MEASURE THEORY

Volume 5

Part I

D.H.Fremlin

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MEASURE THEORY

Volume 5

Set-theoretic Measure Theory

Part I

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Dedicated by the Author

to the Publisher

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Part II

General introduction In this treatise I aim to give a comprehensive description of modern abstract measure theory, with some indication of its principal applications. The first two volumes are set at an introductory level; they are intended for students with a solid grounding in the concepts of real analysis, but possibly with rather limited detailed knowledge. As the book proceeds, the level of sophistication and expertise demanded will increase; thus for the volume on topological measure spaces, familiarity with general topology will be assumed. The emphasis throughout is on the mathematical ideas involved, which in this subject are mostly to be found in the details of the proofs.

My intention is that the book should be usable both as a first introduction to the subject and as a reference work. For the sake of the first aim, I try to limit the ideas of the early volumes to those which are really essential to the development of the basic theorems. For the sake of the second aim, I try to express these ideas in their full natural generality, and in particular I take care to avoid suggesting any unnecessary restrictions in their applicability. Of course these principles are to to some extent contradictory. Nevertheless, I find that most of the time they are very nearly reconcilable, *provided* that I indulge in a certain degree of repetition. For instance, right at the beginning, the puzzle arises: should one develop Lebesgue measure first on the real line, and then in spaces of higher dimension, or should one go straight to the multidimensional case? I believe that there is no single correct answer to this question. Most students will find the one-dimensional case easier, and it therefore seems more appropriate for a first introduction, since even in that case the technical problems can be daunting. But certainly every student of measure theory must at a fairly early stage come to terms with Lebesgue area and volume as well as length; and with the correct formulations, the multidimensional case differs from the one-dimensional case only in a definition and a (substantial) lemma. So what I have done is to write them both out (§§114-115). In the same spirit, I have been uninhibited, when setting out exercises, by the fact that many of the results I invite students to look for will appear in later chapters; I believe that throughout mathematics one has a better chance of understanding a theorem if one has previously attempted something similar alone.

The plan of the work is as follows:

Volume 1: The Irreducible Minimum
Volume 2: Broad Foundations
Volume 3: Measure Algebras
Volume 4: Topological Measure Spaces
Volume 5: Set-theoretic Measure Theory.

Volume 1 is intended for those with no prior knowledge of measure theory, but competent in the elementary techniques of real analysis. I hope that it will be found useful by undergraduates meeting Lebesgue measure for the first time. Volume 2 aims to lay out some of the fundamental results of pure measure theory (the Radon-Nikodým theorem, Fubini's theorem), but also gives short introductions to some of the most important applications of measure theory (probability theory, Fourier analysis). While I should like to believe that most of it is written at a level accessible to anyone who has mastered the contents of Volume 1, I should not myself have the courage to try to cover it in an undergraduate course, though I would certainly attempt to include some parts of it. Volumes 3 and 4 are set at a rather higher level, suitable to postgraduate courses; while Volume 5 assumes a wide-ranging competence over large parts of analysis and set theory.

There is a disclaimer which I ought to make in a place where you might see it in time to avoid paying for this book. I make no real attempt to describe the history of the subject. This is not because I think the history uninteresting or unimportant; rather, it is because I have no confidence of saying anything which would not be seriously misleading. Indeed I have very little confidence in anything I have ever read concerning the history of ideas. So while I am happy to honour the names of Lebesgue and Kolmogorov and Maharam in more or less appropriate places, and I try to include in the bibliographies the works which I have myself consulted, I leave any consideration of the details to those bolder and better qualified than myself.

For the time being, at least, printing will be in short runs. I hope that readers will be energetic in commenting on errors and omissions, since it should be possible to correct these relatively promptly. An inevitable consequence of this is that paragraph references may go out of date rather quickly. I shall be most flattered if anyone chooses to rely on this book as a source for basic material; and I am willing to attempt to maintain a concordance to such references, indicating where migratory results have come to rest for the moment, if authors will supply me with copies of papers which use them.

I mention some minor points concerning the layout of the material. Most sections conclude with lists of 'basic exercises' and 'further exercises', which I hope will be generally instructive and occasionally entertaining. How many of these you should attempt must be for you and your teacher, if any, to decide, as no two students will have quite the same needs. I mark with a **>** those which seem to me to be particularly important. But while you may not need to write out solutions to all the 'basic exercises', if you are in any doubt as to your capacity to do so you should take this as a warning to slow down a bit. The 'further exercises' are unbounded in difficulty, and are unified only by a presumption that each has at least one solution based on ideas already introduced. Occasionally I add a final 'problem', a question to which I do not know the answer and which seems to arise naturally in the course of the work.

The impulse to write this book is in large part a desire to present a unified account of the subject. Cross-references are correspondingly abundant and wide-ranging. In order to be able to refer freely across the whole text, I have chosen a reference system which gives the same code name to a paragraph wherever it is being called from. Thus 132E is the fifth paragraph in the second section of the third chapter of Volume 1, and is referred to by that name throughout. Let me emphasize that cross-references are supposed to help the reader, not distract her. Do not take the interpolation '(121A)' as an instruction, or even a recommendation, to lift Volume 1 off the shelf and hunt for §121. If you are happy with an argument as it stands, independently of the reference, then carry on. If, however, I seem to have made rather a large jump, or the notation has suddenly become opaque, local cross-references may help you to fill in the gaps.

Each volume will have an appendix of 'useful facts', in which I set out material which is called on somewhere in that volume, and which I do not feel I can take for granted. Typically the arrangement of material in these appendices is directed very narrowly at the particular applications I have in mind, and is unlikely to be a satisfactory substitute for conventional treatments of the topics touched on. Moreover, the ideas may well be needed only on rare and isolated occasions. So as a rule I recommend you to ignore the appendices until you have some direct reason to suppose that a fragment may be useful to you.

During the extended gestation of this project I have been helped by many people, and I hope that my friends and colleagues will be pleased when they recognise their ideas scattered through the pages below. But I am especially grateful to those who have taken the trouble to read through earlier drafts and comment on obscurities and errors.

There is a particular debt which may not be obvious from the text, and which I ought to acknowledge. From 1984 to 2006 the biennial CARTEMI conferences, organized by the Department of Mathematics of the University Federico II of Naples, were the principal meeting place of European measure theorists, and a clearing house for ideas from all over the world. I had the good fortune to attend nearly all the meetings from 1988 onwards. I do not think it is a coincidence that I should have started work on this book in 1992; and from then on every meeting has contributed something to its content. It would have been very different, probably shorter, but certainly duller, without this regular stimulation. Now the CARTEMI conferences, while of course dependent on the energies and talents of many people, were essentially the creation of one man, whose vision and determination maintained a consistent level of quality and variety. So while the dedication on the title page must remain to my wife, without whose support and forbearance the project would have been simply impossible, I should like to offer a second dedication here, to my friend Paolo de Lucia.

Introduction to Volume 5

For the final volume of this treatise, I have collected results which demand more sophisticated set theory than elsewhere. The line is not sharp, but typically we are much closer to questions which are undecidable in ZFC. Only in Chapter 55 are these brought to the forefront of the discussion, but elsewhere much of the work depends on formulations carefully chosen to express, as arguments in ZFC, ideas which arose in contexts in which some special axiom – Martin's axiom, for instance – was being assumed. This has forced the development of concepts – e.g., cardinal functions of structures – which have taken on vigorous lives of their own, and which stand outside the territory marked by the techniques of earlier volumes.

In terms of the classification I have used elsewhere, this volume has one preparatory chapter and five working chapters. There is practically no measure theory in Chapter 51, which is an introduction to some of the methods which have been devised to make sense of abstract analysis in the vast range of alternative mathematical worlds which have become open to us in the last fifty years. It is centered on a study of partially ordered sets, which provide a language in which many of the most important principles can be expressed. Chapter 52 looks at manifestations of these ideas in measure theory. In Chapter 53 I continue the work of Volumes 3 and 4, examining questions which arise more or less naturally if we approach the topics of those volumes with the new techniques.

The Banach-Ulam problem got a mention in Volume 2, a paragraph in Volume 3 and a section in Volume 4; at last, in Chapter 54 of the present volume, I try to give a proper account of the extraordinary ideas to which it has led. I have regretfully abandoned the idea of describing even a representative sample of the forcing models which have been devised to show that measure-theoretic propositions are consistent, but in Chapter 55 I set out some of the basic properties of random real forcing. Finally, in Chapter 56, I look at what measure theory becomes in ZF alone, with countable or dependent choice, and with the axiom of determinacy.

While I should like to believe that most of the material of this volume will be accessible to those who have learnt measure theory from other sources, it has obviously been written with earlier volumes constantly in mind, and I have to advise you to make sure that Volumes 3 and 4, at least, will be available in case of need. Apart from these, I do of course assume that readers will be at ease with modern set theory. It is not so much that I demand a vast amount of knowledge – §§5A1-5A2 have a good many proofs to help cover any gaps – as that I present arguments without much consideration for the inexperienced, and some of them may be indigestible at first if you have not cut your teeth on Just & Weese 96 or Jech 78. What you do not need is any prior knowledge of forcing. But of course for Chapter 55 you will have to take a proper introduction to forcing, e.g., Kunen 80, in parallel with §5A3, since nothing here will make sense without an acquaintance with forcing languages and the fundamental theorem of forcing.

Note on second printing

There has been the usual crop of errors (most, but not all, minor) to be corrected, and I have added a few new results. The most important is P.Larson's proof that it is relatively consistent with ZFC to suppose that there is no medial limit. In the process of preparing new editions of Volumes 1-4, I have I hope covered all the items listed in the old §5A6 ('Later editions only'), which I have therefore dropped, even though there are one or two further entries under this heading. As before, these can be found on the Web edition at `http://www.essex.ac.uk/maths/people/fremlin/mtcont.htm`.

Chapter 51

Cardinal functions

The primary object of this volume is to explore those topics in measure theory in which questions arise which are undecided by the ordinary axioms of set theory. We immediately face a new kind of interaction between the propositions we consider. If two statements are undecidable, we can ask whether either implies the other. Almost at once we find ourselves trying to make sense of a bewildering tangle of uncoordinated patterns. The most successful method so far found of listing the multiple connexions present is to reduce as many arguments as possible to investigations of the relationships between specially defined cardinal numbers. In any particular model of set theory (so long as we are using the axiom of choice) these numbers must be in a linear order, so we can at least estimate the number of potential configurations, and focus our attention on the possibilities which seem most accessible or most interesting. At the very beginning of the theory, for instance, we can ask whether $\mathfrak{c} = 2^\omega$ is equal to ω_1, or ω_2, or ω_{ω_1}, or 2^{ω_1}. For Lebesgue measure, perhaps the first question to ask is: if $\langle E_\xi\rangle_{\xi<\omega_1}$ is a family of measurable sets, is $\bigcup_{\xi<\omega_1} E_\xi$ necessarily measurable? If the continuum hypothesis is true, certainly not; but if $\mathfrak{c} > \omega_1$, either 'yes' or 'no' becomes possible. The way in which it is now customary to express this is to say that '$\omega_1 \le \operatorname{add}\mathcal{N} \le \mathfrak{c}$, and $\omega_1 \le \operatorname{add}\mathcal{N} < \mathfrak{c}$, $\omega_1 < \operatorname{add}\mathcal{N} \le \mathfrak{c}$ and $\omega_1 < \operatorname{add}\mathcal{N} < \mathfrak{c}$ are all possible', where $\operatorname{add}\mathcal{N}$ is defined as the least cardinal of any family $\mathcal{E}$ of Lebesgue measurable sets such that $\bigcup\mathcal{E}$ is not measurable. (Actually it is not usually defined in quite this way, but that is what it comes to.)

At this point I suggest that you turn to 522B, where you will find a classic picture ('Cichoń's diagram') of the relationships between ten cardinals intermediate between ω_1 and $\mathfrak{c}$, with $\operatorname{add}\mathcal{N}$ immediately above ω_1. As this diagram already makes clear, one can define rather a lot of cardinal numbers. Furthermore, the relationships between them are not entirely expressible in terms of the partial order in which we say that $\kappa_{\mathfrak{a}} \preccurlyeq \kappa_{\mathfrak{b}}$ if we can prove in ZFC that $\kappa_{\mathfrak{a}} \le \kappa_{\mathfrak{b}}$. Even in Cichoń's diagram we have results of the type $\operatorname{add}\mathcal{M} = \min(\mathfrak{b}, \operatorname{cov}\mathcal{M})$ in which three cardinals are involved. It is clear that the framework which has been developed over the last thirty-five years is only a beginning. Nevertheless, I am confident that it will maintain a leading role as the theory evolves. The point is that at least some of the cardinals ($\operatorname{add}\mathcal{N}$, $\mathfrak{b}$ and $\operatorname{cov}\mathcal{M} = \mathfrak{m}_{\text{countable}}$, for instance) describe such important features of such important structures that they appear repeatedly in arguments relating to diverse topics, and give us a chance to notice unexpected connexions.

The first step is to list and classify the relevant cardinals. This is the purpose of the present chapter. In fact the definitions here are mostly of a general type. Associated with any ideal of sets, for instance, we have four cardinals ('additivity', 'cofinality', 'unformity' and 'covering number'; see 511F). Most of the cardinals examined in this volume can be defined by one of a limited number of processes from some more or less naturally arising structure; thus $\operatorname{add}\mathcal{N}$, already mentioned, is normally defined as the additivity of the ideal of Lebesgue negligible subsets of $\mathbb{R}$, and $\operatorname{cov}\mathcal{M}$ is the covering number of the ideal of meager subsets of $\mathbb{R}$. Another important type of definition is in terms of whole classes of structure: thus Martin's cardinal $\mathfrak{m}$ can be regarded as the least Martin number (definition: 511Dg) of any ccc Boolean algebra.

§511 lists some of the cardinals associated with partially ordered sets, Boolean algebras, topological spaces and ideals of sets. Which structures count as 'naturally arising' is a matter of taste and experience, but it turns out that many important ideas can be expressed in terms of cardinals associated with relations, and some of these are investigated in §512. The core ideas of the chapter are most clearly manifest in their application to partially ordered sets, which I look at in §513. In §514 I run through the elementary results connecting the cardinal functions of topological spaces and associated Boolean algebras and partially ordered sets. §515 is a brief excursion into abstract Boolean algebra. §516 is a discussion of 'precalibers'. §517 is an introduction to the theory of 'Martin numbers', which (following the principles I have just tried to explain) I will use as vehicles for the arguments which have been used to make deductions from Martin's axiom. §518 gives results on Freese-Nation numbers and tight filtrations of Boolean algebras which can be expressed in general terms and are relevant to questions in measure theory.

511 Definitions

A large proportion of the ideas of this volume will be expressed in terms of cardinal numbers associated with the structures of measure theory. For any measure space (X,Σ,μ) we have, at least, the structures (X,Σ), $(X,\Sigma,\mathcal{N}(\mu))$ (where $\mathcal{N}(\mu)$ is the null ideal of μ) and the measure algebra $\mathfrak{A} = \Sigma/\Sigma\cap\mathcal{N}(\mu)$; each of these types of structure has a family of cardinal functions associated with it, starting from the obvious ones $\#(X)$, $\#(\Sigma)$ and $\#(\mathfrak{A})$. For the measure algebra $\mathfrak{A}$, we quickly find that we have cardinals naturally associated with its Boolean structure and others naturally associated with the topological structure of its Stone space; of course the most important ones are those which can be described in both languages. The actual measure $\mu : \Sigma \to [0, \infty]$, and its daughter $\bar{\mu} : \mathfrak{A} \to [0, \infty]$, will be less conspicuous here; for most of the questions addressed in this volume, replacing a measure by another with the same measurable sets and the same negligible sets will make no difference.

In this section I list the definitions on which the rest of the chapter depends, with a handful of elementary results to give you practice with the definitions.

511A Pre-ordered sets When we come to the theory of forcing in Chapter 55, there will be technical advantages in using a generalization of the concept of 'partial order'. A **pre-ordered set** is a set P together with a relation $\le$ on P such that

if $p \le q$ and $q \le r$ then $p \le r$,

$p \le p$ for every $p \in P$;

that is, $\le$ is transitive and reflexive but need not be antisymmetric. As with partial orders, I will write $p \ge q$ to mean $q \le p$; $[p, q] = \{r : p \le r \text{ and } r \le q\}$; $[p, \infty[= \{q : p \le q\}$, $]-\infty, p] = \{q : q \le p\}$. An **upper** (resp. **lower**) **bound** for a set $A \subseteq P$ will be a $p \in P$ such that $q \le p$ (resp. $p \le q$) for every $q \in A$. If $(Q, \le)$ is another pre-ordered set, I will say that $f : P \to Q$ is **order-preserving** if $f(p) \le f(p')$ whenever $p \le p'$ in P. If $\langle (P_i, \le_i) \rangle_{i \in I}$ is a family of pre-ordered sets, their **product** is the pre-ordered set $(P, \le)$ where $P = \prod_{i \in I} P_i$ and, for p, $q \in P$, $p \le q$ iff $p(i) \le_i q(i)$ for every $i \in I$ (cf. 315C).

If $(P, \le)$ is a pre-ordered set, we have an equivalence relation $\sim$ on P defined by saying that $p \sim q$ if $p \le q$ and $q \le p$. Now we have a canonical partial order on the set $\tilde{P}$ of equivalence classes defined by saying that $p^\bullet \le q^\bullet$ iff $p \le q$. For all ordinary purposes, $(P, \le)$ and $(\tilde{P}, \le)$ carry the same structural information, and the move to the true partial order is natural and convenient. It occasionally happens (see 512Ee below, for instance, and also the theory of iterated forcing in Kunen 80, chap. VIII) that it is helpful to have a language which enables us to dispense with this step, thereby simplifying some basic definitions. However the extra generality leads to no new ideas, and I expect that most readers will prefer to do nearly all their thinking in the context of partially ordered sets.

511B Definitions Let $(P, \le)$ be any pre-ordered set.

(a) A subset Q of P is **cofinal** with P if for every $p \in P$ there is a $q \in Q$ such that $p \le q$. The **cofinality** of P, $\operatorname{cf} P$, is the least cardinal of any cofinal subset of P.

(b) The **additivity** of P, $\operatorname{add} P$, is the least cardinal of any subset of P with no upper bound in P. If there is no such set, write $\operatorname{add} P = \infty$.

(c) A subset Q of P is **coinitial** with P if for every $p \in P$ there is a $q \in Q$ such that $q \le p$. The **coinitiality** of P, $\operatorname{ci} P$, is the least cardinal of any coinitial subset of P.

(d) Two elements p, p' of P are **compatible upwards** if $[p, \infty[\cap [p', \infty[\ne \emptyset$, that is, if $\{p, p'\}$ has an upper bound in P; otherwise they are **incompatible upwards**. A subset A of P is an **up-antichain** if no two distinct elements of A are compatible upwards. The **upwards cellularity** of P is $c^\uparrow(P) = \sup\{\#(A) : A \subseteq P \text{ is an up-antichain in } P\}$; the **upwards saturation** of P, $\operatorname{sat}^\uparrow(P)$, is the least cardinal κ such that there is no up-antichain in P of size κ. P is called **upwards-ccc** if it has no uncountable up-antichain, that is, $c^\uparrow(P) \le \omega$, that is, $\operatorname{sat}^\uparrow(P) \le \omega_1$.

(e) Two elements p, p' of P are **compatible downwards** if $]-\infty, p] \cap]-\infty, p'] \ne \emptyset$, that is, if $\{p, p'\}$ has a lower bound in P; otherwise they are **incompatible downwards**. A subset A of P is a **down-antichain** if no two distinct elements of A are compatible downwards. The **downwards cellularity** of P is $c^\downarrow(P) = \sup\{\#(A) : A \subseteq P \text{ is a down-antichain in } P\}$; the **downwards saturation** of P, $\operatorname{sat}^\downarrow(P)$, is the least κ such that there is no down-antichain in P with cardinal κ. P is called **downwards-ccc** if it has no uncountable down-antichain, that is, $c^\downarrow(P) \le \omega$, that is, $\operatorname{sat}^\downarrow(P) \le \omega_1$.

(f) If κ is a cardinal, a subset A of P is **upwards-$<\kappa$-linked** in P if every subset of A of cardinal less than κ is bounded above in P. The **upwards $<\kappa$-linking number** of P, $\operatorname{link}^\uparrow_{<\kappa}(P)$, is the smallest cardinal of any cover of P by upwards-$<\kappa$-linked sets.

A subset A of P is **upwards-κ-linked** in P if it is upwards-$<\kappa^+$-linked, that is, every member of $[A]^{\le\kappa}$ is bounded above in P. The **upwards κ-linking number** of P, $\operatorname{link}^\uparrow_\kappa(P) = \operatorname{link}^\uparrow_{<\kappa^+}(P)$, is the smallest cardinal of any cover of P by upwards-κ-linked sets.

Similarly, a subset A of P is **downwards-$<\kappa$-linked** if every member of $[A]^{<\kappa}$ has a lower bound in P, and **downwards-κ-linked** if it is downwards-$<\kappa^+$-linked; the **downwards $<\kappa$-linking number** of P, $\operatorname{link}^\downarrow_{<\kappa}(P)$, is the smallest cardinal of any cover of P by downwards-$<\kappa$-linked sets, and $\operatorname{link}^\downarrow_\kappa(P) = \operatorname{link}^\downarrow_{<\kappa^+}(P)$.

(g) The most important cases of (f) above are $\kappa = 2$ and $\kappa = \omega$. A subset A of P is **upwards-linked** if any two members of A are compatible upwards in P, and **upwards-centered** if it is upwards-$<\omega$-linked, that is, any finite subset of A has an upper bound in P. The **upwards linking number** of P, $\operatorname{link}^{\uparrow}(P) = \operatorname{link}^{\uparrow}_2(P)$, is the least cardinal of any cover of P by upwards-linked sets, and the **upwards centering number** of P, $d^{\uparrow}(P) = \operatorname{link}^{\uparrow}_{<\omega}(P)$, is the least cardinal of any cover of P by upwards-centered sets.

Similarly, $A \subseteq P$ is **downwards-linked** if any two members of A are compatible downwards in P, and **downwards-centered** if any finite subset of A has a lower bound in P; the **downwards linking number** of P is $\operatorname{link}^{\downarrow}(P) = \operatorname{link}^{\downarrow}_2(P)$, and the **downwards centering number** of P is $d^{\downarrow}(P) = \operatorname{link}^{\downarrow}_{<\omega}(P)$.

If $\operatorname{link}^{\uparrow}(P) \le \omega$ (resp. $\operatorname{link}^{\downarrow}(P) \le \omega$) we say that P is **σ-linked upwards** (resp. **downwards**). If $d^{\uparrow}(P) \le \omega$ (resp. $d^{\downarrow}(P) \le \omega$) we say that P is **σ-centered upwards** (resp. **downwards**).

(h) The **upwards Martin number** $\mathfrak{m}^{\uparrow}(P)$ of P is the smallest cardinal of any family $\mathcal{Q}$ of cofinal subsets of P such that there is some $p \in P$ such that no upwards-linked subset of P containing p meets every member of $\mathcal{Q}$; if there is no such family $\mathcal{Q}$, write $\mathfrak{m}^{\uparrow}(P) = \infty$.

Similarly, the **downwards Martin number** $\mathfrak{m}^{\downarrow}(P)$ of P is the smallest cardinal of any family $\mathcal{Q}$ of coinitial subsets of P such that there is some $p \in P$ such that no downwards-linked subset of P containing p meets every member of $\mathcal{Q}$, or ∞ if there is no such $\mathcal{Q}$.

(i) A **Freese-Nation function** on P is a function $f : P \to \mathcal{P}P$ such that whenever $p \le q$ in P then $[p,q] \cap f(p) \cap f(q)$ is non-empty. The **Freese-Nation number** of P, $\mathrm{FN}(P)$, is the least κ such that there is a Freese-Nation function $f : P \to [P]^{<\kappa}$. The **regular Freese-Nation number** of P, $\mathrm{FN}^*(P)$, is the least regular infinite κ such that there is a Freese-Nation function $f : P \to [P]^{<\kappa}$. If Q is a subset of P, the **Freese-Nation index** of Q in P is the least cardinal κ such that $\operatorname{cf}(Q \cap \mathopen]-\infty, p]) < \kappa$ and $\operatorname{ci}(Q \cap [p, \infty\mathclose[) < \kappa$ for every $p \in P$.

(j) The **(principal) bursting number** $\operatorname{bu} P$ of P is the least cardinal κ such that there is a cofinal subset Q of P such that

$$\#(\{q : q \in Q,\ q \le p,\ p \not\le q\}) < \kappa$$

for every $p \in P$.

(k) It will be convenient to have a phrase for the following phenomenon. I will say that P is **separative upwards** if whenever p, $q \in P$ and $p \not\le q$ there is a $q' \ge q$ which is incompatible upwards with p. Similarly, of course, P is **separative downwards** if whenever p, $q \in P$ and $p \not\ge q$ there is a $q' \le q$ which is incompatible downwards with p.

511C On the symbol ∞ I note that in the definitions above I have introduced expressions of the form '$\operatorname{add} P = \infty$'. The '$\infty$' here must be rigorously distinguished from the '∞' of ordinary measure theory, which can be regarded as a top point added to the set of real numbers. The '∞' of 511B is rather a top point added to the class of ordinals. But it is convenient, and fairly safe, to use formulae like '$\operatorname{add} P \le \operatorname{add} Q$' on the understanding that $\operatorname{add} P \le \infty$ for every pre-ordered set P, while $\infty \le \operatorname{add} Q$ only when $\operatorname{add} Q = \infty$. Of course we have to be careful to distinguish between '$\operatorname{add} P < \infty$' (meaning that there is a subset of P with no upper bound in P) and '$\operatorname{add} P$ is finite' (meaning that $\operatorname{add} P < \omega$).

511D Definitions Let $\mathfrak{A}$ be a Boolean algebra. I write $\mathfrak{A}^+$ for the set $\mathfrak{A} \setminus \{0\}$ of non-zero elements of $\mathfrak{A}$ and $\mathfrak{A}^-$ for $\mathfrak{A} \setminus \{1\}$, so that the partially ordered sets $(\mathfrak{A}^-, \subseteq)$ and $(\mathfrak{A}^+, \supseteq)$ are isomorphic.

(a) The **Maharam type** $\tau(\mathfrak{A})$ of $\mathfrak{A}$ is the smallest cardinal of any subset B of $\mathfrak{A}$ which τ-generates $\mathfrak{A}$ in the sense that the order-closed subalgebra of $\mathfrak{A}$ including B is $\mathfrak{A}$ itself. (See Chapter 33.)

(b) The **cellularity** of $\mathfrak{A}$ is

$$c(\mathfrak{A}) = c^{\uparrow}(\mathfrak{A}^-) = c^{\downarrow}(\mathfrak{A}^+) = \sup\{\#(C) : C \subseteq \mathfrak{A}^+ \text{ is disjoint}\}.$$

The **saturation** of $\mathfrak{A}$ is

$$\operatorname{sat}(\mathfrak{A}) = \operatorname{sat}^{\uparrow}(\mathfrak{A}^-) = \operatorname{sat}^{\downarrow}(\mathfrak{A}^+) = \sup\{\#(C)^+ : C \subseteq \mathfrak{A}^+ \text{ is disjoint}\},$$

that is, the smallest cardinal κ such that there is no disjoint family of size κ in $\mathfrak{A}^+$.

(c) The **π-weight** or **density** $\pi(\mathfrak{A})$ of $\mathfrak{A}$ is $\operatorname{cf}\mathfrak{A}^- = \operatorname{ci}\mathfrak{A}^+$, that is, the smallest cardinal of any order-dense subset of $\mathfrak{A}$.

(d) Let κ be a cardinal. A subset A of $\mathfrak{A}^+$ is **$<\kappa$-linked** if it is downwards-$<\kappa$-linked in $\mathfrak{A}^+$, that is, no $B \in [A]^{<\kappa}$ has infimum 0, and **κ-linked** if it is $<\kappa^+$-linked, that is, every $B \in [A]^{\le\kappa}$ has a non-zero lower bound. The **$<\kappa$-linking number** $\operatorname{link}_{<\kappa}(\mathfrak{A})$ of $\mathfrak{A}$ is $\operatorname{link}^{\downarrow}_{<\kappa}(\mathfrak{A}^+)$, the least cardinal of any family of $<\kappa$-linked sets covering $\mathfrak{A}^+$; and the **κ-linking number** $\operatorname{link}_\kappa(\mathfrak{A})$ of $\mathfrak{A}$ is $\operatorname{link}_{<\kappa^+}(\mathfrak{A})$, that is, the least cardinal of any cover of $\mathfrak{A}^+$ by κ-linked sets.

(e) As in 511Bg, I say that $A \subseteq \mathfrak{A}^+$ is **linked** if no two members of A are disjoint; the **linking number** of $\mathfrak{A}$ is $\operatorname{link}(\mathfrak{A}) = \operatorname{link}_2(\mathfrak{A})$, the least cardinal of any cover of $\mathfrak{A}^+$ by linked sets. Similarly, $A \subseteq \mathfrak{A}^+$ is **centered** if $\inf I \ne 0$ for any finite $I \subseteq A$; that is, if A is downwards-centered in $\mathfrak{A}^+$. The **centering number** $d(\mathfrak{A})$ of $\mathfrak{A}$ is $d^{\uparrow}(\mathfrak{A}^-) = d^{\downarrow}(\mathfrak{A}^+)$, that is, the smallest cardinal of any cover of $\mathfrak{A}^+$ by centered sets. $\mathfrak{A}$ is **σ-m-linked** if $\operatorname{link}_m(\mathfrak{A}) \le \omega$; in particular, it is **σ-linked** iff $\operatorname{link}(\mathfrak{A}) \le \omega$. $\mathfrak{A}$ is **σ-centered** if $d(\mathfrak{A}) \le \omega$.

(f) If κ is any cardinal, $\mathfrak{A}$ is **weakly (κ,∞)-distributive** if whenever $\langle A_\xi\rangle_{\xi<\kappa}$ is a family of partitions of unity in $\mathfrak{A}$, there is a partition B of unity such that $\{a : a \in A_\xi,\ a \cap b \ne 0\}$ is finite for every $b \in B$ and $\xi < \kappa$. Now the **weak distributivity** $\operatorname{wdistr}(\mathfrak{A})$ of $\mathfrak{A}$ is the least cardinal κ such that $\mathfrak{A}$ is not weakly (κ,∞)-distributive. (If there is no such cardinal, write $\operatorname{wdistr}(\mathfrak{A}) = \infty$.)

(g) The **Martin number** $\mathfrak{m}(\mathfrak{A})$ of $\mathfrak{A}$ is the downwards Martin number of $\mathfrak{A}^+$, that is, the smallest cardinal of any family $\mathcal{B}$ of coinitial subsets of $\mathfrak{A}^+$ for which there is some $a \in \mathfrak{A}^+$ such that no linked subset of $\mathfrak{A}$ containing a meets every member of $\mathcal{B}$; or ∞ if there is no such $\mathcal{B}$.

(h) The **Freese-Nation number** of $\mathfrak{A}$, $\operatorname{FN}(\mathfrak{A})$, is the Freese-Nation number of the partially ordered set $(\mathfrak{A},\subseteq)$. The **regular Freese-Nation number** $\operatorname{FN}^*(\mathfrak{A})$ of $\mathfrak{A}$ is the regular Freese-Nation number of $(\mathfrak{A},\subseteq)$, that is, the smallest regular infinite cardinal greater than or equal to $\operatorname{FN}(\mathfrak{A})$.

(i) If κ is a cardinal, a **tight κ-filtration** of $\mathfrak{A}$ is a family $\langle a_\xi\rangle_{\xi<\zeta}$ in $\mathfrak{A}$, where ζ is an ordinal, such that, writing $\mathfrak{A}_\alpha$ for the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \alpha\}$, (α) $\mathfrak{A}_\zeta = \mathfrak{A}$ (β) for every $\alpha < \zeta$, the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}$ is at most κ. If $\mathfrak{A}$ has a tight κ-filtration, I will say that it is **tightly κ-filtered**.

511E Precalibers (a) Let $(P,\le)$ be a pre-ordered set.

(i) I will say that $(\kappa,\lambda,<\theta)$ is an **upwards precaliber triple** of P if κ, λ and θ are cardinals, and whenever $\langle p_\xi\rangle_{\xi<\kappa}$ is a family in P then there is a set $\Gamma \in [\kappa]^\lambda$ such that $\{p_\xi : \xi \in I\}$ has an upper bound in P for every $I \in [\Gamma]^{<\theta}$.

Similarly, $(\kappa,\lambda,<\theta)$ is a **downwards precaliber triple** of P if κ, λ and θ are cardinals and whenever $\langle p_\xi\rangle_{\xi<\kappa}$ is a family in P then there is a set $\Gamma \in [\kappa]^\lambda$ such that $\{p_\xi : \xi \in I\}$ has a lower bound in P for every $I \in [\Gamma]^{<\theta}$.

(ii) An **upwards precaliber pair** of P is a pair (κ,λ) of cardinals such that $(\kappa,\lambda,<\omega)$ is an upwards precaliber triple of P, that is, whenever $\langle p_\xi\rangle_{\xi<\kappa}$ is a family in P there is a $\Gamma \in [\kappa]^\lambda$ such that $\{p_\xi : \xi \in \Gamma\}$ is upwards-centered in P.

A **downwards precaliber pair** of P is a pair (κ,λ) of cardinals such that $(\kappa,\lambda,<\omega)$ is a downwards precaliber triple of P.

(iii) An **up-** (resp. **down-**) **precaliber** of P is a cardinal κ such that (κ,κ) is an upwards (resp. downwards) precaliber pair of P.

(b) Let $(X,\mathfrak{T})$ be a topological space. Then $(\kappa,\lambda,<\theta)$ is a **precaliber triple** of X if it is a downwards precaliber triple of $\mathfrak{T}\setminus\{\emptyset\}$; (κ,λ) is a **precaliber pair** of X if it is a downwards precaliber pair of $\mathfrak{T}\setminus\{\emptyset\}$; and κ is a **precaliber** of X if it is a down-precaliber of $\mathfrak{T}\setminus\{\emptyset\}$.

(c) Let $\mathfrak{A}$ be a Boolean algebra. Then $(\kappa,\lambda,<\theta)$ is a **precaliber triple** of $\mathfrak{A}$ if it is a downwards precaliber triple of $\mathfrak{A}^+$; (κ,λ) is a **precaliber pair** of $\mathfrak{A}$ if it is a downwards precaliber pair of $\mathfrak{A}^+$; and κ is a **precaliber** of $\mathfrak{A}$ if it is a down-precaliber of $\mathfrak{A}^+$.

(d) If $(\mathfrak{A},\bar\mu)$ is a measure algebra, then $(\kappa,\lambda,<\theta)$ is a **measure-precaliber triple** of $(\mathfrak{A},\bar\mu)$ if whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa}\bar\mu a_\xi > 0$, then there is a $\Gamma \in [\kappa]^\lambda$ such that $\{a_\xi : \xi \in I\}$ has a non-zero lower bound for every $I \in [\Gamma]^{<\theta}$. Now (κ,λ) is a **measure-precaliber pair** of $(\mathfrak{A},\bar\mu)$ if $(\kappa,\lambda,<\omega)$ is a measure-precaliber triple, and κ is a **measure-precaliber** of $(\mathfrak{A},\bar\mu)$ if (κ,κ) is a measure-precaliber pair.

(e) In this context, I will say that $(\kappa, \lambda, \theta)$ is a precaliber triple (in any sense) if $(\kappa, \lambda, {<}\theta^+)$ is a precaliber triple as defined above; and similarly for measure-precaliber triples.

(f) I will say that one of the structures here satisfies **Knaster's condition** if it has $(\omega_1, \omega_1, 2)$ as a precaliber triple, that is, if every uncountable set has an uncountable linked subset. (For pre-ordered sets I will speak of 'Knaster's condition upwards' or 'Knaster's condition downwards'.)

511F Definitions Let X be a set and $\mathcal{I}$ an ideal of subsets of X.

(a) Taking $\mathcal{I}$ to be partially ordered by $\subseteq$, we can speak of $\operatorname{add}\mathcal{I}$ and $\operatorname{cf}\mathcal{I}$ in the sense of 511B. $\mathcal{I}$ is called **κ-additive** or **κ-complete** if $\kappa \le \operatorname{add}\mathcal{I}$, that is, if $\bigcup\mathcal{E} \in \mathcal{I}$ for every $\mathcal{E} \in [\mathcal{I}]^{<\kappa}$.

In addition we have three other cardinals which will be important to us.

(b) The **uniformity** of $\mathcal{I}$ is

$$\operatorname{non}\mathcal{I} = \min\{\#(A) : A \subseteq X,\ A \notin \mathcal{I}\},$$

or ∞ if there is no such set A. (Note the hidden variable X in this notation; if any confusion seems possible, I will write $\operatorname{non}(X, \mathcal{I})$. Many authors prefer $\operatorname{unif}\mathcal{I}$.)

(c) The **shrinking number** of $\mathcal{I}$, $\operatorname{shr}\mathcal{I}$, is the smallest cardinal κ such that whenever $A \in \mathcal{P}X \setminus \mathcal{I}$ there is a $B \in [A]^{\le\kappa} \setminus \mathcal{I}$. (Again, we need to know X as well as $\mathcal{I}$ to determine $\operatorname{shr}\mathcal{I}$, and if necessary I will write $\operatorname{shr}(X, \mathcal{I})$.) The **augmented shrinking number** $\operatorname{shr}^+(\mathcal{I})$ is the smallest κ such that whenever $A \in \mathcal{P}X \setminus \mathcal{I}$ there is a $B \in [A]^{<\kappa} \setminus \mathcal{I}$.

(d) The **covering number** of $\mathcal{I}$ is

$$\operatorname{cov}\mathcal{I} = \min\{\#(\mathcal{E}) : \mathcal{E} \subseteq \mathcal{I},\ \bigcup\mathcal{E} = X\},$$

or ∞ if there is no such set $\mathcal{E}$. (Once more, X is a hidden variable here, and I may write $\operatorname{cov}(X, \mathcal{I})$.)

511G Definition Let (X, Σ, μ) be a measure space.

(a) If κ is a cardinal, μ is **κ-additive** if $\bigcup\mathcal{E} \in \Sigma$ and $\mu(\bigcup\mathcal{E}) = \sum_{E\in\mathcal{E}} \mu E$ for every disjoint family $\mathcal{E} \in [\Sigma]^{<\kappa}$. The **additivity** $\operatorname{add}\mu$ of μ is the largest cardinal κ such that μ is κ-additive, or ∞ if μ is κ-additive for every κ.

(b) The **π-weight** $\pi(\mu)$ of μ is the coinitiality of $\Sigma \setminus \mathcal{N}(\mu)$, where $\mathcal{N}(\mu)$ is the null ideal of μ.

(c) Recall that the Maharam type $\tau(\mu)$ of μ is the Maharam type of the measure algebra of μ (331Fc).

511H Elementary facts: pre-ordered sets Let P be a pre-ordered set.

(a) If $\tilde{P}$ is the partially ordered set of equivalence classes in P, as described in 511A, all the cardinal functions defined in 511B have the same values for P and $\tilde{P}$. (The point is that p is an upper bound for $A \subseteq P$ iff $p^\bullet$ is an upper bound for $\{q^\bullet : q \in A\} \subseteq \tilde{P}$.) Similarly, P and $\tilde{P}$ will have the same triple precalibers, precaliber pairs and precalibers.

(b) Obviously, $c^\uparrow(P) \le \operatorname{sat}^\uparrow(P)$. (In fact $c^\uparrow(P)$ is determined by $\operatorname{sat}^\uparrow(P)$; see 513Bc below.) If $\kappa \le \lambda$ are cardinals then

$$\operatorname{link}^\uparrow_{<\kappa}(P) \le \operatorname{link}^\uparrow_{<\lambda}(P) \le \operatorname{cf} P,$$

because every upwards-${<}\lambda$-linked set is upwards-${<}\kappa$-linked and every set $]-\infty, p]$ is upwards-${<}\lambda$-linked. $c^\uparrow(P) \le \operatorname{link}^\uparrow(P)$, because if $A \subseteq P$ is an up-antichain then no upwards-linked set can contain more than one point of A. It follows that

$$\operatorname{link}^\uparrow(P) = \operatorname{link}^\uparrow_{<3}(P) \le \operatorname{link}^\uparrow_{<\omega}(P) = d^\uparrow(P) \le \operatorname{cf} P.$$

Of course $\operatorname{cf} P \le \#(P)$. Similarly,

$$\operatorname{link}^\downarrow_{<\kappa}(P) \le \operatorname{link}^\downarrow_{<\lambda}(P) \le \operatorname{ci} P$$

whenever $\kappa \le \lambda$, and

$$c^\downarrow(P) \le \operatorname{link}^\downarrow(P) \le d^\downarrow(P) \le \operatorname{ci} P \le \#(P), \quad c^\downarrow(P) \le \operatorname{sat}^\downarrow P.$$

(c) P is empty iff $\operatorname{cf} P = 0$ iff $\operatorname{ci} P = 0$ iff $\operatorname{add} P = 0$ iff $d^{\uparrow}(P) = 0$ iff $d^{\downarrow}(P) = 0$ iff $\operatorname{link}^{\uparrow}(P) = 0$ iff $\operatorname{link}^{\downarrow}(P) = 0$ iff $c^{\uparrow}(P) = 0$ iff $c^{\downarrow}(P) = 0$ iff $\operatorname{sat}^{\uparrow}(P) = 1$ iff $\operatorname{sat}^{\downarrow}(P) = 1$ iff $\operatorname{FN}(P) = 0$.

(d) P is upwards-directed iff $c^{\uparrow}(P) \le 1$ iff $\operatorname{sat}^{\uparrow}(P) \le 2$ iff $\operatorname{link}^{\uparrow}(P) \le 1$ iff $d^{\uparrow}(P) \le 1$. Similarly, P is downwards-directed iff $c^{\downarrow}(P) \le 1$ iff $\operatorname{sat}^{\downarrow}(P) \le 2$ iff $\operatorname{link}^{\downarrow}(P) \le 1$ iff $d^{\downarrow}(P) \le 1$.

If P is not empty, it is upwards-directed iff $\operatorname{add} P > 2$ iff $\operatorname{add} P \ge \omega$.

(e) If P is partially ordered, it has a greatest element iff $\operatorname{cf} P = 1$ iff $\operatorname{add} P = \infty$. Otherwise, $\operatorname{add} P \le \operatorname{cf} P$, since no cofinal subset of P can have an upper bound in P.

(f) If P is totally ordered, then $\operatorname{cf} P \le \operatorname{add} P$. **P** If $A \subseteq P$ has no upper bound in P it must be cofinal with P. **Q**

(g) If $\langle P_i\rangle_{i\in I}$ is a non-empty family of non-empty pre-ordered sets with product P, then $\operatorname{add} P = \min_{i\in I} \operatorname{add} P_i$. **P** A set $A \subseteq P$ lacks an upper bound in P iff there is an $i \in I$ such that $\{p(i) : p \in A\}$ is unbounded above in P_i. **Q**

511I Elementary facts: Boolean algebras Let $\mathfrak{A}$ be a Boolean algebra.

(a)

$$\operatorname{link}_{<\kappa}(\mathfrak{A}) \le \operatorname{link}_{<\lambda}(\mathfrak{A}) \le \pi(\mathfrak{A})$$

whenever $\kappa \le \lambda$,

$$c(\mathfrak{A}) \le \operatorname{link}(\mathfrak{A}) \le d(\mathfrak{A}) \le \pi(\mathfrak{A}) \le \#(\mathfrak{A}), \quad c(\mathfrak{A}) \le \operatorname{sat}(\mathfrak{A}).$$

In addition, $\tau(\mathfrak{A}) \le \pi(\mathfrak{A})$ because any order-dense subset of $\mathfrak{A}$ τ-generates $\mathfrak{A}$.

(b) $\mathfrak{A} = \{0\}$ iff $\pi(\mathfrak{A}) = 0$ iff $\operatorname{link}(\mathfrak{A}) = 0$ iff $d(\mathfrak{A}) = 0$ iff $c(\mathfrak{A}) = 0$ iff $\operatorname{sat}(\mathfrak{A}) = 1$.

(c) If $\mathfrak{A}$ is finite, then $c(\mathfrak{A}) = \operatorname{link}(\mathfrak{A}) = d(\mathfrak{A}) = \pi(\mathfrak{A})$ is the number of atoms of $\mathfrak{A}$, $\operatorname{sat}(\mathfrak{A}) = c(\mathfrak{A}) + 1$ and $\#(\mathfrak{A}) = 2^{c(\mathfrak{A})}$, while $\tau(\mathfrak{A}) = \lceil \log_2 c(\mathfrak{A})\rceil$, unless $\mathfrak{A} = \{0\}$, in which case $\tau(\mathfrak{A}) = 0$. If $\mathfrak{A}$ is infinite then $c(\mathfrak{A})$, $\operatorname{link}(\mathfrak{A})$, $d(\mathfrak{A})$, $\pi(\mathfrak{A})$, $\operatorname{sat}(\mathfrak{A})$ and $\tau(\mathfrak{A})$ are all infinite.

(d) Note that $\mathfrak{A}$ is 'ccc' just when $c(\mathfrak{A}) \le \omega$, that is, $\operatorname{sat}(\mathfrak{A}) \le \omega_1$. $\mathfrak{A}$ is weakly (σ,∞)-distributive, in the sense of 316G, iff $\operatorname{wdistr}(\mathfrak{A}) \ge \omega_1$.

(e)(i) If $\mathfrak{A}$ is purely atomic, $\operatorname{wdistr}(\mathfrak{A}) = \infty$. **P** Suppose that $\langle A_\xi\rangle_{\xi<\kappa}$ is any family of partitions of unity in $\mathfrak{A}$. Then the set B of atoms of $\mathfrak{A}$ is a partition of unity, and $\{a : a \in A_\xi,\ a \cap b \ne 0\}$ has just one member for every $b \in B$ and $\xi < \kappa$. As $\langle A_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{wdistr}(\mathfrak{A}) = \infty$. **Q**

(ii) If $\mathfrak{A}$ is not purely atomic, $\operatorname{wdistr}(\mathfrak{A}) \le \pi(\mathfrak{A})$. **P** Let $c \in \mathfrak{A}^+$ be disjoint from every atom of $\mathfrak{A}$, and $D \subseteq \mathfrak{A}$ an order-dense set of size $\pi(\mathfrak{A})$; let D' be $\{d : d \in D,\ d \subseteq c\}$. For $d \in D'$, there is a disjoint sequence of non-zero elements included in d; let A_d be a partition of unity in $\mathfrak{A}$ including such a sequence. If B is any partition of unity in $\mathfrak{A}$, there are a $b \in B$ such that $b \cap c \ne 0$, and a $d \in D'$ such that $d \subseteq b \cap c$; now $\{a : a \in A_d,\ b \cap a \ne 0\}$ is infinite. So $\langle A_d\rangle_{d\in D'}$ witnesses that $\operatorname{wdistr}(\mathfrak{A}) \le \#(D') \le \pi(\mathfrak{A})$. **Q**

(f) $\mathfrak{m}(\mathfrak{A}) = \infty$ iff $\mathfrak{A}$ is purely atomic. **P** Write $\mathcal{B}$ for the family of all coinitial subsets of $\mathfrak{A}^+$. (i) If $\mathfrak{A}$ is purely atomic and $a \in \mathfrak{A}^+$, then there is an atom $d \subseteq a$; now $d \in B$ for every $B \in \mathcal{B}$, so $\{d, a\}$ is a linked subset of $\mathfrak{A}$ meeting every member of $\mathcal{B}$. Accordingly $\mathfrak{m}(\mathfrak{A}) = \infty$. (ii) If $\mathfrak{A}$ is not purely atomic, let $a \in \mathfrak{A}^+$ be such that no atom of $\mathfrak{A}$ is included in a. **?** If A is a linked subset of $\mathfrak{A}$ containing a and meeting every member of $\mathcal{B}$, set $B = \mathfrak{A}^+ \setminus A$. If $b \in \mathfrak{A}^+$, then either $b \cap a = 0$ and $b \in B$, or there are non-zero disjoint b', $b'' \subseteq b \cap a$ and one of b', b'' must belong to B. So $B \in \mathcal{B}$, which is impossible. **X** So $\mathfrak{m}(\mathfrak{A}) \le \#(\mathcal{B}) < \infty$. **Q**

511J Elementary facts: ideals of sets Let X be a set and $\mathcal{I}$ an ideal of subsets of X.

(a) $\operatorname{add}\mathcal{I} \ge \omega$, by the definition of 'ideal of sets'.

(b) $\operatorname{shr}\mathcal{I} = \sup\{\operatorname{non}(A, \mathcal{I} \cap \mathcal{P}A) : A \in \mathcal{P}X \setminus \mathcal{I}\}$, counting $\sup\emptyset$ as 0; $\operatorname{shr}\mathcal{I} \le \#(X)$; $\operatorname{shr}\mathcal{I} \le \operatorname{shr}^+\mathcal{I} \le (\operatorname{shr}\mathcal{I})^+$; if $\operatorname{shr}\mathcal{I}$ is a successor cardinal, $\operatorname{shr}^+\mathcal{I} = (\operatorname{shr}\mathcal{I})^+$.

(c) Suppose that $\mathcal{I}$ covers X but does not contain X. Then $\operatorname{add}\mathcal{I} \le \operatorname{cov}\mathcal{I} \le \operatorname{cf}\mathcal{I}$ and $\operatorname{add}\mathcal{I} \le \operatorname{non}\mathcal{I} \le \operatorname{shr}\mathcal{I} \le \operatorname{cf}\mathcal{I}$.
P Let $\mathcal{J}$ be a subset of $\mathcal{I}$ with cardinal $\operatorname{cov}\mathcal{I}$ covering X; let $\mathcal{K}$ be a cofinal subset of $\mathcal{I}$ with cardinal $\operatorname{cf}\mathcal{I}$; let $A \in \mathcal{P}X \setminus \mathcal{I}$ be such that $\#(A) = \operatorname{non}\mathcal{I}$. (i) $\mathcal{J}$ cannot have an upper bound in $\mathcal{I}$, so $\operatorname{add}\mathcal{I} \le \#(\mathcal{J}) = \operatorname{cov}\mathcal{I}$. (ii) $\bigcup\mathcal{K} = \bigcup\mathcal{I} = X$, so $\operatorname{cov}\mathcal{I} \le \#(\mathcal{K}) = \operatorname{cf}\mathcal{I}$. (iii) For each $x \in A$ we can find an $I_x \in \mathcal{I}$ containing x; now $\{I_x : x \in A\}$ cannot have an upper bound in $\mathcal{I}$, so $\operatorname{add}\mathcal{I} \le \#(A) = \operatorname{non}\mathcal{I}$. (iv) By (b), $\operatorname{shr}\mathcal{I} \ge \operatorname{non}\mathcal{I}$. (v) Take any $B \subseteq X$ such that $B \notin \mathcal{I}$. Then for each $K \in \mathcal{K}$ we can find an $x_K \in B \setminus K$; now $B' = \{x_K : K \in \mathcal{K}\}$ is not included in any member of $\mathcal{K}$, so cannot belong to $\mathcal{I}$, while $B' \subseteq B$ and $\#(B') \le \#(\mathcal{K}) = \operatorname{cf}\mathcal{I}$. As B is arbitrary, $\operatorname{shr}\mathcal{I} \le \operatorname{cf}\mathcal{I}$. **Q**

(d) Suppose that $X \in \mathcal{I}$. Then $\operatorname{add}\mathcal{I} = \operatorname{non}\mathcal{I} = \infty$, $\operatorname{cov}\mathcal{I} \le 1$ (with $\operatorname{cov}\mathcal{I} = 0$ iff $X = \emptyset$) and $\operatorname{shr}\mathcal{I} = 0$.

(e) Suppose that $\mathcal{I}$ has a greatest member which is not X. Then $\operatorname{add}\mathcal{I} = \operatorname{cov}\mathcal{I} = \infty$ and $\operatorname{non}\mathcal{I} = \operatorname{shr}\mathcal{I} = \operatorname{cf}\mathcal{I} = 1$.

(f) Suppose that $\mathcal{I}$ has no greatest member and does not cover X. Then $\operatorname{add}\mathcal{I} \le \operatorname{cf}\mathcal{I}$ (511He), $\operatorname{non}\mathcal{I} = \operatorname{shr}\mathcal{I} = 1$ and $\operatorname{cov}\mathcal{I} = \infty$.

(g) Suppose that $Y \subseteq X$, and set $\mathcal{I}_Y = \mathcal{I} \cap \mathcal{P}Y$, regarded as an ideal of subsets of Y. Then $\operatorname{add}\mathcal{I}_Y \ge \operatorname{add}\mathcal{I}$, $\operatorname{non}\mathcal{I}_Y \ge \operatorname{non}\mathcal{I}$, $\operatorname{shr}\mathcal{I}_Y \le \operatorname{shr}\mathcal{I}$, $\operatorname{shr}^+\mathcal{I}_Y \le \operatorname{shr}^+\mathcal{I}$, $\operatorname{cov}\mathcal{I}_Y \le \operatorname{cov}\mathcal{I}$ and $\operatorname{cf}\mathcal{I}_Y \le \operatorname{cf}\mathcal{I}$.

511X Basic exercises (a) Let P be a partially ordered set and $\kappa \ge 3$ a cardinal. Show that $\operatorname{add}P \ge \kappa$ iff $(\lambda, \lambda, \lambda)$ is an upwards precaliber triple of P for every $\lambda < \kappa$.

>(b) Let X be a compact Hausdorff space. Show that a pair (κ, λ) of cardinals is a precaliber pair of X iff whenever $\langle G_\xi\rangle_{\xi<\kappa}$ is a family of non-empty open subsets of X there is an $x \in X$ such that $\{\xi : x \in G_\xi\}$ has cardinal at least λ.

(c) Let (X, Σ, μ) be a measure space. For $A \subseteq X$ write μ_A for the subspace measure on A, and $\mathcal{N}(\mu)$, $\mathcal{N}(\mu_A)$ for the corresponding null ideals. Show that $\operatorname{shr}(X, \mathcal{N}(\mu)) = \sup\{\operatorname{non}(A, \mathcal{N}(\mu_A)) : A \in \mathcal{P}X \setminus \mathcal{N}(\mu)\}$.

(d) Let (X, Σ, μ) be a measure space, and let $\hat\mu$, $\tilde\mu$ be the completion and c.l.d. version of μ. (i) Let $\mathcal{N}(\mu) = \mathcal{N}(\hat\mu)$ and $\mathcal{N}(\tilde\mu)$ be the corresponding null ideals. Show that $\operatorname{add}\mathcal{N}(\mu) \le \operatorname{add}\mathcal{N}(\tilde\mu)$, $\operatorname{cov}\mathcal{N}(\mu) \ge \operatorname{cov}\mathcal{N}(\tilde\mu)$, $\operatorname{non}\mathcal{N}(\mu) \le \operatorname{non}\mathcal{N}(\tilde\mu)$, $\operatorname{shr}\mathcal{N}(\mu) \ge \operatorname{shr}\mathcal{N}(\tilde\mu)$ and $\operatorname{shr}^+\mathcal{N}(\mu) \ge \operatorname{shr}^+\mathcal{N}(\tilde\mu)$. (ii) Show that $\operatorname{add}\mu \le \operatorname{add}\hat\mu \le \operatorname{add}\tilde\mu$, $\pi(\mu) = \pi(\hat\mu) \le \pi(\tilde\mu)$ and $\tau(\mu) = \tau(\hat\mu) \ge \tau(\tilde\mu)$.

(e) Show that if P is a partially ordered set and $c^\uparrow(P) < \omega$ then $c^\uparrow(P) = \operatorname{link}^\uparrow(P) = d^\uparrow(P)$ and $\mathfrak{m}^\uparrow(P) = \infty$.

(f) Let P be a partially ordered set. Show that ω is an up-precaliber of P iff $c^\uparrow(P) < \omega$.

>(g)(i) Show that if P is a partially ordered set and κ is an up-precaliber of P, then $\operatorname{cf}\kappa$ is also an up-precaliber of P. (ii) Show that if κ is a cardinal and $\operatorname{cf}\kappa > \operatorname{cf}P$ then κ is an up-precaliber of P.

>(h) Give $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Q}$ their usual total orders. Show that $\mathrm{FN}(\mathbb{N}) = \mathrm{FN}(\mathbb{Q}) = \omega$ and that $\mathrm{FN}(\mathbb{R}) = \omega_1$.

(i) Show that if P is a partially ordered set and $\#(P) \ge 3$ then $\mathrm{FN}(P) \le \#(P)$. (*Hint*: consider separately the cases P infinite, P finite with no greatest member, and P finite with greatest and least members.)

(j) Let P be a partially ordered set and $\mathcal{Q}$ a family of subsets of P with $\#(\mathcal{Q}) < \operatorname{add}P$. Show that if $\bigcup\mathcal{Q}$ is cofinal with P then one of the members of $\mathcal{Q}$ is cofinal with P.

(k) Let U be a Riesz space and κ a cardinal. Then U is **weakly** (κ, ∞)**-distributive** if whenever $\langle A_\xi\rangle_{\xi<\kappa}$ is a family of non-empty downwards-directed subsets of U^+, each with infimum 0, and $\bigcup_{\xi<\kappa} A_\xi$ has an upper bound in U, then

$$\{u : u \in U, \text{ for every } \xi < \kappa \text{ there is a } v \in A_\xi \text{ such that } v \le u\}$$

has infimum 0 in U. Show that an Archimedean Riesz space is weakly (κ, ∞)-distributive iff its band algebra is. (*Hint*: 368R.)

(l) Let X be a set and $\mathcal{I}$ an ideal of subsets of X. Show that the coinitiality $\operatorname{ci}(\mathcal{P}X \setminus \mathcal{I})$ is at most $\#(X)^{\operatorname{shr}\mathcal{I}}$.

511Y Further exercises (a)(i) Show that $d^\uparrow(P) \le 2^{\operatorname{link}^\uparrow(P)}$ for every partially ordered set P. (ii) Show that there is a partially ordered set P such that $d^\uparrow(P) = \omega$ but P cannot be covered by countably many upwards-directed sets.

(b) Let κ be an infinite cardinal, with its usual well-ordering. Show that $\mathrm{FN}(\kappa) = \kappa$.

(c)(i) Find a semi-finite measure space (X, Σ, μ) such that $\mathrm{cf}\,\mathcal{N}(\mu) < \mathrm{cf}\,\mathcal{N}(\tilde\mu)$, where $\mathcal{N}(\mu)$ and $\mathcal{N}(\tilde\mu)$ are the null ideals of μ and its c.l.d. version. (ii) Find a semi-finite measure space (X, Σ, μ) such that $\mathrm{add}\,\mathcal{N}(\tilde\mu) > \mathrm{add}\,\mathcal{N}(\mu)$ and $\mathrm{cf}\,\mathcal{N}(\tilde\mu) < \mathrm{cf}\,\mathcal{N}(\mu)$.

511 Notes and comments Because $(P, \ge)$ is a pre-ordered set whenever $(P, \le)$ is, any cardinal function on pre-ordered sets is bound to appear in two mirror-image forms. It does not quite follow that we have to set up a language with a complete set of mirror pairs of definitions, and indeed I have omitted the reflections of 'additivity' and 'bursting number'; but the naturally arising pre-ordered sets to which we shall want to apply these ideas may appear in either orientation. The most natural conversions to topological spaces and Boolean algebras use the families of non-empty open sets and non-zero elements, which are 'active downwards', so that we have such formulae as $\pi(\mathfrak{A}) = \mathrm{ci}\,\mathfrak{A}^+$ and $c(X) = c^\downarrow(\mathfrak{T} \setminus \{\emptyset\})$; but we could equally well say that $\pi(\mathfrak{A}) = \mathrm{cf}\,\mathfrak{A}^-$ or that $c(X)$ is the upwards-cellularity of the partially ordered set of proper closed subsets of X.

Most readers, especially those acquainted with Volumes 3 and 4 of this treatise, will be more familiar with topological spaces and Boolean algebras than with general pre-ordered sets, and will prefer to approach the concepts here through the formulations in 5A4A and 511D. But even in the present chapter we shall be looking at questions which demand substantial fragments of the theory of general partially ordered sets, and I think it is useful to grapple with these immediately. The list of definitions above is a long one, and the functions here vary widely in importance; but I hope you will come to agree that all are associated with interesting questions.

I apologise for introducing two cardinal functions to represent the 'breadth' of a pre-ordered set (or topological space or Boolean algebra), its 'cellularity' and 'saturation'. It turns out that the saturation of a space determines its cellularity (513Bc), which seems to render the concept of 'cellularity' unnecessary; but it is well-established and makes some formulae simpler. This is an example of a standard problem: whenever we give a name to a supremum, we find ourselves asking whether the supremum is attained. The question of whether cellularity is attained turns out to be rather interesting (513B again). In the case of shrinking numbers, the ordinary shrinking number $\mathrm{shr}\,\mathcal{I}$ is the one which has been most studied, but I shall have some results which are more elegantly expressed in terms of the augmented shrinking number $\mathrm{shr}^+\,\mathcal{I}$.

I give very little space here to the functions $\mathfrak{m}()$ and $\mathrm{wdistr}()$ and to precalibers; these are bound to be a bit mysterious. Later in the chapter I will explore their relations with each other and with other cardinal functions. You may recognise them as belonging to the general area associated with Martin's axiom (Fremlin 84a, or §517 below). 'Precaliber pairs' have a slightly more direct description in the context of compact Hausdorff spaces (511Xb). 'Freese-Nation numbers' relate to quite different aspects of the structure of ordered sets. As will be made clear in the next two sections, all the other cardinal functions defined in 511B refer to the cofinal (or coinitial) structure of a partially ordered set; the Freese-Nation number, by contrast, tells us something about the nature of intervals inside it. We see a difference already in the formula for the Freese-Nation number of a Boolean algebra, which refers to the whole algebra $\mathfrak{A}$ rather than to $\mathfrak{A}^+$. Another signal is the fact that it is not a trivial matter to calculate the Freese-Nation number of a finite partially ordered set.

The only cardinal functions I have explicitly defined for measure spaces are the additivity and π-weight of a measure (511G), and even these are, in the most important cases, reducible to the additivity of the null ideal (521A) and the π-weight of the measure algebra (521Da). I give a pair of warming-up exercises (511Xc-511Xd), but we shall hardly see 'measure' again until Chapter 52. For the questions studied in this volume, the important cardinals associated with a measure μ are those defined from its measure algebra together with the four cardinals $\mathrm{add}\,\mathcal{N}(\mu)$, $\mathrm{cov}\,\mathcal{N}(\mu)$, $\mathrm{non}\,\mathcal{N}(\mu)$ and $\mathrm{cf}\,\mathcal{N}(\mu)$. In particular, the additivity of Lebesgue measure will have a special position. In the case of a topological measure space, of course, we can investigate relationships between the cardinal functions of the topology and the cardinal functions of the measure. I will come to such questions in Chapter 53.

512 Galois-Tukey connections

One of the most powerful methods of relating the cardinals associated with two partially ordered sets P and Q is to identify a 'Tukey function' from one to the other (513D). It turns out that the idea can be usefully generalized to other relational structures through the concept of 'Galois-Tukey connection' (512A). In this section I give the elementary theory of these connections and their effect on simple cardinal functions.

512A Definitions (a) A **supported relation** is a triple (A, R, B) where A and B are sets and R is a subset of $A \times B$.

It will be convenient, and I think not dangerous, to abuse notation by writing $(A, \in, B)$ or $(A, \subseteq, B)$ to mean (A, R, B) where R is $\{(a, b) : a \in A,\ b \in B,\ a \in b\}$ or $\{(a, b) : a \in A,\ b \in B,\ a \subseteq b\}$.

(b) If (A, R, B) is a supported relation, its **dual** is the supported relation $(A, R, B)^\perp = (B, S, A)$ where

$$S = (B \times A) \setminus R^{-1} = \{(b, a) : a \in A,\ b \in B,\ (a, b) \notin R\}.$$

(c) If (A, R, B) and (C, S, D) are supported relations, a **Galois-Tukey connection** from (A, R, B) to (C, S, D) is a pair (ϕ, ψ) such that $\phi : A \to C$ and $\psi : D \to B$ are functions and $(a, \psi(d)) \in R$ whenever $(\phi(a), d) \in S$.

(d) (Vojtáš 93) If (A, R, B) and (C, S, D) are supported relations, I write $(A, R, B) \preccurlyeq_{\mathrm{GT}} (C, S, D)$ if there is a Galois-Tukey connection from (A, R, B) to (C, S, D), and $(A, R, B) \equiv_{\mathrm{GT}} (C, S, D)$ if $(A, R, B) \preccurlyeq_{\mathrm{GT}} (C, S, D)$ and $(C, S, D) \preccurlyeq_{\mathrm{GT}} (A, R, B)$.

512B Definitions (a) If (A, R, B) is a supported relation, its **covering number** $\mathrm{cov}(A, R, B)$ (sometimes called **norm** $\|(A, R, B)\|$) is the least cardinal of any set $C \subseteq B$ such that $A \subseteq R^{-1}[C]$; or ∞ if $A \not\subseteq R^{-1}[B]$. Its **additivity** is $\mathrm{add}(A, R, B) = \mathrm{cov}(A, R, B)^\perp$, that is, the smallest cardinal of any subset $C \subseteq A$ such that $C \not\subseteq R^{-1}[\{b\}]$ for any $b \in B$; or ∞ if there is no such C.

Note that $\mathrm{add}(A, R, B) = 0$ iff $B = \emptyset$, and that $\mathrm{add}(A, R, B) = 1$ iff $B \neq \emptyset$ and $\mathrm{cov}(A, R, B) = \infty$.

(b) If (A, R, B) is a supported relation, its **saturation** $\mathrm{sat}(A, R, B)$ is the least cardinal κ such that whenever $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in A then there are distinct ξ, $\eta < \kappa$ and a $b \in B$ such that (a_ξ, b) and (a_η, b) both belong to R; if there is no such κ (that is, if $\mathrm{cov}(A, R, B) = \infty$) I write $\mathrm{sat}(A, R, B) = \infty$.

(c) If (A, R, B) is a supported relation and κ is a cardinal, say that a subset A' of A is **$<\kappa$-linked** if for every $I \in [A']^{<\kappa}$ there is a $b \in B$ such that $I \subseteq R^{-1}[\{b\}]$, and **$\kappa$-linked** if it is $<\kappa^+$-linked, that is, for every $I \in [A']^{\le\kappa}$ there is a $b \in B$ such that $I \subseteq R^{-1}[\{b\}]$. Now the **$<\kappa$-linking number** $\mathrm{link}_{<\kappa}(A, R, B)$ of (A, R, B) is the least cardinal of any cover of A by $<\kappa$-linked sets, if there is such a cover, and otherwise is ∞; and the **κ-linking number** $\mathrm{link}_\kappa(A, R, B)$ of (A, R, B) is $\mathrm{link}_{<\kappa^+}(A, R, B)$, that is, the least cardinal of any cover of A by κ-linked sets.

If $\kappa \le \lambda$, then every $<\lambda$-linked set is $<\kappa$-linked, so $\mathrm{link}_{<\kappa}(A, R, B) \le \mathrm{link}_{<\lambda}(A, R, B)$. Note also that $\mathrm{link}_\kappa(A, R, B)$ is equal to $\mathrm{cov}(A, R, B)$ for every $\kappa \ge \#(A)$, so that $\mathrm{link}_{<\theta}(A, R, B) \le \mathrm{cov}(A, R, B)$ for every θ.

512C There are two things which should be done at once: to plainly state enough of the elementary theory to show at least that the definitions here lead to a coherent structure; and to give examples. I begin with the theory, which really is elementary.

Theorem Let (A, R, B), (C, S, D) and (E, T, F) be supported relations.

(a) $(A, R, B)^{\perp\perp} = (A, R, B)$.

(b) If (ϕ, ψ) is a Galois-Tukey connection from (A, R, B) to (C, S, D) and (ϕ', ψ') is a Galois-Tukey connection from (C, S, D) to (E, T, F), then $(\phi'\phi, \psi\psi')$ is a Galois-Tukey connection from (A, R, B) to (E, T, F).

(c) If (ϕ, ψ) is a Galois-Tukey connection from (A, R, B) to (C, S, D), then (ψ, ϕ) is a Galois-Tukey connection from $(C, S, D)^\perp$ to $(A, R, B)^\perp$.

(d) $(A, R, B) \preccurlyeq_{\mathrm{GT}} (A, R, B)$.

(e) If $(A, R, B) \preccurlyeq_{\mathrm{GT}} (C, S, D)$ and $(C, S, D) \preccurlyeq_{\mathrm{GT}} (E, T, F)$ then $(A, R, B) \preccurlyeq_{\mathrm{GT}} (E, T, F)$.

(f) $\equiv_{\mathrm{GT}}$ is an equivalence relation on the class of supported relations.

(g) If $(A, R, B) \preccurlyeq_{\mathrm{GT}} (C, S, D)$ then $(C, S, D)^\perp \preccurlyeq_{\mathrm{GT}} (A, R, B)^\perp$. So if $(A, R, B) \equiv_{\mathrm{GT}} (C, S, D)$ then $(A, R, B)^\perp \equiv_{\mathrm{GT}} (C, S, D)^\perp$.

proof (a)-(c) are immediate from the definitions. (d) is trivial because the identity functions from A and B to themselves form a Galois-Tukey connection from (A, R, B) to itself. (e) follows from (b), and (g) from (c). (f) is immediate from (d) and (e) and the symmetry of the definition of $\equiv_{\mathrm{GT}}$.

512D Theorem Let (A,R,B) and (C,S,D) be supported relations such that $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$. Then
(a) $\operatorname{cov}(A,R,B)\le\operatorname{cov}(C,S,D)$;
(b) $\operatorname{add}(C,S,D)\le\operatorname{add}(A,R,B)$;
(c) $\operatorname{sat}(A,R,B)\le\operatorname{sat}(C,S,D)$;
(d) $\operatorname{link}_{<\kappa}(A,R,B)\le\operatorname{link}_{<\kappa}(C,S,D)$ for every cardinal κ.

proof Let (ϕ,ψ) be a Galois-Tukey connection from (A,R,B) to (C,S,D). If $D_0\subseteq D$ is such that $C=S^{-1}[D_0]$, then $A=R^{-1}[\psi[D_0]]$; this shows that $\operatorname{cov}(A,R,B)\le\operatorname{cov}(C,S,D)$. If $\kappa=\operatorname{sat}(C,S,D)$ and $\langle a_\xi\rangle_{\xi<\kappa}$ is any family in A, then there are a $d\in D$ and distinct ξ, $\eta<\kappa$ such that $(\phi(a_\xi),d)\in S$ and $(\phi(a_\eta),d)\in S$, in which case $(a_\xi,\psi(d))$ and $(a_\eta,\psi(d))$ both belong to R; so $\operatorname{sat}(A,R,B)\le\kappa$. If $\mathcal{C}$ is a cover of C by $<\kappa$-linked sets, then $\{\phi^{-1}[C']:C'\in\mathcal{C}\}$ is a cover of A by $<\kappa$-linked sets; this shows that $\operatorname{link}_{<\kappa}(A,R,B)\le\operatorname{link}_{<\kappa}(C,S,D)$.

Finally, $(C,S,D)^\perp\preccurlyeq_{\mathrm{GT}}(A,R,B)^\perp$, by 512Cc, so

$$\operatorname{add}(C,S,D)=\operatorname{cov}(C,S,D)^\perp\le\operatorname{cov}(A,R,B)^\perp=\operatorname{add}(A,R,B).$$

512E Examples Of course 'supported relations' appear everywhere in mathematics. They are important to us here because covering numbers, saturation and linking numbers, as defined above, correspond to important cardinal functions as defined in §511, and because surprising Galois-Tukey connections exist, as we shall see in Chapter 52. The simplest examples are the following.

(a) Let $(P,\le)$ be a pre-ordered set. Then $(P,\le,P)$ and $(P,\ge,P)$ are supported relations, with duals $(P,\not\ge,P)$ and $(P,\not\le,P)$. $\operatorname{cov}(P,\le,P)=\operatorname{cf}P$, $\operatorname{cov}(P,\ge,P)=\operatorname{ci}P$, $\operatorname{add}(P,\le,P)=\operatorname{add}P$ and $\operatorname{sat}(P,\le,P)=\operatorname{sat}^\uparrow(P)$. For any cardinal κ, a subset of P is upwards-$<\kappa$-linked in the sense of 511Bf iff it is $<\kappa$-linked in $(P,\le,P)$ in the sense of 512Bc. So $\operatorname{link}^\uparrow_{<\kappa}(P)=\operatorname{link}_{<\kappa}(P,\le,P)$. In particular, $d^\uparrow(P)=\operatorname{link}_{<\omega}(P,\le,P)$ (511Bg).

(b) Let $(X,\mathfrak{T})$ be a topological space. Then

$$\pi(X)=\operatorname{cov}(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\}),$$

$$d(X)=\operatorname{cov}(\mathfrak{T}\setminus\{\emptyset\},\ni,X)=\operatorname{add}(X,\notin,\mathfrak{T}\setminus\{\emptyset\}),$$

$$\operatorname{sat}(X)=\operatorname{sat}(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})=\operatorname{sat}(\mathfrak{T}\setminus\{\emptyset\},\ni,X),$$

$$n(X)=\operatorname{cov}(X,\in,\mathcal{N}\mathrm{wd}(X))=\operatorname{cov}(X,\mathcal{N}\mathrm{wd}(X))$$

where $\mathcal{N}\mathrm{wd}(X)$ is the ideal of nowhere dense subsets of X. Note that if $\mathcal{M}(X)$ is the ideal of meager subsets of X, then $\operatorname{cov}(X,\mathcal{M}(X))=n(X)$ unless $n(X)=\omega$, in which case $\operatorname{cov}(X,\mathcal{M}(X))=1$.

(c) Let $\mathfrak{A}$ be a Boolean algebra. Write $\mathfrak{A}^+$ for $\mathfrak{A}\setminus\{0\}$ and $\mathfrak{A}^-$ for $\mathfrak{A}\setminus\{1\}$. Then

$$\pi(\mathfrak{A})=\operatorname{cov}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{cov}(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-),$$

$$\operatorname{sat}(\mathfrak{A})=\operatorname{sat}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{sat}(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-),$$

$$d(\mathfrak{A})=\operatorname{link}_{<\omega}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{link}_{<\omega}(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-),$$

$$\operatorname{link}(\mathfrak{A})=\operatorname{link}_2(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{link}_2(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-)$$

and generally

$$\operatorname{link}_{<\kappa}(\mathfrak{A})=\operatorname{link}_{<\kappa}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{link}_{<\kappa}(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-),$$

$$\operatorname{link}_{\kappa}(\mathfrak{A})=\operatorname{link}_{\kappa}(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)=\operatorname{link}_{\kappa}(\mathfrak{A}^-,\subseteq,\mathfrak{A}^-)$$

for every cardinal κ.

(d) Let X be a set and $\mathcal{I}$ an ideal of subsets of X. Then the dual of $(X,\in,\mathcal{I})$ is $(\mathcal{I},\not\ni,X)$; $\operatorname{cov}(X,\in,\mathcal{I})=\operatorname{cov}\mathcal{I}$ and $\operatorname{add}(X,\in,\mathcal{I})=\operatorname{non}\mathcal{I}$.

(e) For a Boolean algebra $\mathfrak{A}$, write $\operatorname{Pou}(\mathfrak{A})$ for the set of partitions of unity in $\mathfrak{A}$. For C, $D\in\operatorname{Pou}(\mathfrak{A})$, say that $C\sqsubseteq^* D$ if every element of D meets only finitely many members of C. Then $\sqsubseteq^*$ is a pre-order on $\operatorname{Pou}(\mathfrak{A})$. Translating the definition 511Df into this language, we see that $\operatorname{wdistr}(\mathfrak{A})=\operatorname{add}\operatorname{Pou}(\mathfrak{A})$.

512F I now turn to some constructions involving supported relations and Galois-Tukey connections which will be useful later.

Dominating sets For any supported relation (A,R,B) and any cardinal κ, we can form a corresponding supported relation $(A,R',[B]^{<\kappa})$, where

$$R' = \{(a,I) : I \in [B]^{<\kappa},\ a \in R^{-1}[I]\}.$$

The most important cases to us will be $\kappa=\omega$ and $\kappa=\omega_1$. When κ is a successor cardinal I will normally write $(A,R',[B]^{\le\lambda})$ rather than $(A,R',[B]^{<\lambda^+})$. Purists may wish to revise the definition of R' so that it is no longer a proper class.

512G Proposition Let (A,R,B) and (C,S,D) be supported relations and κ, λ cardinals.

(a) (A,R,B) is isomorphic to $(A,R',[B]^1)$.

(b) If $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$ and $\lambda\le\kappa$ then $(A,R',[B]^{<\kappa})\preccurlyeq_{\mathrm{GT}}(C,S',[D]^{<\lambda})$.

(c) In particular, $(A,R',[B]^{<\kappa})\preccurlyeq_{\mathrm{GT}}(A,R,B)$ if $\kappa\ge 2$.

(d) If $\operatorname{cf}\kappa\ge\lambda$ and $(A,R',[B]^{<\kappa})\preccurlyeq_{\mathrm{GT}}(C,S,D)$ then $(A,R',[B]^{<\kappa})\preccurlyeq_{\mathrm{GT}}(C,S',[D]^{<\lambda})$.

(e)(i) If $\operatorname{cov}(A,R,B)=\infty$ then $\operatorname{add}(A,R',[B]^{<\kappa})\le 1$.

(ii) If $\operatorname{cov}(A,R,B)<\infty$ then $\operatorname{add}(A,R',[B]^{<\kappa})\ge\kappa$.

(f) $\operatorname{cov}(A,R,B)\le\max(\omega,\kappa,\operatorname{cov}(A,R',[B]^{\le\kappa}))$; if $\kappa\ge 1$ and $\operatorname{cov}(A,R,B)>\max(\kappa,\omega)$ then $\operatorname{cov}(A,R,B)=\operatorname{cov}(A,R',[B]^{\le\kappa})$.

proof (a) is trivial.

(b) If (ϕ,ψ) is a Galois-Tukey connection from (A,R,B) to (C,S,D), then (ϕ,ψ') is a Galois-Tukey connection from $(A,R',[B]^{<\kappa})$ to $(C,S',[D]^{<\lambda})$, where $\psi'(J)=\psi[J]$ for every $J\in[D]^{<\lambda}$.

(c) Setting $\phi(a)=a$ for $a\in A$ and $\psi(b)=\{b\}$ for $b\in B$, (ψ,ϕ) is a Galois-Tukey connection from $(A,R',[B]^{<\kappa})$ to (A,R,B).

(d) Let (ϕ,ψ) be a Galois-Tukey connection from $(A,R',[B]^{<\kappa})$ to (C,S,D). Set $\psi'(I)=\bigcup_{d\in I}\psi(d)$ for $I\in[D]^{<\lambda}$; then (ϕ,ψ') is a Galois-Tukey connection from $(A,R',[B]^{<\kappa})$ to $(C,S',[D]^{<\lambda})$.

(e)(i) There is an $a\in A\setminus R^{-1}[B]$; now $(a,I)\notin R'$ for any $I\in[B]^{<\kappa}$, so $\operatorname{add}(A,R',[B]^{<\kappa})\le 1$.

(ii) For every $a\in A$ there is a $b_a\in B$ such that $(a,b_a)\in R$. If $A'\subseteq A$ and $\#(A')<\kappa$, then $I=\{b_a : a\in A'\}$ belongs to $[B]^{<\kappa}$, and $(a,I)\in R'$ for every $a\in A'$; as A' is arbitrary, $\operatorname{add}(A,R',[B]^{<\kappa})\ge\kappa$.

(f) If $\lambda=\operatorname{cov}(A,R',[B]^{\le\kappa})$ is not ∞, let $\mathcal{D}\subseteq[B]^{\le\kappa}$ be a set of size λ such that $A=(R')^{-1}[\mathcal{D}]$, and set $D=\bigcup\mathcal{D}$; then $A\subseteq R^{-1}[D]$, so $\operatorname{cov}(A,R,B)\le\#(D)\le\max(\omega,\kappa,\lambda)$.

If $\kappa\ge 1$, then $\operatorname{cov}(A,R',[B]^{\le\kappa})\le\operatorname{cov}(A,R,B)$, by (c) and 512Da, so if the latter is greater than $\max(\kappa,\omega)$ they are equal1111.

512H Simple products (a) If $\langle(A_i,R_i,B_i)\rangle_{i\in I}$ is any family of supported relations, its **simple product** is $(\prod_{i\in I}A_i,T,\prod_{i\in I}B_i)$ where $T=\{(a,b) : (a(i),b(i))\in R_i$ for every $i\in I\}$.

(b) Let $\langle(A_i,R_i,B_i)\rangle_{i\in I}$ and $\langle(C_i,S_i,D_i)\rangle_{i\in I}$ be two families of supported relations, with simple products (A,R,B) and (C,S,D). If $(A_i,R_i,B_i)\preccurlyeq_{\mathrm{GT}}(C_i,S_i,D_i)$ for every i, then $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$. **P** For each i, let (ϕ_i,ψ_i) be a Galois-Tukey connection from (A_i,R_i,B_i) to (C_i,S_i,D_i). Define $\phi:A\to C$ and $\psi:D\to B$ by setting $\phi(\langle a_i\rangle_{i\in I})=\langle\phi_i(a_i)\rangle_{i\in I}$, $\psi(\langle d_i\rangle_{i\in I})=\langle\psi_i(d_i)\rangle_{i\in I}$ for $\langle a_i\rangle_{i\in I}\in A$, $\langle d_i\rangle_{i\in I}\in D$; then

$$\begin{aligned}(\phi(\langle a_i\rangle_{i\in I}),\langle d_i\rangle_{i\in I})\in S &\Longrightarrow (\phi_i(a_i),d_i)\in S_i \text{ for every } i\in I\\ &\Longrightarrow (a_i,\psi_i(d_i))\in R_i \text{ for every } i\in I\\ &\Longrightarrow (\langle a_i\rangle_{i\in I},\psi(\langle d_i\rangle_{i\in I}))\in R.\end{aligned}$$

So (ϕ,ψ) is a Galois-Tukey connection and $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$. **Q**

(c) Let $\langle(A_i,R_i,B_i)\rangle_{i\in I}$ be a family of supported relations with simple product (A,R,B). Suppose that no A_i is empty. Then $\operatorname{add}(A,R,B)=\min_{i\in I}\operatorname{add}(A_i,R_i,B_i)$, interpreting $\min\emptyset$ as ∞ if $I=\emptyset$. **P** Set $\kappa=\operatorname{add}(A,R,B)$ and $\kappa'=\min_{i\in I}\operatorname{add}(A_i,R_i,B_i)$. If $I=\emptyset$ then $A=B=\{\emptyset\}$ and $R=\{(\emptyset,\emptyset)\}$ so $\operatorname{add}(A,R,B)=\infty$. Otherwise, if $C\subseteq A$ and $\#(C)<\kappa'$, then, for each i, $\#(\{c(i):c\in C\})<\operatorname{add}(A_i,R_i,B_i)$, so there is a $b_i\in B_i$ such that $(c(i),b_i)\in R_i$ for every $c\in C$; now $(c,\langle b_i\rangle_{i\in I})\in R$ for every $c\in C$; as C is arbitrary, $\kappa\ge\kappa'$. In the other direction, if $i\in I$ and $C'\in[A_i]^{<\kappa}$, then (because no A_j is empty) there is a $C\in[A]^{<\kappa}$ such that $C'=\{c(i):c\in C\}$. Now there is a $b\in B$ such that $(c,b)\in R$ for every $c\in C$, so that $(c',b(i))\in R_i$ for every $c'\in C'$. As i and C' are arbitrary, $\kappa'\le\kappa$. **Q**

(d) Suppose that (A,R,B) and (C,S,D) are supported relations with simple product $(A\times C,T,B\times D)$. Let κ be an infinite cardinal and define $(A,R',[B]^{<\kappa})$, $(C,S',[D]^{<\kappa})$ and $(A\times C,T',[B\times D]^{<\kappa})$ as in 512F. Then

$$(A,R',[B]^{<\kappa})\times(C,S',[D]^{<\kappa})\equiv_{\mathrm{GT}}(A\times C,T',[B\times D]^{<\kappa}).$$

P Express $(A,R',[B]^{<\kappa})\times(C,S',[D]^{<\kappa})$ as $(A\times C,\tilde T,[B]^{<\kappa}\times[D]^{<\kappa})$.

(i) Set $\phi(a,c)=(a,c)$ for all $a\in A$, $c\in C$, and for $I\in[B\times D]^{<\kappa}$ set

$$\psi(I)=(\pi_1[I],\pi_2[I])\in[B]^{<\kappa}\times[D]^{<\kappa},$$

where $\pi_1(b,d)=b$ and $\pi_2(b,d)=d$ for $b\in B$, $d\in D$. If $a\in A$, $c\in C$ and $I\in[B\times D]^{<\kappa}$ are such that $(\phi(a,c),I)\in T'$, then there must be a $(b,d)\in I$ such that $((a,c),(b,d))\in T$, that is, $(a,b)\in R$ and $(c,d)\in S$; now $b\in\pi_1[I]$ and $d\in\pi_2[I]$, so $(a,\pi_1[I])\in R'$ and $(c,\pi_2[I])\in S'$ and

$$((a,c),\psi(I))=((a,c),(\pi_1[I],\pi_2[I]))\in\tilde T.$$

As a, c and I are arbitrary, (ϕ,ψ) is a Galois-Tukey connection and

$$(A,R',[B]^{<\kappa})\times(C,S',[D]^{<\kappa})\preccurlyeq_{\mathrm{GT}}(A\times C,T',[B\times D]^{<\kappa}).$$

(ii) In the other direction, given $(J,K)\in[B]^{<\kappa}\times[D]^{<\kappa}$ set $\psi'(J,K)=J\times K\in[B\times D]^{<\kappa}$. (This is where we need to suppose that κ is infinite.) If now $(\phi(a,c),(J,K))\in\tilde T$, that is, $(a,J)\in R'$ and $(c,K)\in S'$, there are $b\in J$ and $d\in K$ such that $(a,b)\in R$ and $(c,d)\in S$, so that $((a,c),(b,d))\in T$ and $((a,c),\psi'(J,K))\in T'$. As a, c, J and K are arbitrary, (ϕ,ψ') is a Galois-Tukey connection and

$$(A\times C,T',[B\times D]^{<\kappa})\preccurlyeq_{\mathrm{GT}}(A,R',[B]^{<\kappa})\times(C,S',[D]^{<\kappa}).\ \mathbf{Q}$$

(e) If $\langle(P_i,\le_i)\rangle_{i\in I}$ is a family of pre-ordered sets, with product $(P,\le)$ (511A), then $(P,\le,P)$ is just $\prod_{i\in I}(P_i,\le_i,P_i)$ in the sense here.

512I Sequential compositions Let (A,R,B) and (C,S,D) be supported relations. Their **sequential composition** $(A,R,B)\ltimes(C,S,D)$ is $(A\times C^B,T,B\times D)$, where

$$T=\{((a,f),(b,d)):(a,b)\in R,\ f\in C^B,\ (f(b),d)\in S\}.$$

Their **dual sequential composition** $(A,R,B)\rtimes(C,S,D)$ is $(A\times C,\tilde T,B\times D^A)$ where

$$\begin{aligned}\tilde T=\{((a,c),(b,g)):a\in A,\ b\in B,\ c\in C,\ g\in D^A\\ \text{and either }(a,b)\in R\text{ or }(c,g(a))\in S\}.\end{aligned}$$

512J Proposition Let (A,R,B) and (C,S,D) be supported relations.

(a) $(A,R,B)\rtimes(C,S,D)=((A,R,B)^{\perp}\ltimes(C,S,D)^{\perp})^{\perp}$.

(b) $\mathrm{cov}((A,R,B)\ltimes(C,S,D))$ is the cardinal product $\mathrm{cov}(A,R,B)\cdot\mathrm{cov}(C,S,D)$ unless $B=C=\emptyset\ne A$, if we use the interpretations

$$0\cdot\infty=\infty\cdot0=0,\quad \kappa\cdot\infty=\infty\cdot\kappa=\infty\cdot\infty=\infty\text{ for every cardinal }\kappa\ge1.$$

(c) $\mathrm{add}((A,R,B)\ltimes(C,S,D))=\min(\mathrm{add}(A,R,B),\mathrm{add}(C,S,D))$ unless $A\times C=\emptyset\ne B\times D$.

proof (a) is just a matter of disentangling the definitions.

(b) Define $T\subseteq(A\times C^B)\times(B\times D)$ as in 512I.

(i) Suppose first that neither A nor C is empty, that $A\subseteq R^{-1}[B]$ and that $C\subseteq S^{-1}[D]$. If $B_0\subseteq B$ and $D_0\subseteq D$ are such that $A\subseteq R^{-1}[B_0]$ and $C\subseteq S^{-1}[D_0]$, then for any $a\in A$ and $f\in C^B$ there are $b\in B_0$ and $d\in D_0$ such that $(a,b)\in R$ and $(f(b),d)\in S$, so that $(a,f)\in T^{-1}[B_0\times D_0]$. So $\mathrm{cov}((A,R,B)\ltimes(C,S,D))\le\mathrm{cov}(A,R,B)\cdot\mathrm{cov}(C,S,D)$.

On the other hand, if $H\subseteq B\times D$ is such that $A\times C^B\subseteq T^{-1}[H]$, set $B_0=\{b:C\subseteq S^{-1}[H[\{b\}]]\}$. Then $\#(H[\{b\}])\ge\mathrm{cov}(C,S,D)$ for $b\in B_0$. Also $A\subseteq R^{-1}[B_0]$. **P?** Otherwise, take $a\in A\setminus R^{-1}[B_0]$. For $b\in B\setminus B_0$, choose $f(b)\in C\setminus S^{-1}[H[\{b\}]]$; for $b\in B_0$, take $f(b)$ to be any member of C. There is supposed to be a member (b,d) of H such that $((a,f),(b,d))\in T$, that is, $(a,b)\in R$ and $(f(b),d)\in S$. But now $b\notin B_0$, by the choice of a, and $(f(b),d)\notin S$, by the choice of f; so we have a contradiction. **XQ**

So $\#(B_0)\ge\mathrm{cov}(A,R,B)$ and $\#(H)\ge\mathrm{cov}(A,R,B)\cdot\mathrm{cov}(C,S,D)$; as H is arbitrary, $\mathrm{cov}((A,R,B)\ltimes(C,S,D))\ge\mathrm{cov}(A,R,B)\cdot\mathrm{cov}(C,S,D)$.

(ii) If $A=\emptyset$ then $A\times C^B=\emptyset$ so $\operatorname{cov}(A,R,B)$ and $\operatorname{cov}((A,R,B)\ltimes(C,S,D))$ are both zero. If $C=\emptyset$ and $B\neq\emptyset$ then $\operatorname{cov}(C,S,D)=\operatorname{cov}((A,R,B)\ltimes(C,S,D))=0$. If A and C are non-empty and $A\not\subseteq R^{-1}[B]$, then $A\times C^B\not\subseteq T^{-1}[B\times D]$, so $\operatorname{cov}(A,R,B)=\operatorname{cov}((A,R,B)\ltimes\operatorname{cov}(C,S,D))=\infty$, while $\operatorname{cov}(C,S,D)\geq 1$. If A and C are non-empty and $A\subseteq R^{-1}[B]$ and $C\not\subseteq S^{-1}[D]$, then $B\neq\emptyset$; if we take $c\in C\setminus S^{-1}[D]$ and any member a of A, and set $f(b)=c$ for every $b\in B$, then $(a,f)\notin T^{-1}[B\times D]$, so $\operatorname{cov}((A,R,B)\ltimes(C,S,D))=\operatorname{cov}(C,S,D)=\infty$, while $\operatorname{cov}(A,R,B)\geq 1$. So with the single exception of $B=C=\emptyset\neq A$ (in which case the empty function belongs to C^B, so that $\operatorname{cov}((A,R,B)\ltimes(C,S,D))=\infty$, while $\operatorname{cov}(C,S,D)=0$) we have $\operatorname{cov}((A,R,B)\ltimes(C,S,D))=\operatorname{cov}(A,R,B)\cdot\operatorname{cov}(C,S,D)$.

(c) Assume throughout that either $A\times C\neq\emptyset$ (so that $A\times C^B\neq\emptyset$) or that $B\times D=\emptyset$.

(i) $\operatorname{add}((A,R,B)\ltimes(C,S,D))\leq\operatorname{add}(A,R,B)$. **P** If $\operatorname{add}(A,R,B)=\infty$ the result is trivial. If $B\times D=\emptyset$ then $\operatorname{add}((A,R,B)\ltimes(C,S,D))=0\leq\operatorname{add}(A,R,B)$. Otherwise, our hypothesis ensures that C is not empty; take $A_0\subseteq A$ such that $\#(A_0)=\operatorname{add}(A,R,B)$ and $A_0\not\subseteq R^{-1}[\{b\}]$ for any $b\in B$, take any $b_0\in B$ and any $f_0\in C^B$; then there is no $(b,d)\in B\times D$ such that $((a,f_0(b_0)),(b,d))\in T$ for every $a\in A_0$, so $\operatorname{add}((A,R,B)\ltimes(C,S,D))\leq\#(A_0)=\operatorname{add}(A,R,B)$. **Q**

(ii) $\operatorname{add}((A,R,B)\ltimes(C,S,D))\leq\operatorname{add}(C,S,D)$. **P** Again, if $\operatorname{add}(C,S,D)=\infty$ or $B\times D=\emptyset$ the result is immediate. Otherwise, $A\neq\emptyset$. Take $C_0\subseteq C$ such that $\#(C_0)=\operatorname{add}(C,S,D)$ and there is no $d\in D$ such that $C_0\subseteq S^{-1}[\{d\}]$, for $c\in C_0$ set $f_c(b)=c$ for every $b\in B$, and fix any $a_0\in A$; then there is no $(b,d)\in B\times D$ such that $((a_0,f_c),(b,d))\in T$ for every $c\in C_0$, so $\operatorname{add}((A,R,B)\ltimes(C,S,D))\leq\#(C_0)=\operatorname{add}(C,S,D)$. **Q**

(iii) $\operatorname{add}((A,R,B)\ltimes(C,S,D))\geq\min(\operatorname{add}(A,R,B),\operatorname{add}(C,S,D))$. **P** If $H\subseteq A\times C^B$ and $\#(H)$ is less than $\min(\operatorname{add}(A,R,B),\operatorname{add}(C,S,D))$, set $A_0=\{a:(a,f)\in H\}$ and $F=\{f:(a,f)\in H\}$. Then there are a $b\in B$ such that $(a,b)\in R$ for any $a\in A_0$, and a $d\in D$ such that $(f(b),d)\in S$ for any $f\in F$, so that $((a,f),(b,d))\in T$ for any $(a,f)\in H$. As H is arbitrary, $\operatorname{add}((A,R,B)\ltimes(C,S,D))\geq\min(\operatorname{add}(A,R,B),\operatorname{add}(C,S,D))$. **Q**

512K The following fact will be used in §526.

Lemma Suppose that (A,R,B) and (C,S,D) are supported relations, and P is a partially ordered set. Suppose that $\langle A_p\rangle_{p\in P}$ is a family of subsets of A such that

$$(A_p,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)\text{ for every }p\in P,$$

$$A_p\subseteq A_q\text{ whenever }p\leq q\text{ in }P,\quad\bigcup_{p\in P}A_p=A.$$

Then $(A,R,B)\preccurlyeq_{\mathrm{GT}}(P,\leq,P)\ltimes(C,S,D)$.

proof If $C=\emptyset$ the result is trivial, since every A_p is empty and B can be empty only if D is. So we may suppose that $C\neq\emptyset$. For each $p\in P$, let (ϕ_p,ψ_p) be a Galois-Tukey connection from (A_p,R,B) to (C,S,D). For $a\in A$, let $r(a)\in P$ be such that $a\in A_{r(a)}$, and set $f_a(p)=\phi_p(a)$ whenever $p\in P$ and $a\in A_p$; for other $p\in P$ take $f_a(p)$ to be any member of C. Set $\phi(a)=(r(a),f_a)$ for $a\in A$. For $q\in P$, $d\in D$ set $\psi(q,d)=\psi_q(d)\in B$. Now (ϕ,ψ) is a Galois-Tukey connection from (A,R,B) to $(P,\leq,P)\ltimes(C,S,D)$. **P** Suppose that $a\in A$ and $(q,d)\in P\times D$ are such that $r(a)\leq q$ and $(f_a(q),d)\in S$. Then $a\in A_{r(a)}\subseteq A_q$ so $f_a(q)=\phi_q(a)$. Because (ϕ_q,ψ_q) is a Galois-Tukey connection, $(a,\psi(q,d))=(a,\psi_q(d))\in R$. **Q**

So we have the result.

512X Basic exercises (a)(i) Suppose that $A\subseteq A'$, $B'\subseteq B$ and that R is any relation. Show that $(A,R,B)\preccurlyeq_{\mathrm{GT}}(A',R,B')$. (ii) Show that $(\emptyset,\emptyset,\{\emptyset\})\preccurlyeq_{\mathrm{GT}}(A,R,B)\preccurlyeq_{\mathrm{GT}}(\{\emptyset\},\emptyset,\emptyset)$ for every supported relation (A,R,B).

(b) Let (A,R,B) be any supported relation. Show that $\operatorname{sat}(A,R,B)\leq(\operatorname{link}_2(A,R,B))^+$.

(c) Let $(X,\mathfrak{T})$ be a topological space and $(Y,\mathfrak{T}_Y)$ an open subspace. Show that $(\mathfrak{T}_Y\setminus\{\emptyset\},\supseteq,\mathfrak{T}_Y\setminus\{\emptyset\})\preccurlyeq_{\mathrm{GT}}(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})$.

(d) Let $(X,\mathfrak{T})$ and $(Y,\mathfrak{S})$ be topological spaces. (i) Show that if Y is a continuous image of X, $(\mathfrak{S}\setminus\{\emptyset\},\ni,Y)\preccurlyeq_{\mathrm{GT}}(\mathfrak{T}\setminus\{\emptyset\},\ni,X)$. (ii) Show that if X and Y are compact and Hausdorff and there is an irreducible continuous surjection from X onto Y, then $(\mathfrak{T}\setminus\{\emptyset\},\ni,X)\equiv_{\mathrm{GT}}(\mathfrak{S}\setminus\{\emptyset\},\ni,Y)$ and $(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})\equiv_{\mathrm{GT}}(\mathfrak{S}\setminus\{\emptyset\},\supseteq,\mathfrak{S}\setminus\{\emptyset\})$, so $d(X)=d(Y)$ and $\pi(X)=\pi(Y)$.

(e) Let $\langle (A_i, R_i, B_i)\rangle_{i\in I}$ be a family of supported relations with simple product (A, R, B). Show that $(A, R, B)^{\perp}$ can be naturally identified with the simple product of $\langle (A_i, R_i, B_i)^{\perp}\rangle_{i\in I}$.

(f) Let (A, R, B) be a supported relation and $\kappa > 0$ a cardinal. Show that $(A, R', [B]^{\le\kappa}) \preccurlyeq_{\mathrm{GT}} (A, R, B)^{\kappa}$, where $(A, R, B)^{\kappa}$ is the simple product of κ copies of (A, R, B) and $R' = \{(a, J) : a \in R^{-1}[J]\}$ as usual.

(g) Let (A, R, B) and (C, S, D) be supported relations, and $(A\times C, T, B\times D)$ their simple product. (i) Show that if $C \ne \emptyset$, then $(A, R, B) \preccurlyeq_{\mathrm{GT}} (A\times C, T, B\times D)$. (ii) Show that $(A\times C, T, B\times D) \preccurlyeq_{\mathrm{GT}} (A, R, B) \ltimes (C, S, D)$. (iii) Show that (using the conventions of 512Jb) $\mathrm{cov}(A\times C, T, B\times D) = \mathrm{cov}(A, R, B)\cdot\mathrm{cov}(C, S, D)$.

(h) Let (A_0, R_0, B_0), (A_1, R_1, B_1), (C_0, S_0, D_0) and (C_1, S_1, D_1) be supported relations such that $(A_0, R_0, B_0) \preccurlyeq_{\mathrm{GT}} (A_1, R_1, B_1)$ and $(C_0, S_0, D_0) \preccurlyeq_{\mathrm{GT}} (C_1, S_1, D_1)$. Show that

$$(A_0, R_0, B_0) \ltimes (C_0, S_0, D_0) \preccurlyeq_{\mathrm{GT}} (A_1, R_1, B_1) \ltimes (C_1, S_1, D_1),$$

$$(A_0, R_0, B_0) \rtimes (C_0, S_0, D_0) \preccurlyeq_{\mathrm{GT}} (A_1, R_1, B_1) \rtimes (C_1, S_1, D_1).$$

512 Notes and comments Much of this section is cluttered by the repeated names (A, R, B) of 'supported relations'. In fact these could probably be dispensed with. While I am reluctant to alter the general convention I use in this book, that a 'relation' is neither more nor less than a class of ordered pairs, it is clear that in all significant cases our supported relation (A, R, B) will be such that $A = \{a : (a, b) \in R\}$ and $B = \{b : (a, b) \in R\}$, so that A and B can be recovered from the set R. But this would make impossible the very useful convention that '$(X, \in, \mathcal{A})$' is to be interpreted as '$(X, \{(x, A) : A \in \mathcal{A},\, x \in X\cap A\}, \mathcal{A})$', and since nearly every mathematical argument in this context demands names for the domains and codomains of the relations, it seems easier to write these in each time.

An important feature of the theory here is that while it is very common for our relations to be reasonably well-behaved by some criterion (for instance, we may have Polish spaces A and B and a coanalytic set $R \subseteq A\times B$), the functions in a Galois-Tukey connection are not required to have any properties beyond those declared in the definition. Of course the most important Galois-Tukey connections are those which are 'natural' in some sense, and are constructed in a way which does not involve totally unscrupulous use of the axiom of choice. I will return to this question in the next section.

513 Partially ordered sets

In §§511-512 I have given long lists of definitions. It is time I filled in details of the most elementary relationships between the various concepts introduced. Here I treat some of those which can be expressed in the language of partially ordered sets. I begin with notes on cofinality and saturation, with the Erdős-Tarski theorem (513B). In this context, Galois-Tukey connections take on particularly direct forms (513D-513E); for directed sets, we have an alternative definition of Tukey equivalence (513F). The majority of the cardinal functions defined so far on partially ordered sets are determined by their cofinal structure (513G, 513If; see also 516Ga below).

In the last third of the section (513K-513O), I discuss Tukey functions between directed sets with a special kind of topological structure, which I call 'metrizably compactly based'; the point is that for Polish metrizably compactly based directed sets, if there is any Tukey function between them, there must be one which is measurable in an appropriate sense (513O).

513A It will help to have an elementary lemma on maximal antichains.

Lemma Let P be a partially ordered set.

(a) If $Q \subseteq P$ is cofinal and $A \subseteq Q$ is an up-antichain, there is a maximal up-antichain A' in P such that $A \subseteq A' \subseteq Q$. In particular, Q includes a maximal up-antichain.

(b) If $A \subseteq P$ is a maximal up-antichain, $Q = \bigcup_{q\in A} [q, \infty[$ is cofinal with P.

proof (a) Let $A' \subseteq Q$ be maximal subject to being an up-antichain in P including A. Then for any $p \in P\setminus A'$, there are a $q \in Q$ such that $p \le q$, and an $r \in A'$ such that

$$\emptyset \ne [r, \infty[\cap [q, \infty[\subseteq [r, \infty[\cap [p, \infty[,$$

so $A' \cup \{p\}$ is not an up-antichain. But this means that A' is a maximal up-antichain in P.

Starting from $A = \emptyset$, we see that Q includes a maximal up-antichain.

(b) If $p \in P$, then either $p \in A \subseteq Q$, or $A \cup \{p\}$ is not an up-antichain, so that there is some $q \in A$ such that

$$\emptyset \neq [p, \infty[\cap [q, \infty[\subseteq Q \cap [p, \infty[.$$

513B Theorem Let P be a partially ordered set.
(a) $\operatorname{bu} P \le \operatorname{cf} P \le \#(P)$.
(b) $\operatorname{sat}^{\uparrow}(P)$ is either finite or a regular uncountable cardinal.
(c) $c^{\uparrow}(P)$ is the predecessor of $\operatorname{sat}^{\uparrow}(P)$ if $\operatorname{sat}^{\uparrow}(P)$ is a successor cardinal, and otherwise is equal to $\operatorname{sat}^{\uparrow}(P)$.

proof (a) To see that $\operatorname{cf} P \le \#(P)$ all we have to note is that P is a cofinal subset of itself. To see that $\operatorname{bu} P \le \operatorname{cf} P$, set $\kappa = \operatorname{cf} P$ and let $\langle p_\xi \rangle_{\xi<\kappa}$ enumerate a cofinal subset of P. Set

$$A = \{\xi : \xi < \kappa,\, p_\xi \not\le p_\eta \text{ for any } \eta < \xi\}, \quad Q = \{p_\xi : \xi \in A\}.$$

If $p \in P$ there is a least $\xi < \kappa$ such that $p \le p_\xi$, and now $\xi \in A$; so Q is cofinal with P. If $p \in P$, there is some $\xi \in A$ such that $p \le p_\xi$, and now $\{q : q \in Q,\, q \le p,\, p \not\le q\} \subseteq \{p_\eta : \eta < \xi\}$ has cardinal less than κ, so that Q witnesses that $\operatorname{bu} P \le \kappa$.

(b)(i) Set $\kappa = \operatorname{sat}^{\uparrow}(P)$. For $p \in P$, set $\theta(p) = \operatorname{sat}^{\uparrow}([p, \infty[)$. Note that if $p \in P$ and $B \subseteq P$ is any up-antichain, then $B_p = \{q : q \in B,\, [q, \infty[\cap [p, \infty[\neq \emptyset\}$ has less than $\theta(p)$ members. **P** For $q \in B_p$, choose $q' \in [q, \infty[\cap [p, \infty[$. Because B is an up-antichain, $\{q' : q \in B_p\}$ is an up-antichain and $q \mapsto q'$ is injective; so $\#(B_p) = \#(\{q' : q \in B_p\}) < \theta(p)$. **Q**

If $p \le q$ in P, any up-antichain in $[q, \infty[$ is also an up-antichain in $[p, \infty[$, so $\theta(q) \le \theta(p)$. It follows that $Q = \{p : p \in P,\, \theta(q) = \theta(p) \text{ for every } q \ge p\}$ is cofinal with P. Let $A \subseteq Q$ be a maximal up-antichain (513Aa); then $\#(A) < \kappa$.

(ii) ? Suppose, if possible, that $\kappa = \omega$.

case 1 Suppose there is a $p \in A$ such that $\theta(p) = \omega$. Then we can choose $\langle p_n \rangle_{n\in\mathbb{N}}$, $\langle q_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. $p_0 = p$. Given that $p_n \in Q$ and $\theta(p_n) = \omega$, there must be $p_{n+1}, q_n \in [p_n, \infty[$ such that $[p_{n+1}, \infty[\cap [q_n, \infty[= \emptyset$; now $p_{n+1} \in Q$ and $\theta(p_{n+1}) = \omega$. Continue. At the end of the induction, $\{q_n : n \in \mathbb{N}\}$ is an infinite up-antichain in P, which is impossible.

case 2 Suppose that $\theta(p) < \omega$ for every $p \in A$. Then $n = \sum_{p\in A} \theta(p)$ is finite. Let $B \subseteq P$ be any up-antichain. For each $p \in A$, set $B_p = \{q : q \in B,\, [q, \infty[\cap [p, \infty[\neq \emptyset\}$; as noted in (i), $\#(B_p) < \theta(p)$ for every $p \in A$, so $\#(\bigcup_{p\in A} B_p) \le n$. But $B = \bigcup_{p\in A} B_p$, because A is a maximal up-antichain, so $\#(B) \le n$. As B is arbitrary, $\operatorname{sat}^{\uparrow}(P) \le n + 1 < \omega$. **X**

Thus $\kappa \neq \omega$.

(iii) ? Suppose, if possible, that κ is a singular infinite cardinal. Set $\lambda = \operatorname{cf} \kappa$ and let $\langle \kappa_\xi \rangle_{\xi<\lambda}$ be a strictly increasing family of cardinals with supremum κ.

case 1 Suppose there is a $p \in Q$ such that $\theta(p) = \kappa$. Then, because $\lambda < \kappa$, there is an up-antichain $B \subseteq [p, \infty[$ with cardinal λ; enumerate B as $\langle p_\xi \rangle_{\xi<\lambda}$. For each $\xi < \lambda$, $\theta(p_\xi) > \kappa_\xi$, so there is an up-antichain $C_\xi \subseteq [p_\xi, \infty[$ with cardinal κ_ξ. Now $C = \bigcup_{\xi<\lambda} C_\xi$ is an up-antichain in P with cardinal κ, which is supposed to be impossible.

case 2 Suppose that $\theta(p) < \kappa$ for every $p \in Q$.

case 2a Suppose that $\sup_{p\in A} \theta(p) < \kappa$. Then there is an up-antichain $C \subseteq P$ such that $\#(C)$ is greater than $\max(\omega, \#(A), \sup_{p\in A} \theta(p))$. For each $p \in A$ set $C_p = \{q : q \in C,\, [q, \infty[\cap [p, \infty[\neq \emptyset\}$, so that $\#(C_p) < \theta(p)$. It follows that $C \neq \bigcup_{p\in A} C_p$. But if $q \in C \setminus \bigcup_{p\in A} C_p$, there is a $q' \in Q$ such that $q' \ge q$, and now $A \cup \{q'\}$ is an up-antichain in Q strictly including A, which is impossible.

case 2b Suppose that $\sup_{p\in A} \theta(p) = \kappa$. Then we can choose inductively a family $\langle p_\xi \rangle_{\xi<\lambda}$ in A such that $\theta(p_\xi) > \max(\kappa_\xi, \sup_{\eta<\xi} \theta(p_\eta))$ for each ξ; the point being that when we come to choose p_ξ, $\langle \theta(p_\eta) \rangle_{\eta<\xi}$ is a family of fewer than $\operatorname{cf} \kappa$ cardinals less than κ, so has supremum less than κ. Now all the p_ξ must be distinct. For each ξ, let $B_\xi \subseteq [p_\xi, \infty[$ be an up-antichain of size κ_ξ; then $\bigcup_{\xi<\lambda} B_\xi$ is an up-antichain in P of size κ, which is impossible. **X**

Thus κ cannot be a singular infinite cardinal.

(c) All we need to know is that $c^{\uparrow}(P) = \sup\{\kappa : \kappa < \operatorname{sat}^{\uparrow}(P)\}$.

Remark (b) is sometimes called the **Erdős-Tarski theorem**.

513C Cofinalities of cardinal functions We can say a little about the possible cofinalities of the cardinals which have appeared so far.

Proposition (a) Let P be a partially ordered set with no greatest member.

(i) If $\operatorname{add} P$ is greater than 2 (that is, P is upwards-directed), it is a regular infinite cardinal, and there is a family $\langle p_\xi\rangle_{\xi<\operatorname{add} P}$ in P such that $p_\eta < p_\xi$ whenever $\eta<\xi<\operatorname{add} P$, but $\{p_\xi:\xi<\operatorname{add} P\}$ has no upper bound in P.

(ii) If $\operatorname{cf} P$ is infinite, its cofinality is at least $\operatorname{add} P$.

(b) Let $\mathcal{I}$ be an ideal of subsets of a set X such that $\bigcup\mathcal{I}=X\notin\mathcal{I}$.

(i) $\operatorname{cf}(\operatorname{add}\mathcal{I})=\operatorname{add}\mathcal{I}\le\operatorname{cf}(\operatorname{cf}\mathcal{I})$.

(ii) $\operatorname{cf}(\operatorname{non}\mathcal{I})\ge\operatorname{add}\mathcal{I}$.

(iii) If $\operatorname{cov}\mathcal{I}=\operatorname{cf}\mathcal{I}$ then $\operatorname{cf}(\operatorname{cf}\mathcal{I})\ge\operatorname{non}\mathcal{I}$.

proof (a)(i) By 511Hd, $\operatorname{add} P\ge\omega$; by 511He, $\operatorname{add} P<\infty$, so $\operatorname{add} P$ is an infinite cardinal. Let $\langle q_\xi\rangle_{\xi<\operatorname{add} P}$ be a family in P with no upper bound in P. Choose $\langle p_\xi\rangle_{\xi<\operatorname{add} P}$ inductively, as follows. Given p_ξ, where $\xi<\operatorname{add} P$, there is a $p'_\xi\in P$ such that $p'_\xi\not\le p_\xi$; let $p_{\xi+1}$ be an upper bound of $\{p_\xi,p'_\xi,q_\xi\}$. For a limit ordinal $\xi<\operatorname{add} P$, let p_ξ be an upper bound of $\{p_\eta:\eta<\xi\}$. This will ensure that $p_\eta<p_\xi$ whenever $\xi<\eta<\operatorname{add} P$ and that $\{p_\xi:\xi<\operatorname{add} P\}$ has no upper bound, since such a bound would have to be a bound for $\{q_\xi:\xi<\operatorname{add} P\}$.

? If $\operatorname{add} P$ is singular, express it as $\sup_{\xi<\lambda}\kappa_\xi$, where $\lambda<\operatorname{add} P$ and $\kappa_\xi<\operatorname{add} P$ for each $\xi<\lambda$. Then for each $\xi<\lambda$, $\{p_\eta:\eta<\kappa_\xi\}$ has an upper bound r_ξ in P; but now $\{r_\xi:\xi<\lambda\}$ has an upper bound in P, which is also an upper bound of $\{p_\eta:\eta<\operatorname{add} P\}$. **X** Thus $\operatorname{add} P$ is regular.

(ii) ? If $\operatorname{cf}(\operatorname{cf} P)<\operatorname{add} P$, express $\operatorname{cf} P$ as $\sup_{\xi<\lambda}\kappa_\xi$ where $\lambda<\operatorname{add} P$ and $\kappa_\xi<\operatorname{cf} P$ for each $\xi<\lambda$. Let $\langle p_\eta\rangle_{\eta<\operatorname{cf} P}$ enumerate a cofinal subset of P. Then $\{p_\eta:\eta<\kappa_\xi\}$ is never cofinal with P, so there is for each $\xi<\lambda$ a $q_\xi\in P$ such that $q_\xi\not\le p_\eta$ for every $\eta<\kappa_\xi$. But now there is a $q\in P$ which is an upper bound for $\{q_\xi:\xi<\lambda\}$, and $q\not\le p_\eta$ for any $\eta<\operatorname{cf} P$. **X**

(b)(i) Because $\bigcup\mathcal{I}=X\notin\mathcal{I}$, $\operatorname{cf}\mathcal{I}$ and $\operatorname{add}\mathcal{I}$ are both infinite, so this is just a special case of (a).

(ii) ? If $\operatorname{cf}(\operatorname{non}\mathcal{I})<\operatorname{add}\mathcal{I}$, express $\operatorname{non}\mathcal{I}$ as $\sup_{\xi<\lambda}\kappa_\xi$ where $\lambda<\operatorname{add}\mathcal{I}$ and $\kappa_\xi<\operatorname{non}\mathcal{I}$ for each $\xi<\lambda$. Let $\langle x_\eta\rangle_{\eta<\operatorname{non}\mathcal{I}}$ enumerate a subset of X not belonging to $\mathcal{I}$. Then $I_\xi=\{x_\eta:\eta<\kappa_\xi\}$ belongs to $\mathcal{I}$ for each $\xi<\lambda$; but this means that $\bigcup_{\xi<\lambda}I_\xi=\{x_\eta:\eta<\operatorname{non}\mathcal{I}\}$ belongs to $\mathcal{I}$. **X**

(iii) Set $\operatorname{cf}(\operatorname{cf}\mathcal{I})=\kappa$. Let $\langle\mathcal{A}_\xi\rangle_{\xi<\kappa}$ be a family of subsets of $\mathcal{I}$, all with cardinal less than $\operatorname{cf}\mathcal{I}=\operatorname{cov}\mathcal{I}$, such that $\bigcup_{\xi<\kappa}\mathcal{A}_\xi$ is cofinal with $\mathcal{I}$. Because $\#(\mathcal{A}_\xi)<\operatorname{cov}\mathcal{I}$, there is an $x_\xi\in X\setminus\bigcup\mathcal{A}_\xi$ for each $\xi<\kappa$. Now $\{x_\xi:\xi<\kappa\}$ is not included in any member of $\mathcal{A}_\xi$ for any ξ, so cannot belong to $\mathcal{I}$ and witnesses that $\operatorname{non}\mathcal{I}\le\kappa$.

513D Galois-Tukey connections between partial orders have some distinctive features which make a special language appropriate.

Definition Let P and Q be pre-ordered sets. A function $\phi:P\to Q$ is a **Tukey function** if $\phi^{-1}[B]$ is bounded above in P whenever $B\subseteq Q$ is bounded above in Q. A function $\psi:Q\to P$ is a **dual Tukey function** (also called 'cofinal function', 'convergent function') if $\psi[B]$ is cofinal with P whenever $B\subseteq Q$ is cofinal with Q.

If P and Q are pre-ordered sets, I will write '$P\preccurlyeq_{\mathrm{T}}Q$' if $(P,\le,P)\preccurlyeq_{\mathrm{GT}}(Q,\le,Q)$, and '$P\equiv_{\mathrm{T}}Q$' if $(P,\le,P)\equiv_{\mathrm{GT}}(Q,\le,Q)$; in the latter case I say that P and Q are **Tukey equivalent**. It follows immediately from 512C that $\preccurlyeq_{\mathrm{T}}$ is reflexive and transitive, and of course $P\equiv_{\mathrm{T}}Q$ iff $P\preccurlyeq_{\mathrm{T}}Q$ and $Q\preccurlyeq_{\mathrm{T}}P$.

513E Theorem Let P and Q be pre-ordered sets.

(a) If (ϕ,ψ) is a Galois-Tukey connection from $(P,\le,P)$ to $(Q,\le,Q)$ then $\phi:P\to Q$ is a Tukey function and $\psi:Q\to P$ is a dual Tukey function.

(b)(i) A function $\phi:P\to Q$ is a Tukey function iff there is a function $\psi:Q\to P$ such that (ϕ,ψ) is a Galois-Tukey connection from $(P,\le,P)$ to $(Q,\le,Q)$.

(ii) A function $\psi:Q\to P$ is a dual Tukey function iff there is a function $\phi:P\to Q$ such that (ϕ,ψ) is a Galois-Tukey connection from $(P,\le,P)$ to $(Q,\le,Q)$.

(iii) If $\psi:Q\to P$ is order-preserving and $\psi[Q]$ is cofinal with P, then ψ is a dual Tukey function.

(c) The following are equiveridical, that is, if one is true so are the others:

(i) $P\preccurlyeq_{\mathrm{T}}Q$;

(ii) there is a Tukey function $\phi:P\to Q$;

(iii) there is a dual Tukey function $\psi:Q\to P$.

(d)(i) Let $f : P \to Q$ be such that $f[P]$ is cofinal with Q and, for $p, p' \in P$, $f(p) \le f(p')$ iff $p \le p'$. Then $P \equiv_{\mathrm{T}} Q$.

(ii) Suppose that $A \subseteq P$ is cofinal with P. Then $A \equiv_{\mathrm{T}} P$

(iii) For $p, q \in P$ say that $p \equiv q$ if $p \le q$ and $q \le p$; let $\tilde{P}$ be the partially ordered set of equivalence classes in P under the equivalence relation $\equiv$ (511A, 511Ha). Then $P \equiv_{\mathrm{T}} \tilde{P}$.

(e) Suppose now that $P \preccurlyeq_{\mathrm{T}} Q$. Then

(i) $\operatorname{cf} P \le \operatorname{cf} Q$;

(ii) $\operatorname{add} P \ge \operatorname{add} Q$;

(iii) $\operatorname{sat}^{\uparrow}(P) \le \operatorname{sat}^{\uparrow}(Q)$, $c^{\uparrow}(P) \le c^{\uparrow}(Q)$;

(iv) $\operatorname{link}^{\uparrow}_{<\kappa}(P) \le \operatorname{link}^{\uparrow}_{<\kappa}(Q)$ for any cardinal κ;

(v) $\operatorname{link}^{\uparrow}(P) \le \operatorname{link}^{\uparrow}(Q)$, $d^{\uparrow}(P) \le d^{\uparrow}(Q)$.

(f) If P and Q are Tukey equivalent, then $\operatorname{cf} P = \operatorname{cf} Q$ and $\operatorname{add} P = \operatorname{add} Q$.

(g) If $\langle P_i \rangle_{i \in I}$ and $\langle Q_i \rangle_{i \in I}$ are families of pre-ordered ordered sets such that $P_i \preccurlyeq_{\mathrm{T}} Q_i$ for every i, then $\prod_{i \in I} P_i \preccurlyeq_{\mathrm{T}} \prod_{i \in I} Q_i$.

(h) If $0 < \kappa < \operatorname{add} P$ then $P \equiv_{\mathrm{T}} P^{\kappa}$. In particular, if P is upwards-directed then $P \equiv_{\mathrm{T}} P \times P$.

proof (a) To say that (ϕ, ψ) is a Galois-Tukey connection from $(P, \le, P)$ to $(Q, \le, Q)$ means just that $p \le \psi(q)$ whenever $\phi(p) \le q$. Now if $B \subseteq Q$ has an upper bound q, $\psi(q)$ is an upper bound for $\phi^{-1}[B]$; as B is arbitrary, ϕ is a Tukey function. Similarly, if $B \subseteq Q$ is cofinal, then for any $p \in P$ there is a $q \in B$ such that $\phi(p) \le q$ and $p \le \psi(q)$, so $\psi[B]$ is cofinal with P. As B is arbitrary, ψ is a dual Tukey function.

(b)(i) If $\phi : P \to Q$ is a Tukey function, then for each $q \in Q$ set $A_q = \{p : p \in P,\ \phi(p) \le q\}$. A_q must be bounded above in P; take $\psi(q)$ to be any upper bound for A_q in P. Then we see that $p \le \psi(q)$ whenever $\phi(p) \le q$, so that (ϕ, ψ) is a Galois-Tukey connection from $(P, \le, P)$ to $(Q, \le, Q)$. Together with (a), this proves (i).

(ii) If $\psi : Q \to P$ is a dual Tukey function, then for each $p \in P$ set $B_p = \{q : q \in Q,\ \psi(q) \not\ge p\}$. Then $\psi[B_p]$ is not cofinal with P, so B_p cannot be cofinal with Q, and there must be a $\phi(p) \in P$ such that $\phi(p) \not\le q$ for any $q \in B_p$. Turning this round, if $\phi(p) \le q$ then $q \notin B_p$ and $p \le \psi(q)$; so (ϕ, ψ) is a Galois-Tukey connection from $(P, \le, P)$ to $(Q, \le, Q)$. Together with (a), this proves (ii).

(iii) Because $\psi[Q]$ is cofinal with P, we have a function $\phi : P \to Q$ such that $p \le \psi\phi(p)$ for every $p \in P$. Now, for any $p \in P$ and $q \in Q$,

$$\phi(p) \le q \Longrightarrow p \le \psi\phi(p) \le \psi(q),$$

so (ϕ, ψ) is a Galois-Tukey connection and ψ is a dual Tukey function.

(c) This follows immediately from (a) and (b).

(d)(i) f is a Tukey function. **P** If $A \subseteq P$ and $f[A]$ is bounded above in Q, let q be an upper bound for $f[A]$. Because $f[P]$ is cofinal with Q, there is a $p_0 \in P$ such that $q \le f(p_0)$. If now $p \in A$, we have $f(p) \le q \le f(p_0)$ so $p \le p_0$; thus A is bounded above in P. As A is arbitrary, f is a Tukey function. **Q** So $P \preccurlyeq_{\mathrm{T}} Q$.

f is a dual Tukey function. **P** If $A \subseteq P$ is cofinal with P, and $q \in Q$, there are a $p \in P$ such that $q \le f(p)$, and a $p' \in A$ such that $p \le p'$; in which case

$$q \le f(p) \le f(p') \in f[A].$$

Thus $f[A]$ is cofinal with Q; as A is arbitrary, f is a dual Tukey function. **Q** So $Q \preccurlyeq_{\mathrm{T}} P$ and $P \equiv_{\mathrm{T}} Q$.

(ii) Apply (i) to the identity map from A to P.

(iii) Apply (i) to the canonical map from P to $\tilde{P}$.

(e) This is just a restatement of the results in 512D, using the identifications listed in 512Ea. The only omission concerns cellularities, for which I have not set out a formal definition in the context of supported relations; but if $A \subseteq P$ is an up-antichain and $\phi : P \to Q$ is a Tukey function, then $\{\phi(a), \phi(a')\}$ can have no upper bound in Q for any distinct $a, a' \in A$, so $\phi[A]$ is an up-antichain in Q with the same cardinality as A, and $\#(A) \le c^{\uparrow}(Q)$. As A is arbitrary, $c^{\uparrow}(P) \le c^{\uparrow}(Q)$ and $\operatorname{sat}^{\uparrow}(P) \le \operatorname{sat}^{\uparrow}(Q)$.

(f) follows at once from (e).

(g) This is a special case of 512H.

(h) Let $Q \subseteq P^{\kappa}$ be the set of constant functions. Because $\kappa \ge 1$, Q is isomorphic to P; because $\kappa < \operatorname{add} P$, Q is cofinal with P^{κ}; so $P \cong Q \equiv_{\mathrm{GT}} P^{\kappa}$.

513F Theorem (Tukey 40) Suppose that P and Q are upwards-directed partially ordered sets. Then P and Q are Tukey equivalent iff there is a partially ordered set R such that P and Q are both isomorphic, as partially ordered sets, to cofinal subsets of R.

proof (a) Suppose that P and Q are Tukey equivalent. Then there are Tukey functions $\phi : P \to Q$ and $\psi : Q \to P$. Set $S = (P \times \{0\}) \cup (Q \times \{1\})$, with a relation $\le$ defined by saying that

$(p, 0) \le (q, 1)$ iff (α) there is a $p' \ge p$ in P such that $\phi(p') \le q$ in Q (β) $q' \le q$ in Q whenever $q' \in Q$ and $\psi(q') \le p$ in P,

$(q, 1) \le (p, 0)$ iff (α) there is a $q' \ge q$ in Q such that $\psi(q') \le p$ in P (β) $p' \le p$ in P whenever $p' \in P$ and $\phi(p') \le q$ in Q,

$(p', 0) \le (p, 0)$ iff $p' \le p$ in P,

$(q', 1) \le (q, 1)$ iff $q' \le q$ in Q.

Of course $\le$ is reflexive. To see that it is transitive, observe that if $(p, 0) \le (q, 1) \le (\tilde{p}, 0)$ then there is a $p' \ge p$ such that $\phi(p') \le q$, and now $p' \le \tilde{p}$, so $(p, 0) \le (\tilde{p}, 0)$. Similarly, $(q, 1) \le (\tilde{q}, 1)$ whenever $(q, 1) \le (p, 0) \le (\tilde{q}, 1)$. The other cases to check are equally easy. It is *not* necessarily the case that $\le$ is antisymmetric, since it is possible to have $(p, 0) \le (q, 1) \le (p, 0)$; but we have an equivalence relation $\cong$ on S defined by saying that $s \cong t$ if $s \le t$ and $t \le s$, and a natural partial order on the set R of equivalence classes defined by saying that $s^\bullet \le t^\bullet$ iff $s \le t$.

The map $p \mapsto (p, 0)^\bullet : P \to R$ is an order-isomorphism between P and its image $\tilde{P} \subseteq R$. Now for any $q \in Q$ there is a $p \in P$ such that $(q, 1) \le (p, 0)$. **P** Since ϕ is a Tukey function, $A = \{p' : \phi(p') \le q\}$ must be bounded above in P; let p_0 be an upper bound for A. Now because P is upwards-directed, there is a $p \in P$ such that $p_0 \le p$ and $\psi(q) \le p$, and in this case $(q, 1) \le (p, 0)$. **Q** This is what we need to see that $\tilde{P}$ is cofinal with R. Similarly, Q is order-isomorphic to its canonical image in R, and this too is cofinal with R. So both P and Q are isomorphic to cofinal subsets of R.

(b) Conversely, if P and Q are both isomorphic to cofinal subsets of a partially ordered set R, then P, R and Q are all Tukey equivalent, by 513Ed.

513G We shall repeatedly want to use some elementary facts about cofinal subsets.

Proposition Let P be a pre-ordered set and Q a cofinal subset of P. Then

(a) $\operatorname{add} Q = \operatorname{add} P$;

(b) $\operatorname{cf} Q = \operatorname{cf} P$;

(c) $\operatorname{sat}^\uparrow(Q) = \operatorname{sat}^\uparrow(P)$, $c^\uparrow(Q) = c^\uparrow(P)$;

(d) $\operatorname{link}^\uparrow_{<\kappa}(Q) = \operatorname{link}^\uparrow_{<\kappa}(P)$ for any cardinal κ; in particular, $\operatorname{link}^\uparrow(Q) = \operatorname{link}^\uparrow(P)$ and $d^\uparrow(Q) = d^\uparrow(P)$;

(e) $\operatorname{bu} Q = \operatorname{bu} P$.

proof All except (e) are consequences of 513Ed and 513Ee. As for bursting numbers, every cofinal subset of Q is also cofinal with P, so $\operatorname{bu} P \le \operatorname{bu} Q$. For the reverse inequality, let Q_1 be a cofinal subset of P such that $\#(\{q : q \in Q_1,\ q \le p,\ p \not\le q\}) < \operatorname{bu} P$ for every $p \in p$. Let $\phi : Q_1 \to Q$ be any function such that $\phi(q) \ge q$ for every $q \in Q_1$, so that $\phi[Q_1]$ is cofinal with Q. If $q \in Q$, then

$$\{q' : q' \in \phi[Q_1],\ q' \le q,\ q \not\le q'\} \subseteq \{\phi(q'') : q'' \in Q_1,\ q'' \le q,\ q \not\le q''\}$$

has cardinal less than $\operatorname{bu} P$, and $\phi[Q_1]$ witnesses that $\operatorname{bu} Q \le \operatorname{bu} P$.

513H Definition Let P be a partially ordered set. Its **σ-additivity** $\operatorname{add}_\omega P$ is the smallest cardinal of any subset A of P such that $A \not\subseteq \bigcup_{q \in D} \left]-\infty, q\right]$ for any countable set $D \subseteq P$. If there is no such set, that is, if $\operatorname{cf} P \le \omega$, I write $\operatorname{add}_\omega P = \infty$.

513I Proposition Let P be a partially ordered set. As in 512F, write $p \le' A$, for $p \in P$ and $A \subseteq P$, if there is a $q \in A$ such that $p \le q$.

(a) $\operatorname{add}_\omega P = \operatorname{add}(P, \le', [P]^{\le\omega})$.

(b) $\max(\omega_1, \operatorname{add} P) \le \operatorname{add}_\omega(P)$.

(c) If $\operatorname{add}_\omega P$ is an infinite cardinal, it is regular.

(d) If $2 \le \kappa \le \operatorname{add} P$, then $(P, \le', [P]^{<\kappa}) \equiv_{\mathrm{GT}} (P, \le, P)$. So if $\operatorname{add} P > \omega$, $\operatorname{add}_\omega(P) = \operatorname{add} P$.

(e) If Q is another partially ordered set and $(P, \le', [P]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (Q, \le', [Q]^{\le\omega})$ (in particular, if $P \preccurlyeq_{\mathrm{T}} Q$) then $\operatorname{add}_\omega P \ge \operatorname{add}_\omega Q$.

(f) If $Q \subseteq P$ is cofinal with P, then $\operatorname{add}_\omega Q = \operatorname{add}_\omega P$.

(g) If $\kappa \le \operatorname{cf} P$ then $\operatorname{add}(P, \le', [P]^{<\kappa}) \le \operatorname{cf} P$. So if $\operatorname{cf} P > \omega$ then $\operatorname{add}_\omega P \le \operatorname{cf} P$.
(h) If $\operatorname{cf}(\operatorname{cf} P) > \omega$ then $\operatorname{cf}(\operatorname{cf} P) \ge \operatorname{add}_\omega P$.

proof (a) All we have to do is to disentangle the definitions in 512Ba, 512F and 513H.

(b) is immediate from the definition of $\operatorname{add}_\omega$.

(c) ? Suppose, if possible, that $\operatorname{add}_\omega P = \kappa$ where $\kappa > \max(\omega, \operatorname{cf}\kappa)$. Express κ as $\sup_{\xi<\lambda} \kappa_\xi$ where $\kappa_\xi < \kappa$ for every $\xi < \lambda = \operatorname{cf}\kappa$. Let $A \subseteq P$ be a set with cardinal κ such that $A \not\subseteq \bigcup_{q\in D}]-\infty, q]$ for any countable set $D \subseteq P$. Express A as $\bigcup_{\xi<\lambda} A_\xi$ where $\#(A_\xi) = \kappa_\xi$ for each $\xi < \lambda$. For each $\xi < \lambda$, there is a countable set $D_\xi \subseteq P$ such that $A_\xi \subseteq \bigcup_{q\in D_\xi}]-\infty, q]$. Set $B = \bigcup_{\xi<\lambda} D_\xi$; then $\#(B) \le \lambda < \kappa$, so there is a countable set $D \subseteq P$ such that $B \subseteq \bigcup_{q\in D}]-\infty, q]$. But now $A \subseteq \bigcup_{q\in D}]-\infty, q]$. **X**

(d) By 512Gc, $(P, \le', [P]^{<\kappa}) \preccurlyeq_{\mathrm{GT}} (P, \le, P)$. In the other direction, because $\kappa \le \operatorname{add} P$, we have a function $\psi : [P]^{<\kappa} \to P$ such that $I \subseteq]-\infty, \psi(I)]$ for every $I \in [P]^{<\kappa}$; so if we set $\phi(p) = p$ for $p \in P$, (ϕ, ψ) will be a Galois-Tukey connection from $(P, \le, P)$ to $(P, \le', [P]^{<\kappa})$, and $(P, \le, P) \preccurlyeq_{\mathrm{GT}} (P, \le', [P]^{<\kappa})$.

Now if $\operatorname{add} P > \omega$,

$$\operatorname{add}_\omega P = \operatorname{add}(P, \le', [P]^{\le\omega}) = \operatorname{add}(P, \le, P) = \operatorname{add} P.$$

(e) Use (a) with 512Db and 512Gb.

(f) Use (e) and 513Ed.

(g) Let $Q \subseteq P$ be a cofinal subset of P with cardinal $\operatorname{cf} P$. If $A \subseteq P$ is such that every member of Q is dominated by a member of A, then A also is cofinal, so $\#(A) \ge \kappa$; thus Q witnesses that $\operatorname{add}(P, \le', [P]^{<\kappa}) \le \operatorname{cf} P$. Putting $\kappa = \omega_1$ we see that if $\operatorname{cf} P > \omega$ then $\operatorname{add}_\omega P \le \operatorname{cf} P$.

(h) ? If $\omega < \operatorname{cf}(\operatorname{cf} P) = \lambda < \operatorname{add}_\omega P$ let $Q \subseteq P$ be a cofinal set of size $\operatorname{cf} P$ and express Q as $\bigcup_{\xi<\lambda} Q_\xi$ where $\#(Q_\xi) < \operatorname{cf} P$ and $Q_\xi \subseteq Q_\eta$ whenever $\xi \le \eta < \lambda$. For each $\xi < \lambda$, Q_ξ cannot be cofinal with P, so there is a $p_\xi \in P$ such that $p_\xi \not\le q$ for any $q \in Q_\xi$. Now $A = \{p_\xi : \xi < \lambda\}$ has cardinal less than $\operatorname{add}_\omega P$, so there is a countable set $D \subseteq P$ such that $A \subseteq \bigcup_{r\in D}]-\infty, r]$. For each $r \in D$ there is a $q_r \in Q$ such that $r \le q_r$; let $\xi_r < \lambda$ be such that $q_r \in Q_{\xi_r}$. Because λ is uncountable and regular (being the cofinality of a cardinal), $\zeta = \sup_{r\in D} \xi_r$ is less than λ, and $q_r \in Q_\zeta$ for every $r \in D$. But now there is an $r \in D$ such that $p_\zeta \le r \le q_r \in Q_\zeta$, contrary to the choice of p_ζ. **X**

Remark The point of (b) and (d) here is that there are significant cases in which $\operatorname{add} P < \omega_1 < \operatorname{add}_\omega P$.

***513J Cofinalities of products** It is easy to find the additivity of a product of partially ordered sets (511Hg). Calculating the cofinality of a product of partially ordered sets is surprisingly difficult, and there are some extraordinary results in this area. (See Burke & Magidor 90; there is a taster in 542J below.) Here I will give just one special fact which will be useful.

Proposition Suppose that the generalized continuum hypothesis is true. Let $\langle P_i \rangle_{i\in I}$ be a family of non-empty partially ordered sets with product P. Set

$$\kappa = \#(\{i : i \in I,\ \operatorname{cf} P_i > 1\}), \qquad \lambda = \sup_{i\in I} \operatorname{cf} P_i.$$

Then
(i) if κ and λ are both finite, $\operatorname{cf} P$ is the cardinal product $\prod_{i\in I} \operatorname{cf} P_i$;
(ii) if $\lambda > \kappa$ and there is some $\gamma < \lambda$ such that $\operatorname{cf}\lambda > \#(\{i : i \in I,\ \operatorname{cf} P_i > \gamma\})$, then $\operatorname{cf} P = \lambda$;
(iii) otherwise, $\operatorname{cf} P = \max(\kappa^+, \lambda^+)$.

proof (a) For each $i \in I$, let $Q_i \subseteq P_i$ be a cofinal set of size $\operatorname{cf} P_i$. Then $Q = \prod_{i\in I} Q_i$ is cofinal with $P = \prod_{i\in I} P_i$, so $\operatorname{cf} P \le \#(Q)$. If $\lambda < \omega$, then every Q_i must be just the set of maximal elements of P_i, so Q is the set of maximal elements of P, and $\operatorname{cf} P = \#(Q)$. This deals with case (i).

(b) $\operatorname{cf} P > \kappa$. **P** Set $J = \{i : i \in I,\ \operatorname{cf} P_i > 1\}$. If $\langle p_i \rangle_{i\in J}$ is any family in P, then we can choose $q \in P$ such that $q(i) \not\le p_i(i)$ for every $i \in J$; accordingly $\{p_i : i \in J\}$ cannot be cofinal with P. **Q** So if $\max(\omega, \lambda) \le \kappa$,

$$\operatorname{cf} P \ge \kappa^+ = 2^\kappa$$

(because we are assuming the generalized continuum hypothesis)

$$= 2^{\max(\kappa,\lambda)} \ge \#(\mathcal{P}\{(i,q) : i \in J,\, q \in Q_i\}) \ge \#(\prod_{i\in J} Q_i) = \#(Q)$$

(because $\#(Q_i) = 1$ for $i \in I \setminus J$)

$$\ge \operatorname{cf} P,$$

and $\operatorname{cf} P = \kappa^+ = \max(\kappa^+, \lambda^+)$, as required by (iii).

(c) Note that $\operatorname{cf} P \ge \lambda$, because if $R \subseteq P$ is cofinal with P then $\{p(i) : p \in R\}$ is cofinal with P_i for each i. So if κ is finite and λ is infinite,

$$\lambda \le \operatorname{cf} P \le \#(Q) \le \max(\omega, \sup_{i\in J} \#(Q_i)) = \lambda$$

and $\operatorname{cf} P = \lambda$, as required by (ii).

(d) If λ is infinite and $\kappa < \operatorname{cf}\lambda$ then every function from κ to λ is bounded above in λ. So

$$\lambda \le \operatorname{cf} P \le \#(Q) \le \#(\lambda^\kappa)$$

(where λ^κ, for once, denotes the set of functions from κ to λ)

$$= \#(\bigcup_{\zeta<\lambda} \zeta^\kappa) \le \max(\omega, \lambda, \sup_{\zeta<\lambda} \#(\zeta^\kappa)) \le \max(\omega, \lambda, \sup_{\zeta<\lambda} 2^{\max(\zeta,\kappa)}) = \lambda,$$

again using GCH. Thus in this case also we have $\operatorname{cf} P = \lambda$, as required by (ii).

(e) So we are left with the case in which $\operatorname{cf}\lambda = \theta \le \kappa < \lambda$. Let $\langle\lambda_\eta\rangle_{\eta<\theta}$ be a family of cardinals less than λ with supremum λ.

(α) Suppose that we are in case (iii), that is, $\#(\{i : i \in I,\, \operatorname{cf} P_i > \gamma\}) \ge \theta$ for every $\gamma < \lambda$. Then $\operatorname{cf} P > \lambda$. **P** We can choose $\langle i(\eta)\rangle_{\eta<\theta}$ inductively in I so that $\operatorname{cf} P_{i(\eta)} > \lambda_\eta$ and $i(\eta) \ne i(\xi)$ when $\xi < \eta < \theta$. If $\langle p_\xi\rangle_{\xi<\lambda}$ is any family in P, we can find $q \in P$ such that $q(i(\eta)) \not\le p_\xi(i(\eta))$ for any $\eta < \theta$ and $\xi < \lambda_\eta$, so that $q \not\le p_\xi$ for any $\xi < \lambda$. As $\langle p_\xi\rangle_{\xi<\lambda}$ is arbitrary, $\operatorname{cf} P > \lambda$. **Q** Now

$$\operatorname{cf} P \le \#(Q) \le \#(\lambda^\kappa) \le 2^{\max(\kappa,\lambda)} = \lambda^+ \le \operatorname{cf} P,$$

so $\operatorname{cf} P = \lambda^+ = \max(\kappa^+, \lambda^+)$, as required.

(β) Otherwise, we are in case (ii), and there is a cardinal $\gamma < \lambda$ such that $\#(K) < \theta$, where $K = \{i : i \in I,\, \operatorname{cf} P_i > \gamma\}$. Then $\sup_{i\in K} \operatorname{cf} P_i = \lambda$, so (d) tells us that $\operatorname{cf}(\prod_{i\in K} P_i) = \lambda$. On the other hand,

$$\#(\prod_{i\in I\setminus K} \operatorname{cf} P_i) \le 2^{\max(\gamma,\kappa)} \le \lambda.$$

Since we can identify P with the product of $\prod_{i\in K} P_i$ and $\prod_{i\in I\setminus K} P_i$, $\operatorname{cf} P \le \#(\lambda\times\lambda) = \lambda$. But we noted in (c) that $\operatorname{cf} P \ge \lambda$, so $\operatorname{cf} P = \lambda$, as required. This completes the proof.

***513K** I remarked in the notes to §512 that Galois-Tukey correspondences are not required to have any special properties, and of course the same is true of Tukey functions. But it is also the case that the 'natural' Tukey functions arising in Chapter 52 can in many cases be derived from Borel functions between Polish spaces. I now present some ideas taken from Solecki & Todorčević 04 which may be regarded as a partial explanation of the phenomenon.

Definition I will say that a **metrizably compactly based directed set** is a partially ordered set P endowed with a metrizable topology such that

(i) $p \vee q = \sup\{p, q\}$ is defined for all $p, q \in P$, and $\vee : P \times P \to P$ is continuous;

(ii) $\{p : p \le q\}$ is compact for every $q \in P$;

(iii) every convergent sequence in P has a subsequence which is bounded above.

In this context, I will say that P is 'separable' or 'analytic' if it is separable, or analytic, in the topological sense.

I leave it to you to check that many significant partially ordered sets are compactly based in the sense defined here (513Xj-513Xn, 513Yg).

***513L Proposition** Let P be a metrizably compactly based directed set.

(a) The ordering of P is a closed subset of $P \times P$.

(b) P is Dedekind complete.

(c)(i) A non-decreasing sequence in P has an upper bound iff it is topologically convergent, and in this case its supremum is its limit.

(ii) A non-increasing sequence in P converges topologically to its infimum.

(d) If $p \in P$ and $\langle p_i\rangle_{i\in\mathbb{N}}$ is a sequence in P, then $\langle p_i\rangle_{i\in\mathbb{N}}$ is topologically convergent to p iff for every $I \in [\mathbb{N}]^\omega$ there is a $J \in [I]^\omega$ such that $p = \inf_{n\in\mathbb{N}} \sup_{i\in J\setminus n} p_i$.

(e) Suppose that $p \in P$ and a double sequence $\langle p_{ni}\rangle_{n,i\in\mathbb{N}}$ in P are such that $\lim_{i\to\infty} p_{ni} = p_n$ is defined in P and less than or equal to p for each n. Then there is a $q \in P$ such that $\{i : p_{ni} \le q\}$ is infinite for every $n \in \mathbb{N}$.

proof Let ρ be a metric on P inducing its topology.

(a) We have only to observe that $\{(p,q) : p \le q\} = \{(p,q) : p \vee q = q\}$.

(b) Suppose that $A \subseteq P$ is non-empty and bounded above. Let B be the set of upper bounds of A. Then $\mathcal{E} = \{[p,q] : p \in A, q \in B\}$ is a non-empty family of compact sets with the finite intersection property, because any non-empty finite subset of A has a least upper bound. So there is a $q_0 \in \bigcap \mathcal{E}$ and now q_0 must be the supremum of A.

(c)(i) Suppose that $\langle p_i\rangle_{i\in\mathbb{N}}$ is a non-decreasing sequence in P. (α) If it has a topological limit p, then

$$p \vee p_j = \lim_{i\to\infty} p_i \vee p_j = \lim_{i\to\infty} p_i = p$$

for each j, so p is an upper bound for $\{p_i : i \in \mathbb{N}\}$; while if q is an upper bound for $\{p_i : i \in \mathbb{N}\}$ then $p \le q$ by (a). Thus $p = \sup_{i\in\mathbb{N}} p_i$. (β) If $\{p_i : i \in \mathbb{N}\}$ is bounded above, then it has a least upper bound p, by (b). Now $]-\infty, p]$ is compact, therefore sequentially compact, and every subsequence of $\langle p_i\rangle_{i\in\mathbb{N}}$ has a convergent sub-subsequence; by (α), the limit of this sub-subsequence is always its supremum, which must be p; so $\langle p_i\rangle_{i\in\mathbb{N}}$ itself converges to p.

(ii) Suppose that $\langle p_i\rangle_{i\in\mathbb{N}}$ is a non-increasing sequence in P. Then it lies in the compact set $]-\infty, p_0]$ so has a convergent subsequence $\langle p'_i\rangle_{i\in\mathbb{N}}$ with limit p say. As in (i) just above,

$$p \vee p'_j = \lim_{i\to\infty} p'_i \vee p'_j = \lim_{i\to\infty} p'_j = p'_j$$

for each j, so p is a lower bound for $\{p'_i : i \in \mathbb{N}\}$; while if q is a lower bound for $\{p'_i : i \in \mathbb{N}\}$ then $q \le p$ by (a). Thus $p = \inf_{i\in\mathbb{N}} p'_i = \inf_{i\in\mathbb{N}} p_i$. What this shows is that $\inf_{i\in\mathbb{N}} p_i$ is the only cluster point of $\langle p_i\rangle_{i\in\mathbb{N}}$ and is therefore its topological limit.

(d)(i) Suppose that $p = \lim_{i\to\infty} p_i$. Note first that if $q \in P$ then

$$\limsup_{i\to\infty} \rho(q \vee p_i, p) \le \lim_{i\to\infty} \rho(q \vee p_i, q \vee p) + \rho(q \vee p, p) = \rho(q \vee p, p),$$

$$\limsup_{i\to\infty} \rho(p \vee q \vee p_i, p) \le \rho((p \vee q) \vee p, p) = \rho(q \vee p, p)$$

because $\vee$ is continuous. Now let $I \subseteq \mathbb{N}$ be infinite. By 513K(iii), there is an infinite $I' \subseteq I$ such that $\{p_i : i \in I'\}$ is bounded above. We can choose inductively a strictly increasing sequence $\langle i_n\rangle_{n\in\mathbb{N}}$ in I' such that

$$\rho(\sup_{j\le n\le k} p_{i_n}, p) < 2^{-j}, \quad \rho(p \vee \sup_{j\le n\le k} p_{i_n}, p) < 2^{-j}$$

whenever $j \le k$ in $\mathbb{N}$. Set $J = \{i_n : n \in \mathbb{N}\}$; then $\rho(\sup_{j\in J, m\le j\le k} p_j, p) < 2^{-m}$ whenever $m \le k \in \mathbb{N}$ and $[m,k]$ meets J. For each m, $q_m = \sup_{j\in J\setminus m} p_j$ is defined in P, by (b) above; moreover, (c-i) tells us that $q_m = \lim_{k\to\infty} \sup_{j\in J, m\le j\le k} p_j$ so

$$\rho(q_m, p) = \lim_{k\to\infty} \rho(\sup_{j\in J, m\le j\le k} p_j, p) \le 2^{-m}.$$

But this means that p is the topological limit of the non-increasing sequence $\langle q_m\rangle_{m\in\mathbb{N}}$ and must be $\inf_{m\in\mathbb{N}} q_m$. Thus $\langle p_i\rangle_{i\in\mathbb{N}}$ satisfies the condition proposed.

(ii) Now suppose that for every $I \in [\mathbb{N}]^\omega$ there is a $J \in [I]^\omega$ such that $p = \inf_{n\in\mathbb{N}} \sup_{i\in J\setminus n} p_i$. Then any convergent subsequence of $\langle p_i\rangle_{i\in\mathbb{N}}$ has limit p. **P** Suppose the subsequence is $\langle p_{i_n}\rangle_{n\in\mathbb{N}}$ where $\langle i_n\rangle_{n\in\mathbb{N}}$ is strictly increasing. Set $I = \{i_n : n \in \mathbb{N}\}$. Then we must have an infinite $J \subseteq \mathbb{N}$ such that $p = \inf_{m\in\mathbb{N}} \sup_{k\in J\setminus m} p_{i_k}$. Now (i) tells us that we also have an infinite $K \subseteq J$ such that the limit p' of $\langle p_{i_n}\rangle_{n\in\mathbb{N}}$ is $\inf_{m\in\mathbb{N}} \sup_{k\in K\setminus m} p_{i_k}$. Since $\sup_{k\in K\setminus m} p_{i_k} \le \sup_{k\in J\setminus m} p_{i_k}$ for every m, $p' \le p$. On the other hand, we also have an infinite $L \subseteq K$ such that $p = \inf_{m\in\mathbb{N}} \sup_{k\in L\setminus m} p_{i_k}$; so that $p \le p'$ and $p = p'$. **Q**

Since the condition tells us also that every subsequence of $\langle p_i\rangle_{i\in\mathbb{N}}$ has a sub-subsequence which is bounded above, and therefore has a convergent sub-sub-subsequence, p is actually the limit of $\langle p_i\rangle_{i\in\mathbb{N}}$.

(e) Note first that if $\langle q_i\rangle_{i\in\mathbb{N}}$ is a sequence in P converging to $q^*\in P$, and $\epsilon>0$, there is a $q'\in P$ such that $\rho(q',q^*)\le\epsilon$ and $\{i:q_i\le q'\}$ is infinite. **P** By (d), there is an infinite $J\subseteq\mathbb{N}$ such that $q^*=\inf_{n\in\mathbb{N}}\sup_{i\in J\setminus n}q_i$; by (c-ii), we can take $q'=\sup_{i\in J\setminus n}q_i$ for some n. **Q**

For m, $i\in\mathbb{N}$, set $q_{mi}=p\vee\sup_{n\le m}p_{ni}$. Then $\lim_{i\to\infty}q_{mi}=p\vee\sup_{n\le m}p_n=p$ for each m. We can therefore find, for each $m\in\mathbb{N}$, a $q_m\in P$ such that $\rho(q_m,p)\le 2^{-m}$ and $\{i:q_{mi}\le q_m\}$ is infinite. As $\langle q_m\rangle_{m\in\mathbb{N}}\to p$, there is a $q\in P$ such that $\{m:q_m\le q\}$ is infinite. Now, for any $n\in\mathbb{N}$, there is an $m\ge n$ such that $q_m\le q$, so that

$$\{i:p_{ni}\le q\}\supseteq\{i:q_{mi}\le q_m\}$$

is infinite.

***513M Proposition** Let P be a separable metrizably compactly based directed set, and give the set $\mathcal{C}$ of closed subsets of P its Vietoris topology. Let $\mathcal{K}_b\subseteq\mathcal{C}$ be the family of non-empty compact subsets of P which are bounded above in P. Then $K\mapsto\sup K:\mathcal{K}_b\to P$ is Borel measurable.

proof Writing $\mathcal{K}$ for the family of non-empty compact subsets of P, we have a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of Borel measurable functions from $\mathcal{K}$ to P such that $K=\overline{\{f_n(K):n\in\mathbb{N}\}}$ for every $K\in\mathcal{K}$ (5A4Dc). Set $g_n(K)=\sup_{i\le n}f_i(K)$ for each $K\in\mathcal{K}$ and $n\in\mathbb{N}$; because P is separable, every g_n is Borel measurable (put 418Bd and 418Ac together). For $K\in\mathcal{K}_b$, $\langle g_n(K)\rangle_{n\in\mathbb{N}}$ is a non-decreasing bounded sequence, so converges to $g(K)\in P$, by 513L(c-i); now $g:\mathcal{K}_b\to P$ is Borel measurable (418Ba). Since $\{q:q\le g(K)\}$ is a closed set including $\{f_i(K):i\in\mathbb{N}\}$, it includes K, and $g(K)$ is an upper bound for K; because $g(K)=\sup_{i\in\mathbb{N}}f_i(K)$, $g(K)=\sup K$. So we have the result.

***513N Lemma** Let P and Q be non-empty metrizably compactly based directed sets of which P is separable, and $\phi:P\to Q$ a Tukey function. Set

$$R=\overline{\{(q,p):p\in P,\,q\in Q,\,\phi(p)\le q\}}.$$

Then

(a) $R[\,]-\infty,q]\,]$ is bounded above in P for every $q\in Q$;

(b) $R\subseteq Q\times P$ is usco-compact.

proof (a) Because P is non-empty, we need consider only the case in which $R[\,]-\infty,q]\,]$ is non-empty. Let $\langle p_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a dense subset of $R[\,]-\infty,q]\,]$. For each $n\in\mathbb{N}$ we have sequences $\langle p_{ni}\rangle_{i\in\mathbb{N}}$ in P and $\langle q_{ni}\rangle_{n\in\mathbb{N}}$ in Q such that $\phi(p_{ni})\le q_{ni}$, $\lim_{i\to\infty}p_{ni}=p_n$ and $\lim_{i\to\infty}q_{ni}=q_n\le q$. By 513Le, there is a $q'\in Q$ such that $I_n=\{i:q_{ni}\le q'\}$ is infinite for every $n\in\mathbb{N}$. Because ϕ is a Tukey function, there is a $p'\in P$ such that $p_{ni}\le p'$ whenever $n\in\mathbb{N}$ and $i\in I_n$. But now $\{p:p\le p'\}$ is closed, so it contains every p_n and p' is an upper bound for $R[\,]-\infty,q]\,]$.

(b) In particular, for any $q\in Q$, $R[\{q\}]$ is bounded above in P, therefore relatively compact; since R is closed, every $R[\{q\}]$ is closed and therefore compact. Now suppose that $F\subseteq P$ is closed and that $\langle q_n\rangle_{n\in\mathbb{N}}$ is a sequence in $R^{-1}[F]$ converging to $q\in Q$. Then there is a $q^*\in Q$ such that $J=\{n:q_n\le q^*\}$ is infinite. For $n\in J$, let $p_n\in F$ be such that $(q_n,p_n)\in R$. Then $\{p_n:n\in J\}$ is included in the order-bounded set $R^{-1}[\,]-\infty,q^*]\,]$, so is relatively compact, and $\langle p_n\rangle_{n\in\mathbb{N}}$ has a cluster point p say. Of course $p\in F$; also (q,p) is a cluster point of $\langle(q_n,p_n)\rangle_{n\in\mathbb{N}}$, so belongs to $\overline{R}=R$, and $q\in R^{-1}[F]$. As q is arbitrary, $R^{-1}[F]$ is closed; as F is arbitrary, R is usco-compact.

***513O Theorem** (Solecki & Todorčević 04) Let P and Q be metrizably compactly based directed sets such that $P\preccurlyeq_{\mathrm{T}}Q$. Let Σ be the σ-algebra of subsets of P generated by the Souslin-F sets.

(a) If P is separable, there is a Borel measurable dual Tukey function $\psi:Q\to P$.

(b) If P is separable and Q is analytic, there is a Σ-measurable Tukey function $\phi:P\to Q$.

proof If either P or Q is empty, so is the other, and the result is trivial; suppose that they are non-empty.

(a) Let $\phi_0:P\to Q$ be a Tukey function, and set $R=\overline{\{(q,p):p\in P,\,q\in Q,\,\phi_0(p)\le q\}}$, so that R is usco-compact (513N). Let $\mathcal{C}$ be the set of closed subsets of P with its Vietoris topology; then $q\mapsto R[\{q\}]$ is Borel measurable (5A4Db). Since $\emptyset$ is an isolated point of $\mathcal{C}$, $Q_0=\{y:R[\{y\}]\ne\emptyset\}$ is a Borel set in Q. If $q\in Q_0$, then $R[\{q\}]$ is a non-empty compact subset of P which is bounded above (513Na), so 513M tells us that $q\mapsto\sup R[\{q\}]:Q_0\to P$ is Borel measurable. Fix any $p_0\in P$ and set $\psi(q)=\sup R[\{q\}]$ if $q\in Q_0$, p_0 if $q\in Q\setminus Q_0$. Then ψ is Borel measurable. If $p\in P$, $q\in Q$ and $\phi_0(p)\le q$, then $(q,p)\in R$, $p\in R[\{q\}]$ and $p\le\psi(q)$; thus (ϕ_0,ψ) is a Galois-Tukey connection and ψ is a dual Tukey function.

(b) $R \subseteq Q \times P$ is a closed set (422Da) and $R[Q] = P$. Because P and Q are separable and metrizable, R can be obtained by Souslin's operation from products of closed sets. By the von Neumann-Jankow selection theorem (423M), there is a Σ-measurable $\phi : P \to Q$ such that $(\phi(p), p) \in R$ for every $p \in P$. If $q \in Q$, then $\{p : \phi(p) \le q\} \subseteq R[\,]-\infty, q]\,]$ is bounded above in P, so ϕ is a Tukey function.

513P The last result in this section is entirely unconnected with the rest, and is a standard trick; but it will be useful later and contains an implicit challenge (513Yj).

Lemma Let P and Q be non-empty partially ordered sets, and suppose that (i) every non-decreasing sequence in P has an upper bound in P (ii) there is no strictly increasing family $\langle q_\xi \rangle_{\xi<\omega_1}$ in Q. Let $f : P \to Q$ be an order-preserving function. Then there is a $p \in P$ such that $f(p') = f(p)$ whenever $p' \in P$ and $p' \ge p$.

proof ? Otherwise, we can choose $\langle p_\xi \rangle_{\xi<\omega_1}$ inductively so that

$p_0 \in P$,

$p_{\xi+1} \ge p_\xi$ and $f(p_{\xi+1}) > f(p_\xi)$ for every $\xi < \omega_1$,

p_ξ is an upper bound for $\{p_\eta : \eta < \xi\}$ for every non-zero limit ordinal $\xi < \omega_1$.

But now $\langle f(p_\xi) \rangle_{\xi<\omega_1}$ is strictly increasing, which is impossible. **X**

513X Basic exercises (a) Let P be a partially ordered set, and $\mathcal{A}$ the family of subsets of P which are not cofinal with P. Show that $(\mathcal{A}, \not\ni, P) \preccurlyeq_{\mathrm{GT}} (P, \le, P)$. Explain the relation of this fact to 511Xj, 513C(a-ii) and 513Xb.

(b) Let P be a partially ordered set such that $\operatorname{bu} P \ge \omega$. Show that $\operatorname{cf}(\operatorname{bu} P) \ge \operatorname{add} P$.

(c) Let P, Q and R be partially ordered sets. (i) Show that if $\phi_1 : P \to Q$ and $\phi_2 : Q \to R$ are Tukey functions, then $\phi_2\phi_1 : P \to R$ is a Tukey function. (ii) Show that if $\psi_1 : P \to Q$ and $\psi_2 : Q \to R$ are dual Tukey functions, then $\psi_2\psi_1 : P \to R$ is a dual Tukey function.

(d) Let P and Q be partially ordered sets, and $g : Q \to P$ a function. Show that g is a dual Tukey function iff for every $p_0 \in P$ there is a $q_0 \in Q$ such that $g(q) \ge p_0$ for every $q \ge q_0$.

(e)(i) Show that if P, Q are partially ordered sets, P is Dedekind complete and $P \preccurlyeq_{\mathrm{T}} Q$, there is an order-preserving dual Tukey function from Q to P. (ii) Set $P = [\{0,1,2\}]^{\le 2}$ and $Q = [\{0,1,2\}]^2$. Show that there is no order-preserving Tukey function from P to Q.

(f) Suppose that P is a partially ordered set and $\operatorname{add} P = \operatorname{cf} P = \kappa \ge \omega$. Show that $P \equiv_{\mathrm{T}} \kappa$.

(g) Prove (a)-(d) of 513G directly, without mentioning Tukey functions or Galois-Tukey connections.

>(h)(i) Show that if P and Q are two partially ordered sets such that $\operatorname{sat}^\uparrow(P) = \#(P)^+ = \#(Q)^+ = \operatorname{sat}^\uparrow(Q)$ then P and Q are Tukey equivalent. (*Hint*: if $B \subseteq Q$ is an up-antichain, any injective function $\phi : P \to B$ is a Tukey function from P to Q.) (ii) Give an example of such a pair P, Q such that $\mathfrak{m}(P) \ne \mathfrak{m}(Q)$ and $\operatorname{bu} P \ne \operatorname{bu} Q$.

(i) Let P, Q_1, Q_2 be partially ordered sets such that $(P, \le, P) \preccurlyeq_{\mathrm{GT}} (Q_1, \le, Q_1) \ltimes (Q_2, \le, Q_2)$ (definition: 512I). Show that $\operatorname{add}_\omega P \ge \min(\operatorname{add}_\omega Q_1, \operatorname{add}_\omega Q_2)$.

(j) Show that $\mathbb{N}^{\mathbb{N}}$, with its usual ordering and topology, is a metrizably compactly based directed set.

(k) Let X be a set, $1 \le p < \infty$ and P the positive cone $(\ell^p(X))^+$ of the Banach lattice $\ell^p(X)$, with the topology induced by the norm of $\ell^p(X)$. Show that P is a metrizably compactly based directed set.

(l) Let $\mathcal{Z}$ be the ideal of subsets of $\mathbb{N}$ with asymptotic density zero, with its natural ordering and the topology induced by the metric $(a, b) \mapsto \sup_{n\ge1} \frac{1}{n}\#((a \triangle b) \cap n)$. Show that $\mathcal{Z}$ is a metrizably compactly based directed set.

(m) Let X be a metrizable space, and $\mathcal{F}$ the set of nowhere dense compact subsets of X. Show that $\mathcal{F}$, with its natural ordering and its Vietoris topology, is a metrizably compactly based directed set. (*Hint*: use a Hausdorff metric.)

(n) Let X be a metrizable space, $\mathcal{K}$ the family of compact subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X. Show that $\mathcal{K}\cap\mathcal{I}$, with the natural partial order and the Vietoris topology, is a metrizably compactly based directed set.

(o) Let $\langle P_i\rangle_{i\in I}$ be a countable family of metrizably compactly based directed sets, with product P. Show that P is metrizably compactly based.

(p) Let P be a metrizably compactly based directed set. (i) Show that P is a lattice iff it has a least element. (ii) Show that if we adjoin a least element $-\infty$ to P as an isolated point, $P\cup\{-\infty\}$ is a metrizably compactly based directed set.

(q) Let $\langle P_i\rangle_{i\in I}$ be a family of partially ordered sets, and P their product. (i) Show that $\operatorname{cf} P$ is at most the cardinal product $\prod_{i\in I}\operatorname{cf} P_i$, with equality if I is finite. (ii) Show that if $P\ne\emptyset$ then $\sup_{i\in I}\operatorname{cf} P_i\le\operatorname{cf} P$. (iii) Show that if $P\ne\emptyset$ and for every $i\in I$ there is a $j\in I$ such that $\operatorname{cf} P_i<\operatorname{cf} P_j$, then $\sup_{i\in I}\operatorname{cf} P_i<\operatorname{cf} P$.

513Y Further exercises (a) Show that for a cardinal κ, there is a partially ordered set P such that $c^{\uparrow}(P)=\operatorname{sat}^{\uparrow}(P)=\kappa$ iff κ is weakly inaccessible. (*Hint*: for such a κ, take X to be a product of discrete spaces, one of each cardinality less than κ, and P the family of proper closed subsets of X.)

(b) Show that for any cardinal $\kappa>0$ there is a supported relation (A,R,B) such that $\operatorname{sat}(A,R,B)=\kappa$.

(c) For a non-empty upwards-directed set P, a topological space X and $A\subseteq X$, write $\operatorname{cl}_P(A)$ for the set of points $x\in X$ for which there is a function $f:P\to A$ such that $x\in\overline{f[C]}$ for every cofinal set $C\subseteq P$; equivalently, $f[[\mathcal{F}^{\uparrow}(P)]]\to x$, where $\mathcal{F}^{\uparrow}(P)$ is the filter on P generated by sets of the form $[p,\infty[$ as p runs over P. Now let P and Q be upwards-directed sets. Show that $P\preccurlyeq_{\mathrm{T}} Q$ iff $\operatorname{cl}_P(A)\subseteq\operatorname{cl}_Q(A)$ for any subset A of any topological space.

(d) For partially ordered sets P and Q, say that $P\approx Q$ if there is a partially ordered set R into which both P and Q can be embedded as cofinal subsets. (i) Show that $P\approx Q$ iff there is a Galois-Tukey connection (ϕ,ψ) from $(P,\le,P)$ to $(Q,\le,Q)$ such that (ψ,ϕ) is a Galois-Tukey connection from $(Q,\le,Q)$ to $(P,\le,P)$. (ii) Show that if P, R and R' are partially ordered sets such that P can be embedded as a cofinal subset into both R and R', then $R\approx R'$. (iii) Show that $\approx$ is an equivalence relation on the class of partially ordered sets. (iv) Show that if $\mathcal{P}$ is a set of partially ordered sets, and $P\approx P'$ for all P, $P'\in\mathcal{P}$, then there is a partially ordered set R such that every member of $\mathcal{P}$ can be embedded into R as a cofinal set. (v) Give an example of partially ordered sets P and Q such that $P\equiv_{\mathrm{T}} Q$ but $P\not\approx Q$.

(e) For a cardinal κ and a supported relation (A,R,B) set $\operatorname{add}_{<\kappa}(A,R,B)=\operatorname{add}(A,\boldsymbol{R'},[B]^{<\kappa})$. Which of the ideas of 513I can be extended to the general context?

(f) Show that there are two families $\langle(A_i,R_i,B_i)\rangle_{i\in I}$ and $\langle(C_i,S_i,D_i)\rangle_{i\in I}$ of supported relations, with simple products (A,R,B) and (C,S,D) respectively, such that $\operatorname{cov}(A_i,R_i,B_i)=\operatorname{cov}(C_i,S_i,D_i)$ for each i, but $\operatorname{cov}(A,R,B)\ne\operatorname{cov}(C,S,D)$. (*Hint*: examine the proof of 513J.)

(g) Let X be a set and U a solid linear subspace of $\mathbb{R}^X$ with an order-continuous norm under which it is a Banach lattice. Show that its positive cone, with its norm topology, is a metrizably compactly based directed set.

(h) Explore possible definitions of 'compactly based' partially ordered set which do not require the topology to be metrizable.

(i) Let P be an analytic metrizably compactly based directed set. Show that P is Polish. (*Hint*: Solecki & Todorčević 04.)

(j) For partially ordered sets P and Q, say that $Q\maltese P$ if for every order-preserving $f:P\to Q$ there is a $p\in P$ such that $f(p')=f(p)$ for every $p'\ge p$. Explore the properties of the relation $\maltese$.

513 Notes and comments Most of the first part of this section consists of elementary verifications; an exception is the Erdős-Tarski theorem on the cellularity and saturation of a partially ordered set (513Bb-513Bc), which can equally well be regarded as a theorem about topological spaces or Boolean algebras (see 514Da and 514Nc). As usual, I have presented the ideas of the last two sections in an ahistorical manner; the original objective of Tukey 40 was to classify directed sets from the point of view of net-convergence (513Yc).

I have starred 513K-513O because I do not expect to rely on them in the rest of this work. Nevertheless I think that they give a useful support to the ideas here, particularly in the context of §526, where these 'compactly based' partial orders appear naturally. Note that 513Ld tells us that if a directed set P is metrizably compactly based, there is a unique witnessing topology; every topological property of P must be a reflection of a property of the ordering.

514 Boolean algebras

The cardinal functions of Boolean algebras and topological spaces are intimately entwined; necessarily so, because we have a functorial connexion between Boolean algebras and zero-dimensional compact Hausdorff spaces (312Q). In this section I run through the elementary ideas. In 514D-514E I list properties of cardinal functions of Boolean algebras, corresponding to the relatively familiar results in 5A4B for topological spaces; Stone spaces (514B), regular open algebras (514H) and category algebras (514I) provide links of different kinds between the two theories. It turns out that some of the most important features of the cofinal structure of a pre-ordered set can also be described in terms of its 'up-topology' (514L-514M) and the associated regular open algebra (514N-514S). I conclude with a brief note on finite-support products (514T-514U).

514A I put a special property of locally compact spaces into the language of this chapter.

Lemma Let $(X,\mathfrak{T})$ be a topological space. Then $d(X)\ge d^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})$. If X is locally compact and Hausdorff, then $d(X)=d^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})$.

proof If $x\in X$, then $\{G:x\in G\in\mathfrak{T}\}$ is downwards-centered in $\mathfrak{T}\setminus\{\emptyset\}$. So

$$d^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})\le\operatorname{cov}(\mathfrak{T}\setminus\{\emptyset\},\ni,X)=d(X).$$

Now suppose that X is locally compact and Hausdorff. Set $\kappa=d^{\downarrow}(\mathfrak{T}\setminus\{\emptyset\})$, and let $\langle\mathcal{H}_\xi\rangle_{\xi<\kappa}$ be a cover of $\mathfrak{T}\setminus\{\emptyset\}$ by downwards-centered sets. For $\xi<\kappa$ set $F_\xi=\bigcap\{\overline{H}:H\in\mathcal{H}_\xi\}$, and let $D\subseteq X$ be a set with cardinal at most κ such that $D\cap F_\xi\ne\emptyset$ whenever $\xi<\kappa$ and $F_\xi\ne\emptyset$. If $G\subseteq X$ is a non-empty open set, then there is a non-empty relatively compact open set H_0 such that $\overline{H}_0\subseteq G$ (recall that X, being locally compact and Hausdorff, is certainly regular). There is some $\xi<\kappa$ such that $H_0\in\mathcal{H}_\xi$; because $\{\overline{H}:H\in\mathcal{H}_\xi\}$ is a family of closed sets with the finite intersection property containing the compact set $\overline{H}_0$, its intersection F_ξ is not empty. Also $F_\xi\subseteq\overline{H}_0\subseteq G$, so $D\cap G\supseteq D\cap F_\xi$ is non-empty. As G is arbitrary, D is dense, and $d(X)\le\#(D)\le\kappa$. We know already that $\kappa\le d(X)$, so they are equal.

514B Stone spaces Necessarily, any cardinal function ζ of topological spaces corresponds to a cardinal function $\tilde\zeta$ of Boolean algebras, taking $\tilde\zeta(\mathfrak{A})=\zeta(Z)$ where Z is the Stone space of $\mathfrak{A}$. Working through the functions described in 5A4A and 511D, we have the following results.

Theorem Let $\mathfrak{A}$ be any Boolean algebra and Z its Stone space. For $a\in\mathfrak{A}$ let $\widehat{a}$ be the corresponding open-and-closed subset of Z.

(a) $\#(\mathfrak{A})$ is $2^{w(Z)}=2^{\#(Z)}$ if $\mathfrak{A}$ is finite, $w(Z)$ otherwise.

(b) $\operatorname{sat}(\mathfrak{A})=\operatorname{sat}(Z)$, $c(\mathfrak{A})=c(Z)$.

(c) $\pi(\mathfrak{A})=\pi(Z)$.

(d) $d(\mathfrak{A})=d(Z)$.

(e) Let $\mathcal{N}\mathrm{wd}(Z)$ be the ideal of nowhere dense subsets of Z. Then $\operatorname{wdistr}(\mathfrak{A})=\operatorname{add}\mathcal{N}\mathrm{wd}(Z)$.

proof Let $\mathcal{E}$ be the algebra of open-and-closed subsets of Z, so that $a\mapsto\widehat{a}$ is an isomorphism from $\mathfrak{A}$ to $\mathcal{E}$. The essential fact here is that $\mathcal{E}\setminus\{\emptyset\}$ is coinitial with $\mathfrak{T}\setminus\{\emptyset\}$, where $\mathfrak{T}$ is the topology of Z, so that (writing $\mathfrak{A}^+$ for $\mathfrak{A}\setminus\{0\}$, as usual)

$$(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)\cong(\mathcal{E}\setminus\{\emptyset\},\supseteq,\mathcal{E}\setminus\{\emptyset\})\equiv_{\mathrm{GT}}(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})$$

by 513E(d-ii), inverted.

(a) If $\mathfrak{A}$ is finite, so is Z, and in this case $\mathfrak{A}\cong\mathcal{P}Z$ has cardinal $2^{\#(Z)}$. If $\mathfrak{A}$ is infinite, so are Z and $w(Z)$. Because $\mathcal{E}$ is a base for the topology of Z, $w(Z)\le\#(\mathcal{E})=\#(\mathfrak{A})$. On the other hand, let $\mathcal{U}$ be a base for the topology of Z with $\#(\mathcal{U})=w(Z)$. Then every member of $\mathcal{E}$ is expressible as the union of a finite subset of $\mathcal{U}$, so

$$\#(\mathfrak{A})=\#(\mathcal{E})\le\#([\mathcal{U}]^{<\omega})\le\max(\omega,\#(\mathcal{U}))=w(Z).$$

(b)-(c)

$$c(\mathfrak{A}) = c(\mathcal{E}) = c^{\downarrow}(\mathcal{E} \setminus \{\emptyset\}) = c^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\}) = c(Z),$$

$$\operatorname{sat}(\mathfrak{A}) = \operatorname{sat}(\mathcal{E}) = \operatorname{sat}^{\downarrow}(\mathcal{E} \setminus \{\emptyset\}) = \operatorname{sat}^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\}) = \operatorname{sat}(Z),$$

$$\pi(\mathfrak{A}) = \pi(\mathcal{E}) = \operatorname{ci}(\mathcal{E} \setminus \{\emptyset\}) = \operatorname{ci}(\mathfrak{T} \setminus \{\emptyset\}) = \pi(Z)$$

using 513Gb, inverted, to move between $\mathcal{E} \setminus \{\emptyset\}$ and $\mathfrak{T} \setminus \{\emptyset\}$.

(d)

$$d(\mathfrak{A}) = d^{\downarrow}(\mathfrak{A}^+) = d^{\downarrow}(\mathcal{E} \setminus \{\emptyset\}) = d^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\})$$

(513Gd, inverted)

$$= d(Z)$$

because Z is compact and Hausdorff (514A).

(e) Let $(\operatorname{Pou}(\mathfrak{A}), \sqsubseteq^*)$ be the pre-ordered set of partitions of unity in $\mathfrak{A}$ as described in 512Ee. For $C \in \operatorname{Pou}(\mathfrak{A})$, set

$$f(C) = Z \setminus \bigcup_{c \in C} \widehat{c}.$$

Then $f(C) \in \mathcal{N}\mathrm{wd}(Z)$. **P?** Otherwise, since $f(C)$ is certainly closed, its interior is non-empty, and there is a non-zero $a \in \mathfrak{A}$ such that $\widehat{a} \subseteq f(C)$; but in this case $a \cap c = 0$ for every $c \in C$ and C is not a partition of unity. **XQ**

If C, $D \in \operatorname{Pou}(\mathfrak{A})$ and $C \sqsubseteq^* D$ then $f(C) \subseteq f(D)$. **P** If $d \in D$, $C_0 = \{c : c \in C,\ c \cap d \neq 0\}$ is finite and $d \subseteq \sup C_0$; so $\widehat{d} \subseteq \bigcup_{c \in C_0} \widehat{c}$ is disjoint from $f(C)$. Thus $Z \setminus f(D) \subseteq Z \setminus f(C)$ and $f(C) \subseteq f(D)$. **Q**

If C, $D \in \operatorname{Pou}(\mathfrak{A})$ and $f(C) \subseteq f(D)$ then $C \sqsubseteq^* D$. **P** If $d \in D$ then the compact set $\widehat{d}$ is included in the open set $\bigcup_{c \in C} \widehat{c}$. So there is a finite set $C_0 \subseteq C$ such that $\widehat{d} \subseteq \bigcup_{c \in C_0} \widehat{c}$ and $\{c : c \in C,\ d \cap c \neq 0\} \subseteq C_0$ is finite. **Q**

$f[\operatorname{Pou}(\mathfrak{A})]$ is cofinal with $\mathcal{N}\mathrm{wd}(Z)$. **P** If $F \in \mathcal{N}\mathrm{wd}(Z)$, let $C \subseteq \mathfrak{A}$ be a maximal disjoint set such that $F \cap \widehat{c} = \emptyset$ for every $c \in C$. **?** If C is not a partition of unity in $\mathfrak{A}$, let $a \in \mathfrak{A}^+$ be such that $a \cap c = 0$ for every $c \in C$. Then $\widehat{a} \setminus F$ is a non-empty open set, so there is a non-zero $b \in \mathfrak{A}$ such that $\widehat{b} \subseteq \widehat{a} \setminus F$; in which case we ought to have added b to C. **X** So $C \in \operatorname{Pou}(\mathfrak{A})$ and $F \subseteq f(C)$. **Q**

By 513E(d-i), $\operatorname{Pou}(\mathfrak{A})$ and $\mathcal{N}\mathrm{wd}(Z)$ are Tukey equivalent, and

$$\operatorname{add} \mathcal{N}\mathrm{wd}(Z) = \operatorname{add} \operatorname{Pou}(\mathfrak{A}) = \operatorname{wdistr}(\mathfrak{A})$$

as remarked in 512Ee.

514C I begin the detailed study of cardinal functions of Boolean algebras with two elementary remarks.

Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) $d(\mathfrak{A})$ is the smallest cardinal κ such that $\mathfrak{A}$ is isomorphic, as Boolean algebra, to a subalgebra of $\mathcal{P}\kappa$.

(b) $\operatorname{link}(\mathfrak{A})$ is the smallest cardinal κ such that $\mathfrak{A}$ is isomorphic, as partially ordered set, to a subset of $\mathcal{P}\kappa$.

proof (a)(i) If we have an isomorphism π from $\mathfrak{A}$ to a subalgebra of $\mathcal{P}\kappa$, then $A_\xi = \{a : \xi \in \pi a\}$ is centered for each $\xi < \kappa$, and $\bigcup_{\xi<\kappa} A_\xi = \mathfrak{A}^+$; so $d(\mathfrak{A}) \leq \kappa$.

(ii) Let Z be the Stone space of $\mathfrak{A}$, and for $a \in \mathfrak{A}$ let $\widehat{a} \subseteq Z$ be the corresponding open-and-closed set. There is a dense set $D \subseteq Z$ of size $d(\mathfrak{A})$ (514Bd), and $a \mapsto D \cap \widehat{a} : \mathfrak{A} \to \mathcal{P}D$ is an injective Boolean homomorphism; so $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathcal{P}D \cong \mathcal{P}(d(\mathfrak{A}))$.

(b)(i) $\kappa \leq \operatorname{link}(\mathfrak{A})$. **P** Let $\langle A_\xi \rangle_{\xi < \operatorname{link}(\mathfrak{A})}$ be a family of linked subsets of $\mathfrak{A}^+$ covering $\mathfrak{A}^+$. Set $A'_\xi = \{b : \exists\, a \in A_\xi,\ b \supseteq a\}$; then each A'_ξ is still linked in $\mathfrak{A}$. Define $h : \mathfrak{A} \to \mathcal{P}\kappa$ by setting $h(a) = \{\xi : a \in A'_\xi\}$. Then h is order-preserving. Now if a, $b \in \mathfrak{A}$ and $a \not\subseteq b$, there is a $\xi < \kappa$ such that $a \setminus b \in A_\xi$, in which case $\xi \in h(a) \setminus h(b)$. Thus h is an embedding and $\kappa \leq \operatorname{link}(\mathfrak{A})$. **Q**

(ii) $\operatorname{link}(\mathfrak{A}) \leq \kappa$. **P** Let $h : \mathfrak{A} \to \mathcal{P}\kappa$ be an order-isomorphism between $\mathfrak{A}$ and a subset of $\mathcal{P}\kappa$. For each ξ, set

$$A_\xi = \{a : a \in \mathfrak{A},\ \xi \in h(a) \setminus h(1 \setminus a)\}.$$

If a, $b \in A_\xi$ then $\xi \in h(b) \setminus h(1 \setminus a)$ so $b \not\subseteq 1 \setminus a$ and $a \cap b \neq 0$; thus A_ξ is linked. If $a \in \mathfrak{A}^+$ then $a \not\subseteq 1 \setminus a$ so $h(a) \not\subseteq h(1 \setminus a)$ and there is a $\xi < \kappa$ such that $\xi \in h(a) \setminus h(1 \setminus a)$; thus $\mathfrak{A}^+ = \bigcup_{\xi<\kappa} A_\xi$ and $\operatorname{link}(\mathfrak{A}) \leq \kappa$. **Q**

514D Theorem Let $\mathfrak{A}$ be a Boolean algebra.

(a)

$$c(\mathfrak{A}) \le \mathrm{link}(\mathfrak{A}) \le d(\mathfrak{A}) \le \pi(\mathfrak{A}) \le \#(\mathfrak{A}) \le 2^{\mathrm{link}(\mathfrak{A})}, \quad \tau(\mathfrak{A}) \le \pi(\mathfrak{A}),$$

$\mathrm{sat}(\mathfrak{A}) = c(\mathfrak{A})^+$ unless $\mathrm{sat}(\mathfrak{A})$ is weakly inaccessible, in which case $\mathrm{sat}(\mathfrak{A}) = c(\mathfrak{A})$.

(b) If $A \subseteq \mathfrak{A}$, there is a $B \in [A]^{<\mathrm{sat}(\mathfrak{A})}$ with the same upper bounds as A; similarly, there is a $B \in [A]^{<\mathrm{sat}(\mathfrak{A})}$ with the same lower bounds as A.

(c) $\mathrm{link}_{c(\mathfrak{A})}(\mathfrak{A}) = \mathrm{link}_{<\mathrm{sat}(\mathfrak{A})}(\mathfrak{A}) = \pi(\mathfrak{A})$.

(d) If $\mathfrak{A}$ is not purely atomic, $\mathrm{wdistr}(\mathfrak{A}) \le \min(d(\mathfrak{A}), 2^{\tau(\mathfrak{A})})$ is a regular infinite cardinal.

(e) $\#(\mathfrak{A}) \le \max(4, \sup_{\lambda<\mathrm{sat}(\mathfrak{A})} \tau(\mathfrak{A})^\lambda)$, where $\tau(\mathfrak{A})^\lambda$ is the cardinal power.

proof Let Z be the Stone space of $\mathfrak{A}$; for $a \in \mathfrak{A}$, let $\widehat{a} \subseteq Z$ be the corresponding open-and-closed set.

(a) This is mostly a repetition of 511Ia. By 514Cb, $\#(\mathfrak{A}) \le 2^{\mathrm{link}(\mathfrak{A})}$. By 513Bc, inverted, and the definitions in 511Db,

$$\mathrm{sat}(\mathfrak{A}) = \mathrm{sat}^{\downarrow}(\mathfrak{A}^+) = c^{\downarrow}(\mathfrak{A}^+)^+ = c(\mathfrak{A})^+$$

unless $\mathrm{sat}(\mathfrak{A})$ is a regular uncountable limit cardinal, that is, is weakly inaccessible, and otherwise $\mathrm{sat}(\mathfrak{A}) = c(\mathfrak{A})$. (See also 5A4Ba.)

(b) By 5A4Bd, applied to $\{\widehat{a} : a \in A\}$, there is a $B \in [A]^{<\mathrm{sat}(\mathfrak{A})}$ such that $\overline{\bigcup_{b\in B}\widehat{b}} = \overline{\bigcup_{a\in A}\widehat{a}}$. Now if c is an upper bound of B, then $\widehat{c}$ is a closed set including $\widehat{b}$ for every $b \in B$, so it also includes $\widehat{a}$ for every $a \in A$, and c is an upper bound of A.

Applying this to $\{1 \setminus a : a \in A\}$ we see that there is a set $B' \in [A]^{<\mathrm{sat}(\mathfrak{A})}$ with the same lower bounds as A.

(c) Set $\kappa = \mathrm{link}_{<\mathrm{sat}(\mathfrak{A})}(\mathfrak{A})$. By 511Ia, $\kappa \le \mathrm{link}_{c(\mathfrak{A})}(\mathfrak{A}) \le \pi(\mathfrak{A})$. On the other hand, if $A \subseteq \mathfrak{A}^+$ is $<\mathrm{sat}(\mathfrak{A})$-linked, it has a lower bound in $\mathfrak{A}^+$. **P** By (b), there is a set $B \subseteq A$, with the same lower bounds as A, such that $\#(B) < \mathrm{sat}(\mathfrak{A})$. Now B has a non-zero lower bound because A is $<\mathrm{sat}(\mathfrak{A})$-linked, so A also has a non-zero lower bound. **Q** We have a cover $\langle A_\xi\rangle_{\xi<\kappa}$ of $\mathfrak{A}^+$ by $<\mathrm{sat}(\mathfrak{A})$-linked sets; each A_ξ has a non-zero lower bound a_ξ say; and $\{a_\xi : \xi < \kappa\}$ is a π-base for $\mathfrak{A}$, so $\pi(\mathfrak{A}) \le \kappa$.

(d)(i) Because $\mathrm{wdistr}(\mathfrak{A}) = \mathrm{add}\,\mathcal{N}\mathit{wd}(Z)$, where $\mathcal{N}\mathit{wd}(Z)$ is the ideal of nowhere dense subsets of Z (514Be), and is not ∞ (511Ie), it must be a regular infinite cardinal (513C(a-i)). (Or argue directly from 511Df.)

(ii) As for the upper bound for $\mathrm{wdistr}(\mathfrak{A})$, suppose that $a \in \mathfrak{A}^+$ includes no atom and that $D \in [\mathfrak{A}]^{\tau(\mathfrak{A})}$ τ-generates $\mathfrak{A}$. Since $\mathfrak{A}$ and $\tau(\mathfrak{A})$ are surely infinite, the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by $D \cup \{a\}$ is still of size $\tau(\mathfrak{A})$ (331Gc). For $B \subseteq \mathfrak{B}$ set $E_B = Z \cap \bigcap_{b\in B}\widehat{b}$, and set $\mathcal{C} = \{B : B \subseteq \mathfrak{B}, E_B$ is nowhere dense$\}$. Then $\bigcup_{B\in\mathcal{C}} E_B \supseteq \widehat{a}$. **P** Take any $z \in \widehat{a}$. Set $B = \{b : b \in \mathfrak{B}, z \in \widehat{b}\}$. **?** If E_B has non-empty interior, it includes $\widehat{c}$ for some non-zero $c \subseteq a$. But now, for any $d \in D$, either $d \in B$ and $c \subseteq d$, or $1 \setminus d \in B$ and $c \cap d = 0$. So the order-closed subalgebra $\{d :$ either $c \subseteq d$ or $c \cap d = 0\}$ includes D and must be the whole of $\mathfrak{A}$, and $c \subseteq a$ is an atom. **X** So $\mathrm{int}\,E_B = \emptyset$, $B \in \mathcal{C}$ and $z \in E_B$. As z is arbitrary, $\widehat{a} \subseteq \bigcup_{B\in\mathcal{C}} E_B$. **Q**

By 514Be, with 514Bd,

$$\mathrm{wdistr}(\mathfrak{A}) \le \#(\mathcal{C}) \le 2^{\#(\mathfrak{B})} = 2^{\tau(\mathfrak{A})}.$$

At the same time, if $Y \subseteq Z$ is any dense set of size $d(Z)$, then $\{\{y\} : y \in Y \cap \widehat{a}\}$ is a family of nowhere dense sets with no upper bound in the ideal of nowhere dense subsets of Z; so 514Be also tells us that

$$\mathrm{wdistr}(\mathfrak{A}) \le \#(Y \cap \widehat{a}) \le d(Z) = d(\mathfrak{A}).$$

(e) (Compare 4A1O.) Set $\kappa = \sup_{\lambda<\mathrm{sat}(\mathfrak{A})} \tau(\mathfrak{A})^\lambda$. If $\#(\mathfrak{A}) > 4$ then $\tau(\mathfrak{A}) \ge 2$ so $\kappa \ge \sup_{\lambda<\mathrm{sat}(\mathfrak{A})} 2^\lambda$, and the result is immediate from 511Ic if $\mathfrak{A}$ is finite. If $\mathfrak{A}$ is infinite, so is $\mathrm{sat}(\mathfrak{A})$, while $\lambda < \kappa$ for every $\lambda < \mathrm{sat}(\mathfrak{A})$, so $\mathrm{sat}(\mathfrak{A}) \le \kappa$. Let $D \subseteq \mathfrak{A}$ be a set with cardinal $\tau(\mathfrak{A})$ which τ-generates $\mathfrak{A}$. Define $\langle D_\xi\rangle_{\xi<\kappa}$ inductively by setting

$$D_0 = D, \quad D_\xi = \{1 \setminus a : a \in \mathfrak{A},\ a = \sup C \text{ for some } C \subseteq \textstyle\bigcup_{\eta<\xi} D_\eta\}$$

for $\xi < \kappa$. Then $\#(D_\xi) \le \kappa$ for every $\xi < \kappa$. **P** The point is that, by (b), every member of D_ξ is expressible in the form $1 \setminus \sup C$ for some $C \in [\bigcup_{\eta<\xi} D_\eta]^{<\mathrm{sat}(\mathfrak{A})}$. But the inductive hypothesis tells us that $\bigcup_{\eta<\xi} D_\eta$ has cardinal at most κ, so the number of its subsets of cardinal less than $\mathrm{sat}(\mathfrak{A})$ is also κ (5A1Ef, because $\mathrm{sat}(\mathfrak{A})$ is regular), and $\#(D_\xi) \le \kappa$. **Q**

At the end of the induction, set $\mathfrak{B} = \bigcup_{\xi<\mathrm{sat}(\mathfrak{A})} D_\xi$. Then $1 \setminus (b \cup b') \in \mathfrak{B}$ for every b, $b' \in \mathfrak{B}$, so $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. Also it is order-closed. **P** If $B \subseteq \mathfrak{B}$ has a supremum $a \in \mathfrak{A}$, there is a $C \subseteq B$ such that $\#(C) < \mathrm{sat}(\mathfrak{A})$ and $a = \sup C$. Now there must be some set $J \subseteq \mathrm{sat}(\mathfrak{A})$ such that $\#(J) < \mathrm{sat}(\mathfrak{A})$ and $C \subseteq \bigcup_{\eta\in J} D_\eta$. Since $\mathrm{sat}(\mathfrak{A})$ is regular (513Bb), $\zeta = \sup J$ is less than $\mathrm{sat}(\mathfrak{A})$. Now $1 \setminus a \in D_{\zeta+1}$ and $a \in \mathfrak{B}$. **Q**

By the choice of D, $\mathfrak{B} = \mathfrak{A}$, so $\#(\mathfrak{A}) = \#(\mathfrak{B}) \le \kappa$.

514E Subalgebras, homomorphic images, products: Theorem Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then

$$\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) \le c(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A})$$

for every $\kappa \le \omega$, in particular,

$$d(\mathfrak{B}) \le d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) \le \mathrm{link}(\mathfrak{A}).$$

(b) If $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$ then, in addition, $\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A})$ for $\kappa > \omega$, $\pi(\mathfrak{B}) \le \pi(\mathfrak{A})$ and $\mathrm{wdistr}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{B})$.

(c) If $\mathfrak{B}$ is a Boolean algebra and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then

$$\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) \le c(\mathfrak{A}), \quad \pi(\mathfrak{B}) \le \pi(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A}) \text{ for every cardinal } \kappa,$$

$$d(\mathfrak{B}) \le d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) \le \mathrm{link}(\mathfrak{A}),$$

and also

$$\mathrm{wdistr}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{B}), \quad \tau(\mathfrak{B}) \le \tau(\mathfrak{A}).$$

(d) If $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$, then

$$\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) \le c(\mathfrak{A}), \quad \pi(\mathfrak{B}) \le \pi(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A}) \text{ for every } \kappa,$$

$$d(\mathfrak{B}) \le d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) \le \mathrm{link}(\mathfrak{A});$$

moreover,

$$\mathrm{wdistr}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{B}), \quad \tau(\mathfrak{B}) \le \tau(\mathfrak{A}).$$

(e) If $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$ then

$$\mathrm{sat}(\mathfrak{B}) = \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) = c(\mathfrak{A}), \quad \pi(\mathfrak{B}) = \pi(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) = \mathrm{link}_{<\kappa}(\mathfrak{A}) \text{ for every } \kappa,$$

$$d(\mathfrak{B}) = d(\mathfrak{A}), \quad \mathrm{link}(\mathfrak{B}) = \mathrm{link}(\mathfrak{A}),$$

and finally

$$\mathrm{wdistr}(\mathfrak{B}) = \mathrm{wdistr}(\mathfrak{A}), \quad \tau(\mathfrak{A}) \le \tau(\mathfrak{B}).$$

(f) If $\mathfrak{A}$ is the simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras, then

$$\tau(\mathfrak{A}) \le \max(\omega, \sup_{i\in I} \tau(\mathfrak{A}_i), \min\{\lambda : \#(I) \le 2^\lambda\}),$$

$$\mathrm{sat}(\mathfrak{A}) \le \max(\omega, \#(I)^+, \sup_{i\in I} \mathrm{sat}(\mathfrak{A}_i)),$$

$$c(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} c(\mathfrak{A}_i)),$$

$$\pi(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} \pi(\mathfrak{A}_i)),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} \mathrm{link}_{<\kappa}(\mathfrak{A}_i)) \text{ for every } \kappa,$$

$$\mathrm{link}(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} \mathrm{link}(\mathfrak{A}_i)),$$

$$d(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} d(\mathfrak{A}_i)),$$

and

$$\mathrm{wdistr}(\mathfrak{A}) = \min_{i\in I} \mathrm{wdistr}(\mathfrak{A}_i).$$

proof Write Z for the Stone space of $\mathfrak{A}$.

