n)}$, then $E\in\Sigma_{\zeta(m)}$ for every $m\ge n$, so $g_{Em}=_{\text{a.e.}}\chi E$ for every $m\ge n$; $H_{km}(E)\triangle E$ is negligible for $k\ge 1$, $m\ge n$;

$$\tilde H_{km}(E)=\underline{\phi}_mE=\underline{\phi}_nE$$

for $k\ge 1$, $m\ge n$; and $\underline{\phi}E=\underline{\phi}_nE$. Thus $\underline{\phi}$ extends every $\underline{\phi}_n$.

(viii) I have still to check the translation-invariance of $\underline{\phi}$. If $E\in\Sigma_\xi$ and $x\in X$, consider g'_n, defined by setting

$$g'_n(y)=g_{En}(y-x)$$

for every $y\in X$, $n\in\mathbb{N}$; that is, g'_n is the composition $g_{En}\psi$, where $\psi(y)=y-x$ for $y\in X$. (I am not sure whether it is more, or less, confusing to distinguish between the operations of addition and subtraction in X. Of course $y-x=y+(-x)=y+x$ for every y.) Because ψ is a measure space automorphism, and in particular is inverse-measure-preserving, we have

$$\int_{F+x}g'_n=\int_{\psi^{-1}[F]}g'_n=\int_F g_{En}=\nu_\kappa(E\cap F)$$

whenever $F\in\Sigma_{\zeta(n)}$ (235Gc[3]). But because $\Sigma_{\zeta(n)}$ is itself translation-invariant, we can apply this to $F-x$ to get

[3]Formerly 235I.

$$\int_F g'_n = \nu_\kappa(E \cap (F - x)) = \nu_\kappa((E + x) \cap F)$$

for every $F \in \Sigma_{\zeta(n)}$. Moreover, for any $\alpha \in \mathbb{R}$,

$$\{y : g'_n(y) \geq \alpha\} = \{y : g_{En}(y) \geq \alpha\} + x \in \Sigma_{\zeta(n)}$$

for every α, and g'_n is $\Sigma_{\zeta(n)}$-measurable. So g'_n is a conditional expectation of $\chi(E + x)$ on $\Sigma_{\zeta(n)}$, and must be equal almost everywhere to $g_{E+x,n}$.

This means that if we set

$$H'_{kn} = \{y : g'_n(y) \geq 1 - 2^{-k}\} = H_{kn}(E) + x$$

for k, $n \in \mathbb{N}$, we shall have $H'_{kn} \in \Sigma_{\zeta(n)}$ and $H'_{kn} \triangle H_{kn}(E + x)$ will be negligible, so

$$\begin{aligned}\tilde{H}_{kn}(E + x) &= \underline{\phi}_n(H_{kn}(E + x)) = \underline{\phi}_n(H'_{kn}) \\ &= \underline{\phi}_n(H_{kn}(E) + x) = \underline{\phi}_n(H_{kn}(E)) + x = \tilde{H}_{kn}(E) + x.\end{aligned}$$

Consequently

$$\begin{aligned}\underline{\phi}(E + x) &= \bigcap_{k\geq 1} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \tilde{H}_{kn}(E + x) \\ &= \bigcap_{k\geq 1} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \tilde{H}_{kn}(E) + x = \underline{\phi}E + x.\end{aligned}$$

As E and x are arbitrary, $\underline{\phi}$ is translation-invariant and belongs to Φ_ξ. **Q**

(d) We are now ready for the proof that there is a translation-invariant lower density for ν_κ. **P** Build inductively a family $\langle\underline{\phi}_\xi\rangle_{\xi\leq\kappa}$ such that (α) $\underline{\phi}_\xi \in \Phi_\xi$ for each ξ (β) $\underline{\phi}_\xi$ extends $\underline{\phi}_\eta$ whenever $\eta \leq \xi \leq \kappa$. The induction starts with $\Sigma_0 = \{\emptyset, X\}$, $\underline{\phi}_0\emptyset = \emptyset$, $\underline{\phi}_0 X = X$. The inductive step to a successor ordinal is dealt with in (b), and the inductive step to a non-zero ordinal of countable cofinality is dealt with in (c). If $\xi \leq \kappa$ has uncountable cofinality, then $\Sigma_\xi = \bigcup_{\eta<\xi} \Sigma_\eta$, so we can (and must) take $\underline{\phi}_\xi$ to be the unique common extension of all the previous $\underline{\phi}_\eta$.

The induction ends with $\underline{\phi}_\kappa : \Sigma_\kappa \to \mathrm{T}_\kappa$. Note that Σ_κ is not in general the whole of T_κ. But for every $E \in \mathrm{T}_\kappa$ there is an $F \in \Sigma_\kappa$ such that $E\triangle F$ is negligible (254Ff). So we can extend $\underline{\phi}_\kappa$ to a function $\underline{\phi}$ defined on the whole of T_κ by setting

$$\underline{\phi}E = \underline{\phi}_\kappa F \text{ whenever } E \in \mathrm{T}_\kappa,\ F \in \Sigma_\kappa \text{ and } \nu_\kappa(E\triangle F) = 0$$

(the point being that $\underline{\phi}_\kappa F = \underline{\phi}_\kappa F'$ if F, $F' \in \Sigma_\kappa$ and $\nu_\kappa(E\triangle F) = \nu_\kappa(E\triangle F') = 0$). It is easy to check that $\underline{\phi}$ is a lower density, and it is translation-invariant because if $E \in \mathrm{T}_\kappa$, $x \in X$, $F \in \Sigma_\kappa$ and $E\triangle F$ is negligible, then $(E + x)\triangle(F + x) = (E\triangle F) + x$ is negligible, so

$$\underline{\phi}(E + x) = \underline{\phi}_\kappa(F + x) = \underline{\phi}_\kappa F + x = \underline{\phi}E + x. \text{ **Q**}$$

(e) The rest of the argument is exactly that of parts (b)-(e) of the proof of 345B; you have to change $\mathbb{R}^r$ into X wherever it appears, but otherwise you can use it word for word, interpreting '$\mathbf{0}$' as the identity of the group X, that is, the constant function with value 0.

345D Translation-invariant liftings are of great importance, and I will return to them in §447 with a theorem dramatically generalizing the results above. Here I shall content myself with giving one of their basic properties, set out for the two kinds of translation-invariant lifting we have seen.

Proposition Let (X, Σ, μ) be *either* Lebesgue measure on $\mathbb{R}^r$ *or* the usual measure on $\{0, 1\}^I$ for some set I, and let $\phi : \Sigma \to \Sigma$ be a translation-invariant lifting. Then for any open set $G \subseteq X$ we must have $G \subseteq \phi G \subseteq \overline{G}$, and for any closed set F we must have $\operatorname{int} F \subseteq \phi F \subseteq F$.

proof (a) Suppose that $G \subseteq X$ is open and that $x \in G$. Then there is an open set U such that $\mathbf{0} \in U$ and $x + U - U = \{x + y - z : y, z \in U\} \subseteq G$. **P** ($\alpha$) If $X = \mathbb{R}^r$, take $\delta > 0$ such that $\{y : \|y - x\| \leq \delta\} \subseteq G$, and set $U = \{y : \|y - x\| < \frac{1}{2}\delta\}$. ($\beta$) If $X = \{0, 1\}^I$, then there is a finite set $K \subseteq I$ such that $\{y : y{\restriction}K = x{\restriction}K\} \subseteq G$ (3A3K); set $U = \{y : y(i) = 0 \text{ for every } i \in K\}$. **Q**

It follows that $x \in \phi G$. **P** Consider $H = x + U$. Then $\mu H = \mu U > 0$ so $H \cap \phi H \neq \emptyset$. Let $y \in U$ be such that $x + y \in \phi H$. Then

$$x = (x + y) - y \in \phi(H - y) \subseteq \phi G$$

because

$$H - y \subseteq x + U - U \subseteq G. \ \mathbf{Q}$$

(b) Thus $G \subseteq \phi G$ for every open set $G \subseteq X$. But it follows at once that if G is open and F is closed,

$$\operatorname{int} F \subseteq \phi(\operatorname{int} F) \subseteq \phi F,$$

$$\overline{G} = X \setminus \operatorname{int}(X \setminus G) \supseteq X \setminus \phi(X \setminus G) = \phi G,$$

$$F = X \setminus (X \setminus F) \supseteq X \setminus \phi(X \setminus F) = \phi F.$$

345E I remarked in 341Lg that it is undecidable in ordinary set theory whether there is a lifting for Borel measure on $\mathbb{R}$. It is however known that there can be no translation-invariant Borel lifting. The argument depends on the following fact about measurable sets in $\{0,1\}^{\mathbb{N}}$.

Lemma Give $X = \{0,1\}^{\mathbb{N}}$ its usual measure $\nu_{\mathbb{N}}$, and let $E \subseteq X$ be any non-negligible measurable set. Then there is an $n \in \mathbb{N}$ such that for every $k \ge n$ there are x, $x' \in E$ which differ at k and nowhere else.

proof By 254Fe, there is a set F, determined by coordinates in a finite set, such that $\nu_{\mathbb{N}}(E \triangle F) \le \frac{1}{4}\nu_{\mathbb{N}}E$; we have $\nu_{\mathbb{N}}F \ge \frac{3}{4}\nu_{\mathbb{N}}E$, so $\nu_{\mathbb{N}}(E \triangle F) \le \frac{1}{3}\nu_{\mathbb{N}}F$. Let $n \in \mathbb{N}$ be such that F is determined by coordinates in $\{0, \dots, n-1\}$. Take any $k \ge n$. Then the map $\psi : X \to X$, defined by setting $(\psi x)(k) = 1 - x(k)$, $(\psi x)(i) = x(i)$ for $i \ne k$, is a measure space automorphism, and

$$\nu_{\mathbb{N}}(\psi^{-1}[E \triangle F] \cup (E \triangle F)) \le 2\nu_{\mathbb{N}}(E \triangle F) < \nu_{\mathbb{N}}F.$$

Take any $x \in F \setminus ((E \triangle F) \cup \psi^{-1}[E \triangle F])$. Then $x' = \psi x$ differs from x at k, and only there; but also $x' \in F$, by the choice of n, so both x and x' belong to E.

345F Proposition Let μ be the restriction of Lebesgue measure to the algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$. Then μ is translation-invariant, but has no translation-invariant lifting.

proof (a) To see that μ is translation-invariant all we have to know is that $\mathcal{B}$ is translation-invariant and that Lebesgue measure is translation-invariant. I have already cited 134A for the proof that Lebesgue measure is invariant, and $\mathcal{B}$ is invariant because $G + x$ is open for every open set G and every $x \in \mathbb{R}$.

(b) The argument below is most easily expressed in terms of the geometry of the Cantor set C. Recall that C is defined as the intersection $\bigcap_{n\in\mathbb{N}} C_n$ of a sequence of closed subsets of $[0,1]$; each C_n consists of 2^n closed intervals of length 3^{-n}; C_{n+1} is obtained from C_n by deleting the middle third of each interval of C_n. Any point of C is uniquely expressible as $f(e) = \frac{2}{3}\sum_{n=0}^{\infty} 3^{-n}e(n)$ for some $e \in \{0,1\}^{\mathbb{N}}$. (See 134Gb.) Let $\nu_{\mathbb{N}}$ be the usual measure of $\{0,1\}^{\mathbb{N}}$. Because the map $e \mapsto e(n) : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ is measurable for each n, $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is measurable.

We can label the closed intervals constituting C_n as $\langle J_z \rangle_{z\in\{0,1\}^n}$, taking $J_\emptyset$ to be the unit interval $[0,1]$ and, for $z \in \{0,1\}^n$, taking $J_{z^\frown<0>}$ to be the left-hand third of J_z and $J_{z^\frown<1>}$ to be the right-hand third of J_z. (If the notation here seems odd to you, there is an explanation in 3A1H.)

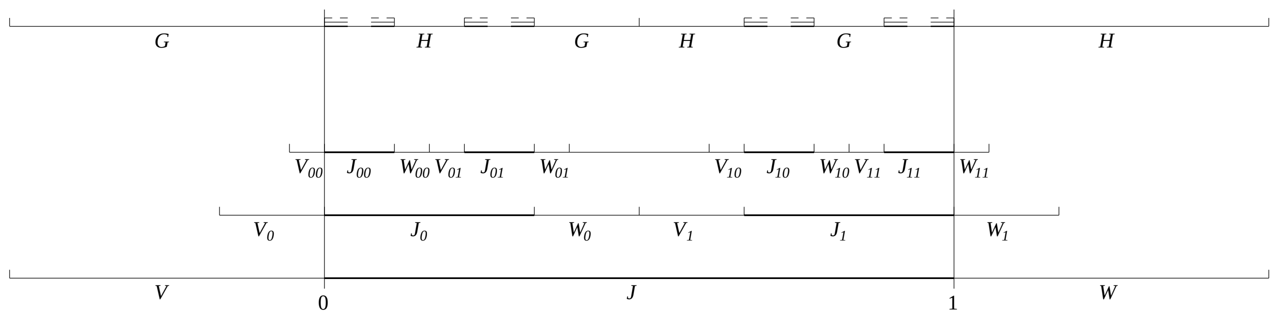
For $n \in \mathbb{N}$ and $z \in \{0,1\}^n$, let J'_z be the open interval with the same centre as J_z and twice the length. Then $J'_z \setminus J_z$ consists of two open intervals of length $3^{-n}/2$ on either side of J_z; call the left-hand one V_z and the right-hand one W_z. Thus $V_{z^\frown<1>}$ is the right-hand half of the middle third of J_z, and $W_{z^\frown<0>}$ is the left-hand half of the middle third of J_z.

Construct sets G, $H \subseteq \mathbb{R}$ as follows.

> G is to be the union of the intervals V_z where z takes the value 1 an even number of times, together with the intervals W_z where z takes the value 0 an odd number of times;
>
> H is to be the union of the intervals V_z where z takes the value 1 an odd number of times, together with the intervals W_z where z takes the value 0 an even number of times.

G and H are open sets. The intervals V_z, W_z between them cover the whole of the interval $\left]-\frac{1}{2}, \frac{3}{2}\right[$ with the exception of the set C and the countable set of midpoints of the intervals J_z; so that $\left]-\frac{1}{2}, \frac{3}{2}\right[\setminus (G \cup H)$ is negligible. We have to observe that $G \cap H = \emptyset$. $\mathbf{P}$ For each z, $J'_{z^\frown<0>}$ and $J'_{z^\frown<1>}$ are disjoint subsets of J'_z. Consequently $J'_z \cap J'_w$ is non-empty just when one of z, w extends the other, and we need consider only the intersections of the four sets V_z, W_z, V_w, W_w when w is a proper extension of z; say $w \in \{0,1\}^n$ and $z = w{\restriction}m$, where $m < n$. (α) If in the extension $(w(m), \dots, w(n-1))$ both values 0 and 1 appear, J'_w will be a subset of J_z, and certainly the four sets will all be disjoint. (β) If $w(i) = 0$ for $m \le i < n$, then $W_w \subseteq J_z$ is disjoint from the rest, while $V_w \subseteq V_z$; but z and w take the value 1 the same number of times, so V_w is assigned to G iff V_z is, and otherwise both are assigned to H. (γ) Similarly, if $w(i) = 1$ for $m \le i < n$, $V_w \subseteq J_z$, $W_w \subseteq W_z$ and z, w take the value 0 the same number of times, so W_z and W_w are assigned to the same set. **Q**

The following diagram may help you to see what is supposed to be happening:



The assignment rule can be restated as follows:

$V = V_\emptyset$ is assigned to G, $W = W_\emptyset$ is assigned to H;

$V_{z^\frown<0>}$ is assigned to the same set as V_z, and $V_{z^\frown<1>}$ to the other;

$W_{z^\frown<1>}$ is assigned to the same set as W_z, and $W_{z^\frown<0>}$ to the other.

(c) Now take any $n \in \mathbb{N}$ and $z \in \{0,1\}^n$. Consider the two open intervals $I_0 = J'_{z^\frown<0>}$, $I_1 = J'_{z^\frown<1>}$. These are both of length $\gamma = 2 \cdot 3^{-n-1}$ and abut at the centre of J_z, so I_1 is just the translate $I_0 + \gamma$. I claim that $I_1 \cap H = (I_0 \cap G) + \gamma$. **P** Let A be the set

$$\bigcup_{m>n}\{w : w \in \{0,1\}^m,\ w \text{ extends } z^\frown<0>\},$$

and for $w \in A$ let w' be the finite sequence obtained from w by changing $w(n) = 0$ into $w'(n) = 1$ but leaving the other values of w unaltered. Then $V_{w'} = V_w + \gamma$ and $W_{w'} = W_w + \gamma$ for every $w \in A$. Now

$$\begin{aligned} I_0 \cap G = &\bigcup\{V_w : w \in A,\ w \text{ takes the value 1 an even number of times}\} \\ &\cup \bigcup\{W_w : w \in A,\ w \text{ takes the value 0 an odd number of times}\}, \end{aligned}$$

so

$$\begin{aligned} (I_0 \cap G) + \gamma &= \bigcup\{V_{w'} : w \in A,\ w \text{ takes the value 1 an even number of times}\} \\ &\qquad \cup \bigcup\{W_{w'} : w \in A,\ w \text{ takes the value 0 an odd number of times}\} \\ &= \bigcup\{V_{w'} : w \in A,\ w' \text{ takes the value 1 an odd number of times}\} \\ &\qquad \cup \bigcup\{W_{w'} : w \in A,\ w' \text{ takes the value 0 an even number of times}\} \\ &= I_1 \cap H. \ \mathbf{Q} \end{aligned}$$

(d) ? Now suppose, if possible, that $\phi : \mathcal{B} \to \mathcal{B}$ is a translation-invariant lifting. Note first that $U \subseteq \phi U$ for every open $U \subseteq \mathbb{R}$. **P** The argument is exactly that of 345D as applied to $\mathbb{R} = \mathbb{R}^1$. **Q** Consequently

$$J'_\emptyset = \left]-\tfrac{1}{2}, \tfrac{3}{2}\right[\subseteq \phi J'_\emptyset.$$

But as $J'_\emptyset \setminus (G \cup H)$ is negligible,

$$C \subseteq \left]-\tfrac{1}{2}, \tfrac{3}{2}\right[\subseteq \phi G \cup \phi H.$$

Consider the sets $E = f^{-1}[\phi G]$, $F = \{0,1\}^{\mathbb{N}} \setminus E = f^{-1}[\phi H]$. Because f is measurable and ϕG, ϕH are Borel sets, E and F are measurable subsets of $\{0,1\}^{\mathbb{N}}$, and at least one of them has positive measure for $\nu_{\mathbb{N}}$. There must therefore be e, $e' \in \{0,1\}^{\mathbb{N}}$, differing at exactly one coordinate, such that either both belong to E or both belong to F (345E).

Let us suppose that n is such that $e(n) = 0$, $e'(n) = 1$ and $e(i) = e'(i)$ for $i \neq n$. Set $z = e\restriction n = e'\restriction n$. Then $f(e)$ belongs to the open interval $I_0 = J'_{z^\frown<0>}$, so $f(e) \in \phi I_0$ and $f(e) \in \phi G$ iff $f(e) \in \phi(I_0 \cap G)$. But now

$$f(e') = f(e) + 2 \cdot 3^{-n-1} \in I_1 = J'_{z^\frown<1>},$$

so

$$\begin{aligned} e \in E &\iff f(e) \in \phi G \iff f(e) \in \phi(I_0 \cap G) \\ &\iff f(e') \in \phi((I_0 \cap G) + 2 \cdot 3^{-n-1}) \end{aligned}$$

(because ϕ is translation-invariant)

$$\iff f(e') \in \phi(I_1 \cap H)$$

(by (c) above)

$$\iff f(e') \in \phi H$$

(because $f(e') \in I_1 \subseteq \phi I_1$)

$$\iff e' \in F.$$

But this contradicts the choice of e. **X**

Thus there is no translation-invariant lifting for μ.

Remark This result is due to Johnson 80; the proof here follows Talagrand 82b. For references to various generalizations see Burke 93, §3.

345X Basic exercises (a) In 345Ab I wrote 'It follows at once that the map $y \mapsto y + x : X \to X$ is a measure space automorphism'. Write the details out in full, using 254G or otherwise.

(b) Let S^1 be the unit circle in $\mathbb{R}^2$, and let μ be one-dimensional Hausdorff measure on S^1 (§§264-265). Show that μ is translation-invariant, if S^1 is given its usual group operation corresponding to complex multiplication (255M), and that it has a translation-invariant lifting ϕ. (*Hint*: Identifying S^1 with $]-\pi, \pi]$ with the group operation $+_{2\pi}$, show that we can set $\phi E = \,]-\pi, \pi] \cap \phi'(\bigcup_{n\in\mathbb{Z}} E + 2\pi n)$, where ϕ' is any translation-invariant lifting for Lebesgue measure.)

>(c) Show that there is no lifting ϕ of Lebesgue measure on $\mathbb{R}$ which is 'symmetric' in the sense that $\phi(-E) = -\phi E$ for every measurable set E, writing $-E = \{-x : x \in E\}$. (*Hint*: can 0 belong to $\phi([0, \infty[)$?)

>(d) Let μ be Lebesgue measure on $X = \mathbb{R} \setminus \{0\}$. Show that there is a lifting ϕ of μ such that $\phi(xE) = x\phi E$ for every $x \in X$ and every measurable $E \subseteq X$, writing $xE = \{xy : y \in E\}$.

(e) Let ν_I be the usual measure on $X = \{0, 1\}^I$, for some set I, T_I its domain, and $(\mathfrak{B}_I, \bar{\nu}_I)$ its measure algebra. (i) Show that we can define $\pi_x(a) = a + x$, for $a \in \mathfrak{B}_I$ and $x \in X$, by the formula $E^\bullet + x = (E + x)^\bullet$; and that $x \mapsto \pi_x$ is a group homomorphism from X to the group of measure-preserving automorphisms of $\mathfrak{A}$. (ii) Define Σ_ξ as in the proof of 345C, and set $\mathfrak{A}_\xi = \{E^\bullet : E \in \Sigma_\xi\}$. Say that a partial lifting $\underline{\theta} : \mathfrak{A}_\xi \to \mathrm{T}_I$ is translation-invariant if $\underline{\theta}(a + x) = \underline{\theta}a + x$ for every $a \in \mathfrak{A}_\xi$ and $x \in X$. Show that any such partial lifting can be extended to a translation-invariant partial lifting on $\mathfrak{A}_{\xi+1}$. (iii) Write out a proof of 345C in the language of 341F-341H.