(a) Any disjoint subset of $\mathfrak{B}^+$ is a disjoint subset of $\mathfrak{A}^+$, so $\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A})$ and $c(\mathfrak{B}) \le c(\mathfrak{A})$. If $\kappa \le \omega$ and $\mathcal{A}$ is a cover of $\mathfrak{A}^+$ by sets which are downwards $<\kappa$-linked in $\mathfrak{A}^+$, then $A \cap \mathfrak{B}$ is downwards $<\kappa$-linked in $\mathfrak{B}^+$ for each $A \in \mathcal{A}$, so $\mathrm{link}_{<\kappa}(\mathfrak{B}) \le \mathrm{link}_{<\kappa}(\mathfrak{A})$.

(b) For each non-zero $a \in \mathfrak{A}$, the set $B_a = \{b : b \in \mathfrak{B},\ a \subseteq b\}$ does not have infimum 0 in $\mathfrak{A}$ so cannot have infimum 0 in $\mathfrak{B}$; let $\psi(a) \in \mathfrak{B}^+$ be a lower bound for B_a. If now we set $\phi(b) = b$ for $b \in \mathfrak{B}$, (ϕ, ψ) is a Galois-Tukey connection from $(\mathfrak{B}^+, \supseteq, \mathfrak{B}^+)$ to $(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+)$. It follows at once that

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) = \mathrm{link}_{<\kappa}(\mathfrak{B}^+, \supseteq, \mathfrak{B}^+) \le \mathrm{link}_{<\kappa}(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) = \mathrm{link}_{<\kappa}(\mathfrak{A})$$

for arbitrary κ (512Dd), and that

$$\pi(\mathfrak{B}) = \mathrm{cov}(\mathfrak{B}^+, \supseteq, \mathfrak{B}^+) \le \mathrm{cov}(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) = \pi(\mathfrak{A})$$

(512Da).

Now suppose that $\kappa < \mathrm{wdistr}(\mathfrak{A})$ and that $\langle B_\xi \rangle_{\xi<\kappa}$ is a family of partitions of unity in $\mathfrak{B}$. Then

$$D = \{d : d \in \mathfrak{B},\ \{b : b \in B_\xi,\ b \cap d \ne 0\} \text{ is finite for every } \xi < \kappa\}$$

is order-dense in $\mathfrak{B}$. **P** Take any non-zero $d \in \mathfrak{B}$. $\sup B_\xi = 1$ in $\mathfrak{A}$, that is, B_ξ is still a partition of unity in $\mathfrak{A}$, for each ξ. So there is a partition A of unity in $\mathfrak{A}$ such that $\{b : b \in B_\xi,\ b \cap a \ne 0\}$ is finite for every $\xi < \kappa$ and $a \in A$. Let $a \in A$ be such that $d \cap a \ne 0$, and set $e_\xi = \sup\{b : b \in B_\xi,\ b \cap a \ne 0\}$ for each $\xi < \kappa$. Then $a \subseteq e_\xi \in \mathfrak{B}$ for each ξ. This means that $\{d\} \cup \{e_\xi : \xi < \kappa\}$ has a non-zero lower bound $d \cap a$ in $\mathfrak{A}$; as $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$, there is a non-zero $d' \subseteq d$ which is also a lower bound for $\{e_\xi : \xi < \kappa\}$. But this means that $d' \in D$. As d is arbitrary, D is order-dense in $\mathfrak{B}$. **Q**

There is therefore a partition of unity included in D. As $\langle B_\xi \rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{wdistr}(\mathfrak{B}) \ge \mathrm{wdistr}(\mathfrak{A})$.

(c)(i) For any $b \in \mathfrak{B}^+$ there is a $\psi(b) \in \mathfrak{A}^+$ such that $\phi\psi(b) \subseteq b$ and $a = 0$ whenever $a \subseteq \psi(b)$ and $\phi a = 0$. **P** Consider $D = \{d : d \in \mathfrak{A},\ \phi d \supseteq b\}$. This is a non-empty downwards-directed subset of $\mathfrak{A}$ and b is a non-zero lower bound of $\phi[D]$. As ϕ is supposed to be order-continuous, D must have a non-zero lower bound in $\mathfrak{A}$; let $\psi(b)$ be such a lower bound. Since there is a $d \in \mathfrak{A}$ such that $\phi d = b$, and now $d \in D$, we must have $\phi\psi(b) \subseteq \phi d = b$. If $a \subseteq \psi(b)$ and $\phi a = 0$, then $\phi(1 \subseteq a) = 1 \supseteq b$, $1 \setminus a \in D$ and $a \subseteq \psi(b) \subseteq 1 \setminus a$, so $a = 0$. **Q**

(ii) If $\kappa = \mathrm{sat}(\mathfrak{A})$ and $\langle b_\xi \rangle_{\xi<\kappa}$ is a family in $\mathfrak{B}^+$, then $\langle \psi(b_\xi) \rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}^+$ and there are distinct ξ, $\eta < \kappa$ such that $a = \psi(b_\xi) \cap \psi(b_\eta)$ is non-zero. Now

$$0 \ne \phi a \subseteq \phi\psi(b_{xi}) \cap \phi\psi(b_\eta) \subseteq b_\xi \cap b_\eta.$$

As $\langle b_\xi \rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{sat}(\mathfrak{B}) \le \mathrm{sat}(\mathfrak{A})$. By a similar argument, or using 514Da, we see that $c(\mathfrak{B}) \le c(\mathfrak{A})$.

(iii) Let A be a coinitial subset of $\mathfrak{A}^+$ of cardinal $\pi(\mathfrak{A})$. Set $B = \phi[A] \setminus \{0\}$. If $b \in \mathfrak{B}^+$, there is an $a \in A$ such that $a \subseteq \psi(b)$, and now $\phi a \in B$ and $\phi a \subseteq \phi\psi(b) \subseteq b$. So B is cofinal with $\mathfrak{B}^+$ and

$$\pi(\mathfrak{B}) \le \#(B) \le \#(A) = \pi(\mathfrak{A}).$$

(iv) Let W be the Stone space of $\mathfrak{B}$. Write $\mathcal{N}\mathrm{wd}(Z)$, $\mathcal{N}\mathrm{wd}(W)$ for the ideals of nowhere dense subsets of Z and W, so that $\mathrm{add}\,\mathcal{N}\mathrm{wd}(Z) = \mathrm{wdistr}(\mathfrak{A})$ and $\mathrm{add}\,\mathcal{N}\mathrm{wd}(W) = \mathrm{wdistr}(\mathfrak{B})$ (514Be). Corresponding to $\phi : \mathfrak{A} \to \mathfrak{B}$ we have an injective continuous function $\theta : W \to Z$ such that $\theta^{-1}[E] \in \mathcal{N}\mathrm{wd}(W)$ for every $E \in \mathcal{N}\mathrm{wd}(Z)$ (312Sb, 313R). Also $\theta[F] \in \mathcal{N}\mathrm{wd}(Z)$ for every $F \in \mathcal{N}\mathrm{wd}(W)$. **P?** Otherwise, because $\theta[\overline{F}]$ is compact, therefore closed, there is a non-empty open set $G \subseteq \theta[\overline{F}]$. Now $\theta^{-1}[G]$ is a non-empty open set, and is included in $\overline{F}$, because θ is injective; but this is impossible. **XQ** So if $\mathcal{J}_0 \subseteq \mathcal{N}\mathrm{wd}(W)$ and $\#(\mathcal{J}_0) < \mathrm{wdistr}(\mathfrak{A})$, $E = \bigcup\{\theta[F] : F \in \mathcal{J}_0\}$ belongs to $\mathcal{N}\mathrm{wd}(Z)$ and $\bigcup \mathcal{J}_0 \subseteq \theta^{-1}[E]$ belongs to $\mathcal{N}\mathrm{wd}(W)$. This shows that $\mathrm{add}\,\mathcal{N}\mathrm{wd}(W) \ge \mathrm{add}\,\mathcal{N}\mathrm{wd}(Z)$, so that $\mathrm{wdistr}(\mathfrak{B}) \ge \mathrm{wdistr}(\mathfrak{A})$.

(v) As for $\tau(\mathfrak{B})$, we have only to recall that if $D \subseteq \mathfrak{A}$ is a τ-generating set of size $\tau(\mathfrak{A})$, the order-closed subalgebra of $\mathfrak{B}$ generated by $\phi[D]$ includes $\phi[\mathfrak{A}] = \mathfrak{B}$ (313Mb), and

$$\tau(\mathfrak{B}) \le \#(\phi[D]) \le \tau(\mathfrak{A}).$$

(d) If $\mathfrak{B}$ is the principal ideal generated by b, then $a \mapsto a \cap b : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous surjection, so we can repeat the list in (c).

(e)(i) Because $\mathfrak{B}^+$ is coinitial with $\mathfrak{A}^+$ we can use 513Gc, inverted, to see that

$$\mathrm{sat}(\mathfrak{B}) = \mathrm{sat}^{\downarrow}(\mathfrak{B}^+) = \mathrm{sat}^{\downarrow}(\mathfrak{A}^+) = \mathrm{sat}(\mathfrak{A}), \quad c(\mathfrak{B}) = c(\mathfrak{A}),$$

$$\pi(\mathfrak{B}) = \mathrm{ci}(\mathfrak{B}^+) = \mathrm{ci}(\mathfrak{A}^+) = \pi(\mathfrak{A}),$$

$$\mathrm{link}_{<\kappa}(\mathfrak{B}) = \mathrm{link}^{\downarrow}_{<\kappa}(\mathfrak{B}^+) = \mathrm{link}^{\downarrow}_{<\kappa}(\mathfrak{A}^+) = \mathrm{link}_{<\kappa}(\mathfrak{A}).$$

(ii) From (b) we know that $\mathrm{wdistr}(\mathfrak{B}) \ge \mathrm{wdistr}(\mathfrak{A})$. For the reverse inequality, suppose that $\kappa < \mathrm{wdistr}(\mathfrak{B})$ and that $\langle A_\xi \rangle_{\xi<\kappa}$ is any family of partitions of unity in $\mathfrak{A}$. For each $\xi < \kappa$ set $B_\xi = \{b : b \in \mathfrak{B}, \exists\, a \in A_\xi, b \subseteq a\}$. Then B_ξ is order-dense in $\mathfrak{B}$ and includes a partition of unity B'_ξ (313K). Now there is a partition C of unity in $\mathfrak{B}$ such that $D'_{\xi c} = \{b : b \in B'_\xi,\ b \cap c \ne 0\}$ is finite for any $\xi < \kappa$ and $c \in C$. C is still a partition of unity in $\mathfrak{A}$, and $D_{\xi c} = \{a : a \in A_\xi,\ a \cap c \ne 0\}$ is finite for every $c \in C$ and $\xi < \kappa$. (For if a, a' are distinct elements of $D_{\xi c}$, then $\{b : b \in D'_{\xi c},\ b \subseteq a\}$ and $\{b : b \in D'_{\xi c},\ b \subseteq a'\}$ are disjoint and not empty.) As $\langle A_\xi \rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{wdistr}(\mathfrak{B}) \le \mathrm{wdistr}(\mathfrak{A})$.

(iii) If $D \subseteq \mathfrak{B}$ τ-generates $\mathfrak{B}$, then D also τ-generates $\mathfrak{A}$. **P** Applying 313Mb to the identity map from $\mathfrak{B}$ to $\mathfrak{A}$, we see that the order-closed subalgebra $\mathfrak{D}$ of $\mathfrak{A}$ generated by D includes $\mathfrak{B}$; but as any member of $\mathfrak{A}$ is the supremum of a subset of $\mathfrak{B}$, $\mathfrak{D} = \mathfrak{A}$. **Q** So $\tau(\mathfrak{A}) \le \tau(\mathfrak{B})$.

(f) We can identify each $\mathfrak{A}_i$ with the principal ideal of $\mathfrak{A}$ generated by an element a_i, where $\langle a_i \rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ (315E). If $\kappa = \max(\omega, \sup_{i\in I} \tau(\mathfrak{A}_i), \min\{\lambda : \#(I) \le 2^\lambda\})$, then for each $i \in I$ choose $\langle a_{i\xi} \rangle_{\xi<\kappa}$ in $\mathfrak{A}_i$ such that $\{a_{i\xi} : \xi < \kappa\}$ τ-generates $\mathfrak{A}_i$, and let $\phi : I \to \mathcal{P}\kappa$ be injective. For $\xi < \kappa$, set

$$b_\xi = \sup_{i\in I} a_{i\xi}, \quad c_\xi = \sup_{i\in\phi(\xi)} a_i.$$

Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{b_\xi : \xi < \kappa\} \cup \{c_\xi : \xi < \kappa\}$. Then

$$a_i = \inf\{c_\xi : \xi \in \phi(i)\} \setminus \sup\{c_\xi : \xi \in \kappa \setminus \phi(i)\} \in \mathfrak{B}$$

for each i. Because $\{b : b \in \mathfrak{B}, b \subseteq a_i\}$ is an order-closed subalgebra of $\mathfrak{A}_i$ containing $b_\xi \cap a_i = a_{i\xi}$ for every $\xi < \kappa$, it is the whole of $\mathfrak{A}_i$, so $\mathfrak{A}_i \subseteq \mathfrak{B}$ for every $i \in I$. It follows at once that $\mathfrak{B} = \mathfrak{A}$, so that $\tau(\mathfrak{A}) \le \kappa$.

The other parts are all elementary.

514F For measure algebras, Maharam type is not only the cardinal function which gives most information, but is also, as a rule, easy to calculate. For other Boolean algebras, it may not be obvious what the Maharam type is. The following result sometimes helps.

Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and $\langle a_{ij} \rangle_{i\in I, j\in J}$ a τ-generating family in $\mathfrak{A}$ such that

$$\langle a_{ij} \rangle_{j\in J} \text{ is disjoint for every } i \in I, \quad \sup_{i\in I} a_{ij} = 1 \text{ for every } j \in J.$$

Then $\tau(\mathfrak{A}) \le \max(\omega, \#(I))$.

proof (a) We may suppose that $J = \kappa$ is a cardinal. For i, $j \in I$ set

$$a_i^* = \sup_{\xi<\kappa} a_{i\xi}, \quad b_{ij} = \sup_{\xi<\eta<\kappa} a_{i\eta} \cap a_{j\xi}.$$

Then

$$\sup_{\eta\le\zeta} a_{i\eta} = a_i^* \setminus \sup_{j\in I}(b_{ij} \setminus \sup_{\xi<\zeta} a_{j\xi})$$

whenever $i \in I$ and $\zeta < \kappa$. **P** (i) If $\eta \le \zeta$ and $j \in I$, then $a_{i\eta} \subseteq a_i^*$ and

$$a_{i\eta} \cap b_{ij} = \sup_{\xi<\theta<\kappa} a_{i\eta} \cap a_{i\theta} \cap a_{j\xi} = \sup_{\xi<\eta} a_{i\eta} \cap a_{j\xi}$$

(because $\langle a_{i\theta} \rangle_{\theta<\kappa}$ is disjoint)

$$\subseteq \sup_{\xi<\zeta} a_{j\xi},$$

so

$$\sup_{\eta\le\zeta} a_{i\eta} \subseteq a_i^* \setminus \sup_{j\in I}(b_{ij} \setminus \sup_{\xi<\zeta} a_{j\xi}).$$

(ii) If $0 \ne c \subseteq a_i^*$ and $c \cap a_{i\eta} = 0$ for every $\eta \le \zeta$, there are an $\eta > \zeta$ such that $c' = c \cap a_{i\eta}$ is non-zero, and a $j \in I$ such that $c'' = c' \cap a_{j\zeta}$ is non-zero. In this case $c'' \subseteq b_{ij} \setminus \sup_{\xi<\zeta} a_{j\xi}$, so $c \cap b_{ij} \setminus \sup_{\xi<\zeta} a_{j\xi} \ne 0$. Accordingly

$$\sup_{\eta\le\zeta} a_{i\eta} \supseteq a_i^* \setminus \sup_{j\in I}(b_{ij} \setminus \sup_{\xi<\zeta} a_{j\xi}).\ \mathbf{Q}$$

(b) Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_i^* : i \in I\} \cup \{b_{ij} : i, j \in I\}$. Using (a) for the inductive step, we see that $\sup_{\xi\le\zeta} a_{i\xi} \in \mathfrak{B}$ for every $i \in I$ and $\zeta < \kappa$. Consequently $a_{i\zeta} = \sup_{\xi\le\zeta} a_{i\xi} \setminus \sup_{\xi<\zeta} a_{i\xi}$ belongs to $\mathfrak{B}$ whenever $i \in I$ and $\zeta < \kappa$, and $\mathfrak{A} = \mathfrak{B}$ is τ-generated by $\{a_i^* : i \in I\} \cup \{b_{ij} : i, j \in I\}$, so has Maharam type at most $\max(\omega, \#(I))$.

514G Order-preserving functions of Boolean algebras (a) Let F be an ordinal function of Boolean algebras, that is, a function defined on the class of Boolean algebras, taking ordinal values, and such that $F(\mathfrak{A}) = F(\mathfrak{B})$ whenever $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. We say that F is **order-preserving** if $F(\mathfrak{B}) \le F(\mathfrak{A})$ whenever $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$. It is easy to check that all the cardinal functions defined in 511D are order-preserving; see 514Ed. Now a Boolean algebra $\mathfrak{A}$ is F**-homogeneous** if $F(\mathfrak{B}) = F(\mathfrak{A})$ for every non-zero principal ideal $\mathfrak{B}$ of $\mathfrak{A}$. Of course any principal ideal of an F-homogeneous Boolean algebra is again F-homogeneous.

We have already seen 'Maharam-type-homogeneous' algebras in Chapter 33. I mention **cellularity-homogeneous** algebras as a class which will be used later. The proof of the Erdős-Tarski theorem in 513Bb is based on the idea of upwards-saturation-homogeneous partially ordered set. Of course all the most important ordinal functions of Boolean algebras actually take cardinal values.

(b) If F is any order-preserving ordinal function of Boolean algebras, and $\mathfrak{A}$ is a Boolean algebra, then (writing $\mathfrak{A}_a$ for the principal ideal generated by a) $\{a : a \in \mathfrak{A},\ \mathfrak{A}_a$ is F-homogeneous$\}$ is order-dense in $\mathfrak{A}$. $\mathbf{P}$ If $a \in \mathfrak{A}^+$, set $\xi = \min\{F(\mathfrak{A}_b) : 0 \ne b \subseteq a\}$, and let b be such that $0 \ne b \subseteq a$ and $F(\mathfrak{A}_b) = \xi$; then $\mathfrak{A}_b$ is F-homogeneous. $\mathbf{Q}$ So if $\mathfrak{A}$ is a Dedekind complete Boolean algebra, it is isomorphic to a simple product of F-homogeneous Boolean algebras. (Argue as in the proof of 332B.)

(c) Similarly, if $F_0, \dots, F_n$ are order-preserving ordinal functions of Boolean algebras, and $\mathfrak{A}$ is any Boolean algebra, then $\{a : \mathfrak{A}_a$ is F_i-homogeneous for every $i \le n\}$ is order-dense in $\mathfrak{A}$; and if $\mathfrak{A}$ is Dedekind complete, it is isomorphic to a simple product of Boolean algebras all of which are F_i-homogeneous for every $i \le n$.

(d) Of course any Boolean algebra which is homogeneous in the full sense (316N) is F-homogeneous for every function F of Boolean algebras. Maharam's theorem tells us that a Maharam-type-homogeneous *measurable* algebra is homogeneous (331N).

514H Regular open algebras: Proposition Let $(X, \mathfrak{T})$ be a topological space and $\mathrm{RO}(X)$ its regular open algebra (314O *et seq.*).

(a)(i) $(\mathrm{RO}(X)^+, \supseteq, \mathrm{RO}(X)^+) \preccurlyeq_{\mathrm{GT}} (\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$.

(ii) If X is regular, $(\mathrm{RO}(X)^+, \supseteq, \mathrm{RO}(X)^+) \equiv_{\mathrm{GT}} (\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$.

(b)(i) $\mathrm{sat}(\mathrm{RO}(X)) = \mathrm{sat}(X)$, $c(\mathrm{RO}(X)) = c(X)$, $\pi(\mathrm{RO}(X)) \le \pi(X)$ and $d(\mathrm{RO}(X)) \le d(X)$.

(ii) If X is regular, $\pi(\mathrm{RO}(X)) = \pi(X)$.

(iii) If X is locally compact and Hausdorff, $d(\mathrm{RO}(X)) = d(X)$.

(c) Let $\mathcal{N}\mathrm{wd}(X)$ be the ideal of nowhere dense subsets of X.

(i) If X is regular, $\mathrm{wdistr}(\mathrm{RO}(X)) \le \mathrm{add}\,\mathcal{N}\mathrm{wd}(X)$.

(ii) If X is locally compact and Hausdorff, $\mathrm{wdistr}(\mathrm{RO}(X)) = \mathrm{add}\,\mathcal{N}\mathrm{wd}(X)$.

(d) If $Y \subseteq X$ is dense, then $G \mapsto G \cap Y$ is a Boolean isomorphism from $\mathrm{RO}(X)$ to $\mathrm{RO}(Y)$.

proof (a) For $G \in \mathfrak{T} \setminus \{\emptyset\}$, set $\psi(G) = \mathrm{int}\,\overline{G}$. If we set $\phi(G) = G$ for $G \in \mathrm{RO}(X)^+$, then (ϕ, ψ) is a Galois-Tukey connection from $(\mathrm{RO}(X)^+, \supseteq, \mathrm{RO}(X)^+)$ to $(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$.

If X is regular, then $\mathrm{RO}(X)^+$ is coinitial with $\mathfrak{T} \setminus \{\emptyset\}$, so 513Ed, inverted, shows that they are equivalent.

(b)(i) Any disjoint family in $\mathrm{RO}(X)^+$ is a disjoint family of non-empty open subsets of X, so $c(\mathrm{RO}(X)) \le c(X)$ and $\mathrm{sat}(\mathrm{RO}(X)) \le \mathrm{sat}(X)$. On the other hand, if $\mathcal{G}$ is a disjoint family of non-empty open subsets of X, then $\langle \mathrm{int}\,\overline{G} \rangle_{G\in\mathcal{G}}$ is a disjoint family in $\mathrm{RO}(X)^+$, so $c(X) \le c(\mathrm{RO}(X))$ and $\mathrm{sat}(\mathrm{RO}(X)) \le \mathrm{sat}(X)$.

By (a) and 513Ee, inverted,

$$\pi(\mathrm{RO}(X)) = \mathrm{ci}(\mathrm{RO}(X)^+) \le \mathrm{ci}(\mathfrak{T} \setminus \{\emptyset\}) = \pi(X),$$

$$d(\mathrm{RO}(X)) = d^{\downarrow}(\mathrm{RO}(X)^+) \le d^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\} \le d(X)$$

by 514A.

(ii) If X is regular, $\mathrm{RO}(X)^+$ is coinitial with $\mathfrak{T} \setminus \{\emptyset\}$, so $\mathrm{ci}(\mathrm{RO}(X)^+) = \mathrm{ci}(\mathfrak{T} \setminus \{\emptyset\})$ and $\pi(\mathrm{RO}(X)) = \pi(X)$.

(iii) If X is locally compact and Hausdorff it is also regular, so $\mathrm{RO}(X)^+$ is coinitial with $\mathfrak{T} \setminus \{\emptyset\}$, and

$$\begin{aligned} d(X) &= d^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\}) \\ &= d^{\downarrow}(\mathrm{RO}(X)^+) = d(\mathrm{RO}(X)). \qquad \text{(514A)} \end{aligned}$$

(c)(i) Let $\langle E_\xi \rangle_{\xi<\kappa}$ be a family of nowhere dense sets in X, where $\kappa < \mathrm{wdistr}(\mathrm{RO}(X))$. For each $\xi < \kappa$, set $\mathcal{G}_\xi = \{G : G \in \mathrm{RO}(X), \overline{G} \cap E_\xi = \emptyset\}$. Then $\mathcal{G}_\xi$ is upwards-directed, and $\bigcup \mathcal{G}_\xi = X \setminus \overline{E}_\xi$, because any point of $X \setminus \overline{E}_\xi$ belongs to a regular open set with closure disjoint from E_ξ. But this means that $\sup \mathcal{G}_\xi = X$ in $\mathrm{RO}(X)$ (314P), and there is a partition $\mathcal{G}'_\xi$ of unity included in $\mathcal{G}_\xi$. Because $\kappa < \mathrm{wdistr}(\mathrm{RO}(X))$, there is a partition $\mathcal{H}$ of unity in $\mathrm{RO}(X)$ such that $\{G : G \in \mathcal{G}'_\xi,\, G \cap H \ne \emptyset\}$ is finite for each ξ and $H \in \mathcal{H}$. It follows that $H \subseteq \bigcup\{\overline{G} : G \in \mathcal{G}'_\xi\}$ is disjoint from E_ξ whenever $\xi < \kappa$ and $H \in \mathcal{H}$. Accordingly $\bigcup_{\xi<\kappa} E_\xi$ is disjoint from $\bigcup \mathcal{H}$ and is nowhere dense. As $\langle E_\xi \rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{add}\,\mathcal{N}\mathfrak{wd}(X) \ge \mathrm{wdistr}(\mathrm{RO}(X))$.

(ii) If X is locally compact and Hausdorff, suppose that $\kappa < \mathrm{add}\,\mathcal{N}\mathfrak{wd}(X)$ and that $\langle \mathcal{G}_\xi \rangle_{\xi<\kappa}$ is a family of partitions of unity in $\mathrm{RO}(X)$. Then $E_\xi = X \setminus \bigcup \mathcal{G}_\xi$ is a nowhere dense closed set for each ξ (314P again). So $E = \bigcup_{\xi<\kappa} E_\xi$ is nowhere dense. Set

$$\mathcal{U} = \{U : U \subseteq X \text{ is open, } \overline{U} \subseteq X \setminus E \text{ is compact}\};$$

then $\mathcal{U}$ is an upwards-directed family with union $X \setminus \overline{E}$, so includes a partition $\mathcal{G}$ of unity. But if $H \in \mathcal{G}$ and $\xi < \kappa$, $\overline{H}$ is a compact set disjoint from E_ξ, so must be included in the union of some finite subfamily from $\mathcal{G}_\xi$, and $\{G : G \in \mathcal{G}_\xi,\, G \cap H \ne \emptyset\}$ is finite. As $\langle \mathcal{G}_\xi \rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{wdistr}(\mathrm{RO}(X)) \ge \mathrm{add}\,\mathcal{N}\mathfrak{wd}(X)$ and we have equality.

(d) If $Y \subseteq X$ is dense, and we write int_Y, $\overline{}^{(Y)}$ for interior and closure in the subspace topology of Y, we have

$$\mathrm{int}_Y \overline{G \cap Y}^{(Y)} = \mathrm{int}_Y(Y \cap \overline{G \cap Y}) = \mathrm{int}_Y(Y \cap \overline{G}) = Y \cap \mathrm{int}\,\overline{G}$$

for every open set $G \subseteq X$. Let $f : Y \to X$ be the identity map. Then f is continuous and $f^{-1}[M] = Y \cap M$ is nowhere dense in Y whenever $M \subseteq X$ is nowhere dense in X, so we have a corresponding Boolean homomorphism $\pi : \mathrm{RO}(X) \to \mathrm{RO}(Y)$ defined by setting

$$\pi G = \mathrm{int}_Y \overline{f^{-1}[G]}^{(Y)} = \mathrm{int}_Y \overline{G \cap Y}^{(Y)} = Y \cap \mathrm{int}\,\overline{G} = G \cap Y$$

for every $G \in \mathrm{RO}(X)$ (314Ra). Because Y is dense, $\pi G \ne \emptyset$ for every non-empty G, and π is injective. If $H \in \mathrm{RO}(Y) \setminus \{\emptyset\}$, then there is an open set $G \subseteq X$ such that $H = G \cap Y$, so that

$$\pi(\mathrm{int}\,\overline{G}) = Y \cap \mathrm{int}\,\overline{G} = \mathrm{int}_Y \overline{G \cap Y}^{(Y)} = H;$$

thus π is surjective and is an isomorphism.

514I Category algebras For many topological spaces, their regular open algebras can be understood better through their expressions as quotients of Baire-property algebras. It is time I brought this approach into the main line of the argument.

(a) Let X be a topological space, and $\mathcal{M}$ the σ-ideal of meager subsets of X. Recall that the Baire-property algebra of X is the σ-algebra $\widehat{\mathcal{B}} = \{G \triangle A : G \subseteq X \text{ is open, } A \in \mathcal{M}\}$, and that the category algebra of X is the quotient Boolean algebra $\mathfrak{G} = \widehat{\mathcal{B}}/\mathcal{M}$ (4A3Q). Note that if $G \subseteq X$ is any open set, then $\overline{G} \setminus G$ and $\overline{G} \setminus \mathrm{int}\,\overline{G}$ are nowhere dense, so

$$G^\bullet = \overline{G}^\bullet = (\mathrm{int}\,\overline{G})^\bullet$$

in $\mathfrak{G}$.

(b) For $G \in \mathrm{RO}(X)$, set $\pi G = G^\bullet \in \mathfrak{G}$. Then $\pi : \mathrm{RO}(X) \to \mathfrak{G}$ is an order-continuous surjective Boolean homomorphism. **P** (i) If G, $H \in \mathrm{RO}(X)$, then

$$G \cap_{\mathrm{RO}(X)} H = G \cap H, \quad X \setminus_{\mathrm{RO}(X)} G = X \setminus \overline{G},$$

(314P), so

$$(G \cap_{\mathrm{RO}(X)} H)^\bullet = (G \cap H)^\bullet = G^\bullet \cap H^\bullet,$$

$$(X \setminus_{\mathrm{RO}(X)} G)^\bullet = (X \setminus \overline{G})^\bullet = 1 \setminus \overline{G}^\bullet = 1 \setminus G^\bullet.$$

By 312H(ii), this is enough to show that π is a Boolean homomorphism. (ii) If $E \in \widehat{\mathcal{B}}$, let $G_0 \subseteq X$ be an open set such that $G_0 \triangle E \in \mathcal{M}$; then $G = \operatorname{int} \overline{G}_0$ belongs to $\mathrm{RO}(X)$ and

$$\pi G = G^\bullet = G_0^\bullet = E^\bullet.$$

Thus π is surjective. (iii) There is a regular open set W such that $X \setminus W$ is meager and every non-empty open subset of W is non-meager (4A3Ra); now the kernel of π is just $\{G : G \in \mathrm{RO}(X),\ G \cap W = \emptyset\}$ which has a largest member $\operatorname{int}(X \setminus W)$. This shows that the kernel of π is order-closed, so that π is order-continuous (313P(a-ii)). **Q**

(c) From the last part of the proof of (b), we see that the kernel of π is the principal ideal of $\mathrm{RO}(X)$ generated by $X \setminus \overline{W}$, so that in fact π includes an isomorphism between the complementary principal ideal generated by W and $\mathfrak{G}$.

In particular, being isomorphic to a principal ideal in the Dedekind complete Boolean algebra $\mathrm{RO}(X)$, $\mathfrak{G}$ is Dedekind complete (314Xd, 314Ea).

(d) It is useful to know that if $G \subseteq X$ is open, then the category algebra of G can be identified with the principal ideal of $\mathfrak{G}$ generated by $G^\bullet$; this is because a subset of G is nowhere dense regarded as a subset of G iff it is nowhere dense regarded as a subset of X, so that $\mathcal{M} \cap \mathcal{P}G$ is exactly the ideal of meager subsets of G for the subspace topology, while the Borel σ-algebra of G is $\{G \cap E : E \subseteq X \text{ is Borel}\}$ (4A3Ca).

(e) Recall from 431Fa that every $A \subseteq X$ has a Baire-property envelope, that is, a set $E \in \widehat{\mathcal{B}}$ such that $A \subseteq E$ and $E \setminus F$ is meager whenever $F \in \widehat{\mathcal{B}}$ and $A \subseteq F$. If $\langle A_n \rangle_{n\in\mathbb{N}}$ is any sequence of subsets of X, and E_n is a Baire-property envelope of A_n for each n, then $E = \bigcup_{n\in\mathbb{N}} E_n$ is a Baire-property envelope of $A = \bigcup_{n\in\mathbb{N}} A_n$. **P** Of course $A \subseteq E \in \widehat{\mathcal{B}}$. If $A \subseteq F \in \widehat{\mathcal{B}}$, then $A_n \subseteq F$ for every n, so $E_n \setminus F$ is meager for every n and $E \setminus F$ is meager. **Q**

If $A \subseteq X$, we can define $\psi(A) \in \mathfrak{G}$ by setting $\psi(A) = \inf\{F^\bullet : A \subseteq F \in \widehat{\mathcal{B}}\}$, because $\mathfrak{G}$ is Dedekind complete. Note that $\psi(A) = E^\bullet$ for any Baire-property envelope E of A. It follows that $\psi(\bigcup_{n\in\mathbb{N}} A_n) = \sup_{n\in\mathbb{N}} \psi(A_n)$ for any sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of subsets of X; also $\psi(A) = 0$ in $\mathfrak{G}$ iff A is meager.

(f) The construction here is most useful when X is a Baire space, so that no non-empty open set is meager, π is injective and is an isomorphism between $\mathrm{RO}(X)$ and $\mathfrak{G}$.

(g) If X is a zero-dimensional space, then the algebra $\mathcal{E}$ of open-and-closed sets in X is an order-dense subalgebra of $\mathrm{RO}(X)$, so that $\mathrm{RO}(X)$ can be identified with the Dedekind completion of $\mathcal{E}$; and if X is a zero-dimensional compact Hausdorff space, then the category algebra of X can equally be identified with the Dedekind completion of $\mathcal{E}$.

(h) Finally, I note that if X is an extremally disconnected compact Hausdorff space, so that its algebra $\mathcal{E}$ of open-and-closed sets is already Dedekind complete (314S), then $\mathcal{E} = \mathrm{RO}(X)$. So if X is the Stone space of a Dedekind complete Boolean algebra $\mathfrak{A}$, we have a Boolean isomorphism $a \mapsto \widehat{a}^\bullet$ from $\mathfrak{A}$ to $\mathfrak{G}$, writing $\widehat{a}$ for the open-and-closed subset of X corresponding to $a \in \mathfrak{A}$.

514J Now we have the following.

Proposition Let X be a topological space and $\mathfrak{C}$ its category algebra.

(a) $\mathrm{sat}(\mathfrak{C}) \le \mathrm{sat}(X)$, $c(\mathfrak{C}) \le c(X)$, $\pi(\mathfrak{C}) \le \pi(X)$ and $d(\mathfrak{C}) \le d(X)$.

(b) If X is a Baire space, $\mathrm{sat}(\mathfrak{C}) = \mathrm{sat}(X)$ and $c(\mathfrak{C}) = c(X)$.

(c) If X is regular, $\mathrm{wdistr}(\mathfrak{C}) \le \operatorname{add} \mathcal{N}\mathit{wd}(X)$, where $\mathcal{N}\mathit{wd}(X)$ is the ideal of nowhere dense subsets of X.

proof All we need to know is that $\mathfrak{C}$ is isomorphic to a principal ideal of $\mathrm{RO}(X)$, which is the whole of $\mathrm{RO}(X)$ if X is a Baire space (514Ic, 514If), and apply 514H and 514Ed.

514K Later in this volume, we shall see that the Lebesgue measure algebra, in particular, can have weak distributivity large compared with its cellularity and its Maharam type. For such algebras the following result gives us significant information.

Proposition Let $\mathfrak{A}$ be a Boolean algebra such that $\mathrm{sat}(\mathfrak{A}) \le \mathrm{wdistr}(\mathfrak{A})$. Then whenever $A \subseteq \mathfrak{A}$ and $\#(A) < \mathrm{wdistr}(\mathfrak{A})$ there is a set $C \subseteq \mathfrak{A}$ such that $\#(C) \le \max(c(\mathfrak{A}), \tau(\mathfrak{A}))$ and $a = \sup\{c : c \in C, c \subseteq a\}$ for every $a \in A$.

proof (a) If $\mathfrak{A}$ is finite we can take C to be the set of atoms of $\mathfrak{A}$; so let us henceforth suppose that $\mathfrak{A}$ is infinite. Let $D \subseteq \mathfrak{A}$ be a τ-generating set of cardinal $\tau(\mathfrak{A})$, and $\mathfrak{D}$ the subalgebra of $\mathfrak{A}$ generated by D, so that (because $\mathfrak{A}$ is infinite) $\#(\mathfrak{D}) = \tau(\mathfrak{A})$. For any $a \in \mathfrak{A}$, write

$$Q(a) = \{b : b \in \mathfrak{A},\ \exists\ d \in \mathfrak{D},\ (a \triangle d) \cap b = 0\},$$

$$\mathcal{E}(a) = \{B : B \text{ is a maximal antichain, } \sup B' \in Q(a) \text{ for every finite } B' \subseteq B\}.$$

Now the first fact to establish is that $\mathcal{E}(a) \ne \emptyset$ for any $a \in \mathfrak{A}$.

P Set $E = \{a : \mathcal{E}(a) \ne \emptyset\}$. Then $\mathfrak{D} \subseteq E$, because $1 \in Q(d)$ and $\{1\} \in \mathcal{E}(d)$ for every $d \in \mathfrak{D}$. If $a \in E$, then $Q(1 \setminus a) = Q(a)$ (because $1 \setminus d \in \mathfrak{D}$ for every $d \in \mathfrak{D}$), so $\mathcal{E}(1 \setminus a) = \mathcal{E}(a)$ is non-empty, and $1 \setminus a \in E$. If $F \subseteq E$ is non-empty and has supremum $a \in \mathfrak{A}$, then there is a non-empty set $F_0 \subseteq F$, still with supremum a, such that $\#(F_0) < \mathrm{sat}(\mathfrak{A})$ (514Db). For each $c \in F_0$ choose $B_c \in \mathcal{E}(c)$. Because $\#(F_0) < \mathrm{wdistr}(\mathfrak{A})$, there is a maximal antichain $B \subseteq \mathfrak{A}$ such that $\{e : e \in B_c,\ e \cap b \ne 0\}$ is finite for every $c \in F_0$. If $B' \subseteq B$ is finite and $c \in F_0$, then $\sup B' \subseteq \sup B'_c$ where $B'_c = \{e : e \in B_c,\ e \cap \sup B' \ne 0\}$, so $\sup B' \in Q(c)$. Set

$$\tilde{D} = \{b : \text{there are } b' \in B \text{ and } c \in F_0 \text{ such that } b \subseteq b' \setminus (a \setminus c)\}.$$

Because $\sup F_0 = a$ and $\sup B = 1$, $\sup \tilde{D} = 1$ and there is a maximal antichain $\tilde{B} \subseteq \tilde{D}$. If $B' \subseteq \tilde{B}$ is finite, with supremum b^*, there are $c_0, \dots, c_n \in F_0$ such that b^* is disjoint from $a \setminus \sup_{i \le n} c_i$; also $b^* \in Q(c_i)$ for each i. So we can find $d_i \in \mathfrak{D}$ such that $c_i \triangle d_i$ is disjoint from b^* for each $i \le n$; accordingly $c \triangle d$ is disjoint from b^*, where $c = \sup_{i \le n} c_i$ and $d = \sup_{i \le n} d_i$, and

$$a \triangle d \subseteq (a \triangle c) \cup (c \triangle d) \subseteq (a \setminus c) \cup (c \triangle d) \subseteq 1 \setminus b^*,$$

while $d \in \mathfrak{D}$. This shows that $b^* \in Q(a)$; as B' is arbitrary, $\tilde{B} \in \mathcal{E}(a)$ and $a \in E$.

This shows that E is closed under complements and arbitrary suprema. It is therefore an order-closed subalgebra of $\mathfrak{A}$ (312B(iii), 313E(a-i)); since it includes $\mathfrak{D}$, it is the whole of $\mathfrak{A}$, which is what we need to know. **Q**

(b) Now turn to the given set A. For each $a \in A$ choose $B_a \in \mathcal{E}(a)$. Then there is a maximal antichain B such that $\{e : e \in B_a,\ e \cap b \ne 0\}$ is finite for every $b \in B$ and $a \in A$. Of course $\#(B) < \mathrm{sat}(\mathfrak{A})$. Set $C = \{d \cap b : d \in \mathfrak{D},\ b \in B\}$. Then

$$\#(C) \le \max(\omega, \#(B), \#(\mathfrak{D})) \le \max(c(\mathfrak{A}), \tau(\mathfrak{A})).$$

? Suppose that $a \in A$ is not the supremum of $C' = \{c : c \in C,\ c \subseteq a\}$. Let $a' \subseteq a$ be non-zero and disjoint from every member of C'. Then there is a $b \in B$ such that $b \cap a' \ne 0$. As b is covered by finitely many members of B_a it belongs to $Q(a)$, and there is a $d \in D$ such that $(a \triangle d) \cap b = 0$; which means that

$$0 \ne a' \cap b \subseteq a \cap b = d \cap b,$$

while $d \cap b \in C$. Thus $d \cap b \in C'$; but a' is supposed to be disjoint from every member of C'. **X**

Thus C has the properties we need.

514L The regular open algebra of a pre-ordered set Many important features of pre-ordered sets, at least in those aspects which are of concern to us here, can be related to the regular open algebras of suitable topologies.

Definitions (a) For any pre-ordered set P, a subset G of P is **up-open** if $[p, \infty[\subseteq G$ whenever $p \in G$. The family of such sets is a topology on P, the **up-topology**. Similarly, the **down-topology** of P is the family of **down-open** sets H such that $p \le q \in H \Rightarrow p \in H$. Note that $G \subseteq P$ is up-open iff it is closed for the down-topology, and vice versa. In particular, the intersection of any non-empty family of up-open sets is again up-open..

(b) I will write $\mathrm{RO}^{\uparrow}(P)$ for the regular open algebra of P when P is given its up-topology, and $\mathrm{RO}^{\downarrow}(P)$ for the regular open algebra of P when P is given its down-topology.

514M These up- and down-topologies, entirely unrelated to the usual 'order topology' on a totally ordered set (4A2A) and the ideas of order-convergence considered in Volume 3, take a bit of getting used to. Their characteristic property is that every point p has a smallest neighbourhood $[p,\infty[$; see 514Xj. I begin with an elementary lemma for practice.

Lemma Let P be a pre-ordered set endowed with its up-topology.

(a)(i) For any $A\subseteq P$, $\overline{A}=\{p:A\cap[p,\infty[\ne\emptyset\}$.

(ii) For any $p\in P$, $\overline{[p,\infty[}$ is the set of elements of P which are compatible upwards with p.

(iii) For any p, $q\in P$, the following are equiveridical: (α) $q\in\operatorname{int}\overline{[p,\infty[}$; ($\beta$) every member of $[q,\infty[$ is compatible upwards with p; (γ) q is incompatible upwards with every $r\in P$ which is incompatible upwards with p.

(b) A subset of P is dense iff it is cofinal.

(c) If Q is another pre-ordered set with its up-topology, a function $f:P\to Q$ is continuous iff it is order-preserving.

(d)(i) A subset G of P is a regular open set iff

$$G=\{p:G\cap[q,\infty[\ne\emptyset\text{ for every }q\ge p\}.$$

(ii) If $\mathcal{G}$ is a non-empty family of regular open subsets of P, then $\bigcap\mathcal{G}$ is a regular open subset of P, and is $\inf\mathcal{G}$ in the regular open algebra $\mathrm{RO}^{\uparrow}(P)$.

(e) P is separative upwards iff all the sets $[p,\infty[$ are regular open sets.

(f) If P is separative upwards and $A\subseteq P$ has a supremum p in P, then $[p,\infty[=\inf_{q\in A}[q,\infty[$ in $\mathrm{RO}^{\uparrow}(P)$.

proof (a) For (i), we need only note that $[p,\infty[$ is the smallest open set containing p. Now (ii) amounts to a restatement of the definition of 'compatible upwards'. As for (iii),

$$\begin{aligned}q\in\operatorname{int}\overline{[p,\infty[}&\iff[q,\infty[\subseteq\overline{[p,\infty[}\\&\iff[q',\infty[\cap[p,\infty[\ne\emptyset\text{ for every }q'\ge q\end{aligned}$$

(by (i))

$$\iff[q,\infty[\cap[r,\infty[=\emptyset\text{ whenever }[r,\infty[\cap[p,\infty[=\emptyset$$

because

$$P\setminus\overline{[p,\infty[}=\bigcup\{[r,\infty[:[r,\infty[\cap[p,\infty[=\emptyset\}.$$

(b) $\mathcal{U}=\{[p,\infty[:p\in P\}$ is a base for the up-topology, so a subset of P is dense iff it meets every member of $\mathcal{U}$; but this is the same thing as saying that it is cofinal.

(c) If f is order-preserving and $H\subseteq Q$ is up-open, then

$$p'\ge p\in f^{-1}[H]\Longrightarrow f(p')\ge f(p)\in H\Longrightarrow f(p')\in H,$$

so $f^{-1}[H]$ is up-open; as H is arbitrary, f is continuous. If f is continuous and $p\le p'$ in P, then $H=[f(p),\infty[$ is up-open, so $f^{-1}[H]$ is up-open and must contain p', that is, $f(p')\ge f(p)$; as p and p' are arbitrary, f is order-preserving.

(d)(i) For any set $A\subseteq P$,

$$\{p:A\cap[q,\infty[\ne\emptyset\text{ for every }q\ge p\}=\{p:[p,\infty[\subseteq\overline{A}\}=\operatorname{int}\overline{A}$$

(using (a)).

(ii) As noted in 514L, $\bigcap\mathcal{G}$ is open, so is equal to its interior; but 314P tells us that $\operatorname{int}\bigcap\mathcal{G}$ is $\inf\mathcal{G}$ in $\mathrm{RO}(P)$.

(e)

$$\begin{aligned}&P\text{ is separative upwards}\\&\iff\forall\ p,\,q\in P,\text{ either }p\le q\text{ or }\exists\ r,\,r\ge q,\ [r,\infty[\cap[p,\infty[=\emptyset\end{aligned}$$

(511Bk)

$$\iff\forall\ p,\,q\in P,\text{ either }q\in[p,\infty[\text{ or }q\notin\operatorname{int}\overline{[p,\infty[}$$

((a-iii) above)

$$\begin{aligned}&\iff\forall\ p\in P,\ \operatorname{int}\overline{[p,\infty[}\subseteq[p,\infty[\\&\iff\forall\ p\in P,\ [p,\infty[\text{ is a regular open set.}\end{aligned}$$

(f) $[p,\infty[$ is actually the intersection $\bigcap_{q\in A}[q,\infty[$.

514N Proposition Let $(P,\le)$ be a pre-ordered set, and write $\mathfrak{T}^\uparrow$ for the up-topology of P and $\mathrm{RO}^\uparrow(P)$ for the regular open algebra of $(P,\mathfrak{T}^\uparrow)$.

(a) $(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+)\preccurlyeq_{\mathrm{GT}}(\mathfrak{T}^\uparrow\setminus\{\emptyset\},\supseteq,\mathfrak{T}^\uparrow\setminus\{\emptyset\})\equiv_{\mathrm{GT}}(P,\le,P)$. If P is separative upwards, then $(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+)\equiv_{\mathrm{GT}}(P,\le,P)$.

(b) $\pi(\mathrm{RO}^\uparrow(P))\le\pi(P,\mathfrak{T}^\uparrow)=d(P,\mathfrak{T}^\uparrow)=\mathrm{cf}\,P$. If P is separative upwards, then we have equality.

(c) $\mathrm{sat}^\uparrow(P,\le)=\mathrm{sat}(P,\mathfrak{T}^\uparrow)=\mathrm{sat}(\mathrm{RO}^\uparrow(P))$ and $c^\uparrow(P,\le)=c(P,\mathfrak{T}^\uparrow)=c(\mathrm{RO}^\uparrow(P))$.

(d) For any cardinal κ,

$$\mathrm{link}_{<\kappa}(\mathrm{RO}^\uparrow(P))\le\mathrm{link}^\uparrow_{<\kappa}(P,\le),$$

with equality if *either* P is separative upwards *or* $\kappa\le\omega$. In particular, we always have

$$\mathrm{link}^\uparrow(P,\le)=\mathrm{link}(\mathrm{RO}^\uparrow(P)),\quad d^\uparrow(P,\le)=d(\mathrm{RO}^\uparrow(P)).$$

(e) If $Q\subseteq P$ is cofinal, then $\mathrm{RO}^\uparrow(Q)\cong\mathrm{RO}^\uparrow(P)$.

(f) If $A\subseteq P$ is a maximal up-antichain, then $\mathrm{RO}^\uparrow(P)\cong\prod_{a\in A}\mathrm{RO}^\uparrow([a,\infty[)$.

(g) If $\tilde P$ is the partially ordered set of equivalence classes associated with P, then $\mathrm{RO}^\uparrow(\tilde P)\cong\mathrm{RO}^\uparrow(P)$.

proof (a) By 514Ha,

$$(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+)\preccurlyeq_{\mathrm{GT}}(\mathfrak{T}^\uparrow\setminus\{\emptyset\},\supseteq,\mathfrak{T}^\uparrow\setminus\{\emptyset\}).$$

Next, observe that $\mathcal{U}=\{[p,\infty[\,:p\in P\}$ is a base for $\mathfrak{T}^\uparrow$, so that

$$(\mathfrak{T}^\uparrow\setminus\{\emptyset\},\supseteq,\mathfrak{T}^\uparrow\setminus\{\emptyset\})\equiv_{\mathrm{GT}}(\mathcal{U},\supseteq,\mathcal{U})$$

by 513Ed (inverted, as usual). If we set $\phi(p)=[p,\infty[$ for $p\in P$, and choose $\psi(U)\in P$ such that $U=[\psi(U),\infty[$ for $U\in\mathcal{U}$, then (ϕ,ψ) is a Galois-Tukey connection from $(P,\le,P)$ to $(\mathcal{U},\supseteq,\mathcal{U})$, while (ψ,ϕ) is a Galois-Tukey connection in the reverse direction; so $(P,\le)\equiv_{\mathrm{GT}}(\mathcal{U},\supseteq)$.

If P is separative upwards, then $\mathcal{U}$ is included in $\mathrm{RO}^\uparrow(P)$ (514Me) and is coinitial with $\mathrm{RO}^\uparrow(P)^+$, so

$$(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+)\equiv_{\mathrm{GT}}(\mathcal{U},\supseteq,\mathcal{U})\equiv_{\mathrm{GT}}(P,\le,P).$$

(b) Now

$$\pi(\mathrm{RO}^\uparrow(P))\le\pi(P,\mathfrak{T}^\uparrow)$$

(514H(b-i))

$$=\mathrm{ci}(\mathfrak{T}^\uparrow\setminus\{\emptyset\})=\mathrm{ci}\,\mathcal{U}=\mathrm{cf}\,P,$$

defining $\mathcal{U}$ as in (a) above. By 514Mb, $\mathrm{cf}\,P=d(P,\mathfrak{T}^\uparrow)$. If P is separative upwards, then $\pi(\mathrm{RO}^\uparrow(P))=\mathrm{cf}\,P$ because $(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+)\equiv_{\mathrm{GT}}(P,\le,P)$.

(c) Similarly, again using 514H(b-i), and with 512Dc at the last step,

$$\mathrm{sat}(\mathrm{RO}^\uparrow(P))=\mathrm{sat}(P,\mathfrak{T}^\uparrow)=\mathrm{sat}^\downarrow(\mathfrak{T}^\uparrow\setminus\{\emptyset\})=\mathrm{sat}^\downarrow(\mathcal{U})=\mathrm{sat}^\uparrow(P).$$

Now we saw in 514Da and 513Bc that cellularity is determined by saturation both for partially ordered sets and for Boolean algebras, so $c(\mathrm{RO}^\uparrow(P))=c^\uparrow(P)$. (Of course this is easily shown by a direct argument.)

(d) Using (a) and 512Dd, we see that

$$\begin{aligned}\mathrm{link}_{<\kappa}(\mathrm{RO}^\uparrow(P))&=\mathrm{link}_{<\kappa}(\mathrm{RO}^\uparrow(P)^+,\supseteq,\mathrm{RO}^\uparrow(P)^+)\\&\le\mathrm{link}_{<\kappa}(P,\le,P)=\mathrm{link}^\uparrow_{<\kappa}(P,\le),\end{aligned}$$

with equality if P is separative upwards. For other P, if $\kappa\le\omega$, set $\lambda=\mathrm{link}_{<\kappa}(\mathrm{RO}^\uparrow(P))$ and let $\langle\mathcal{H}_\xi\rangle_{\xi<\lambda}$ be a cover of $\mathrm{RO}^\uparrow(P)^+$ by $<\kappa$-linked sets. Set $A_\xi=\{p:\mathrm{int}\,\overline{[p,\infty[}\in\mathcal{H}_\xi\}$ for each $\xi<\kappa$. Then any A_ξ is upwards-$<\kappa$-linked in P. **P?** Otherwise, there is an $I\in[A_\xi]^{<\kappa}$ which has no upper bound in P, that is, $\bigcap_{p\in I}[p,\infty[\,=\emptyset$. Now

$$\bigcap_{p\in I}\mathrm{int}\,\overline{[p,\infty[}\subseteq\bigcup_{i\in I}(\overline{[p,\infty[}\setminus[p,\infty[)$$

is an open set covered by finitely many nowhere dense sets and is therefore empty, so we have a finite subset of $\mathcal{H}_\xi$ with empty intersection. **XQ** So $\langle A_\xi\rangle_{\xi<\lambda}$ witnesses that $\mathrm{link}^\uparrow_{<\kappa}(P,\le)\le\lambda$ and again we have equality. In particular,

$$\mathrm{link}(\mathrm{RO}^{\uparrow}(P)) = \mathrm{link}_{<3}(\mathrm{RO}^{\uparrow}(P)) = \mathrm{link}^{\uparrow}_{<3}(P,\le) = \mathrm{link}^{\uparrow}(P,\le),$$

$$d(\mathrm{RO}^{\uparrow}(P)) = \mathrm{link}_{<\omega}(\mathrm{RO}^{\uparrow}(P)) = \mathrm{link}^{\uparrow}_{<\omega}(P,\le) = d^{\uparrow}(P,\le).$$

(e) Put 514Mb and 514Hd together.

(f) Because A is an up-antichain, $\langle [a,\infty[\,\rangle_{a\in A}$ is a disjoint family of open sets in P; because A is maximal, $\bigcup_{a\in A}[a,\infty[$ is cofinal, therefore dense. So 315H gives the result.

(g) Let $Q \subseteq P$ be a set meeting each equivalence class in just one point, so that $q \mapsto q^{\bullet} : Q \to \tilde{P}$ is a bijection. Then Q is cofinal with P, while with its subspace ordering Q is isomorphic to $\tilde{P}$. So

$$\mathrm{RO}^{\uparrow}(\tilde{P}) \cong \mathrm{RO}^{\uparrow}(Q) \cong \mathrm{RO}^{\uparrow}(P)$$

by (e).

514O Of course we very much want to be able to recognise cases in which two partially ordered sets have isomorphic regular open algebras; and it is also important to know when one $\mathrm{RO}^{\uparrow}(P)$ can be regularly embedded in another. The next four results give some of the known sufficient conditions for these.

Proposition Suppose that P and Q are pre-ordered sets and $f : P \to Q$ is an order-preserving function such that $f^{-1}[Q_0]$ is cofinal with P for every up-open cofinal $Q_0 \subseteq Q$. Then there is an order-continuous Boolean homomorphism $\pi : \mathrm{RO}^{\uparrow}(Q) \to \mathrm{RO}^{\uparrow}(P)$ defined by setting $\pi H = \mathrm{int}\,\overline{f^{-1}[H]}$ (taking the closure and interior with respect to the up-topology on P) for every $H \in \mathrm{RO}^{\uparrow}(Q)$. If $f[P]$ is cofinal with Q then π is injective, so is a regular embedding of $\mathrm{RO}^{\uparrow}(Q)$ in $\mathrm{RO}^{\uparrow}(P)$.

proof By 514Mc, f is continuous for the up-topologies. Moreover, $f^{-1}[M]$ is nowhere dense in P whenever $M \subseteq Q$ is nowhere dense in Q. **P** $Q_0 = Q \setminus \overline{M}$ is up-open and dense, therefore cofinal (514Mb), so $f^{-1}[Q_0]$ is up-open and dense, and $f^{-1}[M] \subseteq P \setminus f^{-1}[Q_0]$ is nowhere dense. **Q**

By 314Ra again, there is an order-continuous Boolean homomorphism $\pi : \mathrm{RO}^{\uparrow}(Q) \to \mathrm{RO}^{\uparrow}(P)$ defined by setting $\pi H = \mathrm{int}\,\overline{f^{-1}[H]}$ for every $H \in \mathrm{RO}^{\uparrow}(Q)$. Now

$$\begin{aligned} f[P] \text{ is cofinal } &\iff f[P] \text{ is dense}\\ &\implies f[P]\cap H \ne \emptyset \text{ for every } H \in \mathrm{RO}^{\uparrow}(Q)\setminus\{\emptyset\}\\ &\iff f^{-1}[H] \ne \emptyset \text{ for every } H \in \mathrm{RO}^{\uparrow}(Q)\setminus\{\emptyset\}\\ &\iff \pi H \ne \emptyset \text{ for every } H \in \mathrm{RO}^{\uparrow}(Q)\setminus\{\emptyset\} \iff \pi \text{ is injective.}\end{aligned}$$

So in this case π is a regular embedding of $\mathrm{RO}^{\uparrow}(Q)$ in $\mathrm{RO}^{\uparrow}(P)$.

514P Corollary Suppose that P and Q are pre-ordered sets, that $f : P \to Q$ is an order-preserving function and whenever $p \in P$, $q \in Q$ and $f(p) \le q$, there is a $p' \ge p$ such that $f(p') \ge q$. If $f[P]$ is *either* cofinal with Q *or* coinitial with Q, then $\mathrm{RO}^{\uparrow}(Q)$ can be regularly embedded in $\mathrm{RO}^{\uparrow}(P)$.

proof If $Q_0 \subseteq Q$ is up-open and cofinal, then $f^{-1}[Q_0]$ is cofinal with P. **P** Take any $p \in P$. Then there are a $q \in Q_0$ such that $q \ge f(p)$ and a $p' \ge p$ such that $f(p') \ge q$; as Q_0 is up-open, $p' \in f^{-1}[Q_0]$; as p is arbitrary, $f^{-1}[Q_0]$ is cofinal. **Q** So if $f[P]$ is cofinal with Q, we can use 514O. On the other hand, if $f[P]$ is coinitial with Q it is also cofinal with Q. **P** For $q \in Q$ there is a $p \in P$ such that $f(p) \le q$; now our main hypothesis tells us that there is a $p' \in P$ such that $f(p') \ge q$. **Q** So we have the result in this case also.

514Q Proposition Let P and Q be pre-ordered sets, endowed with their up-topologies, and $f : P \to Q$ a function such that

> whenever $A \subseteq P$ is a maximal up-antichain then $f{\restriction}A$ is injective and $f[A]$ is a maximal up-antichain in Q.

Then there is an injective order-continuous Boolean homomorphism $\pi : \mathrm{RO}^{\uparrow}(P) \to \mathrm{RO}^{\uparrow}(Q)$ defined by setting $\pi(\mathrm{int}\,\overline{[p,\infty[}) = \mathrm{int}\,\overline{[f(p),\infty[}$ for every $p \in P$. In particular, $\mathrm{RO}^{\uparrow}(P)$ can be regularly embedded in $\mathrm{RO}^{\uparrow}(Q)$. If $f[P]$ is cofinal with Q, then π is an isomorphism.

proof (a) For $p \in P$, set $H_p = \mathrm{int}\,\overline{[f(p),\infty[} \in \mathrm{RO}^{\uparrow}(Q)$. If $A \subseteq P$ is a maximal up-antichain, $\langle [f(p),\infty[\,\rangle_{p\in A}$ is a disjoint family of up-open subsets of Q with dense union, so $\langle H_p\rangle_{p\in A}$ is a partition of unity in $\mathrm{RO}^{\uparrow}(Q)$. It follows that $\langle H_p\rangle_{p\in A}$ must be disjoint for every up-antichain $A\subseteq P$. Moreover, if $p_0\in P$ and $p_1\in\operatorname{int}\overline{[p_0,\infty[}$ in P, we have a maximal up-antichain A containing p_0, and $A'=(A\setminus\{p_0\})\cup\{p_1\}$ is an up-antichain; as $H_{p_1}\cap\bigcup_{p\in A,p\ne p_0}H_p=\emptyset$, H_{p_1} must be included in H_{p_0}.

(b) For $G\in\mathrm{RO}^\uparrow(P)$, set $\pi G=\sup\{H_p:p\in G\}$, the supremum being taken in $\mathrm{RO}^\uparrow(Q)$. If G, $G'\in\mathrm{RO}^\uparrow(P)$ are disjoint, then p and p' are incompatible upwards, so H_p and $H_{p'}$ are disjoint, whenever $p\in G$ and $p'\in G'$; accordingly πG and $\pi G'$ must be disjoint.

(c) If $p\in P$, then of course $H_p\subseteq\pi(\operatorname{int}\overline{[p,\infty[})$. On the other hand, if $p'\in\operatorname{int}\overline{[p,\infty[}$, then we saw in (a) that $H_{p'}\subseteq H_p$, so that $\pi(\operatorname{int}\overline{[p,\infty[})$ must be exactly H_p.

(d) If $\mathcal{G}\subseteq\mathrm{RO}^\uparrow(P)$ has supremum G_0 in $\mathrm{RO}^\uparrow(P)$, $\pi G_0=\sup_{G\in\mathcal{G}}\pi G$ in $\mathrm{RO}^\uparrow(Q)$. **P** Of course $\pi G_0\supseteq\pi G$ for every $G\in\mathcal{G}$. Let A be maximal among the up-antichains included in $\bigcup\mathcal{G}$, and extend A to a maximal up-antichain $A'\subseteq P$. Then $\langle H_p\rangle_{p\in A'}$ is a partition of unity in $\mathrm{RO}^\uparrow(Q)$, so $H=\sup_{p\in A}H_p$ and $H'=\sup_{p\in A'\setminus A}H_p$ are complementary elements of $\mathrm{RO}^\uparrow(Q)$. For every $p\in A$ there is a $G\in\mathcal{G}$ with $p\in G$, so that $H_p\subseteq\pi G$; accordingly $H\subseteq\sup_{G\in\mathcal{G}}\pi G$. On the other hand, take any $p\in G_0$. By the maximality of A, $G\cap[p',\infty[\,=\emptyset$ for every $p'\in A'\setminus A$ and $G\in\mathcal{G}$, so $[p,\infty[\,\cap[p',\infty[\,\subseteq G_0\cap[p',\infty[\,=\emptyset$ for every $p'\in A'\setminus A$ and $H_p\cap H_{p'}=\emptyset$ for every $p'\in A'\setminus A$, that is, $H_p\cap H'=\emptyset$ and $H_p\subseteq H$. As p is arbitrary,

$$\pi G_0\subseteq H\subseteq\sup\nolimits_{G\in\mathcal{G}}\pi G\subseteq\pi G_0$$

and we have equality. **Q**

(e) Now we see that

$$\pi\emptyset=\emptyset,$$

$$\pi P=Q$$

(because if we take any maximal up-antichain $A\subseteq P$, πP includes $\sup_{p\in A}H_p$),

$$\pi G\cap\pi H=\emptyset \text{ whenever } G,\,H\in\mathrm{RO}^\uparrow(P) \text{ and } G\cap H=\emptyset,$$

$$\pi(\sup\mathcal{G})=\sup\pi[\mathcal{G}] \text{ for every } \mathcal{G}\subseteq\mathrm{RO}^\uparrow(P).$$

By 312H(iv), π is a Boolean homomorphism, and by 313L(b-iv) it is order-continuous. Finally, $\pi G\ne\emptyset$ whenever $G\in\mathrm{RO}^\uparrow(P)\setminus\{\emptyset\}$, so π is injective and is a regular embedding.

(f) If $f[P]$ is cofinal with Q, then $\pi[\mathrm{RO}^\uparrow(P)]$ is order-dense in $\mathrm{RO}^\uparrow(Q)$. **P** Let $H\in\mathrm{RO}^\uparrow(Q)$ be non-empty. As $f[P]$ is dense, there is a $p\in P$ such that $f(p)\in H$. Now

$$\emptyset\ne\pi(\operatorname{int}\overline{[p,\infty[})=\operatorname{int}\overline{[f(p),\infty[}\subseteq\operatorname{int}\overline{H}=H;$$

as H is arbitrary, we have the result. **Q** By 314Ia, π is an isomorphism. This completes the proof.

514R Corollary Let P and Q be pre-ordered sets. Suppose that there is a function $f:P\to Q$ such that $f[P]$ is cofinal with Q and, for p, $p'\in P$, p and p' are compatible upwards in P iff $f(p)$ and $f(p')$ are compatible upwards in Q. Then $\mathrm{RO}^\uparrow(P)\cong\mathrm{RO}^\uparrow(Q)$.

proof The point is that f satisfies the condition of 514Q. **P** Suppose that $A\subseteq P$ is a maximal up-antichain. If p, p' are distinct elements of A, then p and p' are incompatible upwards in P, so $f(p)$ and $f(p')$ are incompatible upwards in Q. This shows simultaneously that $f\restriction A$ is injective and that $f[A]$ is an up-antichain in Q. If q is any element of Q, there is a $p\in P$ such that $f(p)\ge q$; now there must be a $p'\in A$ such that p' is compatible upwards with p, in which case $f(p')$ is compatible upwards with $f(p)$ and therefore with q. So $f[A]$ is maximal; as A is arbitrary, we have the result. **Q**

So 514Q gives the result.

514S Proposition (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and P a pre-ordered set. Suppose that we have a function $f:P\to\mathfrak{A}^+$ such that, for $p,q\in P$,

$$f(p)\subseteq f(q) \text{ whenever } p\le q,$$

$$f(p)\cap f(q)=0 \text{ whenever } p \text{ and } q \text{ are incompatible downwards in } P,$$

$$f[P] \text{ is order-dense in } \mathfrak{A}.$$

Then $\mathrm{RO}^{\downarrow}(P) \cong \mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $D \subseteq \mathfrak{A}$ an order-dense set not containing 0. Give D the ordering $\subseteq$, and write $\mathrm{RO}^{\downarrow}(D)$ for the regular open algebra of D with its down-topology. Then $\mathrm{RO}^{\downarrow}(D) \cong \mathfrak{A}$.

(c) Let $(X, \mathfrak{T})$ be a topological space and P a pre-ordered set. Suppose we have a function $g : P \to \mathfrak{T} \setminus \{\emptyset\}$ such that, for p, $q \in P$,

$$g(p) \subseteq g(q) \text{ whenever } p \le q,$$

$$g(p) \cap g(q) = \emptyset \text{ whenever } p \text{ and } q \text{ are incompatible downwards in } P,$$

$$g[P] \text{ is a } \pi\text{-base for } \mathfrak{T}.$$

Then $\mathrm{RO}^{\downarrow}(P) \cong \mathrm{RO}(X)$.

(d) Let $(X, \mathfrak{T})$ be a topological space and $\mathcal{U}$ a π-base for the topology of X not containing $\{\emptyset\}$. Give $\mathcal{U}$ the ordering $\subseteq$. Then $\mathrm{RO}^{\downarrow}(\mathcal{U}) \cong \mathrm{RO}(X)$.

proof (a)(i) The key is the following fact: if $p \in P$, $a \in \mathfrak{A}$ and $a \cap f(p) \ne 0$, then there is a $q \le p$ such that $f(q) \subseteq a$. **P** There is a $q_0 \in P$ such that $f(q_0) \subseteq a \cap f(p)$. Now q_0 and p cannot be incompatible downwards, so there is a $q \in \left]-\infty, q_0\right] \cap \left]-\infty, p\right]$, and in this case $f(q) \subseteq f(q_0) \subseteq a$. **Q**

(ii) For $G \in \mathrm{RO}^{\downarrow}(P)$, set $\pi G = \sup f[G]$ in $\mathfrak{A}$. Then $\pi : \mathrm{RO}^{\downarrow}(P) \to \mathfrak{A}$ is order-preserving. Of course $\pi(\emptyset) = 0$.

$\pi(G \cap H) = \pi G \cap \pi H$ for all G, $H \in \mathrm{RO}^{\downarrow}(P)$. **P** Because π is order-preserving, $\pi(G \cap H) \subseteq \pi G \cap \pi H$. **?** If $a = \pi G \cap \pi H \setminus \pi(G \cap H) \ne 0$, take $p \in G$ such that $a \cap f(p) \ne 0$; then there is a $p' \le p$ such that $f(p') \subseteq a$. Next, there must be a $q \in H$ such that $f(q) \cap f(p') \ne 0$, and a $q' \le q$ such that $f(q') \subseteq f(p')$. But now $q' \in \left]-\infty, p\right] \cap \left]-\infty, q\right] \subseteq G \cap H$, so $f(q') \subseteq \pi(G \cap H)$; while at the same time $f(q') \subseteq a$. **X** Thus $\pi(G \cap H) = \pi G \cap \pi H$. **Q**

$\pi(P \setminus \overline{G}) = 1 \setminus \pi G$ for every $G \in \mathrm{RO}^{\downarrow}(P)$. **P** (Perhaps I should say that $\overline{G}$ here is the closure of G for the down-topology of P.) Set $H = P \setminus \overline{G}$. Then $\pi G \cap \pi H = \pi(G \cap H) = 0$ by what we have just seen. **?** If $a = 1 \setminus (\pi G \cup \pi H)$ is non-zero, let $p_0 \in P$ be such that $f(p_0) \subseteq a$. Then $\left]-\infty, p_0\right]$ is a non-empty open set so must meet one of G, H. But if $p \in G \cup H$ then $f(p_0) \cap f(p) = 0$ so p_0 and p are incompatible downwards and, in particular, $p \not\le p_0$. **XQ**

So π is a Boolean homomorphism, and it is injective because $\pi G \supseteq f(p) \ne 0$ whenever $p \in G \in \mathrm{RO}^{\downarrow}(P)$. Finally, π is surjective. **P** If $a \in \mathfrak{A}$, set $G = \{p : f(p) \subseteq a\}$. Then G is down-open. If $q \notin G$, $f(q) \setminus a \ne 0$, so there is a $q_1 \le q$ such that $f(q_1) \cap a = 0$ and $\left]-\infty, q_1\right]$ does not meet G; accordingly $\left]-\infty, q\right] \not\subseteq \overline{G}$ and $q \notin \operatorname{int} \overline{G}$. So $G \in \mathrm{RO}^{\downarrow}(P)$. Because $f[P]$ is order-dense, $a = \sup f[G] = \pi G$ belongs to $\pi[\mathrm{RO}^{\downarrow}(P)]$. **Q**

Thus we have an isomorphism between $\mathrm{RO}^{\downarrow}(P)$ and $\mathfrak{A}$.

(b) Apply (a) to the identity map from D to $\mathfrak{A}$.

(c) Apply (a) to the map $p \mapsto \operatorname{int} \overline{g(p)} : P \to \mathrm{RO}(X)$.

(d) Apply (c) to the identity function from $\mathcal{U}$ to $\mathfrak{T}$.

514T Finite-support products At many points in this chapter we find ourselves seeking to relate partially ordered sets to Boolean algebras and topological spaces. In 511D and 512Eb I sought to describe the cardinal functions of topological spaces and Boolean algebras in terms of naturally associated partially ordered sets, and in 514L and 514N of this section I described constructions of topologies and Boolean algebras from partial orders. One of the most important constructions of general topology is that of 'product'. The matching construction in Boolean algebra is that of 'free product' (315I). I now come to the corresponding idea for partial orders.

Definition Let $\langle P_i \rangle_{i \in I}$ be a family of non-empty partially ordered sets. The **upwards finite-support product** $\bigotimes^{\uparrow}_{i \in I} P_i$ of $\langle P_i \rangle_{i \in I}$ is the set $\bigcup \{\prod_{i \in J} P_i : J \in [I]^{<\omega}\}$, ordered by saying that $p \le q$ iff $\operatorname{dom} p \subseteq \operatorname{dom} q$ and $p(i) \le q(i)$ for every $i \in \operatorname{dom} p$. Similarly, the **downwards finite-support product** $\bigotimes^{\downarrow}_{i \in I} P_i$ of $\langle P_i \rangle_{i \in I}$ is the same set $\bigcup \{\prod_{i \in J} P_i : J \in [I]^{<\omega}\}$, but ordered by saying that $p \le q$ iff $\operatorname{dom} q \subseteq \operatorname{dom} p$ and $p(i) \le q(i)$ for every $i \in \operatorname{dom} q$.

514U Proposition Let $\langle P_i \rangle_{i \in I}$ be a family of non-empty partially ordered sets, with upwards finite-support product $P = \bigotimes^{\uparrow}_{i \in I} P_i$.

(a) The regular open algebra $\mathrm{RO}^{\uparrow}(P)$ is isomorphic to the regular open algebra of $P^* = \prod_{i \in I} P_i$ when every P_i is given its up-topology.

(b) If I is finite, P^* is a cofinal subset of P, and the ordering of P^*, regarded as a subset of P, is the usual product partial order on P^*.

(c) If $Q_i \subseteq P_i$ is cofinal for each $i \in I$, then $\bigcup_{J\in[I]^{<\omega}} \prod_{i\in J} Q_i$ is cofinal with P. So $\operatorname{cf} P \le \max(\omega, \#(I), \sup_{i\in I} \operatorname{cf} P_i)$.

(d) $c^{\uparrow}(P) = \sup_{J\in[I]^{<\omega}} c^{\uparrow}(\prod_{i\in J} P_i)$.

proof (a) For $p \in P$, set

$$G_p = \{q : q \in P^*,\, q(i) \ge p(i) \text{ for every } i \in \operatorname{dom} p\}.$$

Then G_p is a non-empty open set in P^*. If $p \le p'$ in P, then $G_p \supseteq G_{p'}$. If p, $p' \in P$ are incompatible upwards in P, there must be an $i \in \operatorname{dom} p \cap \operatorname{dom} p'$ such that $p(i)$ and $p'(i)$ are incompatible upwards in P_i, in which case $G_p \cap G_{p'}$ is empty. If $V \subseteq P^*$ is a non-empty open set, take any $q \in V$. There is a finite set $J \subseteq I$ such that $V \supseteq \{q' : q' \in P^*,\, q'(i) \ge q(i) \text{ for every } i \in J\}$. Set $p = q{\restriction}J$; then $G_p \subseteq V$. So $p \mapsto G_p$ satisfies the conditions of 514Sc, inverted, and $\mathrm{RO}^{\uparrow}(P)$ is isomorphic to $\mathrm{RO}(P^*)$.

(b)-(c) These are immediate from the definition of the ordering of P. For the estimate of the cofinality of P, just take cofinal sets $Q_i \subseteq P_i$ such that $\#(Q_i) = \operatorname{cf} P_i$ for each i, and estimate $\#(\bigcup_{J\in[I]^{<\omega}} \prod_{i\in J} Q_i)$.

(d) We have

$$c^{\uparrow}(P) = c(\mathrm{RO}^{\uparrow}(P))$$

(514Nc)

$$= c(\mathrm{RO}^{\uparrow}(P^*))$$

((a) above))

$$= c(P^*)$$

(514Hb)

$$= \sup_{J\in[I]^{<\omega}} c(\prod_{i\in J} P_i)$$

(5A4Be, here taking the product topology on $\prod_{i\in I} P_i$)

$$= \sup_{J\in[I]^{<\omega}} c^{\uparrow}(\prod_{i\in J} P_i)$$

because if J is finite then the up-topology $\mathfrak{T}_J^{\uparrow}$ on $\prod_{i\in J} P_i$ is just the product of the up-topologies on the P_i, so we can use the identification of $c(\prod_{i\in J} P_i, \mathfrak{T}_J^{\uparrow})$ with $c^{\uparrow}(\prod_{i\in J} P_i)$.

514X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. Show that $\mathrm{link}_{<\kappa}(\mathfrak{A}) = \pi(\mathfrak{A})$ for any $\kappa \ge \mathrm{sat}(\mathfrak{A})$.

(b) Let $\langle \mathfrak{A}_i \rangle_{i\in I}$ be a family of Boolean algebras and $\mathfrak{A}$ their free product. Show that

$$d(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} d(\mathfrak{A}_i)), \quad \pi(\mathfrak{A}) \le \max(\omega, \#(I), \sup_{i\in I} \pi(\mathfrak{A}_i)), \quad c(\mathfrak{A}) \le \max(\omega, \sup_{i\in I} 2^{c(\mathfrak{A}_i)}).$$

(*Hint*: 5A1Ga.)

(c) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a chargeable Boolean algebra (definition: 391Bb). Suppose that $\mathfrak{A}\setminus\{1\} \preccurlyeq_{\mathrm{T}} \mathfrak{B}\setminus\{1\}$. Show that $\mathfrak{A}$ is chargeable. (*Hint*: 391J.)

(d) Let κ be a cardinal and $\mathfrak{A}$ a Boolean algebra of size at most 2^κ. (i) Show that $\mathfrak{A}$ is a homomorphic image of a κ-centered Boolean algebra. (*Hint*: if κ is infinite, $\{0,1\}^{2^\kappa}$ has density κ.) (ii) Show that if $\mathfrak{A}$ is Dedekind complete it is a homomorphic image of $\mathcal{P}\kappa$. (*Hint*: 514Ca, 314K.)

(e) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ *either* a regularly embedded subalgebra of $\mathfrak{A}$ *or* a quotient $\mathfrak{A}/I$ where I is an order-closed ideal in $\mathfrak{A}$. Let $\mathrm{Pou}(\mathfrak{A})$, $\mathrm{Pou}(\mathfrak{B})$ be the pre-ordered sets of partitions of unity in $\mathfrak{A}$, $\mathfrak{B}$ respectively (512Ee). Show that $\mathrm{Pou}(\mathfrak{B}) \preccurlyeq_{\mathrm{T}} \mathrm{Pou}(\mathfrak{A})$, and hence that $\mathrm{wdistr}(\mathfrak{B}) \ge \mathrm{wdistr}(\mathfrak{A})$.

(f) Let X be a set. Show that $\tau(\mathcal{P}X)$ is the least cardinal λ such that $\#(X) \le 2^\lambda$.

(g) Let Σ be the countable-cocountable algebra of ω_1. Show that Σ is an order-dense subalgebra of $\mathcal{P}\omega_1$, that $\tau(\Sigma) = \omega_1$, and that $\tau(\mathcal{P}\omega_1) = \omega$.

(h) For a Boolean algebra $\mathfrak{A}$, write $\mathrm{hc}(\mathfrak{A}) = \min\{c(\mathfrak{B}) : \mathfrak{B}$ is a non-zero principal ideal of $\mathfrak{A}\}$, counting $\min\emptyset$ as ∞. (i) Show that if $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then $\mathrm{hc}(\mathfrak{B}) \le \mathrm{hc}(\mathfrak{A})$. (ii) Show that if $\mathfrak{B}$ is a Boolean algebra and there is a surjective order-continuous Boolean homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, then $\mathrm{hc}(\mathfrak{B}) \le \mathrm{hc}(\mathfrak{A})$. (iii) Show that if $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$ then $\mathrm{hc}(\mathfrak{B}) \ge \mathrm{hc}(\mathfrak{A})$. (iv) Show that if $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$ then $\mathrm{hc}(\mathfrak{B}) = \mathrm{hc}(\mathfrak{A})$. (v) Show that if $\mathfrak{A}$ is the simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras, then $\mathrm{hc}(\mathfrak{A}) = \min_{i\in I} \mathrm{hc}(\mathfrak{A}_i)$.

>(i)(i) (Solovay 66) Let I be any set, with its discrete topology, and $X = I^{\mathbb{N}}$ with the product topology. Show that $\tau(\mathrm{RO}(X)) = \omega$. (ii) Show that the subalgebra $\mathfrak{B}$ of $\mathrm{RO}(X)$ generated by $\{\{x : x(n) = i\} : n \in \mathbb{N}, i \in I\}$ is an order-dense subalgebra of $\mathrm{RO}(X)$ and that $\tau(\mathfrak{B}) \ge \#(I)$ if $\#(I) > 1$.

(j) Let $(X, \mathfrak{T})$ be a T_0 topological space. Show that we have a partial order on X defined by saying that $x \le y$ iff $x \in \overline{\{y\}}$. Show that $\mathfrak{T}$ is the up-topology on X iff the family of $\mathfrak{T}$-closed sets is a topology.

(k) Let P be a partially ordered set. Show that a subset of P is a regular open set for the up-topology iff it is of the form $\bigcap_{q\in A}\{p : p \in P,\, [p,\infty[\, \cap\, [q,\infty[\, = \emptyset\}$ for some set $A \subseteq P$.

>(l) Rewrite the statement and proof of the Erdős-Tarski theorem (513Bb) (i) in terms of topological spaces (ii) in terms of Boolean algebras.

(m) Find partially ordered sets P and Q such that the regular open algebras of P and Q for their up-topologies are isomorphic, but $\operatorname{add} P \ne \operatorname{add} Q$ and $\operatorname{cf} P \ne \operatorname{cf} Q$.

(n) Let P be a non-empty partially ordered set such that its regular open algebra $\mathrm{RO}^{\uparrow}(P)$ for the up-topology is atomless, and let Q be a set of the same size as P with the trivial partial order in which $q \le q'$ iff $q = q'$. Show that Q and the product partially ordered set $P \times Q$ are Tukey equivalent but $\mathrm{RO}^{\uparrow}(P \times Q)$ is atomless, while $\mathrm{RO}^{\uparrow}(Q)$ is purely atomic.

(o) Let P be a partially ordered set and κ an infinite cardinal. Show that $\kappa < \mathrm{wdistr}(\mathrm{RO}^{\uparrow}(P))$ iff for every family $\langle Q_\xi\rangle_{\xi<\kappa}$ of cofinal subsets of P there is a cofinal $Q \subseteq P$ such that for every $q \in Q$ and $\xi < \kappa$ there is an $I \in [Q_\xi]^{<\omega}$ such that for every $p \ge q$ there is an $r \in I$ which is compatible upwards with p.

(p) Suppose that P is a partially ordered set and that $A \subseteq P$ is such that

$$Q = \{q : q \in P,\, q = \sup\{a : a \in A,\, a \le q\}\}$$

is cofinal with P. Show that if P is separative upwards, then $\tau(\mathrm{RO}^{\uparrow}(P)) \le \#(A)$.

(q) Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure. Show that the simple products $\{0,1\} \times \mathfrak{A}$ and $\mathcal{P}\mathbb{N} \times \mathfrak{A}$ are not isomorphic, but that each can be regularly embedded in the other.

(r) Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological spaces. Suppose we have a π-base $\mathcal{U}$ for $\mathfrak{T}$ and a function $f : \mathcal{U} \to \mathfrak{S}$ such that $f[\mathcal{U}]$ is a π-base for $\mathfrak{S}$ and, for $U, U' \in \mathcal{U}$, $U \cap U' = \emptyset$ iff $f(U) \cap f(U') = \emptyset$. Show that $\mathrm{RO}(X) \cong \mathrm{RO}(Y)$.

(s) Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets and $\langle I_j\rangle_{j\in J}$ a partition (that is, disjoint cover) of I. Show that the upwards finite-support product $\bigotimes^{\uparrow}_{i\in I} P_i$ can be naturally identified with $\bigotimes^{\uparrow}_{j\in J} \bigotimes^{\uparrow}_{i\in I_j} P_i$.

514Y Further exercises (a) For a partially ordered set P, its **order-dimension** is the smallest cardinal κ such that P is isomorphic, as partially ordered set, to a subset of a product $\prod_{\xi<\kappa} X_\xi$ where every X_ξ is a totally ordered set (and the product is given its product partial order, as in 315C). Show that the order-dimension of a Boolean algebra $\mathfrak{A}$ is $\mathrm{link}(\mathfrak{A})$.

(b) Show that $\mathcal{P}\mathbb{N}$ has a subalgebra with uncountable π-weight. (*Hint*: 515H.)

(c) Let $\mathfrak{A}$ be a Boolean algebra such that $c(\mathfrak{A}) \ne 1$, and A a subset of $\mathfrak{A}$. Show that there is a $B \in [A]^{\le c(\mathfrak{A})}$ with the same upper and lower bounds as A.

(d) Show that for any cardinal κ there are a ccc Boolean algebra $\mathfrak{A}$ and an ideal $\mathcal{I}$ of $\mathfrak{A}$ such that $c(\mathfrak{A}/\mathcal{I}) = \kappa$.

(e) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, $\mathbb{P}$ the forcing notion $(\mathfrak{A}^+, \subseteq, 1, \downarrow)$ (5A3M), and κ a cardinal. Show that the following are equiveridical: (i) there is an atomless order-closed subalgebra of $\mathfrak{A}$ with Maharam type at most κ; (ii) $\Vdash_{\mathbb{P}} \mathcal{P}\check{\kappa} \ne (\mathcal{P}\kappa)\check{}$.

(f) Let $\mathfrak{A}$ be a Boolean algebra and κ a cardinal. I will say that $\mathfrak{A}$ has the **$<\kappa$-interpolation property** if whenever A, $B \subseteq \mathfrak{A}$, $a \subseteq b$ whenever $a \in A$ and $b \in B$, and $\#(A \cup B) < \kappa$, then there is a $c \in \mathfrak{A}$ such that $a \subseteq c \subseteq b$ for every $a \in A$, $b \in B$. (Thus the σ-interpolation property of 466G is the $<\omega_1$-interpolation property.) (i) Suppose that $\mathfrak{A}$ has the $<\kappa$-interpolation property and I is an ideal of $\mathfrak{A}$ such that $\kappa \le (\operatorname{add} I)^+$. Show that the quotient $\mathfrak{A}/I$ has the $<\kappa$-interpolation property. (ii) Suppose that $\mathfrak{A}$ has the $<\kappa$-interpolation property, $\mathfrak{B}$ is a Boolean algebra of size at most κ, $\mathfrak{C}$ is a subalgebra of $\mathfrak{B}$ and $\phi : \mathfrak{C} \to \mathfrak{A}$ is a Boolean homomorphism. Show that ϕ has an extension to a Boolean homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$. (Compare 314K.) (iii) Show that if $\mathfrak{A}$ has the $<\operatorname{sat}(\mathfrak{A})$-interpolation property it is Dedekind complete.

514 Notes and comments With any mathematical object, the set-theorist's first concern is simply to establish its cardinality. There is therefore a natural distinction to make between cardinal invariants which control the cardinality of a space, as linking number, centering number and π-weight do for Boolean algebras (514Da), and others, like weak distributivity, which are measures of some kind of complexity not directly linked with cardinality. Observe that for general Boolean algebras $\mathfrak{A}$ not even the cellularity is controlled by the Maharam type (514Xi); in fact, of the cardinals here, only $\operatorname{wdistr}(\mathfrak{A})$ is controlled by $\tau(\mathfrak{A})$ alone (514Dd). Maharam type and cellularity together control the size of the algebra (514De), and for measurable algebras, of course, Maharam type almost completely determines the algebra and even the measure (see Chapter 33).

I use the language of Galois-Tukey connections in many of the proofs of this section. This is not because there is any real need for it (there is no depth to any of the results I quote) but because I think that it shows some common strands running through a rather long list of facts. Also it points up the proofs which are *not* reducible to simple applications of ideas in §512; for instance, those relating to weak distributivity. And, finally, it will provide useful practice for the ideas of Chapter 52.

I have deliberately arranged the lists of cardinal functions of topological spaces and Boolean algebras in such a way that the cardinals of Boolean algebras and their Stone spaces will naturally correspond. There are of course important exceptions. The Maharam type of a Boolean algebra, and the tightness of a topological space, do not seem to have significant natural analogues in the other category. Note that the correspondences depend to a significant degree on the compactness of Stone spaces. This is perhaps more important than their zero-dimensionality. The point about the open-and-closed algebra of a zero-dimensional space is that it is order-dense in the regular open algebra, and that our cardinal functions of Boolean algebras are nearly all unchanged by Dedekind completion (514Ee). For arbitrary topological spaces, we can still investigate their regular open algebras, and we find that the cardinal functions of a regular open algebra are much more closely related to those of the topological space if the space is locally compact (514A, 514H(b)-(c)).

You will not be surprised to recognise some of the results and arguments of this section as direct generalizations of special cases already treated; thus 316B becomes 514Bb, 316E (or 215B(iv)) becomes 514Db, 316I becomes 514Be and 4A1O becomes 514De.

I have to admit that there are rather more pages than ideas in this section. What it is really here for is to provide a compendium of useful facts in the language which I wish to use in the rest of the volume. Perhaps I should say 'languages', because much of the space is taken up by repeating results in three forms, as they apply to partially ordered sets, to Boolean algebras and to topological spaces. The point is of course that we frequently find that a fact which is obvious in one of its three manifestations is a surprise in another. And some care is needed in the translations. The theory of finite-support products of partially ordered sets (514T-514U), for instance, is supposed to mimic the theory of products of topological spaces. But actually it reflects the theory of π-bases of topologies rather than the theory of spaces-with-points. And while we have straightforward functors between the *categories* of Boolean algebras and topological spaces, with Boolean homomorphisms corresponding to continuous functions (312Q-312S), such results as we have concerning functions between partially ordered sets and their actions on the corresponding regular open algebras are partial and delicate (514O-514R).

The Tukey classification (513D) and the regular open algebras of 514N are both attempts to reduce the multitudinous variety of partially ordered sets to relatively coherent schemes. They carry rather different information; the Tukey classification tells us about additivity and cofinality (513E) and precalibers (516C below), while the regular open algebra determines linking numbers (514N). It is easy to find partially ordered sets with the same regular open algebras but different additivity and cofinality (514Xm), or with the same Tukey classification but different regular open algebras (514Xn). The regular open algebras studied here are primarily of interest in relation to the use of partially ordered sets in the theory of forcing; I hope to return to such questions later in this volume.

Of the cardinal functions of Boolean algebras defined in §511, I have not mentioned Martin numbers or Freese-Nation numbers. These will be dealt with at length in §§517-518.

515 The Balcar-Franěk theorem

I interpolate a section to give two basic results on Dedekind complete Boolean algebras: the Balcar-Franěk theorem (515H) on independent sets and the Pierce-Koppelberg theorem (515L) on cardinalities.

515A Definition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$.

(a) I say that a family $\langle\mathfrak{B}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is **Boolean-independent** if $\inf_{i\in J} b_i \ne 0$ whenever $J \subseteq I$ is finite and $b_i \in \mathfrak{B}_i^+ = \mathfrak{B}_i \setminus \{0\}$ for every $i \in J$.

(b) I say that a family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ is **Boolean-independent** if $\inf_{j\in J} a_j \setminus \sup_{k\in K} a_k$ is non-zero whenever J, $K \subseteq I$ are disjoint finite sets. Similarly, a set $B \subseteq \mathfrak{A}$ is **Boolean-independent** if $\inf J \setminus \sup K \ne 0$ for any disjoint finite sets J, $K \subseteq B$.

(c) I say that a family $\langle D_i\rangle_{i\in I}$ of partitions of unity in $\mathfrak{A}$ is **Boolean-independent** if $\inf_{i\in J} d_i \ne 0$ whenever $J \subseteq I$ is finite and $d_i \in D_i$ for every $i \in J$.

(Many authors write 'independent' rather than 'Boolean-independent'. But in this book it is more often natural to read 'independent' as 'stochastically independent', as in 458L and 525H.)

515B Lemma (Compare 272D.) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$.

(a) A family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ is Boolean-independent iff no a_i is 0 or 1 and $\langle\{0, a_i, 1\setminus a_i, 1\}\rangle_{i\in I}$ is a Boolean-independent family of subalgebras of $\mathfrak{A}$.

(b) Let $\langle\mathfrak{B}_i\rangle_{i\in I}$ be a family of subalgebras of $\mathfrak{A}$. Let $\mathfrak{B}$ be the free product of $\langle\mathfrak{B}_i\rangle_{i\in I}$, and $\varepsilon_i : \mathfrak{B}_i \to \mathfrak{B}$ the canonical homomorphism for each $i \in I$ (315I). Then we have a unique Boolean homomorphism $\phi : \mathfrak{B} \to \mathfrak{A}$ such that $\phi\varepsilon_i(b) = b$ whenever $i \in I$ and $b \in \mathfrak{B}_i$, and $\langle\mathfrak{B}_i\rangle_{i\in I}$ is Boolean-independent iff ϕ is injective; in which case $\mathfrak{B}$ is isomorphic to the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{B}_i$.

(c) If $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a Boolean-independent family of subalgebras of $\mathfrak{A}$, $\langle I_j\rangle_{j\in J}$ is a disjoint family of subsets of I, and $\mathfrak{C}_j$ is the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I_j}\mathfrak{B}_i$ for each j, then $\langle\mathfrak{C}_j\rangle_{j\in J}$ is Boolean-independent.

(d) Suppose that $B \subseteq \mathfrak{A}$ is a Boolean-independent set and that $\langle C_j\rangle_{j\in J}$ is a disjoint family of subsets of B. For $j \in J$ write $\mathfrak{C}_j$ for the subalgebra of $\mathfrak{A}$ generated by C_j. Then $\langle\mathfrak{C}_j\rangle_{j\in J}$ is Boolean-independent.

(e) Suppose that $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a Boolean-independent family of subalgebras of $\mathfrak{A}$, and that for each $i \in I$ we have a Boolean-independent subset B_i of $\mathfrak{B}_i$. Then $\langle B_i\rangle_{i\in I}$ is disjoint and $\bigcup_{i\in I} B_i$ is Boolean-independent.

(f) Let $\langle D_i\rangle_{i\in I}$ be a family of partitions of unity in $\mathfrak{A}$, none containing 0. For each $i \in I$ let $\mathfrak{B}_i$ be the order-closed subalgebra of $\mathfrak{A}$ generated by D_i. Then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is Boolean-independent iff $\langle D_i\rangle_{i\in I}$ is Boolean-independent.

proof (a) The point is just that in 515Aa we need consider only $b_i \in \mathfrak{B}_i \setminus \{0, 1\}$, while in 515Ab we have

$$\inf_{j\in J} a_j \setminus \sup_{k\in K} a_k = \inf_{j\in J} a_j \cap \inf_{i\in K} 1 \setminus a_k.$$

(b) 315Jb, applied to the identity maps from the $\mathfrak{B}_i$ to $\mathfrak{A}$, assures us that there is a unique Boolean homomorphism $\phi : \mathfrak{B} \to \mathfrak{A}$ such that $\phi\varepsilon_i(b) = b$ for every $i \in I$ and $b \in \mathfrak{B}_i$.

(i) By 315K(e-ii), $\langle\varepsilon_i[\mathfrak{B}_i]\rangle_{i\in I}$ is a Boolean-independent family of subalgebras of $\mathfrak{B}$. So if ϕ is injective, $\langle\mathfrak{B}_i\rangle_{i\in I} = \langle\pi_i[\varepsilon_i[\mathfrak{B}_i]]\rangle_{i\in I}$ is Boolean-independent in $\phi[\mathfrak{B}]$ and therefore in $\mathfrak{A}$. In this case, because $\mathfrak{B}$ is the subalgebra of itself generated by $\bigcup_{i\in I}\varepsilon_i[\mathfrak{B}_i]$ (315Ka), the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{B}_i$ is $\phi[\mathfrak{B}]$ and is isomorphic to $\mathfrak{B}$.

(ii) If $\langle\mathfrak{B}_i\rangle_{i\in I}$ is Boolean-independent and $b \in \mathfrak{B}^+$, there are a finite $J \subseteq I$ and a family $\langle b_j\rangle_{j\in J} \in \prod_{j\in J}\mathfrak{B}_j^+$ such that $b \supseteq \inf_{j\in J}\varepsilon_j(b_j)$ (315Kb). Now $\phi(b) \supseteq \inf_{j\in J} b_j$ is non-zero; as b is arbitrary, ϕ is injective.

(c) Let $L \subseteq J$ be a finite set and suppose that $c_j \in \mathfrak{C}_j^+$ for each $j \in L$. As observed in (b), the embeddings $\mathfrak{B}_i \subsetneqq \mathfrak{C}_j$ identify $\mathfrak{C}_j$ with the free product of $\langle\mathfrak{B}_i\rangle_{i\in I_j}$, so 315Kb tells us that there must be a finite set $K_j \subseteq I_j$ and elements $b_i \in \mathfrak{B}_i^+$, for $i \in K_j$, such that $\inf_{i\in K_j} b_i \subseteq c_j$. Now $\inf_{j\in L} c_j \supseteq \inf\{b_i : i \in \bigcup_{j\in L} K_j\}$ is non-zero. As $\langle c_j\rangle_{j\in L}$ is arbitrary, $\langle\mathfrak{C}_j\rangle_{j\in J}$ is Boolean-independent. (Compare 315L.)

(d) Set $\mathfrak{B}_b = \{0, b, 1\setminus b, 1\}$ for $b \in B$, so that $\langle\mathfrak{B}_b\rangle_{b\in B}$ is Boolean-independent, by (a); now apply (c).

(e)(i) If i, $j \in I$ are distinct, $b \in B_i$ and $b' \in B_j$, then $b \in \mathfrak{B}_i^+$ and $1 \setminus b' \in \mathfrak{B}_j^+$, so $b \setminus b' \neq 0$ and $b \neq b'$.

(ii) If J, K are disjoint finite subsets of $\bigcup_{i \in I} B_i$, then $J \cap B_i$ and $K \cap B_i$ are disjoint finite subsets of B_i, so that

$$b_i = \inf(J \cap B_i) \setminus \sup(K \cap B_i) \in \mathfrak{B}_i^+$$

for each $i \in I$. Let $L \subseteq I$ be a finite set such that $J \cup K \subseteq \bigcup_{i \in L} B_i$; then

$$\inf J \setminus \sup K = \inf_{i \in L} b_i \neq 0.$$

As J and K are arbitrary, $\bigcup_{i \in I} B_i$ is Boolean-independent.

(f) Since $D_i \subseteq \mathfrak{B}_i$, $\langle D_i \rangle_{i \in I}$ must be Boolean-independent if $\langle \mathfrak{B}_i \rangle_{i \in I}$ is.

On the other hand, each D_i is order-dense in $\mathfrak{B}_i$. **P** For $d \in D_i$, the set $\{b : d \subseteq b \text{ or } d \cap b = 0\}$ is an order-closed subalgebra of $\mathfrak{A}$ including D_i, so includes $\mathfrak{B}_i$. If $b \in \mathfrak{B}_i^+$, then (because $\sup D_i = 1$ in $\mathfrak{A}$) there must be a $d \in D_i$ such that $b \cap d \neq 0$, in which case $0 \neq d \subseteq b$. As b is arbitrary, D_i is order-dense in $\mathfrak{B}_i$. **Q**

Now suppose that $\langle D_i \rangle_{i \in I}$ is Boolean-independent, $J \subseteq I$ is finite and $b_i \in \mathfrak{B}_i^+$ for each $i \in J$. Then we have non-zero $d_i \in D_i$ such that $d_i \subseteq b_i$ for each i. So $\inf_{i \in J} b_i \supseteq \inf_{i \in J} d_i$ is non-zero. As $\langle b_i \rangle_{i \in J}$ is arbitrary, $\langle \mathfrak{B}_i \rangle_{i \in I}$ is Boolean-independent.

515C Proposition Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and κ a cardinal.

(a) There is a Boolean-independent subset of $\mathfrak{A}$ with cardinal κ iff there is a subalgebra of $\mathfrak{A}$ which is isomorphic to the algebra of open-and-closed subsets of $\{0,1\}^\kappa$.

(b) If $\mathfrak{A}$ is Dedekind complete, there is a Boolean-independent subset of $\mathfrak{A}$ with cardinal κ iff there is a subalgebra of $\mathfrak{A}$ which is isomorphic to the regular open algebra of $\{0,1\}^\kappa$.

proof Set $Z = \{0,1\}^\kappa$; write $\mathcal{E}$ for the algebra of open-and-closed subsets of Z and $\mathfrak{G}$ for the regular open algebra of Z.

(a)(i) Suppose that $\mathfrak{A}$ has a Boolean-independent subset of cardinal κ, enumerated as $\langle a_\xi \rangle_{\xi < \kappa}$. Setting $\mathfrak{A}_\xi = \{0, a_\xi, 1 \setminus a_\xi, 1\}$ for each ξ, $\langle \mathfrak{A}_\xi \rangle_{\xi < \kappa}$ is a Boolean-independent family of subalgebras of $\mathfrak{A}$, and the subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\bigcup_{\xi < \kappa} \mathfrak{A}_\xi$ can be identified with the free product of $\langle \mathfrak{A}_\xi \rangle_{\xi < \kappa}$ (515Bb). But since the Stone space of each $\mathfrak{A}_\xi$ has just two points, the construction of 315I makes it plain that the Stone space of $\mathfrak{C}$ is homeomorphic to Z, so that $\mathfrak{C}$ is isomorphic to $\mathcal{E}$.

(ii) In the other direction, the sets $E_\xi = \{z : z \in Z, z(\xi) = 1\}$ are Boolean-independent in $\mathcal{E}$, so if $\mathcal{E}$ can be embedded in $\mathfrak{A}$ there must be a Boolean-independent subset of $\mathfrak{A}$ with cardinal κ.

(b) Now suppose that $\mathfrak{A}$ is Dedekind complete. $\mathcal{E}$ is an order-dense subalgebra of $\mathfrak{G}$ (314T). So if $\mathfrak{A}$ has a subalgebra isomorphic to $\mathfrak{G}$ it certainly has one isomorphic to $\mathcal{E}$. On the other hand, if $\mathfrak{A}$ has a subalgebra isomorphic to $\mathcal{E}$, so that there is an injective Boolean homomorphism $\pi : \mathcal{E} \to \mathfrak{A}$, then (because $\mathfrak{A}$ is Dedekind complete) π has an extension to a Boolean homomorphism $\pi_1 : \mathfrak{G} \to \mathfrak{A}$ (314K); because $\mathcal{E}$ is order-dense in $\mathfrak{G}$ and π is injective, π_1 is injective, so that $\pi_1[\mathfrak{G}]$ is a subalgebra of $\mathfrak{A}$ isomorphic to $\mathfrak{G}$.

Putting this together with (a), we see that $\mathfrak{A}$ has a Boolean-independent subset with cardinal κ iff it has a subalgebra isomorphic to $\mathfrak{G}$.

515D Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$. Then there is an $a^* \in \mathfrak{A} \setminus \{0,1\}$ such that $\mathfrak{B}$ and $\{0, a^*, 1 \setminus a^*, 1\}$ are Boolean-independent subalgebras of $\mathfrak{A}$.

Remark Recall from 331A that a Boolean algebra $\mathfrak{A}$ is 'relatively atomless' over an order-closed subalgebra $\mathfrak{B}$ if for every $a \in \mathfrak{A}^+$ there is a $c \subseteq a$ which is not of the form $a \cap b$ for any $b \in \mathfrak{B}$.

proof Set

$$C = \{c : c \in \mathfrak{A}, c \neq 0, \mathfrak{B} \cap \mathfrak{A}_c = \{0\}\},$$

where $\mathfrak{A}_c$ is the principal ideal of $\mathfrak{A}$ generated by c. Then C is order-dense in $\mathfrak{A}$. **P** If $a \in \mathfrak{A}^+$, there is a $c \in \mathfrak{A}_a \setminus \{a \cap b : b \in \mathfrak{B}\}$. Set $b_0 = \sup\{b : b \in \mathfrak{B}, b \subseteq c\}$; then $c \setminus b_0 \subseteq a$ and $c \setminus b_0 \in C$. **Q**

For $a \in \mathfrak{A}$ set $\operatorname{upr}(a, \mathfrak{B}) = \inf\{b : a \subseteq b \in \mathfrak{B}\}$, as in 313S. Set $E = \{\operatorname{upr}(c, \mathfrak{B}) : c \in C\}$. Then E is order-dense in $\mathfrak{B}$, because if $b \in \mathfrak{B}^+$ there is a $c \in C$ such that $c \subseteq b$, and now $\operatorname{upr}(c, \mathfrak{B})$ belongs to E and is included in b. So there is a partition D of unity in $\mathfrak{B}$ included in E (313K). For each $d \in D$ choose $c_d \in C$ such that $d = \operatorname{upr}(c_d, \mathfrak{B})$, and set $a^* = \sup\{c_d : d \in D\}$. If $b \in \mathfrak{B}^+$, there is a $d \in D$ such that $b \cap d \neq 0$, that is, $d \not\subseteq 1 \setminus b$, so $c_d \not\subseteq 1 \setminus b$ and $b \cap c_d \neq 0$; so $b \cap a^* \neq 0$. Also, because $c_d \in C$, $b \cap d \not\subseteq c_d = a^* \cap d$, so $b \not\subseteq a^*$ and $b \cap (1 \setminus a^*) \neq 0$. As b is arbitrary, $\mathfrak{B}$ and $\{0, a^*, 1 \setminus a^*, 1\}$ are Boolean-independent.

515E Lemma (Balcar & Vojtáš 77) Let $\mathfrak{A}$ be a Boolean algebra. Suppose that $C \subseteq \mathfrak{A}^+$ and that $\#(C) < c(\mathfrak{A}_c)$ for every $c \in C$, where $\mathfrak{A}_c$ is the principal ideal of $\mathfrak{A}$ generated by c. Then there is a partition D of unity in $\mathfrak{A}$ such that every member of C includes a non-zero member of D.

proof Enumerate C as $\langle c_\xi \rangle_{\xi < \kappa}$. For each $\xi < \kappa$, let B_ξ be a disjoint set in $\mathfrak{A}^+_{c_\xi}$ of size κ^+, and set

$$A_\xi = \{\eta : \eta < \kappa,\ \#(\{b : b \in B_\xi,\ b \cap c_\eta \neq 0\}) \leq \kappa\},$$

$$B'_\xi = B_\xi \setminus \bigcup_{\eta \in A_\xi} \{b : b \in B_\xi,\ b \cap c_\eta \neq 0\}.$$

Then B'_ξ is a disjoint set in $\mathfrak{A}_{c_\xi}$, $\#(B'_\xi) = \kappa^+$, and $\{b : b \in B'_\xi,\ b \cap c_\eta \neq 0\}$ is empty if $\eta \in A_\xi$ and has cardinal κ^+ otherwise. Now define $A \subseteq \kappa$ inductively by saying that $\xi \in A$ iff $\xi \in A_\eta$ whenever $\eta \in A \cap \xi$, and set $B = \bigcup_{\xi \in A} B'_\xi$.

B is disjoint. **P** If η, $\xi \in A$, $\eta \leq \xi$, $b \in B'_\eta$, $b' \in B'_\xi$ and $b \neq b'$, then either $\eta = \xi$ and $b \cap b' = 0$ because B'_ξ is disjoint, or $\eta < \xi$ and $\xi \in A_\eta$ and $b \cap b' \subseteq b \cap c_\xi = 0$. **Q** Also $D_\xi = \{b : b \in B,\ b \cap c_\xi \neq 0\}$ has cardinal κ^+ for every $\xi < \kappa$. **P** If $\xi \in A$, then $D_\xi \supseteq B'_\xi$ has cardinal κ^+. If $\xi \notin A$ there is some $\eta \in A \cap \xi$ such that $\xi \notin A_\eta$ and $D_\xi \supseteq \{b : b \in B'_\eta,\ b \cap c_\xi \neq 0\}$ has cardinal κ^+. **Q**

We can therefore find an injection $\xi \mapsto b_\xi : \kappa \to B$ such that $c_\xi \cap b_\xi \neq 0$ for every ξ. Let D be any partition of unity including $\{c_\xi \cap b_\xi : \xi < \kappa\}$; this works.

515F Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that $c(\mathfrak{A}) = \mathrm{sat}(\mathfrak{A})$ and $\mathfrak{A}$ is cellularity-homogeneous. Then there is a Boolean-independent family $\langle D_i \rangle_{i \in I}$ of partitions of unity in $\mathfrak{A}$ such that $\#(I) = \sup_{i \in I} \#(D_i) = c(\mathfrak{A})$.

proof Write κ for $c(\mathfrak{A}) = \mathrm{sat}(\mathfrak{A})$. Choose $\langle D_\xi \rangle_{\xi < \kappa}$ inductively, as follows. Given D_η for $\eta < \xi$, let $\mathfrak{C}_\xi$ be the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{\eta < \xi} D_\eta$; then $\#(\mathfrak{C}_\xi) < \kappa$. (Recall from 513Bb that $\kappa = \mathrm{sat}^\downarrow(\mathfrak{A}^+)$ must be a regular uncountable cardinal, while of course $\#(D_\eta) < \kappa$ for every η.) By 515E we have a partition D of unity in $\mathfrak{A}$, not containing $\{0\}$, such that every non-zero element of $\mathfrak{C}_\xi$ includes an element of D. For each $d \in D$ the principal ideal of $\mathfrak{A}$ generated by d has cellularity $\kappa > \#(\xi)$ so there is a disjoint family $\langle b_{d\eta} \rangle_{\eta \leq \xi}$ of non-zero elements with supremum d. Set $b_\eta = \sup_{d \in D} b_{d\eta}$ for $\eta \leq \xi$, and $D_\xi = \{b_\eta : \eta \leq \xi\}$; then D_ξ is a partition of unity in $\mathfrak{A}$.

The construction ensures that whenever $d \in D_\xi$ and $c \in \mathfrak{C}^+_\xi$ then $d \cap c \neq 0$. It follows that $\langle D_\xi \rangle_{\xi < \kappa}$ is Boolean-independent. **P** I show by induction on $\#(J)$ that if $J \subseteq \kappa$ is finite and $d_\xi \in D_\xi$ for each $\xi \in J$, then $\inf_{\xi \in J} d_\xi \neq 0$. If J is empty this is trivial. For the inductive step to $\#(J) = n + 1$, set $\xi = \max J$ and $J' = \xi \cap J$. By the inductive hypothesis, $c = \inf_{\eta \in J'} d_\eta$ is non-zero; but $c \in \mathfrak{C}_\xi$, so, by the construction of D_ξ, $c \cap d_\xi = \inf_{\eta \in J} d_\eta$ is non-empty. So the induction proceeds. **Q**

515G Lemma Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a non-empty family of Boolean algebras with simple product $\mathfrak{A}$. Suppose that for each $i \in I$ the algebra $\mathfrak{A}_i$ has a Boolean-independent set with cardinal $\kappa_i \geq \omega$. Then $\mathfrak{A}$ has a Boolean-independent set with cardinal $\kappa = \#(\prod_{i \in I} \kappa_i)$.

proof For each $i \in I$ let B_i be a Boolean-independent set in $\mathfrak{A}_i$ with cardinal κ_i. Let $\langle a_\xi \rangle_{\xi < \kappa}$ be a family in $\prod_{i \in I} B_i \subseteq \mathfrak{A}$ such that for every finite $J \subseteq \kappa$ there is an $i \in I$ such that $a_\xi(i) \neq a_\eta(i)$ whenever ξ, $\eta \in J$ are distinct (5A1K). Now $\langle a_\xi \rangle_{\xi < \kappa}$ is Boolean-independent in $\mathfrak{A}$. **P** Suppose that J, $K \subseteq \kappa$ are finite and disjoint. Then there is an $i \in I$ such that $a_\xi(i) \neq a_\eta(i)$ whenever ξ, $\eta \in J \cup K$ are distinct. But this means that $\langle a_\xi(i) \rangle_{\xi \in J \cup K}$ is Boolean-independent in $\mathfrak{A}_i$, so that, setting $a = \inf_{\xi \in J} a_\xi \setminus \sup_{\xi \in K} a_\xi$,

$$a(i) = \inf_{\xi \in J} a_\xi(i) \setminus \sup_{\xi \in K} a_\xi(i) \neq 0,$$

and $a \neq 0$. As J and K are arbitrary, $\langle a_\xi \rangle_{\xi < \kappa}$ is a Boolean-independent family. **Q**

Accordingly $\{a_\xi : \xi < \kappa\}$ is a Boolean-independent set of size κ.

515H The Balcar-Franěk theorem (Balcar & Franěk 82) Let $\mathfrak{A}$ be an infinite Dedekind complete Boolean algebra. Then there is a Boolean-independent set $A \subseteq \mathfrak{A}$ such that $\#(A) = \#(\mathfrak{A})$.

proof Set $\kappa = \#(\mathfrak{A})$. For $a \in \mathfrak{A}$ write $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a.

(a) Suppose that $\mathfrak{A}$ is purely atomic. Then $\mathfrak{A}$ has an independent set of size κ. **P** Let B be the set of its atoms; because $\mathfrak{A}$ is infinite, so is B; set $\lambda = \#(B)$, so that

$$\mathfrak{A} \cong \prod_{b\in B} \mathfrak{A}_b \cong \mathcal{P}\lambda$$

(315F(iii)), and $\kappa = 2^\lambda$. There is a dense subset D of $\{0,1\}^\kappa$ with $\#(D) = \lambda$ (5A4Be); let $f : B \to D$ be a surjection. For $\xi < \kappa$ set

$$a_\xi = \sup\{b : b \in B,\ f(b)(\xi) = 1\}.$$

If J, $K \subseteq \kappa$ are disjoint finite sets, the set

$$G = \{x : x \in \{0,1\}^\kappa,\ x(\xi) = 1\ \forall\ \xi \in J,\ x(\eta) = 0\ \forall\ \eta \in K\}$$

is a non-empty open set, so there is a $b \in B$ such that $f(b) \in G$; but this means that $\inf_{\xi\in J} a_\xi \setminus \sup_{\eta\in K} a_\eta \supseteq b$ is non-zero. As J and K are arbitrary, $\{a_\xi : \xi < \kappa\}$ is an independent subset of $\mathfrak{A}$ with cardinal κ. **Q**

(b) Suppose that $\mathfrak{A}$ is Maharam-type-homogeneous, and that $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ with Maharam type less than $\tau(\mathfrak{A})$. Then there is a subalgebra $\mathfrak{C}$ of $\mathfrak{A}$, Boolean-independent of $\mathfrak{B}$, such that $\mathfrak{C}$ has a Boolean-independent subset with cardinal $\tau(\mathfrak{A})$. **P** Let $B \subseteq \mathfrak{B}$ be a set with cardinal less than $\tau(\mathfrak{A})$ which τ-generates $\mathfrak{B}$. Choose $\langle c_\xi\rangle_{\xi<\tau(\mathfrak{A})}$ inductively, as follows. Given $\langle c_\eta\rangle_{\eta<\xi}$, where $\xi < \tau(\mathfrak{A})$, let $\mathfrak{B}_\xi$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $B \cup \{c_\eta : \eta < \xi\}$. If $a \in \mathfrak{A}^+$, the order-closed subalgebra $\mathfrak{D} = \{a \cap b : b \in \mathfrak{B}_\xi\}$ of $\mathfrak{A}_a$ is τ-generated by $\{a \cap c_\eta : \eta < \xi\} \cup \{a \cap b : b \in B\}$ (314Hb), so $\tau(\mathfrak{D}) < \tau(\mathfrak{A}) = \tau(\mathfrak{A}_a)$ and $\mathfrak{D} \neq \mathfrak{A}_a$. Thus $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}_\xi$; by 515D, there is a $c_\xi \in \mathfrak{A} \setminus \{0,1\}$ such that $\mathfrak{B}_\xi$ and $\{0, c_\xi, 1 \setminus c_\xi, 1\}$ are Boolean-independent. Continue. Now an easy induction on $\#(J \cup K)$ (as in the last part of the proof of 515F) shows that if J, K are disjoint finite subsets of $\tau(\mathfrak{A})$, and $b \in \mathfrak{B}$ is non-zero, $b \cap \inf_{\xi\in J} c_\xi \setminus \sup_{\eta\in K} c_\eta \neq 0$. So if we take $\mathfrak{C}$ to be the subalgebra of $\mathfrak{A}$ generated by $C = \{c_\xi : \xi < \tau(\mathfrak{A})\}$, $\mathfrak{C}$ and $\mathfrak{B}$ are Boolean-independent and $C \subseteq \mathfrak{C}$ is a Boolean-independent set with cardinal $\tau(\mathfrak{A})$. **Q**

(c) Suppose that $\mathfrak{A}$ is Maharam-type-homogeneous and that $c(\mathfrak{A}) < \mathrm{sat}(\mathfrak{A})$. Then $\mathfrak{A}$ has a Boolean-independent subset with cardinal κ. **P** Because $\mathfrak{A}$ is infinite, $c(\mathfrak{A})$ is infinite. Let $D \subseteq \mathfrak{A}^+$ be a disjoint set with cardinal $c(\mathfrak{A})$; adding $1 \setminus \sup D$ if necessary, we may suppose that D is a partition of unity. For each $d \in D$, $\mathfrak{A}_d$ has a Boolean-independent set with cardinal $\tau(\mathfrak{A}_d) = \tau(\mathfrak{A})$ (apply (b) above to $\mathfrak{A}_d$, with $\mathfrak{D} = \{0, d\}$). By 315F(iii) again, $\mathfrak{A} \cong \prod_{d\in D} \mathfrak{A}_d$; by 515G, $\prod_{d\in D} \mathfrak{A}_d$ has a Boolean-independent subset of size the cardinal power $\tau(\mathfrak{A})^{\#(D)} = \tau(\mathfrak{A})^{c(\mathfrak{A})}$, so $\mathfrak{A}$ also has. But

$$\kappa \le \sup_{\lambda<\mathrm{sat}(\mathfrak{A})} \tau(\mathfrak{A})^\lambda = \tau(\mathfrak{A})^{c(\mathfrak{A})}$$

by 514De, so $\mathfrak{A}$ has a Boolean-independent set of cardinal κ. **Q**

(d) Suppose that $\mathfrak{A}$ is cellularity-homogeneous and Maharam-type-homogeneous and $c(\mathfrak{A}) = \mathrm{sat}(\mathfrak{A})$. Then $\mathfrak{A}$ has a Boolean-independent subset with cardinal κ.

P (i) By 515F, we can find a Boolean-independent family $\langle D_i\rangle_{i\in I}$ of partitions of unity in $\mathfrak{A}$ such that $\#(I) = \sup_{i\in I} \#(D_i) = \mathrm{sat}(\mathfrak{A})$. We know that $\mathrm{sat}(\mathfrak{A}) \ge \omega_1$, so we can suppose that all the D_i are infinite. For each $i \in I$, let $\mathfrak{D}_i$ be the order-closed subalgebra of $\mathfrak{A}$ generated by D_i. By (a) above, $\mathfrak{D}_i$ has a Boolean-independent subset B_i with cardinal $2^{\#(D_i)}$, so that $B = \bigcup_{i\in I} B_i$ has cardinal $\sup_{\lambda<\mathrm{sat}(\mathfrak{A})} 2^\lambda$. By 515Df, $\langle\mathfrak{D}_i\rangle_{i\in I}$ is Boolean-independent. By 515De, B is Boolean-independent.

(ii) If $\#(B) = \kappa$, we can stop. Otherwise, let $\mathfrak{D}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $D = \bigcup_{i\in I} D_i$. Because

$$\sup_{\lambda<\mathrm{sat}(\mathfrak{A})} \#(B)^\lambda = \#(B)$$

(5A1Ef)

$$< \kappa \le \sup_{\lambda<\mathrm{sat}(\mathfrak{A})} \tau(\mathfrak{A})^\lambda$$

(514Be), we must have

$$\tau(\mathfrak{A}) > \#(B) = \sup_{\lambda<\mathrm{sat}(\mathfrak{A})} 2^\lambda \ge \mathrm{sat}(\mathfrak{A}) = \#(D) \ge \tau(\mathfrak{D}).$$

By (b), we have a subalgebra $\mathfrak{C}$ of $\mathfrak{A}$, Boolean-independent of $\mathfrak{D}$, such that $\mathfrak{C}$ has a Boolean-independent subset C with cardinal $\tau(\mathfrak{A})$. Let $\langle C_i\rangle_{i\in I}$ be a disjoint family of subsets of C all with cardinal $\tau(\mathfrak{A})$.

For $i \in I$, let $\mathfrak{C}_{i0}$ be the subalgebra of $\mathfrak{A}$ generated by D_i and $\mathfrak{C}_{i1}$ the subalgebra generated by C_i. Let $\mathfrak{E}_i$ be the subalgebra generated by $\mathfrak{C}_{i0} \cup \mathfrak{C}_{i1}$ and $\widehat{\mathfrak{E}}_i$ its Dedekind completion (314T-314U). In $\widehat{\mathfrak{E}}_i$ we have the partition of unity D_i and the Boolean-independent set C_i with cardinal $\tau(\mathfrak{A})$. For each $b \in D_i$, the principal ideal $(\widehat{\mathfrak{E}}_i)_b$ of $\widehat{\mathfrak{E}}_i$ generated by b has a Boolean-independent set $\{b \cap c : c \in C_i\}$ with cardinal $\tau(\mathfrak{A})$. Because $\widehat{\mathfrak{E}}_i$ is Dedekind complete, it is isomorphic to $\prod_{b \in D_i}(\widehat{\mathfrak{E}}_i)_b$, and has a Boolean-independent subset with cardinal $\tau(\mathfrak{A})^{\#(D_i)}$ (515G again).

Because $\mathfrak{A}$ is Dedekind complete, the embedding $\mathfrak{E}_i \subsetneq \mathfrak{A}$ extends to a Boolean homomorphism $\pi_i : \widehat{\mathfrak{E}}_i \to \mathfrak{A}$ (314K). Because $\mathfrak{E}_i$ is order-dense in $\widehat{\mathfrak{E}}_i$, π_i is injective. So $\mathfrak{E}_i^* = \pi_i[\widehat{\mathfrak{E}}_i]$ is a subalgebra of $\mathfrak{A}$ isomorphic to $\widehat{\mathfrak{E}}_i$, and has a Boolean-independent subset E_i with cardinal $\tau(\mathfrak{A})^{\#(D_i)}$.

(iii) By 515Df and 515Dd, $\langle \mathfrak{C}_{i0}\rangle_{i \in I}$ and $\langle \mathfrak{C}_{i1}\rangle_{i \in I}$ are both Boolean-independent families; because $\mathfrak{C}_{i0} \subseteq \mathfrak{D}$ and $\mathfrak{C}_{j1} \subseteq \mathfrak{C}$ whenever i, $j \in I$, and $\mathfrak{D}$ and $\mathfrak{C}$ are Boolean-independent, $\langle \mathfrak{C}_{ij}\rangle_{i \in I, j \in \{0,1\}}$ is Boolean-independent, so $\langle \mathfrak{E}_i\rangle_{i \in I}$ is Boolean-independent (515Dc). If $J \subseteq I$ is finite, and $e_i \in (\mathfrak{E}_i^*)^+$ for each $i \in J$, then there are $e_i' \in \mathfrak{E}_i$ such that $0 \neq e_i' \subseteq e_i$ for each i. Now $\inf_{i \in J} e_i \supseteq \inf_{i \in J} e_i' \neq 0$. As $\langle e_i\rangle_{i \in J}$ is arbitrary, $\langle \mathfrak{E}_i^*\rangle_{i \in I}$ is Boolean-independent. But this means that $E = \bigcup_{i \in I} E_i$ is Boolean-independent (515De), while

$$\#(E) \geq \sup_{i \in I} \tau(\mathfrak{A})^{\#(D_i)} = \sup_{\lambda < \mathrm{sat}(\mathfrak{A})} \tau(\mathfrak{A})^{\lambda} \geq \kappa.$$

Of course $\#(E) \leq \#(\mathfrak{A}) = \kappa$, so we have a Boolean-independent set with cardinal κ in this case also. **Q**

(e) If $\mathfrak{A}$ is atomless it has a Boolean-independent subset with cardinal κ. **P** Because Maharam type and cellularity are both order-preserving cardinal functions (514Ed), $\mathfrak{A}$ is isomorphic to the product of a family $\langle \mathfrak{A}_i\rangle_{i \in I}$ of Maharam-type-homogeneous cellularity-homogeneous algebras, none of them $\{0\}$ (514Gc). Now, for each i, $\mathfrak{A}_i$ is an atomless (therefore infinite) Maharam-type-homogeneous cellularity-homogeneous Dedekind complete Boolean algebra, so by (c)-(d) above has a Boolean-independent set with cardinal $\#(\mathfrak{A}_i)$. By 515G once more, $\mathfrak{A}$ has a Boolean-independent set with cardinal $\#(\prod_{i \in I} \mathfrak{A}_i) = \kappa$. **Q**

(f) Finally, for the general case, let A be the set of atoms of A and set $c = \sup A$, so that the principal ideal $\mathfrak{A}_c$ is purely atomic and the principal ideal $\mathfrak{A}_{1 \setminus c}$ is atomless. Because $\mathfrak{A} \cong \mathfrak{A}_c \times \mathfrak{A}_{1 \setminus c}$ is infinite, one of $\mathfrak{A}_c$, $\mathfrak{A}_{1 \setminus c}$ has cardinal κ, and therefore (by (a) or (f)) has a Boolean-independent subset with cardinal κ; which is now a Boolean independent subset of $\mathfrak{A}$ with cardinal κ.

This completes the proof.

515I Corollary If $\mathfrak{A}$ is an infinite Dedekind complete Boolean algebra and $\kappa \leq \#(\mathfrak{A})$, $\mathfrak{A}$ has a subalgebra isomorphic to the regular open algebra of $\{0,1\}^{\kappa}$.

proof By 515H, $\mathfrak{A}$ has a Boolean-independent family $\langle a_\xi\rangle_{\xi < \kappa}$. By 515Cb, $\mathfrak{A}$ has a subalgebra isomorphic to the regular open algebra of $\{0,1\}^{\kappa}$.

515J Corollary If $\mathfrak{A}$ is an infinite Dedekind complete Boolean algebra with Stone space Z, then $\#(Z) = 2^{\#(\mathfrak{A})}$.

proof Since Z may be identified with the set of uniferent ring homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$ (311E), $\#(Z) \leq 2^{\#(\mathfrak{A})}$. On the other hand, writing $W = \{0,1\}^{\#(\mathfrak{A})}$, we have a subalgebra of $\mathfrak{A}$ isomorphic to the algebra $\mathcal{E}$ of open-and-closed subsets of W (515I). If $\pi : \mathcal{E} \to \mathfrak{A}$ is an injective Boolean homomorphism, it corresponds to a surjective continuous function $\psi : Z \to W$ (312Sa), so that $\#(Z) \geq \#(W) = 2^{\#(\mathfrak{A})}$.

515K I extract part of the proof of the next theorem as a lemma.

Lemma Let $\mathfrak{A}$ be an infinite Boolean algebra with the σ-interpolation property.

(a) Let $\langle a_n\rangle_{n \in \mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$. Then $\#(\mathfrak{A}) \geq \#(\prod_{n \in \mathbb{N}} \mathfrak{A}_{a_n})$, writing $\mathfrak{A}_d$ for the principal ideal of $\mathfrak{A}$ generated by d, as usual.

(b) Set $\kappa = \#(\mathfrak{A})$, and let I be the set of those $a \in \mathfrak{A}$ such that $\#(\mathfrak{A}_a) < \kappa$. Then I is an ideal of $\mathfrak{A}$, and
either $\mathfrak{A}/I$ is infinite,
or there is a set $J \subseteq I$ with cardinal κ such that every sequence in J has an upper bound in J,
or $\#(\prod_{n \in \mathbb{N}} \mathfrak{A}_{a_n}) = \kappa$ for some sequence $\langle a_n\rangle_{n \in \mathbb{N}}$ in I.

Remark Recall from 466G that $\mathfrak{A}$ has the 'σ-interpolation property' if whenever A, $B \subseteq \mathfrak{A}$ are countable and $a \subseteq b$ for every $a \in A$ and $b \in B$, then there is a $c \in \mathfrak{A}$ such that $a \subseteq c \subseteq b$ for every $a \in A$ and $b \in B$. See also 514Yf above.

proof (a) The point is that the map $a \mapsto \langle a \cap a_n \rangle_{n\in\mathbb{N}} : \mathfrak{A} \to \prod_{n\in\mathbb{N}} \mathfrak{A}_{a_n}$ is surjective. **P** If $\langle b_n \rangle_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathfrak{A}_{a_n}$, there must be an $a \in \mathfrak{A}$ such that $b_n \subseteq a \subseteq 1 \setminus (a_n \setminus b_n)$ for every n, so that $a \cap a_n = b_n$ for every n. **Q** The result follows at once.

(b) If a, $b \in I$ then $(c, d) \mapsto c \cup d$ is a surjection from $\mathfrak{A}_a \times \mathfrak{A}_b$ onto $\mathfrak{A}_{a\cup b}$, so $a \cup b \in I$; of course $b \in I$ whenever $b \subseteq a \in I$, so I is an ideal of $\mathfrak{A}$.

? Suppose, if possible, that all three alternatives are false. Then $\mathfrak{A}/I$ is finite; let $v_0, \dots, v_m$ be its atoms. Let $c_0, \dots, c_m \in \mathfrak{A}$ be such that $c_i^\bullet = v_i$ for every i. Observe that $\mathfrak{A}$ is the union of finitely many sets of size $\#(I)$, so I itself must have cardinal κ, and there is a sequence $\langle b'_n \rangle_{n\in\mathbb{N}}$ in I with no upper bound in I; setting $b_n = b'_n \setminus \sup_{m<n} b'_m$ for each n, we get a disjoint sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ in I with no upper bound in I. Now there is some $k \le m$ such that $\langle b_n \cap c_k \rangle_{n\in\mathbb{N}}$ has no upper bound in I. Set $K = \{d : d \subseteq c_k,\ d \cap b_n = 0 \text{ for every } n \in \mathbb{N}\}$. Then $K \lhd \mathfrak{A}_{c_k}$. If $d \in K$, $c_k \setminus d$ is an upper bound for $\{b_n \cap c_k : n \in \mathbb{N}\}$, so does not belong to I; as $c_k^\bullet$ is an atom in $\mathfrak{A}/I$, d must belong to I. Thus $K \subseteq I$. The function $d \mapsto \langle d \cap b_n \rangle_{n\in\mathbb{N}} : \mathfrak{A}_{c_k} \to \prod_{n\in\mathbb{N}} \mathfrak{A}_{c_k \cap b_n}$ is a Boolean homomomorphism with kernel K, so

$$\#(\mathfrak{A}_{c_k}/K) \le \#(\textstyle\prod_{n\in\mathbb{N}} \mathfrak{A}_{c_k\cap b_n}) < \kappa$$

(since the third alternative is false, and $\#(\prod_{n\in\mathbb{N}} \mathfrak{A}_{c_k\cap b_n}) \le \kappa$ by (a)); as $\#(\mathfrak{A}_{c_k}) = \kappa$, $\#(K) = \kappa$. There is therefore a sequence $\langle d_n \rangle_{n\in\mathbb{N}}$ in K with no upper bound in K. But there is a $d \in \mathfrak{A}$ such that $d_n \subseteq d \subseteq 1 \setminus b_n$ for every $n \in \mathbb{N}$, because $\mathfrak{A}$ has the σ-interpolation property; so that $d \cap c_k \in K$ is an upper bound for $\{d_n : n \in \mathbb{N}\}$. **X**

515L Theorem (Koppelberg 75) If $\mathfrak{A}$ is an infinite Boolean algebra with the σ-interpolation property, then $\#(\mathfrak{A})$ is equal to the cardinal power $\#(\mathfrak{A})^\omega$.

proof Induce on $\kappa = \#(\mathfrak{A})$.

(a) If $\kappa \le \mathfrak{c}$, then (because $\mathfrak{A}$ is infinite) there is a disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}^+$, so that

$$\mathfrak{c} \le \#(\textstyle\prod_{n\in\mathbb{N}} \mathfrak{A}_{a_n}) \le \#(\mathfrak{A})$$

by 515Ka, and $\kappa = \mathfrak{c}$. So $\kappa^\omega = (2^\omega)^\omega = \kappa$.

(b) For the inductive step to $\kappa > \mathfrak{c}$, set $I = \{a : a \in \mathfrak{A},\ \#(\mathfrak{A}_a) < \kappa\}$, as in 515Kb. It is easy to see that every principal ideal of $\mathfrak{A}$ has the σ-interpolation property, so that $\#(\mathfrak{A}_a)^\omega \le \max(\mathfrak{c}, \#(\mathfrak{A}_a))$ for every $a \in I$. Now consider the three possibilities of 515Kb.

case 1 If the quotient algebra $\mathfrak{A}/I$ is infinite, then $\kappa^\omega = \kappa$. **P** There is a disjoint sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}/I$. For each $n \in \mathbb{N}$ take $a_n \in \mathfrak{A}$ such that $a_n^\bullet = u_n$; now setting $a'_n = a_n \setminus \sup_{i<n} a_i$ for each n, $\langle a'_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A} \setminus I$. So

$$\kappa \le \kappa^\omega = \#(\textstyle\prod_{n\in\mathbb{N}} \mathfrak{A}_{a'_n}) \le \kappa$$

by 515Ka again. **Q**

case 2 Suppose that there is a set $J \subseteq I$ such that $\#(J) = \kappa$ and every sequence in J has an upper bound in J. Then $\kappa^\omega = \kappa$. **P**

$$\kappa^\omega = \#(J^{\mathbb{N}}) \le \#(\bigcup_{a\in J} \mathfrak{A}_a^{\mathbb{N}})$$

(because every sequence in J is included in $\mathfrak{A}_a$ for some $a \in J$)

$$\le \max(\omega, \#(J), \sup_{a\in I} \#(\mathfrak{A}_a^{\mathbb{N}})) \le \max(\kappa, \sup_{a\in J} \#(\mathfrak{A}_a)) = \kappa \le \kappa^\omega. \text{ **Q**}$$

case 3 Suppose there is a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in I such that $\#(\prod_{n\in\mathbb{N}} \mathfrak{A}_{a_n}) = \kappa$. Then $\kappa^\omega = \kappa$. **P** Set $L = \{n : n \in \mathbb{N},\ \mathfrak{A}_{a_n} \text{ is infinite}\}$. Then

$$\kappa^\omega = \#((\prod_{n\in\mathbb{N}} \mathfrak{A}_{a_n})^{\mathbb{N}}) = \#(\prod_{n\in\mathbb{N}\setminus L} \mathfrak{A}_{a_n}^{\mathbb{N}} \times \prod_{n\in L} \mathfrak{A}_{a_n}^{\mathbb{N}})$$

$$\le \#(\mathfrak{c} \times \prod_{n\in L} \mathfrak{A}_{a_n}) \le \max(\mathfrak{c}, \kappa) = \kappa. \text{ **Q**}$$

Thus in all three cases we have $\kappa^\omega = \kappa$, and the induction proceeds.

515M Corollary (a) If $\mathfrak{A}$ is an infinite ccc Dedekind σ-complete Boolean algebra then $\#(\mathfrak{A}) = \tau(\mathfrak{A})^\omega$.

(b) If $\mathfrak{A}$ is any infinite Dedekind σ-complete Boolean algebra, then $\#(L^0(\mathfrak{A})) = \#(L^\infty(\mathfrak{A})) = \#(\mathfrak{A})$.

proof (a) Of course $\mathfrak{A}$, being Dedekind σ-complete, has the σ-interpolation property, as noted in 466G. So by 515L and 514De,

$$\tau(\mathfrak{A})^\omega \le \#(\mathfrak{A})^\omega = \#(\mathfrak{A}) \le \tau(\mathfrak{A})^\omega.$$

(b) $a \mapsto \chi a : \mathfrak{A} \to L^\infty(\mathfrak{A})$ and $u \mapsto \langle [\![u > q]\!] \rangle_{q\in\mathbb{Q}} : L^0(\mathfrak{A}) \to \mathfrak{A}^{\mathbb{Q}}$ are injective, so

$$\#(\mathfrak{A}) \le \#(L^\infty(\mathfrak{A})) \le \#(L^0(\mathfrak{A})) \le \#(\mathfrak{A})^\omega = \#(\mathfrak{A}).$$

515N It will be convenient at one point later to know a little more about the regular open algebras of powers of $\{0,1\}$.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, and κ a cardinal. Then $\mathfrak{A}$ is isomorphic to the regular open algebra $\mathrm{RO}(\{0,1\}^\kappa)$ iff it is Dedekind complete and there is a Boolean-independent family $\langle e_\xi \rangle_{\xi<\kappa}$ in $\mathfrak{A}$ such that the subalgebra generated by $\{e_\xi : \xi < \kappa\}$ is order-dense in $\mathfrak{A}$.

proof Write $\mathcal{E}$ for the algebra of open-and-closed subsets of $\{0,1\}^\kappa$.

(a) $\mathrm{RO}(\{0,1\}^\kappa)$ is Dedekind complete (314P) and $\mathcal{E}$ is order-dense in $\mathrm{RO}(\{0,1\}^\kappa)$, because $\{0,1\}^\kappa$ is zero-dimensional. Setting $e_\xi = \{x : x \in \{0,1\}^\kappa,\ x(\xi) = 1\}$, $\langle e_\xi \rangle_{\xi<\kappa}$ is Boolean-independent and generates $\mathcal{E}$. So $\mathrm{RO}(\{0,1\}^\kappa)$ has the declared properties.

(b) If $\mathfrak{A}$ satisfies the conditions, then, as in (a-i) of the proof of 515C, the subalgebra of $\mathfrak{A}$ generated by $\{e_\xi : \xi < \kappa\}$ is isomorphic to $\mathcal{E}$. Now both $\mathfrak{A}$ and $\mathrm{RO}(\{0,1\}^\kappa)$ are Dedekind completions of $\mathcal{E}$, so they are isomorphic (314U).

515X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\langle D_i \rangle_{i\in I}$ a family of partitions of unity in $\mathfrak{A}$, none containing 0. Show that the following are equiveridical: (i) $\langle D_i \rangle_{i\in I}$ is Boolean-independent; (ii) $\langle \mathfrak{B}_i \rangle_{i\in I}$ is Boolean-independent, where $\mathfrak{B}_i$ is the subalgebra of $\mathfrak{A}$ generated by D_i for each $i \in I$.

(b) Give an example of a Boolean algebra $\mathfrak{A}$ with Boolean-independent subalgebras $\mathfrak{B}$, $\mathfrak{C}$ such that the order-closed subalgebras generated by $\mathfrak{B}$ and $\mathfrak{C}$ are not Boolean-independent.

(c) For a Boolean algebra $\mathfrak{A}$, not $\{0\}$, write $\mathrm{ind}(\mathfrak{A})$ for $\sup\{\#(A) : A \subseteq \mathfrak{A}$ is Boolean-independent$\}$. (If $\mathfrak{A} = \{0\}$, say $\mathrm{ind}(\mathfrak{A}) = 0$.) (i) Show that if $\mathfrak{B}$ is either a subalgebra or a principal ideal or a homomorphic image of $\mathfrak{A}$ then $\mathrm{ind}(\mathfrak{B}) \le \mathrm{ind}(\mathfrak{A})$. (ii) Show that $\mathfrak{A}$ is infinite iff $\mathrm{ind}(\mathfrak{A})$ is infinite. (iii) Show that if $\mathfrak{A}$ is finite and not $\{0\}$ then $\mathrm{ind}(\mathfrak{A})$ is the largest n such that $2^{2^n} \le \#(\mathfrak{A})$. (iv) Show that if $\mathfrak{A}$ is the finite-cofinite algebra of subsets of an infinite set X, then $\mathrm{ind}(\mathfrak{A}) = \omega$ but $\mathfrak{A}$ has no infinite Boolean-independent set. (v) Show that if $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras then $\mathrm{ind}(\mathfrak{A} \times \mathfrak{B})$ is at most the cardinal sum $\mathrm{ind}(\mathfrak{A}) + \mathrm{ind}(\mathfrak{B})$. (vi) Show that if $\mathfrak{A}$ is infinite and has the σ-interpolation property then $\mathrm{ind}(\mathfrak{A}) \ge \mathfrak{c}$.

(d) Let Z be an infinite extremally disconnected compact Hausdorff space. Show that there is a continuous surjection from Z onto $\{0,1\}^{w(Z)}$.

(e) Let $\mathfrak{A}$ be a Boolean algebra with the σ-interpolation property. Show that any homomorphic image of $\mathfrak{A}$ has the σ-interpolation property.

(f) Let κ be an infinite cardinal. Show that the following are equiveridical: (i) there is a measure algebra with cardinal κ; (ii) there is a measurable algebra with cardinal κ; (iii) $\kappa^\omega = \kappa$.

515Y Further exercises (a)(i) Show that if $\mathfrak{A}$ is any Boolean algebra, other than $\{0\}$, with cardinal at most ω_1, it is isomorphic to a subalgebra of $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$. (ii) Show that an atomless Boolean algebra with cardinal ω_1 and the σ-interpolation property is isomorphic to $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$. (This is a version of **Parovičenko's theorem**.)

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and $\kappa \le \#(\mathfrak{A})$ a regular uncountable cardinal. Show that there is a strictly increasing family $\langle \mathfrak{A}_\xi \rangle_{\xi<\kappa}$ of subalgebras of $\mathfrak{A}$ with union $\mathfrak{A}$. (Compare 494Yk.)

515 Notes and comments The material of this section is taken from Koppelberg 89, where you can find a good deal more. I have picked out the results which are essential to a proper understanding of measure algebras. Of course there are short cuts, using Maharam's theorem (332B), if we know that we are dealing with a localizable measure algebra; but I should not like to leave you with the impression that the theorems here are restricted to measure algebras.

Any theorem about Boolean algebras is also a theorem about zero-dimensional compact Hausdorff spaces; thus 515H and 515Xd have an equal right to be called the Balcar-Franěk theorem. 515D and part (b) of the proof of 515H may be regarded as a simple form of some of the ideas of §331.

Clearly some of the ideas of this section can be expressed in terms of the independence number $\operatorname{ind}(\mathfrak{A})$ (515Xc). But the expression is complicated by the fact that (like cellularity) the independence number may not be attained (see 515Xc(iv)), while the theorems here mostly need actual independent families. Since $\operatorname{ind}(\mathfrak{A}) = \#(\mathfrak{A})$ for infinite Dedekind complete Boolean algebras (515H), we shall not have to grapple with these difficulties.

516 Precalibers

In this section I will try to display the elementary connexions between 'precalibers', as defined in 511E, and the cardinal functions we have looked at so far. The first step is to generalize the idea of precaliber from partially ordered sets to supported relations (516A); the point is that Galois-Tukey connections give us information on precalibers (516C), and in particular give quick proofs that partially ordered sets, topological spaces and Boolean algebras related in the canonical ways explored in §514 have many of the same precalibers (516G, 516H, 516M). Much of the section is taken up with lists of expected facts, but for some results the hypotheses need to be chosen with care. I end with a fundamental theorem on the saturation of product spaces (516T).

516A Definition If (A, R, B) is a supported relation, a **precaliber triple** of (A, R, B) is a triple $(\kappa, \lambda, {<}\theta)$ where κ, λ and θ are cardinals and whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in A then there is a set $\Gamma \in [\kappa]^\lambda$ such that $\langle a_\xi\rangle_{\xi\in\Gamma}$ is ${<}\theta$-linked in the sense of 512Bc, that is, for every $I \in [\Gamma]^{<\theta}$ there is a $b \in B$ such that $(a_\xi, b) \in R$ for every $\xi \in I$. Similarly, $(\kappa, \lambda, \theta)$ is a precaliber triple of (A, R, B) if whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in A then there is a set $\Gamma \in [\kappa]^\lambda$ such that $\langle a_\xi\rangle_{\xi\in\Gamma}$ is θ-linked; that is, if $(\kappa, \lambda, {<}\theta^+)$ is a precaliber triple.

Now (κ, λ) is a **precaliber pair** of (A, R, B) if $(\kappa, \lambda, {<}\omega)$ is a precaliber triple of (A, R, B), and κ is a **precaliber** of (A, R, B) if (κ, κ) is a precaliber pair.

516B Elementary remarks I ought perhaps to spell out the following immediate consequences of the definitions. Let (A, R, B) be a supported relation.

(a) If $\kappa' \ge \kappa$, $\lambda' \le \lambda$, $\theta' \le \theta$ and $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of (A, R, B), then $(\kappa', \lambda', {<}\theta')$ is a precaliber triple of (A, R, B). So if $\kappa' \ge \kappa$, $\lambda' \le \lambda$ and (κ, λ) is a precaliber pair of (A, R, B), then (κ', λ') is a precaliber pair of (A, R, B).

(b) If $\theta > 0$, then $(0, 0, {<}\theta)$ is a precaliber triple of (A, R, B) iff $B \ne \emptyset$. If $A = \emptyset$ then $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of (A, R, B) whenever $\kappa \ge 1$. If $A \ne \emptyset$ and $A \ne R^{-1}[B]$, that is, $\operatorname{cov}(A, R, B) = \infty$, then the only precaliber triples of (A, R, B) are of the form $(\kappa, 0, {<}\theta)$. If $A \ne \emptyset$ and $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of (A, R, B), then $\lambda \le \kappa$. $\operatorname{cov}(A, R, B) = \infty$ iff 1 is not a precaliber of (A, R, B).

(c) If $(\kappa, \lambda, \lambda)$ is a precaliber triple of (A, R, B) then $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of (A, R, B) for every θ; in particular, (κ, λ) is a precaliber pair of (A, R, B).

(d) If $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of (A, R, B), so is $(\operatorname{cf}\kappa, \operatorname{cf}\kappa, {<}\theta)$. **P** If $\operatorname{cf}\kappa = \kappa$ there is nothing to prove. If $2 \le \kappa < \omega$ and A is empty the result is trivial. If $2 \le \kappa < \omega$ and A is not empty, then B is not empty, so if $\theta \le 1$ the result is trivial. If $2 \le \kappa < \omega$ and A is not empty and $\theta > 1$, then $R^{-1}[B] = A$ so $(\operatorname{cf}\kappa, \operatorname{cf}\kappa, {<}\theta) = (1, 1, {<}\theta)$ is a precaliber triple of (A, R, B).

If $\kappa > \operatorname{cf}\kappa$ is infinite, let $\langle\gamma_\xi\rangle_{\xi<\operatorname{cf}\kappa}$ be a strictly increasing family with supremum κ. For $\eta < \kappa$, set $f(\eta) = \min\{\xi : \eta \le \gamma_\xi\}$. If $\langle a_\xi\rangle_{\xi<\operatorname{cf}\kappa}$ is a family in A, set $a'_\eta = a_{f(\eta)}$ for each $\eta < \kappa$. Then there is a $\Gamma \in [\kappa]^\kappa$ such that $\langle a'_\eta\rangle_{\eta\in\Gamma}$ is ${<}\theta$-linked. Set $\Gamma' = \{f(\eta) : \eta \in \Gamma\}$; then $\langle a_\xi\rangle_{\xi\in\Gamma'}$ is ${<}\theta$-linked. Also Γ must be cofinal with κ, so Γ' is cofinal with $\operatorname{cf}\kappa$ and $\#(\Gamma') = \operatorname{cf}\kappa$. As $\langle a_\xi\rangle_{\xi<\operatorname{cf}\kappa}$ is arbitrary, $(\operatorname{cf}\kappa, \operatorname{cf}\kappa, {<}\theta)$ is a precaliber triple of (A, R, B). **Q**

In particular, if κ is a precaliber of (A, R, B), so is $\operatorname{cf}\kappa$.

516C Theorem Suppose that (A,R,B) and (C,S,D) are supported relations, and that $(A,R,B)\preccurlyeq_{\mathrm{GT}}(C,S,D)$. Then $(\kappa,\lambda,{<}\theta)$ or (κ,λ,θ) is a precaliber triple of (A,R,B) whenever it is a precaliber triple of (C,S,D), so (κ,λ) is a precaliber pair of (A,R,B) whenever it is a precaliber pair of (C,S,D), and κ is a precaliber of (A,R,B) whenever it is a precaliber of (C,S,D).

proof Let (ϕ,ψ) be a Galois-Tukey connection from (A,R,B) to (C,R,S). If $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of (C,S,D), and $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in A, then there is a set $\Gamma\in[\kappa]^\lambda$ such that whenever $I\in[\Gamma]^{<\theta}$ there is a $d\in D$ such that $(f(a_\xi),d)\in S$ for every $\xi\in I$, and now $(a_\xi,g(d))\in R$ for every $\xi\in I$. Thus $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of (A,R,B). The results for precaliber pairs and precalibers follow at once.

516D Corollary If $(A,R,B)\equiv_{\mathrm{GT}}(C,S,D)$ then (A,R,B) and (C,S,D) have the same precaliber triples, the same precaliber pairs and the same precalibers.

516E Remark Because all the definitions in 516A start from precaliber triples $(\kappa,\lambda,{<}\theta)$, any theorem about such precaliber triples is likely to lead at once to corresponding results concerning precaliber triples (κ,λ,θ), precaliber pairs and precalibers. In the rest of this section I shall not always take the space to spell these out systematically, and when later I wish to use a fact about precalibers I may direct you, without comment, to a fact about precaliber triples or pairs from which it may be deduced.

516F The next step is to check the connexion between the definition in 516A and those of §511. But this is elementary.

Proposition (a) If P is a partially ordered set, $(\kappa,\lambda,{<}\theta)$ or (κ,λ,θ) is a precaliber triple of $(P,\le,P)$ iff it is an upwards precaliber triple of P.

(b) If $\mathfrak{A}$ is a Boolean algebra, then $\mathfrak{A}$ and $(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)$ have the same precaliber triples, where $\mathfrak{A}^+=\mathfrak{A}\setminus\{0\}$.

(c) If $(X,\mathfrak{T})$ is a topological space, then X and $(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})$ have the same precaliber triples.

proof Read the definitions in 511E and 516A.

516G Corollary Let $(P,\le)$ be a partially ordered set.

(a) If Q is a cofinal subset of P, then P and Q have the same upwards precaliber triples.

(b) Let $\mathfrak{T}^\uparrow$ be the up-topology of P (definition: 514L). Then $(\kappa,\lambda,{<}\theta)$ is an upwards precaliber triple for $(P,\le)$ iff it is a precaliber triple for $(P,\mathfrak{T}^\uparrow)$.

proof (a) By 513E(d-ii), $(P,\le,P)\equiv_{\mathrm{GT}}(Q,\le,Q)$.

(b) By 514Na, $(P,\le,P)\equiv_{\mathrm{GT}}(\mathfrak{T}^\uparrow\setminus\{\emptyset\},\supseteq,\mathfrak{T}^\uparrow\setminus\{\emptyset\})$.

516H Corollary Let $\mathfrak{A}$ be a Boolean algebra.

(a) If Z is the Stone space of $\mathfrak{A}$, then $\mathfrak{A}$ and Z have the same precaliber triples.

(b) If $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ and $\mathfrak{B}$ have the same precaliber triples.

proof (a) Write $\mathfrak{T}$ for the topology of Z and $\mathcal{E}$ for the algebra of open-and-closed sets. Because Z is zero-dimensional, $\mathcal{E}^+$ is coinitial with $\mathfrak{T}\setminus\{\emptyset\}$, so $(\mathfrak{A}^+,\supseteq,\mathfrak{A}^+)\cong(\mathcal{E}^+,\supseteq,\mathcal{E}^+)$ and $(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})$ have the same precaliber triples, by 516Ga, inverted.

(b) $\mathfrak{B}^+$ is coinitial with $\mathfrak{A}^+$, so we can use the same idea.

516I Corollary Let $(X,\mathfrak{T})$ be a topological space.

(a) If Y is an open subspace of X, then every precaliber triple of X is a precaliber triple of Y.

(b) If Y is a dense subspace of X, then every precaliber triple of X is a precaliber triple of Y.

(c) If X is regular and Y is a dense subspace of X, then X and Y have the same precaliber triples.

(d) Suppose that Y is a topological space, and that there is a continuous surjection $f:X\to Y$ such that $\operatorname{int}f[G]\ne\emptyset$ whenever $G\subseteq X$ is a non-empty open set. Then every precaliber triple of X is a precaliber triple of Y.

proof (a) Write $\mathfrak{S}$ for the topology of Y. For $H\in\mathfrak{S}\setminus\{\emptyset\}$, set $\phi(H)=H$; for $G\in\mathfrak{T}\setminus\{\emptyset\}$, set $\psi(G)=G\cap Y$ if this is non-empty, G otherwise. Then (ϕ,ψ) is a Galois-Tukey connection from $(\mathfrak{S}\setminus\{\emptyset\},\supseteq,\mathfrak{S}\setminus\{\emptyset\})$ to $(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})$, so 516C and 516Fc give the result.

(b) Again write $\mathfrak{S}$ for the topology of Y. For $H \in \mathfrak{S} \setminus \{\emptyset\}$, set $\phi(H) = X \setminus \overline{Y \setminus H}$, where the closure is taken in X; for $G \in \mathfrak{T} \setminus \{\emptyset\}$, set $\psi(G) = G \cap Y$. Then (ϕ, ψ) is a Galois-Tukey connection from $(\mathfrak{S} \setminus \{\emptyset\}, \supseteq, \mathfrak{S} \setminus \{\emptyset\})$ to $(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$, so again we have the result.

(c) If now X is regular, then for each $G \in \mathfrak{T} \setminus \{\emptyset\}$ choose $V_G \in \mathfrak{T} \setminus \{\emptyset\}$ such that $\overline{V}_G \subseteq G$ and set $\psi'(G) = V_G \cap Y$. Then (ψ', ϕ) is a Galois-Tukey connection from $(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$ to $(\mathfrak{S} \setminus \{\emptyset\}, \supseteq, \mathfrak{S} \setminus \{\emptyset\})$, so every precaliber triple of Y is a precaliber triple of X.

(d) Once more writing $\mathfrak{S}$ for the topology of Y, set $\phi(H) = f^{-1}[H]$ for every $H \in \mathfrak{S} \setminus \{\emptyset\}$ and $\psi(G) = \operatorname{int} f[G]$ for every $G \in \mathfrak{T} \setminus \{\emptyset\}$; then again (ϕ, ψ) is a Galois-Tukey connection from $(\mathfrak{S} \setminus \{\emptyset\}, \supseteq, \mathfrak{S} \setminus \{\emptyset\})$ to $(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$.

Remark For variations on (b) and (d) here, see 516Xh and 516Oa.

516J Straightforward counting arguments give us some connexions between precalibers and other cardinal functions, as follows.

Proposition Let (A, R, B) be a supported relation.

(a) $\operatorname{sat}(A, R, B)$ is the least cardinal κ, if there is one, such that $(\kappa, 2)$ is a precaliber pair of (A, R, B); if there is no such κ, $\operatorname{sat}(A, R, B) = \infty$. In particular, if $\kappa \geq 2$ is a precaliber of (A, R, B), then $\kappa \geq \operatorname{sat}(A, R, B)$.

(b) If $\kappa > \max(\omega, \lambda, \operatorname{link}_{<\theta}(A, R, B))$ then $(\kappa, \lambda^+, {<\theta})$ is a precaliber triple of (A, R, B). In particular, if $\kappa > \max(\omega, \lambda, \operatorname{cov}(A, R, B))$ then $(\kappa, \lambda^+, {<\theta})$ is a precaliber triple of (A, R, B) for every θ.

(c) If $\operatorname{cf} \kappa > \operatorname{link}_{<\theta}(A, R, B)$ then $(\kappa, \kappa, {<\theta})$ is a precaliber triple of (A, R, B).

proof (a) If $(\kappa, 2)$ is a precaliber pair of (A, R, B), and $\langle a_\xi \rangle_{\xi<\kappa}$ is any family in A, then there must be a $\Gamma \in [\kappa]^2$ such that for every finite $I \subseteq \Gamma$ there is a $b \in B$ such that $(a_\xi, b) \in R$ for every $\xi \in I$. But this means that if $\Gamma = \{\xi, \eta\}$ then ξ, η are distinct members of κ such that, for some $b \in B$, both (a_ξ, b) and (a_η, b) belong to R. As $\langle a_\xi \rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{sat}(A, R, B) \leq \kappa$.

Conversely, any witness that $(\kappa, 2)$ is not a precaliber pair of (A, R, B) will provide a witness that $\operatorname{sat}(A, R, B) > \kappa$.

Now if $\kappa \geq 2$ is a precaliber of (A, R, B), that is, (κ, κ) is a precaliber pair, then $(\kappa, 2)$ is a precaliber pair of (A, R, B), by 516Ba, so $\kappa \geq \operatorname{sat}(A, R, B)$.

(b) Write δ for $\operatorname{link}_{<\theta}(A, R, B)$, and let $\langle A_\eta \rangle_{\eta<\delta}$ be a cover of A by ${<\theta}$-linked sets. Let $\langle a_\xi \rangle_{\xi<\kappa}$ be any family in A. For $\eta < \delta$ set $C_\eta = \{\xi : a_\xi \in A_\eta\}$; then $\kappa = \bigcup_{\eta<\delta} C_\eta$ so there must be some $\eta < \delta$ such that $\#(C_\eta) > \lambda$. Now if $\Gamma \subseteq C_\eta$ is a set of size λ^+, $\{a_\xi : \xi \in \Gamma\}$ is ${<\theta}$-linked in (A, R, B). As $\langle a_\xi \rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \lambda^+, {<\theta})$ is a precaliber triple of (A, R, B).

The special case is now elementary, if we remember that $\operatorname{link}_{<\theta}(A, R, B) \leq \operatorname{cov}(A, R, B)$ for every θ (512Bc).

(c) If $\operatorname{link}_{<\theta}(A, R, B) = 0$ then $A = \emptyset$ and the result is trivial. Otherwise, $\operatorname{cf} \kappa \geq \omega$. Choose δ and $\langle C_\eta \rangle_{\eta<\delta}$ as in (b) above. Let $\langle a_\xi \rangle_{\xi<\kappa}$ be any family in A. Then there must be some $\eta < \delta$ such that $\#(C_\eta) = \kappa$, and $\{a_\xi : \xi \in C_\eta\}$ is ${<\theta}$-linked in (A, R, B). As $\langle a_\xi \rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \kappa, {<\theta})$ is a precaliber triple of (A, R, B).

516K For partially ordered sets, we have translations of the results above, and a further useful fact.

Proposition Let P be a partially ordered set.

(a) $\operatorname{sat}^\uparrow(P)$ is the least cardinal κ such that $(\kappa, 2)$ is an upwards precaliber pair of P.

(b) If $\kappa > \max(\omega, \lambda, \operatorname{link}^\uparrow_{<\theta}(P))$ then $(\kappa, \lambda^+, {<\theta})$ is an upwards precaliber triple of P. In particular, if $\kappa > \max(\omega, \lambda, \operatorname{cf} P)$ then $(\kappa, \lambda^+, {<\theta})$ is an upwards precaliber triple of P for every θ, and if $\kappa > \max(\omega, \lambda, d^\uparrow(P))$ then (κ, λ^+) is an upwards precaliber pair of P.

(c) If $\operatorname{cf} \kappa > \operatorname{cf} P$ then $(\kappa, \kappa, {<\theta})$ is an upwards precaliber triple of P for every θ. If $\operatorname{cf} \kappa > d^\uparrow(P)$ then κ is an up-precaliber of P.

(d) If $\operatorname{sat}^\uparrow(P) \geq \omega$, $(\operatorname{sat}^\uparrow(P), \omega)$ is an upwards precaliber pair of P.

proof (a)-(c) We need only identify $\operatorname{cf} P$ with $\operatorname{cov}(P, \leq, P) \geq \sup_\theta \operatorname{link}_{<\theta}(P, \leq, P)$ (512Bc) and $d^\uparrow(P)$ with the centering number $\operatorname{link}_{<\omega}(P, \leq, P)$, as in 512Ea.

(d)(i) Set $\kappa = \operatorname{sat}^\uparrow(P)$. By 513Bb, κ is a regular uncountable cardinal. The first thing to note is that if $\langle p_\xi \rangle_{\xi<\kappa}$ is any family in P, then there is a $\zeta < \kappa$ such that $\{\xi : \xi < \kappa,\ p_\xi \text{ and } p_\zeta \text{ are compatible upwards in } P\}$ has cardinal κ. **P?** Otherwise, for each $\zeta < \kappa$ there is an $\alpha_\zeta < \kappa$ such that p_ζ and p_ξ are upwards-incompatible for every $\xi \geq \alpha_\zeta$. Set $C = \{\xi : \xi < \kappa,\ \alpha_\eta \leq \xi \text{ for every } \eta < \xi\}$. Then $\#(C) = \kappa$ and $\langle p_\xi \rangle_{\xi \in C}$ is an up-antichain in P, which is impossible. **XQ**

(ii) Now let $\langle p_\xi\rangle_{\xi<\kappa}$ be a family in P. Choose inductively sets $A_n \in [\kappa]^\kappa$, ordinals $\zeta_n \in A_n$ and families $\langle p_{n\xi}\rangle_{\xi\in A_n}$ in P, as follows. $A_0 = \kappa$, $p_{0\xi} = p_\xi$ for each $\xi < \kappa$. Given $\langle p_{n\xi}\rangle_{\xi\in A_n}$, then by (i) there is a $\zeta_n \in A_n$ such that

$$A_{n+1} = \{\xi : \xi \in A_n,\ \xi \neq \zeta_n,\ p_{n\xi} \text{ is compatible upwards with } p_{n,\zeta_n}\}$$

has cardinal κ. Now, for $\xi \in A_{n+1}$, let $p_{n+1,\xi}$ be an upper bound of $\{p_{n,\zeta_n}, p_{n\xi}\}$; continue.

At the end of the induction, observe that $\langle p_{n,\zeta_n}\rangle_{n\in\mathbb{N}}$ is non-decreasing. At the same time, we see that $p_\xi \le p_{n\xi}$ whenever $n \in \mathbb{N}$ and $\xi \in A_n$. So $\{p_{\zeta_n} : n \in \mathbb{N}\}$ is upwards-centered. Also the ζ_n are all different, so $\Gamma = \{\zeta_n : n \in \mathbb{N}\}$ is infinite. As $\langle p_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\mathrm{sat}^\uparrow(P), \omega)$ is an upwards precaliber pair of P.

Remark There will be a stronger form of (d) in 517Fa below.

516L Corollary Let $\mathfrak{A}$ be a Boolean algebra.

(a) $\mathrm{sat}(\mathfrak{A})$ is the least cardinal κ such that $(\kappa, 2)$ is a precaliber pair of $\mathfrak{A}$.

(b) If $\kappa > \max(\omega, \lambda, \mathrm{link}_{<\theta}(\mathfrak{A}))$ then $(\kappa, \lambda^+, {<}\theta)$ is a precaliber triple of $\mathfrak{A}$. In particular, if $\kappa > \max(\omega, \lambda, \pi(\mathfrak{A}))$ then $(\kappa, \lambda^+, {<}\theta)$ is a precaliber triple of $\mathfrak{A}$ for every θ, and if $\kappa > \max(\omega, \lambda, d(\mathfrak{A}))$ then (κ, λ^+) is a precaliber pair of $\mathfrak{A}$.

(c) If $\mathrm{cf}\,\kappa > d(\mathfrak{A})$ then κ is a precaliber of $\mathfrak{A}$.

(d) If $\mathfrak{A}$ is infinite, $(\mathrm{sat}(\mathfrak{A}), \omega)$ is a precaliber pair of $\mathfrak{A}$.

proof Apply 516K, inverted, to $\mathfrak{A}^+$, recalling that $\pi(\mathfrak{A}) = \mathrm{ci}(\mathfrak{A}^+)$.

516M When we turn to topological spaces, we can refine the results slightly, using the following elementary facts.

Lemma Let $(X, \mathfrak{T})$ be a topological space and $\mathrm{RO}(X)$ its regular open algebra. If κ, λ and θ are cardinals, and $\theta \le \omega$, then the following are equiveridical:

(i) $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $(X, \mathfrak{T})$;

(ii) $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $(\mathfrak{T} \setminus \{\emptyset\}, \ni, X)$;

(iii) $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $\mathrm{RO}(X)$.

proof (a)(i)⇒(ii) If we set $\phi(G) = G$ and choose a point $\psi(G) \in G$ for every non-empty open set $G \subseteq X$, then (ϕ, ψ) is a Galois-Tukey connection from $(\mathfrak{T} \setminus \{\emptyset\}, \ni, X)$ to $(\mathfrak{T} \setminus \{\emptyset\}, \supseteq, \mathfrak{T} \setminus \{\emptyset\})$, so any precaliber triple of the latter is a precaliber triple of the former.

(b)(ii)⇒(iii) Assume (ii), and let $\langle G_\xi\rangle_{\xi<\kappa}$ be a family in $\mathrm{RO}(X)^+$. Then there is a $\Gamma \in [\kappa]^\lambda$ such that $\bigcap_{\xi\in I} G_\xi \neq \emptyset$ for every $I \in [\Gamma]^{<\theta}$. But in this case, because I is finite, $\bigcap_{\xi\in I} G_\xi$ is a lower bound for $\{G_\xi : \xi \in I\}$ in $\mathrm{RO}(X)^+$. As $\langle G_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $\mathrm{RO}(X)$.

(c)(iii)⇒(i) Assume (iii), and let $\langle G_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{T}\setminus\{\emptyset\}$. Then there is a $\Gamma \in [\kappa]^\lambda$ such that $\bigcap_{\xi\in I} \mathrm{int}\,\overline{G}_\xi \neq \emptyset$ for every $I \in [\Gamma]^{<\theta}$. But in this case, because I is finite, $\bigcap_{\xi\in I} G_\xi$ is not empty, and is a lower bound for $\{G_\xi : \xi \in I\}$ in $\mathfrak{T} \setminus \{\emptyset\}$. As $\langle G_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $(X, \mathfrak{T})$.

516N Corollary Let X be a topological space.

(a) $\mathrm{sat}(X)$ is the least cardinal κ such that $(\kappa, 2)$ is a precaliber pair of X.

(b) If $\kappa > \max(\omega, \lambda, d(X))$ then (κ, λ^+) is a precaliber pair of X.

(c) If $\mathrm{cf}\,\kappa > d(X)$ then κ is a precaliber of X.

(d) If $\mathrm{sat}(X)$ is infinite, then $(\mathrm{sat}(X), \omega)$ is a precaliber pair of X.

proof Here we need to know that $\mathrm{sat}(X) = \mathrm{sat}(RO(X))$ and $d(X) \ge d(RO(X))$ (514H(b-i)).

516O The idea of 516M leads to further results about precalibers of topological spaces.

Proposition Let $(X, \mathfrak{T})$ be a topological space.

(a) If Y is a continuous image of X and $\theta \le \omega$, then $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of Y whenever it is a precaliber triple of X.

(b) Suppose that X is the product of a family $\langle X_i\rangle_{i\in I}$ of topological spaces. If $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of every X_i and *either* I is finite *or* $\theta \le \omega$ and κ is a regular uncountable cardinal, then $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of X.

proof (a) Let $f : X \to Y$ be a continuous surjection. Writing $\mathfrak{S}$ for the topology of Y, we have a Galois-Tukey connection (ϕ, f) from $(\mathfrak{S} \setminus \{\emptyset\}, \ni, Y)$ to $(\mathfrak{T} \setminus \{\emptyset\}, \ni, X)$, if we set $\phi(H) = f^{-1}[H]$ for $H \in \mathfrak{S} \setminus \{\emptyset\}$. Now if $\theta \le \omega$ and $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $(X, \mathfrak{T})$, it is a precaliber triple of $(\mathfrak{T} \setminus \{\emptyset\}, \ni, X)$, $(\mathfrak{S} \setminus \{\emptyset\}, \ni, Y)$ and $(Y, \mathfrak{S})$, using 516M and 516C.

(b) If $X = \emptyset$ then $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of X just because $X = X_i$ for some i; so let us suppose that $X \neq \emptyset$.

(i) If $I = \{0, 1\}$ then $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of X. **P** Let $\langle W_\xi \rangle_{\xi<\kappa}$ be a family of non-empty open sets in X. For each $\xi < \kappa$, let $G_{\xi 0} \subseteq X_0$ and $G_{\xi 1} \subseteq X_1$ be non-empty open sets such that $G_{\xi 0} \times G_{\xi 1} \subseteq W_\xi$. Because $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of X_0, there is a $\Gamma \in [\kappa]^\kappa$ such that $H_K^{(0)} = \operatorname{int}(\bigcap_{\xi \in K} G_{\xi 0})$ is non-empty for every $K \in [\Gamma]^{<\theta}$. Because $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of X_1, there is a $\Delta \in [\Gamma]^\kappa$ such that $H_K^{(1)} = \operatorname{int}(\bigcap_{\xi \in K} G_{\xi 1})$ is non-empty for every $K \in [\Delta]^{<\theta}$. Now $\bigcap_{\xi \in K} W_\xi \supseteq H_K^{(0)} \times H_K^{(1)}$ has non-empty interior for every $K \in [\Delta]^{<\theta}$. As $\langle W_\xi \rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of X. **Q**

(ii) If I is finite, then $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of X. **P** Induce on $\#(I)$, using (i) for the inductive step. **Q**

(iii) Now suppose that I is infinite, κ is regular and uncountable and $\theta \le \omega$. Then $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of X. **P** Let $\langle W_\xi \rangle_{\xi<\kappa}$ be a family of non-empty open sets in X. Let $\mathcal{V}$ be the standard base for the topology of X consisting of sets of the form $\prod_{\xi<\kappa} U_\xi$ where $U_\xi \subseteq X_\xi$ is open for every ξ and $\{\xi : U_\xi \neq X_\xi\}$ is finite. For each $\xi < \kappa$ let $W'_\xi \subseteq W_\xi$ be a non-empty member of $\mathcal{V}$, so that W'_ξ is determined by a coordinates in a finite subset I_ξ of I. By the Δ-system Lemma (4A1Db) there is a set $A \subseteq \kappa$, with cardinal κ, such that $\langle I_\xi \rangle_{\xi \in A}$ is a Δ-system with root J say. For $\xi \in A$ express W'_ξ as $U_\xi \cap V_\xi$ where U_ξ is determined by coordinates in J and V_ξ is determined by coordinates in $I_\xi \setminus J$. Now U_ξ is of the form $\pi_J^{-1}[H_\xi]$ where $H_\xi \subseteq \prod_{i \in J} X_i$ is a non-empty open set and $\pi_J : X \to \prod_{i \in J} X_i$ is the canonical map. By (ii), $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of $\prod_{i \in J} X_i$, so there is a $\Gamma \in [A]^\kappa$ such that $\bigcap_{\xi \in K} H_\xi$ is non-empty whenever $K \in [\Gamma]^{<\theta}$. Now take any $K \in [\Gamma]^{<\theta}$. Then $U = \pi_J^{-1}[\bigcap_{\xi \in K} H_\xi]$ is a non-empty set determined by coordinates in J, while V_ξ is a non-empty open set determined by coordinates in $I_\xi \setminus J$ for each $\xi \in K$; because the $I_\xi \setminus J$ are disjoint and K is finite, $U \cap \bigcap_{\xi \in J} V_\xi$ is non-empty, and $\bigcap_{\xi \in K} W_\xi$ is a non-empty set, necessarily open because K is finite. As $\langle W_\xi \rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \kappa, {<}\theta)$ is a precaliber triple of X. **Q**

516P Corollary Let $\langle P_i \rangle_{i \in I}$ be a family of non-empty partially ordered sets, with upwards finite-support product $P = \bigotimes_{i \in I}^{\uparrow} P_i$ (definition: 514T). If $(\kappa, \kappa, {<}\theta)$ is an upwards precaliber triple of every P_i and either I is finite or $\theta \le \omega$ and κ is a regular uncountable cardinal, then $(\kappa, \kappa, {<}\theta)$ is an upwards precaliber triple of P.

proof Suppose first that θ is countable. By 516Gb and 516M we can identify the relevant upwards precaliber triples of each P_i and P with the precaliber triples of their regular open algebras. But $\mathrm{RO}^\uparrow(P) \cong \mathrm{RO}(\prod_{i \in I} P_i)$ (514Ua), so 516Ob gives the result at once.

For finite I, $P^* = \prod_{i \in I} P_i$ is a cofinal subset of P (514Ub), so that it has the same upwards precaliber triples (516Ga); at the same time, it is easy to see that the up-topology of P^* is just the product of the up-topologies on the P_i. So this time we do not need to look at regular open algebras and can use 516Gb and 516Ob directly.

516Q For locally compact spaces, as usual, we have further results.

Proposition Let X be a locally compact Hausdorff topological space.

(a) (κ, λ) is a precaliber pair of X iff whenever $\langle G_\xi \rangle_{\xi<\kappa}$ is a family of non-empty open subsets of X, then there is an $x \in X$ such that $\#(\{\xi : x \in G_\xi\}) \ge \lambda$.

(b) Suppose that κ is a regular infinite cardinal. Then κ is a precaliber of X iff $\operatorname{sat}(X) \le \kappa$ and whenever $\langle E_\xi \rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense subsets of X then $\bigcup_{\xi<\kappa} E_\xi$ has empty interior.

proof (a)(i) The condition asserts that $(\kappa, \lambda, \lambda)$ is a precaliber triple of $(\mathfrak{T} \setminus \{\emptyset\}, \ni, X)$. It follows at once that (κ, λ) is a precaliber pair of $(\mathfrak{T} \setminus \{\emptyset\}, \ni, X)$ and therefore of $(X, \mathfrak{T})$, by 516M.

(ii) Now suppose that (κ, λ) is a precaliber pair of X, and that $\langle G_\xi \rangle_{\xi<\kappa}$ is a family of non-empty open subsets of X. For each $\xi < \kappa$ choose a non-empty relatively compact open set H_ξ such that $\overline{H}_\xi \subseteq G_\xi$. Then there is a $\Gamma \in [\kappa]^\lambda$ such that $\{H_\xi : \xi \in \Gamma\}$ is centered. In this case, $\{\overline{H}_\xi : \xi \in \Gamma\}$ has the finite intersection property, so has non-empty intersection. If x is any point of this intersection, then $\{\xi : x \in G_\xi\} \supseteq \Gamma$ has cardinal at least λ.

(b)(i) Suppose that κ is a precaliber of X. Then surely $\operatorname{sat}(X) \le \kappa$ (516Ja). If $\langle E_\xi \rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense subsets of X, take any non-empty open set $G \subseteq X$. For each $\xi < \kappa$, $G_\xi = G \setminus \overline{E}_\xi$ is a non-empty open set, so by (a) there is an $x \in X$ such that $\Gamma = \{\xi : x \in G_\xi\}$ has cardinal κ. But as $\langle G_\xi \rangle_{\xi<\kappa}$ is non-increasing, this means that $\Gamma = \kappa$ and $x \in G \setminus \bigcup_{\xi<\kappa} E_\xi$. As G is arbitrary, $\bigcup_{\xi<\kappa} E_\xi$ has empty interior.

(ii) Now suppose that the condition is satisfied. Let $\langle G_\xi \rangle_{\xi<\kappa}$ be a family of non-empty open subsets of X. For $\xi < \kappa$ set $H_\xi = \bigcup_{\eta \ge \xi} G_\eta$, $W_\xi = X \setminus \overline{H}_\xi$. By 5A4Bd, there is a set $I \subseteq \kappa$ such that $\#(I) < \operatorname{sat}(X)$ and

$$\overline{\bigcup_{\xi\in I} W_\xi} = \overline{\bigcup_{\xi<\kappa} W_\xi}.$$

Because $\#(I) < \operatorname{cf}\kappa$, $\zeta = \sup I$ is less than κ, and $H_\zeta \cap W_\xi = \emptyset$ for every $\xi \in I$, so $H_\zeta \cap W_\xi = \emptyset$ for every $\xi < \kappa$, that is, $H_\zeta \subseteq \overline{H}_\xi$ for every $\xi < \kappa$.

Setting

$$E_\xi = H_\zeta \setminus H_\xi \subseteq \overline{H}_\xi \setminus H_\xi$$

for each ξ, $\langle E_\xi \rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense sets, and cannot cover H_ζ. If $x \in H_\zeta \setminus \bigcup_{\xi<\kappa} E_\xi$, then $x \in H_\xi$ for every $\xi < \kappa$, so $\Gamma = \{\eta : x \in G_\eta\}$ is cofinal with κ. Because κ is regular, $\Gamma \in [\kappa]^\kappa$, and $\bigcap_{\xi\in I} G_\xi$ is non-empty for every $I \in [\Gamma]^{<\omega}$. As $\langle G_\xi \rangle_{\xi<\kappa}$ is arbitrary, κ is a precaliber of X, by (a).

516R We can use the last proposition to give corresponding characterizations of precaliber pairs of Boolean algebras in terms of their Stone spaces.

Corollary Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space.

(a) A pair (κ, λ) of cardinals is a precaliber pair of $\mathfrak{A}$ iff whenever $\langle G_\xi \rangle_{\xi<\kappa}$ is a family of non-empty open sets in Z there is a $z \in Z$ such that $\#(\{\xi : z \in G_\xi\}) \ge \lambda$.

(b) Suppose that $\kappa \ge \operatorname{sat}(\mathfrak{A})$ is a regular infinite cardinal. Then κ is a precaliber of $\mathfrak{A}$ iff whenever $\langle E_\xi \rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense subsets of Z then $\bigcup_{\xi<\kappa} E_\xi$ has empty interior.

proof Put 516Ha and 516Q together.

516S I collect some further results relating precalibers to the standard constructions involving Boolean algebras.

Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then every precaliber pair of $\mathfrak{A}$ is a precaliber pair of $\mathfrak{B}$.

(b) If $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.

(c) If $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$, every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.

(d) If $\mathfrak{A}$ is the simple product of a family $\langle \mathfrak{A}_i \rangle_{i\in I}$ of Boolean algebras, $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of every $\mathfrak{A}_i$ and $\#(I) < \operatorname{cf}\kappa$, then $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $\mathfrak{A}$.

proof (a) The Stone space of $\mathfrak{B}$ is a continuous image of the Stone space of $\mathfrak{A}$ (312Sa). So all we have to do is to put 516Ha and 516Oa together.

(b) The embedding of $\mathfrak{B}$ in $\mathfrak{A}$ is order-continuous, so corresponds to a continuous surjection from the Stone space Z of $\mathfrak{A}$ onto the Stone space W of $\mathfrak{B}$ which takes non-empty open sets to sets with non-empty interior (313R). By 516Id, every precaliber triple of Z is a precaliber triple of W, so every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.

(c) The Stone space of $\mathfrak{B}$ can be identified with an open subset of the Stone space of $\mathfrak{A}$ (312T), so 516Ia gives the result.

(d) If $I = \emptyset$ this is trivial; suppose otherwise; then κ is infinite. Let $\langle a_\xi \rangle_{\xi<\kappa}$ be a family of non-zero elements of $\mathfrak{A}$. For each $\xi < \kappa$ there is an $i \in I$ such that $a_\xi(i) \ne 0$; as $\operatorname{cf}\kappa > \#(I)$, there is an $i \in I$ such that $C = \{\xi : a_\xi(i) \ne 0\}$ has cardinal κ. Because $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $\mathfrak{A}_i$, there is a $\Gamma \in [C]^\lambda$ such that $\{a_\xi(i) : \xi \in J\}$ has a non-zero lower bound in $\mathfrak{A}_i$ for every $J \in [\Gamma]^{<\theta}$. But now $\{a_\xi : \xi \in J\}$ has a non-zero lower bound in $\mathfrak{A}$ for every $J \in [\Gamma]^{<\theta}$. As $\langle a_\xi \rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $\mathfrak{A}$.

516T A central problem from the very beginning of set-theoretic topology concerns the saturation of product spaces. Here I describe one of the principal methods of showing that product spaces have small saturation, in a form adapted to partially ordered sets.

Theorem (a) Let P and Q be partially ordered sets, and κ a cardinal such that $(\kappa,\mathrm{sat}^\uparrow(Q),2)$ is an upwards precaliber triple of P. Then $\mathrm{sat}^\uparrow(P\times Q)\le\kappa$.

(b) Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets with upwards finite-support product P. Suppose that κ is a regular uncountable cardinal such that $(\kappa,\kappa,2)$ is an upwards precaliber triple of every P_i. Then $\mathrm{sat}^\uparrow(P)\le\kappa$.

proof (a) ? Otherwise, there is an up-antichain $\langle(p_\xi,q_\xi)\rangle_{\xi<\kappa}$ in $P\times Q$. Let $\Gamma\subseteq\kappa$ be a set of size $\mathrm{sat}^\uparrow(Q)$ such that $\{p_\xi:\xi\in\Gamma\}$ is upwards-linked. Then $\langle q_\xi\rangle_{\xi\in\Gamma}$ must be an up-antichain in Q; but this is impossible. **X**

(b) By 516P, $(\kappa,\kappa,2)$ is an upwards precaliber triple of P. So $(\kappa,2,2)$ and $(\kappa,2,{<}\omega)$ also are (516Ba, 516Bc), and $\mathrm{sat}^\uparrow(P)\le\kappa$ (516Ka).

516U It will be useful to be able to quote what amounts to a simple special case of the above result.

Corollary Let $\mathfrak{A}$ be a Boolean algebra satisfying Knaster's condition (511Ef) and $\mathfrak{B}$ a ccc Boolean algebra. Then their free product $\mathfrak{A}\otimes\mathfrak{B}$ is ccc.

proof By 516Ta, inverted, $(\mathfrak{A}\setminus\{0\})\times(\mathfrak{B}\setminus\{0\})$ is downwards-ccc. But $(a,b)\mapsto a\otimes b$ is an order-preserving bijection between $(\mathfrak{A}\setminus\{0\})\times(\mathfrak{B}\setminus\{0\})$ and an order-dense (that is, coinitial) subset of $(\mathfrak{A}\otimes\mathfrak{B})\setminus\{0\}$ (315Kb); so $(\mathfrak{A}\otimes\mathfrak{B})\setminus\{0\}$ is downwards-ccc (513Gc, inverted), that is, $\mathfrak{A}\otimes\mathfrak{B}$ is ccc.

516X Basic exercises (a) Let (A,R,B) be a supported relation, and $n\ge1$ an integer. Show that n is a precaliber of (A,R,B) iff $\mathrm{add}(A,R,B)>n$.

(b) Let P and Q be partially ordered sets, and $f:P\to Q$ a surjection such that, for any finite set $I\subseteq P$, I is bounded above in P iff $f[I]$ is bounded above in Q. Show that P and Q have the same upwards precaliber pairs.

(c)(i) Show that if P is a partially ordered set and $\kappa>\mathrm{cf}\,P$ is an infinite cardinal such that $\mathrm{cf}\,\kappa$ is an up-precaliber of P, then κ is an up-precaliber of P. (ii) Show that if $\mathfrak{A}$ is a Boolean algebra and $\kappa>\pi(\mathfrak{A})$ is an infinite cardinal such that $\mathrm{cf}\,\kappa$ is a precaliber of $\mathfrak{A}$, then κ is a precaliber of $\mathfrak{A}$.

(d) Let P be a partially ordered set and κ an infinite cardinal. Show that $\mathrm{sat}^\uparrow(P)\le\kappa$ iff (κ,ω) is an upwards precaliber pair of P. (*Hint*: if $\kappa=\mathrm{sat}^\uparrow(P)$ and $\langle p_\xi\rangle_{\xi<\kappa}$ is a family in P, choose ξ_n, q_n such that $p_{\xi_i}\le q_n$ for $i\le n$ and $\{\xi:q_n$ is compatible upwards with $p_\xi\}$ is always cofinal with κ.)

(e) Let $(X,\mathfrak{T})$ be a topological space, $\mathrm{RO}(X)$ its regular open algebra and $\mathfrak{G}$ its category algebra (definition: 514I). (i) Show that any precaliber triple of $(X,\mathfrak{T})$ is also a precaliber triple of $\mathrm{RO}(X)$, $(\mathfrak{T}\setminus\{\emptyset\},\ni,X)$ and $\mathfrak{G}$. (ii) Show that if $(X,\mathfrak{T})$ is regular, then $(X,\mathfrak{T})$ and $\mathrm{RO}(X)$ have the same precaliber triples. (iii) Show that if $(X,\mathfrak{T})$ is locally compact and Hausdorff, then $(X,\mathfrak{T})$ and $\mathfrak{G}$ have the same precaliber triples.

(f) Let $(P,\le)$ be the totally ordered set ω_1, $\mathfrak{T}^\uparrow$ its up-topology and $\mathrm{RO}^\uparrow(P)$ the regular open algebra of $(P,\mathfrak{T}^\uparrow)$. Show that $(\omega_1,\omega_1,\omega_1)$ is a precaliber triple of $\mathrm{RO}^\uparrow(P)$ but not of $(P,\le)$ or $(P,\mathfrak{T}^\uparrow)$.

(g) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras and $\mathfrak{A}$ their free product. Show that if $(\kappa,\kappa,{<}\theta)$ is a precaliber triple of every $\mathfrak{A}_i$ and either I is finite or $\theta\le\omega$ and κ is a regular infinite cardinal, then $(\kappa,\kappa,{<}\theta)$ is a precaliber triple of $\mathfrak{A}$.

(h) Suppose that X is a topological space and Y is a dense subset of X and $\theta\le\omega$. Show that $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of Y iff it is a precaliber triple of X.

(i) Let X be a locally compact Hausdorff space, and κ a precaliber of X. Show that whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of nowhere dense subsets of X then $\bigcup_{\xi<\kappa}E_\xi$ has empty interior.

(j) Prove 516Sa-516Sc without mentioning Stone spaces.

(k)(i) Let X and Y be topological spaces, and κ a cardinal such that $(\kappa,\mathrm{sat}(Y),2)$ is a precaliber triple of X. Show that $\mathrm{sat}(X\times Y)\le\kappa$. (ii) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces with product X. Suppose that κ is a regular uncountable cardinal such that $(\kappa,\kappa,2)$ is a precaliber triple of every X_i. Show that $\mathrm{sat}(X)\le\kappa$.

(l)(i) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\mathfrak{A}\otimes\mathfrak{B}$ their free product. Suppose that κ is a cardinal such that $(\kappa,\mathrm{sat}(\mathfrak{B}),2)$ is a precaliber triple of $\mathfrak{A}$. Show that $\mathrm{sat}(\mathfrak{A}\otimes\mathfrak{B})\le\kappa$. (ii) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras with free product $\mathfrak{A}$. Suppose that κ is a regular uncountable cardinal such that $(\kappa,\kappa,2)$ is a precaliber triple of every $\mathfrak{A}_i$. Show that $\mathrm{sat}(\mathfrak{A})\le\kappa$.

(m) Let X and Y be topological spaces. Show that if $(\kappa,\kappa',<\theta)$ is a precaliber triple of X and $(\kappa',\lambda,<\theta)$ is a precaliber triple of Y, then $(\kappa,\lambda,<\theta)$ is a precaliber triple of $X\times Y$.

516 Notes and comments 'Precaliber triples' are visibly complex. With three cardinals in action, there is a promise of a powerful method of describing special features of a partially ordered set or Boolean algebra, but at the same time a threat of alarming demands on our memory. In fact none of the arguments in this section are deep, and they are here mainly for reference. Some of the results depend in not-quite-obvious ways on the exact hypotheses, and it will be useful later to have clear statements to hand. In the proofs I have emphasized Galois-Tukey connections whenever possible; at the cost of possibly tedious repetititions of such formulae as $(\mathfrak{T}\setminus\{\emptyset\},\supseteq,\mathfrak{T}\setminus\{\emptyset\})$ naming the supported relations involved, they can save us the trouble of negotiating the quantifiers in the definition

$$\forall\ \langle a_\xi\rangle_{\xi<\kappa}\in A^\kappa\ \exists\ \Gamma\in[\kappa]^\lambda\ \forall\ I\in[\Gamma]^{<\theta}\ \exists\ b\in B\ldots.$$

But of course it is a useful exercise to find proofs from first principles, not mentioning supported relations and not (for instance) using Stone spaces to deal with Boolean algebras.

'Supported relations' form a materially more various class of structures than partially ordered sets, topological spaces or Boolean algebras. But the constructions already developed in this book (Stone spaces, regular open algebras, up-topologies) give us functorial relations between the last three categories which mean that from the point of view of this section they are nearly the same. So such results as 516T can be expected to apply to topological spaces and Boolean algebras as well (516Xk, 516Xl). (But note 516Xf.)

Precaliber triples belong with saturation and linking numbers as parameters describing the 'breadth' of a topological space or Boolean algebra; see Comfort & Negrepontis 82. In the first place, they address a classic problem: when is the product of ccc topological spaces ccc? (This is the case $\kappa=\omega_1$ of 516Xk.) But with the exception of saturation, there do not appear to be simple connexions between precalibers and the cardinal functions we have looked at so far. Precalibers seem to correspond to new features of the structures considered here. When we come to look at the most important objects of measure theory (in particular, measure algebras), we shall find that their precalibers are relatively fluid; I mean that while cellularity, Maharam types and many linking numbers, for instance, are determined by simple formulae in ZFC, precalibers are not.

517 Martin numbers

I devote a section to the study of 'Martin numbers' of partially ordered sets and Boolean algebras. Like additivity and cofinality they enable us to frame as theorems of ZFC some important arguments which were first used in special models of set theory, and to pose challenging questions on the relationships between classical structures in analysis. I begin with some general remarks on the Martin numbers of partially ordered sets (517A-517E); most of these are perfectly elementary but the equivalent conditions of 517B, in particular, are useful and not all obvious. Much of the importance of Martin numbers comes from their effect on precalibers (517F, 517H) and hence on saturation of products (517G). The same ideas can be expressed in terms of Boolean algebras, with no surprises (517I). I have not set out a definition of 'Martin number' for a topological space, but the Novák number of a locally compact Hausdorff space is closely related to the Martin numbers of its regular open algebra and its algebra of open-and-closed sets (517J-517K). Consequently we have connexions between the Martin number and the weak distributivity of a Boolean algebra (517L). A striking fact, which will have a prominent role in the next chapter, is that non-trivial countable partially ordered sets all have the same Martin number $\mathfrak{m}_{\mathrm{countable}}$ (517P).

517A Proposition For any partially ordered set P, $\mathfrak{m}^\uparrow(P)\ge\omega_1$.

proof If $\mathcal{Q}$ is a countable family of cofinal subsets of P and $p_0\in P$, let $\langle Q_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{Q}\cup\{P\}$, and choose $\langle p_n\rangle_{n\ge1}$ inductively such that $p_{n+1}\ge p_n$ and $p_{n+1}\in Q_n$ for every $n\in\mathbb{N}$. Then $\{p_n:n\in\mathbb{N}\}$ is an upwards-linked subset of P meeting every member of $\mathcal{Q}$.

517B Lemma Let P be a partially ordered set, and κ a cardinal. Then the following are equiveridical:

(i) $\kappa < \mathfrak{m}^{\uparrow}(P)$;

(ii) whenever $p_0 \in P$ and $\mathcal{Q}$ is a family of up-open cofinal subsets of P with $\#(\mathcal{Q}) \le \kappa$, there is an upwards-linked subset of P which contains p_0 and meets every member of $\mathcal{Q}$;

(iii) whenever $p_0 \in P$ and $\mathcal{A}$ is a family of maximal up-antichains in P with $\#(\mathcal{A}) \le \kappa$, there is an upwards-linked subset of P which contains p_0 and meets every member of $\mathcal{A}$;

(iv) whenever $p_0 \in P$ and $\mathcal{Q}$ is a family of cofinal subsets of P with $\#(\mathcal{Q}) \le \kappa$, there is an upwards-directed subset of P which contains p_0 and meets every member of $\mathcal{Q}$;

(v) whenever $p_0 \in P$ and $\mathcal{Q}$ is a family of up-open cofinal subsets of P with $\#(\mathcal{Q}) \le \kappa$, there is an upwards-directed subset of P which contains p_0 and meets every member of $\mathcal{Q}$;

(vi) whenever $p_0 \in P$ and $\mathcal{A}$ is a family of maximal up-antichains in P with $\#(\mathcal{A}) \le \kappa$, there is an upwards-directed subset of P which contains p_0 and meets every member of $\mathcal{A}$.

proof (a) Most of the circuit is elementary.

(vi)⇒(iv) because every cofinal subset of P includes a maximal up-antichain (513Aa).

(iv)⇒(v)⇒(ii) are trivial.

(ii)⇒(i) Assuming (ii), let $\mathcal{Q}$ be a family of cofinal subsets of P with $\#(\mathcal{Q}) \le \kappa$. For each $Q \in \mathcal{Q}$, $U_Q = \bigcup_{q\in Q} [q,\infty[$ is up-open and cofinal with P (513Ab). If $p_0 \in P$, (ii) tells us that there is an upwards-linked subset R_0 of P containing p_0 and meeting U_Q for every $Q \in \mathcal{Q}$. Set $R = \bigcup_{p\in R_0}]-\infty,p]$; then R is an upwards-linked subset of P containing p_0 and meeting every member of $\mathcal{Q}$. As $\mathcal{Q}$ and p_0 are arbitrary, (i) is true.

(i)⇒(iii) Assuming (i), let $\mathcal{A}$ be a family of maximal up-antichains in P with $\#(\mathcal{A}) \le \kappa$. For each $A \in \mathcal{A}$, $U_A = \bigcup_{p\in A} [p,\infty[$ is cofinal with P. So (i) tells us that if $p_0 \in P$ there is an upwards-linked subset R_0 of P containing p_0 and meeting U_A for every $A \in \mathcal{A}$. As just above, set $R = \bigcup_{p\in R_0}]-\infty,p]$; then R is upwards-linked, contains p_0 and meets every member of $\mathcal{A}$. As $\mathcal{A}$ and p_0 are arbitrary, (iii) is true.

(b) So we are left with (iii)⇒(vi). Assume (iii), and take $p_0 \in P$ and a family $\mathcal{A}$ of maximal up-antichains in P with $\#(\mathcal{A}) \le \kappa$. Let $\mathcal{C}$ be the set of all maximal up-antichains in P. For $A \in \mathcal{C}$, set $U_A = \bigcup_{q\in A} [q,\infty[$. Then U_A is cofinal with P. Consequently $U_A \cap U_B$ is cofinal with P for any A, $B \in \mathcal{C}$, because if $p \in P$ there are $q \in U_A$ and $r \in U_B$ such that $p \le q \le r$, and now $r \in U_A \cap U_B$. It follows that $U_A \cap U_B$ includes a maximal up-antichain $D(A,B)$.