>(f) Let $\underline{\phi}$ be a lower density for Lebesgue measure on $\mathbb{R}^r$ which is translation-invariant in the sense that $\underline{\phi}(E + x) = \underline{\phi}E + x$ for every $x \in \mathbb{R}^r$ and every measurable set E. Show that $\underline{\phi}G \supseteq G$ for every open set $G \subseteq \mathbb{R}^r$.

(g) Let μ be 1-dimensional Hausdorff measure on S^1, as in 345Xb. Show that there is no translation-invariant lifting ϕ of μ such that ϕE is a Borel set for every $E \in \operatorname{dom}\mu$.

345Y Further exercises (a) Let (X, Σ, μ) be a complete measure space, and suppose that X has a group operation $(x, y) \mapsto xy$ (not necessarily abelian!) such that μ is left-translation-invariant, in the sense that $xE = \{xy : y \in E\} \in \Sigma$ and $\mu(xE) = \mu E$ whenever $E \in \Sigma$ and $x \in X$. Suppose that $\underline{\phi} : \Sigma \to \Sigma$ is a lower density which is left-translation-invariant in the sense that $\underline{\phi}(xE) = x(\underline{\phi}E)$ for every $E \in \Sigma$ and $x \in X$. Show that there is a left-translation-invariant lifting $\phi : \Sigma \to \Sigma$ such that $\underline{\phi}E \subseteq \phi E$ for every $E \in \Sigma$.

(b) Write Σ for the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$, and $\mathcal{L}^0(\Sigma)$ for the linear space of Σ-measurable functions from $\mathbb{R}$ to itself. Show that there is a linear operator $T : L^0(\mu) \to \mathcal{L}^0(\Sigma)$ such that (α) $(Tu)^\bullet = u$ for every $u \in L^0(\mu)$ (β) $\sup_{x\in\mathbb{R}} |(Tu)(x)| = \|u\|_\infty$ for every $u \in L^\infty(\mu)$ (γ) $Tu \ge 0$ whenever $u \in L^\infty(\mu)$ and $u \ge 0$ (δ) T is translation-invariant in the sense that $T(S_xf)^\bullet = S_xTf^\bullet$ for every $x \in \mathbb{R}$ and $f \in \mathcal{L}^0(\Sigma)$, where $(S_xf)(y) = f(x+y)$ for $f \in \mathcal{L}^0(\Sigma)$ and $x, y \in \mathbb{R}$ (ϵ) T is reflection-invariant in the sense that $T(Rf)^\bullet = RTf^\bullet$ for every $f \in \mathcal{L}^0(\Sigma)$, where $(Rf)(x) = f(-x)$ for $f \in \mathcal{L}^0(\Sigma)$ and $x \in \mathbb{R}$. (*Hint*: for $f \in \mathcal{L}^0(\Sigma)$, set

$$p(f^\bullet) = \inf\{\alpha : \alpha \in [0,\infty],\ \lim_{\delta\downarrow 0} \frac{1}{2\delta}\mu\{x : |x| \le \delta,\ |f(x)| > \alpha\} = 0\}.$$

Set $V = \{u : u \in L^0(\mu),\ p(u) < \infty\}$ and show that V is a linear subspace of $L^0(\mu)$ and that $p\restriction V$ is a seminorm. Let $h_0 : V \to \mathbb{R}$ be a linear functional such that $h_0(\chi\mathbb{R})^\bullet = 1$ and $h_0(u) \le p(u)$ for every $u \in V$. Extend h_0 arbitrarily to a linear functional $h_1 : L^0(\mu) \to \mathbb{R}$; set $h(f^\bullet) = \frac{1}{2}(h_1(f^\bullet) + h_1(Rf)^\bullet)$. Set $(Tf^\bullet)(x) = h(S_{-x}f)^\bullet$. You will need 223C.) Show that there must be a $u \in L^1(\mu)$ such that $u \ge 0$ but $Tu \not\ge 0$.

(c) Show that there is no translation-invariant lifting ϕ of the usual measure on $\{0,1\}^\mathbb{N}$ such that ϕE is a Borel set for every measurable set E.

345 Notes and comments I have taken a great deal of care over the concept of 'translation-invariance'. I hope that you are already a little impatient with some of the details as I have written them out; but while it is very easy to guess at the structure of such arguments as part (e) of the proof of 345B, or (b-iii) and (c-viii) in the proof of 345C, I am not sure that one can always be certain of guessing correctly. A fair test of your intuition will be how quickly you can generate the formulae appropriate to a non-abelian group operation, as in 345Ya.

Part (b) of the proof of 345C is based on the same idea as the proof of 341F. There is a useful simplification because the set E_ξ in 345C, corresponding to the set E of the proof of 341F, is independent of the algebra Σ_ξ in a very strong sense, so that the expression of an element of $\Sigma_{\xi+1}$ in the form $(F \cap E_\xi) \cup (G \setminus E_\xi)$ is unique. Interpreted in the terms of 341F, we have $w = v = 1$, so that the formula

$$\underline{\theta}_1((a \cap e) \cup (b \setminus e)) = \big(\underline{\theta}((a \cap v) \cup (b \setminus v)) \cap E\big) \cup \big(\underline{\theta}((a \setminus w) \cup (b \cap w)) \setminus E\big)$$

used there becomes

$$\underline{\theta}_1((a \cap e) \cup (b \setminus e)) = (\underline{\theta}a \cap E) \cup (\underline{\theta}b \setminus E),$$

matching the formula for $\underline{\phi}_1$ in the proof of 345C.

The results of this section are satisfying and natural; they have obvious generalizations, many of which are true. The most important measure spaces come equipped with a variety of automorphisms, and we can always ask which of these can be preserved by a lifting. The answers are not always obvious; I offer 345Xc and 346Xc as warnings, and 345Xd as an encouragement. 345Yb is striking (I have made it as striking as I can), but slightly off the most natural target; the sting is in the last sentence (see 341Xg).

346 Consistent liftings

I turn now to a different type of condition which we should naturally prefer our liftings to satisfy. If we have a product measure μ on a product $X = \prod_{i\in I} X_i$ of probability spaces, then we can look for liftings ϕ which 'respect coordinates', that is, are compatible with the product structure in the sense that they factor through subproducts (346A). There seem to be obstacles in the way of the natural conjecture (346Za), and I give the partial results which are known. For Maharam-type-homogeneous spaces X_i, there is always a lifting which respects coordinates (346E), and indeed the translation-invariant liftings of §345 on $\{0,1\}^I$ already have this property (346C). There is always a lower density for the product measure which respects coordinates, and we can ask for a little more (346G); using the full strength of 346G, we can enlarge this lower density to a lifting which respects single coordinates and initial segments of a well-ordered product (346H). In the case in which all the factors are copies of each other, we can arrange for the induced liftings on the factors to be copies also (346I, 346J, 346Ye). I end the section with an important fact about Stone spaces which is relevant here (346K-346L).

346A Definition Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X, Σ, μ). I will say that a lifting $\phi : \Sigma \to \Sigma$ **respects coordinates** if ϕE is determined by coordinates in J whenever $E \in \Sigma$ is determined by coordinates in $J \subseteq I$.

Remark Recall that a set $E \subseteq X$ is 'determined by coordinates in J' if $x' \in E$ whenever $x \in E$, $x' \in X$ and $x'\restriction J = x\restriction J$; that is, if E is expressible as $\pi_J^{-1}[F]$ for some $F \subseteq \prod_{i\in J} X_i$, where $\pi_J(x) = x\restriction J$ for every $x \in X$; that is, if $E = \pi_J^{-1}[\pi_J[E]]$. See 254M. Recall also that in this case, if E is measured by the product measure on X, then $\pi_J[E]$ is measured by the product measure on $\prod_{i\in J} X_i$ (254Ob).

346B Lemma (a) Let (X, Σ, μ) be a measure space with a lifting $\phi : \Sigma \to \Sigma$. Suppose that Y is a set and $f : X \to Y$ a surjective function such that whenever $E \in \Sigma$ is such that $f^{-1}[f[E]] = E$, then $f^{-1}[f[\phi E]] = \phi E$. Then we have a lifting ψ for the image measure μf^{-1} defined by the formula

$$f^{-1}[\psi F] = \phi(f^{-1}[F]) \text{ whenever } F \subseteq Y \text{ and } f^{-1}[F] \in \Sigma.$$

(b) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (Z, Λ, λ). For $J \subseteq I$ let $(Z_J, \Lambda_J, \lambda_J)$ be the product of $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in J}$, and $\pi_J : Z \to Z_J$ the canonical map. Let $\phi : \Lambda \to \Lambda$ be a lifting. If $J \subseteq I$ is such that ϕW is determined by coordinates in J whenever $W \in \Lambda$ is determined by coordinates in J, then ϕ induces a lifting $\phi_J : \Lambda_J \to \Lambda_J$ defined by the formula

$$\pi_J^{-1}[\phi_J E] = \phi(\pi_J^{-1}[E]) \text{ for every } E \in \Lambda_J.$$

proof (a) Set $\psi F = f[\phi(f^{-1}[F])]$ for $F \in \mathrm{dom}(\mu f^{-1})$. Because f is surjective, $\psi Y = Y$, and it is now elementary to check that ψ is a lifting for μf^{-1}.

(b) By 254Oa, λ_J is the image measure $\lambda\pi_J^{-1}$, so we can use (a).

Remark Of course we frequently wish to use part (b) here with a singleton set $J = \{j\}$. In this case we must remember that $(Z_J, \Sigma_J, \lambda_J)$ corresponds to the *completion* of the probability space (X_j, Σ_j, μ_j).

346C Theorem Let I be any set, and ν_I the usual measure on $X = \{0,1\}^I$. Then any translation-invariant lifting for ν_I respects coordinates.

proof Suppose that $E \subseteq X$ is a measurable set determined by coordinates in $J \subseteq I$; take $x \in \phi E$ and $x' \in X$ such that $x'\restriction J = x\restriction J$. Set $y = x' - x$; then $y(i) = 0$ for $i \in J$, so that $E + y = y$. Now

$$x' = x + y \in \phi E + y = \phi(E + y) = \phi E$$

because ϕ is translation-invariant. As x, x' are arbitrary, ϕE is determined by coordinates in J. As E and J are arbitrary, ϕ respects coordinates.

346D I describe a standard method of constructing liftings from other liftings.

Lemma Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with measure algebras $\mathfrak{A}$, $\mathfrak{B}$; suppose that $f : X \to Y$ represents an isomorphism $F^\bullet \mapsto f^{-1}[F]^\bullet : \mathfrak{B} \to \mathfrak{A}$. Then if $\phi : \mathrm{T} \to \mathrm{T}$ is a lifting for ν, there is a corresponding lifting $\phi' : \Sigma \to \Sigma$ given by the formula

$$\phi' E = f^{-1}[\phi F] \text{ whenever } \mu(E \triangle f^{-1}[F]) = 0.$$

proof If we say that $\pi : \mathfrak{B} \to \mathfrak{A}$ is the isomorphism induced by f, then

$$\phi' E = f^{-1}[\theta(\pi^{-1}E^\bullet)],$$

where $\theta : \mathfrak{B} \to \mathrm{T}$ is the lifting corresponding to $\phi : \mathrm{T} \to \mathrm{T}$. Since θ, π^{-1} and $F \mapsto f^{-1}[F]$ are all Boolean homomorphisms, so is ϕ', and it is easy to check that $(\phi' E)^\bullet = E^\bullet$ for every $E \in \Sigma$ and that $\phi' E = \emptyset$ if $\mu E = 0$.