Take any $A_0 \in \mathcal{C}$ such that $p_0 \in A_0$. Let $\mathcal{A}^* \subseteq \mathcal{C}$ be such that $\{A_0\} \cup \mathcal{A} \subseteq \mathcal{A}^*$, $D(A,B) \in \mathcal{A}^*$ for all A, $B \in \mathcal{A}^*$, and $\#(\mathcal{A}^*) \le \max(\omega,\kappa)$ (5A1Fb). Then there is an upwards-linked subset R_0 of P containing p_0 and meeting every member of $\mathcal{A}^*$. **P** If $\#(\mathcal{A}^*) \le \kappa$, this is immediate from (iii); if $\#(\mathcal{A}^*) \le \omega$, it is because $\omega < \mathfrak{m}^{\uparrow}(P)$, by 517A, and (i)⇒(iii). **Q**

Set $R = R_0 \cap \bigcup \mathcal{A}^*$. Then R contains p_0 (because $p_0 \in R_0 \cap A_0$) and R meets every member of $\mathcal{A}$. Also R is upwards-directed. **P** If p, $q \in R$, take A, $B \in \mathcal{A}^*$ such that $p \in A$ and $q \in B$. Then $D(A,B) \in \mathcal{A}^*$, so there is an $r \in R_0 \cap D(A,B)$, and $r \in R$. As $r \in U_A \cap U_B$, there must be $p' \in A$ and $q' \in B$ such that $p' \le r$ and $q' \le r$. But R_0 is upwards-linked, so

$$\emptyset \ne [p,\infty[\cap [r,\infty[\subseteq [p,\infty[\cap [p',\infty[;$$

as A is an up-antichain, $p = p'$. Similarly, $q = q'$ and $r \in R$ is an upper bound of $\{p,q\}$. As p and q are arbitrary, R is upwards-directed. **Q**

So we have a set R of the kind required by (vi).

517C Lemma Let P_0 and P_1 be partially ordered sets, and suppose that there is a relation $S \subseteq P_0 \times P_1$ such that $S[P_0]$ is cofinal with P_1, $S^{-1}[Q]$ is cofinal with P_0 for every cofinal $Q \subseteq P_1$, and $S[R]$ is upwards-linked in P_1 for every upwards-linked $R \subseteq P_0$. Then $\mathfrak{m}^{\uparrow}(P_1) \ge \mathfrak{m}^{\uparrow}(P_0)$.

proof Suppose that $p_1 \in P_1$ and that $\mathcal{Q}$ is a family of cofinal subsets of P_1 with $\#(\mathcal{Q}) < \mathfrak{m}^{\uparrow}(P_0)$. Then there is a pair $(p_0,p_1') \in S$ such that $p_1' \ge p_1$. Now $S^{-1}[Q]$ is cofinal with P_0 for every $Q \in \mathcal{Q}$, so there is an upwards-linked $R \subseteq P_0$ containing p_0 and meeting $S^{-1}[Q]$ for every $Q \in \mathcal{Q}$. In this case $p_1' \in S[R]$ and $S[R]$ is upwards-linked, so $\{p_1\} \cup S[R]$ is an upwards-linked subset of P_1 containing p_1 and meeting every member of $\mathcal{Q}$. As p_1 and $\mathcal{Q}$ are arbitrary, $\mathfrak{m}^{\uparrow}(P_1) \ge \mathfrak{m}^{\uparrow}(P_0)$.

517D Proposition (a) If P is a partially ordered set and Q is a cofinal subset of P, then $\mathfrak{m}^{\uparrow}(P)=\mathfrak{m}^{\uparrow}(Q)$.

(b) If P is any partially ordered set and $\mathrm{RO}^{\uparrow}(P)$ is its regular open algebra when it is given its up-topology, then $\mathfrak{m}^{\uparrow}(P)=\mathfrak{m}(\mathrm{RO}^{\uparrow}(P))$.

(c) If P is a partially ordered set and $p_0\in P$, then $\mathfrak{m}^{\uparrow}([p_0,\infty[)\ge\mathfrak{m}^{\uparrow}(P)$.

proof (a) Let P_0, P_1 be cofinal subsets of P, and set $S=\{(p_0,p_1):p_0\in P_0,\ p_1\in P_1,\ p_0\ge p_1\}$. Then S satisfies the conditions of 517C, so $\mathfrak{m}^{\uparrow}(P_1)\ge\mathfrak{m}^{\uparrow}(P_0)$. It follows at once that all cofinal subsets of P, including P itself, have the same Martin number.

(b)(i) Setting $S=\{(p,G):p\in G\in\mathrm{RO}^{\uparrow}(P)\}$, S satisfies the conditions of 517C with $P_0=(P,\le)$ and $P_1=(\mathrm{RO}^{\uparrow}(P)^+,\supseteq)$, so $\mathfrak{m}(\mathrm{RO}^{\uparrow}(P))\ge\mathfrak{m}^{\uparrow}(P)$.

(ii) Setting $S'=\{(G,p):G\in\mathrm{RO}^{\uparrow}(P)^+,\ p\in P,\ G\subseteq\overline{[p,\infty[}\}$, S' satisfies the conditions of 517C with $P_0=(\mathrm{RO}^{\uparrow}(P)^+,\supseteq)$ and $P_1=(P,\le)$, so $\mathfrak{m}^{\uparrow}(P)\ge\mathfrak{m}(\mathrm{RO}^{\uparrow}(P))$.

(c) Let $\mathcal{Q}$ be a family of upwards-cofinal subsets of $[p_0,\infty[$ with $\#(\mathcal{Q})<\mathfrak{m}^{\uparrow}(P)$, and $p_1\in[p_0,\infty[$. For each $Q\in\mathcal{Q}$, set $Q'=Q\cup\{p:p\in P,\ [p,\infty[\cap[p_0,\infty[=\emptyset\}$. Then every Q' is cofinal with P. So there is an upwards-linked set $R\subseteq P$ containing p_1 and meeting Q' for every $Q\in\mathcal{Q}$. If $Q\in\mathcal{Q}$ and $r\in R\cap Q'$, then $[r,\infty[\cap[p_0,\infty[\supseteq[r,\infty[\cap[p_1,\infty[$ is non-empty, so $r\in Q$. Thus $R\cap[p_0,\infty[$ is an upwards-linked subset of $[p_0,\infty[$ containing p_1 and meeting every member of $\mathcal{Q}$. As $\mathcal{Q}$ and p_1 are arbitrary, $\mathfrak{m}^{\uparrow}([p_0,\infty[)\ge\mathfrak{m}^{\uparrow}(P)$.

517E Corollary Let P be a partially ordered set such that $\mathfrak{m}^{\uparrow}(P)$ is not ∞. Then $\mathfrak{m}^{\uparrow}(P)\le 2^{\operatorname{cf}P}$.

proof Let Q_0 be a cofinal subset of P with $\#(Q_0)=\operatorname{cf}P$. Then $\mathfrak{m}^{\uparrow}(Q_0)=\mathfrak{m}^{\uparrow}(P)<\infty$. So there are $q_0\in Q_0$ and a family $\mathcal{Q}$ of cofinal subsets of Q_0 such that no upwards-linked subset Q_0 containing p_0 can meet every member of $\mathcal{Q}$. Now

$$\mathfrak{m}^{\uparrow}(P)=\mathfrak{m}^{\uparrow}(Q_0)\le\#(\mathcal{Q})\le 2^{\#(Q_0)}=2^{\operatorname{cf}P}.$$

517F Proposition Let P be a non-empty partially ordered set.

(a) Suppose that κ and λ are cardinals such that $\operatorname{sat}^{\uparrow}(P)\le\operatorname{cf}\kappa$, $\lambda\le\kappa$ and $\lambda<\mathfrak{m}^{\uparrow}(P)$. Then (κ,λ) is an upwards precaliber pair of P.

(b) In particular, if $\operatorname{sat}^{\uparrow}(P)\le\operatorname{cf}\kappa\le\kappa<\mathfrak{m}^{\uparrow}(P)$ then κ is an up-precaliber of P.

proof (a) Since P is not empty, $\operatorname{sat}^{\uparrow}(P)\ge 2$ and κ is infinite. Write θ for $\operatorname{sat}^{\uparrow}(P)$. Let $\langle p_\xi\rangle_{\xi<\kappa}$ be a family in P. For $I\subseteq\kappa$, set

$$U_I=\bigcup_{\xi\in I}[p_\xi,\infty[,\quad V_I=\{q:q\in P,\ [q,\infty[\cap[p_\xi,\infty[=\emptyset\text{ for every }\xi\in I\}.$$

Then for every $J\subseteq\kappa$ there is an $I\in[J]^{<\theta}$ such that $V_J\cup U_I$ is cofinal with P. **P** $V_J\cup U_J$ is cofinal with P, so there is a maximal up-antichain $A\subseteq V_J\cup U_J$. Now $\#(A\cap U_J)<\operatorname{sat}^{\uparrow}(P)=\theta$, so there is a set $I\in[J]^{<\theta}$ such that $A\cap U_J\subseteq U_I$, and $A\subseteq V_J\cup U_I$. Now $V_J\cup U_I\supseteq\bigcup_{q\in A}[q,\infty[$, so is cofinal with P. **Q**

Next, $Q=\bigcup\{V_{\kappa\setminus I}:I\in[\kappa]^{<\kappa}\}$ is not cofinal with P. **P?** If it were, there would be a maximal up-antichain $A\subseteq Q$. For each $q\in A$, let $I_q\in[\kappa]^{<\kappa}$ be such that $q\in V_{\kappa\setminus I_q}$. Because $\#(A)<\theta\le\operatorname{cf}\kappa$, $\bigcup_{q\in A}I_q\ne\kappa$, and there is a $\xi\in\kappa\setminus\bigcup_{q\in A}I_q$. But now $[q,\infty[\cap[p_\xi,\infty[=\emptyset$ for every $q\in A$, and A is not a maximal antichain. **XQ**

Let $q_0\in P$ be such that $Q\cap[q_0,\infty[=\emptyset$. Choose $\langle I_\xi\rangle_{\xi<\lambda}$ inductively in such a way that, writing $J_\xi=\kappa\setminus\bigcup_{\eta<\xi}I_\eta$, $I_\xi\in[J_\xi]^{<\theta}$ and $Q_\xi=V_{J_\xi}\cup U_{I_\xi}$ is cofinal with P for every $\xi<\lambda$. Because $\lambda<\mathfrak{m}^{\uparrow}(P)$, there is an upwards-directed set $R\subseteq P$ containing q_0 and meeting every Q_ξ. Set $\Gamma=\{\eta:\eta<\kappa,\ R\cap[p_\eta,\infty[\ne\emptyset\}$; then $\{p_\eta:\eta\in\Gamma\}$ is upwards-centered. Next, $\kappa\setminus J_\xi=\bigcup_{\eta<\xi}I_\eta$ has cardinal less than κ for every $\xi<\lambda$. **P** If $\theta=\kappa$ or $\kappa=\omega$, this is because $\#(\xi)<\kappa$ and $\#(I_\eta)<\kappa$ for every $\eta<\xi$ and κ is regular (use 513Bb). Otherwise it's because $\max(\omega,\theta,\#(\xi))<\kappa$. **Q**

This means that $V_{J_\xi}\cap[q_0,\infty[=\emptyset$ and $R\cap V_{J_\xi}$ must be empty, for every $\xi<\lambda$. We must therefore have $R\cap U_{I_\xi}\ne\emptyset$ for each $\xi<\lambda$, so that $\Gamma\cap I_\xi\ne\emptyset$; as $\langle I_\xi\rangle_{\xi<\lambda}$ is disjoint, $\#(\Gamma)\ge\lambda$.

As $\langle p_\xi\rangle_{\xi<\kappa}$ is arbitrary, (κ,λ) is an upwards precaliber pair of P.

(b) This follows at once, setting $\lambda=\kappa$.

517G Corollary (a) If P and Q are partially ordered sets and $\operatorname{sat}^\uparrow(Q) < \mathfrak{m}^\uparrow(P)$, then $\operatorname{sat}^\uparrow(P \times Q)$ is at most $\max(\omega, \operatorname{sat}^\uparrow(P), \operatorname{sat}^\uparrow(Q))$.

(b) Let $\langle P_i \rangle_{i \in I}$ be a family of non-empty partially ordered sets with upwards finite-support product P. Let κ be a regular uncountable cardinal such that $\operatorname{sat}^\uparrow(P_i) \le \kappa < \mathfrak{m}^\uparrow(P_i)$ for every $i \in I$. Then $\operatorname{sat}^\uparrow(P) \le \kappa$.

proof (a) Set $\lambda = \operatorname{sat}^\uparrow(Q)$, $\kappa = \max(\omega, \operatorname{sat}^\uparrow(P), \operatorname{sat}^\uparrow(Q))$. Then κ is regular (513Bb again), $\lambda \le \kappa$ and $\lambda < \mathfrak{m}^\uparrow(P)$, so (κ, λ) is an upwards precaliber pair of P and $(\kappa, \lambda, 2)$ is an upwards precaliber triple of P. By 516Ta, $\operatorname{sat}^\uparrow(P \times Q) \le \kappa$.

(b) By 517Fb, κ is an up-precaliber of P_i for every i, so $(\kappa, \kappa, 2)$ is an upwards precaliber triple of every P_i, and we can use 516Tb.

517H Proposition Let P be a non-empty partially ordered set, and let P^* be the upwards finite-support product of the family $\langle P_n \rangle_{n \in \mathbb{N}}$ where $P_n = P$ for every n. Suppose that $\kappa < \mathfrak{m}^\uparrow(P^*)$.

(a) Every subset of P with κ or fewer members can be covered by a sequence of upwards-directed sets.

(b) In particular, if κ is uncountable then (κ, λ) is an upwards precaliber pair of P for every $\lambda < \kappa$, and if κ has uncountable cofinality then κ is an up-precaliber of P.

proof (a) Let $A \in [P]^{<\kappa}$. For each $p \in A$, set $Q_p = \{q : q \in P^*, \exists \ n \in \operatorname{dom} q,\, q(n) = p\}$; then Q_p is cofinal with P^*. So there is an upwards-directed set $R \subseteq P^*$ such that $R \cap Q_p \ne \emptyset$ for every $p \in A$. For each $n \in \mathbb{N}$, set $R_n = \{q(n) : q \in R,\, n \in \operatorname{dom} q\}$. Then $A \subseteq \bigcup_{n \in \mathbb{N}} R_n$. If $n \in \mathbb{N}$ and r, $r' \in R_n$, there are q, $q' \in R$ such that $q(n) = r$ and $q'(n) = r'$. Now there is a $q'' \in R$ such that $q'' \ge q$ and $q'' \ge q'$, in which case $q''(n)$ belongs to $R_n \cap [r, \infty[\cap [r', \infty[$. As r and r' are arbitrary, R_n is upwards-directed. Thus $\langle R_n \rangle_{n \in \mathbb{N}}$ is an appropriate sequence.

(b) Suppose that κ is uncountable and that either $\lambda < \kappa$ or $\operatorname{cf} \kappa > \omega$ and $\lambda = \kappa$. Let $\langle p_\xi \rangle_{\xi < \kappa}$ be any family in P. Let $\langle R_n \rangle_{n \in \mathbb{N}}$ be a sequence of upwards-directed sets covering $\{p_\xi : \xi < \kappa\}$, and for each $n \in \mathbb{N}$ set $\Gamma_n = \{\xi : p_\xi \in R_n\}$. There must be some n such that $\#(\Gamma_n) \ge \lambda$, and $\{p_\xi : \xi \in \Gamma_n\}$ is upwards-centered.

517I Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then $\mathfrak{m}(\mathfrak{B}) \ge \mathfrak{m}(\mathfrak{A})$.

(b) If $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$, then $\mathfrak{m}(\mathfrak{B}) \ge \mathfrak{m}(\mathfrak{A})$.

(c) If $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$, then $\mathfrak{m}(\mathfrak{B}) = \mathfrak{m}(\mathfrak{A})$.

(d) If $\widehat{\mathfrak{A}}$ is the Dedekind completion of $\mathfrak{A}$, then $\mathfrak{m}(\widehat{\mathfrak{A}}) = \mathfrak{m}(\mathfrak{A})$.

(e) If $D \subseteq \mathfrak{A}$ is non-empty and $\sup D = 1$, then $\mathfrak{m}(\mathfrak{A}) = \min_{d \in D} \mathfrak{m}(\mathfrak{A}_d)$, where $\mathfrak{A}_d$ is the principal ideal generated by d.

(f) If $\mathfrak{A}$ is the simple product of a non-empty family $\langle \mathfrak{A}_i \rangle_{i \in I}$ of Boolean algebras, then $\mathfrak{m}(\mathfrak{A}) = \min_{i \in I} \mathfrak{m}(\mathfrak{A}_i)$.

(g) Suppose that κ and λ are infinite cardinals such that $\operatorname{sat}(\mathfrak{A}) \le \operatorname{cf} \kappa$, $\lambda \le \kappa$ and $\lambda < \mathfrak{m}(\mathfrak{A})$. Then (κ, λ) is a precaliber pair of $\mathfrak{A}$.

proof (a) Setting $S = \{(a, b) : a \in \mathfrak{A}^+,\, a \subseteq b \in \mathfrak{B}\}$, S satisfies the conditions of 517C for $P_0 = (\mathfrak{A}^+, \supseteq)$ and $P_1 = (\mathfrak{B}^+, \supseteq)$. **P** The only non-trivial part is the check that if Q is coinitial with $\mathfrak{B}^+$ then $S^{-1}[Q]$ is coinitial with $\mathfrak{A}^+$. But $\sup Q = 1$ in $\mathfrak{B}$; as $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$, $\sup Q = 1$ in $\mathfrak{A}$. So if $a \in \mathfrak{A}^+$, there is a $b \in Q$ such that $a \cap b \ne 0$, and now $a \cap b \in S^{-1}[Q]$ and $a \cap b \subseteq a$. As a is arbitrary, $S^{-1}[Q]$ is coinitial with $\mathfrak{A}^+$. **Q** So $\mathfrak{m}(\mathfrak{B}) \ge \mathfrak{m}(\mathfrak{A})$.

(b) If $\mathfrak{A}_a$ is the principal ideal generated by $a \in \mathfrak{A}^+ = \mathfrak{A} \setminus \{0\}$, we have

$$\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}^\downarrow(\mathfrak{A}^+) \le \mathfrak{m}^\downarrow(]0, a])$$

(by 517Dc, inverted)

$$= \mathfrak{m}(\mathfrak{A}_a).$$

On my definitions the trivial ideal $\{0\}$ also is a principal ideal, but of course $\mathfrak{m}(\{0\}) = \infty \ge \mathfrak{m}(\mathfrak{A})$.

(c) Apply 517Da (inverted) to $\mathfrak{A}^+$ and $\mathfrak{B}^+$.

(d) This follows from (c), because $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$.

(e) By (b), $\mathfrak{m}(\mathfrak{A}) \le \mathfrak{m}(\mathfrak{A}_d)$ for every d. In the other direction, let $\mathcal{Q}$ be a family of coinitial subsets of $\mathfrak{A}^+$ such that $\#(\mathcal{Q}) < \min_{d \in D} \mathfrak{m}(\mathfrak{A}_d)$, and take any $c \in \mathfrak{A}^+$. Then there is a $d \in D$ such that $c \cap d \ne 0$. For $Q \in \mathcal{Q}$ set $Q' = \{a \cap d : a \in Q\} \setminus \{0\}$; then Q' is coinitial with $\mathfrak{A}_d^+$. Since $\#(\{Q' : Q \in \mathcal{Q}\}) < \mathfrak{m}(\mathfrak{A}_d)$, there is a downwards-linked set $R'\subseteq\mathfrak{A}_d^+$ meeting every Q' and containing $c\cap d$. Set $R=\{a:a\in\mathfrak{A},\ a\cap d\in R'\}$; then R is a downwards-linked subset of $\mathfrak{A}^+$ meeting every member of $\mathcal{Q}$ and containing c. As c and $\mathcal{Q}$ are arbitrary, $\mathfrak{m}(\mathfrak{A})\ge\min_{d\in D}\mathfrak{m}(\mathfrak{A}_d)$.

(f) This is, in effect, a special case of (e), since we can identify the $\mathfrak{A}_i$ with principal ideals of $\mathfrak{A}$ (315E).

(g) Apply 517Fa (inverted) to $\mathfrak{A}^+$.

517J Proposition Let X be a locally compact Hausdorff space, and κ a cardinal. Then the following are equiveridical:

(i) $\kappa<\mathfrak{m}(\mathrm{RO}(X))$, where $\mathrm{RO}(X)$ is the regular open algebra of X;
(ii) $X\cap\bigcap\mathcal{G}$ is dense in X whenever $\mathcal{G}$ is a family of dense open subsets of X and $\#(\mathcal{G})\le\kappa$;
(iii) $\kappa<n(H)$ for every non-empty open set $H\subseteq X$.

proof (i)⇒(iii) Suppose that $\kappa<\mathfrak{m}(\mathrm{RO}(X))$. Let $H\subseteq X$ be a non-empty open set and $\mathcal{E}$ a family of nowhere dense subsets of H with $\#(\mathcal{E})\le\kappa$. Note that every member of $\mathcal{E}$ is nowhere dense in X. Because X is locally compact and regular, we have a non-empty regular open set H_0 such that $K=\overline{H}_0$ is compact and included in H. For each $E\in\mathcal{E}$, set $\mathcal{G}_E=\{G:G\in\mathrm{RO}(X)^+,\ \overline{G}\cap E=\emptyset\}$; then $\mathcal{G}_E$ is coinitial with $\mathrm{RO}(X)^+$. Because $\kappa<\mathfrak{m}^{\downarrow}(\mathrm{RO}(X)^+)$, there is a centered $\mathcal{G}\subseteq\mathrm{RO}(X)^+$ containing H_0 and meeting every $\mathcal{G}_E$. But in this case $\{K\}\cup\{\overline{G}:G\in\mathcal{G}\}$ is a family of closed sets in X containing the compact set K and with the finite intersection property, so has non-empty intersection F, which is included in $H\setminus\bigcup\mathcal{E}$. As H and $\mathcal{E}$ are arbitrary, (iii) is true.

(iii)⇒(ii) This is easy. If (iii) is true, $\mathcal{G}$ is a family of dense open subsets of X with $\#(\mathcal{G})\le\kappa$, and $H\subseteq X$ is a non-empty open set, then $\mathcal{E}=\{H\setminus G:G\in\mathcal{G}\}$ is a family of nowhere dense subsets of H, so cannot cover H, and $H\cap\bigcap\mathcal{G}\ne\emptyset$. As $\mathcal{G}$ and H are arbitrary, (ii) is true.

(ii)⇒(i) Suppose that (ii) is true. Take $H\in\mathrm{RO}(X)^+$ and a family $\mathfrak{G}$ of coinitial subsets of $\mathrm{RO}(X)^+$ with $\#(\mathfrak{G})\le\kappa$. For each $\mathcal{G}\in\mathfrak{G}$, $\bigcup\mathcal{G}$ is a dense open subset of X. Accordingly there is a point $x\in H\cap\bigcap_{\mathcal{G}\in\mathfrak{G}}\bigcup\mathcal{G}$. Set $R=\{G:G\in\mathrm{RO}(X),\ x\in G\}$. Then R is a downwards-linked subset of $\mathrm{RO}(X)^+$ containing H and meeting every member of $\mathfrak{G}$. As H and $\mathfrak{G}$ are arbitrary, $\kappa<\mathfrak{m}(\mathrm{RO}(X))$.

517K Corollary Let $\mathfrak{A}$ be a Boolean algebra with Stone space Z.

(a) $\mathfrak{m}(\mathfrak{A})=\mathfrak{m}(\mathrm{RO}(Z))$.
(b) For any cardinal κ, the following are equiveridical:
 (i) $\kappa<\mathfrak{m}(\mathfrak{A})$;
 (ii) $Z\cap\bigcap\mathcal{G}$ is dense in Z whenever $\mathcal{G}$ is a family of dense open subsets of Z and $\#(\mathcal{G})\le\kappa$;
 (iii) $\kappa<n(H)$ for every non-empty open set $H\subseteq Z$.

proof (a) $\mathfrak{A}$ is isomorphic to the algebra of open-and-closed subsets of Z, which is an order-dense subalgebra of $\mathrm{RO}(Z)$ (314Ta). So $\mathfrak{m}(\mathfrak{A})=\mathfrak{m}(\mathrm{RO}(Z))$ by 517Ic.

(b) now follows from 517J.

517L These identifications make the following results easy.

Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) $\mathrm{wdistr}(\mathfrak{A})\le\mathfrak{m}(\mathfrak{A})$.
(b) If $\mathrm{wdistr}(\mathfrak{A})$ is a precaliber of $\mathfrak{A}$ then $\mathrm{wdistr}(\mathfrak{A})<\mathfrak{m}(\mathfrak{A})$.

proof (a) Let Z be the Stone space of $\mathfrak{A}$ and $\mathcal{N}\mathcal{W}\mathrm{d}$ the ideal of nowhere dense subsets of Z. Then $\mathrm{wdistr}(\mathfrak{A})=\mathrm{add}\,\mathcal{N}\mathcal{W}\mathrm{d}$ (514Be), while $\mathfrak{m}(\mathfrak{A})$ is the least cardinal of any subset of $\mathcal{N}\mathcal{W}\mathrm{d}$ covering a non-empty open subset of Z, if there is one (517Kb). Since no non-empty open subset of Z can belong to $\mathcal{N}\mathcal{W}\mathrm{d}$, $\mathrm{wdistr}(\mathfrak{A})\le\mathfrak{m}(\mathfrak{A})$.

(b) Because $\mathrm{wdistr}(\mathfrak{A})=\mathrm{add}\,\mathcal{N}\mathcal{W}\mathrm{d}\ge\omega$, it is a regular infinite cardinal (513C(a-i)). If $\langle G_\xi\rangle_{\xi<\mathrm{wdistr}(\mathfrak{A})}$ is a family of dense open subsets of Z, and $H\subseteq Z$ is open and not empty, then $H_\xi=H\cap\mathrm{int}(\bigcap_{\eta<\xi}G_\eta)$ is non-empty for every $\xi<\mathrm{wdistr}(\mathfrak{A})$. So if also $\mathrm{wdistr}(\mathfrak{A})$ is a precaliber of $\mathfrak{A}$ and therefore of Z (516Ha), there is a point z of Z such that $\{\xi:z\in H_\xi\}$ has cardinal $\mathrm{wdistr}(\mathfrak{A})$ (516Qb) and is therefore cofinal with $\mathrm{wdistr}(\mathfrak{A})$; which means that $z\in H\cap\bigcap_{\xi<\mathrm{wdistr}(\mathfrak{A})}G_\xi$. Thus $n(H)>\mathrm{wdistr}(\mathfrak{A})$; as H is arbitrary, $\mathfrak{m}(\mathfrak{A})>\mathrm{wdistr}(\mathfrak{A})$, by 517Kb again.

517M It is worth extracting an idea from the proofs just above as a general result.

Proposition Let X be any topological space. Then the Novák number $n(X)$ of X (5A4Af) is at most $\sup\{\mathfrak{m}(\mathrm{RO}(G)) : G \subseteq X$ is open and not empty$\}$, where $\mathrm{RO}(G)$ is the regular open algebra of G.

proof (a) If there is a non-empty open subset G of X such that $\mathfrak{m}(\mathrm{RO}(G)) = \infty$, the result is trivial; suppose otherwise. Set $\kappa = \sup\{\mathfrak{m}(\mathrm{RO}(G)) : G \subseteq X$ is open and not empty$\}$. Then for any non-empty open set $G \subseteq X$ there is a family $\langle E_\xi\rangle_{\xi<\kappa}$ of nowhere dense sets such that $\#(\mathcal{E}) \le \kappa$ and $G \cap \mathrm{int}(\bigcup_{\xi<\kappa} E_\xi) \ne \emptyset$. **P** We have a family $\langle \mathcal{Q}_\xi\rangle_{\xi<\kappa}$ of order-dense subsets of $\mathrm{RO}(G)^+$ and an $H \in \mathrm{RO}(G)^+$ such that there is no downwards-directed family in $\mathrm{RO}(G)^+$ containing H and meeting every $\mathcal{Q}_\xi$. Set $E_\xi = G \setminus \bigcup \mathcal{Q}_\xi$ for each ξ; then E_ξ must be nowhere dense in the topological sense because any open set meeting G at all must meet some member of $\mathcal{Q}_\xi$. If $x \in H$, then $\mathcal{R} = \{U : U \in \mathrm{RO}(G), x \in U\}$ is a downwards-directed family in $\mathrm{RO}(G)^+$ containing H, so does not meet every $\mathcal{Q}_\xi$, and there must be a $\xi < \kappa$ such that $x \notin \bigcup \mathcal{Q}_\xi$, that is, $x \in E_\xi$. As x is arbitrary, $G \cap \mathrm{int}(\bigcup_{\xi<\kappa} E_\xi) \supseteq H$ is not empty. **Q**

(b) Let $\langle H_i\rangle_{i\in I}$ be a maximal disjoint family of non-empty open sets in X such that every H_i can be covered by a family of at most κ nowhere dense sets. By (a), $\bigcup_{i\in I} H_i$ is dense. For each $i \in I$, let $\langle E_{i\xi}\rangle_{\xi<\kappa}$ be a family of nowhere dense sets covering H_i. Set $E_\xi = \bigcup_{i\in I} H_i \cap E_{i\xi}$ for each $\xi < \kappa$; then E_ξ is nowhere dense (5A4Ea). Also $\bigcup_{\xi<\kappa} E_\xi = \bigcup_{i\in I} H_i$ is a dense open set, so that $\{E_\xi : \xi < \kappa\} \cup (X \setminus \bigcup_{i\in I} H_i)$ is a cover of X by nowhere dense sets, and $n(X) \le \kappa$. (Of course κ is infinite, by 517A, except in the trivial case $X = \emptyset$.)

517N Corollary If $\mathfrak{A}$ is a Martin-number-homogeneous Boolean algebra with Stone space Z, then $\mathfrak{m}(\mathfrak{A}) = n(Z)$.

proof By 517Kb(i)$\Rightarrow$(iii), $\mathfrak{m}(\mathfrak{A}) \le n(Z)$. In the other direction, given $a \in \mathfrak{A}$, write $\widehat{a}$ for the open-and-closed subset of Z corresponding to a, and $\mathfrak{A}_a$ for the principal ideal generated by a. If $G \subseteq Z$ is a non-empty regular open set, let $a \in \mathfrak{A} \setminus \{0\}$ be such that $\widehat{a} \subseteq G$. Then

$$\mathfrak{m}(\mathrm{RO}(G)) \le \mathfrak{m}(\mathrm{RO}(\widehat{a}))$$

(by 517Ib, because $\mathrm{RO}(\widehat{a})$ can be regarded as a principal ideal of $\mathrm{RO}(G)$)

$$= \mathfrak{m}(\mathfrak{A}_a)$$

(because we can identify $\widehat{a}$ with the Stone space of $\mathfrak{A}_a$, by 312T, and use 517Ka)

$$= \mathfrak{m}(\mathfrak{A}).$$

By 517M, $n(Z) \le \mathfrak{m}(\mathfrak{A})$.

517O Martin cardinals (a) For any class $\mathcal{P}$ of partially ordered sets, we have an associated cardinal

$$\mathfrak{m}^\uparrow_{\mathcal{P}} = \min\{\mathfrak{m}^\uparrow(P) : P \in \mathcal{P}\}.$$

Much the most important of these is the cardinal

$$\mathfrak{m} = \min\{\mathfrak{m}^\uparrow(P) : P \text{ is upwards-ccc}\}.$$

Others of great interest are

$$\mathfrak{p} = \min\{\mathfrak{m}^\uparrow(P) : P \text{ is } \sigma\text{-centered upwards}\},$$

$$\mathfrak{m}_{\mathrm{K}} = \min\{\mathfrak{m}^\uparrow(P) : P \text{ satisfies Knaster's condition upwards}\},$$

$$\mathfrak{m}_{\mathrm{countable}} = \min\{\mathfrak{m}^\uparrow(P) : P \text{ is a countable partially ordered set}\}.$$

Two more which are worth examining are

$$\mathfrak{m}_{\sigma\text{-linked}} = \min\{\mathfrak{m}^\uparrow(P) : P \text{ is } \sigma\text{-linked upwards}\},$$

$$\mathfrak{m}_{\mathrm{pc}\omega_1} = \min\{\mathfrak{m}^\uparrow(P) : \omega_1 \text{ is an up-precaliber of } P\}.$$

(b) These cardinals are related as follows:

$$\begin{tikzcd}
 & & \mathfrak{m}_{\sigma\text{-linked}} \arrow[r, dash] & \mathfrak{p} \arrow[r, dash] & \mathfrak{m}_{\mathrm{countable}} \arrow[r, dash] & \mathfrak{c} \\
\omega_1 \arrow[r, dash] & \mathfrak{m} \arrow[r, dash] & \mathfrak{m}_{\mathrm{K}} \arrow[u, dash] \arrow[r, dash] & \mathfrak{m}_{\mathrm{pc}\omega_1} \arrow[u, dash] & &
\end{tikzcd}$$

The numbers here increase from bottom left to top right; that is,

$$\omega_1 \le \mathfrak{m} \le \mathfrak{m}_{\mathrm{K}} \le \mathfrak{m}_{\mathrm{pc}\omega_1} \le \mathfrak{p} \le \mathfrak{m}_{\mathrm{countable}} \le \mathfrak{c},$$

$$\mathfrak{m}_{\mathrm{K}} \le \mathfrak{m}_{\sigma\text{-linked}} \le \mathfrak{p}.$$

From 517A we see that $\omega_1 \le \mathfrak{m}$. For the proof that $\mathfrak{m}_{\mathrm{countable}} \le \mathfrak{c}$, see 517P below. As for the intermediate inequalities involving Martin cardinals, they follow directly from inclusions between the corresponding classes of partially ordered set. These are all immediate from the definitions; I give references to the general results of this chapter which cover the relevant facts, as follows.

(i) Every partially ordered set satisfying Knaster's condition upwards is ccc. (If $(\omega_1, 2)$ is an upwards precaliber pair of P, then $\mathrm{sat}^{\uparrow}(P) \le \omega_1$ (516Ka).)

(ii) If ω_1 is an up-precaliber of P, then P satisfies Knaster's condition upwards. (If $(\omega_1, \omega_1, {<}\omega)$ is a triple precaliber of P, so is $(\omega_1, 2, {<}\omega)$, by 516Ba.)

(iii) If P is σ-linked upwards, it satisfies Knaster's condition upwards. (As $\omega_1 > \max(\omega, \omega, \mathrm{link}^{\uparrow}(P))$, $(\omega_1, \omega_1, {<}3)$ is an upwards precaliber triple of P (516Kb), so $(\omega_1, 2, {<}3)$ and $(\omega_1, 2, {<}\omega)$ also are, by 516Ba again.)

(iv) If P is σ-centered upwards, it is σ-linked upwards. ($\mathrm{link}(P) \le \mathrm{link}_{<\omega}(P)$, by 511Hb.)

(v) If P is σ-centered upwards, ω_1 is an up-precaliber of P. (As $\omega_1 > \max(\omega, \omega, \mathrm{link}^{\uparrow}_{<\omega}(P))$, (ω_1, ω_1) is an upwards precaliber pair of P, by 516Kb again.)

(vi) If P is countable, it is σ-centered upwards. (Singleton subsets are centered.)

(c) I should note a special feature of the bottom row of this diagram. In the chain $\omega_1 \le \mathfrak{m} \le \mathfrak{m}_{\mathrm{K}} \le \mathfrak{m}_{\mathrm{pc}\omega_1}$, at most one of the inequalities can be strict. **P** Suppose that P is upwards-ccc and $\mathfrak{m}^{\uparrow}(P) > \omega_1$. Then ω_1 is an up-precaliber of P (517Fb), so $\mathfrak{m}^{\uparrow}(P) \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$. So if, for instance, $\mathfrak{m}_{\mathrm{K}} > \omega_1$ and P satisfies Knaster's condition upwards, $\mathfrak{m}^{\uparrow}(P) > \omega_1$ and $\mathfrak{m}^{\uparrow}(P) \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$; as P is arbitrary, $\mathfrak{m}_{\mathrm{K}} \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$. Similarly, if $\mathfrak{m} > \omega_1$ then $\mathfrak{m} = \mathfrak{m}_{\mathrm{pc}\omega_1}$. **Q**

(d) Now **Martin's Axiom** is the assertion

$$\text{‘}\mathfrak{m} = \mathfrak{c}\text{’}.$$

From the diagram above, we see that this is a consequence of the continuum hypothesis (‘$\omega_1 = \mathfrak{c}$’), and fixes all the intermediate cardinals.

(e) All the partially ordered sets considered in (b) are ccc, which is why $\mathfrak{m}$ appears at bottom left. The same idea can be applied to larger classes, e.g. ‘proper’ or ‘stationary-set-preserving’ partial orders. For the moment I will not even define these classes; I mention them only for the sake of readers who are already familiar with them and may be expecting a reference here. There is an important difference, however, in that if the cardinal which we might call

$$\mathfrak{m}_{\mathrm{proper}} = \min\{\mathfrak{m}^{\uparrow}(P) : P \text{ is upwards-proper}\}$$

is greater than ω_1, then $\mathfrak{c} = \mathfrak{m}_{\mathrm{proper}} = \omega_2$ (Veličković 92, or Moore 05); so that we have only to say whether the Proper Forcing Axiom (‘$\mathfrak{m}_{\mathrm{proper}} > \omega_1$’) is true or false to determine the value of $\mathfrak{m}_{\mathrm{proper}}$.

517P All the cardinals here have special features, but the ones I will concentrate on just now are the two largest, $\mathfrak{m}_{\mathrm{countable}}$ and $\mathfrak{p}$.

Proposition (a) $\omega_1 \le \mathfrak{m}_{\mathrm{countable}} \le \mathfrak{c}$.

(b) Let $\mathfrak{A}$ be a Boolean algebra with countable π-weight. If $\mathfrak{A}$ is purely atomic, then $\mathfrak{m}(\mathfrak{A}) = \infty$; otherwise, $\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}_{\mathrm{countable}}$.

(c) If P is a partially ordered set of countable cofinality and $\mathfrak{m}^{\uparrow}(P)$ is not ∞, then $\mathfrak{m}^{\uparrow}(P) = \mathfrak{m}_{\mathrm{countable}}$.

(d)(i) Let X be a topological space such that its category algebra is atomless and has countable π-weight. Then $n(X) \le \mathfrak{m}_{\mathrm{countable}}$.

(ii) If X is a non-empty locally compact Hausdorff space with countable π-weight and no isolated points, then $n(X) = \mathfrak{m}_{\mathrm{countable}}$.

(iii) If X is a non-empty Polish space with no isolated points, then $n(X) = \mathfrak{m}_{\mathrm{countable}}$.

proof Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. The argument will go more smoothly if I prove (a)-(c) with $\mathfrak{m}(\mathfrak{B})$ in place of $\mathfrak{m}_{\mathrm{countable}}$, and at an appropriate moment point out that I have shown that the two are equal.

(a) $\mathfrak{m}(\mathfrak{B}) = \mathfrak{m}^{\downarrow}(\mathfrak{B}^+)$ is uncountable, by 517A. To see that $\mathfrak{m}(\mathfrak{B}) \le \mathfrak{c}$, let $\mathcal{Q}$ be the set of all coinitial subsets of $\mathfrak{B}^+$; then $\#(\mathcal{Q}) \le \mathfrak{c}$ because $\mathfrak{B}$ is countable. **?** If $\mathfrak{m}(\mathfrak{B}) > \mathfrak{c}$, there must be a linked set $R \subseteq \mathfrak{B}^+$ meeting every member of $\mathcal{Q}$. But now consider $Q = \mathfrak{B}^+ \setminus R$. If $a \in \mathfrak{B}^+$, there are disjoint non-zero a', $a'' \subseteq a$ which cannot both belong to R, so at least one belongs to Q. But this means that Q is order-dense in $\mathfrak{B}$ and ought to meet R. **X**

(Compare 517E.)

(b)(i) If $\mathfrak{A}$ is purely atomic, $\mathfrak{m}(\mathfrak{A}) = \infty$, by 511If.

(ii) Suppose that $\mathfrak{A}$ is not purely atomic. Because $\pi(\mathfrak{A})$ is countable, there is a countable order-dense set $C \subseteq \mathfrak{A}$. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by C, so that $\mathfrak{C}$ is a countable order-dense subalgebra of $\mathfrak{A}$, and is not purely atomic. Consider the free product $\mathfrak{C} \otimes \mathfrak{B}$ (315N). This is a countable atomless Boolean algebra (use 315O), so is isomorphic to $\mathfrak{B}$ (316M). Also we have an injective order-continuous Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{C} \otimes \mathfrak{B}$ (315K), so that $\mathfrak{C}$ is isomorphic to a regularly embedded subalgebra of $\mathfrak{B}$ and $\mathfrak{m}(\mathfrak{C}) \ge \mathfrak{m}(\mathfrak{B})$ (517Ia).

Next, $\mathfrak{C}$ has a non-trivial atomless principal ideal $\mathfrak{C}_a$ say. Because $\mathfrak{C}_a$ is still countable, it is itself isomorphic to $\mathfrak{B}$. So 517Ib tells us that $\mathfrak{m}(\mathfrak{C}) \le \mathfrak{m}(\mathfrak{C}_a) = \mathfrak{m}(\mathfrak{B})$, and $\mathfrak{m}(\mathfrak{C}) = \mathfrak{m}(\mathfrak{B})$.

Finally, $\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}(\mathfrak{C})$ by 517Ic.

(c) We know that $\mathfrak{m}^{\uparrow}(P) = \mathfrak{m}(\mathrm{RO}^{\uparrow}(P))$ (517Db) and that $\pi(\mathrm{RO}^{\uparrow}(P)) \le \operatorname{cf} P$ (514Nb) is countable. Let $D \subseteq \mathrm{RO}^{\uparrow}(P)^+$ be a countable order-dense set, and $\mathfrak{A}$ the subalgebra of $\mathrm{RO}^{\uparrow}(P)$ generated by D. Then $\mathfrak{m}(\mathrm{RO}^{\uparrow}(P)) = \mathfrak{m}(\mathfrak{A})$ by 517Ic, and $\mathfrak{A}$ is countable. By (b), $\mathfrak{m}^{\uparrow}(P) = \mathfrak{m}(\mathfrak{A})$ is either ∞ or $\mathfrak{m}(\mathfrak{B})$; since the former is ruled out by hypothesis, we are left with the latter.

What this shows, however, is that

$$\begin{aligned}\mathfrak{m}(\mathfrak{B}) &\le \min\{\mathfrak{m}^{\uparrow}(P) : \operatorname{cf} P \le \omega\} \le \min\{\mathfrak{m}^{\uparrow}(P) : \#(P) \le \omega\} \\ &= \mathfrak{m}_{\mathrm{countable}} \le \mathfrak{m}^{\downarrow}(\mathfrak{B}^+) = \mathfrak{m}(\mathfrak{B})\end{aligned}$$

so that $\mathfrak{m}_{\mathrm{countable}} = \mathfrak{m}(\mathfrak{B})$ and we can rewrite the results so far in the forms given in the statement of the proposition.

(d)(i) Consider first the case in which X is a non-empty Baire space, so that its category algebra is isomorphic to $\mathrm{RO}(X)$ (514If). Since $\mathrm{RO}(X)$ is atomless and not $\{\emptyset\}$, and in particular is not purely atomic, but has countable π-weight, $\mathfrak{m}(\mathrm{RO}(X)) = \mathfrak{m}_{\mathrm{countable}}$, by (b). The same applies to any non-empty open subset G of X, recalling that the category algebra of G can be identified with a principal ideal of the category algebra of X (514Id). So $n(X) \le \mathfrak{m}_{\mathrm{countable}}$ by 517M.

If X is not a Baire space, then it has a smallest comeager regular open set H, which is itself a Baire space (4A3Ra), and X and H have isomorphic category algebras (514Ic), so we see from the argument just above that $n(H) \le \mathfrak{m}_{\mathrm{countable}}$. But $X \setminus H$ is a countable union of nowhere dense subsets of X, and every subset of H which is nowhere dense in H is also nowhere dense in X, so $n(X) \le \max(\omega, n(H)) \le \mathfrak{m}_{\mathrm{countable}}$.

(ii) Because X is Hausdorff and has no isolated points, $\mathrm{RO}(X)$ is atomless. Next, $\pi(\mathrm{RO}(X)) \le \pi(X)$ is countable (514H(b-i)), and $\mathrm{RO}(X)$ is isomorphic to the category algebra of X, by Baire's theorem. So the first part of the proof of (i) tells us that $n(X) \le \mathfrak{m}_{\mathrm{countable}} = \mathfrak{m}(\mathrm{RO}(X))$. From 517J we now see that

$$\mathfrak{m}(\mathrm{RO}(X)) = \min\{n(H) : H \subseteq X \text{ is a non-empty open set}\} \le n(X),$$

so $n(X) = \mathfrak{m}_{\mathrm{countable}}$ exactly.

(iii) Now suppose that X is a non-empty Polish space without isolated points. As in (ii), the category algebra of X is atomless and has countable π-weight, so $n(X) \le \mathfrak{m}_{\mathrm{countable}}$. In the other direction, suppose that $\kappa < \mathfrak{m}_{\mathrm{countable}}$ and that $\langle E_\xi \rangle_{\xi<\kappa}$ is a family of nowhere dense subsets of X. Let ρ be a metric defining the topology of X under which X is complete, and $\mathcal{U}$ a countable base for the topology of X, not containing $\emptyset$. For $\xi < \kappa$, set $\mathcal{Q}_\xi = \{U : U \in \mathcal{U},$

$\overline{U} \cap E_\xi = \emptyset\}$; for $n \in \mathbb{N}$ set $\mathcal{Q}'_n = \{U : U \in \mathcal{U},\, \operatorname{diam} U \le 2^{-n}\}$. Then every $\mathcal{Q}_\xi$ and every $\mathcal{Q}'_n$ is coinitial with $\mathcal{U}$. By (c) above,

$$\mathfrak{m}^{\downarrow}(\mathcal{U}) \ge \mathfrak{m}_{\mathrm{countable}} > \max(\kappa, \omega),$$

so there is a downwards-directed $\mathcal{V} \subseteq \mathcal{U}$ meeting every $\mathcal{Q}_\xi$ and every $\mathcal{Q}_n$. Now $\{\overline{V} : V \in \mathcal{V}\}$ is a downwards-directed set containing sets of arbitrarily small diameter, so generates a Cauchy filter and (because (X, ρ) is complete) has non-empty intersection. Take any $x \in \bigcap_{V \in \mathcal{V}} \overline{V}$. Because $\mathcal{V}$ meets every $\mathcal{Q}_\xi$, $x \notin \bigcup_{\xi<\kappa} E_\xi$ and $\langle E_\xi\rangle_{\xi<\kappa}$ does not cover X. As $\langle E_\xi\rangle_{\xi<\kappa}$ is arbitrary, $n(X) \ge \mathfrak{m}_{\mathrm{countable}}$ and the two are equal.

517Q Lemma If P is any partially ordered set, $\mathfrak{m}^{\uparrow}(P) \ge \min(\operatorname{add}_\omega P, \mathfrak{m}_{\mathrm{countable}})$.

proof (For the definition of $\operatorname{add}_\omega P$, see 513H.) Take $\kappa < \min(\operatorname{add}_\omega P, \mathfrak{m}_{\mathrm{countable}})$, $p_0 \in P$ and a family $\langle Q_\xi\rangle_{\xi<\kappa}$ of cofinal subsets of P. Choose $\langle R_n\rangle_{n\in\mathbb{N}}$ and $\langle Q_{n\xi}\rangle_{n\in\mathbb{N},\xi<\kappa}$ as follows. $R_0 = \{p_0\}$. Given that $R_n \subseteq P$ is countable, then for each $\xi < \kappa$ choose a countable set $Q_{n\xi} \subseteq Q_\xi$ such that for every $p \in R_n$ there is a $q \in Q_{n\xi}$ such that $p \le q$. Now, because $\operatorname{add}_\omega P > \kappa$ (and, of course, $\operatorname{add}_\omega P > \omega$, as noted in 513Ib), we can find a countable set $R_{n+1} \subseteq P$ such that whenever $q \in \bigcup_{\xi<\kappa} Q_{n\xi}$ there is an $r \in R_{n+1}$ such that $q \le r$. This will ensure that whenever $p \in R_n$ and $\xi < \kappa$ there are $q \in Q_\xi$ and $p' \in R_{n+1}$ such that $p \le q \le p'$.

At the end of the induction, consider the countable partially ordered set $R = \bigcup_{n\in\mathbb{N}} R_n$. For $\xi < \kappa$ set

$$Q'_\xi = \{r : r \in R, \exists\, q \in Q_\xi,\, q \le r\};$$

then Q'_ξ is cofinal with R. Because $\kappa < \mathfrak{m}_{\mathrm{countable}}$, there is an upwards-linked subset S of R meeting every Q'_ξ and containing p_0. But now $\{p : p \in P, \exists\, s \in S,\, p \le s\}$ is an upwards-linked subset of p containing p_0 and meeting every Q_ξ. As p_0 and $\langle Q_\xi\rangle_{\xi<\kappa}$ are arbitrary, $\mathfrak{m}^{\uparrow}(P) \ge \min(\operatorname{add}_\omega P, \mathfrak{m}_{\mathrm{countable}})$.

517R Proposition (a) ('Booth's Lemma'; see Booth 70) Suppose that $\mathcal{A}$ is a family of subsets of $\mathbb{N}$ such that $\#(\mathcal{A}) < \mathfrak{p}$ and $\bigcap \mathcal{J}$ is infinite for every finite $\mathcal{J} \subseteq \mathcal{A}$. Then there is an infinite $I \subseteq \mathbb{N}$ such that $I \setminus A$ is finite for every $A \in \mathcal{A}$.

(b) $2^\kappa \le \mathfrak{c}$ for every $\kappa < \mathfrak{p}$.

(c) Suppose that X is a set and $\#(X) < \mathfrak{p}$. Then there is a countable set $\mathcal{A} \subseteq \mathcal{P}X$ such that $\mathcal{P}X$ is the σ-algebra generated by $\mathcal{A}$.

proof (a) Let P be $[\mathbb{N}]^{<\omega} \times [\mathcal{A}]^{<\omega}$, ordered by saying that $(K, \mathcal{J}) \le (K', \mathcal{J}')$ if $K \subseteq K' \subseteq K \cup \bigcap \mathcal{J}$ and $\mathcal{J} \subseteq \mathcal{J}'$. If $(K, \mathcal{J}) \le (K', \mathcal{J}') \le (K'', \mathcal{J}'')$ then of course $K \subseteq K''$ and $\mathcal{J} \subseteq \mathcal{J}''$; also

$$K'' \subseteq K' \cup \bigcap \mathcal{J}' \subseteq K \cup \bigcap \mathcal{J} \cup \bigcap \mathcal{J}' \subseteq K \cup \bigcap \mathcal{J}.$$

So $\le$ is a partial ordering of P. For any $K \in [\mathbb{N}]^{<\omega}$, $\{(K, \mathcal{J}) : J \in [\mathcal{A}]^{<\omega}\}$ is upwards-centered; so P is σ-centered upwards.

For each $A \in \mathcal{A}$, set $Q_A = \{(K, \mathcal{J}) : (K, \mathcal{J}) \in P,\, A \in \mathcal{J}\}$; since $(K, \mathcal{J}) \le (K, \mathcal{J} \cup \{A\})$ whenever $(K, \mathcal{J}) \in P$, Q_A is cofinal with P. For $n \in \mathbb{N}$, set $Q'_n = \{(K, \mathcal{J}) : (K, \mathcal{J}) \in P,\, K \not\subseteq n\}$. If $(K, \mathcal{J}) \in P$, $\bigcap \mathcal{J}$ must be infinite, and there is an $m \in \mathbb{N} \cap \bigcap \mathcal{J} \setminus n$; now $(K, \mathcal{J}) \le (K \cup \{m\}, \mathcal{J}) \in Q'_n$. So Q'_n is cofinal with P.

Because $\max(\omega, \#(\mathcal{A})) < \mathfrak{p}$ (517Ob), there is an upwards-linked $R \subseteq P$ meeting every Q_A and every Q'_n. Set $I = \bigcup\{K : (K, \mathcal{J}) \in R\}$. If $n \in \mathbb{N}$, there is a $(K, \mathcal{J}) \in R \cap Q'_n$; now $K \not\subseteq n$ and $K \subseteq I$, so $I \not\subseteq n$; as n is arbitrary, I is infinite. If $A \in \mathcal{A}$, there is $(K_0, \mathcal{J}_0) \in R \cap Q_A$. **?** If $I \not\subseteq K_0 \cup A$, there is a $(K, \mathcal{J}) \in R$ such that $K \not\subseteq K_0 \cup A$. Now there is a $(K', \mathcal{J}') \in P$ such that $(K, \mathcal{J}) \le (K', \mathcal{J}')$ and $(K_0, \mathcal{J}_0) \le (K', \mathcal{J}')$. But in this case

$$K \subseteq K' \subseteq K_0 \cup \bigcap \mathcal{J}_0 \subseteq K_0 \cup A. \text{ **X**}$$

So $I \setminus A \subseteq K_0$ is finite. As A is arbitrary, we have a suitable I.

(b) We may suppose that κ is infinite. By 515H, or otherwise, there is a Boolean-independent family $\langle J_\xi\rangle_{\xi<\kappa}$ in $\mathcal{P}\mathbb{N}$. Note that $I = \bigcap_{\xi\in K} J_\xi \setminus \bigcup_{\xi\in L} J_\xi$ must be infinite whenever K, $L \subseteq \kappa$ are disjoint finite sets, because $\langle I \cap J_\xi\rangle_{\xi\in\kappa\setminus(K\cup L)}$ is Boolean-independent. For $C \subseteq \kappa$ set

$$\mathcal{A}_C = \{J_\xi : \xi \in C\} \cup \{\mathbb{N} \setminus J_\xi : \xi \in \kappa \setminus C\}.$$

By (a), there is an infinite $I_C \subseteq \mathbb{N}$ such that $I_C \setminus A$ is finite for every $A \in \mathcal{A}_C$. If C, $D \subseteq \kappa$ and $\xi \in C \setminus D$, then $I_C \setminus J_\xi$ and $I_D \cap J_\xi$ are finite, so $I_C \cap I_D$ is finite and $I_C \ne I_D$. Thus $C \mapsto I_C$ is injective and $2^\kappa \le \mathfrak{c}$.

(c) Let $\langle I_x\rangle_{x\in X}$ be a family of infinite subsets of $\mathbb{N}$ such that $I_x \cap I_y$ is finite for all distinct x, $y \in X$ (5A1Fa). Set $A_n = \{x : n \in I_x\}$ for $n \in \mathbb{N}$.

Take any $A \subseteq X$ and set $P_A = \mathrm{Fn}_{<\omega}(\mathbb{N};\{0,1\}) \times [X \setminus A]^{<\omega}$, partially ordered by saying that

$(f,J) \le (f',J')$ if f' extends f, $J' \supseteq J$ and whenever $x \in J$ and $i \in I_x \cap \mathrm{dom}\, f' \setminus \mathrm{dom}\, f$, then $f'(i) = 0$.

Then P_A is σ-centered upwards because $\{(f,J) : J \in [X \setminus A]^{<\omega}\}$ is upwards-centered for every $f \in \mathrm{Fn}_{<\omega}(\mathbb{N};\{0,1\})$. For $x \in A$ and $m \in \mathbb{N}$ set

$$Q_{xm} = \{(f,J) : (f,J) \in P_A,\ f(i) = 1 \text{ for some } i \in I_x \setminus m\};$$

for $x \in X \setminus A$ set

$$Q'_x = \{(f,J) : (f,J) \in P_A,\ x \in J\}.$$

Then every Q_{xm} and every Q'_x is cofinal with P_A. Because $\#(X) < \mathfrak{p}$, there is an upwards-directed $R \subseteq P_A$ meeting every Q_{xm} and every Q'_x. Set $L = \bigcup_{(f,J)\in R}\{i : f(i) = 1\}$. Now

— if $x \in A$ and $m \in \mathbb{N}$ then $L \cap I_x \setminus m$ is non-empty, so $L \cap I_x$ is infinite,

— if $x \in X \setminus A$, there is a pair $(f_0,J_0) \in R$ such that $x \in J_0$; now $f(i) = 0$ whenever $(f,J) \in R$ and $i \in \mathrm{dom}\, f \setminus \mathrm{dom}\, f_0$, so $L \cap I_x \subseteq \mathrm{dom}\, f_0$.

Accordingly

$$A = \{x : x \in X,\ I_x \cap L \text{ is infinite}\} = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in L\setminus n} A_m$$

belongs to the σ-algebra generated by $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$, and we have a suitable family.

517S Proposition Let P be a partially ordered set which satisfies Knaster's condition upwards. If $A \subseteq P$ and $\#(A) < \mathfrak{m}_{\mathrm{K}}$, then A can be covered by a sequence of upwards-directed subsets of P.

proof By 516P, the upwards finite-support product P^* of countably many copies of P also satisfies Knaster's condition upwards. So we can use 517Ha.

517X Basic exercises (a) Let P be a partially ordered set and κ a cardinal. Show that the following are equiveridical: (i) $\kappa < \mathfrak{m}^{\uparrow}(P)$; (ii) whenever $p_0 \in P$ and $\mathcal{Q}$ is a family of cofinal subsets of P with $\#(\mathcal{Q}) \le \kappa$, there is an upwards-centered subset of P which contains p_0 and meets every member of $\mathcal{Q}$; (iii) whenever $p_0 \in P$ and $\mathcal{Q}$ is a family of up-open cofinal subsets of P with $\#(\mathcal{Q}) \le \kappa$, there is an upwards-centered subset of P which contains p_0 and meets every member of $\mathcal{Q}$; (iv) whenever $p_0 \in P$ and $\mathcal{A}$ is a family of maximal up-antichains in P with $\#(\mathcal{A}) \le \kappa$, there is an upwards-centered subset of P which contains p_0 and meets every member of $\mathcal{A}$.

(b) Let P be a partially ordered set and A a maximal up-antichain in P. Show that

$$\mathfrak{m}^{\uparrow}(P) = \min_{p\in A} \mathfrak{m}^{\uparrow}([p,\infty[).$$

(c)(i) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$. For $a \in \mathfrak{A}$ let $\mathfrak{A}_a$ be the corresponding principal ideal. Show that there is an $a \in \mathfrak{A}^+$ such that $\mathfrak{m}(\mathfrak{A}_b) = \mathfrak{m}(\mathfrak{A}_a)$ whenever $0 \ne b \subseteq a$. (ii) Show that any Dedekind complete Boolean algebra is isomorphic to a simple product of Martin-number-homogeneous Boolean algebras.

>(d) Let P be a partially ordered set. Show that $\mathfrak{m}^{\uparrow}(P) = \infty$ iff $\{p : [p,\infty[\text{ is upwards-linked}\}$ is cofinal with P.

(e) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras and $\mathfrak{A}$ its free product; suppose that κ is a regular uncountable cardinal such that $\mathrm{sat}(\mathfrak{A}_i) \le \kappa < \mathfrak{m}(\mathfrak{A}_i)$ for every $i \in I$. Show that $\mathrm{sat}(\mathfrak{A}) \le \kappa$.

(f) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ the free product of a sequence of copies of $\mathfrak{A}$. Suppose that $\kappa < \mathfrak{m}(\mathfrak{C})$. (i) Show if $A \in [\mathfrak{A}^+]^{\le\kappa}$ then A can be covered by a sequence of centered subsets of $\mathfrak{A}^+$. (ii) Show that if $\mathrm{cf}\,\kappa \ge \omega_1$ then κ is a precaliber of $\mathfrak{A}$.

(g) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. Show that $\mathfrak{m}(\mathfrak{A}) = \min\{n(Y) : Y \subseteq Z \text{ is a non-meager set with the Baire property}\}$.

>(h) Let P be a non-empty partially ordered set and P^* the upwards finite-support product of a sequence of copies of P. Show that if $\mathfrak{m}^{\uparrow}(P^*) > \omega_1$ then P must be upwards-ccc.

(i) Let X be any topological space. Show that $\mathfrak{m}(\mathrm{RO}(X)) \ge \min\{n(G) : G \subseteq X \text{ is a non-empty open set}\}$.

(j) Let X be a locally compact Hausdorff space such that $\mathrm{RO}(X)$ is Martin-number-homogeneous. Show that $\mathfrak{m}(\mathrm{RO}(X)) = n(X)$.

(k)(i) Let P be a partially ordered set which is σ-linked upwards. Show that if $A \subseteq P$ and $\#(A) < \mathfrak{m}_{\sigma\text{-linked}}$, then A can be covered by a sequence of upwards-directed subsets of P. (ii) Let P be a partially ordered set such that ω_1 is an up-precaliber of P. Show that if $A \subseteq P$ and $\#(A) < \mathfrak{m}_{\mathrm{pc}\omega_1}$, then A can be covered by a sequence of upwards-directed subsets of P. (iii) Let P be a partially ordered set which is σ-centered upwards. Show that if $A \subseteq P$ and $\#(A) < \mathfrak{p}$, then A can be covered by a sequence of upwards-directed subsets of P.

517Y Further exercises (a) For a partially ordered set P, write A_P for the family of upwards-linked subsets of P, B_P for the family of cofinal subsets of P, and T_P for $\{(R,Q) : R \in A_P,\, Q \in B_P,\, R \cap Q = \emptyset\}$. (i) Show that $\mathfrak{m}^{\uparrow}(P) = \min_{p \in P} \operatorname{cov}(A_{[p,\infty[}, T_{[p,\infty[}, B_{[p,\infty[})$. (ii) Show that if Q is another partially ordered set and there is a relation $S \subseteq P \times Q$ with the properties described in 517C, then for every $q \in Q$ there is a $p \in P$ such that $(A_{[p,\infty[}, T_{[p,\infty[}, B_{[p,\infty[}) \preccurlyeq_{\mathrm{GT}} (A_{[q,\infty[}, T_{[q,\infty[}, B_{[q,\infty[})$.

(b) Show that for every infinite regular cardinal κ there is a partially ordered set with Martin number κ^+.

(c) Show that $\mathfrak{m}(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}) \ge \mathfrak{p}$.

517 Notes and comments The study of 'Martin numbers' is a natural extension of investigations into consequences of Martin's axiom. Most of the results here are straightforward expressions of techniques developed for deducing consequences from $\mathfrak{m} = \mathfrak{c}$ or $\mathfrak{m} > \omega_1$. In particular, 517F, 517G and 517H correspond to the now-classical theorems that if $\mathfrak{m} > \omega_1$ then ω_1 is a precaliber of every ccc partially ordered set, the product of any family of ccc topological spaces is ccc, and a ccc partially ordered set with cardinal ω_1 is a countable union of directed sets (see Fremlin 84a, §41). For those familiar with the use of Martin's axiom there are no surprises here, though some refinements in the arguments are necessary. The cardinal $\mathfrak{m}_{\mathrm{countable}}$ is probably most commonly known as the Novák number of $\mathbb{R}$ (517Pd), the covering number of the ideal of meager subsets of $\mathbb{R}$. In countable partially ordered sets, most of the arguments above short-circuit to some degree; precalibers become trivial, finite-support products are automatically ccc, and directed sets have cofinal totally ordered subsets, so that the ideas take on new colours.

In Fremlin 84a I found that focusing on the cardinals $\mathfrak{p}$, $\mathfrak{m}_{\mathrm{K}}$ and $\mathfrak{m}$ broke the arguments up into reasonably balanced chapters. Within the chapter on $\mathfrak{m}_{\mathrm{K}}$, however, there is a natural division between arguments applying to $\mathfrak{m}_{\mathrm{pc}\omega_1}$ and those applying to $\mathfrak{m}_{\sigma\text{-linked}}$, which in the present book I intend to make explicit. The notation $\mathfrak{p}$ is the standard name for the cardinal $\mathfrak{m}_{\sigma\text{-centered}}$; its special position comes in part from the fact that it had been studied under a different, combinatorial, definition for a decade before M.G.Bell showed that it could also be described by the definition here (Bell 81, or Fremlin 84a, 14C).

518 Freese-Nation numbers

I run through those elements of the theory of Freese-Nation numbers, as developed by S.Fuchino, S.Geschke, S.Koppelberg, S.Shelah and L.Soukup, which seem relevant to questions concerning measure spaces and measure algebras. The first part of the section (518A-518K) examines the calculation of Freese-Nation numbers of familiar partially ordered sets and Boolean algebras. In 518L-518S I look at 'tight filtrations', which are of interest to us because of their use in lifting theorems (518L, §535).

For the definitions of 'Freese-Nation number' and 'Freese-Nation index' see 511Bi and 511Dh.

518A Proposition (Fuchino Koppelberg & Shelah 96) Let P be a partially ordered set.

(a) $\mathrm{FN}(P) \le \max(3, \#(P))$.

(b) $\mathrm{FN}(P, \ge) = \mathrm{FN}(P, \le)$.

(c) If P has no maximal element, then $\operatorname{add} P \le \mathrm{FN}(P)$.

proof (a)(i) Suppose first that P is finite and totally ordered. If $\#(P) \le 2$, set $f(p) = P$ for every $p \in P$. Otherwise, take $p_0 \in P$ such that $]-\infty, p_0[$ and $]p_0, \infty[$ are both non-empty, and set $f(p) = [p_0, \infty[$ if $p \ge p_0$, $]-\infty, p_0]$ if $p < p_0$; then f is a Freese-Nation function witnessing that $\mathrm{FN}(P) \le \#(P)$.

(ii) Next suppose that P is finite and not totally ordered. For $p \in P$ set $A_p =]-\infty, p] \cup [p, \infty[$, and take $B = \{p : A_p = P\}$; then $B \neq P$. Set $f(p) = A_p$ for $p \in P \setminus B$, B for $p \in B$; then f is a Freese-Nation function so again $\mathrm{FN}(P) \leq \#(P)$.

(iii) If P is infinite, enumerate it as $\langle p_\xi \rangle_{\xi < \#(P)}$ and set $f(p_\xi) = \{p_\eta : \eta \leq \xi\}$ for each ξ; once more we have a Freese-Nation function witnessing that $\mathrm{FN}(P) \leq \#(P)$.

(b) A function $f : P \to \mathcal{P}P$ is a Freese-Nation function for $\leq$ iff it is a Freese-Nation function for the reverse ordering $\geq$.

(c) Set $\kappa = \mathrm{FN}(P)$. Then we have a Freese-Nation function $f : P \to [P]^{<\kappa}$.

(i) I had better sort out the trivial cases. If P is empty, then $\kappa = \operatorname{add} P = 0$. Otherwise, $p \in f(p)$ for every $p \in P$, so $\kappa \geq 2$; if $\operatorname{add} P \leq 2$ we can stop. So we may suppose that $\operatorname{add} P > 2$, that is, that P is upwards-directed.

(ii) ? If $\kappa < \operatorname{add} P$, choose $\langle p_\xi \rangle_{\xi \leq \kappa}$ inductively, as follows. Given $\langle p_\eta \rangle_{\eta < \xi}$, where $\xi \leq \kappa$, then $\bigcup_{\eta < \xi} f(p_\eta)$ has an upper bound p'_ξ in P. **P** If κ is infinite, this is because $\#(\bigcup_{\eta<\xi} f(p_\eta)) \leq \kappa < \operatorname{add} P$. If κ is finite, it is because $\#(\bigcup_{\eta<\xi} f(p_\eta)) < \omega \leq \operatorname{add} P$. **Q**

As P has no maximal element, we can find $p_\xi > p'_\xi$, and continue. At the end of the induction, we have $p_\xi < p_\kappa$, so there is a $q_\xi \in f(p_\xi) \cap f(p_\kappa) \cap [p_\xi, p_\kappa]$, for each $\xi < \kappa$. If $\eta < \xi < \kappa$, then

$$q_\eta \leq p'_\xi < p_\xi \leq q_\xi$$

and $q_\eta \neq q_\xi$. But this means that $f(p_\kappa) \supseteq \{q_\xi : \xi < \kappa\}$ has at least κ elements. **X**

518B Proposition Let P be a partially ordered set and Q a subset of P.

(a) If Q is order-convex (that is, $[q, q'] \subseteq Q$ whenever q, $q' \in Q$), then $\mathrm{FN}(Q) \leq \mathrm{FN}(P)$.

(b) If Q is a retract of P (that is, there is an order-preserving $h : P \to Q$ such that $h(q) = q$ for every $q \in Q$), then $\mathrm{FN}(Q) \leq \mathrm{FN}(P)$.

(c) If Q is, in itself, Dedekind complete (that is, every non-empty subset of Q with an upper bound in Q has a supremum in Q for the induced ordering), then $\mathrm{FN}(Q) \leq \mathrm{FN}(P)$.

proof (a) If $f : P \to \mathcal{P}P$ is a Freese-Nation function on P, then $q \mapsto Q \cap f(q) : Q \to \mathcal{P}Q$ is a Freese-Nation function on Q.

(b) If f is a Freese-Nation function on P, then $q \mapsto h[f(q)]$ is a Freese-Nation function on Q.

(c) Set $Q_1 = \bigcup_{q,q' \in Q} [q, q']$, so that Q_1 is an order-convex subset of P and $\mathrm{FN}(Q_1) \leq \mathrm{FN}(P)$. For $p \in Q_1$, set $h(p) = \sup(Q \cap]-\infty, p])$, the supremum being taken in Q; then $h : Q_1 \to Q$ is a retraction, so $\mathrm{FN}(Q) \leq \mathrm{FN}(Q_1)$.

518C Corollary (a) If $\mathfrak{A}$ is an infinite Dedekind σ-complete Boolean algebra then $\mathrm{FN}(\mathfrak{A}) \geq \mathrm{FN}(\mathcal{P}\mathbb{N})$.

(b) Let $\mathfrak{A}$ be an infinite Dedekind complete Boolean algebra. Then

$$\mathrm{FN}(\mathrm{RO}(\{0,1\}^{\#(\mathfrak{A})})) \leq \mathrm{FN}(\mathfrak{A}) \leq \mathrm{FN}(\mathcal{P}(\mathrm{link}(\mathfrak{A}))) \leq \max(3, 2^{\mathrm{link}(\mathfrak{A})}).$$

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. If $\mathfrak{B}$ is either an order-closed subalgebra or a principal ideal of $\mathfrak{A}$, then $\mathrm{FN}(\mathfrak{B}) \leq \mathrm{FN}(\mathfrak{A})$.

proof (a) Take any disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$; then $I \mapsto \sup_{n \in I} a_n$ is an embedding of the partially ordered set $\mathcal{P}\mathbb{N}$ into $\mathfrak{A}$. As $\mathcal{P}\mathbb{N}$ is Dedekind complete, 518Bc tells us that $\mathrm{FN}(\mathcal{P}\mathbb{N}) \leq \mathrm{FN}(\mathfrak{A})$.

(b)(i) By 515I, $\mathfrak{A}$ has a subalgebra $\mathfrak{B}$ isomorphic to the regular open algebra $\mathrm{RO}(\{0,1\}^{\#(\mathfrak{A})})$; by 518Bc, $\mathrm{FN}(\mathfrak{B}) \leq \mathrm{FN}(\mathfrak{A})$.

(ii) By 514Cb, we have a subset Q of $\mathcal{P}(\mathrm{link}(\mathfrak{A}))$ which is order-isomorphic to $\mathfrak{A}$, and 518Bc tells us that $\mathrm{FN}(Q) \leq \mathrm{FN}(\mathcal{P}(\mathrm{link}(\mathfrak{A})))$.

(iii) By 518Aa, $\mathrm{FN}(\mathcal{P}(\mathrm{link}(\mathfrak{A}))) \leq 2^{\mathrm{link}(\mathfrak{A})}$ except in the trivial case $\mathfrak{A} = \{0, 1\}$.

(c) Immediate from 518Bc.

518D Corollary The following sets all have the same Freese-Nation number:

(i) $\mathcal{P}\mathbb{N}$;

(ii) $\mathbb{N}^{\mathbb{N}}$, with its usual ordering $\le$;

(iii) any infinite σ-linked Dedekind complete Boolean algebra;

(iv) the family of open subsets of any infinite Hausdorff second-countable topological space.

proof (a) The map $I \mapsto \chi I : \mathcal{P}\mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ is an order-preserving embedding; because $\mathcal{P}\mathbb{N}$ is Dedekind complete, $\mathrm{FN}(\mathcal{P}\mathbb{N}) \le \mathrm{FN}(\mathbb{N}^{\mathbb{N}})$ (518Bc).

(b) The map $f \mapsto \{(i,j) : j \le f(i)\} : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N}\times\mathbb{N})$ is an order-preserving embedding; because $\mathbb{N}^{\mathbb{N}}$ is Dedekind complete,

$$\mathrm{FN}(\mathbb{N}^{\mathbb{N}}) \le \mathrm{FN}(\mathcal{P}(\mathbb{N}\times\mathbb{N})) = \mathrm{FN}(\mathcal{P}\mathbb{N}).$$

(c) Now let $\mathfrak{A}$ be an infinite σ-linked Dedekind complete Boolean algebra. By 518Ca, $\mathrm{FN}(\mathcal{P}\mathbb{N}) \le \mathrm{FN}(\mathfrak{A})$; by 518Cb, $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(\mathcal{P}\mathbb{N})$.

(d) Let $(X,\mathfrak{T})$ be an infinite Hausdorff second-countable space. (α) Because X is Hausdorff and infinite, it has a disjoint sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of non-empty open sets; now $I \mapsto \bigcup_{n\in I} G_n$ is an embedding of $\mathcal{P}\mathbb{N}$ in $\mathfrak{T}$, so $\mathrm{FN}(\mathcal{P}\mathbb{N}) \le \mathrm{FN}(\mathfrak{T})$. (β) Let $\mathcal{U}$ be a countable base for $\mathfrak{T}$. Then $G \mapsto \{U : U \in \mathcal{U},\, U \subseteq G\}$ is an embedding of $\mathfrak{T}$ in $\mathcal{P}\mathcal{U}$; as $\mathfrak{T}$, regarded as a partially ordered set, is Dedekind complete, $\mathrm{FN}(\mathfrak{T}) \le \mathrm{FN}(\mathcal{P}\mathcal{U}) = \mathrm{FN}(\mathcal{P}\mathbb{N})$.

518E There is a simple result in general topology which will be used a couple of times in the next chapter.

Lemma Let $(X,\mathfrak{T})$ be a T_1 topological space without isolated points, and $\mathcal{N}\mathrm{wd}(X)$ the ideal of nowhere dense sets. Then there is a set $A \subseteq X$, with cardinal $\mathrm{cov}\,\mathcal{N}\mathrm{wd}(X)$, such that $\#(A\cap F) < \mathrm{FN}^*(\mathfrak{T})$ for every $F \in \mathcal{N}\mathrm{wd}(X)$.

Remark Perhaps I should say here that $\mathrm{FN}^*(\mathfrak{T})$ is the regular Freese-Nation number of the partially ordered set $(\mathfrak{T},\subseteq)$.

proof As X has no isolated points, $\mathrm{cov}\,\mathcal{N}\mathrm{wd}(X) \le \#(X)$. Set $\kappa = \mathrm{cov}\,\mathcal{N}\mathrm{wd}(X)$ and $\lambda = \mathrm{FN}^*(\mathfrak{T})$. If $\kappa < \lambda$ the result is trivial and we can stop. Otherwise, let $f : \mathfrak{T} \to [\mathfrak{T}]^{<\lambda}$ be a Freese-Nation function. Then we can choose $\langle x_\xi\rangle_{\xi<\kappa}$ inductively so that whenever $\eta < \xi$ and $G \in f(X\setminus\{x_\eta\})$ is dense, then $x_\xi \in G$. **P** When we come to choose x_ξ, set $\theta = \#(\bigcup_{\eta<\xi}\{G : G \in f(X\setminus\{x_\eta\})$ is dense$\})$. If $\lambda < \kappa$ then $\theta \le \max(\#(\xi),\omega,\lambda) < \kappa$. If $\lambda = \kappa$ then κ is regular and infinite and $\#(f(X\setminus\{x_\eta\}) < \kappa$ for every $\eta < \xi$ so again $\theta < \kappa$. So we have fewer than $\mathrm{cov}\,\mathcal{N}\mathrm{wd}(X)$ dense open sets and can find a point x_ξ in all of them. **Q**

Note that as $X\setminus\{x_\eta\}$ is itself dense for every $\eta < \xi$, and $H \in f(H)$ for every $H \in \mathfrak{T}$, all the x_ξ must be distinct, and $A = \{x_\xi : \xi < \kappa\}$ has cardinal κ. Now suppose that $F \in \mathcal{N}\mathrm{wd}(X)$ and set $B = \{\xi : \xi < \kappa,\, x_\xi \in F\}$. For each $\xi \in B$, $X\setminus\overline{F} \subseteq X\setminus\{x_\xi\}$, so there is a $G_\xi \in f(X\setminus\overline{F})\cap f(X\setminus\{x_\xi\})$ such that $X\setminus\overline{F} \subseteq G_\xi$ and $x_\xi \notin G_\xi$. If $\eta, \xi \in B$ and $\eta < \xi$, then $G_\eta \in f(X\setminus\{x_\eta\})$ is dense, so contains x_ξ, and cannot be equal to G_ξ. Thus $\xi \mapsto G_\xi$ is an injective function from B to $f(X\setminus\overline{F})$, and $\#(B) < \lambda$. Thus we have an appropriate set A.

518F Lemma Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ and κ an infinite cardinal.

(a) If $\mathrm{cf}(\mathfrak{B}\cap[0,a]) < \kappa$ for every $a \in \mathfrak{A}$, then the Freese-Nation index of $\mathfrak{B}$ in $\mathfrak{A}$ is at most κ.

(b) Suppose that $I \in [\mathfrak{A}]^{<\mathrm{cf}\,\kappa}$ and $\mathfrak{B}_I$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}\cup I$. If the Freese-Nation index of $\mathfrak{B}$ in $\mathfrak{A}$ is less than or equal to κ, so is the Freese-Nation index of $\mathfrak{B}_I$.

(c) If $\mathfrak{B}$ is expressible as the union of fewer than κ order-closed subalgebras of $\mathfrak{A}$, each of them Dedekind complete in itself, then the Freese-Nation index of $\mathfrak{B}$ in $\mathfrak{A}$ is at most κ.

proof (a) For any $a \in \mathfrak{A}$,

$$\mathrm{ci}(\mathfrak{B}\cap[a,1]) = \mathrm{cf}(\mathfrak{B}\cap[0,1\setminus a]) < \kappa.$$

(b)(i) Suppose first that I is a singleton $\{d\}$. In this case

$$\mathfrak{B}_I = \{(b\cap d)\cup(c\setminus d) : b,\, c \in \mathfrak{B}\}.$$

Take $a \in \mathfrak{A}$. Then there are sets B, $C \subseteq \mathfrak{B}$, with cardinal less than κ, which are cofinal in

$$\mathfrak{B}\cap[0,a\cup(1\setminus d)], \qquad \mathfrak{B}\cap[0,a\cup d]$$

respectively. Set $D = \{b\cap d : b \in B\}\cup\{c\setminus d : c \in C\}$, so that $D \subseteq \mathfrak{B}_I\cap[0,a]$ and $\#(D) < \kappa$. If b, $c \in \mathfrak{B}$ and $(b\cap d)\cup(c\setminus d) \subseteq a$, then $b \subseteq a\cup(1\setminus d)$ and $c \subseteq a\cup d$, so there are $b' \in B$, $c' \in C$ such that $b \subseteq b'$ and $c \subseteq c'$; now

$$(b \cap d) \cup (c \setminus d) \subseteq (b' \cap d) \cup (c' \setminus d) \in D.$$

Thus D witnesses that $\mathrm{cf}(\mathfrak{C} \cap [0,a]) < \kappa$. By (a), this is enough to show that the Freese-Nation index of $\mathfrak{B}_I$ in $\mathfrak{A}$ is at most κ.

(ii) An elementary induction now shows that the Freese-Nation index of $\mathfrak{B}_I$ in $\mathfrak{A}$ is at most κ for every finite subset I of $\mathfrak{A}$. If $\omega \le \#(I) < \mathrm{cf}\,\kappa$ and $a \in \mathfrak{A}$, then $\mathfrak{B}_I = \bigcup\{\mathfrak{B}_J : J \in [I]^{<\omega}\}$. For each $J \in [I]^{<\omega}$, let B_J be a cofinal subset of $\mathfrak{B}_J \cap [0,a]$ of size less than κ. Then $B = \bigcup\{B_J : J \in [I]^{<\omega}\}$ is cofinal in $\mathfrak{B}_I \cap [0,a]$; and as $\#([I]^{<\omega}) < \mathrm{cf}\,\kappa$, $\#(B) < \kappa$. So again we have $\mathrm{cf}(\mathfrak{B}_I \cap [0,a]) < \kappa$ for every $a \in \mathfrak{A}$, and the Freese-Nation index of $\mathfrak{B}_I$ in $\mathfrak{A}$ is at most κ.

(c) Suppose that $\langle \mathfrak{B}_\xi \rangle_{\xi<\lambda}$ is a family of order-closed subalgebras with union $\mathfrak{B}$, where $\lambda < \kappa$. If $a \in \mathfrak{A}$, then $b_\xi = \sup(\mathfrak{B}_\xi \cap [0,a])$ is defined in $\mathfrak{B}_\xi$, and belongs to $[0,a]$, for each $\xi < \lambda$, and $\{b_\xi : \xi < \lambda\}$ is cofinal with $\mathfrak{B} \cap [0,a]$; so we can apply (a).

518G Lemma (Fuchino Koppelberg & Shelah 96) Let P be a partially ordered set, ζ an ordinal, and $\langle A_\xi \rangle_{\xi<\zeta}$ a family with union P; set $P_\alpha = \bigcup_{\xi<\alpha} A_\xi$ for each $\alpha \le \zeta$. Let κ be a regular infinite cardinal such that, for each $\alpha < \zeta$, $\mathrm{FN}(P_{\alpha+1}) \le \kappa$ and the Freese-Nation index of P_α in $P_{\alpha+1}$ is at most κ. Then $\mathrm{FN}(P) \le \kappa$.

proof For each $\alpha < \zeta$ set $A'_\alpha = A_\alpha \setminus P_\alpha$ and choose a Freese-Nation function $f_\alpha : P_{\alpha+1} \to [P_{\alpha+1}]^{<\kappa}$. For $p \in P$, let $\gamma(p)$ be that $\alpha < \zeta$ such that $p \in A'_\alpha$, and let $D_p \subseteq P_{\gamma(p)}$ be a set of size less than κ such that $D_p \cap \,]-\infty, p]$ is cofinal with $P_{\gamma(p)} \cap \,]-\infty, p]$ and $D_p \cap [p, \infty[$ is cofinal with $P_{\gamma(p)} \cap [p, \infty[$. Define g inductively, on each A'_α in turn, by setting $g(p) = f_{\gamma(p)}(p) \cup \bigcup_{q \in D_p} g(q)$ for every $p \in P$. Because κ is regular, g is a function from P to $[P]^{<\kappa}$.

Now g is a Freese-Nation function on P. **P** I induce on α to show that if p, $q \in P$ and $\max(\gamma(p), \gamma(q)) = \alpha$ then $g(p) \cap g(q) \cap [p,q] \ne \emptyset$. For the inductive step to $\alpha < \zeta$, if $\gamma(p) = \gamma(q) = \alpha$ then

$$g(p) \cap g(q) \cap [p,q] \supseteq f_\alpha(p) \cap f_\alpha(q) \cap [p,q] \ne \emptyset.$$

If $\gamma(p) < \gamma(q) = \alpha$, then there is an $r \in D_q$ such that $p \le r \le q$; now $\max(\gamma(p), \gamma(r)) < \alpha$, so

$$g(p) \cap g(q) \cap [p,q] \supseteq g(p) \cap g(r) \cap [p,r] \ne \emptyset$$

by the inductive hypothesis. The same argument works if $\gamma(q) < \gamma(p)$. **Q**

518H Lemma Suppose that κ is an uncountable cardinal of countable cofinality such that $\Box_\kappa$ is true and $\mathrm{cf}[\lambda]^{\le\omega} \le \lambda^+$ for every $\lambda \le \kappa$. Then there are families $\langle M_{\alpha n} \rangle_{\alpha<\kappa^+, n\in\mathbb{N}}$, $\langle M_\alpha \rangle_{\alpha<\kappa^+}$ of sets and a function sk such that

(i) $\#(M_{\alpha n}) < \kappa$ whenever $\alpha < \kappa^+$ and $n \in \mathbb{N}$;

(ii) $\langle M_{\alpha n} \rangle_{n\in\mathbb{N}}$ is non-decreasing for each $\alpha < \kappa^+$;

(iii) $\langle M_\alpha \rangle_{\alpha<\kappa^+}$ is a non-decreasing family, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for every non-zero limit ordinal $\alpha < \kappa^+$, and $\kappa^+ \subseteq \bigcup_{\alpha<\kappa^+} M_\alpha$;

(iv) if $\alpha < \kappa^+$ has uncountable cofinality, $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha n}$;

(v) $X \subseteq \mathrm{sk}(X)$ for every set X;

(vi) $\mathrm{sk}(X)$ is countable whenever X is countable;

(vii) $A \subseteq \mathrm{sk}(X)$ whenever $A \in \mathrm{sk}(X)$ is countable;

(viii) $\mathrm{sk}(X) \subseteq \mathrm{sk}(Y)$ whenever $X \subseteq \mathrm{sk}(Y)$;

(ix) for every $\alpha < \kappa^+$ of uncountable cofinality there is an $m \in \mathbb{N}$ such that whenever $n \ge m$ and $A \subseteq M_{\alpha n}$ is countable there is a countable set $D \in M_{\alpha n}$ such that $A \subseteq \mathrm{sk}(D)$;

(x) $\bigcup_{\alpha<\kappa^+} M_\alpha \cap [\kappa]^{\le\omega}$ is cofinal with $[\kappa]^{\le\omega}$.

proof (a) There is a strictly increasing sequence $\langle \kappa_n \rangle_{n\in\mathbb{N}}$ of cardinals with supremum κ; since

$$\mathrm{cf}[\kappa_n^+]^{\le\omega} = \max(\kappa_n^+, \mathrm{cf}[\kappa_n]^{\le\omega}) = \kappa_n^+$$

for each n (5A1E(e-iv)), we can suppose that in fact $\mathrm{cf}[\kappa_n]^{\le\omega} = \kappa_n$ for every n. Take $\langle C_\alpha \rangle_{\alpha<\kappa^+}$ witnessing $\Box_\kappa$, so that

for every $\alpha < \kappa^+$, $C_\alpha \subseteq \alpha$ is a closed cofinal set in α of order type at most κ,

whenever $\delta < \alpha < \kappa^+$ and $\delta = \sup(\delta \cap C_\alpha)$, then $C_\delta = \delta \cap C_\alpha$

(5A6D(a-ii)). For $\alpha < \kappa^+$ set

$$C'_\alpha = \{\delta : \delta < \alpha,\ \delta = \sup(\delta \cap C_\alpha)\} \subseteq C_\alpha$$

and

$$C'_{\alpha n} = \{\delta : \delta \in C'_\alpha,\ \mathrm{otp}(\delta \cap C_\alpha) < \kappa_n\}$$

for each n. Because $\mathrm{otp}(C_\alpha) \le \kappa$, $C'_\alpha = \bigcup_{n\in\mathbb{N}} C'_{\alpha n}$, while $\#(C'_{\alpha n}) \le \kappa_n$ for each n. Note that if α has uncountable cofinality, C'_α will be cofinal with α.

(b) Let $g : \kappa^+ \to [\kappa]^{\le\omega}$ be such that $g[\kappa^+]$ is cofinal with $[\kappa]^{\le\omega}$. For each non-zero $\alpha < \kappa^+$, fix on a surjective function $f_\alpha : \kappa \to \alpha$. For each $n \in \mathbb{N}$, let $g_n : \kappa_n \to [\kappa_n]^{\le\omega}$ be such that $g_n[\kappa_n]$ is cofinal with $[\kappa_n]^{\le\omega}$. For each $\alpha < \kappa^+$, let $h_\alpha : \#(C_\alpha) \to C_\alpha$ be a bijection. Now, for any set X, write $\mathrm{sk}(X)$ for the smallest set including X and such that

$g(\alpha) \in \mathrm{sk}(X)$ whenever $\alpha \in \mathrm{sk}(X) \cap \kappa^+$,
$f_\alpha(\xi) \in \mathrm{sk}(X)$ whenever $0 < \alpha < \kappa^+$, $\xi < \kappa$ and α, $\xi \in \mathrm{sk}(X)$,
$g_n(\xi) \in \mathrm{sk}(X)$ whenever $n \in \mathbb{N}$ and $\xi \in \kappa_n \cap \mathrm{sk}(X)$,
$h_\alpha[A] \in \mathrm{sk}(X)$ whenever $\alpha \in \mathrm{sk}(X) \cap \kappa^+$ and $A \in \mathrm{sk}(X)$,
$A \cup B \in \mathrm{sk}(X)$ whenever A, $B \in \mathrm{sk}(X)$,
$A \subseteq \mathrm{sk}(X)$ whenever $A \in \mathrm{sk}(X)$ is countable.

Of course we always have

$$\mathrm{sk}(\mathrm{sk}(X)) = \mathrm{sk}(X) = \bigcup\{\mathrm{sk}(I) : I \in [X]^{<\omega}\}$$

and $\#(\mathrm{sk}(X)) \le \max(\omega, \#(X))$, so (v)-(viii) are all true.

(c) For each $\alpha < \kappa^+$ and $n \in \mathbb{N}$, set $M_{\alpha n} = \mathrm{sk}(\kappa_n \cup C'_{\alpha n})$ and $M_\alpha = \mathrm{sk}(\kappa \cup \alpha)$. Then

$$\#(M_{\alpha n}) \le \max(\omega, \kappa_n, \#(C'_{\alpha n})) < \kappa.$$

Also $\langle M_\alpha\rangle_{\alpha<\kappa^+}$ is non-decreasing and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ whenever $\alpha < \kappa$ is a non-zero limit ordinal, while $\kappa^+ \subseteq \bigcup_{\alpha<\kappa^+} M_\alpha$. This deals with (i)-(iii).

(d) Now for (iv): $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha n}$ whenever $\alpha < \kappa^+$ has uncountable cofinality. **P** Of course $M_{\alpha n} \subseteq M_\alpha$ for every n just because $\mathrm{sk}(X) \subseteq \mathrm{sk}(Y)$ whenever $X \subseteq Y$. On the other hand, if $\beta < \alpha$, take $\delta \in C'_\alpha$ such that $\beta < \delta$, and $\xi < \kappa$ such that $f_\delta(\xi) = \beta$; then if $n \in \mathbb{N}$ is such that $\xi < \kappa_n$ and $\mathrm{otp}(\delta \cap C_\alpha) < \kappa_n$, β will be in $M_{\alpha n}$. So $\bigcup_{n\in\mathbb{N}} M_{\alpha_n} \supseteq \alpha$. Moreover,

$$\kappa = \bigcup_{n\in\mathbb{N}} \kappa_n \subseteq \bigcup_{n\in\mathbb{N}} M_{\alpha n}.$$

So $\alpha \cup \kappa \subseteq \bigcup_{n\in\mathbb{N}} M_{\alpha n}$ and M_α must be exactly $\bigcup_{n\in\mathbb{N}} M_{\alpha n}$. **Q**

(e) Again suppose that $\alpha < \kappa^+$ has uncountable cofinality. Then there must be an $m \in \mathbb{N}$ such that $C'_{\alpha m}$ is cofinal with α. Suppose that $n \ge m$ and $A \subseteq M_{\alpha n}$ is countable. Then there must be a countable set $C \subseteq \kappa_n \cup C'_{\alpha n}$ such that $A \subseteq \mathrm{sk}(C)$. Let $\delta \in C'_{\alpha m}$ be such that $C \cap \alpha \subseteq \delta$. Then $C_\delta = \delta \cap C_\alpha$ has cardinal less than $\kappa_m \le \kappa_n$, so $(C \cap \kappa_n) \cup h_\delta^{-1}[C]$ is a countable subset of κ_n and is included in $g_n(\xi)$ for some $\xi < \kappa_n$. Now ξ and δ belong to $M_{\alpha n}$, so $g_n(\xi)$ and $h_\delta[g_n(\xi)]$ and $D = g_n(\xi) \cup h_\delta[g_n(\xi)]$ all belong to $M_{\alpha n}$, and are countable. But $C \subseteq D$, so $A \subseteq \mathrm{sk}(D)$, as required by (ix).

(f) Finally, if $A \subseteq \kappa$ is countable, there is a $\beta < \kappa^+$ such that $g(\beta) \supseteq A$, and now $g(\beta) \in M_{\beta+1}$. So (x) is true.

518I Theorem (Fuchino & Soukup 97) Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra. Suppose that

(α) $\mathrm{cf}[\lambda]^{\le\omega} \le \lambda^+$ for every cardinal $\lambda \le \tau(\mathfrak{A})$,
(β) $\square_\lambda$ is true for every uncountable cardinal $\lambda \le \tau(\mathfrak{A})$ of countable cofinality.

Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra, and κ a regular uncountable cardinal such that $\mathrm{FN}(\mathfrak{B}) \le \kappa$ for every countably generated order-closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. Then $\mathrm{FN}(\mathfrak{A}) \le \kappa$.

proof Induce on the Maharam type $\tau(\mathfrak{A})$ of $\mathfrak{A}$.

(a) If $\tau(\mathfrak{A}) \le \omega$ the result is trivial.

(b) For the inductive step to $\tau(\mathfrak{A}) = \lambda$, where λ is an infinite cardinal of uncountable cofinality, let $\langle a_\xi\rangle_{\xi<\lambda}$ enumerate a τ-generating subset of $\mathfrak{A}$. For each $\beta < \lambda$, let $\mathfrak{B}_\beta$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \beta\}$, and for $\alpha \le \lambda$ set $\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{B}_\beta$. By the inductive hypothesis, $\mathrm{FN}(\mathfrak{B}_\beta) \le \kappa$ for every $\beta < \lambda$. Also, for $\alpha < \kappa$, either α has uncountable cofinality, in which case (because $\mathfrak{A}$ is ccc) $\mathfrak{A}_\alpha = \mathfrak{B}_\alpha$ is order-closed, or α has countable cofinality, in which case $\mathfrak{A}_\alpha$ is a countable union of order-closed subalgebras. In either case, the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}_{\alpha+1}$ is countable (518Fc). Because $\mathrm{cf}\,\lambda > \omega$, $\mathfrak{A} = \mathfrak{A}_\lambda$. By 518G, $\mathrm{FN}(\mathfrak{A}) \le \kappa$.

(c)(i) For the inductive step to $\tau(\mathfrak{A}) = \lambda$, where λ is an uncountable cardinal of countable cofinality, we may use the method of Lemma 518H to construct $\langle M_{\alpha n}\rangle_{\alpha<\lambda^+,n\in\mathbb{N}}$, $\langle M_\alpha\rangle_{\alpha<\lambda^+}$ and sk as described there. Enumerate a τ-generating set in $\mathfrak{A}$ as $\langle a_\xi\rangle_{\xi<\lambda}$, and for any set X write $\mathfrak{B}_X$ for the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi \in X \cap \lambda\}$. For each $\alpha < \lambda^+$ set $\mathfrak{C}_\alpha = \bigcup\{\mathfrak{B}_{\mathrm{sk}(D)} : D \in M_\alpha$ is countable$\}$.

(ii) If $\alpha < \lambda^+$ has uncountable cofinality, then $\mathfrak{C}_\alpha$ is the union of a non-decreasing sequence of order-closed subalgebras of $\mathfrak{A}$ with Maharam type less than λ. **P** By (ix) of 518H, there is an $m \in \mathbb{N}$ such that whenever $n \ge m$ and $\mathcal{D} \subseteq M_{\alpha n}$ is countable there is a countable set $F \in M_{\alpha n}$ such that $\mathcal{D} \subseteq \mathrm{sk}(F)$. For each $n \ge m$, set $\mathfrak{C}_n = \bigcup\{\mathfrak{B}_{\mathrm{sk}(D)} : D \in M_{\alpha n}$ is countable$\}$. Then for any countable set $C \subseteq \mathfrak{C}_n$, there is a countable set $\mathcal{D}$ of countable sets belonging to $M_{\alpha n}$ such that $C \subseteq \bigcup_{D\in\mathcal{D}} \mathfrak{B}_{\mathrm{sk}(D)}$. So there is a countable set $F \in M_{\alpha n}$ such that $\mathcal{D} \subseteq \mathrm{sk}(F)$; by 518H(vii), $D \subseteq \mathrm{sk}(F)$ and $\mathfrak{B}_{\mathrm{sk}(D)} \subseteq \mathfrak{B}_{\mathrm{sk}(F)}$ (518H(viii)) for every $D \in \mathcal{D}$. But this means that $C \subseteq \mathfrak{B}_{\mathrm{sk}}(F) \subseteq \mathfrak{C}_n$, while $\mathfrak{B}_{\mathrm{sk}}(F)$ is an order-closed subalgebra of $\mathfrak{A}$. Because $\mathfrak{A}$ is ccc, this is enough to show that $\mathfrak{C}_n$ is an order-closed subalgebra of $\mathfrak{A}$; by 518H(ii) and 518H(iv), $\langle\mathfrak{C}_n\rangle_{n\ge m}$ is non-decreasing and has union $\mathfrak{C}_\alpha$. Each $\mathfrak{C}_n$ is τ-generated by

$$\{a_\eta : \text{there is a countable } D \in M_{\alpha n} \text{ such that } \eta \in \mathrm{sk}(D) \cap \lambda^+\},$$

so (using 518H(vi))

$$\tau(\mathfrak{C}_n) \le \max(\omega, \#(M_{\alpha n})) < \lambda. \textbf{ Q}$$

It follows from 518Fc again that the Freese-Nation index of $\mathfrak{C}_\alpha$ in $\mathfrak{A}$ is countable, and from the inductive hypothesis we see also that $\mathrm{FN}(\mathfrak{C}_n) \le \kappa$ for every $n \ge m$, so that (using 518G, as usual) $\mathrm{FN}(\mathfrak{C}_\alpha) \le \kappa$.

(iii) If $\alpha < \lambda^+$ is the union of a sequence $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ of ordinals of uncountable cofinality, then $M_\alpha = \bigcup_{n\in\mathbb{N}} M_{\alpha_n}$ (518H(iii)), so $\mathfrak{C}_\alpha = \bigcup_{n\in\mathbb{N}} \mathfrak{C}_{\alpha_n}$ is again a countable union of order-closed subalgebras of $\mathfrak{A}$, and the Freese-Nation index of $\mathfrak{C}_\alpha$ in $\mathfrak{A}$ is at most ω. Moreover, because $\mathrm{FN}(\mathfrak{C}_{\alpha_n}) \le \kappa$ for each n, $\mathrm{FN}(\mathfrak{C}_\alpha) \le \kappa$.

(iv) If $a \in \mathfrak{A}$, there is a countable set $D \subseteq \lambda$ such that $a \in \mathfrak{B}_D$. But now there is an $\alpha < \lambda^+$, of uncountable cofinality, such that $D \subseteq D'$ for some countable $D' \in M_\alpha$ (518H(x)), and

$$a \in \mathfrak{B}_D \subseteq \mathfrak{B}_{D'} \subseteq \mathfrak{B}_{\mathrm{sk}(D')} \subseteq \mathfrak{C}_\alpha,$$

by 518H(v).

(v) Let $F \subseteq \lambda^+$ be the set of ordinals which are either of uncountable cofinality, or the union of a sequence of such ordinals; so that F is a closed cofinal set in λ^+, and $\mathfrak{C}_\alpha$ has countable Freese-Nation index in $\mathfrak{A}$ for every $\alpha \in F$. By (iv), $\bigcup_{\alpha\in F} \mathfrak{C}_\alpha = \mathfrak{A}$. So if we enumerate F in ascending order as $\langle\alpha_\xi\rangle_{\xi<\lambda^+}$ and set $P_\xi = \mathfrak{C}_{\alpha_\xi}$ for each ξ, $P_{\lambda^+} = \mathfrak{A}$ then $\langle P_\xi\rangle_{\xi\le\lambda^+}$ satisfies the conditions of 518G, so $\mathrm{FN}(\mathfrak{A}) \le \kappa$, and the induction proceeds.

518J Lemma Let λ be an infinite cardinal and $\mathfrak{G}$ the regular open algebra of $\{0,1\}^\lambda$. Suppose that κ is the least cardinal of uncountable cofinality greater than or equal to $\mathrm{FN}(\mathfrak{G})$. Then $\kappa \le \mathfrak{c}^+$ and we have a family $\mathcal{V} \subseteq [\lambda]^{\le\mathfrak{c}}$, cofinal with $[\lambda]^{\le\mathfrak{c}}$, such that $\#(\{A \cap V : V \in \mathcal{V}\}) < \kappa$ for every countable set $A \subseteq \lambda$.

proof Actually it is more convenient to work with $\mathfrak{G} = \mathrm{RO}(\{0,1\}^{\lambda\times\mathbb{N}})$; of course this makes no difference.

(a) I will use the phrase 'cylinder set' to mean a subset of $X = \{0,1\}^{\lambda\times\mathbb{N}}$ of the form $\{x : x\restriction J = z\}$, where $J \subseteq \lambda \times \mathbb{N}$ is finite. For $I \subseteq \lambda$, let $\mathfrak{G}_I$ be the order-closed subalgebra of $\mathfrak{G}$ consisting of those regular open sets determined by coordinates in $I \times \mathbb{N}$. For $G \in \mathfrak{G}$, there is a smallest subset $J(G)$ of λ such that $G \in \mathfrak{G}_{J(G)}$ (use 4A2B(g-ii)). Recall that $J(G)$ is always countable (use 4A2E(b-i)), so that $\#(\mathfrak{G}_I) \le \mathfrak{c}$ whenever $\#(I) \le \mathfrak{c}$.

(b) The function $G \mapsto \mathfrak{G}_{J(G)}$ is a Freese-Nation function. **P** Suppose that $G_1 \subseteq G_2$ in $\mathfrak{G}$. Set $K = J(G_1)$ and $L = J(G_2)$, and let $\phi : X \to \{0,1\}^{L\times\mathbb{N}}$ be the canonical map, so that $\phi^{-1}[\phi[G_2]] = G_2$. Set $H = \phi^{-1}[\mathrm{int}\,\overline{\phi[G_1]}]$; because ϕ is continuous and open (4A2B(f-i)), $H = \mathrm{int}\,\overline{\phi^{-1}[\phi[G_1]]}$ (4A2B(f-ii)). In particular, H is a regular open set; at the same time, $H \supseteq G_1$ and $H \subseteq \mathrm{int}\,\overline{\phi^{-1}[\phi[G_2]]} = G_2$ and H is determined by coordinates in $L \times \mathbb{N}$, so $H \in \mathfrak{G}_L$. Next, $\phi[G_1] \subseteq \{0,1\}^{L\times\mathbb{N}}$ is determined by coordinates in $(K \cap L) \times \mathbb{N}$, so $\mathrm{int}\,\overline{\phi[G_1]}$ also is (4A2B(g-i)) and H is determined by coordinates in $K \times \mathbb{N}$. Thus $H \in \mathfrak{G}_{J(G_1)} \cap \mathfrak{G}_{J(G_2)}$, which is what we need. **Q**

Since $\#(\mathfrak{G}_{J(G)}) \le \mathfrak{c}$ for every G, $\mathrm{FN}(\mathfrak{G}) \le \mathfrak{c}^+$; as $\mathrm{cf}\,\mathfrak{c}^+$ is surely uncountable, $\kappa \le \mathfrak{c}^+$.

(c) Now let $f : \mathfrak{G} \to [\mathfrak{G}]^{<\kappa}$ be a Freese-Nation function. Let $\mathcal{V}$ be the family of those sets $V \in [\lambda]^{\le\mathfrak{c}}$ such that $f(G) \subseteq \mathfrak{G}_V$ for every $G \in \mathfrak{G}_V$; because $\#(f(G)) \le \mathfrak{c}$ for every G, and $\#(\mathfrak{G}_V) \le \mathfrak{c}$ whenever $V \in [\lambda]^{\le\mathfrak{c}}$, $\mathcal{V}$ is cofinal with $[\lambda]^{\le\mathfrak{c}}$.

(d) Fix a countable set $A \subseteq \lambda$ and $\zeta \in A$ for the moment. Let $\langle C_\xi\rangle_{\xi\in A}$ be a disjoint family of non-empty cylinder sets determined by coordinates in $\{\zeta\} \times \mathbb{N}$; for each $\xi \in A$, set $C'_\xi = \{x : x \in X,\ x(\xi, 0) = 1\}$. Set

$$G^* = \sup_{\xi\in A} C_\xi \cap C'_\xi \in \mathfrak{G}_A.$$

Next, for $V \in \mathcal{V}$, set

$$G_V = \sup_{\xi\in A\cap V} C_\xi \cap C'_\xi, \quad G'_V = \sup\{H : H \in \mathfrak{G}_V,\, H \subseteq G^*\}$$

so that $G_V \subseteq G^*$ and $G'_V \in \mathfrak{G}_V$. Now if $\zeta \in V \in \mathcal{V}$, $G_V = G'_V$. **P** Since $C_\xi \cap C'_\xi \in \mathfrak{G}_V$ for every $\xi \in V \cap A$, $G_V \in \mathfrak{G}_V$ and $G_V \subseteq G'_V$. **?** Suppose, if possible, that $G_V \neq G'_V$. Then $G'_V \setminus \overline{G}_V$ is a non-empty set belonging to $\mathfrak{G}_V$, so includes a non-empty cylinder set D determined by coordinates in $V \times \mathbb{N}$. Express D as $D' \cap D''$, where D' is determined by coordinates in $(V \cap A) \times \mathbb{N}$ and D'' by coordinates in $(V \setminus A) \times \mathbb{N}$. As $D' \cap D'' \subseteq G^* \in \mathfrak{G}_A$, $D' \subseteq G^*$, so $D' \cap C_\xi \subseteq C'_\xi$ for $\xi \in A$.

If $\xi \in A \setminus V$, $D \cap C'_\xi$ is determined by coordinates in a set not containing $\{(\xi, 0)\}$, but is included in C'_ξ, so must be empty. Thus

$$D \subseteq D' = \sup_{\xi\in A\cap V} D' \cap C_\xi \cap C'_\xi \subseteq G_V,$$

which is impossible. **X** Accordingly $G_V = G'_V$, as claimed. **Q**

Note next that if V, $V' \in \mathcal{V}$ and $V \cap A \neq V' \cap A$, then $G_V \neq G_{V'}$, because if $\xi \in A \cap (V \triangle V')$ then $C_\xi \cap C'_\xi \subseteq G_V \triangle G_{V'}$.

At this point, consider $f(G^*)$. For each $V \in \mathcal{V}$ such that $\zeta \in V$, there must be some $H_V \in f(G^*) \cap f(G_V)$ such that $G_V \subseteq H_V \subseteq G^*$. By the definition of $\mathcal{V}$, $H_V \in \mathfrak{G}_V$ so $H_V \subseteq G'_V = G_V$ and $H_V = G_V$. But this shows that

$$\#(\{V \cap A : \zeta \in V \in \mathcal{V}\}) \leq \#(\{G_V : \zeta \in V \in \mathcal{V}\}) \leq \#(f(G^*)) < \kappa.$$

(e) Now take any countable $A \subseteq \lambda$. By (d), we see that $\#(\{A \cap V : \zeta \in V \in \mathcal{V}\}) < \kappa$ for every $\zeta \in A$. But now

$$\{A \cap V : V \in \mathcal{V}\} \subseteq \{\emptyset\} \cup \bigcup_{\zeta\in A} \{A \cap V : \zeta \in V \in \mathcal{V}\}$$

has size less than κ, because $\operatorname{cf}\kappa > \omega$. This completes the proof.

518K Theorem (Fuchino Geschke Shelah & Soukup 01) Suppose that $\lambda > \mathfrak{c}$ is a cardinal of countable cofinality such that $\mathrm{CTP}(\lambda^+, \lambda)$ is true (definition: 5A6F). Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra of size at least λ. Then $\mathrm{FN}(\mathfrak{A}) \geq \omega_2$.

proof (a) By 518Cb and 518Cc, it is enough to show that $\mathrm{FN}(\mathfrak{G}) \geq \omega_2$, where $\mathfrak{G}$ is the regular open algebra of $\{0,1\}^\lambda$.

(b) ? Suppose, if possible, that $\mathrm{FN}(\mathfrak{G}) \leq \omega_1$. Let $\mathcal{V} \subseteq [\lambda]^{\leq\mathfrak{c}}$ be as in 518J, with $\kappa = \omega_1$. Note first that if $\mathcal{V}' \subseteq \mathcal{V}$ and $\#(\mathcal{V}') \leq \lambda$ then there is an $A \in [\lambda]^{\leq\omega}$ such that $A \not\subseteq V$ for every $V \in \mathcal{V}'$. **P** Let $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ be a sequence of cardinals less than λ with supremum λ. Express $\mathcal{V}'$ as $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ where $\#(\mathcal{V}_n) \leq \lambda_n$ for each n. For each $n \in \mathbb{N}$, $\#(\bigcup \mathcal{V}_n) < \lambda$, so we can find an $\alpha_n \in \lambda \setminus \bigcup \mathcal{V}_n$; now $A = \{\alpha_n : n \in \mathbb{N}\}$ is not included in any member of $\mathcal{V}'$. **Q**

Choose $\langle A_\xi\rangle_{\xi<\lambda^+}$ and $\langle V_\xi\rangle_{\xi<\lambda^+}$ inductively, as follows. Given $V_\eta \in \mathcal{V}$ for $\eta < \xi$, choose $A_\xi \in [\lambda]^{\leq\omega}$ such that $A_\xi \not\subseteq V_\eta$ for every $\eta < \xi$; now take $V_\xi \in \mathcal{V}$ such that $A_\xi \subseteq V_\xi$, and continue.

Because $\mathrm{CTP}(\lambda^+, \lambda)$ is true, there is an uncountable set $B \subseteq \lambda^+$ such that $C = \bigcup_{\xi\in B} A_\xi$ is countable (5A6F(b-ii)). If η, $\xi \in B$ and $\eta < \xi$, then $A_\xi = A_\xi \cap C \cap V_\xi \not\subseteq V_\eta$, so $C \cap V_\xi \neq C \cap V_\eta$. But this means that $\{C \cap V : V \in \mathcal{V}\}$ is uncountable, contrary to the choice of $\mathcal{V}$. **X**

Thus $\mathrm{FN}(\mathfrak{G}) \geq \omega_2$, and the proof is complete.

Remark Compare Fuchino & Soukup 97, Theorem 12, where it is shown that if the generalized continuum hypothesis and $\mathrm{CTP}(\omega_{\omega+1}, \omega_\omega)$ are both true the Freese-Nation number of $[\omega_\omega]^{\leq\omega}$ is greater than ω_1, and also Fuchino Geschke Shelah & Soukup 01, Theorem 4.2, where a different special axiom is used to find a ccc Dedekind complete Boolean algebra of size $\omega_{\omega+1}$ with Freese-Nation number greater than ω_1.

518L I turn now to the associated idea of 'tight filtration' (511Di). Before discussing conditions ensuring the existence of such filtrations, I give the application of the idea which is most important for this book.

Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $\mathfrak{B}$ a tightly ω_1-filtered Boolean algebra, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a surjective sequentially order-continuous Boolean homomorphism; suppose that $\mathfrak{B} \neq \{0\}$. Then there is a Boolean homomorphism $\theta : \mathfrak{B} \to \mathfrak{A}$ such that $\pi\theta b = b$ for every $b \in \mathfrak{B}$.

proof Let $\langle b_\xi\rangle_{\xi<\zeta}$ be a tight ω_1-filtration in $\mathfrak{B}$; for $\alpha \leq \zeta$, write $\mathfrak{C}_\alpha$ for the subalgebra of $\mathfrak{B}$ generated by $\{b_\xi : \xi < \alpha\}$. Define Boolean homomorphisms $\theta_\alpha : \mathfrak{C}_\alpha \to \mathfrak{A}$ inductively, as follows. Start with $\mathfrak{C}_0 = \{0, 1\}$, $\theta_0 0 = \emptyset$, $\theta_0 1 = 1$. Given

θ_α, let B, $B' \subseteq \mathfrak{C}_\alpha$ be countable sets such that B is a cofinal subset of $\{b : b \in \mathfrak{C}_\alpha,\, b \subseteq b_\alpha\}$ and B' is a cofinal subset of $\{b : b \in \mathfrak{C}_\alpha,\, b \subseteq 1 \setminus b_\alpha\}$. Choose any $a \in \mathfrak{A}$ such that $\pi a = b_\alpha$ and set

$$a_\alpha = (a \cup \sup_{b\in B} \theta_\alpha b) \setminus \sup_{b\in B'} \theta_\alpha b.$$

Because B and B' are both countable and $\mathfrak{A}$ is Dedekind σ-complete, a_α is defined in $\mathfrak{A}$. Because B and B' are cofinal with $\{b : b \in \mathfrak{C}_\alpha,\, b \subseteq b_\alpha\}$ and $\{b : b \in \mathfrak{C}_\alpha,\, b \subseteq 1 \setminus b_\alpha\}$ respectively, $\theta b \subseteq a_\alpha$ whenever $b \in \mathfrak{C}_\alpha$ and $b \subseteq b_\alpha$, while $\theta b \cap a_\alpha = \emptyset$ whenever $b \in \mathfrak{C}_\alpha$ and $b \subseteq 1 \setminus b_\alpha$. This means that we can define a Boolean homomorphism $\theta_{\alpha+1} : \mathfrak{C}_{\alpha+1} \to \mathfrak{A}$ by setting

$$\theta_{\alpha+1}((b \cap b_\alpha) \cup (c \setminus b_\alpha)) = (\theta_\alpha b \cap a_\alpha) \cup (\theta_\alpha c \setminus a_\alpha)$$

for all b, $c \in \mathfrak{C}_\alpha$ (312O).

This is the inductive step to a successor ordinal. For the inductive step to a non-zero limit ordinal $\alpha \le \zeta$, $\mathfrak{C}_\alpha = \bigcup_{\xi<\alpha} \mathfrak{C}_\xi$ and we can define θ_α by setting $\theta_\alpha a = \theta_\xi a$ whenever $\xi < \alpha$ and $a \in \mathfrak{C}_\xi$.

An easy induction (using the hypothesis that π is sequentially order-continuous) now shows that $c = \pi\theta_\alpha c$ whenever $\alpha \le \zeta$ and $c \in \mathfrak{C}_\alpha$, so that $\pi\theta_\zeta$ is the identity homomorphism on $\mathfrak{C}_\zeta = \mathfrak{B}$.

518M Theorem Let $\mathfrak{A}$ be a Boolean algebra and κ a regular infinite cardinal such that $\mathrm{FN}(\mathfrak{A}) \le \kappa$ and $\#(\mathfrak{A}) \le \kappa^+$. Then $\mathfrak{A}$ is tightly κ-filtered.

proof (a) Let $\langle a_\xi \rangle_{\xi<\kappa^+}$ run over $\mathfrak{A}$, and let $f : \mathfrak{A} \to [\mathfrak{A}]^{<\kappa}$ be a Freese-Nation function. For each $\alpha < \kappa^+$, let $\mathfrak{A}_\alpha$ be the smallest subalgebra of $\mathfrak{A}$ containing a_ξ for every $\xi < \alpha$ and such that $f(a) \subseteq \mathfrak{A}_\alpha$ for every $a \in \mathfrak{A}_\alpha$. Then $\langle \mathfrak{A}_\alpha \rangle_{\alpha<\kappa^+}$ is a non-decreasing family with union $\mathfrak{A}$, and $\#(\mathfrak{A}_\alpha) \le \kappa$ for every $\alpha < \kappa^+$.

(b)(i) If $\alpha < \kappa^+$, the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}$ is at most κ. **P** If $a \in \mathfrak{A}$, then whenever $b \in \mathfrak{A}_\alpha$ and $b \subseteq a$, there is a $c \in f(a) \cap f(b) \cap [b, a]$. Now $c \in f(a) \cap \mathfrak{A}_\alpha$. This shows that $f(a) \cap \mathfrak{A}_\alpha \cap [0, a]$ is cofinal with $\mathfrak{A}_\alpha \cap [0, a]$, so that $\mathrm{cf}(\mathfrak{A}_\alpha \cap [0, a]) < \kappa$. By 518Fa, this is what we need to know. **Q**

(ii) If $\alpha < \kappa^+$, $I \in [\mathfrak{A}]^{<\kappa}$ and $\mathfrak{B}$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\alpha \cup I$, then the Freese-Nation index of $\mathfrak{B}$ in $\mathfrak{A}$ is at most κ, by 518Fb.

(c) For each $\alpha < \kappa^+$ enumerate $\mathfrak{A}_{\alpha+1} \setminus \mathfrak{A}_\alpha$ as $\langle a_{\alpha\xi} \rangle_{\xi<\kappa_\alpha}$, where $\kappa_\alpha \le \kappa$. Well-order $\mathfrak{A}$ by setting $a \preccurlyeq a'$ if *either* there is some $\alpha < \kappa^+$ such that $a \in \mathfrak{A}_\alpha$ and $a' \notin \mathfrak{A}_\alpha$ *or* there are $\alpha < \kappa^+$ and $\xi \le \eta < \kappa_\alpha$ such that $a = a_{\alpha\xi}$ and $a' = a_{\alpha\eta}$. Let $\zeta \in \mathrm{On}$ be the order type of this well-ordering and $\langle b_\xi \rangle_{\xi<\zeta}$ the corresponding enumeration of $\mathfrak{A}$. For each $\beta \le \zeta$, let $\mathfrak{B}_\beta$ be the subalgebra of $\mathfrak{A}$ generated by $\{b_\xi : \xi < \beta\}$. Then the Freese-Nation index of $\mathfrak{B}_\beta$ in $\mathfrak{A}$ is at most κ. **P** If $\beta < \zeta$, there is a largest $\alpha < \kappa^+$ such that $\mathfrak{A}_\alpha \subseteq B_\beta$, and in this case $\mathfrak{A}_\alpha = \mathfrak{B}_\gamma$ for some $\gamma \le \beta$, while $\mathfrak{A}_{\alpha+1} = \mathfrak{B}_{\gamma'}$ for some $\gamma' > \beta$; moreover, $\#(\beta \setminus \gamma) < \kappa_\alpha \le \kappa$, because $\mathrm{otp}(\gamma' \setminus \gamma) = \kappa_\alpha$. But this means that $\mathfrak{B}$, which is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\alpha \cup \{b_\xi : \gamma \le \xi < \beta\}$, has Freese-Nation index at most κ, by (b-ii) above. **Q**

Thus $\langle b_\xi \rangle_{\xi<\zeta}$ is a tight κ-filtration of $\mathfrak{A}$.

518N Definition Let $\mathfrak{A}$ be a Boolean algebra and κ a cardinal. Then a **κ-Geschke system** for $\mathfrak{A}$ is a family $\mathbb{G}$ of subalgebras of $\mathfrak{A}$ such that

(α) every element of $\mathfrak{A}$ belongs to an element of $\mathbb{G}$ of size less than κ;

(β) for any $\mathbb{G}_0 \subseteq \mathbb{G}$, the subalgebra of $\mathfrak{A}$ generated by $\bigcup \mathbb{G}_0$ belongs to $\mathbb{G}$;

(γ) whenever $\mathfrak{B}_1$, $\mathfrak{B}_2 \in \mathbb{G}$, $a \in \mathfrak{B}_1$, $b \in \mathfrak{B}_2$ and $a \subseteq b$, then there is a $c \in \mathfrak{B}_1 \cap \mathfrak{B}_2$ such that $a \subseteq c \subseteq b$.

(Of course (γ) can be rephrased as '$\mathfrak{B}_1 \cap \mathfrak{B}_2 \cap [0, b]$ is cofinal with $\mathfrak{B}_1 \cap [0, b]$ whenever $\mathfrak{B}_1$, $\mathfrak{B}_2 \in \mathbb{G}$ and $b \in \mathfrak{B}_2$'.)

518O Lemma Let $\mathfrak{A}$ be a Boolean algebra, κ a cardinal and $\mathbb{G}$ a κ-Geschke system for $\mathfrak{A}$. Suppose that $\lambda \ge \kappa$ is a regular uncountable cardinal and that $f : [\mathfrak{A}]^{<\omega} \to [\mathfrak{A}]^{<\lambda}$ is a function. Then there is a $\mathfrak{B} \in \mathbb{G}$ such that $\#(\mathfrak{B}) < \lambda$ and $f(I) \subseteq \mathfrak{B}$ whenever $I \in [\mathfrak{B}]^{<\omega}$.

proof Enlarging f if necessary, we may suppose that $f(I)$ always includes the subalgebra of $\mathfrak{A}$ generated by I, and that $f(\{a\})$ includes a member of $\mathbb{G}$, of size less than κ and containing a, for every $a \in \mathfrak{A}$. If now we take $A_0 = \emptyset$ and $A_{n+1} = \bigcup\{f(I) : I \in [A_n]^{<\omega}\}$ for each $n \in \mathbb{N}$, $\mathfrak{B} = \bigcup_{n\in\mathbb{N}} A_n$ will be a subalgebra of $\mathfrak{A}$, of size less than λ, and a union of members of $\mathbb{G}$, so belongs to $\mathbb{G}$; while $f(I) \subseteq \mathfrak{B}$ for every $I \in [\mathfrak{B}]^{<\omega}$.

518P Lemma (Geschke 02) Let κ be a regular uncountable cardinal and $\mathfrak{A}$ a Boolean algebra. Then $\mathfrak{A}$ is tightly κ-filtered iff there is a κ-Geschke system for $\mathfrak{A}$.

proof (a) Suppose that $\mathfrak{A}$ is tightly κ-filtered.

(i) Let $\langle a_\xi \rangle_{\xi<\zeta}$ be a tight κ-filtration of $\mathfrak{A}$. For $I \subseteq \zeta$ let $\mathfrak{A}_I$ be the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi \in I\}$. For $\alpha < \zeta$, there must be subsets U_α, V_α of $\mathfrak{A}_\alpha$, of size less than κ, such that U_α is cofinal with $\mathfrak{A}_\alpha \cap [0, a_\alpha]$ and V_α is cofinal with $\mathfrak{A}_\alpha \cap [0, 1 \setminus a_\alpha]$. Let $K_\alpha \in [\alpha]^{<\kappa}$ be such that $U_\alpha \cup V_\alpha \subseteq \mathfrak{A}_{K_\alpha}$.

Write $\mathcal{M}$ for the family of those subsets M of ζ such that $K_\alpha \subseteq M$ for every $\alpha \in M$.

(ii) If M, $N \in \mathcal{M}$, $\gamma \le \zeta$, $a \in \mathfrak{A}_{M\cap\gamma}$, $b \in \mathfrak{A}_{N\cap\gamma}$ and $a \subseteq b$, then there is a $c \in \mathfrak{A}_{M\cap N\cap\gamma}$ such that $a \subseteq c \subseteq b$. **P** Induce on γ.

(α) If $\gamma = 0$ then

$$\mathfrak{A}_{M\cap\gamma} = \mathfrak{A}_{N\cap\gamma} = \mathfrak{A}_{M\cap N\cap\gamma} = \{0, 1\}$$

and the result is trivial.

(β) For the inductive step to $\gamma = \alpha + 1$, consider the following cases.

case 1 If $\alpha \notin M \cup N$ then $a \in \mathfrak{A}_{M\cap\alpha}$ and $b \in \mathfrak{A}_{N\cap\alpha}$, so the inductive hypothesis gives us a $c \in \mathfrak{A}_{M\cap N\cap\alpha}$ such that $a \subseteq c \subseteq b$.

case 2 If $\alpha \in N \setminus M$, then $a \in \mathfrak{A}_{M\cap\alpha}$ and b is of the form $(b' \cap a_\alpha) \cup (b'' \setminus a_\alpha)$ where b', $b'' \in \mathfrak{A}_{N\cap\alpha}$. Now $a \setminus b' \in \mathfrak{A}_\alpha$ and $a \setminus b' \subseteq 1 \setminus a_\alpha$, so there is a $v \in V_\alpha$ such that $a \setminus b' \subseteq v$. Since $K_\alpha \subseteq N \cap \alpha$, $v \in \mathfrak{A}_{N\cap\alpha}$. Similarly, there is a $u \in U_\alpha \subseteq \mathfrak{A}_{N\cap\alpha}$ such that $a \setminus b'' \subseteq u$. We have

$$a \subseteq (u \cap b') \cup (v \cap b'') \cup (b' \cap b'') \in \mathfrak{A}_{N\cap\alpha},$$

so the inductive hypothesis tells us that there is a $c \in \mathfrak{A}_{M\cap N\cap\alpha}$ such that

$$a \subseteq c \subseteq (u \cap b') \cup (v \cap b'') \cup (b' \cap b'') \subseteq b.$$

case 3 Similarly, if $\alpha \in M \setminus N$, then we express a as $(a' \cap a_\alpha) \cup (a'' \setminus a_\alpha)$ where a', $a'' \in \mathfrak{A}_{M\cap\alpha}$, and find $v \in V_\alpha \subseteq \mathfrak{A}_{M\cap\alpha}$, $u \in U_\alpha \subseteq \mathfrak{A}_{M\cap\alpha}$, $c \in \mathfrak{A}_{M\cap N\cap\alpha}$ such that

$$a' \setminus b \subseteq v, \quad a'' \setminus b \subseteq u,$$

$$a \subseteq c \subseteq (b \cup u) \cap (b \cup v) \subseteq b.$$

case 4 Finally, if $\alpha \in M \cap N$, express a as $(a' \cap a_\alpha) \cup (a'' \setminus a_\alpha)$ and b as $(b' \cap a_\alpha) \cup (b'' \setminus a_\alpha)$ where a', a'' belong to $\mathfrak{A}_{M\cap\alpha}$ and b', b'' belong to $\mathfrak{A}_{N\cap\alpha}$. As $a' \setminus b'$ belongs to $\mathfrak{A}_\alpha$ and is included in $1 \setminus a_\alpha$, there is a $v \in V_\alpha$ such that $a' \setminus b' \subseteq v$; as $K_\alpha \subseteq M \cap N \cap \alpha$, $v \in \mathfrak{A}_{M\cap N\cap\alpha}$. Now $a' \setminus v \in \mathfrak{A}_{M\cap\alpha}$, $b' \setminus v \in \mathfrak{A}_{N\cap\alpha}$ and $a' \setminus v \subseteq b' \setminus v$, so the inductive hypothesis tells us that there is a $c' \in \mathfrak{A}_{M\cap N\cap\alpha}$ such that $a' \setminus v \subseteq c' \subseteq b' \setminus v$; in which case $c' \cap a_\alpha \in \mathfrak{A}_{M\cap N\cap\gamma}$ and

$$a' \cap a_\alpha = a' \cap a_\alpha \setminus v \subseteq c' \cap a_\alpha \subseteq b' \cap a_\alpha \setminus v = b' \cap a_\alpha.$$

Similarly, there are $u \in \mathfrak{A}_{M\cap N\cap\alpha}$, $c'' \in \mathfrak{A}_{M\cap N\cap\gamma}$ such that

$$a'' \setminus b'' \subseteq v, \quad a'' \setminus u \subseteq c'' \subseteq b'' \setminus u, \quad a'' \setminus a_\alpha \subseteq c'' \setminus a_\alpha \subseteq b'' \setminus a_\alpha.$$

Putting these together, $c = (c' \cap a_\alpha) \cup (c'' \setminus a_\alpha)$ belongs to $\mathfrak{A}_{M\cap N\cap\gamma}$ and $a \subseteq c \subseteq b$.

Thus the induction proceeds to a successor ordinal γ.

(γ) If $\gamma > 0$ is a limit ordinal, $a \in \mathfrak{A}_{M\cap\gamma}$ and $b \in \mathfrak{A}_{N\cap\gamma}$ and $a \subseteq b$, then there is some $\alpha < \gamma$ such that $a \in \mathfrak{A}_{M\cap\alpha}$ and $b \in \mathfrak{A}_{N\cap\alpha}$, so the inductive hypothesis gives us a $c \in \mathfrak{A}_{M\cap N\cap\alpha} \subseteq \mathfrak{A}_{M\cap N\cap\gamma}$ with $a \subseteq c \subseteq b$, and again the induction proceeds. **Q**

(ii) Now set $\mathbb{G} = \{\mathfrak{A}_M : M \in \mathcal{M}\}$, and consider the conditions (α)-(γ) of 518N.

(α) For any $a \in \mathfrak{A}$, there is a finite set $I \subseteq \zeta$ such that $a \in \mathfrak{A}_I$. Let M be the smallest element of $\mathcal{M}$ including I; then (because κ is regular and uncountable) $\#(M) < \kappa$, so $\#(\mathfrak{A}_M) < \kappa$, while $a \in \mathfrak{A}_M \in \mathbb{G}$.

(β) If $\mathbb{G}_0 \subseteq \mathbb{G}$, consider $\mathcal{M}^* = \{M : M \in \mathcal{M}, \mathfrak{A}_M \in \mathbb{G}_0\}$. Then $M^* = \bigcup \mathcal{M}^*$ belongs to $\mathcal{M}$, and $\mathfrak{A}_{M^*} \in \mathbb{G}$ must be the subalgebra of $\mathfrak{A}$ generated by $\bigcup \mathbb{G}_0$.

(γ) Finally, condition (γ) is just (ii) above with $\gamma = \zeta$.

So $\mathbb{G}$ is a κ-Geschke system for $\mathfrak{A}$.

(b) Suppose that $\mathfrak{A}$ has a κ-Geschke system $\mathbb{G}$. I seek to use the ideas of the proof of 518M.

(i) Enumerate $\mathfrak{A}$ as $\langle a_\xi \rangle_{\xi<\lambda}$, and for each $\xi < \lambda$ let $\mathfrak{C}_\xi \in \mathbb{G}$ be such that $a_\xi \in \mathfrak{C}_\xi$ and $\#(\mathfrak{C}_\xi) < \kappa$. For $\alpha \le \lambda$ let $\mathfrak{A}_\alpha$ be the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{\xi<\alpha} \mathfrak{C}_\xi$, so that $\mathfrak{A}_\alpha \in \mathbb{G}$. Set $C_\alpha = \mathfrak{C}_\alpha \setminus \mathfrak{A}_\alpha$ for each $\alpha < \lambda$. An easy induction shows that, for any $\alpha \le \lambda$, $\mathfrak{A}_\alpha$ is the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{\xi<\alpha} C_\xi$.

(ii) If $\alpha \le \lambda$, the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}$ is at most κ. **P** For any $\xi < \lambda$ and $b \in \mathfrak{A}_\alpha \cap [0, a_\xi]$ there must be a $c \in \mathfrak{A}_\alpha \cap \mathfrak{C}_\xi$ such that $b \subseteq c \subseteq a_\xi$, because both $\mathfrak{A}_\alpha$ and $\mathfrak{C}_\xi$ belong to $\mathbb{G}$; so $\mathfrak{C}_\xi \cap \mathfrak{A}_\alpha \cap [0, a_\xi]$ is cofinal with $\mathfrak{A}_\alpha \cap [0, a_\xi]$ and $\mathrm{cf}(\mathfrak{A}_\alpha \cap [0, a_\xi]) \le \#(\mathfrak{C}_\xi) < \kappa$. Similarly, $\mathfrak{C}_\xi \cap \mathfrak{A}_\alpha \cap [a_\xi, 1]$ is coinitial with $\mathfrak{A}_\alpha \cap [a_\xi, 1]$ and $\mathrm{ci}(\mathfrak{A}_\alpha \cap [a_\xi, 1]) < \kappa$. **Q**

(iii) List $\bigcup_{\alpha<\lambda} C_\alpha$ as $\langle b_\xi \rangle_{\xi<\zeta}$, where ζ is an ordinal, in such a way that whenever $\xi \le \eta < \zeta$, $b_\xi \in C_\alpha$ and $b_\eta \in C_\beta$, then $\alpha \le \beta$. Then $\{b_\xi : \xi < \zeta\}$ generates $\mathfrak{A}$. If $\beta < \zeta$ and $\mathfrak{B}_\beta$ is the subalgebra of $\mathfrak{A}$ generated by $\{b_\xi : \xi < \beta\}$, let α be such that $b_\xi \in C_\alpha$; then $\mathfrak{A}_\alpha = \mathfrak{B}_\gamma$ for some $\gamma \le \beta$, $\#(\beta \setminus \gamma) < \#(C_\alpha) < \kappa$ and $\mathfrak{B}_\beta$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\alpha \cup \{b_\xi : \gamma \le \xi < \beta\}$, so has Freese-Nation index at most κ in $\mathfrak{A}$, by 518Fb. This shows that $\langle b_\xi \rangle_{\xi<\zeta}$ is a tight κ-filtration of $\mathfrak{A}$, and $\mathfrak{A}$ is tightly κ-filtered.

518Q Corollary Let κ be a regular uncountable cardinal and $\mathfrak{A}$ a tightly κ-filtered Boolean algebra.

(a) If $\mathfrak{C}$ is a retract of $\mathfrak{A}$ (that is, $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$ and there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\pi c = c$ for every $c \in \mathfrak{C}$), then $\mathfrak{C}$ is tightly κ-filtered.

(b) If $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$ which is (in itself) Dedekind complete, then $\mathfrak{C}$ is tightly κ-filtered.

proof (a) By 518P there is a κ-Geschke system $\mathbb{G}$ for $\mathfrak{A}$. Let $\mathbb{G}_1$ be the set of those $\mathfrak{B} \in \mathbb{G}$ such that $\pi[\mathfrak{B}] \subseteq \mathfrak{B}$. Then $\mathbb{G}_1$ is a κ-Geschke system. **P** Of course $\mathbb{G}_1$ satisfies (γ) of 518N, just because $\mathbb{G}_1 \subseteq \mathbb{G}$. As for (β), if $\mathbb{G}_0 \subseteq \mathbb{G}_1$ and $\mathfrak{B}^*$ is the subalgebra generated by $\bigcup \mathbb{G}_0$, then $\mathfrak{B}^* \in \mathbb{G}$ and $\pi[\mathfrak{B}^*]$ must be the subalgebra generated by $\bigcup_{\mathfrak{B}\in\mathbb{G}_0} \pi[\mathfrak{B}] \subseteq \mathfrak{B}^*$, so $\pi[\mathfrak{B}^*] \subseteq \mathfrak{B}^*$ and $\mathfrak{B}^* \in \mathbb{G}_1$. Finally, if $a \in \mathfrak{A}$, 518O (taking $\lambda = \kappa$ and $f(I) = \{a\} \cup \pi[I]$) tells us that there is a $\mathfrak{B} \in \mathbb{G}_1$ containing a and of size less than κ. **Q**

Observe next that because $\pi c = c$ for every $c \in \mathfrak{C}$, $\pi[\mathfrak{B}] = \mathfrak{B} \cap \mathfrak{C}$ for every $\mathfrak{B} \in \mathbb{G}_1$. Set $\mathbb{H} = \{\mathfrak{B} \cap \mathfrak{C} : \mathfrak{B} \in \mathbb{G}_1\}$. Then $\mathbb{H}$ is a κ-Geschke system for $\mathfrak{C}$. **P** (α) If $c \in \mathfrak{C}$ there is a $\mathfrak{B} \in \mathbb{G}_1$ such that $c \in \mathfrak{B}$ and $\#(\mathfrak{B}) < \kappa$; now $c \in \mathfrak{B} \cap \mathfrak{C} \in \mathbb{H}$ and $\#(\mathfrak{B} \cap \mathfrak{C}) < \kappa$. (β) If $\mathbb{H}' \subseteq \mathbb{H}$, set $\mathbb{G}_1' = \{\mathfrak{B} : \mathfrak{B} \in \mathbb{G}_1,\ \mathfrak{B} \cap \mathfrak{C} \in \mathbb{H}'\}$. Then the subalgebra $\mathfrak{B}^*$ generated by $\bigcup \mathbb{G}_1'$ belongs to $\mathbb{G}_1$, and $\pi[\mathfrak{B}^*] \in \mathbb{H}$ is the subalgebra generated by $\bigcup\{\pi[\mathfrak{B}] : \mathfrak{B} \in \mathbb{G}_1'\} = \bigcup \mathbb{H}$. (γ) If $b_1 \in \mathfrak{D}_1 \in \mathbb{H}$, $b_2 \in \mathfrak{D}_2 \in \mathbb{H}$ and $b_1 \subseteq b_2$, express $\mathfrak{D}_1$, $\mathfrak{D}_2$ as $\mathfrak{B}_1 \cap \mathfrak{C}$, $\mathfrak{B}_2 \cap \mathfrak{C}$ where $\mathfrak{B}_1$ and $\mathfrak{B}_2$ belong to $\mathbb{G}_1$. Then there is a $b \in \mathfrak{B}_1 \cap \mathfrak{B}_2$ such that $b_1 \subseteq b \subseteq b_2$; in which case $\pi b \in \mathfrak{D}_1 \cap \mathfrak{D}_2$ and

$$b_1 = \pi b_1 \subseteq \pi b \subseteq \pi b_2 = b_2.$$

Thus $\mathbb{H}$ satisfies (γ) of 518N and is a κ-Geschke system for $\mathfrak{C}$. By 518P in the other direction, $\mathfrak{C}$ is tightly κ-filtered. **Q**

(b) In this case, the identity map from $\mathfrak{C}$ to itself extends to a Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{C}$ (314K), so we can use (a).

518R Lemma (a) Let I be a set and $\mathfrak{G}$ the regular open algebra of $\{0,1\}^I$. For $J \subseteq I$ let $\mathfrak{G}_J$ be the order-closed subalgebra of $\mathfrak{G}$ consisting of regular open sets determined by coordinates in J. Suppose that J and K are disjoint subsets of I, and $\langle a_q \rangle_{q\in\mathbb{Q}}$, $\langle b_q \rangle_{q\in\mathbb{Q}}$ disjoint families in $\mathfrak{G}_J \setminus \{\emptyset\}$ and $\mathfrak{G}_K \setminus \{\emptyset\}$ respectively. For $t \in \mathbb{R}$ set $w_t = \sup_{p,q\in\mathbb{Q},p\le t\le q} a_q \cap b_p$, the supremum being taken in $\mathfrak{G}$; set $w = \sup_{p,q\in\mathbb{Q},p\le q} a_q \cap b_p$. If $w' \subseteq w$ belongs to the subalgebra of $\mathfrak{G}$ generated by $\mathfrak{G}_{I\setminus K} \cup \mathfrak{G}_{I\setminus J}$, then $\{t : w_t \subseteq w'\}$ is finite.

(b) If $I = \omega_3$ then $\mathfrak{G}$ is not tightly ω_1-filtered.

proof (a)(i) I had better explain why each $\mathfrak{G}_J$ is an order-closed subalgebra; the point is just that if $A \subseteq \{0,1\}^I$ is determined by coordinates in $J \subseteq I$ then so are its closure and interior (4A2B(g-i) again), so that the operations $\mathcal{H} \mapsto \mathrm{int}(\bigcap \mathcal{H})$, $\mathcal{H} \mapsto \mathrm{int}\overline{\bigcup \mathcal{H}}$ take subsets of $\mathfrak{G}_J$ to members of $\mathfrak{G}_J$.

(ii) w' must be expressible in the form $\sup_{i<n} u_i \cap v_i$ where $u_i \in \mathfrak{G}_{I\setminus K}$ and $v_i \in \mathfrak{G}_{I\setminus J}$ for each i. **?** Suppose, if possible, that there are $t_0 < t_1 < \ldots < t_n$ in $\mathbb{R}$ such that $w_{t_j} \subseteq \sup_{i<n} u_i \cap v_i$ for every j. Take rational numbers q_j and q_j', for $j \le n$, such that $q_0 \le t_0 \le q_0' < q_1 \le t_1 \le q_1' < \ldots < q_n \le t_n \le q_n'$. Set $e_{-1} = \{0,1\}^I$. Choose i_j, e_j, c_j, c_j' and c_j'' inductively, for $j \le n$, as follows. Given that $e_{j-1} \in \mathfrak{G}_{I\setminus(J\cup K)}$ is non-empty, where $j \le n$, then $a_{q_j'}$, b_{q_j} and e_{j-1} are non-empty sets determined by coordinates in J, K and $I \setminus (J \cup K)$ respectively, so have non-empty intersection; also $a_{q_j'} \cap b_{q_j} \subseteq w_{t_j} \subseteq \sup_{i<n} u_i \cap v_i$. There is therefore an $i_j < n$ such that $a_{q_j'} \cap b_{q_j} \cap e_{j-1} \cap u_{i_j} \cap v_{i_j}$ is non-empty, and includes a basic cylinder set c_j say. Now we can express c_j as $c_j' \cap c_j'' \cap e_j$ where c_j' is determined by coordinates in J, c_j'' by coordinates in K and e_j by coordinates in $I \setminus (J \cup K)$; note that $e_j \subseteq e_{j-1}$, and continue.

At the end of this process, there must be $j < k \le n$ such that $i_j = i_k = i$ say. Now $q'_j < q_k$, so

$$a_{q'_j} \cap b_{q_k} \cap u_i \cap v_i \subseteq a_{q'_j} \cap b_{q_k} \cap w = \emptyset.$$

(Recall that $\langle a_q\rangle_{q\in\mathbb{Q}}$ and $\langle b_q\rangle_{q\in\mathbb{Q}}$ are disjoint.) On the other hand, $c'_j \cap c''_j \cap e_j \subseteq u_i$, which is determined by coordinates in $I \setminus K$, so $c'_j \cap e_j \subseteq u_i$; similarly, $c''_k \cap e_k \subseteq v_i$; so

$$capc'_j \cap e_j \cap c''_k \cap e_k \subseteq a_{q'_j} \cap b_{q_k} \cap u_i \cap v_i = \emptyset.$$

But c'_j, c''_k and $e_j \cap e_k$ are all non-empty and determined by coordinates in J, K and $I \setminus (J \cup K)$ respectively, so this is impossible. **X**

Thus $\{t : w_t \subseteq w'\}$ has at most n members, and is finite.

(b) As in 518J, I will work with $I = \omega_3 \times \mathbb{N}$.

(i) Note that every member of $\mathfrak{G}$ belongs to $\mathfrak{G}_J$ for some countable J (4A2E(b-i) again), so we can choose for each $c \in \mathfrak{G}$ a countable $J(c) \subseteq \omega_3$ such that $c \in \mathfrak{G}_{J(c)\times\mathbb{N}}$ for some countable I. For each $\xi < \omega_3$, let $\langle a_{\xi q}\rangle_{q\in\mathbb{Q}}$ be a disjoint family of non-zero elements of $\mathfrak{G}_{\{\xi\}\times\mathbb{N}}$. For $t \in \mathbb{R}$, $\xi < \omega_3$ set $c'_{\xi t} = \sup_{q\in\mathbb{Q},q\le t} a_{\xi q}$, $c''_{\xi t} = \sup_{q\in\mathbb{Q},q\ge t} a_{\xi q}$. Let $T \subseteq \mathbb{R}$ be a set of size ω_1. For $D \subseteq \mathfrak{G}$ set $\tilde J(D) = \bigcup_{c\in D} J(c)$.

(ii) ? Suppose, if possible, that $\mathfrak{G}$ is tightly ω_1-filtered. Then it has an ω_1-Geschke system $\mathbb{B}$ say (518P). By 518O, with $\lambda = \omega_3$ and

$$f(\emptyset) = \{c''_{\xi t} : \xi < \omega_2,\, t \in T\} = C$$

say, there is a $\mathfrak{B}_1 \in \mathbb{B}$ such that $C \subseteq \mathfrak{B}_1$ and $\#(\mathfrak{B}_1) \le \omega_2$; take $\xi \in \omega_3 \setminus \tilde J(\mathfrak{B}_1)$, and let $\mathfrak{B}_2 \in \mathbb{B}$ be such that $\mathfrak{B}_2$ is countable and $a_{\xi p} \in \mathfrak{B}_2$ for every $p \in \mathbb{Q}$. Then $\tilde J(\mathfrak{B}_2)$ is countable, so there is an $\eta \in \omega_2 \setminus \tilde J(\mathfrak{B}_2)$.

Set $w = \sup_{p,q\in\mathbb{Q},p\le q} a_{\xi p} \cap a_{\eta q}$, and for $t \in T$ set $w_t = c'_{\xi t} \cap c''_{\eta t}$. Then w belongs to a countable $\mathfrak{B}_0 \in \mathbb{B}$, while the subalgebra $\mathfrak{B}^*$ of $\mathfrak{G}$ generated by $\mathfrak{B}_1 \cup \mathfrak{B}_2$ belongs to $\mathbb{B}$. But if we set $J = \{\xi\} \times \mathbb{N}$, $K = \{\eta\} \times \mathbb{N}$ then we see that $\mathfrak{B}_1 \subseteq \mathfrak{G}_{(\omega_3\times\mathbb{N})\setminus J}$ and $\mathfrak{B}_2 \subseteq \mathfrak{G}_{(\omega_3\times\mathbb{N})\setminus K}$. So (a) tells us that any member of $\mathfrak{B}^*$ included in w can include only finitely many w_t, while $w_t \in \mathfrak{B}^* \cap [0, w]$. Thus $\mathrm{cf}(\mathfrak{B}^* \cap [0, w]) \ge \omega_1$. On the other hand, by (γ) of 518N, the countable set $\mathfrak{B}_0 \cap \mathfrak{B}^* \cap [0, w]$ is cofinal with $\mathfrak{B}^* \cap [0, w]$. **X**

This contradiction proves the result.

518S Theorem (Geschke 02) If $\mathfrak{A}$ is a tightly ω_1-filtered Dedekind complete Boolean algebra then $\#(\mathfrak{A}) \le \omega_2$.

proof ? Otherwise, by 515I, $\mathfrak{A}$ has a subalgebra $\mathfrak{C}$ isomorphic to the regular open algebra of $\{0,1\}^{\omega_3}$. By 518Rb, $\mathfrak{C}$ is not tightly ω_1-filtered; by 518Qb, nor is $\mathfrak{A}$. **X**

518X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a principal ideal of $\mathfrak{A}$. Show that $\mathrm{FN}(\mathfrak{B}) \le \mathrm{FN}(\mathfrak{A})$.

(b) Show that $\mathrm{FN}(\alpha) = \#(\alpha)$ for every infinite ordinal α.

(c) Show that if P and Q are partially ordered sets, then $\mathrm{FN}(P\times Q)$ is at most the cardinal product $\mathrm{FN}(P)\cdot\mathrm{FN}(Q)$.

>(d) Show that $\mathrm{FN}(\mathcal{P}\mathbb{N}) \ge \omega_1$.

(e) Show that $\mathrm{FN}(\mathbb{Q}) = \omega$ and $\mathrm{FN}(\mathbb{R}) = \omega_1$.

(f) Show that $\mathrm{FN}^*(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}) = \mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

(g) Let P be a partially ordered set and Q a subset of P with Freese-Nation index κ in P. Show that if $\lambda \ge \max(\kappa, \mathrm{FN}(P))$ is a regular infinite cardinal then $\mathrm{FN}(Q) \le \kappa$.

(h) Let P be a partially ordered set and $\langle P_\xi\rangle_{\xi<\zeta}$ a non-decreasing family of subsets of P such that $P_\xi = \bigcup_{\eta<\xi} P_\eta$ for every non-zero limit ordinal $\xi \le \zeta$. Suppose that κ is a regular infinite cardinal such that the Freese-Nation index of P_ξ in $P_{\xi+1}$ is at most κ for every $\xi < \zeta$. Show that the Freese-Nation index of P_0 in P_ζ is at most κ.

(i) Let $\mathfrak{A}$ be a Boolean algebra, κ a regular infinite cardinal and $\langle a_\xi\rangle_{\xi<\zeta}$ a family in $\mathfrak{A}$. For each $\alpha \le \zeta$ let $\mathfrak{A}_\alpha$ be the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \alpha\}$. Suppose that $\mathfrak{A}_\zeta = \mathfrak{A}$ and that the Freese-Nation index of $\mathfrak{A}_\alpha$ in $\mathfrak{A}_{\alpha+1}$ is at most κ for every $\alpha < \zeta$. Show that $\langle a_\xi\rangle_{\xi<\zeta}$ is a tight κ-filtration of $\mathfrak{A}$.

(j) Suppose that $\mathfrak{c} = \omega_1$. Show that any Dedekind complete ccc Boolean algebra with cardinal at most $\mathfrak{c}^+ = \omega_2$ is tightly ω_1-filtered.

(k) Let $\mathfrak{A}$ be a Boolean algebra, $\kappa \le \lambda$ cardinals and $\mathbb{G}$ a κ-Geschke system for $\mathfrak{A}$. Show that $\mathbb{G}$ is a λ-Geschke system for $\mathfrak{A}$.

(l) Let $\kappa \le \mathfrak{c}$ be a regular uncountable cardinal. Show that if $\mathfrak{A}$ is a tightly κ-filtered Dedekind complete Boolean algebra then $\#(\mathfrak{A}) \le \kappa^+$.

518Y Further exercises (a) Show that if P is a finite partially ordered set then $\mathrm{FN}(P) \le 2 + \frac{1}{2}\#(P)$.

(b) Show that $\mathrm{FN}(\mathcal{P}I) > \#(I)$ for every infinite set I. (*Hint*: Fuchino Koppelberg & Shelah 96.)

(c)(i) Let $\mathfrak{A}$ be an infinite Boolean algebra. Show that $\mathrm{FN}(S(\mathfrak{A})) = \mathrm{FN}(\mathfrak{A})$. (ii) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. Show that $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(L^0(\mathfrak{A})) \le \mathrm{FN}(\mathfrak{A}^{\mathbb{N}})$.

518 Notes and comments 'Freese-Nation numbers' are a relatively recent topic, beginning with the investigation of partially ordered sets with Freese-Nation numbers at most ω (the 'Freese-Nation property') in Freese & Nation 78 and those with Freese-Nation numbers at most ω_1 (the 'weak Freese-Nation property') in Fuchino Koppelberg & Shelah 96. There are interesting puzzles concerning the Freese-Nation numbers of finite and countable partially ordered sets which I pass over here. Unlike most of the cardinals discussed in this chapter, Freese-Nation numbers refer to the internal, rather than cofinal, structure of a partially ordered set.

The Freese-Nation number $\mathrm{FN}(\mathcal{P}\mathbb{N})$ appears in many contexts besides the identifications of 518D. I will mention it again in 522U. I do not know whether it is consistent to suppose that its cofinality is countable.

Of the special axioms used in 518I, (α) has a more familiar aspect; for instance, it is a consequence of GCH, regardless of the value of $\tau(\mathfrak{A})$ (5A6Ab). (β) is believed not to be a consequence of GCH (see 555Yf), but is true in 'ordinary' models of set theory (5A6Db, 5A6Bc). In 518K I call on a form of Chang's transfer principle; this is *false* in ordinary models of set theory (5A6Fc), but is believed to be relatively consistent with ZFC + GCH (5A6Fa). Freese-Nation numbers are therefore a little exceptional among those appearing in measure theory, in that they are not fixed by the generalized continuum hypothesis.

Chapter 52

Cardinal functions of measure theory

From the point of view of this book, the most important cardinals are those associated with measures and measure algebras, especially, of course, Lebesgue measure and the usual measure ν_I of $\{0,1\}^I$. In this chapter I try to cover the principal known facts about these which are theorems of ZFC. I start with a review of the theory for general measure spaces in §521, including some material which returns to the classification scheme of Chapter 21, exploring relationships between (strict) localizability, magnitude and Maharam type. §522 examines Lebesgue measure and the surprising connexions found by Bartoszyński 84 and Raisonnier & Stern 85 between the cardinals associated with the Lebesgue null ideal and the corresponding ones based on the ideal of meager subsets of $\mathbb{R}$. §523 looks at the measures ν_I for uncountable sets I, giving formulae for the additivities and cofinalities of their null ideals, and bounds for their covering numbers, uniformities and shrinking numbers. Remarkably, these cardinals are enough to tell us most of what we want to know concerning the cardinal functions of general Radon measures and semi-finite measure algebras (§524). These three sections are heavily dependent on the Galois-Tukey connections and Tukey functions of §§512-513. Precalibers do not seem to fit into this scheme, and the relatively partial information I have is in §525. The second half of the chapter deals with special topics which can be approached with the methods so far developed. In §526 I return to the ideal of subsets of $\mathbb{N}$ with asymptotic density zero, seeking to locate it in the Tukey classification. Further σ-ideals which are of interest in measure theory are the 'skew products' of §527. In §528 I examine some interesting Boolean algebras, the 'amoeba algebras' first introduced by Martin & Solovay 70, giving the results of Truss 88 on the connexions between different amoeba algebras and localization posets. Finally, in §529, I look at a handful of other structures, concentrating on results involving cardinals already described.

521 Basic theory

In the first half of this section (down to 521L) I collect facts about the cardinal functions add, cf, non, cov, shr and shr$^+$ when applied to the null ideal $\mathcal{N}(\mu)$ of a measure μ, and also the π-weight of a measure. In particular I look at their relations with the constructions introduced earlier in this treatise: measure algebras and function spaces (521B), subspace measures (521F), direct sums (521G), inverse-measure-preserving functions and image measures (521H), products (521J), perfect measures (521K) and compact measures (521L). The list is long just because I have four volumes' worth of miscellaneous concepts to examine; nearly all the individual arguments are elementary.

In the second half of the section, I give a handful of easy results which may clarify some patterns from earlier volumes. In 521M-521P I look again at 'strict localizability' as considered in Chapter 21, importing the concept of 'magnitude' of a measure space from §332, hoping to throw light on the examples of §216. In 521E I consider the topological densities of measure algebras. In 521R-521S I explore possibilities for the 'countably separated' measure spaces of §§343-344, examining in particular their Maharam types. Finally, in 521T, I review some measures which arose in §464 while analyzing the L-space $\ell^\infty(I)^*$.

521A Proposition Let (X,Σ,μ) be a measure space.
(a) If $\mathcal{E}\subseteq\Sigma$ and $\#(\mathcal{E})<\operatorname{add}\mu$ then $\bigcup\mathcal{E}\in\Sigma$ and

$$\mu(\bigcup\mathcal{E})=\sup\{\mu(\bigcup\mathcal{E}_0):\mathcal{E}_0\subseteq\mathcal{E}\text{ is finite}\}.$$

(b) $\omega_1\le\operatorname{add}\mu\le\operatorname{add}\mathcal{N}(\mu)$.
(c) If μ is the measure defined by Carathéodory's method from an outer measure θ on X, then $\operatorname{add}\mu=\operatorname{add}\mathcal{N}(\mu)$.
(d) If μ is complete and locally determined, $\operatorname{add}\mu=\operatorname{add}\mathcal{N}(\mu)$.

proof (a) Induce on $\#(\mathcal{E})$. If $\mathcal{E}$ is finite, the result is trivial. For the inductive step to $\#(\mathcal{E})=\kappa\ge\omega$, enumerate $\mathcal{E}$ as $\langle E_\xi\rangle_{\xi<\kappa}$. For each $\xi<\kappa$, set $H_\xi=E_\xi\setminus\bigcup_{\eta<\xi}E_\eta$ for each $\xi<\kappa$. Then the inductive hypothesis tells us that $H_\xi\in\Sigma$ for every ξ. Set $E=\bigcup\mathcal{E}=\bigcup_{\xi<\kappa}H_\xi$; because $\langle H_\xi\rangle_{\xi<\kappa}$ is disjoint, and $\kappa<\operatorname{add}\mu$, $E\in\Sigma$ and

$$\mu E=\sum_{\xi<\kappa}\mu H_\xi=\sup_{I\subseteq\kappa\text{ is finite}}\mu(\bigcup_{\xi\in I}H_\xi)\le\sup_{\mathcal{E}_0\subseteq\mathcal{E}\text{ is finite}}\mu(\bigcup\mathcal{E})\le\mu E.$$

(b) By the definition of 'measure' (112A), μ is ω_1-additive. Suppose that $\mathcal{A}\subseteq\mathcal{N}(\mu)$ and $\#(\mathcal{A})<\operatorname{add}\mu$. For each $A\in\mathcal{A}$, choose a measurable negligible $E_A\supseteq A$. Then (a) tells us that $E=\bigcup_{A\in\mathcal{A}}E_A$ has measure zero, so $\bigcup\mathcal{A}\subseteq E$ is negligible. As $\mathcal{A}$ is arbitrary, $\operatorname{add}\mathcal{N}(\mu)\ge\operatorname{add}\mu$.

(c) Now suppose that μ is defined by Carathéodory's method from θ. Let $\langle E_i\rangle_{i\in I}$ be a disjoint family in Σ, where $\#(I) < \operatorname{add}\mathcal{N}(\mu)$, with union E.

Let $A \subseteq X$ be any set. Then $\theta(A\cap E) = \sum_{i\in I}\theta(A\cap E_i)$. **P** Of course

$$\theta(A\cap E) \ge \sup_{J\subseteq I \text{ is finite}} \theta(A\cap\bigcup_{i\in J} E_i) = \sup_{J\subseteq I \text{ is finite}} \sum_{i\in J}\theta(A\cap E_i)$$

(induce on $\#(J)$, using the fact that $\theta B = \theta(B\cap E_i) + \theta(B\setminus E_i)$ for every $B\subseteq X$ and $i\in J$)

$$= \sum_{i\in I}\theta(A\cap E_i).$$

If $\sum_{i\in I}\theta(A\cap E_i)$ is infinite, we can stop. Otherwise, recalling that $\mathcal{N}(\mu) = \theta^{-1}[\{0\}]$, $J = \{i : A\cap E_i \notin \mathcal{N}(\mu)\}$ is countable, and $\bigcup_{i\in I\setminus J} A\cap E_i$ is negligible, because $\#(I) < \operatorname{add}\mathcal{N}(\mu)$; so

$$\theta(A\cap E) = \theta(A\cap \textstyle\bigcup_{i\in J} E_i) \le \sum_{i\in J}\theta(A\cap E_i) = \sum_{i\in I}\theta(A\cap E_i)$$

and we have equality. **Q**

It follows that $\theta(A\cap E) + \theta(A\setminus E) \le \theta A$. **P** For any finite $J\subseteq I$,

$$\theta(A\setminus E) + \sum_{i\in J}\theta(A\cap E_i) = \theta(A\setminus E) + \theta(A\cap\bigcup_{i\in J}E_i)$$

$$\le \theta(A\setminus\bigcup_{i\in J}E_i) + \theta(A\cap\bigcup_{i\in J}E_i) = \theta A.$$

Taking the supremum over J, we have the result. **Q**

As A is arbitrary, $E\in\Sigma$; and setting $A = E$, we see that $\mu E = \sum_{i\in I}\mu E_i$. As $\langle E_i\rangle_{i\in I}$ is arbitrary, $\operatorname{add}\mu \ge \operatorname{add}\mathcal{N}(\mu)$ and the two additivities are equal.

(d) Now this follows immediately from (c), by 213C.

521B Proposition Let (X,Σ,μ) be a measure space and $(\mathfrak{A},\bar\mu)$ its measure algebra.

(a) If $\mathcal{E}\subseteq\Sigma$ and $\#(\mathcal{E}) < \operatorname{add}\mu$, then $(\bigcup\mathcal{E})^\bullet = \sup_{E\in\mathcal{E}} E^\bullet$ and $(X\cap\bigcap\mathcal{E})^\bullet = \inf_{E\in\mathcal{E}} E^\bullet$ in $\mathfrak{A}$.

(b) Suppose that $A\subseteq[-\infty,\infty]^X$ is a non-empty family of Σ-measurable functions with $\#(A) < \operatorname{add}\mu$, and that $g(x) = \sup_{f\in A} f(x)$ in $[-\infty,\infty]$ for every $f\in A$. Then g is Σ-measurable.

(c) Write $\mathcal{L}^0$ for the family of μ-virtually measurable real-valued functions defined almost everywhere in X, and L^0 for the corresponding space of equivalence classes, as in §241. Suppose that $A\subseteq\mathcal{L}^0$ is such that $0 < \#(A) < \operatorname{add}\mu$ and $\{f^\bullet : f\in A\}$ is bounded above in L^0. Set $g(x) = \sup_{f\in A} f(x)$ whenever this is defined in $\mathbb{R}$; then $g\in\mathcal{L}^0$ and $g^\bullet = \sup_{f\in A} f^\bullet$ in L^0.

(d)(i) If, in (b), A consists of non-negative integrable functions and is upwards-directed, then $\int g\,d\mu = \sup_{f\in A}\int f d\mu$.

(ii) If, in (b), $f_1\wedge f_2 = 0$ a.e. for all distinct f_1, $f_2\in A$, then $\int g\,d\mu = \sum_{f\in A}\int f d\mu$.

proof (a) As in 521Aa, $\bigcup\mathcal{E}\in\Sigma$, and of course $(\bigcup\mathcal{E})^\bullet$ is an upper bound for $\{E^\bullet : E\in\mathcal{E}\}$. If $F\in\Sigma$ and $F^\bullet$ is an upper bound for $\{E^\bullet : E\in\mathcal{E}\}$, then, applying 521Aa to $\{E\setminus F : E\in\mathcal{E}\}$, we see that $\bigcup\mathcal{E}\setminus F$ is negligible, so $(\bigcup\mathcal{E})^\bullet \subseteq F^\bullet$. Thus $(\bigcup\mathcal{E})^\bullet$ is the least upper bound of $\{E^\bullet : E\in\mathcal{E}\}$.

Applying this to $\{X\setminus E : E\in\mathcal{E}\}$ we see that $(X\cap\bigcap\mathcal{E})^\bullet = \inf_{E\in\mathcal{E}} E^\bullet$.

(b) For any $\alpha\in\mathbb{R}$,

$$\{x : g(x) > \alpha\} = \textstyle\bigcup_{f\in A}\{x : f(x) > \alpha\}\in\Sigma$$

by 521Aa.

(c) Take any $h\in\mathcal{L}^0$ such that $f^\bullet \le h^\bullet$ for every $f\in A$. For each $f\in A$, let E_f be a conegligible measurable subset of $\{x : x\in\operatorname{dom}f\cap\operatorname{dom}h,\ f(x)\le h(x)\}$ such that $f\restriction E_f$ is measurable. Set $E = \bigcap_{f\in A}E_f$; then E is measurable and g is defined everywhere in E and $g\restriction E$ is measurable (as in (b)). Also E is conegligible, so $g\in\mathcal{L}^0$, and of course $f^\bullet\le g^\bullet$ for every $f\in A$, while $g^\bullet\le h^\bullet$. But this argument works for every h such that $h^\bullet$ is an upper bound for $\{f^\bullet : f\in A\}$, so $g^\bullet$ must be actually the supremum of $\{f^\bullet : f\in A\}$.

(d)(i) If $\sup_{f\in A}\int f d\mu$ is infinite, this is trivial. Otherwise, $\{f^\bullet : f \in A\}$ is bounded above in L^1 and therefore in L^0. By (c), $g^\bullet$ is its supremum in L^0, therefore in L^1; so

$$\int g = \int g^\bullet = \sup_{f\in A}\int f^\bullet = \sup_{f\in A}\int f,$$

as in 365Df.

(ii) Apply (i) to $A^* = \{\sup I : I \in [A]^{<\omega}\}$.

521C Just because null ideals are σ-ideals of sets, we can read off some of the elementary properties of their cardinal functions from 511J. But the presence of a measure gives us a new way to use shrinking numbers, which will be useful later.

Proposition Let (X, Σ, μ) be a measure space, and $A \subseteq X$.

(a) If $\gamma < \mu^*A$ there is a $B \subseteq A$ such that $\#(B) < \mathrm{shr}^+ \mathcal{N}(\mu)$ and $\mu^*B > \gamma$.

(b) There is a $B \subseteq A$ such that $\#(B) \le \max(\omega, \mathrm{shr}\,\mathcal{N}(\mu))$ and $\mu^*B = \mu^*A$.

proof (a) Set $\kappa = \mathrm{shr}^+ \mathcal{N}(\mu)$. Let $\mathcal{E}$ be the family of those measurable subsets of X such that there is a $B \in [A\cap E]^{<\kappa}$ with $\mu^*B = \mu E$. Then $\mathcal{E}$ is closed under finite unions (132Ed). **?** If $\mu^*B \le \gamma$ for every $B \in [A]^{<\kappa}$, then $\mu E \le \gamma$ for every $E \in \mathcal{E}$. By 215Ab, there is a non-decreasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $E \setminus \bigcup_{n\in\mathbb{N}} E_n$ is negligible for every $E \in \mathcal{E}$. Now $\mu(\bigcup_{n\in\mathbb{N}} E_n) \le \gamma < \mu^*A$ and $A' = A \setminus \bigcup_{n\in\mathbb{N}} E_n$ is not negligible. Let $B \in [A']^{<\kappa}$ be a non-negligible set. Then $\mu^*B \le \gamma$ is finite, so B has a measurable envelope F (132Ee), which belongs to $\mathcal{E}$; but $F \setminus \bigcup_{n\in\mathbb{N}} E_n \supseteq B$ is not negligible. **X** So we have a $B \in [A]^{<\kappa}$ with $\mu^*B > \gamma$, as required.

(b) If $\mu^*A = 0$ take $B = \emptyset$. Otherwise, let $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ be a sequence in $[0, \mu^*A[$ with supremum μ^*A. For each $n \in \mathbb{N}$, (a) tells us that there is a set $B_n \subseteq A$ such that $\#(B_n) \le \mathrm{shr}(\mu)$ and $\mu^*B_n > \gamma_n$; set $B = \bigcup_{n\in\mathbb{N}} B_n$.

521D Proposition Let (X, Σ, μ) be a measure space and $(\mathfrak{A}, \bar\mu)$ its measure algebra.

(a) $\pi(\mathfrak{A}) \le \pi(\mu) \le \max(\pi(\mathfrak{A}), \mathrm{cf}\,\mathcal{N}(\mu))$ (definitions: 511Dc, 511Gb).

(b) If $\mu X > 0$, then $\mathrm{non}\,\mathcal{N}(\mu) \le \pi(\mu)$.

(c) If (X, Σ, μ) has locally determined negligible sets (definition: 213I), then $\mathrm{shr}\,\mathcal{N}(\mu) \le \pi(\mu)$.

(d) Suppose that there is a topology $\mathfrak{T}$ on X such that $(X, \mathfrak{T}, \Sigma, \mu)$ is a quasi-Radon measure space. Then, writing $\mathfrak{A}^+$ for $\mathfrak{A} \setminus \{0\}$, the partially ordered sets $(\Sigma \setminus \mathcal{N}(\mu), \supseteq)$ and $(\mathfrak{A}^+, \supseteq)$ are Tukey equivalent and $\pi(\mu) = \pi(\mathfrak{A})$.

proof Let $\mathcal{H} \subseteq \Sigma \setminus \mathcal{N}(\mu)$ be a coinitial set of size $\pi(\mu)$.

(a)(i) If $a \in \mathfrak{A}$ is non-zero, there is an $E \in \Sigma$ such that $E^\bullet = a$, and now E is not negligible, so there is an $H \in \mathcal{H}$ such that $H \subseteq E$ and $0 \ne H^\bullet \subseteq a$. Thus $\{H^\bullet : H \in \mathcal{H}\}$ is coinitial with $\mathfrak{A}^+$ and witnesses that $\pi(\mathfrak{A}) \le \#(\mathcal{H}) = \pi(\mu)$.

(ii) Let $B \subseteq \mathfrak{A}^+$ be a coinitial set of size $\pi(\mathfrak{A})$, and $\mathcal{E}$ a cofinal subset of $\mathcal{N}(\mu)$ of size $\mathrm{cf}\,\mathcal{N}(\mu)$. For $b \in B$, let $F_b \in \Sigma$ be such that $F_b^\bullet = b$, and consider $\mathcal{G} = \{F_b \setminus E : b \in B,\ E \in \mathcal{E}\}$. Then $\mathcal{G} \subseteq \Sigma \setminus \mathcal{N}(\mu)$ is coinitial with $\Sigma \setminus \mathcal{N}(\mu)$. **P** If $\mu F > 0$, there is a $b \in B$ such that $b \subseteq F^\bullet$. In this case, $F_b \setminus F$ is negligible, so there is an $E \in \mathcal{E}$ such that $F_b \setminus F \subseteq E$ and $F \supseteq F_b \setminus E \in \mathcal{G}$. **Q**

It follows that $\pi(\mu) \le \#(\mathcal{G}) \le \#(B \times \mathcal{E})$ is at most the cardinal product $\pi(\mathfrak{A}) \cdot \mathrm{cf}\,\mathcal{N}(\mu) \le \max(\omega, \pi(\mathfrak{A}), \mathrm{cf}\,\mathcal{N}(\mu))$. But if $\mathrm{cf}\,\mathcal{N}(\mu)$ is finite it is 1, so in fact $\pi(\mu) \le \pi(\mathfrak{A}) \cdot \mathrm{cf}\,\mathcal{N}(\mu) = \max(\pi(\mathfrak{A}), \mathrm{cf}\,\mathcal{N}(\mu))$.

(b) For each $H \in \mathcal{H}$ choose $x_H \in \mathcal{H}$. Then $A = \{x_H : H \in \mathcal{H}\}$ must meet every non-negligible measurable set, so (as $\mu X > 0$) cannot itself be negligible. Thus

$$\mathrm{non}\,\mathcal{N}(\mu) \le \#(A) \le \#(\mathcal{H}) = \pi(\mu).$$

(c) Suppose that $B \subseteq X$ is non-negligible. Because (X, Σ, μ) has locally determined negligible sets there is an $E \in \Sigma$ such that $\mu E > 0$ and $B \cap E$ is not negligible, and now $B \cap E$ has a measurable envelope F say (132Ee again). Set $\mathcal{H}' = \{H : H \in \mathcal{H},\ B \cap H \ne \emptyset\}$ and for $H \in \mathcal{H}'$ choose $x_H \in B \cap H$; set $A = \{x_H : H \in \mathcal{H}'\}$, so that $A \subseteq B$ and $\#(A) \le \pi(\mu)$. **?** If A is negligible, then $F \setminus A$ includes a non-negligible measurable set so includes a member H of $\mathcal{H}$. As $\mu H > 0$ and F is a measurable envelope of B, H meets B and belongs to $\mathcal{H}'$, and $x_H \in A \cap H$. **X** Thus A is not negligible. As B is arbitrary, $\mathrm{shr}\,\mathcal{N}(\mu) \le \pi(\mu)$.

(d) For $E \in \Sigma \setminus \mathcal{N}(\mu)$ let F_E be a closed non-negligible subset of E and set $\phi(E) = F_E^\bullet \in \mathfrak{A}^+$; for $a \in \mathfrak{A}^+$, let $\psi(a)$ be a self-supporting measurable set such that $\psi(a)^\bullet = a$ (414F). Then if $\phi(E) \supseteq a$, $\psi(a) \setminus F_E$ is negligible so $E \supseteq F_E \supseteq \psi(a)$. Thus (ϕ, ψ) is a Galois-Tukey connection and $(\Sigma \setminus \mathcal{N}(\mu), \supseteq, \Sigma \setminus \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (\mathfrak{A}^+, \supseteq, \mathfrak{A}^+)$.

Moreover, if $\psi(a) \supseteq E$, then $a \supseteq \phi(E)$, so (ψ, ϕ) also is a Galois-Tukey connection and $(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) \preccurlyeq_{\mathrm{GT}} (\Sigma \setminus \mathcal{N}(\mu), \supseteq, \Sigma \setminus \mathcal{N}(\mu))$.

Thus $(\Sigma \setminus \mathcal{N}(\mu), \supseteq, \Sigma \setminus \mathcal{N}(\mu)) \equiv_{\mathrm{GT}} (\mathfrak{A}^+, \supseteq, \mathfrak{A}^+)$, that is, $(\Sigma \setminus \mathcal{N}(\mu), \supseteq) \equiv_{\mathrm{T}} (\mathfrak{A}^+, \supseteq)$. By 513E(e-i), inverted,

$$\pi(\mu) = \mathrm{ci}(\Sigma \setminus \mathcal{N}(\mu)) = \mathrm{ci}(\mathfrak{A}^+) = \pi(\mathfrak{A}).$$

521E It will be useful later in the chapter to be able to calculate the topological density of measure-algebra topologies.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra.

(a) Give $\mathfrak{A}$ its measure-algebra topology (323A).

(i) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, it is topologically dense iff it τ-generates $\mathfrak{A}$, that is, $\mathfrak{A}$ is the order-closed subalgebra of itself generated by $\mathfrak{B}$.

(ii) If $\mathfrak{A}$ is finite, then its topological density is $\#(\mathfrak{A})$; if $\mathfrak{A}$ is infinite, its topological density is equal to its Maharam type $\tau(\mathfrak{A})$.

(b) Let $\mathfrak{A}^f$ be the set of elements of $\mathfrak{A}$ with finite measure, with its strong measure-algebra topology (323Ad). Then the topological density of $\mathfrak{A}^f$ is $\#(\mathfrak{A}^f) = \#(\mathfrak{A})$ if $\mathfrak{A}$ is finite, and $\max(c(\mathfrak{A}), \tau(\mathfrak{A}))$ if $\mathfrak{A}$ is infinite.

proof (a)(i)($\boldsymbol{\alpha}$) Suppose that $\mathfrak{B}$ is topologically dense. Let $\mathfrak{C}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$. If $a \in \mathfrak{A}^f$ and $c \in \mathfrak{A}$, there is a $b \in \mathfrak{C}$ such that $b \cap a = c \cap a$. **P** For each $n \in \mathbb{N}$, there is an $a_n \in \mathfrak{B}$ such that $\bar{\mu}(a \cap (a_n \triangle c)) \le 2^{-n}$. Set $b = \inf_{n\in\mathbb{N}} \sup_{m\ge n} a_m \in \mathfrak{C}$; then $b \cap a = c \cap a$ (apply 323F to $\langle a \cap a_n\rangle_{n\in\mathbb{N}}$). **Q**

It follows that $\mathfrak{A}^f \subseteq \mathfrak{C}$. **P** If $c \in \mathfrak{A}^f$, then whenever $c \subseteq a \in \mathfrak{A}^f$ there is a $b_a \in \mathfrak{C}$ such that $b_a \cap a = c$. Now (because $\bar{\mu}$ is semi-finite) $c = \inf\{b_a : c \subseteq a \in \mathfrak{A}^f\} \in \mathfrak{C}$. **Q**

Finally, again because $\bar{\mu}$ is semi-finite,

$$c = \sup\{a : a \in \mathfrak{A}^f,\, a \subseteq c\} \in \mathfrak{C}$$

for every $c \in \mathfrak{A}$, and $\mathfrak{A} = \mathfrak{C}$. Thus $\mathfrak{B}$ τ-generates $\mathfrak{A}$.

($\boldsymbol{\beta}$) Suppose that $\mathfrak{B}$ τ-generates $\mathfrak{A}$. Then the topological closure of $\mathfrak{B}$ is order-closed (323D(c-i)) and a subalgebra (323B), so must be $\mathfrak{A}$, and $\mathfrak{B}$ is topologically dense.

(ii)($\boldsymbol{\alpha}$) If $\mathfrak{A}$ is finite, this is trivial, just because the measure-algebra topology is Hausdorff (323Ga). So let us henceforth suppose that $\mathfrak{A}$ is infinite, so that both $\tau(\mathfrak{A})$ and the topological density $d_{\mathfrak{T}}(\mathfrak{A})$ of $\mathfrak{A}$ are infinite.

($\boldsymbol{\beta}$) Let $A \subseteq \mathfrak{A}$ be a set with cardinal $\tau(\mathfrak{A})$ which τ-generates $\mathfrak{A}$, and let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by A. Then $\#(\mathfrak{B}) = \#(A) = \tau(\mathfrak{A})$ (331Gc), and $\mathfrak{B}$ is topologically dense in $\mathfrak{A}$, by (i); so $d_{\mathfrak{T}}(\mathfrak{A}) \le \tau(\mathfrak{A})$.

($\boldsymbol{\gamma}$) Let $A \subseteq \mathfrak{A}$ be a topologically dense set with cardinal $d_{\mathfrak{T}}(\mathfrak{A})$, and $\mathfrak{B}$ the subalgebra of $\mathfrak{A}$ generated by A. Then $\mathfrak{B}$ is topologically dense, so it τ-generates $\mathfrak{A}$, and

$$\tau(\mathfrak{A}) \le \#(\mathfrak{B}) = \#(A) = d_{\mathfrak{T}}(\mathfrak{A});$$

with (β), this means that we have equality, as claimed.

(b)(i) The case of finite $\mathfrak{A}$ is again trivial; suppose that $\mathfrak{A}$ is infinite. Let $\langle a_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ consisting of non-zero elements of finite measure.

(ii) The topological density $d_{\mathrm{top}}(\mathfrak{A}^f)$ is at most $\max(c(\mathfrak{A}), \tau(\mathfrak{A}))$. **P** For each i, the topological density of $\mathfrak{A}_{a_i}$, with its measure-algebra topology, is at most $\max(\omega, \tau(\mathfrak{A}_{a_i})) \le \tau(\mathfrak{A})$ ((a) above and 514Ed); let $B_i \subseteq \mathfrak{A}_{a_i}$ be a dense subset of this size or less. Set $B = \bigcup_{i\in I} B_i$, $D = \{\sup B' : B' \in [B]^{<\omega}\}$. Then the metric closure $\overline{D}$ of D in $\mathfrak{A}^f$ is closed under $\cup$ and includes $\mathfrak{A}_{a_i}$ for every i. If now $a \in \mathfrak{A}^f$, $a = \sup_{i\in I} a \cap a_i \in \overline{D}$. So

$$d_{\mathrm{top}}(\mathfrak{A}^f) \le \#(D) \le \max(\omega, \#(I), \tau(\mathfrak{A})) \le \max(c(\mathfrak{A}), \tau(\mathfrak{A})). \text{ **Q**}$$

(iii) $c(\mathfrak{A}) \le d_{\mathrm{top}}(\mathfrak{A}^f)$. **P** Let $\langle b_j\rangle_{j\in J}$ be any disjoint family in $\mathfrak{A}^+$. For each j, let $b'_j \subseteq b_j$ be a non-zero element of non-zero finite measure. Set $G_j = \{a : a \in \mathfrak{A}^f,\, \bar{\mu}(a \triangle b'_j) < \bar{\mu}b'_j\}$ for $j \in J$. Then $\langle G_j\rangle_{j\in J}$ is a disjoint family of non-empty open sets in $\mathfrak{A}^f$, so $\#(J) \le d_{\mathrm{top}}(\mathfrak{A}^f)$ (5A4Ba). As $\langle b_j\rangle_{j\in J}$ is arbitrary, $c(\mathfrak{A}) \le d_{\mathrm{top}}(\mathfrak{A}^f)$. **Q**

(iv) $\tau(\mathfrak{A}) \le d_{\mathrm{top}}(\mathfrak{A}^f)$. **P** Let $A \subseteq \mathfrak{A}^f$ be a dense set of size $d_{\mathrm{top}}(\mathfrak{A}^f)$. Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $B = A \cup \{a_i : i \in I\}$. For any $i \in I$, set $A_i = \{a \cap a_i : a \in A\}$. Now A_i is topologically dense in $\mathfrak{A}_{a_i}$ (use 3A3Eb), so the order-closed subalgebra of $\mathfrak{A}_{a_i}$ it generates is the whole of $\mathfrak{A}_{a_i}$ (323H); by 314H, $\mathfrak{A}_{a_i} = \{b \cap a_i : b \in \mathfrak{B}\}$. As $a_i \in A \subseteq \mathfrak{B}$, $\mathfrak{B}$ includes $\mathfrak{A}_{a_i}$. As $\sup_{i\in I} a_i = 1$, $\mathfrak{B} = \mathfrak{A}$. Thus

$$\tau(\mathfrak{A}) \le \#(B) \le \max(\omega, \#(I), d_{\mathrm{top}}(\mathfrak{A}^f)) = d_{\mathrm{top}}(\mathfrak{A}^f)$$

(using (iii) for the last equality). **Q**

(v) Putting these together, we have the result.

521F Proposition Let (X, Σ, μ) be a measure space, A a subset of X and μ_A the subspace measure on A.
(a) $\mathcal{N}(\mu_A) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$, so $\operatorname{add}\mathcal{N}(\mu_A) \ge \operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{cf}\mathcal{N}(\mu_A) \le \operatorname{cf}\mathcal{N}(\mu)$.
(b) $(A, \in, \mathcal{N}(\mu_A)) \preccurlyeq_{\mathrm{GT}} (X, \in, \mathcal{N}(\mu))$, so $\operatorname{non}\mathcal{N}(\mu_A) \ge \operatorname{non}\mathcal{N}(\mu)$ and $\operatorname{cov}\mathcal{N}(\mu_A) \le \operatorname{cov}\mathcal{N}(\mu)$.
(c) $\operatorname{add}\mu_A \ge \operatorname{add}\mu$.
(d) $\operatorname{shr}\mathcal{N}(\mu_A) \le \operatorname{shr}\mathcal{N}(\mu)$ and $\operatorname{shr}^+\mathcal{N}(\mu_A) \le \operatorname{shr}^+\mathcal{N}(\mu)$.
(e) If either $A \in \Sigma$ or (X, Σ, μ) has locally determined negligible sets, $\pi(\mu_A) \le \pi(\mu)$.
(f) If μ_A is semi-finite, then $\tau(\mu_A) \le \tau(\mu)$.

proof (a) Because $\mathcal{N}(\mu_A) = \mathcal{P}A \cap \mathcal{N}(\mu)$ (214Cb), the embedding $\mathcal{N}(\mu_A) \subsetneqq \mathcal{N}(\mu)$ is a Tukey function, and $\mathcal{N}(\mu_A) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$. By 513Ee, $\operatorname{add}\mathcal{N}(\mu_A) \ge \operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{cf}\mathcal{N}(\mu_A) \le \operatorname{cf}\mathcal{N}(\mu)$.

(b) Next, setting $\phi(x) = x$ for $x \in A$ and $\psi(F) = F \cap A$ for $F \in \mathcal{N}(\mu)$, (ϕ, ψ) witnesses that $(A, \in, \mathcal{N}(\mu_A)) \preccurlyeq_{\mathrm{GT}} (X, \in, \mathcal{N}(\mu))$. By 512D and 512Ed,

$$\operatorname{non}\mathcal{N}(\mu_A) = \operatorname{add}(A, \in, \mathcal{N}(\mu_A)) \ge \operatorname{add}(X, \in, \mathcal{N}(\mu)) = \operatorname{non}\mathcal{N}(\mu),$$

$$\operatorname{cov}\mathcal{N}(\mu_A) = \operatorname{cov}(A, \in, \mathcal{N}(\mu_A)) \le \operatorname{cov}(X, \in, \mathcal{N}(\mu)) = \operatorname{cov}\mathcal{N}(\mu).$$

(c) If $\langle F_\xi \rangle_{\xi<\kappa}$ is a disjoint family in $\Sigma_A = \operatorname{dom}\mu_A$, where $\kappa < \operatorname{add}\mu$, then for each $\xi < \kappa$ we have an $E_\xi \in \Sigma$ such that $F_\xi = A \cap E_\xi$ and $\mu_A F_\xi = \mu E_\xi$ (214Ca). Set $E'_\xi = E_\xi \setminus \bigcup_{\eta<\xi} E_\eta$ for $\xi < \kappa$; then $E'_\xi \in \Sigma$ for each ξ, and $\langle E'_\xi \rangle_{\xi<\kappa}$ is disjoint. Set $E = \bigcup_{\xi<\kappa} E'_\xi = \bigcup_{\xi<\kappa} E_\xi$ and $F = A \cap E = \bigcup_{\xi<\kappa} F_\xi$. Then

$$\sum_{\xi<\kappa} \mu_A F_\xi \le \mu_A F \le \mu E = \sum_{\xi<\kappa} \mu E'_\xi \le \sum_{\xi<\kappa} \mu E_\xi = \sum_{\xi<\kappa} \mu_A F_\xi,$$

and we have equality. As $\langle F_\xi \rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{add}\mu_A \ge \operatorname{add}\mu$.

(d) If $B \in \mathcal{P}A \setminus \mathcal{N}(\mu_A)$, there is a $C \subseteq B$ such that $C \notin \mathcal{N}(\mu)$ and $\#(C) \le \operatorname{shr}\mathcal{N}(\mu)$ (resp. $\#(C) < \operatorname{shr}^+\mathcal{N}(\mu)$); now $C \notin \mathcal{N}(\mu_A)$; as B is arbitrary, $\operatorname{shr}\mathcal{N}(\mu_A) \le \operatorname{shr}\mathcal{N}(\mu)$ (resp. $\operatorname{shr}^+\mathcal{N}(\mu_A) \le \operatorname{shr}^+\mathcal{N}(\mu)$).

(e) Let $\mathcal{H} \subseteq \Sigma \setminus \mathcal{N}(\mu)$ be a coinitial set of size $\pi(\mu)$. Set $\mathcal{G} = \{H \cap A : H \in \mathcal{H}\} \setminus \mathcal{N}(\mu)$. Then $\mu_A G$ is defined and non-zero for every $G \in \mathcal{G}$. Now $\mathcal{G}$ is coinitial with $\operatorname{dom}\mu_A \setminus \mathcal{N}(\mu_A)$. **P** If $\mu_A B > 0$, there is an $E \in \Sigma$ such that $B = E \cap A$. If $A \in \Sigma$, then $B \in \Sigma$ and there is an $H \in \mathcal{H}$ such that $H \subseteq B$, while of course $H \in \mathcal{G}$. If (X, Σ, μ) has locally determined negligible sets, then, as in the proof of 521Dc, there is a non-negligible set $F \in \Sigma$ which is a measurable envelope of a subset of B. Now there is an $H \in \mathcal{H}$ included in $F \cap E$, in which case $H \cap A$ is included in B and belongs to $\mathcal{G}$. **Q** So

$$\pi(\mu_A) \le \#(\mathcal{G}) \le \#(\mathcal{H}) = \pi(\mu).$$

(f) Writing $\mathfrak{A}$, $\mathfrak{A}_A$ for the measure algebras of μ and μ_A, we have a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}_A$ defined by saying that $\pi E^\bullet = (F \cap A)^\bullet$ for every $E \in \Sigma$. (The point is just that $F \cap A \in \mathcal{N}(\mu_A)$ whenever $F \in \mathcal{N}(\mu)$.) Now π is order-continuous. **P** Suppose that $C \subseteq \mathfrak{A}$ is non-empty and downwards-directed and $\inf C = 0$ in $\mathfrak{A}$. **?** If $b \in \mathfrak{A}_A$ is a non-zero lower bound of $\pi[C]$, then, because ν_A is semi-finite, there is a $G \in \operatorname{dom}\mu_A$ such that $0 < \mu_A G < \infty$ and $G^\bullet \subseteq b$. Let $E \in \Sigma$ be such that $G = E \cap A$ and $\mu E = \mu_A G$ (214Ca). Then $E^\bullet$ cannot be a lower bound of C; let $a \in C$ be such that $E^\bullet \setminus a \ne 0$. In this case, there is an $F \in \Sigma$ such that $F \subseteq E$ and $F^\bullet = E^\bullet \setminus a$, so that $\pi F^\bullet$ is disjoint from $\pi a \supseteq b$, and $F \cap G = (F \cap A) \cap G$ must be negligible. We know that $\mu F > 0$, so $\mu(E \setminus F) < \mu E$; but also $G \setminus (E \setminus F)$ is negligible, so

$$\mu_A G = \mu^* G \le \mu(E \setminus F) < \mu E = \mu_A G. \text{ **X**}$$

It follows that $\inf \pi[C] = 0$ in $\mathfrak{A}_A$; as C is arbitrary, π is order-continuous. **Q**

Now let $B \subseteq \mathfrak{A}$ be such that B τ-generates $\mathfrak{A}$ and $\#(B) = \tau(\mathfrak{A})$. Writing $\mathfrak{B}$ for the order-closed subalgebra of $\mathfrak{A}_A$ generated by $\pi[B]$, we see that $\pi^{-1}[\mathfrak{B}]$ is an order-closed subalgebra of $\mathfrak{A}$ including B, so must be the whole of $\mathfrak{A}$, and $\mathfrak{A}_A = \pi[\mathfrak{A}] = \mathfrak{B}$. Accordingly

$$\tau(\mu_A) = \tau(\mathfrak{A}_A) \le \#(\pi[B]) \le \#(B) = \tau(\mathfrak{A}) = \tau(\mu),$$

as claimed.

521G Proposition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a non-empty family of measure spaces with direct sum (X,Σ,μ). Then

$$\operatorname{add}\mathcal{N}(\mu)=\min_{i\in I}\operatorname{add}\mathcal{N}(\mu_i),\quad \operatorname{add}\mu=\min_{i\in I}\operatorname{add}\mu_i,$$

$$\operatorname{cov}\mathcal{N}(\mu)=\sup_{i\in I}\operatorname{cov}\mathcal{N}(\mu_i),\quad \operatorname{non}\mathcal{N}(\mu)=\min_{i\in I}\operatorname{non}\mathcal{N}(\mu_i),$$

$$\operatorname{shr}\mathcal{N}(\mu)=\sup_{i\in I}\operatorname{shr}\mathcal{N}(\mu_i),\quad \operatorname{shr}^+\mathcal{N}(\mu)=\sup_{i\in I}\operatorname{shr}^+\mathcal{N}(\mu_i)$$

and $\pi(\mu)$ is the cardinal sum $\sum_{i\in I}\pi(\mu_i)$. If I is finite, then

$$\operatorname{cf}\mathcal{N}(\mu)=\max_{i\in I}\operatorname{cf}\mathcal{N}(\mu_i).$$

proof Concerning each of $\operatorname{add}\mathcal{N}(\mu)$, $\operatorname{add}\mu$, $\operatorname{cov}\mathcal{N}(\mu)$, $\operatorname{non}\mathcal{N}(\mu)$, $\operatorname{shr}\mathcal{N}(\mu)$ and $\operatorname{shr}^+\mathcal{N}(\mu)$, 521F provides an inequality in one direction. The reverse inequalities are equally straightforward, especially if we note that $\mathcal{N}(\mu)\cong\prod_{i\in I}\mathcal{N}(\mu_i)$, so that 512Hc is relevant.

As for $\pi(\mu)$, if for each $i\in I$ we choose a coinitial set $\mathcal{H}_i$ of $\Sigma_i\setminus\mathcal{N}(\mu_i)$ of size $\pi(\mu_i)$, then

$$\mathcal{H}=\{H\times\{i\}:i\in I,\,H\in\mathcal{H}_i\}$$

is coinitial with $\Sigma\setminus\mathcal{N}(\mu)$ and witnesses that $\pi(\mu)\le\sum_{i\in I}\pi(\mu_i)$. (As in 214L, I am thinking of X as $\bigcup_{i\in I}X_i\times\{i\}$.) Conversely, if $\mathcal{H}$ is coinitial with $\Sigma\setminus\mathcal{N}(\mu)$ and for each $i\in I$ we set $\mathcal{H}_i=\{H:H\times\{i\}\in\mathcal{H}\}$, we shall have $\mathcal{H}_i$ coinitial with $\Sigma_i\setminus\mathcal{N}(\mu_i)$, so that

$$\textstyle\sum_{i\in I}\pi(\mu_i)\le\sum_{i\in I}\#(\mathcal{H}_i)\le\#(\mathcal{H})=\pi(\mu)$$

and $\pi(\mu)=\sum_{i\in I}\pi(\mu_i)$.

521H Proposition Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $f:X\to Y$ an inverse-measure-preserving function.

(a)(i) $(X,\in,\mathcal{N}(\mu))\preccurlyeq_{\mathrm{GT}}(Y,\in,\mathcal{N}(\nu))$, so $\operatorname{non}\mathcal{N}(\mu)\ge\operatorname{non}\mathcal{N}(\nu)$ and $\operatorname{cov}\mathcal{N}(\mu)\le\operatorname{cov}\mathcal{N}(\nu)$.

(ii) If there is a topology on Y such that ν is a topological measure inner regular with respect to the closed sets, then $\pi(\nu)\le\pi(\mu)$.

(iii) If ν is σ-finite, then $\tau(\nu)\le\tau(\mu)$.

(b) If ν is the image measure μf^{-1}, then $\operatorname{add}\nu\ge\operatorname{add}\mu$. If, moreover, μ is complete, $\mathcal{N}(\nu)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\mu)$, so $\operatorname{add}\mathcal{N}(\mu)\le\operatorname{add}\mathcal{N}(\nu)$ and $\operatorname{cf}\mathcal{N}(\mu)\ge\operatorname{cf}\mathcal{N}(\nu)$; also $\operatorname{shr}\mathcal{N}(\mu)\ge\operatorname{shr}\mathcal{N}(\nu)$ and $\operatorname{shr}^+\mathcal{N}(\mu)\ge\operatorname{shr}^+\mathcal{N}(\nu)$.

proof (a)(i) Set $\psi(F)=f^{-1}[F]$ for $F\in\mathcal{N}(\nu)$. Then (f,ψ) is a Galois-Tukey connection from $(X,\in,\mathcal{N}(\mu))$ to $(Y,\in,\mathcal{N}(\nu))$, so

$$\operatorname{cov}\mathcal{N}(\mu)=\operatorname{cov}(X,\in,\mathcal{N}(\mu))\le\operatorname{cov}(Y,\in,\mathcal{N}(\nu))=\operatorname{cov}\mathcal{N}(\nu),$$

$$\operatorname{non}\mathcal{N}(\mu)=\operatorname{add}(X,\in,\mathcal{N}(\mu))\ge\operatorname{add}(Y,\in,\mathcal{N}(\nu))=\operatorname{non}\mathcal{N}(\nu)$$

(512D, 512Ed again).