Remark Compare the construction in 341P.

346E Theorem Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of Maharam-type-homogeneous probability spaces, with product (X, Σ, μ). Then there is a lifting for μ which respects coordinates.

proof (a) Replacing each μ_i by its completion does not change μ (254I), so we may suppose that all the μ_i are complete. In this case there is for each i an isomorphism between the measure algebra $(\mathfrak{A}_i, \bar\mu_i)$ of μ_i and the measure algebra $(\mathfrak{B}_{J_i}, \bar\nu_{J_i})$ of some $\{0,1\}^{J_i}$ with its usual measure ν_{J_i} (331L). We may suppose that the sets J_i are disjoint. Each ν_{J_i} is compact (342Jd), so the isomorphisms are represented by inverse-measure-preserving functions $f_i : X_i \to \{0,1\}^{J_i}$ (343Ca).

Set $K = \bigcup_{i\in I} J_i$, and let ν_K be the usual measure on $Y = \{0,1\}^K$, T_K its domain. We have a natural bijection between $\prod_{i\in I}\{0,1\}^{J_i}$ and Y, so we obtain a function $f : X \to Y$; literally speaking,

$$f(x)(j) = f_i(x(i))(j)$$

for $i \in I$, $j \in J_i$ and $x \in X$.

(b) Now f is inverse-measure-preserving and induces an isomorphism between the measure algebras $\mathfrak{A}$, $\mathfrak{B}_K$ of μ, ν_K.

P(i) If $L \subseteq K$ is finite and $z \in \{0,1\}^L$, then, setting $L_i = L \cap J_i$ for $i \in I$,

$$\begin{aligned}\mu\{x : x \in X,\ f(x)\restriction L = z\} &= \mu\Bigl(\prod_{i\in I}\{w : w \in X_i,\ f_i(w)\restriction L_i = z\restriction L_i\}\Bigr)\\ &= \prod_{i\in I}\mu_i\{w : w \in X_i,\ f_i(w)\restriction L_i = z\restriction L_i\}\\ &= \prod_{i\in I}\nu_{J_i}\{v : v \in \{0,1\}^{J_i},\ v\restriction L_i = z\restriction L_i\}\end{aligned}$$

(because every f_i is inverse-measure-preserving)

$$= \prod_{i\in I} 2^{-\#(L_i)} = 2^{-\#(L)} = \nu_K\{y : y \in Y,\ y\restriction L = z\}.$$

So $\mu f^{-1}[C] = \nu_K C$ for every basic cylinder set $C \subseteq Y$. By 254G, f is inverse-measure-preserving.

(ii) Accordingly f induces a measure-preserving homomorphism $\pi : \mathfrak{B}_K \to \mathfrak{A}$. To see that π is surjective, consider

$$\Lambda' = \{E : E \text{ is } \Sigma\text{-measurable},\ E^\bullet \in \pi[\mathfrak{B}_K]\}.$$

Because $\pi[\mathfrak{B}_K]$ is a closed subalgebra of $\mathfrak{A}$ (324Kb), Λ' is a σ-subalgebra of the domain Λ of μ, and of course it contains all μ-negligible sets. If $i \in J$ and $G \in \Sigma_i$, then there is an $H \subseteq \{0,1\}^{J_i}$ such that $G\triangle f_i^{-1}[H]$ is μ_i-negligible. Now if $E = \{x : x \in X,\ x(i) \in G\}$ and $F = \{y : y \in Y,\ y\restriction J_i \in H\}$,

$$E\triangle f^{-1}[F] = \{x : x(i) \in G\triangle f_i^{-1}[H]\}$$

is μ-negligible, and $E \in \Lambda'$. But this means that $\Lambda' \supseteq \widehat{\bigotimes}_{i\in I}\Sigma_i$, and must therefore be the whole of Λ (254Ff). **Q**

(c) By 345C, there is a translation-invariant lifting ϕ for ν_K; by 346C, this respects coordinates. By 346D, we have a corresponding lifting ϕ' for μ such that

$$\phi' f^{-1}[F] = f^{-1}[\phi F]$$

for every $F \in \mathrm{T}_K$. Now suppose that $E \in \Lambda$ is determined by coordinates in $L \subseteq I$. Then there is an E' belonging to the σ-algebra Λ'_L generated by

$$\{\{x : x(i) \in G\} : i \in L,\ G \in \Sigma_i\}$$

such that $\mu(E\triangle E') = 0$ (254Ob). Write T_L for the family of sets in T_K determined by coordinates in $\bigcup_{i\in L} J_i$. Then, just as in (b-ii), every member of Λ'_L differs by a negligible set from some set of the form $f^{-1}[F]$ with $F \in \mathrm{T}_L$. So there is an $F \in \mathrm{T}_L$ such that $E\triangle f^{-1}[F]$ is μ-negligible. Consequently

$$\phi' E = \phi' f^{-1}[F] = f^{-1}[\phi F].$$

But ϕ respects coordinates, so ϕF is determined by coordinates in $\bigcup_{i\in L} J_i$. It follows at once that $f^{-1}[\phi F]$ is determined by coordinates in L; that is, that $\phi' E$ is determined by coordinates in L. As E and L are arbitrary, ϕ' respects coordinates, and witnesses the truth of the theorem.

346F It seems to be unknown whether 346E is true of arbitrary probability spaces (346Za); I give some partial results in this direction. The following general method of constructing lower densities will be useful.

Lemma Let (X, Σ, μ) and (Y, T, ν) be complete probability spaces, with product $(X \times Y, \Lambda, \lambda)$. If $\underline{\phi} : \Lambda \to \Lambda$ is a lower density, then we have a lower density $\underline{\phi}_1 : \Sigma \to \Sigma$ defined by saying that

$$\underline{\phi}_1 E = \{x : x \in X,\ \{y : (x, y) \in \underline{\phi}(E \times Y)\} \text{ is conegligible in } Y\}$$

for every $E \in \Sigma$.

proof For $E \in \Sigma$, $(E \times Y)\triangle\underline{\phi}(E \times Y)$ is negligible, so that

$$H_x = \{y : (x, y) \in (E \times Y)\triangle\underline{\phi}(E \times Y)\}$$

is ν-negligible for almost every $x \in X$ (252D). Now $E\triangle\underline{\phi}_1 E = \{x : H_x$ is not negligible$\}$ is negligible, so $\underline{\phi}_1 E \in \Sigma$. If E, $F \in \Sigma$, then

$$\underline{\phi}((E \cap F) \times Y) = \underline{\phi}((E \times Y) \cap (F \times Y)) = \underline{\phi}(E \times Y) \cap \underline{\phi}(F \times Y),$$

so that

$$\{y : (x, y) \in \underline{\phi}((E \cap F) \times Y)\} = \{y : (x, y) \in \underline{\phi}(E \times Y)\} \cap \{y : (x, y) \in \underline{\phi}(F \times Y)\}$$

is conegligible iff both $\{y : (x, y) \in \underline{\phi}(E\times Y)\}$ and $\{y : (x, y) \in \underline{\phi}(F\times Y)\}$ are conegligible, and $\underline{\phi}_1(E\cap F) = \underline{\phi}_1 E\cap\underline{\phi}_1 F$.

The rest is easy. Of course $\underline{\phi}(\emptyset \times Y) = \emptyset$ so $\underline{\phi}_1\emptyset = \emptyset$. If E, $F \in \Sigma$ and $E\triangle F$ is negligible, then $(E \times Y)\triangle(F \times Y)$ is negligible, $\underline{\phi}(E \times Y) = \underline{\phi}(F \times Y)$ and $\underline{\phi}_1 E = \underline{\phi}_1 F$. So $\underline{\phi}_1$ is a lower density, as claimed.

346G Theorem Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces with product (X, Σ, μ). For $J \subseteq I$ let Σ_J be the set of members of Σ which are determined by coordinates in J. Then there is a lower density $\underline{\phi} : \Sigma \to \Sigma$ such that

(i) whenever $J \subseteq I$ and $E \in \Sigma_J$ then $\underline{\phi}E \in \Sigma_J$,

(ii) whenever J, $K \subseteq I$ are disjoint, $E \in \Sigma_J$ and $F \in \Sigma_K$ then $\underline{\phi}(E \cup F) = \underline{\phi}E \cup \underline{\phi}F$.

proof For each $i \in I$, set $Y_i = X_i^{\mathbb{N}}$, with the product measure ν_i; set $Y = \prod_{i\in I} Y_i$, with its product measure ν; set $Z_i = X_i \times Y_i$, with its product measure λ_i, and $Z = \prod_{i\in I} Z_i$, with its product measure λ. Then the natural identification of $Z = \prod_{i\in I} X_i \times Y_i$ with $\prod_{i\in I} X_i \times \prod_{i\in I} Y_i = X \times Y$ makes λ correspond to the product of μ and ν (254N).

Each (Z_i, λ_i) can be identified with an infinite power of (X_i, μ_i), and is therefore Maharam-type-homogeneous (334E). Consequently there is a lifting $\phi : \Lambda \to \Lambda$ which respects coordinates (346E). Regarding (Z, λ) as the product of (X, μ) and (Y, ν), we see that ϕ induces a lower density $\underline{\phi} : \Sigma \to \Sigma$ by the formula of 346F.

If $J \subseteq I$ and $E \in \Sigma$ is determined by coordinates in J, then $E\times Y$ (regarded as a subset of $\prod_{i\in I} Z_i$) is determined by coordinates in J, so $\phi(E \times Y)$ also is. Now suppose that $x \in \underline{\phi}E$, $x' \in X$ and $x\restriction J = x'\restriction J$. Then for any $y \in Y$, $(x\restriction J, y\restriction J) = (x'\restriction J, y\restriction J)$, so $(x, y) \in \phi(E \times Y)$ iff $(x', y) \in \phi(E \times Y)$. Thus

$$\{y : (x', y) \in \phi(E \times Y)\} = \{y : (x, y) \in \phi(E \times Y)\}$$

is conegligible in Y, and $x' \in \underline{\phi}E$. This shows that $\underline{\phi}E$ is determined by coordinates in J.

Now suppose that J and K are disjoint subsets of I, that E, $F \in \Sigma$ are determined by coordinates in J, K respectively, and that $x \notin \underline{\phi}E \cup \underline{\phi}F$. Then $A = \{y : (x, y) \notin \phi(E \times Y)\}$ and $B = \{y : (x, y) \notin \phi(F \times Y)\}$ are non-negligible. As noted just above, $\phi(E\times Y)$ is determined by coordinates in J, so A is determined by coordinates in J, and can be expressed as $\{y : y\restriction J \in A'\}$, where $A' \subseteq Y_J = \prod_{i\in J} Y_i$. Because $y \mapsto y\restriction J : Y \to Y_J$ is inverse-measure-preserving, A' cannot be negligible in Y_J. Similarly, B can be expressed as $\{y : y\restriction K \in B'\}$ for some non-negligible $B' \subseteq Y_K$.