(ii) Let $\mathcal{H}$ be a coinitial subset of $\Sigma\setminus\mathcal{N}(\mu)$ of size $\pi(\mu)$. Set $\mathcal{G}=\{\overline{f[H]}:H\in\mathcal{H}\}$. Because ν is a topological measure, $\mathcal{G}\subseteq\mathrm{T}$; and if $H\in\mathcal{H}$, then

$$\nu\overline{f[H]}=\mu f^{-1}[\overline{f[H]}]\ge\mu H>0,$$

so $\mathcal{G}\subseteq\mathrm{T}\setminus\mathcal{N}(\nu)$. If $F\in\mathrm{T}\setminus\mathcal{N}(\nu)$, there is a closed set $F'\subseteq F$ such that $0<\nu F'=\mu f^{-1}[F']$; there is an $H\in\mathcal{H}$ such that $H\subseteq f^{-1}[F']$; now $G=\overline{f[H]}$ belongs to $\mathcal{G}$ and is included in $F'\subseteq F$. So $\mathcal{G}$ is coinitial with $\mathrm{T}\setminus\mathcal{N}(\nu)$ and

$$\pi(\nu)\le\#(\mathcal{G})\le\#(\mathcal{H})=\pi(\mu).$$

(iii) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be the measure algebras of μ, ν respectively. Then we have a sequentially order-continuous measure-preserving Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by setting $\pi F^\bullet=f^{-1}[F]^\bullet$ for every $F\in\operatorname{dom}\nu$ (324M). If $\mathfrak{A}$ is finite then $\mathfrak{B}$ must be finite with $\#(\mathfrak{B})\le\#(\mathfrak{A})$, and consequently $\tau(\mathfrak{B})\le\tau(\mathfrak{A})$ (331Xc, or otherwise). So let us suppose that $\mathfrak{A}$ is infinite.

Writing $\mathfrak{A}^f$, $\mathfrak{B}^f$ for the respective ideals of elements of finite measure, $\pi{\restriction}\mathfrak{B}^f$ is a function from $\mathfrak{B}^f$ to $\mathfrak{A}^f$ which is an isometry for the measure metrics on $\mathfrak{B}^f$ and $\mathfrak{A}^f$. So the topological density $d_{\mathrm{top}}(\mathfrak{B}^f)$ is equal to $d_{\mathrm{top}}(\pi[\mathfrak{B}^f])$ and less than or equal to $d_{\mathrm{top}}(\mathfrak{A}^f)$ (5A4B(h-ii)).

Observe next that $(\mathfrak{B},\bar\nu)$ is σ-finite because ν is, and that $(\mathfrak{A},\bar\mu)$ therefore also is (324Kd). So we get

$$\tau(\nu)=\tau(\mathfrak{B})\le\max(\omega,c(\mathfrak{B}),\tau(\mathfrak{B}))=\max(\omega,d_{\text{top}}(\mathfrak{B}^f))$$

(521Eb)

$$\le\max(\omega,d_{\text{top}}(\mathfrak{A}^f))=\max(\omega,c(\mathfrak{A}),\tau(\mathfrak{A}))=\tau(\mathfrak{A})=\tau(\mu),$$

as required.

(b) If $\langle F_\xi\rangle_{\xi<\kappa}$ is a disjoint family in T, where $\kappa<\operatorname{add}\mu$, then $\langle f^{-1}[F_\xi]\rangle_{\xi<\kappa}$ is a disjoint family in Σ, so

$$\nu(\bigcup_{\xi<\kappa}F_\xi)=\mu f^{-1}[\bigcup_{\xi<\kappa}F_\xi]=\mu(\bigcup_{\xi<\kappa}f^{-1}[F_\xi]=\sum_{\xi<\kappa}\mu f^{-1}[F_\xi]=\sum_{\xi<\kappa}\nu F_\xi.$$

As $\langle F_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{add}\nu\ge\operatorname{add}\mu$.

Now suppose that μ is complete. In this case, $F\in\mathrm{T}$ whenever $F\subseteq Y$ and $f^{-1}[F]\in\mathcal{N}(\mu)$, so that $\mathcal{N}(\nu)$ is precisely $\{F:F\subseteq Y,\ f^{-1}[F]\in\mathcal{N}(\mu)\}$. It is now easy to check that $F\mapsto f^{-1}[F]:\mathcal{N}(\nu)\to\mathcal{N}(\mu)$ is a Tukey function. So $\operatorname{add}\mathcal{N}(\nu)\ge\operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{cf}\mathcal{N}(\nu)\le\operatorname{cf}\mathcal{N}(\mu)$, by 513Ee again.

Take any non-negligible $A\subseteq Y$. Then $f^{-1}[A]\notin\mathcal{N}(\mu)$, so there is a set $B\subseteq f^{-1}[A]$ such that $\#(B)\le\operatorname{shr}\mathcal{N}(\mu)$ and $B\notin\mathcal{N}(\mu)$. In this case, $f[B]\subseteq A$, $f[B]\notin\mathcal{N}(\nu)$ and $\#(f[B])\le\operatorname{shr}\mathcal{N}(\mu)$. As A is arbitrary, $\operatorname{shr}\mathcal{N}(\nu)\le\operatorname{shr}\mathcal{N}(\mu)$. The same argument, with $<$ instead of $\le$ at appropriate points, shows that $\operatorname{shr}^+\mathcal{N}(\nu)\le\operatorname{shr}^+\mathcal{N}(\mu)$.

521I Corollary Let (X,Σ,μ) be an atomless strictly localizable measure space. Then $\operatorname{non}\mathcal{N}(\mu)\ge\operatorname{non}\mathcal{N}$ and $\operatorname{cov}\mathcal{N}(\mu)\le\operatorname{cov}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.

proof (a) If $\mu X=0$ this is trivial.

(b) If $0<\mu X<\infty$, let ν be the completion of the normalized measure $\frac{1}{\mu X}\mu$. Then ν is complete and atomless, so by 343Cb there is an inverse-measure-preserving function from (X,ν) to $([0,1],\mu_1)$, where μ_1 is Lebesgue measure on $[0,1]$. Also $\mathcal{N}(\nu)=\mathcal{N}(\mu)$. By 521Ha, $\operatorname{non}\mathcal{N}(\nu)\ge\operatorname{non}\mathcal{N}(\mu_1)$ and $\operatorname{cov}\mathcal{N}(\nu)\le\operatorname{cov}\mathcal{N}(\mu_1)$. Now $([0,1],\mathcal{N}(\mu_1))$ is isomorphic to $(\mathbb{R},\mathcal{N})$. **P** Take a bijection $h:\mathbb{R}\to[0,1]$ such that $h(x)=\frac{1}{2}(1+\tanh x)$ for $x\in\mathbb{R}\setminus\mathbb{Q}$; then h is a suitable isomorphism. **Q** So $\operatorname{non}\mathcal{N}(\mu)=\operatorname{non}\mathcal{N}(\nu)\ge\operatorname{non}\mathcal{N}$ and $\operatorname{cov}\mathcal{N}(\mu)\le\operatorname{cov}\mathcal{N}$.

(c) If X has infinite measure, let $\langle X_i\rangle_{i\in I}$ be a decomposition of X into sets of finite measure. For each $i\in I$ let μ_i be the subspace measure on X_i. Then every μ_i is atomless, so, putting (b) and 521G together,

$$\operatorname{non}\mathcal{N}(\mu)=\min_{i\in I}\operatorname{non}\mathcal{N}(\mu_i)\ge\operatorname{non}\mathcal{N},$$

$$\operatorname{cov}\mathcal{N}(\mu)=\sup_{i\in I}\operatorname{cov}\mathcal{N}(\mu_i)\le\operatorname{cov}\mathcal{N}.$$

521J For product spaces the situation is more complicated, because the product measure introduces 'new' negligible sets which are not directly definable in terms of the null ideals of the factors. In the next three sections, however, we shall find out quite a lot about the cardinal functions of Radon measures, and this information, when it comes, can be used to give results about general products of probability measures.

Proposition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a non-empty family of probability spaces with product (X,Σ,μ).

(a)

$$\operatorname{non}\mathcal{N}(\mu)\ge\sup_{i\in I}\operatorname{non}\mathcal{N}(\mu_i),\quad\operatorname{cov}\mathcal{N}(\mu)\le\min_{i\in I}\operatorname{cov}\mathcal{N}(\mu_i),$$

$$\operatorname{add}\mu=\operatorname{add}\mathcal{N}(\mu)\le\min_{i\in I}\operatorname{add}\mathcal{N}(\mu_i),\quad\operatorname{cf}\mathcal{N}(\mu)\ge\sup_{i\in I}\operatorname{cf}\mathcal{N}(\mu_i),$$

$$\operatorname{shr}\mathcal{N}(\mu)\ge\sup_{i\in I}\operatorname{shr}\mathcal{N}(\mu_i),\quad\operatorname{shr}^+\mathcal{N}(\mu)\ge\sup_{i\in I}\operatorname{shr}^+\mathcal{N}(\mu_i),$$

$$\pi(\mu)\ge\sup_{i\in I}\pi(\mu_i).$$

(b) Set $\kappa=\#(\{i:i\in I,\ \Sigma_i\ne\{\emptyset,X_i\}\}$. Then $[\kappa]^{\le\omega}\preccurlyeq_{\mathrm{T}}\mathcal{N}(\mu)$; consequently $\operatorname{add}\mu=\operatorname{add}\mathcal{N}(\mu)$ is ω_1 if κ is uncountable, while $\operatorname{cf}\mathcal{N}(\mu)$ is at least $\operatorname{cf}[\kappa]^{\le\omega}$.

(c) Now suppose that I is countable and that we have for each $i \in I$ a probability space $(Y_i, \mathrm{T}_i, \nu_i)$ and an inverse-measure-preserving function $f_i : X_i \to Y_i$ which represents an isomorphism of the measure algebras of μ_i and ν_i. Let (Y, T, ν) be the product of $\langle (Y_i, \mathrm{T}_i, \nu_i)\rangle_{i\in I}$. Then

$$\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\nu) \times \prod_{i\in I} \mathcal{N}(\mu_i).$$

Consequently

$$\operatorname{add}\mathcal{N}(\mu) \ge \min(\operatorname{add}\mathcal{N}(\nu), \min_{i\in I} \operatorname{add}\mathcal{N}(\mu_i)),$$

and if I is finite

$$\operatorname{cf}\mathcal{N}(\mu) \le \max(\operatorname{cf}\mathcal{N}(\nu), \max_{i\in I} \operatorname{cf}\mathcal{N}(\mu_i)).$$

(d) If I is finite, then

$$\operatorname{non}\mathcal{N}(\mu) = \max_{i\in I} \operatorname{non}\mathcal{N}(\mu_i), \quad \operatorname{cov}\mathcal{N}(\mu) = \min_{i\in I} \operatorname{cov}\mathcal{N}(\mu_i).$$

proof (a) Note that $\operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu)$ by 521Ad. Now with one exception the inequalities are immediate if we apply 521H to the canonical maps from X to X_i. The odd one out is the last, because we do not have a simple general result concerning the π-weight of an image measure. But in the present case we can argue as follows. Let $\mathcal{H} \subseteq \Sigma \setminus \mathcal{N}(\mu)$ be a coinitial set of size $\pi(\mu)$, and take $i \in I$. Then we can identify (X, Σ, μ) with $(X', \Sigma', \mu') \times (X_i, \Sigma_i, \mu_i)$ where (X', Σ', μ') is the product of the family $\langle (X_j, \Sigma_j, \mu_j)\rangle_{j\in I, j\ne i}$ (254N). If $H \in \mathcal{H}$, $\mu(X \setminus H) = \int \mu_i^*(X_i \setminus H[\{x'\}])\mu'(dx')$ (252D) is less than 1, so there is an $x'_H \in X'$ such that $\mu_i^*(X_i \setminus H[\{x'_H\}]) < 1$, $(\mu_i)_*H[\{x'_H\}] > 0$ and there is a $G_H \in \Sigma_i \setminus \mathcal{N}(\mu_i)$ such that $G_H \subseteq H[\{x'_H\}]$. Set $\mathcal{G} = \{G_H : H \in \mathcal{H}\}$. If $\mu_i F > 0$, then $\mu(X' \times F) > 0$ and there is an $H \in \mathcal{H}$ included in $X' \times F$; in which case $G_H \subseteq H[\{x'_H\}] \subseteq F$. So $\mathcal{G}$ is coinitial with $\Sigma_i \setminus \mathcal{N}(\mu_i)$ and

$$\pi(\mu_i) \le \#(\mathcal{G}) \le \#(\mathcal{H}) \le \pi(\mu).$$

As i is arbitrary, $\sup_{i\in I} \pi(\mu_i) \le \pi(\mu)$, as claimed.

(b)(i) If $\kappa \le \omega$ then the constant function with value $\emptyset$ is a Tukey function from $[\kappa]^{\le\omega}$ to $\mathcal{N}(\mu)$. Otherwise, set $J = \{i : i \in I,\ \Sigma_i \ne \{\emptyset, X_i\}\}$ and for $i \in J$ choose a non-empty $C_i \in \Sigma_i$ such that $\mu_i C_i \le \frac{1}{2}$. Index J as $\langle i_{\xi n}\rangle_{\xi<\kappa, n\in\mathbb{N}}$ and for $\xi < \kappa$ set $E_\xi = \{x : x(i_{\xi n}) \in C_{i_{\xi n}}$ for every $n \in \mathbb{N}\}$, so that $\mu E_\xi = \prod_{n\in\mathbb{N}} \mu_{i_{\xi n}} C_{i_{\xi n}} = 0$. Define $\phi : [\kappa]^{\le\omega} \to \mathcal{N}(\mu)$ by setting $\phi K = \bigcup_{\xi\in K} E_\xi$ for countable $K \subseteq \kappa$. Then ϕ is a Tukey function. **P** If $E \in \mathcal{N}(\mu)$, there is a negligible $E' \supseteq E$ which is determined by a countable set I' of coordinates (254Oc). Set $L = \{\xi : \xi < \kappa,\ i_{\xi n} \in I'$ for some $n \in \mathbb{N}\}$; then L is countable. If $\xi < \kappa$ and $E_\xi \subseteq E'$, E_ξ is determined by coordinates in $\{i_{\xi n} : n \in \mathbb{N}\}$; as neither E_ξ nor $X \setminus E'$ is empty, this must meet I', and $\xi \in L$. So $\{K : K \in [\kappa]^{\le\omega},\ \phi K \subseteq E\}$ is bounded above by $L \in [\kappa]^{\le\omega}$. As E is arbitrary, ϕ is a Tukey function. **Q** Accordingly $[\kappa]^{\le\omega} \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu)$.

(ii) It follows that $\operatorname{add}\mathcal{N}(\mu) \le \operatorname{add}[\kappa]^{\le\omega} \le \omega_1$ if κ is uncountable, and that $\operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}[\kappa]^{\le\omega}$.

(c)(i) Recall that any inverse-measure-preserving function f between measure spaces induces a measure-preserving Boolean homomorphism $F^\bullet \mapsto (f^{-1}[F])^\bullet$ between the measure algebras (324M). For $i \in I$ and $C \in \Sigma_i$ choose $\psi_i(C) \in \mathrm{T}_i$ such that $(f_i^{-1}[\psi_i(C)])^\bullet = C^\bullet$ in the measure algebra of μ_i, that is, $C \triangle f_i^{-1}[\psi_i(C)] \in \mathcal{N}(\mu_i)$. Next, for $E \in \mathcal{N}(\mu)$ and $n \in \mathbb{N}$, choose a family $\langle C_{Enmi}\rangle_{m\in\mathbb{N}, i\in I}$ such that $C_{Enmi} \in \Sigma_i$ for every $m \in \mathbb{N}$ and $i \in I$, $E \subseteq \bigcup_{m\in\mathbb{N}} \prod_{i\in I} C_{Enmi}$, and $\sum_{m\in\mathbb{N}} \prod_{i\in I} \mu_i C_{Enmi} \le 2^{-n}$; set

$$\phi(E) = \bigl(\bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \prod_{i\in I} \psi_i(C_{Enmi}), \langle \bigcup_{m,n\in\mathbb{N}} (C_{Enmi} \setminus f_i^{-1}[\psi_i(C_{Enmi})])\rangle_{i\in I}\bigr).$$

Because

$$\begin{aligned}\nu(\bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \prod_{i\in I} \psi_i(C_{Enmi})) &\le \inf_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \prod_{i\in I} \nu_i \psi_i(C_{Enmi}) \\ &= \inf_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \prod_{i\in I} \mu_i C_{Enmi} = 0,\end{aligned}$$

ϕ is a function from $\mathcal{N}(\nu)$ to $\mathcal{N}(\nu) \times \prod_{i\in I} \mathcal{N}(\mu_i)$.

Now ϕ is a Tukey function. **P** Suppose that $W \in \mathcal{N}(\nu)$ and that $E_i \in \mathcal{N}(\mu_i)$ for every $i \in I$. Define $f : X \to Y$ by setting $f(x) = \langle f_i(x(i))\rangle_{i\in I}$ for $x \in X$; then f is inverse-measure-preserving (254H). So $V = f^{-1}[W] \cup \bigcup_{i\in I}\{x : x(i) \in E_i\}$ is negligible. (This is where we need to know that I is countable.) Suppose that $E \in \mathcal{N}(\mu)$ is such that $\phi(E) \le (W, \langle E_i\rangle_{i\in I})$; take $x \in E$ such that $x(i) \notin E_i$ for every $i \in I$, and $n \in \mathbb{N}$. Then there is an $m \in \mathbb{N}$ such that $x \in \prod_{i\in I} C_{Enmi}$. For each $i \in I$,

$$C_{Enmi} \setminus f_i^{-1}[\psi_i(C_{Enmi})] \subseteq E_i, \quad x(i) \in C_{Enmi} \setminus E_i,$$

so $f_i(x(i)) \in \psi_i(C_{Enmi})$; thus $f(x) \in \prod_{i\in I} \psi_i(C_{Enmi})$. As n is arbitrary,

$$f(x) \in \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \prod_{i\in I} \psi_i(C_{Enmi}) \subseteq W$$

and $x \in V$. As x is arbitrary, $E \subseteq V$. As $(W, \langle E_i\rangle_{i\in I})$ is arbitrary, ϕ is a Tukey function. **Q**

So $\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\nu) \times \prod_{i\in I} \mathcal{N}(\mu_i)$.

(ii) Accordingly

$$\operatorname{add}\mathcal{N}(\mu) \ge \operatorname{add}(\mathcal{N}(\nu) \times \prod_{i\in I}\mathcal{N}(\mu_i)) = \min(\operatorname{add}\mathcal{N}(\nu), \min_{i\in I}\operatorname{add}\mathcal{N}(\mu_i))$$

and

$$\operatorname{cf}\mathcal{N}(\mu) \le \operatorname{cf}(\mathcal{N}(\nu) \times \prod_{i\in I}\mathcal{N}(\mu_i)) = \max(\operatorname{cf}\mathcal{N}(\nu), \max_{i\in I}\operatorname{cf}\mathcal{N}(\mu_i))$$

if I is finite.

(d)(i) For each $i \in I$ let $A_i \subseteq X_i$ be a non-negligible set of size $\operatorname{non}\mathcal{N}(\mu_i)$. Then $A = \prod_{i\in I} A_i$ is not negligible (251Wm), while $\#(A) \le \max(\omega, \max_{i\in I}\operatorname{non}\mathcal{N}(\mu_i))$. If all the $\operatorname{non}\mathcal{N}(\mu_i)$ are finite, then they are all equal to 1, and A is a singleton. So we must in any case have $\#(A) = \max_{i\in I}\operatorname{non}\mathcal{N}(\mu_i)$, and $\operatorname{non}\mathcal{N}(\mu) \le \max_{i\in I}\operatorname{non}\mathcal{N}(\mu_i)$. By (a), we have equality.

(ii) Suppose that $I = \{0,1\}$, and that $\mathcal{E}$ is a cover of $X = X_0 \times X_1$ by negligible sets. For each $E \in \mathcal{E}$, set $C_E = \{x : x \in X_0, E[\{x\}] \notin \mathcal{N}(\mu_1)\}$; then C_E is negligible. If $\#(\mathcal{E}) < \operatorname{cov}\mathcal{N}(\mu_0)$, then there is an $x \in X_0 \setminus \bigcup_{E\in\mathcal{E}} C_E$; in which case $\{E[\{x\}] : E \in \mathcal{E}\}$ witnesses that $\operatorname{cov}\mathcal{N}(\mu_1) \le \#(\mathcal{E})$. So $\#(\mathcal{E})$ must be at least $\min(\operatorname{cov}\mathcal{N}(\mu_0), \operatorname{cov}\mathcal{N}(\mu_1))$. As $\mathcal{E}$ is arbitrary, $\operatorname{cov}\mathcal{N}(\mu) \ge \min(\operatorname{cov}\mathcal{N}(\mu_0), \operatorname{cov}\mathcal{N}(\mu_1))$.

Now an induction on $\#(I)$ (using the associative law 254N) shows that $\operatorname{cov}\mathcal{N}(\mu) \ge \min_{i\in I}\operatorname{cov}\mathcal{N}(\mu_i)$ whenever I is finite. Using (a) again, we have equality here also.

Remark The simplest applications of (c) here will be when the μ_i are Maharam-type-homogeneous, so that we can take the ν_i to be the usual measures on powers $\{0,1\}^{\kappa_i}$ of $\{0,1\}$, and ν will be isomorphic to the usual measure on $\{0,1\}^\kappa$ where κ is the cardinal sum $\sum_{i\in I}\kappa_i$. The cardinal functions of these measures are dealt with in §523. For non-homogeneous μ_i we shall still be able to arrange for the ν_i to be completion regular Radon measures on dyadic spaces, so that the product measure ν is again a Radon measure Y (532F), and (once we have identified its measure algebra – see 334E, 334Ya) approachable by the methods of §524.

521K I turn now to 'perfect' and 'compact' measure spaces. (See §451 for the basic theory of these.)

Proposition Let (X, Σ, μ) be a perfect semi-finite measure space which is not purely atomic. Then

$$\operatorname{add}\mathcal{N}(\mu) \le \operatorname{add}\mathcal{N}, \quad \operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}\mathcal{N},$$

$$\operatorname{shr}\mathcal{N}(\mu) \ge \operatorname{shr}\mathcal{N}, \quad \operatorname{shr}^+\mathcal{N}(\mu) \ge \operatorname{shr}^+\mathcal{N}, \quad \pi(\mu) \ge \pi(\mu_L)$$

where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$ and μ_L is Lebesgue measure on $\mathbb{R}$.

proof (a) Suppose first that μ is a complete atomless probability measure. Then there is a function $f : X \to [0,1]$ which is inverse-measure-preserving for μ and Lebesgue measure μ_1 on $[0,1]$ (343Cb again); and in fact μ_1 is the image measure μf^{-1}. **P** By 451O, μf^{-1} is a Radon measure; since it extends μ_1 it must actually be equal to μ_1, by 415H. **Q** So $\operatorname{add}\mathcal{N}(\mu) \le \operatorname{add}\mathcal{N}(\mu_1)$, $\operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}\mathcal{N}(\mu_1)$, $\operatorname{shr}\mathcal{N}(\mu) \ge \operatorname{shr}\mathcal{N}(\mu_1)$, and $\operatorname{shr}^+\mathcal{N}(\mu) \ge \operatorname{shr}^+\mathcal{N}(\mu_1)$ and $\pi(\mu) \ge \pi(\mu_1)$, by 521H. As in the proof of 521I, $([0,1], \mathcal{N}(\mu_1))$ is isomorphic to $(\mathbb{R}, \mathcal{N})$. Of course μ_1 is not isomorphic to μ_L. But μ_L is isomorphic to a direct sum of countably many copies of μ_1, so by 521G we know that $\pi(\mu_L)$ is the cardinal product $\omega \cdot \pi(\mu_1)$; as $\pi(\mu_1)$ is surely infinite, this is $\pi(\mu_1)$ again. So we have the result in the special case.

(b) Now suppose that (X, Σ, μ) is any semi-finite perfect measure space which is not purely atomic. Then the completion $\hat\mu$ of μ is still a semi-finite perfect measure which is not purely atomic (212Gd, 451G(c-i)), and $\mathcal{N}(\mu) = \mathcal{N}(\hat\mu)$ (212Eb). Because $\hat\mu$ is semi-finite and not purely atomic, there is a set $E \in \Sigma$ of non-zero finite measure such that the subspace measure $\hat\mu_E$ is atomless. Set $\nu = \frac{1}{\mu E}\hat\mu_E$, so that ν is an atomless complete perfect probability measure on E, while $\mathcal{N}(\nu) = \mathcal{N}(\mu_E)$. Putting (a) together with 521F, we get

$$\operatorname{add}\mathcal{N}(\mu) = \operatorname{add}\mathcal{N}(\hat\mu) \le \operatorname{add}\mathcal{N}(\hat\mu_E) = \operatorname{add}\mathcal{N}(\nu) \le \operatorname{add}\mathcal{N}$$

and similarly for cf, shr and shr^+.

521L Proposition (a) Let (X,Σ,μ) be a strictly localizable measure space and (Y,T,ν) a locally compact semi-finite measure space, and suppose that they have isomorphic measure algebras. Then $(X,\in,\mathcal{N}(\mu))\preccurlyeq_{\mathrm{GT}}(Y,\in,\mathcal{N}(\nu))$; consequently $\operatorname{cov}\mathcal{N}(\mu)\le\operatorname{cov}\mathcal{N}(\nu)$ and $\operatorname{non}\mathcal{N}(\nu)\le\operatorname{non}\mathcal{N}(\mu)$.

(b) Let (X,Σ,μ) be a Maharam-type-homogeneous compact probability space with Maharam type κ. Then $\operatorname{cov}\mathcal{N}(\mu)=\operatorname{cov}\mathcal{N}_\kappa$ and $\operatorname{non}\mathcal{N}(\mu)=\operatorname{non}\mathcal{N}_\kappa$, where $\mathcal{N}_\kappa$ is the null ideal of the usual measure ν_κ on $\{0,1\}^\kappa$.

(c) Let (X,Σ,μ) be a compact strictly localizable measure space with measure algebra $\mathfrak{A}$. Then

$$d(\mathfrak{A})=\min\{\#(A):A\subseteq X\text{ has full outer measure}\}.$$

proof (a) This follows immediately from 521Ha, because by 343B there is an inverse-measure-preserving function from X to Y.

(b) The point is that ν_κ is a compact measure (342Jd, 451Ja), so that we can apply (a) in both directions to see that $\operatorname{cov}\mathcal{N}(\mu)=\operatorname{cov}\mathcal{N}_\kappa$ and $\operatorname{non}\mathcal{N}(\mu)=\operatorname{non}\mathcal{N}_\kappa$.

(c) The case $\mu X=0$ is trivial; suppose that $\mu X>0$. Let $\mathcal{K}$ be a compact class such that μ is inner regular with respect to $\mathcal{K}$.

(i) Suppose that $\langle C_\xi\rangle_{\xi<d(\mathfrak{A})}$ is a family of centered sets in $\mathfrak{A}$ covering $\mathfrak{A}^+$. For each $\xi<d(\mathfrak{A})$, set $\mathcal{K}_\xi=\{K:K\in\mathcal{K}\cap\Sigma,\,K^\bullet\in C_\xi\}$; then $\mathcal{K}_\xi$ has the finite intersection property so there is a point $x_\xi\in X\cap\bigcap\mathcal{K}_\xi$. Set $A=\{x_\xi:\xi<d(\mathfrak{A})\}$. If $K\in\mathcal{K}\cap\Sigma$ and $K\cap A=\emptyset$, then $K\notin\bigcup_{\xi<d(\mathfrak{A})}\mathcal{K}_\xi$ so $K^\bullet=0$; it follows that every measurable subset of $X\setminus A$ is negligible and A has full outer measure, while $\#(A)\le d(\mathfrak{A})$.

(ii) Let $\hat\mu$ be the completion of μ, $\hat\Sigma$ its domain and $\theta:\mathfrak{A}\to\hat\Sigma$ a lifting (341K, 212Gb, 322Da). Take any $A\subseteq X$ of full outer measure for μ; then it also has full outer measure for $\hat\mu$ (212Eb). For $x\in A$, set $C_x=\{a:a\in\mathfrak{A},\,x\in\theta a\}$; then $\langle C_x\rangle_{x\in A}$ is a family of centered sets in $\mathfrak{A}$ with union $\mathfrak{A}^+$, so $d(\mathfrak{A})\le\#(A)$.

521M Proposition Let (X,Σ,μ) be a complete locally determined measure space of magnitude at most $\operatorname{add}\mu$. Then it is strictly localizable.

proof Write κ for $\operatorname{add}\mu$. Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ. Then there is a partition of unity $D\subseteq\mathfrak{A}$ consisting of elements of finite measure; as $\#(D)\le c(\mathfrak{A})\le\kappa$, there is a family $\langle a_\xi\rangle_{\xi<\kappa}$ running over $D\cup\{0\}$. For each $\xi<\kappa$, choose $E_\xi\in\Sigma$ such that $E_\xi^\bullet=a_\xi$, and set $F_\xi=E_\xi\setminus\bigcup_{\eta<\xi}E_\eta$. Because $E_\xi\setminus F_\xi=\bigcup_{\eta<\xi}E_\xi\cap E_\eta$ is the union of fewer than $\operatorname{add}\mu$ negligible sets, it is negligible, and $F_\xi\in\Sigma$, with $F_\xi^\bullet=a_\xi$. Now $\langle F_\xi\rangle_{\xi<\kappa}$ is a disjoint family of sets of finite measure. If $E\in\Sigma$ and $\mu E>0$, there is some $\xi<\kappa$ such that $E^\bullet\cap a_\xi\ne0$, and now $\mu(E\cap F_\xi)>0$. Thus $\langle F_\xi\rangle_{\xi<\kappa}$ satisfies the condition of 213Oa, and μ is strictly localizable.

521N Proposition Let (X,Σ,μ) be a complete locally determined localizable measure space of magnitude at most $\mathfrak{c}$. Then it is strictly localizable.

proof Again let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ, and take a partition of unity $D\subseteq\mathfrak{A}$ consisting of elements of finite measure; as $\#(D)\le c(\mathfrak{A})\le\mathfrak{c}$, there is an injective function $h:D\to\mathcal{P}\mathbb{N}$. This time, because $\mathfrak{A}$ is Dedekind complete, we can set $b_n=\sup\{d:d\in D,\,n\in h(d)\}$ for each $n\in\mathbb{N}$. If $d\in D$, then $d=\inf_{n\in h(d)}b_n\setminus\sup_{n\in\mathbb{N}\setminus h(d)}b_n$. So if we choose $E_n\in\Sigma$ such that $E_n^\bullet=b_n$ for each n, and set $F_d=\bigcap_{n\in h(d)}E_n\setminus\bigcup_{n\in\mathbb{N}\setminus h(d)}E_n$ for $d\in D$, $\langle F_d\rangle_{d\in D}$ will be a disjoint family in Σ and $F_d^\bullet=d$ for every d. Now $\mu F_d=\bar\mu d$ is always finite; and if $E\in\Sigma$ is non-negligible, there is a $d\in D$ such that $0\ne\bar\mu(E^\bullet\cap d)=\mu(E\cap F_d)$. Thus $\langle F_d\rangle_{d\in D}$ satisfies the condition of 213Oa, and μ is strictly localizable.

521O Proposition (a) If (X,Σ,μ) is a semi-finite measure space, its magnitude is at most $\max(\omega,2^{\#(X)})$.

(b) If (X,Σ,μ) is a strictly localizable measure space, its magnitude is at most $\max(\omega,\#(X))$.

(c) There is an infinite semi-finite measure space (X,Σ,μ) with magnitude $2^{\#(X)}$.

proof (a)-(b) These are elementary. If (X,Σ,μ) is semi-finite, with measure algebra $\mathfrak{A}$, then

$$c(\mathfrak{A})\le\#(\mathfrak{A})\le\#(\Sigma)\le\#(\mathcal{P}X)=2^{\#(X)}.$$

If μ is strictly localizable, with decomposition $\langle X_i\rangle_{i\in I}$, then $\langle X_i^\bullet\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ consisting of elements of finite measure, so

$$c(\mathfrak{A})\le\max(\omega,\#(\{i:i\in I,\,\mu X_i>0\}))\le\max(\omega,\#(X))$$

by 332E.

(c) Let $\langle X_\xi\rangle_{\xi<\omega_1}$ be a disjoint family of sets such that $\#(X_\xi)=\#(\mathcal{P}(\bigcup_{\eta<\xi}X_\eta))$ for every $\xi<\omega_1$; for each ξ, let $h_\xi:\mathcal{P}(\bigcup_{\eta<\xi}X_\eta)\to X_\xi$ be an injection. Set $X=\bigcup_{\xi<\omega_1}X_\xi$. For $A\subseteq X$ define $f_A:\omega_1\to X$ by setting $f(\xi)=h_\xi(A\cap\bigcup_{\eta<\xi}X_\eta)$ for each ξ; let J_A be $f_A[\omega_1]$ and μ_A the countable-cocountable measure on J_A. Observe that $\#(J_A)=\omega_1$ for every $A\subseteq X$, and that if A, $B\subseteq X$ are distinct then $J_A\cap J_B$ is countable. So if we set $\mu E=\sum_{A\subseteq X}\mu_A(E\cap J_A)$ whenever $E\subseteq X$ is such that $E\cap J_A$ is countable or cocountable in J_A for every A, then μ will be a complete locally determined measure on X. Since $\mu J_A=1$ and $\mu(J_A\cap J_B)=0$ whenever A, $B\subseteq X$ are distinct, μ has magnitude $2^{\#(X)}$.

521P Proposition (a) If $2^\lambda<2^\kappa$ whenever $\mathfrak{c}\le\lambda<\kappa$ and $\operatorname{cf}\lambda>\omega$, then the magnitude $\operatorname{mag}\mu$ of μ is at most $\max(\omega,\#(X))$ for every localizable measure space (X,Σ,μ).

(b) Suppose that $2^{\mathfrak{c}}=2^{\mathfrak{c}^+}$. Then there is a localizable measure space (Y,T,ν) with $\#(Y)=\mathfrak{c}$ and $\operatorname{mag}\nu=\mathfrak{c}^+$.

Remark TEX, for once, is obscure; $2^{\mathfrak{c}^+}$ here is $\#(\mathcal{P}(\mathfrak{c}^+))$, not $(2^{\mathfrak{c}})^+$.

proof (a) If $\operatorname{mag}\mu\le\omega$ we can stop. Otherwise, set $\kappa=\operatorname{mag}\mu$. Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ, so that $\kappa=c(\mathfrak{A})$ and there is a disjoint family $\langle a_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}^+$ (332F). If $\tilde\mu$ is the c.l.d. version of μ, we can identify $\mathfrak{A}$ with the measure algebra of $\tilde\mu$ (322Db).

case 1 If $\kappa\le\mathfrak{c}$, $\tilde\mu$ is strictly localizable (521N), so has a lifting θ (341K again); but now $\langle\theta a_\xi\rangle_{\xi<\kappa}$ is a disjoint family of non-empty subsets of X, so $\#(X)\ge\kappa$.

case 2 If $\kappa>\mathfrak{c}$, of course X is uncountable (521Oa). For $\xi<\kappa$, choose $E_\xi\in\Sigma$ such that $E_\xi^\bullet=a_\xi$. **?** If $\#(X)<\kappa$, there is a set $Y\subseteq X$ such that $\#(Y)$ has uncountable cofinality and $I_Y=\{\xi:\xi<\kappa,\ \mu^*(E_\xi\cap Y)>0\}$ has cardinal greater than $\max(\mathfrak{c},\#(X))$. **P** If $\operatorname{cf}(\#(X))$ is uncountable, take $Y=X$. Otherwise, let $\langle Y_n\rangle_{n\in\mathbb{N}}$ be an increasing sequence of subsets of X, with union X, such that $\#(Y_n)$ is an uncountable successor cardinal less than $\#(X)$ for every n. If $\xi<\kappa$, there is some n such that $E_\xi\cap Y_n$ is non-negligible, that is, $\xi\in I_{Y_n}$. So the non-decreasing sequence $\langle I_{Y_n}\rangle_{n\in\mathbb{N}}$ has union κ, and there is some $n\in\mathbb{N}$ such that $\#(I_{Y_n})>\max(\mathfrak{c},\#(X))$. Now we can take $Y=Y_n$. **Q**

For every $J\subseteq I_Y$, set $b_J=\sup_{\xi\in J}a_\xi$ and let $F_J\in\Sigma$ be such that $F_J^\bullet=b_J$. If J, $K\subseteq I_Y$ are distinct, there is a $\xi\in J\triangle K$, in which case $a_\xi\subseteq b_J\triangle b_K$, $E_\xi\setminus(F_J\triangle F_K)$ is negligible and $Y\cap(F_J\triangle F_K)$ is non-empty. Thus $J\mapsto Y\cap F_J:\mathcal{P}I_Y\to\mathcal{P}Y$ is injective, and

$$2^{\#(I_Y)}\le2^{\#(Y)}\le2^{\#(X)}\le2^{\#(I_Y)}.$$

Setting $\lambda=\max(\mathfrak{c},\#(Y))$, $\kappa'=\#(I_Y)$ we now have $\mathfrak{c}\le\lambda<\kappa'$, $\operatorname{cf}\lambda>\omega$ and $2^\lambda=2^{\kappa'}$, which is supposed to be impossible. **X**

So in this case also we have $\#(X)\ge\kappa$.

(b)(i) Set $I=\mathcal{P}\mathfrak{c}^+$ and $X=\{0,1\}^I\cong\{0,1\}^{2^{\mathfrak{c}}}$. Putting 5A4Be and 5A4C(a-ii) together, we see that there is a set $Y\subseteq X$, with cardinal at most $\mathfrak{c}^\omega=\mathfrak{c}$, which meets every non-empty G_δ subset of X. In particular, if $\mathcal{K}\subseteq I$ is countable and $x\in X$ there is a $y\in Y$ such that $y\restriction\mathcal{K}=x\restriction\mathcal{K}$.

(ii) Let μ be the complete locally determined localizable measure on X described in 216E, with $C=\mathfrak{c}^+$. Then Y has full outer measure in X. **P** (I follow the notation and argument of 216E.) If $\mu E>0$, then, by the argument of part (g) of the proof of 216E, there are a $\gamma<\mathfrak{c}^+$ and a $K\in[I]^{\le\omega}$ such that $F_{\gamma K}\subseteq E$, where $F_{\gamma K}=\{x:x\restriction K=x_\gamma\restriction K\}$. But Y was chosen to meet every such set. As E is arbitrary, Y has full outer measure. **Q**

(iii) $\operatorname{mag}\mu=\mathfrak{c}^+$. **P** In the language of 216E, we have a family $\langle G_{\{\gamma\}}\rangle_{\gamma<\mathfrak{c}^+}$ of μ-atoms of measure 1, each pair with negligible intersection, and every non-negligible measurable set meets some $G_{\{\gamma\}}$ in a non-negligible set. **Q**

(iv) Now let ν be the subspace measure on Y. By 214Ie, ν is complete, locally determined and localizable. By 322I, we can identify the measure algebras of μ and ν, so $\operatorname{mag}\nu=\operatorname{mag}\mu=\mathfrak{c}^+$, while $\#(Y)=\mathfrak{c}$.

521Q Free products We have some simple calculations associated with the measure algebra free products of §325.

Proposition (a) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras and $(\mathfrak{C},\bar\lambda)$ their localizable measure algebra free product. Then

$$c(\mathfrak{C})\le\max(\omega,c(\mathfrak{A}),c(\mathfrak{B})),$$

$$\tau(\mathfrak{C})\le\max(\omega,\tau(\mathfrak{A}),\tau(\mathfrak{B})).$$

(b) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, and $(\mathfrak{C},\bar\lambda)$ their probability algebra free product. Then

$$\tau(\mathfrak{C})\le\max(\omega,\#(I),\sup_{i\in I}\tau(\mathfrak{A}_i)).$$

proof (a)(i) Let $A\subseteq\mathfrak{A}$, $B\subseteq\mathfrak{B}$ be partitions of unity consisting of elements of finite measure (322Ea). Then $C=\{a\otimes b:a\in A,\,b\in B\}$ is a disjoint family in $\mathfrak{C}$, and

$$\sup C=\sup\{(a\otimes 1)\cap(1\otimes b):a\in A,\,b\in B\}=(\sup_{a\in A}a\otimes 1)\cap(\sup_{b\in B}1\otimes b)$$

(313Bc)

$$=(\sup A\otimes 1)\cap(1\otimes\sup B)$$

(325Da)

$$=(1\otimes 1)\cap(1\otimes 1)=1;$$

that is, C is a partition of unity. As every member of C has finite measure,

$$c(\mathfrak{C})\le\max(\omega,\#(C))=\max(\omega,\#(A),\#(B))=\max(\omega,c(\mathfrak{A}),c(\mathfrak{B}))$$

by 332E.

(ii) As for Maharam types, I am just repeating the result stated and proved in 334B.

(b) This is 334D.

521R Proposition If (X,Σ,μ) is any measure space, its Maharam type is at most $2^{\#(X)}$.

proof If $\mathfrak{A}$ is the measure algebra of μ,

$$\tau(\mathfrak{A})\le\#(\mathfrak{A})\le\#(\Sigma)\le\#(\mathcal{P}X)=2^{\#(X)}.$$

521S Proposition (a) A countably separated measure space has Maharam type at most $2^{\mathfrak{c}}$.
(b) There is a countably separated quasi-Radon probability space with Maharam type $2^{\mathfrak{c}}$.
(c) A countably separated semi-finite measure space has magnitude at most $2^{\mathfrak{c}}$.
(d) There is a countably separated semi-finite measure space with magnitude $2^{\mathfrak{c}}$.

proof Set $\kappa=2^{\mathfrak{c}}$.

(a) If (X,Σ,μ) is countably separated, there is an injective function from X to $\mathbb{R}$ (343E), so $\#(X)\le\mathfrak{c}$; now use 521R.

(b) As in (b-i) of the proof of 521P, there is a set $Y\subseteq X=\{0,1\}^\kappa$, with cardinal $\mathfrak{c}$, which meets every non-empty G_δ subset of X, and therefore has full outer measure for the usual measure ν_κ of X.

In $[0,1]$ let $\langle C_y\rangle_{y\in Y}$ be a disjoint family of sets of full outer measure for Lebesgue measure μ_1 on $[0,1]$ (419I), and set $C=\{(y,t):y\in Y,\,t\in C_y\}\subseteq Z=X\times[0,1]$. Now C has full outer measure for the product measure λ on Z. **P** Suppose that $W\subseteq Z$ and $\lambda W>0$. Then $\int\mu_1 W[\{x\}]\nu_\kappa(dx)>0$ (252D), so $\{x:\mu_1W[\{x\}]>0\}$ has non-zero measure and meets Y. Taking $y\in Y$ such that $\mu_1W[\{y\}]>0$, $\{y\}\times(C_y\cap W[\{y\}])$ is a non-empty subset of $C\cap W$. **Q**

The measure algebra $\mathfrak{A}$ of the subspace measure λ_C on C can therefore be identified with the measure algebra of λ (322Jb), and has Maharam type κ. Because $\langle C_y\rangle_{y\in Y}$ is disjoint, each horizontal section of C is a singleton and C is separated by the measurable sets $C\cap(X\times[0,q])$ for $q\in\mathbb{Q}\cap[0,1]$. Thus λ_C is countably separated.

If we give Z the product topology, then λ is a Radon measure (417T, or otherwise), so λ_C is quasi-Radon for the subspace topology (415B).

(c) As in (a), $\#(X)\le\mathfrak{c}$, so we can use 521Oa.

(d)(i) The first step is to build a measure space of magnitude $2^{\mathfrak{c}}$ and cardinal $\mathfrak{c}$, as follows. Let $h:\mathfrak{c}\to([\mathfrak{c}]^{\le\omega})^2$ be a surjection; take its two components to be h_1 and h_2. For $D\subseteq\mathfrak{c}$ set $F_D=\{\xi:\xi<\mathfrak{c},\,h_2(\xi)=D\cap h_1(\xi)\}$. For $I\in[\mathfrak{c}]^{\le\omega}$ set $A_I=\{\xi:\xi<\mathfrak{c},\,I\not\subseteq h_1(\xi)\}$, and set $\mathcal{A}=\bigcup\{\mathcal{P}A_I:I\in[\mathfrak{c}]^{\le\omega}\}$; note that $\mathcal{A}$ is a σ-ideal of subsets of $\mathfrak{c}$.

If $D\subseteq\mathfrak{c}$, $F_D\notin\mathcal{A}$. **P** If $I\in[\mathfrak{c}]^{\le\omega}$, there is a $\xi<\mathfrak{c}$ such that $h(\xi)=(I,I\cap D)$, Now $\xi\in F_D\setminus A_I$; as I is arbitrary, $F_D\notin\mathcal{A}$. **Q** So we can define a measure ν_D on $\mathfrak{c}$ by saying that

$$\begin{aligned}\nu_D(E) &= 1 \text{ if } E \subseteq \mathfrak{c} \text{ and } F_D \setminus E \in \mathcal{A},\\ &= 0 \text{ if } E \subseteq \mathfrak{c} \text{ and } F_D \cap E \in \mathcal{A},\\ &\text{undefined otherwise,}\end{aligned}$$

and $\nu_D F_D = 1$.

If D, $D' \subseteq \mathfrak{c}$ are distinct, $F_D \cap F_{D'} \in \mathcal{A}$. **P** Take $\eta \in D \triangle D'$. If $\xi \in F_D \cap F_{D'}$, then $D \cap h_1(\xi) = h_2(\xi) = D' \cap h_1(\xi)$, so $\eta \notin h_1(\xi)$ and $\xi \in A_{\{\eta\}}$. Thus $F_D \cap F_{D'} \subseteq A_{\{\eta\}} \in \mathcal{A}$. **Q**

So if we set $\nu = \sum_{D \subseteq \mathfrak{c}} \nu_D$, as defined in 234G, ν is a measure on $\mathfrak{c}$ such that $\nu F_D = 1$ and $\nu(F_D \cap F_{D'}) = 0$ for all distinct D, $D' \subseteq \mathfrak{c}$. Also ν is semi-finite, because if $\nu E > 0$ there is a $D \subseteq \mathfrak{c}$ such that $\nu_D E > 0$, in which case $\nu(E \cap F_D) = \nu_D(E \cap F_D) = 1$. So ν is a semi-finite measure on $\mathfrak{c}$ with magnitude $\#(\mathcal{P}\mathfrak{c}) = 2^{\mathfrak{c}}$. Because every ν_D is complete, so is ν (234Ha).

(ii) As in (b), let $\langle C_\xi \rangle_{\xi < \mathfrak{c}}$ be a disjoint family of subsets of $[0,1]$ all with full outer measure for Lebesgue measure μ_1. Set $Z = \mathfrak{c} \times [0,1]$ with its c.l.d. product measure $\lambda = \nu \times \mu_1$, and $C = \{(\xi, t) : \xi < \mathfrak{c},\, t \in C_\xi\} \subseteq Z$. Then C has full outer measure, by the argument of (b) above. So, as in (b), the measure algebra $\mathfrak{A}$ of the subspace measure λ_C on C can be identified with the measure algebra of λ. The map $E \mapsto C \cap (E \times [0,1])$ induces a measure-preserving homomorphism from the measure algebra of ν to $\mathfrak{A}$, so $\operatorname{mag} \lambda_C = c(\mathfrak{A})$ is at least $2^{\mathfrak{c}}$; by (c), it is exactly $2^{\mathfrak{c}}$. Also as in (b), λ_C is countably separated.

521T In §464 I looked at the L-space M of bounded additive functionals on $\mathcal{P}I$ for infinite sets I, of which $I = \mathbb{N}$ is of course by far the most important, and found a band decomposition of M as $M_\tau \oplus (M_{\mathrm{m}} \cap M_\tau^\perp) \oplus M_{\mathrm{pnm}}$, where M_τ consists of the 'completely additive' functionals (and may be identified with $\ell^1(I)$), M_{m} consists of the 'measurable' functionals (that is, those integrated by the usual measure on $\mathcal{P}I$), and $M_{\mathrm{pnm}} = M_{\mathrm{m}}^\perp$ consists of the 'purely non-measurable' functionals. Any non-negative functional $\theta \in M$ can be identified with a Radon measure μ_θ on the Stone-Čech compactification βI (464P). The purely atomic measures on I correspond to members of M_τ. Among the others, the general rule is that 'simple' measures must correspond to functionals in M_{pnm}; see 464Pa and 464Xa. The next proposition, strengthening 464Qb, shows that this rule is followed by Maharam types.

Proposition Let I be a set, and suppose that a non-zero $\theta \in (M_{\mathrm{m}} \cap M_\tau^\perp)^+$, as defined in §464, corresponds to the Radon measure μ_θ on βI. Let ν be the usual measure on $\mathcal{P}I$. Then the Maharam type of μ_θ is at least $\operatorname{cov} \mathcal{N}(\nu)$.

proof Of course I has to be infinite, since not every additive functional on $\mathcal{P}I$ is completely additive; so $\operatorname{cov} \mathcal{N}(\nu)$ is not ∞. By 464Qc, we know that

$$\{(a,b) : a,\, b \subseteq I,\, \theta a = \tfrac{1}{2}\theta I,\, \theta(a \cap b) = \tfrac{1}{4}\theta I\}$$

is conegligible for the product measure $\nu \times \nu$ on $(\mathcal{P}I)^2$. Set

$$A_0 = \{a : a \subseteq I,\, \theta a = \tfrac{1}{2},\, \{b : \theta(a \cap b) = \tfrac{1}{4}\theta I\} \text{ is } \nu\text{-conegligible}\};$$

then A_0 is ν-conegligible. Now take a set $A \subseteq A_0$ which is maximal subject to the requirement that $\theta(a \cap b) = \frac{1}{4}\theta I$ for all distinct a, $b \in A$. If a is any subset of I, then either $a \in A$, or $a \notin A_0$, or there is a $b \in A \setminus \{a\}$ such that $\theta(a \cap b) \neq \frac{1}{4}\theta I$; so $\mathcal{P}I$ is the union of

$$(\mathcal{P}I \setminus A_0) \cup \bigcup_{b \in A} \{a : \theta(a \cap b) \neq \tfrac{1}{4}\theta I\}$$

and $\operatorname{cov} \mathcal{N}(\nu) \leq 1 + \#(A)$. As $\operatorname{cov} \mathcal{N}(\nu)$ is surely infinite, it is in fact less than or equal to $\#(A)$.

Now consider the open-and-closed sets $\widehat{a} \subseteq \beta I$ for $a \in A$. If a, $b \in A$ are distinct,

$$\mu_\theta(\widehat{a} \triangle \widehat{b}) = \mu_\theta(\widehat{a \triangle b}) = \theta(a \triangle b) = \tfrac{1}{2}\theta I > 0.$$

So in the measure algebra $\mathfrak{A}$ of μ_θ, $\{\widehat{a}^\bullet : a \in A\}$ is a discrete set of size at least $\operatorname{cov} \mathcal{N}(\nu)$, and the topological density of $\mathfrak{A}$ is at least $\operatorname{cov} \mathcal{N}(\nu)$ (5A4B(h-ii) again). By 521E, $\tau(\mu_\theta) = \tau(\mathfrak{A}) \geq \operatorname{cov} \mathcal{N}(\nu)$.

521X Basic exercises (a) Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra of $\mathbb{R}$, and μ the restriction of Lebesgue measure to $\mathcal{B}(\mathbb{R})$. Show that $\operatorname{add} \mu = \omega_1$. (*Hint*: if $\mathfrak{c} = \omega_1$, $[\mathbb{R}]^{\omega_1} \not\subseteq \mathcal{B}(\mathbb{R})$; if $\mathfrak{c} > \omega_1$, $[\mathbb{R}]^{\omega_1} \cap \mathcal{B}(\mathbb{R}) = \emptyset$; or use 423L and 423P.)

(b) Let (X, Σ, μ) be a semi-finite locally compact measure space. Show that $\operatorname{add} \mu$ is the least cardinal of any set $\mathcal{E} \subseteq \Sigma$ such that $\bigcup \mathcal{E} \notin \Sigma$, or ∞ if there is no such $\mathcal{E}$. (*Hint*: 451Q.)

(c) Let (X,Σ,μ) be a complete locally determined measure space, and κ a cardinal such that $\kappa < \operatorname{cov}\mathcal{N}(\mu_E)$ for every non-negligible measurable set $E \subseteq X$, writing μ_E for the subspace measure. Suppose that $A \subseteq X$ is such that both A and $X \setminus A$ are expressible as the union of at most κ members of Σ. Show that $A \in \Sigma$.

>(d)(i) Find a probability space (X,Σ,μ), with measure algebra $\mathfrak{A}$, such that $\pi(\mathfrak{A}) < \pi(\mu)$. (ii) Find a probability space (X,Σ,μ), with null ideal $\mathcal{N}(\mu)$, such that $\operatorname{cf}\mathcal{N}(\mu) < \pi(\mu)$. (iii) Find a probability space (X,Σ,μ) such that $\pi(\mu) < \operatorname{cf}\mathcal{N}(\mu)$. (*Hint*: 513X(q-iii).)

(e) Let (X,Σ,μ) be a measure space and ν an indefinite-integral measure over μ. Show that $\operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}(\mu)$, $\operatorname{cf}\mathcal{N}(\nu) \le \operatorname{cf}\mathcal{N}(\mu)$, $\operatorname{non}\mathcal{N}(\nu) \ge \operatorname{non}\mathcal{N}(\mu)$, $\operatorname{cov}\mathcal{N}(\nu) \le \operatorname{cov}\mathcal{N}(\mu)$, $\operatorname{shr}\mathcal{N}(\nu) \le \operatorname{shr}\mathcal{N}(\mu)$, $\operatorname{shr}^+\mathcal{N}(\nu) \le \operatorname{shr}^+\mathcal{N}(\mu)$, $\pi(\nu) \le \pi(\mu)$, $\tau(\nu) \le \tau(\mu)$.

(f) Let (X,Σ,μ) be a semi-finite measure space which is not purely atomic. Show that $\pi(\mu) \ge \pi(\mu_L)$, where μ_L is Lebesgue measure on $\mathbb{R}$.

(g) Let (X,Σ,μ) be an atomless measure space with locally determined negligible sets (definition: 213I). Show that $\operatorname{non}\mathcal{N}(\mu) \ge \operatorname{non}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure.

(h) Let (X,Σ,μ) and (Y,T,ν) be complete locally determined measure spaces, neither of measure 0, and $\mu\times\nu$ the c.l.d. product measure on $X \times Y$. Show that

$$\operatorname{non}\mathcal{N}(\mu\times\nu) = \max(\operatorname{non}\mathcal{N}(\mu), \operatorname{non}\mathcal{N}(\nu)),$$

$$\operatorname{cov}\mathcal{N}(\mu\times\nu) \le \min(\operatorname{cov}\mathcal{N}(\mu), \operatorname{cov}\mathcal{N}(\nu))$$

with equality if either μ or ν is strictly localizable,

$$\operatorname{add}(\mu\times\nu) = \operatorname{add}\mathcal{N}(\mu\times\nu) \le \min(\operatorname{add}\mathcal{N}(\mu), \operatorname{add}\mathcal{N}(\nu)),$$

$$\operatorname{cf}\mathcal{N}(\mu\times\nu) \ge \max(\operatorname{cf}\mathcal{N}(\mu), \operatorname{cf}\mathcal{N}(\nu)),$$

$$\operatorname{shr}\mathcal{N}(\mu\times\nu) \ge \max(\operatorname{shr}\mathcal{N}(\mu), \operatorname{shr}\mathcal{N}(\nu)),$$

$$\operatorname{shr}^+\mathcal{N}(\mu\times\nu) \ge \max(\operatorname{shr}^+\mathcal{N}(\mu), \operatorname{shr}^+\mathcal{N}(\nu)),$$

$$\pi(\mu\times\nu) \ge \max(\pi(\mu), \pi(\nu)).$$

(i) Let (X,Σ,μ) be a probability space, and $\mu^{\mathbb{N}}$ the product measure on $X^{\mathbb{N}}$. (i) Show that X has a set of full outer measure of size at most $\operatorname{non}\mathcal{N}(\mu^{\mathbb{N}})$. (ii) Show that if $\mathcal{A} \subseteq \Sigma \setminus \mathcal{N}(\mu)$ and $\#(\mathcal{A}) < \operatorname{cov}\mathcal{N}(\mu^{\mathbb{N}})$, then there is a countable set which meets every member of $\mathcal{A}$.

(j) Show that the direct sum of $\mathfrak{c}$ or fewer countably separated measure spaces is countably separated.

(k) Show that $2^{\mathfrak{c}} < 2^{\mathfrak{c}^+}$ iff every countably separated complete locally determined localizable measure space is strictly localizable. (*Hint*: 521P, 521S, 252Yp.)

(l) Show that if (X,Σ,μ) is a purely atomic countably separated semi-finite measure space then its magnitude is at most $\max(\omega, \#(X))$ and its Maharam type is countable.

(m) Suppose that $2^{\kappa} \le \mathfrak{c}$ for every $\kappa < \mathfrak{c}$. Show that there is a countably separated semi-finite measure space with magnitude $2^{\mathfrak{c}}$.

521Y Further exercises (a) Find a probability space (X,Σ,μ), a set Y and a function $f : X \to Y$ such that, setting $\nu = \mu f^{-1}$, $\operatorname{add}\mathcal{N}(\mu) > \operatorname{add}\mathcal{N}(\nu)$, $\operatorname{cf}\mathcal{N}(\mu) < \operatorname{cf}\mathcal{N}(\nu)$, $\operatorname{shr}\mathcal{N}(\mu) < \operatorname{shr}\mathcal{N}(\nu)$ and $\pi(\mu) < \pi(\nu)$.

(b) Find a strictly localizable measure space (X,Σ,μ), a set Y, and a function $f : X \to Y$ such that, setting $\nu = \mu f^{-1}$, ν is semi-finite and $\tau(\mu) < \tau(\nu)$.

(c) Let (X,Σ,μ) and (Y,T,ν) be localizable measure spaces, and suppose that $\max(\operatorname{mag}(\nu), \tau(\nu)) \le \mathfrak{c}$. Show that the c.l.d. product measure on $X \times Y$ is localizable.

(d) Show that there is a probability space (X,Σ,μ) with Maharam type greater than $\#(X)$. (*Hint*: 523Ib.)

(e) Let κ be an infinite cardinal. Let us say that a measure space (X,Σ,μ) is **κ-separated** if there is a family $\mathcal{E}\subseteq\Sigma$, with cardinal at most κ, separating the points of X. (i) Show that there is a disjoint family $\mathcal{A}$ of subsets of $\{0,1\}^\kappa$, all of full outer measure for the usual measure of $\{0,1\}^\kappa$, such that $\#(\mathcal{A})=2^\kappa$. (ii) Show that every κ-separated measure space has Maharam type at most 2^{2^κ}, and that there is a κ-separated quasi-Radon probability space with Maharam type 2^{2^κ}. (iii) Show that every semi-finite κ-separated measure space has magnitude at most 2^{2^κ}, and that there is a semi-finite κ-separated measure space with magnitude greater than 2^κ. (iv) Suppose that $\mathfrak{c}\le\kappa\le\lambda$ and $2^\kappa=2^\lambda$. Show that the usual measure on $\{0,1\}^\lambda$ is κ-separated.

521 Notes and comments The cardinal functions of an ideal can be thought of as measures of the 'complexity' of that ideal. In a measure space, it is natural to suppose that a subspace measure (at least, on a measurable subspace) will be 'simpler' than the original measure; in 521F we see that the additivity and uniformity tend to rise and the covering number, cofinality, shrinking number and π-weight tend to fall. Similarly, an image measure ought to be simpler than its parent; but here, while additivity rises and cofinality and shrinking number fall, uniformity falls and covering number rises (521H). Also there is a trap if the original measure is not complete (521Ya), and π-weight is more complicated (521H(a-ii)). There is a similar problem concerning topological π-weight, which led to the concept of network weight (5A4Ai, 5A4Bc); and just as network weight matches topological weight for compact Hausdorff spaces (5A4C(a-i)), an appropriate hypothesis on our measures can make their π-weights more coherent (521H(a-ii)).

Direct sums should not be more complex than their most complex component; 521G confirms this prejudice except in respect of cofinality. Since we are looking, in effect, at the cofinality of a product of partially ordered sets, we can expect at least as many difficulties as are to be found in pcf theory (§5A2). We should like to be able to bound the complexity of a product in terms of the complexities of the factors; here there seem to be some interesting questions, and 521J and 521Xh are, I hope, only a start.

Consider the statement

$$(\dagger)\ \text{`}\operatorname{mag}\mu\le\#(X)\text{ for every localizable measure space }(X,\Sigma,\mu)\text{'.}$$

From 521P we see that the generalized continuum hypothesis implies (†), and also that there are simple models of set theory in which (†) is false (Kunen 80, VIII.4.7; Jech 03, 15.18). I do not know whether there is a natural combinatorial statement equiveridical with (†). If we amend (†) to

$$\text{`}\operatorname{mag}\mu\le\#(X)\text{ for every countably separated localizable measure space }(X,\Sigma,\mu)\text{'}$$

we find ourselves with a statement equiveridical with '$2^{\mathfrak{c}}=2^{\mathfrak{c}^+}$' (cf. 521Xk).

I give space to 'countably separated' measures because these can be identified with the topological measures on subsets of $\mathbb{R}$, and I do not think it is immediately apparent just how complicated these can be. In fact, as shown by the proofs of parts (b) and (d) in 521S, most of the phenomena which can arise in any measure space with cardinal less than or equal to $\mathfrak{c}$ can appear in countably separated measure spaces. In 521Sb I add 'quasi-Radon' to show that the very strong restrictions on countably separated Radon probability measures (522Wa) depend on their perfectness, not on their τ-additivity.

The constructions in 521Oc and 521Sd both depend on almost-disjoint families of sets. Those described here are elementary. In many models of set theory, we have much more striking results, of which 521Xm is a simple example.

Some new considerations intrude rather abruptly in 521T, but the argument here is both elementary and important, quite apart from its use in helping us to understand the classification scheme in §464.

522 Cichoń's diagram

In this section I describe some extraordinary relationships between the cardinals associated with the ideals of meager and negligible sets in the real line. I concentrate on the strikingly symmetric pattern of Cichoń's diagram (522B); most of the section is taken up with proofs of the facts encapsulated in this diagram. I include a handful of results characterizing some of the most important cardinals here (522C, 522M, 522S), notes on the Martin cardinals associated with the diagram (522T) and the Freese-Nation number of $\mathcal{P}\mathbb{N}$ (522U), and a brief discussion of cofinalities (522V).

522A Notation In this section, I will use the symbols $\mathcal{M}$ and $\mathcal{N}$ for the ideals of meager and negligible subsets of $\mathbb{R}$ respectively. Associated with these we have the eight cardinals $\operatorname{add}\mathcal{M}$, $\operatorname{cov}\mathcal{M}$, $\operatorname{non}\mathcal{M}$, $\operatorname{cf}\mathcal{M}$, $\operatorname{add}\mathcal{N}$, $\operatorname{cov}\mathcal{N}$, $\operatorname{non}\mathcal{N}$

and $\mathrm{cf}\,\mathcal{N}$. In addition we have two cardinals associated with the partially ordered set $\mathbb{N}^{\mathbb{N}}$: the **bounding number** $\mathfrak{b} = \mathrm{add}_\omega\,\mathbb{N}^{\mathbb{N}}$ (see 513H for the definition of add_ω, and 522C for alternative descriptions of $\mathfrak{b}$) and the **dominating number** $\mathfrak{d} = \mathrm{cf}\,\mathbb{N}^{\mathbb{N}}$; and finally I should include $\mathfrak{c}$ itself as an eleventh cardinal in the list to be examined here. I use the notions of Galois-Tukey connection and Tukey function, and the associated relations $\preccurlyeq_{\mathrm{GT}}$, $\equiv_{\mathrm{GT}}$ and $\preccurlyeq_{\mathrm{T}}$, as described in §§512-513.

522B Cichoń's diagram The diagram itself is the following:

$$\begin{tikzcd}
 & \mathrm{cov}\,\mathcal{N} \arrow[r, dash] \arrow[dd, dash] & \mathrm{non}\,\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \mathrm{cf}\,\mathcal{M} \arrow[r, dash] \arrow[d, dash] & \mathrm{cf}\,\mathcal{N} \arrow[r, dash] \arrow[dd, dash] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[r, dash] \arrow[d, dash] & \mathfrak{d} \arrow[d, dash] & & \\
\omega_1 \arrow[r, dash] & \mathrm{add}\,\mathcal{N} \arrow[r, dash] & \mathrm{add}\,\mathcal{M} \arrow[r, dash] & \mathrm{cov}\,\mathcal{M} \arrow[r, dash] & \mathrm{non}\,\mathcal{N} &
\end{tikzcd}$$

The cardinals here increase from bottom left to top right; that is,

$$\omega_1 \le \mathrm{add}\,\mathcal{N} \le \mathrm{add}\,\mathcal{M} \le \mathfrak{b} \le \mathfrak{d} \le \mathrm{cf}\,\mathcal{M} \le \mathrm{cf}\,\mathcal{N} \le \mathfrak{c},$$

etc. In addition, we have two equalities:

$$\mathrm{add}\,\mathcal{M} = \min(\mathfrak{b}, \mathrm{cov}\,\mathcal{M}), \quad \mathrm{cf}\,\mathcal{M} = \max(\mathfrak{d}, \mathrm{non}\,\mathcal{M}).$$

In the rest of this section I will prove all the inequalities declared here, seeking to demonstrate reasons for the remarkable symmetry of the diagram. I will make heavy use of the ideas of §512. Of course many of the elementary results can be proved directly without difficulty; but for the most interesting part of the argument (522K-522Q below) Tukey functions seem to be the right way to proceed.

I start with the easiest results. It will be helpful to have descriptions of $\mathfrak{b}$ and $\mathfrak{d}$ in terms of other partially ordered sets.

522C Lemma (i) On $\mathbb{N}^{\mathbb{N}}$ define a relation $\le^*$ by saying that $f \le^* g$ if if $\{n : f(n) > g(n)\}$ is finite. Then $\le^*$ is a pre-order on $\mathbb{N}^{\mathbb{N}}$; $\mathrm{add}(\mathbb{N}^{\mathbb{N}}, \le^*) = \mathfrak{b}$ and $\mathrm{cf}(\mathbb{N}^{\mathbb{N}}, \le^*) = \mathfrak{d}$.

(ii) On $\mathbb{N}^{\mathbb{N}}$ define a relation $\preceq$ by saying that $f \preceq g$ if either $f \le g$ or $\{n : g(n) \le f(n)\}$ is finite. Then $\preceq$ is a partial order on $\mathbb{N}^{\mathbb{N}}$, $\mathrm{add}(\mathbb{N}^{\mathbb{N}}, \preceq) = \mathfrak{b}$ and $\mathrm{cf}(\mathbb{N}^{\mathbb{N}}, \preceq) = \mathfrak{d}$.

proof (a) The checks that $\le^*$ is a pre-order and that $\preceq$ is a partial order are elementary. Write ι for the identity map from $\mathbb{N}^{\mathbb{N}}$ to itself.

(b) For $f \in \mathbb{N}^{\mathbb{N}}$ and $A \subseteq \mathbb{N}^{\mathbb{N}}$ say that $f \le' A$ if there is a $g \in A$ such that $f \le g$ (see 512F). Now $(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}})$. **P** For $g \in \mathbb{N}^{\mathbb{N}}$, set $\psi(g) = \{h : g \le^* h \le^* g\} \in [\mathbb{N}^{\mathbb{N}}]^\omega$. If $f \le^* g$, then $f \le f \vee g \in \psi(g)$, so $f \le' \psi(g)$. Thus (ι, ψ) is a Galois-Tukey connection and $(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}})$. **Q**

(c) $(\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}})$. **P** If $f, g \in \mathbb{N}^{\mathbb{N}}$ and $f \preceq g$, then $f \le^* g$; so (ι, ι) is a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}})$ to $(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}})$. **Q**

(d) If $A \subseteq \mathbb{N}^{\mathbb{N}}$ is countable, there is a $\psi(A) \in \mathbb{N}^{\mathbb{N}}$ such that $g \preceq \psi(A)$ for every $g \in A$. **P** If A is empty, this is trivial. Otherwise, let $\langle g_n \rangle_{n\in\mathbb{N}}$ be a sequence running over A, and set $\psi(A)(i) = 1 + \max_{n \le i} g_n(i)$ for every $i \in \mathbb{N}$. **Q**

It follows that $(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega})$. **P** If $A \in [\mathbb{N}^{\mathbb{N}}]^{\le\omega}$ and $f \le' A$, then there is some $g \in A$ such that $f \le g$, so that $f \preceq \psi(A)$. Thus (ι, ψ) is a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}})$ to $(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega})$. **Q**

(e) By 512D,

$$\mathrm{add}(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) = \mathrm{add}(\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}}) = \mathrm{add}(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}}),$$

$$\mathrm{cov}(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) = \mathrm{cov}(\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}}) = \mathrm{cov}(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}}).$$

But by 513Ia we have

$$\mathrm{add}(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) = \mathrm{add}_\omega(\mathbb{N}^{\mathbb{N}}) = \mathfrak{b},$$

so $\mathfrak{b} = \mathrm{add}(\mathbb{N}^{\mathbb{N}}, \le^*) = \mathrm{add}(\mathbb{N}^{\mathbb{N}}, \preceq)$. In the other direction,

$$\mathfrak{d} = \operatorname{cf} \mathbb{N}^{\mathbb{N}} = \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}) \le \max(\omega, \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}))$$

(512Gf)

$$\le \max(\omega, \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}))$$

(512Gc)

$$= \max(\omega, \mathfrak{d}) = \mathfrak{d}.$$

So

$$\mathfrak{d} = \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega}) = \operatorname{cf}(\mathbb{N}^{\mathbb{N}}, \le^*) = \operatorname{cf}(\mathbb{N}^{\mathbb{N}}, \preceq),$$

and the proof is complete.

522D Proposition $\mathfrak{b} \le \mathfrak{d}$.

proof Use 511He and 522C.

522E Proposition $\operatorname{add} \mathcal{N} \le \operatorname{cov} \mathcal{N}$, $\operatorname{add} \mathcal{M} \le \operatorname{cov} \mathcal{M}$, $\operatorname{non} \mathcal{M} \le \operatorname{cf} \mathcal{M}$ and $\operatorname{non} \mathcal{N} \le \operatorname{cf} \mathcal{N}$.

proof We need only observe that both $\mathcal{M}$ and $\mathcal{N}$ are proper ideals of $\mathcal{P}\mathbb{R}$ with union $\mathbb{R}$, and use 511Jc.

522F Proposition $\omega_1 \le \operatorname{add} \mathcal{N}$ and $\operatorname{cf} \mathcal{N} \le \mathfrak{c}$.

proof Of course $\omega_1 \le \operatorname{add} \mathcal{N}$ because $\mathcal{N}$ is a σ-ideal of sets. As for $\operatorname{cf} \mathcal{N}$, we know that the family of negligible Borel sets is cofinal with $\mathcal{N}$ (134Fb) and has at most $\mathfrak{c}$ members (4A3Fa), so $\operatorname{cf} \mathcal{N} \le \mathfrak{c}$.

522G Proposition (Rothberger 38a) $\operatorname{cov} \mathcal{N} \le \operatorname{non} \mathcal{M}$ and $\operatorname{cov} \mathcal{M} \le \operatorname{non} \mathcal{N}$.

proof The point is just that there is a comeager negligible set $E \subseteq \mathbb{R}$. **P** Enumerate $\mathbb{Q}$ as $\langle q_n \rangle_{n\in\mathbb{N}}$, and set

$$E = \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n}]q_n - 2^{-n}, q_n + 2^{-n}[. \text{ } \mathbf{Q}$$

Because $x \mapsto a+x$ and $x \mapsto a-x$ are measure-preserving homeomorphisms, $a+E$ is negligible and $a-E$ is comeager for every $a \in \mathbb{R}$. Let $A \subseteq \mathbb{R}$ be a non-meager set of size $\operatorname{non} \mathcal{M}$. Then $A \cap (a-E) \ne \emptyset$ for every $a \in \mathbb{R}$, that is, $\{x+E : x \in A\}$ covers $\mathbb{R}$; so $\operatorname{cov} \mathcal{N} \le \#(A) = \operatorname{non} \mathcal{M}$.

For the other inequality, note that $F = \mathbb{R} \setminus E$ is conegligible and meager; so the same argument shows that $\operatorname{cov} \mathcal{M} \le \operatorname{non} N$.

522H Proposition $\operatorname{add} \mathcal{M} \le \mathfrak{b}$ and $\mathfrak{d} \le \operatorname{cf} \mathcal{M}$.

proof (a) For $f \in \mathbb{N}^{\mathbb{N}}$ and $B \subseteq \mathbb{N}^{\mathbb{N}}$ say that $f \le' B$ if there is a $g \in B$ such that $f \le g$. I seek functions $\phi : \mathbb{N}^{\mathbb{N}} \to \mathcal{M}$ and $\psi : \mathcal{M} \to [\mathbb{N}^{\mathbb{N}}]^{\le\omega}$ which will form a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega})$ to $(\mathcal{M}, \subseteq, \mathcal{M})$. Start by choosing a countable base $\mathcal{U}$ for the topology of $\mathbb{R}$, not containing $\emptyset$, and enumerate it as $\langle U_k \rangle_{k\in\mathbb{N}}$. For each $k \in \mathbb{N}$ let $\langle V_{kl} \rangle_{l\in\mathbb{N}}$ be a disjoint sequence of non-empty open subsets of U_k; finally, enumerate $V_{kl} \cap \mathbb{Q}$ as $\langle x_{kli} \rangle_{i\in\mathbb{N}}$ for each k, $l \in \mathbb{N}$.

(b) Fix $f \in \mathbb{N}^{\mathbb{N}}$ for the moment. Set $E_k(f) = \{x_{kli} : l \in \mathbb{N}, i \le f(l)\} \subseteq U_k$ for each $k \in \mathbb{N}$. This is nowhere dense because if G is a non-empty open set, either $G \cap \bigcup_{l\in\mathbb{N}} V_{kl} = \emptyset$ and $G \cap E_k(f) = \emptyset$, or there is an l such that $G \cap V_{kl}$ is non-empty, in which case $G \cap V_{kl} \cap E_k(f)$ is finite and $G \setminus \overline{E_k(f)} \supseteq G \cap V_{kl} \setminus E_k(f)$ is non-empty.

Now choose $\langle k_n \rangle_{n\in\mathbb{N}}$, $\langle l_n \rangle_{n\in\mathbb{N}}$ inductively as follows. $k_0 = 0$. Given $\langle k_i \rangle_{i\le n}$, $\bigcup_{i\le n} E_{k_i}(f)$ is nowhere dense, so there is an $l_n \in \mathbb{N}$ such that $\overline{U}_{l_n} \subseteq U_n \setminus \bigcup_{i\le n} E_{k_i}(f)$. Now if $U_n \subseteq \overline{\bigcup_{i\le n} U_{l_i}}$, set $k_n = 0$; otherwise, take k_n such that $U_{k_n} \subseteq U_n \setminus \bigcup_{i\le n} U_{l_i}$, and continue. At the end of the induction, set $\phi(f) = \overline{\bigcup_{n\in\mathbb{N}} E_{k_n}}$.

The construction ensures that $\overline{U}_{l_n} \cap E_{k_m} = \emptyset$ for all m and n, so that U_{l_n} is always a non-empty open subset of $U_n \setminus \phi(f)$; accordingly $\phi(f)$ is nowhere dense. If $G \subseteq \mathbb{R}$ is a non-empty open set meeting $\phi(f)$, there is a $k \in \mathbb{N}$ such that $E_k(f) \subseteq G \cap \phi(f)$. **P** Let $n \ge 1$ be such that $U_n \subseteq G$ and $U_n \cap \phi(f) \ne \emptyset$. Then there is an $i \in \mathbb{N}$ such that $U_n \cap E_{k_i}(f) \ne \emptyset$; as $E_{k_i}(f) \cap \overline{U}_{l_j}$ is empty for every j, $U_n \not\subseteq \overline{\bigcup_{j\le n} U_{l_j}}$ and

$$E_{k_n}(f) \subseteq U_{k_n} \subseteq U_n \subseteq G,$$

so we can take $k = k_n$. **Q**

(c) In the other direction, given $M \in \mathcal{M}$, choose a sequence $\langle F_n(M)\rangle_{n\in\mathbb{N}}$ of closed nowhere dense closed sets covering M. For k, l, $n \in \mathbb{N}$ set $g_{nk}(l) = \min\{j : x_{klj} \notin F_n(M)\}$, and set $\psi(M) = \{g_{nk} : n, k \in \mathbb{N}\}$.

(d) Now (ϕ, ψ) is a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega})$ to $(\mathcal{M}, \subseteq, \mathcal{M})$. **P** Suppose that $f \in \mathbb{N}^{\mathbb{N}}$ and $M \in \mathcal{M}$ are such that $\phi(f) \subseteq M$. Because $\phi(f)$ is closed and not empty and included in $\bigcup_{n\in\mathbb{N}} F_n(M)$, Baire's theorem (3A3G or 4A2Ma) tells us that there are $n \in \mathbb{N}$ and an open set G such that $\emptyset \ne G \cap \phi(f) \subseteq F_n(M)$. By the last remark in (b), there is a $k \in \mathbb{N}$ such that $E_k(f) \subseteq G \cap \phi(f)$. But this means that, for any $l \in \mathbb{N}$, $x_{kli} \in G \cap \phi(f)$ for every $i \le f(l)$, while if $j = g_{nk}(l)$ then $x_{klj} \notin G \cap \phi(f)$. So $f(l) \le g_{nk}(l)$ for every l, and $f \le' \psi(M)$. **Q**

(e) It follows at once that

$$\operatorname{add}\mathcal{M} = \operatorname{add}(\mathcal{M}, \subseteq, \mathcal{M})$$

(512Ea)

$$\le \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega})$$

(512Db)

$$= \operatorname{add}_\omega \mathbb{N}^{\mathbb{N}}$$

(513Ia again)

$$= \mathfrak{b}.$$

On the other hand,

$$\mathfrak{d} = \operatorname{cf}\mathbb{N}^{\mathbb{N}} = \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}})$$

(512Ea)

$$= \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le', [\mathbb{N}^{\mathbb{N}}]^{\le\omega})$$

(512Gf, since $\omega_1 \le \operatorname{add}\mathcal{M} \le \mathfrak{b} \le \mathfrak{d}$)

$$\le \operatorname{cov}(\mathcal{M}, \subseteq, \mathcal{M})$$

(512Da)

$$= \operatorname{cf}\mathcal{M}$$

(512Ea again).

522I Proposition $\mathfrak{b} \le \operatorname{non}\mathcal{M}$ and $\operatorname{cov}\mathcal{M} \le \mathfrak{d}$.

proof Let $\preceq$ be the partial order on $\mathbb{N}^{\mathbb{N}}$ described in 522C(ii). Then $([0,1]\setminus\mathbb{Q}, \in, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}})$. **P** Let $\phi : [0,1]\setminus\mathbb{Q} \to \mathbb{N}^{\mathbb{N}}$ be a homeomorphism (4A2Ub). For $f \in \mathbb{N}^{\mathbb{N}}$, set $K_f = \{g : g \le f\}$; then K_f is compact, so $\phi^{-1}[K_f]$ is compact. Because $\phi^{-1}[K_f]$ is disjoint from $\mathbb{Q}$, it is nowhere dense. Set

$$\psi(f) = \bigcup\{\phi^{-1}[K_g] : g \in \mathbb{N}^{\mathbb{N}},\ \{n : g(n) \ne f(n)\} \text{ is finite}\}.$$

Because there are only countably many functions eventually equal to f, $\psi(f) \in \mathcal{M}$.

Suppose that $x \in [0,1]\setminus\mathbb{Q}$ and $f \in \mathbb{N}^{\mathbb{N}}$ are such that $\phi(x) \preceq f$. Set $g = \phi(x) \vee f$; then $g(n) = f(n)$ for all but finitely many n, and $\phi(x) \le g$, so $x \in \phi^{-1}[K_g] \subseteq \psi(f)$. This shows that (ϕ, ψ) is a Galois-Tukey connection from $([0,1]\setminus\mathbb{Q}, \in, \mathcal{M})$ to $(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}})$, so that $([0,1]\setminus\mathbb{Q}, \in, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}})$. **Q**

It follows (using 522C(ii)) that

$$\begin{aligned}\mathfrak{b} &= \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}}) \le \operatorname{add}([0,1]\setminus\mathbb{Q}, \in, \mathcal{M})\\ &= \min\{\#(A) : A \subseteq [0,1]\setminus\mathbb{Q},\ A \notin \mathcal{M}\} \le \operatorname{non}\mathcal{M},\end{aligned}$$

$$\mathfrak{d} = \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \preceq, \mathbb{N}^{\mathbb{N}}) \ge \operatorname{cov}([0,1]\setminus\mathbb{Q}, \in, \mathcal{M}) = \min\{\mathcal{A} : \mathcal{A} \subseteq \mathcal{M},\ [0,1]\setminus\mathbb{Q} \subseteq \bigcup\mathcal{A}\}.$$

But if we take $\mathcal{A} \subseteq \mathcal{M}$ such that $[0,1]\setminus\mathbb{Q} \subseteq \bigcup\mathcal{A}$ and $\#(\mathcal{A}) \le \mathfrak{d}$, and set

$$\mathcal{A}' = \{A + n : A \in \mathcal{A},\ n \in \mathbb{Z}\} \cup \{\{q\} : q \in \mathbb{Q}\},$$

then $\mathcal{A}' \subseteq \mathcal{M}$ is a cover of $\mathbb{R}$ and

$$\operatorname{cov}\mathcal{M} \le \#(\mathcal{A}') \le \max(\#(\mathcal{A}), \omega) \le \mathfrak{d}.$$

522J Theorem (see TRUSS 77 and MILLER 81) $\operatorname{add}\mathcal{M} = \min(\mathfrak{b}, \operatorname{cov}\mathcal{M})$ and $\operatorname{cf}\mathcal{M} = \max(\mathfrak{d}, \operatorname{non}\mathcal{M})$.

proof My aim this time is to prove that

$$(\mathcal{M}, \subseteq, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathbb{R}, \in, \mathcal{M})^{\perp} \ltimes (\mathbb{N}^{\mathbb{N}}, \leq', [\mathbb{N}^{\mathbb{N}}]^{\leq\omega}),$$

defining $\leq'$ as in the proof of 522H and $\ltimes$ as in 512I.

(a) Let $\langle q_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathbb{Q}$ with cofinal repetitions. For $f \in \mathbb{N}^{\mathbb{N}}$, set

$$E_f = \mathbb{R} \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} \left]q_m - 2^{-f(m)}, q_m + 2^{-f(m)}\right[,$$

so that E_f is a meager set disjoint from $\mathbb{Q}$. Observe that if $\langle H_n\rangle_{n\in\mathbb{N}}$ is any sequence of closed sets disjoint from $\mathbb{Q}$, then there is an $f \in \mathbb{N}^{\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} H_n \subseteq E_f$. **P** For each $n \in \mathbb{N}$, let $f(n)$ be such that $\left]q_n - 2^{-f(n)}, q_n + 2^{-f(n)}\right[$ does not meet $\bigcup_{m\leq n} H_m$. **Q**

For $M \in \mathcal{M}$, choose a sequence $\langle F_n(M)\rangle_{n\in\mathbb{N}}$ of nowhere dense closed sets covering M. For $x \in \mathbb{R}$, if $\mathbb{Q} \cap (\bigcup_{n\in\mathbb{N}} F_n(M) - x)$ is not empty, set $p_M(x)(n) = 0$ for every $n \in \mathbb{N}$; otherwise, take $p_M(x) = f$ for some $f \in \mathbb{N}^{\mathbb{N}}$ such that $E_f \supseteq \bigcup_{n\in\mathbb{N}} F_n(M) - x$. Now set $\phi(M) = (\bigcup_{n\in\mathbb{N}} F_n(M) + \mathbb{Q}, p_M)$. This defines $\phi : \mathcal{M} \to \mathcal{M} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{R}}$.

(b) In the other direction, define $\psi : \mathbb{R} \times [\mathbb{N}^{\mathbb{N}}]^{\leq\omega} \to \mathcal{M}$ by setting $\psi(x, B) = \bigcup_{f\in B}(x + E_f)$ for $x \in \mathbb{R}$ and $B \in [\mathbb{N}^{\mathbb{N}}]^{\leq\omega}$. Now (ϕ, ψ) is a Galois-Tukey connection from $(\mathcal{M}, \subseteq, \mathcal{M})$ to $(\mathbb{R}, \in, \mathcal{M})^{\perp} \ltimes (\mathbb{N}^{\mathbb{N}}, \leq', [\mathbb{N}^{\mathbb{N}}]^{\leq\omega})$. **P** $(\mathbb{R}, \in, \mathcal{M})^{\perp} = (\mathcal{M}, \not\ni, \mathbb{R})$, so

$$(\mathbb{R}, \in, \mathcal{M})^{\perp} \ltimes (\mathbb{N}^{\mathbb{N}}, \leq', [\mathbb{N}^{\mathbb{N}}]^{\leq\omega}) = (\mathcal{M} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{R}}, T, \mathbb{R} \times [\mathbb{N}^{\mathbb{N}}]^{\leq\omega}),$$

where $((M, p), (x, B)) \in T$ iff $x \notin M$ and $p(x) \leq g$ for some $g \in B$. Now suppose that $M \in \mathcal{M}$ and $(x, B) \in \mathbb{R}\times[\mathbb{N}^{\mathbb{N}}]^{\leq\omega}$ are such that $(\phi(M), (x, B)) \in T$. Then $x \notin \bigcup_{n\in\mathbb{N}} F_n(M) + \mathbb{Q}$, so $\mathbb{Q} \cap (\bigcup_{n\in\mathbb{N}} F_n(M) - x) = \emptyset$, while $p_M(x) \leq g$ for some $g \in B$. But this means that

$$E_g \supseteq E_{p_M(x)} \supseteq \bigcup_{n\in\mathbb{N}} F_n(M) - x \supseteq M - x, \quad M \subseteq E_g + x \subseteq \psi(x, B).$$

As M and (x, B) are arbitrary, (ϕ, ψ) is a Galois-Tukey connection, as claimed. **Q**

(c) It follows that

$$\operatorname{cf}\mathcal{M} = \operatorname{cov}(\mathcal{M}, \subseteq, \mathcal{M})$$

(512Ea)

$$\leq \operatorname{cov}((\mathbb{R}, \in, \mathcal{M})^{\perp} \ltimes (\mathbb{N}^{\mathbb{N}}, \leq', [\mathbb{N}^{\mathbb{N}}]^{\leq\omega}))$$
$$= \max(\operatorname{cov}(\mathbb{R}, \in, \mathcal{M})^{\perp}, \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \leq', [\mathbb{N}^{\mathbb{N}}]^{\leq\omega}))$$

(512Jb)

$$= \max(\operatorname{add}(\mathbb{R}, \in, \mathcal{M}), \mathfrak{d}) = \max(\operatorname{non}\mathcal{M}, \mathfrak{d})$$

by 512Ed and the calculation in part (e) of the proof of 522H. On the other hand

$$\begin{aligned}\min(\operatorname{cov}\mathcal{M}, \mathfrak{b}) &= \min(\operatorname{cov}\mathcal{M}, \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \leq', [\mathbb{N}^{\mathbb{N}}]^{\leq\omega}))\\ &= \min(\operatorname{cov}(\mathbb{R}, \in, \mathcal{M}), \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \leq', [\mathbb{N}^{\mathbb{N}}]^{\leq\omega}))\\ &= \min(\operatorname{add}(\mathbb{R}, \in, \mathcal{M})^{\perp}, \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \leq', [\mathbb{N}^{\mathbb{N}}]^{\leq\omega}))\\ &= \operatorname{add}((\mathbb{R}, \in, \mathcal{M})^{\perp} \ltimes \operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \leq', [\mathbb{N}^{\mathbb{N}}]^{\leq\omega}))\end{aligned}$$

(512Jc)

$$\leq \operatorname{add}(\mathcal{M}, \subseteq, \mathcal{M})$$

((b) above and 512Db)

$$= \operatorname{add}\mathcal{M}$$

(512Ea). Since we already know from 522E and 522H that $\operatorname{add}\mathcal{M} \leq \min(\mathfrak{b}, \operatorname{cov}\mathcal{M})$ and that $\max(\mathfrak{d}, \operatorname{non}\mathcal{M}) \leq \operatorname{cf}\mathcal{M}$, we have the result.

522K Localization The last step in proving the facts announced in 522B depends on the following construction. Let I be any set. Write $\mathcal{S}_I$ for the family of sets $S \subseteq \mathbb{N} \times I$ such that each vertical section $S[\{n\}]$ has at most 2^n members. For $f \in I^{\mathbb{N}}$ and $S \subseteq \mathbb{N} \times I$ say that $f \subseteq^* S$ if $\{n : n \in \mathbb{N},\, (n, f(n)) \notin S\}$ is finite; that is, $f \setminus S$ is finite, if we identify f with its graph. I will say that the supported relation $(I^{\mathbb{N}}, \subseteq^*, \mathcal{S}_I)$ is the ***I*-localization relation**. By far the most important case (and the only one needed in this section) is when I is countably infinite; when $I = \mathbb{N}$ I will generally write $\mathcal{S}$ rather than $\mathcal{S}_{\mathbb{N}}$.

Members of $\mathcal{S}_I$, or similar sets, are sometimes called **slaloms**. The particular formula '$\#(S[\{n\}]) \le 2^n$' is convenient for the results of this section, but it is worth knowing that any other function diverging to ∞ will give rise to equivalent partially ordered sets.

***522L Lemma** Let I be an infinite set. For any $\alpha \in \mathbb{N}^{\mathbb{N}}$ write

$$\mathcal{S}_I^{(\alpha)} = \{S : S \subseteq \mathbb{N} \times I,\ \#(S[\{n\}]) \le \alpha(n) \text{ for every } n \in \mathbb{N}\}.$$

Then $(I^{\mathbb{N}}, \subseteq^*, \mathcal{S}_I^{(\alpha)}) \equiv_{\mathrm{GT}} (I^{\mathbb{N}}, \subseteq^*, \mathcal{S}_I^{(\beta)})$ whenever α, $\beta \in \mathbb{N}^{\mathbb{N}}$ and $\lim_{n\to\infty} \alpha(n) = \lim_{n\to\infty} \beta(n) = \infty$.

proof Let $g \in \mathbb{N}^{\mathbb{N}}$ be a strictly increasing sequence such that $\beta(n) \le \alpha(i)$ whenever $n \in \mathbb{N}$ and $i \ge g(n)$, and let $h_n : I \to I^{g(n+1)\setminus g(n)}$ be a bijection for each n. Define $\phi : I^{\mathbb{N}} \to I^{\mathbb{N}}$ by setting $\phi(f)(n) = h_n^{-1}(f{\restriction}g(n+1) \setminus g(n))$ for $f \in I^{\mathbb{N}}$ and $n \in \mathbb{N}$. Define $\psi : \mathcal{S}_I^{(\beta)} \to \mathcal{P}(\mathbb{N} \times I)$ by setting $\psi(S) = \bigcup_{(n,i)\in S} h_n(i)$, identifying each $h_n(i) \in I^{g(n+1)\setminus g(n)} \subseteq (g(n+1) \setminus g(n)) \times I$ with a subset of $\mathbb{N} \times I$. Now for $g(n) \le j < g(n+1)$, $\psi(S)[\{j\}] = \{h_n(i)(j) : i \in S[\{n\}]\}$ has at most $\beta(n) \le \alpha(j)$ members, while $\psi(S)[\{j\}] = \emptyset$ for $j < g(0)$, so $\psi(S) \in \mathcal{S}_I^{(\alpha)}$ for every $S \in \mathcal{S}_I^{(\beta)}$.

If $f \in I^{\mathbb{N}}$ and $S \in \mathcal{S}_I^{(\beta)}$ and $\phi(f) \subseteq^* S$, then there is an $n_0 \in \mathbb{N}$ such that $\phi(f)(n) \in S[\{n\}]$ for every $n \ge n_0$. So

$$f{\restriction}g(n+1) \setminus g(n) = h_n(\phi(f)(n)) \subseteq \psi(S)$$

for every $n \ge n_0$, $(m, f(m)) \in \psi(S)$ for every $m \ge g(n_0)$ and $f \subseteq^* \psi(S)$. This means that (ϕ, ψ) is a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}_I^{(\alpha)})$ to $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}_I^{(\beta)})$. Similarly, $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}_I^{(\beta)}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}_I^{(\alpha)})$ and the two supported relations are equivalent.

522M Proposition Let $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ be the $\mathbb{N}$-localization relation. Then $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}) \equiv_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})$.

proof (a) Let $\langle G_{ij} \rangle_{i,j\in\mathbb{N}}$ be a stochastically independent family of open subsets of $[0,1]$ such that the Lebesgue measure μG_{ij} of G_{ij} is 2^{-i} for all i, $j \in \mathbb{N}$. For $f \in \mathbb{N}^{\mathbb{N}}$, set $\phi(f) = \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} G_{m,f(m)}$. Then $\phi(f)$ is negligible.

For each $E \in \mathcal{N}$, choose a non-empty compact self-supporting set $K_E \subseteq [0,1] \setminus E$ (416Dc). Let $\langle W_{En} \rangle_{n\in\mathbb{N}}$ enumerate a base for the relative topology on K_E not containing $\emptyset$; because K_E is self-supporting, no W_{En} is negligible. Set

$$I_{Eni} = \{j : j \in \mathbb{N},\ W_{En} \cap G_{ij} = \emptyset\}$$

for n, $i \in \mathbb{N}$. Then

$$\sum_{i=0}^{\infty} 2^{-i}\#(I_{Eni}) = \sum\{\mu G_{ij} : i,\ j \in \mathbb{N},\ G_{ij} \cap W_{En} = \emptyset\}$$

is finite, by the Borel-Cantelli lemma (273K). For each n, let $k(E,n) \in \mathbb{N}$ be such that $2^{-i}\#(I_{Eni}) \le 2^{-n-1}$ for $i \ge k(E,n)$, and set

$$\psi(E) = \bigcup_{n\in\mathbb{N}} \{(i,j) : i,\ j \in \mathbb{N},\ i \ge k(E,n),\ j \in I_{Eni}\}.$$

Then

$$\#(\{j : (i,j) \in \psi(E)\}) \le \sum_{n\in\mathbb{N}, k(E,n)\le i} \#(I_{Eni}) \le \sum_{n\in\mathbb{N}, k(E,n)\le i} 2^{-n-1}2^i \le 2^i$$

for every $i \in \mathbb{N}$, so $\psi(E) \in \mathcal{S}$.

Now (ϕ, ψ) is a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ to $(\mathcal{N}, \subseteq, \mathcal{N})$. **P** Suppose that $f \in \mathbb{N}^{\mathbb{N}}$ and $E \in \mathcal{N}$ are such that $\phi(f) \subseteq E$. Then $K_E \cap \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} G_{m,f(m)} = \emptyset$. By Baire's theorem, there is some $m \in \mathbb{N}$ such that $\bigcup_{i\ge m} G_{i,f(i)} \cap K_E$ is not dense in K_E, that is, there is an $n \in \mathbb{N}$ such that $W_{En} \cap \bigcup_{i\ge m} G_{i,f(i)} = \emptyset$ and $f(i) \in I_{Eni}$ for every $i \ge m$. But this means that $(i, f(i)) \in \psi(E)$ for every $i \ge \max(m, k(E,n))$, so that $f \subseteq^* \psi(E)$. As f and E are arbitrary, we have the result. **Q**

Thus $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})$.

(b) Let $\mathcal{H}$ be the family of finite unions of bounded open intervals in $\mathbb{R}$ with rational endpoints. Then $\mathcal{H}$ is countable. For each $n \in \mathbb{N}$, let $\langle H_{ni}\rangle_{i\in\mathbb{N}}$ be an enumeration of $\{H : H \in \mathcal{H},\, \mu H \le 4^{-n}\}$. Now for each $E \in \mathcal{N}$ there is an $f \in \mathbb{N}^{\mathbb{N}}$ such that $E \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} H_{m,f(m)}$. **P** For each $n \in \mathbb{N}$, let $\langle J_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence of open intervals with rational endpoints such that $E \subseteq \bigcup_{i\in\mathbb{N}} J_{ni}$ and $\sum_{i=0}^{\infty} \mu J_{ni} \le 2^{-n-1}$. Re-enumerating $\langle J_{ni}\rangle_{n\in\mathbb{N},i\in\mathbb{N}}$ as $\langle J_i\rangle_{i\in\mathbb{N}}$, we have a sequence of open intervals with rational endpoints such that $\sum_{i=0}^{\infty} \mu J_i \le 1$ and $E \subseteq \bigcup_{i\ge n} J_i$ for every n. Let $\langle k(n)\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence such that $k(0) = 0$ and $\sum_{i=k(n)}^{\infty} \mu J_i \le 4^{-n}$ for every $n \in \mathbb{N}$. Then $V_n = \bigcup_{k(n)\le i<k(n+1)} J_i$ belongs to $\mathcal{H}$ and has measure at most 4^{-n} for each n, so we can define $f \in \mathbb{N}^{\mathbb{N}}$ by saying that $H_{n,f(n)} = V_n$ for each n, and we shall have an appropriate function. **Q**

We can therefore find a function $\phi : \mathcal{N} \to \mathbb{N}^{\mathbb{N}}$ such that $E \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} H_{m,\phi(E)(m)}$ for every $E \in \mathcal{N}$. In the reverse direction, define

$$\psi(S) = \bigcap_{n\in\mathbb{N}} \bigcup\{H_{mi} : m \ge n,\, (m,i) \in S\}$$

for $S \in \mathcal{S}$; because

$$\sum_{(m,i)\in S} \mu H_{mi} \le \sum_{m=0}^{\infty} 2^m 4^{-m} < \infty,$$

$\psi(S) \in \mathcal{N}$.

Now (ϕ,ψ) is a Galois-Tukey connection from $(\mathcal{N},\subseteq,\mathcal{N})$ to $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$. **P** If $E \in \mathcal{N}$ and $S \in \mathcal{S}$ are such that $\phi(E) \subseteq^* S$, then

$$E \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} H_{m,\phi(E)(m)} \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n,(m,i)\in S} H_{mi} = \psi(S). \textbf{ Q}$$

So $(\mathcal{N},\subseteq,\mathcal{N}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$ and the proof is complete.

522N Lemma Let X be a topological space with a countable π-base. Then there is for each $n \in \mathbb{N}$ a countable family $\mathcal{U}_n$ of open subsets of X such that $\bigcap \mathcal{V} \ne \emptyset$ for every $\mathcal{V} \in [\mathcal{U}_n]^{\le n}$ and every dense open subset of X includes some member of $\mathcal{U}_n$.

proof Induce on n. Start by taking $\mathcal{U}$ to be a countable π-base for the topology of X which is closed under finite unions. Set $\mathcal{U}_0 = \{\emptyset\}$. For the inductive step to $n+1$, let $\langle H_i\rangle_{i\in\mathbb{N}}$ be a sequence running over $\mathcal{U}_n$, and set

$$\mathcal{J}_i = \{J : J \subseteq i,\, \bigcap_{j\in J} H_j \ne \emptyset\}$$

for $i \in \mathbb{N}$,

$$\mathcal{U}_{n+1} = \{U \cup H_i : i \in \mathbb{N},\, U \in \mathcal{U},\, U \cap \bigcap_{j\in J} H_j \ne \emptyset \text{ whenever } J \in \mathcal{J}_i\}.$$

Then $\mathcal{U}_{n+1}$ is a countable family of open sets. If $G \subseteq X$ is a dense open set, let $i \in \mathbb{N}$ be such that $H_i \subseteq G$. Then $\mathcal{J}_i$ is finite, so we can find a $U \in \mathcal{U}$ such that $U \subseteq G$ and $U \cap \bigcap_{j\in J} U_j \ne \emptyset$ for every $J \in \mathcal{J}_i$; then $U \cup H_i$ belongs to $\mathcal{U}_{n+1}$ and is included in G. If $\mathcal{V} \subseteq \mathcal{U}_{n+1}$ and $\#(\mathcal{V}) \le n+1$, then if $\mathcal{V}$ is empty we certainly have $\bigcap \mathcal{V} \ne \emptyset$. Otherwise, express $\mathcal{V}$ as $\{U_k \cup H_{i(k)} : k \le n\}$ where $U_k \cap \bigcap_{j\in J} H_j \ne \emptyset$ whenever $J \in \mathcal{J}_{i(k)}$; do this in such a way that $i(k) \le i(n)$ for every $k < n$. By the inductive hypothesis, $\bigcap_{k<n} H_{i(k)} \ne \emptyset$; if $i(k) = i(n)$ for some $k < n$, then of course $\bigcap_{k\le n} H_{i(k)} \ne \emptyset$; otherwise, $U_n \cap \bigcap_{k<n} H_{i(k)} \ne \emptyset$. In either case, $\bigcap \mathcal{V}$ is non-empty. So $\mathcal{U}_{n+1}$ has the required properties and the induction continues.

522O Proposition Let $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$ be the $\mathbb{N}$-localization relation. Then $(\mathcal{M},\subseteq,\mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$.

proof Let $\langle U_n\rangle_{n\in\mathbb{N}}$ enumerate a π-base for the topology of $\mathbb{R}$ not containing $\emptyset$. By 522N, there is for each $n \in \mathbb{N}$ a countable family $\mathcal{V}_n$ of open subsets of U_n such that $\bigcap \mathcal{V} \ne \emptyset$ for every $\mathcal{V} \in [\mathcal{V}_n]^{\le 2^n}$ and every dense open subset of U_n includes some member of $\mathcal{V}_n$. Enumerate $\mathcal{V}_n$ as $\langle U_{nm}\rangle_{m\in\mathbb{N}}$.

For each $M \in \mathcal{M}$, let $\langle F_n(M)\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of nowhere dense sets covering M, and let $\phi(M) \in \mathbb{N}^{\mathbb{N}}$ be such that $F_n(M) \cap U_{n,\phi(M)(n)} = \emptyset$ for every n. In the other direction, for $S \in \mathcal{S}$ set

$$\psi(S) = \mathbb{R} \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} (U_m \cap \bigcap_{i\in S[\{m\}]} U_{mi});$$

then because $\bigcap_{i\in S[\{m\}]} U_{mi}$ is non-empty for every n, $\bigcup_{m\ge n}(U_m \cap \bigcap_{i\in S[\{m\}]} U_{mi})$ is a dense open set for every n, and $\psi(S)$ is meager.

Now (ϕ,ψ) is a Galois-Tukey connection from $(\mathcal{M},\subseteq,\mathcal{M})$ to $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$. **P** Suppose that $M \in \mathcal{M}$ and $S \in \mathcal{S}$ are such that $\phi(M) \subseteq^* S$. Let $n \in \mathbb{N}$ be such that $\phi(M)(k) \in S[\{k\}]$ for every $k \ge n$. Then

$$F_m(M) \cap \bigcap_{i\in S[\{k\}]} U_{ki} \subseteq F_k(M) \cap U_{k,\phi(M)(k)} = \emptyset$$

whenever $k \ge \max(m,n)$, so

$$F_m(M) \subseteq \mathbb{R} \setminus \bigcup_{k \ge \max(m,n)} \bigcap_{i \in S[\{k\}]} U_{ki} \subseteq \psi(S)$$

for every m, and $M \subseteq \psi(S)$. **Q**

So we have the result.

522P Corollary $\mathcal{M} \preccurlyeq_{\mathrm{T}} \mathcal{N}$.

proof Putting 522M and 522O and 512Cb together, we see that $(\mathcal{M}, \subseteq, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})$, that is, $\mathcal{M} \preccurlyeq_{\mathrm{T}} \mathcal{N}$.

522Q Theorem (Bartoszyński 84, Raisonnier & Stern 85) $\operatorname{add}\mathcal{N} \le \operatorname{add}\mathcal{M}$ and $\operatorname{cf}\mathcal{M} \le \operatorname{cf}\mathcal{N}$.

proof 522P, 513Ee.

522R The exactness of Cichoń's diagram The list of inequalities displayed in Cichoń's diagram is complete in the following sense: it is known that all assignments of the values ω_1, ω_2 to the eleven cardinals of the diagram which are allowed by the diagram together with the two equalities $\operatorname{add}\mathcal{M} = \min(\mathfrak{b}, \operatorname{cov}\mathcal{M})$, $\operatorname{cf}\mathcal{M} = \max(\mathfrak{d}, \operatorname{non}\mathcal{M})$ are relatively consistent with the axioms of ZFC. So, for instance, it is possible to have

$$\omega_1 = \operatorname{add}\mathcal{N} = \operatorname{cov}\mathcal{N} = \operatorname{add}\mathcal{M} = \mathfrak{b} = \operatorname{non}\mathcal{M},$$

$$\operatorname{cov}\mathcal{M} = \mathfrak{d} = \operatorname{cf}\mathcal{M} = \operatorname{non}\mathcal{N} = \operatorname{cf}\mathcal{N} = \mathfrak{c} = \omega_2.$$

In §§552 and 554 below I will describe forcing constructions demonstrating a few of these combinations; for the rest, I refer you to Bartoszyński & Judah 95, §§5.2, 7.5 and 7.6. I remark also that the forcing methods so far known are not all effective beyond ω_2, so that if we allow $\mathfrak{c} = \omega_3$ then some puzzles remain.

522S The cardinal $\operatorname{cov}\mathcal{M}$ All the cardinals in Cichoń's diagram appear in many different ways in set-theoretic real analysis. But $\operatorname{add}\mathcal{N}$, the additivity of Lebesgue measure, the bounding number $\mathfrak{b}$, and $\operatorname{cov}\mathcal{M}$, the Novák number of $\mathbb{R}$, seem to be particularly important. The additivity of measure will play a large role in the next section. Here I will give two striking characterizations of $\operatorname{cov}\mathcal{M}$.

Theorem (a) $n(\mathbb{R}) = \operatorname{cov}\mathcal{M} = \mathfrak{m}_{\mathrm{countable}}$.

(b) (Bartoszyński 87) $\operatorname{cov}\mathcal{M}$ is the least cardinal of any set $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that for every $g \in \mathbb{N}^{\mathbb{N}}$ there is an $f \in A$ such that $f(n) \ne g(n)$ for every $n \in \mathbb{N}$.

proof (a) Because $\mathbb{R}$ is a Baire space, the Novák number $n(\mathbb{R})$ is equal to $\operatorname{cov}\mathcal{M}$. By 517P(d-ii) or 517P(d-iii), $n(\mathbb{R}) = \mathfrak{m}_{\mathrm{countable}}$.

(b) Let κ be the smallest cardinal of any $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that for every $g \in \mathbb{N}^{\mathbb{N}}$ there is an $f \in A$ such that $f \cap g = \emptyset$, identifying the functions f and g with their graphs in $\mathbb{N} \times \mathbb{N}$.

(i) Suppose that $A \subseteq \mathbb{N}^{\mathbb{N}}$ and that $\#(A) < \operatorname{cov}\mathcal{M}$. Set $P = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$, ordered by extension of functions. Then P is a non-empty countable partially ordered set. For each $f \in A$ set $Q_f = \{p : p \in P,\, p \cap f \ne \emptyset\}$; then Q_f is cofinal with P. Set $\mathcal{Q} = \{Q_f : f \in A\}$. Then

$$\#(\mathcal{Q}) \le \#(A) < \operatorname{cov}\mathcal{M} = \mathfrak{m}_{\mathrm{countable}} \le \mathfrak{m}^{\uparrow}(P),$$

so there is an upwards-linked $R \subseteq P$ meeting every member of $\mathcal{Q}$. Now $g_0 = \bigcup R \subseteq \mathbb{N} \times \mathbb{N}$ is a function; taking $g \in \mathbb{N}^{\mathbb{N}}$ to be any extension of g_0 to the whole of $\mathbb{N}$, $g \cap f \ne \emptyset$ for every $f \in A$.

As A is arbitrary, this shows that $\kappa \ge \operatorname{cov}\mathcal{M}$.

In particular, $\kappa \ge \omega_1$, as can also be seen by elementary arguments.

(ii) Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be any sequence of non-empty countable sets, and write F for the set of all functions f such that $\operatorname{dom} f$ is an infinite subset of $\mathbb{N}$ and $f(n) \in K_n$ for every $n \in \operatorname{dom} f$. Then if $A \in [F]^{<\kappa}$ there is a $g \in \prod_{n \in \mathbb{N}} K_n$ such that $f \cap g \ne \emptyset$ for every $f \in A$. **P** For each $n \in \mathbb{N}$, let F_n be the countably infinite set $\bigcup\{\prod_{n \in I} K_n : I \in [\mathbb{N}]^{n+1}\}$. For $f \in F$ and $n \in \mathbb{N}$ take any $n+1$-element subset of f and call it $p_f(n)$, so that $p_f(n) \in F_n$. Now each F_n is countably infinite, and

$$A' = \{p_f : f \in A\} \subseteq \prod_{n \in \mathbb{N}} F_n \cong \mathbb{N}^{\mathbb{N}}$$

has cardinal less than κ, so there is a $\phi \in \prod_{n \in \mathbb{N}} F_n$ such that $\phi \cap p_f \ne \emptyset$ for every $f \in A$.

Now choose $\langle i_k\rangle_{k\in\mathbb{N}}$ inductively so that $i_k \in \operatorname{dom}\phi(k)\setminus\{i_j : j<k\}$ for each $k\in\mathbb{N}$, and take $g\in\prod_{n\in\mathbb{N}}K_n$ such that $g(i_k)=\phi(k)(i_k)$ for every k. Then for any $f\in A$ there is a $k\in\mathbb{N}$ such that $\phi(k)=p_f(k)\subseteq f$, so that $g(i_k)=f(i_k)$ and $f\cap g\ne\emptyset$, as required. **Q**

(iii) If $A\subseteq\mathbb{N}^{\mathbb{N}}$ and $f_0\in\mathbb{N}^{\mathbb{N}}$ and $\#(A)<\kappa$, then there is a function $g\in\mathbb{N}^{\mathbb{N}}$ such that $g(n+1)\ge f_0(g(n))$ for every n and $\{n : f(g(n))\le g(n+1)\}$ is infinite for every $f\in A$. **P** For $f\in A$ set

$$f^*(0)=0,\quad \tilde f(n)=\max_{i\le n}f(i),\quad f^*(n+1)=n+\tilde f(\tilde f(f^*(n)))$$

for each n, so that $f\le\tilde f$, $\tilde f$ and f^* are non-decreasing, and f^* is unbounded. Consider $B=\{f^*\restriction\mathbb{N}\setminus n : f\in A,\ n\in\mathbb{N}\}$; then $\#(B)\le\max(\#(A),\omega)<\kappa$, so by (ii) (or otherwise) there is an $h\in\mathbb{N}^{\mathbb{N}}$ meeting every member of B. Now $h\cap f^*$ is infinite for every $f\in A$. Set

$$g(0)=1+h(0),\quad g(n+1)=1+\max_{i\le n+1}h(i)+\max_{i\le n}f_0(g(i))$$

for $n\in\mathbb{N}$, so that $h(n)<g(n)$ and $f_0(g(n))\le g(n+1)$ for every n, and g is non-decreasing.

? Suppose, if possible, that $f\in A$ is such that $\{n : f(g(n))\le g(n+1)\}$ is finite. Let $n_0\in\mathbb{N}$ be such that $f(g(n))\ge g(n+1)$ for every $n\ge n_0$. If $i\ge n_0$ then

$$\tilde f(g(i))\ge f(g(i))\ge g(i+1),$$

so if $i\ge n_0$ and $j\in\mathbb{N}$ are such that $f^*(j)\ge g(i)$, then

$$f^*(j+1)=\tilde f(\tilde f(f^*(j)))\ge\tilde f(\tilde f(g(i)))\ge\tilde f(g(i+1))\ge g(i+2)$$

because $\tilde f$ is non-decreasing. But f^* is also unbounded; taking k such that $f^*(k)\ge g(n_0)$, we have $f^*(k+i)\ge g(n_0+2i)$ for every $i\in\mathbb{N}$; because both f^* and g are non-decreasing, this means that $f^*(n)\ge g(n)$ whenever $n\ge\max(k,2k-n_0)$. But there must be such an n with $f^*(n)=h(n)<g(n)$, so this is impossible. **X**

Thus g has the required property. **Q**

(iv) Now suppose that P is a countable partially ordered set, $\mathcal{Q}$ is a family of cofinal subsets of P with $\#(\mathcal{Q})<\kappa$, and $p_0\in P$. Let $\langle p_i\rangle_{i\ge1}$ be such that $\langle p_i\rangle_{i\in\mathbb{N}}$ runs over P with cofinal repetitions. Let $f\in\mathbb{N}$ be a strictly increasing function such that whenever $n\in\mathbb{N}$ and $i<n$ then there is a $j\in f(n)\setminus n$ such that $p_i\le p_j$. For each $Q\in\mathcal{Q}$ let $f_Q\in\mathbb{N}^{\mathbb{N}}$ be a strictly increasing function such that whenever $n\in\mathbb{N}$ and $i<n$ there is a $j\in f_Q(n)\setminus n$ such that $p_i\le p_j\in Q$. By (iii), we can find a strictly increasing $g\in\mathbb{N}^{\mathbb{N}}$ such that $g(n+1)\ge f(g(n))$ for every n and $I_Q=\{n : g(n+1)\ge f_Q(g(n))\}$ is infinite for every $Q\in\mathcal{Q}$.

For each $n\in\mathbb{N}$, set $J_n=g(n+1)\setminus g(n)$, and let Φ_n be the set of functions $h:g(n)\to J_n$ such that $p_i\le p_{h(i)}$ for every $i\in J_n$; because $g(n+1)\ge f(g(n))$ this is non-empty. For $Q\in\mathcal{Q}$ and $n\in I_Q$ let $\phi_Q(n)\in\Phi_n$ be such that $p_{\phi_Q(n)(i)}\in Q$ for every $i<g(n)$; such a function exists because $g(n+1)\ge f_Q(g(n))$. Now all the Φ_n are countable (indeed finite), so (ii) tells us that there is a $\phi\in\prod_{n\in\mathbb{N}}\Phi_n$ such that $\phi\cap\phi_Q$ is non-empty for every $Q\in\mathcal{Q}$.

Define $\langle i_n\rangle_{n\in\mathbb{N}}$ by setting $i_0=0$ and $i_{n+1}=\phi(n)(i_n)$ for $n\in\mathbb{N}$; because $\operatorname{dom}\phi(n)=g(n)$ and $\phi(n)(i)<g(n+1)$ whenever $i<g(n)$, i_n is well-defined for each n. Because $\phi(n)\in\Phi_n$ for each n, $p_{i_n}\le p_{i_{n+1}}$ for each n. If $Q\in\mathcal{Q}$ there is some n such that $\phi(n)=\phi_Q(n)$, so that

$$p_{i_{n+1}}=p_{\phi(n)(i_n)}=p_{\phi_Q(n)(i_n)}\in Q.$$

But this means that $R=\{p_{i_k} : k\in\mathbb{N}\}$ is an upwards-linked (indeed, totally ordered) subset of P meeting every member of $\mathcal{Q}$ and containing p_0. As p_0 and $\mathcal{Q}$ are arbitrary, $\mathfrak{m}^{\uparrow}(P)\ge\kappa$. As P is arbitrary, $\mathfrak{m}_{\text{countable}}\ge\kappa$ and $\kappa=\mathfrak{m}_{\text{countable}}=\operatorname{cov}\mathcal{M}$, as claimed.

522T Martin numbers Following the identification of $\operatorname{cov}\mathcal{M}$ with $\mathfrak{m}_{\text{countable}}$, we can amalgamate the diagrams in 522B and 517Ob, as follows:

$$\begin{tikzcd}
& & \operatorname{cov}\mathcal{N} \arrow[r,dash] \arrow[dd,dash] & \operatorname{non}\mathcal{M} \arrow[r,dash]\arrow[d,dash] & \operatorname{cf}\mathcal{M}\arrow[r,dash]\arrow[d,dash] & \operatorname{cf}\mathcal{N}\arrow[r,dash]\arrow[dd,dash] & \mathfrak{c} \\
& & & \mathfrak{b}\arrow[r,dash]\arrow[d,dash] & \mathfrak{d}\arrow[d,dash] & & \\
& & \operatorname{add}\mathcal{N}\arrow[r,dash]\arrow[d,dash] & \operatorname{add}\mathcal{M}\arrow[r,dash]\arrow[d,dash] & \operatorname{cov}\mathcal{M}\arrow[r,dash] & \operatorname{non}\mathcal{N} & \\
& & \mathfrak{m}_{\sigma\text{-linked}}\arrow[r,dash]\arrow[d,dash] & \mathfrak{p}\arrow[d,dash] & & & \\
\omega_1\arrow[r,dash] & \mathfrak{m}\arrow[r,dash] & \mathfrak{m}_{\mathrm{K}}\arrow[r,dash] & \mathfrak{m}_{\mathrm{pc}\omega_1} & & &
\end{tikzcd}$$

proof The two new inequalities to be proved are $\mathfrak{m}_{\sigma\text{-linked}} \le \operatorname{add}\mathcal{N}$ and $\mathfrak{p} \le \operatorname{add}\mathcal{M}$.

(a) Let $\mathcal{S}^\infty$ be the 'localization poset'

$$\{p : p \subseteq \mathbb{N}\times\mathbb{N},\ \#(p[\{n\}]) \le 2^n \text{ for every } n,\ \sup_{n\in\mathbb{N}}\#(p[\{n\}]) < \infty\},$$

ordered by $\subseteq$. For $p \in \mathcal{S}^\infty$ set $\|p\| = \max_{n\in\mathbb{N}}\#(p[\{n\}])$. Then $\mathcal{S}^\infty$ is σ-linked upwards. **P** If p, $q \in \mathcal{S}^\infty$, $\|p\| \le n$, $\|q\| \le n$ and $p[\{i\}] = q[\{i\}]$ for every $i \le n$, then $p \cup q \in \mathcal{S}^\infty$. So for any $n \in \mathbb{N}$ and $\langle J_i\rangle_{i\le n} \in \prod_{i\le n}[\mathbb{N}]^{\le 2^i}$ we have an upwards-linked set

$$\{p : p \in \mathcal{S}^\infty,\ \|p\| \le n,\ p[\{i\}] = J_i \text{ for every } i \le n\};$$

as there are only countably many such families $\langle J_i\rangle_{i\le n}$, $\mathcal{S}^\infty$ is σ-linked upwards. **Q**

Accordingly $\mathfrak{m}_{\sigma\text{-linked}} \le \mathfrak{m}^\uparrow(\mathcal{S}^\infty)$. Next, $\mathfrak{m}^\uparrow(\mathcal{S}^\infty) \le \operatorname{add}(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S})$, where $(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S})$ is the $\mathbb{N}$-localization relation. **P** Suppose that $A \subseteq \mathbb{N}^\mathbb{N}$ and $\#(A) < \mathfrak{m}^\uparrow(\mathcal{S}^\infty)$. For each $f \in A$, set $Q_f = \{p : p \in \mathcal{S}^\infty,\ f \subseteq^* p\}$. If $p \in \mathcal{S}^\infty$ and $\|p\| = n$, then $p \subseteq p \cup \{(i, f(i)) : i \ge n\} \in Q_f$; so Q_f is cofinal with $\mathcal{S}^\infty$. As $\#(\{Q_f : f \in A\}) < \mathfrak{m}^\uparrow(\mathcal{S}^\infty)$, there is an upwards-directed $R \subseteq \mathcal{S}^\infty$ meeting Q_f for every $f \in A$. Set $S = \bigcup R$. For each $n \in \mathbb{N}$, $\{p[\{n\}] : p \in R\}$ is an upwards-directed family of subsets of $\mathbb{N}$, all of size at most 2^n, with union $S[\{n\}]$. So $\#(S[\{n\}]) \le 2^n$; as n is arbitrary, $S \in \mathcal{S}$. If $f \in A$, there is a $p \in R \cap Q_f$, and now $f \subseteq^* p \subseteq S$. As A is arbitrary, we have the result. **Q**

Now

$$\mathfrak{m}_{\sigma\text{-linked}} \le \mathfrak{m}^\uparrow(\mathcal{S}^\infty) \le \operatorname{add}(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S}) = \operatorname{add}(\mathcal{N}, \subseteq, \mathcal{N})$$

(522M, 512Db)

$$= \operatorname{add}\mathcal{N},$$

as required.

(b)(i) Let $\mathcal{U}$ be a countable base for the topology of $\mathbb{R}$, not containing $\emptyset$. Consider the set P of pairs (σ, F) where $\sigma \in \bigcup_{n\in\mathbb{N}}\mathcal{U}^n$ and $F \subseteq \mathbb{R}$ is nowhere dense, together with the relation $\le$ where $(\sigma, F) \le (\sigma', F')$ if σ' extends σ, $F' \supseteq F$ and $F \cap \sigma'(i) = \emptyset$ whenever $i \in \operatorname{dom}\sigma' \setminus \operatorname{dom}\sigma$. Then $\le$ is a partial order on P. **P** If $(\sigma, F) \le (\sigma', F') \le (\sigma'', F'')$ then we surely have $\sigma \subseteq \sigma' \subseteq \sigma''$ and $F \subseteq F' \subseteq F''$. If $i \in \operatorname{dom}\sigma'' \setminus \operatorname{dom}\sigma$, then either $i \in \operatorname{dom}\sigma' \setminus \operatorname{dom}\sigma$ and $\sigma''(i) = \sigma'(i)$ must be disjoint from F, or $i \in \operatorname{dom}\sigma'' \setminus \operatorname{dom}\sigma'$ and $\sigma''(i)$ must be disjoint from $F' \supseteq F$. Thus in either case $F \cap \sigma''(i) = \emptyset$; as i is arbitrary, $(\sigma, F) \le (\sigma'', F'')$. Thus $\le$ is transitive. Evidently it is also reflexive and anti-symmetric, so it is a partial order. **Q**

(ii) $(P, \le)$ is σ-centered upwards. **P** If $(\sigma, F_0), \dots, (\sigma, F_k)$ are members of P with a common first member, then they have a common upper bound $(\sigma, \bigcup_{i\le k} F_i)$ in P. So for any $n \in \mathbb{N}$ and $\sigma \in \mathcal{U}^n$ the set $\{(\sigma, F) : F \subseteq \mathbb{R}$ is nowhere dense$\}$ is upwards-centered in P; as $\bigcup_{n\in\mathbb{N}}\mathcal{U}^n$ is countable, P is σ-centered upwards. **Q**

(iii) For each $V \in \mathcal{U}$ and $n \in \mathbb{N}$ set

$$Q_{nV} = \{(\sigma, F) : (\sigma, F) \in P,\ V \cap \textstyle\bigcup_{n\le i<\operatorname{dom}\sigma}\sigma(i) \ne \emptyset\}.$$

Then Q_{nV} is cofinal with P. **P** If $(\sigma, F) \in P$, set $m = \max(n, \operatorname{dom}\sigma) + 1$, and take $U \in \mathcal{U}$ such that $U \subseteq V \setminus F$. Setting

$$\sigma'(i) = \sigma(i) \text{ for } i < \operatorname{dom}\sigma,$$
$$= U \text{ for } \operatorname{dom}\sigma \le i < m,$$

we find that $(\sigma, F) \le (\sigma', F) \in Q_{nV}$. **Q**

For each nowhere dense set $H \subseteq \mathbb{R}$,

$$Q'_H = \{(\sigma, F) : (\sigma, F) \in P,\ H \subseteq F\}$$

is cofinal with P. **P** For any $(\sigma, F) \in P$, we have $(\sigma, F) \le (\sigma, F \cup H) \in Q'_H$. **Q**

(iv) Now suppose that $\mathcal{A} \subseteq \mathcal{M}$ and $\#(\mathcal{A}) < \mathfrak{p}$. Then each member of $\mathcal{A}$ is covered by a sequence of nowhere dense sets, so there is a family $\mathcal{H}$ of nowhere dense sets with the same union as $\mathcal{A}$ and with $\#(\mathcal{H}) \le \max(\omega, \#(\mathcal{A}))$. In this case

$$\mathcal{Q} = \{Q_{nV} : n \in \mathbb{N},\ V \in \mathcal{U}\} \cup \{Q'_H : H \in \mathcal{H}\}$$

is a family of cofinal subsets of P and

$$\#(\mathcal{Q}) \le \max(\omega, \#(\mathcal{A})) < \mathfrak{p} \le \mathfrak{m}^{\uparrow}(P).$$

There is therefore an upwards-directed $R \subseteq P$ meeting every member of $\mathcal{Q}$. If (σ, F) and (σ', F') belong to R, they must be upwards-compatible in P, and in particular σ and σ' have a common extension; we therefore have a function $\phi = \bigcup_{(\sigma,F)\in R} \sigma$ from a subset of $\mathbb{N}$ to $\mathcal{U}$. If $n \in \mathbb{N}$ and $V \in \mathcal{U}$, then there is a $(\sigma, F) \in R \cap Q_{nV}$, so that there is some $i \ge n$ such that $\phi(i) = \sigma(i)$ meets V. As V is arbitrary, the open set $W_n = \bigcup_{i \in \operatorname{dom}\phi, i \ge n} \phi(i)$ is dense; as n is arbitrary, $M = \mathbb{R} \setminus \bigcap_{n\in\mathbb{N}} W_n$ is meager. Now $H \subseteq M$ for every $H \in \mathcal{H}$. **P** There is a $(\sigma, F) \in R \cap Q'_H$. Set $n = \operatorname{dom}\sigma$. If $i \in \operatorname{dom}\phi \setminus n$, there is a $(\sigma', F') \in R$ such that $i \in \operatorname{dom}\sigma'$; because R is upwards-directed, we may suppose that $(\sigma, F) \le (\sigma', F')$. But in this case $\phi(i) = \sigma'(i)$ must be disjoint from F and therefore from H. As i is arbitrary, $H \cap W_n = \emptyset$ and $H \subseteq M$. **Q**

As H is arbitrary, $\bigcup \mathcal{A} = \bigcup \mathcal{H} \subseteq M$. As $\mathcal{A}$ is arbitrary, $\operatorname{add}\mathcal{M} \ge \mathfrak{p}$, as claimed.

Remark In fact $\mathfrak{m}^{\uparrow}(\mathcal{S}^{\infty})$ is exactly equal to $\operatorname{add}\mathcal{N}$; see 528N.

***522U FN($\mathcal{P}\mathbb{N}$)** For any cardinal which is known to lie between ω_1 and $\mathfrak{c}$, it is natural, and often profitable, to try to locate it on Cichoń's diagram. For the Freese-Nation number of $\mathcal{P}\mathbb{N}$, which appeared more than once in §518, we have the following results.

Proposition (Fuchino Koppelberg & Shelah 96, Fuchino Geschke & Soukup 01) (a) $\mathrm{FN}(\mathcal{P}\mathbb{N}) \ge \mathfrak{b}$.

(b) $\mathrm{FN}(\mathcal{P}\mathbb{N}) \ge \operatorname{cov}\mathcal{N}$.

(c) If $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ then $\operatorname{shr}\mathcal{M} = \omega_1$, so

$$\mathfrak{m} = \mathfrak{m}_{\mathrm{K}} = \mathfrak{m}_{\mathrm{pc}\omega_1} = \mathfrak{m}_{\sigma\text{-linked}} = \mathfrak{p} = \operatorname{add}\mathcal{N} = \operatorname{add}\mathcal{M} = \mathfrak{b} = \operatorname{cov}\mathcal{N} = \operatorname{non}\mathcal{M} = \omega_1.$$

(d) If $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ and $\kappa \ge \mathfrak{m}_{\mathrm{countable}}$ is such that $\operatorname{cf}[\kappa]^{\le\omega} \le \kappa \le \mathfrak{c}$, then $\kappa = \mathfrak{c}$. So if $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ and $\mathfrak{m}_{\mathrm{countable}} < \omega_\omega$, then

$$\mathfrak{m}_{\mathrm{countable}} = \operatorname{non}\mathcal{N} = \mathfrak{d} = \operatorname{cf}\mathcal{M} = \operatorname{cf}\mathcal{N} = \mathfrak{c}.$$

(e) There is a set $A \subseteq \mathbb{R}$ with cardinal $\mathfrak{m}_{\mathrm{countable}}$ such that every meager set meets A in a set with cardinal less than $\mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

proof (a) Let $\le^*$ and $\preccurlyeq$ be the pre-order and partial order on $\mathbb{N}^{\mathbb{N}}$ described in 522C, so that $\mathfrak{b} = \operatorname{add}(\mathbb{N}^{\mathbb{N}}, \preccurlyeq)$. Write κ for $\mathrm{FN}(\mathcal{P}\mathbb{N})$; by 518D, $\kappa = \mathrm{FN}(\mathbb{N}^{\mathbb{N}}, \le)$ and we have a Freese-Nation function $\phi : \mathbb{N}^{\mathbb{N}} \to [\mathbb{N}^{\mathbb{N}}]^{<\kappa}$ for $\le$. For $f \in \mathbb{N}^{\mathbb{N}}$, set $\psi(f) = \bigcup\{\phi(g) : g \le^* f \le^* g\}$; then $\#(\psi(f)) \le \kappa$.

? Suppose, if possible, that $\kappa < \mathfrak{b}$. Choose a family $\langle f_\xi\rangle_{\xi\le\kappa}$ in $\mathbb{N}^{\mathbb{N}}$ inductively, as follows. Given $\langle f_\eta\rangle_{\eta<\xi}$ where $\xi \le \kappa$, $\bigcup_{\eta<\xi} \psi(f_\eta)$ has cardinal at most $\kappa < \mathfrak{b}$, so has a $\preccurlyeq$-upper bound f'_ξ; now set $f_\xi(i) = f'_\xi(i) + 1$ for every i, and continue.

Now choose $\langle h_\xi\rangle_{\xi<\kappa}$ in $\phi(f_\kappa)$ as follows. For each $\xi < \kappa$, $f_\xi \preccurlyeq f_\kappa$, so if we set $g_\xi = f_\xi \wedge f_\kappa$ then $g_\xi \le^* f_\xi \le^* g_\xi$ and $g_\xi \le f_\kappa$. There is therefore an $h_\xi \in \phi(g_\xi) \cap \phi(f_\kappa)$ such that $g_\xi \le h_\xi \le f_\kappa$. Now if $\eta < \xi < \kappa$, $h_\eta \phi(g_\eta) \subseteq\in \psi(f_\eta)$ so $h_\eta \preccurlyeq f'_\xi$. Accordingly

$$\{i : h_\xi(i) \le h_\eta(i)\} \subseteq \{i : f'_\xi(i) < h_\eta(i)\} \cup \{i : g_\xi(i) < f_\xi(i)\}$$

is finite and $h_\xi \ne h_\eta$. But this means that $\{h_\xi : \xi < \kappa\}$ has cardinal κ and $\#(\phi(f_\kappa)) = \kappa$, contrary to the choice of ϕ. **X**

Thus $\mathfrak{b} \le \kappa = \mathrm{FN}(\mathcal{P}\mathbb{N})$, as claimed. In particular, $\mathrm{FN}(\mathcal{P}\mathbb{N})$ is uncountable.

(b)(i) We need to know the following fact: if $\mathcal{E}$ is a family of non-negligible Lebesgue measurable subsets of $\mathbb{R}$, and $\#(\mathcal{E}) < \mathrm{cov}\,\mathcal{N}$, there is a countable set meeting every member of $\mathcal{E}$. **P** For each $E \in \mathcal{E}$, $\mathbb{R} \setminus (\mathbb{Q} + E)$ is negligible (439Eb), so there is an $x \in \mathbb{R} \cap \bigcap_{E\in\mathcal{E}} \mathbb{Q} + E$; now $\mathbb{Q} + x$ is countable and meets every member of $\mathcal{E}$. **Q**

(ii) Set $\kappa = \mathrm{FN}(\mathcal{P}\mathbb{N})$. If $\mathcal{C}$ is the family of closed sets in $\mathbb{R}$, then $(\mathcal{C}, \subseteq) \cong (\mathfrak{T}, \supseteq)$, so $\mathrm{FN}(\mathcal{C}) = \mathrm{FN}(\mathfrak{T}) = \kappa$ (518D). Let $f : \mathcal{C} \to [\mathcal{C}]^{<\kappa}$ be a Freese-Nation function.

(iii) ? If $\kappa < \mathrm{cov}\,\mathcal{N}$, write K for the set of infinite successor cardinals $\lambda \le \kappa$, and for $\lambda \in K$ set $D_\lambda = \{x : x \in \mathbb{R}, \#(f(\{x\})) < \lambda\}$. As $\mathbb{R} = \bigcup_{\lambda\in K} D_\lambda$, there must be some $\lambda \in K$ such that D_λ cannot be covered by κ negligible sets. Choose $\langle M_\xi\rangle_{\xi\le\lambda}$ and $\langle H_{\xi n}\rangle_{\xi<\lambda,n\in\mathbb{N}}$ inductively, as follows. $M_\xi = \emptyset$. Given that $M_\xi \subseteq \mathcal{C}$ and $\#(M_\xi) \le \kappa$, (i) tells us that there is a countable set $A_\xi \subseteq \mathbb{R}$ meeting every non-negligible member of M_ξ; let $\langle H_{\xi n}\rangle_{n\in\mathbb{N}}$ be a sequence of closed subsets of $\mathbb{R} \setminus A_\xi$ such that $\bigcup_{n\in\mathbb{N}} H_{\xi n}$ is conegligible. Now set

$$M_{\xi+1} = M_\xi \cup \{H_{\xi n} : n \in \mathbb{N}\} \cup \bigcup_{F\in M_\xi} f(F) \in [\mathcal{C}]^{\le\kappa}.$$

At non-zero limit ordinals $\xi \le \lambda$, set $M_\xi = \bigcup_{\eta<\xi} M_\eta$.

By the choice of λ, there is an $x \in D_\lambda$ which does not belong to any negligible set belonging to M_λ, nor to any of the sets $\mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} H_{\xi n}$ for $\xi < \lambda$. Now $\#(M_\lambda \cap f(\{x\})) < \lambda$; because λ is regular, there is a $\xi < \lambda$ such that $M_\lambda \cap f(\{x\}) \subseteq M_\xi$. Let $n \in \mathbb{N}$ be such that $x \in H_{\xi n}$. Then there must be an $F \in f(\{x\}) \cap f(H_{\xi n})$ such that $x \in F \subseteq H_{\xi n}$. In this case, $F \in M_{\xi+2} \subseteq M_\lambda$, so in fact $F \in M_\xi$. Because $x \in F$, F cannot be negligible, so $A_\xi \cap F \ne \emptyset$; but $H_{\xi n}$ was chosen to be disjoint from A_ξ. **X**

(iv) Thus $\kappa \ge \mathrm{cov}\,\mathcal{N}$, as claimed.

(c) Let $A \subseteq \mathbb{R}$ be a non-meager set.

(i) By 518D, $\mathrm{FN}(\mathfrak{T}) = \omega_1$, where $\mathfrak{T}$ is the topology of $\mathbb{R}$. Let $f : \mathfrak{T} \to [\mathfrak{T}]^{\le\omega}$ be a Freese-Nation function. There is a set M such that

(α) whenever $G \in M \cap \mathfrak{T}$ then $f(G) \subseteq M$;
(β) whenever $t \in M \cap \mathbb{R}$ then $\mathbb{R} \setminus \{t\} \in M$;
(γ) whenever $\mathcal{G} \subseteq M$ is a countable family of dense open subsets of $\mathbb{R}$, $M \cap A \cap \bigcap \mathcal{G}$ is non-empty;
(δ) $\#(M) \le \omega_1$.

P Build a non-decreasing family $\langle M_\xi\rangle_{\xi<\omega_1}$ of countable sets as follows. $M_0 = \emptyset$. Given that M_ξ is countable, let $M_{\xi+1}$ be a countable set including M_ξ such that

(α) whenever $G \in M_\xi \cap \mathfrak{T}$ then $f(G) \subseteq M_{\xi+1}$;
(β) whenever $t \in M_\xi \cap \mathbb{R}$ then $\mathbb{R} \setminus \{t\} \in M_{\xi+1}$;
(γ) $M_{\xi+1} \cap A \cap \bigcap\{G : G \in M_\xi$ is a dense open subset of $\mathbb{R}\}$ is not empty.

For limit ordinals $\xi > 0$, set $M_\xi = \bigcup_{\eta<\xi} M_\eta$. At the end of the construction, set $M = \bigcup_{\xi<\omega_1} M_\xi$. **Q**

(ii) If $H \subseteq \mathbb{R}$ is an open set, there is a countable family $\mathcal{G} \subseteq M \cap \mathfrak{T}$ such that $M \cap \mathbb{R} \cap \bigcap \mathcal{G} \subseteq H \subseteq \bigcap \mathcal{G}$. **P** Set $\mathcal{G} = \{G : G \in f(H) \cap M, H \subseteq G\}$; then certainly $H \subseteq \bigcap \mathcal{G}$ and $\mathcal{G}$ is countable. If $t \in M \cap \mathbb{R} \setminus H$, then $H \subseteq \mathbb{R} \setminus \{t\}$ so there is a $G \in f(H) \cap f(\mathbb{R} \setminus \{t\})$ such that $H \subseteq G \subseteq \mathbb{R} \setminus \{t\}$; since $\mathbb{R} \setminus \{t\} \in M$, $G \in M$; and $t \notin G$. As t is arbitrary, $M \cap \mathbb{R} \cap \bigcap \mathcal{G} \subseteq H$. **Q**

(iii) Now consider $B = A \cap M$. Then $\#(B) \le \omega_1$. **?** If B is meager, let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a sequence of dense open sets such that $B \cap \bigcap_{n\in\mathbb{N}} H_n = \emptyset$. For each $n \in \mathbb{N}$, let $\mathcal{G}_n$ be a countable family of dense open sets belonging to M such that $M \cap \mathbb{R} \cap \bigcap \mathcal{G}_n \subseteq H_n$ (using (ii)). Set $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$; then $\mathcal{G} \subseteq M$ is a countable family of dense open sets, so there is a $t \in M \cap A \cap \bigcap \mathcal{G}$, by condition ($\gamma$) in the specification of M. But now $t \in M \cap A \cap \bigcap \mathcal{G}_n \subseteq H_n$ for each n, so $t \in B \cap \bigcap_{n\in\mathbb{N}} H_n$, which is impossible. **X**

Thus A has a non-meager subset of size at most ω_1; as A is arbitrary, $\mathrm{shr}\,\mathcal{M} = \omega_1$.

(d)(i) Again let $\mathfrak{T}$ be the topology of $\mathbb{R}$ and $f : \mathfrak{T} \to [\mathfrak{T}]^{\le\omega}$ a Freese-Nation function. This time, we can find a set M such that

($\dagger$) for every $g \in \mathbb{N}^\mathbb{N}$ there is an $h \in M \cap \mathbb{N}^\mathbb{N}$ such that $g(n) \ne h(n)$ for every $n \in \mathbb{N}$;
(α) whenever $G \in M \cap \mathfrak{T}$ then $f(G) \subseteq M$;
(β) $M \cap [M]^{\le\omega}$ is cofinal with $[M]^{\le\omega}$;

(γ) whenever $D \in M$ is countable, then there is a double sequence $\langle G_{ij} \rangle_{i,j\in\mathbb{N}}$ belonging to M such that every G_{ij} belongs to $\mathfrak{T}$, $\langle G_{ij} \rangle_{j\in\mathbb{N}}$ is disjoint for each $i \in \mathbb{N}$ and whenever $G \in D$ is an open subset of $\mathbb{R}$ with infinite complement, there is an $i \in \mathbb{N}$ such that $G_{ij} \setminus G$ is non-empty for every $j \in \mathbb{N}$;

(δ) whenever $\langle G_{ij} \rangle_{i,j\in\mathbb{N}} \in M$ is a double sequence of open subsets of $\mathbb{R}$, and $h \in M \cap \mathbb{N}^{\mathbb{N}}$, then $\bigcup_{i\in\mathbb{N}} G_{i,h(i)} \in M$;

(ϵ) $\#(M) \le \kappa$.

P Build a non-decreasing family $\langle M_\xi \rangle_{\xi<\omega_1}$ of sets of size κ as follows. Start with a set $M_0 \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\#(M_0) = \kappa$ and for every $g \in \mathbb{N}^{\mathbb{N}}$ there is an $h \in M_0$ such that $g(n) \ne h(n)$ for every $n \in \mathbb{N}$ (using 522Sb). Given that $\#(M_\xi) = \kappa$, let $M_{\xi+1} \supseteq M_\xi$ be such that

(α) whenever $G \in M_\xi \cap \mathfrak{T}$ then $f(G) \subseteq M_{\xi+1}$;

(β) $M_{\xi+1} \cap [M_\xi]^{\le\omega}$ is cofinal with $[M_\xi]^{\le\omega}$;

(γ) whenever $D \in M_\xi$ is countable, then there is a double sequence $\langle G_{ij} \rangle_{i,j\in\mathbb{N}}$ belonging to M such that every G_{ij} is an open set, $\langle G_{ij} \rangle_{j\in\mathbb{N}}$ is disjoint for each $i \in \mathbb{N}$ and whenever $G \in D$ is an open subset of $\mathbb{R}$ with infinite complement, there is an $i \in \mathbb{N}$ such that $G_{ij} \setminus G$ is non-empty for every $j \in \mathbb{N}$;

(δ) whenever $\langle G_{ij} \rangle_{i,j\in\mathbb{N}} \in M_\xi$ is a double sequence of open subsets of $\mathbb{R}$, and $h \in M_\xi \cap \mathbb{N}^{\mathbb{N}}$, then $\bigcup_{i\in\mathbb{N}} G_{i,h(i)} \in M_{\xi+1}$;

(ϵ) $\#(M_{\xi+1}) = \kappa$.

For limit ordinals $\xi > 0$, set $M_\xi = \bigcup_{\eta<\xi} M_\eta$. At the end of the construction, set $M = \bigcup_{\xi<\omega_1} M_\xi$. Then

$$M \cap [M]^{\le\omega} = \bigcup_{\xi<\omega_1} M_{\xi+1} \cap [M_\xi]^{\le\omega}$$

is cofinal with $\bigcup_{\xi<\omega_1} [M_\xi]^{\le\omega} = [M]^{\le\omega}$, and it is easy to see that the other conditions are satisfied. **Q**

(ii) ? Now suppose, if possible, that there is a $t \in \mathbb{R}$ such that $\mathbb{R} \setminus I \notin M$ for any finite set I containing t. Set $\mathcal{G} = f(\mathbb{R} \setminus \{t\}) \cap M$. Then $\mathcal{G}$ is a countable subset of M and $\mathbb{R} \setminus G$ is infinite for every $G \in \mathcal{G}$. Let $D \in M$ be a countable set including $\mathcal{G}$. Then we have a double sequence $\langle G_{ij} \rangle_{i,j\in\mathbb{N}} \in M$ such that $\langle G_{ij} \rangle_{j\in\mathbb{N}}$ is a disjoint sequence of open sets for each $i \in \mathbb{N}$ and whenever $G \in D$ is an open subset of $\mathbb{R}$ with infinite complement, there is an $i \in \mathbb{N}$ such that $G_{ij} \setminus G$ is non-empty for every $j \in \mathbb{N}$. In particular, this last clause is true for every $G \in \mathcal{G}$. For each $i \in \mathbb{N}$ choose $g(i) \in \mathbb{N}$ such that $t \notin G_{ij}$ for any $j \ne g(i)$; let $h \in M \cap \mathbb{N}^{\mathbb{N}}$ be such that $h(i) \ne g(i)$ for every i, and set $H = \bigcup_{i\in\mathbb{N}} G_{i,h(i)} \in M$; note that $t \notin H$. Now there is a $G \in f(H) \cap f(\mathbb{R} \setminus \{t\})$ such that $H \subseteq G$ and $t \notin G$. As $f(H) \subseteq M$, $G \in M$, so $G \in \mathcal{G}$. But this means that $G_{i,h(i)} \subseteq G$ for every $i \in \mathbb{N}$; and we chose $\langle G_{ij} \rangle_{i,j\in\mathbb{N}}$ so that this could not be so. **X**

Thus $\mathcal{I} = \{I : I \in [\mathbb{R}]^{<\omega}, \mathbb{R} \setminus I \in M\}$ covers $\mathbb{R}$. As $\#(\mathcal{I}) \le \#(M) \le \kappa$, $\#(\mathbb{R}) \le \kappa$ and $\kappa = \mathfrak{c}$, as claimed.

(iii) Finally, if $\mathfrak{m}_{\text{countable}} < \omega_\omega$, then we can take $\kappa = \mathfrak{m}_{\text{countable}}$, by 5A1E(e-iv), and get $\mathfrak{m}_{\text{countable}} = \ldots = \mathfrak{c}$.

(e) Because $\mathrm{FN}(\mathfrak{T}) = \mathrm{FN}(\mathcal{P}\mathbb{N})$, 518E tells us that there is a set $A \subseteq \mathbb{R}$, with cardinal $n(\mathbb{R}) = \mathfrak{m}_{\text{countable}}$, such that $\#(A \cap F) < \mathrm{FN}^*(\mathfrak{T}) = \mathrm{FN}^*(\mathcal{P}\mathbb{N})$ for every nowhere dense set $F \subseteq \mathbb{R}$. As $\mathrm{FN}^*(\mathcal{P}\mathbb{N})$ certainly has uncountable cofinality, A meets every meager set in a set of size less than $\mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

522V Cofinalities For any cardinal associated with a mathematical structure, we can ask whether there are any limitations on what that cardinal can be. The commonest form of such limitations, when they appear, is a restriction on the possible cofinalities of the cardinal. I run through the known results concerning the cardinals of Cichoń's diagram. Most are elementary, but part (f) requires a substantial argument.

Proposition (a) $\mathrm{cf}\,\mathfrak{c} \ge \mathfrak{p}$.

(b) $\mathrm{add}\,\mathcal{N}$, $\mathrm{add}\,\mathcal{M}$ and $\mathfrak{b}$ are regular.

(c) $\mathrm{cf}(\mathrm{cf}\,\mathcal{N}) \ge \mathrm{add}\,\mathcal{N}$, $\mathrm{cf}(\mathrm{cf}\,\mathcal{M}) \ge \mathrm{add}\,\mathcal{M}$ and $\mathrm{cf}\,\mathfrak{d} \ge \mathfrak{b}$.

(d) $\mathrm{cf}(\mathrm{non}\,\mathcal{N}) \ge \mathrm{add}\,\mathcal{N}$, $\mathrm{cf}(\mathrm{non}\,\mathcal{M}) \ge \mathrm{add}\,\mathcal{M}$.

(e) If $\mathrm{cf}\,\mathcal{M} = \mathfrak{m}_{\text{countable}}$ then $\mathrm{cf}(\mathrm{cf}\,\mathcal{M}) \ge \mathrm{non}\,\mathcal{M}$; if $\mathrm{cf}\,\mathcal{N} = \mathrm{cov}\,\mathcal{N}$, then $\mathrm{cf}(\mathrm{cf}\,\mathcal{N}) \ge \mathrm{non}\,\mathcal{N}$.

(f) (Bartoszyński & Judah 89) $\mathrm{cf}(\mathfrak{m}_{\text{countable}}) \ge \mathrm{add}\,\mathcal{N}$.

proof (a) If $\omega \le \kappa < \mathfrak{p}$ then $2^\kappa = \mathfrak{c}$, by 517Rb, so $\mathrm{cf}\,\mathfrak{c} > \kappa$ by 5A1Ed.

(b) Use 513C(a-i); to see that $\mathfrak{b}$ is regular, use its characterization as the additivity of a partially ordered set in 522C(ii).

(c) Use 513C(a-ii); this time, we need to know that $\mathfrak{d}$ is the cofinality of a partially ordered set for which $\mathfrak{b}$ is the additivity.

(d)-(e) 513Cb.

(f)(i) Write $\mathcal{M}_1$ for the ideal of meager subsets of $\mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}^{\mathbb{N}}$ is given its usual topology. Let $\mathcal{S}^{(0)}$ be the family of subsets S of $\mathbb{N}\times\mathbb{N}$ such that $\#(S[\{n\}])\le 2^n$ for every n and $\lim_{n\to\infty}2^{-n}\#(S[\{n\}])=0$.

I will write $\mathtt{finint}$ and $\mathtt{disj}$ for the relations $\{(A,B):A\cap B$ is finite$\}$, $\{(A,B):A\cap B=\emptyset\}$. Following the same mild abuse of notation as in 512Aa and elsewhere, I will write $(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^{\mathbb{N}})$ and $(\mathbb{N}^{\mathbb{N}},\mathtt{disj},\mathbb{N}^{\mathbb{N}})$ for the supported relations $(\mathcal{S}^{(0)},R_1,\mathbb{N}^{\mathbb{N}})$ and $(\mathbb{N}^{\mathbb{N}},R_2,\mathbb{N}^{\mathbb{N}})$, where

$$R_1=\{(S,f):S\in\mathcal{S}^{(0)},\ f\in\mathbb{N}^{\mathbb{N}},\ \{n:(n,f(n))\in S\}\text{ is finite}\},$$

$$R_2=\{(f,g):f,\ g\in\mathbb{N}^{\mathbb{N}},\ f(n)\ne g(n)\text{ for every }n\}.$$

(ii)(α) $(\mathbb{N}^{\mathbb{N}},\in,\mathcal{M}_1)\preccurlyeq_{\mathrm{GT}}(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^{\mathbb{N}})$. **P** For $f\in\mathbb{N}^{\mathbb{N}}$, set $\phi(f)=f$ (identifying f with its graph, as usual); for $g\in\mathbb{N}^{\mathbb{N}}$, set $\psi(g)=\{h:h\in\mathbb{N}^{\mathbb{N}},\ h\cap g$ is finite$\}$. Then $\phi(f)\in\mathcal{S}^{(0)}$ for every $f\in\mathbb{N}^{\mathbb{N}}$, and $\psi(g)\in\mathcal{M}_1$ for every $g\in\mathbb{N}^{\mathbb{N}}$, because all the sets $\{h:h\cap g\subseteq n\}$ are nowhere dense. If f, $g\in\mathbb{N}^{\mathbb{N}}$ and $(\phi(f),g)\in\mathtt{finint}$, then $f\cap g$ is finite so $f\in\psi(g)$; thus (ϕ,ψ) is a Galois-Tukey connection from $(\mathbb{N}^{\mathbb{N}},\in,\mathcal{M}_1)$ to $(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^{\mathbb{N}})$. **Q**

(β) $(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^{\mathbb{N}})\preccurlyeq_{\mathrm{GT}}(\mathbb{N}^{\mathbb{N}},\mathtt{disj},\mathbb{N}^{\mathbb{N}})$. **P** Let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ such that $\#(I_n)=2^n$ for each n. For $n\in\mathbb{N}$, let $\theta_n:\mathbb{N}^{I_n}\to\mathbb{N}$ be a bijection. For $S\in\mathcal{S}^{(0)}$, choose $\phi(S)\in\mathbb{N}^{\mathbb{N}}$ such that whenever $(n,i)\in S$ then $\phi(S)\cap\theta_n^{-1}(i)\ne\emptyset$, where once again both the function $\phi(S)$ and the function $\theta_n^{-1}(i)$ are identified with their graphs; this is possible because on each set I_n there are at most 2^n functions with domain I_n that $\phi(S)$ has to meet. For $g\in\mathbb{N}^{\mathbb{N}}$, define $\psi(g)\in\mathbb{N}^{\mathbb{N}}$ by saying that $\psi(g)(n)=\theta_n(g\restriction I_n)$ for every n.

Now suppose that $S\in\mathcal{S}^{(0)}$ and $g\in\mathbb{N}^{\mathbb{N}}$ are such that $S\cap\psi(g)$ is infinite. Then there is certainly an n such that $(n,\psi(g)(n))\in S$. In this case,

$$\emptyset\ne\phi(S)\cap\theta_n^{-1}(\psi(g)(n))=\phi(S)\cap g\restriction I_n,$$

so $\phi(S)\cap g$ is non-empty. Turning this round, if $(\phi(S),g)\in\mathtt{disj}$ then $(S,\psi(g))\in\mathtt{finint}$; that is, (ϕ,ψ) is a Galois-Tukey connection from $(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^{\mathbb{N}})$ to $(\mathbb{N}^{\mathbb{N}},\mathtt{disj},\mathbb{N}^{\mathbb{N}})$. **Q**

(γ) $\operatorname{cov}(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^{\mathbb{N}})=\mathfrak{m}_{\mathrm{countable}}$. **P**

$$\mathfrak{m}_{\mathrm{countable}}=n(\mathbb{N}^{\mathbb{N}})=\operatorname{cov}\mathcal{M}_1$$

(517Pd)

$$=\operatorname{cov}(\mathbb{N}^{\mathbb{N}},\in,\mathcal{M}_1)\le\operatorname{cov}(\mathcal{S}^{(0)},\mathtt{finint},\mathbb{N}^{\mathbb{N}})$$

(512Da and (α) above)

$$\le\operatorname{cov}(\mathbb{N}^{\mathbb{N}},\mathtt{disj},\mathbb{N}^{\mathbb{N}})$$

((β) above)

$$=\mathfrak{m}_{\mathrm{countable}}$$

(522Sb). **Q**

(iii) Suppose that $\kappa<\operatorname{add}\mathcal{N}$ and that $\langle S_\xi\rangle_{\xi<\kappa}$ is any family in $\mathcal{S}^{(0)}$. Then there is an $S^*\in\mathcal{S}^{(0)}$ such that $S_\xi\setminus S^*$ is finite for every $\xi<\kappa$. **P** For $\xi<\kappa$, $n\in\mathbb{N}$ let $f_\xi(n)\in\mathbb{N}$ be such that $\#(S_\xi[\{i\}])\le 2^{i-2n}$ for every $i\ge f_\xi(n)$. Because $\kappa<\operatorname{add}\mathcal{N}\le\mathfrak{b}$, there is an $f\in\mathbb{N}^{\mathbb{N}}$ such that $\{n:f_\xi(n)>f(n)\}$ is finite for every $\xi<\kappa$ (522C(ii)); of course we may suppose that $f(0)=0$ and that f is strictly increasing and that $f(n)\ge 2n$ for every n. Set $J_n=f(n+1)\setminus f(n)$ for each n. For each $\xi<\kappa$, let m_ξ be such that $f_\xi(n)\le f(n)$ for every $n\ge m_\xi$; set $S'_\xi=\{(i,j):(i,j)\in S_\xi,\ i\ge f(m_\xi)\}$. Then $S_\xi\setminus S'_\xi$ is finite and $\#(S'_\xi[\{i\}])\le 2^{i-2n}$ whenever $i\in J_n$.

For each $n\in\mathbb{N}$, let $\mathcal{K}_n$ be the family of those sets $K\subseteq J_n\times\mathbb{N}$ such that $\#(K[\{i\}])\le 2^{i-2n}$ for every $i\in J_n$, and $\theta_n:\mathcal{K}_n\to\mathbb{N}$ a bijection; set $h_\xi(n)=\theta_n(S'_\xi\cap(J_n\times\mathbb{N}))$ for each $\xi<\kappa$.

Let $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$ be the $\mathbb{N}$-localization relation. By 522M, $\operatorname{add}(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})=\operatorname{add}\mathcal{N}$ is greater than κ, so there is an $S\in\mathcal{S}$ such that $h_\xi\subseteq^* S$ for every $\xi<\kappa$. Set $S^*=\bigcup_{(n,j)\in S}\theta_n^{-1}(j)$. For any $n\in\mathbb{N}$ and $i\in J_n$, $\#(\theta_n^{-1}(j)[\{i\}])\le 2^{i-2n}$ for every $j\in\mathbb{N}$, so that $S^*[\{i\}]=\bigcup_{(n,j)\in S}\theta_n^{-1}(j)[\{i\}]$ has cardinal at most 2^{i-n}. This means that $S^*\in\mathcal{S}^{(0)}$.

Take any $\xi<\kappa$. As $h_\xi\subseteq^* S$, there is some $m\in\mathbb{N}$ such that $(n,h_\xi(n))\in S$, that is, $(n,\theta_n(S'_\xi\cap(J_n\times\mathbb{N})))\in S$, for every $n\ge m$. But this means that $S'_\xi\cap(J_n\times\mathbb{N})\subseteq S^*$ for every $n\ge m$, so $S'_\xi\setminus S^*$ is finite; it follows at once that $S_\xi\setminus S^*$ is finite. Thus we have a suitable S^*. **Q**

(iv) ? Now suppose, if possible, that $\mathrm{cf}(\mathfrak{m}_{\text{countable}}) = \kappa < \operatorname{add}\mathcal{N}$. By (ii-$\gamma$), there is a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ of size $\mathfrak{m}_{\text{countable}}$ such that for every $S \in \mathcal{S}^{(0)}$ there is an $f \in A$ such that $S \cap f$ is finite. Express A as $\bigcup_{\xi<\kappa} A_\xi$ where $\#(A_\xi) < \mathfrak{m}_{\text{countable}}$ for every $\xi < \kappa$. Then for each $\xi < \kappa$ we can find an $S_\xi \in \mathcal{S}^{(0)}$ such that $S_\xi \cap f$ is infinite for every $f \in A_\xi$. By (iii), there is an $S^* \in \mathcal{S}^{(0)}$ such that $S_\xi \setminus S^*$ is finite for every $\xi < \kappa$. But this means that $S^* \cap f$ must be infinite for every $f \in A_\xi$ and every $\xi < \kappa$; which contradicts the choice of A. **X**

So we are forced to conclude that $\mathrm{cf}(\mathfrak{m}_{\text{countable}}) \ge \operatorname{add}\mathcal{N}$, as stated.

522W Other spaces All the theorems above refer to the specific σ-ideals $\mathcal{M}$ and $\mathcal{N}$ of subsets of $\mathbb{R}$ or the specific partially ordered set $\mathbb{N}^{\mathbb{N}}$. Of course the structures involved appear in many other guises. In particular, we have the following results.

(a)(i) Let (X, Σ, μ) be an atomless countably separated (definition: 343D) σ-finite perfect (definition: 342K) measure space of non-zero measure, and $\mathcal{N}(\mu)$ the null ideal of μ. Then $(X, \mathcal{N}(\mu))$ is isomorphic to $(\mathbb{R}, \mathcal{N})$; in particular, $\operatorname{add}\mathcal{N}(\mu) = \operatorname{add}\mathcal{N}$, $\operatorname{cov}\mathcal{N}(\mu) = \operatorname{cov}\mathcal{N}$, $\operatorname{non}\mathcal{N}(\mu) = \operatorname{non}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}(\mu) = \operatorname{cf}\mathcal{N}$. **P** The first thing to note is that because μ is σ-finite there is a probability measure ν on X with the same measurable sets and the same negligible sets as μ (215B(vii)); and of course ν is still atomless, countably separated and perfect. Next, the completion $\hat{\nu}$ of ν is again atomless, countably separated and perfect (212Gd, 343H(vi), 451G(c-i)) and has the same negligible sets as ν (212Eb). In the same way, starting from Lebesgue measure instead of μ, we have a complete atomless countably separated perfect probability measure λ on $\mathbb{R}$ with the same negligible sets as Lebesgue measure. But now $(X, \hat{\nu})$ and $(\mathbb{R}, \lambda)$ are isomorphic (344I), so that $(X, \mathcal{N}(\mu))$ and $(\mathbb{R}, \mathcal{N})$ are isomorphic. **Q**

(ii) The most important examples of spaces satisfying the conditions of (i) are Lebesgue measure on the unit interval and the usual measure on $\{0,1\}^{\mathbb{N}}$. But the ideas go much farther. On a Hausdorff space with a countable network (e.g., any separable metrizable space, or any analytic Hausdorff space), any topological measure is countably separated (433B). So any non-zero atomless Radon measure on such a space will have a null ideal isomorphic to $\mathcal{N}$. (The measure will be σ-finite because it is a locally finite measure on a Lindelöf space, and perfect by 416Wa.)

(iii) As we shall see in §523, there are many more measure spaces (X, μ) for which $\mathcal{N}(\mu)$ is close enough to $\mathcal{N}$ to have the same additivity and cofinality, and even uniformity and covering number match in a number of interesting cases.

(b)(i) Similarly, the structure $(\mathbb{R}, \mathcal{M})$ is duplicated in any non-empty Polish space X without isolated points, in the sense that $(X, \mathcal{B}(X), \mathcal{M}(X)) \cong (\mathbb{R}, \mathcal{B}, \mathcal{M})$, where $\mathcal{B}$ and $\mathcal{B}(X)$ are the Borel σ-algebras of $\mathbb{R}$ and X respectively, and $\mathcal{M}(X)$ is the ideal of meager subsets of X. **P** Note first that $\mathbb{N}^{\mathbb{N}}$, with its usual topology, has an uncountable nowhere dense closed set; e.g., $\{f : f(2n) = 0 \text{ for every } n\}$. Now we know that X has a dense G_δ set X_1 homeomorphic to $\mathbb{N}^{\mathbb{N}}$ (5A4Ie), and X_1 must also have an uncountable nowhere dense closed set F_1; since $X_1 \setminus F_1$ is again a non-empty Polish space without isolated points (4A2Qd), it too has a dense G_δ set X_2 homeomorphic to $\mathbb{N}^{\mathbb{N}}$, and X_2 is a dense G_δ set in X with uncountable complement. Similarly, $\mathbb{R}$ has a dense G_δ subset H which is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and has uncountable complement.

Let $\mathcal{M}(X_2)$, $\mathcal{M}(H)$ be the ideals of meager subsets of X_2 and H when they are given their subspace topologies. Because X_2 is dense, a closed subset of X is nowhere dense in X iff its intersection with X_2 is nowhere dense in X_2; accordingly $\mathcal{M}(X_2)$ is precisely $\{M \cap X_2 : M \in \mathcal{M}(X)\}$. Similarly, $\mathcal{M}(H) = \{M \cap H : M \in \mathcal{M}\}$.

Consider the complements $X \setminus X_2$, $\mathbb{R} \setminus H$. These are uncountable Borel subsets of Polish spaces. They are therefore Borel isomorphic (424G, 424Cb); let $\phi : X \setminus X_2 \to \mathbb{R} \setminus H$ be a Borel isomorphism. Next, X_2 and H are homeomorphic to $\mathbb{N}^{\mathbb{N}}$, therefore to each other; let $\psi : X_2 \to H$ be a homeomorphism. Finally, set $\theta = \psi \cup \phi$, so that $\theta : X \to \mathbb{R}$ is a Borel isomorphism. For $M \subseteq X$,

$$M \in \mathcal{M}(X) \iff M \cap X_2 \in \mathcal{M}(X)$$

(because $X \setminus X_2$ is meager)

$$\iff M \cap X_2 \in \mathcal{M}(X_2) \iff \psi[M \cap X_2] \in \mathcal{M}(H)$$
$$\iff \theta[M] \cap H \in \mathcal{M}(H) \iff \theta[M] \in \mathcal{M}.$$

So θ is an isomorphism between the structures $(X, \mathcal{B}(X), \mathcal{M}(X))$ and $(\mathbb{R}, \mathcal{B}, \mathcal{M})$. **Q**

(ii) Again, the most important special cases here are $X = [0,1]$, $X = \{0,1\}^{\mathbb{N}}$ and $X = \mathbb{N}^{\mathbb{N}}$.

522X Basic exercises >(a) Let $\mathcal{K}$ be the σ-ideal of subsets of $\mathbb{N}^{\mathbb{N}}$ generated by the compact sets. Show that $(\mathcal{K}, \subseteq)$ is Tukey equivalent to the pre-ordered sets of 522C, so that $\operatorname{add}\mathcal{K} = \mathfrak{b}$ and $\operatorname{cf}\mathcal{K} = \mathfrak{d}$.

(b) Let (X, Σ, μ) be an atomless semi-finite measure space with $\mu X > 0$. Show that $\#(X) \ge \operatorname{non}\mathcal{N}$. (*Hint*: 343Cb.)

(c) Show that $(\mathbb{R}, \in, \mathcal{M}) \equiv_{\mathrm{GT}} (\mathbb{R}, \in, \mathcal{N})^{\perp}$. (*Hint*: there is a comeager negligible set.) Use this to prove 522G.

>(d) Show that there are just 23 assignments of values to the cardinals of Cichoń's diagram which are allowed by the results in 522D-522Q and have $\mathfrak{c} = \omega_2$.

(e) Let (X, Σ, μ) be a complete locally determined space, and suppose that κ is the least cardinal of any cover of a non-negligible measurable set by negligible sets. Let $A \subseteq X$ be such that both A and $X \setminus A$ can be expressed as the union of fewer than κ measurable sets. Show that $A \in \Sigma$.

(f) Show that if $\operatorname{cov}\mathcal{N} > \omega_1$ then every $\mathbf{\Delta}^1_2$ (= PCA-&-CPCA) set in a Polish space is universally measurable. (*Hint*: 423Rb, 521Xc.)

(g) Let Z be the Stone space of the measure algebra $\mathfrak{A}$ of Lebesgue measure. Show that the Novák number $n(Z)$ of Z and the Martin number $\mathfrak{m}(\mathfrak{A})$ of $\mathfrak{A}$ are both equal to $\operatorname{cov}\mathcal{N}$.

(h) (O.Kalenda) Set $P = \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ with the product of the usual order on $\mathbb{N}$ and the partial ordering $\preceq$ of 522C(ii). Show that $\mathbb{N} \preccurlyeq_{\mathrm{T}} P \preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$, $P \not\preccurlyeq_{\mathrm{T}} \mathbb{N}$ and $\mathbb{N}^{\mathbb{N}} \not\preccurlyeq_{\mathrm{T}} P$.

522Y Further exercises (a) Show that if $\operatorname{add}\mathcal{N} = \operatorname{cf}\mathcal{N}$ then $(\mathbb{R}, \mathcal{M})$ and $(\mathbb{R}, \mathcal{N})$ are isomorphic, in the sense that there is a permutation $f : \mathbb{R} \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ is meager iff $f[A]$ is Lebesgue negligible.

(b) Show that if $\operatorname{cov}\mathcal{N} > \omega_1$ then $\operatorname{cov}\mathcal{N} \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$. (*Hint*: 525Td.)

(c) Let P and Q be partially ordered sets such that Q has no greatest member, $\sim$ an equivalence relation on P, and $\pi : P \to Q$ a surjective function such that, for p_0, $p_1 \in P$, $\pi(p_0) \le \pi(p_1)$ iff there is a $p \sim p_0$ such that $p \le p_1$. Suppose that κ is a cardinal such that no $\sim$-equivalence class has cardinal greater than κ. Show that $\operatorname{add}(Q) \le \max(\mathrm{FN}(P), \kappa)$.

(d) Suppose that $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$. Show that whenever $A \subseteq \mathbb{R}$ is non-meager there is a set $B \in [A]^{\omega_1}$ such that every uncountable subset of B is non-meager.

(e) Suppose that $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \mathfrak{p}$ and that $\kappa \ge \mathfrak{m}_{\mathrm{countable}}$ is such that $\operatorname{cf}[\kappa]^{<\mathfrak{p}} \le \kappa$. Show that $\kappa \ge \mathfrak{c}$.

(f) (S.Geschke) Show that if $\mathrm{FN}^*(\mathcal{P}\mathbb{N}) \le \mathfrak{m}_{\mathrm{countable}}$ then $\operatorname{non}\mathcal{M} \le \mathrm{FN}^*(\mathcal{P}\mathbb{N})$. (*Hint*: proof of 522Uc.)

(g) Let $\mathcal{S}^{(0)}$ be the family described in the proof of 522Vf. For any sets A, B say that $A \subseteq^* B$ if $A \setminus B$ is finite, and define $\le^*$ as in 522C. Show that $(\mathcal{S}^{(0)}, \subseteq^*, \mathcal{S}^{(0)}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \le^*, \mathbb{N}^{\mathbb{N}}) \ltimes (\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}^{(0)})$.

(h) Suppose that we have supported relations (A, R, B) and (A, S, A) such that $R \circ S \subseteq R$, that is, $(a, b) \in R$ whenever $(a, a') \in S$ and $(a', b) \in R$. Show that if $\omega \le \operatorname{cov}(A, R, B) < \infty$ then $\operatorname{cf}(\operatorname{cov}(A, R, B)) \ge \operatorname{add}(A, S, A)$.

(i) Let X be any topological space with countable π-weight and write $\mathcal{M}(X)$ for the family of meager subsets of X. Show that there is a Tukey function from $\mathcal{M}(X)$ to $\mathcal{M}$, and that if the category algebra of X is not purely atomic then $\mathcal{M}(X)$ and $\mathcal{M}$ are Tukey equivalent.

(j) Show that a cardinal κ is less than $\operatorname{non}\mathcal{M}$ iff whenever $A \subseteq \mathbb{N}^{\mathbb{N}}$ and $\#(A) \le \kappa$ then there is a $g \in \mathbb{N}^{\mathbb{N}}$ such that $\{n : f(n) = g(n)\}$ is finite for every $f \in A$. (*Hint*: Bartoszyński & Judah 95, 2.4.7.)

522 Notes and comments All the significant ideas of this section may be found in Bartoszyński & Judah 95, with a good deal more.

For many years it appeared that 'measure' and 'category' on the real line, or at least the structures $(\mathbb{R}, \mathcal{B}, \mathcal{N})$ and $(\mathbb{R}, \mathcal{B}, \mathcal{M})$ where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$, were in a symmetric duality. It was perfectly well understood that the algebras $\mathfrak{A} = \mathcal{B}/\mathcal{B} \cap \mathcal{N}$ and $\mathfrak{G} = \mathcal{B}/\mathcal{B} \cap \mathcal{M}$ – what in this book I call the 'Lebesgue measure algebra' and the 'category algebra of $\mathbb{R}$' – are very different, but their complexities seemed to be balanced, and such results as 522G encouraged us to suppose that anything provable in ZFC relating measure to category ought to respect the symmetry. It therefore came as a surprise to most of us when Bartoszyński and Raisonnier & Stern (independently, but both drawing inspiration from ideas of Shelah 84, themselves responding to a difficulty noted in Solovay 70) showed that $\operatorname{add}\mathcal{N} \le \operatorname{add}\mathcal{M}$ in all models of set theory. (It was already known that $\operatorname{add}\mathcal{N}$ could be strictly less than $\operatorname{add}\mathcal{M}$.)

The diagram in its present form emphasizes a new dual symmetry, corresponding to the duality of Galois-Tukey connections (512Ab). No doubt this also is only part of the true picture. It gives no hint, for instance, of a striking difference between $\operatorname{cov}\mathcal{M}$ and $\operatorname{cov}\mathcal{N}$. While $\operatorname{cov}\mathcal{M} = \mathfrak{m}_{\text{countable}}$ must have uncountable cofinality (522Vf), $\operatorname{cov}\mathcal{N}$ can be ω_ω (Shelah 00).

I have hardly mentioned shrinking numbers here. This is because while $\operatorname{shr}\mathcal{M}$ and $\operatorname{shr}\mathcal{N}$ can be located in Cichoń's diagram (we have $\operatorname{non}\mathcal{M} \le \operatorname{shr}\mathcal{M} \le \operatorname{cf}\mathcal{M}$ and $\operatorname{non}\mathcal{N} \le \operatorname{shr}\mathcal{N} \le \operatorname{cf}\mathcal{N}$, by 511Jc), they are not known to be connected organically with the rest of the diagram. I will return to them in a more general context in 523M. I have also not said where the π-weight of Lebesgue measure (see 511Gb) fits in; this is in fact equal to $\operatorname{cf}\mathcal{N}$, as will appear in 524P.

In 522T I give two classic 'Martin's axiom' arguments. They are typical in that the structure of the proof is to establish that there is a suitable partially ordered set for which a 'generic' upwards-directed subset will provide an object to witness the truth of some assertion. 'Generic', in this context, means 'meeting sufficiently many cofinal sets'. If there were any more definite method of finding the object sought, we would use it; these constructions are always even more ethereal than those which depend on unscrupulous use of the axiom of choice. 'Really' they are names in a suitable forcing language, since (as a rule) we can lift Martin numbers above ω_1 only by entering a universe created by forcing. But in this chapter, at least, I will try to avoid such considerations, and use arguments which are expressible in the ordinary language of ZFC, even though their non-trivial applications depend on assumptions beyond ZFC.

Of the partially ordered sets $\mathcal{S}^\infty$ and P in the proof of 522T, the former comes readily to hand as soon as we cast the problem in terms of the supported relation $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$; we need only realize that we can express members of $\mathcal{S}$ as limits of upwards-directed subsets of a subfamily in which there is some room to manoeuvre, so that we have enough cofinal sets. The latter is more interesting. It belongs to one of the standard types in that the partially ordered set is made up of pairs (σ, F) in which σ is the 'working part', from which the desired meager set

$$M = \mathbb{R} \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in R, i\ge n} \sigma(i)$$

will be constructed, and F is a 'side condition', designed to ensure that the partial order of P interacts correctly with the problem. In such cases, there is generally a not-quite-trivial step to be made in proving that the ordering is transitive ((b-i) of the proof of 522T). Note that we have two classes of cofinal set to declare in (b-iii) of the proof here; the Q_{nV} are there to ensure that M is meager, and the Q'_H to ensure that it includes every member of $\mathcal{H}$. And a final element which must appear in every proof of this kind, is the check that the partial order found is of the correct type, σ-linked in (a) and σ-centered in (b).

In 522U I suggest that it is natural to try to locate any newly defined cardinal among those displayed in Cichoń's diagram. Of course there is no presumption that it will be possible to do this tidily, or that we can expect any final structure to be low-dimensional; the picture in 522T is already neater than we are entitled to expect, and the complications in 522U (and 522Yd-522Yf) are a warning that our luck may be running out. However, we can surprisingly often find relationships like the ones between $\operatorname{FN}(\mathcal{P}\mathbb{N})$, $\mathfrak{b}$, $\operatorname{shr}\mathcal{M}$ and $\mathfrak{m}_{\text{countable}}$ here, which is one of my reasons for using this approach. It is very remarkable that under fairly weak assumptions on cardinal arithmetic (the hypothesis '$\mathfrak{m}_{\text{countable}} < \omega_\omega$' in 522Ud is much stronger than is necessary, since in 'ordinary' models of set theory we have $\operatorname{cf}[\kappa]^{\le\omega} = \kappa$ whenever $\operatorname{cf}\kappa > \omega$ – see 5A6Bc and 5A6C), the axiom '$\operatorname{FN}(\mathcal{P}\mathbb{N}) = \omega_1$' splits Cichoń's diagram neatly into two halves. For an explanation of why it was worth looking for such a split, see Fuchino Geschke & Soukup 01.

For the sake of exactness and simplicity, I have maintained rigorously the convention that $\mathcal{M}$ and $\mathcal{N}$ are the ideals of meager and negligible sets in $\mathbb{R}$ with Lebesgue measure. But from the point of view of the diagram, they are 'really' representatives of classes of ideals defined on non-empty Polish spaces without isolated points, on the one hand, and on atomless countably separated σ-finite perfect measure spaces of non-zero measure on the other (522W). The most natural expressions of the duality between the supported relations $(\mathbb{R}, \in, \mathcal{M})$ and $(\mathbb{R}, \in, \mathcal{N})$ (522G, 522Xc) depend, of course, on the fact that both structures are invariant under translation; but even this is duplicated in $\mathbb{R}^r$ and in infinite compact metrizable groups like $\{0,1\}^{\mathbb{N}}$.

At some stage I ought to mention a point concerning the language of this chapter. It is natural to think of such expressions as $\operatorname{add}\mathcal{N}$ as names for objects which exist in some ideal universe. Starting from such a position, the sentence ‘it is possible that $\operatorname{add}\mathcal{N} < \operatorname{add}\mathcal{M}$’ has to be interpreted as ‘there is a possible mathematical universe in which $\operatorname{add}\mathcal{N} < \operatorname{add}\mathcal{M}$’. But this can make sense only if ‘$\operatorname{add}\mathcal{N}$’ can refer to different objects in different universes, and has a meaning independent of any particular incarnation. I think that in fact we have to start again, and say that the expression $\operatorname{add}\mathcal{N}$ is not a name for an object, but an abbreviation of a definition. We can then speak of the interpretations of that definition in different worlds. In fact we have to go much farther back than the names for cardinals in this section. $\mathcal{P}\mathbb{N}$ and $\mathbb{R}$ also have to be considered primarily as definitions. The set $\mathbb{N}$ itself has a relatively privileged position; but even here it is perhaps safest to regard the symbol $\mathbb{N}$ as a name for a formula in the language of set theory rather than anything else. Fortunately, one can do mathematics without aiming at perfect consistency or logical purity, and I will make no attempt to disinfect my own language beyond what seems to be demanded by the ideas I am trying to express at each moment; but you should be aware that there are possibilities for confusion here, and that at some point you will need to find your own way of balancing among them. My own practice, when the path does not seem clear, is to re-read Kunen 80.

523 The measure of $\{0,1\}^I$

In §522 I tried to give an account of current knowledge concerning the most important cardinals associated with Lebesgue measure. The next step is to investigate the usual measure ν_I on $\{0,1\}^I$ for an arbitrary set I. Here I discuss the cardinals associated with these measures. Obviously they depend only on $\#(I)$, and are trivial if I is finite. I start with the basic diagram relating the cardinal functions of ν_κ and ν_λ for different cardinals κ and λ (523B). I take the opportunity to mention some simple facts about the measures ν_I (523C-523D). Then I look at additivities (523E), covering numbers (523F-523G), uniformities (523H-523L), shrinking numbers (523M) and cofinalities (523N). I end with a description of these cardinals under the generalized continuum hypothesis (523P).

523A Notation For any measure μ, write $\mathcal{N}(\mu)$ for the null ideal of μ. For any set I, I will write ν_I for the usual measure on $\{0,1\}^I$ and $\mathcal{N}_I = \mathcal{N}(\nu_I)$ for its null ideal. Recall that $(\{0,1\}^\omega, \mathcal{N}_\omega)$ is isomorphic to $(\mathbb{R}, \mathcal{N})$, where $\mathcal{N}$ is the Lebesgue null ideal (522Wa).

523B The basic diagram Suppose that κ and λ are infinite cardinals, with $\kappa \le \lambda$. Then we have the following diagram for the additivity, covering number, uniformity, shrinking number and cofinality of the ideals $\mathcal{N}_\kappa$ and $\mathcal{N}_\lambda$:

$$\begin{tikzcd}
 & \operatorname{cov}\mathcal{N}_\lambda \arrow[r, dash] \arrow[dd, dash] & \operatorname{cov}\mathcal{N}_\kappa \arrow[r, dash] \arrow[dd, dash] & \operatorname{cf}\mathcal{N}_\kappa \arrow[r, dash] \arrow[d, dash] & \operatorname{cf}\mathcal{N}_\lambda \arrow[r, dash] \arrow[d, dash] & \lambda^\omega \\
 & & & \operatorname{shr}\mathcal{N}_\kappa \arrow[r, dash] \arrow[d, dash] & \operatorname{shr}\mathcal{N}_\lambda \arrow[d, dash] & \\
\omega_1 \arrow[r, dash] & \operatorname{add}\mathcal{N}_\lambda \arrow[r, dash] & \operatorname{add}\mathcal{N}_\kappa \arrow[r, dash] & \operatorname{non}\mathcal{N}_\kappa \arrow[r, dash] & \operatorname{non}\mathcal{N}_\lambda &
\end{tikzcd}$$

(As in 522B, the cardinals here increase from bottom left to top right.)

proof For the inequalities relating two cardinals associated with the same ideal, see 511Jc; all we need to know is that $\mathcal{N}_\kappa$ and $\mathcal{N}_\lambda$ are proper ideals containing singletons. For the inequalities relating the cardinal functions of the two different ideals, use 521H; ν_κ is the image of ν_λ under the map $x \mapsto x{\restriction}\kappa : \{0,1\}^\lambda \to \{0,1\}^\kappa$, by 254Oa. Of course $\omega_1 \le \operatorname{add}\mathcal{N}_\lambda$. I leave the final inequality $\operatorname{cf}\mathcal{N}_\lambda \le \lambda^\omega$ for the moment, since this will be part of Theorem 523N below.

523C In the next few paragraphs I will say what is known about the cardinals here. It will be convenient to begin with two easy lemmas.

Lemma Let I be any set, and $\mathcal{J}$ a family of subsets of I such that every countable subset of I is included in some member of $\mathcal{J}$. Then a subset A of $\{0,1\}^I$ belongs to $\mathcal{N}_I$ iff there is some $J \in \mathcal{J}$ such that $\{x{\restriction}J : x \in A\} \in \mathcal{N}_J$.

proof For $J \subseteq I$, $x \in \{0,1\}^I$ set $\pi_J(x) = x{\restriction}J \in \{0,1\}^J$. Then ν_J is the image measure $\nu_I\pi_J^{-1}$ (254Oa), so $A \in \mathcal{N}_I$ whenever there is some $J \in \mathcal{J}$ such that $\pi_J[A] \in \mathcal{N}_J$. On the other hand, if $A \in \mathcal{N}_I$, there is a countable set $K \subseteq I$ such that $\pi_K[A] \in \mathcal{N}_K$ (254Od). Now there is a $J \in \mathcal{J}$ such that $K \subseteq J$, so that $\pi_J^{-1}[\pi_J[A]] \subseteq \pi_K^{-1}[\pi_K[A]] \in \mathcal{N}_I$ and $\pi_J[A] \in \mathcal{N}_J$.

523D Because the measures ν_I are homogeneous in a strong sense, we have the following facts which are occasionally useful.

Proposition Let κ be an infinite cardinal, and T the domain of ν_κ. For $A \subseteq \{0,1\}^\kappa$ write T_A for the subspace σ-algebra on A.

(a) If $E \subseteq \{0,1\}^\kappa$ is measurable and not negligible, then $(E, \mathrm{T}_E, \mathcal{N}_\kappa \cap \mathcal{P}E)$ is isomorphic to $(\{0,1\}^\kappa, \mathrm{T}, \mathcal{N}_\kappa)$.

(b) If $\mathcal{E} \subseteq \mathcal{N}_\kappa$ and $\#(\mathcal{E}) < \operatorname{cov}\mathcal{N}_\kappa$, then $(\nu_\kappa)_*(\bigcup\mathcal{E}) = 0$.

(c) If $A \subseteq \{0,1\}^\kappa$ is non-negligible, then there is a set $B \subseteq \{0,1\}^\kappa$, of full outer measure, such that $(A, \mathrm{T}_A, \mathcal{N}_\kappa \cap \mathcal{P}A)$ is isomorphic to $(B, \mathrm{T}_B, \mathcal{N}_\kappa \cap \mathcal{P}B)$.

(d) There is a set $A \subseteq \{0,1\}^\kappa$ with cardinal $\operatorname{non}\mathcal{N}_\kappa$ which has full outer measure.

proof (a) In fact the subspace measure on E is isomorphic to a scalar multiple of ν_I (344L).

(b) ? Otherwise, let $F \subseteq \bigcup\mathcal{E}$ be a non-negligible measurable set; then $\{F \cap E : E \in \mathcal{E}\}$ witnesses that $\operatorname{cov}(F, \mathcal{N}_\kappa \cap \mathcal{P}F) < \operatorname{cov}\mathcal{N}_\kappa$, which contradicts (a). **X**

(c) Let E be a measurable envelope of A. By (a), there is a bijection $f : E \to \{0,1\}^\kappa$ which is an isomorphism of the structures $(E, \mathrm{T}_E, \mathcal{N}_\kappa \cap \mathcal{P}E)$ and $(\{0,1\}^\kappa, \mathrm{T}, \mathcal{N}_\kappa)$. Set $B = f[A]$. Then $f \restriction A$ is an isomorphism of the structures $(A, \mathrm{T}_A, \mathcal{N}_\kappa \cap \mathcal{P}A)$ and $(B, \mathrm{T}_B, \mathcal{N}_\kappa \cap \mathcal{P}B)$. Moreover, since A meets every member of $\mathrm{T}_E \setminus \mathcal{N}_\kappa$, B meets every member of $\mathrm{T} \setminus \mathcal{N}_\kappa$, that is, B has full outer measure.

(d) Let $A_0 \subseteq \{0,1\}^\kappa$ be a non-negligible set of cardinal $\operatorname{non}\mathcal{N}_\kappa$. By (c), there is a set A of full outer measure which is isomorphic to A_0 in the sense described there; in particular, $\#(A) = \operatorname{non}\mathcal{N}_\kappa$.

523E Additivities Because the function $\kappa \mapsto \operatorname{add}\mathcal{N}_\kappa$ is non-increasing, it must stabilize, that is, there is some first κ_a such that $\operatorname{add}\mathcal{N}_\kappa = \operatorname{add}\mathcal{N}_{\kappa_a}$ for every $\kappa \geq \kappa_a$. But in fact it stabilizes almost immediately. If κ is any uncountable cardinal, then $\operatorname{add}\mathcal{N}_\kappa = \operatorname{add}\nu_\kappa = \omega_1$, by 521Jb. Thus among the additivities $\operatorname{add}\mathcal{N}_\kappa$, only $\operatorname{add}\mathcal{N}_\omega = \operatorname{add}\mathcal{N}$, the additivity of Lebesgue measure, can have any surprises for us.

523F Covering numbers Still on the left-hand side of the diagram, we again have a non-increasing function $\kappa \mapsto \operatorname{cov}\mathcal{N}_\kappa$, and a critical value κ_c after which it is constant. We can locate this value to some extent through the following simple fact. If $\theta = \operatorname{cov}\mathcal{N}_{\kappa_c} = \min\{\operatorname{cov}\mathcal{N}_\kappa : \kappa \text{ is a cardinal}\}$, then $\operatorname{cov}\mathcal{N}_\theta = \theta$. **P** Let κ be such that $\operatorname{cov}\mathcal{N}_\kappa = \theta$. For $I \subseteq \kappa$, set $\pi_I(x) = x \restriction I$ for $x \in \{0,1\}^\kappa$. Let $\mathcal{E} \subseteq \mathcal{N}_\kappa$ be a cover of $\{0,1\}^\kappa$ of cardinality θ. For each $E \in \mathcal{E}$, let $J_E \subseteq \kappa$ be a countable set such that $\pi_{J_E}[E] \in \mathcal{N}_{J_E}$. Set $I = \bigcup_{E \in \mathcal{E}} J_E$, so that $\#(I) \leq \theta$ and $\pi_I[E] \in \mathcal{N}_I$ for every $E \in \mathcal{E}$. Then $\{\pi_I[E] : E \in \mathcal{E}\}$ is a cover of $\{0,1\}^I$ by at most $\operatorname{cov}\mathcal{N}_\kappa$ sets, and $\operatorname{cov}\mathcal{N}_I \leq \operatorname{cov}\mathcal{N}_\kappa$. Since $(\{0,1\}^I, \mathcal{N}_I)$ is isomorphic to $(\{0,1\}^{\#(I)}, \mathcal{N}_{\#(I)})$, we also have

$$\operatorname{cov}\mathcal{N}_\theta \leq \operatorname{cov}\mathcal{N}_{\#(I)} \leq \operatorname{cov}\mathcal{N}_\kappa \leq \operatorname{cov}\mathcal{N}_\theta,$$

and $\operatorname{cov}\mathcal{N}_\theta = \operatorname{cov}\mathcal{N}_\kappa = \theta$. **Q**

What this means is that

$$\omega \leq \kappa_c \leq \operatorname{cov}\mathcal{N}_{\kappa_c} \leq \operatorname{cov}\mathcal{N}_\omega = \operatorname{cov}\mathcal{N} \leq \mathfrak{c}.$$

Another way of putting the same idea is to say that

if λ is a cardinal such that $\operatorname{cov}\mathcal{N}_\lambda \geq \lambda$ then $\operatorname{cov}\mathcal{N}_\kappa \geq \lambda$ for every κ

(since $\theta \geq \lambda$).

523G When the additivity of Lebesgue measure is large we have a further constraint on covering numbers.

Proposition (Kraszewski 01) If κ is an infinite cardinal and $\operatorname{cov}\mathcal{N}_\kappa < \operatorname{add}\mathcal{N}$, then $\operatorname{cov}\mathcal{N}_\kappa \leq \operatorname{cf}[\kappa]^{\leq\omega}$.

proof Let $\mathcal{E}$ be a subset of $\mathcal{N}_\kappa$ of size $\operatorname{cov}\mathcal{N}_\kappa$ with union $\{0,1\}^\kappa$, and $\mathcal{J}$ a cofinal subset of $[\kappa]^\omega$ of size $\operatorname{cf}[\kappa]^{\leq\omega}$. For $J \in \mathcal{J}$ and $x \in \{0,1\}^\kappa$ set $\pi_J(x) = x \restriction J$, so that $\pi_J : \{0,1\}^\kappa \to \{0,1\}^J$ is inverse-measure-preserving. For $J \in \mathcal{J}$ set $\mathcal{E}_J = \{E : E \in \mathcal{E},\ \pi_J[E] \in \mathcal{N}_J\}$, $H_J = \bigcup\mathcal{E}_J$. Since

$$\#(\mathcal{E}_J) \leq \#(\mathcal{E}) = \operatorname{cov}\mathcal{N}_\kappa < \operatorname{add}\mathcal{N} = \operatorname{add}\mathcal{N}_\omega = \operatorname{add}\mathcal{N}_J,$$

$F_J = \bigcup\{\pi_J[E] : E \in \mathcal{E}_J\} \in \mathcal{N}_J$ and $H_J \subseteq \pi_J^{-1}[F_J] \in \mathcal{N}_\kappa$. Since $\bigcup_{J \in \mathcal{J}} \mathcal{E}_J = \mathcal{E}$ (523C) covers $\{0,1\}^\kappa$, $\{H_J : J \in \mathcal{J}\}$ covers $\{0,1\}^\kappa$ and $\operatorname{cov}\mathcal{N}_\kappa \leq \#(\mathcal{J}) = \operatorname{cf}[\kappa]^{\leq\omega}$.

523H Uniformities On the other side of the diagram we have non-decreasing functions. To get upper bounds for $\operatorname{non}\mathcal{N}_\kappa$ we have the following method.

Lemma (Kraszewski 01) Suppose that I is a set and F a family of functions with domain I such that for every countable $J \subseteq I$ there is an $f \in F$ such that $f\restriction J$ is injective. Then

$$\operatorname{non}\mathcal{N}_I \le \max(\#(F), \sup_{f\in F}\operatorname{non}\mathcal{N}_{f[I]}).$$

proof If I is finite the result is trivial. Otherwise, for each $f \in F$ take a non-negligible subset A_f of $\{0,1\}^{f[I]}$ of size $\operatorname{non}\mathcal{N}_{f[I]}$. For $y \in A_f$ set $x_{fy} = yf \in \{0,1\}^I$. Set $A = \{x_{fy} : f \in F, y \in A_f\}$. **?** If $A \in \mathcal{N}_I$, there is a countable set $J \subseteq I$ such that $\{x\restriction J : x \in A\} \in \mathcal{N}_J$. Let $f \in F$ be such that $f\restriction J$ is injective. Then we have a function $\phi : \{0,1\}^{f[I]} \to \{0,1\}^J$ defined by saying that $\phi(z) = zf\restriction J$ for every $z \in \{0,1\}^{f[I]}$, and (because $f\restriction J$ is injective) ϕ is inverse-measure-preserving for $\nu_{f[I]}$ and ν_J, so $\phi[A_f]$ cannot be ν_J-negligible. But if $y \in A_f$ then $\phi(y)(\xi) = y(f(\xi)) = x_{fy}(\xi)$ for every $\xi \in J$, so $\phi[A_f] \subseteq \{x\restriction J : x \in A\}$, which is supposed to be negligible. **X**

Thus A is not negligible, and

$$\operatorname{non}\mathcal{N}_I \le \#(A) \le \max(\omega, \#(F), \sup_{f\in F}\#(A_f)) = \max(\#(F), \sup_{f\in F}\operatorname{non}\mathcal{N}_{f[I]})$$

because we are supposing that I is infinite, so there is some $f \in F$ such that $f[I]$ is infinite.

523I Theorem (a) For any cardinal κ,

(i) $\operatorname{non}\mathcal{N}_\kappa \le \max(\operatorname{non}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega})$,

(ii) $\operatorname{non}\mathcal{N}_{2^\kappa} \le \max(\mathfrak{c}, \operatorname{cf}[\kappa]^{\le\omega})$,

(iii) $\operatorname{non}\mathcal{N}_{2^{\kappa^+}} \le \max(\kappa^+, \operatorname{non}\mathcal{N}_{2^\kappa})$.

(b) If $\operatorname{cf}\kappa > \omega$, then $\operatorname{non}\mathcal{N}_{\kappa^+} \le \max(\operatorname{cf}\kappa, \sup_{\lambda<\kappa}\operatorname{non}\mathcal{N}_\lambda)$.

proof (a)(i) Let $\mathcal{J} \subseteq [\kappa]^{\le\omega}$ be a cofinal set of size $\operatorname{cf}[\kappa]^{\le\omega}$, and for $J \in \mathcal{J}$ let f_J be the identity function on J. Applying 523H with $F = \{f_J : J \in \mathcal{J}\}$ we get

$$\begin{aligned}\operatorname{non}\mathcal{N}_\kappa &\le \max(\#(\mathcal{J}), \sup_{J\in\mathcal{J}}\operatorname{non}\mathcal{N}_J)\\ &\le \max(\operatorname{non}\mathcal{N}_\omega, \operatorname{cf}[\kappa]^{\le\omega}) = \max(\operatorname{non}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega}).\end{aligned}$$

(ii) Take $\mathcal{J}$ as in (i). This time, for $J \in \mathcal{J}$, define $f_J : \mathcal{P}\kappa \to \mathcal{P}J$ by setting $f_J(A) = A \cap J$ for every $A \subseteq \kappa$. Applying 523H with $F = \{f_J : J \in \mathcal{J}\}$ we get

$$\begin{aligned}\operatorname{non}\mathcal{N}_{2^\kappa} = \operatorname{non}\mathcal{N}_{\mathcal{P}\kappa} &\le \max(\#(\mathcal{J}), \sup_{J\in\mathcal{J}}\operatorname{non}\mathcal{N}_{\mathcal{P}J}) \le \max(\operatorname{cf}[\kappa]^{\le\omega}, \operatorname{non}\mathcal{N}_{\mathfrak{c}})\\ &\le \max(\operatorname{cf}[\kappa]^{\le\omega}, \operatorname{non}\mathcal{N}, \operatorname{cf}[\mathfrak{c}]^{\le\omega}) = \max(\operatorname{cf}[\kappa]^{\le\omega}, \mathfrak{c})\end{aligned}$$

(5A1E(c-ii)).