By 251S/251Wm, $A' \times B' \times Y_{I\setminus(J\cup K)}$, regarded as a subset of Y, is non-negligible, that is,

$$C = \{y : y \in Y,\ y\restriction J \in A',\ y\restriction K \in B'\}$$

is non-negligible. But

$$C = A \cap B = \{y : (x, y) \notin \phi(E \times Y) \cup \phi(F \times Y)\} = \{y : (x, y) \notin \phi((E \cup F) \times Y\}.$$

So $x \notin \underline{\phi}(E \cup F)$. As x is arbitrary, $\underline{\phi}(E \cup F) \subseteq \underline{\phi}E \cup \underline{\phi}F$; but of course $\underline{\phi}E \cup \underline{\phi}F \subseteq \underline{\phi}(E \cup F)$, because $\underline{\phi}$ is a lower density, so that $\underline{\phi}(E \cup F) = \underline{\phi}E \cup \underline{\phi}F$, as required.

Remark See Macheras Musiał & Strauss 99 for an alternative proof.

346H Theorem Let ζ be an ordinal, and $\langle(X_\xi, \Sigma_\xi, \mu_\xi)\rangle_{\xi<\zeta}$ a family of probability spaces, with product (Z, Λ, λ). For $J \subseteq \zeta$ let Λ_J be the set of those $W \in \Lambda$ which are determined by coordinates in J. Then there is a lifting $\phi : \Lambda \to \Lambda$ such that $\phi W \in \Lambda_J$ whenever $W \in \Lambda_J$ and J is *either* a singleton subset of ζ *or* an initial segment of ζ.

proof (a) Let P be the set of all lower densities $\underline{\phi}:\Lambda\to\Lambda$ such that, for every $\xi<\zeta$, (i) whenever $E\in\Lambda_\xi$ then $\underline{\phi}E\in\Lambda_\xi$ (ii) whenever $E\in\Lambda_{\{\xi\}}$ then $\underline{\phi}E\in\Lambda_{\{\xi\}}$ (iii) whenever $E\in\Lambda_\xi$ and $F\in\Lambda_{\zeta\setminus\xi}$ then $\underline{\phi}(E\cup F)=\underline{\phi}E\cup\underline{\phi}F$. By 346G, P is not empty. Order P by saying that $\underline{\phi}\le\underline{\phi}'$ if $\underline{\phi}E\subseteq\underline{\phi}'E$ for every $E\in\Lambda$; then P is a partially ordered set. Note that if $\underline{\phi}\in P$ then $\underline{\phi}Z=Z$ (because $\Lambda_0=\{\emptyset,Z\}$).

(b) Any non-empty totally ordered subset Q of P has an upper bound in P. **P** Define $\underline{\phi}^*:\Lambda\to\mathcal{P}X$ by setting $\underline{\phi}^*E=\bigcup_{\underline{\phi}\in Q}\underline{\phi}E$ for every $E\in\Lambda$. (i)

$$\underline{\phi}^*\emptyset=\textstyle\bigcup_{\underline{\phi}\in Q}\emptyset=\emptyset.$$

(ii) If E, $F\in\Lambda$ and $\lambda(E\triangle F)=0$ then $\underline{\phi}E=\underline{\phi}F$ for every $\underline{\phi}\in Q$ so $\underline{\phi}^*E=\underline{\phi}^*F$. (iii) If E, $F\in\Lambda$ and $E\subseteq F$ then $\underline{\phi}E\subseteq\underline{\phi}F$ for every $\underline{\phi}\in Q$ so $\underline{\phi}^*E\subseteq\underline{\phi}^*F$. (iv) If E, $F\in\Lambda$ and $x\in\underline{\phi}^*E\cap\underline{\phi}^*F$, then there are $\underline{\phi}_1$, $\underline{\phi}_2\in Q$ such that $x\in\underline{\phi}_1E\cap\underline{\phi}_2F$; now either $\underline{\phi}_1\le\underline{\phi}_2$ or $\underline{\phi}_2\le\underline{\phi}_1$, so that

$$x\in(\underline{\phi}_1E\cap\underline{\phi}_1F)\cup(\underline{\phi}_2E\cap\underline{\phi}_2F)=\underline{\phi}_1(E\cap F)\cup\underline{\phi}_2(E\cap F)\subseteq\underline{\phi}^*(E\cap F).$$

Accordingly $\underline{\phi}^*E\cap\underline{\phi}^*F\subseteq\underline{\phi}^*(E\cap F)$ and $\underline{\phi}^*E\cap\underline{\phi}^*F=\underline{\phi}^*(E\cap F)$. (v) Taking any $\underline{\phi}_0\in Q$, we have $\underline{\phi}_0E\subseteq\underline{\phi}^*E$ for every $E\in\Lambda$, so (because λ is complete) $\underline{\phi}^*$ is a lower density, by 341Ib. (vi) Now suppose that $J\subseteq I$ is either a singleton $\{\xi\}$ or an initial segment ξ, and that $E\in\Lambda_J$. Then $\underline{\phi}E$ is determined by coordinates in J for every $\underline{\phi}\in Q$, so $\underline{\phi}^*E$ is determined by coordinates in J. (vii) Finally, suppose that $\xi<\zeta$ and that $E\in\Lambda_\xi$, $F\in\Lambda_{\zeta\setminus\xi}$. If $x\in\underline{\phi}^*(E\cup F)$ then there is a $\underline{\phi}\in Q$ such that

$$x\in\underline{\phi}(E\cup F)=\underline{\phi}E\cup\underline{\phi}F\subseteq\underline{\phi}^*E\cup\underline{\phi}^*F.$$

So $\underline{\phi}^*(E\cup F)\subseteq\underline{\phi}^*E\cup\underline{\phi}^*F$ and (using (iii) again) $\underline{\phi}^*(E\cup F)=\underline{\phi}^*E\cup\underline{\phi}^*F$. Thus $\underline{\phi}^*$ belongs to P and is an upper bound for Q in P. **Q**

By Zorn's Lemma, P has a maximal element $\underline{\tilde\phi}$.

(c) For any $H\in\Lambda$ we may define a function $\underline{\phi}_H$ as follows. Set $A_H=Z\setminus(\underline{\tilde\phi}H\cup\underline{\tilde\phi}(Z\setminus H))$,

$$\underline{\phi}_HE=\underline{\tilde\phi}E\cup(A_H\cap\underline{\tilde\phi}(H\cup E))$$

for $E\in\Lambda$. Then $\underline{\phi}_H$ is a lower density. **P** (i) Because $H\triangle\underline{\tilde\phi}H$ and $(Z\setminus H)\triangle\underline{\tilde\phi}(Z\setminus H)$ are both negligible, A_H is negligible and $\underline{\phi}_HE$ is measurable and $(\underline{\phi}_HE)^\bullet=(\underline{\tilde\phi}E)^\bullet=E^\bullet$ for every $E\in\Lambda$. (ii) Because $A_H\cap\underline{\tilde\phi}H=\emptyset$, $\underline{\phi}_H\emptyset=\emptyset$. (iii) If E, $F\in\Lambda$ and $\lambda(E\triangle F)=0$ then $\underline{\tilde\phi}E=\underline{\tilde\phi}F$ and $\underline{\tilde\phi}(E\cup H)=\underline{\tilde\phi}(F\cup H)$, so $\underline{\phi}_HE=\underline{\phi}_HF$. (iv) If E, $F\in\Lambda$ and $E\subseteq F$ then $\underline{\tilde\phi}E\subseteq\underline{\tilde\phi}F$ and $\underline{\tilde\phi}(E\cup H)\subseteq\underline{\tilde\phi}(F\cup H)$, so $\underline{\phi}_HE\subseteq\underline{\phi}_HF$. (v) If E, $F\in\Lambda$ and $x\in\underline{\phi}_HE\cap\underline{\phi}_HF$, then

(α) if $x\notin A_H$,

$$x\in\underline{\tilde\phi}E\cap\underline{\tilde\phi}F=\underline{\tilde\phi}(E\cap F)\subseteq\underline{\phi}_H(E\cap F),$$

(β) if $x\in A_H$,

$$x\in\underline{\tilde\phi}(E\cup H)\cap\underline{\tilde\phi}(F\cup H)=\underline{\tilde\phi}((E\cap F)\cup H)\subseteq\underline{\phi}_H(E\cap F).$$

Thus $\underline{\phi}_HE\cap\underline{\phi}_HF\subseteq\underline{\phi}_H(E\cap F)$ and $\underline{\phi}_HE\cap\underline{\phi}_HF=\underline{\phi}_H(E\cap F)$. **Q**

(d) It is worth noting the following.

(i) If E, $H\in\Lambda$ and $\underline{\tilde\phi}(E\cup H)=\underline{\tilde\phi}E\cup\underline{\tilde\phi}H$ then $\underline{\phi}_HE=\underline{\tilde\phi}E$. **P** We have

$$\underline{\phi}_HE=\underline{\tilde\phi}E\cup(A_H\cap\underline{\tilde\phi}(E\cup H))=\underline{\tilde\phi}E\cup(A_H\cap\underline{\tilde\phi}E)\cup(A_H\cap\underline{\tilde\phi}H)=\underline{\tilde\phi}E$$

because $A_H\cap\underline{\tilde\phi}H=\emptyset$. **Q**

(ii) If $H\in\Lambda$ and $\underline{\phi}_H\in P$ then $\underline{\tilde\phi}H\cup\underline{\tilde\phi}(Z\setminus H)=Z$. **P** By the maximality of $\underline{\tilde\phi}$, we must have $\underline{\phi}_H=\underline{\tilde\phi}$. But

$$A_H=\underline{\phi}_H(Z\setminus H)\setminus\underline{\tilde\phi}(Z\setminus H),$$

so $A_H=\emptyset$, that is, $\underline{\tilde\phi}H\cup\underline{\tilde\phi}(Z\setminus H)=Z$. **Q**

(iii) If E, $F\in\Lambda$ and $\underline{\tilde\phi}E\cup\underline{\tilde\phi}(Z\setminus E)=Z$, then $\underline{\tilde\phi}(E\cup F)=\underline{\tilde\phi}E\cup\underline{\tilde\phi}F$. **P**

$$\underline{\tilde\phi}(E\cup F)\setminus\underline{\tilde\phi}E=\underline{\tilde\phi}(E\cup F)\cap\underline{\tilde\phi}(Z\setminus E)=\underline{\tilde\phi}((E\cup F)\cap(Z\setminus E))=\underline{\tilde\phi}(F\setminus E)\subseteq\underline{\tilde\phi}F,$$

so $\underline{\tilde\phi}(E\cup F)\subseteq\underline{\tilde\phi}E\cup\underline{\tilde\phi}F$; as the reverse inclusion is true for all E and F, we have the result. **Q**

(e) If $\xi<\zeta$ and $H\in\Lambda_{\{\xi\}}$, then $\underline{\phi}_H\in P$.

P (i) If $J\subseteq I$ is either a singleton or an inital segment, and $E\in\Lambda_J$, then

(α) if $\xi\in J$, $E\cup H$ and $\underline{\tilde\phi}E$ and $\underline{\tilde\phi}(E\cup H)$ and A_H all belong to Λ_J, so $\underline{\phi}_H E\in\Lambda_J$.

(β) If $\xi\notin J$, $\underline{\tilde\phi}(E\cup H)=\underline{\tilde\phi}E\cup\underline{\tilde\phi}H$, because there is some η such that $J\subseteq\eta$ and $\{\xi\}\subseteq\zeta\setminus\eta$; so $\underline{\phi}_H E=\underline{\tilde\phi}E\in\Lambda_J$ by (d-i).

(ii) If $\eta<\zeta$, $E\in\Lambda_\eta$ and $F\in\Lambda_{\zeta\setminus\eta}$, then

if $\xi<\eta$, $E\cup H\in\Lambda_\eta$ so $\underline{\tilde\phi}(E\cup F\cup H)=\underline{\tilde\phi}(E\cup H)\cup\underline{\tilde\phi}F$, and

$$\begin{aligned}\underline{\phi}_H(E\cup F)&=\underline{\tilde\phi}(E\cup F)\cup(A_H\cap\underline{\tilde\phi}(E\cup F\cup H))\\&=\underline{\tilde\phi}E\cup\underline{\tilde\phi}F\cup(A_H\cap\underline{\tilde\phi}(E\cup H))\cup(A_H\cap\underline{\tilde\phi}F)\subseteq\underline{\phi}_H E\cup\underline{\phi}_H F;\end{aligned}$$

if $\eta\le\xi$, $F\cup H\in\Lambda_{\zeta\setminus\eta}$ so $\underline{\tilde\phi}(E\cup F\cup H)=\underline{\tilde\phi}(E)\cup\underline{\tilde\phi}(F\cup H)$, and

$$\begin{aligned}\underline{\phi}_H(E\cup F)&=\underline{\tilde\phi}(E\cup F)\cup(A_H\cap\underline{\tilde\phi}(E\cup F\cup H))\\&=\underline{\tilde\phi}E\cup\underline{\tilde\phi}F\cup(A_H\cap\underline{\tilde\phi}E))\cup(A_H\cap\underline{\tilde\phi}(F\cup H))\subseteq\underline{\phi}_H E\cup\underline{\phi}_H F;\end{aligned}$$

accordingly $\underline{\phi}_H(E\cup F)=\underline{\phi}_H E\cup\underline{\phi}_H F$. **Q**

By (d-ii) we have

$$\underline{\tilde\phi}H\cup\underline{\tilde\phi}(Z\setminus H)=Z$$

whenever $\xi<\zeta$ and $H\in\Lambda_{\{\xi\}}$.

(f) If $\xi\le\zeta$ and $H\in\Lambda_\xi$, then $\underline{\phi}_H\in P$. **P** Induce on ξ. For $\xi=0$, $H\in\Lambda_0=\{\emptyset,Z\}$ so $\underline{\tilde\phi}H$ is either $\emptyset$ or Z, $A_H=\emptyset$ and $\underline{\phi}_H=\underline{\tilde\phi}$ belongs to P. For the inductive step to $\xi\le\zeta$, we have the following.

(i) If $\eta<\zeta$ and $E\in\Lambda_\eta$, then

(α) if $\xi\le\eta$, $E\cup H$ and $\underline{\tilde\phi}E$ and $\underline{\tilde\phi}(E\cup H)$ and A_H all belong to Λ_η, so $\underline{\phi}_H E\in\Lambda_\eta$.

(β) if $\eta<\xi$, then, by the inductive hypothesis, $\underline{\phi}_E\in P$, $\underline{\tilde\phi}E=Z\setminus\underline{\tilde\phi}(Z\setminus E)$ and $\underline{\tilde\phi}(E\cup H)=\underline{\tilde\phi}E\cup\underline{\tilde\phi}H$, by (d-ii) and (d-iii) above; so $\underline{\phi}_H E=\underline{\tilde\phi}E\in\Lambda_\eta$ by (d-i).

(ii) If $\eta<\zeta$ and $E\in\Lambda_{\{\eta\}}$, then, by (e), $\underline{\tilde\phi}E\cup\underline{\tilde\phi}(Z\setminus E)=Z$, so that $\underline{\tilde\phi}(E\cup H)=\underline{\tilde\phi}E\cup\underline{\tilde\phi}H$, by (d-iii), and $\underline{\phi}_H E=\underline{\tilde\phi}E\in\Lambda_{\{\eta\}}$, by (d-i).

(iii) If $\eta<\zeta$, $E\in\Lambda_\eta$ and $F\in\Lambda_{\zeta\setminus\eta}$, then

(α) if $\xi\le\eta$, then $E\cup H\in\Lambda_\eta$ and $F\in\Lambda_{\zeta\setminus\eta}$, so that $\underline{\tilde\phi}(E\cup F\cup H)=\underline{\tilde\phi}(E\cup H)\cup\underline{\tilde\phi}F$, and

$$\begin{aligned}\underline{\phi}_H(E\cup F)&=\underline{\tilde\phi}(E\cup F)\cup(A_H\cap\underline{\tilde\phi}(E\cup F\cup H))\\&=\underline{\tilde\phi}E\cup\underline{\tilde\phi}F\cup(A_H\cap\underline{\tilde\phi}(E\cup H))\cup(A_H\cap\underline{\tilde\phi}F)\subseteq\underline{\phi}_H E\cup\underline{\phi}_H F,\end{aligned}$$

as in (e-ii) above, and accordingly $\underline{\phi}_H(E\cup F)=\underline{\phi}_H E\cup\underline{\phi}_H F$.

(β) If $\eta<\xi$ then, as in (ii), using the inductive hypothesis, we have $\underline{\tilde\phi}(E\cup F\cup H)=\underline{\tilde\phi}E\cup\underline{\tilde\phi}(F\cup H)$, and (just as in ($\alpha$)) we get $\underline{\phi}_H(E\cup F)=\underline{\phi}_H E\cup\underline{\phi}_H F$.

Thus $\underline{\phi}_H\in P$ and the induction continues. **Q**

(g) But the case $\xi=\zeta$ of (f) just tells us that

$$\underline{\tilde\phi}H\cup\underline{\tilde\phi}(Z\setminus H)=Z$$

for every $H\in\Lambda$. This means that $\underline{\tilde\phi}$ is actually a lifting (since it preserves intersections and complements). And the definition of P is just what is needed to ensure that it is a lifting of the right type.

Remark This result is due to Macheras & Strauss 96b.

346I Theorem Let (X,Σ,μ) be a complete probability space. For any set I, write λ_I for the product measure on X^I, Λ_I for its domain and $\pi_{Ii}(x)=x(i)$ for $x\in X^I$, $i\in I$. Then there is a lifting $\psi:\Sigma\to\Sigma$ such that for every set I there is a lifting $\phi:\Lambda_I\to\Lambda_I$ such that $\phi(\pi_{Ii}^{-1}[E])=\pi_{Ii}^{-1}[\psi E]$ whenever $E\in\Sigma$ and $i\in I$.

proof ? Suppose, if possible, otherwise.

Let Ψ be the set of all liftings for μ. We are supposing that for every $\psi\in\Psi$ there is a set I_ψ for which there is no lifting for λ_{I_ψ} consistent with ψ in the sense above. Let κ be a cardinal greater than $\max(\omega,\#(\Psi),\sup_{\psi\in\Psi}\#(I_\psi))$. Let $\phi_0:\Lambda_\kappa\to\Lambda_\kappa$ be a lifting satisfying the conditions of 346H. 346Bb tells us that for every $\xi<\kappa$ we have a lifting ψ for μ defined by the formula $\pi_{\kappa\xi}^{-1}[\psi E]=\phi_0(\pi_{\kappa\xi}^{-1}[E])$. For $\psi\in\Psi$ set

$$K_\psi=\{\xi:\xi<\kappa,\ \phi_0(\pi_{\kappa\xi}^{-1}[E])=\pi_{\kappa\xi}^{-1}[\psi E]\text{ for every }E\in\Sigma\}.$$

Then $\bigcup_{\psi\in\Psi}K_\psi=\kappa$, so $\kappa\le\max(\omega,\#(\Psi),\sup_{\psi\in\Psi}\#(K_\psi))$ and there is some $\psi\in\Psi$ such that $\#(K_\psi)>\#(I_\psi)$. Take $I\subseteq K_\psi$ such that $\#(I)=\#(I_\psi)$.

We may regard X^κ as $X^I\times X^{\kappa\setminus I}$, and in this form we can use the method of 346F to obtain a lower density $\underline{\phi}:\Lambda_I\to\Lambda_I$ from $\phi_0:\Lambda_\kappa\to\Lambda_\kappa$. Now

$$\underline{\phi}(\pi_{I\xi}^{-1}[E])=\pi_{I\xi}^{-1}[\psi E]\text{ for every }E\in\Sigma,\ \xi\in I.$$

P The point is that $\pi_{I\xi}^{-1}[E]\times X^{\kappa\setminus I}$ corresponds to $\pi_{\kappa\xi}^{-1}[E]\subseteq X^\kappa$, while $\phi_0(\pi_{\kappa\xi}^{-1}[E])=\pi_{\kappa\xi}^{-1}[\psi E]$ can be identified with $\pi_{I\xi}^{-1}[\psi E]\times X^{\kappa\setminus I}$. Now the construction of 346F obviously makes $\underline{\phi}(\pi_{I\xi}^{-1}[E])$ equal to $\pi_{I\xi}^{-1}[\psi E]$. **Q**

By 341Jb, there is a lifting $\phi:\Lambda_I\to\Lambda_I$ such that $\phi W\supseteq\underline{\phi}W$ for every $W\in\Lambda_I$. But now we must have

$$\begin{aligned}\pi_{I\xi}^{-1}[\psi E]&=\underline{\phi}(\pi_{I\xi}^{-1}[E])\subseteq\phi(\pi_{I\xi}^{-1}[E])\\&=X^I\setminus\phi(\pi_{I\xi}^{-1}[X\setminus E])\subseteq X^I\setminus\underline{\phi}(\pi_{I\xi}^{-1}[X\setminus E])\\&=X^I\setminus\pi_{I\xi}^{-1}[\psi(X\setminus E)]=X^I\setminus\pi_{I\xi}^{-1}[X\setminus\psi E]=\pi_{I\xi}^{-1}[\psi E]\end{aligned}$$

and $\phi(\pi_{I\xi}^{-1}[E])=\pi_{I\xi}^{-1}[\psi E]$ for every $E\in\Sigma$ and $\xi\in I$. But since $\#(I)=\#(I_\psi)$, this must be impossible, by the choice of I_ψ. **X**

This contradiction proves the theorem.

346J Consistent liftings Let (X,Σ,μ) be a measure space. A lifting $\psi:\Sigma\to\Sigma$ is **consistent** if for every $n\ge1$ there is a lifting ϕ_n of the product measure on X^n such that $\phi_n(E_1\times\ldots\times E_n)=\psi E_1\times\ldots\times\psi E_n$ for all $E_1,\ldots,E_n\in\Sigma$. Thus 346I tells us, in part, that every complete probability space has a consistent lifting; it follows that every non-trivial complete totally finite measure space has a consistent lifting.