(iii) Set $f_\xi(A) = A \cap \xi$ for $\xi < \kappa^+$ and $A \subseteq \kappa^+$. If $\mathcal{J} \subseteq \mathcal{P}\kappa^+$ is countable, there is a $\xi < \kappa^+$ such that $A \cap \xi \ne A' \cap \xi$ for all distinct A, $A' \in \mathcal{J}$, that is, $f_\xi\restriction\mathcal{J}$ is injective. So 523H tells us that

$$\begin{aligned}\operatorname{non}\mathcal{N}_{2^{\kappa^+}} = \operatorname{non}\mathcal{N}_{\mathcal{P}(\kappa^+)} &\le \max(\kappa^+, \sup_{\xi<\kappa^+}\operatorname{non}\mathcal{N}_{f_\xi[\kappa^+]})\\ &\le \max(\kappa^+, \sup_{\xi<\kappa^+}\operatorname{non}\mathcal{N}_{\mathcal{P}\xi}) = \max(\kappa^+, \operatorname{non}\mathcal{N}_{2^\kappa}).\end{aligned}$$

(b)(i) If $\kappa = \theta^+$ then $\operatorname{non}\mathcal{N}_\kappa \le \max(\kappa, \operatorname{non}\mathcal{N}_\theta)$. **P** For each $\xi < \kappa$ let $f_\xi : \kappa \to \theta$ be a function which is injective on ξ, and set $F = \{f_\xi : \xi < \kappa\}$. By 523H,

$$\operatorname{non}\mathcal{N}_\kappa \le \max(\kappa, \sup_{\xi<\kappa}\operatorname{non}\mathcal{N}_{f_\xi[\kappa]}) \le \max(\kappa, \operatorname{non}\mathcal{N}_\theta). \text{ **Q**}$$

In fact $\operatorname{non}\mathcal{N}_{\kappa^+} \le \max(\kappa, \operatorname{non}\mathcal{N}_\theta)$. **P** Choose an injective function $h_\zeta : \zeta \to \kappa$ for each $\zeta < \kappa^+$. For $\xi < \kappa$ define $f_\xi : \kappa^+ \to \kappa$ by saying that

$$f_\xi(\zeta) = \min(\kappa \setminus \{f_\xi(\eta) : \eta < \zeta, h_\zeta(\eta) \le \xi\})$$

for $\zeta < \kappa^+$. If $J \subseteq \kappa^+$ is countable, then $\xi = \sup_{\eta,\zeta\in J,\eta<\zeta} h_\zeta(\eta)$ is less than κ, and $f_\xi(\eta) \ne f_\xi(\zeta)$ for all distinct η, $\zeta \in J$. Applying 523H with $F = \{f_\xi : \xi < \kappa\}$, we get

$$\operatorname{non}\mathcal{N}_{\kappa^+} \le \max(\kappa, \operatorname{non}\mathcal{N}_\kappa) \le \max(\kappa, \operatorname{non}\mathcal{N}_\theta).\ \mathbf{Q}$$

So $\operatorname{non}\mathcal{N}_{\kappa^+} \le \max(\operatorname{cf}\kappa, \sup_{\lambda<\kappa}\operatorname{non}\mathcal{N}_\lambda)$ if κ is an infinite successor cardinal.

(ii) Now suppose that κ is an uncountable limit cardinal with uncountable cofinality. Again choose an injective function $h_\zeta : \zeta \to \kappa$ for each $\zeta < \kappa^+$. This time, let $K \subseteq \kappa$ be a cofinal set of size $\operatorname{cf}\kappa$ consisting of cardinals, and for $\lambda \in K$ define $f_\lambda : \kappa^+ \to \lambda^+$ by the formula

$$f_\lambda(\zeta) = \min\{\lambda^+ \setminus \{f_\lambda(\eta) : \eta < \zeta,\ h_\zeta(\eta) \le \lambda\})$$

for $\zeta < \kappa^+$. If $J \subseteq \kappa^+$ is countable, then there is a $\lambda \in K$ such that $\lambda \ge \sup_{\eta,\zeta\in J,\eta<\zeta} h_\zeta(\eta)$, and $f_\lambda(\eta) \ne f_\lambda(\zeta)$ for all distinct η, $\zeta \in J$. Applying 523H with $F = \{f_\lambda : \lambda \in K\}$, we get

$$\operatorname{non}\mathcal{N}_{\kappa^+} \le \max(\#(F), \sup_{f\in F}\operatorname{non}\mathcal{N}_{f[\kappa^+]}) \le \max(\operatorname{cf}\kappa, \sup_{\lambda<\kappa}\operatorname{non}\mathcal{N}_\lambda).$$

523J Corollary (Kraszewski 01) (a) $\operatorname{non}\mathcal{N}_{\omega_2} = \operatorname{non}\mathcal{N}_{\omega_1} = \operatorname{non}\mathcal{N}$.
(b) For any $n \in \mathbb{N}$, $\operatorname{non}\mathcal{N}_{\omega_{n+1}} \le \max(\omega_n, \operatorname{non}\mathcal{N})$.
(c) $\operatorname{non}\mathcal{N}_{2^{\omega_1}} = \operatorname{non}\mathcal{N}_{\mathfrak{c}}$.
(d) If $n \in \mathbb{N}$ then $\operatorname{non}\mathcal{N}_{2^{\omega_n}} \le \max(\omega_n, \operatorname{non}\mathcal{N}_{\mathfrak{c}})$.

proof (a) We have

$$\operatorname{non}\mathcal{N} = \operatorname{non}\mathcal{N}_\omega \le \operatorname{non}\mathcal{N}_{\omega_1} \le \operatorname{non}\mathcal{N}_{\omega_2}$$

(523B)

$$\le \max(\omega_1, \operatorname{non}\mathcal{N}_\omega)$$

(523Ib)

$$= \operatorname{non}\mathcal{N}.$$

(b) Induce on n, using 523Ib for the inductive step.

(c) By 523I(a-iii), $\operatorname{non}\mathcal{N}_{2^{\omega_1}} \le \max(\omega_1, \operatorname{non}\mathcal{N}_{\mathfrak{c}})$; since

$$\omega_1 \le \operatorname{non}\mathcal{N}_{\mathfrak{c}} \le \operatorname{non}\mathcal{N}_{2^{\omega_1}},$$

we have the result.

(d) Induce on n, using 523I(a-iii) for the inductive step.

523K Corollary (Burke n05) For any sets I, K let $\Upsilon_\omega(I,K)$ be the least cardinal of any family F of functions from I to K such that for every countable $J \subseteq I$ there is an $f \in F$ which is injective on J. (If $\#(K) < \min(\omega, \#(I))$ take $\Upsilon_\omega(I,K) = \infty$.) Then
(a) $\operatorname{non}\mathcal{N}_I \le \max(\Upsilon_\omega(I,K), \operatorname{non}\mathcal{N}_K)$ for all sets I and K;
(b) if $\kappa \ge \mathfrak{c}$ is a cardinal, then $\operatorname{non}\mathcal{N}_\kappa = \max(\Upsilon_\omega(\kappa,\mathfrak{c}), \operatorname{non}\mathcal{N}_{\mathfrak{c}})$.

proof (a) This is just a slightly weaker version of 523H.

(b) The point is that $\Upsilon_\omega(\kappa,\mathfrak{c}) \le \operatorname{non}\mathcal{N}_\kappa$. $\mathbf{P}$ Let $A \subseteq \{0,1\}^{\kappa\times\omega}$ be a non-negligible set of cardinal $\operatorname{non}\mathcal{N}_{\kappa\times\omega}$. For $x \in \{0,1\}^{\kappa\times\omega}$ define $f_x : \kappa \to \{0,1\}^\omega$ by setting $f_x(\xi) = \langle x(\xi,n)\rangle_{n\in\mathbb{N}}$ for each $\xi < \kappa$. If ξ, $\eta < \kappa$ are distinct, then $\{x : f_x(\xi) = f_x(\eta)\}$ is negligible, so if $J \subseteq \kappa$ is countable then $\{x : f_x\restriction J \text{ is injective}\}$ is conegligible and meets A. Accordingly $\{f_x : x \in A\}$ witnesses that

$$\Upsilon_\omega(\kappa,\mathfrak{c}) = \Upsilon_\omega(\kappa,\{0,1\}^\omega) \le \#(A) = \operatorname{non}\mathcal{N}_{\kappa\times\omega} = \operatorname{non}\mathcal{N}_\kappa.\ \mathbf{Q}$$

Since we already know that

$$\operatorname{non}\mathcal{N}_{\mathfrak{c}} \le \operatorname{non}\mathcal{N}_\kappa \le \max(\Upsilon_\omega(\kappa,\mathfrak{c}), \operatorname{non}\mathcal{N}_{\mathfrak{c}}),$$

we have the result.

523L On the other side we can find lower bounds which give a notion of the rate of growth of the numbers $\operatorname{non}\mathcal{N}_\kappa$ as κ increases.

Proposition (a) If λ and κ are infinite cardinals with $\kappa > 2^\lambda$, then $\mathrm{non}\,\mathcal{N}_\kappa > \lambda$.

(b) If κ is a strong limit cardinal of countable cofinality then $\mathrm{non}\,\mathcal{N}_\kappa > \kappa$.

proof (a) Let $A \subseteq \{0,1\}^\kappa$ be any set of size at most λ. For $\xi < \kappa$ set $B_\xi = \{x : x \in A, x(\xi) = 1\}$. Because $\kappa > 2^{\#(A)}$, there is some $B \subseteq A$ such that $I = \{\xi : B_\xi = B\}$ is infinite. But what this means is that if $\xi \in I$ then $x(\xi) = 1$ for every $\xi \in B$ and $x(\xi) = 0$ for every $x \in A \setminus B$, and $A \subseteq \{x : x$ is constant on $I\}$ is negligible. As A is arbitrary, $\mathrm{non}\,\mathcal{N}_\kappa > \lambda$.

(d) By (a), $\mathrm{non}\,\mathcal{N}_\kappa > \lambda$ for every $\lambda < \kappa$, so $\mathrm{non}\,\mathcal{N}_\kappa \ge \kappa$; but also $\mathrm{cf}(\mathrm{non}\,\mathcal{N}_\kappa) \ge \mathrm{add}\,\mathcal{N}_\kappa$ (513C(b-ii)), so $\mathrm{non}\,\mathcal{N}_\kappa$ has uncountable cofinality and must be greater than κ.

523M Shrinking numbers As with $\mathrm{non}\,\mathcal{N}_\bullet$, the functions $\kappa \mapsto \mathrm{shr}\,\mathcal{N}_\kappa$ and $\kappa \mapsto \mathrm{shr}\,\mathcal{N}_\kappa$ are non-decreasing, by 521Hb. Some of the ideas used in 523I can be adapted to this context, but the pattern as a whole is rather different.

Proposition (a)(i) For any non-zero cardinals κ and λ,

$$\mathrm{shr}\,\mathcal{N}_\kappa \le \max(\mathrm{cov}_{\mathrm{Sh}}(\kappa, \lambda, \omega_1, 2), \sup_{\theta<\lambda} \mathrm{shr}\,\mathcal{N}_\theta).$$

(ii) For any infinite cardinal κ, $\mathrm{shr}\,\mathcal{N}_\kappa \le \max(\mathrm{shr}\,\mathcal{N}, \mathrm{cf}[\kappa]^{\le\omega})$.

(iii) If $\mathrm{cf}\,\kappa > \omega$, then $\mathrm{shr}\,\mathcal{N}_\kappa \le \max(\kappa, \sup_{\theta<\kappa} \mathrm{shr}\,\mathcal{N}_\theta)$.

(b) For any infinite cardinal κ,

(i) $\mathrm{shr}\,\mathcal{N}_\kappa \ge \kappa$;

(ii) $\mathrm{cf}(\mathrm{shr}\,\mathcal{N}_\kappa) > \omega$;

(iii) $\mathrm{cf}(\mathrm{shr}^+\,\mathcal{N}_\kappa) > \kappa$.

Remark For the definition of $\mathrm{cov}_{\mathrm{Sh}}$, see 5A2Da.

proof (a)(i) If $\mathrm{cov}_{\mathrm{Sh}}(\kappa, \lambda, \omega_1, 2) = \infty$ or κ is finite this is trivial. Otherwise, $\lambda \ge \omega_1$. Take a non-negligible $A \subseteq \{0,1\}^\kappa$. Let $\mathcal{J} \subseteq [\kappa]^{<\lambda}$ be a set of size $\mathrm{cov}_{\mathrm{Sh}}(\kappa, \lambda, \omega_1, 2)$ such that for every $I \in [\kappa]^{<\omega_1}$ there is a $\mathcal{D} \in [\mathcal{J}]^{<2}$ such that $I \subseteq \bigcup \mathcal{D}$, that is, there is a $J \in \mathcal{J}$ such that $I \subseteq J$. For each $J \in \mathcal{J}$, $A_J = \{x\restriction J : x \in A\}$ is non-negligible; let $B_J \subseteq A_J$ be a non-negligible set of size at most $\mathrm{shr}\,\mathcal{N}_J$. Let $B \subseteq A$ be a set of size at most $\max(\omega, \#(\mathcal{J}), \sup_{J\in\mathcal{J}} \mathrm{shr}\,\mathcal{N}_J)$ such that $B_J \subseteq \{x\restriction J : x \in B\}$ for every $J \in \mathcal{J}$. If $I \subseteq \kappa$ is countable, there is a $J \in \mathcal{J}$ such that $I \subseteq J$, so $\{x\restriction I : x \in B\} \supseteq \{y\restriction I : y \in B_J\}$ is non-negligible; it follows that B is non-negligible, while $\#(B) \le \max(\mathrm{cov}_{\mathrm{Sh}}(\kappa, \lambda, \omega_1, 2), \sup_{\theta<\lambda} \mathrm{shr}\,\mathcal{N}_\theta)$.

(ii) Taking $\lambda = \omega_1$ in (i),

$$\mathrm{shr}\,\mathcal{N}_\kappa \le \max(\mathrm{cov}_{\mathrm{Sh}}(\kappa, \omega_1, \omega_1, 2), \mathrm{shr}\,\mathcal{N}_\omega) = \max(\mathrm{cf}[\kappa]^{\le\omega}, \mathrm{shr}\,\mathcal{N}).$$

(iii) Take $\lambda = \kappa$ in (i); as $[\kappa]^{\le\omega} = \bigcup_{\xi<\kappa} [\xi]^{\le\omega}$,

$$\mathrm{shr}\,\mathcal{N}_\kappa \le \max(\mathrm{cov}_{\mathrm{Sh}}(\kappa, \kappa, \omega_1, 2), \sup_{\theta<\kappa} \mathrm{shr}\,\mathcal{N}_\theta) = \max(\kappa, \sup_{\theta<\kappa} \mathrm{shr}\,\mathcal{N}_\theta).$$

(b)(i) Induce on κ. If $\kappa = \omega$ the result is trivial. For the inductive step to κ^+, consider the set

$$A = \{x : x \in \{0,1\}^{\kappa^+}, \exists\, \xi < \kappa^+, x(\eta) = 0 \text{ for every } \eta \ge \xi\}.$$

Then the only set which includes A and is determined by a countable set of coordinates is $\{0,1\}^{\kappa^+}$, so A has full outer measure. On the other hand, if $B \subseteq A$ and $\#(B) \le \kappa$, then there is some $\zeta < \kappa^+$ such that $x(\xi) = 0$ for every $x \in B$ and every $\xi \ge \zeta$, so B is negligible. Thus A witnesses that $\mathrm{shr}\,\mathcal{N}_{\kappa^+} \ge \kappa^+$. Because $\kappa \mapsto \mathrm{shr}\,\mathcal{N}_\kappa$ is non-decreasing (523B), the inductive step to limit cardinals κ is trivial.

(ii) ? Now suppose, if possible, that $\mathrm{cf}(\mathrm{shr}\,\mathcal{N}_\kappa) = \omega$. Then there is a sequence $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ of cardinals less than $\mathrm{shr}\,\mathcal{N}_\kappa$ with supremum $\mathrm{shr}\,\mathcal{N}_\kappa$. For each $n \in \mathbb{N}$ set $I_n = \kappa \times \{n\}$, and let $A_n \subseteq \{0,1\}^{I_n}$ be a non-negligible set such that every non-negligible subset of A_n has more than λ_n members. By 523Dc, there is a set $B_n \subseteq \{0,1\}^{I_n}$ of full outer measure such that every non-negligible subset of B_n has more than λ_n members. Set

$$B = \{x : x \in \{0,1\}^{\kappa\times\mathbb{N}}, x\restriction I_n \in B_n \text{ for every } n \in \mathbb{N}\}.$$

Then the natural isomorphism between $\{0,1\}^{\kappa\times\mathbb{N}}$ and $\prod_{n\in\mathbb{N}} \{0,1\}^{I_n}$ identifies B with $\prod_{n\in\mathbb{N}} B_n$, so B has full outer measure in $\{0,1\}^{\kappa\times\mathbb{N}}$ (254Lb). There must therefore be a set $C \subseteq B$, of non-zero measure, such that $\#(C) \le \mathrm{shr}\,\mathcal{N}_\kappa$. Express C as $\bigcup_{n\in\mathbb{N}} C_n$ where $\#(C_n) \le \lambda_n$ for every n. Then there is an $n \in \mathbb{N}$ such that C_n is not negligible, in which case $D_n = \{x\restriction I_n : x \in C_n\}$ is non-negligible. But $D_n \subseteq B_n$ and $\#(D_n) \le \lambda_n$, so this is impossible. **X**

(iii) The argument of (i) shows that if κ is a successor cardinal, then $\operatorname{shr}^+\mathcal{N}_\kappa > \kappa$. So we need consider only the case in which κ is a limit cardinal. **?** If $\operatorname{cf}(\operatorname{shr}^+\mathcal{N}_\kappa) \le \kappa$, then there is a family $\langle\lambda_\xi\rangle_{\xi<\kappa}$ of cardinals less than $\operatorname{shr}^+\mathcal{N}_\kappa$ with supremum $\operatorname{shr}^+\mathcal{N}_\kappa$. I use the same method as in (ii). For each $\xi < \kappa$ set $I_\xi = \kappa \times \{\xi\}$, and let $B_\xi \subseteq \{0,1\}^{I_\xi}$ be a set of full outer measure such that every non-negligible subset of B_ξ has at least λ_ξ members. Set

$$B = \{x : x \in \{0,1\}^{\kappa\times\kappa},\ x\restriction I_\xi \in B_\xi \text{ for every } \xi < \kappa.$$

Then B has full outer measure in $\{0,1\}^{\kappa\times\kappa}$. There must therefore be a set $C \subseteq B$, of non-zero measure, such that $\#(C) < \operatorname{shr}^+\mathcal{N}_\kappa$. Let $\xi < \kappa$ be such that $\#(C) < \lambda_\xi$. Then $D = \{x\restriction I_\xi : x \in C\}$ is non-negligible. But $D \subseteq B_\xi$ and $\#(D_\xi) < \lambda_\xi$, so this is impossible. **X**

523N Cofinalities For the cardinals $\operatorname{cf}\mathcal{N}_\kappa$ the pattern from 523I(a-i) and 523Mb continues, and indeed we have an exact formula.

Theorem For any infinite cardinal κ,

$$\kappa \le \operatorname{cf}\mathcal{N}_\kappa = \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega}) \le \kappa^\omega.$$

proof (a) $\operatorname{cf}\mathcal{N}_\kappa \le \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega})$. **P** Let $\mathcal{J}$ be a cofinal family in $[\kappa]^\omega$ with cardinal $\operatorname{cf}[\kappa]^{\le\omega}$. For each $J \in \mathcal{J}$, write $\pi_J(x) = x\restriction J$ for $x \in \{0,1\}^\kappa$. Let $\mathcal{E}_J$ be a cofinal subset of $\mathcal{N}_J$ with cardinal $\operatorname{cf}\mathcal{N}_J = \operatorname{cf}\mathcal{N}_\omega = \operatorname{cf}\mathcal{N}$. Consider $\mathcal{E} = \{\pi_J^{-1}[E] : J \in \mathcal{J},\ E \in \mathcal{E}_J\}$. By 523C, $\mathcal{E}$ is cofinal with $\mathcal{N}_\kappa$, so that

$$\operatorname{cf}\mathcal{N}_\kappa \le \#(\mathcal{E}) \le \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega}). \text{ } \mathbf{Q}$$

(b) We know that $\operatorname{cf}[\kappa]^{\le\omega} \le \operatorname{cf}\mathcal{N}_\kappa$ (521Jb) and that $\operatorname{cf}\mathcal{N} = \operatorname{cf}\mathcal{N}_\omega \le \operatorname{cf}\mathcal{N}_\kappa$ (523B). So $\operatorname{cf}\mathcal{N}_\kappa = \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\kappa]^{\le\omega})$.

(c) For the inequalities, note that if κ is uncountable then

$$\operatorname{cf}[\kappa]^{\le\omega} \ge \operatorname{cov}(\kappa, [\kappa]^{\le\omega}) = \kappa.$$

On the other side, $\operatorname{cf}\mathcal{N} \le \mathfrak{c} \le \kappa^\omega$ and $\operatorname{cf}[\kappa]^{\le\omega} \le \#([\kappa]^{\le\omega}) = \kappa^\omega$.

523O Cofinalities of the cardinals In 523Mb I have shown that $\operatorname{shr}\mathcal{N}_\kappa$ has uncountable cofinality for infinite κ, and rather more about $\operatorname{shr}^+\mathcal{N}_\kappa$. From 513Cb we have a little information concerning the cofinalities of $\operatorname{add}\mathcal{N}_\kappa$, $\operatorname{cov}\mathcal{N}_\kappa$, $\operatorname{non}\mathcal{N}_\kappa$ and $\operatorname{cf}\mathcal{N}_\kappa$; but except when $\kappa = \omega$ we learn only that $\operatorname{cf}\mathcal{N}_\kappa$ and $\operatorname{non}\mathcal{N}_\kappa$ have uncountable cofinality, and that if $\operatorname{cov}\mathcal{N}_\kappa = \operatorname{cf}\mathcal{N}_\kappa$ then their common cofinality is at least $\operatorname{non}\mathcal{N}_\kappa$. This last remark can apply only to 'small' κ, since $\operatorname{cf}\mathcal{N}_\kappa \ge \kappa$ (if κ is infinite) and $\operatorname{cov}\mathcal{N}_\kappa \le \operatorname{cov}\mathcal{N}$.

523P The generalized continuum hypothesis In this chapter I am trying to present arguments in forms which show their full strength and are not tied to particular axioms beyond those of ZFC. However it is perhaps worth mentioning that in one of the standard universes the pattern is particularly simple.

Proposition Suppose that the generalized continuum hypothesis is true. Then, for any infinite cardinal κ,

$$\operatorname{add}\mathcal{N}_\kappa = \operatorname{add}\nu_\kappa = \operatorname{cov}\mathcal{N}_\kappa = \omega_1;$$

$$\begin{aligned}\operatorname{non}\mathcal{N}_\kappa &= \lambda \text{ if } \kappa = \lambda^+ \text{ where } \operatorname{cf}\lambda > \omega,\\ &= \kappa^+ \text{ if } \operatorname{cf}\kappa = \omega,\\ &= \kappa \text{ otherwise;}\end{aligned}$$

$$\begin{aligned}\operatorname{shr}\mathcal{N}_\kappa = \operatorname{cf}\mathcal{N}_\kappa &= \kappa^+ \text{ if } \operatorname{cf}\kappa = \omega,\\ &= \kappa \text{ otherwise;}\end{aligned}$$

$$\begin{aligned}\operatorname{shr}^+\mathcal{N}_\kappa = (\operatorname{shr}\mathcal{N}_\kappa)^+ &= \kappa^{++} \text{ if } \operatorname{cf}\kappa = \omega,\\ &= \kappa^+ \text{ otherwise.}\end{aligned}$$

proof Since

$$\omega_1 \le \operatorname{add}\mathcal{N}_\kappa = \operatorname{add}\nu_\kappa \le \operatorname{cov}\mathcal{N}_\kappa \le \operatorname{cov}\mathcal{N} \le \mathfrak{c} = \omega_1,$$

the additivity and covering number are always ω_1.

If $\operatorname{cf}\kappa = \omega$, then

$$\kappa^+ \le \operatorname{shr}\mathcal{N}_\kappa \le \operatorname{cf}\mathcal{N}_\kappa = \max(\omega_1, \operatorname{cf}[\kappa]^{\le\omega}) \le 2^\kappa = \kappa^+$$

by 523M(b-ii) and 523N. If $\operatorname{cf}\kappa > \omega$ then

$$\kappa \le \operatorname{shr}\mathcal{N}_\kappa \le \operatorname{cf}\mathcal{N}_\kappa \le \kappa$$

by 523M(b-i), 523N and 5A6Ab. This deals with $\operatorname{shr}\mathcal{N}_\kappa$ and $\operatorname{cf}\mathcal{N}_\kappa$. For the augmented shrinking numbers, we know that $\kappa < \operatorname{shr}^+\mathcal{N}_\kappa \le (\operatorname{shr}\mathcal{N}_\kappa)^+$ (523M(b-iii)), with equality if $\operatorname{shr}\mathcal{N}_\kappa$ is a successor cardinal, so in the present case we must have $\operatorname{shr}^+\mathcal{N}_\kappa = (\operatorname{shr}\mathcal{N}_\kappa)^+$.

As for $\operatorname{non}\mathcal{N}_\kappa$, if $\kappa = \lambda^+$ where $\operatorname{cf}\lambda > \omega$, then $\kappa > 2^\theta$ for every $\theta < \lambda$, so we have

$$\lambda \le \operatorname{non}\mathcal{N}_\kappa \le \max(\mathfrak{c}, \operatorname{cf}[\lambda]^\omega) = \lambda$$

(523I(a-ii), 523La, 5A6Ab). If $\kappa = \lambda^+$ where $\operatorname{cf}\lambda = \omega$, then

$$\lambda \le \operatorname{non}\mathcal{N}_\kappa \le \lambda^\omega \le 2^\lambda = \kappa;$$

but as $\operatorname{non}\mathcal{N}_\kappa$ has uncountable cofinality (513C(b-ii) again), $\operatorname{non}\mathcal{N}_\kappa$ must be κ. If κ is a limit cardinal, then $\kappa > 2^\theta$ for every $\theta < \kappa$, so

$$\kappa \le \operatorname{non}\mathcal{N}_\kappa \le \max(\omega_1, \operatorname{cf}[\kappa]^{\le\omega});$$

if $\operatorname{cf}\kappa > \omega$ this is already enough to show that $\operatorname{non}\mathcal{N}_\kappa = \kappa$; if $\operatorname{cf}\kappa = \omega$ then $\operatorname{non}\mathcal{N}_\kappa$ cannot be κ so must be $\kappa^+ = \kappa^\omega$.

523X Basic exercises (a) Show that

$$(\mathcal{N}_\kappa, \not\ni, \{0,1\}^\kappa) \preccurlyeq_{\mathrm{GT}} ([\kappa]^{\le\omega}, \subseteq, [\kappa]^{\le\omega}) \ltimes (\mathcal{N}, \not\ni, \mathbb{R})$$

for every infinite cardinal κ. (See 512I for the definition of $\ltimes$.) Use this to prove 523I(a-i).

(b) Let κ be an infinite cardinal, and $\mathcal{J}$ a family of subsets of κ such that every countable subset of κ is included in some member of $\mathcal{J}$. Show that $\operatorname{non}\mathcal{N}_\kappa \le \max(\#(\mathcal{J}), \sup_{J\in\mathcal{J}} \operatorname{non}\mathcal{N}_J)$, $\operatorname{non}\mathcal{N}_{\mathcal{P}\kappa} \le \max(\#(\mathcal{J}), \sup_{J\in\mathcal{J}} \operatorname{non}\mathcal{N}_{\mathcal{P}J})$, $\operatorname{shr}\mathcal{N}_\kappa \le \max(\#(\mathcal{J}), \sup_{J\in\mathcal{J}} \operatorname{shr}\mathcal{N}_J)$ and $\operatorname{cf}\mathcal{N}_\kappa \le \max(\#(\mathcal{J}), \sup_{J\in\mathcal{J}} \operatorname{cf}\mathcal{N}_J)$.

(c) Show that

$$(\mathcal{N}_\kappa, \subseteq, \mathcal{N}_\kappa) \preccurlyeq_{\mathrm{GT}} ([\kappa]^{\le\omega}, \subseteq, [\kappa]^{\le\omega}) \ltimes (\mathcal{N}, \subseteq, \mathcal{N})$$

for every infinite cardinal κ. Use this to prove 523N.

(d) Let (X, Σ, μ) be any probability space, and for a set I write $\mathcal{N}(\mu^I)$ for the null ideal of the product measure on X^I. Show that all the results of 523E-523J, 523L-523N are valid with $\mathcal{N}(\mu^I)$ in place of $\mathcal{N}_I$ and $\mathcal{N}(\mu^\omega)$ in place of $\mathcal{N}$, except that

— in 523F we can no longer be sure that $\operatorname{cov}\mathcal{N}(\mu^\omega) \le \mathfrak{c}$;

— in 523I(a-ii) we need to write '$\operatorname{non}\mathcal{N}(\mu^{2^\kappa}) \le \max(\mathfrak{c}, \operatorname{non}\mathcal{N}(\mu^\omega), \operatorname{cf}[\kappa]^{\le\omega})$';

— in 523L and 523Mb we have to assume that the measure algebra of μ is not $\{0,1\}$, so that the product measure $\mu^{\mathbb{N}}$ is atomless;

— in 523N we can no longer be sure that $\operatorname{cf}\mathcal{N}(\mu^\omega) \le \kappa^\omega$.

523Y Further exercises (a) Set $\mathfrak{A} = \mathcal{P}\mathbb{R}/\mathcal{N}$. (i) Show that $\mathfrak{c} \le c(\mathfrak{A}) \le \pi(\mathfrak{A}) \le 2^{\operatorname{shr}\mathcal{N}}$. (ii) Show that if $\operatorname{shr}^+\mathcal{N} \ge \mathfrak{c}$ and $2^\lambda \le \mathfrak{c}$ for every $\lambda < \mathfrak{c}$, then $c(\mathfrak{A}) = 2^\mathfrak{c}$.

(b) Let κ be an infinite cardinal. Show that there is a family $\mathcal{J} \subseteq [\kappa]^{\le\omega}$ such that $\#(\mathcal{J}) \le \operatorname{shr}\mathcal{N}_\kappa$ and every infinite subset of κ meets some member of $\mathcal{J}$ in an infinite set.

(c) Suppose that $\kappa \ge \omega$ and that $[\kappa]^{\le\omega}$ has bursting number at most $\operatorname{add}\mathcal{N}$. Show that $\mathcal{N}_\kappa \equiv_{\mathrm{T}} [\kappa]^{\le\omega} \times \mathcal{N}$. (*Hint*: Fremlin 91..)

(d) Show that

$$(\omega_1, \le, \omega_1) \ltimes (\omega_1, \le, \omega_1) \not\preccurlyeq_{\mathrm{GT}} (\omega_1, \le, \omega_1) \times (\omega_1, \le, \omega_1).$$

(e) For infinite cardinals κ, write $\mathcal{M}_\kappa$ for the ideal of meager subsets of $\{0,1\}^\kappa$. Show that under the same conventions as in 522B and 523B we have the diagrams

$$\begin{tikzcd}
 & \operatorname{cov}\mathcal{M}_\lambda \arrow[r, dash] \arrow[dd, dash] & \operatorname{cov}\mathcal{M}_\kappa \arrow[r, dash] \arrow[dd, dash] & \operatorname{cf}\mathcal{M}_\kappa \arrow[r, dash] \arrow[d, dash] & \operatorname{cf}\mathcal{M}_\lambda \arrow[r, dash] \arrow[d, dash] & \lambda^\omega \\
 & & & \operatorname{shr}\mathcal{M}_\kappa \arrow[r, dash] \arrow[d, dash] & \operatorname{shr}\mathcal{M}_\lambda \arrow[d, dash] & \\
\omega_1 \arrow[r, dash] & \operatorname{add}\mathcal{M}_\lambda \arrow[r, dash] & \operatorname{add}\mathcal{M}_\kappa \arrow[r, dash] & \operatorname{non}\mathcal{M}_\kappa \arrow[r, dash] & \operatorname{non}\mathcal{M}_\lambda &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \operatorname{cov}\mathcal{N}_\kappa \arrow[r, dash] \arrow[d, dash] & \operatorname{non}\mathcal{M}_\kappa \arrow[r, dash] \arrow[d, dash] & \operatorname{cf}\mathcal{M}_\kappa \arrow[r, dash] \arrow[d, dash] & \operatorname{cf}\mathcal{N}_\kappa \arrow[r, dash] \arrow[d, dash] & \kappa^\omega \\
\omega_1 \arrow[r, dash] & \operatorname{add}\mathcal{N}_\kappa \arrow[r, dash] & \operatorname{add}\mathcal{M}_\kappa \arrow[r, dash] & \operatorname{cov}\mathcal{M}_\kappa \arrow[r, dash] & \operatorname{non}\mathcal{N}_\kappa &
\end{tikzcd}$$

whenever $\omega \le \kappa \le \lambda$. Show moreover that all the results of 523E-523P have parallel forms referring to $\mathcal{M}_\kappa$.

(f) In the language of 523Ye, show that (i) $\mathfrak{m}_{\mathrm{pc}\omega_1} \le \operatorname{cov}\mathcal{M}_\kappa$ for every infinite κ (ii) $\mathfrak{m}_{\sigma\text{-linked}} \le \operatorname{cov}\mathcal{N}_\kappa$ if $\omega \le \kappa \le \mathfrak{c}$. (*Hint*: ω_1 is a precaliber of $\mathrm{RO}(\{0,1\}^\kappa)$, and the measure algebra of ν_κ is σ-linked if $\kappa \le \mathfrak{c}$.)

(g) Show that Ostaszewski's $\clubsuit$ implies that $\operatorname{cov}\mathcal{N}_{\omega_1} = \operatorname{cov}\mathcal{M}_{\omega_1} = \omega_1$.

523Z Problem Is there a proof in ZFC that $\operatorname{shr}\mathcal{N}_\kappa \ge \operatorname{cf}[\kappa]^{\le\omega}$ for every cardinal κ?

523 Notes and comments The basic diagram 523B is natural and easy to establish. Of course it leaves a great deal of room, especially on the right-hand side, where we have the increasing functions $\operatorname{non}\mathcal{N}_\bullet$, $\operatorname{shr}\mathcal{N}_\bullet$ and $\operatorname{cf}\mathcal{N}_\bullet$, and rather weak constraints

$$\lambda < \operatorname{non}\mathcal{N}_\kappa \le \operatorname{shr}\mathcal{N}_\kappa \le \operatorname{cf}\mathcal{N}_\kappa \le \kappa^\omega \text{ whenever } 2^\lambda < \kappa$$

to control them. However the generalized continuum hypothesis is sufficient to determine exact values for all the cardinals considered here (523P).

The combinatorics of $\operatorname{cf}[\kappa]^{\le\omega}$ and almost-disjoint families of functions are extremely complex, and depend in surprising ways on special axioms; I think it possible that the results of 523I-523J can be usefully extended. However 523N at least reduces the measure-theoretic problem of determining $\operatorname{cf}\mathcal{N}_\kappa$ to a standard, if difficult, question in infinitary combinatorics. I do not know if there are corresponding results concerning $\operatorname{non}\mathcal{N}_\kappa$ and $\operatorname{shr}\mathcal{N}_\kappa$ (see 523Kb and 523Z).

All the ideas in this section up to and including 523P can be applied to ideals of meager sets (523Ye) and indeed to other classes of ideals satisfying the fundamental lemma 523C; see Kraszewski 01.

524 Radon measures

It is a remarkable fact that for a Radon measure the principal cardinal functions are determined by its measure algebra (524J), so can in most cases be calculated in terms of the cardinals of the last section (524P-524Q). The proof of this seems to require a substantial excursion involving not only measure algebras but also the Banach lattices $\ell^1(\kappa)$ and/or the κ-localization relation (524D, 524E). The same machinery gives us formulae for the cardinal functions of measurable algebras (524M). The results of §518 can be translated directly to give partial information on the Freese-Nation numbers of measurable algebras (524O). For covering number and uniformity, we can see from 521L that strictly localizable compact measures follow Radon measures. I know of no such general results for any other class of measure, but there are some bounds for cardinal functions of countably compact and quasi-Radon measures, which I give in 524R-524T.

524A Notation If (X,Σ,μ) is a measure space, $\mathcal{N}(\mu)$ will be the null ideal of μ. For any cardinal κ, ν_κ will be the usual measure on $\{0,1\}^\kappa$, T_κ its domain and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ its measure algebra. As in §§522-523, I will write $\mathcal{N}_\kappa$ for $\mathcal{N}(\nu_\kappa)$ and $\mathcal{N}$ for the null ideal of Lebesgue measure on $\mathbb{R}$, so that $(\mathbb{R},\mathcal{N})$ and $(\{0,1\}^\omega,\mathcal{N}_\omega)$ are isomorphic (522Wa). If $\mathfrak{A}$ is any Boolean algebra, I write $\mathfrak{A}^+$ for $\mathfrak{A}\setminus\{0\}$ and $\mathfrak{A}^-$ for $\mathfrak{A}\setminus\{1\}$. If (A,R,B) is a supported relation, R' is the relation $\{(a,I) : a \in R^{-1}[I]\}$ (see 512F). For any cardinal κ, $(\kappa^{\mathbb{N}},\subseteq^*,\mathcal{S}_\kappa)$ will be the κ-localization relation (522K).

524B Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite Radon measure space with Maharam type κ. Then $\mathcal{N}(\mu)\preccurlyeq_{\mathrm{T}}\mathfrak{B}_\kappa^-$.

proof (a) Suppose, to begin with, that $\mu X=1$ and $\kappa\ge\omega$. Let $\mathfrak{A}$ be the measure algebra of the Radon product measure $\tilde\lambda$ on $Y=X^{\mathbb{N}}$. Then $\mathfrak{A}\cong\mathfrak{B}_\kappa$. **P** By 417E(ii), $\mathfrak{A}$ is isomorphic to the measure algebra of the usual product measure λ on Y, which by 334E is isomorphic to $\mathfrak{B}_\kappa$. **Q**

For $E\in\mathcal{N}(\mu)$, let $\langle F_{Ei}\rangle_{i\in\mathbb{N}}$ be a sequence of closed subsets of X such that $E\cap F_{Ei}=\emptyset$ and $\mu F_{Ei}\ge 1-2^{-i-1}$ for every $n\in\mathbb{N}$. Then

$$\tilde\lambda(\prod_{i\in\mathbb{N}}F_{Ei})=\lambda(\prod_{i\in\mathbb{N}}F_{Ei})\ge\prod_{i\in\mathbb{N}}(1-2^{-i-1})>0;$$

set

$$\phi(E)=(Y\setminus\prod_{i\in\mathbb{N}}F_{Ei})^\bullet\in\mathfrak{A}^-.$$

For $b\in\mathfrak{A}^-$ let $K_b\subseteq Y$ be a non-empty compact self-supporting set such that $K_b^\bullet\cap b=0$. Set $\pi_i(y)=y(i)$ for $i\in\mathbb{N}$ and $y\in Y$. Then each $\pi_i[K_b]\subseteq X$ is compact and $K_b\subseteq\prod_{i\in\mathbb{N}}\pi_i^{-1}[\pi_i[K_b]]$, so $\prod_{i\in\mathbb{N}}\mu\pi_i[K_b]>0$ and $\sup_{i\in\mathbb{N}}\mu\pi_i[K_b]=1$; set

$$\psi(b)=X\setminus\bigcup_{i\in\mathbb{N}}\pi_i[K_b]\in\mathcal{N}(\mu).$$

If $E\in\mathcal{N}(\mu)$ and $b\in\mathfrak{A}^-$ and $\phi(E)\subseteq b$ and $j\in\mathbb{N}$, then

$$K_b\setminus\pi_j^{-1}[F_{Ej}]\subseteq K_b\setminus\prod_{i\in\mathbb{N}}F_{Ei}$$

is negligible. As K_b is self-supporting, $K_b\setminus\pi_j^{-1}[F_{Ej}]$ is empty and $\pi_j[K_b]\subseteq F_{Ej}$. But this means that $\pi_j[K_b]\cap E=\emptyset$ for every $j\in\mathbb{N}$, so that $E\subseteq\psi(b)$.

This shows that ϕ is a Tukey function, so that $\mathcal{N}(\mu)\preccurlyeq_{\mathrm{T}}\mathfrak{A}^-\cong\mathfrak{B}_\kappa^-$.

(b) If κ is finite, $\mathcal{N}(\mu)$ has a greatest member and the constant function with value 0 is a Tukey function from $\mathcal{N}(\mu)$ to $\mathfrak{B}_\kappa^-$ and the result is trivial. If κ is infinite and $\mu X\ne 1$, then, because μ is σ-finite and not trivial, there is a function $f:X\to\,]0,\infty[$ such that $\int f d\mu=1$ (215B(ix)). Let ν be the corresponding indefinite-integral measure; then ν is a Radon probability measure (416S) with the same measurable sets and the same negligible sets as μ (234L), so $\Sigma/\mathcal{N}(\nu)=\Sigma/\mathcal{N}(\mu)$ has Maharam type κ. In this case, (a) tells us that $\mathcal{N}(\mu)=\mathcal{N}(\nu)\preccurlyeq_{\mathrm{T}}\mathfrak{B}_\kappa^-$.

524C Lemma Let P be a partially ordered set such that $p\vee q=\sup\{p,q\}$ is defined for all p, $q\in P$. Suppose that ρ is a metric on P such that P is complete (as a metric space) and $\vee:P\times P\to P$ is uniformly continuous with respect to ρ. Let $Q\subseteq P$ be an open set, and $\kappa\ge d(Q)$ a cardinal. Then $(Q,\le',[Q]^{<\omega})\preccurlyeq_{\mathrm{GT}}(\ell^1(\kappa),\le,\ell^1(\kappa))$. If Q is upwards-directed, then $Q\preccurlyeq_{\mathrm{T}}\ell^1(\kappa)$.

proof (a) If Q is finite, then we can set $\phi(q)=0$ for every $q\in Q$, $\psi(x)=Q$ for every $x\in\ell^1(\kappa)$ and (ϕ,ψ) will be a Galois-Tukey connection from $(Q,\le',[Q]^{<\omega})$ to $(\ell^1(\kappa),\le,\ell^1(\kappa))$. So let us suppose that Q and κ are infinite.

(b) Let $\langle q_\xi\rangle_{\xi<\kappa}$ run over a dense subset of Q. For each $q\in Q$ let $m(q)\in\mathbb{N}$ be such that $\{p:p\in P,\ \rho(p,q)\le 2^{-m(q)}\}\subseteq Q$. For each $n\in\mathbb{N}$, let $\delta_n>0$ be such that $\rho(\sup I,\sup J)\le 2^{-n}$ whenever $\emptyset\ne I\subseteq J\subseteq P$ and $\#(J)\le 2^n$ and $\max_{q\in J}\min_{p\in I}\rho(p,q)\le 2\delta_n$; such exists because $\langle p_i\rangle_{i<k}\mapsto\sup_{i<k}p_i:P^k\to P$ is uniformly continuous whenever $k>0$, and in particular when $k=2^n$. Reducing the δ_n if necessary, we may suppose that $\delta_{n+1}\le\delta_n\le 2^{-n}$ for every n.

(c) Define $\phi:Q\to\ell^1(\kappa)$ as follows. Given $p\in Q$, choose a sequence $\langle\xi(p,n)\rangle_{n\in\mathbb{N}}$ in κ such that $\rho(p,q_{\xi(p,n)})\le\delta_{n+1}$ for every n. Take $\phi(p)\in\ell^1(\kappa)$ such that

$$\phi(p)(m(p))\ge 1,\quad \phi(p)(\xi(p,n))\ge 2^{-n}\text{ for every }n\in\mathbb{N}$$

(regarding $m(p)$ as a finite ordinal).

(d) Define $\psi:\ell^1(\kappa)\to[Q]^{<\omega}$ as follows. Given $x\in\ell^1(\kappa)$, set $K_n(x)=\{q_\xi:\xi<\kappa,\ x(\xi)\ge 2^{-n}\}$ for $n\in\mathbb{N}$. Then

$$\sum_{n=0}^\infty 2^{-n}\#(K_n(x))\le\sum_{\xi<\kappa}\sum\{2^{-n}:x(\xi)\ge 2^{-n}\}\le 2\|x\|_1<\infty,$$

so there is a $k(x)\in\mathbb{N}$ such that $x(n)<1$ for $n\in\omega\setminus k(x)$ and also $\#(K_n(x))\le 2^n$ for $n\ge k(x)$. Set $\tilde K(x)=K_{k(x)}(x)$. For $s\in\tilde K(x)$ set

$$I(x,s,k(x))=\{s\},\quad I(x,s,n+1)=\{q:q\in K_{n+1}(x),\ \rho(q,I(x,s,n))\le 2\delta_{n+1}\}$$

for $n \ge k(x)$, writing $\rho(q,I)$ for $\inf_{q' \in I} \rho(q,q')$. Because $\langle K_n(x)\rangle_{n\in\mathbb{N}}$ is non-decreasing, so is $\langle I(x,s,n)\rangle_{n\ge k(x)}$. Set $r_{xsn} = \sup I(x,s,n)$ in P for $n \ge k(x)$; then $\rho(r_{x,s,n+1}, r_{xsn}) \le 2^{-n-1}$ for every $n \ge k(x)$, by the choice of δ_{n+1}, so $r_{xs} = \lim_{n\to\infty} r_{xsn}$ is defined in P. Set $\psi(x) = Q \cap \{r_{xs} : s \in \tilde{K}(x)\}$.

(e) Now (ϕ,ψ) is a Galois-Tukey connection from $(Q, \le', [Q]^{<\omega})$ to $(\ell^1(\kappa), \le, \ell^1(\kappa))$. **P** Suppose that $p \in Q$ and $x \in \ell^1(\kappa)$ are such that $\phi(p) \le x$. Then $q_{\xi(p,n)} \in K_n(x)$ for every n, so $s = q_{\xi(p,k(x))} \in \tilde{K}(x)$. Also $q_{\xi(p,n)} \in I(x,s,n)$ for every $n \ge k(x)$, because

$$\rho(q_{\xi(p,n+1)}, q_{\xi(p,n)}) \le \delta_{n+2} + \delta_{n+1} \le 2\delta_{n+1}$$

for every n. So $q_{\xi(p,n)} \le r_{xsn}$ for every $n \ge k(x)$. It follows that

$$p \vee r_{xs} = \lim_{n\to\infty} q_{\xi(p,n)} \vee r_{xsn} = \lim_{n\to\infty} r_{xsn} = r_{xs}$$

and $p \le r_{xs}$.

By the choice of $k(x)$, we also have $\phi(p)(n) < 1$ for $n \ge k(x)$, so that $m(p) < k(x)$. We therefore have

$$\rho(r_{xs}, p) \le \rho(q_{\xi(p,k(x))}, p) + \sum_{n=k(x)}^{\infty} \rho(r_{x,s,n+1}, r_{xsn})$$

(because $s = q_{\xi(p,k(x))}$ is the unique member of $I(x,s,k(x))$, so is equal to $r_{x,s,k(x)}$)

$$\le \delta_{k(x)+1} + \sum_{n=k(x)}^{\infty} 2^{-n-1} \le 2^{-k(x)-1} + 2^{-k(x)} \le 2^{-m(p)}$$

and $r_{xs} \in Q$. So $p \le r_{xs} \in \psi(x)$ and $p \le' \psi(x)$. As p and x are arbitrary, (ϕ,ψ) is a Galois-Tukey connection. **Q**

(f) So $(Q, \le', [Q]^{<\omega}) \preccurlyeq_{\text{GT}} (\ell^1(\kappa), \le, \ell^1(\kappa))$, as claimed.

Finally, if Q is upwards-directed, then $\operatorname{add} Q \ge \omega$, so $(Q, \le, Q) \equiv_{\text{GT}} (Q, \le', [Q]^{<\omega})$ (513Id) and $(Q, \le, Q) \preccurlyeq_{\text{GT}} (\ell^1(\kappa), \le, \ell^1(\kappa))$, that is, $Q \preccurlyeq_{\text{T}} \ell^1(\kappa)$.

524D Proposition If κ is any cardinal,

$$(\mathfrak{B}_\kappa^-, \subseteq', [\mathfrak{B}_\kappa^-]^{<\omega}) \preccurlyeq_{\text{GT}} (\ell^1(\kappa), \le, \ell^1(\kappa)).$$

proof If κ is finite then $\mathfrak{B}_\kappa^-$ is finite and the result is trivial. Otherwise, if we give $\mathfrak{B}_\kappa$ its measure metric ρ (323Ad), then it is a complete metric space in which $\cup$ is uniformly continuous (323Gc, 323B) and $\mathfrak{B}_\kappa^- = \mathfrak{B}_\kappa \setminus \{1\}$ is an open set. Now the topological density of $\mathfrak{B}_\kappa^-$ and $\mathfrak{B}_\kappa$ is κ, by 521E; so 524C gives the result.

524E Proposition Let κ be an infinite cardinal. Then

$$(\ell^1(\kappa), \le', [\ell^1(\kappa)]^{\le\omega}) \preccurlyeq_{\text{GT}} (\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa).$$

proof (a) For each $i \in \mathbb{N}$, let $\langle z_{i\xi}\rangle_{\xi<\kappa}$ run over a norm-dense subset of $\{x : x \in \ell^1(\kappa)^+,\ \|x\|_1 \le 4^{-i}\}$. Now there is a function $\phi : \ell^1(\kappa) \to \kappa^{\mathbb{N}}$ such that

for every $x \in \ell^1(\kappa)$, $n \in \mathbb{N}$ there is a $k \in \mathbb{N}$ such that $x \le k\sum_{i=n}^{\infty} z_{i,\phi(x)(i)}$.

P Given $x \in \ell^1(\kappa)$, choose $\langle x_n\rangle_{n\in\mathbb{N}}$, $\langle \xi_n\rangle_{n\in\mathbb{N}}$, $\langle k_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Take $k_0 \ge 1$ such that $\|x^+\|_1 \le k_0$; set $x_0 = k_0^{-1}x^+$ and take $\xi_0 < \kappa$ such that $\|x_0 - z_{0\xi_0}\|_1 < \frac{1}{4}$. Given that $x_n \in \ell^1(\kappa)^+$, $\xi_n < \kappa$ are such that $\|x_n - z_{n\xi_n}\|_1 < 4^{-n-1}$, let $k_{n+1} \ge 1$ be such that $\|x_{n+1}\|_1 \le 4^{-n-1}$ where $x_{n+1} = (x_n - z_{n\xi_n})^+ + k_{n+1}^{-1}x^+$, and take $\xi_{n+1} < \kappa$ such that $\|x_{n+1} - z_{n+1,\xi_{n+1}}\|_1 < 4^{-n-2}$; continue. At the end of the process, set $\phi(x) = \langle \xi_n\rangle_{n\in\mathbb{N}}$.

Now, for any $n \in \mathbb{N}$, we have $x \le x^+ \le k_n x_n$. But we also have, for any $m \ge n$, $x_{m+1} \ge x_m - z_{m\xi_m}$, so that $x_n \le x_m + \sum_{i=n}^{m-1} z_{i\xi_i}$ for every $m \ge n$. Since $\|x_m\|_1 \le 4^{-m}$ for every m, $\lim_{m\to\infty} x_m = 0$ and

$$x \le k_n x_n \le k_n \sum_{i=n}^{\infty} z_{i,\phi(x)(i)}.$$

As x and n are arbitrary, ϕ is a suitable function. **Q**

(b) Define $\psi_0 : \mathcal{S}_\kappa \to \ell^1(\kappa)$ by setting $\psi_0(S) = \sum_{(i,\xi)\in S} z_{i\xi}$; because

$$\sum_{(i,\xi)\in S} \|z_{i\xi}\|_1 \le \sum_{i=0}^{\infty} 4^{-i}\#(S[\{i\}]) \le \sum_{i=0}^{\infty} 2^{-i}$$

is finite, $\psi_0(S)$ is well defined in $\ell^1(\kappa)$ for every $S \in \mathcal{S}_\kappa$ (4A4Ie). Now define $\psi : \mathcal{S}_\kappa \to [\ell^1(\kappa)]^{\le\omega}$ by setting $\psi(S) = \{k\psi_0(S) : k \in \mathbb{N}\}$ for $S \in \mathcal{S}_\kappa$.

(c) If $x \in \ell^1$ and $S \in \mathcal{S}_\kappa$ are such that $\phi(x) \subseteq^* S$, then $x \le' \psi(S)$. **P** Let $n \in \mathbb{N}$ be such that $(i, \phi(x)(i)) \in S$ for every $i \ge n$. Then there is a $k \in \mathbb{N}$ such that

$$x \le k\sum_{i=n}^{\infty} z_{i,\phi(x)(i)} \le k\sum_{(i,\xi)\in S} z_{i\xi} = k\psi_0(S) \in \psi(S). \text{ } \mathbf{Q}$$

(d) Thus (ϕ, ψ) is a Galois-Tukey connection and

$$(\ell^1(\kappa), \le', [\ell^1(\kappa)]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa).$$

524F Lemma Let (X, Σ, μ) be a countably compact measure space with Maharam type κ.

(a) If μ is a Maharam-type-homogeneous probability measure, there is a family $\langle E_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}(\mu)$ such that $\bigcup_{\xi\in A} E_\xi$ has full outer measure for every uncountable $A \subseteq \kappa$.

(b) If μ is σ-finite, there is a family $\langle E_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}(\mu)$ such that $\bigcup_{\xi\in A} E_\xi$ is non-negligible for every uncountable $A \subseteq \kappa$.

proof Let $\mathfrak{A}$ be the measure algebra of (X, Σ, μ).

(a) If κ is countable, we can take $E_\xi = \emptyset$ for every ξ. Otherwise, $\mathfrak{A}$ is τ-generated by a stochastically independent family $\langle a_\xi\rangle_{\xi<\kappa}$ of elements of measure $\frac{1}{2}$, and for every $G \in \Sigma$ there is a smallest countable set $I_G \subseteq \kappa$ such that $G^\bullet$ is in the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi \in I_G\}$ (254Rd or 325Mb). For each $\xi < \kappa$ choose $F_\xi \in \Sigma$ such that $F_\xi^\bullet = a_\xi$. Let $\mathcal{K}$ be a countably compact class such that μ is inner regular with respect to $\mathcal{K}$.

Let $\langle J_\xi\rangle_{\xi<\kappa}$ be a disjoint family of subsets of κ all with cardinal ω_1. For each $\xi < \kappa$ choose $\langle K_{\xi n}\rangle_{n\in\mathbb{N}}$, $\langle \alpha_{\xi n}\rangle_{n\in\mathbb{N}}$ inductively, as follows. $\alpha_{\xi 0} = \min J_\xi$. Given $\alpha_{\xi n}$ and $\langle K_{\xi i}\rangle_{i<n}$, let $K_{\xi n} \in \mathcal{K}$ be such that $K_{\xi n} \cap F_{\alpha_{\xi n}} = \emptyset$ and $\mu(K_{\xi n}) \ge \frac{1}{2}(1 - 3^{-n-2})$; now let $\alpha_{\xi,n+1}$ be a member of J_ξ not belonging to $I_{K_{\xi i}} \cup \{\alpha_{\xi i}\}$ for any $i \le n$. Continue. Set

$$E_\xi = \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} K_{\xi m} \subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} (X \setminus F_{\alpha_{\xi m}}),$$

so that E_ξ is negligible, because all the $\alpha_{\xi m}$ are different, so that $\langle F_{\alpha_{\xi m}}\rangle_{m\in\mathbb{N}}$ is stochastically independent.

Now suppose that $A \subseteq \kappa$ is uncountable, and that $F \subseteq X$ is measurable and not negligible. Let $K \in \mathcal{K}$ be such that $K \subseteq F$ and $\mu K > 0$; let $\xi \in A$ be such that $I_K \cap J_\xi = \emptyset$; let $n \in \mathbb{N}$ be such that $\mu K \ge 3^{-n-1}$. Set $G_m = K \cap \bigcap_{n\le i<m} K_{\xi i}$ for $m \ge n$. Then $I_{G_m} \subseteq I_K \cup \bigcup_{i<m} I_{K_{\xi i}}$ does not contain $\alpha_{\xi m}$, for any m. This means that

$$\mu G_{m+1} = \mu(G_m \cap K_{\xi m}) \ge \mu(G_m \setminus F_{\alpha_{\xi m}}) - \frac{3^{-m-2}}{2} = \frac{1}{2}(\mu G_m - 3^{-m-2})$$

for every $m \ge n$, and an easy induction shows that $\mu G_m \ge 3^{-m-1}$ for every m. But this tells us that every G_m is non-empty; because $\mathcal{K}$ is a countably compact class, $K \cap E_\xi \supseteq \bigcap_{m\ge n} G_m$ is non-empty, and F meets E_ξ.

As F is arbitrary, $\bigcup_{\xi\in A} E_\xi$ has full outer measure.

(b) For the general case, because μ is σ-finite, there is a countable partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that all the principal ideals $\mathfrak{A}_{a_i}$ are totally finite and Maharam-type-homogeneous (use 332A), and we can find a partition $\langle X_i\rangle_{i\in I}$ of X into measurable sets such that $X_i^\bullet = a_i$ for each i. Moreover, the subspace measure μ_{X_i} on X_i is countably compact (451Db). Writing κ_i for the Maharam type of $\mathfrak{A}_{a_i}$, there is a family $\langle E_{i\xi}\rangle_{\xi<\kappa_i}$ of negligible subsets of X_i such that $\{\xi : \xi < \kappa_i,\ E_{i\xi} \subseteq E\}$ is countable for every negligible set E. (Apply (a) to a scalar multiple of μ_{X_i}.) Now we know from 332S that $\kappa = \sup_{i\in I} \kappa_i = \#(\{(i,\xi) : i \in I,\ \xi < \kappa_i\})$. On the other hand, for any negligible set $E \subseteq X$, $\{(i,\xi) : i \in I,\ \xi < \kappa_i,\ E_{i\xi} \subseteq E\}$ is countable. So if we re-enumerate $\langle E_{i\xi}\rangle_{i\in I,\xi<\kappa_i}$ as $\langle E_\xi\rangle_{\xi<\kappa}$ we shall have an appropriate family.

524G Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Maharam-type-homogeneous Radon probability space with Maharam type $\kappa \ge \omega$. Then $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu))$.

proof (Compare 522M.)

(a) By 524F, there is a family $\langle E_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}(\mu)$ such that $\{\xi : E_\xi \subseteq E\}$ is countable for every $E \in \mathcal{N}(\mu)$. Next, because the measure algebra of μ is isomorphic to the measure algebra of the usual measure on $[0,1]^{\mathbb{N}\times\kappa}$, there is a stochastically independent family $\langle G_{i\xi}\rangle_{i\in\mathbb{N},\xi<\kappa}$ in Σ such that $\mu G_{i\xi} = 2^{-i}$ for every $i \in \mathbb{N}$ and $\xi < \kappa$. For $f \in \kappa^{\mathbb{N}}$ set

$$\phi(f) = \bigcup_{n\in\mathbb{N}} E_{f(n)} \cup \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} G_{m,f(m)} \in \mathcal{N}(\mu).$$

(b) Take $E \in \mathcal{N}(\mu)$ and set $I_E = \{\xi : E_\xi \subseteq E\}$, so that I_E is countable. Define $\pi_E : X \to \{0,1\}^{\mathbb{N}\times I_E}$ by setting $\pi_E(x)(i,\xi) = 1$ if $x \in G_{i\xi}$, 0 otherwise. Then there is a non-empty compact self-supporting set K_E such that $\pi_E{\restriction}K_E$ is continuous. **P** Then π_E is measurable, therefore almost continuous (418J), and there is a non-negligible measurable set $H \subseteq X \setminus E$ such that $\pi_E{\restriction}H$ is continuous. Because μ is inner regular with respect to the compact self-supporting sets, there is a non-negligible compact self-supporting $K_E \subseteq H$, and this has the required property. **Q**

$\pi_E[K_E]$ is compact. Let $\langle W_n(E)\rangle_{n\in\mathbb{N}}$ run over the family of open-and-closed subsets W of $\{0,1\}^{\mathbb{N}\times I_E}$ meeting $\pi_E[K_E]$. Then $\pi_E^{-1}[W_n(E)]$ is a non-empty relatively open subset of K_E for every n; because K_E is self-supporting, $\pi_E^{-1}[W_n(E)]$ is never negligible. Set

$$J(E,n,i) = \{\xi : \xi \in I_E,\ \pi_E^{-1}[W_n(E)] \cap G_{i\xi} = \emptyset\}$$

for n, $i \in \mathbb{N}$. Because $\langle G_{i\xi}\rangle_{i\in\mathbb{N},\xi\in I_E}$ is stochastically independent,

$$\textstyle\sum_{i=0}^\infty 2^{-i}\#(J(E,n,i)) = \sum\{\mu G_{i\xi} : i \in \mathbb{N},\ \xi \in I_E,\ G_{i\xi} \cap \pi_E^{-1}[W_n(E)] = \emptyset\}$$

is finite, by the Borel-Cantelli lemma (273K). For each n, let $k(E,n) \in \mathbb{N}$ be such that $2^{-i}\#(J(E,n,i)) \le 2^{-n-1}$ for $i \ge k(E,n)$, and set

$$\psi(E) = \textstyle\bigcup_{n\in\mathbb{N}}\{(i,\xi) : i \ge k(E,n),\ \xi \in J(E,n,i)\} \subseteq \mathbb{N}\times\kappa.$$

Then

$$\begin{aligned}\#(\{\xi : (i,\xi) \in \psi(E)\} &\le \sum_{n\in\mathbb{N},k(E,n)\le i} \#(J(E,n,i))\\ &\le \sum_{n\in\mathbb{N},k(E,n)\le i} 2^{-n-1}2^i \le 2^i\end{aligned}$$

for every $i \in \mathbb{N}$, and $\psi(E) \in \mathcal{S}_\kappa$.

(c) Now (ϕ,ψ) is a Galois-Tukey connection from $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa)$ to $(\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu))$. **P** Suppose that $f \in \kappa^{\mathbb{N}}$ and $E \in \mathcal{N}(\mu)$ are such that $\phi(f) \subseteq E$. Because $E_{f(n)} \subseteq \phi(f)$, $f(n) \in I_E$ for every $n \in \mathbb{N}$. Next, K_E does not meet $\phi(f)$, so $K_E \cap \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n} G_{m,f(m)}$ is empty, that is,

$$\pi_E[K_E] \cap \textstyle\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}\{w : w \in \{0,1\}^{\mathbb{N}\times I_E},\ w(m,f(m)) = 1\} = \emptyset.$$

By Baire's theorem, there is some $m \in \mathbb{N}$ such that

$$\pi_E[K_E] \cap \textstyle\bigcup_{i\ge m}\{w : w \in \{0,1\}^{\mathbb{N}\times I_E},\ w(i,f(i)) = 1\}$$

is not dense in $\pi_E[K_E]$, and there is an $n \in \mathbb{N}$ such that

$$W_n(E) \cap \textstyle\bigcup_{i\ge m}\{w : w \in \{0,1\}^{\mathbb{N}\times I_E},\ w(i,f(i)) = 1\} = \emptyset.$$

In this case, $f(i) \in J(E,n,i)$ for every $i \ge m$. But this means that $(i,f(i)) \in \psi(E)$ for every $i \ge \max(m,k(E,n))$, so that $f \subseteq^* \psi(E)$. As f and E are arbitrary, (ϕ,ψ) is a Galois-Tukey connection. **Q**

(d) Thus ϕ and ψ witness that $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu))$, as claimed.

524H Corollary Let κ be an infinite cardinal, and μ a Maharam-type-homogeneous Radon probability measure with Maharam type κ. Then $(\mathfrak{B}_\kappa^+, \supseteq', [\mathfrak{B}_\kappa^+]^{\le\omega})$, $(\ell^1(\kappa), \le', [\ell^1(\kappa)]^{\le\omega})$, $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa)$ and $(\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu))$ are Galois-Tukey equivalent.

proof By 512Gb, 524D and 524B,

$$(\mathfrak{B}_\kappa^-, \subseteq', [\mathfrak{B}_\kappa^-]^{<\omega_1}) \preccurlyeq_{\mathrm{GT}} (\mathfrak{B}_\kappa^-, \subseteq', [\mathfrak{B}_\kappa^-]^{<\omega}) \preccurlyeq_{\mathrm{GT}} (\ell^1(\kappa), \le, \ell^1(\kappa)),$$

$$(\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (\mathfrak{B}_\kappa^-, \subseteq, \mathfrak{B}_\kappa^-).$$

So

$$\begin{aligned}(\mathfrak{B}_\kappa^+, \supseteq', [\mathfrak{B}_\kappa^+]^{\le\omega}) &\cong (\mathfrak{B}_\kappa^-, \subseteq', [\mathfrak{B}_\kappa^-]^{\le\omega}) = (\mathfrak{B}_\kappa^-, \subseteq', [\mathfrak{B}_\kappa^-]^{<\omega_1})\\ &\preccurlyeq_{\mathrm{GT}} (\ell^1(\kappa), \le', [\ell^1(\kappa)]^{<\omega_1})\end{aligned}$$

(512Gd)

$$= (\ell^1(\kappa), \leq', [\ell^1(\kappa)]^{\leq\omega}) \preccurlyeq_{\mathrm{GT}} (\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa)$$

(524E)

$$\preccurlyeq_{\mathrm{GT}} (\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu))$$

(524G)

$$\equiv_{\mathrm{GT}} (\mathcal{N}(\mu), \subseteq', [\mathcal{N}(\mu)]^{\leq\omega})$$

(513Id again)

$$\preccurlyeq_{\mathrm{GT}} (\mathfrak{B}_\kappa^-, \subseteq', [\mathfrak{B}_\kappa^-]^{\leq\omega})$$

by 512Gb again.

524I Corollary Let μ be a Maharam-type-homogeneous Radon probability measure with infinite Maharam type κ. Then

$$\operatorname{add}\mathcal{N}(\mu) = \operatorname{add}\mathcal{N}_\kappa = \operatorname{add}_\omega \ell^1(\kappa),$$

$$\operatorname{cf}\mathcal{N}(\mu) = \operatorname{cf}\mathcal{N}_\kappa = \operatorname{cf}\ell^1(\kappa).$$

proof By 524H and 512Db,

$$\begin{aligned}\operatorname{add}(\ell^1(\kappa), \leq', [\ell^1(\kappa)]^{\leq\omega}) &= \operatorname{add}(\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu)) \\ &= \operatorname{add}(\mathcal{N}(\nu_\kappa), \subseteq, \mathcal{N}(\nu_\kappa)) = \operatorname{add}(\mathcal{N}_\kappa, \subseteq, \mathcal{N}_\kappa).\end{aligned}$$

But $\operatorname{add}(\ell^1(\kappa), \leq', [\ell^1(\kappa)]^{\leq\omega}) = \operatorname{add}_\omega \ell^1(\kappa)$ (513Ia), while $\operatorname{add}(\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu)) = \operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{add}(\mathcal{N}_\kappa, \subseteq, \mathcal{N}_\kappa) = \operatorname{add}\mathcal{N}_\kappa$ (512Ea). So

$$\operatorname{add}_\omega \ell^1(\kappa) = \operatorname{add}\mathcal{N}(\mu) = \operatorname{add}\mathcal{N}_\kappa.$$

On the other side, 512Da tells us that

$$\operatorname{cov}(\mathcal{N}_\kappa, \subseteq, \mathcal{N}_\kappa) = \operatorname{cov}(\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu)) = \operatorname{cov}(\ell^1(\kappa), \leq', [\ell^1(\kappa)]^{\leq\omega}).$$

But

$$\operatorname{cov}(\mathcal{N}_\kappa, \subseteq, \mathcal{N}_\kappa) = \operatorname{cf}\mathcal{N}_\kappa, \quad \operatorname{cov}(\mathcal{N}(\mu), \subseteq, \mathcal{N}(\mu)) = \operatorname{cf}\mathcal{N}(\mu)$$

(512Ea). Next, $\operatorname{cf}\ell^1(\kappa) > \omega$. **P** If $\langle x_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\ell^1(\kappa)$, then (because κ is infinite) $F_n = \{x : x \leq x_n\}$ is nowhere dense (for the norm topology) for any $n \in \mathbb{N}$, so $\langle F_n\rangle_{n\in\mathbb{N}}$ cannot cover $\ell^1(\kappa)$ (4A2Ma) and $\{x_n : n \in \mathbb{N}\}$ cannot be cofinal. **Q** So 512Gf tells us that

$$\operatorname{cov}(\ell^1(\kappa), \leq', [\ell^1(\kappa)]^{\leq\omega}) = \operatorname{cov}(\ell^1(\kappa), \leq, \ell^1(\kappa)) = \operatorname{cf}\ell^1(\kappa).$$

Putting these together,

$$\operatorname{cf}\mathcal{N}(\mu) = \operatorname{cf}\mathcal{N}_\kappa = \operatorname{cf}\ell^1(\kappa)$$

as required.

524J Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be Radon measure spaces with non-zero measure and isomorphic measure algebras.

(a) $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$ are Tukey equivalent, so $\operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu) = \operatorname{add}\mathcal{N}(\nu) = \operatorname{add}\nu$ and $\operatorname{cf}\mathcal{N}(\mu) = \operatorname{cf}\mathcal{N}(\nu)$.

(b) $(X, \in, \mathcal{N}(\mu))$ and $(Y, \in, \mathcal{N}(\nu))$ are Galois-Tukey equivalent, so $\operatorname{cov}\mathcal{N}(\mu) = \operatorname{cov}\mathcal{N}(\nu)$ and $\operatorname{non}\mathcal{N}(\mu) = \operatorname{non}\mathcal{N}(\nu)$.

proof (a) Let $\mathfrak{A}$, $\mathfrak{B}$ be the measure algebras of μ and ν. Let $\langle a_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}^+$ such that all the principal ideals $\mathfrak{A}_{a_i}$ are homogeneous and totally finite, and $\langle b_i\rangle_{i\in I}$ a matching family in $\mathfrak{B}$, so that $\mathfrak{A}_{a_i} \cong \mathfrak{B}_{b_i}$ for every i. Because (X, Σ, μ) and (Y, T, ν) are strictly localizable (416B), there are decompositions $\langle X_i\rangle_{i\in I}$ and $\langle Y_i\rangle_{i\in I}$ of X, Y respectively such that $X_i^\bullet = a_i$ and $Y_i^\bullet = b_i$ for every i (322M). Write μ_{X_i}, ν_{Y_i} for the corresponding subspace measures; of course these are Radon measures (416Rb). Then $\mathcal{N}(\mu_{X_i})$ and $\mathcal{N}(\nu_{Y_i})$ are Tukey equivalent for every i. **P** If the common Maharam type of $\mathfrak{A}_{a_i}$ and $\mathfrak{B}_{b_i}$ is infinite, this is a consequence of 524H. If $\mathfrak{A}_{a_i} = \{0, a_i\}$, then μ_{X_i} is purely atomic and there is a single point x of X_i such that $\mu\{x\} = \mu X_i$ (414G). In this case $\mathcal{N}(\mu_{X_i})$ has a greatest member $X_i \setminus \{x\}$, and similarly $\mathcal{N}(\nu_{\kappa_i})$ has a greatest member, so they have Tukey equivalent cofinal subsets and are Tukey equivalent (513E(d-ii)). **Q**

Now $E \mapsto \langle E \cap X_i\rangle_{i\in I}$ is a partially-ordered-set isomorphism between $\mathcal{N}(\mu)$ and $\prod_{i\in I}\mathcal{N}(\mu_{X_i})$. Similarly, $\mathcal{N}(\nu)$ is isomorphic to $\prod_{i\in I}\mathcal{N}(\nu_{Y_i})$. It now follows from 513Eg that $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$ are Tukey equivalent. Accordingly $\operatorname{add}\mathcal{N}(\mu) = \operatorname{add}\mathcal{N}(\nu)$ and $\operatorname{cf}\mathcal{N}(\mu) = \operatorname{cf}\mathcal{N}(\nu)$. By 521Ad, $\operatorname{add}\mu = \operatorname{add}\mathcal{N}(\mu)$ and $\operatorname{add}\nu = \operatorname{add}\mathcal{N}(\nu)$.

(b) Immediate from 521La, applied in both directions.

524K Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces with measure algebras $\mathfrak{A}$, $\mathfrak{B}$ respectively. If $\mathfrak{A}$ can be regularly embedded in $\mathfrak{B}$, then $\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\nu)$.

proof As usual, write $\bar\mu$ and $\bar\nu$ for the functionals on $\mathfrak{A}$, $\mathfrak{B}$ respectively defined from μ and ν, and let $\pi:\mathfrak{A}\to\mathfrak{B}$ be a regular embedding, that is, an order-continuous injective Boolean homomorphism.

(a) Consider first the case in which μ is totally finite and π is measure-preserving for $\bar\mu$ and $\bar\nu$. Let $(\tilde X,\tilde{\mathfrak{T}},\tilde\Sigma,\tilde\mu)$ and $(\tilde Y,\tilde{\mathfrak{S}},\tilde{\mathrm{T}},\tilde\nu)$ be the Stone spaces of $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ respectively. Then π corresponds to a continuous function $f:\tilde Y\to\tilde X$ (312Q). By 418I, the image measure $\tilde\nu f^{-1}$ is a Radon measure on $\tilde X$. If $a\in\mathfrak{A}$ and $\widehat{a}$ is the corresponding open-and-closed set in $\tilde X$, then

$$\tilde\nu f^{-1}[\widehat a] = \tilde\nu(\widehat{\pi a}) = \bar\nu(\pi a) = \bar\mu a = \tilde\mu\widehat a.$$

By 415H(v), $\tilde\nu f^{-1} = \tilde\mu$. By 521Hb, $\mathcal{N}(\tilde\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\tilde\nu)$. But now 524Ja tells us that

$$\mathcal{N}(\mu) \equiv_{\mathrm{T}} \mathcal{N}(\tilde\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\tilde\nu) \equiv_{\mathrm{T}} \mathcal{N}(\nu).$$

(b) Next, consider the case in which μ and ν are totally finite but π is not necessarily measure-preserving. As it is (sequentially) order-continuous, we have a measure μ' on X defined by saying that $\mu'E = \bar\nu(\pi E^\bullet)$ for $E\in\Sigma$, and $\mathcal{N}(\mu') = \mathcal{N}(\mu)$. Because μ' is absolutely continuous with respect to μ, it is an indefinite-integral measure over μ (234O) and is a Radon measure on X (416S). Taking $\bar\mu'$ to be the corresponding functional on $\mathfrak{A}$, $(\mathfrak{A},\bar\mu')$ is the measure algebra of μ' and π is measure-preserving for $\bar\mu'$ and $\bar\nu$. So (a) tells us that

$$\mathcal{N}(\mu) = \mathcal{N}(\mu') \preccurlyeq_{\mathrm{T}} \mathcal{N}(\nu).$$

(c) Thirdly, suppose that μ is totally finite, but ν might not be. Set $\mathfrak{B}^f = \{b : b\in\mathfrak{B},\ \bar\nu b<\infty\}$. For $b\in\mathfrak{B}^f$, set $c_b = \sup\{a : a\in\mathfrak{A},\ b\cap\pi a = 0\}$; then $b\cap\pi c_b = 0$, because π is order-continuous. If $a\in\mathfrak{A}\setminus\{0\}$, there is a $b\in\mathfrak{B}^f$ such that $b\cap\pi a\ne 0$, so that $a\not\subseteq c_b$. Accordingly $\sup_{b\in\mathfrak{B}^f} 1\setminus c_b = 1$ in $\mathfrak{A}$; as $\mathfrak{A}$ is ccc, there is a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}^f$ such that $\sup_{n\in\mathbb{N}} 1\setminus c_{b_n} = 1$, that is, $\bar\nu(a\cap\sup_{n\in\mathbb{N}} b_n)>0$ for every non-zero $a\in\mathfrak{A}$.

For each $n\in\mathbb{N}$, choose $F_n\in\mathrm{T}$ such that $F_n^\bullet = b_n$ in $\mathfrak{B}$, and set $Y' = \bigcup_{n\in\mathbb{N}} F_n$. The subspace measure $\nu_{Y'}$ is σ-finite, so there is a totally finite measure ν' on Y', an indefinite-integral measure over $\nu_{Y'}$, with the same null ideal as $\nu_{Y'}$ (use 215B(ix)). The measures $\nu_{Y'}$ and ν' are both Radon measures (416Rb, 416S). Setting $b = \sup_{n\in\mathbb{N}} b_n$ in $\mathfrak{B}$, the principal ideal $\mathfrak{B}_b$ can be identified with the measure algebra of $\nu_{Y'}$ (322I) and ν'. Moreover, the map $a\mapsto b\cap\pi a:\mathfrak{A}\to\mathfrak{B}_b$ is an injective order-continuous Boolean homomorphism. By (b) and 521Fa,

$$\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\nu') = \mathcal{N}(\nu_{Y'}) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\nu).$$

(d) For the general case, let $\langle a_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ such that $\bar\mu a_i$ is finite for every i, and set $b_i = \pi a_i$ for each i, so that $\langle b_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{B}$. As in the proof of 524J, we have corresponding partitions $\langle X_i\rangle_{i\in I}$, $\langle Y_i\rangle_{i\in I}$ of X, Y into measurable sets; as before, 322M tells us that $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$ can be identified with $\prod_{i\in I}\mathcal{N}(\mu_{X_i})$ and $\prod_{i\in I}\mathcal{N}(\nu_{Y_i})$ respectively. Now, for each i, we can identify the principal ideals $\mathfrak{A}_{a_i}$, $\mathfrak{B}_{b_i}$ with the measure algebras of the subspace measures μ_{X_i} and ν_{Y_i}, and $\pi{\restriction}\mathfrak{A}_{a_i}$ is an order-continuous embedding of $\mathfrak{A}_{a_i}$ in $\mathfrak{B}_{b_i}$. So (c) tells us that $\mathcal{N}(\mu_{X_i}) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\nu_{Y_i})$. Accordingly

$$\mathcal{N}(\mu) \cong \prod_{i\in I}\mathcal{N}(\mu_{X_i}) \preccurlyeq_{\mathrm{T}} \prod_{i\in I}\mathcal{N}(\nu_{Y_i}) \cong \mathcal{N}(\nu)$$

(513Eg again), and the proof is complete.

524L So far we have been looking at cardinals defined from null ideals. Of course there is an equally important series based on measurable algebras, which turns out to be similarly strongly associated with the cardinal functions of the ideals $\mathcal{N}_\kappa$. I have already developed a good deal of the machinery in the arguments of this section. But for 'linking numbers' we need a new idea, which is most clearly expressed in the context of homogeneous algebras.

Proposition (Dow & Steprāns 94) Let κ be an infinite cardinal. Then for any $n\ge 2$ the n-linking number $\operatorname{link}_n(\mathfrak{B}_\kappa)$ is the least λ such that $\kappa\le 2^\lambda$.

proof Let λ be the least cardinal such that $\kappa \le 2^\lambda$.

(a) By 514Cb, $\mathfrak{B}_\kappa$ is isomorphic, as partially ordered set, to a subset of $\mathcal{P}(\mathrm{link}(\mathfrak{B}_\kappa))$, so we must have

$$2^{\mathrm{link}(\mathfrak{B}_\kappa)} \ge \#(\mathfrak{B}_\kappa) \ge \kappa$$

and $\mathrm{link}(\mathfrak{B}_\kappa) \ge \lambda$. It follows at once that $\mathrm{link}_n(\mathfrak{B}_\kappa) \ge \lambda$ for every $n \ge 2$ (511Ia).

(b) Now let $n \ge 2$, and take an injective function $\phi : \kappa \to \{0,1\}^\lambda$. Let $\mathcal{C}$ be the family of measurable cylinders in $\{0,1\}^\kappa$, that is, sets of the form $\{x : x \in \{0,1\}^\kappa, x{\restriction}I = z\}$, where $I \subseteq \kappa$ is finite and $z \in \{0,1\}^I$. For each $E \in \mathrm{T}_\kappa \setminus \mathcal{N}_\kappa$ we can find disjoint finite sets I'_E, I''_E, $J_E \subseteq \kappa$ and $G_E \in \mathrm{T}_\kappa$ such that

setting $C_E = \{x : x \in \{0,1\}^\kappa, x(\xi) = 0$ for $\xi \in I'_E$ and $x(\xi) = 1$ for $\xi \in I''_E\}$, and $k_E = \#(I'_E) + \#(I''_E)$, $\nu_\kappa(C_E \setminus E) \le \frac{1}{4n}\nu_\kappa C_E = \frac{1}{4n} \cdot 2^{-k_E}$;

G_E is determined by coordinates in J_E and $\nu_\kappa(C_E \cap (E \triangle G_E)) \le \frac{1}{4n} \cdot 2^{-nk_E}$;

$\nu_\kappa G_E \ge 1 - \frac{1}{2n}$.

P By 254Fe, there is a set W, expressible as the union of finitely many measurable cylinders, such that $\nu_\kappa(E \triangle W) \le \frac{1}{5n}\nu_\kappa E$. Now $\nu_\kappa W \ge \frac{9}{10}\nu_\kappa E$ so $\nu_\kappa(W \setminus E) \le \frac{1}{4n}\nu_\kappa W$. W is determined by coordinates in a finite set, so is expressible as a disjoint union of non-empty measurable cylinders, and for at least one of these we must have $\nu_\kappa(C \setminus E) \le \frac{1}{4n}\nu_\kappa C$; take such a one for C_E. Express C_E as $\{x : x{\restriction}I_E = z_E\}$, where $I_E \subseteq \kappa$ is finite and $z_E \in \{0,1\}^{I_E}$, and set $I'_E = \{\xi : \xi \in I_E, z_E(\xi) = 0\}$ and $I''_E = \{\xi : \xi \in I_E, z_E(\xi) = 1\}$; then $\nu_\kappa C_E = 2^{-k_E}$ and $\nu_\kappa(C_E \setminus E) \le \frac{1}{4}n \cdot 2^{-k_E}$.

Next, take a set $W' \subseteq \{0,1\}^\kappa$, determined by coordinates in a finite subset J of κ, such that $\nu_\kappa(E \triangle W') \le \frac{1}{4n} \cdot 2^{-nk_E}$. Set

$$G_E = \{x : x \in \{0,1\}^\kappa, \exists\, y \in W' \cap C_E, x{\restriction}\kappa \setminus I_E = y{\restriction}\kappa \setminus I_E\},$$

so that G_E is determined by coordinates in $J_E = J \setminus I_E$ and $G_E \cap C_E = W' \cap C_E$; accordingly

$$\nu_\kappa(C_E \cap (E \triangle G_E)) = \nu_\kappa(C_E \cap (E \triangle W')) \le \nu_\kappa(E \triangle W') \le \frac{1}{4n} \cdot 2^{-nk_E}.$$

Note that G_E and C_E are stochastically independent, so that

$$\begin{aligned}\nu_\kappa C_E(1 - \nu_\kappa G_E) = \nu_\kappa(C_E \setminus G_E) &\le \nu_\kappa(C_E \setminus E) + \nu_\kappa(C_E \cap (E \setminus G_E))\\ &\le \frac{1}{4n}\nu_\kappa C_E + \frac{1}{4n}(\nu_\kappa C_E)^n \le \frac{1}{2n}\nu_\kappa C_E\end{aligned}$$

and $\nu_\kappa G_E \ge 1 - \frac{1}{2n}$. **Q**

(c) Let Q be the set of all quadruples (k, U, V, W) where $k \in \mathbb{N}$ and U, V, W are disjoint open-and-closed subsets of $\{0,1\}^\lambda$ in its usual topology. For $q = (k, U, V, W) \in Q$, set

$$\mathcal{E}_q = \{E : E \in \mathrm{T}_\kappa \setminus \mathcal{N}_\kappa,\ k_E = k,\ \phi[I'_E] \subseteq U,\ \phi[I''_E] \subseteq V,\ \phi[J_E] \subseteq W\}.$$

For any $E \in \mathrm{T}_\kappa \setminus \mathcal{N}_\kappa$, I'_E, I''_E and J_E, as chosen in (b) above, are disjoint finite sets, so $\phi[I'_E]$, $\phi[I''_E]$ and $\phi[J_E]$ also are, and there is a $q \in Q$ such that $E \in \mathcal{E}_q$. Now if $q = (k, U, V, W) \in Q$ and $E_i \in \mathcal{E}_q$ for $i < n$, then $\nu_\kappa(\bigcap_{i<n} E_i) > 0$. **P** Set $I' = \bigcup_{i<n} I'_{E_i}$, $I'' = \bigcup_{i<n} I''_{E_i}$ and $J = \bigcup_{i<n} J_{E_i}$. Then $\phi[I'] \subseteq U$, $\phi[I''] \subseteq V$ and $\phi[J] \subseteq W$, so that I', I'' and J must be disjoint. Set

$$C = \bigcap_{i<n} C_{E_i} = \{x : x \in \{0,1\}^\kappa,\ x(\xi) = 0 \text{ for } \xi \in I',\ x(\xi) = 1 \text{ for } \xi \in I''\};$$

then $\nu_\kappa C = 2^{-\#(I' \cup I'')} \ge 2^{-nk}$. Next, setting $G = \bigcap_{i<n} G_{E_i}$,

$$\nu_\kappa G \ge 1 - \sum_{i=0}^{n-1}(1 - \nu_\kappa G_{E_i}) \ge \frac{1}{2},$$

and G is stochastically independent of C, so that $\nu_\kappa(C \cap G) \ge 2^{-nk-1}$. Finally,

$$\nu_\kappa(C \cap G \setminus E_i) \le \nu_\kappa(C_{E_i} \cap G_{E_i} \setminus E_i) \le \frac{1}{4n} \cdot 2^{-nk}$$

for each i, so

$$\nu_\kappa(C\cap G\setminus\bigcap_{i<n}E_i)\le 2^{-nk-2}<\nu_\kappa(C\cap G)$$

and $\nu_\kappa(\bigcap_{i<n}E_i)>0$. **Q**

(d) This means that if we set $A_q=\{E^\bullet : E\in\mathcal{E}_q\}$ for each $q\in Q$, then every A_q is an n-linked set in $\mathfrak{B}_\kappa$ and $\bigcup_{q\in Q}A_q=\mathfrak{B}_\kappa^+$. Because $\{0,1\}^\lambda$ is a compact topological space with a subbase of size $\lambda\ge\omega$, it has λ open-and-closed sets and $\#(Q)=\lambda$. So $\langle A_q\rangle_{q\in Q}$ witnesses that $\operatorname{link}_n(\mathfrak{B}_\kappa)\le\lambda$, and the proof is complete.

524M Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Let K be the set of infinite cardinals κ such that $\mathfrak{A}$ has a homogeneous principal ideal with Maharam type κ.

(a)
$$\begin{aligned}\#(\mathfrak{A})&=2^{c(\mathfrak{A})}\text{ if }\mathfrak{A}\text{ is finite,}\\&=\tau(\mathfrak{A})^\omega\text{ if }\mathfrak{A}\text{ is ccc and infinite.}\end{aligned}$$

(b)
$$\begin{aligned}\operatorname{wdistr}(\mathfrak{A})&=\infty\text{ if }\mathfrak{A}\text{ is purely atomic,}\\&=\operatorname{add}\mathcal{N}\text{ if }K=\{\omega\},\\&=\omega_1\text{ otherwise.}\end{aligned}$$

(c)
$$\begin{aligned}\pi(\mathfrak{A})&=c(\mathfrak{A})\text{ if }\mathfrak{A}\text{ is purely atomic,}\\&=\max(c(\mathfrak{A}),\operatorname{cf}\mathcal{N},\sup_{\kappa\in K}\operatorname{cf}[\kappa]^{\le\omega})\text{ otherwise.}\end{aligned}$$

(d)
$$\begin{aligned}\mathfrak{m}(\mathfrak{A})&=\infty\text{ if }\mathfrak{A}\text{ is purely atomic,}\\&=\min_{\kappa\in K}\operatorname{cov}\mathcal{N}_\kappa\text{ otherwise.}\end{aligned}$$

(e)
$$\begin{aligned}d(\mathfrak{A})&=c(\mathfrak{A})\text{ if }\mathfrak{A}\text{ is purely atomic,}\\&=\max(c(\mathfrak{A}),\sup_{\kappa\in K}\operatorname{non}\mathcal{N}_\kappa)\text{ otherwise.}\end{aligned}$$

(f) For $2\le n<\omega$,

$$\begin{aligned}\operatorname{link}_n(\mathfrak{A})&=c(\mathfrak{A})\text{ if }\mathfrak{A}\text{ is purely atomic,}\\&=\max(c(\mathfrak{A}),\min\{\lambda:\tau(\mathfrak{A})\le 2^\lambda\})\text{ otherwise.}\end{aligned}$$

proof The case $\mathfrak{A}=\{0\}$ is trivial, so I shall assume henceforth that $\mathfrak{A}\ne\{0\}$. Let $\langle a_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}^+$ such that all the principal ideals $\mathfrak{A}_{a_i}$ are homogeneous and totally finite. For each $i\in I$, set $\kappa_i=\tau(\mathfrak{A}_{a_i})$, so that $\mathfrak{A}_{a_i}\cong\mathfrak{B}_{\kappa_i}$, and let (Z_i,λ_i) be the Stone space of $(\mathfrak{A}_{a_i},\bar\mu\restriction\mathfrak{A}_{a_i})$. Let $(\widehat{\mathfrak{A}},\tilde\mu)$ be the localization of $(\mathfrak{A},\bar\mu)$ (322Q). $\mathfrak{A}$ can be identified with an order-dense Boolean subalgebra of $\widehat{\mathfrak{A}}$, so that $\langle a_i\rangle_{i\in I}$ is still a partition of unity in $\widehat{\mathfrak{A}}$. Because $\mathfrak{A}^f=\widehat{\mathfrak{A}}^f$ (322P), $\mathfrak{A}_{a_i}$ is still a principal ideal of $\widehat{\mathfrak{A}}$, and $\widehat{\mathfrak{A}}$ can be identified with the simple product $\prod_{i\in I}\mathfrak{A}_{a_i}$ (315F).

(a) This is elementary if $\mathfrak{A}$ is finite (see 511Ic). If $\mathfrak{A}$ is infinite, then 515Ma tells us that $\#(\mathfrak{A})=\tau(\mathfrak{A})^\omega$.

(b)

$$\operatorname{wdistr}(\mathfrak{A})=\operatorname{wdistr}(\widehat{\mathfrak{A}})$$

(514Ee)

$$=\min_{i\in I}\operatorname{wdistr}(\mathfrak{A}_{a_i})$$

(514Ef)

$$=\min_{i\in I}\operatorname{wdistr}(\mathfrak{B}_{\kappa_i})=\min_{i\in I}\operatorname{add}(\mathcal{N}(\lambda_i))$$

(514Be, because $\mathcal{N}(\lambda_i)$ is the ideal of nowhere dense subsets of Z, by 322R)

$$=\min_{i\in I}\operatorname{add}(\mathcal{N}_{\kappa_i})$$

(524Ja)

$$\begin{aligned}&=\infty\text{ if }K=\emptyset,\\&=\operatorname{add}\mathcal{N}\text{ if }K=\{\omega\},\\&=\omega_1\text{ otherwise}\end{aligned}$$

(523E).

(c)(i) Consider first an algebra $\mathfrak{B}_\kappa$, where $\kappa \geq \omega$. Then $\operatorname{ci}\mathfrak{B}_\kappa^+ > \omega$. **P** If $\langle b_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{B}_\kappa^+$, then (because $\mathfrak{B}_\kappa$ is atomless) we can choose $c_n \subseteq b_n$ such that $0 < \bar\nu_\kappa c_n \leq 2^{-n-2}$ for each $n \in \mathbb{N}$. Set $c = \sup_{n\in\mathbb{N}} c_n$, $b = 1\setminus c$; then $b \neq 0$ and $b_n \not\subseteq b$ for every n, so $\{b_n : n \in \mathbb{N}\}$ is not coinitial with $\mathfrak{B}_\kappa^+$. **Q**

It follows that

$$\operatorname{ci}\mathfrak{B}_\kappa^+ = \operatorname{cov}(\mathfrak{B}_\kappa^+, \supseteq, \mathfrak{B}_\kappa^+) = \operatorname{cov}(\mathfrak{B}_\kappa^+, \supseteq', [\mathfrak{B}_\kappa^+]^{\leq\omega})$$

(512Gf)

$$= \operatorname{cov}(\mathcal{N}_\kappa, \subseteq, \mathcal{N}_\kappa)$$

(524H)

$$= \operatorname{cf}\mathcal{N}_\kappa.$$

(ii) If $\mathfrak{A}$ is purely atomic, then $\mathfrak{A}_{a_i} = \{0, a_i\}$ for every i, and $\pi(\mathfrak{A}) = \#(I) = c(\mathfrak{A})$. Otherwise,

$$\max(c(\mathfrak{A}), \sup_{i\in I}\pi(\mathfrak{A}_{a_i})) \leq \pi(\mathfrak{A})$$

(514Da, 514Ed)

$$\leq \max(\omega, c(\mathfrak{A}), \sup_{i\in I}\pi(\mathfrak{A}_{a_i}))$$

(514Ef)

$$= \max(c(\mathfrak{A}), \sup_{\kappa\in K}\pi(\mathfrak{B}_\kappa)) = \max(c(\mathfrak{A}), \sup_{\kappa\in K}\operatorname{cf}\mathcal{N}_\kappa)$$

(by (i))

$$= \max(c(\mathfrak{A}), \operatorname{cf}\mathcal{N}, \sup_{\kappa\in K}\operatorname{cf}[\kappa]^{\leq\omega})$$

by 523N.

(d) If $\mathfrak{A}$ is purely atomic, then $\mathfrak{m}(\mathfrak{A}) = \infty$ (511If). Otherwise,

$$\mathfrak{m}(\mathfrak{A}) = \mathfrak{m}(\widehat{\mathfrak{A}})$$

(517Id)

$$= \min_{i\in I}\mathfrak{m}(\mathfrak{A}_{a_i}) = \min_{i\in I} n(Z_i)$$

(517N)

$$= \min_{i\in I}\operatorname{cov}\mathcal{N}(\lambda_i)$$

(again because $\mathcal{N}(\lambda_i)$ is the ideal of nowhere dense subsets of Z_i)

$$= \min_{i\in I}\operatorname{cov}\mathcal{N}_{\kappa_i}$$

(524Jb)

$$= \min_{\kappa\in K}\operatorname{cov}\mathcal{N}_\kappa,$$

as claimed.

(e)(i) I note first that $d(\mathfrak{A}_{a_i}) = \operatorname{non}\mathcal{N}_{\kappa_i}$ for each i. **P** Let $A \in \mathcal{P}Z_i \setminus \mathcal{N}(\lambda_i)$ be a set with cardinal $\operatorname{non}\mathcal{N}(\lambda_i)$. Then $H = \operatorname{int}\overline{A}$ is not empty. Let $a \in \mathfrak{A}_{a_i}^+$ be such that the corresponding open-and-closed set $\widehat{a}$ is included in H. Then $\widehat{a}$ can be identified with the Stone space of $\mathfrak{A}_a$ (312T); because $\mathfrak{A}_{a_i}$ is homogeneous, and $A\cap\widehat{a}$ is dense in $\widehat{a}$,

$$d(\mathfrak{A}_{a_i}) = d(\mathfrak{A}_a) = d(\widehat{a})$$

(514Bd)

$$\leq \#(A \cap \widehat{a}) \leq \operatorname{non}\mathcal{N}(\lambda_i) = \operatorname{non}\mathcal{N}_{\kappa_i}$$

(524Jb)

$$\leq d(Z_i)$$

(because $\mathcal{N}(\lambda_i)$ is the ideal of nowhere dense subsets of Z_i, so surely contains no dense set)

$$= d(\mathfrak{A}_{a_i})$$

by 514Bd again. **Q**

(ii) If $\mathfrak{A}$ is purely atomic, $d(\mathfrak{A}) = c(\mathfrak{A})$. Otherwise,

$$\max(c(\mathfrak{A}), \sup_{i \in I} d(\mathfrak{A}_{a_i})) \leq d(\mathfrak{A})$$

(514Da, 514Ed)

$$= d(\widehat{\mathfrak{A}})$$

(514Ee)

$$\leq \max(\omega, c(\mathfrak{A}), \sup_{i \in I} d(\mathfrak{A}_{a_i}))$$

(514Ef)

$$= \max(c(\mathfrak{A}), \sup_{i \in I} \operatorname{non}\mathcal{N}_{\kappa_i}) = \max(c(\mathfrak{A}), \sup_{\kappa \in K} \operatorname{non}\mathcal{N}_\kappa).$$

(f) If $\mathfrak{A}$ is purely atomic, this is elementary, since any linked subset of $\mathfrak{A}^+$ can contain at most one atom. Otherwise, set

$$\theta = \max(c(\mathfrak{A}), \min\{\lambda : \tau(\mathfrak{A}) \leq 2^\lambda\}), \quad \theta' = \operatorname{link}_n(\mathfrak{A}).$$

For any $i \in I$, $\kappa_i \leq \tau(\mathfrak{A})$ (514Ed), so $\kappa_i \leq 2^\theta$ and $\operatorname{link}_n(\mathfrak{A}_{a_i}) = \operatorname{link}_n(\mathfrak{B}_{\kappa_i}) \leq \theta$ (524L; of course the case $\kappa_i = 0$ is trivial here). Accordingly

$$\theta' = \operatorname{link}_n(\widehat{\mathfrak{A}})$$

(514Ee)

$$\leq \max(\omega, c(\mathfrak{A}), \sup_{i \in I} \operatorname{link}_n(\mathfrak{A}_{a_i}))$$

(514Ef)

$$\leq \theta.$$

On the other hand, $c(\mathfrak{A}) \leq \theta'$ (514Da). For each $i \in I$, $\operatorname{link}_n(\mathfrak{A}_i) \leq \theta'$ (514Ed), so $\kappa_i \leq 2^{\theta'}$ (524L, in the other direction). Let A_i be a τ-generating subset of $\mathfrak{A}_{a_i}$ of size κ_i. Now the order-closed subalgebra of $\mathfrak{A}$ generated by $A = \{a_i : i \in I\} \cup \bigcup_{i \in I} A_i$ is $\mathfrak{A}$, so

$$\tau(\mathfrak{A}) \leq \#(A) = \max(c(\mathfrak{A}), \sup_{i \in I} \kappa_i) \leq \max(\theta', 2^{\theta'}) = 2^{\theta'}.$$

But this means that $\theta \leq \theta'$ and the two are equal.

Remark For the corresponding calculation of $\tau(\mathfrak{A})$, when $(\mathfrak{A}, \bar{\mu})$ is localizable, see 332S.

524N Corollary (a) If (X, Σ, μ) is a semi-finite locally compact measure space, with $\mu X > 0$, then $\operatorname{cov}\mathcal{N}(\mu) \geq \mathfrak{m}_{\sigma\text{-linked}}$.

(b) If $\mathfrak{A}$ is any measurable algebra, then $\mathfrak{m}(\mathfrak{A}) \geq \mathfrak{m}_{\sigma\text{-linked}}$.

proof (a) Because μ is semi-finite and $\mu X > 0$, there is an $E \in \Sigma$ such that $0 < \mu E < \infty$. The subspace measure μ_E on E is compact, so $\nu = \frac{1}{\mu E}\mu_E$ is a compact probability measure. Set $\kappa = \max(\omega, \tau(\nu))$. Because ν is a compact measure, there is a function $f : \{0,1\}^\kappa \to E$ which is inverse-measure-preserving for ν_κ and ν (343Cd). Now

$$\mathfrak{m}_{\sigma\text{-linked}} \le \mathfrak{m}(\mathfrak{B}_{\mathfrak{c}})$$

(because $\mathfrak{B}_{\mathfrak{c}}$ is σ-linked, by 524Mf)

$$= \operatorname{cov} \mathcal{N}_{\mathfrak{c}}$$

(524Md)

$$\le \operatorname{cov} \mathcal{N}_{\kappa}$$

(523F)

$$\le \operatorname{cov} \mathcal{N}(\nu)$$

(521Ha)

$$= \operatorname{cov} \mathcal{N}(\mu_E) \le \operatorname{cov} \mathcal{N}(\mu)$$

(521Fb).

(b) This is now immediate from 524Md.

524O Freese-Nation numbers I spell out those facts about Freese-Nation numbers of measure algebras which can be read off from the results in §518.

Proposition (a) Let $(\mathfrak{A}, \bar{\mu})$ be an infinite measure algebra. Then $\mathrm{FN}(\mathfrak{A}) \ge \mathrm{FN}(\mathcal{P}\mathbb{N})$.

(b) Let $\mathfrak{A}$ be a measurable algebra.

(i) $\mathrm{FN}(\mathfrak{A}) \le \mathfrak{c}^+$.

(ii) If $\tau(\mathfrak{A}) \le \mathfrak{c}$ then $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(\mathcal{P}\mathbb{N})$.

(iii) If

(α) $\mathrm{cf}([\lambda]^{\le\omega}) \le \lambda^+$ for every cardinal $\lambda \le \tau(\mathfrak{A})$,

(β) $\square_\lambda$ is true for every uncountable cardinal $\lambda \le \tau(\mathfrak{A})$ of countable cofinality,

then $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

(c) Suppose that the continuum hypothesis and $\mathrm{CTP}(\omega_{\omega+1}, \omega_\omega)$ are both true. If $\mathfrak{A}$ is a measurable algebra, then

$$\begin{aligned} \mathrm{FN}(\mathfrak{A}) &= \mathfrak{c} = \omega_1 \text{ if } \omega \le \tau(\mathfrak{A}) < \omega_\omega, \\ &= \mathfrak{c}^+ = \omega_2 \text{ otherwise }. \end{aligned}$$

proof (a) This is a special case of 518Ca.

(b)(i) Consider first the case $\mathfrak{A} = \mathfrak{B}_\kappa$ for some cardinal κ. For $I \subseteq \kappa$, let $\mathfrak{C}_I$ be the closed subalgebra of $\mathfrak{B}_\kappa$ consisting of those $a \in \mathfrak{B}_\kappa$ expressible in the form $E^\bullet$ for some measurable $E \subseteq \{0,1\}^\kappa$ determined by coordinates in I. For $a \in \mathfrak{B}_\kappa$, there is a smallest subset I_a of λ such that $a \in \mathfrak{C}_I$ (325M again); I_a is always countable.

For each $a \in \mathfrak{B}_\kappa$, set

$$f(a) = \{b : I_b \subseteq I_a\}.$$

Then $\#(f(a)) \le \mathfrak{c}$. If $a \subseteq b$, then there is a $c \in \mathfrak{B}_\kappa$ such that $a \subseteq c \subseteq b$ and $I_c \subseteq I_a \cap I_b$ (325M(b-ii)). So f is a Freese-Nation function. This shows that $\mathrm{FN}(\mathfrak{B}_\kappa) \le \mathfrak{c}^+$.

In general, $\mathfrak{A}$ is either $\{0\}$ or isomorphic to a closed subalgebra of $\mathfrak{B}_\kappa$ where $\kappa = \max(\omega, \tau(\mathfrak{A}))$, so $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(\mathfrak{B}_\kappa) \le \mathfrak{c}^+$ by 518Cc.

(ii) $\mathfrak{A}$ is σ-linked (524Mf), so 518D(iii) tells us that $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(\mathcal{P}\mathbb{N})$.

(iii) If $\mathfrak{B} \subseteq \mathfrak{A}$ is a countably generated order-closed subalgebra, then $\mathrm{FN}(\mathfrak{B}) \le \mathrm{FN}(\mathcal{P}\mathbb{N})$, by (ii); so 518I tells us that $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}^*(\mathcal{P}\mathbb{N})$.

(c) If $\tau(\mathfrak{A}) < \omega_\omega$ then $\mathrm{cf}[\lambda]^{\le\omega} = \lambda$ for $\omega_1 \le \lambda \le \tau(\mathfrak{A})$ (5A1E(e-iv)), so we can use (a) and (b-iii); otherwise use (b-i) and 518K.

524P The Maharam classification If the cardinal functions of a Radon measure space are determined by its measure algebra, there ought to be some way of calculating them directly from the classification of measure algebras in §332. In many cases this is straightforward.

Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, and $\mathfrak{A}$ its measure algebra. Let K be the set of infinite cardinals κ such that the Maharam-type-κ component of $\mathfrak{A}$ is non-zero.

(a)
$$\begin{aligned}\operatorname{add}\mu=\operatorname{add}\mathcal{N}(\mu)&=\infty \text{ if } K=\emptyset,\\ &=\operatorname{add}\mathcal{N} \text{ if } K=\{\omega\},\\ &=\omega_1 \text{ otherwise.}\end{aligned}$$

(b)
$$\begin{aligned}\pi(\mu)=\pi(\mathfrak{A})&=c(\mathfrak{A}) \text{ if } K=\emptyset,\\ &=\max(c(\mathfrak{A}),\operatorname{cf}\mathcal{N},\sup_{\kappa\in K}\operatorname{cf}[\kappa]^{\le\omega}) \text{ otherwise.}\end{aligned}$$

(c)
$$\begin{aligned}\operatorname{cov}\mathcal{N}(\mu)&=1 \text{ if } \mathfrak{A}=\{0\},\\ &=\infty \text{ if } \mathfrak{A} \text{ has an atom},\\ &=\operatorname{cov}\mathcal{N}_{\min K} \text{ otherwise.}\end{aligned}$$

(d)
$$\begin{aligned}\operatorname{non}\mathcal{N}(\mu)&=\infty \text{ if } \mathfrak{A}=\{0\},\\ &=1 \text{ if } \mathfrak{A} \text{ has an atom},\\ &=\operatorname{non}\mathcal{N}_{\min K} \text{ otherwise.}\end{aligned}$$

(e)
$$\begin{aligned}\operatorname{shr}\mathcal{N}(\mu)&=0 \text{ if } \mathfrak{A}=\{0\},\\ &=1 \text{ if } \mathfrak{A} \text{ has an atom},\\ &\ge\operatorname{shr}\mathcal{N} \text{ otherwise.}\end{aligned}$$

(f) If μ is σ-finite,
$$\begin{aligned}\operatorname{cf}\mathcal{N}(\mu)&=1 \text{ if } K=\emptyset,\\ &=\max(\operatorname{cf}\mathcal{N},\operatorname{cf}[\tau(\mathfrak{A})]^{\le\omega}) \text{ otherwise.}\end{aligned}$$

proof If $\mu X=0$ all these results are trivial, so let us suppose henceforth that $\mu X>0$. As in part (a) of the proof of 524J, there is a decomposition $\langle X_i\rangle_{i\in I}$ of X such that the subspace measures μ_{X_i} are all Maharam-type-homogeneous and non-zero. Note that $\max(\omega,\#(I))=\max(\omega,c(\mathfrak{A}))$ (332E). For each $i\in I$, let κ_i be the Maharam type of μ_{X_i}.

(a) By 521Ad, $\operatorname{add}\mu=\operatorname{add}\mathcal{N}(\mu)$. The map $E\mapsto\langle E\cap X_i\rangle_{i\in I}$ identifies $\mathcal{N}(\mu)$, as partially ordered set, with the product of the family $\langle\mathcal{N}(\mu_{X_i})\rangle_{i\in I}$. So $\operatorname{add}\mathcal{N}(\mu)=\min_{i\in I}\operatorname{add}\mathcal{N}(\mu_{X_i})$ (511Hg). Now if $i\in I$ and $\kappa_i=0$, X_i is an atom of (X,Σ,μ), so there is an $x_i\in X_i$ such that $\mu(X_i\setminus\{x_i\})=0$ (414G again). In this case, $X_i\setminus\{x_i\}$ is the largest member of $\mathcal{N}(\mu_{X_i})$ and $\operatorname{add}\mathcal{N}(\mu_{X_i})=\infty$. If κ_i is infinite, then $\operatorname{add}\mathcal{N}(\mu_{X_i})=\operatorname{add}\mathcal{N}_{\kappa_i}$, by 524I applied to a scalar multiple of μ_{X_i}. So $\operatorname{add}\mathcal{N}(\mu)=\min_{\kappa\in K}\operatorname{add}\mathcal{N}_\kappa$, interpreting this as ∞ if $K=\emptyset$. But we know from 523E that $\operatorname{add}\mathcal{N}_\kappa=\omega_1$ if $\kappa>\omega$, while of course $\operatorname{add}\mathcal{N}_\omega=\operatorname{add}\mathcal{N}$. It follows at once that

$$\begin{aligned}\operatorname{add}\mathcal{N}(\mu)=\min_{i\in I}\operatorname{add}\mathcal{N}(\mu_{X_i})&=\infty \text{ if } K=\emptyset,\\ &=\operatorname{add}\mathcal{N} \text{ if } K=\{\omega\},\\ &=\omega_1 \text{ otherwise.}\end{aligned}$$

(b) By 521Dd, $\pi(\mu)=\pi(\mathfrak{A})$; and 524Mc gives us the formula for $\pi(\mathfrak{A})$.

(c) If $\mathcal{E}$ is a cover of X by negligible sets, and $i\in I$, then $\{E\cap X_i:E\in\mathcal{E}\}$ is a cover of X_i by negligible sets; thus $\operatorname{cov}\mathcal{N}(\mu)\ge\sup_{i\in I}\operatorname{cov}\mathcal{N}(\mu_{X_i})$. By 524Jb, $\operatorname{cov}\mathcal{N}(\mu)\ge\sup_{i\in I}\operatorname{cov}\mathcal{N}_{\kappa_i}$. If any of the κ_i is zero, that is, if $\mathfrak{A}$ has an atom, this is ∞, and we can stop.

Otherwise, for each $i\in I$,

$$\operatorname{cov}\mathcal{N}(\mu_{X_i})=\operatorname{cov}\mathcal{N}_{\kappa_i}\le\operatorname{cov}\mathcal{N}_{\min K}=\lambda$$

say, by 523B. So we have a family $\langle E_{i\xi}\rangle_{\xi<\lambda}$ of negligible subsets of X_i covering X_i; setting $E_\xi=\bigcup_{i\in I}E_{i\xi}$ for each ξ, we have a family $\langle E_\xi\rangle_{\xi<\lambda}$ in $\mathcal{N}(\mu)$ covering X, so $\operatorname{cov}\mathcal{N}(\mu)\le\operatorname{cov}\mathcal{N}_{\min K}$. But we already know that

$$\operatorname{cov}\mathcal{N}(\mu)\ge\sup_{i\in I}\operatorname{cov}\mathcal{N}_{\kappa_i}\ge\operatorname{cov}\mathcal{N}_{\min K},$$

so $\operatorname{cov}\mathcal{N}(\mu)=\operatorname{cov}\mathcal{N}_{\min K}$.

(d) A set $A \subseteq X$ is non-negligible iff $A \cap X_i$ is non-negligible for some $i \in I$. It follows at once that $\operatorname{non}\mathcal{N}(\mu) = \min_{i\in I}\operatorname{non}\mathcal{N}(\mu_{X_i})$. If any of the X_i is an atom, it contains a point of non-zero measure, so that $\operatorname{non}\mathcal{N}(\mu) = 1$. If $\kappa_i \ge \omega$ for every i, then we have

$$\operatorname{non}\mathcal{N}(\mu) = \min_{i\in I}\operatorname{non}\mathcal{N}_{\kappa_i} = \operatorname{non}\mathcal{N}_{\min K}$$

by 524Jb and 523B again.

(e) If $\mathfrak{A}$ is purely atomic, then μ is point-supported, so $\operatorname{shr}\mathcal{N}(\mu) = 1$. Otherwise, let E be a measurable set of non-zero finite measure such that the subspace measure μ_E is atomless; let ν be the normalized subspace measure $\frac{1}{\mu E}\mu_E$; then ν, like μ_E, is a Radon measure. By 343Cb, there is a function $f : E \to \{0,1\}^\omega$ which is inverse-measure-preserving for ν and ν_ω; because $\{0,1\}^\omega$ is separable and metrizable, νf^{-1} is a Radon measure (451O, or 418I-418J) and must be equal to ν_ω (416Eb). By 521Fd and 521Hb,

$$\operatorname{shr}\mathcal{N}(\mu) \ge \operatorname{shr}\mathcal{N}(\mu_E) = \operatorname{shr}\mathcal{N}(\nu) \ge \operatorname{shr}\mathcal{N}(\nu_\omega) = \operatorname{shr}\mathcal{N}.$$

(f)(i) If $K = \emptyset$ then (a) tells us that $\mathcal{N}(\mu)$ has a greatest member, so that $\operatorname{cf}\mathcal{N}(\mu) = 1$.

(ii) Now suppose that K is not empty. Then 524Fb tells us that there is a family $\langle E_\xi\rangle_{\xi<\tau(\mathfrak{A})}$ in $\mathcal{N}(\mu)$ such that $\{\xi : E_\xi \subseteq E\}$ is countable for every $E \in \mathcal{N}(\mu)$. In this case, $J \mapsto \bigcup_{\xi\in J} E_\xi : [\tau(\mathfrak{A})]^{\le\omega} \to \mathcal{N}(\mu)$ is a Tukey function, so $\operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}[\tau(\mathfrak{A})]^{\le\omega}$. At the same time, there is an $i \in I$ such that $\kappa_i \ge \omega$. The identity map from $\mathcal{N}(\mu_{X_i})$ to $\mathcal{N}(\mu)$ is a Tukey function; but this means that

$$\operatorname{cf}\mathcal{N}(\mu) \ge \operatorname{cf}\mathcal{N}(\mu_{X_i}) = \operatorname{cf}\mathcal{N}_{\kappa_i}$$

(524I again)

$$\ge \operatorname{cf}\mathcal{N}_\omega = \operatorname{cf}\mathcal{N}$$

(523B). Thus $\operatorname{cf}\mathcal{N}(\mu) \ge \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\tau(\mathfrak{A})]^{\le\omega})$.

(iii) In the other direction, we know from 524H (again, applied to a scalar multiple of μ_{X_i}) that $(\mathcal{N}(\mu_{X_i}), \subseteq, \mathcal{N}(\mu_{X_i}) \equiv_{\mathrm{GT}} (\kappa_i^{\mathbb{N}}, \subseteq^*, \mathcal{S}_{\kappa_i})$ whenever κ_i is infinite. Now $\tau(\mathfrak{A}) \ge \kappa_i$, so the maps

$$\text{identity}: \kappa_i^{\mathbb{N}} \to \tau(\mathfrak{A})^{\mathbb{N}}, \qquad S \mapsto S \cap (\mathbb{N}\times\kappa_i) : \mathcal{S}_{\tau(\mathfrak{A})} \to \mathcal{S}_{\kappa_i}$$

form a Galois-Tukey connection from $(\kappa_i^{\mathbb{N}}, \subseteq^*, \mathcal{S}_{\kappa_i})$ to $(\tau(\mathfrak{A})^{\mathbb{N}}, \subseteq^*, \mathcal{S}_{\tau(\mathfrak{A})})$. Accordingly we have

$$\begin{aligned}(\mathcal{N}(\mu_{X_i}), \subseteq, \mathcal{N}(\mu_{X_i})) &\equiv_{\mathrm{GT}} (\kappa_i^{\mathbb{N}}, \subseteq^*, \mathcal{S}_{\kappa_i}) \\ &\preccurlyeq_{\mathrm{GT}} (\tau(\mathfrak{A})^{\mathbb{N}}, \subseteq^*, \mathcal{S}_{\tau(\mathfrak{A})}) \equiv_{\mathrm{GT}} (\mathcal{N}_{\tau(\mathfrak{A})}, \subseteq, \mathcal{N}_{\tau(\mathfrak{A})}),\end{aligned}$$

and $\mathcal{N}(\mu_{X_i}) \preccurlyeq_{\mathrm{T}} \mathcal{N}_{\tau(\mathfrak{A})}$.

The arguments quoted assume that κ_i is infinite; but of course it is still true that $\mathcal{N}(\mu_{X_i}) \preccurlyeq_{\mathrm{T}} \mathcal{N}_{\tau(\mathfrak{A})}$ when $\kappa_i = 0$, since then any constant function from $\mathcal{N}(\mu_{X_i})$ to $\mathcal{N}_{\tau(\mathfrak{A})}$ is a Tukey function. It follows that

$$\mathcal{N}(\mu) \cong \prod_{i\in I}\mathcal{N}(\mu_{X_i}) \preccurlyeq_{\mathrm{T}} \mathcal{N}^I_{\tau(\mathfrak{A})}$$

(513Eg once more).

(iv) At this point observe that as we are assuming that $K \ne \emptyset$, $\tau(\mathfrak{A})$ is infinite; and as μ is supposed to be σ-finite, I is countable. So we can find a disjoint family $\langle F_i\rangle_{i\in I}$ of measurable subsets of $\{0,1\}^{\tau(\mathfrak{A})}$ such that all the subspace measures $(\nu_{\tau(\mathfrak{A})})_{F_i}$ are isomorphic to scalar multiples of $\nu_{\tau(\mathfrak{A})}$. (Take $F_i = \{x : x(n_i) = 1, x(m) = 0$ for $m < n_i\}$ where $i \mapsto n_i : I \to \mathbb{N}$ is injective.) In this case, the map

$$\langle E_i\rangle_{i\in I} \mapsto \bigcup_{i\in I} E_i : \prod_{i\in I}\mathcal{N}((\nu_{\tau(\mathfrak{A})})_{F_i}) \to \mathcal{N}(\nu_{\tau(\mathfrak{A})})$$

is a Tukey function, while $\mathcal{N}^I_{\tau(\mathfrak{A})}$ is isomorphic to $\prod_{i\in I}\mathcal{N}((\nu_{\tau(\mathfrak{A})})_{F_i})$. Putting these together,

$$\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}^I_{\tau(\mathfrak{A})} \cong \prod_{i\in I}\mathcal{N}((\nu_{\tau(\mathfrak{A})})_{F_i}) \preccurlyeq_{\mathrm{T}} \mathcal{N}_{\tau(\mathfrak{A})}.$$

It follows that

$$\operatorname{cf}\mathcal{N}(\mu) \le \operatorname{cf}\mathcal{N}_{\tau(\mathfrak{A})} = \max(\mathcal{N}, \operatorname{cf}[\tau(\mathfrak{A})]^{\le\omega}).$$

So we have inequalities in both directions and $\operatorname{cf}\mathcal{N}(\mu) = \max(\mathcal{N}, \operatorname{cf}[\tau(\mathfrak{A})]^{\le\omega})$, as claimed.

***524Q** I do not know how to calculate $\operatorname{cf}\mathcal{N}(\mu)$ for non-σ-finite Radon measures μ without special assumptions. In the presence of GCH, however, we have the following result.

Proposition Suppose that the generalized continuum hypothesis is true. Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space and $(\mathfrak{A},\bar\mu)$ its measure algebra. For each cardinal κ, write e_κ for the Maharam-type-κ component of $\mathfrak{A}$, and $\mathfrak{C}_\kappa$ for the principal ideal of $\mathfrak{A}$ generated by $\sup_{\kappa'>\kappa}e_{\kappa'}$; set $\lambda=\sup\{\kappa:e_\kappa\ne 0\}$. Then $\operatorname{cf}\mathcal{N}(\mu)=\max(c(\mathfrak{C}_0)^+,\lambda^+)$ unless $\lambda>c(\mathfrak{C}_0)$ and there is some $\gamma<\lambda$ such that $\operatorname{cf}\lambda>c(\mathfrak{C}_\gamma)$, in which case $\operatorname{cf}\mathcal{N}(\mu)=\lambda$.

proof (a) Write

$$\begin{aligned}\theta&=\lambda \text{ if } \lambda>c(\mathfrak{C}_0) \text{ and } \operatorname{cf}\lambda>\min_{\gamma<\lambda}c(\mathfrak{C}_\gamma),\\&=\max(\lambda^+,c(\mathfrak{C}_0)^+)\text{ otherwise.}\end{aligned}$$

If μ is purely atomic, it is point-supported, so $\lambda=0$ and $\mathfrak{C}_0=\{0\}$ and $\theta=1=\operatorname{cf}\mathcal{N}(\mu)$. So let us suppose henceforth that μ is not purely atomic, that is, $\mathfrak{C}_0\ne\{0\}$ and $\lambda\ge\omega$. As in the proofs of 524J and 524P, there is a decomposition $\langle X_i\rangle_{i\in I}$ of X such that the subspace measures μ_{X_i} are all Maharam-type-homogeneous and non-zero. Let κ_i be the Maharam type of μ_{X_i} for each i, so that $\lambda=\sup_{i\in I}\kappa_i$. Now $\mathcal{N}(\mu)\cong\prod_{i\in I}\mathcal{N}(\mu_{X_i})$ (see the proof of 524Ja). For $i\in I$, $\operatorname{cf}\mathcal{N}(\mu_{X_i})=1$ if $\kappa_i=0$, and otherwise is $\max(\operatorname{cf}\mathcal{N},\operatorname{cf}[\kappa_i]^{\le\omega})=\max(\omega_1,\operatorname{cf}[\kappa_i]^{\le\omega})$ (524Ja, 523N). By 5A6Ab,

$$\begin{aligned}\operatorname{cf}\mathcal{N}(\mu_{X_i})&=1 \text{ if } \kappa_i=0,\\&=\kappa_i \text{ if } \operatorname{cf}\kappa_i>\omega,\\&=\kappa_i^+ \text{ if } \operatorname{cf}\kappa_i=\omega.\end{aligned}$$

(b) For each cardinal κ, set $J_\kappa=\{i:i\in I,\ \operatorname{cf}\mathcal{N}(\mu_{X_i})>\kappa\}$, and set

$$\begin{aligned}\lambda_1=\sup_{i\in I}\operatorname{cf}\mathcal{N}(\mu_{X_i})&=\lambda^+ \text{ if there is an } i\in I \text{ such that } \kappa_i=\lambda \text{ and } \operatorname{cf}\kappa_i=\omega,\\&=\lambda\text{ otherwise.}\end{aligned}$$

Then 513J tells us that if $\lambda_1>\#(J_1)$ and there is some $\gamma<\lambda_1$ such that $\operatorname{cf}\lambda_1>\#(J_\gamma)$, then $\operatorname{cf}\mathcal{N}(\mu)=\lambda_1$, and that otherwise $\operatorname{cf}\mathcal{N}(\mu)=\max(\#(J_1)^+,\lambda_1^+)$. As we are supposing that μ is not purely atomic, $c(\mathfrak{C}_0)\ge\omega$ and $c(\mathfrak{C}_0)=\max(\omega,\#(J_1))$; also $\lambda^+\ge\lambda_1\ge\lambda\ge\omega$.

case 1 Suppose $\lambda_1\le\#(J_1)$. Then J_1 is infinite, so $c(\mathfrak{C}_0)=\#(J_1)\ge\lambda$, and

$$\operatorname{cf}\mathcal{N}(\mu)=\#(J_1)^+=c(\mathfrak{C}_0)^+=\theta$$

as required.

case 2 Suppose $\lambda_1>\max(\lambda,\#(J_1))$. Then there must be some $i\in I$ such that $\operatorname{cf}\mathcal{N}(\mu_{X_i})>\lambda\ge\omega$, in which case $\kappa_i=\lambda$ has countable cofinality and $\lambda_1=\lambda^+$. In this case, $\operatorname{cf}\lambda_1=\lambda_1>\#(J_1)$, so $\operatorname{cf}\mathcal{N}(\mu)=\lambda_1$. If $\gamma<\lambda$, then $\mathfrak{C}_\gamma$ is non-trivial, and $\operatorname{cf}\lambda=\omega\le c(\mathfrak{C}_\gamma)$; so

$$\theta=\max(\lambda^+,\#(J_1)^+)=\lambda_1=\operatorname{cf}\mathcal{N}(\mu).$$

case 3 Suppose $\lambda_1=\lambda>\#(J_1)$ has countable cofinality. In this case we must have $\kappa_i<\lambda_1$ for every i, so $\#(J_\gamma)\ge\omega=\operatorname{cf}\lambda_1$ for every $\gamma<\lambda_1$, and $\operatorname{cf}\mathcal{N}(\mu)=\lambda_1^+$. At the same time, $\operatorname{cf}\lambda=\omega\le c(\mathfrak{C}_\gamma)$ for every $\gamma<\lambda$, so

$$\theta=\max(\lambda^+,\#(J_1)^+)=\max(\lambda_1^+,\#(J_1)^+)=\operatorname{cf}\mathcal{N}(\mu).$$

case 4 Suppose $\lambda_1=\lambda>\#(J_1)$ has uncountable cofinality. In this case we have $\lambda>\max(\omega,\#(J_1))=c(\mathfrak{C}_0)$, so

$$\begin{aligned}\operatorname{cf}\mathcal{N}(\mu)=\lambda_1&\iff\#(J_\gamma)<\operatorname{cf}\lambda_1\text{ for some }\gamma<\lambda_1\\&\iff\max(\omega,\#(J_\gamma))<\operatorname{cf}\lambda_1\text{ for some }\gamma<\lambda_1\\&\iff c(\mathfrak{C}_\gamma)<\operatorname{cf}\lambda\text{ for some }\gamma<\lambda\\&\iff\theta=\lambda\iff\theta=\lambda_1,\end{aligned}$$

and otherwise

$$\operatorname{cf}\mathcal{N}(\mu)=\lambda_1^+=\max(\lambda^+,c(\mathfrak{C}_0)^+)=\theta.$$

Thus $\operatorname{cf}\mathcal{N}(\mu)=\theta$ in all cases.

524R The results above show that most of the most important cardinal functions of measurable algebras and Radon measures are readily calculable from the cardinal functions of the ideals $\mathcal{N}_\kappa$ studied in §523. There are no such simple formulae for other classes of space such as compact or quasi-Radon measures (524Xj, 524Xk). However I can give a handful of partial results, as follows.

Proposition Let (X,Σ,μ) be a countably compact σ-finite measure space with Maharam type κ. Then $[\kappa]^{\le\omega}\preccurlyeq_{\mathrm{T}}\mathcal{N}(\mu)$. Consequently $\mathrm{cf}[\kappa]^{\le\omega}\le\mathrm{cf}\,\mathcal{N}(\mu)$, and if κ is uncountable then $\mathrm{add}\,\mathcal{N}(\mu)=\omega_1$ and $\mathrm{cf}\,\mathcal{N}(\mu)\ge\mathrm{cf}\,\mathcal{N}_\kappa$.

proof If $\langle E_\xi\rangle_{\xi<\kappa}$ is a family as in 524Fb, then $I\mapsto\bigcup_{\xi\in I}E_\xi:[\kappa]^{\le\omega}\to\mathcal{N}(\mu)$ is a Tukey function, if both $[\kappa]^{\le\omega}$ and $\mathcal{N}(\mu)$ are given their natural partial orderings of inclusion. By 513Ee, $\mathrm{cf}[\kappa]^{\le\omega}\le\mathrm{cf}\,\mathcal{N}(\mu)$ and $\mathrm{add}[\kappa]^{\le\omega}\ge\mathrm{add}\,\mathcal{N}(\mu)$. But if κ is uncountable, $\mathrm{add}[\kappa]^{\le\omega}=\omega_1$ so $\mathrm{add}\,\mathcal{N}(\mu)$ is also ω_1. At the same time, $\mathrm{cf}\,\mathcal{N}(\mu)\ge\mathrm{cf}\,\mathcal{N}$ (521K), so

$$\mathrm{cf}\,\mathcal{N}(\mu)\ge\max(\mathrm{cf}\,\mathcal{N},\mathrm{cf}[\kappa]^{\le\omega})=\mathrm{cf}\,\mathcal{N}_\kappa.$$

524S In a different direction, there is something we can say about quasi-Radon measures.

Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, with $\mu X>0$, and $(Y,\mathfrak{S},\mathrm{T},\nu)$ a quasi-Radon measure space such that the measure algebras of μ and ν are isomorphic. Then

(a) $\mathcal{N}(\nu)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\mu)$, so $\mathrm{add}\,\nu=\mathrm{add}\,\mathcal{N}(\nu)\ge\mathrm{add}\,\mathcal{N}(\mu)=\mathrm{add}\,\mu$ and $\mathrm{cf}\,\mathcal{N}(\nu)\le\mathrm{cf}\,\mathcal{N}(\mu)$;

(b) $(Y,\in,\mathcal{N}(\nu))\preccurlyeq_{\mathrm{GT}}(X,\in,\mathcal{N}(\mu))$, so $\mathrm{cov}\,\mathcal{N}(\nu)\le\mathrm{cov}\,\mathcal{N}(\mu)$ and $\mathrm{non}\,\mathcal{N}(\nu)\ge\mathrm{non}\,\mathcal{N}(\mu)$.

proof (a) Let $(Z,\mathfrak{U},\Lambda,\lambda)$ be the Stone space of the measure algebra $\mathfrak{B}$ of (Y,T,ν), and $R\subseteq Z\times Y$ the relation described in 415Q/416V, so that $R^{-1}[F]\in\mathcal{N}(\lambda)$ for every $F\in\mathcal{N}(\nu)$. Let $W\subseteq Z$ be the union of the open sets of finite measure. Then the subspace measure λ_W is a Radon measure and its measure algebra is isomorphic to the measure algebras of ν and μ (411Pf).

Now $F\mapsto W\cap R^{-1}[F]:\mathcal{N}(\nu)\to\mathcal{N}(\lambda_W)$ is a Tukey function. **P?** Otherwise, there is a family $\mathcal{A}\subseteq\mathcal{N}(\nu)$ such that $\bigcup\mathcal{A}\notin\mathcal{N}(\nu)$ but $\{W\cap R^{-1}[A]:A\in\mathcal{A}\}$ is bounded above in $\mathcal{N}(\lambda_W)$. Because W is conegligible, $B=\bigcup_{A\in\mathcal{A}}R^{-1}[A]$ is negligible in Z. Let $E\in\mathrm{T}$ be a measurable envelope of $\bigcup\mathcal{A}$ (213J/213L). Then the open-and-closed set $E^*\subseteq Z$ corresponding to $E^\bullet\in\mathfrak{B}$ is not negligible; as λ is inner regular with respect to the open-and-closed sets (411Pb), there must be a non-empty open-and-closed set $V\subseteq E^*$ which is disjoint from $\bigcup_{A\in\mathcal{A}}R^{-1}[A]$. Express V as F^* where $F\in\mathrm{T}$. Then $R[V]=R[F^*]$ is disjoint from $\bigcup\mathcal{A}$. But $R[F^*]$ is measurable and $F\setminus R[F^*]$ is negligible (415Qb), while $F\setminus E$ must also be negligible, so $E\cap R[F^*]$ is a non-negligible measurable subset of $E\setminus\bigcup\mathcal{A}$, which is impossible. **XQ**

This shows that $\mathcal{N}(\nu)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\lambda_W)$. But λ_W and μ are Radon measures with isomorphic non-zero measure algebras, so $\mathcal{N}(\lambda_W)\equiv_{\mathrm{T}}\mathcal{N}(\mu)$ (524J) and $\mathcal{N}(\nu)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\mu)$. Accordingly $\mathrm{add}\,\mathcal{N}(\nu)\ge\mathrm{add}\,\mathcal{N}(\mu)$ and $\mathrm{cf}\,\mathcal{N}(\nu)\le\mathrm{cf}\,\mathcal{N}(\mu)$

(b) This is a special case of 521La.

524T Corollary Let $(Y,\mathfrak{S},\mathrm{T},\nu)$ be a quasi-Radon measure space, and $\mathfrak{B}$ its measure algebra. Let K be the set of infinite cardinals κ such that the Maharam-type-κ component of $\mathfrak{B}$ is non-zero.

(a)
$$\begin{aligned}\mathrm{add}\,\nu=\mathrm{add}\,\mathcal{N}(\nu)&=\infty\text{ if }K=\emptyset,\\&\ge\mathrm{add}\,\mathcal{N}\text{ if }K=\{\omega\}.\end{aligned}$$

(b)
$$\begin{aligned}\pi(\nu)=\pi(\mathfrak{B})&=c(\mathfrak{B})\text{ if }K=\emptyset,\\&=\max(c(\mathfrak{B}),\mathrm{cf}\,\mathcal{N},\sup_{\kappa\in K}\mathrm{cf}[\kappa]^{\le\omega})\text{ otherwise.}\end{aligned}$$

(c)
$$\begin{aligned}\mathrm{cov}\,\mathcal{N}(\nu)&=1\text{ if }\mathfrak{B}=\{0\},\\&=\infty\text{ if }\mathfrak{B}\text{ has an atom},\\&\le\mathrm{cov}\,\mathcal{N}_{\min K}\text{ otherwise.}\end{aligned}$$

(d)
$$\begin{aligned}\mathrm{non}\,\mathcal{N}(\nu)&=\infty\text{ if }\mathfrak{B}=\{0\},\\&=1\text{ if }\mathfrak{B}\text{ has an atom},\\&\ge\mathrm{non}\,\mathcal{N}_{\min K}\text{ otherwise.}\end{aligned}$$

(e) If ν is σ-finite,

$$\begin{aligned}\operatorname{cf}\mathcal{N}(\nu) &= 1 \text{ if } K=\emptyset,\\ &\le \max(\operatorname{cf}\mathcal{N}, \operatorname{cf}[\tau(\mathfrak{B})]^{\le\omega}) \text{ otherwise.}\end{aligned}$$

proof Parts (a), (c), (d) and (e) are mostly a matter of putting 524P and 524S together. If there are atoms for μ, they may no longer include singletons of non-zero measure; but they do include minimal non-negligible closed sets, so there are non-negligible singletons and $\operatorname{cov}\mathcal{N}(\mu)$, $\operatorname{non}\mathcal{N}(\mu)$ are ∞ and 1 respectively. As for (b), the proof of 524Pb still works.

524X Basic exercises (a) Suppose that $(\mathfrak{A},\bar\mu)$ is a probability algebra and that $\kappa=\operatorname{link}_n(\mathfrak{A})$, where $2\le n<\omega$. Show that there are families $\langle A_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}\setminus\{0\}$ and $\langle\epsilon_\xi\rangle_{\xi<\kappa}$ in $]0,1]$ such that $\bar\mu(\inf I)\ge\epsilon_\xi$ whenever $I\in[A_\xi]^n$ and $\bigcup_{\xi<\kappa}A_\xi=\mathfrak{A}\setminus\{0\}$. (*Hint*: proof of 524L.)

(b) Let (X,Σ,μ) be a semi-finite measure space with measure algebra $\mathfrak{A}$, and $\mathcal{A}$ a family of non-negligible (not necessarily measurable) subsets of X such that every non-negligible member of Σ includes a member of $\mathcal{A}$. Show that $\#(\mathcal{A})\ge\pi(\mathfrak{A})$.

(c) Show that if κ is uncountable, there is no function $f:[0,1]^\kappa\to\{0,1\}^\kappa$ which is almost continuous and inverse-measure-preserving for the usual measures on these spaces. (*Hint*: if $K\subseteq[0,1]^\kappa$ is a zero set, any continuous function from K to $\{0,1\}^\kappa$ is determined by coordinates in a countable set.)

(d) Let $I^{\parallel}$ be the split interval and μ its usual measure (343J). Show that there are $f:\{0,1\}^\omega\to I^{\parallel}$ and $g:I^{\parallel}\to\{0,1\}^\omega$ such that $\mu=\nu_\omega f^{-1}$ and $\nu_\omega=\mu g^{-1}$. (*Hint*: let $A\subseteq[0,1]$ be a non-measurable set; define $f_0:[0,1]^2\to I^{\parallel}$ by setting $f_0(x,y)=y^+$ if $x\in A$, y^- otherwise.)

(e) Let (Z,μ) be the Stone space of $(\mathfrak{B}_\omega,\bar\nu_\omega)$. Show that there is no $f:\{0,1\}^\omega\to Z$ such that $\mu=\nu_\omega f^{-1}$. (*Hint*: use 515J and 322Ra to show that every non-negligible measurable subset of Z has cardinal $2^{\mathfrak{c}}$.)

(f) Let X be a Hausdorff space with a compact topological probability measure μ with Maharam type κ, and suppose that $w(X)<\operatorname{cov}\mathcal{N}_\kappa$. (i) Show that there is an equidistributed sequence for μ. (*Hint*: 491Eb.) (ii) Show that if μ is strictly positive then X is separable.

(g) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon probability space with a strong lifting, and (Z,ν) the Stone space of its measure algebra. Show that $\operatorname{shr}\mathcal{N}(\mu)\le\operatorname{shr}\mathcal{N}(\nu)$ and $\operatorname{shr}^+\mathcal{N}(\mu)\le\operatorname{shr}^+\mathcal{N}(\nu)$. (*Hint*: 453Mb.)

(h) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, and K the set of infinite cardinals κ such that the Maharam-type-κ component of its measure algebra $\mathfrak{A}$ is non-zero. Show that

$$\min\{\#(A):A\subseteq X\text{ has full outer measure}\}=\sup(\{c(\mathfrak{A})\}\cup\{\operatorname{non}\mathcal{N}_\kappa:\kappa\in K\}).$$

(i) Show that for any σ-ideal $\mathcal{I}$ of sets there is a compact probability measure μ such that $\mathcal{I}=\mathcal{N}(\mu)$. (*Hint*: set $X=\bigcup\mathcal{I}\cup\{x_0\}$.)

(j) Show that for any non-zero measurable algebra $\mathfrak{B}$ and any cardinal κ, there is a complete compact probability measure μ such that the measure algebra of μ is isomorphic to $\mathfrak{B}$, $\operatorname{add}\mathcal{N}(\mu)=\omega_1$ and $\operatorname{cf}\mathcal{N}(\mu)\ge\kappa$. (*Hint*: 524Xi.)

(k) Suppose that $\operatorname{non}\mathcal{N}_{\mathfrak{c}}=\operatorname{cov}\mathcal{N}_{\mathfrak{c}}=\operatorname{cf}\mathfrak{c}=\mathfrak{c}$. Show that there is a quasi-Radon probability measure μ with Maharam type $\mathfrak{c}$ such that $\operatorname{add}\mathcal{N}(\mu)=\mathfrak{c}$.

524Y Further exercises (a) Show that if $m\ge 2$ and $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a family of σ-m-linked Boolean algebras, with $\#(I)\le\mathfrak{c}$, then the free product of $\langle\mathfrak{A}_i\rangle_{i\in I}$ is σ-m-linked.

(b) Let $\mathfrak{A}$ be a measurable algebra with Maharam type λ. Show that there is a family $\mathcal{V}\subseteq[\lambda]^{\le\mathfrak{c}}$, cofinal with $[\lambda]^{\le\mathfrak{c}}$, such that $\#(\{A\cap V:V\in\mathcal{V}\})<\operatorname{FN}^*(\mathfrak{A})$ for every countable set $A\subseteq\lambda$.

(c) For a Boolean algebra $\mathfrak{A}$ and a cardinal θ, write $\psi_\theta(\mathfrak{A})$ for the smallest size of any subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ such that $d(\mathfrak{C})\ge\theta$. (If $\theta>d(\mathfrak{A})$ set $\psi_\theta(\mathfrak{A})=\infty$.) (i) Show that if Z is the Stone space of $\mathfrak{A}$, $\mathcal{I}$ is the ideal of nowhere dense sets in Z, and $\theta \geq 2$ then $\psi_\theta(\mathfrak{A}) \leq \mathrm{cov}([Z]^{<\theta}, \subseteq, \mathcal{I})$. (ii) Show that if (X, Σ, μ) is a Maharam-type-homogeneous compact probability space with Maharam type κ, and θ is uncountable, then

$$\psi_\theta(\mathfrak{B}_\kappa) = \mathrm{cov}([X]^{<\theta}, \subseteq, \mathcal{N}(\mu)) = \mathrm{add}(\Sigma \setminus \mathcal{N}(\mu), \mathtt{meet}, [X]^{<\theta}),$$

where $\mathtt{meet}$ is the relation $\{(A, B) : A \cap B \neq \emptyset\}$. (*Hint*: start with $\mu = \nu_\kappa$.) (iii) Show that if (X, Σ, μ) is a semi-finite locally compact measure space with measure algebra $\mathfrak{A}$ then $\psi_{\omega_1}(\mathfrak{A}) \leq \mathrm{cf}([\mathrm{cov}\,\mathcal{N}(\mu)]^{\leq\omega})$. (iv) Show that if (X, Σ, μ) is any probability space, with measure algebra $\mathfrak{A}$, and λ is the product probability measure on $X^{\mathbb{N}}$, then $\mathrm{cov}\,\mathcal{N}(\lambda) \leq \psi_{\omega_1}(\mathfrak{A})$. (v) Show that $\psi_{\mathrm{add}\,\mathcal{M}}(\mathfrak{B}_\omega) \leq \mathrm{non}\,\mathcal{M}$, where $\mathcal{M}$ is the ideal of meager subsets of $\mathbb{R}$.

524Z Problems (a) Let (Z, μ) be the Stone space of $(\mathfrak{B}_\omega, \bar{\nu}_\omega)$. Is $\mathrm{shr}\,\mathcal{N}(\mu)$ necessarily equal to $\mathrm{shr}\,\mathcal{N}$?

(b) Can there be a quasi-Radon probability measure μ with Maharam type greater than $\mathfrak{c}$ such that $\mathrm{add}\,\mathcal{N}(\mu) > \omega_1$?

524 Notes and comments The ideas of this section are derived primarily from Bartoszyński 84, Fremlin 84b and Fremlin 91. Of course it is not necessary to pass through both $\ell^1(\kappa)$ and the κ-localization relation $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa)$. I bring $\ell^1(\kappa)$ into the argument (following Bartoszyński 84) because it will be useful when we come to look at other structures in later in the chapter, and $\mathcal{S}_\kappa$ because it echoes the ideas of §522. But note that 524G seems to need a new idea (the family $\langle E_\xi \rangle_{\xi<\kappa}$ from 524F) not required in 522M.

The difficulties of the work above arise from the fact that while there are many inverse-measure-preserving functions between Radon measure spaces, immediately linking covering numbers and uniformities, there are far fewer continuous inverse-measure-preserving functions; for instance, there is no almost continuous inverse-measure-preserving function from the unit interval to the split interval, let alone to the Stone space of its measure algebra. And the straightforward Tukey functions between the ideals $\mathcal{N}_\kappa$ of §523 depend on measures being images of each other, which is something we can rely on only when our functions are almost continuous. (But see 524Xd.) I do not know of any direct construction of a Tukey function from the null ideal of the Stone space of the Lebesgue measure algebra to $\mathcal{N}$, for instance. This is why there is nearly nothing about shrinking numbers in this section (see 524Za).

There is a significant gap in the calculations in 524P; for the cofinality of the null ideal I need to assume that the measure is σ-finite. I have no useful general recipe for $\mathrm{cf}\,\mathcal{N}(\mu)$, valid in ZFC, when μ is a non-σ-finite Radon measure. The point is that although we can identify $\mathcal{N}(\mu)$ with the product of a family $\mathcal{N}(\mu_{X_i})$ of partially ordered sets to which the arguments of this section apply (524Q), this is not in itself enough to determine its cofinality in the absence of special axioms.

525 Precalibers

I continue the discussion of precalibers in §516 with results applying to measure algebras. I start with connexions between measure spaces and precalibers of their measure algebras (525B-525C). The next step is to look at measure-precalibers. Elementary facts are in 525D-525G. When we come to ask which cardinals are precalibers of which measure algebras, there seem to be real difficulties; partial answers, largely based on infinitary combinatorics, are in 525I-525O. 525P is a note on a particular pair of cardinals. Finally, 525T deals with precaliber triples (κ, κ, k) where k is finite; I approach it through a general result on correlations in uniformly bounded families of random variables (525S).

525A Notation If (X, Σ, μ) is a measure space, $\mathcal{N}(\mu)$ will be the null ideal of μ. For any set I, ν_I will be the usual measure on $\{0,1\}^I$, T_I its domain, $\mathcal{N}_I = \mathcal{N}(\nu_I)$ its null ideal and $(\mathfrak{B}_I, \bar{\nu}_I)$ its measure algebra. In this context, set $e_i = \{x : x \in \{0,1\}^I,\, x(i) = 1\}^\bullet$ in $\mathfrak{B}_I$ for $i \in I$. Then $\langle e_i \rangle_{i \in I}$ is a stochastically independent family of elements of measure $\frac{1}{2}$ in $\mathfrak{B}_I$, and $\{e_i : i \in I\}$ τ-generates $\mathfrak{B}_I$; I will say that $\langle e_i \rangle_{i \in I}$ is the **standard generating family** in $\mathfrak{B}_I$.

525B Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, and $\mathfrak{A}$ its measure algebra. Then the downwards precaliber triples of the partially ordered set $(\Sigma \setminus \mathcal{N}(\mu), \subseteq)$ are just the precaliber triples of the Boolean algebra $\mathfrak{A}$.

proof Put 521Dd and 516C together.

525C Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space and $(\mathfrak{A},\bar\mu)$ its measure algebra.

(a) A pair (κ,λ) of cardinals is a precaliber pair of $\mathfrak{A}$ iff whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a family in $\Sigma\setminus\mathcal{N}(\mu)$ there is an $x\in X$ such that $\#(\{\xi:x\in E_\xi\})\ge\lambda$.

(b) A pair (κ,λ) of cardinals is a measure-precaliber pair of $(\mathfrak{A},\bar\mu)$ iff whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a family in $\Sigma\setminus\mathcal{N}(\mu)$ such that $\inf_{\xi<\kappa}\mu E_\xi>0$ then there is an $x\in X$ such that $\#(\{\xi:x\in E_\xi\})\ge\lambda$.

(c) Suppose that $\kappa\ge\operatorname{sat}(\mathfrak{A})$ is an infinite regular cardinal. Then the following are equiveridical:

(i) κ is a precaliber of $\mathfrak{A}$;

(ii) $\mu_*(\bigcup_{\xi<\kappa}E_\xi)=0$ whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family in $\mathcal{N}(\mu)$;

(iii) whenever $\langle A_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of sets such that $\bigcup_{\xi<\kappa}A_\xi=X$, then there is some $\xi<\kappa$ such that A_ξ has full outer measure in X.

proof (a)(i) Suppose that (κ,λ) is a precaliber pair of $\mathfrak{A}$ and $\langle E_\xi\rangle_{\xi<\kappa}$ is a family in $\Sigma\setminus\mathcal{N}(\mu)$. For each $\xi<\kappa$, let $K_\xi\subseteq E_\xi$ be a non-negligible compact set. Then there is a $\Gamma\in[\kappa]^\lambda$ such that $\{K_\xi^\bullet:\xi\in\Gamma\}$ is centered in $\mathfrak{A}$. But in this case $\{X\}\cup\{K_\xi:\xi\in\Gamma\}$ has the finite intersection property, and must have non-empty intersection. If x is any point of this intersection, $\{\xi:x\in E_\xi\}$ includes Γ and has size at least λ.

(ii) Suppose that whenever $\langle E_\xi\rangle_{\xi<\kappa}$ is a family in $\Sigma\setminus\mathcal{N}(\mu)$ there is an $x\in X$ such that $\#(\{\xi:x\in E_\xi\})\ge\lambda$. Because μ is complete and strictly localizable (416B), it has a lifting $\psi:\mathfrak{A}\to\Sigma$ (341K). Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{A}\setminus\{0\}$; then there is an $x\in X$ such that $\Gamma=\{\xi:x\in\psi a_\xi\}$ has cardinal at least λ. But now $\{\psi a_\xi:\xi\in\Gamma\}$ is centered in Σ so $\{a_\xi:\xi\in\Gamma\}$ is centered in $\mathfrak{A}$. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, (κ,λ) is a precaliber pair of $\mathfrak{A}$.

(b) We can use exactly the same argument, provided that in part (i) we make sure that $\mu K_\xi\ge\frac{1}{2}\mu E_\xi$, so that $\inf_{\xi<\kappa}\bar\mu K_\xi^\bullet>0$.

(c)(i)⇒(iii) Suppose that κ is a precaliber of $\mathfrak{A}$ and $\langle A_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family of sets with union X. **?** If no A_ξ has full outer measure, then we can choose, for each $\xi<\kappa$, a non-negligible compact set $K_\xi\subseteq X\setminus A_\xi$. Because κ is a precaliber of $\mathfrak{A}$, there is a set $\Gamma\in[\kappa]^\kappa$ such that $\{K_\xi^\bullet:\xi\in\Gamma\}$ is centered. Now $\{K_\xi:\xi\in\Gamma\}$ has the finite intersection property and there is some $x\in\bigcap_{\xi\in\Gamma}K_\xi$, in which case $x\notin\bigcup_{\xi\in\Gamma}A_\xi$. But since Γ must be cofinal with κ, $\bigcup_{\xi\in\Gamma}A_\xi=X$. **X** As $\langle A_\xi\rangle_{\xi<\kappa}$ is arbitrary, (iii) is true.

(iii)⇒(ii) Suppose that (iii) is true, and that $\langle E_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family in $\mathcal{N}(\mu)$. **?** If $\bigcup_{\xi<\kappa}E_\xi$ has non-zero inner measure, let $E\subseteq\bigcup_{\xi<\kappa}E_\xi$ be a non-negligible measurable set. Set $A_\xi=E_\xi\cup(X\setminus E)$ for each ξ; then $\langle A_\xi\rangle_{\xi<\kappa}$ is a non-decreasing family with union X, so there is some ξ such that A_ξ has full outer measure. But $E\setminus E_\xi$ is a non-negligible measurable set disjoint from A_ξ. **X** As $\langle E_\xi\rangle_{\xi<\kappa}$ is arbitrary, (ii) is true.

(ii)⇒(i) Let Z be the Stone space of $\mathfrak{A}$ and ν its usual measure (411P). Because μ has a lifting, there is an inverse-measure-preserving function $f:X\to Z$ (341P).

Let $\langle F_\xi\rangle_{\xi<\kappa}$ be a non-decreasing family of nowhere dense subsets of Z. Then they are all ν-negligible (411Pa), so $\langle f^{-1}[F_\xi]\rangle_{\xi<\kappa}$ is a non-decreasing family in $\mathcal{N}(\mu)$ and $\mu_*(\bigcup_{\xi<\kappa}f^{-1}[F_\xi])=0$. But this means that if $G=\operatorname{int}(\bigcup_{\xi<\kappa}F_\xi)$, $\nu G=\mu f^{-1}[G]=0$ and G is empty. By 516Rb, κ is a precaliber of $\mathfrak{A}$.

525D Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) Any precaliber triple of $\mathfrak{A}$ is a measure-precaliber triple of $(\mathfrak{A},\bar\mu)$.

(b) If $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{A},\bar\mu)$ and κ has uncountable cofinality, then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{A}$.

(c) If κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$, so is $\operatorname{cf}\kappa$.

proof (a) is immediate from the definitions in 511E.

(b) If $\langle a_\xi\rangle_{\xi<\kappa}$ is any family in $\mathfrak{A}^+$, then there is a $\delta>0$ such that $\Gamma=\{\xi:\bar\mu a_\xi\ge\delta\}$ has cardinal κ; and now there is a $\Gamma'\in[\Gamma]^\lambda$ such that $\{a_\xi:\xi\in I\}$ has a non-zero lower bound for every $I\in[\Gamma']^{<\theta}$.

(c) The point is that κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$ iff it is a precaliber of the supported relation $(A_\delta,\supseteq,\mathfrak{A}^+)$ for every $\delta>0$, where $A_\delta=\{a:a\in\mathfrak{A},\ \bar\mu a\ge\delta\}$; so this is just a special case of 516Bd.

525E Proposition (a) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and κ an infinite cardinal. Then κ is a precaliber of $\mathfrak{A}$ iff either $\mathfrak{A}$ is finite or κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$ and $\operatorname{cf}\kappa>\omega$.

(b) An infinite cardinal κ is a precaliber of every measurable algebra iff it is a measure-precaliber of every probability algebra and has uncountable cofinality.

proof (a) If κ is a precaliber of $\mathfrak{A}$, of course κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$. Also $\operatorname{cf}\kappa$ is a precaliber of $\mathfrak{A}$ (516Bd again), so $\operatorname{cf}\kappa \ge \operatorname{sat}(\mathfrak{A})$ (516Ja); and if $\mathfrak{A}$ is infinite, $\operatorname{cf}\kappa > \omega$.

If $\mathfrak{A}$ is finite, then any infinite cardinal is a precaliber of $\mathfrak{A}$ (516Lc). If κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$ and $\operatorname{cf}\kappa > \omega$, then κ is a precaliber of $\mathfrak{A}$ by 525Db.

(b) Recall that an algebra $\mathfrak{A}$ is 'measurable' iff either $\mathfrak{A} = \{0\}$ or there is a functional $\bar\mu$ such that $(\mathfrak{A},\bar\mu)$ is a probability algebra (391B). So the result follows directly from (a).

525F Proposition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra.

(a) ω is a measure-precaliber of $(\mathfrak{A},\bar\mu)$.

(b) If $\omega \le \kappa < \mathfrak{m}(\mathfrak{A})$, then κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$.

proof (a) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}}\bar\mu a_n = \delta > 0$. Set $a = \inf_{n\in\mathbb{N}}\sup_{m\ge n} a_m$; then $\bar\mu a = \inf_{n\in\mathbb{N}}\bar\mu(\sup_{m\ge n} a_m) \ge \delta > 0$, so $a \ne 0$. If $0 \ne b \subseteq a$ and $n \in \mathbb{N}$, there is an $m \ge n$ such that $b \cap a_m \ne 0$. We can therefore choose inductively a strictly increasing sequence $\langle n_i\rangle_{i\in\mathbb{N}}$ such that $a \cap \inf_{j\le i} a_{n_j} \ne 0$ for every i, so that $\langle a_{n_i}\rangle_{i\in\mathbb{N}}$ is centered. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ω is a measure-precaliber of $(\mathfrak{A},\bar\mu)$.

(b) If $\kappa = \omega$, this is (a). Otherwise, let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{A}$ with $\inf_{\xi<\kappa}\bar\mu a_\xi = \delta > 0$. Set

$$c = \inf_{J\subseteq\kappa,\#(J)<\kappa}\sup_{\xi\in\kappa\setminus J} a_\xi;$$

then

$$\bar\mu c = \inf_{J\subseteq\kappa,\#(J)<\kappa}\bar\mu(\sup_{\xi\in\kappa\setminus J} a_\xi) \ge \delta.$$

Choose $\langle I_\xi\rangle_{\xi<\kappa}$ inductively so that, for each $\xi < \kappa$, I_ξ is a countable subset of $\kappa \setminus \bigcup_{\eta<\xi} I_\eta$ and $c \subseteq \sup_{\eta\in I_\xi} a_\eta$.

For $\xi < \kappa$, set

$$Q_\xi = \{b : 0 \ne b \subseteq c,\, \exists\, \eta \in I_\xi,\, b \subseteq a_\eta\}.$$

Then Q_ξ is coinitial with $\mathfrak{A}_c^+$. Because $\kappa < \mathfrak{m}(\mathfrak{A}) \le \mathfrak{m}(\mathfrak{A}_c)$, there is a centered $R \subseteq \mathfrak{A}_c^+$ meeting every Q_ξ. Now

$$\Gamma = \{\eta : \eta < \kappa,\, \exists\, b \in R,\, b \subseteq a_\eta\}$$

meets every I_ξ so has cardinal κ, and $\{a_\eta : \eta \in \Gamma\}$ is centered. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, κ is a measure-precaliber of $(\mathfrak{A},\bar\mu)$.

525G As is surely to be expected, questions about precalibers of measurable algebras can generally be reduced to questions about precalibers of the algebras $\mathfrak{B}_\kappa$. Some of these can be quickly answered in terms of the cardinals examined earlier in this chapter.

Proposition (a) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra. Let K be the set of infinite cardinals κ' such that the Maharam-type-κ' component of $\mathfrak{A}$ is non-zero (cf. 524M). If κ, λ and θ are cardinals, of which κ is infinite, then $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{A},\bar\mu)$ iff it is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'},\bar\nu_{\kappa'})$ for every $\kappa' \in K$.

(b) Suppose that $\omega \le \kappa < \operatorname{cov}\mathcal{N}_{\kappa'}$. Then κ is a measure-precaliber of $\mathfrak{B}_{\kappa'}$.

(c) For any cardinal κ', ω_1 is a precaliber of $\mathfrak{B}_{\kappa'}$ iff $\operatorname{cov}\mathcal{N}_{\kappa'} > \omega_1$.

(d) If κ, κ' are cardinals such that $\operatorname{non}\mathcal{N}_{\kappa'} < \operatorname{cf}\kappa$, then κ is a precaliber of $\mathfrak{B}_{\kappa'}$.

proof (a)(i) Suppose that $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{A},\bar\mu)$, $\kappa' \in K$ and $\langle b_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{B}_{\kappa'}$ with $\inf_{\xi<\kappa}\bar\nu_{\kappa'}b_\xi = \delta > 0$. Let $a \in \mathfrak{A}$ be such that the principal ideal $\mathfrak{A}_a$ is homogeneous with Maharam type κ', so that there is an isomorphism $\pi : \mathfrak{B}_{\kappa'} \to \mathfrak{A}_a$ with $\frac{1}{\bar\mu a}\bar\mu(\pi b) = \bar\nu_{\kappa'}b$ for every $b \in \mathfrak{B}_{\kappa'}$ (331L). Then $\inf_{\xi<\kappa}\bar\mu(\pi b_\xi) = \delta\bar\mu a > 0$, so there is a $\Gamma \in [\kappa]^\lambda$ such that $\inf_{\xi\in I}\pi b_\xi$ and $\inf_{\xi\in I} b_\xi$ are non-zero for every $I \in [\Gamma]^{<\theta}$. As $\langle b_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'},\bar\nu_{\kappa'})$.

(ii) Suppose that $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'},\bar\nu_{\kappa'})$ for every $\kappa' \in K$ and $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ with $\inf_{\xi<\kappa}\bar\mu a_\xi = \delta > 0$. Let $D \subseteq \mathfrak{A}\setminus\{0\}$ be a partition of unity in $\mathfrak{A}$ such that all the principal ideals $\mathfrak{A}_d$, for $d \in D$, are homogeneous. Let $C \subseteq D$ be a finite set such that $\sum_{d\in D\setminus C}\bar\mu d \le \frac{1}{2}\delta$. Then for every $\xi < \kappa$ there is a $c \in C$ such that $\bar\mu(a_\xi \cap c) \ge \frac{1}{2}\delta\bar\mu c$, so (because κ is infinite) there are $c \in C$ and $\Gamma_0 \in [\kappa]^\kappa$ such that $\bar\mu(a_\xi \cap c) \ge \frac{1}{2}\delta\bar\mu c$ for every $\xi \in \Gamma_0$. If c is an atom then $\inf_{\xi\in I} a_\xi \supseteq c$ is non-zero for every $I \subseteq \Gamma_0$. Otherwise, the Maharam type κ' of $\mathfrak{A}_c$ belongs to K. Let $\pi : \mathfrak{B}_{\kappa'} \to \mathfrak{A}_c$ be an isomorphism with $\bar\mu(\pi b) = \bar\mu c\cdot\bar\nu_{\kappa'}b$ for every $b \in \mathfrak{B}_{\kappa'}$. Set $b_\xi = \pi^{-1}(a_\xi \cap c)$; then $\bar\nu_{\kappa'}b_\xi \ge \frac{1}{2}\delta$ for every $\xi \in \Gamma_0$. There is therefore a $\Gamma \in [\Gamma_0]^\lambda$ such that $\inf_{\xi\in I} b_\xi$ and $\inf_{\xi\in I} a_\xi$ are non-zero for every $I \in [\Gamma]^{<\theta}$. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa,\lambda,{<}\theta)$ is a measure-precaliber triple of $(\mathfrak{A},\bar\mu)$.

(b) We have $\operatorname{cov}\mathcal{N}_{\kappa'} = \mathfrak{m}(\mathfrak{B}_{\kappa'})$ (524Md), so we can use 525Fb.

(c) If $\operatorname{cov}\mathcal{N}_{\kappa'} > \omega_1$ then (b) tells us that ω_1 is a precaliber of $\mathfrak{B}_{\kappa'}$. If $\operatorname{cov}\mathcal{N}_{\kappa'} = \omega_1$, let $\langle E_\xi\rangle_{\xi<\omega_1}$ be a cover of $\{0,1\}^{\kappa'}$ by negligible sets; then $\langle\bigcup_{\eta<\xi} E_\eta\rangle_{\xi<\omega_1}$ is a non-decreasing family in $\mathcal{N}_{\kappa'}$ with union of non-zero inner measure, so 525Cc tells us that ω_1 is not a precaliber of $\mathfrak{B}_{\kappa'}$.

(d) If κ' is finite this is elementary. Otherwise, $d(\mathfrak{B}_{\kappa'}) = \operatorname{non}\mathcal{N}_{\kappa'}$ (524Me). By 516Lc, κ is a precaliber of $\mathfrak{B}_{\kappa'}$.

525H The structure of $\mathfrak{B}_I$ Several of the arguments below will depend on the following ideas. Let I be any set and $\langle e_i\rangle_{i\in I}$ the standard generating family in $\mathfrak{B}_I$. If $a \in \mathfrak{B}_I$, there is a smallest countable set $J \subseteq I$ such that a belongs to the closed subalgebra $\mathfrak{C}_J$ of $\mathfrak{B}_I$ generated by $\{e_i : i \in J\}$ (254Rd, 325Mb). (Of course $\mathfrak{C}_J$ is canonically isomorphic to $\mathfrak{B}_J$; see 325Ma.)

Now suppose that $\langle a_\xi\rangle_{\xi\in\Gamma}$ is a family in $\mathfrak{B}_I$, that for each $\xi \in \Gamma$ we are given a set $I_\xi \subseteq I$ such that $a_\xi \in \mathfrak{C}_{I_\xi}$, and that $J \subseteq I$ is such that $I_\xi \cap I_\eta \subseteq J$ for all distinct ξ, $\eta \in \Gamma$. Then $\langle a_\xi\rangle_{\xi\in\Gamma}$ is relatively stochastically independent over $\mathfrak{C}_J$. **P** $\langle\mathfrak{C}_{I_\xi\setminus J}\rangle_{\xi\in\Gamma}$ is stochastically independent, because $\langle I_\xi \setminus J\rangle_{\xi\in\Gamma}$ is disjoint; moreover, $\mathfrak{C}_J$ is independent of $\mathfrak{C}_{I\setminus J} \supseteq \bigcup_{\xi\in\Gamma}\mathfrak{C}_{I_\xi\setminus J}$, and $\mathfrak{C}_{I_\xi\cup J}$ is the closed subalgebra generated by $\mathfrak{C}_{I_\xi\setminus J}\cup\mathfrak{C}_J$ for each ξ. So 458Lg tells us that $\langle\mathfrak{C}_{I_\xi\cup J}\rangle_{\xi\in\Gamma}$ is relatively stochastically independent over $\mathfrak{C}_J$; *a fortiori*, $\langle a_\xi\rangle_{\xi\in\Gamma}$ is relatively stochastically independent over $\mathfrak{C}_J$. **Q** It follows that if $\Delta \subseteq \Gamma$ is finite and $\inf_{\xi\in\Delta}\operatorname{upr}(a_\xi, \mathfrak{C}_J) \ne 0$, then $\inf_{\xi\in\Delta} a_\xi \ne 0$ (458Lf); in particular, if $\langle\operatorname{upr}(a_\xi, \mathfrak{C}_J)\rangle_{\xi\in\Gamma}$ is centered, so is $\langle a_\xi\rangle_{\xi\in\Gamma}$.

525I Theorem (a)(i) If $\kappa > 0$ and $(\kappa, \lambda, {<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$, then it is a measure-precaliber triple of every probability algebra.

(ii) If $\kappa > 0$ and $(\kappa, \lambda, {<}\theta)$ is a precaliber triple of $\mathfrak{B}_\kappa$, then it is a precaliber triple of every measurable algebra.

(b) Suppose that $\operatorname{cf}\kappa \ge \omega_2$. If (κ, λ) is a precaliber pair of $\mathfrak{B}_{\kappa'}$ for every $\kappa' < \kappa$, then it is a precaliber pair of every measurable algebra.

(c) Suppose that $(\kappa, \lambda, {<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_\omega, \bar\nu_\omega)$ and that κ' is such that $\operatorname{cf}[\kappa']^{\le\omega} < \operatorname{cf}\kappa$. Then $(\kappa, \lambda, {<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'}, \bar\nu_{\kappa'})$.

proof (a)(i) Let $(\mathfrak{A}, \bar\mu)$ be any probability algebra and $\langle a_\xi\rangle_{\xi<\kappa}$ a family in $\mathfrak{A}^+$ such that $\inf_{\xi<\kappa}\bar\mu a_\xi > 0$. Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \kappa\}$. Then $(\mathfrak{B}, \bar\mu\restriction\mathfrak{B})$ is a probability algebra with Maharam type at most κ, so is isomorphic to a closed subalgebra of $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ (332N). Since $(\kappa, \lambda, {<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ it is a measure-precaliber triple of $(\mathfrak{B}, \bar\mu\restriction\mathfrak{B})$ (cf. 516Sb), and there is a $\Gamma \in [\kappa]^\lambda$ such that $\{a_\xi : \xi \in I\}$ is bounded below in $\mathfrak{B}^+$ and therefore in $\mathfrak{A}^+$ for every $I \in [\Gamma]^{<\theta}$. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \lambda, {<}\theta)$ is a measure-precaliber triple of $(\mathfrak{A}, \bar\mu)$.

(ii) The same argument applies, deleting the phrase '$\inf_{\xi<\kappa}\bar\mu a_\xi > 0$', since if $\mathfrak{A}$ is a measurable algebra other than $\{0\}$ there is a functional $\bar\mu$ such that $(\mathfrak{A}, \bar\mu)$ is a probability algebra.

(b) By (a-ii), it is enough to prove that (κ, λ) is a precaliber pair of $\mathfrak{B}_\kappa$. Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_\kappa^+$. For each $I \subseteq \kappa$, let $\mathfrak{C}_I$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_i : i \in I\}$, as in 525H. Then for each $\xi < \kappa$ we have a countable set $I_\xi \subseteq \kappa$ such that $a_\xi \in \mathfrak{C}_{I_\xi}$. Because $\operatorname{cf}\kappa \ge \omega_2$, there are a $\Gamma \in [\kappa]^\kappa$ and a $J \in [\kappa]^{<\kappa}$ such that $I_\xi \cap I_\eta \subseteq J$ for all distinct ξ, $\eta \in \Gamma$ (5A1I(a-i)). Because $\#(J) < \kappa$, (κ, λ) is a precaliber pair of $\mathfrak{B}_J \cong \mathfrak{C}_J$, so there is a $\Gamma' \in [\Gamma]^\lambda$ such that $\langle\operatorname{upr}(a_\xi, \mathfrak{C}_J)\rangle_{\xi\in\Gamma'}$ is centered. It follows that $\langle a_\xi\rangle_{\xi\in\Gamma'}$ is centered (525H). As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, we have the result.

(c) Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_{\kappa'}$ such that $\bar\nu_{\kappa'}a_\xi \ge \delta > 0$ for every $\xi < \kappa$. Fix a cofinal family $\mathcal{J}$ in $[\kappa']^{\le\omega}$ with cardinal less than $\operatorname{cf}\kappa$. For each $\xi < \kappa$ let $J_\xi \in \mathcal{J}$ be such that $a_\xi \in \mathfrak{C}_{J_\xi}$, where this time $\mathfrak{C}_{J_\xi}$ is interpreted as a subalgebra of $\mathfrak{B}_{\kappa'}$. Then there must be some $J \in \mathcal{J}$ such that $A = \{\xi : J_\xi = J\}$ has cardinal κ. Now $(\mathfrak{C}_J, \bar\nu_{\kappa'}\restriction\mathfrak{C}_J)$ is isomorphic to a subalgebra of $(\mathfrak{B}_\omega, \bar\nu_\omega)$, so has $(\kappa, \lambda, {<}\theta)$ as a measure-precaliber triple, and there is a $\Gamma \in [A]^\lambda$ such that $\{a_\xi : \xi \in I\}$ has a non-zero lower bound for every $I \in [\Gamma]^{<\theta}$. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, $(\kappa, \lambda, {<}\theta)$ is a measure-precaliber triple of $(\mathfrak{B}_{\kappa'}, \bar\nu_{\kappa'})$.

525J Corollary Suppose that κ is an infinite cardinal and $\kappa < \operatorname{cov}\mathcal{N}_\kappa$. Then κ is a measure-precaliber of every probability algebra.

proof By 525Gb, κ is a measure-precaliber of $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$; by 525Ia, it is a measure-precaliber of every probability algebra.

525K Proposition Let $\kappa > \operatorname{non}\mathcal{N}_\omega$ be a regular cardinal such that $\operatorname{cf}[\lambda]^{\le\omega} < \kappa$ for every $\lambda < \kappa$ (e.g., $\kappa = \mathfrak{c}^+$, $(\mathfrak{c}^+)^+$, etc.; or $\kappa = \omega_2$ if $\operatorname{non}\mathcal{N}_\omega = \omega_1$). Then κ is a precaliber of every measurable algebra.

proof The point is that κ is a precaliber of $\mathfrak{B}_\lambda$ for every $\lambda < \kappa$. **P** If λ is finite, this is trivial. Otherwise,

$$d(\mathfrak{B}_\lambda) = \operatorname{non}\mathcal{N}_\lambda \le \max(\operatorname{non}\mathcal{N}_\omega, \operatorname{cf}[\lambda]^{\le\omega}) < \kappa = \operatorname{cf}\kappa$$

by 524Me and 523I(a-i); it follows that κ is a precaliber of $\mathfrak{B}_\lambda$ (516Lc once more). **Q**

By 525Ib, κ is a precaliber of all measurable algebras.

525L If $\kappa > \mathfrak{c}$ is not a strong limit cardinal we can do a little better than 525K.

Proposition (Džamonja & Plebanek 04) Suppose that λ and κ are infinite cardinals such that $\lambda^\omega < \operatorname{cf}\kappa \le \kappa \le 2^\lambda$, where λ^ω is the cardinal power. Then κ is a precaliber of every measurable algebra.

proof By 525Eb and 525I(a-ii), it is enough to show that κ is a precaliber of $\mathfrak{B}_\kappa$. Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_\kappa \setminus \{0\}$. Let $\theta : \mathfrak{B}_\kappa \to \mathrm{T}_\kappa$ be a lifting, and for each $\xi < \kappa$ let K_ξ be a non-empty closed subset of θa_ξ which is determined by coordinates in a countable set I_ξ. We may suppose that each I_ξ is infinite; let $h_\xi : \mathbb{N} \to I_\xi$ be a bijection, and set $g_\xi(x) = xh_\xi$ for $x \in \{0,1\}^\kappa$, so that $g_\xi : \{0,1\}^\kappa \to \{0,1\}^{\mathbb{N}}$ is continuous and $K_\xi = g_\xi^{-1}[g_\xi[K_\xi]]$. As $\mathfrak{c} < \operatorname{cf}\kappa$, there is an $L \subseteq \{0,1\}^{\mathbb{N}}$ such that $\Gamma_0 = \{\xi : \xi < \kappa,\ g_\xi[K_\xi] = L\}$ has cardinal κ.

Because $\kappa \le 2^\lambda$, there is an $f : \kappa \times \lambda^\omega \to \mathbb{N}$ such that whenever $\langle\xi_n\rangle_{n\in\mathbb{N}}$ is a sequence of distinct elements of κ there is an $\eta < \lambda^\omega$ such that $f(\xi_n, \eta) = n$ for every n (5A1Eg). For each $\eta < \lambda^\omega$, set $A_\eta = \{\xi : \xi < \kappa,\ f(h_\xi(n), \eta) = n$ for every $n \in \mathbb{N}\}$; then $\bigcup_{\eta<\lambda^\omega} A_\eta = \kappa$, while $\operatorname{cf}\kappa > \lambda^\omega$, so there is an $\eta^* < \lambda^\omega$ such that $\Gamma = \Gamma_0 \cap A_{\eta^*}$ has κ members.

Fix $z \in L$. For ξ, $\eta \in \Gamma$ and i, $j \in \mathbb{N}$,

$$h_\xi(i) = h_\eta(j) \Longrightarrow i = f(h_\xi(i), \eta^*) = f(h_\eta(j), \eta^*) = j.$$

So we can find an $x \in \{0,1\}^\kappa$ such that $x(h_\xi(i)) = z(i)$ whenever $\xi \in \Gamma$ and $i \in \mathbb{N}$; that is, $g_\xi(x) = z$ for every $\xi \in \Gamma$. But this means that $x \in g_\xi^{-1}[L] = K_\xi$ for every $\xi \in \Gamma$. It follows that whenever $I \in [\Gamma]^{<\omega}$ then $\bigcap_{\xi\in I} \theta a_\xi \ne \emptyset$ and $\inf_{\xi\in I} a_\xi \ne 0$; that is, that $\{a_\xi : \xi \in \Gamma\}$ is centered. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, κ is a precaliber of $\mathfrak{B}_\kappa$.

525M Proposition Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and κ an infinite cardinal such that $\operatorname{cf}\kappa$ is a measure-precaliber of $(\mathfrak{A}, \bar\mu)$ and $\lambda^\omega < \kappa$ for every $\lambda < \kappa$. Then κ is a measure-precaliber of $(\mathfrak{A}, \bar\mu)$.

proof (a) Suppose first that $\mathfrak{A} = \mathfrak{B}_I$ for some set I; let $\langle e_i\rangle_{i\in I}$ be the standard generating family in $\mathfrak{B}_I$. If κ is regular, the result is trivial. Otherwise, let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{A}^+$ such that $\inf_{\xi<\kappa} \bar\nu_I a_\xi = \delta > 0$. There is a strictly increasing family $\langle\kappa_\alpha\rangle_{\alpha<\operatorname{cf}\kappa}$ of regular uncountable cardinals with supremum κ such that $\kappa_0 > \operatorname{cf}\kappa$ and if $\alpha < \operatorname{cf}\kappa$ and $\lambda < \kappa_\alpha$ then $\lambda^\omega < \kappa_\alpha$. **P** All we need to know is that if $\theta < \kappa$ there is a regular uncountable cardinal θ' such that $\theta \le \theta' < \kappa$ and $\lambda^\omega < \theta'$ whenever $\lambda < \theta'$; and $\theta' = (\theta^\omega)^+$ has this property. **Q**

For each $\xi < \kappa$, let $I_\xi \subseteq I$ be a countable set such that a_ξ belongs to the closed subalgebra of $\mathfrak{A}$ generated by $\{e_i : i \in I_\xi\}$. By the Δ-system Lemma (5A1I(a-ii)), there is for each $\alpha < \operatorname{cf}\kappa$ a set $\Gamma_\alpha \subseteq \kappa_{\alpha+1} \setminus \kappa_\alpha$ such that $\#(\Gamma_\alpha) = \kappa_{\alpha+1}$ and $\langle I_\xi\rangle_{\xi\in\Gamma_\alpha}$ is a Δ-system with root J_α say. Set $J = \bigcup_{\alpha<\operatorname{cf}\kappa} J_\alpha$, so that $\#(J) \le \operatorname{cf}\kappa$, and

$$\Gamma'_\alpha = \{\xi : \xi \in \Gamma_\alpha,\ (I_\xi \setminus J_\alpha) \cap (J \cup \bigcup_{\eta<\kappa_\alpha} I_\eta) = \emptyset\};$$

then $\#(\Gamma'_\alpha) = \kappa_{\alpha+1}$ for every $\alpha < \operatorname{cf}\kappa$, and $I_\xi \cap I_\eta \subseteq J$ for all distinct ξ, $\eta \in \Gamma' = \bigcup_{\alpha<\operatorname{cf}\kappa} \Gamma'_\alpha$. Let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{e_i : i \in J\}$. For $\xi \in \Gamma'$, set $b_\xi = \operatorname{upr}(a_\xi, \mathfrak{C}_J)$. By 515Ma,

$$\#(\mathfrak{C}_J) \le \max(\omega, \#(J))^\omega < \kappa_{\alpha+1} = \operatorname{cf}\kappa_{\alpha+1},$$

there is for each $\alpha < \operatorname{cf}\kappa$ a $c_\alpha \in \mathfrak{C}_J$ such that $\Gamma''_\alpha = \{\xi : \xi \in \Gamma'_\alpha,\ b_\xi = c_\alpha\}$ has cardinal $\kappa_{\alpha+1}$. Note that

$$\bar\nu_I c_\alpha = \bar\nu_I b_\xi \ge \bar\nu_I a_\xi \ge \delta$$

whenever $\alpha < \operatorname{cf}\kappa$ and $\xi \in \Gamma''_\alpha$.

Now recall that we are supposing that $\operatorname{cf}\kappa$ is a measure-precaliber of $\mathfrak{A}$. So there is a $\Delta \in [\operatorname{cf}\kappa]^{\operatorname{cf}\kappa}$ such that $\{c_\alpha : \alpha \in \Delta\}$ is centered in $\mathfrak{A}$. Now $\Gamma'' = \bigcup_{\alpha\in\Delta} \Gamma''_\alpha$ has cardinal κ, and $\langle b_\xi\rangle_{\xi\in\Gamma''}$ is centered. It follows that $\langle a_\xi\rangle_{\xi\in\Gamma''}$ is centered (525H).

As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, κ is a measure-precaliber of $\mathfrak{A}$.

(b) For the general case, observe that by Maharam's theorem (332B) $\mathfrak{A}$ is isomorphic to the simple product $\prod_{k\in K} \mathfrak{A}_{d_k}$ of a countable family of homogeneous principal ideals, where $\langle d_k\rangle_{k\in K}$ is a partition of unity in $\mathfrak{A}$. Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa} \bar\mu a_\xi = \delta > 0$. Let $L \subseteq K$ be a finite set such that $\sum_{k\in K\setminus L} \bar\mu d_k = \delta' < \delta$. Then there is some $k \in L$ such that

$$\Gamma_k = \{\xi : \xi < \kappa,\, \bar{\mu}(a_\xi \cap d_k) \ge \tfrac{\delta-\delta'}{\#(L)}\}$$

has cardinal κ. Since $\operatorname{cf}\kappa$ is a measure-precaliber of $\mathfrak{A}$, it is also a measure-precaliber of $\mathfrak{A}_{d_k}$ (cf. 516Sc). Since $(\mathfrak{A}_{d_k}, \bar{\mu}\restriction\mathfrak{A}_{d_k})$ is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_I, \bar{\nu}_I)$ for some I, (a) tells us that κ is a measure-precaliber of $\mathfrak{A}_{d_k}$. There is therefore a set $\Gamma \in [\Gamma_k]^\kappa$ such that $\langle a_\xi \cap d_k\rangle_{\xi\in\Gamma}$ and $\langle a_\xi\rangle_{\xi\in\Gamma}$ are centered. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, κ is a measure-precaliber of $\mathfrak{A}$.

525N Proposition (Argyros & Tsarpalias 82) Let κ be either ω or a strong limit cardinal of countable cofinality, and suppose that $2^\kappa = \kappa^+$. Then κ^+ is not a precaliber of $\mathfrak{B}_\kappa$.

proof By 523Lb, $\operatorname{non}\mathcal{N}_\kappa > \kappa$. So if we enumerate $\{0,1\}^\kappa$ as $\langle x_\xi\rangle_{\xi<\kappa^+}$ and set $E_\xi = \{x_\eta : \eta < \xi\}$ for $\xi < \kappa^+$, $\langle E_\xi\rangle_{\xi<\kappa^+}$ is an increasing family in $\mathcal{N}_\kappa$ with union $\{0,1\}^\kappa$. By 525Cc, κ^+ is not a precaliber of $\mathfrak{B}_\kappa$.

525O As in 523P, GCH decides the most important questions.

Proposition Suppose that the generalized continuum hypothesis is true.

(a) An infinite cardinal κ is a measure-precaliber of every probability algebra iff $\operatorname{cf}\kappa$ is not the successor of a cardinal of countable cofinality.

(b) An infinite cardinal κ is a precaliber of every measurable algebra iff $\operatorname{cf}\kappa$ is neither ω nor the successor of a cardinal of countable cofinality.

proof (a)(i) If κ is a measure-precaliber of every probability algebra, so is $\operatorname{cf}\kappa$ (525Dc). By 525N, $\operatorname{cf}\kappa$ cannot be the successor of a cardinal of countable cofinality.

(ii) Now suppose that $\operatorname{cf}\kappa$ is not the successor of a cardinal of countable cofinality. If $\kappa = \omega$, then certainly κ is a measure-precaliber of every probability algebra (525Fa). Otherwise, $\kappa > \lambda^\omega$ for every $\lambda < \kappa$ and $\operatorname{cf}\kappa > \lambda^\omega$ for every $\lambda < \operatorname{cf}\kappa$ (5A6Ac). By 525K, $\operatorname{cf}\kappa$ is a measure-precaliber of every probability algebra; by 525M, so is κ.

(b) Put (a) and 525Eb together.

***525P** As in 522U, the Freese-Nation number of $\mathcal{P}\mathbb{N}$ is relevant to the questions here.

Proposition $(\mathfrak{m}_{\text{countable}}, \mathrm{FN}^*(\mathcal{P}\mathbb{N}))$ is not a precaliber pair of $\mathfrak{B}_\omega$.

proof By 518D(iv), the Freese-Nation number of the topology of $\{0,1\}^\omega$ is $\mathrm{FN}(\mathcal{P}\mathbb{N})$; the regular Freese-Nation numbers are therefore also equal. We know that $\mathfrak{m}_{\text{countable}}$ is the covering number of the meager ideal of $\mathbb{R}$ (522Sa), and therefore also of the meager ideal of $\{0,1\}^\omega$ (522Wb) and of the nowhere dense ideal of $\{0,1\}^\omega$. By 518E, there is a set $A \subseteq \{0,1\}^\omega$, with cardinal $\mathfrak{m}_{\text{countable}}$, such that $\#(A\cap F) < \mathrm{FN}^*(\mathcal{P}\mathbb{N})$ for every nowhere dense set $F \subseteq \{0,1\}^\omega$.

Fix a nowhere dense compact set $K \subseteq \{0,1\}^\omega$ of non-zero measure. For each $x \in A$, set $a_x = (K+x)^\bullet$ in $\mathfrak{B}_\omega$, where $+$ here is the usual group operation corresponding to the identification $\{0,1\}^\omega \cong \mathbb{Z}_2^\omega$. Then every a_x is non-zero. If $B \subseteq A$ and $\{a_x : x \in B\}$ is centered, then $\{K + x : x \in B\}$ has the finite intersection property, so there is a y in its intersection; now $B \subseteq A \cap (K + y)$, and $K + y$ is nowhere dense, so $\#(B) < \mathrm{FN}^*(\mathcal{P}\mathbb{N})$. Thus $\langle a_x\rangle_{x\in A}$ has no centered subfamily of size $\mathrm{FN}^*(\mathcal{P}\mathbb{N})$ and witnesses that $(\mathfrak{m}_{\text{countable}}, \mathrm{FN}^*(\mathcal{P}\mathbb{N}))$ is not a precaliber pair of $\mathfrak{B}_\omega$.

525Q I turn now to some results which may be interpreted as information on precaliber triples in which the third cardinal is *finite*.

Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, $\langle u_n\rangle_{n\in\mathbb{N}}$ a $\|\;\|_2$-bounded sequence in $L^2 = L^2(\mathfrak{A}, \bar{\mu})^+$, and $\mathcal{F}$ a non-principal ultrafilter on $\mathbb{N}$. Suppose that $p \in [0,\infty[$ is such that $\sup_{n\in\mathbb{N}}\|u_n^p\|_2$ is finite, and set $v = \lim_{n\to\mathcal{F}} u_n$, $w = \lim_{n\to\mathcal{F}} u_n^p$, the limits being taken for the weak topology in L^2. Then $v^p \le w$.

proof Of course the positive cone of L^2 is closed for the weak topology so $v \ge 0$ and we can speak of v^p. **?** If $v^p \not\le w$, there are α, $\beta \ge 0$ such that $\alpha^p > \beta$ and

$$a = [\![v > \alpha]\!] \setminus [\![w > \beta]\!] \ne 0.$$

Let $b \subseteq a$ be such that $0 < \bar{\mu}b < \infty$ and consider $u = \frac{1}{\bar{\mu}b}\chi b$. Then, setting $q = p/(p-1)$ (of course $p > 1$),

$$(\alpha\bar{\mu}b)^p \le (\int_b v)^p = \lim_{n\to\mathcal{F}}(\int u_n \times \chi b \times \chi b)^p \le \lim_{n\to\mathcal{F}}(\|u_n \times \chi b\|_p \|\chi b\|_q)^p$$

(by Hölder's inequality, 244Eb)

$$= \lim_{n\to\mathcal{F}} (\bar\mu b)^{p/q} \int_b u_n^p = (\bar\mu b)^{p-1} \int_b w \le \beta(\bar\mu b)^p < \alpha^p(\bar\mu b)^p,$$

which is absurd. **X** So $v^p \le w$.

525R Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\langle u_n\rangle_{n\in\mathbb{N}}$ a $\| \|_\infty$-bounded sequence in $L^\infty(\mathfrak{A},\bar\mu)^+$ such that $\delta = \inf_{n\in\mathbb{N}} \int u_n > 0$. Let $k_0,\dots,k_m$ be strictly positive integers with sum k. Suppose that $\gamma < \delta^k$.

(a) There are integers $n_0 < n_1 < \ldots < n_m$ such that $\int \prod_{j=0}^m u_{n_j}^{k_j} \ge \gamma$.

(b) In fact, there is an infinite set $I \subseteq \mathbb{N}$ such that $\int \prod_{j=0}^m u_{n_j}^{k_j} \ge \gamma$ whenever $n_0,\dots,n_m$ belong to I and $n_0 < n_1 < \ldots < n_m$.

proof (a) Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$. For each $j \le m$, let v_j be the limit $\lim_{n\to\mathcal{F}} u_n^{k_j}$ for the weak topology on $L^2(\mathfrak{A},\bar\mu)$; let v be the limit $\lim_{n\to\mathcal{F}} u_n$. By 525Q,

$$\int \prod_{j=0}^m v_j \ge \int \prod_{j=0}^m v^{k_j} = \int v^k = (\|\chi 1\|_q \|v\|_k)^k$$

(where $q = \frac{k}{k-1}$, or ∞ if $k = 1$)

$$\ge \Bigl(\int v \times \chi 1\Bigr)^k$$

(by Hölder's inequality again, if $k > 1$)

$$= \Bigl(\int v\Bigr)^k = \lim_{n\to\mathcal{F}} \Bigl(\int u_n\Bigr)^k \ge \delta^k > \gamma.$$

(Or use 244Xd to show more directly that $\int v^k \ge (\int v)^k$.) We can therefore choose $n_0,\dots,n_m$ inductively so that

$$\int \prod_{j=0}^s u_{n_j}^{k_j} \times \prod_{j=s+1}^m v_j > \gamma$$

for each $s \le m$ (interpreting the final product $\prod_{j=m+1}^m v_j$ as $\chi 1$), since when we come to choose n_s we shall be able to use any member of

$$\{n : n > n_j \text{ for } j < s,\ \int u_n^{k_s} \times \prod_{j=0}^{s-1} u_{n_j}^{k_j} \times \prod_{j=s+1}^m v_j > \gamma\},$$

which belongs to $\mathcal{F}$ so is not empty. At the end of the induction we shall have a sequence $n_0 < \ldots < n_m$ such that $\int \prod_{j=0}^m u_{n_j}^{k_j} \ge \gamma$, as required.

(b) Let $\mathcal{J} \subseteq [\mathbb{N}]^{m+1}$ be the family of all sets of the form $\{n_0,\dots,n_m\}$ where $n_0 < \ldots < n_m$ and $\int \prod_{j=0}^m u_{n_j}^{k_j} \ge \gamma$. By (a), applied to subsequences of $\langle u_n\rangle_{n\in\mathbb{N}}$, every infinite subset of $\mathbb{N}$ includes some member of $\mathcal{J}$. By Ramsey's theorem (4A1G), there is an infinite $I \subseteq \mathbb{N}$ such that $[I]^{m+1} \subseteq \mathcal{J}$, which is what we need.

525S Theorem (Fremlin 88) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and κ an infinite cardinal. Let $\langle u_\xi\rangle_{\xi<\kappa}$ be a $\| \|_\infty$-bounded family in $L^\infty(\mathfrak{A})^+$. Set $\delta = \inf_{\xi<\kappa} \int u_\xi$. Then for any $k \in \mathbb{N}$ and $\gamma < \delta^{k+1}$ there is a $\Gamma \in [\kappa]^\kappa$ such that $\int \prod_{i=0}^k u_{\xi_i} \ge \gamma$ for all $\xi_0,\dots,\xi_k \in \Gamma$.

proof (a) It will be helpful to note straight away that it will be enough to consider the case $(\mathfrak{A},\bar\mu) = (\mathfrak{B}_I,\bar\nu_I)$ for some set I. **P** There is always a $(\mathfrak{B}_I,\bar\nu_I)$ in which $(\mathfrak{A},\bar\mu)$ can be embedded. In this case, $L^\infty(\mathfrak{A})$ can be identified, as f-algebra, with a subspace of $L^\infty(\mathfrak{B}_I)$, and the embedding respects integrals. So we can regard $\langle u_\xi\rangle_{\xi<\kappa}$ as a family in $L^\infty(\mathfrak{B}_I)$ and perform all calculations there. **Q**

At the same time, the case $\delta = 0$ is trivial, so let us suppose henceforth that $\delta > 0$.

(b) Next, having fixed on a suitable set I, let $\langle e_i\rangle_{i\in I}$ be the standard generating family in $\mathfrak{B}_I$, and for $J \subseteq I$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{B}_I$ generated by $\{e_i : i \in J\}$; following 325N, I will say that a member of $\mathfrak{C}_J$ is 'determined by coordinates in J'. For $J \subseteq I$ let $P_J : L^1(\mathfrak{B}_I,\bar\nu_I) \to L^1(\mathfrak{C}_J,\bar\nu_I\restriction\mathfrak{C}_J)$ be the conditional expectation operator. Note that if J, $K \subseteq I$ then $P_J P_K = P_{J\cap K}$ (254Ra/458M(iii)).

It will be useful to start by looking at a particular subset W of $L^\infty(\mathfrak{B}_I)$, being the set of linear combinations $\sum_{i=0}^n \alpha_i \chi c_i$ where every α_i is rational and every c_i is determined by coordinates in a finite set. Now $P_J[W] \subseteq W$ for every $J \subseteq I$. **P** If $c \in \mathfrak{C}_K$ where $K \subseteq I$ is finite, then

$$P_J(\chi c) = P_J P_K(\chi c) = P_{J\cap K}(\chi c) = \sum_{d \text{ is an atom of } \mathfrak{C}_{J\cap K}} \frac{\bar{\nu}_I(c\cap d)}{\bar{\nu}_I d}\chi d \in W.$$

As P_J is linear, this is enough. **Q** Observe also that if $K \subseteq I$ is finite, then $P_K[W]$ is countable, being the set of rational linear combinations of $\{\chi c : c \in \mathfrak{C}_K\}$.

(c) Suppose, therefore, that we have a set I, a $\|\,\|_\infty$-bounded family $\langle u_\xi\rangle_{\xi<\kappa}$ in $L^\infty(\mathfrak{B}_I)^+$ with $\inf_{\xi<\kappa}\int u_\xi = \delta > 0$, a $k \in \mathbb{N}$ and a $\gamma < \delta^{k+1}$. To begin with, let us suppose further that

(α) $u_\xi \le \chi 1$ for every $\xi < \kappa$,

(β) $u_\xi \in W$, as described in (b), for each $\xi < \kappa$;

for each $\xi < \kappa$, let $I_\xi \in [I]^{<\omega}$ be such that $P_{I_\xi}u_\xi = u_\xi$.

(i) Suppose that $\kappa = \omega$. Because there are only finitely many sequences $k_0, \dots, k_m$ of strictly positive integers with sum equal to $k+1$, we can use 525Rb a finite number of times to find an infinite $\Gamma \subseteq \omega$ such that $\int \prod_{j=0}^m u_{n_j}^{k_j} \ge \gamma$ whenever $\sum_{j=0}^m k_j = k+1$ and $n_0 < \ldots < n_m$ belong to Γ. But in this case we surely have $\int \prod_{i=0}^k u_{n_i} \ge \gamma$ for all $n_0, \dots, n_k \in \Gamma$.

(ii) Next, suppose that $\kappa > \omega$ is regular. By the Δ-system Lemma (4A1Db) there is a $\Delta \in [\kappa]^\kappa$ such that $\langle I_\xi\rangle_{\xi\in\Delta}$ is a Δ-system with root J say. Since $P_J[W]$ is countable, there is a v such that $\Gamma = \{\xi : \xi \in \Delta,\ P_J u_\xi = v\}$ has cardinal κ. Of course

$$\int v = \int u_\xi \ge \delta$$

for every $\xi \in \Gamma$.

By 458Lg again, $\langle \mathfrak{C}_{I_\xi}\rangle_{\xi\in\Delta}$ is relatively independent over $\mathfrak{C}_J$. Now suppose that $\xi_0, \dots, \xi_k$ belong to Γ. Then

$$\int \prod_{i=0}^k u_{\xi_i} \ge \int \prod_{i=0}^k P_J u_{\xi_i}$$

(458Lh)

$$= \int v^{k+1} \ge \Bigl(\int v\Bigr)^{k+1}$$

(as in the proof of 525Ra)

$$\ge \delta^{k+1} \ge \gamma,$$

and this is what we need to know.

(iii) Finally, suppose that $\kappa > \operatorname{cf}\kappa \ge \omega$. Set $\lambda = \operatorname{cf}\kappa$ and let $\langle \kappa_\zeta\rangle_{\zeta<\lambda}$ be a strictly increasing family of regular cardinals greater than λ and with supremum κ. For each $\zeta < \lambda$ let $\Delta_\zeta \subseteq \kappa_{\zeta+1} \setminus \kappa_\zeta$ be a set of size $\kappa_{\zeta+1}$ such that $\langle I_\xi\rangle_{\xi\in\Delta_\zeta}$ is a Δ-system with root J_ζ say. Set $J = \bigcup_{\zeta<\lambda} J_\zeta$; note that $\#(J) \le \lambda < \kappa_{\zeta+1}$ for every ζ, so that

$$\Delta'_\zeta = \{\xi : \xi \in \Delta_\zeta,\ (I_\xi \setminus J_\zeta) \cap (J \cup \textstyle\bigcup_{\eta<\kappa_\zeta} I_\eta) = \emptyset\}$$

still has cardinal $\kappa_{\zeta+1}$.

If $\zeta < \lambda$ and $\xi \in \Delta'_\zeta$, $I_\xi \cap J$ is included in the finite set J_ζ; so $\{P_J u_\xi : \xi \in \Delta'_\zeta\}$ is countable, and there is a v_ζ such that $\Delta''_\zeta = \{\xi : \xi \in \Delta'_\zeta,\ P_J u_\xi = v_\zeta\}$ has cardinal $\kappa_{\zeta+1}$. Note that

$$\int v_\zeta = \int u_\xi \ge \delta$$

whenever $\zeta < \lambda$ and $\xi \in \Delta''_\zeta$.

Because λ is regular, we can apply (i) or (ii) above to find an $A \in [\lambda]^\lambda$ such that $\int \prod_{i=0}^k v_{\zeta_i} \ge \gamma$ whenever $\zeta_0, \dots, \zeta_k \in A$. Set $\Gamma = \bigcup_{\zeta\in A} \Delta''_\zeta$; because A must be cofinal with λ, $\#(\Gamma) = \kappa$.

If ξ, $\eta \in \Gamma$ are distinct, then $I_\xi \cap I_\eta \subseteq J$. So $\langle \mathfrak{C}_{I_\xi}\rangle_{\xi\in\Gamma}$ is relatively independent over $\mathfrak{C}_J$. Take any $\xi_0, \dots, \xi_k \in \Gamma$; for each $i \le k$, let $\zeta_i \in A$ be such that $\xi_i \in \Delta''_{\zeta_i}$. By 458Lh again,

$$\int \prod_{i=0}^n u_{\xi_i} \ge \int \prod_{i=0}^n P_J u_{\xi_i} = \int \prod_{i=0}^n v_{\zeta_i} \ge \gamma,$$

so we are done (provided (α)-(β) are true).

(d) Now let us unwind these conditions from the bottom.

(i) If (α) is true, but (β) is not, take $\epsilon \in]0,\delta[$ such that $(\delta-\epsilon)^{k+1} > \gamma+(k+1)\epsilon$. For each $\xi<\kappa$, let $u'_\xi \in W$ be such that $u'_\xi \le \chi 1$ and $\int |u_\xi - u'_\xi| \le \epsilon$. (Such a u'_ξ exists because $\bigcup\{\mathfrak{C}_K : K \in [I]^{<\omega}\}$ is topologically dense in $\mathfrak{B}_I$ and $u \wedge \chi 1 \in W$ for every $u \in W$.) Then $\int u'_\xi \ge \delta - \epsilon$ for each ξ, so we can apply (c) to $\langle u'_\xi\rangle_{\xi<\kappa}$ to see that there is a $\Gamma \in [\kappa]^\kappa$ such that $\int \prod_{i=0}^k u'_{\xi_i} \ge \gamma + (k+1)\epsilon$ for all $\xi_0,\dots,\xi_k \in \Gamma$. Now (because u_ξ and u'_ξ all lie between 0 and $\chi 1$) we have

$$|\prod_{i=0}^k u_{\xi_i} - \prod_{i=0}^k u'_{\xi_i}| \le \sum_{i=0}^k |u_{\xi_i} - u'_{\xi_i}|$$

(see 285O), so that

$$\int \prod_{i=0}^k u_{\xi_i} \ge \int \prod_{i=0}^k u'_{\xi_i} - \sum_{i=0}^k \int |u_{\xi_i} - u'_{\xi_i}| \ge \gamma$$

whenever $\xi_0,\dots,\xi_n \in \Gamma$, and the theorem is still true.

(ii) Finally, for the general case, set $M = 1 + \sup_{\xi<\kappa} \|u_\xi\|_\infty$, and $u'_\xi = \frac{1}{M} u_\xi$ for $\xi < \kappa$. Then every u'_ξ belongs to $[0,\chi 1]$ and $\int u'_\xi \ge \frac{\delta}{M}$. By (i) there is a $\Gamma \in [\kappa]^\kappa$ such that $\int \prod_{i=0}^k u'_{\xi_i} \ge \frac{\gamma}{M^{k+1}}$ for all $\xi_0,\dots,\xi_k \in \Gamma$; in which case $\int \prod_{i=0}^k u_{\xi_i} \ge \gamma$ for all $\xi_0,\dots,\xi_k \in \Gamma$.

This completes the proof.

525T Corollary (Argyros & Kalamidas 82) (a) If κ is an infinite cardinal and $k \in \mathbb{N}$, (κ,κ,k) is a measure-precaliber triple of every probability algebra.

(b) If κ is a cardinal of uncountable cofinality and $k \in \mathbb{N}$, (κ,κ,k) is a precaliber triple of every measurable algebra. In particular, every measurable algebra satisfies Knaster's condition.

(c) If κ is a cardinal of uncountable cofinality, $(\mathfrak{A},\bar\mu)$ is a probability algebra, $k \ge 1$ and $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}\setminus\{0\}$, then there are a $\delta > 0$ and a $\Gamma \in [\kappa]^\kappa$ such that $\bar\mu(\inf_{\xi\in I} a_\xi) \ge \delta$ for every $I \in [\Gamma]^k$.

(d) For any measurable algebra $\mathfrak{A}$, $\mathfrak{m}(\mathfrak{A}) \ge \mathfrak{m}_{\mathrm{K}}$; and if $\mathfrak{m}(\mathfrak{A}) > \omega_1$, then $\mathfrak{m}(\mathfrak{A}) \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$. So if $\omega \le \kappa < \mathfrak{m}_K$, κ is a measure-precaliber of every probability algebra.

proof Really this is just the special case of 525S in which every u_ξ belongs to $\{\chi a : a \in \mathfrak{A}\}$.

(a) If $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa} \bar\mu a_\xi = \delta > 0$, take any $\gamma \in \left]0,\delta^k\right[$. Setting $u_\xi = \chi a_\xi$ for each ξ, $\int u_\xi \ge \delta$ for each ξ, so there is a $\Gamma \in [\kappa]^\kappa$ such that $\int \prod_{i=1}^k u_{\xi_i} \ge \gamma$ for every $\xi_1,\dots,\xi_k \in \Gamma$; in which case $\inf_{\xi\in J} a_\xi \ne 0$ for every $J \in [\Gamma]^k$. As $\langle a_\xi\rangle_{\xi<\kappa}$ is arbitrary, (κ,κ,k) is a measure-precaliber triple of $(\mathfrak{A},\bar\mu)$.

(b) This now follows at once from 525Db, since any non-zero measurable algebra can be given a probability measure. Taking $\kappa = \omega_1$ and $k = 2$, we have Knaster's condition.

(c) For the quantitative version, we have only to note that there must be some $\alpha > 0$ such that $\#(\{\xi : \bar\mu a_\xi \ge \alpha\}$ has cardinal κ, and take $\delta < \alpha^k$.

(d) By (b), $\mathfrak{A}$ satisfies Knaster's condition; it follows at once that $\mathfrak{m}(\mathfrak{A}) \ge \mathfrak{m}_{\mathrm{K}}$, while $\mathrm{sat}(\mathfrak{A}) \le \omega_1$. If $\mathfrak{m}(\mathfrak{A}) > \omega_1$, then ω_1 is a precaliber of $\mathfrak{A}$ (517Ig) so $\mathfrak{m}(\mathfrak{A}) \ge \mathfrak{m}_{\mathrm{pc}\omega_1}$. By 525Fb, every infinite cardinal less than $\mathfrak{m}_{\mathrm{K}}$ is a measure-precaliber of every probability algebra.

525X Basic exercises (a) Let (X,Σ,μ) be any measure space and $\mathfrak{A}$ its measure algebra. (i) Show that $(\mathfrak{A}^+,\supseteq) \preccurlyeq_{\mathrm{T}} (\Sigma\setminus\mathcal{N}(\mu),\supseteq)$. (ii) Show that a pair (κ,λ) is a downwards precaliber pair of $\Sigma\setminus\mathcal{N}(\mu)$ iff it is a precaliber pair of $\mathfrak{A}$.

>(b) Let $\mathfrak{A}$ be a measurable algebra. Show that ω_1 is a precaliber of $\mathfrak{A}$ iff *either* $\mathfrak{A}$ is purely atomic *or* $\tau(\mathfrak{A}) \le \omega$ and $\mathrm{cov}\,\mathcal{N}_\omega > \omega_1$ *or* $\mathrm{cov}\,\mathcal{N}_{\omega_1} > \omega_1$. (*Hint*: 525G, 523F.)

>(c) (i) Suppose that $\mathrm{add}\,\mathcal{N}_\omega = \mathrm{cov}\,\mathcal{N}_\omega = \kappa$. Show that κ is not a precaliber of $\mathfrak{B}_\omega$. (ii) Suppose that $\mathrm{non}\,\mathcal{N}_\omega = \mathfrak{c}$. Show that $\mathfrak{c}$ is not a precaliber of $\mathfrak{B}_\omega$.

(d) Let (X,Σ,μ) be a complete strictly localizable measure space and $\mathfrak{A}$ its measure algebra. Show that the supported relation $(\Sigma\setminus\mathcal{N}(\mu),\ni,X)$ has the same precaliber pairs as the Boolean algebra $\mathfrak{A}$.

(e) Suppose that (κ,λ) is a precaliber pair of every measurable algebra, that I is a set, and that $X\subseteq\mathbb{R}^I$ is a compact set such that $\#(\{i:x(i)\ne 0\})<\lambda$ for every $x\in X$. Show that $\#(\{i:x(i)\ne 0\})<\kappa$ for every x belonging to the closed convex hull of X in $\mathbb{R}^I$. (*Hint*: 461I.)

(f) Suppose that $\lambda\le\kappa$ are infinite cardinals, $(\mathfrak{A},\bar\mu)$ is a homogeneous probability algebra, and that $\gamma<1$ is such that whenever $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ and $\bar\mu a_\xi\ge\gamma$ for every $\xi<\kappa$, there is a $\Gamma\in[\kappa]^\lambda$ such that $\{a_\xi:\xi\in\Gamma\}$ is centered. Show that (κ,λ) is a measure-precaliber pair of $(\mathfrak{A},\bar\mu)$. (*Hint*: given that $\inf_{\xi<\kappa}\bar\mu a_\xi>0$, take $(\mathfrak{C},\bar\lambda)=\widehat{\bigotimes}_m(\mathfrak{A},\bar\mu)\cong(\mathfrak{A},\bar\mu)$ to be the probability algebra free product of a large finite number of copies of $(\mathfrak{A},\bar\mu)$, and consider $c_\xi=\sup_{j<m}\varepsilon_j a_\xi$ for $\xi<\kappa$.)

(g) Let $\mathfrak{A}$ be a Boolean algebra, and λ, κ cardinals such that (κ,λ) is a measure-precaliber pair of every probability algebra. Suppose that $A\subseteq\mathfrak{A}\setminus\{0\}$ has positive intersection number. Show that if $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in A, then there is a $\Gamma\in[\kappa]^\lambda$ such that $\{a_\xi:\xi\in\Gamma\}$ is centered.

(h) Let κ be a cardinal such that (α) $\lambda^\omega<\kappa$ for every $\lambda<\kappa$ (β) $\lambda^\omega<\operatorname{cf}\kappa$ for every $\lambda<\operatorname{cf}\kappa$. Show that κ is a measure-precaliber of every probability algebra.

525Z Problem Can we, in ZFC, find an infinite cardinal κ which is not a measure-precaliber of all probability algebras? From 525N we see that a negative answer will require a model of set theory in which $2^\kappa>\kappa^+$ for all strong limit cardinals κ of countable cofinality; for such models see Foreman & Woodin 91, Cummings 92.

525 Notes and comments There seem to be three methods of proving that a cardinal is a precaliber of a measure algebra. First, we have the counting arguments of 516L; since we know something about the centering numbers of measure algebras (524Me), this gives us a start (see the proof of 525K). Next, we can try to use Martin numbers, as in 517Ig and 525F; since we can relate the Martin number of a measure algebra to the cardinals of §523 (524Md), we get the formulation 525J. In third place, we have arguments based on the special structure of measure algebras, using 525H to apply Δ-system theorems from infinitary combinatorics. Subject to the generalized continuum hypothesis, these ideas are enough to answer the most natural questions (525O). Without this simplification, they leave conspicuous gaps. The most important seems to be 525Z. Even if we know all the cardinals $\operatorname{add}\mathcal{N}_\kappa$, $\operatorname{cov}\mathcal{N}_\kappa$, $\operatorname{non}\mathcal{N}_\kappa$ and $\operatorname{cf}\mathcal{N}_\kappa$ of §523, we may still not be able to determine which cardinals are precalibers; 525Xb is an exceptional special case.

I have presented this section with a bias towards measure-precalibers rather than precalibers. When there is a difference, the former search deeper. 'Cofinality ω_1' has a rather special position in this theory (525Ib), deriving from the combinatorial arguments of 5A1H.

526 Asymptotic density zero

In §491, I devoted some paragraphs to the ideal $\mathcal{Z}$ of subsets of $\mathbb{N}$ with asymptotic density zero, as part of an investigation into equidistributed sequences in topological measure spaces. Here I return to $\mathcal{Z}$ to examine its place in the Tukey ordering of partially ordered sets. We find that it lies strictly between $\mathbb{N}^\mathbb{N}$ and ℓ^1 (526B, 526J, 526L) but in some sense is closer to ℓ^1 (526Ga). On the way, I mention the ideal $\mathcal{N}\mathit{wd}$ of nowhere dense subsets of $\mathbb{N}^\mathbb{N}$ (526H-526L) and ideals of sets with negligible closures (526I-526M).

526A Proposition For $I\subseteq\mathbb{N}$, set $\nu I=\sup_{n\ge1}\frac{1}{n}\#(I\cap n)$.

(a) ν is a strictly positive submeasure (definition: 392A) on $\mathcal{P}\mathbb{N}$. We have a metric ρ on $\mathcal{P}\mathbb{N}$ defined by setting $\rho(I,J)=\nu(I\triangle J)$ for all I, $J\subseteq\mathbb{N}$, under which the Boolean operations $\cup$, $\cap$, $\triangle$ and $\setminus$ and upper asymptotic density $d^*:\mathcal{P}\mathbb{N}\to[0,1]$ are uniformly continuous and $\mathcal{P}\mathbb{N}$ is complete.

(b) $\mathcal{Z}$ is a separable closed subset of $\mathcal{P}\mathbb{N}$.

(c) If $\mathcal{I}\subseteq\mathcal{Z}$ is such that $\sum_{I\in\mathcal{I}}\nu I$ is finite, then $\bigcup\mathcal{I}\in\mathcal{Z}$.

(d) With the subspace topology, $(\mathcal{Z},\subseteq)$ is a metrizably compactly based directed set (definition: 513K).

proof (a) It is elementary to check that ν is a strictly positive submeasure. By 392H, ρ is a metric under which the Boolean operations are uniformly continuous. Since

$$|d^*(I)-d^*(J)|\le d^*(I\triangle J)\le\nu(I\triangle J)$$

for all I, $J \subseteq \mathbb{N}$, d^* is uniformly continuous. Let $\langle I_j \rangle_{j\in\mathbb{N}}$ be a sequence in $\mathcal{P}\mathbb{N}$ such that $\rho(I_j, I_{j+1}) \le 2^{-j}$ for every $j \in \mathbb{N}$. Set $I = \bigcap_{m\in\mathbb{N}} \bigcup_{j\ge m} I_j$. For any $m \in \mathbb{N}$ and $n \ge 1$,

$$\frac{1}{n}\#(n \cap (I_m \triangle I)) \le \frac{1}{n}\sum_{j=m}^{\infty} \#(n \cap (I_j \triangle I_{j+1})) \le \sum_{j=m}^{\infty} \rho(I_j, I_{j+1}) \le 2^{-m+1},$$

so $\rho(I_m, I) \le 2^{-m+1}$. Thus $\langle I_j \rangle_{j\in\mathbb{N}}$ is convergent to I; as $\langle I_j \rangle_{j\in\mathbb{N}}$ is arbitrary, $\mathcal{P}\mathbb{N}$ is complete (cf. 2A4E).

(b) $\mathcal{Z}$ is a closed subset of $\mathcal{P}\mathbb{N}$. **P** If I belongs to the closure $\overline{\mathcal{Z}}$ of $\mathcal{Z}$, and $\epsilon > 0$, let $J \in \mathcal{Z}$ be such that $\rho(I, J) \le \frac{1}{2}\epsilon$, and let $m \ge 1$ be such that $\frac{1}{n}\#(J \cap n) \le \frac{1}{2}\epsilon$ for every $n \ge m$; then $\frac{1}{n}\#(I \cap n) \le \epsilon$ for every $n \ge m$. As ϵ is arbitrary, $I \in \mathcal{Z}$; as I is arbitrary, $\mathcal{Z}$ is closed. **Q**

$\mathcal{Z}$ is separable because $[\mathbb{N}]^{<\omega}$ is a countable dense set. (If $I \in \mathcal{Z}$ and $n \in \mathbb{N}$, $\rho(a, a \cap n) \le \sup_{m>n} \frac{1}{m}\#(m \cap I)$.)

(c) Let $\epsilon > 0$. Then there is a finite $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\sum_{I\in\mathcal{I}\setminus\mathcal{I}_0} \nu I \le \epsilon$. Set $J = \bigcup \mathcal{I}$, $J_0 = \bigcup \mathcal{I}_0$; then $J_0 \in \mathcal{Z}$ so there is an $n_0 \in \mathbb{N}$ such that $\#(J_0 \cap n) \le n\epsilon$ for every $n \ge n_0$. If $n \ge n_0$, then

$$\#(J \cap n) \le \#(J_0 \cap n) + \sum_{I\in\mathcal{I}\setminus\mathcal{I}_0} \#(I \cap n) \le n\epsilon + \sum_{I\in\mathcal{I}\setminus\mathcal{I}_0} n\nu I \le 2n\epsilon.$$

As ϵ is arbitrary, $J \in \mathcal{Z}$.

(d) $\mathcal{Z}$ is closed under $\cup$, so is a directed set under $\subseteq$, and $\cup : \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ is continuous. If $a \in \mathcal{Z}$, then on $\{b : b \subseteq a\}$ the topology $\mathfrak{T}_\rho$ induced by ρ agrees with the usual compact Hausdorff topology $\mathfrak{S}$ of $\mathcal{P}\mathbb{N} \cong \{0,1\}^{\mathbb{N}}$. **P** If $n \in \mathbb{N}$ and $\rho(b, c) < \frac{1}{n+1}$, then $b \cap n = c \cap n$; so $\mathfrak{T}_\rho$ is finer than $\mathfrak{S}$ on $\mathcal{P}\mathbb{N}$. If $\epsilon > 0$, there is an $m \in \mathbb{N}$ such that $\#(a \cap n) \le n\epsilon$ whenever $n \ge m$; now $\rho(b, c) \le \epsilon$ whenever b, $c \subseteq a$ and $b \cap m = c \cap m$. So $\mathfrak{S}$ is finer than $\mathfrak{T}_\rho$ on $\{b : b \subseteq a\}$. **Q** Since $\{b : b \subseteq a\}$ is $\mathfrak{S}$-compact, it is $\mathfrak{T}_\rho$-compact.

Now suppose that $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{Z}$ with $\mathfrak{T}_\rho$-limit a. Then it has a subsequence $\langle a_{n_k} \rangle_{k\in\mathbb{N}}$ such that $\rho(a, a_{n_k}) \le 2^{-k}$ for every k. Set $b = \bigcup_{k\in\mathbb{N}} a_{n_k}$. Then, given $\epsilon > 0$, let r, $m \in \mathbb{N}$ be such that $2^{-r} \le \epsilon$ and $\#(n \cap (a \cup \bigcup_{k\le r} a_{n_k})) \le n\epsilon$ for every $n \ge m$; then

$$\begin{aligned}\#(n \cap b) &\le \#(n \cap (a \cup \bigcup_{k\le r} a_{n_k})) + \sum_{k=r+1}^{\infty} \#(n \cap a_{n_k} \setminus a) \\ &\le n\epsilon + \sum_{k=r+1}^{\infty} 2^{-k} n \le 2n\epsilon\end{aligned}$$

for every $n \ge m$. So $b \in \mathcal{Z}$ and $\{a_{n_k} : k \in \mathbb{N}\}$ is bounded above in $\mathcal{Z}$.

526B Proposition (Fremlin 91) $\mathbb{N}^{\mathbb{N}} \preccurlyeq_{\mathrm{T}} \mathcal{Z} \preccurlyeq_{\mathrm{T}} \ell^1$.

proof (a) For $\alpha \in \mathbb{N}^{\mathbb{N}}$, set

$$\phi(\alpha) = \{2^n i : n \in \mathbb{N},\ i \le \alpha(n)\}.$$

Then $\phi(\alpha) \in \mathcal{Z}$, because if $k \in \mathbb{N}$ then

$$\#(m \cap \phi(\alpha)) \le \sum_{n=0}^{k} \alpha(n) + \lceil 2^{-k} m \rceil$$

for every m. Also $\phi : \mathbb{N}^{\mathbb{N}} \to \mathcal{Z}$ is a Tukey function, because if $\phi(\alpha) \subseteq a \in \mathcal{Z}$ then $\alpha(n) \le \min\{i : 2^n i \notin a\}$ for every $n \in \mathbb{N}$. So $\mathbb{N}^{\mathbb{N}} \preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

(b) Give $\mathcal{Z}$ the metric ρ of 526A. Then $\mathcal{Z}$ is complete and separable and the lattice operation $\cup$ is uniformly continuous (526Aa). By 524C, $(\mathcal{Z}, \subseteq', [\mathcal{Z}]^{<\omega}) \preccurlyeq_{\mathrm{GT}} (\ell^1(\omega), \le, \ell^1(\omega))$. Since $\mathcal{Z}$ is upwards-directed, $(\mathcal{Z}, \subseteq, \mathcal{Z}) \equiv_{\mathrm{GT}} (\mathcal{Z}, \subseteq', [\mathcal{Z}]^{<\omega})$ (513Id) and $(\mathcal{Z}, \subseteq, \mathcal{Z}) \preccurlyeq_{\mathrm{GT}} (\ell^1, \le, \ell^1)$, that is, $\mathcal{Z} \preccurlyeq_{\mathrm{T}} \ell^1$.

526C The next three lemmas are steps on the way to Theorem 526F. I give them in much more generality than is required by that theorem because a couple of them will be useful later, and I think they are interesting in themselves. But if you are reading this primarily for the sake of 526F, you might save time by looking ahead to the proof there and working backwards, extracting arguments adequate for the special case of 526E which is actually required.

Lemma Let $\langle (\mathfrak{A}_n, \bar{\mu}_n) \rangle_{n\in\mathbb{N}}$ be a sequence of purely atomic probability algebras, and $\mathfrak{A} = \prod_{n\in\mathbb{N}} \mathfrak{A}_n$ the simple product algebra. Then there is an order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathcal{P}\mathbb{N}$ such that $\limsup_{n\to\infty} \bar{\mu}_n a(n)$ is the upper asymptotic density $d^*(\pi a)$ for every $a \in \mathfrak{A}$; consequently, $\lim_{n\to\infty} \bar{\mu}_n a(n)$ is the asymptotic density $d(\pi a)$ of πa if either is defined.

proof (a) For each $n\in\mathbb{N}$, let C_n be the set of atoms of $\mathfrak{A}_n$, and choose rational numbers $\alpha_n(c)$ such that $\alpha_n(c)\le\bar\mu_n c$ for each $c\in C_n$, $\sum_{c\in C_n}\alpha_n(c)>1-2^{-n}$, and $\{c:c\in C_n,\,\alpha_n(c)>0\}$ is finite. Express $\alpha_n(c)$ as $r_n(c)/s_n$ for each $c\in C_n$, where $r_n(c)\in\mathbb{N}$ and $s_n\in\mathbb{N}\setminus\{0\}$; let $\langle I_n(c)\rangle_{c\in C_n}$ be a disjoint family of subsets of $\mathbb{N}$ with $\#(I_n(c))=r_n(c)$ for each c, and set $J_n=\bigcup_{c\in C_n}I_n(c)$; let $\pi_n:\mathfrak{A}_n\to\mathcal{P}J_n$ be the Boolean homomorphism such that $\pi_n(c)=I_n(c)$ for each $c\in C_n$. Then

$$(1-2^{-n})s_n<s_n\textstyle\sum_{c\in C_n}\alpha_n(c)=\sum_{c\in C_n}r_n(c)=\#(J_n)\le s_n.$$

Note that $\#(J_n)>0$. Also

$$\begin{aligned}\#(J_n)(\bar\mu_n d-2^{-n})&\le\#(J_n)\sum_{c\in C_n,c\subseteq d}\alpha_n(c)\\&\le s_n\sum_{c\in C_n,c\subseteq d}\alpha_n(c)=\sum_{c\in C_n,c\subseteq d}r_n(c)=\#(\pi_n d)\end{aligned}$$

and

$$(1-2^{-n})\#(\pi_n d)=(1-2^{-n})s_n\textstyle\sum_{c\in C_n,c\,\subseteq\,d}\alpha_n(c)\le\#(J_n)\cdot\bar\mu_n d$$

for every $d\in\mathfrak{A}_n$. So, for $a\in\mathfrak{A}$,

$$\limsup_{n\to\infty}\bar\mu_n a(n)=\limsup_{n\to\infty}\tfrac{\#(\pi_n a(n))}{\#(J_n)}.$$

(b) Let $\langle m_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that

$$m_n\#(J_n)\ge 2^n\max(\#(J_{n+1}),\textstyle\sum_{i<n}m_i\#(J_i))$$

for every n. Set $n_k=n$ if $\sum_{i<n}m_i\le k<\sum_{i\le n}m_i$ where $n\in\mathbb{N}$; then $\lim_{k\to\infty}n_k=\infty$. Set $l_k=\sum_{i<k}\#(J_{n_i})$, so that $l_{k+1}-l_k=\#(J_{n_k})$ for each k, and let $\phi_k:\mathcal{P}J_{n_k}\to\mathcal{P}(l_{k+1}\setminus l_k)$ be a Boolean isomorphism; set

$$\pi a=\textstyle\bigcup_{k\in\mathbb{N}}\phi_k\pi_{n_k}a(n_k)$$

for $a\in\mathfrak{A}$, so that $\pi:\mathfrak{A}\to\mathcal{P}\mathbb{N}$ is an order-continuous Boolean isomorphism.

(c) Let $a\in\mathfrak{A}$, and set

$$\gamma=\limsup_{n\to\infty}\bar\mu_n a(n)=\limsup_{n\to\infty}\tfrac{\#(\pi_n a(n))}{\#(J_n)},$$

$$\gamma'=\limsup_{l\to\infty}\frac{1}{l}\#(l\cap\pi a).$$

Then $\gamma\le\gamma'$. **P** Setting $l'_n=\sum_{i<n}m_i\#(J_i)$, we have $\#(l'_{n+1}\cap\pi a)\ge m_n\#(\pi_n a(n))$, while $l'_{n+1}=l'_n+m_n\#(J_n)\le(1+2^{-n})m_n\#(J_n)$ for each n; but this means that

$$\gamma'\ge\limsup_{n\to\infty}\frac{1}{l'_{n+1}}\#(l'_{n+1}\cap\pi a)\ge\limsup_{n\to\infty}\frac{m_n\#(\pi_n a(n))}{(1+2^{-n})m_n\#(J_n)}=\gamma.\ \mathbf{Q}$$

Also $\gamma'\le\gamma$. **P** Let $\epsilon>0$. Let $n^*\ge 1$ be such that $2^{-n^*}\le\epsilon$ and $\#(\pi_n a(n))\le(\gamma+\epsilon)\#(J_n)$ for every $n\ge n^*$. Suppose that $l\ge l'_{n^*+1}$. Then l is of the form $l'_{n+1}+j\#(J_{n+1})+i$ where $n\ge n^*$, $j<m_{n+1}$ and $i<\#(J_{n+1})$. Now $l'_{n+1}=l'_n+m_n\#(J_n)$, so

$$\begin{aligned}\#(l'_{n+1}\cap\pi a)&\le l'_n+m_n\#(\pi_n a(n))\le l'_n+m_n\#(J_n)(\gamma+\epsilon)\\&\le m_n\#(J_n)(\gamma+\epsilon+2^{-n})\le m_n\#(J_n)(\gamma+2\epsilon)\end{aligned}$$

by the choice of m_n. Accordingly

$$\begin{aligned}\#(l\cap\pi a)&\le m_n\#(J_n)(\gamma+2\epsilon)+(j+1)\#(\pi_{n+1}a(n+1))\\&\le m_n\#(J_n)(\gamma+2\epsilon)+j(\gamma+\epsilon)\#(J_{n+1})+\#(J_{n+1})\\&\le(\gamma+2\epsilon)l+\#(J_{n+1})\le(\gamma+2\epsilon)l+2^{-n}m_n\#(J_n)\end{aligned}$$

(by the choice of m_n)

$$\leq (\gamma + 3\epsilon)l.$$

As this is true for any $l \geq l'_{n^*+1}$, $\gamma' \leq \gamma + 3\epsilon$; as ϵ is arbitrary, $\gamma' \leq \gamma$. **Q**

(d) Thus

$$\limsup_{n\to\infty} \bar{\mu}_n a(n) = \limsup_{n\to\infty} \frac{1}{n}\#(n \cap \pi a)$$

for every $a \in \mathfrak{A}$. But as π is a Boolean homomorphism, it follows at once that

$$\liminf_{n\to\infty} \bar{\mu}_n a(n) = \liminf_{n\to\infty} \frac{1}{n}\#(n \cap \pi a)$$

for every a, so that the limits are equal if either is defined.

526D Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and $\kappa \geq \max(\omega, c(\mathfrak{A}), \tau(\mathfrak{A}))$ a cardinal. Let $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ be the measure algebra of the usual measure on $\{0,1\}^\kappa$, and $\gamma > 0$. Then there is a function $\theta : \mathfrak{A} \to \mathfrak{B}_\kappa$ such that

(i) $\theta(\sup A) = \sup \theta[A]$ for every non-empty $A \subseteq \mathfrak{A}$ such that $\sup A$ is defined in $\mathfrak{A}$;

(ii) $\bar{\nu}_\kappa \theta(a) = 1 - e^{-\gamma\bar{\mu}a}$ for every $a \in \mathfrak{A}$, interpreting $e^{-\infty}$ as 0;

(iii) if $\langle a_i \rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$, and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{\theta(a) : a \subseteq a_i\}$ for each i, then $\langle \mathfrak{C}_i \rangle_{i\in I}$ is stochastically independent.

proof (a) By 495J, we have exactly this result for some probability algebra $(\mathfrak{B}, \bar{\lambda})$ in place of $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$. Set $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\}$, and give $\mathfrak{A}^f$ its measure metric ρ (323Ad). Then $\theta{\restriction}\mathfrak{A}^f$ is uniformly continuous for ρ and the measure metric σ of $\mathfrak{B}$. **P** If $\epsilon > 0$, there is a $\delta > 0$ such that $|e^{-\gamma s} - e^{-\gamma t}| \leq \frac{1}{2}\epsilon$ whenever s, $t \in [0, \infty[$ and $|s - t| \leq \delta$. Now if a, $a' \in \mathfrak{A}$ and $\bar{\mu}(a \mathbin{\triangle} a') \leq \delta$, set $b = a \cap a'$; then $\theta(b) \subseteq \theta(a)$ and $\bar{\mu}a - \bar{\mu}b \leq \delta$, so

$$\sigma(\theta(a), \theta(b)) = \bar{\lambda}(\theta(a) \setminus \theta(b)) = \bar{\lambda}\theta(a) - \bar{\lambda}\theta(b) = e^{-\gamma\bar{\mu}b} - e^{-\gamma\bar{\mu}a} \leq \frac{1}{2}\epsilon.$$

Similarly, $\sigma(\theta(a'), \theta(b)) \leq \frac{1}{2}\epsilon$ so $\sigma(\theta(a), \theta(a')) \leq \epsilon$. As ϵ is arbitrary, this gives the result. **Q**

(b) By 521Eb, there is a set $B \subseteq \mathfrak{A}^f$, of cardinal at most κ, which is dense for ρ. Accordingly $\theta[B]$ is dense in $f[\mathfrak{A}^f]$ for σ. Taking $\mathfrak{D}$ to be the closed subalgebra of $\mathfrak{B}$ generated by $\theta[B]$, $\tau(\mathfrak{D}) \leq \kappa$ and $\theta[\mathfrak{A}^f] \subseteq \mathfrak{D}$. But if $a \in \mathfrak{A} \setminus \mathfrak{A}^f$ then $\theta(a) = 1$, so $\theta[\mathfrak{A}] \subseteq \mathfrak{D}$. Now there is a measure-preserving Boolean homomorphism $\phi : \mathfrak{D} \to \mathfrak{B}_\kappa$ (332N), and $\phi\theta : \mathfrak{A} \to \mathfrak{B}_\kappa$ has the properties we need.

526E Lemma Let $\langle(\mathfrak{A}_n, \bar{\mu}_n)\rangle_{n\in\mathbb{N}}$ be a sequence of finite probability algebras and $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ a sequence in $]0, \infty[$. Write P for the set

$$\{p : p \in \prod_{n\in\mathbb{N}} \mathfrak{A}_n,\ \lim_{n\to\infty} \gamma_n \bar{\mu}_n p(n) = 0\},$$

with the ordering inherited from the product partial order on $\prod_{n\in\mathbb{N}} \mathfrak{A}_n$. Then $P \preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

proof (a) By 526D, we can find for each n a probability algebra $(\mathfrak{B}_n, \bar{\nu}_n)$ and a function $\theta_n : \mathfrak{A}_n \to \mathfrak{B}_n$ such that, for all a, $a' \in \mathfrak{A}_n$,

$$\theta_n(a \cup a') = \theta_n(a) \cup \theta_n(a'),$$

$$\bar{\nu}_n\theta_n(a) = 1 - \exp(-\gamma_n\bar{\mu}a).$$

We may suppose that $\mathfrak{B}_n$ is generated by $\theta_n[\mathfrak{A}_n]$, so is itself finite. Set $\mathfrak{A} = \prod_{n\in\mathbb{N}} \mathfrak{A}_n$, $\mathfrak{B} = \prod_{n\in\mathbb{N}} \mathfrak{B}_n$, $\theta(p) = \langle\theta_n(p(n))\rangle_{n\in\mathbb{N}}$ for $p \in \mathfrak{A}$; then $\theta(\sup A) = \sup \theta[A]$ for any non-empty subset A of $\mathfrak{A}$. Set

$$Q = \{q : q \in \prod_{n\in\mathbb{N}} \mathfrak{B}_n,\ \lim_{n\to\infty} \bar{\nu}_n q(n) = 0\}.$$

Then $\theta{\restriction}P$ is a Tukey function from P to Q. **P** $P = f^{-1}[Q]$, because $\lim_{n\to\infty} \gamma_n\xi_n = 0$ iff $\lim_{n\to\infty} 1 - e^{-\gamma_n\xi_n} = 0$. So $\theta{\restriction}P$ is a function from P to Q. If $q \in Q$, $A = \{p : p \in \mathfrak{A},\ \theta(p) \subseteq q\}$ has a supremum $p_0 \in \mathfrak{A}$; now $\theta(p_0) = \sup \theta[A] \subseteq q$, so $\theta(p_0) \in Q$ and $p_0 \in P$ is an upper bound for A in P. **Q**

(b) By 526C, we have an order-continuous Boolean homomorphism $\pi : \mathfrak{B} \to \mathcal{P}\mathbb{N}$ such that $\pi(q) \in \mathcal{Z}$ iff $q \in Q$. Now $\pi{\restriction}Q$ is a Tukey function from Q to $\mathcal{Z}$. **P** If $d \in \mathcal{Z}$, set $B = \{q : q \in \mathfrak{B},\ \pi(q) \subseteq d\}$. Because π is an order-continuous Boolean homomorphism, B contains its supremum, and B is bounded above in Q. **Q**

(c) Thus $\pi\theta{\restriction}P : P \to \mathcal{Z}$ is a Tukey function and $P \preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

526F Theorem $(\ell^1,\le,\ell^1)\preccurlyeq_{\mathrm{GT}}(\mathbb{N}^{\mathbb{N}},\le,\mathbb{N}^{\mathbb{N}})\ltimes(\mathcal{Z},\subseteq,\mathcal{Z})$.

proof (a) Let $Q\subseteq\mathbb{N}^{\mathbb{N}}$ be the set of strictly increasing sequences α such that $\alpha(0)>0$. For $\alpha\in Q$, set

$$\begin{aligned}P_\alpha&=\{x:x\in\ell^1,\,\|x\|_\infty\le\alpha(0),\,\lim_{n\to\infty}2^n\sum_{i=\alpha(n)}^{\infty}x(i)^+=0\}\\&=\{x:x\in\ell^\infty,\,\|x\|_\infty\le\alpha(0),\,\lim_{n\to\infty}2^n\sum_{i=\alpha(n)}^{\alpha(n+1)-1}x(i)^+=0\}\end{aligned}$$

because

$$2^n\sum_{i=\alpha(n)}^{\infty}x(i)^+=\sum_{m=n}^{\infty}2^{n-m}2^m\sum_{i=\alpha(m)}^{\alpha(m+1)-1}x(i)^+\le 2\sup_{m\ge n}2^m\sum_{i=\alpha(m)}^{\alpha(m+1)-1}x(i)^+$$

for every n and x.

The point is that $P_\alpha\preccurlyeq_{\mathrm{T}}\mathcal{Z}$. **P** For each $n\in\mathbb{N}$ set $k_n=2^{2n}(\alpha(n+1)-\alpha(n))$,

$$V_n=(\alpha(n+1)\setminus\alpha(n))\times k_n\alpha(0)\subseteq\mathbb{N}\times\mathbb{N},\quad\mathfrak{A}_n=\mathcal{P}V_n,$$

and let $\bar\mu_n$ be the uniform probability measure on $\mathfrak{A}_n$, so that $\bar\mu_n d=\#(d)/\#(V_n)$ for $d\subseteq V_n$. For $n\in\mathbb{N}$ and $x\in\ell^\infty$ set

$$f_n(x)=\{(i,j):\alpha(n)\le i<\alpha(n+1),\,j<k_n\min(\alpha(0),x(i))\}\subseteq V_n.$$

Then

$$|\#(f_n(x))-k_n\textstyle\sum_{\alpha(n)\le i<\alpha(n+1)}\min(\alpha(0),x(i)^+)|\le\alpha(n+1)-\alpha(n),$$

so if $\|x\|_\infty\le\alpha(0)$ then

$$\begin{aligned}\bigl|2^n\alpha(0)(\alpha(n+1)-\alpha(n))\bar\mu f_n(x)-2^n\sum_{i=\alpha(n)}^{\alpha(n+1)-1}x(i)^+\bigr|&\le 2^n(\alpha(n+1)-\alpha(n))/k_n\\&=2^{-n}.\end{aligned}$$

Accordingly

$$P_\alpha=\{x:x\in\ell^\infty,\,\|x\|_\infty\le\alpha(0),\,\lim\nolimits_{n\to\infty}2^n(\alpha(n+1)-\alpha(n))\bar\mu_n f_n(x)=0\}.$$

Let $\mathfrak{A}=\prod_{n\in\mathbb{N}}\mathfrak{A}_n$ be the simple product of the Boolean algebras $\mathfrak{A}_n$, and I the ideal

$$\{a:a\in\mathfrak{A},\,\lim\nolimits_{n\to\infty}2^n(\alpha(n+1)-\alpha(n))\bar\mu_n a(n)=0\}$$

of $\mathfrak{A}$. For $x\in\ell^\infty$, set $f(x)=\langle f_n(x)\rangle_{n\in\mathbb{N}}$. Observe that $f:\ell^\infty\to\mathfrak{A}$ is supremum-preserving in the sense that $f(\sup A)=\sup f[A]$ for any non-empty bounded subset A of ℓ^∞.

The last formula for P_α shows that $f(x)\in I$ for every $x\in P_\alpha$. But if $a\in I$, $A=\{x:x\in P_\alpha,\,f(x)\subseteq a\}$ is upwards-directed and has a supremum $x_0\in\ell^\infty$, with $\|x_0\|_\infty\le\alpha(0)$. Now $f(x_0)=\sup_{x\in A}f(x)\subseteq a$, so $x_0\in P_\alpha$ and is an upper bound for A in P_α. Thus $f\restriction P_\alpha$ is a Tukey function from P_α to I, and $P_\alpha\preccurlyeq_{\mathrm{T}}I$. By 526E, $I\preccurlyeq_{\mathrm{T}}\mathcal{Z}$, so $P_\alpha\preccurlyeq_{\mathrm{T}}\mathcal{Z}$. **Q**

Thus $(P_\alpha,\le,P_\alpha)\preccurlyeq_{\mathrm{GT}}(\mathcal{Z},\subseteq,\mathcal{Z})$; it follows at once that $(P_\alpha,\le,\ell^1)\preccurlyeq_{\mathrm{GT}}(\mathcal{Z},\subseteq,\mathcal{Z})$.

(b) Now, for $\alpha\in\mathbb{N}^{\mathbb{N}}$, take $\tilde P_\alpha=P_\beta$ where $\beta(n)=1+n+\max_{i\le n}\alpha(i)$ for $n\in\mathbb{N}$. Then $\tilde P_\alpha\subseteq\tilde P_{\alpha'}$ whenever $\alpha\le\alpha'$ in $\mathbb{N}^{\mathbb{N}}$, $\bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}}\tilde P_\alpha=\ell^1$ and $(\tilde P_\alpha,\le,\ell^1)\preccurlyeq_{\mathrm{GT}}(\mathcal{Z},\subseteq,\mathcal{Z})$ for every α; so $(\ell^1,\le,\ell^1)\preccurlyeq_{\mathrm{GT}}(\mathbb{N}^{\mathbb{N}},\le,\mathbb{N}^{\mathbb{N}})\ltimes(\mathcal{Z},\subseteq,\mathcal{Z})$, by 512K.

526G Corollary Let $\mathcal{N}$ be the ideal of Lebesgue negligible subsets of $\mathbb{R}$.

(a) $\operatorname{add}_\omega\mathcal{Z}=\operatorname{add}\mathcal{N}=\operatorname{add}_\omega\ell^1$ and $\operatorname{cf}\mathcal{Z}=\operatorname{cf}\mathcal{N}=\operatorname{cf}\ell^1$.

(b) If $\mathcal{A}\subseteq\mathcal{Z}$ and $\#(\mathcal{A})<\operatorname{add}\mathcal{N}$, there is a $J\in\mathcal{Z}$ such that $I\setminus J$ is finite for every $I\in\mathcal{A}$.

proof (a)(i) Putting 526B and 513Ie together, we see that

$$\operatorname{add}_\omega\mathbb{N}^{\mathbb{N}}\ge\operatorname{add}_\omega\mathcal{Z}\ge\operatorname{add}_\omega\ell^1,$$

that is,

$$\mathfrak{b} \ge \operatorname{add}_\omega \mathcal{Z} \ge \operatorname{add} \mathcal{N}$$

(522A, 524I). Next, we can deduce from 526F that $\operatorname{add}_\omega \ell^1 \ge \min(\operatorname{add}_\omega \mathbb{N}^{\mathbb{N}}, \operatorname{add}_\omega \mathcal{Z})$. **P** Let (ϕ, ψ) be a Galois-Tukey connection from $(\ell^1, \le, \ell^1)$ to

$$(\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}) \ltimes (\mathcal{Z}, \subseteq, \mathcal{Z}) = (\mathbb{N}^{\mathbb{N}} \times \mathcal{Z}^{\mathbb{N}^{\mathbb{N}}}, T, \mathbb{N}^{\mathbb{N}} \times \mathcal{Z}),$$

where

$$T = \{((p, f), (q, a)) : p \le q \text{ in } \mathbb{N}^{\mathbb{N}},\ f(q) \subseteq a \in \mathcal{Z}\}.$$

We can interpret ϕ as a pair (ϕ_1, ϕ_2) where ϕ_1 is a function from ℓ^1 to $\mathbb{N}^{\mathbb{N}}$ and ϕ_2 is a function from $\ell^1 \times \mathbb{N}^{\mathbb{N}}$ to $\mathcal{Z}$, and saying that (ϕ, ψ) is a Galois-Tukey connection means just that

$$\text{if } \phi_1(x) \le q \text{ and } \phi_2(x, q) \subseteq a \text{ then } x \le \psi(q, a).$$

Now suppose that $A \subseteq \ell^1$ and $\#(A) < \min(\operatorname{add}_\omega \mathbb{N}^{\mathbb{N}}, \operatorname{add}_\omega \mathcal{Z})$. Then there is a sequence $\langle q_n \rangle_{n \in \mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ such that for every $x \in A$ there is an $n \in \mathbb{N}$ such that $\phi_1(x) \le q_n$. Next, for each $n \in \mathbb{N}$ there is a sequence $\langle a_{nm} \rangle_{m \in \mathbb{N}}$ in $\mathcal{Z}$ such that for every $x \in A$ there is an $m \in \mathbb{N}$ such that $\phi_2(x, q_n) \subseteq a_{nm}$. In this case, $B = \{\psi(q_n, a_{nm}) : m, n \in \mathbb{N}\}$ is a countable subset of $\mathcal{Z}$, and for every $x \in A$ there are m, $n \in \mathbb{N}$ such that $\phi_1(x) \le q_n$ and $\phi_2(x, q_n) \subseteq a_{nm}$, so that $x \le \psi(q_n, a_{nm}) \in B$. As A is arbitrary, $\operatorname{add}_\omega \ell^1 \ge \min(\operatorname{add}_\omega \mathbb{N}^{\mathbb{N}}, \operatorname{add}_\omega \mathcal{Z})$. **Q**

Thus we have $\operatorname{add} \mathcal{N} \ge \min(\mathfrak{b}, \operatorname{add}_\omega \mathcal{Z}) = \operatorname{add}_\omega \mathcal{Z}$, and $\operatorname{add}_\omega \mathcal{Z} = \operatorname{add} \mathcal{N}$. And we know from 524I, with $\kappa = \omega$ there, that $\operatorname{add} \mathcal{N} = \operatorname{add}_\omega \ell^1$

(ii) On the other hand, 524I, 526F, 512Da and 512Jb tell us that

$$\begin{aligned} \operatorname{cf} \mathcal{N} = \operatorname{cf} \ell^1 = \operatorname{cov}(\ell^1, \subseteq, \ell^1) &\le \operatorname{cov}((\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}) \ltimes (\mathcal{Z}, \subseteq, \mathcal{Z})) \\ &= \max(\operatorname{cov}(\mathbb{N}^{\mathbb{N}}, \le, \mathbb{N}^{\mathbb{N}}), \operatorname{cov}(\mathcal{Z}, \subseteq, \mathcal{Z})) = \max(\mathfrak{d}, \operatorname{cf} \mathcal{Z}). \end{aligned}$$

But from 526B we see that $\mathfrak{d} \le \operatorname{cf} \mathcal{Z} \le \operatorname{cf} \ell^1$, so $\operatorname{cf} \mathcal{Z} = \operatorname{cf} \mathcal{N}$, while 524I tells us that $\operatorname{cf} \mathcal{N} = \operatorname{cf} \ell^1$.

(b) By (a), there is a countable set $\mathcal{D} \subseteq \mathcal{Z}$ such that every member of $\mathcal{A}$ is included in a member of $\mathcal{D}$. By 491Ae, there is a $J \in \mathcal{Z}$ such that $I \setminus J$ is finite for every $I \in \mathcal{D}$; this J serves.

526H I turn now to ideals of nowhere dense sets.

Proposition Let $\mathcal{N}\mathrm{wd}$ be the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{M}$ the ideal of meager subsets of $\mathbb{N}^{\mathbb{N}}$.

(a) $\mathcal{N}\mathrm{wd}$ is isomorphic, as partially ordered set, to $(\mathcal{N}\mathrm{wd})^{\mathbb{N}}$.

(b) $(\mathcal{N}\mathrm{wd}, \subseteq', [\mathcal{N}\mathrm{wd}]^{\le\omega}) \equiv_{\mathrm{GT}} (\mathcal{M}, \subseteq, \mathcal{M})$.

(c) $\mathcal{N}\mathrm{wd} \preccurlyeq_{\mathrm{T}} \ell^1$.

(d) Let X be a set and $\mathcal{V}$ a countable family of subsets of X. Set

$$\mathcal{D} = \{D : D \subseteq X, \text{ for every } V \in \mathcal{V} \text{ there is a } V' \in \mathcal{V} \text{ such that } V' \subseteq V \setminus D\}.$$

Then $\mathcal{D} \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$.

(e) If X is any non-empty Polish space without isolated points, and $\mathcal{N}\mathrm{wd}(X)$ is the ideal of nowhere dense subsets of X, then $\mathcal{N}\mathrm{wd} \equiv_{\mathrm{T}} \mathcal{N}\mathrm{wd}(X)$.

(f) If X is a compact metrizable space and $\mathcal{C}_{\mathrm{nwd}}$ is the family of closed nowhere dense subsets of X with the Fell (or Vietoris) topology (4A2T), then $(\mathcal{C}_{\mathrm{nwd}}, \subseteq)$ is a metrizably compactly based directed set.

Remark Recall that if R is any relation then R' is the relation $\{(x, B) : (x, y) \in R \text{ for some } y \in B\}$; see 512F-512G.

proof Enumerate $S = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ as $\langle \sigma_n \rangle_{n \in \mathbb{N}}$. For $\sigma \in S$ write $I_\sigma = \{\alpha : \sigma \subseteq \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

(a) Define $\phi : \mathcal{N}\mathrm{wd} \to \mathcal{N}\mathrm{wd}^{\mathbb{N}}$ by setting $\phi(F)(n) = \{\alpha : {<}n{>}^\frown\alpha \in F\}$, where ${<}n{>}^\frown\alpha = (n, \alpha(0), \alpha(1), \dots)$ for $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{\mathbb{N}}$. Then ϕ is an isomorphism between $\mathcal{N}\mathrm{wd}$ and $\mathcal{N}\mathrm{wd}^{\mathbb{N}}$.

(b)(i) Choose $\phi : \mathcal{M} \to \mathcal{N}\mathrm{wd}^{\mathbb{N}}$ such that $M \subseteq \bigcup_{n \in \mathbb{N}} \phi(M)(n)$ for every $M \in \mathcal{M}$. Then ϕ is a Tukey function so $\mathcal{M} \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}^{\mathbb{N}} \cong \mathcal{N}\mathrm{wd}$, that is, $(\mathcal{M}, \subseteq, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}\mathrm{wd}, \subseteq, \mathcal{N}\mathrm{wd})$. By 513Id and 512Gb,

$$(\mathcal{M}, \subseteq, \mathcal{M}) \equiv_{\mathrm{GT}} (\mathcal{M}, \subseteq', [\mathcal{M}]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}\mathrm{wd}, \subseteq', [\mathcal{N}\mathrm{wd}]^{\le\omega}).$$

(ii) For $n \in \mathbb{N}$ and $\tau \in \mathbb{N}^n$, define $g_\tau : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by saying that $g_\tau(\alpha) = \tau^\frown\alpha$, that is,

$$\begin{aligned} g_\tau(\alpha)(i) &= \tau(i) \text{ if } i < n, \\ &= \alpha(i-n) \text{ otherwise.} \end{aligned}$$

Note that g_τ is a homeomorphism between $\mathbb{N}^\mathbb{N}$ and I_τ, so that $g_\tau[F]$ and $g_\tau^{-1}[F]$ are nowhere dense whenever F is.

Now for any $F \in \mathcal{N}\mathrm{wd}$ we can find a $\phi(F) \in \mathcal{N}\mathrm{wd}$ such that $F \subseteq \phi(F)$ and for every $\sigma \in S$ either $I_\sigma \cap \phi(F) = \emptyset$ or there is a $\tau \in S$, extending σ, such that $g_\tau[F] \subseteq \phi(F)$. **P** Choose $\langle \tau_n \rangle_{n\in\mathbb{N}}$, $\langle \upsilon_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. Given that $I_{\upsilon_i} \cap (F \cup g_{\tau_j}[F]) = \emptyset$ for all i, $j < n$, set $E = I_{\sigma_n} \cap (F \cup \bigcup_{j<n} g_{\tau_j}[F])$. If $E = \emptyset$ set $\upsilon_n = \sigma_n$ and $\tau_n = \emptyset$, so that $g_{\tau_n}[F] = F$. If E is not empty, it is still nowhere dense, so we can find $\upsilon_n \supseteq \sigma_n$ such that $I_{\upsilon_n} \cap E = \emptyset$. In this case, $\bigcup_{i\le n} I_{\upsilon_i}$ is a closed set not including I_{σ_n}, so we can find a $\tau_n \supseteq \sigma_n$ such that $I_{\tau_n} \cap \bigcup_{i\le n} I_{\upsilon_i} = \emptyset$, and $I_{\upsilon_i} \cap g_{\tau_n}[F] = \emptyset$ for $i \le n$. Thus in both cases we shall have $\bigcup_{i\le n} I_{\upsilon_i} \cap (F \cup \bigcup_{j\le n} g_{\tau_j}[F]) = \emptyset$, and the induction proceeds.

Set $\phi(F) = \overline{F \cup \bigcup_{j\in\mathbb{N}} g_{\tau_j}[F]}$. Because $\upsilon_i \supseteq \sigma_i$ and $\phi(F) \cap I_{\upsilon_i}$ is empty for every $i \in \mathbb{N}$, $\phi(F) \in \mathcal{N}\mathrm{wd}$. If $\sigma \in S$ is such that $\phi(F)$ meets I_σ, there is an $n \in \mathbb{N}$ such that $\sigma = \sigma_n$; now we cannot have $\upsilon_n = \sigma_n$ so we must have $\tau_n \supseteq \sigma_n$ and $g_{\tau_n}[F] \subseteq \phi(F)$. Thus we have found an suitable set $\phi(F)$. **Q**

For each $M \in \mathcal{M}$ let $\mathcal{E}_M$ be a non-empty countable family of closed nowhere dense sets covering M, and set $\psi(M) = \{g_\tau^{-1}[E] : E \in \mathcal{E}_M,\ \tau \in S\}$. Then (ϕ, ψ) is a Galois-Tukey connection from $(\mathcal{N}\mathrm{wd}, \subseteq', [\mathcal{N}\mathrm{wd}]^{\le\omega})$ to $(\mathcal{M}, \subseteq, \mathcal{M})$. **P** Suppose that $F \in \mathcal{N}\mathrm{wd}$ and $M \in \mathcal{M}$ are such that $\phi(F) \subseteq M$. If $F = \emptyset$ then certainly there is an $F' \in \psi(M)$ covering F. Otherwise, $\phi(F)$ is a non-empty closed set included in the union of the countable set $\mathcal{E}_M$ of closed sets. By Baire's theorem, there must be a $\sigma \in S$ and an $E \in \mathcal{E}_M$ such that $\emptyset \ne \phi(F) \cap I_\sigma \subseteq E$. In this case, there is a $\tau \supseteq \sigma$ such that $g_\tau[F] \subseteq \phi(F)$, so that $g_\tau[F] \subseteq E$ and $F \subseteq g_\tau^{-1}[E] \in \psi(M)$ and $F \subseteq' \psi(M)$. As F and M are arbitrary, (ϕ, ψ) is a Galois-Tukey connection. **Q**

(iii) Thus we have

$$(\mathcal{M}, \subseteq, \mathcal{M}) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}\mathrm{wd}, \subseteq', [\mathcal{N}\mathrm{wd}]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\mathcal{M}, \subseteq, \mathcal{M})$$

and $(\mathcal{M}, \subseteq, \mathcal{M}) \equiv_{\mathrm{GT}} (\mathcal{N}\mathrm{wd}, \subseteq', [\mathcal{N}\mathrm{wd}]^{\le\omega})$.

(c) We can use the idea of 522O. Let $\langle U_n \rangle_{n\in\mathbb{N}}$ enumerate a base for the topology of $\mathbb{N}^\mathbb{N}$ not containing $\emptyset$. By 522N, there is for each $n \in \mathbb{N}$ a countable family $\mathcal{V}_n$ of open subsets of U_n such that $\bigcap \mathcal{V} \ne \emptyset$ for every $\mathcal{V} \in [\mathcal{V}_n]^{\le 2^n}$ and every dense open subset of U_n includes some member of $\mathcal{V}_n$. Enumerate $\mathcal{V}_n$ as $\langle U_{nm} \rangle_{m\in\mathbb{N}}$.

For each $F \in \mathcal{N}\mathrm{wd}$ let $f_F : \mathbb{N} \to \mathbb{N}$ be such that $F \cap U_{n,f_F(n)} = \emptyset$ for every $n \in \mathbb{N}$, and for n, $i \in \mathbb{N}$ set

$$\begin{aligned} \phi(F)(2^n(2i+1)-1) &= 2^{-n} \text{ if } f_F(n) = i, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then $\sum_{i=0}^{\infty} \phi(F)(i) = 2$ for each F, so we have a function $\phi : \mathcal{N}\mathrm{wd} \to \ell^1$.

Suppose that $x \in \ell^1$. Set $\mathcal{A} = \{F : F \in \mathcal{N}\mathrm{wd},\ \phi(F) \le x\}$ and $E = \bigcup \mathcal{A}$. The set

$$K = \{n : \#(\{i : x(2^n(2i+1)-1) \ge 2^{-n}\}) \ge 2^n\}$$

is finite; set $k = \sup(\{0\} \cup K)$. If $n > k$, then $\#(\{f_F(n) : F \in \mathcal{A}\}) < 2^n$, so $\bigcap_{F\in\mathcal{A}} U_{n,f_F(n)}$ is a non-empty open subset of U_n disjoint from $\bigcup_{F\in\mathcal{A}} F = E$. Thus $\{n : U_n \subseteq \overline{E}\} \subseteq \{0, \dots, k\}$ is finite, and therefore in fact is empty, that is, $E \in \mathcal{N}\mathrm{wd}$.

As x is arbitrary, $\phi : \mathcal{N}\mathrm{wd} \to \ell^1$ is a Tukey function, and witnesses that $\mathcal{N}\mathrm{wd} \preccurlyeq_{\mathrm{T}} \ell^1$.

(d) If $\mathcal{V} = \emptyset$ then $\mathcal{D} = \mathcal{P}X$ has a greatest element and the result is trivial (any function from $\mathcal{D}$ to $\mathcal{N}\mathrm{wd}$ will be a Tukey function). Otherwise, choose a function $h : S \to \mathcal{V} \cup \{X\}$ such that $h(\emptyset) = X$ and $\langle h(\sigma^\frown{<}i{>}) \rangle_{i\in\mathbb{N}}$ runs over $\{V : h(\sigma) \supseteq V \in \mathcal{V}\}$ for every $\sigma \in S$. Note that $h(\tau) \subseteq h(\sigma)$ whenever $\tau \supseteq \sigma$, and that $\{h(\tau) : \sigma \subseteq \tau \in \mathbb{N}^n\} = \{V : V \in \mathcal{V},\ V \subseteq h(\sigma)\}$ whenever $m \in \mathbb{N}$, $\sigma \in \mathbb{N}^m$ and $n > m$. For each $D \in \mathcal{D}$ we can choose a sequence $\langle \tau_{Dn} \rangle_{n\in\mathbb{N}}$ in S such that $\tau_{Dn} \supseteq \sigma_n$ and $D \cap h(\tau_{Dn})$ is empty and $\#(\tau_{Dn}) \ge n$ for every $n \in \mathbb{N}$. Set $\phi(D) = \mathbb{N}^\mathbb{N} \setminus \bigcup_{n\in\mathbb{N}} I_{\tau_{Dn}}$, so that $\phi(D) \in \mathcal{N}\mathrm{wd}$.

Take any $F \in \mathcal{N}\mathrm{wd}$, and set $D_0 = \bigcup\{D : D \in \mathcal{D},\ \phi(D) \subseteq F\}$. Then $D_0 \in \mathcal{D}$. **P** Let $V \in \mathcal{V}$. Let $\upsilon \in \mathbb{N}^1$ be such that $h(\upsilon) = V$. Take $\tau \supseteq \upsilon$ such that $F \cap I_\tau = \emptyset$. **?** If $D_0 \cap h(\tau) \ne \emptyset$, then there is a $D \in \mathcal{D}$ such that $\phi(D) \subseteq F$ and $D \cap h(\tau) \ne \emptyset$. $I_\tau \cap \phi(D)$ is empty, that is, $I_\tau \subseteq \bigcup_{n\in\mathbb{N}} I_{\tau_{Dn}}$; because $\#(\tau_{Dn}) \ge n$ for every n, this can happen only because there is some $n \in \mathbb{N}$ such that $\tau_{Dn} \subseteq \tau$. But this means that $D \cap h(\tau) \subseteq D \cap h(\tau_{Dn}) = \emptyset$, which is impossible. **X** Thus $D_0 \cap h(\tau)$ is empty, and $h(\tau)$ is a member of $\mathcal{V}$ included in $V \setminus D_0$. As V is arbitrary, $D_0 \in \mathcal{V}$. **Q**

As F is arbitrary, ϕ is a Tukey function and $\mathcal{D} \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$, as claimed.

(e)(i) Taking $\mathcal{V}$ to be a countable base for the topology of X not containing $\emptyset$, we have

$$\mathcal{N}\mathrm{wd}(X) = \{F : F \subseteq X, \text{ for every } V \in \mathcal{V} \text{ there is a } V' \in \mathcal{V} \text{ such that } V' \subseteq V \setminus F\},$$

so (d) tells us that $\mathcal{N}\mathrm{wd}(X) \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$.

(ii) X has a dense subset Y which is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ (5A4Ie). Let $\mathcal{N}\mathrm{wd}(Y)$ be the family of nowhere dense subsets of Y. For $F \in \mathcal{N}\mathrm{wd}(Y)$ let $\phi(F)$ be its closure in X. Then ϕ is a Tukey function from $\mathcal{N}\mathrm{wd}(Y)$ to $\mathcal{N}\mathrm{wd}(X)$, so $\mathcal{N}\mathrm{wd} \cong \mathcal{N}\mathrm{wd}(Y) \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}(X)$.

(f) By 4A2Tg, the Fell topology on the family $\mathcal{C}$ of all closed subsets of X is compact and metrizable. $E \cup F \in \mathcal{C}_{\mathrm{nwd}}$ for all E, $F \in \mathcal{C}_{\mathrm{nwd}}$, and $\cup : \mathcal{C}_{\mathrm{nwd}} \times \mathcal{C}_{\mathrm{nwd}} \to \mathcal{C}_{\mathrm{nwd}}$ is continuous (4A2T(b-ii)). If $F \in \mathcal{C}_{\mathrm{nwd}}$, the set $\{E : E \in \mathcal{C}_{\mathrm{nwd}}, E \subseteq F\} = \{E : E \in \mathcal{C}, E \cup F = F\}$ is closed in $\mathcal{C}$, therefore compact. Now suppose that $\langle E_k \rangle_{k\in\mathbb{N}}$ is a sequence in $\mathcal{C}_{\mathrm{nwd}}$ converging to $E \in \mathcal{C}_{\mathrm{nwd}}$. If $X = \emptyset$ then of course $\{E_k : k \in \mathbb{N}\}$ is bounded above in $\mathcal{C}_{\mathrm{nwd}}$. Otherwise, let $\langle U_n \rangle_{n\in\mathbb{N}}$ run over a base for the topology of X not containing $\emptyset$, and choose $\langle k_n \rangle_{n\in\mathbb{N}}$, $\langle V_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. Given $k_i \in \mathbb{N}$ for $i < n$, let $V_n \subseteq U_n$ be a non-empty open set such that $\overline{V}_n \cap (E \cup \bigcup_{i<n} E_{k_i}) = \emptyset$; given that $E \cap \overline{V}_i = \emptyset$ for $i \le n$, choose $k_n \ge n$ such that $E_{k_n} \cap \bigcup_{i\le n} \overline{V}_i$ is empty. (This is possible because $\bigcup_{i\le n} \overline{V}_i$ is compact, so the family of sets disjoint from it is open in the Fell topology.) Continue. At the end of the induction, $G = \bigcup_{n\in\mathbb{N}} V_n$ is a dense open set disjoint from $\bigcup_{n\in\mathbb{N}} E_{k_n}$, so $X \setminus G$ is an upper bound for $\{E_{k_n} : n \in \mathbb{N}\}$ in $\mathcal{C}_{\mathrm{nwd}}$. Thus all the conditions of 513K are satisfied, and $\mathcal{C}_{\mathrm{nwd}}$ is metrizably compactly based.

526I A related type of ideal is the following. I express the result in more general form because it has some measure theory in it.

Proposition (Fremlin 91) Let X be a second-countable topological space and μ a σ-finite topological measure on X. Let $\mathcal{E}$ be the ideal of subsets of X with negligible closures. Then, writing $\mathcal{N}\mathrm{wd}$ for the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$, $\mathcal{E} \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$ and $\mathcal{E} \preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

proof (a) If $\mu X = 0$ then $\mathcal{E}$ has a greatest element and the result is trivial. Otherwise, there is a probability measure on X with the same measurable sets and the same negligible sets as μ (215B(vii)); so we may suppose that μ itself is a probability measure. Let $\mathcal{U}$ be a countable base for the topology of X, containing X and closed under finite unions.

(b) For $k \in \mathbb{N}$ let $\mathcal{V}_k$ be the countable set $\{V : V \in \mathcal{U}, \mu V > 1 - 2^{-k}\}$. Set

$$\mathcal{E}_k = \{E : E \subseteq X, \text{ for every } V \in \mathcal{V}_k \text{ there is a } U \in \mathcal{V}_k \text{ such that } U \subseteq V \setminus E\}.$$

Then $\mathcal{E} = \bigcap_{k\in\mathbb{N}} \mathcal{E}_k$. **P** Because $X \in \mathcal{V}_k$, $\mu\overline{E} \le 2^{-k}$ for every $E \in \mathcal{E}_k$, so $\bigcap_{k\in\mathbb{N}} \mathcal{E}_k \subseteq \mathcal{E}$. On the other hand, if $E \in \mathcal{E}$ and $k \in \mathbb{N}$ and $V \in \mathcal{V}_k$, then $\mu(V \setminus \overline{E}) > 1 - 2^{-k}$ and $\mathcal{U}' = \{U : U \in \mathcal{U}, U \subseteq V \setminus E\}$ has union $V \setminus \overline{E}$. As $\mathcal{U}'$ is countable, there is a finite $\mathcal{U}'_1 \subseteq \mathcal{U}'$ such that $U = \bigcup \mathcal{U}'_1$ has measure greater than $1 - 2^{-k}$, so that $U \in \mathcal{V}_k$ and $U \subseteq V \setminus E$. As V is arbitrary, $E \in \mathcal{V}_k$; as E and k are arbitrary, $\mathcal{E} \subseteq \bigcap_{k\in\mathbb{N}} \mathcal{E}_k$. **Q**

This means that the map $E \mapsto (E, E, E, \dots)$ is a Tukey function from $\mathcal{E}$ to $\prod_{k\in\mathbb{N}} \mathcal{E}_k$, so that $\mathcal{E} \preccurlyeq_{\mathrm{T}} \prod_{k\in\mathbb{N}} \mathcal{E}_k$. At the same time, $\mathcal{E}_k \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$ for every k, by 526Hd. So $\mathcal{E} \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}^{\mathbb{N}} \cong \mathcal{N}\mathrm{wd}$ (513Eg, 526Ha).

(c) Let $\mathfrak{A}$ be the countable subalgebra of $\mathcal{P}X$ generated by $\mathcal{U}$. Then there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathcal{P}\mathbb{N}$ such that $d(\pi E)$ is defined and equal to μE for every $E \in \mathfrak{A}$. **P** This is easy to prove directly (see 491Xu), but we can also argue as follows. Let $\langle \mathfrak{A}_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of finite subalgebras of $\mathfrak{A}$ with union $\mathfrak{A}$. By 526C, we have a Boolean homomorphism $\pi' : \prod_{n\in\mathbb{N}} \mathfrak{A}_n \to \mathcal{P}\mathbb{N}$ such that $d(\pi'\langle E_n \rangle_{n\in\mathbb{N}}) = \lim_{n\to\infty} \mu E_n$ whenever $E_n \in \mathfrak{A}_n$ for every n and the limit on the right is defined. For each $n \in \mathbb{N}$ let $\pi_n : \mathfrak{A} \to \mathfrak{A}_n$ be a Boolean homomorphism extending the identity homomorphism on $\mathfrak{A}_n$ (314K, or otherwise); set $\pi E = \pi'\langle \pi_n E \rangle_{n\in\mathbb{N}}$ for $E \in \mathfrak{A}$; this works. **Q**

Let $\langle V_n \rangle_{n\in\mathbb{N}}$ be a sequence running over the closed sets belonging to $\mathfrak{A}$. Let $\langle k_n \rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N} \setminus \{0\}$ such that $\frac{1}{k_n}\#(k_n \cap \pi E) \ge \mu E - 2^{-n}$ whenever E belongs to the subalgebra $\mathfrak{B}_n$ of $\mathfrak{A}$ generated by $\{V_i : i \le n\}$. Define $\phi : \mathcal{E} \to \mathcal{P}\mathbb{N}$ by setting

$$\phi(E) = \bigcap\{k_i \cup \pi V_i : i \in \mathbb{N}, E \subseteq V_i\}.$$

Then ϕ is a Tukey function from $\mathcal{E}$ to $\mathcal{Z}$.

P (i) If $E \in \mathcal{E}$ and $\epsilon > 0$ there is a $U \in \mathcal{U}$ such that $U \subseteq X \setminus E$ and $\mu U \ge 1 - \epsilon$. Let $i \in \mathbb{N}$ be such that $X \setminus U = V_i$; then $\phi(E) \subseteq k_i \cup \pi V_i$, so

$$d^*(\phi(E)) \le d^*(\pi V_i) = \mu V_i \le \epsilon.$$

As ϵ is arbitrary, $\phi(E) \in \mathcal{Z}$. Thus ϕ is a function from $\mathcal{E}$ to $\mathcal{Z}$.

(ii) Take any $\mathcal{A} \subseteq \mathcal{E}$, and set $F = \overline{\bigcup \mathcal{A}}$, $a = \bigcup_{E \in \mathcal{E}} \phi(E)$. If $n \in \mathbb{N}$ and $i \in k_n \setminus a$, then $i \notin \phi(E)$ for every $E \in \mathcal{A}$, so for every $E \in \mathcal{A}$ there is a $j < n$ such that $E \subseteq V_j$ and $i \notin \pi V_j$. Set $F_{ni} = \bigcup \{V_j : j < n,\ i \notin \pi V_j\}$, so that $i \notin \pi F_{ni}$, while $\bigcup \mathcal{A} \subseteq F_{ni}$ and $F \subseteq F_{ni}$. Set $F_n = \mathbb{N} \cap \bigcap_{i \in k_n \setminus a} F_{ni}$, so that $F \subseteq F_n$ and no member of $k_n \setminus a$ belongs fo πF_n, that is, $k_n \cap \pi F_n \subseteq a$. Note that $F_n \in \mathfrak{B}_n$. So we have

$$\frac{1}{k_n}\#(k_n \cap a) \ge \frac{1}{k_n}\#(k_n \cap \pi F_n) \ge \mu F_n - 2^{-n} \ge \mu F - 2^{-n}.$$

This means that $d^*(a) \ge \mu F$. So if $\{\phi(E) : E \in \mathcal{A}\}$ is bounded above in $\mathcal{Z}$, $\mathcal{A}$ must be bounded above in $\mathcal{E}$; that is, ϕ is a Tukey function. **Q**

Thus $\mathcal{E} \preccurlyeq_{\mathrm{T}} \mathcal{Z}$ also.

526J Proposition Let $\mathcal{E}_{\mathrm{Leb}}$ be the ideal of subsets of $\mathbb{R}$ whose closures are Lebesgue negligible. Then $\mathbb{N}^{\mathbb{N}} \preccurlyeq_{\mathrm{T}} \mathcal{E}_{\mathrm{Leb}}$ but $\mathcal{E}_{\mathrm{Leb}} \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$; consequently $\mathcal{Z} \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$, $\mathcal{N}\mathrm{wd} \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$ and $\ell^1 \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$.

proof (a) Enumerate $\mathbb{Q} \cap [0,1]$ as $\langle q_i \rangle_{i \in \mathbb{N}}$. Define $\phi : \mathbb{N}^{\mathbb{N}} \to \mathcal{E}_{\mathrm{Leb}}$ by setting $\phi(f)(n) = \{n + q_i : n \in \mathbb{N},\ i \le f(n)\}$. Then it is easy to see that ϕ is a Tukey function, because if $F \subseteq \mathbb{N}^{\mathbb{N}}$ and $\{f(n) : f \in F\}$ is unbounded, then $\bigcup_{f \in F} \phi(f)$ is dense in $[n, n+1]$ so does not belong to $\mathcal{E}_{\mathrm{Leb}}$.

(b) Let $\psi : \mathcal{E}_{\mathrm{Leb}} \to \mathbb{N}^{\mathbb{N}}$ be any function. Let μ be Lebesgue measure on $\mathbb{R}$, and choose $\langle f(n) \rangle_{n \in \mathbb{N}}$ inductively in $\mathbb{N}$ such that $\mu^*\{t : t \in [0,1],\ \psi(\{t\})(i) \le f(i) \text{ for every } i \le n\} > \frac{1}{2}$ for every n. Set

$$A_n = \{t : t \in [0,1],\ \psi(\{t\})(i) \le f(i) \text{ for every } i \le n\}, \quad F = \bigcap_{n \in \mathbb{N}} \overline{A_n}$$

so that $\mu F \ge \frac{1}{2}$. Let $\langle U_n \rangle_{n \in \mathbb{N}}$ enumerate the set of open intervals of $\mathbb{R}$, meeting F, with rational endpoints, and for each $n \in \mathbb{N}$ choose $t_n \in A_n \cap U_n$. Then $\psi(\{t_n\})(i) \le f(i)$ whenever $n \ge i$, so $\{\psi(\{t_n\}) : n \in \mathbb{N}\}$ is bounded above in $\mathbb{N}^{\mathbb{N}}$; but $\overline{\{t_n : n \in \mathbb{N}\}}$ includes F, so $\{\{t_n\} : n \in \mathbb{N}\}$ is not bounded above in $\mathcal{E}_{\mathrm{Leb}}$. Thus ψ cannot be a Tukey function.

(c) Accordingly $\mathcal{E}_{\mathrm{Leb}} \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$; since $\mathcal{E}_{\mathrm{Leb}} \preccurlyeq_{\mathrm{T}} \mathcal{Z} \preccurlyeq_{\mathrm{T}} \ell^1$ and $\mathcal{E}_{\mathrm{Leb}} \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$ (526I, 526B), $\mathcal{Z} \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$, $\mathcal{N}\mathrm{wd} \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$ and $\ell^1 \not\preccurlyeq_{\mathrm{T}} \mathbb{N}^{\mathbb{N}}$.

526K Proposition Let $\mathcal{N}\mathrm{wd}$ be the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$. Then $\mathcal{Z} \not\preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$, so $\mathcal{Z} \not\preccurlyeq_{\mathrm{T}} \mathcal{E}_{\mathrm{Leb}}$ and $\ell^1 \not\preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$.

proof Let $\phi : \mathcal{Z} \to \mathcal{N}\mathrm{wd}$ be any function. Let $\langle U_n \rangle_{n \in \mathbb{N}}$ enumerate a base for the topology of $\mathbb{N}^{\mathbb{N}}$ which contains $\emptyset$ and is closed under finite unions. For each $n \in \mathbb{N}$, set

$$a_n = \{i : i \in \mathbb{N},\ \phi(a) \cap U_n \ne \emptyset \text{ whenever } i \in a \in \mathcal{Z}\}.$$

Set

$$a = \{\min(a_n \setminus n^2) : n \in \mathbb{N},\ a_n \not\subseteq n^2\}$$

(interpreting n^2 in the formula above as a member of $\mathbb{N}$ rather than as a subset of $\mathbb{N}^2$). Then $a \in \mathcal{Z}$ and $a \cap a_n \ne \emptyset$ whenever a_n is infinite. Set $K = \{n : n \in \mathbb{N},\ \phi(a) \cap U_n = \emptyset\}$, so that K is infinite and $\bigcup_{n \in K} U_n = \mathbb{N}^{\mathbb{N}} \setminus \overline{\phi(a)}$ is dense, while a_n is finite for every $n \in K$ (since otherwise there is an $i \in a \cap a_n$, and $\phi(a) \cap U_n$ will not be empty). For $n \in \mathbb{N}$, $\bigcup_{m \in K \cap n} U_m$ belongs to $\mathcal{U}$; let $r(n) \in \mathbb{N}$ be such that $U_{r(n)} = \bigcup_{m \in K \cap n} U_m$. Then $r(n) \in K$ for every n, so $a_{r(n)}$ is always finite. Take a strictly increasing sequence $\langle k_n \rangle_{n \in \mathbb{N}}$ in K such that $a_{r(n)} \subseteq k_n$ for every n. For $i < k_0$, set $b_i = \{i\}$; for $k_n \le i < k_{n+1}$, choose $b_i \in \mathcal{Z}$ such that $i \in b_i$ and $\phi(b_i) \cap U_{r(n)}$ is empty (such exists because $i \notin a_{r(n)}$).

Examine $E = \bigcup_{i \in \mathbb{N}} \phi(b_i) \subseteq \mathbb{N}^{\mathbb{N}}$. If $m \in K$ then $U_m \subseteq U_{r(n)}$ for every $n > m$ so $U_m \cap \phi(b_i) = \emptyset$ for every $i \ge k_{m+1}$ and $U_m \cap E = \bigcup_{i < k_{m+1}} U_m \cap \phi(b_i)$ is nowhere dense. As $\bigcup_{m \in K} U_m$ is a dense open set, E is nowhere dense. On the other hand, $\bigcup_{i \in \mathbb{N}} b_i = \mathbb{N}$. So $\{b : b \in \mathcal{Z},\ \phi(b) \subseteq E\}$ is not bounded above in $\mathcal{Z}$, and ϕ cannot be a Tukey function. As ϕ is arbitrary, $\mathcal{Z} \not\preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$.

Because $\mathcal{E}_{\mathrm{Leb}} \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$ (526I) and $\mathcal{Z} \preccurlyeq_{\mathrm{T}} \ell^1$ (526B), it follows that $\mathcal{Z} \not\preccurlyeq_{\mathrm{T}} \mathcal{E}_{\mathrm{Leb}}$ and $\ell^1 \not\preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$.

526L Proposition (Mátrai p09) $\mathcal{N}\mathrm{wd} \not\preccurlyeq_{\mathrm{T}} \mathcal{Z}$, so $\mathcal{N}\mathrm{wd} \not\preccurlyeq_{\mathrm{T}} \mathcal{E}_{\mathrm{Leb}}$ and $\ell^1 \not\preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

proof (a)(i) Fix a non-empty zero-dimensional compact metrizable space X without isolated points, and write $\mathcal{N}\mathrm{wd}(X)$ for the ideal of nowhere dense subsets of X; the bulk of the argument here will be a proof that $\mathcal{N}\mathrm{wd}(X) \not\preccurlyeq_{\mathrm{T}} \mathcal{Z}$. Let $\mathcal{V}$ be the family of non-empty open-and-closed subsets of X. For $V \in \mathcal{V}$ write $\mathcal{N}\mathrm{wd}(V) = \mathcal{N}\mathrm{wd}(X) \cap \mathcal{P}V$ for the family of nowhere dense subsets of V. As in 526A, set $\nu I = \sup_{n\ge 1} \frac{1}{n}\#(I \cap n)$ for $I \subseteq \mathbb{N}$. Take any function $f : \mathcal{N}\mathrm{wd}(X) \to \mathcal{Z}$.

(ii) Let Q be the set of pairs $\sigma = (m_\sigma, I_\sigma)$ where $I_\sigma \subseteq m_\sigma \in \mathbb{N}$; for σ, $\tau \in Q$, say that $\sigma \le \tau$ if either $\sigma = \tau$ or $2m_\sigma \le m_\tau$ and $I_\sigma = m_\sigma \cap I_\tau$. Then $(Q, \le)$ is a partially ordered set. For $\sigma \in Q$ and $\epsilon > 0$, let $D(\sigma, \epsilon)$ be the set of those $E \subseteq X$ for which there is an $F \in \mathcal{N}\mathrm{wd}(X)$, including E, such that $f(F) \cap m_\sigma \subseteq I_\sigma$ and $\nu(f(F) \setminus I_\sigma) \le \epsilon$.

(iii) If $\sigma \in Q$, $\epsilon > 0$ and $k \ge 2m_\sigma$, then

$$D(\sigma, \epsilon) \subseteq \bigcup\{D(\tau, \epsilon) : \sigma \le \tau \in Q,\ m_\tau = k,\ \nu(I_\tau \setminus I_\sigma) \le \epsilon\}.$$

P If $E \in D(\sigma, \epsilon)$, let $F \in \mathcal{N}\mathrm{wd}(X)$ be such that $E \subseteq F$, $f(F) \cap m_\sigma \subseteq I_\sigma$ and $\nu(f(F) \setminus I_\sigma) \le \epsilon$. Set $\tau = (k, I_\sigma \cup (k \cap f(F))$; then $\sigma \le \tau$ and F witnesses that $E \in D(\tau, \epsilon)$, while $\nu(I_\tau \setminus I_\sigma) \le \epsilon$. **Q**

(iv) If $\sigma \in Q$ and ϵ, $\delta > 0$, then

$$D(\sigma, \epsilon) \subseteq \bigcup\{D(\tau, \delta) : \sigma \le \tau \in Q,\ \nu(I_\tau \setminus I_\sigma) \le \epsilon\}.$$

P If $E \in D(\sigma, \epsilon)$, let $F \in \mathcal{N}\mathrm{wd}(X)$ be such that $E \subseteq F$, $F \cap m_\sigma \subseteq I_\sigma$ and $\nu(f(F) \setminus I_\sigma) \le \epsilon$. As $f(F) \in \mathcal{Z}$, there is a $k \ge 2m_\sigma$ such that $\nu(f(F) \setminus k) \le \delta$. Set $\tau = (k, I_\sigma \cup (k \cap f(F))$; then F witnesses that $E \in D(\tau, \delta)$, while $\nu(I_\tau \setminus I_\sigma) \le \epsilon$. **Q**

(v) Suppose that $n \ge 1$ and that $\langle \sigma_j \rangle_{j \le n}$, $\langle \tau_j \rangle_{j \le n}$ are finite sequences in Q such that $m_{\tau_j} \le m_{\sigma_j}$ for $j \le n$ and $\sigma_j \le \tau_{j+1}$ for $j < n$. Then $\nu(\bigcup_{j<n} I_{\tau_{j+1}} \setminus I_{\sigma_j}) \le 3 \max_{j<n} \nu(I_{\tau_{j+1}} \setminus I_{\sigma_j})$. **P** Note first that we certainly have $m_{\sigma_j} \le m_{\tau_{j+1}} \le m_{\sigma_{j+1}}$ for every $j < n$. Set $K = \bigcup_{j<n} I_{\tau_{j+1}} \setminus I_{\sigma_j}$ and $\epsilon = \max_{j<n} \nu(I_{\tau_{j+1}} \setminus I_{\sigma_j})$. If $m \in \mathbb{N}$, set $J = \{j : j < n,\ \sigma_j \ne \tau_{j+1},\ m_{\sigma_j} \le m\}$, $J' = \{j : j \in J,\ m_{\tau_{j+1}} \le m\}$. Then

$$\#(m \cap K) \le \sum_{j \in J} \#(m \cap I_{\tau_{j+1}} \setminus I_{\sigma_j})$$

(because if $j < n$ and $m \le m_{\sigma_j}$, then $m \cap I_{\tau_{j+1}} \setminus I_{\sigma_j} = \emptyset$)

$$\le \epsilon m + \sum_{j \in J'} \#(I_{\tau_{j+1}} \setminus I_{\sigma_j})$$

(because $\#(J \setminus J') \le 1$)

$$\le \epsilon(m + \sum_{j \in J'} m_{\tau_{j+1}}) \le \epsilon(m + 2m)$$

(because $2m_{\tau_{j+1}} \le 2m_{\sigma_{j+1}} \le 2m_{\sigma_{j'}} \le m_{\tau_{j'+1}} \le m$ whenever j, j' are successive members of J')

$$= 3\epsilon m.$$

As m is arbitrary, $\nu K \le 3\epsilon$. **Q**

(b)(i) Suppose that $V \in \mathcal{V}$ and that $\mathcal{C}_0, \dots, \mathcal{C}_n \subseteq \mathcal{N}\mathrm{wd}(X)$ are such that every nowhere dense subset of V is included in some member of $\bigcup_{i\le n} \mathcal{C}_i$. Then there is an $i \le n$ such that every nowhere dense subset of V is included in some member of $\mathcal{C}_i$. **P?** Otherwise, for each $i \le n$ we can find a nowhere dense subset E_i of V not included in any member of $\mathcal{C}_i$; now $E = \bigcup_{i\le n} E_i$ is a nowhere dense subset of V not included in any member of $\bigcup_{i\le n} \mathcal{C}_i$. **XQ**

(ii) Suppose that $V \in \mathcal{V}$ and that $\langle \mathcal{C}_n \rangle_{n\in\mathbb{N}}$ is a sequence of subsets of $\mathcal{N}\mathrm{wd}(X)$ such that every nowhere dense subset of V is included in some member of $\bigcup_{n\in\mathbb{N}} \mathcal{C}_n$. Then for any $E \in \mathcal{N}\mathrm{wd}(V)$ there are a $U \in \mathcal{V}$ and an $n \in \mathbb{N}$ such that $E \subseteq U$ and every nowhere dense subset of U is included in some member of $\mathcal{C}_n$. **P** As $V \ne \emptyset$ we can suppose that $E \ne \emptyset$. Let $\langle U_n \rangle_{n\in\mathbb{N}}$ be a non-increasing sequence in $\mathcal{V}$ such that $U_0 = V$ and $\bigcap_{n\in\mathbb{N}} U_n = \overline{E}$. **?** If, for every $n \in \mathbb{N}$, there is an $E_n \in \mathcal{N}\mathrm{wd}(U_n)$ not included in any member of $\mathcal{C}_n$, consider $F = \bigcup_{n\in\mathbb{N}} E_n$; then $F \in \mathcal{N}\mathrm{wd}(V)$ but F is not included in any member of any $\mathcal{C}_n$. **X** So some U_n will serve. **Q**

(c) (The key.) Suppose that $V \in \mathcal{V}$, $\sigma \in Q$ and $\epsilon > 0$ are such that $\mathcal{N}\mathrm{wd}(V) \subseteq D(\sigma, \epsilon)$. Let $\langle \epsilon_n \rangle_{n\in\mathbb{N}}$ be a sequence in $]0, \infty[$. Then there are an $n \in \mathbb{N}$, $U_0, \dots, U_n \in \mathcal{V}$ and $\tau \in Q$ such that

$$\sigma \le \tau, \quad \nu(I_\tau \setminus I_\sigma) \le 8\epsilon,$$

$$V \subseteq \bigcup_{j \le n} U_j, \quad \mathcal{N}\mathrm{wd}(U_j) \subseteq D(\tau, \epsilon_j) \text{ for every } j \le n.$$

P It is enough to consider the case in which $\sum_{n=0}^{\infty} \epsilon_n \le \epsilon$. Let $\langle x_n \rangle_{n\in\mathbb{N}}$ run over a dense subset of V. Choose $\langle \sigma_n \rangle_{n\in\mathbb{N}}$, $\langle k_n \rangle_{n\in\mathbb{N}}$, $\langle U_n \rangle_{n\ge 1}$ and $\langle \tau_n \rangle_{n\ge 1}$ inductively, as follows. Start with $\sigma_0 = \sigma$, $k_0 = m_{\sigma_0}$. Given that $\mathcal{N}\mathrm{wd}(V) \subseteq D(\sigma_n, \epsilon)$, we know from (a-iv) that

$$\mathcal{N}\mathrm{wd}(V) \subseteq D(\sigma_n, \epsilon) \subseteq \bigcup\{D(\tau, \epsilon_{n+1}) : \sigma_n \le \tau \in Q,\ \nu(I_\tau \setminus I_{\sigma_n}) \le \epsilon\},$$

so by (b-ii) we can find a $U_{n+1} \in \mathcal{V}$ and a $\tau_{n+1} \ge \sigma_n$ such that $x_n \in U_{n+1}$, $\nu(I_{\tau_{n+1}} \setminus I_{\sigma_n}) \le \epsilon$ and $\mathcal{N}\mathrm{wd}(U_{n+1}) \subseteq D(\tau_{n+1}, \epsilon_{n+1})$. Next, taking $k_{n+1} = \max(m_{\tau_{n+1}}, 2m_{\sigma_n})$, (a-iii) tells us that

$$\mathcal{N}\mathrm{wd}(V) \subseteq D(\sigma_n, \epsilon) \subseteq \bigcup\{D(\tau, \epsilon) : \sigma_n \le \tau \in Q,\ m_\tau = k_{n+1},\ \nu(I_\tau \setminus I_{\sigma_n}) \le \epsilon\},$$

so from (b-i) we see that there is a $\sigma_{n+1} \in Q$ such that $\mathcal{N}\mathrm{wd}(V) \subseteq D(\sigma_{n+1}, \epsilon)$, $m_{\sigma_{n+1}} = k_{n+1}$, $\sigma_n \le \sigma_{n+1}$ and $\nu(I_{\sigma_{n+1}} \setminus I_{\sigma_n}) \le \epsilon$. Continue.

At the end of the induction, set $E = V \setminus \bigcup_{n\in\mathbb{N}} U_{n+1}$. Because $\{x_n : n \in \mathbb{N}\}$ is dense in V, so is $\bigcup_{n\in\mathbb{N}} U_{n+1}$, and $E \in \mathcal{N}\mathrm{wd}(V)$. By (a-iv) and (b-ii) again, there are a $U_0 \in \mathcal{V}$ and a $\tau_0 \ge \sigma$ such that $E \subseteq U_0$, $\mathcal{N}\mathrm{wd}(U_0) \subseteq D(\tau_0, \epsilon_0)$ and $\nu(I_{\tau_0} \setminus I_\sigma) \le \epsilon$. Now $V \subseteq \bigcup_{n\in\mathbb{N}} U_n$; since V is compact, there is an $n \in \mathbb{N}$ such that $V \subseteq \bigcup_{j\le n} U_j$.

I have still to define τ. Set $k = 2\max(k_n, m_{\tau_0})$. For each $j \le n$, (a-iii) and (b-i), as before, show us that there is an $\upsilon_j \in Q$ such that $\tau_j \le \upsilon_j$, $m_{\upsilon_j} = k$, $\nu(I_{\upsilon_j} \setminus I_{\tau_j}) \le \epsilon_j$ and $\mathcal{N}\mathrm{wd}(U_j) \subseteq D(\upsilon_j, \epsilon_j)$. Try setting $\tau = (k, \bigcup_{j\le n} I_{\upsilon_j})$. Then surely $\mathcal{N}\mathrm{wd}(U_j) \subseteq D(\tau, \epsilon_j)$ for each j. To estimate $\nu(I_\tau \setminus I_\sigma)$, set $K = \bigcup_{j<n} I_{\tau_{j+1}} \setminus I_{\sigma_j}$, $K' = \bigcup_{j<n} I_{\sigma_{j+1}} \setminus I_{\sigma_j}$. By (a-v), νK and $\nu K'$ are both at most 3ϵ. Now

$$\begin{aligned} I_\tau \setminus I_\sigma &\subseteq \bigcup_{j\le n} (I_{\upsilon_j} \setminus I_{\tau_j}) \cup (I_{\tau_0} \setminus I_\sigma) \\ &\qquad \cup \bigcup_{j<n} (I_{\tau_{j+1}} \setminus I_{\sigma_j}) \cup \bigcup_{j\le n} (I_{\sigma_j} \setminus I_\sigma) \\ &= \bigcup_{j\le n} (I_{\upsilon_j} \setminus I_{\tau_j}) \cup (I_{\tau_0} \setminus I_\sigma) \cup K \cup K', \end{aligned}$$

and

$$\begin{aligned} \nu(I_\tau \setminus I_\sigma) &\le \sum_{j=0}^{n} \nu(I_{\upsilon_j} \setminus I_{\tau_j}) + \nu(I_{\tau_0} \setminus I_\sigma) + \nu K + \nu K' \\ &\le 7\epsilon + \sum_{j=0}^{n} \epsilon_j \le 8\epsilon, \end{aligned}$$

as required. **Q**

(d) Now we can find $T \subseteq S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$, $\langle \delta_t \rangle_{t\in T}$, $\langle \sigma_t \rangle_{t\in T}$, $\langle \tau_t \rangle_{t\in T}$ and $\langle V_t \rangle_{t\in T}$ such that

T is a tree (that is, $t \restriction k \in T$ whenever $t \in T$ and $k \in \mathbb{N}$),
$\delta_t > 0$ for every $t \in T$, $\sum_{t\in T} \delta_t < \infty$,
$\sigma_t \in Q$, $\tau_t \in Q$, $\sigma_t \le \tau_t$ for every $t \in T$,
$\sigma_t = \tau_{t\restriction n}$ whenever $n \in \mathbb{N}$ and $t \in T \cap \mathbb{N}^{n+1}$,
$\nu(I_{\tau_t} \setminus I_{\sigma_t}) \le \delta_t$ for every $t \in T$,
$V_t \in \mathcal{V}$, $\mathcal{N}\mathrm{wd}(V_t) \subseteq D(\sigma_t, \delta_t)$ for every $t \in T$,
$\bigcup\{V_t : t \in T \cap \mathbb{N}^n\} = X$ for every $n \in \mathbb{N}$.

P Begin by choosing strictly positive δ_t, for $t \in S$, such that $\delta_\emptyset = 1$ and $\sum_{t\in S} \delta_t$ is finite. Now choose $T_n \subseteq \mathbb{N}^n$ and $\langle \sigma_t \rangle_{t\in T_n}$, $\langle V_t \rangle_{t\in T_n}$ inductively, as follows. Start with $T_0 = \{\emptyset\}$, $\sigma_\emptyset = (0, \emptyset)$ and $V_\emptyset = X$. Then

$$\mathcal{N}\mathrm{wd}(V_\emptyset) = \mathcal{N}\mathrm{wd}(X) = D((0, \emptyset), 1) = D(\sigma_\emptyset, \delta_\emptyset),$$

so the process starts. Given that T_n, $\langle \sigma_t \rangle_{t\in T_n}$ and $\langle V_t \rangle_{t\in T_n}$ have been defined, then for each $t \in T_n$ use (c) to find $n_t \in \mathbb{N}$, $\langle V_{t^\frown<i>} \rangle_{i\le n_t} \in \mathcal{V}^{n_t+1}$ and $\tau_t \in Q$ such that $\sigma_t \le \tau_t$, $\nu(I_{\tau_t} \setminus I_{\sigma_t}) \le \delta_t$, $V_t \subseteq \bigcup_{i\le n_t} V_{t^\frown<i>}$ and $\mathcal{N}\mathrm{wd}(V_{t^\frown<i>}) \subseteq D(\tau_t, \delta_{t^\frown<i>})$ for every $i \le n_t$. Set $T_{n+1} = \{t^\frown<i> : t \in T_n,\ i \le n_t\}$ and $\sigma_t = \tau_{t\restriction n}$ for every $t \in T_{n+1}$, and continue. At the end of the construction, set $T = \bigcup_{n\in\mathbb{N}} T_n$. **Q**

(e) Let $\langle y_n \rangle_{n\in\mathbb{N}}$ run over a dense subset of X. For $n \in \mathbb{N}$, take $t_n \in T \cap \mathbb{N}^n$ such that $y_n \in V_{t_n}$. Since $\mathcal{N}\mathrm{wd}(V_{t_n}) \subseteq D(\sigma_{t_n}, \delta_{t_n})$, we can choose an $F_n \in \mathcal{N}\mathrm{wd}(X)$, containing y_n, such that $\nu(f(F_n) \setminus I_{\sigma_{t_n}}) \le \delta_{t_n}$. Now $\{f(F_n) : n \in \mathbb{N}\}$ is bounded above in $\mathcal{Z}$. **P** Set $K = \bigcup_{t\in T} I_{\sigma_t}$. As $I_{\sigma_\emptyset} = \emptyset$,

$$K = \bigcup_{n\in\mathbb{N}} \bigcup_{t\in T\cap\mathbb{N}^{n+1}} I_{\sigma_t} \setminus I_{\sigma_{t\restriction n}} = \bigcup_{t\in T} I_{\tau_t} \setminus I_{\sigma_t};$$

as $\sum_{t\in T} \nu(I_{\tau_t} \setminus I_{\sigma_t})$ is finite, $K \in \mathcal{Z}$ (526Ac). Next,

$$\bigcup_{n\in\mathbb{N}} f(F_n) \setminus K \subseteq \bigcup_{n\in\mathbb{N}} f(F_n) \setminus I_{\sigma_{t_n}};$$

as

$$\sum_{n=0}^{\infty} \nu(f(F_n) \setminus I_{\sigma_{t_n}}) \le \sum_{n=0}^{\infty} \delta_{t_n}$$

is finite, $\bigcup_{n\in\mathbb{N}} f(F_n) \setminus K \in \mathcal{Z}$, so $\bigcup_{n\in\mathbb{N}} f(F_n)$ also belongs to $\mathcal{Z}$, and is an upper bound for $\{f(F_n) : n \in \mathbb{N}\}$. **Q**

(f) On the other hand, $\{F_n : n \in \mathbb{N}\}$ is certainly not bounded above in $\mathcal{N}\mathrm{wd}(X)$, since $\bigcup_{n\in\mathbb{N}} F_n$ includes the dense set $\{y_n : n \in \mathbb{N}\}$. So f cannot be a Tukey function. Since f is arbitrary, $\mathcal{N}\mathrm{wd}(X) \not\preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

(g) Since $\mathcal{N}\mathrm{wd}(X) \equiv_{\mathrm{T}} \mathcal{N}\mathrm{wd}$ (526He), it follows that $\mathcal{N}\mathrm{wd} \not\preccurlyeq_{\mathrm{T}} \mathcal{Z}$. Since $\mathbb{N}^{\mathbb{N}} \preccurlyeq_{\mathrm{T}} \mathcal{E}_{\mathrm{Leb}} \preccurlyeq_{\mathrm{T}} \mathcal{Z}$ (526I, 526J) and $\mathcal{N}\mathrm{wd} \preccurlyeq_{\mathrm{T}} \ell^1$ (526Hc), we see that $\mathcal{N}\mathrm{wd} \not\preccurlyeq_{\mathrm{T}} \mathcal{E}_{\mathrm{Leb}}$ and $\ell^1 \not\preccurlyeq_{\mathrm{T}} \mathcal{Z}$.

Remark A somewhat stronger result is in SOLECKI & TODORČEVIĆ 10.

526M Having introduced ideals of sets with negligible closures, I add a simple result which will be useful later.

Proposition Let X be a second-countable topological space and μ a σ-finite topological measure on X. Let $\mathcal{E}$ be the ideal of subsets of X with negligible closures, $\mathcal{N}(\mu)$ the null ideal of μ, and $\mathcal{M}$ the ideal of meager subsets of $\mathbb{N}^{\mathbb{N}}$. Then

$$(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (\mathcal{M}, \not\ni, \mathbb{N}^{\mathbb{N}});$$

consequently $\mathrm{add}(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) \ge \mathfrak{m}_{\mathrm{countable}}$.

proof (a) Suppose first that μ is a probability measure. Let $\mathcal{U}$ be a countable base for the topology of X, containing $\emptyset$ and closed under finite unions. For each $n \in \mathbb{N}$, let $\langle U_{ni}\rangle_{i\in\mathbb{N}}$ run over $\{U : U \in \mathcal{U},\ \mu U \ge 1 - 2^{-n}\}$. For $f \in \mathbb{N}^{\mathbb{N}}$, set

$$\psi(f) = \bigcap_{n\in\mathbb{N}} \bigcup_{i\ge n} X \setminus U_{i,f(i)} \in \mathcal{N}(\mu).$$

For $E \in \mathcal{E}$, set

$$\phi(E) = \{f : f \in \mathbb{N}^{\mathbb{N}},\ E \not\subseteq \psi(f)\}.$$

Then $\phi(E) \in \mathcal{M}$. **P** Since $X \setminus \overline{E}$ is a conegligible open set, we can find for each $i \in \mathbb{N}$ a $g(i) \in \mathbb{N}$ such that $E \cap U_{i,g(i)} = \emptyset$. Now

$$M = \bigcup_{n\in\mathbb{N}} \bigcap_{i\ge n} \{f : f \in \mathbb{N}^{\mathbb{N}},\ f(i) \ne g(i)\}$$

belongs to $\mathcal{M}$. If $f \in \phi(E)$, there is an $x \in E \setminus \psi(f)$, so that $x \in \bigcap_{i\ge n} U_{i,f(i)}$ for some n; now if $i \ge n$, we have $x \in U_{i,f(i)} \setminus U_{i,g(i)}$, so $f(i) \ne g(i)$; thus $f \in M$. Accordingly $\phi(E) \subseteq M \in \mathcal{M}$. **Q**

Now (ϕ, ψ) is a Galois-Tukey connection from $(\mathcal{E}, \subseteq, \mathcal{N}(\mu))$ to $(\mathcal{M}, \not\ni, \mathbb{N}^{\mathbb{N}})$, and $(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (\mathcal{M}, \not\ni, \mathbb{N}^{\mathbb{N}})$.

(b) If $\mu X = 0$, then of course $(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (\mathcal{M}, \not\ni, \mathbb{N}^{\mathbb{N}})$ (take $\phi(E) = \emptyset$ for every $E \in \mathcal{E}$, $\psi(f) = X$ for every $f \in \mathbb{N}^{\mathbb{N}}$). Otherwise, there is a probability measure ν on X with the same domain and the same null ideal as μ, so (a) tells us that $(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) \preccurlyeq_{\mathrm{GT}} (\mathcal{M}, \not\ni, \mathbb{N}^{\mathbb{N}})$.

(c) Accordingly

$$\mathrm{add}(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) \ge \mathrm{add}(\mathcal{M}, \not\ni, \mathbb{N}^{\mathbb{N}}) = \mathrm{cov}\,\mathcal{M}$$

(512Db). But, writing $\mathcal{M}(\mathbb{R})$ for the ideal of meager subsets of $\mathbb{R}$, $\mathrm{cov}\,\mathcal{M} = \mathrm{cov}\,\mathcal{M}(\mathbb{R}) = \mathfrak{m}_{\mathrm{countable}}$, by 522Wb and 522Sa.

Remark If $X = \mathbb{R}$ and μ is Lebesgue measure, then $\mathrm{add}(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) = \mathfrak{m}_{\mathrm{countable}}$ and $\mathrm{cov}(\mathcal{E}, \subseteq, \mathcal{N}(\mu)) = \mathrm{non}\,\mathcal{M}$; see BARTOSZYŃSKI & SHELAH 92 or BARTOSZYŃSKI & JUDAH 95, 2.6.14.

526X Basic exercises (a) Let $\nu : \mathcal{P}\mathbb{N} \to [0,1]$ be the submeasure described in 526A. Show that $d^*(I) = \lim_{n\to\infty} \nu(I \setminus n)$ for every $I \subseteq \mathbb{N}$.

(b) For I, $J \subseteq \mathbb{N}$ say that $I \subseteq^* J$ if $I \setminus J$ is finite. Show that $(\mathcal{Z}, \subseteq^*, \mathcal{Z}) \equiv_{\mathrm{GT}} (\mathcal{Z}, \subseteq', [\mathcal{Z}]^{\le\omega})$.

(c) Let $\mathcal{N}\mathrm{wd}$ be the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{M}$ the ideal of meager subsets of $\mathbb{N}^{\mathbb{N}}$. Show that $\mathrm{add}_\omega\,\mathcal{N}\mathrm{wd} = \mathrm{add}\,\mathcal{M}$, $\mathrm{non}\,\mathcal{N}\mathrm{wd} = \mathrm{non}\,\mathcal{M}$, $\mathrm{cov}\,\mathcal{N}\mathrm{wd} = \mathfrak{m}_{\mathrm{countable}}$ and $\mathrm{cf}\,\mathcal{N}\mathrm{wd} = \mathrm{cf}\,\mathcal{M}$.

(d) Let X be a topological space with a countable π-base, and $\mathcal{N}\mathrm{wd}(X)$ the ideal of nowhere dense subsets of X. Show that $\mathcal{N}\mathrm{wd}(X) \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$, where $\mathcal{N}\mathrm{wd}$ is the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$.

(e) In 526Hf, show that $\mathcal{C}_{\mathrm{nwd}}$ is a G_δ subset of the family $\mathcal{C}$ of all closed subsets of X with its Fell topology, so is a Polish space in the subspace topology.

(f) Let $\mathcal{C}_{\mathrm{Leb}}$ be the family of closed Lebesgue negligible subsets of $[0,1]$. Show that $\mathcal{C}_{\mathrm{Leb}}$ with its Fell topology is a Polish space and a metrizably compactly based directed set.

(g) Let $\mathcal{E}_{\mathrm{Leb}}$ be the ideal of subsets of $\mathbb{R}$ with negligible closures. (i) Show that it is Tukey equivalent to the partially ordered set $\mathcal{C}_{\mathrm{Leb}}$ of 526Xf. (ii) Show that it is isomorphic to $\mathcal{E}_{\mathrm{Leb}}^{\mathbb{N}}$. (iii) Show that if we write $\mathcal{E}_\sigma$ for the σ-ideal of subsets of $\mathbb{R}$ generated by $\mathcal{E}_{\mathrm{Leb}}$, then $(\mathcal{E}_{\mathrm{Leb}}, \subseteq', [\mathcal{E}_{\mathrm{Leb}}]^{\le\omega}) \equiv_{\mathrm{GT}} (\mathcal{E}_\sigma, \subseteq, \mathcal{E}_\sigma)$. (iv) Show that $\mathrm{add}_\omega \mathcal{E}_{\mathrm{Leb}} = \mathrm{add}\,\mathcal{E}_\sigma$ and $\mathrm{cf}\,\mathcal{E}_{\mathrm{Leb}} = \mathrm{cf}\,\mathcal{E}_\sigma$.

526Y Further exercises (a) Let X be a locally compact separable metrizable space. Let $\mathcal{C}_{\mathrm{nwd}}$ be the family of closed nowhere dense sets in X with its Fell topology. Show that $\mathcal{C}_{\mathrm{nwd}}$ is a metrizably compactly based directed set.

(b) Let $\mathfrak{Z}$ be the asymptotic density algebra $\mathcal{P}\mathbb{N}/\mathcal{Z}$ and define $\bar{d}^* : \mathfrak{Z} \to [0,1]$ by setting $\bar{d}^*(I^\bullet) = d^*(I)$ for every $I \subseteq \mathbb{N}$, as in 491I. Show that if $A \subseteq \mathfrak{Z}$ is non-empty, downwards-directed and has infimum 0, and $\#(A) < \mathfrak{p}$, then $\inf_{a\in A} \bar{d}^*(a) = 0$. (Compare 491Id.)

(c) Show that $\mathrm{wdistr}(\mathfrak{Z}) = \omega_1$.

(d) Show that $\mathfrak{m}(\mathfrak{Z}) \ge \mathfrak{m}_{\sigma\text{-linked}}$.

(e) Show that $\mathrm{FN}(\mathcal{P}\mathbb{N}) \le \mathrm{FN}(\mathfrak{Z}) \le \max(\mathrm{FN}^*(\mathcal{P}\mathbb{N}), (\mathrm{cf}\,\mathcal{N})^+)$.

(f) Show that $\tau(\mathfrak{Z}) \ge \mathfrak{p}$.

526 Notes and comments The 'positive' results of this section are straightforward enough, except perhaps for 526F. As elsewhere in this chapter, I am attempting to describe a framework which will accommodate the many arguments which have been found effective in discussing the cardinal functions of these partially ordered sets. I note that in this section I use the symbol $\mathcal{M}$ to represent the ideal of meager subsets of $\mathbb{N}^{\mathbb{N}}$, rather than the ideal of meager subsets of $\mathbb{R}$, as elsewhere in the chapter. If you miss this point, however, none of the formulae here are dangerous, because the two ideals are Tukey equivalent, and indeed isomorphic (522Wb).

When we come to 'negative' results, we have problems of a new kind. The special character of Tukey functions is that they need not be of any particular type. They are not asked to be order-preserving, and even if we have partially ordered sets with natural Polish topologies (as in 526A, 526Xe and 526Xf, for instance), Tukey functions between them are not required to be Borel functions. This means that in order to show that there is *no* Tukey function between a given pair of partially ordered sets, we have had to consider arbitrary functions, or seek to calculate suitable invariants which we know to be related to the Tukey ordering, like precaliber triples (516C), and show that they are incompatible with the existence of a Tukey function. For a discussion of a class of invariants giving very sharp distinctions, see MÁTRAI p09, §3.

Putting 526B and 526H-526L together, we find that we have a complete description of the Tukey ordering on the set $\{\mathbb{N}^{\mathbb{N}}, \mathcal{E}_{\mathrm{Leb}}, \mathcal{N}\mathrm{wd}, \mathcal{Z}, \ell^1\}$, given by the diagram

$$\begin{tikzcd}[arrows=dash]
 & \mathcal{N}\mathrm{wd} \arrow[r] & \ell^1 \\
\mathbb{N}^{\mathbb{N}} \arrow[r] & \mathcal{E}_{\mathrm{Leb}} \arrow[u] \arrow[r] & \mathcal{Z} \arrow[u]
\end{tikzcd}$$

if we interpret this in the same way as for Cichoń's diagram (522B). Moreover, this is exact, in that no two of the five are Tukey equivalent, and $\mathcal{Z}$ and $\mathcal{N}\mathrm{wd}$ are Tukey incomparable. Note that all five of these partially ordered sets are either themselves metrizably compactly based directed sets (526A, 513Xj, 513Xl) or are Tukey equivalent to metrizably compactly based directed sets (526He-526Hf, 526Xf-526Xg).

In 526Yb-526Yf I list miscellaneous facts about the asymptotic density algebra. A remarkable description of its Dedekind completion is in 556S below.

527 Skew products of ideals

The methods of this chapter can be applied to a large proportion of the partially ordered sets which arise in analysis. In this section I look at skew products of ideals, constructed by a method suggested by Fubini's theorem and the Kuratowski-Ulam theorem (527E).

527A Notation If (X,Σ,μ) is a measure space, $\mathcal{N}(\mu)$ will be the null ideal of μ; $\mathcal{N}$ will be the null ideal of Lebesgue measure on $\mathbb{R}$. If X is a topological space, $\mathcal{B}(X)$ will be the Borel σ-algebra of X and $\mathcal{M}(X)$ the σ-ideal of meager subsets of X; $\mathcal{M}$ will be the ideal $\mathcal{M}(\mathbb{R})$ of meager subsets of $\mathbb{R}$.

527B Skew products of ideals Suppose that $\mathcal{I} \triangleleft \mathcal{P}X$ and $\mathcal{J} \triangleleft \mathcal{P}Y$ are ideals of subsets of sets X, Y respectively.

(a) I will write $\mathcal{I} \ltimes \mathcal{J}$ for their **skew product** $\{W : W \subseteq X \times Y,\ \{x : W[\{x\}] \notin \mathcal{J}\} \in \mathcal{I}\}$. (This use of the symbol $\ltimes$ is unconnected with the usage in §512 except by the vaguest of analogies.) It is easy to check that $\mathcal{I} \ltimes \mathcal{J} \triangleleft \mathcal{P}(X \times Y)$.

Similarly, $\mathcal{I} \rtimes \mathcal{J}$ will be $\{W : W \subseteq X \times Y,\ \{y : W^{-1}[\{y\}] \notin \mathcal{I}\} \in \mathcal{J}\}$.

(b) Suppose that X and Y are not empty and that $\mathcal{I}$ and $\mathcal{J}$ are proper ideals. Then

$$\operatorname{add}(\mathcal{I} \ltimes \mathcal{J}) = \min(\operatorname{add}\mathcal{I}, \operatorname{add}\mathcal{J}), \quad \operatorname{cf}(\mathcal{I} \ltimes \mathcal{J}) \ge \max(\operatorname{cf}\mathcal{I}, \operatorname{cf}\mathcal{J}),$$

$$\operatorname{non}(\mathcal{I} \ltimes \mathcal{J}) = \max(\operatorname{non}\mathcal{I}, \operatorname{non}\mathcal{J}), \quad \operatorname{cov}(\mathcal{I} \ltimes \mathcal{J}) = \min(\operatorname{cov}\mathcal{I}, \operatorname{cov}\mathcal{J}).$$

P (i) If $\langle W_\xi\rangle_{\xi<\kappa}$ is a family in $\mathcal{I} \ltimes \mathcal{J}$ with $\kappa < \min(\operatorname{add}\mathcal{I}, \operatorname{add}\mathcal{J})$, set $W = \bigcup_{\xi<\kappa} W_\xi$. For each $\xi < \kappa$, $H_\xi = \{x : W_\xi[\{x\}] \notin \mathcal{J}\}$ belongs to $\mathcal{I}$; as $\kappa < \operatorname{add}\mathcal{I}$, $H = \bigcup_{\xi<\kappa} H_\xi \in \mathcal{I}$. For any $x \notin H$, $W[\{x\}] = \bigcup_{\xi<\kappa} W_\xi[\{x\}] \in \mathcal{J}$ because $\kappa < \operatorname{add}\mathcal{J}$; so $W \in \mathcal{I} \ltimes \mathcal{J}$. As $\langle W_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\operatorname{add}(\mathcal{I} \ltimes \mathcal{J}) \ge \min(\operatorname{add}\mathcal{I}, \operatorname{add}\mathcal{J})$.

In the other direction, as $X \notin \mathcal{I}$, $\mathcal{J} = \{F : F \subseteq Y,\ X \times F \in \mathcal{I} \ltimes \mathcal{J}\}$, so $F \mapsto X \times F$ is a Tukey function from $\mathcal{J}$ to $\mathcal{I} \ltimes \mathcal{J}$ and $\operatorname{add}\mathcal{J} \ge \operatorname{add}(\mathcal{I} \ltimes \mathcal{J})$, $\operatorname{cf}\mathcal{J} \le \operatorname{cf}(\mathcal{I} \ltimes \mathcal{J})$. Similarly, $E \mapsto E \times Y$ is a Tukey function from $\mathcal{I}$ to $\mathcal{I} \ltimes \mathcal{J}$ and $\operatorname{add}\mathcal{I} \ge \operatorname{add}(\mathcal{I} \ltimes \mathcal{J})$, $\operatorname{cf}\mathcal{I} \le \operatorname{cf}(\mathcal{I} \ltimes \mathcal{J})$.

(ii) Let $A \subseteq X$ and $B \subseteq Y$ be such that $A \notin \mathcal{I}$, $B \notin \mathcal{J}$, $\#(A) = \operatorname{non}\mathcal{I}$ and $\#(B) = \operatorname{non}\mathcal{J}$. Then $A \times B \notin \mathcal{I} \ltimes \mathcal{J}$, so $\operatorname{non}(\mathcal{I} \ltimes \mathcal{J}) \le \#(A \times B)$. But note that as $\mathcal{I}$ and $\mathcal{J}$ are ideals, A and B are either singletons or infinite; so $\#(A \times B) = \max(\#(A), \#(B))$ and $\operatorname{non}(\mathcal{I} \ltimes \mathcal{J}) \le \max(\operatorname{non}\mathcal{I}, \operatorname{non}\mathcal{J})$.

In the other direction, take any $W \in \mathcal{P}(X \times Y) \setminus (\mathcal{I} \ltimes \mathcal{J})$. Set $E = \{x : W[\{x\}] \notin \mathcal{J}\}$. Then $\#(E) \ge \operatorname{non}\mathcal{I}$ and $\#(W[\{x\}]) \ge \operatorname{non}\mathcal{J}$ for every $x \in E$, so $\#(W) \ge \max(\operatorname{non}\mathcal{I}, \operatorname{non}\mathcal{J})$; as W is arbitrary, $\operatorname{non}(\mathcal{I} \ltimes \mathcal{J}) \ge \max(\operatorname{non}\mathcal{I}, \operatorname{non}\mathcal{J})$.

(iii) If $\mathcal{A} \subseteq \mathcal{I}$ covers X, then $\{A \times Y : A \in \mathcal{A}\} \subseteq \mathcal{I} \ltimes \mathcal{J}$ covers $X \times Y$; so $\operatorname{cov}(\mathcal{I} \ltimes \mathcal{J}) \le \operatorname{cov}\mathcal{I}$. Similarly, $\operatorname{cov}(\mathcal{I} \ltimes \mathcal{J}) \le \operatorname{cov}\mathcal{J}$.

Now suppose that $\mathcal{W} \subseteq \mathcal{I} \ltimes \mathcal{J}$ and that $\#(\mathcal{W}) < \min(\operatorname{cov}\mathcal{I}, \operatorname{cov}\mathcal{J})$. For each $W \in \mathcal{W}$ set $E_W = \{x : W[\{x\}] \notin \mathcal{J}\}$; then $E_W \in \mathcal{I}$ for every W, so there is an $x \in X \setminus \bigcup_{W\in\mathcal{W}} E_W$, because $\#(\mathcal{W}) < \operatorname{cov}\mathcal{I}$. Now $W[\{x\}] \in \mathcal{J}$ for every W, so there is a $y \in Y \setminus \bigcup_{W\in\mathcal{W}} W[\{x\}]$, because $\#(\mathcal{W}) < \operatorname{cov}\mathcal{J}$. In this case $(x,y) \in (X \times Y) \setminus \bigcup\mathcal{W}$. As $\mathcal{W}$ is arbitrary, $\operatorname{cov}(\mathcal{I} \ltimes \mathcal{J}) \ge \min(\operatorname{cov}\mathcal{I}, \operatorname{cov}\mathcal{J})$ and we have equality. **Q**

(c) The idea of the operation $\ltimes$ here is that we iterate notions of 'negligible set' in a way indicated by Fubini's theorem: a measurable subset of $\mathbb{R}^2$ is negligible iff almost every vertical section is negligible, that is, iff it belongs to $\mathcal{N} \ltimes \mathcal{N}$. However it is immediately apparent that $\mathcal{N} \ltimes \mathcal{N}$ contains many non-measurable sets, and indeed many sets of full outer measure (527Xa). We are therefore led to the following idea. If Λ is a family of subsets of $X \times Y$, write $\mathcal{I} \ltimes_\Lambda \mathcal{J}$ for the ideal generated by $(\mathcal{I} \ltimes \mathcal{J}) \cap \Lambda$. Note that if $\kappa \le \min(\operatorname{add}\mathcal{I}, \operatorname{add}\mathcal{J})$ and $\bigcup\mathcal{W} \in \Lambda$ for every $\mathcal{W} \in [\Lambda]^{<\kappa}$, then $\operatorname{add}(\mathcal{I} \ltimes_\Lambda \mathcal{J}) \ge \kappa$; in particular, $\mathcal{I} \ltimes_\Lambda \mathcal{J}$ will be a σ-ideal whenever $\mathcal{I}$ and $\mathcal{J}$ are σ-ideals and Λ is a σ-algebra of subsets of $X \times Y$. Typical applications will be with Λ a Borel σ-algebra or an algebra of the form $\Sigma\widehat{\otimes}\mathrm{T}$. Thus 252F tells us that

if (X,Σ,μ) and (Y,T,ν) are measure spaces with c.l.d. product $(X \times Y, \Lambda, \lambda)$ then $\mathcal{N}(\mu) \ltimes_\Lambda \mathcal{N}(\nu) \subseteq \mathcal{N}(\lambda)$.

If μ and ν are σ-finite then we get

$$\mathcal{N}(\lambda) = \mathcal{N}(\mu) \ltimes_{\Sigma\widehat{\otimes}\mathrm{T}} \mathcal{N}(\nu)$$

(252C). If we take $\mathcal{B} = \mathcal{B}(\mathbb{R}^2)$ to be the Borel σ-algebra of $\mathbb{R}^2$, then all four ideals $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N}$, $\mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M}$, $\mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N}$ and $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M}$ become interesting. In the next few paragraphs I will sketch some of the ideas needed to deal with ideals of these kinds.

527C We are already familiar with $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N}$; I begin by repeating a result from §417 in this language.

Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be σ-finite effectively locally finite τ-additive topological measure spaces. Let $\tilde{\lambda}$ be the τ-additive product measure on $X \times Y$ (417C, 417G). Then $\mathcal{N}(\mu) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{N}(\nu) = \mathcal{N}(\tilde{\lambda})$.

proof Completing μ and ν does not change $\mathcal{N}(\mu)$ or $\mathcal{N}(\nu)$, and leaves μ and ν effectively locally finite and τ-additive; so it also does not change $\tilde{\lambda}$ (use the uniqueness assertion in 417Da). We may therefore assume that μ and ν are complete. Now 417H tells us that a Borel subset of $X \times Y$ is $\tilde{\lambda}$-negligible iff it belongs to $\mathcal{N}(\mu) \ltimes \mathcal{N}(\nu)$. On the other hand, because μ and ν are σ-finite, so is $\tilde{\lambda}$, and every $\tilde{\lambda}$-negligible set is included in a $\tilde{\lambda}$-negligible Borel set. **P** Suppose that $W \in \mathcal{N}(\tilde{\lambda})$. Let $\langle W_n\rangle_{n\in\mathbb{N}}$ be a cover of $X \times Y$ by sets of finite measure. Because $\tilde{\lambda}$ is inner regular with respect to the Borel sets (417Da), we can find $V_n \in \mathcal{B}(X \times Y)$ such that $V_n \subseteq W_n \setminus W$ and $\tilde{\lambda}V_n = \tilde{\lambda}W_n$ for each n. Now

$$W \subseteq (X \times Y) \setminus \bigcup_{n\in\mathbb{N}} V_n \in \mathcal{N}(\tilde{\lambda}) \cap \mathcal{B}(X \times Y). \text{ } \mathbf{Q}$$

So $\mathcal{N}(\mu) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{N}(\nu) = \mathcal{N}(\tilde{\lambda})$.

527D The case $\mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M}$ is also well known.

Theorem Let X and Y be topological spaces, with product $X \times Y$. Write $\mathcal{M}^* = \mathcal{M}(X) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{M}(Y)$ and $\mathcal{M}_1^* = \mathcal{M}(X) \ltimes_{\widehat{\mathcal{B}}(X\times Y)} \mathcal{M}(Y)$, writing $\widehat{\mathcal{B}}(X \times Y)$ for the Baire-property algebra of $X \times Y$.

(a) If $\mathcal{M}(X \times Y) \subseteq \mathcal{M}_1^*$, then $\mathcal{M}^* = \mathcal{M}_1^* = \mathcal{M}(X \times Y)$.

(b) Let $\mathfrak{G}$ be the category algebra of Y (514I). If $\pi(\mathfrak{G}) < \operatorname{add}\mathcal{M}(X)$ then $\mathcal{M}^* = \mathcal{M}(X \times Y)$.

proof (a)(i) ? If $\mathcal{M}_1^* \ne \mathcal{M}(X \times Y)$, there is a set $W \in \mathcal{M}_1^* \setminus \mathcal{M}(X \times Y)$; take $W_1 \in \widehat{\mathcal{B}}(X \times Y) \cap (\mathcal{M}(X) \ltimes \mathcal{M}(Y))$ such that $W_1 \supseteq W$. By 4A3Ra, there is an open set $V \subseteq X \times Y$ such that $W_1 \triangle V$ is meager and $V \cap V'$ is empty whenever $V' \subseteq X \times Y$ is open and $V' \cap W_1$ is meager. As $W_1 \notin \mathcal{M}(X \times Y)$, V cannot be empty; let $G \subseteq X$, $H \subseteq Y$ be non-empty open sets such that $G \times H \subseteq V$. In this case, $G \times H$ cannot be meager, so neither G nor H can be meager. (If $F \subseteq X$ is nowhere dense, then $F \times Y$ is nowhere dense in $X \times Y$; so $M \times Y \in \mathcal{M}(X \times Y)$ whenever $M \in \mathcal{M}(X)$; as $G \times Y \notin \mathcal{M}(X \times Y)$, $G \notin \mathcal{M}(X)$.) But now we see that

$$\{x : (G \times H)[\{x\}] \notin \mathcal{M}(Y)\} = G \notin \mathcal{M}(X),$$

so that $G \times H \notin \mathcal{M}_1^*$; but $(G \times H) \setminus W_1$ is meager, so belongs to $\mathcal{M}_1^*$, and W_1 is also supposed to belong to $\mathcal{M}_1^*$. **X**

(ii) So $\mathcal{M}_1^* = \mathcal{M}(X \times Y)$. Of course $\mathcal{M}^* \subseteq \mathcal{M}_1^*$ just because $\mathcal{B}(X \times Y) \subseteq \widehat{\mathcal{B}}(X \times Y)$. In the other direction, if $W \in \mathcal{M}(X\times Y)$ there is a meager F_σ set $W' \supseteq W$. Now W' is a Borel set in $\mathcal{M}(X\times Y) = \mathcal{M}_1^*$, so $W' \in \mathcal{M}(X)\ltimes\mathcal{M}(Y)$ and witnesses that $W \in \mathcal{M}^*$. Thus $\mathcal{M}(X \times Y) \subseteq \mathcal{M}^*$ and the three classes are equal.

(b) By (a), I have only to show that $W \in \mathcal{M}^*$ whenever $W \subseteq X\times Y$ is meager. Let $D \subseteq \mathfrak{G}\setminus\{0\}$ be an order-dense subset with cardinal $\pi(\mathfrak{G})$. Let H be the smallest comeager regular open subset of Y, so that an open subset of Y is meager iff it is disjoint from H (4A3Ra). For each $d \in D$ let $V_d \subseteq Y$ be an open set such that $V_d^\bullet = d$ in $\mathfrak{G}$; since $H^\bullet = 1$, we may suppose that $V_d \subseteq H$. Observe that if $F \subseteq Y$ is a non-meager closed set, then there is a $d \in D$ such that $0 \ne d \subseteq F^\bullet$ in $\mathfrak{G}$, in which case $V_d \setminus F$ is meager; as $V_d \subseteq H$, $V_d \subseteq F$.

If $W \subseteq X \times Y$ is a nowhere dense closed set, it belongs to $\mathcal{M}^*$. **P** Set $E = \{x : W[\{x\}] \text{ is not meager}\}$. For each $d \in D$, the set

$$E_d = \{x : V_d \subseteq W[\{x\}]\} = \{x : (x, y) \in W \text{ for every } y \in V_d\}$$

is a closed set in X and $E_d \times V_d \subseteq W$; so $\operatorname{int} E_d \times V_d$ is an open subset of W. As W is nowhere dense, and $V_d \ne \emptyset$, $\operatorname{int} E_d$ must be empty, and $E_d \in \mathcal{M}(X)$. Next, $E = \bigcup_{d\in D} E_d$ and $\#(D) = \pi(\mathfrak{G}) < \operatorname{add}\mathcal{M}(X)$, so $E \in \mathcal{M}(X)$ and $W \in \mathcal{M}^*$. **Q**

Since $\mathcal{M}^*$ is a σ-ideal, it follows that every meager subset of $X \times Y$ belongs to $\mathcal{M}^*$, as required.

527E Corollary If X and Y are separable metrizable spaces, then $\mathcal{M}(X \times Y) = \mathcal{M}(X) \ltimes_{\mathcal{B}(X\times Y)} \mathcal{M}(Y)$.

proof $\pi(\mathfrak{C}) \le \pi(Y) \le w(Y) \le \omega < \operatorname{add}\mathcal{M}(X)$ (514Ja, 5A4Ba, 4A2P(a-i)).

Remark The case $X = Y = \mathbb{R}$ is the **Kuratowski-Ulam theorem**.

527F If we mix measure and category, as in $\mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N}$ and $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M}$, we encounter some new phenomena. To deal with the first we need the following, which is important for other reasons.

Lemma (see Cichoń & Pawlikowski 86) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X generated by $\Sigma\cap\mathcal{I}$; suppose that the quotient algebra $\Sigma/\Sigma\cap\mathcal{I}$ is non-zero, atomless and has countable π-weight. Let Y be a set, T a σ-algebra of subsets of Y, and $\langle\mathcal{H}_n\rangle_{n\in\mathbb{N}}$ a sequence of finite covers of Y by members of T. Set

$$\mathcal{H}_n^* = \{\bigcup_{m\ge n} H_m : H_m \in \mathcal{H}_m \cup \{\emptyset\} \text{ for every } m \ge n\}$$

for each $n\in\mathbb{N}$. Then there is a sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}^{\mathbb{N}}\times X\times Y$ such that

(i) for every $n\in\mathbb{N}$, W_n is expressible as the union of a sequence of sets of the form $I\times E\times F$ where $I\subseteq\mathbb{N}^{\mathbb{N}}$ is open-and-closed, $E\in\Sigma$ and $F\in\mathrm{T}$;

(ii) whenever $n\in\mathbb{N}$, $\alpha\in\mathbb{N}^{\mathbb{N}}$ and $x\in X$ then $\{y:(\alpha,x,y)\in W_n\}\in\mathcal{H}_n^*$;

(iii) setting $W=\bigcap_{n\in\mathbb{N}} W_n$, the set $\{(\alpha,x):\alpha\in\mathbb{N}^{\mathbb{N}},\, x\in X,\, (\alpha,x,f(x))\notin W\}$ belongs to $[\mathbb{N}^{\mathbb{N}}]^{\le\omega}\ltimes\mathcal{I}$ for every (Σ,T)-measurable function $f:X\to Y$.

proof If $X\in\mathcal{I}$ or Y is empty, we can take every W_n to be $\emptyset$; suppose otherwise.

(a) Set $S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$. There is a family $\langle U_\sigma\rangle_{\sigma\in S}$ such that

every U_σ belongs to $\Sigma\setminus\mathcal{I}$,

for every $\sigma\in S$, $\langle U_{\sigma^\frown<i>}\rangle_{i\in\mathbb{N}}$ is a disjoint sequence of subsets of U_σ and $U_\sigma\setminus\bigcup_{i\in\mathbb{N}}U_{\sigma^\frown<i>}\in\mathcal{I}$,

for every $E\in\Sigma\setminus\mathcal{I}$ there is a $\sigma\in S$ such that $U_\sigma\setminus E\in\mathcal{I}$.

P Let D be a countable order-dense set in $\mathfrak{A}=\Sigma/\Sigma\cap\mathcal{I}$. Then the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by D is countable and atomless and non-trivial. Let $\mathcal{E}$ be the algebra of subsets of $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ generated by the sets $I_\sigma=\{\alpha:\sigma\subseteq\alpha\in\mathbb{N}^{\mathbb{N}}\}$ for $\sigma\in S$. This is also an atomless countable Boolean algebra, and must therefore be isomorphic to $\mathfrak{B}$ (316M). Let $\pi:\mathcal{E}\to\mathfrak{B}$ be an isomorphism, and set $b_\sigma=\pi I_\sigma$ for each $\sigma\in S$. Set $U_\emptyset=X$ and for $n\in\mathbb{N}$, $\sigma\in\mathbb{N}^n$ choose a disjoint sequence $\langle U_{\sigma^\frown<i>}\rangle_{i\in\mathbb{N}}$ of subsets of U_σ such that $U^\bullet_{\sigma^\frown<i>}=b_{\sigma^\frown<i>}$ for every i. This construction ensures that $U_\sigma\in\Sigma\setminus\mathcal{I}$ for every σ. If $E\in\Sigma\setminus\mathcal{I}$, there must be a non-zero $d\in D$ such that $d\subseteq E^\bullet$; now $\pi^{-1}d\in\mathcal{E}\setminus\{\emptyset\}$, so there is a $\sigma\in S$ such that $I_\sigma\subseteq\pi^{-1}d$, $b_\sigma\subseteq E^\bullet$ and $U_\sigma\setminus E\in\mathcal{I}$. Finally, if $\sigma\in S$, set $E=U_\sigma\setminus\bigcup_{i\in\mathbb{N}}U_{\sigma^\frown<i>}$; then for every $\tau\in S$ either $\tau\subseteq\sigma$ and $U_\tau\setminus E\supseteq U_{\sigma^\frown<0>}\notin\mathcal{I}$, or $U_\tau\cap U_\sigma=\emptyset$ and $U_\tau\setminus E\supseteq U_\tau\notin\mathcal{I}$, or there is an $i\in\mathbb{N}$ such that $\tau\supseteq\sigma^\frown<i>$ and again $U_\tau\setminus E\subseteq U_\tau\notin\mathcal{I}$. This means that E must belong to $\mathcal{I}$, so that $\langle U_\sigma\rangle_{\sigma\in S}$ has all the required properties. **Q**

(b) Enumerate S as $\langle\tau_k\rangle_{k\in\mathbb{N}}$. Let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\bigcup_{n\in\mathbb{N}}\mathcal{H}_n$. For $n\in\mathbb{N}$, set

$$K_n=\{(\sigma,k):\sigma\in\mathbb{N}^{n+2},\, k<\#(\tau_{\sigma(n)}),\, \sigma(n+1)=\#(\tau_k),\, H_{\tau_{\sigma(n)}(k)}\in\mathcal{H}_n\},$$

$$V_n=\bigcup_{(\sigma,k)\in K_n}\{(\alpha,x,y):\tau_k\subseteq\alpha\in\mathbb{N}^{\mathbb{N}},\, x\in U_\sigma,\, y\in H_{\tau_{\sigma(n)}(k)}\}.$$

If $\alpha\in\mathbb{N}^{\mathbb{N}}$ and $x\in X$ the section $\{y:(\alpha,x,y)\in V_n\}$ is either empty or $H_{\tau_{\sigma(n)}(k)}$ where $\sigma\in\mathbb{N}^{n+2}$, $x\in U_\sigma$ and $\tau_k=\alpha\restriction\sigma(n+1)$; in either case it belongs to $\mathcal{H}_n\cup\{\emptyset\}$.

So if we now set $W_n=\bigcup_{m\ge n}V_m$, W_n satisfies (i) and (ii) for every n.

(c) Set $W=\bigcap_{n\in\mathbb{N}}W_n$. **?** Suppose, if possible, that $f:X\to Y$ is a (Σ,T)-measurable function such that $\{(\alpha,x):(\alpha,x,f(x))\notin W\}\notin[\mathbb{N}^{\mathbb{N}}]^{\le\omega}\ltimes\mathcal{I}$. Note that

$$\{V:V\subseteq\mathbb{N}^{\mathbb{N}}\times X\times Y,\, \{x:(\alpha,x,f(x))\in V\}\in\Sigma \text{ for every } \alpha\in\mathbb{N}^{\mathbb{N}}\}$$

is a σ-algebra of subsets of $\mathbb{N}^{\mathbb{N}}\times X\times Y$ containing $I\times E\times F$ whenever I is open-and-closed, $E\in\Sigma$ and $F\in\mathrm{T}$, so contains every V_n and every W_n.

Set

$$A_0=\{\alpha:\alpha\in\mathbb{N}^{\mathbb{N}},\, \{x:(\alpha,x,f(x))\notin W\}\notin\mathcal{I}\},$$

so that A_0 is uncountable. For each $\alpha\in A_0$,

$$\bigcup_{n\in\mathbb{N}}\{x:(\alpha,x,f(x))\notin W_n\}=\{x:(\alpha,x,f(x))\notin W\}$$

does not belong to $\mathcal{I}$. So there is an $n\in\mathbb{N}$ such that

$$A_1=\{\alpha:\alpha\in A_0,\, \{x:(\alpha,x,f(x))\notin W_n\}\notin\mathcal{I}\}$$

is uncountable. For each $\alpha\in A_1$, set $G_\alpha=\{x:(\alpha,x,f(x))\notin W_n\}$; then $G_\alpha\in\Sigma\setminus\mathcal{I}$, so there is a $\sigma\in S$ such that $U_\sigma\setminus G_\alpha\in\mathcal{I}$. Let $\sigma\in S$ be such that

$$A_2=\{\alpha:\alpha\in A_1,\, U_\sigma\setminus G_\alpha\in\mathcal{I}\}$$

is uncountable. Set $m=\max(n,\#(\sigma))$, so that $U_\sigma\cap\{x:(\alpha,x,f(x))\in V_m\}\in\mathcal{I}$ for every $\alpha\in A_2$. Set $M=\#(\mathcal{H}_m)$.

Take $k\in\mathbb{N}$ such that $\#(\{\alpha\restriction k:\alpha\in A_2\})\ge M$. Let $\langle\alpha_i\rangle_{i<M}$ be a family in A_2 such that $\alpha_i\restriction k\ne\alpha_j\restriction k$ for distinct i, $j<M$; let $\langle r_i\rangle_{i<M}$, $\langle l_i\rangle_{i<M}$ be such that $\alpha_i\restriction k=\tau_{r_i}$ for each i and $\mathcal{H}_m=\{H_{l_i}:i<M\}$. Let $s\in\mathbb{N}$ be such that $\tau_s(r_i)$ is defined and equal to l_i for $i<M$. Let $\sigma'\in\mathbb{N}^{m+2}$ be such that $\sigma'\supseteq\sigma$, $\sigma'(m)=s$ and $\sigma'(m+1)=k$. Then $U_{\sigma'}\notin\mathcal{I}$ and $U_{\sigma'}\setminus G_\alpha\in\mathcal{I}$ for every $\alpha\in A_2$.

Suppose that $i<M$ and $x\in U_{\sigma'}$. Then

$$\{y:(\alpha_i,x,y)\in V_m\}=H_{\tau_{\sigma'(m)}}(j)=H_{\tau_s(j)}$$

where $(\sigma',j)\in K_m$, that is, j is such that $\tau_j\subseteq\alpha_i$ and $\#(\tau(j))=\sigma'(m+1)=k$. Thus $j=r_i$, $\tau_s(j)=l_i$ and $\{y:(\alpha_i,x,y)\in V_m\}=H_{l_i}$. But this means that, for any $x\in U_{\sigma'}$,

$$\bigcup_{i<M}\{y:(\alpha_i,x,y)\in V_m\}=\bigcup_{i<M}H_{l_i}=Y$$

contains $f(x)$; that is, $U_{\sigma'}\subseteq\bigcup_{i<M}\{x:(\alpha_i,x,f(x))\in V_m\}$. On the other hand,

$$U_{\sigma'}\cap\{x:(\alpha_i,x,f(x))\in V_m\}\subseteq U_\sigma\cap\{x:(\alpha_i,x,f(x))\in V_m\}\in\mathcal{I}$$

for each $i<M$, while $U_{\sigma'}$ itself does not belong to $\mathcal{I}$. So this is impossible. **X**

Thus $\langle W_n\rangle_{n\in\mathbb{N}}$ satisfies (iii).

527G Theorem Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X which is generated by $\Sigma\cap\mathcal{I}$; suppose that the quotient algebra $\Sigma/\Sigma\cap\mathcal{I}$ is non-zero, atomless and has countable π-weight. Let (Y,T,ν) be an atomless perfect semi-finite measure space such that $\nu Y>0$. Set $\mathcal{K}=\mathcal{I}\ltimes_{\Sigma\widehat{\otimes}\mathrm{T}}\mathcal{N}(\nu)$. Then $[\mathfrak{c}]^{\le\omega}\preccurlyeq_{\mathrm{T}}\mathcal{K}$, so $\operatorname{add}\mathcal{K}=\omega_1$ and $\operatorname{cf}\mathcal{K}\ge\mathfrak{c}$.

proof (a) To begin with (down to the end of (d)) suppose that ν is totally finite. Because ν is atomless, we can for each $n\in\mathbb{N}$ find a finite cover $\mathcal{H}_n$ of Y by measurable sets with measures at most 2^{-n}. Let T_0 be the σ-algebra generated by $\bigcup_{n\in\mathbb{N}}\mathcal{H}_n$, so that T_0 is a σ-subalgebra of T. Construct $\langle\mathcal{H}_n^*\rangle_{n\in\mathbb{N}}$, $\langle W_n\rangle_{n\in\mathbb{N}}$ and W from $\langle\mathcal{H}_n\rangle_{n\in\mathbb{N}}$ as in 527F. Then if $f:X\to Y$ is (Σ,T_0)-measurable, $\{(\alpha,x):(\alpha,x,f(x))\notin W\}\in[\mathbb{N}^\mathbb{N}]^{\le\omega}\ltimes\mathcal{I}$. Note that $\nu H\le 2^{-n+1}$ for every $H\in\mathcal{H}_n^*$, so $\nu\{y:(\alpha,x,y)\in W_n\}\le 2^{-n+1}$ for every $\alpha\in\mathbb{N}^\mathbb{N}$, $x\in X$ and $n\in\mathbb{N}$. For each $\alpha\in\mathbb{N}^\mathbb{N}$ set $K_\alpha=\{(x,y):(\alpha,x,y)\in W\}$. Observe that $K_\alpha\in\Sigma\widehat{\otimes}\mathrm{T}$ because $W\in\mathcal{B}(\mathbb{N}^\mathbb{N})\widehat{\otimes}\Sigma\widehat{\otimes}\mathrm{T}$, and that

$$\nu K_\alpha[\{x\}]\le\inf_{n\in\mathbb{N}}\nu\{y:(\alpha,x,y)\in W_n\}=0$$

for every $x\in X$, so $K_\alpha\in\mathcal{K}$ for every $\alpha\in\mathbb{N}^\mathbb{N}$.

(b) Set $\hat\Sigma=\{E\triangle M:E\in\Sigma,\,M\in\mathcal{I}\}$. Then $\hat\Sigma$ is a σ-algebra of subsets of X (cf. 212Ca) and $\mathcal{I}$ is a σ-ideal in $\hat\Sigma$; also the identity embedding of Σ in $\hat\Sigma$ induces an isomorphism between $\Sigma/\Sigma\cap\mathcal{I}$ and $\hat\Sigma/\mathcal{I}$ (cf. 322Da). Consequently $\hat\Sigma/\mathcal{I}$ has countable π-weight, therefore is ccc, and $\hat\Sigma$ is closed under Souslin's operation (431G).

(c) Let $A\subseteq\mathbb{N}^\mathbb{N}$ be an uncountable set, and $V\in\Sigma\widehat{\otimes}\mathrm{T}$ a set disjoint from $\bigcup_{\alpha\in A}K_\alpha$. (I aim to show that $(X\times Y)\setminus V\notin\mathcal{I}\ltimes\mathcal{N}(\nu)$.) There must be sequences $\langle C_n\rangle_{n\in\mathbb{N}}$ in Σ, $\langle F_n\rangle_{n\in\mathbb{N}}$ in T such that V belongs to the σ-algebra generated by $\{C_n\times F_n:n\in\mathbb{N}\}$; we may of course arrange that $\bigcup_{n\in\mathbb{N}}\mathcal{H}_n\subseteq\{F_n:n\in\mathbb{N}\}$. Let T_1 be the σ-subalgebra of T generated by $\{F_n:n\in\mathbb{N}\}$, so that $\mathrm{T}_0\subseteq\mathrm{T}_1$ and $V\in\Sigma\widehat{\otimes}\mathrm{T}_1$. Let $g:Y\to\mathbb{R}$ be the Marczewski functional defined by setting $g=\sum_{n=0}^\infty 3^{-n}\chi F_n$. Because ν is perfect, there is a Borel set $H\subseteq g[Y]$ such that $g^{-1}[H]$ is conegligible. Let $h:H\to Y$ be any function such that $g(h(t))=t$ for every $t\in H$; note that h is $(\mathcal{B}(H),\mathrm{T}_1)$-measurable, where $\mathcal{B}(H)$ is the Borel σ-algebra of H, just because $\overline{g[F_n]}\cap\overline{g[Y\setminus F_n]}$ is empty for every n. Set $V_0=\{(x,t):x\in X,\,t\in H,\,(x,h(t))\in V\}$; then $V_0\in\Sigma\widehat{\otimes}\mathcal{B}(H)$. It follows that V_0 belongs to the class of sets obtainable by Souslin's operation from sets of the form $E\times F$ where $E\in\Sigma$ and $F\subseteq H$ is relatively closed in H. (Use 421F.) Set $\tilde E=V_0^{-1}[H]$. Because H is analytic and $\hat\Sigma$ is closed under Souslin's operation, $\tilde E\in\hat\Sigma$ and there is a $(\hat\Sigma,\mathcal{B}(H))$-measurable function $f_1:\tilde E\to H$ such that $(x,f_1(x))\in V_0$ for every $x\in\tilde E$ (423M). Now $f_2=hf_1:\tilde E\to Y$ is $(\hat\Sigma,\mathrm{T}_1)$-measurable and $(x,f_2(x))\in V$ for every $x\in\tilde E$.

For every $n\in\mathbb{N}$, $E_n=f_1^{-1}[F_n]$ belongs to $\hat\Sigma$, so there is an $E'_n\in\Sigma$ such that $E_n\triangle E'_n\in\mathcal{I}$. Similarly, there is an $\tilde E'\in\Sigma$ such that $\tilde E\triangle\tilde E'\in\mathcal{I}$. Because $\mathcal{I}$ is generated by $\Sigma\cap\mathcal{I}$, there is an $M_0\in\Sigma\cap\mathcal{I}$ including $(\tilde E\triangle\tilde E')\cup\bigcup_{n\in\mathbb{N}}(E_n\triangle E'_n)$. Now $\tilde E\setminus M_0=\tilde E'\setminus M_0$ belongs to Σ. Set $f_3=f_2\restriction\tilde E\setminus M_0$; then $f_3^{-1}[F_n]=E'_n\setminus M_0\in\Sigma$ for every n, so f_3 is (Σ,T_1)-measurable. Take any $y_0\in Y$, and set $f(x)=f_3(x)$ if $x\in\tilde E\setminus M_0$, y_0 for other $x\in X$; then f is (Σ,T_1)-measurable, therefore (Σ,T_0)-measurable.

The set $\{(\alpha,x):(\alpha,x,f(x))\notin W\}$ belongs to $[\mathbb{N}^\mathbb{N}]^{\le\omega}\ltimes\mathcal{I}$, so there must be an $\alpha\in A$ such that $M_1=\{x:(\alpha,x,f(x))\notin W\}$ belongs to $\mathcal{I}$. **?** Suppose, if possible, that $(X\times Y)\setminus V\in\mathcal{I}\ltimes\mathcal{N}(\nu)$. Then there must be an $x\in X\setminus(M_0\cup M_1)$ such that $V[\{x\}]$ is conegligible. In this case, $V[\{x\}]\cap g^{-1}[H]$ is conegligible, so is not empty, and there is a $y \in V[\{x\}] \cap g^{-1}[H]$. Consider $y' = h(g(y))$; then $g(y') = g(y)$, so $\{n : y' \in F_n\} = \{n : y \in F_n\}$, and $\{F : y \in F \iff y' \in F\}$ is a σ-algebra of subsets of Y containing every F_n and therefore containing $V[\{x\}]$. So $y' \in V[\{x\}]$ and $(x, g(y)) \in V_0$. This means that $x \in \tilde{E}$; as $x \notin M_0$, $f(x) = f_3(x) = f_2(x)$ and $(x, f(x)) \in V$. On the other hand, $x \notin M_1$, so $(\alpha, x, f(x)) \in W$ and $(x, f(x)) \in K_\alpha$; contradicting the choice of V as a set disjoint from K_α. **X**

This shows that $(X \times Y) \setminus V \notin \mathcal{I} \ltimes \mathcal{N}(\nu)$. As V is arbitrary, $\bigcup_{\alpha \in A} K_\alpha \notin \mathcal{K}$.

(d) This is true for every uncountable $A \subseteq \mathbb{N}^{\mathbb{N}}$. But this means that $A \mapsto \bigcup_{\alpha \in A} K_\alpha$ is a Tukey function from $[\mathbb{N}^{\mathbb{N}}]^{\le\omega}$ to $\mathcal{K}$, and $[\mathfrak{c}]^{\le\omega} \cong [\mathbb{N}^{\mathbb{N}}]^{\le\omega} \preccurlyeq_{\mathrm{T}} \mathcal{K}$.

(e) Thus the theorem is true if νY is finite. For the general case, let $Y_0 \in \mathrm{T}$ be such that $0 < \nu Y_0 < \infty$. Then the subspace measure ν_{Y_0} is still atomless and perfect (214Ka, 451Dc), so $[\mathfrak{c}]^{\le\omega} \preccurlyeq_{\mathrm{T}} \mathcal{K}_0$, where $\mathcal{K}_0 = \mathcal{I} \ltimes_{\Sigma\widehat{\otimes}(\mathrm{T} \cap \mathcal{P}Y_0)} \mathcal{N}(\nu_{Y_0})$. But $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{P}(X \times Y_0)$, so the identity map from $\mathcal{K}_0$ to $\mathcal{K}$ is a Tukey function, and

$$[\mathfrak{c}]^{\le\omega} \preccurlyeq_{\mathrm{T}} \mathcal{K}_0 \preccurlyeq_{\mathrm{T}} \mathcal{K}$$

in this case also. It follows at once that $\operatorname{add}\mathcal{K} \le \operatorname{add}[\mathfrak{c}]^{\le\omega} = \omega_1$, so that $\operatorname{add}\mathcal{K} = \omega_1$, and that $\operatorname{cf}\mathcal{K} \ge \operatorname{cf}[\mathfrak{c}]^{\le\omega} = \mathfrak{c}$.

527H Corollary $\mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N} \equiv_{\mathrm{T}} [\mathfrak{c}]^{\le\omega}$.

proof By 527G, $[\mathfrak{c}]^{\le\omega} \preccurlyeq_{\mathrm{T}} \mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N}$. In the other direction, all we need to observe is that $\#(\mathcal{B}(\mathbb{R}^2)) = \mathfrak{c}$. Let $\langle W_\xi \rangle_{\xi < \mathfrak{c}}$ run over $\mathcal{B}(\mathbb{R}^2) \cap (\mathcal{M} \ltimes \mathcal{N})$, and for $V \in \mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N}$ choose $\xi_V < \mathfrak{c}$ such that $V \subseteq W_{\xi_V}$; then $V \mapsto \{\xi_V\} : \mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{N} \to [\mathfrak{c}]^{\le\omega}$ is a Tukey function, so $\mathcal{M} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \preccurlyeq_{\mathrm{T}} [\mathfrak{c}]^{\le\omega}$.

527I I now turn to the ideal $\mathcal{N} \ltimes_{\mathcal{B}(\mathbb{R}^2)} \mathcal{M}$.

Lemma Let X be a set, Σ a σ-algebra of subsets of X, and Y a topological space with a countable π-base $\mathcal{H}$. Let $\mathcal{W}$ be the family of subsets of $X \times Y$ of the form $\bigcup_{H \in \mathcal{H}} E_H \times H$, where $E_H \in \Sigma$ for every $H \in \mathcal{H}$, and $\mathcal{D}_0$ the family of sets $D \subseteq X \times Y$ such that $(X \times Y) \setminus D \in \mathcal{W}$ and $D[\{x\}]$ is nowhere dense for every $x \in X$; let $\mathcal{L}_0$ be the σ-ideal of subsets of $X \times Y$ generated by $\mathcal{D}_0$. Then $\Sigma\widehat{\otimes}\mathcal{B}(Y) \subseteq \{W \triangle L : W \in \mathcal{W}, L \in \mathcal{L}_0\}$.

proof Write $\mathcal{V} = \{W \triangle L : W \in \mathcal{W}, L \in \mathcal{L}_0\}$. Then $\mathcal{W}$ and $\mathcal{V}$ are closed under countable unions. Next, $(X \times Y) \setminus W \in \mathcal{V}$ for every $W \in \mathcal{W}$. **P** Express W as $\bigcup_{H \in \mathcal{H}} E_H \times H$ where $E_H \in \Sigma$ for every $H \in \mathcal{H}$. For $H \in \mathcal{H}$, set

$$F_H = X \setminus \bigcup\{E_{H'} : H' \in \mathcal{H}, H' \cap H \ne \emptyset\} \in \Sigma,$$

and set $W' = \bigcup_{H \in \mathcal{H}} F_H \times H$. Then W' and $W \cup W'$ belong to $\mathcal{W}$. Set $D = (X \times Y) \setminus (W \cup W')$. If $x \in X$ and $G \subseteq Y$ is a non-empty open set, let $H \subseteq G$ be a non-empty member of $\mathcal{H}$. Then either $x \in F_H$ and H is a non-empty subset of $G \setminus D[\{x\}]$, or there is an $H' \in \mathcal{H}$ such that $H \cap H' \ne \emptyset$ and $x \in E_{H'}$, in which case $H \cap H'$ is a non-empty subset of $G \setminus D[\{x\}]$. As G is arbitrary, $D[\{x\}]$ is nowhere dense; as x is arbitrary, $D \in \mathcal{D}$. But now observe that $(X \times Y) \setminus W = W' \triangle D$ belongs to $\mathcal{V}$. **Q**

It follows that the complement of any member of $\mathcal{V}$ belongs to $\mathcal{V}$, so $\mathcal{V}$ is a σ-algebra. Now $E \times G \in \mathcal{V}$ for every $E \in \Sigma$ and open $G \subseteq Y$. **P** For $H \in \mathcal{H}$, set $E_H = E$ if $H \subseteq G$, $\emptyset$ otherwise; set $W = \bigcup_{H \in \mathcal{H}} E_H \times H \in \mathcal{W}$. Then $W \subseteq E \times G$. But, defining W' from W as just above, we see that W' is disjoint from $E \times G$. So

$$(E \times G) \triangle W \subseteq (X \times Y) \setminus (W \cup W') \in \mathcal{D}$$

and $E \times G \in \mathcal{V}$. **Q**

Accordingly $\mathcal{V}$ includes the σ-algebra generated by $\{E \times G : E \in \Sigma, G \subseteq Y \text{ is open}\}$, which is $\Sigma\widehat{\otimes}\mathcal{B}(Y)$.

527J Theorem (see Fremlin 91) Let X be a topological space and μ a σ-finite quasi-Radon measure on X with countable Maharam type; let Y be a topological space of countable π-weight. Then $\mathcal{N}(\mu) \ltimes_{\mathcal{B}(X \times Y)} \mathcal{M}(Y) \preccurlyeq_{\mathrm{T}} \mathcal{N}$.

proof Write $\mathcal{L}$ for $\mathcal{N}(\mu) \ltimes_{\mathcal{B}(X \times Y)} \mathcal{M}(Y)$, and fix a countable π-base $\mathcal{H}$ for the topology of Y.

(a) We need to know that for every Borel set $V \subseteq X \times Y$ there are sets V', $V'' \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ such that $V' \subseteq V \subseteq V''$ and $V'' \setminus V' \in \mathcal{L}$. **P** Let $\mathcal{V}^*$ be the family of all subsets of $X \times Y$ with this property. Because $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ is a σ-algebra and $\mathcal{L}$ is a σ-ideal of sets, $\mathcal{V}^*$ is a σ-algebra. If $W \subseteq X \times Y$ is open, set

$$U_H = \bigcup\{G : G \subseteq X \text{ is open}, G \times H \subseteq W\}, \quad U'_H = \{x : H \cap W[\{x\}] \ne \emptyset\}$$

for $H \in \mathcal{H}$, so all the U_H and U'_H are open (U'_H is just the projection of the open set $W \cap (X \times H)$). Set $V_1 = \bigcup_{H\in\mathcal{H}} U_H \times H$ and $V_2 = \bigcup_{H\in\mathcal{H}}((X \setminus U'_H) \times H)$. Then V_1 and V_2 both belong to $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$, $V_1 \subseteq W$ and $W \cap V_2 = \emptyset$.

Let $x \in X$. **?** If the open set $V_1[\{x\}] \cup V_2[\{x\}]$ is not dense, there is a non-empty $H \in \mathcal{H}$ disjoint from both $V_1[\{x\}]$ and $V_2[\{x\}]$. In this case x must belong to U'_H, and there is a point $y \in H \cap W[\{x\}]$. (x, y) belongs to the open set $(X \times H) \cap W$, so there are open sets $G \subseteq X$, $\tilde{H} \subseteq Y$ such that $(x, y) \in G \times \tilde{H} \subseteq (X \times H) \cap W$. Now there is an $H' \in \mathcal{H}$ such that $\emptyset \neq H' \subseteq \tilde{H}$, in which case $x \in U_{H'}$. But this will mean that $H' \subseteq V_1[\{x\}]$ and H' is a non-empty subset of $H \cap V_1[\{x\}]$, which is impossible. **X**

Thus $V_1[\{x\}] \cup V_2[\{x\}]$ is dense for every x, and if we set $V_3 = (X \times Y) \setminus V_2$ we shall have $V_3 \setminus V_1 \in \mathcal{L}$, while both V_1 and V_3 belong to $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$, and $V_1 \subseteq W \subseteq V_3$. So $W \in \mathcal{V}^*$. This is true for every open set $W \subseteq X \times Y$, so the σ-algebra $\mathcal{V}^*$ must contain every Borel set, as required. **Q**

It follows that every member of $\mathcal{L}$ is included in a member of $\mathcal{L} \cap (\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y))$. **P** If $V \in \mathcal{L}$ there is a Borel set $V' \supseteq V$ which belongs to $\mathcal{L}$, and now there is a set $V'' \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ such that $V'' \supseteq V'$ and $V'' \setminus V' \in \mathcal{L}$, in which case $V'' \supseteq V$ also must belong to $\mathcal{L}$. **Q**

Thus $\mathcal{L} = \mathcal{N}(\mu) \ltimes_{\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)} \mathcal{M}(Y)$.

(b) To begin with let us suppose that X is compact and metrizable, μ is totally finite and Y is a Baire space.

(i) Taking $\Sigma = \mathcal{B}(X)$, define $\mathcal{W}$, $\mathcal{D}_0$ and $\mathcal{L}_0$ as in 527I. Now let $\mathcal{D}$ be the family of closed subsets belonging to $\mathcal{D}_0$, and $\mathcal{L}_1$ the σ-ideal of subsets of $X \times Y$ generated by $\{E \times Y : E \in \mathcal{N}(\mu)\} \cup \mathcal{D}$.

(ii) $\mathcal{D}_0 \subseteq \mathcal{L}_1$. **P** If $D \in \mathcal{D}_0$, express $(X \times Y) \setminus D$ as $\bigcup_{H\in\mathcal{H}} E_H \times H$ where $E_H \in \mathcal{B}(X)$ for every $H \in \mathcal{H}$. Because μ is totally finite, μ is outer regular with respect to the open sets (412Wb). So for each $n \in \mathbb{N}$ we can find a family $\langle G_{nH}\rangle_{H\in\mathcal{H}}$ of open sets in X such that $E_H \subseteq G_{nH}$ for every H and $\sum_{H\in\mathcal{H}} \mu(G_{nH} \setminus E_H) \leq 2^{-n}$. Set $D_n = (X\times Y)\setminus\bigcup_{H\in\mathcal{H}}(G_{nH}\times H)$. Then D_n is closed and $D_n \subseteq D \in \mathcal{D}_0$ so $D_n \in \mathcal{D}$. Set $E = \bigcap_{n\in\mathbb{N}} \bigcup_{H\in\mathcal{H}}(G_{nH}\setminus E_H)$; then $E \in \mathcal{N}(\mu)$ and

$$D \subseteq (E \times Y) \cup \bigcup_{n\in\mathbb{N}} D_n \in \mathcal{L}_1. \quad \mathbf{Q}$$

(iii) Of course every member of $\mathcal{D}$ belongs to $\mathcal{L}$, so $\mathcal{L}_1 \subseteq \mathcal{L}$. But in fact $\mathcal{L} = \mathcal{L}_1$. **P** If $V \in \mathcal{L}$, there is a $V' \in (\mathcal{N}(\mu) \ltimes \mathcal{M}(Y)) \cap (\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y))$ such that $V \subseteq V'$, by (a). By 527I, we can express V' as $W \triangle L$ where $W \in \mathcal{W}$ and $L \in \mathcal{L}_0$. By (ii), $\mathcal{L}_0 \subseteq \mathcal{L}_1$, so $W \in \mathcal{L}$. There is therefore a negligible set $E \subseteq X$ such that $W[\{x\}]$ is meager for every $x \in X \setminus E$. But $W[\{x\}]$ is always open, and Y is a Baire space, so $W \subseteq E \times Y \in \mathcal{L}_1$. Accordingly V' and V belong to $\mathcal{L}_1$. As V is arbitrary, $\mathcal{L} \subseteq \mathcal{L}_1$. **Q**

(iv) Let $\mathcal{G}$ be a countable base for the topology of X containing X. Let $\mathcal{U}_0$ be the family of those sets $U \subseteq X\times Y$ such that U is expressible as a finite union of sets of the form $G \times H$ where $G \in \mathcal{G}$ and $H \in \mathcal{H}$, and $\mathcal{U}$ the set of those $U \in \mathcal{U}_0$ such that $\pi_1[U] = X$, where π_1 is the projection from $X \times Y$ onto X. Consider

$$\mathcal{D}' = \{D : D \subseteq X \times Y, \text{ for every } U_0 \in \mathcal{U} \text{ there is a } U \in \mathcal{U} \text{ such that } U \subseteq U_0 \setminus D\}.$$

$\mathcal{D} \subseteq \mathcal{D}'$. **P** Suppose that $D \in \mathcal{D}$ and $U_0 \in \mathcal{U}$, and consider $\mathcal{U}_1 = \{U : U \in \mathcal{U}_0,\ U \subseteq U_0 \setminus D\}$. For every $x \in X$ the section $U_0[\{x\}]$ is open and not empty and the section $D[\{x\}]$ is nowhere dense, so there is a y such that $(x, y) \in U_0 \setminus D$; now there are $G \in \mathcal{G}$, containing x, and an open H containing y such that $G \times H \subseteq U_0 \setminus D$. Let $H' \in \mathcal{H}$ be such that $\emptyset \neq H' \subseteq H$. Then $U = G \times H' \in \mathcal{U}_1$ and $x \in \pi_1[U]$. As x is arbitrary, $\{\pi_1[U] : U \in \mathcal{U}_1\}$ is an open cover of X; as X is compact and $\mathcal{U}_1$ is upwards-directed, there is a $U \in \mathcal{U}_1$ such that $\pi_1[U] = X$; in which case $U \in \mathcal{U}$ and $U \subseteq U_0 \setminus D$. As U is arbitrary, $D \in \mathcal{D}'$; as D is arbitrary, $\mathcal{D} \subseteq \mathcal{D}'$. **Q**

$\mathcal{D}$ is cofinal with $\mathcal{D}'$. **P** Let $D \in \mathcal{D}'$. For each $H \in \mathcal{H} \setminus \{\emptyset\}$, $X \times H \in \mathcal{U}$, so there is a $U_H \in \mathcal{U}$ such that $U_H \subseteq (X\times H)\setminus D$; try $D_1 = (X\setminus Y)\setminus\bigcup_{H\in\mathcal{H}\setminus\{\emptyset\}} U_H$. D_1 is closed. Since $\mathcal{U} \subseteq \mathcal{U}_0 \subseteq \mathcal{W}$, $(X\times Y)\setminus D_1 \in \mathcal{W}$. If $x \in X$, then $D_1[\{x\}]$ is a closed set not including any member of the π-base $\mathcal{H}$, so is nowhere dense in Y; thus $D_1 \in \mathcal{D}_0$ and (being closed) belongs to $\mathcal{D}$. Of course $D \subseteq D_1$. As D is arbitrary, $\mathcal{D}$ is cofinal with $\mathcal{D}'$. **Q**

(v) Because $\mathcal{U}$ is countable, 526Hd tells us that $\mathcal{D}' \preccurlyeq_{\mathrm{T}} \mathcal{N}\mathrm{wd}$, where $\mathcal{N}\mathrm{wd}$ is the ideal of nowhere dense subsets of $\mathbb{N}^{\mathbb{N}}$; while of course $\mathcal{D} \equiv_{\mathrm{T}} \mathcal{D}'$ (513E(d-ii)). Let $\phi : \mathcal{L} \to \mathcal{N}(\mu) \times \mathcal{D}^{\mathbb{N}}$ be such that if $\phi(V) = (E, \langle D_n\rangle_{n\in\mathbb{N}})$ then $V \subseteq (E \times Y) \cup \bigcup_{n\in\mathbb{N}} D_n$; such a function exists by (iii), and is evidently a Tukey function.

Note that the measure algebra of μ, being a totally finite measure algebra with countable Maharam type, can be regularly embedded in the measure algebra of Lebesgue measure on either $[0, 1]$ or on $\mathbb{R}$. Consequently $\mathcal{N}(\mu) \preccurlyeq_{\mathrm{T}} \mathcal{N}$ (524K) and

$$\mathcal{L} \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu) \times \mathcal{D}^{\mathbb{N}} \preccurlyeq_{\mathrm{T}} \mathcal{N} \times \mathcal{N}\mathrm{wd}^{\mathbb{N}} \cong \mathcal{N} \times \mathcal{N}\mathrm{wd}$$

(513Eg, 526Ha). Accordingly

$$(\mathcal{L},\subseteq,\mathcal{L}) \equiv_{\mathrm{GT}} (\mathcal{L},\subseteq',[\mathcal{L}]^{\le\omega})$$

(513Id)

$$\preccurlyeq_{\mathrm{GT}} (\mathcal{N}\times\mathcal{N}\mathit{wd},\le',[\mathcal{N}\times\mathcal{N}\mathit{wd}]^{\le\omega})$$

(512Gb)

$$\equiv_{\mathrm{GT}} (\mathcal{N},\subseteq',[\mathcal{N}]^{\le\omega})\times(\mathcal{N}\mathit{wd},\subseteq',[\mathcal{N}\mathit{wd}]^{\le\omega})$$

(512Hd)

$$\equiv_{\mathrm{GT}} (\mathcal{N},\subseteq\mathcal{N})\times(\mathcal{M},\subseteq,\mathcal{M})$$

(513Id, 526Hb, 512Hb)

$$\preccurlyeq_{\mathrm{GT}} (\mathcal{N},\subseteq,\mathcal{N})\times(\mathcal{N},\subseteq,\mathcal{N})$$

(522P)

$$\equiv_{\mathrm{GT}} (\mathcal{N},\subseteq,\mathcal{N})$$

(513Eh), and $\mathcal{L}\preccurlyeq_{\mathrm{T}}\mathcal{N}$.

(c) This proves the theorem when X is compact and metrizable, μ is totally finite and Y is a Baire space. Now suppose that Y is still a Baire space, while (X,μ) is any totally finite quasi-Radon measure space with countable Maharam type.

(i) There is a compact metrizable Radon measure space (Z,λ) such that λ and μ have isomorphic measure algebras. **P** Because the measure algebra $(\mathfrak{A},\bar\mu)$ of μ is totally finite, it is isomorphic to the simple product of a countable family $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ of homogeneous totally finite measure algebras (332B). Because μ has countable Maharam type, every $\mathfrak{A}_i$ is either $\{0\}$, $\{0,1\}$ or isomorphic to the measure algebra of Lebesgue measure on an interval; in any case it is isomorphic to the measure algebra of a compact Radon measure space (Z_i,λ_i). Take (Z',λ') to be the direct sum of the measure spaces $\langle(Z_i,\lambda_i)\rangle_{i\in I}$; then the measure algebra of (Z',λ') is isomorphic to $\mathfrak{A}$. If we give Z' its disjoint union topology, it is a locally compact σ-compact metrizable space, and its one-point compactification Z is second-countable, therefore metrizable; taking λ to be the trivial extension of λ', (Z,λ) is a compact metrizable Radon measure space with measure algebra $(\mathfrak{B},\bar\lambda)\cong(\mathfrak{A},\bar\mu)$. **Q**

(ii) Let $f:X\to Z$ be an inverse-measure-preserving function inducing an isomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ of the measure algebras (416Wb). By 418J, f is almost continuous, so there is a Borel measurable function which is equal almost everywhere to f; this function will still represent π, so we may suppose that f itself is Borel measurable. Now if $V\in\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$, there is a $V'\in\mathcal{B}(Z)\widehat{\otimes}\mathcal{B}(Y)$ such that $\{x:V[\{x\}]\ne V'[\{f(x)\}]\}\in\mathcal{N}(\lambda)$. **P** Let $\tilde{\mathcal{V}}$ be the family of subsets V of $X\times Y$ for which there is a $V'\in\mathcal{B}(Z)\widehat{\otimes}\mathcal{B}(Y)$ such that $\{x:V[\{x\}]\ne V'[\{f(x)\}]\}\in\mathcal{N}(\lambda)$. Then $\tilde{\mathcal{V}}$ is a σ-algebra. If $E\in\mathcal{B}(X)$ and $H\in\mathcal{B}(Y)$, then there must be an $F\in\mathcal{B}(Z)$ such that $F^\bullet=\pi E^\bullet$ in $\mathfrak{B}$, so that $E\triangle f^{-1}[F]\in\mathcal{N}(\mu)$; now $F\times H$ witnesses that $E\times H$ belongs to $\tilde{\mathcal{V}}$. Accordingly $\tilde{\mathcal{V}}$ must include $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. **Q**

(iii) We know that $\mathcal{N}(\mu)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\lambda)$ (524Sa), so there is a Tukey function $\theta:\mathcal{N}(\mu)\to\mathcal{N}(\lambda)$. Set $\mathcal{L}'=\mathcal{N}(\lambda)\ltimes_{\mathcal{B}(Z)\widehat{\otimes}\mathcal{B}(Y)}\mathcal{M}(Y)$. Define a function $\phi:\mathcal{L}\to\mathcal{L}'$ as follows. First, for $V\in\mathcal{L}$, choose $\phi_0(V)\in\mathcal{L}\cap(\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y))$ including V ((a) above). Next, by (ii) here, we can choose $\phi_1(V)\in\mathcal{B}(Z)\widehat{\otimes}\mathcal{B}(Y)$ such that $N_V=\{x:\phi_0(V)[\{x\}]\ne\phi_1(V)[\{f(x)\}]\}$ belongs to $\mathcal{N}(\mu)$. Set $F=\{z:z\in Z,\ \phi_1(V)[\{z\}]$ is not meager$\}$; then F is a Borel set, by 4A3Sa, and $f^{-1}[F]\subseteq N_V\cup\{x:\phi_0[\{x\}]\notin\mathcal{M}(Y)\}\in\mathcal{N}(\mu)$; so $F\in\mathcal{N}(\lambda)$ and $\phi_1(V)\in\mathcal{L}'$. Finally, set $\phi(V)=(\theta(N_V)\times Y)\cup\phi_1(V)\in\mathcal{L}'$.

ϕ is a Tukey function from $\mathcal{L}$ to $\mathcal{L}'$. **P** Take $W\in\mathcal{L}'$ and consider $\mathcal{E}=\{V:V\in\mathcal{L},\ \phi(V)\subseteq W\}$. If $Y=\emptyset$ then of course $\mathcal{E}$ is bounded above in $\mathcal{L}$. Otherwise, $N^*=\{z:W[\{z\}]=Y\}$ must be negligible, and $\theta(N_V)\subseteq N^*$ for every $V\in\mathcal{E}$; because θ is a Tukey function, $\tilde N=\bigcup\{N_V:V\in\mathcal{E}\}$ is negligible. Take $W_1\in\mathcal{L}'\cap(\mathcal{B}(Z)\widehat{\otimes}\mathcal{B}(Y))$ including W, and set $\tilde W=\{(x,y):(f(x),y)\in W_1\}$; then $\tilde W\in\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ because f is Borel measurable. As

$$\{x:\tilde W[\{x\}]\notin\mathcal{M}(Y)\}=f^{-1}\{z:W_1[\{z\}]\notin\mathcal{M}(Y)\}$$

is negligible, $\tilde W\in\mathcal{L}$. So $V_0=(N^*\times Y)\cup\tilde W$ belongs to $\mathcal{L}$. Now take any $V\in\mathcal{E}$. If $x\in X\setminus N^*$, then $x\notin N_V$, so

$$V[\{x\}]\subseteq\phi_0(V)[\{x\}]=\phi_1(V)[\{f(x)\}]\subseteq W[\{f(x)\}]\subseteq W_1[\{f(x)\}]=\tilde W[\{x\}]=V_0[\{x\}].$$

This shows that $V\subseteq V_0$; as V is arbitrary, V_0 is an upper bound for $\mathcal{E}$ in $\mathcal{L}$; as W is arbitrary, ϕ is a Tukey function. **Q**

(iv) By (b), we know that $\mathcal{L}'\preccurlyeq_{\mathrm{T}}\mathcal{N}$, so (iii) tells us that $\mathcal{L}\preccurlyeq_{\mathrm{T}}\mathcal{N}$, and the theorem is true in this case also.

(d) We are nearly home. If Y is a Baire space and (X,μ) is a σ-finite quasi-Radon measure space with countable Maharam type, which is not totally finite, then there is a measurable function $f:X\to\,]0,\infty[$ such that $\int f d\mu=1$ (215B(ix)). Let ν be the indefinite-integral measure defined by f. Then ν has the same negligible sets as μ (234Lc), and is a quasi-Radon measure (415Ob), so

$$\mathcal{L}=\mathcal{N}(\nu)\ltimes_{\mathcal{B}(X\times Y)}\mathcal{M}(Y)\preccurlyeq_{\mathrm{T}}\mathcal{N},$$

by (c).

(e) Finally, suppose that Y is not a Baire space. In this case, let H^* be the minimal comeager regular open subset of Y (4A3Ra again), and set $\mathcal{L}^*=\mathcal{N}(\mu)\ltimes_{\mathcal{B}(X\times H^*)}\mathcal{M}(H^*)$. Then $\mathcal{L}\preccurlyeq_{\mathrm{T}}\mathcal{L}^*$. **P** For every $V\in\mathcal{L}$, let V' be such that $V\subseteq V'\in\mathcal{B}(X\times Y)\cap(\mathcal{N}(\mu)\ltimes\mathcal{M}(Y))$, and set $\phi(V)=V'\cap(X\times H^*)$. Then $\phi(V)$ is a Borel subset of $X\times H^*$, and $\phi(V)[\{x\}]=V'[\{x\}]\cap H^*$ is meager in H^* whenever $V'[\{x\}]$ is meager in Y, so $\phi(V)\in\mathcal{L}^*$. To see that $\phi:\mathcal{L}:\to\mathcal{L}^*$ is a Tukey function, take any $W\in\mathcal{L}^*$. There is a Borel set $W'\in\mathcal{L}^*$ including W, and now $V'=W'\cup(X\times(Y\setminus H^*))$ is a Borel subset of $X\times Y$; since $V'[\{x\}]$ is meager in Y whenever $W'[\{x\}]$ is meager in H^*, $V'\in\mathcal{L}$. Of course V' is an upper bound of $\{V:V\in\mathcal{L},\,\phi(V)\subseteq W\}$; as W is arbitrary, ϕ is a Tukey function and $\mathcal{L}\preccurlyeq_{\mathrm{T}}\mathcal{L}^*$. **Q**

By (d), $\mathcal{L}\preccurlyeq_{\mathrm{T}}\mathcal{N}$ in this case also, and the proof is complete.

527K Corollary $\mathcal{N}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{M}\equiv_{\mathrm{T}}\mathcal{N}$.

proof By 527J, $\mathcal{N}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{M}\preccurlyeq_{\mathrm{T}}\mathcal{N}$. On the other hand, $E\mapsto E\times\mathbb{R}$ is a Tukey function from $\mathcal{N}$ to $\mathcal{N}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{M}$, so $\mathcal{N}\preccurlyeq_{\mathrm{T}}\mathcal{N}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{M}$.

527L There are some interesting questions concerning the saturation of skew products. Here I give two results which will be useful later.

Theorem Let X be a set, Σ a σ-ideal of subsets of X, and $\mathcal{I}\vartriangleleft\mathcal{P}X$ a σ-ideal; suppose that $\Sigma/\Sigma\cap\mathcal{I}$ is ccc. Let (Y,T,ν) be a σ-finite measure space. Then $(\Sigma\widehat{\otimes}\mathrm{T})/(\Sigma\widehat{\otimes}\mathrm{T})\cap(\mathcal{I}\ltimes\mathcal{N}(\nu))$ is ccc.

proof (a) The case $\nu Y=0$ is trivial, as then $\mathcal{I}\ltimes\mathcal{N}(\nu)=\mathcal{P}(X\times Y)$. Otherwise, there is a probability measure on Y with the same domain and null ideal as ν (215B(vii)), so we may suppose that $\nu Y=1$.

(b) The family $\mathcal{W}$ of sets $W\subseteq X\times Y$ such that $W[\{x\}]\in\mathrm{T}$ for every $x\in X$ and $x\mapsto\nu W[\{x\}]$ is Σ-measurable is a Dynkin class (definition: 136A), and contains $E\times F$ whenever $E\in\Sigma$ and $F\in\mathrm{T}$; by the Monotone Class Theorem (136B) it includes $\Sigma\widehat{\otimes}\mathrm{T}$.

(c) Now suppose that $\langle W_\xi\rangle_{\xi<\omega_1}$ is a disjoint family in $\Sigma\widehat{\otimes}\mathrm{T}$. For $n\in\mathbb{N}$ and $\xi<\kappa$ set

$$E_{n\xi}=\{x:\nu W_\xi[\{x\}]\ge 2^{-n}\};$$

then $\#(\{\xi:x\in E_{n\xi}\})\le 2^{-n}$ for every $x\in X$. It follows that $A_n=\{\xi:\xi<\omega_1,\,E_{n\xi}\notin\mathcal{I}\}$ is countable. **P** **?** Otherwise, write $\mathfrak{A}$ for the ccc algebra $\Sigma/\Sigma\cap\mathcal{I}$, and $a_\xi=E^\bullet_{n\xi}$ for $\xi<\omega_1$. Then $\mathfrak{A}$ is Dedekind complete; set $b_\xi=\sup_{\xi\le\eta<\omega_1}a_\eta$ for $\xi<\omega_1$ and $b=\inf_{\xi<\omega_1}b_\xi$. Because $\mathfrak{A}$ is ccc, there is a $\zeta<\omega_1$ such that $b=b_\xi$ for every $\xi\ge\zeta$; because A_n is uncountable, $b\ne 0$. Choose $\langle c_i\rangle_{i\in\mathbb{N}}$ and $\langle\xi_i\rangle_{i\in\mathbb{N}}$ inductively such that $c_0=b$ and, given that $0\ne c_i\subseteq b$, ξ_i is to be such that $c_{i+1}=a_{\xi_i}\cap c_i\ne 0$ and $\xi_i>\xi_j$ for every $j<i$.
Now $\inf_{i\le 2^n}a_{\xi_i}\supseteq c_{2^n+1}$ is non-zero, so there is an $x\in\bigcap_{i\le 2^n}E_{n\xi_i}$; but this is impossible. **X** **Q**

(d) This is true for every $n\in\mathbb{N}$, so there is a $\xi<\omega_1$ such that $\xi\notin A_n$ for every n, that is, $E_{n\xi}\in\mathcal{I}$ for every n. But in this case

$$\{x:W_\xi[\{x\}]\notin\mathcal{N}(\nu)\}=\bigcup_{n\in\mathbb{N}}E_{n\xi}$$

belongs to $\mathcal{I}$ and $W_\xi\in\mathcal{I}\ltimes\mathcal{N}(\nu)$. As $\langle W_\xi\rangle_{\xi<\omega_1}$ is arbitrary, $(\Sigma\widehat{\otimes}\mathrm{T})\cap(\mathcal{I}\ltimes\mathcal{N}(\nu))$ is ω_1-saturated in $\Sigma\widehat{\otimes}\mathrm{T}$ and $(\Sigma\widehat{\otimes}\mathrm{T})/(\Sigma\widehat{\otimes}\mathrm{T})\cap(\mathcal{I}\ltimes\mathcal{N}(\nu))$ is ccc (316C).

527M The next result provides me with an opportunity to introduce a concept which will be needed in §546.

Definition A Boolean algebra $\mathfrak{A}$ is **harmless** (cf. Just 92) if it is ccc and whenever $\mathfrak{B}$ is a countable subalgebra of $\mathfrak{A}$, there is a countable subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ such that $\mathfrak{B}\subseteq\mathfrak{C}$ and $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$.

527N Lemma (a) If $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{D}$ is a harmless order-dense subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ is harmless.

(b) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra, then it is harmless iff every order-closed subalgebra of $\mathfrak{A}$ with countable Maharam type has countable π-weight.

(c) For any set I, the regular open algebra $\mathrm{RO}(\{0,1\}^I)$ of $\{0,1\}^I$ is harmless, so the category algebra of $\{0,1\}^I$ is harmless.

(d) If $\mathfrak{A}$ has countable π-weight it is harmless.

(e) If $\mathfrak{A}$ is a harmless Boolean algebra, $\mathfrak{B}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then $\mathfrak{B}$ is harmless. In particular, any principal ideal of a harmless Boolean algebra is harmless.

proof (a) By 513E(e-iii), $\mathfrak{A}$ is ccc. Let $\mathfrak{B}$ be a countable subalgebra of $\mathfrak{A}$. For each $b \in \mathfrak{B}$ let $D_b \subseteq \mathfrak{D}$ be a countable set with supremum b (313K, 316E). Let $\mathfrak{D}_0$ be the subalgebra of $\mathfrak{D}$ generated by $\bigcup_{b\in\mathfrak{B}} D_b$. Then $\mathfrak{D}_0$ is countable, so there is a countable subalgebra $\mathfrak{D}_1$ of $\mathfrak{D}$, including $\mathfrak{D}_0$, which is regularly embedded in $\mathfrak{D}$. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \mathfrak{D}_1$. Then $\mathfrak{C}$ is countable. Now every member of $\mathfrak{C}$ is the supremum of the members of $\mathfrak{D}_1$ it includes. **P** Set

$$C = \{c : c \in \mathfrak{C},\, c = \sup\{d : d \in \mathfrak{D}_1,\, d \subseteq c\} = \inf\{d : d \in \mathfrak{D}_1,\, c \subseteq d\}\}.$$

Then C is closed under union (use 313Bd) and complementation (313A), and includes $\mathfrak{B} \cup \mathfrak{D}_1$, so $C = \mathfrak{C}$. **Q**

It follows that $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$, because if $C \subseteq \mathfrak{C}$ has supremum 1 in $\mathfrak{C}$ then $\bigcup_{c\in C}\{d : d \in \mathfrak{D}_1,\, d \subseteq c\}$ must have supremum 1 in $\mathfrak{C}$ and therefore in $\mathfrak{D}_1$ (because $\mathfrak{D}_1 \subseteq \mathfrak{C}$) and in $\mathfrak{D}$ (because $\mathfrak{D}_1$ is regularly embedded in $\mathfrak{D}$) and in $\mathfrak{A}$ (because $\mathfrak{D}$ is regularly embedded in $\mathfrak{A}$). But this means that $\sup C$ must be 1 in $\mathfrak{A}$. As C is arbitrary, $\mathfrak{C}$ is regularly embedded. As $\mathfrak{B}$ is arbitrary, $\mathfrak{A}$ is harmless.

(b)(i) Suppose that $\mathfrak{A}$ is harmless and that $\mathfrak{B} \subseteq \mathfrak{A}$ is an order-closed subalgebra of countable Maharam type. Let $B \subseteq \mathfrak{B}$ be a countable set which τ-generates $\mathfrak{B}$, and $\mathfrak{B}_0$ the algebra generated by B; let $\mathfrak{C}$ be a countable subalgebra of $\mathfrak{A}$, including $\mathfrak{B}_0$, which is regularly embedded in $\mathfrak{A}$. Let $\mathfrak{D}$ be the set

$$\{d : d \in \mathfrak{A},\, d = \sup\{c : c \in \mathfrak{C},\, c \subseteq d\} = \inf\{c : c \in \mathfrak{C},\, d \subseteq c\}\}.$$

Then $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{A}$. **P** As in (a) just above, it is a subalgebra. If $D \subseteq \mathfrak{D}$ is a non-empty set with supremum a in $\mathfrak{A}$, set $C = \{c : c \in \mathfrak{C},\, c \subseteq a\}$, $C' = \{c : c \in \mathfrak{C},\, a \subseteq c\}$. Then a is an upper bound for C and a lower bound for C'. **?** If either a is not the least upper bound of C, or a is not the greatest lower bound of C', then $A = \{c' \setminus c : c' \in C',\, c \in C\}$ is a subset of $\mathfrak{C}$ with a non-zero lower bound in $\mathfrak{A}$, so A has a non-zero lower bound c^* in $\mathfrak{C}$. Now if $d \in D$, $c \in \mathfrak{C}$ and $c \subseteq d$, then $c \in C$ so $c \cap c^* = 0$; as $d = \sup\{c : c \in \mathfrak{C},\, c \subseteq d\}$, $d \cap c^* = 0$. This is true for every $d \in D$, so $a \cap c^* = 0$ and $1 \setminus c^* \in C'$; but c^* was chosen to be included in every member of C'. **X** Thus $a \in \mathfrak{D}$; as D is arbitrary, $\mathfrak{D}$ is order-closed in $\mathfrak{A}$. **Q**

Now $B \subseteq \mathfrak{C} \subseteq \mathfrak{D}$. As $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$ (314Ga), $\mathfrak{B} \cap \mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{B}$ including B, so is the whole of $\mathfrak{B}$, and $\mathfrak{B} \subseteq \mathfrak{D}$. It follows that $\pi(\mathfrak{B}) \le \pi(\mathfrak{D})$ (514Eb). But $\mathfrak{C}$ is countable and order-dense in $\mathfrak{D}$, so $\pi(\mathfrak{D})$ and $\pi(\mathfrak{B})$ are countable. As $\mathfrak{B}$ is arbitrary, $\mathfrak{A}$ satisfies the declared condition.

(ii) Now suppose that $\mathfrak{A}$ satisfies the condition. Note first that $\mathfrak{A}$ is ccc. **P?** Suppose, if possible, otherwise; let $\langle a_\xi\rangle_{\xi<\omega_1}$ be a disjoint family in $\mathfrak{A} \setminus \{0\}$. Replacing a_0 by $a_0 \cup (1 \setminus \sup_{\xi<\omega_1} a_\xi)$ if necessary, we may suppose that $\sup_{\xi<\omega_1} a_\xi = 1$. The map $I \mapsto \sup_{\xi\in I} a_\xi : \mathcal{P}\omega_1 \to \mathfrak{A}$ is an injective order-continuous Boolean homomorphism, so its image $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ isomorphic to $\mathcal{P}\omega_1$. Now $\tau(\mathfrak{B}) = \tau(\mathcal{P}\omega_1) = \omega$ (514Ef, or otherwise), but $\pi(\mathfrak{B}) = \omega_1$; which is supposed to be impossible. **XQ**

If $\mathfrak{B}$ is a countable subalgebra of $\mathfrak{A}$, let $\mathfrak{B}_1$ be the order-closed subalgebra of $\mathfrak{A}$ which it generates. Then $\tau(\mathfrak{B}_1) \le \omega$ so $\pi(\mathfrak{B}_1) \le \omega$, and there is a countable subalgebra $\mathfrak{C}$ of $\mathfrak{B}_1$ which is order-dense in $\mathfrak{B}_1$; of course we may suppose that $\mathfrak{B} \subseteq \mathfrak{C}$. Now the identity maps from $\mathfrak{C}$ to $\mathfrak{B}_1$ and from $\mathfrak{B}_1$ to $\mathfrak{A}$ are both order-continuous, so their composition also is, and $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$. As $\mathfrak{B}$ is arbitrary, $\mathfrak{A}$ is harmless.

(c) All regular open algebras are Dedekind complete. If $\mathfrak{B} \subseteq \mathrm{RO}(\{0,1\}^I)$ is an order-closed subalgebra with countable Maharam type, let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{B}$ which τ-generates $\mathfrak{B}$. Every regular open subset of $\{0,1\}^I$ is determined by coordinates in some countable set (4A2E(b-i)), so there is a countable $J \subseteq I$ such that every G_n is determined by coordinates in J. Let $\pi_J : \{0,1\}^I \to \{0,1\}^J$ be the restriction map; then we have an injective order-continuous Boolean homomorphism $H \mapsto \pi_J^{-1}[H] : \mathrm{RO}(\{0,1\}^J) \to \mathrm{RO}(\{0,1\}^I)$ (4A2B(f-iii)). Let $\mathfrak{C}$ be the image of this homomorphism, so that $\mathfrak{C}$ is an order-closed subalgebra of $\mathrm{RO}(\{0,1\}^I)$. If $H_n = \pi_J[G_n]$ then H_n is regular and open for each n (4A2B(f-iii) again), so $G_n = \pi_J^{-1}[H_n] \in \mathfrak{C}$; accordingly $\mathfrak{B} \subseteq \mathfrak{C}$. Now $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{C}$ so

$$\pi(\mathfrak{B}) \le \pi(\mathfrak{C}) = \pi(\{0,1\}^J) \le \omega.$$

As $\mathfrak{B}$ is arbitrary, $\mathrm{RO}(\{0,1\}^I)$ satisfies the condition of (b) and is harmless.

Of course it follows at once that the category algebra is harmless, because it is isomorphic to the regular open algebra (514If-514Ig).

(d) Let D be a countable order-dense set in $\mathfrak{A}$. If $\mathfrak{B}$ is a countable subalgebra of $\mathfrak{A}$, let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by $D\cup\mathfrak{B}$; then $\mathfrak{C}$ is countable, includes $\mathfrak{B}$ and is order-dense, therefore regularly embedded in $\mathfrak{A}$. As $\mathfrak{B}$ is arbitrary, $\mathfrak{A}$ is harmless.

(e) Let $\mathfrak{D}\subseteq\mathfrak{B}$ be a countable subalgebra. Because $\pi[\mathfrak{A}]=\mathfrak{B}$, there is a countable subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ such that $\pi[\mathfrak{C}]=\mathfrak{D}$. Let $\mathfrak{C}_1\supseteq\mathfrak{C}$ be a countable regularly embedded subalgebra of $\mathfrak{A}$. Then $\mathfrak{D}_1=\pi[\mathfrak{C}_1]$ is regularly embedded in $\mathfrak{B}$. **P** Let $D\subseteq\mathfrak{D}_1$ be a non-empty set such that 1 is not the least upper bound of D in $\mathfrak{B}$. Set $C=\mathfrak{C}_1\cap\pi^{-1}[D\cup\{0\}]$; then 1 is not the least upper bound of $\pi[C]$ in $\mathfrak{B}$, so (because π is order-continuous) 1 is not the least upper bound of C in $\mathfrak{A}$. Because $\mathfrak{C}_1$ is regularly embedded in $\mathfrak{A}$, there is a non-zero $c_0\in\mathfrak{C}_1$ such that $c_0\cap c=0$ for every $c\in C$. In particular, $c_0\notin C$ and $\pi c_0\ne0$. But we also have $\pi c\cap\pi c_0=0$ for every $c\in C$, that is, $d\cap\pi c_0=0$ for every $d\in D$, and 1 is not the least upper bound of D in $\mathfrak{D}_1$. As D is arbitrary, $\mathfrak{D}_1$ is regularly embedded. **Q** Of course $\mathfrak{D}_1$ is countable. As $\mathfrak{D}$ is arbitrary, $\mathfrak{B}$ is harmless.

If $c\in\mathfrak{A}$ then $a\mapsto a\cap c$ is an order-continuous homomorphism onto the principal ideal $\mathfrak{A}_c$ generated by c, so $\mathfrak{A}_c$ is harmless.

527O Theorem Let (X,Σ,μ) be a σ-finite measure space and Y a topological space such that the category algebra $\mathfrak{G}$ of Y is harmless. Write $\mathcal{L}$ for $(\Sigma\widehat{\otimes}\mathcal{B}(Y))\cap(\mathcal{N}(\mu)\ltimes\mathcal{M}(Y))$ and $\mathfrak{A}$ for the measure algebra of μ. Then $\mathfrak{C}=(\Sigma\widehat{\otimes}\mathcal{B}(Y))/\mathcal{L}$ is ccc, and is isomorphic to the Dedekind completion of the free product $\mathfrak{A}\otimes\mathfrak{G}$. If neither $\mathfrak{A}$ nor $\mathfrak{G}$ is trivial, the isomorphism corresponds to embeddings $E^\bullet\mapsto(E\times Y)^\bullet:\mathfrak{A}\to\mathfrak{C}$ and $F^\bullet\mapsto(X\times F)^\bullet:\mathfrak{B}\to\mathfrak{C}$.

proof Write $\mathfrak{S}$ for the topology of Y.

(a) Let $\mathcal{W}$ be the family of all sets of the form $\bigcup_{n\in\mathbb{N}}E_n\times H_n$, where $E_n\in\Sigma$ and $H_n\subseteq Y$ is open for every n. Then for any $W\in\mathcal{W}$ there is a $W'\in\mathcal{W}$ such that $W'\triangle((X\times Y)\setminus W)\in\mathcal{L}$. **P** Express W as $\bigcup_{n\in\mathbb{N}}E_n\times H_n$ where $E_n\in\Sigma$ and $H_n\in\mathfrak{S}$ for each n. Let $\mathfrak{D}$ be the order-closed subalgebra of $\mathfrak{G}$ generated by $\{H_n^\bullet:n\in\mathbb{N}\}$. Because $\mathfrak{G}$ is harmless and Dedekind complete, $\pi(\mathfrak{D})\le\omega$ (527Nb); let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{S}$ such that $\{G_n^\bullet:n\in\mathbb{N}\}$ is a π-base for $\mathfrak{D}$; we may suppose that any non-empty open subset of any G_n is non-meager. Let $\mathfrak{S}_1$ be the second-countable topology on Y generated by $\{H_n:n\in\mathbb{N}\}\cup\{G_n:n\in\mathbb{N}\}$, and $\mathcal{B}_1(Y)\subseteq\mathcal{B}(Y)$ the corresponding Borel σ-algebra. Then $V^\bullet\in\mathfrak{D}$ for every $V\in\mathfrak{S}_1$, because V is the union of a countable family of sets all with images in $\mathfrak{D}$. If $V\in\mathfrak{S}_1$ is dense for $\mathfrak{S}_1$, and $n\in\mathbb{N}$ is such that G_n is non-empty, $V\cap G_n\ne\emptyset$ so $V^\bullet\cap G_n^\bullet\ne0$, by the choice of the G_n. But this means that $V^\bullet=1$, that is, V is comeager for the original topology of Y.

Now W and $(X\times Y)\setminus W$ belong to $\Sigma\widehat{\otimes}\mathcal{B}_1(Y)$. By 527I, there are W' and $\langle D_n\rangle_{n\in\mathbb{N}}$ such that

$((X\times Y)\setminus W)\triangle W'\subseteq\bigcup_{n\in\mathbb{N}}D_n$,

W' is expressible as $\bigcup_{n\in\mathbb{N}}F_n\times V_n$ where $F_n\in\Sigma$ and $V_n\in\mathfrak{S}_1$ for every n,

every D_n belongs to $\Sigma\widehat{\otimes}\mathcal{B}_1(Y)$,

for every $x\in X$ and $n\in\mathbb{N}$, $D_n[\{x\}]$ is closed and nowhere dense for $\mathfrak{S}_1$.