I do not suppose you will be surprised to be told that not all liftings on probability spaces are consistent. What may be surprising is the fact that one of the standard liftings already introduced is not consistent. This depends on a general fact about Stone spaces of measure algebras which has further important applications, so I present it as a lemma.

346K Lemma Let (Z,T,ν) be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, and let λ be the product measure on $Z\times Z$, with Λ its domain. Then there is a set $W\in\Lambda$, with $\lambda W<1$, such that $\lambda^*\tilde W=1$, where

$$\tilde W=\bigcup\{G\times H:G,\ H\subseteq Z\text{ are open-and-closed, }(G\times H)\setminus W\text{ is negligible}\}.$$

Remark For the sake of anybody who has already become acquainted with the alternative measures which can be put on the product of topological measure spaces, I ought to insist that the 'product measure' λ here is, as always in this volume, the ordinary completed product measure as defined in Chapter 25.

proof (a) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence of measurable subsets of $[0,1]$, stochastically independent for Lebesgue measure μ on $[0,1]$, such that $\mu E_n=\frac{1}{n+2}$ for each n. Set $a_n=E_n^\bullet$ in the measure of algebra of μ, and $E_n^*=\widehat{a}_n$ the corresponding compact open subset of Z. Set $W=\bigcup_{n\in\mathbb{N}}E_n^*\times E_n^*$. Then

$$\lambda W\le\textstyle\sum_{n=0}^\infty(\nu E_n)^2=\sum_{n=2}^\infty\frac{1}{n^2}<1.$$

? Suppose, if possible, that $\lambda^*\tilde W<1$. Then there are sequences $\langle G_n\rangle_{n\in\mathbb{N}}$, $\langle H_n\rangle_{n\in\mathbb{N}}$ in T such that $\tilde W\subseteq\bigcup_{n\in\mathbb{N}}G_n\times H_n$ and $\lambda(\bigcup_{n\in\mathbb{N}}G_n\times H_n)<1$. Recall from 322Rc that

$$\nu F = \inf\{\nu G : G \text{ is compact and open, } F \subseteq G\}$$

for every $F \in \mathrm{T}$. Accordingly we can find compact open sets $\tilde{G}_n$, $\tilde{H}_n$ such that $G_n \subseteq \tilde{G}_n$, $H_n \subseteq \tilde{H}_n$ for every $n \in \mathbb{N}$ and

$$\sum_{n=0}^{\infty} \nu(\tilde{G}_n \setminus G_n) + \sum_{n=0}^{\infty} \nu(\tilde{H}_n \setminus H_n) < 1 - \lambda(\bigcup_{n\in\mathbb{N}} G_n \times H_n),$$

so that $\lambda(\bigcup_{n\in\mathbb{N}} \tilde{G}_n \times \tilde{H}_n) < 1$.

Let $\mathcal{U}_0$ be the family

$$\{Z\} \cup \{E_n^* : n \in \mathbb{N}\} \cup \{Z \setminus \tilde{G}_n : n \in \mathbb{N}\} \cup \{Z \setminus \tilde{H}_n : n \in \mathbb{N}\},$$

so that $\mathcal{U}_0$ is a countable subset of T. Let $\mathcal{U}$ be the set of finite intersections $U_0 \cap U_1 \cap \ldots \cap U_n$ where $U_0, \ldots, U_n \in \mathcal{U}_0$, so that $\mathcal{U}$ also is a countable subset of T, and $\mathcal{U}$ is closed under $\cap$.

(b) For $U \in \mathcal{U}$, define $Q(U)$ as follows. If $\nu U = 0$, then $Q(U) = U$. Otherwise,

$$Q(U) = Z \setminus \bigcup\{E_n^* : n \in \mathbb{N},\ \nu(E_n^* \cap U) > 0\}.$$

Then $\nu Q(U)$ is always 0. **P** Of course this is true if $\nu U = 0$, so suppose that $\nu U > 0$. Set $I = \{n : \nu(E_n^* \cap U) = 0\}$. Then we have $\nu U' > 0$, where $U' = U \setminus \bigcup_{n\in I} E_n^*$, and $Z \setminus E_n^* \supseteq U'$ for every $n \in I$. Because $\langle E_n \rangle_{n\in\mathbb{N}}$ is stochastically independent for μ, $\langle E_n^* \rangle_{n\in\mathbb{N}}$ is stochastically independent for ν, while

$$\nu(\bigcup_{n\in I} E_n^*) \le 1 - \nu U' < 1.$$

By the Borel-Cantelli lemma (273K), $\sum_{n\in I} \nu E_n^* < \infty$. Consequently $\sum_{n\in\mathbb{N}\setminus I} \nu E_n^* = \infty$, because $\sum_{n=0}^{\infty} \frac{1}{n+2}$ is infinite, so

$$\nu(Z \setminus Q(U)) = \nu(\bigcup_{n\in\mathbb{N}\setminus I} E_n^*) = 1,$$

and $\nu Q(U) = 0$. **Q**

(c) Set $Q_0 = \bigcup_{U\in\mathcal{U}} Q(U)$; because $\mathcal{U}$ is countable, Q_0 is negligible. Accordingly $(Z \setminus Q_0)^2$ has measure 1 and cannot be included in $\bigcup_{n\in\mathbb{N}} \tilde{G}_n \times \tilde{H}_n$; take $(w, z) \in (Z \setminus Q_0)^2 \setminus \bigcup_{n\in\mathbb{N}} \tilde{G}_n \times \tilde{H}_n$.

(d) We can find sequences $\langle C_n \rangle_{n\in\mathbb{N}}$, $\langle D_n \rangle_{n\in\mathbb{N}}$, $\langle U_n \rangle_{n\in\mathbb{N}}$ and $\langle V_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{U}$ such that

$w \in U_{n+1} \subseteq U_n$, $z \in V_{n+1} \subseteq V_n$, $(U_{n+1} \times V_{n+1}) \cap (\tilde{G}_n \times \tilde{H}_n) = \emptyset$,

$\nu C_n > 0$, $\nu D_n > 0$,

$C_n \subseteq U_n$, $D_n \subseteq V_{n+1}$,

$C_n \times V_{n+1} \subseteq W$, $U_{n+1} \times D_n \subseteq W$

for every $n \in \mathbb{N}$. **P** Build the sequences inductively, as follows. Start with $U_0 = V_0 = Z$. Given that $w \in U_n \in \mathcal{U}$ and $z \in V_n \in \mathcal{U}$, then we know that $(w, z) \notin \tilde{G}_n \times \tilde{H}_n$. If $w \notin \tilde{G}_n$, set $U_n' = U_n \setminus \tilde{G}_n$, $V_n' = V_n$; otherwise set $U_n' = U_n$, $V_n' = V_n \setminus \tilde{H}_n$. In either case, we have $w \in U_n' \in \mathcal{U}$, $z \in V_n' \in \mathcal{U}$ and $(U_n' \times V_n') \cap (\tilde{G}_n \times \tilde{H}_n) = \emptyset$.

Because $U_n' \in \mathcal{U}$, $w \notin Q(U_n')$. But $w \in U_n'$, so this must be because $\nu U_n' > 0$. Now $z \notin Q(U_n')$, so $z \in \bigcup\{E_k^* : k \in \mathbb{N},\ \nu(E_k^* \cap U_n') > 0\}$. Take some $k \in \mathbb{N}$ such that $z \in E_k^*$ and $\nu(E_k^* \cap U_n') > 0$, and set

$$V_{n+1} = V_n' \cap E_k^*, \quad C_n = E_k^* \cap U_n',$$

so that

$$z \in V_{n+1} \in \mathcal{U}, \quad C_n \subseteq U_n, \quad C_n \times V_{n+1} \subseteq E_k^* \times E_k^* \subseteq W, \quad \nu C_n > 0.$$

Next, $z \notin Q(V_{n+1})$ and $\nu V_{n+1} > 0$; also $w \notin Q(V_{n+1})$, so there is an l such that $w \in E_l^*$ and $\nu(E_l^* \cap V_{n+1}) > 0$. Set

$$U_{n+1} = U_n' \cap E_l^*, \quad D_n = E_l^* \cap V_{n+1},$$

so that

$$w \in U_{n+1} \in \mathcal{U}, \quad D_n \subseteq V_{n+1}, \quad U_{n+1} \times D_n \subseteq E_l^* \times E_l^* \subseteq W, \quad \nu D_n > 0,$$

$$(U_{n+1} \times V_{n+1}) \cap (\tilde{G}_n \times \tilde{H}_n) \subseteq (U_n' \times V_n') \cap (\tilde{G}_n \times \tilde{H}_n) = \emptyset,$$

and continue the process. **Q**

(e) Setting $C = \bigcup_{n\in\mathbb{N}} C_n$ and $D = \bigcup_{n\in\mathbb{N}} D_n$ we see that $C \times D \subseteq W$. **P** If $m \le n$, $D_n \subseteq V_{n+1} \subseteq V_{m+1}$, so $C_m \times D_n \subseteq W$. If $m > n$, $C_m \subseteq U_m \subseteq U_{n+1}$, so $C_m \times D_n \subseteq W$. **Q**

Recall from 321K that the measurable sets of Z are precisely those of the form $G\triangle M$ where M is nowhere dense and negligible and G is compact and open. There must therefore be compact open sets G, $H\subseteq Z$ such that $G\triangle C$ and $H\triangle D$ are negligible. Consequently

$$(G\times H)\setminus W\subseteq((G\setminus C)\times Z)\cup(Z\times(H\setminus D))$$

is negligible, and

$$G\times H\subseteq\tilde W\subseteq\bigcup_{n\in\mathbb{N}}\tilde G_n\times\tilde H_n.$$

But because $G\times H$ is compact (3A3J), and all the $\tilde G_n\times\tilde H_n$ are open, there must be some n such that $G\times H\subseteq\bigcup_{k\le n}\tilde G_k\times\tilde H_k=S$ say. Now $(U_{k+1}\times V_{k+1})\cap(\tilde G_k\times\tilde H_k)=\emptyset$ for every k, so

$$(C_{n+2}\times D_{n+2})\cap(G\times H)\subseteq(U_{n+1}\times V_{n+1})\cap S=\emptyset,$$

and either $C_{n+2}\cap G=\emptyset$ or $D_{n+2}\cap H=\emptyset$. Since

$$C_{n+2}\setminus G\subseteq C\setminus G,\quad D_{n+2}\setminus H\subseteq D\setminus H$$

are both negligible, one of C_{n+2}, D_{n+2} is negligible. But the construction took care to ensure that all the C_k, D_k were non-negligible. **X**

(f) Thus $\lambda^*\tilde W=1$, as required.

346L Proposition Let (Z,T,ν) be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$. Let $\psi:\mathrm{T}\to\mathrm{T}$ be the canonical lifting, defined by setting $\psi E=G$ whenever $E\in\mathrm{T}$, G is open-and-closed and $E\triangle G$ is negligible (341O). Then ψ is not consistent.

proof ? Suppose, if possible, that ϕ is a lifting on $Z\times Z$ such that $\phi(E\times F)=\psi E\times\psi F$ for every E, $F\in\mathrm{T}$. Let $W\subseteq Z\times Z$ be a set as in 346K, and consider ϕW. If G, $H\subseteq Z$ are open-and-closed and $(G\times H)\setminus W$ is negligible, then

$$G\times H=\psi G\times\psi H=\phi(G\times H)\subseteq\phi W;$$

that is, in the language of 346K, we must have $\tilde W\subseteq\phi W$. But this means that

$$\lambda(\phi W)\ge\lambda^*\tilde W=1>\lambda W,$$

which is impossible. **X**

Thus ψ fails the first test and cannot be consistent.