Evidently $W'\in\mathcal{W}$; but we have just seen that sets which are closed and nowhere dense for $\mathfrak{S}_1$ are meager for $\mathfrak{S}$. So every D_n belongs to $\mathcal{L}$ and $((X\times Y)\setminus W)\triangle W'\in\mathcal{L}$. **Q**

(b) It follows (as in the proof of 527I) that $\mathcal{V}=\{W\triangle D:W\in\mathcal{W},\,D\in\mathcal{L}\}$ is a σ-algebra of sets, and as $E\times H\in\mathcal{W}$ for every $E\in\Sigma$ and $H\in\mathfrak{S}$, $\mathcal{V}=\Sigma\widehat{\otimes}\mathcal{B}(Y)$.

(c) $\mathfrak{C}$ is ccc. **P?** Otherwise, there is a disjoint family $\langle e_\xi\rangle_{\xi<\omega_1}$ in $\mathfrak{C}\setminus\{0\}$. For each $\xi<\omega_1$, there is a $V_\xi\in(\Sigma\widehat{\otimes}\mathcal{B}(Y))\setminus\mathcal{L}$ such that $V_\xi^\bullet=e_\xi$, and a $W_\xi\in\mathcal{W}$ such that $V_\xi\triangle W_\xi\in\mathcal{L}$. Express W_ξ as $\bigcup_{n\in\mathbb{N}}E_{\xi n}\times H_{\xi n}$; as $W_\xi\notin\mathcal{L}$, there must be an n_ξ such that $E_\xi=E_{\xi,n_\xi}\notin\mathcal{N}(\mu)$ and $H_\xi=H_{\xi,n_\xi}$ is non-meager. Since the measure algebra of μ satisfies Knaster's condition (525Tb), there is an uncountable $A\subseteq\omega_1$ such that $E_\xi\cap E_\eta\notin\mathcal{N}(\mu)$ for all ξ, $\eta\in A$; because $\mathfrak{G}$ is ccc, there are distinct ξ, $\eta\in A$ such that $H_\xi\cap H_\eta$ is non-meager. But also

$$(E_\xi\cap E_\eta)\times(H_\xi\cap H_\eta)\subseteq W_\xi\cap W_\eta\in\mathcal{L}$$

because $(W_\xi\cap W_\eta)^\bullet=e_\xi\cap e_\eta=0$. So this is impossible. **XQ**

Thus $\mathfrak{C}$ is ccc. As it is Dedekind σ-complete (314C), it is Dedekind complete (316Fa).

(d) If either $\mu X=0$ or Y is meager, then $\mathfrak{A}\otimes\mathfrak{G}$ and $\mathfrak{C}$ are trivially isomorphic, and we can stop. Otherwise, the map $E\mapsto(E\times Y)^\bullet:\Sigma\to\mathfrak{C}$ is a Boolean homomorphism with kernel $\Sigma\cap\mathcal{N}(\mu)$, so induces a Boolean homomorphism

$\pi_1 : \mathfrak{A} \to \mathfrak{C}$. Similarly, we have a Boolean homomorphism $\pi_2 : \mathfrak{G} \to \mathfrak{C}$ defined by setting $\pi_2(F^\bullet) = (X \times F)^\bullet$ for $F \in \mathcal{B}(Y)$. These now give us a Boolean homomorphism $\phi : \mathfrak{A} \otimes \mathfrak{G} \to \mathfrak{C}$ defined by saying that

$$\psi(E^\bullet \otimes F^\bullet) = \pi_1(E^\bullet) \cap \pi_2(F^\bullet) = (E \times F)^\bullet$$

for $E \in \Sigma$ and $F \in \mathcal{B}(Y)$ (315Jb). If $E \in \Sigma \setminus \mathcal{N}(\mu)$ and $F \in \mathcal{B}(Y) \setminus \mathcal{M}(X)$, then $E \times F \notin \mathcal{L}$; so ϕ is injective (use 315Kb). If $c \in \mathfrak{C}$ is non-zero, it is expressible as $W^\bullet$ for some $W \in \mathcal{W} \setminus \mathcal{L}$; there must now be $E \in \Sigma$, $F \in \mathcal{B}(Y)$ such that $E \times F \subseteq W$ and $E \times F \notin \mathcal{L}$, so that $\phi(E^\bullet \otimes F^\bullet)$ is non-zero and included in w. Thus $\phi[\mathfrak{A} \otimes \mathfrak{G}]$ is isomorphic to $\mathfrak{A} \otimes \mathfrak{G}$ and is an order-dense subalgebra of the Dedekind complete Boolean algebra $\mathfrak{C}$; it follows that $\mathfrak{C}$ can be identified with the Dedekind completion of $\mathfrak{A} \otimes \mathfrak{G}$.

527X Basic exercises >(a) Show that there is a set belonging to $\mathcal{N} \ltimes \mathcal{N}$ which has full outer measure for Lebesgue measure in the plane. (*Hint*: enumerate the compact non-negligible subsets of the plane as $\langle K_\xi \rangle_{\xi < \mathfrak{c}}$ (4A3Fa); note that the projection L_ξ of K_ξ onto the first coordinate is always non-negligible, therefore uncountable, therefore of cardinal $\mathfrak{c}$ (423K); choose $s_\xi \in L_\xi \setminus \{s_\eta : \eta < \xi\}$ and $t_\xi \in K_\xi[\{s_\xi\}]$ for each ξ; consider $\{(s_\xi, t_\xi) : \xi < \mathfrak{c}\}$.)

>(b) Show that there is a unique construction of iterated skew products $\mathcal{I}_0 \ltimes \mathcal{I}_1 \ltimes \ldots \ltimes \mathcal{I}_n$ such that

(i) whenever $X_0, \ldots, X_n$ are sets and $\mathcal{I}_j$ is an ideal of subsets of X_j for every j, then $\mathcal{I}_0 \ltimes \ldots \ltimes \mathcal{I}_n$ is an ideal of subsets of $X_0 \times \ldots \times X_n$;

(ii) whenever $X_0, \ldots, X_n$ are sets, $\mathcal{I}_j$ is an ideal of subsets of X_j for every j, and $k < n$, then the natural identification of $X_0 \times \ldots \times X_n$ with $(X_0 \times \ldots \times X_k) \times (X_{k+1} \times \ldots \times X_n)$ identifies $\mathcal{I}_0 \ltimes \ldots \ltimes \mathcal{I}_n$ with $(\mathcal{I}_0 \ltimes \ldots \ltimes I_k) \ltimes (\mathcal{I}_{k+1} \ltimes \ldots \ltimes I_n)$ as defined in 527B.

(c) Complete the analysis in 527Bb by describing what happens if one of X, Y is empty or one of the ideals is not proper.

(d) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ an ideal of subsets of X; let Y be a topological space, $\mathcal{B}$ its Borel σ-algebra, $\widehat{\mathcal{B}}$ its Baire-property algebra, and $\mathcal{M}$ its meager ideal. Show that $\mathcal{I} \ltimes_{\Sigma \widehat{\otimes} \mathcal{B}} \mathcal{M} = \mathcal{I} \ltimes_{\Sigma \widehat{\otimes} \widehat{\mathcal{B}}} \mathcal{M}$.

>(e) Let Z be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, and $f : Z \to [0,1]$ the canonical inverse-measure-preserving continuous function (416V). Let $F \subseteq [0,1]$ be a nowhere dense set which is not negligible, and set $W = \{(x,z) : x \in [0,1],\ z \in Z,\ x + f(z) \in F\}$. Show that W is a nowhere dense closed set in $[0,1] \times Z$ but does not belong to $\mathcal{M}([0,1]) \ltimes \mathcal{M}(Z)$. (*Hint*: meager subsets of Z are negligible (321K).)

>(f) Suppose that I and J are sets, $X = \{0,1\}^I$ and $Y = \{0,1\}^J$. Show that $\mathcal{M}(X) \ltimes_{\mathcal{B}(X \times Y)} \mathcal{M}(Y) = \mathcal{M}(X \times Y)$.

(g) Write $\mathfrak{X}$ for the class of topological spaces which have category algebras which are atomless and with countable π-weight. (i) Show that $\mathbb{R}$, with the right-facing Sorgenfrey topology, belongs to $\mathfrak{X}$. (ii) Show that the split interval (419L) belongs to $\mathfrak{X}$. (iii) Show that if the regular open algebra of a topological space X is atomless and has countable π-weight, then $X \in \mathfrak{X}$. (iv) Show that any open subspace of a space in $\mathfrak{X}$ belongs to $\mathfrak{X}$. (v) Show that any dense subspace of a space in $\mathfrak{X}$ belongs to $\mathfrak{X}$. (vi) Show that any comeager subspace of a space in $\mathfrak{X}$ belongs to $\mathfrak{X}$. (vii) Show that the product of countably many spaces in $\mathfrak{X}$ belongs to $\mathfrak{X}$.

(h) Let P be a partially ordered set. Show that if $\kappa \ge \operatorname{cf} P$ and $\lambda \le \operatorname{add} P$ then $P \preccurlyeq_{\mathrm{T}} [\kappa]^{<\lambda}$.

(i) Let X be a topological space with a σ-finite measure μ such that μ has countable Maharam type and every measurable set can be expressed as the symmetric difference of a Borel set and a negligible set. Let Y be a topological space with a countable π-base. Show that $\mathcal{N}(\mu) \ltimes_{\mathcal{B}(X \times Y)} \mathcal{M}(Y) \preccurlyeq_{\mathrm{T}} \mathcal{N}(\mu) \times \mathcal{M}$.

527Y Further exercises (a) Show that $\mathcal{I} \ltimes \mathcal{J} \ne \mathcal{I} \rtimes \mathcal{J}$ for any of the four cases in which $\{\mathcal{I}, \mathcal{J}\} \subseteq \{\mathcal{M}, \mathcal{N}\}$.

(b) Let X be a set, Σ a σ-algebra of subsets of X and $\mathcal{I}$ a σ-ideal of Σ such that $\Sigma/\mathcal{I}$ is ccc. Let (Y, T, ν) be a probability space. Show that $(\Sigma \widehat{\otimes} \mathrm{T})/(\Sigma \widehat{\otimes} \mathrm{T}) \cap (\mathcal{I} \ltimes \mathcal{N}(\nu))$ is ccc. (*Hint*: show that if $V \in \Sigma \widehat{\otimes} \mathrm{T}$ then $x \mapsto \nu V[\{x\}]$ is Σ-measurable, and hence that there is no uncountable disjoint family in $(\Sigma \widehat{\otimes} \mathrm{T}) \setminus (\mathcal{I} \ltimes \mathcal{N}(\nu))$.)

(c) Let $(Y, \mathfrak{T})$ be a topological space. Show that there is a topology $\mathfrak{S}$ on Y, coarser than $\mathfrak{T}$, such that the weight of $(Y, \mathfrak{S})$ is equal to the π-weight of $(Y, \mathfrak{T})$, and the two topologies have the same nowhere dense sets, the same meager ideal and the same Baire-property algebras.

(d) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of harmless Boolean algebras all satisfying Knaster's condition, and $\mathfrak{A}$ their free product (315I). Show that $\mathfrak{A}$ is harmless.

(e) Let μ be a σ-finite Borel probability measure on a topological space X, and Y a topological space such that its category algebra is harmless. Show that $\mathcal{B}(X\times Y)/\mathcal{B}(X\times Y)\cap(\mathcal{N}(\mu)\ltimes\mathcal{M}(Y))$ can be identified with the Dedekind completion of $\mathfrak{A}\otimes\mathfrak{G}$, where $\mathfrak{A}$ is the measure algebra of μ and $\mathfrak{G}$ is the category algebra of Y.

527 Notes and comments Skew products of ideals have been used many times for special purposes, and we are approaching the point at which it would be worth developing a general theory of such products. I am not really attempting to do this here, though the language of 527B is supposed to point to the right questions. My primary aim in this section is to show that $\mathcal{M}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{N}$ and $\mathcal{N}\ltimes_{\mathcal{B}(\mathbb{R}^2)}\mathcal{M}$ are very different (527H, 527K). Of course the difference appears only when the continuum hypothesis is false (513Xf, 527Xh).

The version of the Kuratowski-Ulam theorem given in 527D is a natural one from the point of view of this chapter, but you should be aware that there are many more cases in which $\mathcal{M}^*=\mathcal{M}(X\times Y)$; see 527Xf and Fremlin Natkaniec & Recław 00. The statement of 527J includes the phrase 'quasi-Radon measure'. Actually we do not really need either τ-additivity or inner regularity with respect to closed sets. What we need is a measure μ such that $\mathcal{N}(\mu)\preccurlyeq_{\mathrm{T}}\mathcal{N}$ and the Borel sets generate the measure algebra (527Xi). The argument for 527J betrays its origin in the case $X=Y=[0,1]$, which is of course also the natural home of 527C-527F. Some of the complications of the argument are due to its being written out for spaces of countable π-weight; an alternative approach would start with a reduction to the case in which Y is second-countable (527Yc).

It is interesting that all four of the quotient algebras

$$\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2)\cap(\mathcal{M}\ltimes\mathcal{M}),\quad \mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2)\cap(\mathcal{M}\ltimes\mathcal{N}),$$

$$\mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2)\cap(\mathcal{N}\ltimes\mathcal{M}),\quad \mathcal{B}(\mathbb{R}^2)/\mathcal{B}(\mathbb{R}^2)\cap(\mathcal{N}\ltimes\mathcal{N})$$

are ccc (see 527E, 527Yb, 527O, 527Bc and also 527L). This should not be taken for granted; for a variety of examples of quotient algebras associated with σ-ideals see Fremlin 03.

528 Amoeba algebras

In the course of investigating the principal consequences of Martin's axiom, Martin & Solovay 70 introduced the partially ordered set of open subsets of $\mathbb{R}$ with measure strictly less than γ, for $\gamma>0$ (528O). Elementary extensions of this idea lead us to a very interesting class of partially ordered sets, which I study here in terms of their regular open algebras, the 'amoeba algebras' (528A). Of course the most important ones are those associated with Lebesgue measure, and these are closely related to 'localization posets' (528I), themselves intimately connected with the localization relations of 522K. In the second half of the section I look at the cardinal functions of these algebras, of which the most interesting seems to be Maharam type (528V).

As elsewhere in this chapter, I will write $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$. In any measure algebra $(\mathfrak{A},\bar\mu)$ I will write $\mathfrak{A}^f=\{a:a\in\mathfrak{A},\ \bar\mu a<\infty\}$.

528A Amoeba algebras Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) If $0<\gamma\le\bar\mu 1$, the **amoeba algebra** $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ is the regular open algebra $\mathrm{RO}^\uparrow(P)$ where $P=\{a:a\in\mathfrak{A},\ \bar\mu a<\gamma\}$, ordered by $\subseteq$.

(b) The **variable-measure amoeba algebra** $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$ (Truss 88) is the regular open algebra $\mathrm{RO}^\uparrow(P')$ where

$$P'=\{(a,\alpha):a\in\mathfrak{A},\ \alpha\in\,]\bar\mu a,\bar\mu 1]\},$$

ordered by saying that

$$(a,\alpha)\le(b,\beta)\text{ if }a\subseteq b\text{ and }\beta\le\alpha.$$

528B It may help to have the following simple facts set out straight away.

Lemma Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and $0<\gamma\le\bar{\mu}1$. Set $P=\{a:a\in\mathfrak{A},\ \bar{\mu}a<\gamma\}$.

(a) Two elements a, $b\in P$ are compatible upwards in P iff $\bar{\mu}(a\cup b)<\gamma$.

(b) Suppose that $(\mathfrak{A},\bar{\mu})$ is semi-finite and atomless.

(i) P is separative upwards, so $[a,\infty[\,\in\mathrm{RO}^{\uparrow}(P)$ for every $a\in P$.

(ii) If $A\subseteq P$ is non-empty, then the infimum $\inf_{a\in A}[a,\infty[$ is empty unless $\sup A$ is defined in $\mathfrak{A}$ and belongs to P, and in this case $\inf_{a\in A}[a,\infty[\,=[\sup A,\infty[$.

proof (a) $[a,\infty[\,\cap\,[b,\infty[\,=\{c:a\cup b\subseteq c\in P\}$ is non-empty iff $a\cup b\in P$.

(b)(i) Let a, $b\in P$ be such that $a\not\subseteq b$. If $\bar{\mu}(a\cup b)\ge\gamma$ then a and b are already incompatible upwards. Otherwise, $\bar{\mu}(1\setminus(a\cup b))\ge\gamma-\bar{\mu}(a\cup b)$. Because $(\mathfrak{A},\bar{\mu})$ is atomless and semi-finite, there is a $d\subseteq 1\setminus(a\cup b)$ such that $\bar{\mu}d=\gamma-\bar{\mu}(a\cup b)$. Set $c=b\cup d$. Then

$$\bar{\mu}c=\gamma-\bar{\mu}(a\setminus b)<\gamma=\bar{\mu}(a\cup c),$$

so $c\in[b,\infty[\,\subseteq P$, while a and c are incompatible upwards in P. As a and b are arbitrary, P is separative upwards.

By 514Me, it follows that $[a,\infty[$ is a regular up-open set for every $a\in P$.

(ii) This is a re-phrasing of 514Mf.

528C Proposition Suppose that (X,Σ,μ) is a measure space, $(\mathfrak{A},\bar{\mu})$ its measure algebra and $0<\gamma\le\mu X$. If $\mathcal{E}\subseteq\Sigma$ is any family such that μ is outer regular with respect to $\mathcal{E}$, then $\mathrm{AM}(\mathfrak{A},\bar{\mu},\gamma)$ is isomorphic to $\mathrm{RO}^{\uparrow}(\{E:E\in\mathcal{E},\ \mu E<\gamma\})$.

proof Set $P=\{a:a\in\mathfrak{A},\ \bar{\mu}a<\gamma\}$, $Q=\{E:E\in\mathcal{E},\ \mu E<\gamma\}$. Because μ is outer regular with respect to $\mathcal{E}$, the map $G\mapsto G^{\bullet}:Q\to\mathfrak{A}$ maps Q onto a cofinal subset P' of P. Moreover, two elements E_0 and E_1 of Q are compatible upwards in Q iff $\mu(E_0\cup E_1)<\gamma$ iff $E_0^{\bullet}$ and $E_1^{\bullet}$ are compatible upwards in P. By 514R, $\mathrm{RO}^{\uparrow}(P)$ and $\mathrm{RO}^{\uparrow}(Q)$ are isomorphic.

528D Proposition (a) (Truss 88) Let $(\mathfrak{A},\bar{\mu})$ be an atomless homogeneous probability algebra. Then the amoeba algebras $\mathrm{AM}(\mathfrak{A},\bar{\mu},\gamma)$ and $\mathrm{AM}(\mathfrak{A},\bar{\mu},\gamma')$ are isomorphic for all γ, $\gamma'\in\,]0,1[$.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a non-totally-finite atomless quasi-homogeneous measure algebra (definition: 374G). Then all the amoeba algebras $\mathrm{AM}(\mathfrak{A},\bar{\mu},\gamma)$, for $\gamma>0$, are isomorphic.

proof (a)(i) Set $P=\{a:a\in\mathfrak{A},\ \bar{\mu}a<\gamma\}$, and let κ be the Maharam type of $\mathfrak{A}$. Then the upwards cellularity of P is at most κ. **P?** Otherwise, there is an up-antichain $A\subseteq P$ with cardinality κ^+. Let $\epsilon>0$ be such that $A'=\{a:a\in A,\ \bar{\mu}a\le\gamma-\epsilon\}$ has cardinal κ^+. Because the topological density of $\mathfrak{A}$ is κ (521Ea), there must be distinct a, $a'\in A'$ such that $\bar{\mu}(a\,\triangle\,a')<\epsilon$; but in this case $\bar{\mu}(a\cup a')<\gamma$, so that $a\cup a'$ is an upper bound for $\{a,a'\}$ in P. **XQ**

(ii) If $1-\sqrt{1-\gamma}\le\alpha<\gamma$ and D is a countable subset of $]\alpha,\gamma[$ such that $\sup D=\gamma$, then there is a maximal up-antichain $\langle a_{t\xi}\rangle_{(t,\xi)\in D\times\kappa}$ in P such that $\bar{\mu}a_{t\xi}=t$ for every $t\in D$, $\xi<\kappa$. **P** Start with a stochastically independent family $\langle c_{t\xi}\rangle_{(t,\xi)\in D\times\kappa}$ of elements of $\mathfrak{A}$ with $\bar{\mu}c_{t\xi}=t$ for all $t\in D$, $\xi<\kappa$. Because $\alpha\ge1-\sqrt{1-\gamma}$, $A=\langle c_{t\xi}\rangle_{(t,\xi)\in D\times\kappa}$ is an up-antichain in P. Next, because $\sup D=\gamma$, $Q=\{a:a\in P,\ \bar{\mu}a\in D\}$ is cofinal with P. So there is a maximal up-antichain $A'\supseteq A$ such that $A'\subseteq Q$ (513Aa). Now (because $c^{\uparrow}(P)\le\kappa$) $\{a:a\in A',\ \bar{\mu}a=t\}$ has cardinal κ for every $t\in D$, so we can enumerate A' as $\langle a_{t\kappa}\rangle_{(t,\xi)\in D\times\kappa}$ in P where $\bar{\mu}a_{t\xi}=t$ for every $t\in D$ and $\xi<\kappa$. **Q**

(iii) There are α, $\alpha'\in\,]0,1[$ such that

$$1-\sqrt{1-\gamma}\le\alpha<\gamma,\quad 1-\sqrt{1-\gamma'}\le\alpha'<\gamma',\quad \tfrac{\gamma-\alpha}{1-\alpha}=\tfrac{\gamma'-\alpha'}{1-\alpha'}.$$

P We need consider only the case $\gamma\le\gamma'$. Set

$$\beta=\frac{1}{\sqrt{1-\gamma}}-1,\quad \alpha=\gamma-\beta(1-\gamma),\quad \alpha'=\gamma'-\beta(1-\gamma').$$

Then $\frac{\gamma-\alpha}{1-\gamma}=\beta=\frac{\gamma'-\alpha'}{1-\gamma'}$. Of course $\alpha\le\gamma$ and $\alpha'\le\gamma'$. On the other side, $\alpha=1-\sqrt{1-\gamma}$, while $\beta\le\frac{1}{\sqrt{1-\gamma'}}-1$ so $\alpha'\ge1-\sqrt{1-\gamma'}$. **Q**

(iv) If $a\in P$, $\{b:a\subseteq b\in P\}$ is isomorphic, as partially ordered set, to $\{b:b\in\mathfrak{A},\ \bar{\mu}b<\frac{\gamma-\bar{\mu}a}{1-\bar{\mu}a}\}$. **P** The principal ideal $\mathfrak{A}_{1\setminus a}$ generated by $1\setminus a$ is isomorphic, up to a scalar multiple of the measure, to $\mathfrak{A}$, and $\{b:a\subseteq b\in P\}$ is isomorphic, as partially ordered set, to $\{b:b\subseteq1\setminus a,\ \bar{\mu}b<\gamma-\bar{\mu}a\}$. **Q**

(v) For each $n\in\mathbb{N}$, set $\alpha_n=\gamma-2^{-n}(\gamma-\alpha)$, $\alpha'_n=\gamma'-2^{-n}(\gamma'-\alpha')$; then

$$\frac{1-\alpha'_n}{\gamma'-\alpha'_n}=1+\frac{1-\gamma'}{\gamma'-\alpha'_n}=1+2^n\frac{1-\gamma'}{\gamma'-\alpha'}=\frac{1-\alpha_n}{\gamma-\alpha_n},\quad \frac{\gamma'-\alpha'_n}{1-\alpha'_n}=\frac{\gamma-\alpha_n}{1-\alpha_n}$$

for every $n\in\mathbb{N}$. Set $P'=\{a:a\in\mathfrak{A},\ \bar\mu a<\gamma'\}$. By (b), we have a maximal up-antichain $\langle a_{n\xi}\rangle_{(n,\xi)\in\mathbb{N}\times\xi}$ in P such that $\bar\mu a_{n\xi}=\alpha_n$ for all $n\in\mathbb{N}$ and $\xi<\kappa$; similarly, there is a maximal up-antichain $\langle a'_{n\xi}\rangle_{(n,\xi)\in\mathbb{N}\times\xi}$ in P' such that $\bar\mu a'_{n\xi}=\alpha'_n$ for all $n\in\mathbb{N}$ and $\xi<\kappa$. Now, for each $n\in\mathbb{N}$ and $\xi<\kappa$, $[a_{n\xi},\infty[$, taken in P, is isomorphic, as partially ordered set, to $[a'_{n\xi},\infty[$, taken in P', by (d). So

$$\mathrm{RO}^{\uparrow}(P)\cong\prod_{n\in\mathbb{N},\xi<\kappa}\mathrm{RO}^{\uparrow}([a_{n\xi},\infty[)$$

(514Nf)

$$\cong\prod_{n\in\mathbb{N},\xi<\kappa}\mathrm{RO}^{\uparrow}([a'_{n\xi},\infty[)\cong\mathrm{RO}^{\uparrow}(P').$$

(b) Suppose that β, $\gamma>0$. As in Lemma 332I, we have a partition D of unity in $\mathfrak{A}$ such that $\bar\mu a=\beta$ for every $a\in D$. Similarly, we have a partition D' of unity such that $\bar\mu a=\gamma$ for every $a\in D'$. By 332E, $\#(D)=\#(D')=c(\mathfrak{A})$. Let $h:D\to D'$ be a bijection. If $d\in D$, the principal ideals $\mathfrak{A}_d$, $\mathfrak{A}_{h(d)}$ have the same Maharam type, because $(\mathfrak{A},\bar\mu)$ is quasi-homogeneous (374H), and are therefore isomorphic as measure algebras, up to a scalar factor of the measure; let $\pi_d:\mathfrak{A}_d\to\mathfrak{A}_{h(d)}$ be a Boolean isomorphism such that $\bar\mu(\pi_d a)=\frac{\gamma}{\beta}\bar\mu a$ for every $a\subseteq d$. Now we have a function $\pi:\mathfrak{A}^f\to\mathfrak{A}^f$ defined by saying that $\pi a=\sup_{d\in D}\pi_d(a\cap d)$ whenever $\bar\mu a<\infty$, and π is a Boolean ring automorphism such that $\bar\mu\pi a=\frac{\gamma}{\beta}\bar\mu a$ for every $a\in\mathfrak{A}^f$. But now π includes an isomorphism between the partially ordered sets $\{a:\bar\mu a<\beta\}$ and $\{a:\bar\mu a<\gamma\}$, so their regular open algebras $\mathrm{AM}(\mathfrak{A},\bar\mu,\beta)$ and $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ are isomorphic.

528E Lemma Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra. Then there is a family $\langle c_\alpha\rangle_{\alpha\in[0,\bar\mu1]}$ in $\mathfrak{A}$ such that $c_\alpha\subseteq c_\beta$ and $\bar\mu c_\alpha=\alpha$ whenever $0\le\alpha\le\beta\le\bar\mu1$, and $\alpha\mapsto c_\alpha$ is continuous for the measure-algebra topology of $\mathfrak{A}$.

proof Because $(\mathfrak{A},\bar\mu)$ is semi-finite, there is a non-decreasing sequence $\langle e_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}^f$ such that $\sup_{n\in\mathbb{N}}\bar\mu e_n=\bar\mu1$, starting from $e_0=0$; set $e=\sup_{n\in\mathbb{N}}e_n$, so that $\bar\mu e=\bar\mu1$. Then $(\mathfrak{A}_e,\bar\mu{\restriction}\mathfrak{A}_e)$ is σ-finite and atomless. Let $(\mathfrak{C},\bar\lambda)$ be the measure algebra of Lebesgue measure on $[0,\bar\mu1[$. For each $n\in\mathbb{N}$ set $e'_n=e_{n+1}\setminus e_n$ and $d_n=[\bar\mu e_n,\bar\mu e_{n+1}[^\bullet\in\mathfrak{C}$.

Because $\mathfrak{A}$ is atomless, 332P tells us that there is for each $n\in\mathbb{N}$ a measure-preserving Boolean homomorphism π_n from the principal ideal $\mathfrak{C}_{d_n}$ to a principal ideal of $\mathfrak{A}_{e'_n}$, which must be $\mathfrak{A}_{e'_n}$ itself because $\bar\mu e'_n=\bar\lambda d_n$; by 324Kb, π_n is order-continuous. Assembling these, we have an order-continuous measure-preserving Boolean homomorphism $\pi:\mathfrak{C}\to\mathfrak{A}_e$ defined by setting $\pi d=\sup_{n\in\mathbb{N}}\pi_n(d\cap d_n)$ for every $d\in\mathfrak{C}$. Now set $c_\alpha=\pi\,[0,\alpha[^\bullet$ for $\alpha\le\bar\mu1$. Because π is continuous for the measure-algebra topologies of $\mathfrak{C}$ and $\mathfrak{A}_e$ (324Fc), or otherwise, $\alpha\mapsto c_\alpha$ is continuous.

528F Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and $\gamma\in\,]0,\infty[$.

(a) Suppose that $e\in\mathfrak{A}$ and $\bar\mu e\ge\gamma$. If $\mathfrak{A}_e$ is atomless, then $\mathrm{AM}(\mathfrak{A}_e,\bar\mu{\restriction}\mathfrak{A}_e,\gamma)$ can be regularly embedded in $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$.

(b) Suppose that $\mathfrak{A}$ is atomless, and that $\gamma<\bar\mu1$. Let $\langle e_k\rangle_{k\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}$ with supremum 1, and suppose that $\bar\mu e_k\ge\gamma$ for every $k\in\mathbb{N}$. Then we have a sequence $\langle\pi_k\rangle_{k\in\mathbb{N}}$ such that $\pi_k:\mathrm{AM}(\mathfrak{A}_{e_k},\bar\mu{\restriction}\mathfrak{A}_{e_k},\gamma)\to\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ is a regular embedding for every $k\in\mathbb{N}$, and $\bigcup_{k\in\mathbb{N}}\pi_k[\mathrm{AM}(\mathfrak{A}_{e_k},\bar\mu{\restriction}\mathfrak{A}_{e_k},\gamma)]$ τ-generates $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$.

(c) Now suppose that $(\mathfrak{A},\bar\mu)$ is atomless and quasi-homogeneous, and that $\gamma<\bar\mu1$. Then $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$.

proof (a) Set $P=\{a:a\in\mathfrak{A},\ \bar\mu a<\gamma\}$ and $Q=P\cap\mathfrak{A}_e$. By 528E, we have a continuous order-preserving function $\alpha\mapsto c_\alpha:[0,\bar\mu e]\to\mathfrak{A}_e$ such that $\bar\mu c_\alpha=\alpha$ for each α. If $a\in\mathfrak{A}$, then the function $\beta\mapsto\bar\mu(c_\beta\setminus a)$ is a continuous non-decreasing function from $[0,\bar\mu e]$ onto $[0,\bar\mu(c_{\bar\mu e}\setminus a)]$, and we can set $\delta(a,\alpha)=\min\{\beta:\bar\mu(c_\beta\setminus a)=\alpha\}$ whenever $0\le\alpha\le\bar\mu(c_{\bar\mu e}\setminus a)$. In this case,

$$\bar\mu((a\cap e)\cup c_{\delta(a,\alpha)})=\bar\mu(a\cap e)+\bar\mu(c_{\delta(a,\alpha)}\setminus a)=\alpha+\bar\mu(a\cap e).$$

Note that $\delta(a,\alpha)\le\delta(a',\alpha')$ whenever $a\subseteq a'$ and $\alpha\le\alpha'\le\bar\mu(c_{\bar\mu1}\setminus a')$.

If $a \in P$, then

$$\bar\mu(c_{\bar\mu e} \setminus a) = \bar\mu c_{\bar\mu e} - \bar\mu(a \cap c_{\bar\mu e}) \ge \bar\mu e - \bar\mu(a \cap e) \ge \bar\mu a - \bar\mu(a \cap e) = \bar\mu(a \setminus e).$$

So $\delta(a, \bar\mu(a \setminus e))$ is defined, and we have a function f given by the formula

$$f(a) = (a \cap e) \cup c_{\delta(a,\bar\mu(a\setminus e))}$$

for $a \in P$. In this case $\bar\mu f(a) = \bar\mu a$, so $f(a) \in Q$, for each a, and f, like δ, is order-preserving. Of course $f(a) = a$ for $a \in Q$.

If $a \in P$, $b \in Q$ and $f(a) \subseteq b$, there is an $a' \in P$ such that $a \subseteq a'$ and $b = f(a')$. **P** Set $a' = a \cup (b \setminus f(a))$. Then

$$\bar\mu a' = \bar\mu a + \bar\mu(b \setminus f(a)) = \bar\mu f(a) + \bar\mu(b \setminus f(a)) = \bar\mu b < \gamma,$$

so $a' \in P$. Also $b \subseteq f(a) \cup (a' \cap e) \subseteq f(a')$; as $\bar\mu b = \bar\mu a' = \bar\mu f(a')$, $b = f(a')$. **Q** So if $Q_0 \subseteq Q$ is cofinal with Q, $f^{-1}[Q_0]$ will be cofinal with P (as in the proof of 514P), and we have an order-continuous Boolean homomorphism $\pi : \mathrm{RO}^\uparrow(Q) \to \mathrm{RO}^\uparrow(P)$ defined by setting $\pi H = \operatorname{int}\overline{f^{-1}[H]}$ for every $H \in \mathrm{RO}^\uparrow(Q)$. Finally, $f[P] = f[Q] = Q$. So π is injective and is a regular embedding of $\mathrm{AM}(\mathfrak{A}_e, \bar\mu\restriction\mathfrak{A}_e, \gamma) = \mathrm{RO}^\uparrow(Q)$ into $\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma) = \mathrm{RO}^\uparrow(P)$.

(b)(i) For each $k \in \mathbb{N}$, set $Q_k = P \cap \mathfrak{A}_{e_k}$ and choose functions $f_k : P \to Q_k$ and $\pi_k : \mathrm{RO}^\uparrow(Q_k) \to \mathrm{RO}^\uparrow(P)$ as in (a) above. If we write $[c, \infty[= \{a : c \subseteq a \in P\}$ for every $c \in P$, then $\mathfrak{A}_{e_k} \cap [c, \infty[= \{b : c \subseteq b \in Q_k\}$ for $k \in \mathbb{N}$ and $c \in Q_k$; in this case, $\mathfrak{A}_{e_k} \cap [c, \infty[\in \mathrm{RO}^\uparrow(Q_k)$, by 528B(b-i).

(ii) Let $\mathfrak{G}$ be the order-closed subalgebra of $\mathrm{RO}^\uparrow(P)$ generated by $\bigcup_{k\in\mathbb{N}} \pi_k[\mathrm{RO}^\uparrow(Q_k)]$. If $a \in P$, there is a non-empty $G \in \mathfrak{G}$ included in $[a, \infty[\in \mathrm{RO}^\uparrow(P)$. **P** Because $a \subseteq \sup_{k\in\mathbb{N}} e_k$ and $\langle e_k\rangle_{k\in\mathbb{N}}$ is non-decreasing, there is an infinite $I \subseteq \mathbb{N}$ such that $\sum_{k\in I} \bar\mu(a \setminus e_k) < \gamma - \bar\mu a$. Set $b = \sup_{k\in I} f_k(a)$. Then

$$\bar\mu b \le \bar\mu a + \textstyle\sum_{k\in I} \bar\mu(a \setminus f_k(a)) \le \bar\mu a + \sum_{k\in I} \bar\mu(a \setminus e_k) < \gamma$$

because $f_k(a) \supseteq a \cap e_k$ for every k, by the construction in (a). Thus $b \in P$. Also

$$\bar\mu(a \setminus b) \le \inf_{k\in\mathbb{N}} \bar\mu(a \setminus f_k(a)) = 0,$$

so $a \subseteq b$.

Set

$$V_k = \mathfrak{A}_{e_k} \cap [f_k(a), \infty[\in \mathrm{RO}^\uparrow(Q_k)$$

for every k. Then $\pi_k V_k = \operatorname{int}\overline{f_k^{-1}[V_k]}$ belongs to $\mathfrak{G}$ for each k, and $G = \inf_{k\in\mathbb{N}} \pi_k V_k = \bigcap_{k\in\mathbb{N}} \pi_k V_k$ (514M(d-ii)) belongs to $\mathfrak{G}$. Because every f_k is order-preserving, $f_k(b') \supseteq f_k(a)$ and $f_k(b') \in V_k$ for every $b' \supseteq b$; thus $b \in \operatorname{int} f_k^{-1}[V_k]$ for every k, and $b \in G$. This shows that $G \ne \emptyset$.

? Suppose, if possible, that $G \not\subseteq [a, \infty[$. Then there is a $c \in G$ such that $a \setminus c \ne 0$. If $\bar\mu(c \cup a) > \gamma$, set $c' = c$. Otherwise, let $\delta > 0$ be such that

$$\delta + \bar\mu c < \gamma < \delta + \bar\mu(c \cup a) < \bar\mu 1.$$

Because $(\mathfrak{A}, \bar\mu)$ is atomless and semi-finite, there is a $d \subseteq 1 \setminus (c \cup a)$ such that $\bar\mu d = \delta$. Set $c' = c \cup d$; then $c \subseteq c' \in P$ so $c' \in G$, while $\bar\mu(c' \cup a) > \gamma$, as in the previous case.

Because I is infinite, $\sup_{k\in I} e_k = 1$ and there is a $k \in I$ such that $\bar\mu((c' \cup a) \cap e_k) \ge \gamma$. In this case, $c' \in \pi_k V_k \subseteq \overline{f_k^{-1}[V_k]}$, so $[c', \infty[$ meets $f_k^{-1}[V_k]$ and there is a $c'' \supseteq c'$ such that $c'' \in P$ and $f_k(c'') \in V_k$, that is, $f_k(c'') \supseteq f_k(a)$. Now, however,

$$f_k(c'') \supseteq (c' \cap e_k) \cup (f_k(a) \cap e_k) \supseteq (c' \cup a) \cap e_k$$

has measure at least γ, and cannot belong to Q_k. **XQ**

(iii) Since $\{[a, \infty[: a \in P\}$ is a base for the topology of P, it is a π-base for $\mathrm{RO}^\uparrow(P)$, and $\mathfrak{G}$ includes a π-base for $\mathrm{RO}^\uparrow(P)$. But this means that every member of $\mathrm{RO}^\uparrow(P)$ is the supremum of the members of $\mathfrak{G}$ it includes, and belongs to $\mathfrak{G}$. Thus $\mathfrak{G} = \mathrm{RO}^\uparrow(P)$, as claimed.

(c)(i) This time, let $\langle c_\alpha\rangle_{\alpha\in[0,\bar\mu 1]}$ be a family in $\mathfrak{A}$ such that $c_\alpha \subseteq c_\beta$ and $\bar\mu c_\alpha = \alpha$ whenever $0 \le \alpha \le \beta \le \bar\mu 1$. Set $P = \{(a, \alpha) : a \in \mathfrak{A},\ \alpha \in\,]\bar\mu a, \bar\mu 1]\}$. Let $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence with supremum $\bar\mu 1$ and $\gamma_0 = 0$. For each $n \in \mathbb{N}$, set $P_n = \{(a, \alpha) : \gamma_n \le \bar\mu a < \alpha \le \gamma_{n+1}\}$ and $Q_n = \{a : a \in \mathfrak{A},\ \bar\mu a < \gamma_{n+1}\}$, so that P_n is an up-open set in P. Note that $\bigcup_{n\in\mathbb{N}} P_n$ is dense in P for the up-topology, since if $(a, \alpha) \in P$ then $(a, \min(\alpha, \gamma_{n+1})) \in P_n$ where $\gamma_n \le \bar\mu a < \gamma_{n+1}$. Also

$$\mathrm{RO}^\uparrow(Q_n) = \mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma_{n+1}) \cong \mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma).$$

P If $\bar{\mu}1 = \infty$, this is 528Db. If $(\mathfrak{A}, \bar{\mu})$ is totally finite, then $\mathfrak{A}$ is homogeneous, so we can apply 528Da to an appropriate multiple of the measure $\bar{\mu}$. **Q**

For $a \in \mathfrak{A}^f$, the function $\alpha \mapsto \bar{\mu}(c_\alpha \setminus a) : [0, \bar{\mu}1] \to [0, \infty]$ is continuous and non-decreasing, and

$$\bar{\mu}(c_{\bar{\mu}1} \setminus a) \ge \bar{\mu}c_{\bar{\mu}1} - \bar{\mu}a = \bar{\mu}1 - \bar{\mu}a = \bar{\mu}(1 \setminus a).$$

So we can define $\delta(a, \alpha)$, for $a \in \mathfrak{A}^f$ and $0 \le \alpha \le \bar{\mu}(1 \setminus a)$, by saying that

$$\delta(a, \alpha) = \min\{\beta : \bar{\mu}(c_\beta \setminus a) = \alpha\} = \min\{\beta : \bar{\mu}(a \cup c_\beta) = \bar{\mu}a + \alpha\}.$$

As in (a), $\delta(a, \alpha) \le \delta(a', \alpha')$ whenever $a \subseteq a'$ and $\alpha \le \alpha'$. For $(a, \alpha) \in P_n$, set

$$f_n(a, \alpha) = a \cup c_{\delta(a, \gamma_{n+1} - \alpha)},$$

so that $\bar{\mu}f_n(a, \alpha) = \bar{\mu}a + \gamma_{n+1} - \alpha < \gamma_{n+1}$ and $f_n(a, \alpha) \in Q_n$. Of course $f_n(a, \gamma_{n+1}) = a$ if $(a, \gamma_{n+1}) \in P_n$, that is, if $a \in Q_n$ and $\bar{\mu}a \ge \gamma_n$.

(ii)($\boldsymbol{\alpha}$) $f_n : P_n \to Q_n$ is order-preserving. **P** If $(a, \alpha) \le (a', \alpha')$ in P_n, then $\delta(a, \gamma_{n+1} - \alpha) \le \delta(a', \gamma_{n+1} - \alpha')$, so $f_n(a, \alpha) \subseteq f_n(a', \alpha')$. **Q**

($\boldsymbol{\beta}$) If $p \in P_n$, $b \in Q_n$ and $f_n(p) \subseteq b$, there is a $p' \in P_n$ such that $p \le p'$ and $b \subseteq f_n(p')$. **P** Express p as (a, α). Consider $a' = a \cup (b \setminus f_n(p))$. Then

$$\bar{\mu}a' = \bar{\mu}a + \bar{\mu}b - \bar{\mu}f_n(p) = \bar{\mu}a + \bar{\mu}b - \bar{\mu}a - \gamma_{n+1} + \alpha < \alpha,$$

so $(a', \alpha) \in P$. Of course $(a, \alpha) \le (a', \alpha)$, so $p' = (a', \alpha) \in P_n$. Also $f_n(p') \supseteq f_n(p)$ and

$$f_n(p') \supseteq a' \supseteq b \setminus f_n(p),$$

so $b \subseteq f_n(p')$. **Q**

($\boldsymbol{\gamma}$) $f_n[P_n]$ is cofinal with Q_n. **P** If $b \in Q_n$, take $b' \in \mathfrak{A}$ such that $b \subseteq b'$ and $\gamma_n \le \bar{\mu}b' < \gamma_{n+1}$. Then $(b', \gamma_{n+1}) \in P_n$ and

$$b \subseteq b' = f_n(b', \gamma_{n+1}) \in f_n[P_n]. \textbf{ Q}$$

(iii) By 514P, $\mathrm{RO}^\uparrow(Q_n)$ can be regularly embedded in $\mathrm{RO}^\uparrow(P_n)$. Now $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ is isomorphic to $\mathrm{RO}^\uparrow(Q_n)$, so there is an injective order-continuous Boolean homomorphism $\pi_n : \mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma) \to \mathrm{RO}^\uparrow(P_n)$. Putting these together, we have an injective order-continuous Boolean homomorphism $\pi : \mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma) \to \prod_{n\in\mathbb{N}} \mathrm{RO}^\uparrow(P_n)$ defined by setting $\pi u = \langle \pi_n(u) \rangle_{n\in\mathbb{N}}$ for $u \in \mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$. On the other hand, since $\langle P_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence of up-open subsets of P with dense union,

$$\prod_{n\in\mathbb{N}} \mathrm{RO}^\uparrow(P_n) \cong \mathrm{RO}^\uparrow(P) = \mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$$

by 315H. So we have a regular embedding of $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ into $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$, as claimed.

528G Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $\mathfrak{C}$ a σ-subalgebra of $\mathfrak{A}$ such that $\sup(\mathfrak{C} \cap \mathfrak{A}^f) = 1$ in $\mathfrak{A}$. Then $\mathrm{AM}^*(\mathfrak{C}, \bar{\mu}{\restriction}\mathfrak{C})$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$.

proof (a) For each $a \in \mathfrak{A}^f$ we have a 'conditional expectation' $u_a \in L^1(\mathfrak{C})$ defined by saying that $\int_c u_a = \bar{\mu}(a \cap c)$ for every $c \in \mathfrak{C}^f$. (Apply 365O[1] to the identity map from $\mathfrak{C}^f$ to $\mathfrak{A}^f$.) Note that as the supremum of $\mathfrak{C}^f$ in $\mathfrak{A}$ is 1,

$$\int u_a = \sup_{c\in\mathfrak{C}^f} \int_c u_a = \sup_{c\in\mathfrak{C}^f} \bar{\mu}(a \cap c) = \bar{\mu}a.$$

Also, of course, $0 \le \bar{\mu}(a \cap c) \le \bar{\mu}c$ for every $c \in \mathfrak{C}^f$, so $0 \le u_a \le \chi 1$ in $L^\infty(\mathfrak{C})$. Next, let u_a^* be the decreasing rearrangement of u_a, that is, the element of $L^\infty(\mathfrak{A}_L)$ (where $\mathfrak{A}_L$ is the measure algebra of Lebesgue measure on $[0, \infty[$) such that $[\![u^* > \alpha]\!] = [0, \bar{\mu}[\![u > \alpha]\!][^\bullet$ for every $\alpha \ge 0$ (373D).

(b) Set

$$P = \{(a, \alpha) : a \in \mathfrak{A},\ \alpha \in \,]\bar{\mu}a, \bar{\mu}1]\}, \quad Q = \{(c, \alpha) : c \in \mathfrak{C},\ \alpha \in \,]\bar{\mu}c, \bar{\mu}1]\}.$$

Define a function f on P by saying that $f(a, \alpha) = (c, \beta)$ if

$$c = [\![u_a = 1]\!] = \max\{d : d \in \mathfrak{C},\ d \subseteq a\},$$

$$\beta = \max\{\beta' : \beta' \ge 0,\ \beta' + \int_{\beta'}^\infty u_a^* \le \alpha\}.$$

[1] Formerly 365P.

Note that $\beta > \bar{\mu}c$ because

$$\bar{\mu}c + \int_{\bar{\mu}c}^{\infty} u_a^* = \int u_a^* = \int u_a < \alpha,$$

using 373Fa for the equality in the middle, while $\beta \le \alpha \le \bar{\mu}1$; so (c, β) belongs to Q.

(c)(i) If $p \le p'$ in P, then $f(p) \le f(p')$ in Q. **P** Express p, p', $f(p)$ and $f(p')$ as (a, α), (a', α'), (c, β), (c', β') respectively. Then $c \subseteq a \subseteq a'$ so $c \subseteq c'$. Next, $\chi a \le \chi a'$ so $u_a \le u_{a'}$ and $u_a^* \le u_{a'}^*$ (373Db); accordingly

$$\alpha' \le \alpha = \beta + \int_{\beta}^{\infty} u_a^* \le \beta + \int_{\beta}^{\infty} u_{a'}^*$$

and $\beta' \le \beta$. **Q**

(ii) If $p \in P$, $q \in Q$ and $f(p) \le q$, then there is a $p' \ge p$ such that $f(p') \ge q$. **P** Express p, $f(p)$ and q as (a, α), (c, β) and (d, γ) respectively. Set $a' = a \cup d$. Then

$$\bar{\mu}a' = \bar{\mu}a + \bar{\mu}d - \bar{\mu}(a \cap d) = \int u_a + \bar{\mu}d - \int_d u_a \ge \int u_a^* + \bar{\mu}d - \int_0^{\bar{\mu}d} u_a^*$$

(apply 373E with $v = \chi d$)

$$= \bar{\mu}d + \int_{\bar{\mu}d}^{\infty} u_a^* < \beta + \int_{\beta}^{\infty} u_a^*$$

(because $[\![u_a^* = 1]\!] = [0, \bar{\mu}c]^\bullet$ and $\bar{\mu}c \le \bar{\mu}d < \gamma \le \beta$)

$$= \alpha.$$

So $(a', \alpha) \in P$. Next, computing the integrals $\int_b u_a \vee \chi d$ for b belonging to $\mathfrak{C}^f$ and either included in d or disjoint from it, we see that $u_{a'} = u_a \vee \chi d$ so that $[\![u_{a'} = 1]\!] = [\![u_a = 1]\!] \cup d = d$. Accordingly

$$\bar{\mu}a' = \int u_{a'} = \int u_{a'}^* = \bar{\mu}d + \int_{\bar{\mu}d}^{\infty} u_{a'}^* < \gamma + \int_{\gamma}^{\infty} u_{a'}^*$$

(as noted above for u_a, we have $[\![u_{a'}^* = 1]\!] = [0, \bar{\mu}d]^\bullet$), and if we set $\alpha' = \min(\alpha, \gamma + \int_{\gamma}^{\infty} u_{a'}^*)$ then $p' = (a', \alpha') \ge p$ and $f(p') \ge q$, as required. **Q**

(iii) Since P and Q have a common least element $(0, \bar{\mu}1)$ which is invariant under f, f satisfies the second condition of 514P and $\mathrm{AM}^*(\mathfrak{C}, \bar{\mu}{\restriction}\mathfrak{C}) = \mathrm{RO}^{\uparrow}(Q)$ is regularly embedded in $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu}) = \mathrm{RO}^{\uparrow}(P)$.

528H Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, not $\{0\}$, and let $\kappa \ge \max(\omega, \tau(\mathfrak{A}), c(\mathfrak{A}))$ be a cardinal. Then $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$ can be regularly embedded in $\mathrm{AM}(\mathfrak{B}_\kappa, \bar{\nu}_\kappa, \frac{1}{2})$.

proof (a) To begin with (down to the end of (g) below), assume that $\mathfrak{A}$ is atomless. Let $(\mathfrak{A}^{\mathbb{N}}, \bar{\mu}_\infty)$ be the simple product of a sequence of copies of $(\mathfrak{A}, \bar{\mu})$ (322L), so that $\bar{\mu}_\infty \boldsymbol{a} = \sum_{n=0}^{\infty} \mu a_n$ if $\boldsymbol{a} = \langle a_n \rangle_{n \in \mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$. Note that as $\mathfrak{A}$ is certainly infinite, $\tau(\mathfrak{A}^{\mathbb{N}}) = \tau(\mathfrak{A})$ and $c(\mathfrak{A}^{\mathbb{N}}) = c(\mathfrak{A})$ (514Ef). By 526D, there is a function $\theta : \mathfrak{A}^{\mathbb{N}} \to \mathfrak{B}_\kappa$ such that

$\theta(\sup A) = \sup \theta[A]$ for every non-empty $A \subseteq \mathfrak{A}^{\mathbb{N}}$ with a supremum in $\mathfrak{A}$,

$\bar{\nu}_\kappa \theta(\boldsymbol{a}) = 1 - \exp(-\bar{\mu}_\infty \boldsymbol{a})$ for every $\boldsymbol{a} \in \mathfrak{A}^{\mathbb{N}}$,

whenever $\langle \boldsymbol{a}^{(i)} \rangle_{i \in I}$ is a disjoint family in $\mathfrak{A}^{\mathbb{N}}$ and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{\theta(\boldsymbol{a}) : \boldsymbol{a} \subseteq \boldsymbol{a}^{(i)}\}$ for each i, then $\langle \mathfrak{C}_i \rangle_{i \in I}$ is stochastically independent.

(b) For $b \in \mathfrak{B}_\kappa$, set $g(b) = \sup\{\boldsymbol{a} : \boldsymbol{a} \in \mathfrak{A}^{\mathbb{N}}, \theta(\boldsymbol{a}) \subseteq b\}$.

(i) It is immediate from its definition that $g : \mathfrak{B}_\kappa \to \mathfrak{A}^{\mathbb{N}}$ is order-preserving.

(ii) Because θ is supremum-preserving, $\theta(g(b)) \subseteq b$ for every $b \in \mathfrak{B}_\kappa$.

(iii) If $b \in \mathfrak{B}_\kappa \setminus \{1\}$ then

$$1 - \bar{\nu}_\kappa b \le 1 - \bar{\nu}_\kappa \theta(g(b)) = \exp(-\bar{\mu}_\infty g(b)),$$

so $\bar{\mu}_\infty g(b) \le -\ln(1 - \bar{\nu}_\kappa b)$ is finite.

(iv) $\boldsymbol{a} \subseteq g(\theta(\boldsymbol{a}))$ for every $\boldsymbol{a} \in \mathfrak{A}^{\mathbb{N}}$; and if $\boldsymbol{a} \in \mathfrak{A}^{\mathbb{N}}$ has finite measure then $g(\theta(\boldsymbol{a})) = \boldsymbol{a}$, because if $\boldsymbol{a}' \not\subseteq \boldsymbol{a}$ then $\bar{\nu}_\kappa \theta(\boldsymbol{a} \cup \boldsymbol{a}') > \bar{\nu}_\kappa \theta(\boldsymbol{a})$.

(v) If $b \in \mathfrak{B}_\kappa$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\bar{\mu}_\infty(g(b') \setminus g(b)) \le \epsilon$ whenever $\bar{\nu}_\kappa(b' \setminus b) \le \delta$. **P?** Otherwise, $g(b) \ne 1_{\mathfrak{A}^{\mathbb{N}}}$ so $b \ne 1_{\mathfrak{B}_\kappa}$ and we can find a sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}_\kappa$ such that $\bar{\nu}_\kappa(b_n \setminus b) \le 2^{-n-2}(1 - \bar{\nu}_\kappa b)$ and $\bar{\mu}_\infty(g(b_n) \setminus g(b)) \ge \epsilon$ for every $n \in \mathbb{N}$. For each n, set $b_n^* = b \cup \sup_{m\ge n} b_m$; then

$$\bar{\mu}_\infty g(b_n^*) = \bar{\mu}_\infty g(b) + \bar{\mu}_\infty(g(b_n^*) \setminus g(b)) \ge \bar{\mu}_\infty g(b) + \bar{\mu}_\infty(g(b_n) \setminus g(b)) \ge \epsilon + \bar{\mu}_\infty g(b).$$

Note that $\bar{\nu}_\kappa b_0^* < 1$ so $g(b_0^*)$ has finite measure.

The sequences $\langle b_n^* \rangle_{n\in\mathbb{N}}$, $\langle g(b_n^*)\rangle_{n\in\mathbb{N}}$ and $\langle \theta(g(b_n^*))\rangle_{n\in\mathbb{N}}$ are all non-increasing. Set $\boldsymbol{a} = \inf_{n\in\mathbb{N}} g(b_n^*)$, so that

$$\theta(\boldsymbol{a}) \subseteq \inf_{n\in\mathbb{N}} \theta(g(b_n^*)) \subseteq \inf_{n\in\mathbb{N}} b_n^* = b$$

because $\bar{\nu}_\kappa(b_n^* \setminus b) \le 2^{-n-1}$ for every n. It follows that $\boldsymbol{a} \subseteq g(b)$. At the same time,

$$\bar{\mu}_\infty \boldsymbol{a} = \lim_{n\to\infty} \bar{\mu}_\infty g(b_n^*) > \bar{\mu}_\infty g(b),$$

which is impossible. **XQ**

(c) Define $\psi : \mathfrak{A}^{\mathbb{N}} \to \mathfrak{A}$ by setting $\psi(\boldsymbol{a}) = \sup_{n\in\mathbb{N}} a_n$ whenever $\boldsymbol{a} = \langle a_n \rangle_{n\in\mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$.

(i) ψ is supremum-preserving and $\psi(0) = 0$.

(ii) If $\boldsymbol{a}$, $\boldsymbol{a}' \in \mathfrak{A}^{\mathbb{N}}$ then

$$\bar{\mu}(\psi(\boldsymbol{a}) \mathbin{\triangle} \psi(\boldsymbol{a}')) \le \bar{\mu}_\infty(\boldsymbol{a} \mathbin{\triangle} \boldsymbol{a}') \qquad \bar{\mu}(\psi(\boldsymbol{a}) \setminus \psi(\boldsymbol{a}')) \le \bar{\mu}_\infty(\boldsymbol{a} \setminus \boldsymbol{a}').$$

(iii) Now if $b \in \mathfrak{B}_\kappa$, $a \in \mathfrak{A}$, $a \supseteq \psi(g(b))$ and $\bar{\mu}a < \alpha \in \mathbb{R}$, there is a $b' \supseteq b$ such that $a \subseteq \psi(g(b'))$, $\bar{\mu}\psi(g(b')) < \alpha$ and $\bar{\nu}_\kappa(b' \setminus b) \le 1 - \exp(-\bar{\mu}(a \setminus \psi(g(b))))$.

P Take α' such that $\bar{\mu}a < \alpha' < \alpha$. By (b-v), there is a $\delta > 0$ such that $\bar{\mu}_\infty(g(b') \setminus g(b)) \le \alpha' - \bar{\mu}a$ whenever $\bar{\nu}_\kappa(b' \setminus b) \le \delta$. Set $\boldsymbol{a}^{(n)} = \langle a_{ni} \rangle_{i\in\mathbb{N}}$ for each $n \in \mathbb{N}$, where $a_{ni} = a \setminus \psi(g(b))$ if $i = n$, 0 otherwise. For each $n \in \mathbb{N}$, let $\mathfrak{C}_n$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by

$$\{\theta(\boldsymbol{a}) : \boldsymbol{a} \in \mathfrak{A}^{\mathbb{N}},\ \boldsymbol{a} \cap \boldsymbol{a}^{(m)} = 0 \text{ for every } m \ge n\},$$

and let $T_n : L^1(\mathfrak{B}_\kappa, \bar{\nu}_\kappa) \to L^1(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ be the corresponding conditional-expectation operator (365Q[2]). Then $\langle \mathfrak{C}_n \rangle_{n\in\mathbb{N}}$ is non-decreasing; also $\bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)} \cap c) = \bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)}) \cdot \bar{\nu}_\kappa c$ for every $n \in \mathbb{N}$ and $c \in \mathfrak{C}_n$, by the final clause of (a). By Lévy's martingale theorem (275I, 367Jb), $\langle T_n(\chi b) \rangle_{n\in\mathbb{N}}$ is $\| \|_1$-convergent. We can therefore find an $n \in \mathbb{N}$ such that $\|T_n(\chi b) - T_{n+1}(\chi b)\|_1 \le \delta \exp(-\bar{\mu}a)$. Set $b' = b \cup \theta(\boldsymbol{a}^{(n)})$. Then $g(b') \supseteq g(b) \cup \boldsymbol{a}^{(n)}$, so $\psi(g(b')) \supseteq a_{nn} \cup \psi(g(b)) = a$. Also

$$\bar{\nu}_\kappa(b' \setminus b) \le \bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)}) = 1 - \exp(-\bar{\mu}_\infty \boldsymbol{a}^{(n)}) = 1 - \exp(-\bar{\mu}(a \setminus \psi(g(b))).$$

? If $\bar{\mu}\psi(g(b')) \ge \alpha$, set $\boldsymbol{e} = g(b') \setminus (g(b) \cup \sup_{m\in\mathbb{N}} \boldsymbol{a}^{(m)})$. Since

$$\psi(g(b) \cup \sup_{m\in\mathbb{N}} \boldsymbol{a}^{(m)}) = \psi(g(b)) \cup a = a,$$

$\psi(\boldsymbol{e}) \supseteq \psi(g(b')) \setminus a$ and

$$\bar{\mu}_\infty \boldsymbol{e} \ge \alpha - \bar{\mu}a > \alpha' - \bar{\mu}a;$$

as $\boldsymbol{e} \subseteq g(\theta(\boldsymbol{e}))$ and $\boldsymbol{e} \cap g(b) = 0$, $\bar{\nu}_\kappa(\theta(\boldsymbol{e}) \setminus b) > \delta$. On the other hand,

$$(1 - \bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)}))\bar{\nu}_\kappa(b \cap \theta(\boldsymbol{e})) = (1 - \bar{\nu}_\kappa\theta(\boldsymbol{a}^{(n)})) \int_{\theta(\boldsymbol{e})} T_n(\chi b)$$

(because $\boldsymbol{e} \cap \boldsymbol{a}^{(m)} = 0$ for every m, so $\theta(\boldsymbol{e}) \in \mathfrak{C}_n$)

$$= \int \chi(1 \setminus \theta(\boldsymbol{a}^{(n)})) \cdot \int T_n(\chi b) \times \chi\theta(\boldsymbol{e})$$

$$= \int \chi(1 \setminus \theta(\boldsymbol{a}^{(n)})) \times T_n(\chi b) \times \chi\theta(\boldsymbol{e})$$

(because $T_n(\chi b) \times \chi\theta(\boldsymbol{e}) \in L^0(\mathfrak{C}_n)$ and $\chi(1 \setminus \theta(\boldsymbol{a}^{(n)}))$ are stochastically independent)

[2]Formerly 365R.

$$= \int_{\theta(\boldsymbol{e})\setminus\theta(\boldsymbol{a}^{(n)})} T_n(\chi b),$$

$$(1-\bar\nu_\kappa\theta(\boldsymbol{a}^{(n)}))\bar\nu_\kappa(\theta(\boldsymbol{e})) = \bar\nu_\kappa(\theta(\boldsymbol{e})\setminus\theta(\boldsymbol{a}^{(n)}) = \bar\nu_\kappa(b\cap\theta(\boldsymbol{e})\setminus\theta(\boldsymbol{a}^{(n)})$$

(because $\theta(\boldsymbol{e}) \subseteq \theta(g(b')) \subseteq b' = b\cup\theta(\boldsymbol{a}^{(n)}))$

$$= \int_{\theta(\boldsymbol{e})\setminus\theta(\boldsymbol{a}^{(n)})} T_{n+1}(\chi b)$$

because $\theta(\boldsymbol{e})$ and $\theta(\boldsymbol{a}^{(n)})$ both belong to $\mathfrak{C}_{n+1}$. So

$$\begin{aligned}\delta\exp(-\bar\mu a) &= \delta\exp(-\bar\mu_\infty\boldsymbol{a}^{(n)}) < \bar\nu_\kappa(\theta(\boldsymbol{e})\setminus b)\exp(-\bar\mu_\infty\boldsymbol{a}^{(n)})\\ &= (1-\bar\nu_\kappa\theta(\boldsymbol{a}^{(n)}))\bar\nu_\kappa(\theta(\boldsymbol{e})\setminus b))\\ &= \int_{\theta(\boldsymbol{e})\setminus\theta(\boldsymbol{a}^{(n)})} T_{n+1}(\chi b) - T_n(\chi b)\\ &\le \|T_n(\chi b) - T_{n+1}(\chi b)\|_1 \le \delta\exp(-\bar\mu a),\end{aligned}$$

which is impossible. **X**

So $\bar\mu\psi(g(b')) < \alpha$, as required. **Q**

(d) Fix $c \in \mathfrak{B}_\kappa$ with measure $\frac{1}{2}$; then the principal ideal of $\mathfrak{B}_\kappa$ generated by c is isomorphic to $\mathfrak{B}_\kappa$ with the measure halved. We therefore have a Boolean isomorphism $\pi : \mathfrak{B}_\kappa \to (\mathfrak{B}_\kappa)_c$ such that $\bar\nu_\kappa\pi b = \frac{1}{2}\bar\nu_\kappa b$ for every $b \in \mathfrak{B}_\kappa$. Set $h(b) = \psi(g(\pi^{-1}(b\cap c)))$ for $b \in \mathfrak{B}_\kappa$. Then $h : \mathfrak{B}_\kappa \to \mathfrak{A}$ is order-preserving and $h(b) = h(b\cap c)$ for every $b \in \mathfrak{B}_\kappa$. Translating the results of (b) and (c), we see that

if $b \in \mathfrak{B}_\kappa$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\bar\mu(h(b')\setminus h(b)) \le \epsilon$ whenever $\bar\nu_\kappa(b'\setminus b) \le \delta$,

if $b \in \mathfrak{B}_\kappa$, $a \in \mathfrak{A}$, $a \supseteq h(b)$ and $\bar\mu a < \alpha \in \mathbb{R}$, there is a $b' \supseteq b$ such that $a \subseteq h(b')$, $\bar\mu h(b') < \alpha$ and $\bar\nu_\kappa(b'\setminus b) \le \frac{1}{2}(1-\exp(-\bar\mu(a\setminus h(b)))$.

Note also that

$$\begin{aligned}\bar\mu h(b) &= \bar\mu\psi(g(\pi^{-1}(b\cap c))) \le \bar\mu_\infty g(\pi^{-1}(b\cap c))\\ &\le -\ln(1-\bar\nu_\kappa\pi^{-1}(b\cap c)) \le -\ln(1-2\bar\nu_\kappa(b\cap c))\end{aligned}$$

if we take $\ln(0)$ to be $-\infty$.

(e)(i) Set $\gamma_0 = \frac{1}{2}(1-\exp(-\bar\mu 1))$, interpreting $\exp(-\infty)$ as 0, so that $0 < \gamma_0 \le \frac{1}{2}$. Let P be the partially ordered set $\{(a,\alpha) : a \in \mathfrak{A},\ \alpha \in \,]\bar\mu a, \bar\mu 1]\}$ and Q the partially ordered set $\{b : b \in \mathfrak{B}_\kappa,\ \bar\nu_\kappa b < \gamma_0\}$, so that $\mathrm{AM}^*(\mathfrak{A},\bar\mu) = \mathrm{RO}^\uparrow(P)$ and $\mathrm{AM}(\mathfrak{B}_\kappa,\bar\nu_\kappa,\gamma_0) = \mathrm{RO}^\uparrow(Q)$. For $b \in Q$, set $\alpha_b = \sup\{\bar\mu h(b') : b \subseteq b' \in Q\}$. Then $\alpha_b > \bar\mu(h(b))$. **P** We have

$$\bar\mu h(b) \le -\ln(1-2\bar\nu_\kappa(b\cap c)) < -\ln(1-2\gamma_0) = \bar\mu 1,$$

so $h(b) \ne 1$. Because $\mathfrak{A}$ is atomless, there is an $a \in \mathfrak{A}$, disjoint from $h(b)$, such that $0 < \bar\mu a < -\ln(2\bar\nu_\kappa b)$. Set $\boldsymbol{a} = \langle a_n\rangle_{n\in\mathbb{N}}$ where $a_0 = a$, $a_n = 0$ for $n \ge 1$. Then

$$\bar\nu_\kappa\theta(\boldsymbol{a}) = 1-\exp(-\bar\mu a) < 1-2\bar\nu_\kappa b,$$

so $b' = b\cup\pi\theta(\boldsymbol{a}) \in Q$, while $h(b') \supseteq h(b)\cup a \supset h(b)$. **Q**

(ii) If $b \in Q$ and $\bar\mu h(b) < \alpha$, there is a $b_1 \in Q$ such that $b \subseteq b_1$, $h(b_1) = h(b)$ and $\alpha_{b'} \le \alpha$. **P** Let $\delta > 0$ be such that $\bar\mu(h(b')\setminus h(b)) \le \alpha - \bar\mu h(b)$ whenever $\bar\nu_\kappa(b'\setminus b) \le \delta$. Because $\gamma_0 \le \frac{1}{2}$, there is a $b_1 \in Q$ such that $b \subseteq b_1$, $b\cap c = b_1\cap c$ and $\bar\mu b_1 \ge \gamma_0 - \delta$. Then $h(b_1) = h(b)$. If $b' \in Q$ and $b' \supseteq b_1$, then $\bar\nu_\kappa(b'\cap c\setminus b) \le \delta$, so

$$\bar\mu(h(b')\setminus h(b)) = \bar\mu(h(b'\cap c)\setminus h(b)) \le \alpha - \bar\mu h(b)$$

and $\bar\mu h(b') \le \alpha$; thus $\alpha_{b_2} \le \alpha$. **Q**

(f) By (e-i), we can define $f : Q \to P$ by setting $f(b) = (h(b),\alpha_b)$ for $b \in Q$.

(i) f is order-preserving because h is.

(ii) If $P_1 \subseteq P$ is up-open and cofinal with P, $f^{-1}[P_1]$ is cofinal with Q. **P** Take any $b \in Q$. Set

$$\alpha = \min(\alpha_b, \bar\mu h(b) - \ln(1-2\gamma_0+2\bar\nu_\kappa b)) > \bar\mu h(b),$$

so that $f(b) \le (h(b), \alpha)$ in P. Then there is an $(a, \beta) \in P_1$ such that $(h(b), \alpha) \le (a, \beta)$, that is, $h(b) \subseteq a$ and $\beta \le \alpha$. In this case, there is a $b_1 \in \mathfrak{B}_\kappa$ such that $b_1 \supseteq b$, $h(b_1) \supseteq a$, $\bar\mu h(b_1) < \beta$ and

$$\bar\nu_\kappa(b_1 \setminus b) \le \frac{1}{2}(1 - \exp(-\bar\mu(a \setminus h(b)))) < \gamma_0 - \bar\nu b$$

because $\bar\mu(a \setminus h(b)) < -\ln(1 - 2\gamma_0 + 2\bar\nu_\kappa b)$. So $b_1 \in Q$. By (e-ii), there is a $b_2 \in Q$ such that $b_2 \supseteq b_1$, $h(b_2) = h(b_1)$ and $\alpha_{b_2} \le \beta$. Now $b \subseteq b_2$, while $f(b_2) = (h(b_1), \alpha_{b_2}) \ge (a, \beta)$. As P_1 is up-open, $f(b_2) \in P_1$; as b is arbitrary, $f^{-1}[P_1]$ is cofinal with Q. **Q**

(iii) $f[Q]$ is cofinal with P. **P** Take $(a, \alpha) \in P$. Set $\boldsymbol{a} = \langle a_n \rangle_{n \in \mathbb{N}} \in \mathfrak{A}^{\mathbb{N}}$ where $a_0 = a$ and $a_n = 0$ for $n \ge 1$, and set $b = \pi\theta(\boldsymbol{a})$. Note that

$$\bar\nu_\kappa b = \frac{1}{2}(1 - \exp(-\bar\mu a)) < \gamma_0,$$

so $b \in Q$. Because $\bar\mu_\infty \boldsymbol{a} < \infty$, $g(\pi^{-1}b) = \boldsymbol{a}$ and $h(b) = a$. By (e-ii) again, we can now find a $b_1 \supseteq b$ in Q such that $h(b_1) = h(b)$ and $\alpha_{b_1} \le \alpha$. So $f(b_1) \ge (a, \alpha)$. As (a, α) is arbitrary, $f[Q]$ is cofinal. **Q**

(g) By 514O, $\mathrm{AM}^*(\mathfrak{A}, \bar\mu) = \mathrm{RO}^\uparrow(P)$ can be regularly embedded in

$$\mathrm{RO}^\uparrow(Q) = \mathrm{AM}(\mathfrak{B}_\kappa, \bar\nu_\kappa, \gamma_0) \cong \mathrm{AM}(\mathfrak{B}_\kappa, \nu_\kappa, \tfrac{1}{2})$$

by 528Da.

(h) All this has been done on the assumption that $\mathfrak{A}$ is atomless, as required in (e). For the general case, consider the localizable measure algebra free product $(\mathfrak{C}, \bar\lambda)$ of $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}_\omega, \bar\nu_\omega)$ (325E). By 521Qa, we have

$$\max(\omega, c(\mathfrak{C}), \tau(\mathfrak{C})) \le \max(\omega, c(\mathfrak{A}), c(\mathfrak{B}_\omega), \tau(\mathfrak{A}), \tau(\mathfrak{B}_\omega)) \le \kappa.$$

Also $\mathfrak{C}$ is atomless because $\mathfrak{B}_\omega$ is isomorphic to a closed subalgebra of $\mathfrak{C}$ (325Dd) and is atomless. By (a)-(g), $\mathrm{AM}^*(\mathfrak{C}, \bar\lambda)$ can be regularly embedded in $\mathrm{AM}(\mathfrak{B}_\omega, \bar\nu_\omega, \frac{1}{2})$. Now consider the canonical embedding $\varepsilon_1 : \mathfrak{A} \to \mathfrak{C}$. This is order-continuous and measure-preserving (325Da), so identifies the Dedekind σ-complete Boolean algebra $\mathfrak{A}$ with a σ-subalgebra of $\mathfrak{C}$; also $\mathfrak{A}^f$ has supremum 1 both in $\mathfrak{A}$ and $\mathfrak{C}$. By 528G, $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{C}, \bar\lambda)$ and therefore in $\mathrm{AM}(\mathfrak{B}_\omega, \bar\nu_\omega, \frac{1}{2})$.

528I Definition For any set I, the **I-localization poset** is the set

$$\mathcal{S}_I^\infty = \{p : p \subseteq \mathbb{N} \times I,\ \#(p[\{n\}]) \le 2^n \text{ for every } n,\ \sup_{n \in \mathbb{N}} \#(p[\{n\}]) \text{ is finite}\},$$

ordered by $\subseteq$. For $p \in \mathcal{S}_I^\infty$ set $\|p\| = \max_{n \in \mathbb{N}} \#(p[\{n\}])$. I will write $\mathcal{S}^\infty$ for the $\mathbb{N}$-localization poset $\mathcal{S}_{\mathbb{N}}^\infty$, already introduced in the proof of 522T.

528J Proposition Let κ be an infinite cardinal, $\mathcal{S}_\kappa^\infty$ the κ-localization poset, and $(\mathfrak{A}, \bar\mu)$ a semi-finite measure algebra, not $\{0\}$, with $\kappa \ge \max(\omega, c(\mathfrak{A}), \tau(\mathfrak{A}))$. Then the variable-measure amoeba algebra $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$.

proof (a) To begin with (down to the end of (d) below), suppose that $\mathfrak{A}$ is atomless. Let P be the partially ordered set $\{(a, \alpha) : a \in \mathfrak{A},\ \alpha \in]\bar\mu a, \mu 1]\}$, so that $\mathrm{AM}^*(\mathfrak{A}, \bar\mu) = \mathrm{RO}^\uparrow(P)$. Give $\mathfrak{A}^f$ its measure metric, so that its topological density is at most κ (521Eb). Set $\gamma_0 = \frac{1}{2}\bar\mu 1$ and for $n \ge 1$ set $\gamma_n = 2^{-2n-1}\bar\mu 1$ if $\bar\mu 1 < \infty$, 4^{-n} otherwise. For each n, let D_n be a dense subset of $\{a : a \in \mathfrak{A}^f,\ \bar\mu a \le \gamma_n\}$, containing 0, with cardinal at most κ, and let $\langle d_{n\xi} \rangle_{\xi < \kappa}$ be a family running over D_n with cofinal repetitions.

(b) If $p \in \mathcal{S}_\kappa^\infty$ set

$$a_p = \sup_{(n,\xi) \in p} d_{n\xi}, \quad \alpha_p = \bar\mu a_p + \textstyle\sum_{n=0}^\infty (2^n - \#(p[\{n\}]))\gamma_n.$$

Then

$$\bar\mu a_p < \alpha_p \le \textstyle\sum_{n=0}^\infty 2^n \gamma_n = \bar\mu 1,$$

so we can define $f : \mathcal{S}_\kappa^\infty \to P$ by setting $f(p) = (a_p, \alpha_p)$. f is order-preserving, because if $p \subseteq p'$ in $\mathcal{S}_\kappa^\infty$ then

$$\begin{aligned}\alpha_{p'} &= \bar\mu a_{p'} + \sum_{n=0}^{\infty}(2^n - \#(p'[\{n\}]))\gamma_n\\ &\le \bar\mu a_p + \sum_{(n,\xi)\in p'\setminus p}\bar\mu d_{n\xi} + \sum_{n=0}^{\infty}(2^n - \#(p'[\{n\}]))\gamma_n\\ &\le \bar\mu a_p + \sum_{n=0}^{\infty}\#(p'[\{n\}]\setminus p[\{n\}])\gamma_n + \sum_{n=0}^{\infty}(2^n - \#(p'[\{n\}]))\gamma_n = \alpha_p.\end{aligned}$$

(c) Suppose that $p \in \mathcal{S}_\kappa^\infty$ and $f(p) \le (a,\alpha) \in P$. Take $\alpha' \in \,]\bar\mu a, \alpha[$. For $n \in \mathbb{N}$, set $k_n = 2^n - \#(p[\{n\}])$. Then $a_p \subseteq a$ and

$$\bar\mu a < \alpha \le \alpha_p = \bar\mu a_p + \sum_{n=0}^{\infty} k_n\gamma_n.$$

So there is an $r \in \mathbb{N}$ such that

$$\bar\mu a < \bar\mu a_p + \sum_{n=0}^{\infty}\gamma_n \min(r, k_n);$$

take r so large that, in addition, $\sum_{n=r+1}^{\infty} 2^n\gamma_n \le \alpha - \alpha'$.

For each n, set $k'_n = \min(r, k_n)$ and $C_n = \{\sup D : D \in [D_n]^{\le k'_n}\}$. Then (because $\mathfrak{A}$ is atomless) C_n is dense in $\{c : c \in \mathfrak{A}^f,\ \bar\mu c \le k'_n\gamma_n\}$. We can therefore choose $\langle c_n\rangle_{n\in\mathbb{N}}$ inductively in such a way that

$$c_n \in C_n, \quad \bar\mu(a \cup \sup_{m<n} c_m) < \alpha',$$

$$\bar\mu(a \setminus (a_p \cup \sup_{m<n} c_m)) < \sum_{m=n}^{\infty} k'_m\gamma_m$$

for every $n \in \mathbb{N}$. **P** For the inductive step to $n \ge 0$, set $b = a \setminus (a_p \cup \sup_{m<n} c_m)$. Take $b' \subseteq b$ such that $\bar\mu b' = \min(k'_n\gamma_n, \bar\mu b)$, so that

$$\bar\mu(a \cup \sup_{m<n} c_m \cup b') = \bar\mu(a \cup \sup_{m<n} c_m) < \alpha',$$

$$\bar\mu(b \setminus b') < \sum_{m=n+1}^{\infty} k'_m\gamma_m.$$

Let $c_n \in C_n$ be such that

$$\alpha' > \bar\mu(a \cup \sup_{m<n} c_m \cup b') + \bar\mu(c_n \setminus b') \ge \bar\mu(a \cup \sup_{m\le n} c_m),$$

$$\sum_{m=n+1}^{\infty} k'_m\gamma_m > \bar\mu(b \setminus b') + \bar\mu(b' \setminus c_n) = \bar\mu(a \setminus (a_p \cup \sup_{m\le n} c_m))$$

and the induction proceeds. **Q**

For each n, we can find a set $D'_n \subseteq D_n$, of size k'_n, such that $c_n = \sup D'_n$. Because $\langle d_{n\xi}\rangle_{\xi<\kappa}$ runs over D_n with cofinal repetitions, we can find a set $I_n \subseteq \kappa \setminus p[\{n\}]$ such that $\#(I_n) = k'_n$ and $c_n = \sup_{\xi\in I_n} d_{n\xi}$. Set $q = p \cup \{(n,\xi) : n \in \mathbb{N},\ \xi \in I_n\}$. Then

$$\#(q[\{n\}]) \le \#(p[\{n\}]) + k'_n \le \min(2^n, \|p\| + r)$$

for every n, so $q \in \mathcal{S}_\kappa^\infty$ and $p \subseteq q$. Now

$$a_q = a_p \cup \sup_{n\in\mathbb{N},\xi\in I_n} d_{n\xi} = a_p \cup \sup_{n\in\mathbb{N}} c_n \supseteq a$$

because

$$\bar\mu(a \setminus a_q) \le \inf_{n\in\mathbb{N}} \bar\mu(a \setminus (a_p \cup \sup_{m<n} c_m)) \le \inf_{n\in\mathbb{N}} \sum_{m=n}^{\infty} 2^m\gamma_m = 0.$$

Also

$$\bar\mu a_q = \sup_{n\in\mathbb{N}} \bar\mu(a_p \cup \sup_{m<n} c_m) \le \alpha' < \alpha,$$

while $\#(q[\{n\}]) = \#(p[\{n\}]) + k_n = 2^n$ whenever $n \le r$, so

$$\alpha_q = \bar\mu a_q + \sum_{n=r+1}^{\infty}(2^n - \#(q[\{n\}]))\gamma_n \le \alpha' + \sum_{n=r+1}^{\infty} 2^n\gamma_n \le \alpha.$$

(d) What (c) shows is that if $p \in \mathcal{S}_\kappa^\infty$ and $f(p) \le (a,\alpha)$ in P, then there is a $q \supseteq p$ in $\mathcal{S}_\kappa^\infty$ such that $(a,\alpha) \le f(q)$. Next, $\mathcal{S}_\kappa^\infty$ has a least element $\emptyset$, and $f(\emptyset) = (0, \bar\mu 1)$ is the least element of P. So 514P tells us that $\mathrm{RO}^\uparrow(P) = \mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty)$.

(e) As for the general case, we can use the same trick as in part (h) of the proof of 528H. Let $(\mathfrak{C},\bar\lambda)$ be the localizable measure algebra free product of $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B}_\omega,\bar\nu_\omega)$; as before, $\mathfrak{C}$ is atomless, $\max(\omega,c(\mathfrak{C}),\tau(\mathfrak{C}))\le\kappa$ and $(\mathfrak{A},\bar\mu)$ is embedded in $(\mathfrak{C},\bar\lambda)$ as a σ-subalgebra with sufficient elements of finite measure. So $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$ is regularly embedded in $\mathrm{AM}^*(\mathfrak{C},\bar\lambda)$ and in $\mathrm{RO}^\uparrow(\mathcal{S}^\infty_\kappa)$.

528K Theorem (Truss 88) Let $(\mathfrak{A},\bar\mu)$ be an atomless σ-finite measure algebra in which every non-zero principal ideal has Maharam type κ, and $0<\gamma<\bar\mu 1$. Then each of the algebras

$$\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma),\quad \mathrm{AM}^*(\mathfrak{A},\bar\mu),\quad \mathrm{AM}(\mathfrak{B}_\kappa,\bar\nu_\kappa,\tfrac12)$$

can be regularly embedded in the other two, and all three can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}^\infty_\kappa)$.

proof By 528H, $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$ can be regularly embedded in $\mathrm{AM}(\mathfrak{B}_\kappa,\bar\nu_\kappa,\frac12)$. Take any $e\in\mathfrak{A}$ such that $\gamma<\bar\mu e<\infty$. Then the principal ideal $(\mathfrak{A}_e,\bar\mu\restriction\mathfrak{A}_e)$ is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$, so

$$\mathrm{AM}(\mathfrak{B}_\kappa,\bar\nu_\kappa,\tfrac12)\cong\mathrm{AM}(\mathfrak{B}_\kappa,\bar\nu_\kappa,\frac{\gamma}{\bar\mu e})$$

(528Da)

$$\cong\mathrm{AM}(\mathfrak{A}_e,\bar\mu\restriction\mathfrak{A}_e,\gamma)$$

can be regularly embedded in $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ (528Fa). By 528Fc, $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$. Finally, by 528J, $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$ can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}^\infty_\kappa)$. Because regular embeddability is transitive (313N), these facts are enough to prove the theorem.

528L It is possible without great effort to calculate many of the cardinal functions of these algebras.

Lemma $\mathfrak{m}(\mathrm{AM}(\mathfrak{B}_\omega,\bar\nu_\omega,\frac12))\le\operatorname{add}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.

proof Set $P=\{a:a\in\mathfrak{B}_\omega,\ \bar\nu_\omega a<\frac12\}$. Then $\mathrm{wdistr}(\mathfrak{B}_\omega)\ge\mathfrak{m}^\uparrow(P)$. **P** Take a family $\langle B_\xi\rangle_{\xi<\kappa}$ of maximal antichains in $\mathfrak{B}_\omega$, where $\kappa<\mathfrak{m}^\uparrow(P)$. Let $C\subseteq\mathfrak{B}_\omega$ be a maximal disjoint set such that $\{b:b\in B_\xi,\ b\cap c\ne0\}$ is finite for every $\xi<\kappa$ and $c\in C$. **?** Suppose, if possible, that $c_0=1\setminus\sup C$ is not 0. Take $a_0\in P$ such that $\bar\nu_\omega(a_0\cup c_0)>\frac12$. (If $\bar\nu_\omega c_0>\frac12$, take $a_0=0$; otherwise, take $a_0\subseteq1\setminus c_0$ such that $\frac12-\bar\nu_\omega c_0<\bar\nu_\omega a_0<\frac12$.) For each $\xi<\kappa$, set

$$Q_\xi=\{a:a\in P,\ \{b:b\in B_\xi,\ b\not\subseteq a\}\text{ is finite}\};$$

then Q_ξ is cofinal with P. There is therefore an upwards-directed $R\subseteq P$ such that $a_0\in R$ and R meets every Q_ξ. Set $e=\sup R$; then $\bar\nu_\omega e\le\frac12$ so $c_1=c_0\setminus e=(a_0\cup c_0)\setminus e$ is non-zero.

If $\xi<\kappa$, there is an $a\in R\cap Q_\xi$, so that

$$\{b:b\in B_\xi,\ b\cap c_1\ne0\}\subseteq\{b:b\in B_\xi,\ b\not\subseteq a\}$$

is finite. But this means that we ought to have added c_1 to C. **X**

Thus C is a maximal antichain. As $\langle B_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{wdistr}(\mathfrak{A})\ge\mathfrak{m}^\uparrow(P)$. **Q**

Now 524Mb tells us that $\mathrm{wdistr}(\mathfrak{B}_\omega)=\operatorname{add}\mathcal{N}$, so $\mathfrak{m}^\uparrow(P)\le\operatorname{add}\mathcal{N}$. Finally, by 517Db,

$$\mathfrak{m}(\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma))=\mathfrak{m}^\uparrow(P)\le\operatorname{add}\mathcal{N},$$

as claimed.

528M Lemma $\mathfrak{m}^\uparrow(\mathcal{S}^\infty)\ge\operatorname{add}\mathcal{N}$.

proof (a) Recall the definition of the supported relations $(\mathbb{N}^\mathbb{N},\subseteq^*,\mathcal{S}^{(\alpha)})$ from 522L, where $\mathcal{S}^{(\alpha)}=\{S:S\subseteq\mathbb{N}\times\mathbb{N},\ \#(S[\{n\}])\le\alpha(n)\text{ for every }n\in\mathbb{N}\}$ for $\alpha\in\mathbb{N}^\mathbb{N}$. Putting 522L, 522M and 512Db together, we have $\operatorname{add}(\mathbb{N}^\mathbb{N},\subseteq^*,\mathcal{S}^{(\alpha)})=\operatorname{add}\mathcal{N}$ whenever $\lim_{n\to\infty}\alpha(n)=\infty$.

(b) The core of the argument is the following fact. Suppose that $Q\subseteq\mathcal{S}^\infty$ is cofinal and up-open, $n\in\mathbb{N}$ and $\sigma\in[\mathbb{N}\times\mathbb{N}]^{<\omega}$. Let $G\subseteq\mathbb{N}\times\mathbb{N}$ be a set with finite vertical sections. Then there is a $k\in\mathbb{N}$ such that whenever $\sigma\subseteq p\in\mathcal{S}^\infty$, $p\subseteq\sigma\cup G$ and $\|p\|\le n$, there is a $q\in Q$ such that $p\subseteq q$ and $\|q\|\le k$.

P? Suppose, if possible, otherwise. Then for each $j\in\mathbb{N}$ we can find $p_j\in\mathcal{S}^\infty$ such that $\sigma\subseteq p_j\subseteq\sigma\cup G$, $\|p_j\|\le n$ and $\|q\|>j$ whenever $p\subseteq q\in Q$. Let p be a cluster point of $\langle p_j\rangle_{j\in\mathbb{N}}$ in $\mathcal{P}(\sigma\cup G)$. Then $\#(p[\{i\}])\le\sup_{j\in\mathbb{N}}\#(p_j[\{i\}])\le\min(2^i,n)$ for every i, so $p\in\mathcal{S}^\infty$. Because Q is cofinal with $\mathcal{S}^\infty$, there is a $q\in Q$ such that

$p \subseteq q$. Set $k = n + \|q\|$. Then $(\sigma \cup G) \cap (k \times \mathbb{N})$ is finite, so there is an $i \ge k$ such that $p_i \cap (k \times \mathbb{N}) = p \cap (k \times \mathbb{N}) \subseteq q$. Set $q' = p_i \cup q$. Then

$$\begin{aligned}\#(q'[\{j\}]) &= \#(q[\{j\}]) \le \min(\|q\|, 2^j) \text{ if } j < k,\\ &\le \|p_i\| + \|q\| \le k \le 2^j \text{ otherwise.}\end{aligned}$$

So $q' \in \mathcal{S}^\infty$ and $\|q'\| \le k \le i$; because Q is up-open in $\mathcal{S}^\infty$, $q' \in Q$, while $p_i \subseteq q'$. But we chose p_i so that this could not happen. **XQ**

(c) We need to know that $\mathcal{S}^\infty$ is upwards-ccc. **P** For any $n \in \mathbb{N}$, finite $\sigma \subseteq \mathbb{N}\times\mathbb{N}$ the set $\{p : p \in \mathcal{S}^\infty,\ \|p\| \le 2^{n-1},\ p \cap (n \times \mathbb{N}) = \sigma\}$ is upwards-linked. **Q**

(d) Now let $\langle Q_\xi\rangle_{\xi<\kappa}$ be any family of cofinal subsets of $\mathcal{S}^\infty$, where $\kappa < \operatorname{add}\mathcal{N}$, and $p_0 \in \mathcal{S}^\infty$. For each $\xi < \kappa$ let $A_\xi \subseteq Q_\xi$ be a maximal up-antichain; by (c), A_ξ is countable. Set $Q'_\xi = \bigcup\{[q, \infty[\ : q \in A_\xi\}$, so that Q'_ξ is an up-open cofinal subset of $\mathcal{S}^\infty$. Set $A = \{p_0\} \cup \bigcup_{\xi<\kappa} A_\xi$. For $q \in A$, let $F_q \subseteq \mathbb{N}^\mathbb{N}$ be a finite set such that (identifying each member of F_q with its graph) $q \subseteq \bigcup F_q$; set $F = \bigcup_{q\in A} F_q$, so that

$$\#(F) \le \max(\omega, \kappa) < \operatorname{add}\mathcal{N} \le \mathfrak{b}$$

(522B). Let $g_0 \in \mathbb{N}^\mathbb{N}$ be a strictly increasing function such that $\{i : f(i) > g_0(i)\}$ is finite for every $f \in F$, and also $p_0[\{i\}] \subseteq g_0(i)$ for every i. Set $G = \{(i, j) : i \in \mathbb{N},\ j < g_0(i)\}$, so that $G \subseteq \mathbb{N}\times\mathbb{N}$ has finite vertical sections. Observe that if $q \in A$ then $q \setminus G$ is finite.

For each $\xi < \kappa$, $n \in \mathbb{N}$ and finite $\sigma \subseteq \mathbb{N}\times\mathbb{N}$, let $k(\xi, \sigma, n) \in \mathbb{N}$ be such that whenever $p \in \mathcal{S}^\infty$ and $\sigma \subseteq p \subseteq \sigma \cup G$ then there is a $q \in Q'_\xi$ such that $p \subseteq q$ and $\|q\| \le k(\xi, \sigma, n)$; such a k exists by (b) above. Set $k_\xi(n) = \sup\{k(\xi, \sigma, n) : \sigma \subseteq n \times g_0(n)\}$. Again because $\kappa < \mathfrak{b}$, there is a $g_1 \in \mathbb{N}^\mathbb{N}$ such that $\{n : k_\xi(n) > g_1(n)\}$ is finite for every $\xi < \kappa$. Let $\alpha \in \mathbb{N}^\mathbb{N}$ be a non-decreasing function such that $\lim_{n\to\infty} \alpha(n) = \infty$ and

$$\alpha(2g_1(n)) \le n, \quad \alpha(n) + \#(p_0[\{n\}]) \le 2^n, \quad 2\alpha(n) \le n$$

for every n.

Because $\operatorname{add}(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S}^{(\alpha)}) = \operatorname{add}\mathcal{N}$, there is an $S_0 \in \mathcal{S}^{(\alpha)}$ such that $f \subseteq^* S_0$ for every $f \in F$, so that $q \setminus S_0$ is finite for every $q \in A$. Replacing S_0 by $S_0 \cap G$ if necessary, we may suppose that $S_0 \subseteq G$.

(e) Let $\mathcal{S}$ be the family of subsets S of $\mathbb{N}\times\mathbb{N}$ such that $\#(S[\{n\}]) \le 2^n$ for every n, as in 522K. Note that $p_0 \cup S_0 \in \mathcal{S}$, because $\alpha(n) + \#(p_0(n)) \le 2^n$ for every n. Let C be the family of finite subsets σ of $\mathbb{N}\times\mathbb{N}$ such that $\sigma \cup S_0 \in \mathcal{S}$. For each $\xi < \kappa$, set

$$D_\xi = \{\sigma : \sigma \in C,\ \exists\, q \in A_\xi,\ q \subseteq \sigma \cup S_0\}.$$

Then D_ξ is cofinal with C. **P** Let $\sigma \in C$. Let n_0 be so large that $g_1(2^{n_0}) \ge k_\xi(2^{n_0})$ and $\sigma \subseteq n_0 \times g_0(n_0)$. Set $m = 2g_1(2^{n_0})$, $p = \sigma \cup (S_0 \cap (m \times \mathbb{N})) \in \mathcal{S}^\infty$. Then $\sigma \subseteq p \subseteq \sigma \cup G$ and $\|p\| \le \max(2^{n_0}, \alpha(m)) = 2^{n_0}$, so there is a $q \in Q'_\xi$ such that $p \subseteq q$ and

$$\|q\| \le k(\xi, \sigma, 2^{n_0}) \le k_\xi(2^{n_0}) \le g_1(2^{n_0}) = \frac{m}{2}.$$

Let $q' \in A_\xi$ be such that $q' \subseteq q$. Let $m' \ge \max(m, n_0)$ be such that $q' \subseteq (m' \times \mathbb{N}) \cup S_0$, and set $\tau = q \cap (m' \times \mathbb{N})$, so that $\sigma \subseteq \tau$.

For $n < m$, we have

$$S_0[\{n\}] \subseteq p[\{n\}] \subseteq q[\{n\}] = \tau[\{n\}],$$

so $(\tau \cup S_0)[\{n\}] = q[\{n\}]$ has at most 2^n members. For $m \le n < m'$, we have

$$\#((\tau \cup S_0)[\{n\}]) \le \#(q[\{n\}]) + \#(S_0[\{n\}]) \le \|q\| + \alpha(n) \le \frac{m}{2} + \frac{n}{2} \le 2^n,$$

while for $n \ge m'$ we have

$$\#((\tau \cup S_0)[\{n\}]) = \#(S_0[\{n\}]) \le \alpha(n) \le 2^n.$$

So $\tau \cup S_0 \in \mathcal{S}$ and $\tau \in C$. Since

$$q' \subseteq (q' \cap (m' \times \mathbb{N})) \cup S_0 \subseteq (q \cap (m' \times \mathbb{N})) \cup S_0 = \tau \cup S_0,$$

$\tau \in D_\xi$. As σ is arbitrary, D_ξ is cofinal with C. **Q**

(f) Because $p_0 \cup S_0 \in \mathcal{S}$, $\sigma_0 = p_0 \setminus S_0$ belongs to C. Because $\kappa < \operatorname{add}\mathcal{N} \le \mathfrak{m}_{\text{countable}} \le \mathfrak{m}^\uparrow(C)$, there is an upwards-directed set $E \subseteq C$ meeting every D_ξ and containing σ_0. Set $S_1 = S_0 \cup \bigcup E$. Then, because E is upwards-directed,

$$\#(S_1[\{n\}]) = \sup_{\sigma \in E} \#((\sigma \cup S_0)[\{n\}]) \le 2^n$$

for every n, and $S_1 \in \mathcal{S}$. Set $R = \{p : p \in \mathcal{S}^\infty, p \subseteq S_1\}$; then $R \subseteq \mathcal{S}^\infty$ is upwards-directed (in fact, closed under $\cup$), and $p_0 \in R$ because $\sigma_0 \in E$. Now R meets Q_ξ for every $\xi < \kappa$. **P** There is a $\sigma \in D_\xi \cap E$. But this means that there is a $q \in A_\xi$ such that $q \subseteq \sigma \cup S_0 \subseteq S_1$ and $q \in R \cap Q_\xi$. **Q**

As p_0 and $\langle Q_\xi \rangle_{\xi<\kappa}$ are arbitrary, $\mathfrak{m}^\uparrow(\mathcal{S}^\infty) \ge \operatorname{add}\mathcal{N}$.

528N Theorem (Brendle 00, 2.3.10; Judah & Repický 95) Let $(\mathfrak{A}, \bar\mu)$ be an atomless σ-finite measure algebra with countable Maharam type, and $0 < \gamma < \bar\mu 1$. Then the algebras $\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma)$ and $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ and the $\mathbb{N}$-localization poset $\mathcal{S}^\infty$ (active upwards) all have Martin numbers equal to $\operatorname{add}\mathcal{N}$.

proof By 517Ia and 528K, with 517Db again,

$$\begin{aligned}\mathfrak{m}(\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma)) &= \mathfrak{m}(\mathrm{AM}^*(\mathfrak{A}, \bar\mu)) = \mathfrak{m}(\mathrm{AM}(\mathfrak{B}_\omega, \bar\nu_\omega, \tfrac{1}{2})) \\ &\ge \mathfrak{m}(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \mathfrak{m}^\uparrow(\mathcal{S}^\infty).\end{aligned}$$

As

$$\mathfrak{m}(\mathrm{AM}(\mathfrak{B}_\omega, \bar\nu_\omega, \tfrac{1}{2})) \le \operatorname{add}\mathcal{N} \le \mathfrak{m}^\uparrow(\mathcal{S}^\infty)$$

(528L, 528M), all these are equal to $\operatorname{add}\mathcal{N}$.

528O Corollary Let $\gamma > 0$. Let $\mathcal{G}$ be the partially ordered set

$$\{G : G \subseteq \mathbb{R} \text{ is open}, \mu_L G < \gamma\},$$

where μ_L is Lebesgue measure. Then $\mathfrak{m}^\uparrow(\mathcal{G}) = \operatorname{add}\mathcal{N}$.

proof Put 528C and 528N together.

528P Proposition Let $(\mathfrak{A}, \bar\mu)$ be an atomless semi-finite measure algebra, and $0 < \gamma < \bar\mu 1$.

(a) For any integer $m \ge 2$,

$$c(\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma)) = \operatorname{link}_m(\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma)) = \max(c(\mathfrak{A}), \tau(\mathfrak{A})).$$

(b) $d(\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma)) = \pi(\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma)) = \max(\operatorname{cf}[c(\mathfrak{A})]^{\le\omega}, \pi(\mathfrak{A}))$.

proof Set $P = \{a : a \in \mathfrak{A}, \bar\mu a < \gamma\}$, so that $\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma) = \mathrm{RO}^\uparrow(P)$.

(a) Set $\kappa_0 = \max(c(\mathfrak{A}), \tau(\mathfrak{A}))$, $\kappa_1 = \operatorname{link}_m(\mathrm{RO}^\uparrow(P)) = \operatorname{link}^\uparrow_m(P)$ and $\kappa_2 = c(\mathrm{RO}^\uparrow(P)) = c^\uparrow(P)$ (514N).

(i) The topological density of $\mathfrak{A}^f$ for its measure metric is κ_0 (521Eb), so P has a metrically dense subset D of size at most κ_0. For $d \in D$, set

$$U_d = \{a : a \in P, \bar\mu(a \setminus d) < \frac{1}{m}(\gamma - \bar\mu d)\}.$$

Then U_d is upwards-m-linked in P. Also, if $a \in P$, there is a $d \in D$ such that $\bar\mu(a \,\triangle\, d) < \frac{1}{m+1}(\gamma - \bar\mu a)$, and now $a \in U_d$. So P is κ_0-m-linked upwards and $\kappa_1 \le \kappa_0$.

(ii) By 511Hb or 511Ia, $\kappa_2 \le \kappa_1$.

(iii) We need to check that κ_2 is infinite. **P** Take $a \in \mathfrak{A}$ such that $\bar\mu a = \gamma$. For any $n \ge 1$, we can find disjoint $a_0, \dots, a_n \subseteq a$ all of measure $\frac{1}{n+1}\gamma$; now $\langle a \setminus a_i \rangle_{i \le n}$ is an up-antichain in P. So $\kappa_2 = c^\uparrow(P) \ge n+1$; and this is true for every n. **Q**

Now if $(\mathfrak{A}, \bar\mu)$ is totally finite, then $c(\mathfrak{A}) = \omega \le \kappa_2$. Otherwise, there is a partition D of unity in $\mathfrak{A}$ such that $\bar\mu d = \frac{1}{2}\gamma$ for every $d \in D$; now D is an up-antichain in P and $\kappa_2 \ge \#(D) = c(\mathfrak{A})$. So we see that in all cases $\kappa_2 \ge c(\mathfrak{A})$.

(iv) If $e \in \mathfrak{A}^f$ and the principal ideal $\mathfrak{A}_e$ is homogeneous, then $\tau(\mathfrak{A}_e) \le \kappa_2$. **P?** Otherwise, set $\alpha = \bar\mu e$, $\kappa = \tau(\mathfrak{A}_e)$. Because $\bar\mu 1 > \gamma$, there is a $d \subseteq 1 \setminus e$ such that $\gamma < \bar\mu(e \cup d) < \gamma + \bar\mu e$, that is, $0 < \gamma - \bar\mu d < \alpha$. Set

$\beta = \sqrt{1 - \frac{\gamma - \bar{\mu}d}{\alpha}}$. Because $\mathfrak{A}_e$ is isomorphic, up to a scalar multiple of the measure, to the measure algebra of the usual measure on $[0,1]^\kappa$, there is a family $\langle c_\xi \rangle_{\xi<\kappa}$ in $\mathfrak{A}_e$ such that

$$\bar{\mu}c_\xi = \beta\alpha, \quad \bar{\mu}(c_\xi \cap c_\eta) = \beta^2\alpha$$

whenever ξ, $\eta < \kappa$ are distinct. Set $b_\xi = d \cup (e \setminus c_\xi)$ for $\xi < \kappa$. Then

$$\bar{\mu}(b_\xi \cup b_\eta) = \bar{\mu}d + \alpha - \beta^2\alpha = \gamma,$$

$$\bar{\mu}b_\xi = \bar{\mu}d + \alpha - \beta\alpha < \gamma$$

for all distinct ξ, $\eta < \kappa$. So $\langle b_\xi \rangle_{\xi<\kappa}$ is an up-antichain in P and witnesses that $\kappa_2 \ge \kappa$. **XQ**

(v) Let E be a partition of unity in $\mathfrak{A}$ such that $0 < \bar{\mu}e < \infty$ and $\mathfrak{A}_e$ is homogeneous for every $e \in E$. For $e \in E$, let $A_e \subseteq \mathfrak{A}_e$ be a set of size at most κ_2 which τ-generates $\mathfrak{A}_e$. Then $A = \bigcup_{e\in E} A_e$ τ-generates $\mathfrak{A}$, so that

$$\kappa_0 = \max(c(\mathfrak{A}), \tau(\mathfrak{A})) \le \max(c(\mathfrak{A}), \#(A)) \le \max(c(\mathfrak{A}), \kappa_2) = \kappa_2,$$

and the three cardinals must be equal.

(b) Set $\kappa_3 = \max(\pi(\mathfrak{A}), \mathrm{cf}[c(\mathfrak{A})]^{\le\omega})$, $\kappa_4 = \pi(\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma))$ and $\kappa_5 = d(\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma))$.

(i) By 323Mc, $\mathfrak{A}^f$ is complete in its measure metric. By 323Ma, $\cup : \mathfrak{A}^f \times \mathfrak{A}^f \to \mathfrak{A}^f$ is uniformly continuous for the measure metric, and P is an open subset of $\mathfrak{A}^f$, while the topological density of $\mathfrak{A}^f$ is κ_0. By 524C, $(P, \subseteq', [P]^{<\omega}) \preccurlyeq_{\mathrm{GT}} (\ell^1(\kappa_0), \le, \ell^1(\kappa_0))$, where $\subseteq'$ is defined as in 512F. It follows that

$$\kappa_4 = \pi(\mathrm{RO}^\uparrow(P)) = \mathrm{cf}\, P$$

(514Nb)

$$= \mathrm{cov}(P, \subseteq, P) \le \max(\omega, \mathrm{cov}(P, \subseteq', [P]^{\le\omega}))$$

(512Gf)

$$\le \max(\omega, \mathrm{cov}(P, \subseteq', [P]^{<\omega}))$$

(512Gb)

$$\le \max(\omega, \mathrm{cf}\, \ell^1(\kappa_0))$$

(512Da)

$$= \mathrm{cf}\, \ell^1(\kappa_0) = \mathrm{cf}\, \mathcal{N}_{\kappa_0}$$

(where $\mathcal{N}_{\kappa_0}$ is the null ideal of the usual measure on $\{0,1\}^{\kappa_0}$, as in 524I)

$$= \max(\mathrm{cf}\, \mathcal{N}, \mathrm{cf}[\kappa_0]^{\le\omega})$$

(523N)

$$= \max(\mathrm{cf}\, \mathcal{N}, \mathrm{cf}[\tau(\mathfrak{A})]^{\le\omega}, \mathrm{cf}[c(\mathfrak{A})]^{\le\omega})$$
$$= \max(\mathrm{cf}\, \mathcal{N}, \mathrm{cf}[\tau(\mathfrak{A})]^{\le\omega}, c(\mathfrak{A}), \mathrm{cf}[c(\mathfrak{A})]^{\le\omega}) = \max(\pi(\mathfrak{A}), \mathrm{cf}[c(\mathfrak{A})]^{\le\omega})$$

(524Mc)

$$= \kappa_3.$$

(ii) By 514Nd, $d^\uparrow(P) = \kappa_5$. Let $\langle B_\xi \rangle_{\xi<\kappa_5}$ be a family of upwards-centered sets covering P. For each ξ, $b_\xi = \sup B_\xi$ is defined in $\mathfrak{A}$ (counting $\sup \emptyset$ as 0 if necessary), and

$$\bar{\mu}b_\xi = \sup_{I\in[B_\xi]^{<\omega}} \bar{\mu}(\sup I) \le \gamma.$$

Set $D = \{b_\xi \setminus b_\eta : \xi,\ \eta < \kappa_5\}$. Then D is order-dense in $\mathfrak{A}$. **P** If $a \in \mathfrak{A} \setminus \{0\}$, take $a' \subseteq a$ such that $0 < \bar{\mu}a' < \gamma$. Then $a' \in P$, so there is some $\xi < \kappa_5$ such that $a' \in B_\xi$ and $a' \subseteq b_\xi$. Next, let $c \subseteq 1 \setminus b_\xi$ be such that

$$\gamma - \bar{\mu}b_\xi < \bar{\mu}c < \gamma - \bar{\mu}b_\xi + \bar{\mu}a'.$$

Then $c \cup (b_\xi \setminus a') \in P$, so there is an $\eta < \kappa_5$ such that $c \cup (b_\xi \setminus a') \subseteq b_\eta$. Now $d = b_\xi \setminus b_\eta \subseteq a'$; as $\bar{\mu}(b_\xi \cup c) > \gamma \ge \bar{\mu}b_\eta$, $b_\xi \not\subseteq b_\eta$ and $d \ne 0$. Of course $d \in D$ and $d \subseteq a$; as a is arbitrary, D is order-dense. **Q**

Accordingly $\pi(\mathfrak{A}) \le \#(D) \le \kappa_5$. At the same time, $\mathrm{cf}[c(\mathfrak{A})]^{\le\omega} \le \kappa_5$. **P** There is a disjoint set $E \subseteq \mathfrak{A} \setminus \{0\}$ of size $c(\mathfrak{A})$ (332F). For each $\xi < \kappa_5$, let I_ξ be the countable set $\{e : e \in E,\ e \cap b_\xi \ne 0\}$. If $J \subseteq E$ is countable, let $\langle \epsilon_e \rangle_{e\in J}$ be a strictly positive family of real numbers with sum less than γ. For each $e \in J$ let $a_e \subseteq e$ be such that $0 < \bar{\mu} a_e \le \epsilon_e$, and set $a = \sup_{e \in J} a_e$. Then $a \in P$ so there is a $\xi < \kappa_5$ such that $a \subseteq b_\xi$ and $J \subseteq I_\xi$. As J is arbitrary, $\{I_\xi : \xi < \kappa_5\}$ is cofinal with $[E]^{\le \omega}$, and

$$\mathrm{cf}[c(\mathfrak{A})]^{\le\omega} = \mathrm{cf}[E]^{\le\omega} \le \kappa_5. \ \mathbf{Q}$$

Putting these together, we see that $\kappa_3 \le \kappa_5$.

(iii) By 514Da, $\kappa_5 \le \kappa_4$, so the three cardinals are equal.

528Q Proposition Let $\mathcal{S}^\infty$ be the $\mathbb{N}$-localization poset.
(a) $\pi(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \mathrm{cf}\,\mathcal{S}^\infty = \mathfrak{c}$.
(b) For every $m \ge 2$,

$$c(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = c^\uparrow(\mathcal{S}^\infty) = \mathrm{link}_m(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \mathrm{link}^\uparrow_m(\mathcal{S}^\infty) = \omega.$$

(c) $d(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = d^\uparrow(\mathcal{S}^\infty) = \mathrm{cf}\,\mathcal{N}$.

proof (a) If p, $q \in \mathcal{S}^\infty$ and $p \not\subseteq q$, take $(n,i) \in p \setminus q$; then there is a $q' \in \mathcal{S}^\infty$ such that $q' \supseteq q$, $\#(q'[\{n\}]) = 2^n$ and $(n,i) \notin q'$, in which case p and q' are incompatible upwards in $\mathcal{S}^\infty$. So $\mathcal{S}^\infty$ is separative upwards and 514Nb tells us that $\pi(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \mathrm{cf}\,\mathcal{S}^\infty$.

Next, there is an almost-disjoint family $\langle h_\xi \rangle_{\xi < \mathfrak{c}}$ in $\mathbb{N}^\mathbb{N}$ (5A1Mc). Identifying each h_ξ with its graph, we can regard them as members of $\mathcal{S}^\infty$; and any member of $\mathcal{S}^\infty$ includes only finitely many of them. So $\mathrm{cf}\,\mathcal{S}^\infty \ge \mathfrak{c}$. On the other hand, of course, $\mathrm{cf}\,\mathcal{S}^\infty \le \#(\mathcal{S}^\infty) = \mathfrak{c}$. So $\pi(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \mathrm{cf}\,\mathcal{S}^\infty = \mathfrak{c}$.

(b) If $m \ge 2$, let Q be the countable set of pairs (I,r) where $r \in \mathbb{N}$ and $I \in [\mathbb{N} \times \mathbb{N}]^{<\omega}$, and for $(I,r) \in Q$ set

$$A_{Ir} = \{p : p \in \mathcal{S}^\infty,\ p \cap (r \times \mathbb{N}) = I,\ \|p\| \le \frac{2^r}{m}\}.$$

Then $\bigcup_{i<m} p_i \in \mathcal{S}^\infty$ for any family $\langle p_i \rangle_{i<m}$ in A_{Ir}, that is, A_{Ir} is upwards-m-linked in $\mathcal{S}^\infty$. Also $\bigcup_{(I,r)\in Q} A_{Ir} = \mathcal{S}^\infty$, so $\mathrm{link}^\uparrow_m(\mathcal{S}^\infty) \le \omega$. Of course $c^\uparrow(\mathcal{S}^\infty)$ is infinite, and since $c^\uparrow(\mathcal{S}^\infty) \le \mathrm{link}^\uparrow_m(\mathcal{S}^\infty)$ (511Hb again), both must be ω. Now 514N tells us that

$$c(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \mathrm{link}_m(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = \omega.$$

(c) Consider the localization relation $(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S})$ of 522K. We know from 522M and 512Da that

$$\mathrm{cov}(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S}) = \mathrm{cov}(\mathcal{N}, \subseteq, \mathcal{N}) = \mathrm{cf}\,\mathcal{N}.$$

(i) Let $\mathcal{A} \subseteq \mathcal{S}$ be a set of cardinality $\mathrm{cf}\,\mathcal{N}$ such that for every $f \in \mathbb{N}^\mathbb{N}$ there is an $S \in \mathcal{A}$ such that $f \subseteq^* S$. Let $\mathcal{A}^*$ be

$$\{S : S \in \mathcal{S},\ S \setminus \bigcup \mathcal{A}' \text{ is finite for some finite } \mathcal{A}' \subseteq \mathcal{A}\};$$

then every member of $\mathcal{S}^\infty$ is included in some member of $\mathcal{A}^*$. But if $S \in \mathcal{A}^*$ then $\{p : p \in \mathcal{S}^\infty,\ p \subseteq S\}$ is upwards-directed. So

$$d^\uparrow(\mathcal{S}^\infty) \le \#(\mathcal{A}^*) \le \mathrm{cf}\,\mathcal{N}.$$

(ii) Now let $\mathcal{Q}$ be a family of upwards-centered subsets of $\mathcal{S}^\infty$ covering $\mathcal{S}^\infty$. For each $Q \in \mathcal{Q}$, $S_Q = \bigcup Q$ belongs to $\mathcal{S}$. Also every $f \in \mathbb{N}^\mathbb{N}$ belongs to $\mathcal{S}^\infty$ so is covered by some S_Q. So S_Q witnesses that $\mathrm{cf}\,\mathcal{N} = \mathrm{cov}(\mathbb{N}^\mathbb{N}, \subseteq^*, \mathcal{S}) \le \#(\mathcal{Q})$; as $\mathcal{Q}$ is arbitrary, $\mathrm{cf}\,\mathcal{N} \le d^\uparrow(\mathcal{S}^\infty)$.

(iii) 514Nd tells us that

$$d(\mathrm{RO}^\uparrow(\mathcal{S}^\infty)) = d^\uparrow(\mathcal{S}^\infty),$$

so we have equality throughout.

528R Theorem Let κ be any cardinal, and $\mathcal{S}^\infty_\kappa$ the κ-localization poset. Then $\mathrm{RO}^\uparrow(\mathcal{S}^\infty_\kappa)$ has countable Maharam type.

proof (a) If κ is finite then $\mathrm{cf}\,\mathcal{S}^\infty_\kappa$ is finite and the result is trivial. So let us suppose from now on that κ is infinite.

(b) $\mathcal{S}^\infty_\kappa$ is separative upwards. $\mathbf{P}$ If p, $q \in \mathcal{S}^\infty_\kappa$ and $p \not\subseteq q$, take $(n,\xi) \in p \setminus q$. Let $J \subseteq \kappa \setminus p[\{n\}]$ be a set of size $2^n - \#(q[\{n\}]$, and set $q' = q \cup (\{n\} \times J)$; then $q \subseteq q' \in \mathcal{S}^\infty$ and p, q' are incompatible upwards in $\mathcal{S}^\infty_\kappa$. $\mathbf{Q}$

Accordingly $[p,\infty[\ \in \mathrm{RO}^{\uparrow}(\mathcal{S}_\kappa^\infty)$ for every $p\in\mathcal{S}_\kappa^\infty$ (514Me).

(c) For $n\in\mathbb{N}$, $m<2^n$ and $\xi<\kappa$, set

$$G_{mn\xi}=\sup\{[p,\infty[\ : p\in\mathcal{S}_\kappa^\infty,\ \#(p[\{n\}])=2^n,\ (n,\xi)\in p \text{ and } \#(p[\{n\}]\cap\xi)=m\},$$

the supremum being taken in $\mathrm{RO}^{\uparrow}(\mathcal{S}_\kappa^\infty)$.

(d) If $n\in\mathbb{N}$, $m<2^n$ and $\xi<\eta<\kappa$ then $G_{mn\xi}\cap G_{mn\eta}=\emptyset$. **P** If p, $q\in\mathcal{S}_\kappa^\infty$, $\#(p[\{n\}])=\#(q[\{n\}])=2^n$, $(n,\xi)\in p$, $(n,\eta)\in q$ and $\#(p[\{n\}]\cap\xi)=\#(q[\{n\}])\cap\eta)=m$ then $p[\{n\}]\ne q[\{n\}]$, $\#(p[\{n\}]\cup q[\{n\}])>2^n$ and $[p,\infty[\ \cap\ [q,\infty[$ is empty. **Q**

(e) If $\xi<\kappa$ then $\bigcup\{G_{mn\xi}:n\in\mathbb{N},\ m<2^n\}$ is dense in $\mathcal{S}_\kappa^\infty$. **P** If $p\in\mathcal{S}_\kappa^\infty$, take $n\in\mathbb{N}$ such that $\#(p[\{n\}])<2^n$; then there is a $q\in S_\kappa^\infty$ such that $p\subseteq q$, $\xi\in q[\{n\}]$ and $\#(q[\{n\}])=2^n$. Set $m=\#(q[\{n\}]\cap\xi)$; then $[q,\infty[\ \subseteq\ [p,\infty[\ \cap G_{mn\xi}$. **Q**

Thus $\sup\{G_{mn\xi}:n\in\mathbb{N},\ m<2^n\}=1$ in $\mathrm{RO}^{\uparrow}(\mathcal{S}^\infty)$ whenever $\xi<\kappa$.

(f) Let $\mathfrak{G}$ be the order-closed subalgebra of $\mathrm{RO}^{\uparrow}(\mathcal{S}_\kappa^\infty)$ generated by $\{G_{mn\xi}:n\in\mathbb{N},\ m<2^n,\ \xi<\kappa\}$. For $n\in\mathbb{N}$ and $\xi<\kappa$ set $H_{n\xi}=[\{(n,\xi)\},\infty[$; then $H_{n\xi}=\sup_{m<2^n}G_{mn\xi}$. **P** Certainly $G_{mn\xi}\subseteq H_{n\xi}$ whenever $m<2^n$. If $\{(n,\xi)\}\subseteq p\in\mathcal{S}_\kappa^\infty$, let $q\in\mathcal{S}_\kappa^\infty$ be such that $p\subseteq q$ and $\#(q[\{n\}])=2^n$; set $m=\#(q[\{n\}]\cap\xi)$; then $[q,\infty[\ \subseteq H_{n\xi}\cap G_{mn\xi}$. Thus $\bigcup_{m<2^n}G_{mn\xi}$ is dense in $H_{n\xi}$ and $H_{n\xi}=\sup_{m<2^n}G_{mn\xi}\in\mathfrak{G}$. **Q** Consequently $H_{n\xi}\in\mathfrak{G}$.

(g) If $p\in\mathcal{S}_\kappa^\infty$ then $[p,\infty[\ =\inf_{(n,\xi)\in p}H_{n\xi}$ belongs to $\mathfrak{G}$, by 514Me. So $\mathfrak{G}$ includes an order-dense subset of $\mathrm{RO}^{\uparrow}(\mathcal{S}_\kappa^\infty)$ and must be the whole of $\mathrm{RO}^{\uparrow}(\mathcal{S}_\kappa^\infty)$; that is, $\mathrm{RO}^{\uparrow}(\mathcal{S}_\kappa^\infty)$ is τ-generated by $\{G_{mn\xi}:n\in\mathbb{N},\ m<2^n,\ \xi<\kappa\}$. With (iv) and (v), we see that the conditions of 514F are satisified with $J=\kappa$ and $I=\{(m,n):n\in\mathbb{N},\ m<2^n\}$, so that

$$\tau(\mathrm{RO}^{\uparrow}(\mathcal{S}_\kappa^\infty))\le\max(\omega,\#(I))=\omega.$$

528S The calculation of Maharam types of amoeba algebras seems to be a good deal harder. However it leads through an investigation of the structure of measure algebras, which is one of the things this book is about, so I take the space to give one of the main theorems. It depends on a special property of the standard generating families in algebras $\mathfrak{B}_\kappa$.

Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. I will say that a **well-spread basis** for $\mathfrak{A}$ is a non-decreasing sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of subsets of $\mathfrak{A}$ such that

(i) setting $D=\bigcup_{n\in\mathbb{N}}D_n$, $\#(D)\le\max(\omega,c(\mathfrak{A}),\tau(\mathfrak{A}))$;

(ii) if $a\in\mathfrak{A}$, $\gamma\in\mathbb{R}$ and $\bar\mu a<\gamma$, there is a set $D\subseteq\bigcup_{n\in\mathbb{N}}D_n$ such that $a\subseteq\sup D$ and $\bar\mu(\sup D)<\gamma$;

(iii) if $n\in\mathbb{N}$ and $\langle d_i\rangle_{i\in\mathbb{N}}$ is a sequence in D_n such that $\bar\mu(\sup_{i\in\mathbb{N}}d_i)<\infty$, there is an infinite set $J\subseteq\mathbb{N}$ such that $d=\sup_{i\in J}d_i$ belongs to D_n;

(iv) whenever $n\in\mathbb{N}$, $a\in\mathfrak{A}$ and $\bar\mu a\le\gamma'<\gamma<\bar\mu 1$, there is a $b\in\mathfrak{A}$ such that $a\subseteq b$ and $\gamma'\le\bar\mu b<\gamma$ and $\bar\mu(b\cup d)\ge\gamma$ whenever $d\in D_n$ and $d\not\subseteq a$.

528T Lemma (a) Let κ be an infinite cardinal, and $\langle e_\xi\rangle_{\xi<\kappa}$ the standard generating family in $\mathfrak{B}_\kappa$. For $n\in\mathbb{N}$ let C_n be the set of elements of $\mathfrak{B}_\kappa$ expressible as $\inf_{\xi\in I}e_\xi\cap\inf_{\xi\in J}(1\setminus e_\xi)$ where I, $J\subseteq\kappa$ are disjoint and $\#(I\cup J)\le n$. Then $\langle C_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$. Moreover,

(*) for each $n\ge1$, there is a set $C'_n\subseteq C_n$, with cardinal κ, such that $\bar\nu_\kappa c=2^{-n}$ for every $c\in C'_n$, and whenever $a\in\mathfrak{B}_\kappa\setminus\{1\}$ and $I\subseteq C'_n$ is infinite, there is a $c\in I$ such that $c'\not\subseteq a\cup c$ whenever $c\subset c'\in C_n$.

(b) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $e\in\mathfrak{A}$. If $\langle C_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}_e,\bar\mu\restriction\mathfrak{A}_e)$ and $\langle D_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}_{1\setminus e},\bar\mu\restriction\mathfrak{A}_{1\setminus e})$, then $\langle C_n\cup D_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A},\bar\mu)$.

proof (a)(i) $\langle C_n\rangle_{n\in\mathbb{N}}$ satisfies (i) of Definition 528S just because $\tau(\mathfrak{B}_\kappa)=\#(C_n)=\kappa$ for $n\ge1$, while $C_0=\{1\}$.

(ii) For $J\subseteq\kappa$, let $\mathfrak{C}_J$ be the order-closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{e_\xi:\xi\in J\}$; recall that for every $a\in\mathfrak{B}_\kappa$ there is a countable set $\operatorname{supp}a\subseteq\kappa$ such that $a\in\mathfrak{C}_J$ iff $J\supseteq\operatorname{supp}a$ (254Rd/325Mb). Of course $\#(\operatorname{supp}c)\le n$ whenever $n\in\mathbb{N}$ and $c\in C_n$.

Suppose that $a\in\mathfrak{B}_\kappa$ and $\gamma>\bar\nu_\kappa a$. Then for each $k\in\mathbb{N}$ we can find an $a_k\in\mathfrak{B}_\kappa$, with finite support, such that $\bar\nu_\kappa(a\,\triangle\,a_k)\le2^{-k-2}(\gamma-\bar\nu_\kappa a)$ (254Fe/325Jc). Set $b=\sup_{k\in\mathbb{N}}a_k$; then

$$\bar\nu_\kappa b\le\bar\nu_\kappa a+\textstyle\sum_{k=0}^\infty\bar\nu_\kappa(a_k\setminus a)<\gamma,$$

$$\bar\nu_\kappa(a\setminus b)\le\inf_{k\in\mathbb{N}}\bar\nu_\kappa(a\setminus a_k)=0,$$

so $a\subseteq b$. If $k\in\mathbb{N}$ and $\#(\operatorname{supp}a_k)=n_k$, then $a_k=\sup\{c:c\in C_{n_k},\,c\subseteq a_k\}$, so $b=\sup\{c:c\in\bigcup_{n\in\mathbb{N}}C_n,\,c\subseteq b\}$. Thus 528S(ii) is satisfied.

(iii) If $n\in\mathbb{N}$ and $\langle c_i\rangle_{i\in\mathbb{N}}$ is a sequence in C_n, there is an infinite $I\subseteq\mathbb{N}$ such that $\langle\operatorname{supp}c_i\rangle_{i\in I}$ is a Δ-system with root K say (5A1Ic). For $i\in I$, express c_i as $c'_i\cap c''_i$ where $c'_i\in\mathfrak{C}_K$ and $c''_i\in\mathfrak{C}_{(\operatorname{supp}c_i)\setminus K}$; as $\mathfrak{C}_K$ is finite, there is a c such that $J=\{i:c'_i=c\}$ is infinite. Now $c\in C_n$, and if $m\in\mathbb{N}$ then

$$\sup_{i\in J\setminus m}c_i=c\cap\sup_{i\in J\setminus m}c''_i=c$$

because $\langle c''_i\rangle_{i\in J\setminus m}$ is a stochastically independent family of elements of $\mathfrak{B}_\kappa$ all of measure at least 2^{-n}, so has supremum 1. In particular, 528S(iii) is satisfied.

(iv) Suppose that $n\in\mathbb{N}$ and $a\in\mathfrak{B}_\kappa$. Then there is a $\delta>0$ such that $\bar\nu_\kappa(c\setminus a)\ge\delta$ whenever $c\in C_n$ and $c\not\subseteq a$. **P?** Otherwise, there is a sequence $\langle c_i\rangle_{i\in\mathbb{N}}$ in C_n such that $0<\bar\nu_\kappa(c_i\setminus a)\le 2^{-i}$ for every $i\in\mathbb{N}$. By (iii) just above, there is an infinite set $J\subseteq\mathbb{N}$ such that $c_j\subseteq\sup_{i\in J\setminus m}c_i$ for every $j\in J$. Set $j_0=\min J$, and let m be such that $2^{-m+1}<\bar\nu_\kappa(c_{j_0}\setminus a)$; then

$$2^{-m+1}<\bar\nu_\kappa(\sup_{j\in J\setminus m}c_j\setminus a)\le\sum_{j=m}^\infty\bar\nu_\kappa(c_j\setminus a)\le 2^{-m+1},$$

which is absurd. **XQ**

(v) Suppose that $n\in\mathbb{N}$, $a\in\mathfrak{B}_\kappa$ and $\bar\nu_\kappa a\le\gamma'<\gamma<1$. Pick $\delta>0$, $r>n$, $k^*\in\mathbb{N}$, $\epsilon>0$ and $a'\in\mathfrak{B}_\kappa$ such that

$$\bar\nu_\kappa(c\setminus a)\ge\delta\text{ whenever }c\in C_n\text{ and }c\not\subseteq a,$$

$$2^{-r}\le\gamma-\gamma',\quad(2^{-n}-2^{-r})^n\delta\ge 2^{-r+2},$$

$$(1-2^{-r})^{k^*}\le 1-\gamma,$$

$$\epsilon\le\frac12\delta,\quad\epsilon\le 2^{-r}(1-2^{-r})^{k^*},$$

$$\operatorname{supp}a'\text{ is finite},\quad\bar\nu_\kappa(a\triangle a')\le\epsilon.$$

Let $\langle K_i\rangle_{i\in\mathbb{N}}$ be a disjoint sequence in $[\kappa\setminus\operatorname{supp}a']^r$, and set $c_i=\inf_{\xi\in K_i}e_\xi$ for each $i\in\mathbb{N}$. Then $\sup_{i\in\mathbb{N}}c_i=1$, so there is a first k such that $\bar\nu_\kappa(a\cup\sup_{i\le k}c_i)\ge\gamma$; set $b_1=\sup_{i<k}c_i$ and $b=a\cup b_1$. Surely $a\subseteq b$ and $\bar\nu_\kappa b<\gamma$; also

$$(1-2^{-r})^{k^*}\le 1-\gamma\le 1-\bar\nu_\kappa b_1=(1-2^{-r})^k$$

so $k\le k^*$. Moreover,

$$\gamma-\bar\nu_\kappa b\le\bar\nu_\kappa(b\cup c_k)-\bar\nu_\kappa b\le\bar\nu_\kappa(c_k\setminus b_1)=2^{-r}(1-2^{-r})^k\le 2^{-r}\le\gamma-\gamma',$$

so that, in particular, $\bar\nu_\kappa b\ge\gamma'$.

If $c\in C_n$ and $c\not\subseteq a$ then $\bar\nu_\kappa(c\setminus a)\ge\delta$ so $\bar\nu_\kappa(c\setminus a')\ge\delta-\epsilon$. Express c as $\inf_{i\le k}c'_i$ where $\operatorname{supp}c'_i\subseteq K_i$ for $i<k$ and $\operatorname{supp}c'_k\subseteq\kappa\setminus\bigcup_{i<k}K_i$. Set $J=\{i:i<k,\,c'_i\ne1\}$; then $\#(J)\le n$. Now

$$\begin{aligned}\bar\nu_\kappa(c\setminus(a'\cup b_1))&=\bar\nu_\kappa((c'_k\setminus a')\cap\inf_{i\in J}(c'_i\setminus c_i)\cap\inf_{i\in k\setminus J}(1\setminus c_i))\\&=\bar\nu_\kappa(c'_k\setminus a')\cdot\prod_{i\in J}\bar\nu_\kappa(c'_i\setminus c_i)\cdot\prod_{i\in k\setminus J}\bar\nu_\kappa(1\setminus c_i)\end{aligned}$$

(because $\operatorname{supp}(c'_k\setminus a')\subseteq\operatorname{supp}c'_k\cup\operatorname{supp}a'\subseteq\kappa\setminus\bigcup_{i<k}K_i$, so we are taking an infimum of stochastically independent elements of $\mathfrak{B}_\kappa$)

$$\ge(\delta-\epsilon)\cdot\prod_{i\in J}(2^{-n}-2^{-r})\cdot\prod_{i\in k\setminus J}(1-2^{-r})$$

(of course every c'_i belongs to C_n)

$$\begin{aligned}&\ge\frac12(2^{-n}-2^{-r})^n(1-2^{-r})^k\delta\ge 2^{-r+1}(1-2^{-r})^k\\&\ge 2^{-r}(1-2^{-r})^k+2^{-r}(1-2^{-r})^{k^*}\ge\gamma-\bar\nu_\kappa b+\epsilon,\end{aligned}$$

and

$$\bar{\nu}_\kappa(c \setminus b) \geq \gamma - \bar{\nu}_\kappa b,$$

so $\bar{\nu}_\kappa(b \cup c) \geq \gamma$.

As n, a, γ' and γ are arbitrary, $\langle C_n \rangle_{n\in\mathbb{N}}$ satisfies 528S(iv) and is a well-spread basis.

(vi) As for (*), given $n \geq 1$, take a disjoint family $\langle K_\xi \rangle_{\xi<\kappa}$ in $[\kappa]^n$, and set $c_\xi = \inf_{\eta\in K_\xi} e_\eta$ for $\xi < \kappa$, $C'_n = \{c_\xi : \xi < \kappa\}$. If $I \subseteq \kappa$ is infinite and $\bar{\nu}_\kappa a < 1$, take $\delta > 0$ such that $\delta < 2^{-n}(1 - \bar{\nu}_\kappa a - \delta)$, and $a' \in \mathfrak{B}_\kappa$ such that $\operatorname{supp} a'$ is finite and $\bar{\nu}_\kappa(a \triangle a') \leq \delta$. Then there is a $\xi \in I$ such that $K_\xi \cap \operatorname{supp} a_\xi = \emptyset$. **?** If $c \in C_n$ is such that $c_\xi \subset c \subseteq a \cup c_\xi$, there must be a $d \in \mathfrak{B}_\kappa$, with support K_ξ, included in $c \setminus c_\xi$. But now $d \subseteq a$ and $\operatorname{supp} d \cap \operatorname{supp} a' = \emptyset$, so

$$2^{-n}(1 - \bar{\nu}_\kappa a - \delta) \leq 2^{-n}(1 - \bar{\nu}_\kappa a') = \bar{\nu}_\kappa(d \setminus a') \leq \delta + \bar{\nu}_\kappa(d \setminus a) = \delta. \text{ **X**}$$

(b)(i) We have

$$\begin{aligned}\#(\bigcup_{n\in\mathbb{N}} C_n \cup D_n) &\leq \max(\omega, \#(\bigcup_{n\in\mathbb{N}} C_n), \#(\bigcup_{n\in\mathbb{N}} D_n)) \\ &\leq \max(\omega, c(\mathfrak{A}_e), c(\mathfrak{A}_{1\setminus e}), \tau(\mathfrak{A}_e), \tau(\mathfrak{A}_{1\setminus e})) = \max(\omega, c(\mathfrak{A}), \tau(\mathfrak{A}))\end{aligned}$$

by 514E.

(ii) Suppose that $a \in \mathfrak{A}$ and $\bar{\mu}a < \gamma$. Then there are γ_1, γ_2 such that $\bar{\mu}(a \cap e) < \gamma_1$, $\bar{\mu}(a \setminus e) < \gamma_2$ and $\gamma_1 + \gamma_2 \leq \gamma$. Let $C \subseteq \bigcup_{n\in\mathbb{N}} C_n$, $D \subseteq \bigcup_{n\in\mathbb{N}} D_n$ be such that $a \cap e \subseteq \sup C$, $a \setminus e \subseteq \sup D$, $\bar{\mu}(\sup C) < \gamma_1$ and $\bar{\mu}(\sup D) < \gamma_2$. Then $C \cup D \subseteq \bigcup_{n\in\mathbb{N}} C_n \cup D_n$, $a \subseteq \sup(C \cup D)$ and $\bar{\mu}(\sup(C \cup D)) < \gamma$.

(iii) Suppose that $n \in \mathbb{N}$ and $\langle c_i \rangle_{i\in\mathbb{N}}$ is a sequence in $C_n \cup D_n$ such that $\bar{\mu}(\sup_{i\in\mathbb{N}} c_i) < \infty$. Then there is an infinite $J \subseteq \mathbb{N}$ such that either $c_i \in C_n$ for every $i \in J$, or $c_i \in D_n$ for every $i \in I$. In either case, there is an infinite $I \subseteq J$ such that $\sup_{i\in I} c_i$ belongs to $C_n \cup D_n$.

(iv) Thus $\langle C_n \cup D_n \rangle_{n\in\mathbb{N}}$ satisfies (i)-(iii) of Definition 528S. As for 528S(iv), suppose that $n \in \mathbb{N}$, $a \in \mathfrak{A}$ and $\bar{\mu}a \leq \gamma' < \gamma < \bar{\mu}1$. We need to find a $b \in \mathfrak{A}$ such that $a \subseteq b$ and

$$\gamma' \leq \bar{\mu}b < \gamma \leq \bar{\mu}(b \cup c)$$

whenever $c \in C_n \cup D_n$ and $c \not\subseteq a$.

case 1 If $e \subseteq a$, then $\bar{\mu}e$ is finite and

$$\bar{\mu}(a \setminus e) \leq \gamma' - \bar{\mu}e < \gamma - \bar{\mu}e < \bar{\mu}(1 \setminus e).$$

So there is a $b_2 \in \mathfrak{A}_{1\setminus e}$ such that $a \setminus e \subseteq b_2$ and

$$\gamma' - \bar{\mu}e \leq \bar{\mu}b_2 < \gamma - \bar{\mu}e \leq \bar{\mu}(b_2 \cup d)$$

whenever $d \in D_n$ and $d \not\subseteq a \setminus e$; that is,

$$\gamma' \leq \bar{\mu}(e \cup b_2) < \gamma \leq \bar{\mu}(e \cup b_2 \cup d)$$

whenever $d \in D_n$ and $d \not\subseteq a$. Since $c \subseteq a$ for every $c \in C_n$, we have $\bar{\mu}(e \cup b_2 \cup c) \geq \gamma$ whenever $c \in C_n \cup D_n$ and $c \not\subseteq a$, and can take $b = e \cup b_2$.

case 2 Similarly, if $a \supseteq 1 \setminus e$, we can take $b = (1 \setminus e) \cup b_1$ for a suitable $b_1 \subseteq e$.

case 3 If neither e nor $1 \setminus e$ is included in a, we have

$$\max(\bar{\mu}(a \cap e), \gamma - \bar{\mu}(1 \setminus e)) < \min(\bar{\mu}e, \gamma - \bar{\mu}(a \setminus e)),$$

so we can find γ'_1, γ_1 such that

$$\max(\bar{\mu}(a \cap e), \gamma - \bar{\mu}(1 \setminus e)) < \gamma'_1 < \gamma_1 < \min(\bar{\mu}e, \gamma - \bar{\mu}(a \setminus e))$$

and $\gamma_1 - \gamma'_1 < \gamma - \gamma'$. Let $b_1 \in \mathfrak{A}_e$ be such that $a \cap e \subseteq b_1$ and

$$\gamma'_1 \leq \bar{\mu}b_1 < \gamma_1 \leq \bar{\mu}(b_1 \cup c)$$

whenever $c \in C_n$ and $c \not\subseteq a \cap e$. Set $\gamma'_2 = \gamma - \gamma_1$ and $\gamma_2 = \gamma - \bar{\mu}b_1$, so that

$$\bar{\mu}(a \setminus e) < \gamma'_2 < \gamma_2 \leq \gamma - \gamma'_1 < \bar{\mu}(1 \setminus e).$$

Let $b_2 \in \mathfrak{A}_{1\setminus e}$ be such that $a \setminus e \subseteq b_2$ and

$$\gamma_2' \le \bar{\mu} b_2 < \gamma_2 \le \bar{\mu}(b_2 \cup d)$$

whenever $d \in D_n$ and $d \not\subseteq a \setminus e$.

Try $b = b_1 \cup b_2$. Then $a \subseteq b$ and $\bar{\mu} b = \bar{\mu} b_1 + \bar{\mu} b_2$ belongs to

$$[\bar{\mu} b_1 + \gamma_2', \bar{\mu} b_1 + \gamma_2[\subseteq [\gamma_1' + \gamma - \gamma_1, \gamma[\subseteq [\gamma', \gamma[.$$

If $c \in C_n$ and $c \not\subseteq a$, then $c \not\subseteq a \cap e$, so

$$\bar{\mu}(b \cup c) = \bar{\mu}(b_1 \cup c) + \bar{\mu} b_2 \ge \gamma_1 + \gamma_2' = \gamma;$$

while if $d \in D_n$ and $d \not\subseteq a$, then

$$\bar{\mu}(b \cup d) = \bar{\mu} b_1 + \bar{\mu}(b_2 \cup d) \ge \mu b_1 + \gamma_2 = \gamma.$$

So in this case also we have found a suitable b.

528U Lemma Let $(\mathfrak{A}, \bar{\mu})$ be an atomless semi-finite measure algebra and $0 < \gamma < \bar{\mu} 1$. Let E, ϵ, $\preccurlyeq$ and $\mathcal{F}$ be such that

E is a partition of unity in $\mathfrak{A}$ such that $\mathfrak{A}_e$ is homogeneous and $0 < \epsilon \le \bar{\mu} e < \infty$ for every $e \in E$;

$\preccurlyeq$ is a well-ordering of E such that $\tau(\mathfrak{A}_e) \le \tau(\mathfrak{A}_{e'})$ whenever $e \preccurlyeq e'$ in E;

$\mathcal{F}$ is a partition of E such that each member of $\mathcal{F}$ is either a singleton or a countable set with no $\preccurlyeq$-greatest member.

Let P_0 be

$$\{a : a \in \mathfrak{A},\ \bar{\mu} a < \gamma,\ \gamma \le \bar{\mu}(a \cup e) \text{ whenever } \{e\} \in \mathcal{F}\},$$

ordered by $\subseteq$. Then $\mathrm{RO}^{\uparrow}(P_0)$ has countable Maharam type.

proof (a)(i) For every $e \in E$, $(\mathfrak{A}_e, \bar{\mu} \restriction \mathfrak{A}_e)$ is a non-zero atomless homogeneous totally finite measure algebra, so is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ for some infinite cardinal κ (331L). So we can copy the well-spread basis for $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ described in 528Ta into a well-spread basis $\langle D_{en} \rangle_{n \in \mathbb{N}}$ for $(\mathfrak{A}_e, \bar{\mu} \restriction \mathfrak{A}_e)$ such that

$\#(\bigcup_{n \in \mathbb{N}} D_{en}) = \tau(\mathfrak{A}_e)$,

$\bar{\mu} d \ge 2^{-n} \bar{\mu} e$ whenever $n \in \mathbb{N}$ and $d \in D_{en}$,

$D_{e0} = \{e\}$,

for each $n \ge 1$ there is a set $D'_{en} \subseteq D_{en}$, with cardinal $\tau(\mathfrak{A}_e)$, such that $\bar{\mu} d = 2^{-n} \bar{\mu} e$ for every $d \in D'_{en}$, and whenever $a \in \mathfrak{A}_e \setminus \{e\}$ and $I \subseteq D'_{en}$ is infinite, there is a $d \in I$ such that $d' \not\subseteq a \cup d$ whenever $d' \in D_{en}$ and $d' \supset d$,

$(\bigcup_{n \in \mathbb{N}} D_{en}) \setminus (\bigcup_{n \ge 1} D'_{en})$ has cardinal $\tau(\mathfrak{A}_e)$.

(The last item is not mentioned in 528T, but is clearly achievable by thinning the sets D'_{en} appropriately, besides being automatic if we use the construction in (a-vi) of the proof of 528T.) Note that $\langle D'_{en} \rangle_{n \ge 1}$ is a disjoint sequence of subsets of $\mathfrak{A}_e$ for each e, so $\langle D'_{en} \rangle_{e \in E, n \ge 1}$ is disjoint.

(ii) For $e \in F \in \mathcal{F}$, set

$$D_e = \bigcup_{n \in \mathbb{N}} D_{en} \setminus \bigcup_{n \ge 1} D'_{en}, \quad D_e^* = \bigcup_{e' \in F, e' \preccurlyeq e} D_e.$$

Because F is countable and $\tau(\mathfrak{A}_{e'}) \le \tau(\mathfrak{A}_e)$ whenever $e' \preccurlyeq e$, $\#(D_e^*) = \tau(\mathfrak{A}_e) = \#(D'_{en})$ for every $n \ge 1$. We therefore have a partition $\langle I_{ed} \rangle_{d \in D_e^*}$ of $\bigcup_{n \ge 1} D'_{en}$ into countably infinite sets such that $I_{ed} \cap D'_{en}$ is infinite whenever $d \in D_e^*$ and $n \ge 1$.

Let θ be a limit ordinal such that the set Ω of limit ordinals less than θ has cardinal $\#(\bigcup_{e \in E} D_e)$. (Of course we can take θ to be either an uncountable cardinal or the ordinal product $\omega \cdot \omega$ or 0.) Again because every member of $\mathcal{F}$ is countable, we have an enumeration $\langle d_\xi \rangle_{\xi < \theta}$ of $\bigcup_{e \in E, n \in \mathbb{N}} D_{en}$ such that whenever $\xi \in \Omega$ then there are $F \in \mathcal{F}$ and $e \in F$ such that

$$d_\xi \in D_e, \quad \{d_{\xi+i} : i \ge 1\} = \bigcup_{e' \in F, e' \succcurlyeq e} I_{e' d_\xi}.$$

This will mean that whenever $\xi \in \Omega$ and $F \in \mathcal{F}$, $e \in F$ are such that $d_\xi \in \mathfrak{A}_e$, then $\{i : d_{\xi+i} \in D'_{e'n}\}$ is infinite whenever $e' \in F$, $e \preccurlyeq e'$ and $n \in \mathbb{N}$.

(b)(i) Setting $P = \{a : a \in \mathfrak{A},\ \bar{\mu} a < \gamma\}$, $P_0 \in \mathrm{RO}^{\uparrow}(P)$. **P** Evidently P_0 is up-open. If $a \in P \setminus P_0$, that is, there is some e such that $\{e\} \in \mathcal{F}$ and $\bar{\mu}(a \cup e) < \gamma$, set $b = a \cup e$; then $a \subseteq b \in P$, while $\bar{\mu}(b' \cup e) = \bar{\mu} b' < \gamma$

whenever $b' \in [b,\infty[$, so $[b,\infty[$ does not meet P_0. Accordingly $[a,\infty[\not\subseteq \overline{P}_0$ and $a \notin \operatorname{int}\overline{P}_0$. As a is arbitrary, $P_0 = \operatorname{int}\overline{P}_0 \in \mathrm{RO}^{\uparrow}(P)$. **Q**

It follows that $\mathrm{RO}^{\uparrow}(P_0)$ is the principal ideal of $\mathrm{RO}^{\uparrow}(P)$ generated by P_0 (314R(b-ii)). Moreover, for $a \in P_0$, $[a,\infty[$ is the same whether taken in P or P_0, and belongs to $\mathrm{RO}^{\uparrow}(P)$ by 528B(b-i).

(ii) For $a \in P_0$ and $n \in \mathbb{N}$, set $A_n(a) = \{d : d \in \bigcup_{e\in E} D_{en},\ d \subseteq a\}$. Then any sequence in $A_n(a)$ has a subsequence with an upper bound in $A_n(a)$. **P** Set $L = \{e : e \in E,\ \bar\mu(a\cap e) \ge 2^{-n}\epsilon\}$; then L is finite. If $e \in E\setminus L$ and $d \in D_{en}$, then $d \subseteq e$ and

$$\bar\mu d \ge 2^{-n}\bar\mu e \ge 2^{-n}\epsilon > \bar\mu(a\cap e) \ge \bar\mu(a\cap d),$$

so $d \not\subseteq a$. Thus $A_n(a) \subseteq \bigcup_{e\in L} D_{en}$. It follows that if $\langle c_i\rangle_{i\in\mathbb{N}}$ is any sequence in $A_n(a)$, there is an $e \in L$ such that $J = \{i : c_i \in D_{en}\}$ is infinite. Now there is an infinite $I \subseteq J$ such that $c = \sup_{i\in I} c_i$ belongs to D_{en}. In this case, $c \subseteq a$ so $c \in A_n(a)$ is an upper bound of $\{c_i : i \in I\}$. **Q**

It follows that $A_n(a)$ has only finitely many maximal elements, and any non-decreasing sequence in $A_n(a)$ has an upper bound in $A_n(a)$. Consequently, every member of $A_n(a)$ is included in a maximal element of $A_n(a)$. **P?** Otherwise, we should be able to find a strictly increasing family $\langle c_\xi\rangle_{\xi<\omega_1}$ in $A_n(a)$; but now there must be a $\xi < \omega_1$ such that $\bar\mu c_\xi = \bar\mu c_{\xi+1} < \gamma$ and $c_\xi = c_{\xi+1}$. **XQ**

Set $E_n(a) = \{\xi : d_\xi$ is a maximal element of $A_n(a)\}$, so that $E_n(a)$ is a finite subset of θ.

(iii) For $n \in \mathbb{N}$, set

$$Q_n = \{b : b \in P_0,\ A_n(b) = A_n(b') \text{ whenever } b \subseteq b' \in P_0\}.$$

Then whenever $a \in P_0$ and $n \in \mathbb{N}$ there is a $b \in Q_n$ such that $a \subseteq b$ and $A_n(a) = A_n(b)$. **P** Let L be a finite subset of E including $\{e : \bar\mu(a\cap e) \ge 2^{-n-1}\epsilon\}$ and such that $\bar\mu(\sup L) > \gamma$. Then $\langle\bigcup_{e\in L} D_{em}\rangle_{m\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}_{\sup L}, \bar\mu\restriction\mathfrak{A}_{\sup L})$. (Induce on $\#(L)$, using 528Tb for the inductive step.) Since

$$\bar\mu(a\cap\sup L) < \gamma - \bar\mu(a\setminus\sup L) < \bar\mu(\sup L),$$

there is a a $b_0 \in \mathfrak{A}_{\sup L}$, including $a\cap\sup L$, such that

$$\gamma - \bar\mu(a\setminus\sup L) - 2^{-n-1}\epsilon \le \bar\mu b_0 < \gamma - \bar\mu(a\setminus\sup L) \le \bar\mu(b_0\cup d)$$

whenever $d \in \bigcup_{e\in L} D_{en}$ and $d \not\subseteq a$. Set $b = b_0\cup a$. Then $\bar\mu b = \bar\mu b_0 + \bar\mu(a\setminus\sup L) < \gamma$, so $b \in P_0$. If $b \subseteq b' \in P_0$ and $d \in \bigcup_{e\in E} D_{en}\setminus A_n(a)$, then either $e \in L$ and

$$\bar\mu(b'\cup d) \ge \bar\mu(b\cup d) + \bar\mu(a\setminus\sup L) \ge \gamma > \bar\mu b',$$

or $e \notin L$,

$$\bar\mu(d\setminus a) \ge \bar\mu d - \bar\mu(a\cap e) \ge 2^{-n}\bar\mu e - 2^{-n-1}\epsilon \ge 2^{-n-1}\epsilon$$

and

$$\bar\mu(b'\cup d) \ge \bar\mu b_0 + \bar\mu(a\setminus\sup L) + 2^{-n-1}\epsilon \ge \gamma > \bar\mu b';$$

in either case $d \not\subseteq b'$. Thus $A_n(b') = A_n(a) = A_n(b)$ whenever $b \subseteq b' \in P_0$, and $b \in Q_n$. **Q**

(c)(i) For m, n, $i \in \mathbb{N}$ and $\xi \in \Omega$, set

$$Q_{nmi\xi} = \{b : b \in Q_n,\ \xi+i \in E_n(b),\ \#(E_n(b)\cap\xi) = m\},$$

$$G_{nmi\xi} = \sup\{[b,\infty[\ : b \in Q_{nmi\xi}\} \in \mathrm{RO}^{\uparrow}(P_0).$$

(ii) For any m, n, $i \in \mathbb{N}$, $\langle G_{nmi\xi}\rangle_{\xi\in\Omega}$ is disjoint. **P** Suppose that $\xi < \eta$ in Ω. If $a \in Q_{nmi\xi}$ and $b \in Q_{nmi\eta}$, we see that $\xi+i < \eta$, $\xi+i \in E_n(a)$ and

$$\#(E_n(b)\cap\eta) = m = \#(E_n(a)\cap\xi) < \#(E_n(a)\cap\eta).$$

So $E_n(a) \ne E_n(b)$ and $A_n(a) \ne A_n(b)$. But both a and b are supposed to belong to Q_n, so $[a,\infty[$ must be disjoint from $[b,\infty[$. As b is arbitrary, $[a,\infty[\,\cap G_{nmi\eta} = \emptyset$; as a is arbitrary, $G_{nmi\xi}\cap G_{nmi\eta} = \emptyset$. **Q**

(iii) For any $\xi \in \Omega$ and $a \in P_0$, there are m, n, $i \in \mathbb{N}$ and $b \in Q_{nmi\xi}$ such that $a \subseteq b$. **P** Let $e \in E$ be such that $d_\xi \subseteq e$; let F be the member of $\mathcal{F}$ containing e. If $F = \{e\}$, then $\bar\mu(a\cup e) \ge \gamma > \bar\mu a$; set $e_0 = e$, so that $e_0 \in F$, $e_0 \succcurlyeq e$ and $a\cap e_0 \ne e_0$. Otherwise, there are infinitely many members of F greater than e for the ordering $\preccurlyeq$, because F has no greatest member, so $\bar\mu(\sup_{e'\in F, e'\succcurlyeq e} e') = \infty$, and there must be an $e_0 \in F$ such that $e_0 \succcurlyeq e$ and $a\cap e_0 \ne e_0$.

Let $n \in \mathbb{N}$ be such that $2^{-n}\bar{\mu}e_0 < \min(\gamma - \bar{\mu}a, \bar{\mu}(e_0 \setminus a))$. Then $\{d_{\xi+i} : i \in \mathbb{N}\}$ meets $D'_{e_0 n}$ in an infinite set, and there is an $i \in \mathbb{N}$ such that $d_{\xi+i} \in D'_{e_0 n}$, $\bar{\mu}d_{\xi+i} = 2^{-n}\bar{\mu}e_0$, and $d \not\subseteq (a \cap e_0) \cup d_{\xi+i}$ whenever $d \in D_{e_0 n}$ and $d \supset d_{\xi+i}$. Set $a' = a \cup d_{\xi+i}$; then $d_{\xi+i}$ is a maximal member of $A_n(a')$. Let $b \in Q_n$ be such that $a' \subseteq b$ and $A_n(b) = A_n(a')$. Then $\xi + i \in E_n(b)$. Set $m = \#(E_n(b) \cap \xi)$. Then $b \in Q_{nmi\xi}$ and $a \subseteq b$. **Q**

Accordingly $b \in [a, \infty[\cap G_{nmi\xi}$. As a is arbitrary, $\bigcup_{m,n,i\in\mathbb{N}} G_{nmi\xi}$ is dense in P_0 and $\sup_{m,n,i\in\mathbb{N}} G_{nmi\xi} = P_0$ in $\mathrm{RO}^{\uparrow}(P_0)$.

(d)(i) Let $\mathfrak{G}$ be the order-closed subalgebra of $\mathrm{RO}^{\uparrow}(P_0)$ generated by $\{G_{nmi\xi} : m, n, i \in \mathbb{N}, \xi \in \Omega\}$. By (c-ii) and (c-iii), the conditions of 514F are satisfied, and $\mathfrak{G}$ has countable Maharam type.

(ii) If $d \in P_0 \cap \bigcup_{e\in E, n\in\mathbb{N}} D_{en}$ then $[d, \infty[\in \mathfrak{G}$. **P** Set

$$H = \sup\{G_{nmi\xi} : m, n, i \in \mathbb{N}, \xi \in \Omega \text{ and } G_{nmi\xi} \subseteq [d, \infty[\} \in \mathrm{RO}^{\uparrow}(P_0).$$

Then $H \in \mathfrak{G}$ and $H \subseteq [d, \infty[$. Suppose that $a \in P_0$ and $a \supseteq d$. Let $n \in \mathbb{N}$ be such that $d \in \bigcup_{e\in E} D_{en}$. Then there is a $b \in Q_n$ such that $a \subseteq b$. In this case, $d \in A_n(b)$ so there is a maximal $d' \in A_n(b)$ including d; let $\xi \in \Omega$, $i \in \mathbb{N}$ be such that $d' = d_{\xi+i}$, and set $m = \#(E_n(b) \cap \xi)$. Then $b \in Q_{nmi\xi}$. On the other hand, for any $b' \in Q_{nmi\xi}$, $d \subseteq d_{\xi+i} \subseteq b'$, so $[b', \infty[\subseteq [d, \infty[$; as b' is arbitrary, $G_{nmi\xi} \subseteq [d, \infty[$ and $G_{nmi\xi} \subseteq H$. Accordingly $b \in H \cap [a, \infty[$. As a is arbitrary, H is dense in $[d, \infty[$ and must be the whole of $[d, \infty[$; thus we have $[d, \infty[= H \in \mathfrak{G}$. **Q**

(iii) If $a \in P_0$ there is a $b \in P_0$ such that $a \subseteq b$ and $[b, \infty[\in \mathfrak{G}$. **P** Let E_0 be a countable subset of E such that $a \subseteq \sup E_0$ and $\bar{\mu}(\sup E_0) > \gamma$. Set $L = \{e : e \in E_0, a \supseteq e\}$. Then $E_0 \setminus L$ is non-empty, and

$$\sum_{e\in E_0\setminus L} \bar{\mu}(a \cap e) = \bar{\mu}a - \bar{\mu}(\sup L) < \gamma - \bar{\mu}(\sup L).$$

We therefore have a family $\langle\gamma_e\rangle_{e\in E_0\setminus L}$ such that $\bar{\mu}(a \cap e) < \gamma_e \le \bar{\mu}e$ for every $e \in E_0 \setminus L$ and $\sum_{e\in E_0\setminus L} \gamma_e < \gamma - \bar{\mu}(\sup L)$. For each $e \in E_0$ there is a $B_e \subseteq \bigcup_{n\in\mathbb{N}} D_{en}$ such that $a \cap e \subseteq \sup B_e$ and $\bar{\mu}(\sup B_e) \le \gamma_e$, by 528S(ii). Set

$$B = L \cup \bigcup_{e\in E_0\setminus L} B_e \subseteq \bigcup_{e\in E, n\in\mathbb{N}} D_{en}$$

and $b = \sup B$. Then $a \subseteq b$ and

$$\bar{\mu}b = \bar{\mu}(\sup L) + \sum_{e\in E_0\setminus L} \bar{\mu}(\sup B_e) \le \bar{\mu}(\sup L) + \sum_{e\in E_0\setminus L} \gamma_e < \gamma,$$

so $b \in P_0$. On the other hand,

$$[b, \infty[= \bigcap_{d\in B} [d, \infty[= \inf_{d\in B} [d, \infty[\in \mathfrak{G},$$

as required. **Q**

(iv) As a is arbitrary, $\mathfrak{G}$ includes a π-base for the Boolean algebra $\mathrm{RO}^{\uparrow}(P_0)$ and must be the whole of $\mathrm{RO}^{\uparrow}(P_0)$. Accordingly

$$\tau(\mathrm{RO}^{\uparrow}(P_0)) = \tau(\mathfrak{G}) \le \omega.$$

This completes the proof.

528V Theorem Let $(\mathfrak{A}, \bar{\mu})$ be an atomless semi-finite measure algebra and $0 < \gamma < \bar{\mu}1$. Then $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ has countable Maharam type.

proof Throughout the proof, P will stand for $\{a : a \in \mathfrak{A}, \bar{\mu}a < \gamma\}$.

(a) Suppose that there are a partition E of unity in $\mathfrak{A}$ and an $\epsilon > 0$ such that $\mathfrak{A}_e$ is homogeneous and $\epsilon \le \bar{\mu}e < \infty$ for every $e \in E$.

(i) Let $\preccurlyeq$ be a well-ordering of E such that $\tau(\mathfrak{A}_e) \le \tau(\mathfrak{A}_{e'})$ whenever $e \preccurlyeq e'$ in E. Let $\mathcal{F}_0$ be a maximal disjoint family of subsets of E of order type ω in the ordering induced by $\preccurlyeq$. Then $M = E \setminus \bigcup \mathcal{F}_0$ must be finite; set $\mathcal{F} = \mathcal{F}_0 \cup \{\{e\} : e \in M\}$.

(ii) For $L \subseteq M$, set

$$P_L = \{a : a \in P, a \supseteq \sup L, \bar{\mu}(a \cup e) \ge \gamma \text{ for } e \in M \setminus L\}.$$

Then $\langle P_L\rangle_{L\subseteq M}$ is a disjoint family of open subsets of P. Also $\bigcup_{L\subseteq M} P_L$ is dense in P. **P** If $a \in P$, let $L \subseteq M$ be a maximal set such that $\bar{\mu}(a \cup \sup L) < \gamma$, and set $b = a \cup \sup L$; then $a \subseteq b \in P_L$. **Q** So $\mathrm{RO}^{\uparrow}(P)$ is isomorphic to the simple product $\prod_{L\subseteq M} \mathrm{RO}^{\uparrow}(P_L)$ (315H again).

(iii) If $L \subseteq M$, then $\mathrm{RO}^\uparrow(P_L)$ has countable Maharam type. **P** If $P_L = \emptyset$ this is trivial. Otherwise there is an $a \in P_L$ and $\bar{\mu}(\sup L) \le \bar{\mu}a < \gamma$. Consider $\mathfrak{A}' = \mathfrak{A}_{1 \setminus \sup L}$, $\gamma' = \gamma - \bar{\mu}(\sup L)$, $E' = E \setminus L$, $\mathcal{F}' = \mathcal{F} \setminus \{\{e\} : e \in L\}$ and $\preccurlyeq' = \preccurlyeq \cap (E' \times E')$. Then $(\mathfrak{A}', \bar{\mu} \restriction \mathfrak{A}')$, γ', E', ϵ, $\preccurlyeq'$ and $\mathcal{F}'$ satisfy the conditions of 528U. Setting

$$Q_0 = \{c : c \in \mathfrak{A}',\ \bar{\mu}c < \gamma' \le \bar{\mu}(c \cup e) \text{ for every } e \in M \setminus L\},$$

$\mathrm{RO}^\uparrow(Q_0)$ has countable Maharam type, by 528U. But the map $c \mapsto c \cup \sup L$ is an order-isomorphism between Q_0 and P_L, so $\mathrm{RO}^\uparrow(P_L)$ has countable Maharam type. **Q**

(iv) Thus $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma) = \mathrm{RO}^\uparrow(P)$ is isomorphic to the product of finitely many Boolean algebras with countable Maharam type, and has countable Maharam type (514Ef).

(b) Now suppose that $(\mathfrak{A}, \bar{\mu})$ is localizable.

(i) In this case, let E be a partition of unity in $\mathfrak{A}$ such that $\mathfrak{A}_e$ is homogeneous and $0 < \bar{\mu}e < \infty$ for every $e \in E$. Let $\epsilon > 0$ be such that $\sum_{e \in E, \bar{\mu}e \ge \epsilon} \bar{\mu}e > \gamma$. For each $k \in \mathbb{N}$, set

$$E_k = \{e : e \in E,\ \bar{\mu}e \ge 2^{-k}\epsilon\}, \quad e_k^* = \sup E_k.$$

By (a), $\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu} \restriction \mathfrak{A}_{e_k^*}, \gamma)$ has countable Maharam type for every k.

(ii) Now 528Fb tells us that we have a sequence $\langle \pi_k \rangle_{k \in \mathbb{N}}$ such that π_k is a regular embedding of $\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu} \restriction \mathfrak{A}_{e_k^*}, \gamma)$ into $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ for each k, and $\bigcup_{k \in \mathbb{N}} \pi_k[\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu} \restriction \mathfrak{A}_{e_k^*}, \gamma)]$ τ-generates $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$. So $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ has countable Maharam type. **P** For each k, we have a countable τ-generating set $D_k \subseteq \mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu} \restriction \mathfrak{A}_{e_k^*}, \gamma)$. Let $\mathfrak{G}$ be the order-closed subalgebra of $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ generated by $D = \bigcup_{k \in \mathbb{N}} \pi_k[D_k]$. For each $k \in \mathbb{N}$, $\pi_k^{-1}[\mathfrak{G}]$ is an order-closed subalgebra of $\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu} \restriction \mathfrak{A}_{e_k^*}, \gamma)$ including D_k, so is the whole of $\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu} \restriction \mathfrak{A}_{e_k^*}, \gamma)$, that is, $\pi_k[\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu} \restriction \mathfrak{A}_{e_k^*}, \gamma)] \subseteq \mathfrak{G}$. Since $\bigcup_{k \in \mathbb{N}} \pi_k[\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar{\mu} \restriction \mathfrak{A}_{e_k^*}, \gamma)]$ τ-generates $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$, $\mathfrak{G} = \mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ and $\tau(\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)) \le \#(D) \le \omega$. **Q**

(c) Thus we have the result when $(\mathfrak{A}, \bar{\mu})$ is localizable. For the general case of atomless semi-finite $(\mathfrak{A}, \bar{\mu})$, let $(\widehat{\mathfrak{A}}, \tilde{\mu})$ be the localization of $(\mathfrak{A}, \bar{\mu})$ (322Q). Since the embedding $\mathfrak{A} \subsetneqq \widehat{\mathfrak{A}}$ identifies $\mathfrak{A}^f$ with $\widehat{\mathfrak{A}}^f$ (322P), $\{a : a \in \widehat{\mathfrak{A}},\ \tilde{\mu}a < \gamma\}$ can be identified with P, and the regular open algebras $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ and $\mathrm{AM}(\widehat{\mathfrak{A}}, \tilde{\mu}, \gamma)$ are isomorphic. Again because $\mathfrak{A}^f$ and $\widehat{\mathfrak{A}}^f$ are isomorphic, $\widehat{\mathfrak{A}}$ is atomless. By (b), the common Maharam type of $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ and $\mathrm{AM}(\widehat{\mathfrak{A}}, \tilde{\mu}, \gamma)$ is countable.

528X Basic exercises (a) Suppose that (X, Σ, μ) is a measure space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Let $\mathcal{E} \subseteq \Sigma$ be a family such that μ is outer regular with respect to $\mathcal{E}$, and P the set $\{(E, \alpha) : E \in \mathcal{E},\ \mu E < \alpha \le \mu X\}$, ordered by saying that $(E, \alpha) \le (F, \beta)$ if $E \subseteq F$ and $\beta \le \alpha$. Show that $\mathrm{RO}^\uparrow(P)$ is isomorphic to $\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless quasi-homogeneous semi-finite measure algebra. Show that $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ is homogeneous whenever $0 < \gamma < \bar{\mu}1$. (*Hint*: first check that $\mathfrak{A} \cong \mathfrak{A}_{1 \setminus a}$ whenever $a \in \mathfrak{A}$ and $0 < \bar{\mu}a < \bar{\mu}1$.)

(c)(i) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra. Show that $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \bar{\mu}1)$ is isomorphic to $\mathfrak{A}$. (ii) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless measure algebra and $e \in \mathfrak{A}$ a non-zero element of finite measure. Show that the principal ideal $\mathfrak{A}_e$ can be regularly embedded in $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \bar{\mu}e)$.

(d) Show that if $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, $0 < \gamma \le 1$ and $\kappa \ge \max(\omega, \tau(\mathfrak{A}))$ then $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$ can be regularly embedded in $\mathrm{AM}(\mathfrak{B}_\kappa, \bar{\nu}_\kappa, \gamma)$.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $(\mathfrak{A}, \bar{\mu})$ its amoeba algebra. Show that if $0 < \gamma < \mu X$ then the additivity of μ is not a precaliber of $\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)$.

(f) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless σ-finite measure algebra and $0 < \gamma < \bar{\mu}1$. Show that $\mathfrak{m}(\mathrm{AM}(\mathfrak{A}, \bar{\mu}, \gamma)) = \mathrm{wdistr}(\mathfrak{A})$.

(g) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless semi-finite measure algebra. (i) Show that

$$c(\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})) = \mathrm{link}_m(\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})) = \max(c(\mathfrak{A}), \tau(\mathfrak{A}))$$

for any integer $m \ge 2$. (ii) Show that

$$d(\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})) = \pi(\mathrm{AM}^*(\mathfrak{A}, \bar{\mu})) = \max(\mathrm{cf}[c(\mathfrak{A})]^{\le\omega}, \pi(\mathfrak{A})).$$

(h) Show that for any cardinal κ there is a probability algebra $(\mathfrak{A},\bar\mu)$ such that $\mathrm{AM}(\mathfrak{A},\bar\mu,\frac{1}{2})$ has Maharam type κ.

528Y Further exercises (a) Let $(\mathfrak{A},\bar\mu)$ be an atomless quasi-homogeneous semi-finite measure algebra. Show that $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$ is homogeneous.

(b) Let $(\mathfrak{A},\bar\mu)$ be an atomless totally finite measure algebra, and suppose that $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ can be regularly embedded in $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$ for every $\gamma\in\,]0,\bar\mu 1[$. Show that $\mathfrak{A}$ is homogeneous.

(c) Show that $\mathfrak{B}_{\omega_1}$ cannot be regularly embedded in $\mathrm{AM}(\mathfrak{B}_\omega,\bar\nu_\omega,\frac{1}{2})$.

(d) Let $(\mathfrak{A},\bar\mu)$ be an atomless probability algebra and $\gamma\in\,]0,1[$. Show that $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ is not weakly (σ,∞)-distributive.

(e) Let κ be an infinite cardinal. Show that (i) $\pi(\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty))=\mathrm{cf}\,\mathcal{S}_\kappa^\infty$ is the cardinal power κ^ω; (ii) for every $m\ge 2$,

$$c(\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty))=c^\uparrow(\mathcal{S}_\kappa^\infty)=\mathrm{link}_m(\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty))=\mathrm{link}_m^\uparrow(\mathcal{S}_\kappa^\infty)=\kappa;$$

(iii) $d(\mathrm{RO}^\uparrow(\mathcal{S}_\kappa^\infty))=d^\uparrow(\mathcal{S}_\kappa^\infty)=\max(\mathrm{cf}\mathcal{N},\mathrm{cf}[\kappa]^{\le\omega})$.

(f) Let $(\mathfrak{A},\bar\mu)$ be a purely atomic semi-finite measure algebra of magnitude at most $\mathfrak{c}$, and $0<\gamma<\bar\mu 1$. Show that $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$ has countable Maharam type.

(g) Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra and $0<\gamma<\bar\mu 1$. Set $\kappa=\max(\omega,c(\mathfrak{A}),\tau(\mathfrak{A}))$ and $P=\{a:a\in\mathfrak{A},\ \bar\mu a<\gamma\}$; let $\mathbb{P}$ be the forcing notion $(P,\subseteq,0,\uparrow)$ (see 5A3A). Show that $\Vdash_{\mathbb{P}}\check\kappa<\omega_1$.

(h) Show that if $(\mathfrak{A},\bar\mu)$ is a measure algebra with at most $\mathfrak{c}$ atoms, then $\tau(\mathrm{AM}^*(\mathfrak{A},\bar\mu))\le\omega$.

528Z Problems (a) Let $(\mathfrak{A}_L,\bar\mu_L)$ be the measure algebra of Lebesgue measure on $\mathbb{R}$. Is the amoeba algebra $\mathrm{AM}(\mathfrak{A}_L,\bar\mu_L,1)$ isomorphic to the amoeba algebra $\mathrm{AM}(\mathfrak{B}_\omega,\bar\nu_\omega,\frac{1}{2})$?

(b) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$, and $0<\gamma<1$. Is it necessarily true that $\mathrm{AM}(\mathfrak{B},\bar\mu\restriction\mathfrak{B},\gamma)$ can be regularly embedded in $\mathrm{AM}(\mathfrak{A},\bar\mu,\gamma)$? (See 528Xd and 528G.)

528 Notes and comments The ideas of 528A-528K are based on Truss 88. The original amoeba algebras of Martin & Solovay 70, used in their proof that $\mathrm{add}\,\mathcal{N}\ge\mathfrak{m}$ (528L), are closest to 528C. For some more about the amoeba algebras derived from Lebesgue measure, see Bartoszyński & Judah 95, §3.4. In this section I have been willing to assume that the measure algebras involved are atomless; amoeba algebras are surely still interesting for other measure algebras, but the new questions seem to be combinatoric rather than measure-theoretic. It seems still to be unknown whether the algebras $\mathrm{AM}(\mathfrak{A}_L,\bar\mu_L,1)$ and $\mathrm{AM}(\mathfrak{B}_\omega,\bar\nu_\omega,\frac{1}{2})$ are actually isomorphic, rather than just mutually embeddable (528K, 528Za).

If we think of the partially ordered sets of 528A and 528I as forcing notions, we can study them in terms of the forcing universes they lead to. This is associated with the prominence of 'regular embeddings' in this section. I will not attempt to use such methods here, but I mention them because results such as 528Yg have been part of the impulse for studying amoeba algebras, and led naturally to 528Ya, 528R and 528U-528V.

529 Further partially ordered sets of measure theory

I end the chapter with notes on some more structures which can be approached by the methods used earlier. The Banach lattices of Chapter 36 are of course partially ordered sets, and many of them can easily be assigned places in the Tukey classification (529C, 529D, 529Xa). More surprising is the fact that the Novák numbers of $\{0,1\}^I$, for large I, are supported by the additivity of Lebesgue measure (529F); this is associated with an interesting property of the localization poset from the last section (529E). There is a similarly unexpected connexion between the covering number of Lebesgue measure and 'reaping numbers' $\mathfrak{r}(\omega_1,\lambda)$ for large λ (529H).

529A Notation As in previous sections, I will write $\mathcal{N}(\mu)$ for the null ideal of μ in a measure space (X,Σ,μ), and $\mathcal{N}$ for the null ideal of Lebesgue measure on $\mathbb{R}$.

529B Proposition Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra.

(a) For $p\in[1,\infty[$, give $L^p=L^p(\mathfrak{A},\bar{\mu})$ (definition: 366A) its norm topology. Then its topological density is

$$\begin{aligned} d(L^p) &= 1 \text{ if } \mathfrak{A}=\{0\},\\ &= \omega \text{ if } 0<\#(\mathfrak{A})<\omega,\\ &= \max(c(\mathfrak{A}),\tau(\mathfrak{A})) \text{ if } \mathfrak{A} \text{ is infinite.}\end{aligned}$$

(b) Give $L^0=L^0(\mathfrak{A})$ its topology of convergence in measure (367L). Then

$$\begin{aligned} d(L^0) &= 1 \text{ if } \mathfrak{A}=\{0\},\\ &= \omega \text{ if } 0<\#(\mathfrak{A})<\omega,\\ &= \tau(\mathfrak{A}) \text{ if } \mathfrak{A} \text{ is infinite.}\end{aligned}$$

proof (a)(i) The case in which $\mathfrak{A}$ is finite is elementary, since in this case $L^p\cong\mathbb{R}^n$, where n is the number of atoms of $\mathfrak{A}$. So henceforth let us suppose that $\mathfrak{A}$ is infinite.

(ii) If $\mathfrak{A}^f$ is the set of elements of $\mathfrak{A}$ of finite measure, we have a natural injection $a\mapsto\chi a:\mathfrak{A}^f\to L^p$, and $\|\chi a-\chi b\|_p=\mu(a\mathbin{\triangle}b)^{1/p}$, so χ is a homeomorphism for the measure metric on $\mathfrak{A}^f$ and the norm metric on L^p. It follows that the density $d(A)$ of $A=\{\chi a:a\in\mathfrak{A}^f\}$ for the norm topology is equal to the density of $\mathfrak{A}^f$ for the strong measure-algebra topology, which is $\max(c(\mathfrak{A}),\tau(\mathfrak{A}))$, by 521Eb. So

$$\max(c(\mathfrak{A}),\tau(\mathfrak{A}))=d(A)\le d(L^p)$$

by 5A4B(h-ii). In the other direction, if A_0 is a dense subset of A of size $d(A)$ and D is the set of rational linear combinations of members of A_0, $\overline{D}\supseteq S(\mathfrak{A}^f)$ is dense in L^p (366C), so

$$d(L^p)\le\#(D)\le\max(\omega,\#(A_0))=\max(c(\mathfrak{A}),\tau(\mathfrak{A})).$$

(b) Again, the case of finite $\mathfrak{A}$ is trivial, so we need consider only infinite $\mathfrak{A}$. In this case, $\tau(\mathfrak{A})$ is equal to the topological density $d_{\mathfrak{T}}(\mathfrak{A})$ of $\mathfrak{A}$ with its measure-algebra topology $\mathfrak{T}$ (521Ea).

(i) Let $A\subseteq\mathfrak{A}$ be a topologically dense set of cardinal $\tau(\mathfrak{A})$. Set

$$D=\{\sum_{i=0}^n q_i\chi a_i:q_0,\dots,q_n\in\mathbb{Q},\ a_0,\dots,a_n\in\mathfrak{A}\},$$

so that $D\subseteq L^0$ has cardinal $\tau(\mathfrak{A})$. Because $a\mapsto\chi a:\mathfrak{A}\to L^0$ is continuous (367R), the closure $\overline{D}$ of D includes $\{\chi a:a\in\mathfrak{A}\}$. Because $\overline{D}$ is a linear subspace of L^0, it includes $S(\mathfrak{A})$. Because $S(\mathfrak{A})$ is dense in L^0 (367Nc), $\overline{D}=L^0$ and $d(L^0)\le\#(D)=\tau(\mathfrak{A})$.

(ii) Let $B\subseteq L^0$ be a dense set with cardinal $d(L^0)$. Set

$$A=\{[\![u>\tfrac{1}{2}]\!]:u\in B\},$$

so that $A\subseteq\mathfrak{A}$ and $\#(A)\le d(L^0)$. Then A is topologically dense in $\mathfrak{A}$. **P** If $c\in\mathfrak{A}$, $a\in\mathfrak{A}^f$ and $\epsilon>0$, there is a $u\in B$ such that $\int|u-\chi c|\wedge\chi a\le\frac{1}{2}\epsilon$. But in this case, setting $b=[\![u>\frac{1}{2}]\!]$, $|\chi(b\mathbin{\triangle}c)|\le 2|u-\chi c|$, so

$$\bar{\mu}(a\cap(b\triangle c))\le 2\int|u-\chi c|\wedge\chi a\le\epsilon.$$

As c, a and ϵ are arbitrary, A is topologically dense in $\mathfrak{A}$. **Q**

Accordingly

$$\tau(\mathfrak{A})=d_{\mathfrak{T}}(\mathfrak{A})\le\#(A)\le d(L^0)$$

and $d(L^0)=\tau(\mathfrak{A})$, as claimed.

529C Theorem (Fremlin 91) Let U be an L-space. Then $U\equiv_{\mathrm{T}}\ell^1(\kappa)$, where $\kappa=\dim U$ if U is finite-dimensional, and otherwise is the topological density of U.

proof (a) The finite-dimensional case is trivial, since in this case U and $\ell^1(\kappa)$ are isomorphic as Banach lattices. So henceforth let us suppose that U is infinite-dimensional. Now $\vee:U\times U\to U$ is uniformly continuous. **P** We have only to observe that $u\vee v=\frac{1}{2}(u+v+|u-v|)$ in any Riesz space, that addition and subtraction are uniformly continuous in any linear topological space, and that $u\mapsto|u|$ is uniformly continuous just because $||u|-|v||\le|u-v|$ (see 354B). **Q** So 524C, with $Q=P=U$, tells us that $U\preccurlyeq_{\mathrm{T}}\ell^1(\kappa)$.

The rest of the proof will therefore be devoted to showing that $\ell^1(\kappa) \preccurlyeq_{\mathrm{T}} U$.

(b) By Kakutani's theorem (369E), there is a localizable measure algebra $(\mathfrak{A}, \bar\mu)$ such that U is isomorphic, as Banach lattice, to $L^1(\mathfrak{A}, \bar\mu)$. Let $\langle a_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ such that $0 < \bar\mu a_i < \infty$ and the principal ideal $\mathfrak{A}_{a_i}$ is homogeneous for each i. Set $\kappa_i = \tau(\mathfrak{A}_{a_i})$, so that κ_i is either 0 or infinite for every i, and $\kappa = \max(\#(I), \sup_{i\in I}\kappa_i)$ by 529Ba.

It will simplify the calculations to follow if we arrange that all the a_i have measure 1. To do this, set $\bar\nu a = \sum_{i\in I}\frac{\bar\mu(a\cap a_i)}{\bar\mu a_i}$ for $a \in \mathfrak{A}$; that is, $\bar\nu a = \int_a w\,d\bar\mu$, where $w = \sup_{i\in I}\frac{1}{\bar\mu a_i}\chi a_i$ in $L^0(\mathfrak{A})$. In this case, $\int v\,d\bar\nu = \int v \times w\,d\bar\nu$ for every $v \in L^1(\mathfrak{A}, \bar\nu)$, while $\int u\,d\bar\mu = \int \frac{1}{w} \times u\,d\bar\nu$ for every $u \in L^1(\mathfrak{A}, \bar\mu)$ (365S[3]). But this means that $u \mapsto u \times \frac{1}{w}$ is a Banach lattice isomorphism between $L^1(\mathfrak{A}, \bar\mu)$ and $L^1(\mathfrak{A}, \bar\nu)$, and U is isomorphic, as L-space, to $L^1 = L^1(\mathfrak{A}, \bar\nu)$; while $\bar\nu a_i = 1$ for every i.

(c) There are a set J, with cardinal κ, and a family $\langle u_j\rangle_{j\in J}$ in L^1 such that $\#(J) = \kappa$, $\|u_j\| \le 2$ for every $j \in J$ and $\|\sup_{j\in K} u_j\| \ge \frac{1}{2}\sqrt{\#(K)}$ for every finite $K \subseteq J$. **P** Set

$$J = \{(i,0) : i \in I,\ \kappa_i = 0\} \cup \{(i,\xi) : i \in I,\ \xi < \kappa_i\}.$$

Then $\#(J) = \kappa$. If $i \in I$ and $\kappa_i = 0$, set $u_{(i,0)} = \chi a_i$. If $i \in I$ and $\kappa_i > 0$, then $(\mathfrak{A}_{a_i}, \bar\nu\restriction\mathfrak{A}_{a_i})$ is a homogeneous probability algebra with Maharam type $\kappa_i \ge \omega$, so is isomorphic to the measure algebra $(\mathfrak{C}_i, \bar\lambda_i)$ of $]0,1]^{\kappa_i}$ with its usual measure λ_i, the product of Lebesgue measure on each copy of $]0,1]$ (334E). For $\xi < \kappa_i$, set

$$h_{i\xi}(t) = \frac{1}{\sqrt{t(\xi)}} \text{ for } t \in\ ]0,1]^{\kappa_i},$$

and let $u_{(i,\xi)} \in L^1$ correspond to $h^\bullet_{i\xi} \in L^1(\lambda_i) \cong L^1(\mathfrak{C}_i, \bar\lambda_i)$ (365B). Of course

$$\|u_{(i,\xi)}\| = \int h_{i\xi}(t)\lambda_i(dt) = \int_0^1 \frac{1}{\sqrt{\alpha}}d\alpha$$

(because the coordinate map $t \mapsto t(\xi)$ is inverse-measure-preserving)

$$= 2.$$

If $L \subseteq \kappa_i$ is finite and not empty, then $\|\sup_{\xi\in L} u_{(i,\xi)}\| = \int g\,d\lambda_i$ where $g = \sup_{\xi\in L} h_{i\xi}$, that is, $g(t) = \sup_{\xi\in L}\frac{1}{\sqrt{t(\xi)}}$ for $t \in\ ]0,1]^{\kappa_i}$. Now, for any $\alpha \ge 1$,

$$\begin{aligned}\lambda_i\{t : g(t) \le \alpha\} &= \lambda_i\{t : \alpha^2 t(\xi) \ge 1 \text{ for every } \xi \in L\}\\ &= (1 - \frac{1}{\alpha^2})^{\#(L)} \le \max(\frac{1}{2}, 1 - \frac{1}{2\alpha^2}\#(L))\end{aligned}$$

(induce on $\#(L)$)

$$= 1 - \frac{1}{2\alpha^2}\#(L) \text{ if } \alpha \ge \sqrt{\#(L)}.$$

So

$$\|\sup_{\xi\in L} u_{(i,\xi)}\| = \int g\,d\lambda_i = \int_0^\infty \lambda_i\{t : g(t) > \alpha\}d\alpha$$

(252O)

$$\begin{aligned}&\ge \int_{\sqrt{\#(L)}}^\infty \lambda_i\{t : g(t) > \alpha\}d\alpha\\ &\ge \int_{\sqrt{\#(L)}}^\infty \frac{1}{2\alpha^2}\#(L)d\alpha = \frac{1}{2}\sqrt{\#(L)}.\end{aligned}$$

[3]Formerly 365T.

What this means is that if $K \subseteq J$ is finite and all the first coordinates of members of K are the same, then $\|\sup_{j\in K} u_j\| \ge \frac{1}{2}\sqrt{\#(K)}$. In general, if $K \subseteq J$ is finite, then for each $i \in I$ set $L_i = \{\xi : (i,\xi) \in K\}$. Set $v_i = 0$ if L_i is empty, $\sup_{\xi\in L_i} u_{(i,\xi)}$ otherwise, so that $\|v_i\| \ge \frac{1}{2}\sqrt{\#(L_i)}$; now $\sup_{j\in K} u_j = \sum_{i\in I} v_i$, so

$$\|\sup_{j\in K} u_j\| = \sum_{i\in I}\|v_i\| \ge \frac{1}{2}\sum_{i\in I}\sqrt{\#(L_i)} \ge \frac{1}{2}\sqrt{\sum_{i\in I}\#(L_i)} = \frac{1}{2}\sqrt{\#(K)},$$

as required. **Q**

(d) We can now apply the idea of the proof of 524C, as follows. The density of $\ell^1(\kappa)$ is of course κ, by 529Ba applied to counting measure on κ, or otherwise. Index a dense subset of $\ell^1(\kappa)$ as $\langle y_j\rangle_{j\in J}$. For each $x \in \ell^1$, choose a sequence $\langle k(x,n)\rangle_{n\in\mathbb{N}}$ in J such that

$$\|x - \sum_{m=0}^{n} y_{k(x,m)}\| \le 8^{-n}$$

for every n. Note that

$$\|y_{k(x,n)}\| \le \|x - \sum_{m=0}^{n} y_{k(x,m)}\| + \|x - \sum_{m=0}^{n-1} y_{k(x,m)}\| \le 9\cdot 8^{-n}$$

for each n. Choose $f(x) \in L^1$ such that $\|f(x)\| \ge \|x\|$ and $f(x) \ge \sum_{n=0}^{\infty} 2^{-n}u_{k(x,n)}$; this is possible because $\{u_{k(x,n)} : n \in \mathbb{N}\}$ is bounded.

(e) f is a Tukey function. **P** Take $v \in L^1$ and set

$$A = \{x : f(x) \le v\}, \quad K_n = \{k(x,n) : x \in A\}$$

for $n \in \mathbb{N}$. Fix n for the moment. If $j \in K_n$, then there is an $x \in A$ such that $j = k(x,n)$ and

$$u_j = u_{k(x,n)} \le 2^n f(x) \le 2^n v,$$

while $\|y_j\| = \|y_{k(x,n)}\| \le 9\cdot 8^{-n}$. If $K \subseteq K_n$ is finite, $\|2^n v\| \ge \frac{1}{2}\sqrt{\#(K)}$, by (c); so $\#(K_n) \le 2^{2n+2}\|v\|^2$. This means that if we set $z_n = \sum_{j\in K_n}|y_j|$ we shall have $\|z_n\| \le 9\cdot 8^{-n}\#(K_n) \le 36\cdot 2^{-n}\|v\|^2$, while $y_{k(x,n)} \le z_n$ for every $x \in A$.

Now $z = \sum_{n=0}^{\infty} z_n$ is defined in $\ell^1(\kappa)$, and if $x \in A$ then $\sum_{m=0}^{n} y_{k(x,m)} \le z$ for every $n \in \mathbb{N}$, so that $x \le z$. Thus A is bounded above in $\ell^1(\kappa)$. As v is arbitrary, f is a Tukey function. **Q**

(f) Accordingly $\ell^1(\kappa) \preccurlyeq_{\mathrm{T}} L^1 \cong U$, and the proof is complete.

529D Theorem (Fremlin 91) Let $\mathfrak{A}$ be a homogeneous measurable algebra with Maharam type $\kappa \ge \omega$. Then $L^0(\mathfrak{A}) \equiv_{\mathrm{T}} \ell^1(\kappa)$.

proof (a) Let $\bar\mu$ be such that $(\mathfrak{A}, \bar\mu)$ is a probability algebra. If we give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure, its density is κ, by 529Bb. Moreover, this topology is defined by the metric $(u,v) \mapsto \int |u-v| \wedge \chi 1$, under which the lattice operation $\vee$ is uniformly continuous. **P** Just as in part (a) of the proof of 529C, we have $u \vee v = \frac{1}{2}(u+v+|u-v|)$ for all u and v, addition and subtraction are uniformly continuous, and $u \mapsto |u|$ is uniformly continuous. **Q** So, just as in 529C, we can use 524C to see that $L^0 \preccurlyeq_{\mathrm{T}} \ell^1(\kappa)$.

(b) For the reverse connection, I repeat ideas from the proof of 529C. $(\mathfrak{A}, \bar\mu)$ is isomorphic to the measure algebra $(\mathfrak{C}, \bar\lambda)$ of $]0,1]^\kappa$ with its usual measure λ. For $\xi < \kappa$ and $t \in\,]0,1]^\kappa$ set $h_\xi(t) = \frac{1}{\sqrt{t(\xi)}}$, and set $u_\xi = h_\xi^\bullet$ in $L^0(\lambda) \cong L^0(\mathfrak{C})$ (364Ic). This time, observe that if $x \in \ell^1(\kappa)^+$ and $\alpha \ge \sqrt{\|x\|}$ then

$$\begin{aligned}\bar\lambda[\![\sup_{\xi<\kappa}\sqrt{x(\xi)}u_\xi \le \alpha]\!] &= \bar\lambda(\inf_{\xi<\kappa}[\![\sqrt{x(\xi)}u_\xi \le \alpha]\!]) = \prod_{\xi<\kappa}\bar\lambda[\![\sqrt{x(\xi)}u_\xi \le \alpha]\!]\\ &= \prod_{\xi<\kappa}\lambda\{t : \sqrt{\tfrac{x(\xi)}{t(\xi)}} \le \alpha\} = \prod_{\xi<\kappa}(1 - \tfrac{x(\xi)}{\alpha^2})\\ &\ge 1 - \frac{1}{\alpha^2}\sum_{\xi<\kappa} x(\xi) \to 0\end{aligned}$$

as $\alpha \to \infty$. This means that $\sup_{\xi<\kappa}\sqrt{x(\xi)}u_\xi$ is defined in $L^0(\lambda)$ (364La). So we can define $f : \ell^1(\kappa) \to L^0(\lambda)$ by saying that $f(x) = \sup_{\xi<\kappa}\sqrt{\max(0,x(\xi))}u_\xi$ for every $x \in \ell^1(\kappa)$.

(c) f is a Tukey function. **P** Take $v \in L^0(\lambda)^+$, and set $A = \{x : f(x) \leq v\}$. Note that $f(x \vee x') = f(x) \vee f(x')$ for all $x, x' \in \ell^1(\kappa)$, so A is upwards-directed. Take $\alpha > 0$ such that $\bar{\lambda}[\![v \leq \alpha]\!] = \beta > \frac{1}{2}$. If $x \in A$ and $x \geq 0$ then $f(x) \geq \sqrt{x(\xi)}\chi 1$ so $x(\xi) \leq \alpha$ for every ξ. Now the calculation in (b) tells us that

$$\begin{aligned}\beta \leq \bar{\lambda}[\![\sup_{\xi<\kappa} \sqrt{x(\xi)}u_\xi \leq \alpha]\!] &= \prod_{\xi<\kappa}(1 - \frac{1}{\alpha^2}x(\xi)) \\ &\leq \max(\frac{1}{2}, 1 - \frac{1}{2\alpha^2}\sum_{\xi<\kappa} x(\xi)) = \max(\frac{1}{2}, 1 - \frac{1}{2\alpha^2}\|x\|),\end{aligned}$$

so $\|x\| \leq 2\alpha^2(1-\beta)$. As A is upwards-directed and norm-bounded and contains 0, it is bounded above in $\ell^1(\kappa)$ (354N). As v is arbitrary, f is a Tukey function. **Q**

(d) Accordingly $\ell^1(\kappa) \preccurlyeq_{\mathrm{T}} L^0(\lambda) \cong L^0(\mathfrak{A})$ and $\ell^1(\kappa)$ and $L^0(\mathfrak{A})$ are Tukey equivalent.

529E Proposition Let $\mathcal{S}^\infty$ be the $\mathbb{N}$-localization poset (528I). Then $\mathrm{RO}(\{0,1\}^{\mathfrak{c}})$ can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}^\infty)$.

proof (a) Let $\langle h_\xi\rangle_{\xi<\mathfrak{c}}$ be a family of eventually-different functions in $\mathbb{N}^\mathbb{N}$ (5A1Mc). Set

$$\begin{aligned} W_0 = \bigcup_{n\in\mathbb{N}\text{ is even}} &\{(h,p) : h \in \mathbb{N}^\mathbb{N},\, p \in \mathcal{S}^\infty,\, \#(p[\{n\}]) = 2^n, \\ &\qquad (n,h(n)) \notin p,\, (i,h(i)) \in p \text{ for every } i > n\} \\ &\cup \{(h,p) : h \in \mathbb{N}^\mathbb{N},\, p \in \mathcal{S}^\infty,\, (i,h(i)) \in p \text{ for every } i \in \mathbb{N}\}, \\ W_1 = \bigcup_{n\in\mathbb{N}\text{ is odd}} &\{(h,p) : h \in \mathbb{N}^\mathbb{N},\, p \in \mathcal{S}^\infty,\, \#(p[\{n\}]) = 2^n, \\ &\qquad (n,h(n)) \notin p,\, (i,h(i)) \in p \text{ for every } i > n\}.\end{aligned}$$

Observe that (i) $W_0 \cap W_1 = \emptyset$ (ii) if $(h,p) \in W_j$, where $j = 0$ or $j = 1$, and $p \subseteq q \in \mathcal{S}^\infty$ then $(h,q) \in W_j$ (iii) if $p \in \mathcal{S}^\infty$ then

$$\#(\{\xi : (h_\xi,p) \in W_0 \cup W_1\}) \leq \|p\|$$

is finite.

(b) Set $Q = \mathrm{Fn}_{<\omega}(\mathfrak{c},\{0,1\})$, the set of functions from finite subsets of $\mathfrak{c}$ to $\{0,1\}$, ordered by extension of functions, so that $(Q,\subseteq)$ is isomorphic to $(\mathcal{U},\supseteq)$ where $\mathcal{U}$ is the usual base of the topology of $\{0,1\}^{\mathfrak{c}}$, and $\mathrm{RO}^\uparrow(Q) \cong \mathrm{RO}^\downarrow(\mathcal{U})$ can be identified with the regular open algebra of $\{0,1\}^{\mathfrak{c}}$ (514Sd). Define $f : \mathcal{S}^\infty \to Q$ by setting $f(p)(\xi) = j$ if $(h_\xi,p) \in W_j$. Then f is order-preserving.