346X Basic exercises (a) Let (X,Σ,μ) be a measure space and $\langle\underline{\phi}_n\rangle_{n\in\mathbb{N}}$ a sequence of lower densities for μ. (i) Show that $E\mapsto\bigcap_{n\in\mathbb{N}}\underline{\phi}_nE$ and $E\mapsto\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\underline{\phi}_mE$ are also lower densities for μ. (ii) Show that if μ is complete and $\mathcal{F}$ is any filter on $\mathbb{N}$, then $E\mapsto\bigcup_{F\in\mathcal{F}}\bigcap_{n\in F}\underline{\phi}_nE$ is a lower density for μ.

(b) Let (X,Σ,μ) be a strictly localizable measure space, and G a countable group of measure space automorphisms from X to itself. Show that there is a lower density $\underline{\phi}:\Sigma\to\Sigma$ which is G-invariant in the sense that $\underline{\phi}(g^{-1}[E])=g^{-1}[\underline{\phi}E]$ for every $E\in\Sigma$ and $g\in G$. (*Hint*: set $\underline{\phi}E=\bigcap_{g\in G}g[\underline{\phi}_0(g^{-1}[E])]$.)

>(c) Show that there is no lifting ϕ of Lebesgue measure on $[0,1]^2$ which is 'symmetric' in the sense that $\phi(E^{-1})=(\phi E)^{-1}$ for every measurable set E, writing $E^{-1}=\{(y,x):(x,y)\in E\}$. (*Hint*: 345Xc.)

(d) Let (X,Σ,μ) be a measure space and $\underline{\phi}$ a lower density for μ. Take $H\in\Sigma$ and set $A=X\setminus(\underline{\phi}H\cup\underline{\phi}(Z\setminus H))$, $\underline{\phi}'E=\underline{\phi}E\cup(A\cap\underline{\phi}(H\cup E))$ for $E\in\Sigma$. Show that $\underline{\phi}'$ is a lower density.

(e) Describe the connections between 346B, 346D and 346F.

>(f) Suppose, in 341H, that (X,Σ,μ) is a product of probability spaces, and that in the proof, instead of taking $\langle a_\xi\rangle_{\xi<\kappa}$ to run over the whole measure algebra $\mathfrak{A}$, we take it to run over the elements of $\mathfrak{A}$ expressible as $E^\bullet$ where $E\in\Sigma$ is determined by a single coordinate. Show that the resulting lower density $\underline{\theta}$ respects coordinates in the sense that $\underline{\theta}E^\bullet$ is determined by coordinates in J whenever $E\in\Sigma$ is determined by coordinates in J. (Compare Macheras & Strauss 95, Theorem 2.)

>(g) Let $\underline{\phi}$ be lower Lebesgue density on $\mathbb{R}$, and ϕ a translation-invariant lifting for Lebesgue measure on $\mathbb{R}$ such that $\phi E \supseteq \underline{\phi}E$ for every measurable set E. Show that ϕ is consistent. (*Hint*: given $n \ge 1$, let $\underline{\phi}_n$ be lower Lebesgue density on $\mathbb{R}^n$. Let $\mathcal{I}$ be the ideal generated by

$$\{W : \mathbf{0} \in \underline{\phi}_n(\mathbb{R}^n \setminus W)\} \cup \bigcup_{i<n}\{\pi_i^{-1}[E] : 0 \in \phi(\mathbb{R} \setminus E)\};$$

show that $\mathbb{R}^n \notin \mathcal{I}$, so that we can use the method of 345B to construct a lifting for Lebesgue measure on $\mathbb{R}^n$.)

(h) Show that Lemma 346K is valid for any (Z, T, ν) which is the Stone space of an atomless probability space.

346Y Further exercises (a) Let $(X_1, \Sigma_1, \mu_1), \dots , (X_n, \Sigma_n, \mu_n)$ be probability spaces with product (X, Σ, μ). Show that there is a lifting for μ which respects coordinates. (Burke n95.)

(b) Let (X, Σ, μ) be a probability space, I any set, and λ the product measure on X^I. Show that there is a lower density for λ which is invariant under transpositions of pairs of coordinates.

(c) Suppose that (X, Σ, μ) and (Y, T, ν) are complete probability spaces with product $(X \times Y, \Lambda, \lambda)$. Show that for any lifting $\psi_1 : \Sigma \to \Sigma$ there are liftings $\psi_2 : \mathrm{T} \to \mathrm{T}$ and $\phi : \Lambda \to \Lambda$ such that $\phi(E \times F) = \psi_1 E \times \psi_2 F$ for all $E \in \Sigma$, $F \in \mathrm{T}$. (*Hint*: use the methods of §341. In the inductive construction of 341H, start with $\underline{\phi}_0(E \times Y) = (\psi_1 E) \times Y$ for every $E \in \Sigma$. Extend each lower density $\underline{\phi}_\xi$ to the algebra generated by $\mathrm{dom}(\underline{\phi}_\xi) \cup \{X \times F_\xi\}$ for some $F_\xi \in \mathrm{T}$. Make sure that $\underline{\phi}_\xi(X \times F)$ is always of the form $X \times F'$, and that $\underline{\phi}_\xi((E \times Y) \cup (X \times F)) = \underline{\phi}_\xi(E \times Y) \cup \underline{\phi}_\xi(X \times F)$; adapt the construction of 341G to maintain this. Use the method of 346H to generate a lifting from the final lower density $\underline{\phi}$. See Macheras & Strauss 96a, Theorem 4.)

(d) Use 346Yc and induction on ζ to prove 346H. (Macheras & Strauss 96b.)

(e) Let (X, Σ, μ) be a complete probability space. Show that there is a lifting $\psi : \Sigma \to \Sigma$ such that whenever $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ is a family of probability spaces, with product measure λ, there is a lifting ϕ for λ such that $\phi(\pi_i^{-1}[E]) = \pi_i^{-1}[\psi E]$ whenever $E \in \Sigma$ and $i \in I$ is such that $(X_i, \Sigma_i, \mu_i) = (X, \Sigma, \mu)$, writing $\pi_i(x) = x(i)$ for $x \in \prod_{i\in I} X_i$.

346Z Problems (a) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X, Σ, μ). Is there always a lifting for μ which respects coordinates in the sense of 346A?

(b) Is there a lower density $\underline{\phi}$ for the usual measure on $\{0, 1\}^{\mathbb{N}}$ which is invariant under all permutations of coordinates?

346 Notes and comments I ought to say at once that in writing this section I have been greatly assisted by M.R.Burke.

The theorem that every complete probability space has a consistent lifting (346J) is due to Talagrand 82a; it is the inspiration for the whole of the section. 'Consistent' liftings were devised in response to some very interesting questions (see Talagrand 84, §6) which I do not discuss here; one will be mentioned in Theorem 465P in Volume 4. My aim here is rather to suggest further ways in which a lifting on a product space can be consistent with the product structure. The labour is substantial and the results achieved are curiously partial. I offer 346Za as the easiest natural question which does not appear amenable to the methods I describe.

The arguments I use are based on the fact that the translation-invariant measures of 345C already respect coordinates (346C). Maharam's theorem now makes it easy to show that any product of Maharam-type-homogeneous probability spaces has a lifting which respects coordinates (346E). A kind of projection argument (346F) makes it possible to obtain a lower density which respects coordinates on any product of probability spaces (346G). In fact the methods of §341, very slightly refined, automatically produce such lower densities (346Xf). But the extra power of 346G lies in the condition (ii): if E and F are 'fully independent' in the sense of being determined by coordinates in disjoint sets, then $\underline{\phi}(E \cup F) = \underline{\phi}E \cup \underline{\phi}F$, that is, $\underline{\phi}$ is making a tentative step towards being a lifting. (Remember that the difference between a lifting and a lower density is mostly that a lifting preserves finite unions as well as finite intersections; see 341Xa.) This can also be achieved by a modification of the previous method, but we have to work harder at one point in the proof.

The next step is to move to liftings which continue, as far as possible, to respect coordinates. Here there seem to be quite new obstacles, and 346H is the best result I know; the lifting respects *individual* coordinates, and also, for a given well-ordering of the index set, initial segments of the coordinates. The treatment of initial segments makes essential use of the well-ordering, which is what leaves 346Za open.

Finally, if all the factors are identical, we can seek lower densities and liftings which are invariant under permutation of coordinates. I give 345Xc and 346Xc as examples to show that we must not just assume that a symmetry in the underlying measure space can be reflected in a symmetry of a lifting. The problems there concern liftings themselves, not lower densities, since we can frequently find lower densities which share symmetries (346Xb, 346Yb). (Even for lower densities there seem to be difficulties if we are more ambitious (346Zb).) However a very simple argument (346I) shows that at least we can make each individual coordinate look more or less the same, as long as we do not investigate its relations with others.

Still on the question of whether, and when, liftings can be 'good', note 346L/346Xh and 346Xg. The most natural liftings for Lebesgue measure are necessarily consistent; but the only example we have of a truly canonical lifting is not consistent in any non-trivial context.

I have deliberately used a variety of techniques here, even though 346H (for instance) has an alternative proof based on the ideas of §341 (346Yc-346Yd). In particular, I give some of the standard methods of constructing liftings and lower densities (346B, 346D, 346F, 346Xd, 346Xa). In fact 346D was one of the elements of Maharam's original proof of the lifting theorem (Maharam 58).

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this volume, and which have since been changed.

314V Upper envelopes The note on upper envelopes, referred to in the 2003 and 2006 editions of Volume 4, has been moved to 313S.

315H Paragraphs 315H-315N, referred to in the 2003 and 2006 printings of Volume 4 and the 2008 printing of Volume 5, are now 315I-315O.

316J Weakly (σ,∞)-distributive algebras The reference to 316J in the 2003 edition of Volume 4 should be changed to 316H.

322K Paragraphs 322K (simple products of measure algebras), 322N (the Stone space of a measure algebra) and 322Q (further properties of Stone spaces), referred to in the 2003 and 2006 editions of Volume 4, are now 322L, 322O and 322R.

326E Countably additive functionals Definition 326E, referred to in the 2003 and 2006 editions of Volume 4 and the 2008 edition of Volume 5, is now 326I.

326G Corollary 326G, referred to in the 2008 edition of Volume 5, is now 326K.

326I Hahn decomposition Theorem 326I, referred to in the 2003 and 2006 editions of Volume 4, is now 326M.

326K Completely additive functionals The notes in 326K, referred to in the 2003 and 2006 editions of Volume 4, have been moved to 326O.

326Q Finitely additive functionals on free products Theorem 326Q, referred to in the 2003 and 2006 editions of Volume 4 and the 2008 edition of Volume 5, is now 326E.

328D Reduced products of probability algebras Paragraph 328D, referred to in the 2008 edition of Volume 5, is now 328E.

341X Exercises 341Xd and 341Xf, referred to in the 2003 and 2006 editions of Volume 4, are now 341Xc and 341Xe.