(c) For $p \in \mathcal{S}^\infty$, set

$$A_0(p) = \{\xi : \xi < \mathfrak{c},\, \{n : n \text{ is even},\, (n,h_\xi(n)) \notin p\} \text{ is finite}\},$$

$$A_1(p) = \{\xi : \xi < \mathfrak{c},\, \{n : n \text{ is odd},\, (n,h_\xi(n)) \notin p\} \text{ is finite}\},$$

$$A(p) = A_0(p) \cup A_1(p),$$

so that $A(p)$ is finite and $\operatorname{dom} f(p) \subseteq A(p)$. Now $P_1 = \{p : p \in \mathcal{S}^\infty,\, A(p) = \operatorname{dom} f(p)\}$ is cofinal with $\mathcal{S}^\infty$. **P** Take $p \in \mathcal{S}^\infty$. Let m be such that

$$2^m \geq \|p\| + \#(A(p)),$$

$$(n,h_\xi(n)) \in p \text{ whenever } \xi \in A_0(p) \text{ and } n > m \text{ is even},$$

$$(n,h_\xi(n)) \in p \text{ whenever } \xi \in A_1(p) \text{ and } n > m \text{ is odd}.$$

Let $p' \in \mathcal{S}^\infty$ be such that

$$\text{for } n \leq m,\ p'[\{n\}] \supseteq p[\{n\}] \text{ and } \#(p'[\{n\}]) = 2^n,$$

$$\text{for } n > m, \ p'[\{n\}] = p[\{n\}] \cup \{h_\xi(n) : \xi \in A(p)\}.$$

Then $p \le p'$ and $A(p') = A(p)$. Also $A(p) = \operatorname{dom} f(p')$, because if $\xi \in A(p)$ then either $(n, h_\xi(n)) \in p'$ for every n and $(h_\xi, p') \in W_0$, or there is a largest n such that $(n, h_\xi(n)) \notin p'$, in which case $n \le m$ and $\#(p'[\{n\}]) = 2^n$, so (h_ξ, p') belongs to W_0 if n is even and W_1 otherwise. **Q**

(d) If $p \in P_1$ and $q \in Q$ extends $f(p)$, there is a $p_1 \in \mathcal{S}^\infty$ such that $p_1 \supseteq p$ and $f(p_1) = q$. **P** Let m be such that $2^m \ge \|p\| + \#(\operatorname{dom} q)$ and $h_\xi(n) \ne h_\eta(n)$ whenever ξ, $\eta \in \operatorname{dom} q$ are distinct and $n \ge m$. For each $\xi \in \operatorname{dom} q \setminus \operatorname{dom} f(p) = \operatorname{dom} q \setminus A(p)$, $\{n : n \text{ is even}, (n, h_\xi(n)) \notin p\}$ and $\{n : n \text{ is odd}, (n, h_\xi(n)) \notin p\}$ are both infinite. So we can find $m' \ge m$ such that all these sets meet $m' \setminus m$. Set

$$\begin{aligned} p' = p \cup \{(n, h_\xi(n)) : n \in m' \setminus m \text{ is odd}, q(\xi) = 0\} \\ \cup \{(n, h_\xi(n)) : n \in m' \setminus m \text{ is even}, q(\xi) = 1\} \\ \cup \{(n, h_\xi(n)) : n \in \mathbb{N} \setminus m', \xi \in \operatorname{dom} q\}, \end{aligned}$$

so that $p \subseteq p' \in \mathcal{S}^\infty$. Let $p_1 \in \mathcal{S}^\infty$ be such that $p_1 \supseteq p'$, $p_1 \setminus p'$ is finite, $\#(p_1[\{n\}]) = 2^n$ for every $n < m'$ and $(n, h_\xi(n)) \notin p_1 \setminus p'$ whenever $n \in \mathbb{N}$ and $\xi \in \operatorname{dom} q$. Now $f(p_1) = q$, while $p \subseteq p_1$. **Q**

(e) Putting (c) and (d) together, we see that $f^{-1}[Q_0]$ must be cofinal with $\mathcal{S}^\infty$ for every cofinal $Q_0 \subseteq Q$; moreover, since $\emptyset \in P_1$ and $f(\emptyset)$ is the empty function, $f[\mathcal{S}^\infty] = Q$. So f satisfies the conditions of 514O, and $\mathrm{RO}^\uparrow(Q) \cong \mathrm{RO}(\{0,1\}^{\mathfrak{c}})$ can be regularly embedded in $\mathrm{RO}^\uparrow(\mathcal{S}^\infty)$.

529F Corollary (Brendle 00, 2.3.10; Brendle 06, Theorem 1) $n(\{0,1\}^I) \ge \operatorname{add} \mathcal{N}$ for every set I.

proof If I is finite, this is trivial. Otherwise, write $\lambda = n(\{0,1\}^I)$. Then $\lambda \ge n(\{0,1\}^{\mathfrak{c}})$. **P** If $J \subseteq I$ is a countably infinite set, then $\{\{x : x{\restriction}J = z\} : z \in \{0,1\}^J\}$ is a cover of $\{0,1\}^I$ by continuum many nowhere dense sets, so $\lambda \le \mathfrak{c}$. Let $\langle E_\xi \rangle_{\xi<\lambda}$ be a cover of $\{0,1\}^\kappa$ by nowhere dense sets. Then each E_ξ is included in a nowhere dense closed set F_ξ determined by coordinates in a countable set $K_\xi \subseteq I$ (4A2E(b-iii)). Set $K = \bigcup_{\xi<\lambda} K_\xi$, so that $\#(K) \le \mathfrak{c}$. Then all the projections $F'_\xi = \{x{\restriction}K : x \in F_\xi\}$ are nowhere dense in $\{0,1\}^K$ (apply 4A2B(f-ii) to the continuous open surjections $x \mapsto x{\restriction}K_\xi : \{0,1\}^I \to \{0,1\}^{K_\xi}$ and $y \mapsto y{\restriction}K_\xi : \{0,1\}^K \to \{0,1\}^K_\xi$), and they cover $\{0,1\}^K$. Next, we have an injection $\phi : K \to \mathfrak{c}$, and the sets $F''_\xi = \{x : x\phi \in F'_\xi\}$ form a cover of $\{0,1\}^{\mathfrak{c}}$ by nowhere dense sets; so $n(\{0,1\}^{\mathfrak{c}}) \le \lambda$. **Q**

Because every non-empty open set $\{0,1\}^{\mathfrak{c}}$ includes an open set homeomorphic to $\{0,1\}^{\mathfrak{c}}$,

$$\begin{aligned} n(\{0,1\}^{\mathfrak{c}}) &= \min\{n(H) : H \subseteq \{0,1\}^{\mathfrak{c}} \text{ is open and not empty}\} \\ &= \mathfrak{m}(\mathrm{RO}(\{0,1\}^{\mathfrak{c}})) \end{aligned}$$

(517J)

$$\ge \mathfrak{m}(\mathrm{RO}^\uparrow(\mathcal{S}^\infty))$$

(where $\mathcal{S}^\infty$ is the $\mathbb{N}$-localization poset, by 529E and 517Ia)

$$= \operatorname{add} \mathcal{N}$$

by 528N.

529G Reaping numbers (following Brendle 00) For cardinals $\theta \le \lambda$ let $\mathfrak{r}(\theta, \lambda)$ be the smallest cardinal of any set $\mathcal{A} \subseteq [\lambda]^\theta$ such that for every $B \subseteq \lambda$ there is an $A \in \mathcal{A}$ such that either $A \subseteq B$ or $A \cap B = \emptyset$.

529H Proposition (Brendle 00, 2.7; Brendle 06, Theorem 5) $\mathfrak{r}(\omega_1, \lambda) \ge \operatorname{cov} \mathcal{N}$ for all uncountable λ.

proof Let $\langle A_\xi \rangle_{\xi<\kappa}$ be a family in $[\lambda]^{\omega_1}$, where $\kappa < \operatorname{cov}(\mathcal{N})$. I seek a $B \subseteq \lambda$ such that $A_\xi \cap B$ and $A_\xi \setminus B$ are non-empty for every $\xi < \kappa$.

(a) If $\kappa \le \omega_1$, then choose $\langle \alpha_\xi \rangle_{\xi<\kappa}$ and $\langle \beta_\xi \rangle_{\xi<\kappa}$ inductively so that

$$\alpha_\xi \in A_\xi \setminus \{\beta_\eta : \eta < \xi\}, \quad \beta_\xi \in A_\xi \setminus \{\alpha_\eta : \eta \le \xi\}$$

for every $\xi < \kappa$, and set $B = \{\beta_\xi : \xi < \kappa\}$; this serves. So henceforth let us suppose that $\kappa > \omega_1$.

(b) For each $\xi<\kappa$ let $A'_\xi\subseteq A_\xi$ be a set of order type ω_1. For each n, let X_n be a set of size $n!$ with its discrete topology and the uniform probability measure which gives measure $\frac{1}{n!}$ to every singleton. Give $X=\prod_{n\in\mathbb{N}}X_n$ its product measure μ and its product topology. Because X is a compact metrizable space and μ is a Radon measure (416U), $\operatorname{cov}\mathcal{N}(\mu)=\operatorname{cov}\mathcal{N}$ (522Wa). We can therefore choose a family $\langle x_\xi\rangle_{\xi<\kappa}$ in X in such a way that each x_ζ is random over its predecessors in the sense that

$$\text{whenever } \xi<\kappa \text{ and } \overline{\{x_\eta:\eta\in A'_\xi\cap\zeta\}} \text{ is negligible, it does not contain } x_\zeta.$$

For distinct x, $y\in X$, set $\Delta(x,y)=\min\{i:x(i)\ne y(i)\}$. For $x\in X$, set $B(x)=\{\eta:x_\eta\ne x,\ \Delta(x_\eta,x)\text{ is even}\}$.

(c) For every $\xi<\kappa$, $\{x:x\in X,\ A_\xi\subseteq B(x)\}$ and $\{x:x\in X,\ A_\xi\cap B(x)=\emptyset\}$ are negligible. **P** There is a $\zeta<\kappa$ such that $\zeta\in A'_\xi$, $A'_\xi\cap\zeta$ is countable and $D=\{x_\eta:\eta\in A'_\xi\cap\zeta\}$ is dense in $\{x_\eta:\eta\in A'_\xi\}$. Since $x_\zeta\in\overline{D}$, $\overline{D}$ has measure greater than 0. By 275I, applied to the sequence $\langle\Sigma_n\rangle_{n\in\mathbb{N}}$ where Σ_n is the finite algebra of subsets of X determined by coordinates less than n, $\overline{D}$ has a point w which is a density point in the sense that

$$\lim_{n\to\infty}\tfrac{\mu\{y:y\restriction n=w\restriction n,\,y\in\overline{D}\}}{\mu\{y:y\restriction n=w\restriction n\}}=1.$$

Consequently, setting

$$J_n=\{y(n):y\in D\}=\{y(n):y\in\overline{D}\}\supseteq\{y(n):y\in\overline{D},\ y\restriction n=w\restriction n\},$$

$$\frac{\#(J_n)}{n!}\ge\tfrac{\mu\{y:y\restriction n=w\restriction n,\,y\in\overline{D}\}}{\mu\{y:y\restriction n=w\restriction n\}}\to 1$$

as $n\to\infty$.

Next note that, for any $y\in X$ and $n\in\mathbb{N}$,

$$\mu\{x:\exists\, i>n,\ x(i)=y(i)\}\le\sum_{i=n+1}^\infty\frac{1}{i!}\le\tfrac{n+2}{(n+1)(n+1)!}.$$

So

$$\mu\{x:\exists\, y\in D, x(n)=y(n),\ x(i)\ne y(i)\text{ for every }i>n\}$$
$$\ge\tfrac{\#(J_n)}{n!}\bigl(1-\tfrac{n+2}{(n+1)(n+1)!}\bigr)\to 1$$

as $n\to\infty$, and

$$\mu\{x:\exists\, y\in D\setminus\{x\},\ \Delta(x,y)\text{ is even}\}=\mu\{x:\exists\, y\in D\setminus\{x\},\ \Delta(x,y)\text{ is odd}\}=1.$$

But if $y\in D\setminus\{x\}$ and $\Delta(x,y)$ is even, we have an $\eta\in A_\xi$ such that $\Delta(x,x_\eta)$ is even, and $\eta\in A_\xi\cap B(x)$; similarly, if there is a $y\in D\setminus\{x\}$ such that $\Delta(x,y)$ is odd, there is an $\eta\in A_\xi\setminus B(x)$. So $\{x:A_\xi\subseteq B(x)\}$ and $\{x:A_\xi\cap B(x)=\emptyset\}$ are both negligible. **Q**

(d) Since $\operatorname{cov}\mathcal{N}(\mu)=\operatorname{cov}\mathcal{N}>\kappa$, there is an $x\in X$ such that both $A_\xi\cap B(x)$ and $A_\xi\setminus B(x)$ are non-empty for every $\xi<\kappa$. So in this case also we have a suitable set B.

529X Basic exercises (a) Let (X,Σ,μ) be a measure space, and $p\in[1,\infty[$. Show that $L^p(\mu)\equiv_{\mathrm{T}}\ell^1(\kappa)$, where $\kappa=\dim L^p(\mu)$ if this is finite, $d(L^p(\mu))$ otherwise.

(b) Let U be an L-space. (i) Show that $\operatorname{add}U=\infty$ if $U=\{0\}$, ω otherwise. (ii) Show that $\operatorname{add}_\omega U=\infty$ if U is finite-dimensional, $\operatorname{add}\mathcal{N}$ if U is separable and infinite-dimensional, ω_1 otherwise. (iii) Show that $\operatorname{cf}U=1$ if $U=\{0\}$, ω if $0<\dim U<\omega$, $\max(\operatorname{cf}\mathcal{N},\operatorname{cf}[d(U)]^{\le\omega})$ otherwise. (iv) Show that $\operatorname{link}^{\uparrow}_{<\kappa}(U)=1$ if $\kappa\le\omega$, $\operatorname{cf}U$ otherwise.

(c) Let U be a separable Banach lattice. Suppose that $\langle u_\xi\rangle_{\xi<\kappa}$ is a family in U, where $\kappa<\operatorname{add}\mathcal{N}$. Show that there is a family $\langle\epsilon_\xi\rangle_{\xi<\kappa}$ of strictly positive real numbers such that $\{\epsilon_\xi u_\xi:\xi<\kappa\}$ is order-bounded in U.

>(d) Let U be a separable Banach lattice, and $D\subseteq U$ a dense set. Let $A\subseteq U$ be a set of size less than $\operatorname{add}\mathcal{N}$. Show that there is a $w\in U$ such that for every $u\in A$ and every $\epsilon>0$ there is a $v\in D$ such that $|u-v|\le\epsilon w$.

(e) Let $\mathcal{I}$ be the ideal of subsets I of $\mathbb{N}$ such that $\sum_{n\in I}\frac{1}{n+1}$ is finite. (See 419A.) Show that $\ell^1\equiv_{\mathrm{T}}\mathcal{I}$, so that $\operatorname{add}_\omega\mathcal{I}=\operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{I}=\operatorname{cf}\mathcal{N}$.

(f) Show that if $\theta \le \theta' \le \lambda' \le \lambda$ are cardinals, then $\mathfrak{r}(\theta, \lambda) \le \mathfrak{r}(\theta', \lambda')$.

(g)(i) Show that $\mathfrak{r}(\omega, \omega) \ge \operatorname{cov} \mathcal{E} \ge \max(\operatorname{cov} \mathcal{N}, \mathfrak{m}_{\text{countable}})$, where $\mathcal{E}$ is the ideal of subsets of $\mathbb{R}$ with Lebesgue negligible closures. (ii) Show that if λ is an infinite cardinal then $\mathfrak{r}(\omega, \lambda) \ge \max(\operatorname{add} \mathcal{N}, \operatorname{cov} \mathcal{N}_\lambda)$, where $\mathcal{N}_\lambda$ is the null ideal of the usual measure on $\{0,1\}^\lambda$. (*Hint*: 529F.)

529Y Further exercises (a) Let X be a Polish space and $\mathcal{K}_\sigma$ the family of K_σ subsets of X. Show that, defining $\le^*$ as in 522C, $(\mathcal{K}_\sigma, \subseteq) \preccurlyeq_{\mathrm{T}} (\mathbb{N}^{\mathbb{N}}, \le^*)$.

(b) Let X be a topological space with a countable network, and $c : \mathcal{P}X \to [0, \infty]$ an outer regular submodular Choquet capacity (definitions: 432J). Show that if $\mathcal{A}$ is an upwards-directed family of subsets of X such that $\#(\mathcal{A}) < \mathfrak{m}_{\sigma\text{-linked}}$, then $c(\bigcup \mathcal{A}) = \sup_{A \in \mathcal{A}} c(A)$.

(c) Let $r \ge 3$ be an integer. (i) Let $c : \mathcal{P}\mathbb{R}^r \to [0, \infty]$ be Choquet-Newton capacity (§479). Show that if $\mathcal{A}$ is an upwards-directed family of subsets of $\mathbb{R}^r$ such that $\#(\mathcal{A}) < \operatorname{add} \mathcal{N}$, then $c(\bigcup \mathcal{A}) = \sup_{A \in \mathcal{A}} c(A)$. (*Hint*: 479Xi.) (ii) Let $\mathcal{I}$ be the ideal of polar sets in $\mathbb{R}^r$. Show that $\operatorname{add} \mathcal{I} = \operatorname{add} \mathcal{N}$.

(d) Show that, for any infinite set I, the regular open algebra $\mathrm{RO}(\{0,1\}^I)$ of $\{0,1\}^I$ is homogeneous, so that $\mathfrak{m}(\mathrm{RO}(\{0,1\}^I)) = n(\{0,1\})^I$.

(e) Show that $\mathfrak{b} \le \mathfrak{r}(\omega, \omega) \le \pi(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega})$.

529 Notes and comments Many of the ideas of the last two chapters were first embodied in forcing arguments. In 529E this becomes particularly transparent. If we have an upwards-directed set $R \subseteq \mathcal{S}^\infty$ which is 'generic' in the sense that it meets all the cofinal subsets of $\mathcal{S}^\infty$ definable in a language $\mathcal{L}$ with terms for all the functions h_ξ, as well as such obvious ones as $\{p : \#(p[\{n\}]) = 2^n\}$ for each n, and we set $S = \bigcup \mathcal{R}$, then S will belong to the set $\mathcal{S} = \mathcal{S}_{\mathbb{N}}$ of 522K, and we shall have $h_\xi \subseteq^* S$ for every ξ; so that we have a corresponding function $\tilde{f}(S) = \bigcup_{p \in R} f(p) \in \{0,1\}^{\mathfrak{c}}$ defined by setting

$$\tilde{f}(S)(\xi) \equiv \sup\{i : (i, h_\xi(i)) \notin S\} \mod 2.$$

Next, if $G \subseteq \{0,1\}^{\mathfrak{c}}$ is a dense open set with a definition in $\mathcal{L}$, then $\tilde{f}(S) \in G$; for, setting $U_q = \{\phi : q \subseteq \phi \in \{0,1\}^{\mathfrak{c}}\}$ when $q \in Q = \mathrm{Fn}_{<\omega}(\mathfrak{c}, \{0,1\})$, $\{q : U_q \subseteq G\}$ is cofinal with Q, so $\{p : U_{f(p)} \subseteq G\}$ is cofinal with $\mathcal{S}^\infty$ (part (d) of the proof of 529E) and meets R. Thus $\tilde{f}(S)$ is 'generic' in the sense that it belongs to every dense open set with a name in $\mathcal{L}$; and it is a commonplace in the theory of forcing that a function which transforms generic objects in one forcing extension into generic objects in another extension corresponds to a regular embedding of the corresponding regular open algebras.

Chapter 53

Topologies and measures III

In this chapter I return to the concerns of earlier volumes, looking for results which can be expressed in the language so far developed in this volume. In Chapter 43 I examined relationships between measure-theoretic and topological properties. The concepts we now have available (in particular, the notion of 'precaliber') make it possible to extend this work in a new direction, seeking to understand the possible Maharam types of measures on a given topological space. §531 deals with general Radon measures; new patterns arise if we restrict ourselves to completion regular Radon measures (§532). In §533 I give a brief account of some further results depending on assumptions concerning the cardinals examined in Chapter 52, including notes on uniformly regular measures and a description of the cardinals κ for which $\mathbb{R}^\kappa$ is measure-compact (533J).

In §534 I set out the elementary theory of 'strong measure zero' ideals in uniform spaces, concentrating on aspects which can be studied in terms of concepts already induced. Here there are some very natural questions which have not I think been answered (534Z). In the same section I run through elementary properties of Hausdorff measures when examined in the light of the concepts in Chapter 52. In §535 I look at liftings and strong liftings, extending the results of §§341 and 453; in particular, asking which non-complete probability spaces have liftings. In §536 I run over what is known about Alexandra Bellow's problem concerning pointwise compact sets of continuous functions, mentioned in §463. With a little help from special axioms, there are some striking possibilities concerning repeated integrals, which I examine in §537. Moving into new territory, I devote a section (§538) to a study of special types of filter on $\mathbb{N}$ associated with measure-theoretic phenomena, and to medial limits. In §539, I complete my account of the result of B.Balcar, T.Jech and T.Pazák that it is consistent to suppose that every Dedekind complete ccc weakly (σ,∞)-distributive Boolean algebra is a Maharam algebra, and work through applications of the methods of Chapter 52 to Maharam submeasures and algebras.

531 Maharam types of Radon measures

In the introduction to §434 I asked

What kinds of measures can arise on what kinds of topological space?

In §§434-435, and again in §438, I considered a variety of topological properties and their relations with measure-theoretic properties of Borel and Baire measures. I passed over, however, some natural questions concerning possible Maharam types, to which I now return. For a given Hausdorff space X, the possible measure algebras of totally finite Radon measures on X can be described in terms of the set $\mathrm{Mah}_{\mathrm{R}}(X)$ of Maharam types of Maharam-type-homogeneous Radon probability measures on X (531F). For $X\neq\emptyset$, $\mathrm{Mah}_{\mathrm{R}}(X)$ is of the form $\{0\}\cup[\omega,\kappa^*[$ for some infinite cardinal κ^* (531Ef). In 531E and 531G I give basic results from which $\mathrm{Mah}_{\mathrm{R}}(X)$ can often be determined; for obvious reasons we are primarily concerned with compact spaces X. In more abstract contexts, there are striking relationships between precalibers of measure algebras, the sets $\mathrm{Mah}_{\mathrm{R}}(X)$ and continuous surjections onto powers of $[0,1]$, which I examine in 531L-531N and 531T. Intertwined with these, we have results relating the character of X to $\mathrm{Mah}_{\mathrm{R}}(X)$ (531O-531P). The arguments here depend on an analysis of the structure of homogeneous measure algebras (531J, 531K, 531R).

531A Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space with measure algebra $(\mathfrak{A},\bar\mu)$.

(a) The Maharam type $\tau(\mathfrak{A})$ of $\mathfrak{A}$ is at most the weight $w(X)$ of X.

(b) The cellularity $c(\mathfrak{A})$ of $\mathfrak{A}$ is at most the hereditary Lindelöf number $\mathrm{hL}(X)$ of X. If μ is locally finite, $c(\mathfrak{A})$ is at most the Lindelöf number $L(X)$ of X.

(c) $\#(\{a:a\in\mathfrak{A},\,\bar\mu a<\infty\})\le\max(1,w(X)^\omega)$, where $w(X)^\omega$ is the cardinal power.

(d) If X is Hausdorff and μ is a Radon measure, then the Maharam type $\tau(\mathfrak{A})$ of $\mathfrak{A}$ is at most the network weight $\mathrm{nw}(X)$ of X.

proof (a) Let $\mathcal{U}$ be a base for $\mathfrak{T}$ with $\#(\mathcal{U})=w(X)$. Set $B=\{U^\bullet:U\in\mathcal{U}\}$ and let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by B; set $\mathrm{T}=\{E:E\in\Sigma,\,E^\bullet\in\mathfrak{B}\}$. Then T is a σ-subalgebra of Σ containing every negligible set.

If $G\subseteq X$ is open, then $G\in\mathrm{T}$. **P** By 414Aa, $G^\bullet=\sup\{U^\bullet:U\in\mathcal{U},\,U\subseteq G\}$ belongs to $\mathfrak{B}$. **Q** So every Borel set belongs to T. If $E\in\Sigma$ and $\mu E<\infty$, then, because μ is inner regular with respect to the Borel sets, there is a Borel subset F of E with the same measure, so F, $E\setminus F$ and E belong to T. Thus $\{a:a\in\mathfrak{A},\,\bar\mu a<\infty\}\subseteq\mathfrak{B}$; because μ is semi-finite, $\mathfrak{B}=\mathfrak{A}$ and $\tau(\mathfrak{A})\le\#(\mathcal{U})=w(X)$.

(b)(i) If $L(X)=n$ is finite, and $F_0,\ldots,F_n\subseteq X$ are disjoint closed sets, then at least one of them is empty. **P** For $i\le n$, set $G_i=X\setminus\bigcup_{j\le n,j\ne i}F_j$; then $\bigcup_{i\le n}G_i=X$, so there is some $k\le n$ such that $\bigcup_{i\ne k}G_i=X$, and now $F_k=\emptyset$. **Q** As μ is inner regular with respect to the closed sets, $c(\mathfrak{A})\le n=L(X)\le\mathrm{hL}(X)$.

(ii) Suppose that $\omega\le L(X)\le\mathrm{hL}(X)$. Let $\mathcal{G}$ be the family of open subsets of X of finite measure. Then there is a set $\mathcal{H}\subseteq\mathcal{G}$, with cardinal at most $\mathrm{hL}(X)$, such that $\bigcup\mathcal{H}=\bigcup\mathcal{G}$ (5A4Bf). Now $\sup_{H\in\mathcal{H}}H^\bullet=1$, because μ is effectively locally finite.

If $D\subseteq\mathfrak{A}\setminus\{0\}$ is disjoint, then for each $d\in D$ take $H_d\in\mathcal{H}$ such that $d\cap H_d^\bullet\ne 0$. If $H\in\mathcal{H}$, then $\{d:H_d=H\}$ must be countable, since $\mu H<\infty$. So $\#(D)\le\max(\omega,\#(\mathcal{H}))$; as D is arbitrary, $c(\mathfrak{A})\le\max(\omega,\mathrm{hL}(X))=\mathrm{hL}(X)$.

(iii) Finally, if $\omega\le L(X)$ and μ is locally finite, then in (ii) above we have $X=\bigcup\mathcal{G}$, so we can take $\mathcal{H}$ to have size at most $L(X)$, and continue as before, ending with $c(\mathfrak{A})\le\max(\omega,\#(\mathcal{H}))=L(X)$.

(c) Again let $\mathcal{U}$ be a base for the topology of X with cardinal $w(X)$. Let T be the σ-subalgebra of Σ generated by $\mathcal{U}$. If $E\in\Sigma$ and $\mu E<\infty$, then for each $n\in\mathbb{N}$ we can find an open set G_n such that $\mu(G_n\triangle E)\le 2^{-n}$; now there is an open set H_n, a finite union of members of $\mathcal{U}$, such that $H_n\subseteq G_n$ and $\mu(G_n\setminus H_n)\le 2^{-n}$. Setting $F=\bigcup_{m\in\mathbb{N}}\bigcap_{n\in\mathbb{N}}H_n$, we see that $F\in\mathrm{T}$ and $E\triangle F$ is negligible. Thus $\{F^\bullet:F\in\mathrm{T}\}\supseteq\{a:\bar\mu a<\infty\}$ and

$$\#(\{a:\bar\mu a<\infty\})\le\#(\mathrm{T})\le\max(1,\#(\mathcal{U})^\omega)=\max(1,w(X)^\omega).$$

(d) If $a\in\mathfrak{A}\setminus\{0\}$ and the principal ideal $\mathfrak{A}_a$ is Maharam-type-homogeneous, then $\tau(\mathfrak{A}_a)\le\mathrm{nw}(X)$. **P** There is a compact set $K\subseteq X$ such that $0\ne K^\bullet\subseteq\mathfrak{A}_a$; let μ_K be the subspace measure on K. Then

$$\tau(\mathfrak{A}_a)=\tau(\mu_K)\le w(K)$$

(by (a))

$$=\mathrm{nw}(K)$$

(5A4C(a-i))

$$\le\mathrm{nw}(X)$$

(5A4Bb). **Q**

By (b), $c(\mathfrak{A})\le\#(\mathfrak{T})\le 2^{\mathrm{nw}(X)}$ (5A4Ba); so 332S tells us that $\tau(\mathfrak{A})\le\mathrm{nw}(X)$.

531B For strictly positive measures we have some easy inequalities in the other direction.

Proposition Let (X,Σ,μ) be a measure space, with measure algebra $\mathfrak{A}$, and $\mathfrak{T}$ a topology on X such that Σ includes a base for $\mathfrak{T}$ and μ is strictly positive.

(a) If X is regular, then $w(X)\le\#(\mathfrak{A})$.

(b) If X is Hausdorff, then $\#(X)\le 2^{\#(\mathfrak{A})}$.

proof Set $\mathcal{V}=\Sigma\cap\mathfrak{T}$, so that $\mathcal{V}$ is a base for $\mathfrak{T}$. If V, $W\in\mathcal{V}$ and $V^\bullet=W^\bullet$ in $\mathfrak{A}$, then $\mathrm{int}\,\overline{V}=\mathrm{int}\,\overline{W}$. **P** $\mu^*(V\setminus\overline{W})\le\mu(V\setminus W)=0$, so (because μ is strictly positive) $V\subseteq\overline{W}$ and $\overline{V}\subseteq\overline{W}$ and $\mathrm{int}\,\overline{V}\subseteq\mathrm{int}\,\overline{W}$. Similarly, $\mathrm{int}\,\overline{W}\subseteq\mathrm{int}\,\overline{V}$. **Q** So if we set $\mathcal{W}=\{\mathrm{int}\,\overline{V}:V\in\mathcal{V}\}$, $\#(\mathcal{W})\le\#(\mathfrak{A})$.

(a) If $\mathfrak{T}$ is regular, $\mathcal{W}$ is a base for $\mathfrak{T}$, so $w(X)\le\#(\mathcal{W})\le\#(\mathfrak{A})$.

(b) If $\mathfrak{T}$ is Hausdorff, then for any distinct x, $y\in X$, there is a $W\in\mathcal{W}$ containing x but not y. **P** Let G, H be disjoint open sets containing x, y respectively. Take $V\in\mathcal{V}$ such that $x\in V\subseteq G$, and set $W=\mathrm{int}\,\overline{V}$. **Q** So $\#(X)\le 2^{\#(\mathcal{W})}\le 2^{\#(\mathfrak{A})}$.

531C Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces with product X, and μ a totally finite quasi-Radon measure on X with Maharam type κ. For each $i\in I$, let μ_i be the marginal measure on X_i, and κ_i its Maharam type. Then κ is at most the cardinal sum $\sum_{i\in I}\kappa_i$.

proof For each $i\in I$, let $\langle E_{i\xi}\rangle_{\xi<\kappa_i}$ be a family in $\mathrm{dom}\,\mu_i$ such that $\{E_{i\xi}^\bullet:\xi<\kappa_i\}$ τ-generates the measure algebra of μ_i. Consider $\mathcal{W}=\{\pi_i^{-1}[E_{i\xi}]:i\in I,\ \xi<\kappa_i\}$, so that $\mathcal{W}\subseteq\mathrm{dom}\,\mu$ and $\#(\mathcal{W})\le\sum_{i\in I}\kappa_i$. Let $\mathfrak{B}$ be the closed subalgebra of the measure algebra $\mathfrak{A}$ of μ generated by $\{W^\bullet:W\in\mathcal{W}\}$.

For each $i\in I$, the canonical map $\pi_i:X\to X_i$ induces a measure-preserving homomorphism ϕ_i from the measure algebra $\mathfrak{A}_i$ of μ_i to $\mathfrak{A}$ (324M). Now $\phi_i^{-1}[\mathfrak{B}]$ is a closed subalgebra of $\mathfrak{A}_i$ containing $E_{i\xi}^\bullet$ for every $\xi<\kappa_i$, so is the whole of $\mathfrak{A}_i$, that is, $\phi_i[\mathfrak{A}_i] \subseteq \mathfrak{B}$. In particular, if $G \subseteq X_i$ is open, $\pi_i^{-1}[G]^\bullet = \phi_i(G^\bullet)$ belongs to $\mathfrak{B}$. Now the family $\mathcal{V}$ of open sets $V \subseteq X$ such that $V^\bullet \in \mathfrak{B}$ is closed under finite intersections and contains $\pi_i^{-1}[G]$ whenever $i \in I$ and $G \subseteq X_i$ is open, so $\mathcal{V}$ is a base for the topology of X. But also $\mathcal{V}$ is closed under arbitrary unions, because $\mathfrak{B}$ is closed and μ is τ-additive (414Aa). So $V^\bullet \in \mathfrak{B}$ for every open set $V \subseteq X$, and therefore for every Borel set $V \subseteq X$; as μ is inner regular with respect to the Borel sets, $\mathfrak{B} = \mathfrak{A}$.

Thus $\{W^\bullet : W \in \mathcal{W}\}$ witnesses that the Maharam type $\tau(\mathfrak{A})$ of μ is at most $\sum_{i \in I} \kappa_i$, as claimed.

531D Definition If X is a Hausdorff space, I write $\mathrm{Mah_R}(X)$ for the set of Maharam types of Maharam-type-homogeneous Radon probability measures on X. Note that $0 \in \mathrm{Mah_R}(X)$ iff X is non-empty, and that any member of $\mathrm{Mah_R}(X)$ is either 0 or an infinite cardinal.

531E Proposition Let X be any Hausdorff space.

(a) $\kappa \le w(X)$ for every $\kappa \in \mathrm{Mah_R}(X)$.

(b) $\mathrm{Mah_R}(Y) \subseteq \mathrm{Mah_R}(X)$ for every $Y \subseteq X$.

(c) $\mathrm{Mah_R}(X) = \bigcup\{\mathrm{Mah_R}(K) : K \subseteq X \text{ is compact}\}$.

(d) If X is compact and Y is a continuous image of X, $\mathrm{Mah_R}(Y) \subseteq \mathrm{Mah_R}(X)$.

(e) $\omega \in \mathrm{Mah_R}(X)$ iff X has a compact subset which is not scattered.

(f) (Haydon 77) If $\omega \le \kappa' \le \kappa \in \mathrm{Mah_R}(X)$ then $\kappa' \in \mathrm{Mah_R}(X)$.

(g) If Y is another Hausdorff space, and neither X nor Y is empty, then $\mathrm{Mah_R}(X \times Y) = \mathrm{Mah_R}(X) \cup \mathrm{Mah_R}(Y)$; generally, for any non-empty finite family $\langle X_i\rangle_{i\in I}$ of non-empty Hausdorff spaces, $\mathrm{Mah_R}(\prod_{i\in I} X_i) = \bigcup_{i\in I} \mathrm{Mah_R}(X_i)$.

proof (a) This is immediate from 531Aa.

(b) If $\kappa \in \mathrm{Mah_R}(Y)$, there a Maharam-type-homogeneous Radon probability measure μ on Y with Maharam type κ. Now μ has a (unique) extension to a Radon probability measure μ' on X, setting $\mu'(Y \setminus X) = 0$. μ' and μ have isomorphic measure algebras, so μ' is Maharam-type-homogeneous and has Maharam type κ, and $\kappa \in \mathrm{Mah_R}(X)$.

(c) By (b), $\mathrm{Mah_R}(K) \subseteq \mathrm{Mah_R}(X)$ for every compact set $K \subseteq X$. In the other direction, if $\kappa \in \mathrm{Mah_R}(X)$, there is a Maharam-type-homogeneous Radon probability measure μ on X with Maharam type κ. Let $K \subseteq X$ be a compact set with $\mu K > 0$. Then the normalized subspace measure $\mu' = (\mu K)^{-1}\mu_K$ is a Radon probability measure on K, and its measure algebra is isomorphic to a principal ideal of the measure algebra of μ, so is Maharam-type-homogeneous with Maharam type κ. Accordingly $\kappa \in \mathrm{Mah_R}(K)$.

(d) Take $\kappa \in \mathrm{Mah_R}(Y)$; then there is a Maharam-type-homogeneous Radon probability measure μ on Y with Maharam type κ. Let $f : X \to Y$ be a continuous surjection. By 418L, there is a Radon measure μ' on X such that f is inverse-measure-preserving for μ' and μ and induces an isomorphism of their measure algebras. So μ' witnesses that $\kappa \in \mathrm{Mah_R}(X)$.

(e)(i) If X has a compact subset K which is not scattered, then there is a continuous surjection from K onto $[0,1]$ (4A2G(j-iv)). Of course Lebesgue measure witnesses that $\omega \in \mathrm{Mah_R}([0,1])$, so (d) and (b) tell us that $\omega \in \mathrm{Mah_R}(K) \subseteq \mathrm{Mah_R}(X)$.

(ii) If every compact subset of X is scattered and μ is a Maharam-type-homogeneous Radon probability measure on X, let K be a compact set of non-zero measure and $Z \subseteq K$ a closed self-supporting set. Then Z has an isolated point z say; in this case, $\mu\{z\} > 0$ so $\{z\}$ is an atom for μ and (because μ is Maharam-type-homogeneous) the Maharam type of μ is 0. As μ is arbitrary, $\omega \notin \mathrm{Mah_R}(X)$.

(f)(i) Suppose first that X is compact. Let μ be a Maharam-type-homogeneous Radon probability measure on X with Maharam type κ. Let $\langle E_\xi\rangle_{\xi<\kappa}$ be a stochastically independent family in $\operatorname{dom}\mu$ with $\mu E_\xi = \frac{1}{2}$ for every ξ. For each $\xi < \kappa'$ and $n \in \mathbb{N}$, let $f_{\xi n} \in C(X)$ be such that $\int |f_{\xi n} - \chi E_\xi| \le 2^{-n}$ (416I). Define $f : X \to \mathbb{R}^{\kappa' \times \mathbb{N}}$ by setting $f(x)(\xi,n) = f_{\xi n}(x)$ for $x \in X$, $\xi < \kappa'$ and $n \in \mathbb{N}$. Then f is continuous, so by 418I the image measure $\nu = \mu f^{-1}$ on the compact set $f[X]$ is a Radon measure. For each $\xi < \kappa'$, the set

$$F_\xi = \{w : w \in f[X],\ \lim_{n\to\infty} w(\xi,n) = 1\}$$

is a Borel set, and $f^{-1}[F_\xi] \triangle E_\xi$ is μ-negligible; so $\langle F_\xi\rangle_{\xi<\kappa'}$ is a stochastically independent family of subsets of $f[X]$ with measure $\frac{1}{2}$. If $\mathfrak{B}$ is the measure algebra of ν, and $\mathfrak{C}$ the closed subalgebra of $\mathfrak{B}$ generated by $\{F_\xi^\bullet : \xi < \kappa'\}$, then $\mathfrak{C}$ is Maharam-type-homogeneous, with Maharam type κ'; at the same time,

$$\tau(\mathfrak{B}) \le w(f[X]) \le w(\mathbb{R}^{\kappa' \times \mathbb{N}}) = \kappa'.$$

By 332N, $\mathfrak{B}$ can be embedded in $\mathfrak{C}$; by 332Q, $\mathfrak{B}$ and $\mathfrak{C}$ are isomorphic, that is, $\mathfrak{B}$ is Maharam-type-homogeneous with Maharam type κ', and ν witnesses that $\kappa' \in \mathrm{Mah}_{\mathrm{R}}(f[X])$. By (d), $\kappa' \in \mathrm{Mah}_{\mathrm{R}}(X)$.

(ii) In general, (c) tells us that there is a compact set $K \subseteq X$ such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(K)$, so $\kappa' \in \mathrm{Mah}_{\mathrm{R}}(K) \subseteq \mathrm{Mah}_{\mathrm{R}}(X)$.

(g) Because neither Y nor X is empty, both X and Y are homeomorphic to subspaces of $X \times Y$, so (b) tells us that $\mathrm{Mah}_{\mathrm{R}}(X \times Y) \supseteq \mathrm{Mah}_{\mathrm{R}}(X) \cup \mathrm{Mah}_{\mathrm{R}}(Y)$. In the other direction, given a Maharam-type-homogeneous Radon probability measure μ on $X \times Y$, let μ_1, μ_2 be the marginal measures on X and Y respectively, so that each μ_k is a Radon probability measure (418I again). Let $\langle E_i\rangle_{i\in I}$, $\langle F_j\rangle_{j\in J}$ be countable partitions of X, Y into Borel sets such that all the subspace measures $(\mu_1)_{E_i}$ and $(\mu_2)_{F_j}$ are Maharam-type-homogeneous. Then there must be $i \in J$, $j \in J$ such that $\mu(E_i \times F_j) > 0$. Let μ' be the subspace measure $\mu_{E_i\times F_j}$; then the Maharam type of μ' is κ, because μ is Maharam-type-homogeneous. Let μ'_1, μ'_2 be the marginal measures of μ' on E_i and F_j respectively. Then μ'_1 is an indefinite-integral measure over $(\mu_1)_{E_i}$ (415Oa), so its measure algebra is isomorphic to a principal ideal of the measure algebra of $(\mu_1)_{E_i}$ (322K), and has the same Maharam type κ_1 say. As in (b) above, $\kappa_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$. Similarly, the Maharam type κ_2 of μ'_2 belongs to $\mathrm{Mah}_{\mathrm{R}}(Y)$. Now 531C tells us that $\kappa \le \kappa_1 + \kappa_2$. Since κ is either zero or infinite, it must be less than or equal to at least one of them, and belongs to $\mathrm{Mah}_{\mathrm{R}}(X) \cup \mathrm{Mah}_{\mathrm{R}}(Y)$ by (f) above.

The result for general finite products now follows easily by induction on $\#(I)$.

531F Proposition Let X be a Hausdorff space. Then a totally finite measure algebra $(\mathfrak{A}, \bar\mu)$ is isomorphic to the measure algebra of a Radon measure on X iff (α) whenever $\mathfrak{A}_a$ is a non-trivial homogeneous principal ideal of $\mathfrak{A}$ then $\tau(\mathfrak{A}_a) \in \mathrm{Mah}_{\mathrm{R}}(X)$ (β) $c(\mathfrak{A}) \le \#(X)$.

proof (a) If μ is a totally finite Radon measure on X with measure algebra $\mathfrak{A}$ and the principal ideal $\mathfrak{A}_a$ generated by $a \in \mathfrak{A}\setminus\{0\}$ is homogeneous, then there are an $E \in \operatorname{dom}\mu$ such that $E^\bullet = a$ and an $F \subseteq E$ such that $0 < \mu F < \infty$. Let ν be the Radon probability measure $(\mu F)^{-1}\mu\llcorner F$, that is, $\nu H = \mu(H \cap F)/\mu F$ whenever $H \subseteq X$ is such that μ measures $H \cap F$. Then the measure algebra of ν is isomorphic to a principal ideal of $\mathfrak{A}_a$ so is homogeneous with the same Maharam type, and ν witnesses that $\tau(\mathfrak{A}_a) \in \mathrm{Mah}_{\mathrm{R}}(X)$. Thus $\mathfrak{A}$ satisfies (α). As for (β), if X is infinite this is trivial (because $(\mathfrak{A}, \bar\mu)$ is totally finite, so $\mathfrak{A}$ is ccc), and otherwise $\mathfrak{A}$ is finite, with

$$c(\mathfrak{A}) = \#(\{a : a \in \mathfrak{A} \text{ is an atom}\}) = \#(\{x : x \in X,\, \mu\{x\} > 0\}) \le \#(X).$$

(b) Now suppose that $(\mathfrak{A}, \bar\mu)$ is a totally finite measure algebra satisfying the conditions. Express it as the simple product of a countable family $\langle(\mathfrak{A}_i, \bar\mu'_i)\rangle_{i\in I}$ of non-zero homogeneous measure algebras (332B); we may suppose that $I \subseteq \mathbb{N}$. For $n \in I$, set $\kappa_n = \tau(\mathfrak{A}_n)$ and $\gamma_n = \bar\mu'_n 1_{\mathfrak{A}_n}$. ($\beta$) tells us that $\#(I) \le \#(X)$; let $\langle x_n\rangle_{n\in I}$ be a family of distinct elements of X.

Set $J = \{n : n \in I,\, \kappa_n \ge \omega\}$. For each $n \in J$, (α) tells us that there is a Maharam-type-homogeneous Radon probability measure μ_n on X with Maharam type κ_n. Now there is a disjoint family $\langle E_n\rangle_{n\in I}$ of Borel subsets of X such that $\mu_n E_n > 0$ for every $n \in J$. **P** Choose $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $F_0 = X \setminus \{x_n : n \in I\}$. Given that F_n is a Borel set and $\mu_j F_n > 0$ for every $j \in J \setminus n$, then if $n \notin J$ set $E_n = \emptyset$ and $F_{n+1} = F_n$. Otherwise, for each $j \in J$ such that $j > n$, we can partition F_n into finitely many Borel sets of μ_n-measure less than $2^{-j}\mu_n F_n$, because μ_n is atomless; take one of these, G_{nj} say, such that $\mu_j G_{nj} > 0$; now set $F_{n+1} = \bigcup_{j\in J, j>n} G_{nj}$ and $E_n = F_n \setminus F_{n+1}$. Continue. **Q** Now set

$$\mu E = \sum\nolimits_{n\in I\setminus J, x_n\in E}\gamma_n + \sum\nolimits_{n\in J}(\mu_n E_n)^{-1}\gamma_n\mu_n(E \cap E_n)$$

whenever $E \subseteq X$ is such that μ_n measures $E \cap E_n$ for every $n \in J$. Of course μ is a measure. Because every μ_n is a topological measure, so is μ; because every μ_n is inner regular with respect to the compact sets, so is μ; because every μ_n is complete, so is μ; thus μ is a Radon measure. Because every subspace measure $(\mu_n)_{E_n}$ is Maharam-type-homogeneous with Maharam type κ_n, the measure algebra of μ is isomorphic to $(\mathfrak{A}, \bar\mu)$.

531G Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of non-empty Hausdorff spaces with product X. Then an infinite cardinal κ belongs to $\mathrm{Mah}_{\mathrm{R}}(X)$ iff *either* $\kappa \le \#(\{i : i \in I,\, \#(X_i) \ge 2\})$ *or* κ is expressible as $\sup_{i\in I}\kappa_i$ where $\kappa_i \in \mathrm{Mah}_{\mathrm{R}}(X_i)$ for every $i \in I$.

proof (a)(i) Suppose that $\kappa = \sup_{i\in I}\kappa_i$ where $\kappa_i \in \mathrm{Mah}_{\mathrm{R}}(X_i)$ for each $i \in I$. For each i, let μ_i be a Maharam-type-homogeneous Radon probability measure on X_i with Maharam type κ_i and compact support (see the proof of 531Ec). Let λ be the ordinary product of the measures μ_i. By 325I, the measure algebra of λ can be identified with the probability algebra free product of the measure algebras of the μ_i. It is therefore isomorphic to the measure algebra of the usual measure on $\{0,1\}^{\kappa'}$, where κ' is the cardinal sum $\sum_{i\in I}\kappa_i$; in particular, it is homogeneous with Maharam type κ' (since we are supposing that $\kappa\ge\omega$). By 417E(ii), the measure algebra of the τ-additive product μ of $\langle\mu_i\rangle_{i\in I}$ can be identified with the measure algebra of λ, while μ is a Radon measure (417Q). So μ witnesses that $\kappa'\in\mathrm{Mah}_{\mathrm{R}}(X)$; by 531Ef, $\kappa\in\mathrm{Mah}_{\mathrm{R}}(X)$.

(ii) Suppose that $\omega\le\kappa\le\#(I')$ where $I'=\{i:i\in I,\,\#(X_i)\ge 2\}$. For $i\in I'$, let x_i, y_i be distinct points of X_i and μ_i the point-supported probability measure on X_i such that $\mu_i\{x_i\}=\mu_i\{y_i\}=\frac{1}{2}$; for $i\in I\setminus I'$, let μ_i be the unique Radon probability measure on X_i. As in (i) above, the Radon measure product of $\langle\mu_i\rangle_{i\in I}$ is Maharam-type-homogeneous, with Maharam type $\#(I')$, so $\#(I')\in\mathrm{Mah}_{\mathrm{R}}(X)$; by 531Ef again, $\kappa\in\mathrm{Mah}_{\mathrm{R}}(X)$.

(b) Now suppose that $\omega\le\kappa\in\mathrm{Mah}_{\mathrm{R}}(X)$ and that $\kappa>\#(I')$. For each $i\in I$, let θ_i be the least cardinal greater than every member of $\mathrm{Mah}_{\mathrm{R}}(X_i)$. Note that $\kappa'\in\mathrm{Mah}_{\mathrm{R}}(X_i)$ whenever $\omega\le\kappa'<\theta_i$. Set

$$I_1=\{i:i\in I,\,\kappa<\theta_i\},\quad Z_1=\prod_{i\in I_1}X_i,$$

$$I_2=\{i:i\in I,\,\theta_i\le\kappa,\,\operatorname{cf}\theta_i>\omega\},\quad Z_2=\prod_{i\in I_2}X_i,$$

$$I_3=\{i:i\in I,\,\theta_i=\kappa,\,\operatorname{cf}\theta_i=\omega\},\quad Z_3=\prod_{i\in I_3}X_i,$$

$$I_4=\{i:i\in I,\,\theta_i<\kappa,\,\operatorname{cf}\theta_i=\omega\},\quad Z_4=\prod_{i\in I_4}X_i,$$

$$I_5=\{i:i\in I,\,\theta_i=1,\,\#(X_i)>1\},\quad Z_5=\prod_{i\in I_5}X_i,$$

$$I_6=\{i:i\in I,\,\#(X_i)=1\},\quad Z_6=\prod_{i\in I_6}X_i.$$

Then X can be identified with $\prod_{1\le k\le 6}Z_k$, so 531Eg tells us that $\kappa\in\mathrm{Mah}_{\mathrm{R}}(Z_k)$ for some k. As Z_6 is a singleton, we actually have $\kappa\in\mathrm{Mah}_{\mathrm{R}}(Z_k)$ for some $k\le 5$.

case 1 Suppose $\kappa\in\mathrm{Mah}_{\mathrm{R}}(Z_1)$. Then, in particular, $I_1\ne\emptyset$ and there is a $j\in I$ such that $\kappa<\theta_j$. In this case, $\kappa\in\mathrm{Mah}_{\mathrm{R}}(X_j)$, and we can set $\kappa_j=\kappa$, $\kappa_i=0$ for $i\ne j$ to find a family in $\prod_{i\in I}\mathrm{Mah}_{\mathrm{R}}(X_i)$ with supremum κ.

case 2 Suppose that $\kappa\in\mathrm{Mah}_{\mathrm{R}}(Z_2)$. Let μ be a Radon probability measure on Z_2 with Maharam type κ. For each $i\in Z_2$, let μ_i' be the marginal measure on X_i, and κ_i' its Maharam type. By 531C, $\kappa\le\sum_{i\in I_2}\kappa_i'$; since $I_2\subseteq I'$, $\kappa>\#(I_2)$; since κ is infinite, it must be less than or equal to $\sup_{i\in I_2}\max(\omega,\kappa_i')$. On the other hand, by 531F, each κ_i' is either finite or the supremum of some countable subset of $\mathrm{Mah}_{\mathrm{R}}(X_i)$; because $\operatorname{cf}\theta_i>\omega$, $\kappa_i'<\theta_i$ and $\max(\omega,\kappa_i')\in\mathrm{Mah}_{\mathrm{R}}(X_i)$. Setting

$$\begin{aligned}\kappa_i&=\operatorname{med}(\kappa_i',\omega,\kappa)\text{ for }i\in I_2,\\&=0\text{ for }i\in I\setminus I_2,\end{aligned}$$

we have $\kappa_i\in\mathrm{Mah}_{\mathrm{R}}(X_i)$ for every $i\in I$ and $\kappa=\sup_{i\in I}\kappa_i$.

case 3 Suppose that $\kappa\in\mathrm{Mah}_{\mathrm{R}}(Z_3)$. Because $\kappa=\theta_i\notin\mathrm{Mah}_{\mathrm{R}}(X_i)$ for $i\in I_3$, 531Eg tells us that I_3 must be infinite. Let $\langle i_n\rangle_{n\in\mathbb{N}}$ be a sequence of distinct elements of I_3. Of course κ itself is uncountable and has countable cofinality, so we can find a sequence κ_n' of infinite cardinals less than κ with supremum κ. Setting $\kappa_{i_n}=\kappa_n'$, $\kappa_i=0$ for $i\in I\setminus\{i_n:n\in\mathbb{N}\}$, we have $\kappa_i\in\mathrm{Mah}_{\mathrm{R}}(X_i)$ for every i and $\kappa=\sup_{i\in I}\kappa_i$.

case 4 Suppose that $\kappa\in\mathrm{Mah}_{\mathrm{R}}(Z_4)$. Following the scheme of case 2 above, let μ be a Radon probability measure on Z_4 with Maharam type κ, and for each $i\in I_4$ let μ_i' be the marginal measure on X_i and κ_i' its Maharam type. Then, as before, $\kappa\le\sup_{i\in I_4}\max(\omega,\kappa_i')$. At the same time, $\kappa_i'\le\theta_i<\kappa$ for every i, so we must have $\kappa=\sup_{i\in I_4}\theta_i$. Set $\delta=\operatorname{cf}\kappa$. Then we can choose $\langle i_\xi\rangle_{\xi<\delta}$ inductively in I_4 so that $\theta_{i_\eta}<\theta_{i_\xi}$ whenever $\eta<\xi<\delta$ and $\sup_{\xi<\delta}\theta_{i_\xi}=\kappa$. Now define $\langle\kappa_i\rangle_{i\in I}$ by saying

$$\begin{aligned}\kappa_{i_{\xi+1}}&=\theta_{i_\xi}\text{ whenever }\xi<\delta,\\\kappa_i&=0\text{ if }i\in I\setminus\{i_{\xi+1}:\xi<\delta\}.\end{aligned}$$

This gives $\kappa_i\in\mathrm{Mah}_{\mathrm{R}}(X_i)$ for every i and $\kappa=\sup_{i\in I}\kappa_i$.

case 5 ? Suppose, if possible, that $\kappa\in\mathrm{Mah}_{\mathrm{R}}(Z_5)$. Once again, we can find a Radon probability measure μ on Z_5 with Maharam type κ, and look at its marginal measures μ_i' for $i\in I_5$. This time, however, every μ_i' must be purely atomic and has Maharam type $\kappa'_i \le \omega$; also $\#(I_5) < \kappa$. So our formula $\kappa \le \sum_{i \in I_5} \kappa'_i$ becomes $\kappa = \omega$. In this case I_5 must be finite and $\kappa \in \bigcup_{i \in I_5} \mathrm{Mah_R}(X_i) = \{0\}$, which is absurd. **X**

Thus this case evaporates and the proof is complete.

531H Remarks The results above already enable us to calculate $\mathrm{Mah_R}(X)$ for many spaces. Of course we begin with compact spaces (531Ec). If X is compact and Hausdorff, and $[0,1]^\kappa$ is a continuous image of X, where κ is an infinite cardinal, then $\kappa \in \mathrm{Mah_R}(X)$ (531Ed); so if $[0,1]^{w(X)}$ is a continuous image of X, then $\mathrm{Mah_R}(X)$ is completely specified, being $\{0\} \cup \{\kappa : \omega \le \kappa \le w(X)\}$ (531Ea, 531Ef). Of course it is not generally true that $w(X) \in \mathrm{Mah_R}(X)$ (531Xc). But it is quite often the case that $[0,1]^\kappa$ is a continuous image of X for every $\kappa \in \mathrm{Mah_R}(X)$, and I will now investigate this phenomenon.

531I Notation For the rest of the section, I will use the following notation, mostly familiar from earlier chapters of this volume. For any set I, let ν_I be the usual measure on $\{0,1\}^I$, T_I its domain, $\mathcal{N}_I$ its null ideal and $(\mathfrak{B}_I, \bar\nu_I)$ its measure algebra. In this context, I will write $\langle e_i \rangle_{i \in I}$ for the standard generating family in $\mathfrak{B}_I$ (525A). For $J \subseteq I$ let $\mathfrak{C}_J$ be the closed subalgebra of $\mathfrak{B}_I$ generated by $\{e_i : i \in J\}$. Now for a new idea. For each $i \in I$, let $\phi_i : \mathfrak{B}_I \to \mathfrak{B}_I$ be the measure-preserving involution corresponding to reversal of the ith coordinate in $\{0,1\}^I$, that is, $\phi_i(e_i) = 1 \setminus e_i$ and $\phi_i(e_j) = e_j$ for $j \ne i$.

531J Lemma Let I be a set, and take $\mathfrak{B}_I$, $\mathfrak{C}_J$, for $J \subseteq I$, and ϕ_i, for $i \in I$, as in 531I.

(a) $\bigcup\{\mathfrak{C}_J : J \in [I]^{<\omega}\}$ is dense in $\mathfrak{B}_I$ for the measure-algebra topology of $\mathfrak{B}_I$.

(b) For every $a \in \mathfrak{B}_I$, there is a (unique) countable $J^*(a) \subseteq I$ such that, for $J \subseteq I$, $a \in \mathfrak{C}_J$ iff $J \supseteq J^*(a)$.

(c) $\phi_i\phi_j = \phi_j\phi_i$ for all i, $j \in I$.

(d) If $J \subseteq I$, $a \in \mathfrak{C}_J$ and $i \in I$, then $a \cap \phi_i a$, $a \cup \phi_i a$ belong to $\mathfrak{C}_{J \setminus \{i\}}$.

(e) For $a \in \mathfrak{B}_I$ and $i \in I$ we have $\phi_i a = a$ iff $i \notin J^*(a)$.

(f) $\phi_i a \in \mathfrak{C}_J$ whenever $J \subseteq I$, $i \in I$ and $a \in \mathfrak{C}_J$.

proof Let $\langle e_i \rangle_{i \in I}$ be the standard generating family in $\mathfrak{B}_I$.

(a) See 254Fe.

(b) See 254Rd or 325Mb.

(c) Because $\{e_k : k \in I\}$ τ-generates $\mathfrak{B}_I$, it is enough to check that $\phi_i\phi_j e_k = \phi_j\phi_i e_k$ for all i, j, $k \in I$, and this is easy.

(d) The subalgebra $\{(c \cap e_i) \cup (c' \setminus e_i) : c,\ c' \in \mathfrak{C}_{J \setminus \{i\}}\}$ generated by $\mathfrak{C}_{J \setminus \{i\}} \cup \{e_i\}$ is closed (323K), so is the whole of $\mathfrak{C}_J$ and contains a. If c, $c' \in \mathfrak{C}_{J \setminus \{i\}}$ are such that $a = (c \cap e_i) \cup (c' \setminus e_i)$, then $\phi_i a = (c \setminus e_i) \cup (c' \cap e_i)$ and $a \cap \phi_i a = c \cap c'$, $a \cup \phi_i a = c \cup c'$ belong to $\mathfrak{C}_{J \setminus \{i\}}$.

(e) If $i \notin J^*(a)$ then $\phi_i a = a$ because $\phi_i(e_j) = e_j$ for every $j \ne i$. If $\phi_i a = a$ then $a = a \cap \phi_i a \in \mathfrak{C}_{I \setminus \{i\}}$, by (d), and $J^*(a) \subseteq I \setminus \{i\}$, that is, $i \notin J^*(a)$.

(f) $\mathfrak{C}_J$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{e_j : j \in J\}$, so $\phi_i[\mathfrak{C}_J]$ is the closed subalgebra generated by $\{\phi_i e_j : j \in J\} \subseteq \mathfrak{C}_J$ (324L).

531K Lemma Let $\kappa \ge \omega_2$ be a cardinal, and $\langle e_\xi \rangle_{\xi < \kappa}$ the standard generating family in $\mathfrak{B}_\kappa$. Suppose that we are given a family $\langle a_\xi \rangle_{\xi < \kappa}$ in $\mathfrak{B}_\kappa$. Then there are a set $\Gamma \in [\kappa]^\kappa$ and a family $\langle c_\xi \rangle_{\xi < \kappa}$ in $\mathfrak{B}_\kappa$ such that

$$c_\xi \subseteq a_\xi, \quad \bar\nu_\kappa c_\xi \ge 2\bar\nu_\kappa a_\xi - 1$$

for every ξ, and

$$\bar\nu_\kappa(\inf_{\xi \in I} c_\xi \cap e_\xi \cap \inf_{\eta \in J} c_\eta \setminus e_\eta) = \frac{1}{2^{\#(I \cup J)}}\bar\nu_\kappa(\inf_{\xi \in I \cup J} c_\xi)$$

whenever I, $J \subseteq \Gamma$ are disjoint finite sets.

proof Let e_ξ, ϕ_ξ, for $\xi < \kappa$, $\mathfrak{C}_L$, for $L \subseteq \kappa$, and $J^*(a)$, for $a \in \mathfrak{B}_\kappa$, be as in 531I-531J. Set $L_\xi = J^*(a_\xi)$ and $c_\xi = a_\xi \cap \phi_\xi a_\xi$ for each ξ; then

$$\bar\nu_\kappa c_\xi = \bar\nu_\kappa a_\xi + \bar\nu_\kappa(\phi_\xi a_\xi) - \bar\nu_\kappa(a_\xi \cup \phi_\xi a_\xi) \ge 2\bar\nu_\kappa a_\xi - 1$$

and $c_\xi \in \mathfrak{C}_{L_\xi\setminus\{\xi\}}$ (531Jd). By Hajnal's Free Set Theorem (5A1I(a-iii)), there is a set $\Gamma \in [\kappa]^\kappa$ such that $\xi \notin L_\eta$ whenever ξ, η are distinct members of Γ. (This is where we use the hypothesis that $\kappa \ge \omega_2$.) Now suppose that I, $J \subseteq \Gamma$ are finite and disjoint. Then $(L_\xi \setminus \{\xi\}) \cap (I \cup J) = \emptyset$, so $c_\xi \in \mathfrak{C}_{\kappa\setminus(I\cup J)}$, for every $\xi \in I \cup J$. Accordingly $c = \inf_{\xi\in I\cup J} c_\xi$ belongs to $\mathfrak{C}_{\kappa\setminus(I\cup J)}$. This means that c and the e_ξ, for $\xi \in I \cup J$, are stochastically independent, and

$$\bar\nu_\kappa(c \cap \inf_{\xi\in I} e_\xi \cap \inf_{\eta\in J}(1 \setminus e_\eta)) = \bar\nu_\kappa c \cdot \prod_{\xi\in I} \bar\nu_\kappa e_\xi \cdot \prod_{\eta\in J} \bar\nu_\kappa(1 \setminus e_\eta) = \frac{1}{2^{\#(I\cup J)}}\bar\nu_\kappa c,$$

as claimed.

531L Theorem Let X be a normal Hausdorff space.

(a) (Haydon 77) If $\omega \in \mathrm{Mah_R}(X)$ then $[0,1]^\omega$ is a continuous image of X.

(b) (Haydon 77, Plebanek 97) If $\kappa \ge \omega_2$ belongs to $\mathrm{Mah_R}(X)$ and $\lambda \le \kappa$ is an infinite cardinal such that (κ, λ) is a measure-precaliber pair of every probability algebra, then $[0,1]^\lambda$ is a continuous image of X.

proof (a) If $\omega \in \mathrm{Mah_R}(X)$ then X has a compact subset K which is not scattered (531Ee) and there is a continuous surjection from K onto $[0,1]$ (4A2G(j-iv) again). By Tietze's theorem (4A2F(d-ix)), this has an extension to a continuous map from X to $[0,1]$ which is still a surjection. As there is a continuous surjection from $[0,1]$ onto $[0,1]^\omega$ (5A4I(b-ii)), there is a continuous surjection from X onto $[0,1]^\omega$.

(b) Let μ be a Maharam-type-homogeneous Radon probability measure on X with Maharam type κ, Σ its domain, and $(\mathfrak{A}, \bar\mu)$ its measure algebra, so that $(\mathfrak{A}, \bar\mu)$ is isomorphic to the measure algebra $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ as discussed in 531I-531K. Let $\langle e_\xi\rangle_{\xi<\kappa}$ be a stochastically independent τ-generating set of elements of measure $\frac{1}{2}$ in $\mathfrak{A}$, so that $(\mathfrak{A}, \langle e_\xi\rangle_{\xi<\kappa})$ is isomorphic to $\mathfrak{B}_\kappa$ with its standard generating family. For each $\xi < \kappa$, let $E_\xi \in \Sigma$ be such that $E_\xi^\bullet = e_\xi$ in $\mathfrak{A}$. Let $K'_\xi \subseteq E_\xi$, $K''_\xi \subseteq X \setminus E_\xi$ be compact sets of measure at least $\frac{1}{3}$, and set $K_\xi = K'_\xi \cup K''_\xi$, $a_\xi = K_\xi^\bullet$ for $\xi < \kappa$. By 531K, copied into $\mathfrak{A}$, there are $\langle c_\xi\rangle_{\xi<\kappa}$ and $\Gamma_0 \in [\kappa]^\kappa$ such that $c_\xi \subseteq a_\xi$ and $\bar\mu c_\xi \ge \frac{1}{3}$ for each ξ, and

$$\bar\nu_\kappa(\inf_{\xi\in I} c_\xi \cap e_\xi \cap \inf_{\eta\in J} c_\eta \setminus e_\eta) = \frac{1}{2^{\#(I\cup J)}}\bar\nu_\kappa(\inf_{\xi\in I\cup J} c_\xi)$$

whenever I, $J \subseteq \Gamma_0$ are disjoint finite sets.

At this point, recall that (κ, λ) is supposed to be a measure-precaliber pair of every probability algebra. So there is a $\Gamma \in [\Gamma_0]^\lambda$ such that $\inf_{\xi\in I} c_\xi \ne 0$ for every finite $I \subseteq \Gamma$. It follows at once that $\inf_{\xi\in I} a_\xi \cap e_\xi \cap \inf_{\eta\in J} a_\eta \setminus e_\eta$ is non-zero for all disjoint finite sets I, $J \subseteq \Gamma$. But this means that $X \cap \bigcap_{\xi\in I} K'_\xi \cap \bigcap_{\eta\in J} K''_\eta$ is non-negligible, therefore non-empty, for all disjoint finite I, $J \subseteq \Gamma$.

Set $K = \bigcap_{\xi\in\Gamma} K_\xi$, so that $K \subseteq X$ is compact. Then we have a continuous function $f : K \to \{0,1\}^\Gamma$ defined by setting

$$\begin{aligned} f(x)(\xi) &= 1 \text{ if } x \in K \cap E_\xi = K \cap K'_\xi, \\ &= 0 \text{ if } x \in K \setminus E_\xi = K \cap K''_\xi. \end{aligned}$$

Now f is surjective. **P** If $w \in \{0,1\}^\Gamma$ and $L \subseteq \Gamma$ is finite, then

$$\begin{aligned} F_L = \{x : x \in X,\ &x \in K'_\xi \text{ whenever } \xi \in L \text{ and } w(\xi) = 1, \\ &x \in K''_\xi \text{ whenever } \xi \in L \text{ and } w(\xi) = 0\} \end{aligned}$$

is a non-empty closed set. The family $\{F_L : L \in [\Gamma]^{<\omega}\}$ is downwards-directed, so has non-empty intersection; and if x is any point of the intersection, $x \in K$ and $f(x) = w$. **Q**

As $\#(\Gamma) = \lambda$, $\{0,1\}^\lambda$ is a continuous image of a closed subset of X and $[0,1]^\lambda$ is a continuous image of X (5A4Fa).

531M Proposition (Plebanek 97) If κ is an infinite cardinal and $[0,1]^\kappa$ is a continuous image of X whenever X is a compact Hausdorff space such that $\kappa \in \mathrm{Mah_R}(X)$, then κ is a measure-precaliber of every probability algebra.

proof It will be enough to show that κ is a measure-precaliber of $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ (525I(a-i)). Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_\kappa$ such that $\inf_{\xi<\kappa} \bar\nu_\kappa a_\xi = \alpha > 0$. Choose $\langle b_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{B}_\kappa$ inductively, as follows. Given $\langle b_\eta\rangle_{\eta<\xi}$, let $\mathfrak{D}_\xi$ be the closed subalgebra of $\mathfrak{B}_\kappa$ generated by $\{b_\eta : \eta < \xi\} \cup \{a_\xi\}$. Because $\mathfrak{B}_\kappa$ is homogeneous with Maharam type $\kappa > \tau(\mathfrak{D}_\xi)$, it is relatively atomless over $\mathfrak{D}_\xi$, and there is a $b \in \mathfrak{B}_\kappa$ such that $\bar\nu_\kappa(b \cap c) = \frac{1}{2}\bar\nu_\kappa c$ for every $c \in \mathfrak{D}_\xi$ (331B). Set $b_\xi = b \cap a_\xi$; then for any $\eta < \xi$ we have

$$\bar\nu_\kappa(b_\xi \triangle b_\eta) = \bar\nu_\kappa b_\xi + \bar\nu_\kappa b_\eta - 2\bar\nu_\kappa(b_\xi \cap b_\eta) = \frac{1}{2}\bar\nu_\kappa a_\xi + \bar\nu_\kappa b_\eta - \bar\nu_\kappa(a_\xi \cap b_\eta) \ge \frac{1}{2}\bar\nu_\kappa a_\xi \ge \frac{\alpha}{2}.$$

Continue.

Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{B}_\kappa$ generated by $\{b_\xi : \xi < \kappa\}$, and X its Stone space. Then $\mathfrak{C}$ is isomorphic to the algebra of open-and-closed subsets of X, so we have a Radon measure μ on X defined by saying that $\mu\widehat{c} = \bar{\nu}_\kappa c$ for every $c \in \mathfrak{C}$, writing $\widehat{c}$ for the open-and-closed subset of X corresponding to c (416Qa). Now μ is strictly positive and we can identify $\mathfrak{C}$ with a topologically dense subalgebra of the measure algebra of μ. It follows that μ has a Maharam-type-homogeneous component of type at least κ. **P?** Otherwise, there would be a set $E \subseteq X$, of measure at least $1 - \frac{1}{4}\alpha$, such that the Maharam type of the subspace measure μ_E was less than κ. But

$$\mu(E \cap \widehat{b}_\xi \triangle \widehat{b}_\eta) \ge \bar{\nu}_\kappa(b_\xi \triangle b_\eta) - \frac{\alpha}{4} \ge \frac{\alpha}{4}$$

whenever $\eta < \xi < \kappa$, so the topological density of the measure algebra of μ_E is at least κ (5A4B(h-ii)) and the Maharam type of μ_E is at least κ (521Ea). **XQ** Thus $\kappa \in \mathrm{Mah_R}(X)$.

Accordingly $[0,1]^\kappa$ is a continuous image of X. By 5A4C(d-iii), there is a non-empty closed subset F of X such that $\chi(x, F) \ge \kappa$ for every $x \in F$. Let $D \subseteq \kappa$ be a maximal set such that $\{F\} \cup \{\widehat{b}_\xi : \xi \in D\}$ has the finite intersection property. Set $Z = F \cap \bigcap_{\xi \in D} \widehat{b}_\xi$; then Z contains a point z say. Because $\{b_\xi : \xi \in D\}$ is centered, so is $\{a_\xi : \xi \in D\}$.

If $x \in X \setminus \{z\}$, then there is a $c \in \mathfrak{C}$ such that $x \in \widehat{c}$ and $z \notin \widehat{c}$; accordingly there is a $\zeta < \kappa$ such that one of x, z belongs to $\widehat{b}_\zeta$ and the other does not. If $\zeta \in D$ then $z \in \widehat{b}_\zeta$ and $x \notin \widehat{b}_\zeta$, so $x \notin Z$. If $\zeta \notin D$ then, by the maximality of D, $Z \cap \widehat{b}_\zeta = \emptyset$, so that $z \notin \widehat{b}_\zeta$, $x \in \widehat{b}_\zeta$ and again $x \notin Z$.

Accordingly $Z = \{z\}$, and $\{z\}$ can be expressed as the intersection of $\#(D)$ relatively open sets in F. By 4A2Gd, it follows that $\#(D) \ge \chi(z, F) \ge \kappa$, and we have already seen that $\{a_\xi : \xi \in D\}$ is centered. As $\langle a_\xi \rangle_{\xi<\kappa}$ is arbitrary, κ is a measure-precaliber of $\mathfrak{B}_\kappa$, as required.

531N From 531L-531M we see that if κ is any infinite cardinal other than ω_1, then the following are equiveridical:

(i) κ is a measure-precaliber of every probability algebra;

(ii) whenever X is a compact Hausdorff space and $\kappa \in \mathrm{Mah_R}(X)$, then $[0,1]^\kappa$ is a continuous image of X.

The following variation on the construction in 531M shows that ω_1 really is a special case.

Proposition (Plebanek 97) Suppose that there is a family $\langle W_\xi \rangle_{\xi<\omega_1}$ in $\mathcal{N}_{\omega_1}$ such that every closed subset of $\{0,1\}^{\omega_1} \setminus \bigcup_{\xi<\omega_1} W_\xi$ is scattered. Then there is a compact Hausdorff space X such that $\omega_1 \in \mathrm{Mah_R}(X)$ but $[0,1]^{\omega_1}$ is not a continuous image of X.

proof (a) Set $E_\xi = \{z : z \in \{0,1\}^{\omega_1},\ z(\xi) = 1\}$ for each $\xi < \omega_1$. Choose a family $\langle K_{\xi n} \rangle_{\xi<\omega_1, n\in\mathbb{N}}$ of compact sets in $\{0,1\}^{\omega_1}$ as follows. Given $\langle K_{\eta n} \rangle_{\eta<\xi, n\in\mathbb{N}}$, where $\xi < \omega_1$, such that $\bigcup_{n\in\mathbb{N}} K_{\eta n}$ is conegligible for every $\eta < \xi$, then for each $j \in \mathbb{N}$ we can find a family $\langle n(\xi, \eta, j) \rangle_{\eta<\xi}$ in $\mathbb{N}$ such that $L_{\xi j} = \bigcap_{\eta<\xi} \bigcup_{i\le n(\xi,\eta,j)} K_{\eta i}$ has measure at least $1 - 2^{-j-4}$. For $j \in \mathbb{N}$ choose a compact set $K'_{\xi j} \subseteq L_{\xi j} \setminus (W_\xi \cup \bigcup_{i<j} K'_{\xi i})$ of measure at least $1 - 2^{-j-3} - \nu_{\omega_1}(\bigcup_{i<j} K'_{\xi i})$. Set

$$K_{\xi,2i} = K'_{\xi i} \cap E_\xi, \quad K_{\xi,2i+1} = K'_{\xi i} \setminus E_\xi$$

for each $i \in \mathbb{N}$, and continue.

(b) At the end of the induction, let $\mathfrak{C}$ be the algebra of subsets of $\{0,1\}^{\omega_1}$ generated by $\{K_{\xi i} : \xi < \omega_1,\ i \in \mathbb{N}\}$, and X its Stone space. Then we have a Radon probability measure μ on X defined by setting $\mu\widehat{C} = \nu_{\omega_1} C$ for every $C \in \mathfrak{C}$, where $\widehat{C}$ is the open-and-closed subset of X corresponding to C. For $\eta < \xi < \omega_1$, we have

$$\begin{aligned}
\mu(\widehat{K}_{\eta 0} \triangle \widehat{K}_{\xi 0}) &= \nu_{\omega_1}(K_{\eta 0} \triangle K_{\xi 0}) \\
&= \nu_{\omega_1}((E_\eta \cap K'_{\eta 0}) \triangle (E_\xi \cap K'_{\xi 0})) \\
&\ge \nu_{\omega_1}(E_\eta \triangle E_\xi) - \nu_{\omega_1}(E_\eta \setminus K'_{\eta 0}) - \nu_{\omega_1}(E_\xi \setminus K'_{\xi 0}) \\
&\ge \frac{1}{2} - \frac{1}{8} - \frac{1}{8} = \frac{1}{4},
\end{aligned}$$

so the Maharam type of μ is at least ω_1 and $\omega_1 \in \mathrm{Mah_R}(X)$.

(c) Let $F \subseteq X$ be a non-scattered closed set. Then there is a $\zeta < \omega_1$ such that $F \not\subseteq \bigcup_{i\in\mathbb{N}} \widehat{K}_{\zeta i}$. **P?** Otherwise, set

$$R = \bigcap_{\xi<\omega_1} \bigcup_{i\in\mathbb{N}} (F \cap \widehat{K}_{\xi i}) \times K_{\xi i} \subseteq X \times \{0,1\}^{\omega_1}.$$

Note that for each $\xi<\omega_1$ the $\widehat{K}_{\xi i}$ are disjoint open-and-compact sets covering the compact set F, so $\{i:F\cap\widehat{K}_{\xi i}\ne\emptyset\}$ is finite and $\bigcup_{i\in\mathbb{N}}(F\cap\widehat{K}_{\xi i})\times K_{\xi i}$ is compact; thus R is compact. If (x,z) and $(x',z')\in R$ and $x\ne x'$, there must be some $C\in\mathfrak{C}$ such that $x\in\widehat{C}$ and $x'\notin\widehat{C}$, so there must be some $\xi<\omega_1$ and $i\in\mathbb{N}$ such that just one of x, x' belongs to $\widehat{K}_{\xi i}$; in this case, only the corresponding one of z, z' can belong to $K_{\xi i}$, and $z\ne z'$.

Conversely, if (x,z) and $(x',z')\in R$ and $z\ne z'$, there is some ξ such that $z(\xi)\ne z'(\xi)$. In this case, if i, $j\in\mathbb{N}$ are such that $(x,z)\in\widehat{K}_{\xi i}\times K_{\xi i}$ and $(x',z')\in\widehat{K}_{\xi j}\times K_{\xi j}$, $i\ne j$ and $x\ne x'$.

This shows that R is the graph of a bijection from F to $R[F]$. Because R is a compact subset of $F\times R[F]$, it is a homeomorphism, and $R[F]$ is not scattered. But, for each $\xi<\omega_1$, $R[F]\subseteq\bigcup_{i\in\mathbb{N}}K_{\xi i}$ is disjoint from W_ξ; and all compact subsets of $\{0,1\}^{\omega_1}\setminus\bigcup_{\xi<\omega_1}W_\xi$ are supposed to be scattered. **XQ**

(d) Take $x\in F\setminus\bigcup_{i\in\mathbb{N}}\widehat{K}_{\zeta i}$. Then $\chi(x,X)\le\omega$. **P** Consider the set

$$V=\bigcap_{\eta\le\zeta,i\in\mathbb{N}}\{x':x'\in X,\ x'\in\widehat{K}_{\eta i}\iff x\in\widehat{K}_{\eta i}\}.$$

This is a G_δ set containing x. **?** If there is an $x'\in V\setminus\{x\}$, there must be some $\xi<\omega_1$ and $j\in\mathbb{N}$ such that just one of x, x' belongs to $\widehat{K}_{\xi j}$. In this case, $\xi>\zeta$, so $K_{\xi j}\subseteq\bigcup_{i\le k}K_{\zeta i}$ and $\widehat{K}_{\xi j}\subseteq\bigcup_{i\le k}\widehat{K}_{\zeta i}$ for some $k\in\mathbb{N}$. But neither x nor x' belongs to $\bigcup_{i\le k}\widehat{K}_{\zeta i}$. **X** Thus $V=\{x\}$; by 4A2Gd again, $\chi(x,X)\le\omega$. **Q**

(e) Thus we see that whenever $F\subseteq X$ is a non-scattered closed set, there is an $x\in F$ such that $\chi(x,X)$ is countable. By 5A4C(d-iii), $[0,1]^{\omega_1}$ is not a continuous image of X.

531O In 531M we have a space X from which there is no surjection onto $[0,1]^\kappa$ because every non-empty closed set has a point of character less than κ. From stronger properties of κ we can get compact spaces with stronger topological properties, as in the next two results.

Proposition Let κ, κ' and λ be infinite cardinals such that (κ,κ') is not a measure-precaliber pair of $(\mathfrak{B}_\lambda,\bar\nu_\lambda)$. Then there is a compact Hausdorff space X such that $\kappa\in\mathrm{Mah}_{\mathrm{R}}(X)$ and $\chi(x,X)<\max(\kappa',\lambda^+)$ for every $x\in X$.

proof Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_\lambda$, with no centered subfamily with cardinal κ', such that $\inf_{\xi<\kappa}\bar\mu a_\xi=\alpha>0$. Let $\psi:\mathfrak{B}_\lambda\to\mathrm{T}_\lambda$ be a lifting; for each $\xi<\kappa$, let $K_\xi\subseteq\psi a_\xi$ be a compact set of measure at least $\frac12\alpha$. If $D\subseteq\kappa$ and $\#(D)=\kappa'$, then there is a finite set $I\subseteq D$ such that $\inf_{\xi\in I}a_\xi=0$, in which case $\bigcap_{\xi\in I}K_\xi\subseteq\bigcap_{\xi\in I}\psi a_\xi=\emptyset$. Thus $\{\xi:x\in K_\xi\}$ has cardinal less than κ' for every $x\in\{0,1\}^\lambda$.

Set

$$X=\bigcap_{\xi<\kappa'}\{(x,y):x\in\{0,1\}^\lambda,\ y\in\{0,1\}^\kappa,\ x\in K_\xi\text{ or }y(\xi)=0\},$$

so that X is a compact subset of $\{0,1\}^\lambda\times\{0,1\}^\kappa$. Now $\chi((x,y),X)<\max(\kappa',\lambda^+)$ for every $(x,y)\in X$. **P** Set $D=\{\xi:\xi<\kappa,\ x\in K_\xi\}$, so that $\#(D)<\kappa'$. For $I\in[\lambda]^{<\omega}$ and $J\in[D]^{<\omega}$ set

$$V_{IJ}=\{(x',y'):(x',y')\in X,\ x'\restriction I=x\restriction I,\ y'\restriction J=y\restriction J\},$$

so that $\mathcal{V}=\{V_{IJ}:I\in[\lambda]^{<\omega},\ J\in[D]^{<\omega}\}$ is a downwards-directed family of closed neighbourhoods of (x,y). If $(x',y')\in\bigcap\mathcal{V}$, then $x'=x$, so $x'\notin K_\xi$ for $\xi\in\kappa\setminus D$, and $y'(\xi)=y(\xi)=0$ for $\xi\notin D$; also $y'\restriction D=y\restriction D$, so $(x',y')=(x,y)$. Thus $\bigcap\mathcal{V}=\{(x,y)\}$; by 4A2Gd once more, $\mathcal{V}$ is a base of neighbourhoods of (x,y), and

$$\chi((x,y),X)\le\#(\mathcal{V})\le\max(\#(D),\lambda)<\max(\kappa',\lambda^+).\ \mathbf{Q}$$

Define $g:\{0,1\}^\lambda\times\{0,1\}^\kappa\to\{0,1\}^\kappa$ and $h:\{0,1\}^\lambda\times\{0,1\}^\kappa\to X$ by setting

$$\begin{aligned}g(x,y)(\xi)&=y(\xi)\text{ if }x\in K_\xi,\\&=0\text{ otherwise},\\h(x,y)&=(x,g(x,y)),\end{aligned}$$

for $\xi<\kappa$, $x\in\{0,1\}^\lambda$ and $y\in\{0,1\}^\kappa$. Write Σ for the domain of the product measure $\nu=\nu_\lambda\times\nu_\kappa$ on $\{0,1\}^\lambda\times\{0,1\}^\kappa$. Then the σ-algebra $\{F:F\subseteq X,\ h^{-1}[F]\in\Sigma\}$ contains all sets of the form $\{(x,y):x(\eta)=1\}$ and $\{(x,y):y(\xi)=1\}$, so includes a base for the topology of X and therefore contains every open-and-closed set. Accordingly we have an additive functional $U\mapsto\nu h^{-1}[U]$ on the algebra of open-and-closed subsets of X, which extends to a Radon probability measure μ on X (416Qa again). Set $F_\xi=\{(x,y):(x,y)\in X,\ y(\xi)=1\}$ for each $\xi<\kappa$; then for any $\eta<\xi<\kappa$,

$$\begin{aligned}\mu(F_\xi \setminus F_\eta) &= \nu h^{-1}[F_\xi \setminus F_\eta] \\ &\ge \nu\{(x,y) : x \in K_\xi,\, y(\xi) = 1,\, y(\eta) = 0\} = \frac{1}{4}\nu_\lambda K_\xi \ge \frac{1}{8}\alpha.\end{aligned}$$

Once again, this shows that the measure algebra of μ must have a homogeneous principal ideal with Maharam type at least κ, and $\kappa \in \mathrm{Mah}(X)$.

531P Putting these ideas together with 531L, we come to the following.

Proposition (Kunen & Mill 95, Plebanek 95) Let κ be a regular infinite cardinal. Then the following are equiveridical:

(i) κ is a measure-precaliber of every measurable algebra;

(ii) if X is a compact Hausdorff space such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then $\chi(x, X) \ge \kappa$ for some $x \in X$.

proof (a) Consider first the case $\kappa \ge \omega_2$.

(i)⇒(ii) If $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then $[0,1]^\kappa$ is a continuous image of X, by 531Lb. By 5A4C(d-iii) again and 5A4Bb, it follows at once that $\chi(x, X) \ge \kappa$ for many points $x \in X$.

not-(i)⇒not-(ii) By 525Ib there is a $\lambda < \kappa$ such that κ is not a precaliber of $\mathfrak{B}_\lambda$. Now 531O tells us that there is a compact Hausdorff space X such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$ and $\chi(x, X) < \max(\kappa, \lambda^+) = \kappa$ for every $x \in X$.

(b) Now suppose that $\kappa = \omega_1$.

(i)⇒(ii) ? Suppose, if possible, that ω_1 is a precaliber of every probability algebra, but that there is a first-countable compact Hausdorff space X with $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$. Let μ be a Maharam-type-homogeneous Radon probability measure on X with Maharam type ω_1, and $(\mathfrak{A}, \bar\mu)$ its measure algebra; let $\langle c_\xi\rangle_{\xi<\omega_1}$ be a τ-generating stochastically independent family of elements of measure $\frac{1}{2}$ in $\mathfrak{A}$. As in 531J, there is for each $a \in \mathfrak{A}$ a countable $J^*(a) \subseteq \omega_1$ such that a belongs to the closed subalgebra of $\mathfrak{A}$ generated by $\{c_\xi : \xi \in J^*(a)\}$.

For each $x \in X$, let $\mathcal{U}_x$ be a countable base of open neighbourhoods of x, and set $A_x = \{U^\bullet : U \in \mathcal{U}_x\}$, $J^\dagger(x) = \bigcup_{a\in A_x} J^*(a)$. Then $J^\dagger(x)$ is countable. For $\xi < \omega_1$, set $D_\xi = \{x : J^\dagger(x) \subseteq \xi\}$; then $\langle D_\xi\rangle_{\xi<\omega_1}$ is a non-decreasing family with union X. Now ω_1 is supposed to be a precaliber of $\mathfrak{A}$, so there must be a $\xi < \omega_1$ such that D_ξ has full outer measure (525Cc).

Let $G \subseteq X$ be open. Then $G^\bullet$ belongs to the closed subalgebra $\mathfrak{C}_\xi$ of $\mathfrak{A}$ generated by $\{c_\eta : \eta < \xi\}$. **P** For each $x \in G \cap D_\xi$, there is a $U_x \in \mathcal{U}_x$ such that $U_x \subseteq G$. Set $H = \bigcup\{U_x : x \in G \cap D_\xi\}$, so that $H \subseteq G$ is open and $G \cap D_\xi = H \cap D_\xi$; as D_ξ has full outer measure, $G \setminus H$ is negligible and $H^\bullet = G^\bullet$. But 414Aa once more tells us that $H^\bullet = \sup_{x\in D_\xi} U_x^\bullet$, and this belongs to $\mathfrak{D}_\xi$, because $J^\dagger(x) \subseteq \xi$ for every $x \in D_\xi$. **Q**

It follows at once that $F^\bullet \in \mathfrak{C}_\xi$ for every closed $F \subseteq X$. Because μ is inner regular with respect to the closed sets, $\mathfrak{C}_\xi$ is order-dense in $\mathfrak{A}$ and $\mathfrak{A} = \mathfrak{C}_\xi$ has Maharam type $\#(\xi) < \omega_1$. **X**

Thus (i)⇒(ii).

not-(i)⇒not-(ii) Suppose that (i) is false.

(α) By 525J, $\operatorname{cov}\mathcal{N}_{\omega_1} = \omega_1$ and there is a family $\langle A_\xi\rangle_{\xi<\omega_1}$ of negligible subsets of $\{0,1\}^{\omega_1}$ covering $\{0,1\}^{\omega_1}$. For each $\xi < \omega_1$, let $A'_\xi \supseteq A_\xi$ be a negligible set determined by coordinates in a countable set $J_\xi \subseteq \omega_1$; set $\tilde A_\xi = \bigcup\{A'_\eta : \eta < \xi,\, J_\eta \subseteq \xi\}$; then $\tilde A_\xi$ is determined by coordinates less than ξ. Set $H_\xi = \{y\restriction\xi : y \in \tilde A_\xi\}$, so that H_ξ is a ν_ξ-negligible subset of $\{0,1\}^\xi$.

We see that $\langle \tilde A_\xi\rangle_{\xi<\omega_1}$ is non-decreasing, and

$$\bigcup_{\xi<\omega_1} \tilde A_\xi = \bigcup_{\xi<\omega_1} A'_\xi = \{0,1\}^{\omega_1}.$$

Consequently $y\restriction\xi \in H_\xi$ whenever $\eta \le \xi < \omega_1$, $y \in \{0,1\}^{\omega_1}$ and $y\restriction\eta \in H_\eta$, while for every $y \in \{0,1\}^{\omega_1}$ there is a $\xi < \omega_1$ such that $y\restriction\xi \in H_\xi$.

(β) Set $Y = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\} \subseteq [0,1]$. For $\xi \le \omega_1$ define $\phi_\xi : Y^\xi \to \{0,1\}^\xi$ by setting

$$\begin{aligned}\phi_\xi(x)(\eta) &= 0 \text{ if } \eta < \xi \text{ and } x(\eta) = 0, \\ &= 1 \text{ for other } \eta < \xi.\end{aligned}$$

Observe that ϕ_ξ is Borel measurable for every $\xi<\omega_1$. Choose $\langle X_\xi\rangle_{\xi<\omega_1}$, and $\langle K_{\xi n}\rangle_{\xi<\omega_1,n\in\mathbb{N}}$ inductively, as follows. The inductive hypothesis will be that X_ξ is a compact subset of Y^ξ, $\phi_\xi[X_\xi]$ is conegligible in $\{0,1\}^\xi$, $\phi_\xi\restriction X_\xi$ is injective and $x\restriction\eta\in X_\eta$ whenever $x\in X_\xi$ and $\eta\le\xi<\omega_1$.

Start with $X_0=Y^0=\{\emptyset\}$ and $\phi_0:X_0\to\{0,1\}^0$ the identity map.

Given $\xi<\omega_1$ and $X_\xi\subseteq Y^\xi$, then 433D tells us that there is a Radon measure μ_ξ on X_ξ such that ν_ξ is the image measure $\mu_\xi\phi_\xi^{-1}$. Let $\langle K_{\xi n}\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of compact subsets of $X_\xi\setminus\phi_\xi^{-1}[H_\xi]$ with μ_ξ-conegligible union. Set

$$\begin{aligned}X_{\xi+1}=\{x:x\in Y^{\xi+1},\,x\restriction\xi\in X_\xi,\,x(\xi)=0\}&\\ \cup\bigcup_{n\in\mathbb{N}}\{x:x\in Y^{\xi+1},\,x\restriction\xi\in K_{\xi n},\,x(\xi)=2^{-n}\}&.\end{aligned}$$

It is easy to see that $X_{\xi+1}$ is compact and $\phi_{\xi+1}\restriction X_{\xi+1}$ is injective, while surely $x\restriction\eta\in X_\eta$ whenever $x\in X_{\xi+1}$ and $\eta\le\xi+1$, just because $x\restriction\xi\in X_\xi$. Also

$$\phi_{\xi+1}[X_{\xi+1}]\supseteq\{y:y\in\{0,1\}^{\xi+1},\,y\restriction\xi\in\bigcup_{n\in\mathbb{N}}\phi_\xi[K_{\xi n}]\}$$

is conegligible for $\nu_{\xi+1}$ because $\phi_\xi[K_{\xi n}]$ must be analytic for every n and

$$\nu_\xi(\bigcup_{n\in\mathbb{N}}\phi_\xi[K_{\xi n}])=\mu_\xi(\bigcup_{n\in\mathbb{N}}K_{\xi n})=1$$

because $\phi_\xi\restriction X_\xi$ is injective.

Given that X_η has been defined for $\eta<\xi$, where $\xi<\omega_1$ is a non-zero limit ordinal, set

$$X_\xi=\{x:x\in Y^\xi,\,x\restriction\eta\in X_\eta\text{ for every }\eta<\xi\}.$$

Of course X_ξ is compact and $\phi_\xi\restriction X_\xi$ is injective. To see that $\phi_\xi[X_\xi]$ is conegligible, observe that

$$W=\bigcap_{\eta<\xi}\{y:y\in\{0,1\}^\xi,\,y\restriction\eta\in\phi_\eta[X_\eta]\}$$

is conegligible. But if $y\in W$ and we choose $x_\eta\in X_\eta$ such that $\phi_\eta(x_\eta)=y\restriction\eta$ for each $\eta<\xi$, then we must have $x_\zeta=x_\eta\restriction\zeta$ whenever $\zeta\le\eta<\xi$, because $\phi_\zeta\restriction X_\zeta$ is injective; so there is an $x\in Y^\xi$ such that $x_\eta=x\restriction\eta$ for every $\eta<\xi$, in which case $x\in X_\xi$ and $\phi_\xi(x)=y$. Thus $\phi_\xi[X_\xi]\supseteq W$ is conegligible.

(γ) At the end of the induction, set

$$X=\{x:x\in Y^{\omega_1},\,x\restriction\xi\in X_\xi\text{ for every }\xi<\omega_1\},\quad\phi=\phi_{\omega_1}\restriction X.$$

As in the limit stage of the construction in (β), we see that X is a closed subset of Y^{ω_1}, so with the subspace topology is a zero-dimensional compact Hausdorff space. This time, we do not expect that $\phi[X]$ should be conegligible in $\{0,1\}^{\omega_1}$, but we find that it has full outer measure. **P** If $K\subseteq\{0,1\}^{\omega_1}$ is a non-negligible closed G_δ set, there is a $\xi<\omega_1$ such that K is determined by coordinates less than ξ. Set $K'=\{y\restriction\xi:y\in K\}$; then $\nu_\xi K'=\nu_{\omega_1}K>0$, so there is an $x_0\in X_\xi$ such that $\phi_\xi(x_0)\in K'$. Extending x_0 to $x\in Y^{\omega_1}$ by setting $x(\eta)=0$ for $\xi\le\eta<\omega_1$, we see by induction on ζ that $x\restriction\zeta\in X_\zeta$ for $\xi\le\zeta<\omega_1$, so $x\in X$; also $\phi(x)\restriction\xi=\phi_\xi(x_0)\in K'$, so $\phi(x)\in K$ and K meets $\phi[X]$. As ν_{ω_1} is completion regular, $\phi[X]$ has full outer measure. **Q**

(δ) X is first-countable. **P** If $x\in X$, $\xi<\omega_1$ and $x(\xi)\ne0$, then $x\restriction(\xi+1)$ belongs to $X_{\xi+1}$, and there must be some $n\in\mathbb{N}$ such that $x(\xi)=2^{-n}$ and $x\restriction\xi\in K_{\xi n}$; in which case $\phi_\xi(x\restriction\xi)\notin H_\xi$. Now take any $x\in X$. Then there is a $\xi<\omega_1$ such that $\phi(x)\in\tilde A_\xi$ and $\phi_\xi(x)=\phi(x)\restriction\xi$ belongs to H_ξ. In this case, $V=\{x':x'\in X,\,x'\restriction\xi=x\restriction\xi\}$ is a G_δ subset of X containing x. But if $x'\in V$ then, for any $\eta\ge\xi$, $\phi_\eta(x'\restriction\eta)\in H_\eta$ and $x'(\eta)=0$. Thus $V=\{x\}$. By 4A2Gd, as usual, x has a countable base of neighbourhoods in X; as x is arbitrary, X is first-countable. **Q**

(ϵ) By 234F, there is a measure λ on X such that ϕ is inverse-measure-preserving for λ and ν_{ω_1}. Of course λ is a probability measure. Now for any $\xi<\omega_1$ and $n\in\mathbb{N}$,

$$\{x:x\in X,\,x(\xi)=0\}=\{x:\phi(x)(\xi)=0\},$$

$$\begin{aligned}\{x:x\in X,\,x(\xi)=2^{-n}\}&=\{x:\phi(x)(\xi)=1,\,x\restriction\xi\in K_{\xi n}\}\\&=\{x:\phi(x)(\xi)=1,\,\phi_\xi(x\restriction\xi)\in\phi_\xi[K_{\xi n}]\}\\&=\{x:\phi(x)(\xi)=1,\,\phi(x)\restriction\xi\in\phi_\xi[K_{\xi n}]\}\end{aligned}$$

are measured by λ. So the domain of λ includes a base for the topology of the zero-dimensional compact Hausdorff space X. By 416Qa, there is a Radon measure μ on X agreeing with λ on the open-and-closed subsets of X; by the

Monotone Class Theorem (136C), μ and λ agree on the σ-algebra generated by the open-and-closed sets, that is, the Baire σ-algebra of X (4A3Oe). In particular, setting $E_\xi = \{x : x \in X, x(\xi) = 0\}$ for $\xi < \omega_1$,

$$\begin{aligned}\mu(E_\xi \cap E_\eta) = \lambda(E_\xi \cap E_\eta) &= \nu_{\omega_1}\{y : y \in \{0,1\}^{\omega_1}, y(\xi) = y(\eta) = 0\} \\ &= \frac{1}{2} \text{ if } \xi = \eta < \omega_1, \\ &= \frac{1}{4} \text{ if } \xi, \eta < \omega_1 \text{ are different.}\end{aligned}$$

It follows that $\mu(E_\xi \triangle E_\eta) = \frac{1}{2}$ for all distinct ξ, $\eta < \omega_1$, so μ has uncountable Maharam type and $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$. Thus X and μ witness that (ii) is false.

(c) Finally, if $\kappa = \omega$, both (i) and (ii) are true for elementary reasons (525Fa).

531Q In 531P we see that if ω_1 is not a precaliber of every measurable algebra then there is a first-countable compact Hausdorff space with a Radon measure with Maharam type ω_1. With a sharper hypothesis, and rather more work, we can get a substantially stronger version, as follows.

Proposition Suppose that $\mathrm{cf}\,\mathcal{N}_\omega = \omega_1$. Then there is a hereditarily separable perfectly normal compact Hausdorff space X such that $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$.

proof For $\eta \le \xi \le \omega_1$ and $x \in \{0,1\}^\xi$, set $\pi_{\xi\eta}(x) = x \restriction \eta$; write π_η for $\pi_{\omega_1\eta} : \{0,1\}^{\omega_1} \to \{0,1\}^\eta$.

(a) It will be helpful to have the following fact out in the open. Let Y be a zero-dimensional metrizable compact Hausdorff space, μ an atomless Radon probability measure on Y, $A \subseteq Y$ a μ-negligible set and $\mathcal{Q}$ a countable family of closed subsets of Y. Then there are closed sets K, $L \subseteq Y$, with union Y, such that

$K \cup L = Y$, $K \cap L \cap A = \emptyset$, $\mu(K \cap L) \ge \frac{1}{2}$,

$K \cap Q = \overline{Q \setminus L}$ and $L \cap Q = \overline{Q \setminus K}$ for every $Q \in \mathcal{Q}$.

P We can of course suppose that $\mathcal{Q}$ is non-empty. For each $Q \in \mathcal{Q}$ let D_Q be a countable dense subset of Q; let $S \subseteq Y \setminus (A \cup \bigcup_{Q \in \mathcal{Q}} D_Q)$ be a closed set of measure at least $\frac{1}{2}$. (This is where we need to know that μ is atomless, so that every D_Q is negligible.) Let $\mathcal{U}$ be a countable base for the topology of Y consisting of open-and-closed sets and let $\langle (U_n, Q_n) \rangle_{n \in \mathbb{N}}$ run over $\mathcal{U} \times \mathcal{Q}$. Choose inductively sequences $\langle G_n \rangle_{n \in \mathbb{N}}$, $\langle H_n \rangle_{n \in \mathbb{N}}$ of open-and-closed subsets of $Y \setminus S$, as follows. Start with $G_0 = H_0 = \emptyset$. Given that G_n and H_n are disjoint from each other and from S, then

— if $U_n \cap S = \emptyset$, take $G_{n+1} = G_n \cup (U_n \setminus H_n)$ and $H_{n+1} = H_n$;

— if $U_n \cap S \cap Q_n \ne \emptyset$, $S \cap Q_n$ is nowhere dense in Q_n (because it is closed and does not meet D_{Q_n}), while $(G_n \cup H_n) \cap Q_n$ is a relatively closed set in Q_n not meeting S, therefore not including $Q_n \cap U_n$; so $Q_n \cap U_n \setminus (S \cup G_n \cup H_n)$ must have at least two points y, y' say, and we can enlarge G_n and H_n to disjoint open-and-closed subsets G_{n+1}, H_{n+1} of $Y \setminus S$ containing y, y' respectively, and therefore both meeting $U_n \cap Q_n$;

— otherwise, take $G_{n+1} = G_n$ and $H_{n+1} = H_n$.

At the end of the induction, set $G = \bigcup_{n \in \mathbb{N}} G_n$ and $H = \bigcup_{n \in \mathbb{N}} H_n$, so that G and H are disjoint open subsets of $Y \setminus S$. Now if y is any point of $Y \setminus S$, there must be some n such that $y \in U_n \subseteq Y \setminus S$, so that $y \in G_{n+1} \cup H_n$; thus $Y = G \cup H \cup S$. Set $K = G \cup S = Y \setminus H$, $L = H \cup S = Y \setminus G$; then K and L are closed sets with union Y, and $K \cap L = S$ has measure at least $\frac{1}{2}$ and is disjoint from A.

If $Q \in \mathcal{Q}$, $y \in S \cap Q$ and U is any neighbourhood of y, there is an $n \in \mathbb{N}$ such that $Q_n = Q$ and $y \in U_n \subseteq U$. In this case, $U_n \cap S \cap Q_n \ne \emptyset$ and G_{n+1} and H_{n+1} meet $U_n \cap Q$. As U is arbitrary,

$$y \in \overline{Q \cap G} \cap \overline{Q \cap H} = \overline{Q \setminus L} \cap \overline{Q \setminus K}.$$

Thus $K \cap L \cap Q \subseteq \overline{Q \setminus L} \cap \overline{Q \setminus K}$, so $K \cap Q \subseteq \overline{Q \setminus L}$ and $L \cap Q \subseteq \overline{Q \setminus K}$. On the other hand, $K \supseteq Q \setminus L$ and $L \supseteq Q \setminus K$, so $K \cap Q = \overline{Q \setminus L}$ and $L \cap Q = \overline{Q \setminus K}$. Thus K and L fulfil all the specifications. **Q**

(b) Choose

$$\langle f_\xi \rangle_{\omega \le \xi \le \omega_1}, \langle \mu_\xi \rangle_{\omega \le \xi < \omega_1}, \langle K_\xi \rangle_{\omega \le \xi < \omega_1}, \langle L_\xi \rangle_{\omega \le \xi < \omega_1}, \langle X_\xi \rangle_{\omega \le \xi \le \omega_1},$$

$$\langle Q'_{\zeta\xi} \rangle_{\omega \le \xi \le \zeta < \omega_1}, \langle Q_{\xi\delta} \rangle_{\omega \le \delta \le \xi < \omega_1}, \langle Q_{\xi\delta\eta} \rangle_{\omega \le \eta \le \delta \le \xi < \omega_1}, \langle A_{\zeta\xi} \rangle_{\omega \le \xi \le \zeta < \omega_1}, \langle A_\xi \rangle_{\omega \le \xi < \omega_1}$$

as follows. $f_\omega(x) = x$ for every $x \in \{0,1\}^\omega = X_\omega$. Given that $\omega \le \xi < \omega_1$ and f_ξ is a Borel measurable surjection from $\{0,1\}^\xi$ onto a compact subset X_ξ of $\{0,1\}^\xi$, let μ_ξ be the image measure $\nu_\xi f_\xi^{-1}$, so that μ_ξ is a Radon measure on $\{0,1\}^\xi$ (433E, 418I); of course $\mu_\xi X_\xi = 1$. Because $\operatorname{cf}\mathcal{N}_\omega = \omega_1$, μ_ξ is inner regular with respect to a family of size at most ω_1 (524Pb), which we may suppose to consist of closed sets; let $\langle Q'_{\zeta\xi}\rangle_{\xi\le\zeta<\omega_1}$ run over such a family. Similarly, there is a family $\langle A_{\zeta\xi}\rangle_{\xi\le\zeta<\omega_1}$ running over a cofinal subset of the null ideal of μ_ξ. Next, for $\omega\le\delta\le\xi$, let $Q_{\xi\delta}\subseteq\pi_{\xi\delta}^{-1}[Q'_{\xi\delta}]$ be a compact μ_ξ-self-supporting set of the same μ_ξ-measure as $\pi_{\xi\delta}^{-1}[Q'_{\xi\delta}]$. Note that, as $\mu_\xi X_\xi = 1$, every $Q_{\xi\delta}$ must be included in X_ξ. Set $Q_{\xi\delta\eta} = X_\xi\cap\pi_{\xi\delta}^{-1}[Q_{\delta\eta}]$ for $\omega\le\eta\le\delta\le\xi$. Set

$$\mathcal{A}_\xi = \{\pi_{\xi\eta}^{-1}[A_{\delta\eta}] : \omega\le\eta\le\delta\le\xi\},\qquad A_\xi = \bigcup\{A : A\in\mathcal{A}_\xi,\ \mu_\xi A = 0\};$$

because $\mathcal{A}_\xi$ is countable, A_ξ is μ_ξ-negligible. By (a) above, we can find closed sets K_ξ, L_ξ covering $\{0,1\}^\xi$ such that $\mu_\xi(K_\xi\cap L_\xi)\ge\frac{1}{2}$, $K_\xi\cap L_\xi\cap A_\xi=\emptyset$, and $K_\xi\cap Q_{\xi\delta\eta} = \overline{Q_{\xi\delta\eta}\setminus L_\xi}$, $L_\xi\cap Q_{\xi\delta\eta} = \overline{Q_{\xi\delta\eta}\setminus K_\xi}$ whenever $\omega\le\eta\le\delta\le\xi$.

Given that $\omega<\xi\le\omega_1$ and that K_η, L_η are closed subsets of $\{0,1\}^\eta$ covering $\{0,1\}^\eta$ for $\omega\le\eta<\xi$, define $f_\xi(x)(\eta)$ inductively, for $x\in\{0,1\}^\xi$ and $\eta<\xi$, by setting

$$\begin{aligned} f_\xi(x)(\eta) &= x(\eta) \text{ if } \eta<\omega \text{ or } \eta\ge\omega \text{ and } x{\restriction}\eta\in K_\eta\cap L_\eta,\\ &= 1 \text{ if } \eta\ge\omega \text{ and } x{\restriction}\eta\notin L_\eta,\\ &= 0 \text{ if } \eta\ge\omega \text{ and } x{\restriction}\eta\notin K_\eta. \end{aligned}$$

Then $f_\xi:\{0,1\}^\xi\to\{0,1\}^\xi$ is Baire measurable. Set

$$X_\xi = \bigcap_{\omega\le\eta<\xi}\{x : x\in\{0,1\}^\xi,\ x(\eta)=1 \text{ or } x{\restriction}\eta\in L_\eta,\ x(\eta)=0 \text{ or } x{\restriction}\eta\in K_\eta\};$$

then $X_\xi\subseteq\{0,1\}^\xi$ is compact, $f_\xi(x)\in X_\xi$ for every $x\in\{0,1\}^\xi$, and $f_\xi(x)=x$ for every $x\in X_\xi$. So $f_\xi[\{0,1\}^\xi]=X_\xi$ and (if $\xi<\omega_1$) the process continues.

(c) At the end of the induction, write f for f_{ω_1} and X for X_{ω_1}. If $z\in\{0,1\}^{\omega_1}$ and $\xi<\omega_1$, the formula for f_ξ in (b) shows that $f(z)(\eta)=f_\xi(z{\restriction}\xi)(\eta)$ for every $\eta<\xi$, that is, that $f(z){\restriction}\xi = f_\xi(z{\restriction}\xi)$. Next, because f is Baire measurable, we have a Radon measure μ on $\{0,1\}^{\omega_1}$ defined by saying that $\mu V = \nu_{\omega_1}f^{-1}[V]$ for every Baire set $V\subseteq\{0,1\}^{\omega_1}$ (432F); of course $\mu V=0$ for every open-and-closed set V disjoint from X, so $\mu X=1$.

(d) A couple of simple facts. I have already observed that $\pi_\xi f = f_\xi\pi_\xi$ for $\omega\le\xi<\omega_1$; consequently

$$X_\xi = f_\xi[\{0,1\}^\xi] = f_\xi[\pi_\xi[\{0,1\}^{\omega_1}]] = \pi_\xi[f[\{0,1\}^{\omega_1}]] = \pi_\xi[X]$$

and

$$\begin{aligned} \mu_\xi V = \nu_\xi f_\xi^{-1}[V] &= (\nu_{\omega_1}\pi_\xi^{-1})f_\xi^{-1}[V] = \nu_{\omega_1}(f_\xi\pi_\xi)^{-1}[V]\\ &= \nu_{\omega_1}(\pi_\xi f)^{-1}[V] = (\nu_{\omega_1}f^{-1})\pi_\xi^{-1}[V] = \mu\pi_\xi^{-1}[V] \end{aligned}$$

for every open-and-closed set $V\subseteq\{0,1\}^\xi$. Thus the Radon measures $\mu\pi_\xi^{-1}$ and μ_ξ are identical. Equally, if $\omega\le\eta\le\xi<\omega_1$,

$$X_\eta = \pi_\eta[X] = \pi_{\xi\eta}[\pi_\xi[X]] = \pi_{\xi\eta}[X_\xi]$$

and

$$\mu_\eta = \mu\pi_\eta^{-1} = \mu(\pi_{\xi\eta}\pi_\xi)^{-1} = \mu_\xi\pi_{\xi\eta}^{-1}.$$

Accordingly, if $\omega\le\eta\le\delta\le\xi<\omega_1$,

$$\mu_\xi\pi_{\xi\eta}^{-1}[A_{\delta\eta}] = \mu_\eta A_{\delta\eta} = 0.$$

Thus in fact $A_\xi = \bigcup\mathcal{A}_\xi$ and $K_\xi\cap L_\xi$ is disjoint from $\pi_{\xi\eta}^{-1}[A_{\delta\eta}]$ whenever $\omega\le\eta\le\delta\le\xi$.

(e) We come now to the first key idea of this construction. If $\omega\le\eta\le\delta\le\xi<\omega_1$, then $g_{\xi\delta\eta}=\pi_{\xi\delta}{\restriction}Q_{\xi\delta\eta}$ is an irreducible surjection onto $Q_{\delta\eta}$. **P** First, let us confirm that if $\delta\le\zeta\le\xi$ then

$$\begin{aligned} \pi_{\xi\zeta}[Q_{\xi\delta\eta}] &= \pi_{\xi\zeta}[X_\xi\cap\pi_{\xi\delta}^{-1}[Q_{\delta\eta}]] = \pi_{\xi\zeta}[X_\xi\cap\pi_{\xi\zeta}^{-1}[\pi_{\zeta\delta}^{-1}[Q_{\delta\eta}]]]\\ &= \pi_{\xi\zeta}[X_\xi]\cap\pi_{\zeta\delta}^{-1}[Q_{\delta\eta}] = X_\zeta\cap\pi_{\zeta\delta}^{-1}[Q_{\delta\eta}] = Q_{\zeta\delta\eta}. \end{aligned}\qquad (*)$$

In particular,

$$\pi_{\xi\delta}[Q_{\xi\delta\eta}] = Q_{\delta\delta\eta} = X_\delta\cap\pi_{\delta\delta}^{-1}[Q_{\delta\eta}] = Q_{\delta\eta}.$$

To see that $g_{\xi\delta\eta}$ is irreducible, induce on ξ. At the start, $g_{\delta\delta\eta}$ is an identity function, so is certainly irreducible. For the inductive step to $\xi+1$, given that $\delta\le\xi<\omega_1$ and $g_{\xi\delta\eta}$ is irreducible, consider $h=\pi_{\xi+1,\xi}\restriction Q_{\xi+1,\delta,\eta}$. By (*), $h[Q_{\xi+1,\delta,\eta}]=Q_{\xi\delta\eta}$. Note that $X_{\xi+1}$ can be identified with

$$\{(x,1):x\in X_\xi\cap K_\xi\}\cup\{(x,0):x\in X_\xi\cap L_\xi\}.$$

If $V\subseteq\{0,1\}^{\xi+1}$ is a cylinder set meeting $Q_{\xi+1,\delta,\eta}$, then $V'=\pi_{\xi+1,\xi}[V]$ is a cylinder set in $\{0,1\}^\xi$ meeting $Q_{\xi\delta\eta}$. Because

$$K_\xi\cap Q_{\xi\delta\eta}=\overline{Q_{\xi\delta\eta}\setminus L_\xi},\quad L_\xi\cap Q_{\xi\delta\eta}=\overline{Q_{\xi\delta\eta}\setminus K_\xi},\quad K_\xi\cup L_\xi\supseteq Q_{\xi\delta\eta},$$

V' must meet both $Q_{\xi\delta\eta}\setminus L_\xi$ and $Q_{\xi\delta\eta}\setminus K_\xi$. Now V must cover at least one of

$$\begin{aligned}\{x:x\in\{0,1\}^{\xi+1},\,x\restriction\xi\in V',\,x(\xi)=1\}&\supseteq\{x:x\in X_{\xi+1},\,x\restriction\xi\in V'\setminus L_\xi\}\\&\supseteq\{x:x\in Q_{\xi+1,\delta,\eta},\,x\restriction\xi\in V'\cap Q_{\xi\delta\eta}\setminus L_\xi\},\end{aligned}$$

$$\begin{aligned}\{x:x\in\{0,1\}^{\xi+1},\,x\restriction\xi\in V',\,x(\xi)=0\}&\supseteq\{x:x\in X_{\xi+1},\,x\restriction\xi\in V'\setminus K_\xi\}\\&\supseteq\{x:x\in Q_{\xi+1,\delta,\eta},\,x\restriction\xi\in V'\cap Q_{\xi\delta\eta}\setminus K_\xi\},\end{aligned}$$

and $h[Q_{\xi+1,\delta,\eta}\setminus V]$ is disjoint from at least one of the non-empty sets $V'\cap Q_{\xi\delta\eta}\setminus L_\xi$, $V'\cap Q_{\xi\delta\eta}\setminus K_\xi$. As V is arbitrary, h is irreducible, and the composition $g_{\xi\delta\eta}h$ is irreducible (5A4C(d-iv)); but this is $g_{\xi+1,\delta,\eta}$.

For the inductive step to a limit ordinal ξ such that $\delta<\xi<\omega_1$, again take a cylinder set $V\subseteq\{0,1\}^\xi$ meeting $Q_{\xi\delta\eta}$. This time, because ξ is a limit ordinal, there is a ζ such that $\delta\le\zeta<\xi$ and V is determined by coordinates less than ζ. Set $V'=\pi_{\xi\zeta}[V]$; then V' is a cylinder set in $\{0,1\}^\zeta$ meeting $\pi_{\xi\zeta}[Q_{\xi\delta\eta}]=Q_{\zeta\delta\eta}$. Now

$$\begin{aligned}\pi_{\xi\delta}[Q_{\xi\delta\eta}\setminus V]&=\pi_{\zeta\delta}[\pi_{\xi\zeta}[Q_{\xi\delta\eta}\setminus\pi_{\xi\zeta}^{-1}[V']]]\\&=\pi_{\zeta\delta}[\pi_{\xi\zeta}[Q_{\xi\delta\eta}]\setminus V']=\pi_{\zeta\delta}[Q_{\zeta\delta\eta}\setminus V']\subset Q_{\delta\eta}\end{aligned}$$

because $g_{\zeta\delta\eta}:Q_{\zeta\delta\eta}\to Q_{\delta\eta}$ is irreducible. Thus the induction continues. **Q**

(f) Repeating the last step of the argument in (e) with $\xi=\omega_1$, we see that in fact $\pi_\delta\restriction X\cap\pi_\delta^{-1}[Q_{\delta\eta}]$ is an irreducible surjection onto $Q_{\delta\eta}$ whenever $\omega\le\eta\le\delta<\omega_1$. The next point to confirm is that if $z,\,z'\in X$, $\omega\le\eta\le\delta<\omega_1$, $z\restriction\delta=z'\restriction\delta$ and $z\restriction\eta\in A_{\delta\eta}$, then $z'=z$. **P** Suppose that $\delta\le\xi<\omega_1$ and $z\restriction\xi=z'\restriction\xi$. Then $K_\xi\cap L_\xi$ does not meet $\pi_{\xi\eta}^{-1}[A_{\delta\eta}]$, so does not contain $z\restriction\xi$. Accordingly

$$z(\xi)=1\implies z\restriction\xi\in K_\xi\implies z\restriction\xi\notin L_\xi\implies z'\restriction\xi\notin L_\xi\implies z'(\xi)=1,$$

and similarly $z(\xi)=0\Rightarrow z'(\xi)=0$. So an easy induction on ξ shows that $z(\xi)=z'(\xi)$ whenever $\delta\le\xi<\omega_1$, and $z=z'$. **Q**

(g) It follows that if $H\subseteq X$ is closed, there is a $\xi<\omega_1$ such that $H=X\cap\pi_\xi^{-1}[\pi_\xi[H]]$ and $\pi_\xi\restriction H$ is irreducible. **P** For $\omega\le\eta\le\xi<\omega_1$,

$$\mu_\eta\pi_\eta[H]=\mu_\xi\pi_{\xi\eta}^{-1}[\pi_\eta[H]]=\mu_\xi\pi_{\xi\eta}^{-1}[\pi_{\xi\eta}[\pi_\xi[H]]]\ge\mu_\xi\pi_\xi[H].$$

So we have an $\eta<\omega_1$ such that $\mu_\eta\pi_\eta[H]=\mu_\xi\pi_\xi[H]$ whenever $\eta\le\xi<\omega_1$. Now recall that μ_η is inner regular with respect to $\{Q'_{\delta\eta}:\eta\le\delta<\omega_1\}$. So there is a countable set $I\subseteq\omega_1\setminus\eta$ such that $\langle Q'_{\delta\eta}\rangle_{\delta\in I}$ is disjoint, $Q'_{\delta\eta}\subseteq\pi_\eta[H]$ for every $\delta\in I$ and $\sum_{\delta\in I}\mu_\eta Q'_{\delta\eta}=\mu\pi_\eta[H]$.

For each $\delta\in I$,

$$\begin{aligned}\mu_\delta(Q_{\delta\eta}\setminus\pi_\delta[H])&\le\mu_\delta(\pi_{\delta\eta}^{-1}[Q'_{\delta\eta}]\setminus\pi_\delta[H])\\&\le\mu_\delta(\pi_{\delta\eta}^{-1}[\pi_\eta[H]]\setminus\pi_\delta[H])\\&=\mu_\delta(\pi_{\delta\eta}^{-1}[\pi_\eta[H]])-\mu_\delta\pi_\delta[H]\end{aligned}$$

(because surely $\pi_\delta[H]\subseteq\pi_{\delta\eta}^{-1}[\pi_\eta[H]]$)

$$=\mu_\eta\pi_\eta[H]-\mu_\delta\pi_\delta[H]=0$$

by the choice of η. Because $Q_{\delta\eta}$ was μ_δ-self-supporting, and $\pi_\delta[H]$ is closed, $Q_{\delta\eta}\subseteq\pi_\delta[H]$. Because $\pi_\delta\restriction X\cap\pi_\delta^{-1}[Q_{\delta\eta}]$ is irreducible, $X\cap\pi_\delta^{-1}[Q_{\delta\eta}]\subseteq H$.

Set $\zeta=\sup(I\cup\{\eta\})<\omega_1$. Since $Q_{\delta\eta}\subseteq\pi_{\delta\eta}^{-1}[Q'_{\delta\eta}]$, $\pi_{\zeta\delta}^{-1}[Q_{\delta\eta}]\subseteq\pi_{\zeta\eta}^{-1}[Q'_{\delta\eta}]$ for each $\delta\in I$; as $\langle Q'_{\delta\eta}\rangle_{\delta\in I}$ is disjoint, so is $\langle\pi_{\zeta\delta}^{-1}[Q_{\delta\eta}]\rangle_{\delta\in I}$; and

$$\begin{aligned}\sum_{\delta\in I}\mu_\zeta\pi_{\zeta\delta}^{-1}[Q_{\delta\eta}]&=\sum_{\delta\in I}\mu_\delta Q_{\delta\eta}=\sum_{\delta\in I}\mu_\delta\pi_{\delta\eta}^{-1}[Q'_{\delta\eta}]\\&=\sum_{\delta\in I}\mu_\eta Q'_{\delta\eta}=\mu_\eta\pi_\eta[H]=\mu_\zeta\pi_\zeta[H].\end{aligned}$$

Because $X\cap\pi_\delta^{-1}[Q_{\delta\eta}]\subseteq H$, $\pi_\zeta[H]\supseteq X_\zeta\cap\pi_{\zeta\delta}^{-1}[Q_{\delta\eta}]$ for every $\delta\in I$. So $\pi_\zeta[H]\setminus\bigcup_{\delta\in I}\pi_{\zeta\delta}^{-1}[Q_{\delta\eta}]$ is μ_ζ-negligible and is included in $A_{\xi\zeta}$ for some $\xi\ge\zeta$. Repeating the arguments of the last two sentences at the new level, we see that

$$X_\xi\cap\bigcup_{\delta\in I}\pi_{\xi\delta}^{-1}[Q_{\delta\eta}]\subseteq\pi_\xi[H]\subseteq\bigcup_{\delta\in I}\pi_{\xi\delta}^{-1}[Q_{\delta\eta}]\cup\pi_{\xi\zeta}^{-1}A_{\xi\zeta}.$$

Now suppose that $V\subseteq\{0,1\}^{\omega_1}$ is an open set meeting H. If there is a z belonging to $V\cap H\cap\pi_\zeta^{-1}[A_{\xi\zeta}]$, then $z\restriction\xi\ne z'\restriction\xi$ for any other $z'\in X$, by (f); so $z\restriction\xi\notin\pi_\xi[H\setminus V]$ and $\pi_\xi[H\setminus V]\ne\pi_\xi[H]$. Otherwise, there is a $\delta\in I$ such that $V\cap\pi_\delta^{-1}[Q_{\delta\eta}]$ is not empty. Because $\pi_\delta\restriction X\cap\pi_\delta^{-1}[Q_{\delta\eta}]$ is irreducible, $\pi_\delta[H\setminus V]$ cannot cover $Q_{\delta\eta}\subseteq\pi_\delta[H]$. Thus $\pi_\delta[H\setminus V]\ne\pi_\delta[H]$; it follows at once that $\pi_\xi[H\setminus V]\ne\pi_\xi[H]$. As V is arbitrary, $\pi_\xi\restriction H$ is irreducible.

I have still to check that $H=X\cap\pi_\xi^{-1}[\pi_\xi[H]]$. If z, $z'\in X$, $z\in H$ and $z'\restriction\xi=z\restriction\xi$, then if $z\in\pi_\zeta^{-1}[A_{\xi\zeta}]$ we have $z'=z\in H$. Otherwise, there is some $\delta\in I$ such that $z\in\pi_\delta^{-1}[Q_{\delta\eta}]$. In this case, $z'\restriction\delta=z\restriction\delta\in Q_{\delta\eta}$; but $X\cap\pi_\delta^{-1}[Q_{\delta\eta}]\subseteq H$, so again $z'\in H$. So we have the result. **Q**

(h) We are within sight of the end. From (g) we see, first, that if $H\subseteq X$ is closed then it is of the form $X\cap\pi_\eta^{-1}[\pi_\eta[H]]$ for some η, so is a zero set in X; accordingly X is perfectly normal, therefore first-countable (5A4Cb). Second, for any closed $H\subseteq X$, there is an irreducible continuous surjection from H onto a compact metrizable space $\pi_\xi[H]$; because $\pi_\xi[H]$ is separable, so is H (5A4C(d-i)). It follows that X is hereditarily separable. **P** If $A\subseteq X$, then $\overline{A}$ is separable; let $D\subseteq\overline{A}$ be a countable dense set. Because X is first-countable, each member of $\overline{A}$ is in the closure of a countable subset of A, and there is a countable $C\subseteq A$ such that $D\subseteq\overline{C}$. Now C is a countable dense subset of A. **Q**

(i) Finally, we need to check that $\omega_1\in\mathrm{Mah_R}(X)$. For $\omega\le\xi<\omega_1$, set $U_\xi=\{z:z\in\{0,1\}^{\omega_1},\ z\restriction\xi\in K_\xi\cap L_\xi,\ z(\xi)=1\}$. Then $\mu(U_\xi\triangle E)\ge\frac14$ whenever $E\subseteq\{0,1\}^{\omega_1}$ is a Baire set determined by coordinates less than ξ. **P** Set $E'=\pi_\xi[E]$, so that $E=\pi_\xi^{-1}[E']$ and E' is a Baire set. Then

$$\begin{aligned}\mu(U_\xi\setminus E)&=\nu_{\omega_1}f^{-1}[U_\xi\setminus E]=\nu_{\omega_1}\{z:f(z)\restriction\xi\in K_\xi\cap L_\xi\setminus E',\ f(z)(\xi)=1\}\\&=\nu_{\omega_1}\{z:f_\xi(z\restriction\xi)\in K_\xi\cap L_\xi\setminus E',\ z(\xi)=1\}\\&=\frac12\nu_{\omega_1}\{z:f_\xi(z\restriction\xi)\in K_\xi\cap L_\xi\setminus E'\}=\frac12\mu_\xi(K_\xi\cap L_\xi\setminus E'),\end{aligned}$$

while

$$\begin{aligned}\mu(E\setminus U_\xi)&=\nu_{\omega_1}f^{-1}[E\setminus U_\xi]\ge\nu_{\omega_1}f^{-1}[E\cap\pi_\xi^{-1}[K_\xi\cap L_\xi]\setminus U_\xi]\\&=\nu_{\omega_1}\{z:f(z)\restriction\xi\in K_\xi\cap L_\xi\cap E',\ f(z)(\xi)=0\}\\&=\nu_{\omega_1}\{z:f_\xi(z\restriction\xi)\in K_\xi\cap L_\xi\cap E',\ z(\xi)=0\}\\&=\frac12\nu_{\omega_1}\{z:f_\xi(z\restriction\xi)\in K_\xi\cap L_\xi\cap E'\}=\frac12\mu_\xi(K_\xi\cap L_\xi\cap E').\end{aligned}$$

Putting these together,

$$\mu(E\triangle U_\xi)\ge\frac12\mu_\xi(K_\xi\cap L_\xi)\ge\frac14.\ \textbf{Q}$$

In particular, $\mu(U_\eta\triangle U_\xi)\ge\frac14$ whenever $\omega\le\eta<\xi<\omega_1$. So μ has uncountable Maharam type. As $\mu X=1$, so has the subspace measure on X, and $\omega_1\in\mathrm{Mah_R}(X)$. Thus X has all the properties announced.

531R Returning to the ideas of 531J, we have the following construction.

Lemma Let I be a set, and let $\mathfrak{B}_I$, $\langle e_i\rangle_{i\in I}$, $\langle\phi_i\rangle_{i\in I}$, $\langle\mathfrak{C}_J\rangle_{J\subseteq I}$ and $J^*:\mathfrak{B}_I\to[I]^{\le\omega}$ be as in 531I-531J. For $a\in\mathfrak{B}_I$ and $J\subseteq I$, set $S_J(a)=\operatorname{upr}(a,\mathfrak{C}_J)=\min\{c:a\subseteq c\in\mathfrak{C}_J\}$, the upper envelope of a in $\mathfrak{C}_J$ (313S).

(a) For all $a\in\mathfrak{B}_I$, $i\in I$ and J, $K\subseteq I$,

(i) $S_I(a)=a$,

(ii) $S_K(a)\subseteq S_J(a)$ if $J\subseteq K$,

(iii) $J^*S_J(a)\subseteq J^*(a)\cap J$,

(iv) $S_{I\setminus\{i\}}(a)=a\cup\phi_i a$,

(v) $S_JS_K(a)=S_{J\cap K}(a)$.

(b) Whenever $a\in\mathfrak{B}_I$, $\epsilon>0$ and $m\in\mathbb{N}$, there is a finite $L\subseteq I$ such that $\bar\nu_I(S_J(a)\setminus a)\le\epsilon$ whenever $L\subseteq J\subseteq I$ and $\#(I\setminus J)\le m$.

proof (a)(i) $\mathfrak{C}_I=\mathfrak{B}_I$ contains a.

(ii) If $J\subseteq K$ then $\mathfrak{C}_K\supseteq\mathfrak{C}_J$ contains $S_J(a)$.

(iii) If $i\in I\setminus(J^*(a)\cap J)$ then $S_J(a)\in\mathfrak{C}_J$ so $\phi_iS_J(a)\in\mathfrak{C}_J$, by 531Jf. Also $\phi_iS_J(a)\supseteq a$. **P** If $i\notin J$ then $\phi_iS_J(a)=S_J(a)\supseteq a$, by 531Je. If $i\notin J^*(a)$ then $\phi_iS_J(a)\supseteq\phi_ia=a$. **Q** So $\phi_iS_J(a)\supseteq S_J(a)$; but they have the same measure, so $\phi_iS_J(a)=S_J(a)$. As i is arbitrary, $J^*S_J(a)\subseteq J^*(a)\cap J$, by 531Je in the other direction.

(iv) By 531Je again, $S_{I\setminus\{i\}}(a)=\phi_iS_{I\setminus\{i\}}(a)\supseteq\phi_ia$; so $S_{I\setminus\{i\}}(a)\supseteq a\cup\phi_ia$. On the other hand, by 531Jd, $a\cup\phi_i(a)$ belongs to $\mathfrak{C}_{I\setminus\{i\}}$, so includes $S_{I\setminus\{i\}}(a)$.

(v) By (iii),

$$J^*S_JS_K(a)\subseteq J^*S_K(a)\cap J\subseteq J^*(a)\cap K\cap J,$$

and $S_JS_K(a)\in\mathfrak{C}_{J\cap K}$; since also $S_JS_K(a)\supseteq S_K(a)\supseteq a$, $S_JS_K(a)\supseteq S_{J\cap K}(a)$. On the other hand, $S_{J\cap K}(a)$ belongs to $\mathfrak{C}_J$ and includes $S_K(a)$, so includes $S_JS_K(a)$.

(b) Induce on m. For $m=0$ the result is immediate from (a-i). For the inductive step to $m+1$, take $L_0\in[I]^{<\omega}$ such that $\bar\nu_I(S_K(a)\setminus a)\le\frac{1}{3}\epsilon$ whenever $L_0\subseteq K$ and $\#(I\setminus K)\le m$. By 531Ja, there are a finite set $L_1\subseteq I$ and a $b\in\mathfrak{C}_{L_1}$ such that $\bar\nu_I(a\,\triangle\,b)\le\frac{1}{3}\epsilon$; set $L=L_0\cup L_1$. Suppose $L\subseteq J$ and $\#(I\setminus J)\le m+1$; take $i\in I\setminus J$ and set $K=J\cup\{i\}$. Then

$$S_J(a)=S_{I\setminus\{i\}}S_K(a)=S_K(a)\cup\phi_iS_K(a)$$

by (a-v) and (a-iv). So

$$\begin{aligned}\bar\nu_I(S_J(a)\setminus a)&\le\bar\nu_I(S_K(a)\setminus a)+\bar\nu_I(\phi_iS_K(a)\setminus a)\\&\le\frac{\epsilon}{3}+\bar\nu_I\phi_i(S_K(a)\setminus a)+\bar\nu_I(\phi_ia\setminus a)\\&\le\frac{\epsilon}{3}+\bar\nu_I(S_K(a)\setminus a)+\bar\nu_I\phi_i(a\setminus b)+\bar\nu_I(\phi_ib\setminus b)+\bar\nu_I(b\setminus a)\\&\le\frac{2\epsilon}{3}+\bar\nu_I(a\setminus b)+\bar\nu_I(b\setminus a)\le\epsilon\end{aligned}$$

because ϕ_i is a measure-preserving Boolean homomorphism and $\phi_ib=b$. Thus the induction continues.

531S Moving on from hypotheses expressible as statements about measure algebras, we have a further result which can be used when Martin's axiom is true.

Lemma Suppose that $\omega_1<\mathfrak{m}_{\mathrm{K}}$ (definition: 517O). Let $\langle e_\xi\rangle_{\xi<\omega_1}$ be the standard generating family in $\mathfrak{B}_{\omega_1}$, and $\langle a_\xi\rangle_{\xi<\omega_1}$ a family of elements of $\mathfrak{B}_{\omega_1}$ of measure greater than $\frac{1}{2}$. Then there is an uncountable set $\Gamma\subseteq\omega_1$ such that $\inf_{\xi\in I}(a_\xi\cap e_\xi)\cap\inf_{\eta\in J}(a_\eta\setminus e_\eta)$ is non-zero whenever I, $J\subseteq\Gamma$ are finite and disjoint.

proof (a) Define $J^*(a)$, for $a\in\mathfrak{B}_{\omega_1}$, and $S_I(a)$, for $a\in\mathfrak{B}_{\omega_1}$ and $I\subseteq\omega_1$, as in 531J and 531R. Let P be the set of pairs (c,I) where $I\subseteq\omega_1$ is finite, $0\ne c\subseteq\inf_{\xi\in I}a_\xi$ and $I\cap J^*(c)=\emptyset$. Order P by saying that $(c,I)\le(c',I')$ if $I\subseteq I'$ and $c'\subseteq c$. Then P is a partially ordered set. For each $\xi<\omega_1$, $a_\xi\cap\phi_\xi a_\xi$ belongs to $\mathfrak{C}_{\kappa\setminus\{\xi\}}$ (531Jd) and is non-zero, so $p_\xi=(a_\xi\cap\phi_\xi a_\xi,\{\xi\})$ belongs to P. The point of the proof is the following fact.

(b) P satisfies Knaster's condition upwards. **P** Let $\langle(c_\xi,I_\xi)\rangle_{\xi<\omega_1}$ be a family in P. Then there are an $\alpha>0$ and an uncountable $A_0\subseteq\omega_1$ such that $\bar\nu_{\omega_1}(c_\xi\cap c_\eta)\ge\alpha$ for all ξ, $\eta\in A_0$ (525Tc). Next, there is an uncountable

$A_1 \subseteq A_0$ such that $\langle I_\xi \rangle_{\xi\in A_1}$ is a Δ-system with root I say (4A1Db); let $m \in \mathbb{N}$ be such that $A_2 = \{\xi : \xi \in A_1, \#(I_\xi \setminus I) = m\}$ is uncountable. Finally, because $J^*(c_\eta)$ is countable for each η, and $\langle I_\xi \setminus I\rangle_{\xi\in A_2}$ is disjoint, we can find an uncountable $A_3 \subseteq A_2$ such that $J^*(c_\eta) \cap I_\xi \setminus I = \emptyset$ whenever η, $\xi \in A_3$ and $\eta < \xi$.

Take a strictly increasing sequence $\langle \eta_k \rangle_{k\in\mathbb{N}}$ in A_3 and a $\zeta \in A_3$ greater than every η_k. By 531Rb, there is a finite set $K \subseteq \omega_1$ such that $\bar\nu_{\omega_1}(S_J(1 \setminus c_\zeta) \setminus (1 \setminus c_\zeta)) < \alpha$ whenever $K \subseteq J \subseteq \omega_1$ and $\#(\omega_1 \setminus J) = m$. Let $k \in \mathbb{N}$ be such that $I_{\eta_k} \setminus I$ does not meet K. Set $c'_\zeta = S_{\kappa\setminus(I_{\eta_k}\setminus I)}(1 \setminus c_\zeta)$. Then

$$\bar\nu_{\omega_1}(c'_\zeta \cap c_\zeta) \le \alpha < \bar\nu_{\omega_1}(c_\zeta \cap c_{\eta_k}),$$

so $c = c_{\eta_k} \setminus c'_\zeta$ is non-zero; as $c'_\zeta \supseteq 1 \setminus c_\zeta$, $c \subseteq c_\zeta$. Set $L = I_{\eta_k} \cup I_\zeta$. Then $J^*(c_{\eta_k})$ is disjoint from I_{η_k} and from $I_\zeta \setminus I$, by the choice of A_3, while

$$J^*(c'_\zeta) \subseteq J^*(1 \setminus c_\zeta) \setminus (I_{\eta_k} \setminus I) = J^*(c_\zeta) \setminus (I_{\eta_k} \setminus I)$$

(531R(a-iii)) is also disjoint from L; so $J^*(c) \subseteq J^*(c_{\eta_k}) \cup J^*(c'_\zeta)$ is disjoint from L. Finally,

$$c \subseteq c_{\eta_k} \cap c_\zeta \subseteq \inf_{\xi\in I_{\eta_k}} a_\xi \cap \inf_{\xi\in I_\zeta} a_\xi = \inf_{\xi\in L} a_\xi,$$

so $(c, L) \in P$. Now (c, L) dominates both (c_{η_k}, I_{η_k}) and (c_ζ, I_ζ).

What this shows is that if we write Q for

$$\{\{\eta, \zeta\} : \eta,\ \zeta \in A_3,\ (c_\eta, I_\eta) \text{ and } (c_\zeta, I_\zeta) \text{ are compatible upwards in } P\},$$

then whenever $\zeta \in A_3$ and $M \subseteq A_3 \cap \zeta$ is infinite there is an $\eta \in M$ such that $\{\eta, \zeta\} \in M$. By 5A1Gb, there is an uncountable $A_4 \subseteq A_3$ such that $[A_4]^2 \subseteq Q$, that is, $\langle (c_\xi, I_\xi)\rangle_{\xi\in A_4}$ is upwards-linked. As $\langle (c_\xi, I_\xi)\rangle_{\xi<\omega_1}$ is arbitrary, P satisfies Knaster's condition upwards. **Q**

(c) By 517S, there is a sequence $\langle R_n \rangle_{n\in\mathbb{N}}$ of upwards-directed subsets of P covering $\{p_\xi : \xi < \omega_1\}$; as ω_1 is uncountable, there must be some n such that $\Gamma = \{\xi : p_\xi \in R_n\}$ has cardinal ω_1. In this case, $\{p_\xi : \xi \in \Gamma\}$ is upwards-centered in P. If I, $J \subseteq \Gamma$ are finite and disjoint, then there must be a $(c, K) \in P$ which is an upper bound for $\{p_\xi : \xi \in I \cup J\}$; now $I \cup J \subseteq K$ does not meet $J^*(c)$, while $c \subseteq \inf_{\xi\in I\cup J} a_\xi$. But this means that

$$\bar\nu_{\omega_1}(c \cap \inf_{\xi\in I} e_\xi \cap \inf_{\eta\in J}(1 \setminus e_\eta)) = 2^{-\#(I\cup J)}\bar\nu_{\omega_1} c > 0,$$

and

$$0 \ne c \cap \inf_{\xi\in I} e_\xi \cap \inf_{\eta\in J}(1 \setminus e_\eta) \subseteq \inf_{\xi\in I}(a_\xi \cap e_\xi) \cap \inf_{\eta\in J}(a_\eta \setminus e_\eta).$$

So we have a set Γ of the kind required.

531T Theorem (Fremlin 97) Suppose that $\omega \le \kappa < \mathfrak{m}_{\mathrm{K}}$. If X is a normal Hausdorff space and $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then $[0, 1]^\kappa$ is a continuous image of X.

proof For $\kappa = 0$ this is trivial. If $\kappa \ge \omega_2$, then κ is a measure-precaliber of all probability algebras (525Fb), so 531Lb gives the result.

If $\kappa = \omega_1$, let μ be a Maharam-type-homogeneous Radon probability measure on X with Maharam type ω_1, and $(\mathfrak{A}, \bar\mu)$ its measure algebra. Let $\langle d_\xi \rangle_{\xi<\omega_1}$ be a τ-generating stochastically independent family of elements of $\mathfrak{A}$ of measure $\frac{1}{2}$. For $\xi < \omega_1$ let $E_\xi \in \operatorname{dom}\mu$ be such that $E^\bullet_\xi = d_\xi$, and $K'_\xi \subseteq E_\xi$, $K''_\xi \subseteq X \setminus E_\xi$ compact sets of measure greater than $\frac{1}{4}$; set $K_\xi = K'_\xi \cup K''_\xi$ and $a_\xi = K^\bullet_\xi$ in $\mathfrak{A}$. Because $(\mathfrak{A}, \bar\mu, \langle d_\xi \rangle_{\xi<\omega_1})$ is isomorphic to $(\mathfrak{B}_{\omega_1}, \bar\nu_{\omega_1}, \langle e_\xi \rangle_{\xi<\omega_1})$, 531S tells us that there is an uncountable set $\Gamma \subseteq \omega_1$ such that

$$0 \ne \inf_{\xi\in I}(a_\xi \cap d_\xi) \cap \inf_{\eta\in J}(a_\eta \setminus d_\eta) = (X \cap \bigcap_{\xi\in I} K'_\xi \cap \bigcap_{\eta\in J} K''_\eta)^\bullet$$

whenever I, $J \subseteq \Gamma$ are finite. Just as in part (b) of the proof of 531L, it follows that there is a continuous surjection from $\bigcap_{\xi\in\Gamma} K_\xi$ onto $\{0, 1\}^\Gamma \cong \{0, 1\}^{\omega_1}$, and therefore a continuous surjection from X onto $[0, 1]^{\omega_1}$.

531X Basic exercises (a) Show that there is a Hausdorff completely regular quasi-Radon probability space $(X, \mathfrak{T}, \Sigma, \mu)$ with Maharam type greater than $\#(X)$. (*Hint*: 523Ib.)

(b) Give an example of a separable Radon measure space with magnitude $2^{\mathfrak{c}}$.

(c) Let $I^{\parallel}$ be the split interval (343J, 419L). Show that $\mathrm{Mah}_{\mathrm{R}}(I) = \{0, \omega\}$.

(d) Let I be an infinite set, and βI the Stone-Čech compactification of the discrete space I. Show that $2^{\#(I)}$ is the greatest member of $\mathrm{Mah}_{\mathrm{R}}(\beta I)$. (*Hint*: 5A4Ia or 515H.)

(e) For a topological space X, write $\mathrm{Mah}_{\mathrm{qR}}(X)$ for the set of Maharam types of Maharam-type-homogeneous quasi-Radon probability measures on X. (i) Show that $\kappa \le w(X)$ for every $\kappa \in \mathrm{Mah}_{\mathrm{qR}}(X)$. (ii) Show that $\mathrm{Mah}_{\mathrm{qR}}(Y) \subseteq \mathrm{Mah}_{\mathrm{qR}}(X)$ for every $Y \subseteq X$. (iii) Show that if Y is another topological space, and neither X nor Y is empty, then $\mathrm{Mah}_{\mathrm{qR}}(X \times Y) = \mathrm{Mah}_{\mathrm{qR}}(X) \cup \mathrm{Mah}_{\mathrm{qR}}(Y)$.

>(f) Let X be a Hausdorff topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra (442H, 443A). Show that $w(X) = \max(c(\mathfrak{A}), \tau(\mathfrak{A}))$. (*Hint*: 443Gf, 529Ba.) Show that if X is σ-compact, locally compact, Hausdorff and not discrete then $w(X) \in \mathrm{Mah}_{\mathrm{R}}(X)$.

(g) Let X be a normal Hausdorff space and κ an infinite cardinal. Suppose that whenever Y is a Hausdorff continuous image of X of weight κ then $\mathrm{Mah}_{\mathrm{R}}(Y) \subseteq \kappa$. Show that $\mathrm{Mah}_{\mathrm{R}}(X) \subseteq \kappa$.

(h) Let X be a Hausdorff space, and $\langle E_i \rangle_{i \in I}$ a family of universally Radon-measurable subsets of X such that $\#(I) < \operatorname{cov} \mathcal{N}_\kappa$ for every κ. Show that $\mathrm{Mah}_{\mathrm{R}}(\bigcup_{i \in I} E_i) = \bigcup_{i \in I} \mathrm{Mah}_{\mathrm{R}}(E_i)$.

(i) Let K be an Eberlein compactum. Show that $\mathrm{Mah}_{\mathrm{R}}(K) \subseteq \{0, \omega\}$. (*Hint*: 467Xj.)

(j) Let $\mathcal{W} \subseteq \mathcal{N}_\omega$ be such that every compact subset of $\{0,1\}^\omega \setminus \bigcup \mathcal{W}$ is scattered. Show that there is a family $\mathcal{W}' \subseteq \mathcal{N}_{\omega_1}$ such that $\#(\mathcal{W}') = \#(\mathcal{W})$ and every compact subset of $\{0,1\}^{\omega_1} \setminus \bigcup \mathcal{W}'$ is scattered.

(k) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Hausdorff quasi-Radon probability space. Show that the Maharam type of μ is at most $\max(\omega, 2^{\chi(X)})$. (*Hint*: 5A4Ba, 5A4Bg.)

(l) In the language of 531R, show that if $a, b \in \mathfrak{B}_\kappa$ and $I \subseteq \kappa \setminus J^*(b)$ is finite, then $\bar{\nu}_\kappa(a \triangle S_{\kappa \setminus I}(a)) \le 2^{\#(I)} \bar{\nu}_\kappa(a \triangle b)$.

(m) Show that if $\mathfrak{m}_{\mathrm{K}} > \omega_1$ and X is a countably tight compact Hausdorff space, then $\omega_1 \notin \mathrm{Mah}_{\mathrm{R}}(X)$.

(n) Let X be an infinite compact Hausdorff space with a strictly positive Radon measure μ, and P the set of Radon probability measures on X with its narrow topology. Show that the topological density of P is at most the Maharam type of μ. (*Hint*: the indefinite-integral measures over μ are dense in P.)

531Y Further exercises (a) Let κ be an infinite cardinal such that $\kappa = \kappa^{\mathfrak{c}}$. Show that there is a set $X \subseteq \{0,1\}^\kappa$, of full outer measure for ν_κ, such that every subset of X with cardinal $\mathfrak{c}$ is discrete. Show that $\mathrm{Mah}_{\mathrm{qR}}(X)$ (531Xe) contains κ but not ω.

(b) Let X and Y be infinite compact Hausdorff spaces, and suppose that there is a norm-preserving linear isomorphism from the dual space $C(X)^*$ to $C(Y)^*$. Show that $\mathrm{Mah}_{\mathrm{R}}(X) = \mathrm{Mah}_{\mathrm{R}}(Y)$.

(c) Let μ be a τ-additive Borel probability measure on a topological space X, and κ a cardinal of uncountable cofinality such that (i) $\chi(x, X) < \operatorname{cf} \kappa$ for every $x \in X$ (ii) no non-negligible measurable set can be covered by $\operatorname{cf} \kappa$ negligible sets. Show that the Maharam type of μ cannot be κ.

531Z Problems (a) Can there be a perfectly normal compact Hausdorff space X such that $\omega_2 \in \mathrm{Mah}_{\mathrm{R}}(X)$? (See 531Q, 554Xd.)

(b) Can there be a hereditarily separable compact Hausdorff space X such that $\omega_2 \in \mathrm{Mah}_{\mathrm{R}}(X)$?

531 Notes and comments This section is directed to Radon measures, studying $\mathrm{Mah}_{\mathrm{R}}(X)$; of course we can look at Maharam types of quasi-Radon measures (531Xe, 531Ya), or Borel or Baire measures for that matter. In the next section I shall have something to say about completion regular measures. The function $X \mapsto \mathrm{Mah}_{\mathrm{R}}(X)$ has a much more satisfying list of basic properties (531E, 531G) than the others.

From 531L and 531T we see that there are many cardinals κ such that whenever X is a compact Hausdorff space and $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then there is a continuous function from X onto $[0,1]^\kappa$. Such cardinals are said to have **Haydon's property**. From 531L, 531M and 531T we see that

ω has Haydon's property (531La);

if $\kappa \ge \omega_2$ and κ is a measure-precaliber of $\mathfrak{B}_\kappa$ then κ has Haydon's property (531Lb);

if $\kappa \ge \omega$ is not a measure-precaliber of $\mathfrak{B}_\kappa$ then κ does not have Haydon's property (531M);

if $\omega_1 < \mathfrak{m}_K$ then ω_1 has Haydon's property (531T).

Thus if $\mathfrak{m}_K > \omega_1$, an infinite cardinal κ has Haydon's property iff it is a measure-precaliber of every probability algebra. ω_1 really is different; it is possible that ω_1 is a precaliber of every probability algebra but does not have Haydon's property. To check this, it is enough to find a model of set theory in which $\operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1$ but there is a family $\langle W_\xi\rangle_{\xi<\omega_1}$ as in 531N; one is described in 553F.

You will observe that the key arguments of this section all depend on analysis of the measure algebras $\mathfrak{B}_\kappa$. We have already seen in §524 that many properties of a Radon measure can be determined from its measure algebra. Here we find that some important topological properties of compact Hausdorff spaces can be determined by the measure algebras of the Radon measures they carry. The results here largely depend for their applications on knowing enough about precalibers; I remind you that it seems to be still unknown whether it is possible that every infinite cardinal is a measure-precaliber of every probability algebra.

The remarks above have concerned the existence of continuous surjections onto $[0,1]^\kappa$; a natural place to start, because measures of Maharam type κ arise immediately from such surjections. In 531O-531Q I look at different measures of the richness of a compact space X. Concerning characters, 531O-531P give us quite a lot of information, slightly irregular at the edges. I ought to offer a remark on the context of 531Q. In some set theories (for instance, when $\mathfrak{m} > \omega_1$), we find not only that ω_1 is a precaliber of every measurable algebra, but also that a compact Hausdorff space is hereditarily separable iff it is hereditarily Lindelöf (Fremlin 84a, 44H); so that, for instance, a hereditarily separable compact Hausdorff space cannot carry a Radon measure of uncountable Maharam type. Typically, the situation is very different if the continuum hypothesis or Jensen's $\diamondsuit$ is true, and 531Q is a descendant of the construction in Kunen 81 of a non-separable hereditarily Lindelöf compact Hausdorff space. See Džamonja & Kunen 93 for further exploration of these questions.

532 Completion regular measures on $\{0,1\}^I$

As I remarked in the introduction to §434, the trouble with topological measure theory is that there are too many questions to ask. In §531 I looked at the problem of determining the possible Maharam types of Radon measures on a Hausdorff space X. But we can ask the same question for any of the other classes of topological measures listed in §411. It turns out that the very narrowly focused topic of completion regular Radon measures on powers of $\{0,1\}$ already leads us to some interesting arguments.

I define the classes $\mathrm{Mah}_{\mathrm{crR}}(X)$, corresponding to the $\mathrm{Mah}_{\mathrm{R}}(X)$ examined in §531, in 532A. They are less accessible, and I almost immediately specialize to the relation $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. This at least is more or less convex (532G, 532K), and can be characterized in terms of the measure algebras $\mathfrak{B}_\lambda$ (532I). On the way it is helpful to extend the treatment of completion regular measures given in §434 (532D, 532E, 532H). For fixed infinite λ, there is a critical cardinal $\kappa_0 \le (2^\lambda)^+$ such that $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ iff $\lambda \le \kappa < \kappa_0$; under certain conditions, when $\lambda = \omega$, we can locate κ_0 in terms of the cardinals of Cichoń's diagram (532P, 532Q). This depends on facts about the Lebesgue measure algebra (532M, 532O) which are of independent interest. Finally, for other λ of countable cofinality, the square principle and Chang's transfer principle are relevant (532R-532S).

532A Definition If X is a topological space, I write $\mathrm{Mah}_{\mathrm{cr}}(X)$ for the set of Maharam types of Maharam-type-homogeneous completion regular topological probability measures on X. If X is a Hausdorff space, I write $\mathrm{Mah}_{\mathrm{crR}}(X)$ for the set of Maharam types of Maharam-type-homogeneous completion regular Radon probability measures on X.

532B Proposition Let X be a Hausdorff space. Then a probability algebra $(\mathfrak{A},\bar\mu)$ is isomorphic to the measure algebra of a completion regular Radon probability measure on X iff (α) $\tau(\mathfrak{A}_a) \in \mathrm{Mah}_{\mathrm{crR}}(X)$ whenever $\mathfrak{A}_a$ is a non-zero homogeneous principal ideal of $\mathfrak{A}$ (β) the number of atoms of $\mathfrak{A}$ is not greater than the number of points $x \in X$ such that $\{x\}$ is a zero set.

proof (a) Suppose that μ is a completion regular Radon probability measure on X and $\mathfrak{A}_a$ is a non-zero homogeneous principal ideal of its measure algebra $\mathfrak{A}$. Let F be such that $F^\bullet = a$ and ν the indefinite-integral measure over μ defined by the function $\frac{1}{\mu F}\chi F$. Then ν is a Radon measure (416S), inner regular with respect to the zero sets (412Q); and its measure algebra is isomorphic, up to a scalar multiple, to $\mathfrak{A}_a$, so is homogeneous with Maharam type $\tau(\mathfrak{A}_a)$. So ν witnesses that $\tau(\mathfrak{A}_a) \in \mathrm{Mah}_{\mathrm{crR}}(X)$. This shows that $\mathfrak{A}$ satisfies condition (α).

As for condition (β), each atom of $\mathfrak{A}$ is of the form $\{x\}^\bullet$ for some $x \in X$ such that $\mu\{x\} > 0$ (414G, or otherwise). In this case, because μ is completion regular, $\{x\}$ must be a zero set. So we have at least as many singleton zero sets as we have atoms in $\mathfrak{A}$.

(b) Now suppose that $(\mathfrak{A},\bar\mu)$ satisfies the conditions. I copy the argment of 531F. Express $(\mathfrak{A},\bar\mu)$ as the simple product of a countable family $\langle(\mathfrak{A}_i,\bar\mu_i')\rangle_{i\in I}$ of non-zero homogeneous measure algebras. For $i\in I$, set $\kappa_i=\tau(\mathfrak{A}_i)$ and $\gamma_i=\bar\mu_i'1_{\mathfrak{A}_i}$. Set $J=\{i:i\in I,\,\kappa_i\ge\omega\}$. (β) tells us that $\#(I\setminus J)$ is less than or equal to the number of singleton zero sets in X; let $\langle x_i\rangle_{i\in I\setminus J}$ be a family of distinct elements of X such that every $\{x_i\}$ is a zero set.

For each $i\in J$, (α) tells us that there is a completion regular Maharam-type-homogeneous Radon probability measure μ_i on X with Maharam type κ_i. Now there is a disjoint family $\langle E_i\rangle_{i\in J}$ of Baire subsets of X such that $\mu_iE_i>0$ for every $i\in J$. **P** We may suppose that $J\subseteq\mathbb{N}$. Choose $\langle E_i\rangle_{i\in\mathbb{N}}$, $\langle F_i\rangle_{i\in\mathbb{N}}$ inductively, as follows. $F_0=X\setminus\{x_i:i\in I\setminus J\}$. Given that F_i is a Baire set and $\mu_jF_i>0$ for every $j\in J\setminus i$, then if $i\notin J$ set $E_i=\emptyset$ and $F_{i+1}=F_i$; otherwise, because μ_i is atomless and completion regular, we can find, for each $j\in J$ such that $j>i$, a Baire set $G_{ij}\subseteq F_i$ such that $\mu_iG_{ij}<2^{-j}\mu_iF_i$ and $\mu_jG_{ij}>0$; set $F_{i+1}=\bigcup_{j\in J,j>i}G_{ij}$ and $E_i=F_i\setminus F_{i+1}$; continue. **Q** Now set

$$\mu E=\sum_{i\in I\setminus J,x_i\in E}\gamma_i+\sum_{i\in J}(\mu_iE_i)^{-1}\gamma_i\mu_i(E\cap E_i)$$

whenever $E\subseteq X$ is such that μ_i measures $E\cap E_i$ for every $i\in J$. Of course μ is a probability measure. Because every μ_i is a topological measure, so is μ; because every μ_i is inner regular with respect to the compact sets, so is μ; because every μ_i is complete, so is μ; so μ is a Radon measure. Because every subspace measure $(\mu_i)_{E_i}$ is Maharam-type-homogeneous with Maharam type κ_i, the measure algebra of μ is isomorphic to $(\mathfrak{A},\bar\mu)$. Because all the $\{x_i\}$ are zero sets and all the μ_i are completion regular, μ is completion regular.

532C Remarks Nearly the whole of this section will be devoted to the usual measures on powers of $\{0,1\}$. Accordingly the following notation will be useful, as previously in this volume. If I is any set, ν_I will be the usual measure on $\{0,1\}^I$, $\mathfrak{B}_I$ its measure algebra and $\mathcal{N}_I$ its null ideal. In this context, $\langle e_i\rangle_{i\in I}$ will be the standard generating family in $\mathfrak{B}_I$ (525A), and for $J\subseteq I$, $\mathfrak{C}_J$ will be the closed subalgebra of $\mathfrak{B}_I$ generated by $\{e_i:i\in J\}$.

If X is a topological space, $\mathcal{B}(X)$ will be its Borel σ-algebra.

Let κ be an infinite cardinal. Then ν_κ is a completion regular Radon probability measure (416U), and $\mathfrak{B}_\kappa$ is homogeneous with Maharam type κ. So $\kappa\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. Next, any Radon measure on $\{0,1\}^\kappa$ can have Maharam type at most $w(\{0,1\}^\kappa)$ (531Aa), so $\lambda\le\kappa$ for every $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. At the bottom end, $0\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ iff $\{0,1\}^\kappa$ has a singleton G_δ set, that is, iff $\kappa=\omega$.

From this we see already that we do not have direct equivalents of any of the results 531Eb-531Ef. However the class $\{(\lambda,\kappa):\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)\}$ is convex in two senses (532G, 532K). For the first of these, it will be useful to have a result left over from §434.

532D Theorem (Fremlin & Grekas 95) Let (X,μ_1) and (Y,μ_2) be effectively locally finite topological measure spaces of which X is quasi-dyadic (definition: 434O), μ_1 is completion regular and μ_2 is τ-additive. Let μ be the c.l.d. product measure on $X\times Y$ as defined in §251. Then μ is a τ-additive topological measure.

proof (a) To begin with (down to the end of (e)) let us suppose that μ_1 and μ_2 are complete and totally finite and inner regular with respect to the Borel sets. Let $\langle X_i\rangle_{i\in I}$ be a family of separable metrizable spaces such that there is a continuous surjection $f:\prod_{i\in I}X_i\to X$. For each $i\in I$, let $\mathcal{U}_i$ be a countable base for the topology of X_i not containing $\emptyset$; for $J\subseteq I$, let $\mathcal{C}_J$ be the family of cylinder sets expressible in the form $\{z:z\in\prod_{i\in I}X_i,\,z(i)\in U_i$ for every $i\in K\}$ where $K\subseteq J$ is finite and $U_i\in\mathcal{U}_i$ for each $i\in K$.

(b) ? Suppose, if possible, that μ is not a topological measure. Let $W\subseteq X\times Y$ be a closed set which is not measured by μ. By 434Q, μ_1 is τ-additive; by 417C, there is a τ-additive topological measure $\tilde\mu$ extending μ, and $\mu^*W=\tilde\mu W$ (apply 417C(iv) to the complement of W).

(c) If $J\subseteq I$ is countable, there are sets H, V, V' such that $H\subseteq Y$ is open, $V\in\mathcal{C}_J$, $V'\in\mathcal{C}_{I\setminus J}$, $f[V\cap V']\times H$ is disjoint from W, and $\mu^*(W\cap(f[V]\times H))>0$. **P** For $V\in\mathcal{C}_J$, set

$$\mathcal{H}_V=\bigcup_{V'\in\mathcal{C}_{I\setminus J}}\{H:H\subseteq Y\text{ is open, }W\cap(f[V\cap V']\times H)=\emptyset\},$$

$$H_V=\bigcup\mathcal{H}_V,$$

and choose a measurable envelope F_V of $f[V]$. As $\mathcal{C}_J$ is countable,

$$W_1=(X\times Y)\setminus\bigcup_{V\in\mathcal{C}_J}F_V\times H_V$$

is measured by μ; also $W_1\subseteq W$ because

$$\{f[V\cap V']\times H:V\in\mathcal{C}_J,\,V'\in\mathcal{C}_{I\setminus J},\,H\subseteq Y\text{ is open}\}$$

is a network for the topology of $X \times Y$. So

$$\tilde{\mu}W_1 = \mu W_1 \leq \mu_* W < \mu^* W = \tilde{\mu}W$$

and $\tilde{\mu}(W \setminus W_1) > 0$. There must therefore be a $V \in \mathcal{C}_J$ such that $\tilde{\mu}(W \cap (F_V \times H_V)) > 0$. Next, because μ_2 is τ-additive, there is a countable $\mathcal{H} \subseteq \mathcal{H}_V$ such that $\mu_2(H_V \setminus \bigcup \mathcal{H}) = 0$, and now $\tilde{\mu}(W \cap (F_V \times \bigcup \mathcal{H})) = \tilde{\mu}(W \cap (F_V \times H_V))$ is non-zero. Accordingly there is an $H \in \mathcal{H}$ such that $\tilde{\mu}(W \cap (F_V \times H)) > 0$. By 417H,

$$\int_{F_V} \mu_2(W[\{x\}] \cap H)\mu_1(dx) = \tilde{\mu}(W \cap (F_V \times H))$$

is greater than 0. But this means that $\mu_1\{x : x \in F_V,\, \mu_2(W[\{x\}] \cap H) > 0\} > 0$. (Recall that we are supposing that μ_1 is complete.) So $\{x : x \in f[V],\, \mu_2(W[\{x\}] \cap H) > 0\}$ is not μ_1-negligible, and $W \cap (f[V] \times H)$ is not μ-negligible. Finally, because $H \in \mathcal{H}_V$, there is a $V' \in \mathcal{C}_{I \setminus J}$ such that $W \cap (f[V \cap V'] \times H) = \emptyset$. **Q**

(d) We may therefore choose inductively families $\langle J_\xi \rangle_{\xi<\omega_1}$, $\langle H_\xi \rangle_{\xi<\omega_1}$, $\langle V_\xi \rangle_{\xi<\omega_1}$, $\langle V'_\xi \rangle_{\xi<\omega_1}$ in such a way that, for every $\xi < \omega_1$,

J_ξ is a countable subset of I,
H_ξ is an open subset of Y,
$V_\xi \in \mathcal{C}_{J_\xi}$, $V'_\xi \in \mathcal{C}_{I \setminus J_\xi}$,
$W \cap (f[V_\xi \cap V'_\xi] \times H_\xi) = \emptyset$,
$\mu^*(W \cap (f[V_\xi] \times H_\xi)) > 0$,
$\bigcup_{\eta<\xi} J_\eta \subseteq J_\xi$,
V_ξ, $V'_\xi \in \mathcal{C}_{J_{\xi+1}}$.

For each $\xi < \omega_1$, let K_ξ be a finite subset of $J_{\xi+1}$ such that V_ξ and V'_ξ are determined by coordinates in K_ξ. By the Δ-system Lemma (4A1Db), there is an uncountable set $A \subseteq \omega_1$ such that $\langle K_\xi \rangle_{\xi\in A}$ is a Δ-system with root K say. Set $\zeta_0 = \min A$. Express each V_ξ as $\tilde{V}_\xi \cap \hat{V}_\xi$ where $\tilde{V}_\xi \in \mathcal{C}_K$ and $\hat{V}_\xi \in \mathcal{C}_{K_\xi \setminus K}$; because $\mathcal{C}_K$ is countable, there is a $\tilde{V}$ such that $B = \{\xi : \xi \in A,\, \xi > \zeta_0,\, \tilde{V}_\xi = \tilde{V}\}$ is uncountable. Note that $\mu_1^* f[\tilde{V}] > 0$, because $\mu_1^* f[\tilde{V}] \geq \mu^*(W \cap (f[V_\xi] \times H_\xi))$ for any $\xi \in B$. Also

$$K \subseteq K_{\zeta_0} \subseteq J_{\zeta_0+1} \subseteq J_\xi,$$

so V'_ξ is determined by coordinates in $K_\xi \setminus J_\xi \subseteq K_\xi \setminus K$, for every $\xi \in B$.

(e) Set $H'_\xi = \bigcup_{\eta\in B\setminus\xi} H_\eta$ for each $\xi < \omega_1$. Then $\langle H'_\xi \rangle_{\xi<\omega_1}$ is non-increasing, so there is a $\zeta < \omega_1$ such that $\mu_2 H'_\xi = \mu_2 H'_\zeta$ whenever $\xi \geq \zeta$. Now consider $F = \{x : \mu_2(W[\{x\}] \cap H'_\zeta) > 0\}$. Applying 417H to the indicator function of $W \cap (X \times H'_\zeta)$, and recalling once more that μ_1 is complete, we see that μ_1 measures F. Also $\mu_1^*(F \cap f[\tilde{V}]) > 0$. **P** Take any $\xi \in B \setminus \zeta$. Then

$$F \cap f[\tilde{V}] \supseteq \{x : x \in f[\tilde{V}_\xi],\, \mu_2(W[\{x\}] \cap H_\xi) > 0\}$$

must be non-μ_1-negligible because $W \cap (f[\tilde{V}_\xi] \times H_\xi)$ is not μ-negligible. **Q**

At this point, recall that we are supposing that μ_1 is completion regular. So there is a zero set $Z \subseteq F$ such that $\mu_1 Z > \mu_1 F - \mu_1^*(F \cap f[\tilde{V}])$, and $Z \cap f[\tilde{V}] \neq \emptyset$, that is, $\tilde{V} \cap f^{-1}[Z]$ is not empty. $f^{-1}[Z]$ is a zero set (4A2C(b-iv)), so there is a countable set $J \subseteq I$ such that $f^{-1}[Z]$ is determined by coordinates in J (4A3Nc); we may suppose that $K \subseteq J$. Because $\langle K_\eta \setminus K \rangle_{\eta\in A}$ is disjoint, there is a $\xi \geq \zeta$ such that $J \cap K_\eta = K$ for every $\eta \in A \setminus \xi$.

Take any $w \in \tilde{V} \cap f^{-1}[Z]$ and modify it to produce $w' \in \prod_{i\in I} X_i$ such that $w' \restriction J = w \restriction J$ and $w' \in \hat{V}_\eta \cap V'_\eta$ for every $\eta \in B \setminus \xi$; this is possible because $\hat{V}_\eta \cap V'_\eta$ is determined by coordinates in $K_\eta \setminus K$ for each η, and J and the $K_\eta \setminus K$ are disjoint. Set $x = f(w')$; then $x \in Z \subseteq F$, so $\mu_2(W[\{x\}] \cap H'_\zeta) > 0$.

$w' \in \tilde{V}$, because $w \in \tilde{V}$ and $\tilde{V}$ is determined by coordinates in $K \subseteq J$; so $w' \in \tilde{V} \cap \hat{V}_\eta \cap V'_\eta = V_\eta \cap V'_\eta$ for every $\eta \in B \setminus \xi$. Accordingly $x \in f[V_\eta \cap V'_\eta]$; as $W \cap (f[V_\eta \cap V'_\eta] \times H_\eta) = \emptyset$, $W[\{x\}]$ does not meet H_η. As η is arbitrary, $W[\{x\}]$ does not meet H'_ξ and $W[\{x\}] \cap H'_\zeta$ is μ_2-negligible. But this is impossible. **X**

(f) This contradiction shows that μ will be a topological measure, at least if μ_1 and μ_2 are complete, totally finite and inner regular with respect to the Borel sets. Now suppose just that μ_1 and μ_2 are totally finite. Let μ'_1 and μ'_2 be the completions of the Borel measures $\mu_1 \restriction \mathcal{B}(X)$ and $\mu_2 \restriction \mathcal{B}(Y)$, and μ' their c.l.d. product. Then $\mu_1 \restriction \mathcal{B}(X)$ and μ'_1 are completion regular topological measures, while $\mu_2 \restriction \mathcal{B}(Y)$ and μ'_2 are τ-additive. So (a)-(e) tell us that μ' measures every open set. Now the completions $\hat{\mu}_1$, $\hat{\mu}_2$ extend μ'_1 and μ'_2, and μ is the c.l.d. product of $\hat{\mu}_1$ and $\hat{\mu}_2$ (251T), so μ extends μ' (251L). Thus we again have a topological product measure μ.

(g) In the general case, let $W \subseteq X \times Y$ be an open set, $E \subseteq X$ a zero set of finite measure, and $F \subseteq Y$ any set of finite measure. Then μ measures $W \cap (E \times F)$. **P** Let $(\mu_1)_E$ and $(\mu_2)_F$ be the subspace measures. Then both are totally finite topological measures, $(\mu_1)_E$ is inner regular with respect to the zero sets (412Pd), E is quasi-dyadic (434Pc), and $(\mu_2)_F$ is τ-additive (414K). So the product $(\mu_1)_E \times (\mu_2)_F$ is a topological measure and measures $W \cap (E \times F)$. By 251Q, μ measures $W \cap (E \times F)$. **Q**

Let $\mathcal{K}$ be the family of zero sets of finite measure in X, $\mathcal{L}$ the family of Borel sets of finite measure in Y, and $\mathcal{M}$ the family of sets $M \subseteq X \times Y$ such that μ measures $W \cap M$. Because μ_1 is inner regular with respect to $\mathcal{K}$, μ_2 is inner regular with respect to $\mathcal{L}$, $E \times F \in \mathcal{M}$ for every $E \in \mathcal{K}$ and $F \in \mathcal{L}$, and $\mathcal{M}$ is a σ-algebra of sets, 412R tells us that μ is inner regular with respect to $\mathcal{M}$. As μ is complete and locally determined, it must measure W (412Ja). As W is arbitrary, μ is a topological measure.

(h) Finally, as noted in (b), μ_1 is τ-additive and there is a τ-additive topological measure $\tilde{\mu}$ on $X \times Y$ extending μ. (434Q and 417C still apply.) So μ too must be τ-additive.

532E Corollary Let $\langle X_i \rangle_{i \in I}$ be a family of regular spaces with countable networks, and Y any topological space. Suppose that we are given a strictly positive topological probability measure μ_i on each X_i, and a τ-additive topological probability measure ν on Y. Let μ be the ordinary product measure on $Z = \prod_{i \in I} X_i \times Y$.

(a) μ is a topological measure.

(b) μ is τ-additive.

(c) If ν is completion regular, and every μ_i is inner regular with respect to the Borel sets, then μ is completion regular.

proof (a) For each i, X_i is hereditarily Lindelöf (4A2Nb), so μ_i is τ-additive (414O); let μ_i' be the completion of the Borel measure $\mu_i \restriction \mathcal{B}(X_i)$. Then μ_i' is a quasi-Radon measure (415C). By 4A2Nb, X_i is perfectly normal, so μ_i' is completion regular. By 434Pb-434Pc, $\prod_{i \in I} X_i$ is quasi-dyadic. The product ν_1 of the μ_i' is a topological measure (453I) and inner regular with respect to the zero sets (412Ub); so the product μ' of ν_1 and ν is a topological measure, by 532D. Now μ' is also the product of the measures $\mu_i \restriction \mathcal{B}(X_i)$ and ν (254I, 254N), so μ extends μ' (254H) and μ also is a topological measure.

(b) Because every μ_i is τ-additive, as is ν, 417E tells us that there is a τ-additive measure extending μ, so μ itself must be τ-additive.

(c) For any $i \in I$, we know from (a) that μ_i' is inner regular with respect to the zero sets. Now every non-μ_i-negligible set includes a non-μ_i-negligible Borel set, which includes a non-μ_i-negligible zero set; accordingly μ_i is completion regular. By 412Ub again, μ is inner regular with respect to the zero sets, so is completion regular.

532F Corollary Let $\langle (X_i, \mu_i) \rangle_{i \in I}$ be a family of quasi-dyadic compact Hausdorff spaces with strictly positive completion regular Radon measures. Then the ordinary product measure μ on $\prod_{i \in I} X_i$ is a completion regular Radon measure.

proof By 532D, the ordinary product measure on $\prod_{i \in J} X_i$ is a topological measure, for every finite $J \subseteq I$. By 417Sc, μ is the τ-additive product measure on $\prod_{i \in I} X_i$, which by 417Q is a Radon measure. By 412Ub once more, μ is completion regular.

532G Proposition Suppose that λ, λ' and κ are cardinals such that $\max(\omega, \lambda) \le \lambda' \le \kappa$ and $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. Then $\lambda' \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

proof Let ν be a completion regular Maharam-type-homogeneous Radon probability measure on $\{0,1\}^\kappa$ with Maharam type λ, and consider the ordinary product measure μ of $\nu_{\lambda'}$ and ν on $X = \{0,1\}^{\lambda'} \times \{0,1\}^\kappa$. Applying 532E with $Y = \{0,1\}^\kappa$ and $X_\xi = \{0,1\}$ for $\xi < \lambda'$, we see that μ is a completion regular topological probability measure on a compact Hausdorff space, therefore (being complete) a Radon measure. By 334A, the Maharam type of μ is at most $\max(\omega, \lambda', \lambda) = \lambda'$, so the measure algebra $(\mathfrak{A}, \bar{\mu})$ of μ can be embedded in $\mathfrak{B}_{\lambda'}$. At the same time, the inverse-measure-preserving projection from X onto $\{0,1\}^{\lambda'}$ induces a measure-preserving embedding of $\mathfrak{B}_{\lambda'}$ into $\mathfrak{A}$. By 332Q, $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}_{\lambda'}, \bar{\nu}_{\lambda'})$ are isomorphic, that is, μ is Maharam-type-homogeneous with Maharam type λ'. So μ witnesses that $\lambda' \in \mathrm{Mah}_{\mathrm{crR}}(X) = \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

532H Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of separable metrizable spaces, and μ a totally finite completion regular topological measure on $X=\prod_{i\in I}X_i$. Then

(a) the support of μ is a zero set;

(b) μ is inner regular with respect to the self-supporting zero sets.

proof (a) Recall from 434Q that μ is τ-additive, so has a support Z. Let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a sequence of zero sets such that $K_n\subseteq Z$ and $\mu K_n\ge\mu X-2^{-n}$ for each n. Then there is a countable set $J\subseteq I$ such that every K_n is determined by coordinates in J (4A3Nc again). So $\bigcup_{n\in\mathbb{N}}K_n$ and $Z'=\overline{\bigcup_{n\in\mathbb{N}}K_n}$ are determined by coordinates in J (4A2B(g-i)), and Z' is a zero set, by 4A3Nc in the other direction. But $Z'\subseteq Z$ and $\mu Z'=\mu Z$ so $Z=Z'$ is a zero set.

(b) If $\mu E>\gamma$ then there is a zero set $K\subseteq E$ such that $\mu K\ge\gamma$. Now $\mu\llcorner K$ (234M) is a totally finite topological measure on X which is completion regular (412Q), so its support Z is a zero set, by (a); and $Z\subseteq K\subseteq E$ is self-supporting for μ with $\mu Z\ge\gamma$.

532I There is a useful general characterization of the sets $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ in terms of the measure algebras $\mathfrak{B}_\lambda$. At the same time, we can check that other products of separable metrizable spaces follow powers of $\{0,1\}$, as follows.

Theorem (Choksi & Fremlin 79) Let $\lambda\le\kappa$ be infinite cardinals. Then the following are equiveridical:

(i) $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$;

(ii) there is a family $\langle X_\xi\rangle_{\xi<\kappa}$ of non-singleton separable metrizable spaces such that $\lambda\in\mathrm{Mah}_{\mathrm{cr}}(\prod_{\xi<\kappa}X_\xi)$;

(iii) there is a Boolean-independent family $\langle b_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{B}_\lambda$ with the following property: for every $a\in\mathfrak{B}_\lambda$ there is a countable set $J\subseteq\kappa$ such that the subalgebras generated by $\{a\}\cup\{b_\xi:\xi\in J\}$ and $\{b_\eta:\eta\in\kappa\setminus J\}$ are Boolean-independent.

proof If $\kappa=\omega$ then $\lambda=\omega$ and (i)-(iii) are all true. So we may assume that κ is uncountable.

(i)⇒(ii) is trivial.

(ii)⇒(iii)(α) If $\lambda\in\mathrm{Mah}_{\mathrm{cr}}(X)$, where every X_ξ is a non-trivial separable metrizable space and $X=\prod_{\xi<\kappa}X_\xi$, let μ be a Maharam-type-homogeneous completion regular topological probability measure on X with Maharam type λ. By 532Ha and 4A3Nc, the support Z of μ is determined by coordinates in a countable subset L of κ.

(β) Let $\mathfrak{A}$ be the measure algebra of μ. For each $\xi<\kappa$, let $f_\xi:X_\xi\to[0,1]$ be a continuous function taking both values 0 and 1; let $t_\xi\in\,]0,1[$ be such that $\mu\{x:x\in X,\ f_\xi(x(\xi))=t_\xi\}=0$. Set $U_\xi=\{x:f_\xi(x(\xi))<t_\xi\}$, $V_\xi=\{x:f_\xi(x)>t_\xi\}$; then U_ξ and V_ξ are disjoint non-empty open sets in X, both determined by coordinates in $\{\xi\}$, and $\mu(U_\xi\cup V_\xi)=1$. Set $b_\xi=U_\xi^\bullet$ in $\mathfrak{A}$. Then $\langle b_\xi\rangle_{\xi<\kappa\setminus L}$ is Boolean-independent. **P** If I, $I'\subseteq\kappa\setminus L$ are disjoint finite sets, then $H=X\cap\bigcap_{\xi\in I}U_\xi\cap\bigcap_{\xi\in I'}V_\xi$ is a non-empty open set in X. As H is determined by coordinates in $I\cup I'$ and Z is determined by coordinates in L, $Z\cap H$ is non-empty and therefore non-negligible; so $\mu H>0$ and $\inf_{\xi\in I}b_\xi\setminus\sup_{\xi\in I'}b_\xi$ is non-zero in $\mathfrak{A}$. **Q**

(γ) If $a\in\mathfrak{A}$ let E be such that $E^\bullet=a$. By 532Hb, we can choose for each $n\in\mathbb{N}$ self-supporting zero sets $K_n\subseteq E$, $\tilde K_n\subseteq X\setminus E$ such that $\mu K_n+\mu\tilde K_n\ge 1-2^{-n}$. Let $J\subseteq\kappa\setminus L$ be a countable set such that every K_n and every $\tilde K_n$ is determined by coordinates in $J\cup L$. Now the subalgebras $\mathfrak{D}_1$, $\mathfrak{D}_2$ generated by $\{a\}\cup\{b_\xi:\xi\in J\}$ and $\{b_\xi:\xi\in(\kappa\setminus L)\setminus J\}$ are Boolean-independent. **P** Take non-zero $d_1\in\mathfrak{D}_1$ and $d_2\in\mathfrak{D}_2$. Suppose for the moment that $d_1\cap a\ne 0$. As in (β), there is an open set G, determined by coordinates in J, such that $0\ne a\cap G^\bullet\subseteq d_1$. There is also an open set H, determined by coordinates in $\kappa\setminus(J\cup L)$, such that $0\ne H^\bullet\subseteq d_2$. Next, as $a=\sup_{n\in\mathbb{N}}K_n^\bullet$, there is an $n\in\mathbb{N}$ such that $0\ne K_n^\bullet\cap G^\bullet$, that is, $K_n\cap G\ne\emptyset$. As $K_n\cap G$ is determined by coordinates in $J\cup L$ and H is determined by coordinates in $\kappa\setminus(J\cup L)$, $K_n\cap G\cap V\ne\emptyset$; as K_n is self-supporting,

$$0\ne(K_n\cap G\cap H)^\bullet\subseteq d_1\cap d_2.$$

In the same way, using K'_n in place of K_n, we see that $d_1\cap d_2\ne 0$ if $d_1\setminus a\ne 0$. As d_1 and d_2 are arbitrary, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are Boolean-independent. **Q**

(δ) As $\#(\kappa\setminus L)=\kappa$ and $\mathfrak{A}\cong\mathfrak{B}_\lambda$, $\langle b_\xi\rangle_{\xi\in\kappa\setminus L}$, suitably reinterpreted, witnesses that (iii) is satisfied.

(iii)⇒(i) Now suppose that the conditions of (iii) are satisfied. Let (Z,ν) be the Stone space of $(\mathfrak{B}_\lambda,\bar\nu_\lambda)$. (See 411P for a summary of the properties of these spaces.) For $b\in\mathfrak{B}_\lambda$ write $\widehat{b}$ for the corresponding open-and-closed subset of Z. Define $\phi:Z\to\{0,1\}^\kappa$ by setting $\phi(z)=\langle\chi\widehat{b}_\xi(z)\rangle_{\xi<\kappa}$ for $z\in Z$. Then ϕ is continuous; let $\mu=\nu\phi^{-1}$ be the image Radon measure on $\{0,1\}^\kappa$ (418I). Now μ is completion regular. **P** Suppose that $K\subseteq\{0,1\}^\kappa$ is compact and self-supporting. Identifying $\mathfrak{B}_\lambda$ with the measure algebra of ν, we have a Boolean homomorphism $\psi:\mathrm{dom}\,\mu\to\mathfrak{B}_\lambda$

defined by setting $\psi E = (\phi^{-1}[E])^\bullet$ whenever μ measures E, and $\bar\nu_\lambda\psi E = \nu\phi^{-1}[E] = \mu E$ for every E; setting $E_\xi = \{x : x \in \{0,1\}^\kappa,\ x(\xi) = 1\}$, $\psi E_\xi = b_\xi$. Set $a = \psi K$. Let $J \subseteq \kappa$ be a countable set such that the subalgebras $\mathfrak{D}_1$, $\mathfrak{D}_2$ generated by $\{a\} \cup \{b_\xi : \xi \in J\}$ and $\{b_\eta : \eta \in \kappa \setminus J\}$ are Boolean-independent. **?** If $x \in K$, $y \in \{0,1\}^\kappa \setminus K$ and $x{\restriction}J = y{\restriction}J$, let U be an open cylinder containing y and disjoint from K. Express U as $U' \cap U''$ where U' is determined by coordinates in J and U'' by coordinates in $\kappa \setminus J$. Then $\psi U' \in \mathfrak{D}_1$ and $\psi U'' \in \mathfrak{D}_2$. As $\langle b_\xi\rangle_{\xi<\kappa}$ is Boolean-independent, $\psi U'' \ne 0$. Now K is self-supporting and $x \in K \cap U'$, so $\mu(K \cap U') > 0$ and $\psi(K \cap U') = a \cap \psi U'$ is non-zero; also $a \cap \psi U' \in \mathfrak{D}_1$; because $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are Boolean-independent, $\psi(K \cap U) = a \cap \psi U' \cap \psi U'' \ne 0$ and $K \cap U$ cannot be empty, contrary to the choice of U. **X**

This shows that K is determined by coordinates in J and is a zero set (4A3Nc, in the other direction). As K is arbitrary, we see that all self-supporting compact sets are zero sets. But as μ is a Radon measure, it is inner regular with respect to the self-supporting compact sets, therefore with respect to the zero sets, and is completion regular. **Q**

The inverse-measure-preserving function ϕ (and, of course, the Boolean homomorphism ψ) correspond to an embedding of the measure algebra of μ into $\mathfrak{B}_\lambda$. So the Maharam type of μ is at most λ. There is therefore a $\lambda' \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ such that $\lambda' \le \lambda$ (532B). By 532G, $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

532J Corollary (a) Suppose that λ, κ are infinite cardinals and $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. Then κ is at most the cardinal power λ^ω.

(b) If κ is an infinite cardinal such that $\lambda^\omega < \kappa$ for every $\lambda < \kappa$ (e.g., $\kappa = \mathfrak{c}^+$), then $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa) = \{\kappa\}$.

proof (a) By 532I, $\kappa \le \#(\mathcal{B}_\lambda)$; by 524Ma, $\#(\mathcal{B}_\lambda) \le \lambda^\omega$.

(b) By (a), no infinite cardinal less than κ can belong to $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. Also κ is uncountable, so the remarks in 532C tell us the rest of what we need.

532K Corollary If $\omega \le \lambda \le \kappa' \le \kappa$ and $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ then $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\kappa'})$.

proof If $\langle b_\xi\rangle_{\xi<\kappa}$ witnesses the truth of 532I(iii) for λ and κ, then its subfamily $\langle b_\xi\rangle_{\xi<\kappa'}$ witnesses the truth of 532I(iii) for λ and κ'. **P** Of course $\langle b_\xi\rangle_{\xi<\kappa'}$ is Boolean-independent. If $a \in \mathfrak{B}_\lambda$, there is a countable set $J \subseteq \kappa$ such that the subalgebras generated by $\{a\} \cup \{b_\xi : \xi \in J\}$ and $\{b_\eta : \eta \in \kappa \setminus J\}$ are Boolean-independent. Now $J' = J \cap \kappa'$ is a countable subset of κ' and the subalgebras generated by $\{a\} \cup \{b_\xi : \xi \in J'\}$ and $\{b_\eta : \eta \in \kappa' \setminus J'\}$ are Boolean-independent. **Q**

532L Corollary If $\omega \le \lambda \le \lambda'$ and $\mathrm{cf}[\lambda']^{\le\lambda} < \mathrm{cf}\,\kappa$ and $\lambda' \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$, then $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

proof Let $\langle b_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}_{\lambda'}$ satisfying (iii) of 532I. Let $\langle e_\eta\rangle_{\eta<\lambda'}$ be the standard generating family in $\mathfrak{B}_{\lambda'}$, and $\mathcal{J}$ a cofinal subset of $[\lambda']^\lambda$ with cardinal less than $\mathrm{cf}\,\kappa$. For each $\xi < \kappa$, there are a countable set $L \subseteq \lambda'$ such that b_ξ belongs to the closed subalgebra $\mathfrak{C}_L$ of $\mathfrak{B}_{\lambda'}$ generated by $\{e_\eta : \eta \in L\}$, and a $J_\xi \in \mathcal{J}$ such that $L \subseteq J_\xi$. Because $\#(J) < \mathrm{cf}\,\kappa$, there is a $J \in \mathcal{J}$ such that $A = \{\xi : \xi < \kappa,\ J_\xi = J\}$ has cardinal κ. Now the closed subalgebra $\mathfrak{C}_J$ of $\mathfrak{B}_{\lambda'}$ generated by $\{e_\eta : \eta \in J\}$ is isomorphic to $\mathfrak{B}_\lambda$, and the Boolean-independent $\langle b_\xi\rangle_{\xi\in A}$ in $\mathfrak{C}_J$ witnesses that 532I(iii) is true of λ and κ, as in the proof of 532K.

532M I turn now to the question of identifying those κ for which $\omega \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. We know from 532C and 532Ja that they all lie between ω and $\mathfrak{c}$. To go farther we need to look at some of the cardinals from §522.

Proposition If $A \subseteq \mathfrak{B}_\omega \setminus \{0\}$ and $\#(A) < \mathfrak{d} = \mathrm{cf}(\mathbb{N}^\mathbb{N})$, then there is a $c \in \mathfrak{B}_\omega$ such that neither c nor $1 \setminus c$ includes any member of A.

proof Let $\langle e_n\rangle_{n\in\mathbb{N}}$ be the standard generating family in $\mathfrak{B}_\omega = \mathfrak{B}_\mathbb{N}$. For $a \in \mathfrak{A}$ and $n \in \mathbb{N}$ let $f_a(n) \in \mathbb{N}$ be such that there is a b in the subalgebra $\mathfrak{C}_{f_a(n)^2}$ generated by $\{e_i : i < f_a(n)^2\}$ such that $\bar\nu_\omega(b \triangle a) < 2^{-n-3}\bar\mu a$. Because $\#(A) < \mathfrak{d}$, there is an $f \in \mathbb{N}^\mathbb{N}$ such that $f \not\le f_a$ for every $a \in \mathfrak{A}$; we may suppose that f is strictly increasing and $f(0) > 0$. Note that

$$f(n)^2 + n + 1 < f(n)^2 + 2f(n) + 1 \le f(n+1)^2$$

for every n. For each $n \in \mathbb{N}$, set

$$I_n = f(n)^2 \subseteq \mathbb{N}, \quad I'_n = I_{n+1} \setminus I_n,$$

$$c'_n = \inf\nolimits_{f(n)^2 \le i \le f(n)^2+n+1} e_i \in \mathfrak{C}_{I'_n};$$

then $\bar{\nu}_\omega c'_n = 2^{-n-2}$ for each n. Define $c_n \in \mathfrak{C}_{I_n}$, for $n \in \mathbb{N}$, by setting $c_0 = 0$ and $c_{n+1} = c_n \triangle c'_n$ for each n. Then $\bar{\nu}_\omega(c_{n+1} \triangle c_n) = 2^{-n-2}$ for every n, so $\langle c_n \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for the measure metric on $\mathfrak{B}_\omega$, and has a limit c. Note that

$$\sum_{i=n}^{m-1} 2^{-i-3} \le \bar{\nu}_\omega(c_m \triangle c_n) \le \sum_{i=n}^{m-1} 2^{-i-2}$$

whenever $m \ge n$. **P** Induce on m. For $m = n$ the result is trivial (interpreting $\sum_{i=n}^{n-1}$ as zero). For the inductive step to $m+1$, $c'_m \in \mathfrak{C}_{I'_m}$ is stochastically independent of $c_m \triangle c_n \in \mathfrak{C}_{I_m}$, so

$$\begin{aligned}
\bar{\nu}_\omega(c_{m+1} \triangle c_n) &= \bar{\nu}_\omega(c'_m \triangle c_m \triangle c_n) \\
&= \bar{\nu}_\omega c'_m + \bar{\nu}_\omega(c_m \triangle c_n) - 2\bar{\nu}_\omega(c'_m \cap (c_m \triangle c_n)) \\
&= 2^{-m-2} + (1 - 2^{-m-1})\bar{\nu}_\omega(c_m \triangle c_n) \\
&\ge 2^{-m-2} + (1 - 2^{-m-1}) \sum_{i=n}^{m-1} 2^{-i-3}
\end{aligned}$$

(by the inductive hypothesis)

$$= \sum_{i=n}^{m-1} 2^{-i-3} + 2^{-m-3}(2 - 4\sum_{i=n}^{m-1} 2^{-i-3}) \ge \sum_{i=n}^{m} 2^{-i-3};$$

on the other hand,

$$\bar{\nu}_\omega(c_{m+1} \triangle c_n) \le 2^{-m-2} + \bar{\nu}_\omega(c_m \triangle c_n) \le \sum_{i=n}^{m} 2^{-i-2}.$$

So the induction proceeds. **Q** Taking the limit as $m \to \infty$, we see that $2^{-n-2} \le \bar{\nu}_\omega(c \triangle c_n) \le 2^{-n-1}$ for every $n \in \mathbb{N}$.

Take any $a \in A$. Let $n \in \mathbb{N}$ be such that $f_a(n) < f(n)$. Then there is a $b \in \mathfrak{C}_{I_n}$ such that $\bar{\nu}_\omega(a \triangle b) < 2^{-n-3}\bar{\mu}a$. Now $c \triangle c_n \in \mathfrak{C}_{\mathbb{N}\setminus I_n}$ is stochastically independent of both $b \setminus c_n$ and $b \cap c_n$, so

$$\begin{aligned}
\bar{\nu}_\omega(b \setminus c) &= \bar{\nu}_\omega(((b \setminus c_n) \setminus (c \triangle c_n)) \cup ((b \cap c_n) \cap (c \triangle c_n))) \\
&= \bar{\nu}_\omega(b \setminus c_n)(1 - \bar{\nu}_\omega(c \triangle c_n)) + \bar{\nu}_\omega(b \cap c_n) \cdot \bar{\nu}_\omega(c \triangle c_n) \\
&\ge \bar{\nu}_\omega(b \setminus c_n)(1 - 2^{-n-1}) + 2^{-n-2}\bar{\nu}_\omega(b \cap c_n) \ge 2^{-n-2}\bar{\nu}_\omega b \ge 2^{-n-3}\bar{\nu}_\omega a.
\end{aligned}$$

So

$$\bar{\nu}_\omega(a \setminus c) \ge 2^{n-3}\bar{\nu}_\omega a - \bar{\nu}_\omega(b \setminus a) > 0,$$

and $a \not\subseteq c$. Similarly,

$$\begin{aligned}
\bar{\nu}_\omega(b \cap c) &= \bar{\nu}_\omega(b \cap c_n)(1 - \bar{\nu}_\omega(c \triangle c_n)) + \bar{\nu}_\omega(b \setminus c_n) \cdot \bar{\nu}_\omega(c \triangle c_n) \\
&\ge \bar{\nu}_\omega(b \cap c_n)(1 - 2^{-n-1}) + 2^{-n-2}\bar{\nu}_\omega(b \setminus c_n) \ge 2^{-n-2}\bar{\nu}_\omega b,
\end{aligned}$$

and $\bar{\nu}_\omega(a \cap c) > 0$.

As a is arbitrary, we have found an appropriate c.

532N It will be useful to have a classic example relevant to a question already examined in 325F.

Lemma There is a Borel set $W \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ such that whenever E, $F \subseteq \{0,1\}^{\mathbb{N}}$ have positive measure for ν_ω then neither $(E \times F) \cap W$ nor $(E \times F) \setminus W$ is negligible for the product measure ν_ω^2 on $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

proof (a) (Cf. 134Jb.) There is a Borel set $H \subseteq \{0,1\}^{\mathbb{N}}$ such that both H and its complement meet every non-empty open set in a set of non-zero measure. **P** For $x \in \{0,1\}^{\mathbb{N}}$ set $I_x = \{n : x(i) = 0 \text{ for } 2^n \le i < 2^{n+1}\}$. Set $H = \{x : I_x$ is finite and not empty and $\max I_x$ is even$\}$. **Q**

(b) Let $+$ be the usual group operation on $\{0,1\}^{\mathbb{N}} \cong \mathbb{Z}_2^{\mathbb{N}}$. In this group, addition and subtraction are identical, as $x + x = 0$ for every x; but the formulae may be easier to read if I use the symbol $-$ when it seems appropriate. Set $W = \{(x,y) : x, y \in \{0,1\}^{\mathbb{N}}, x - y \in H\}$.

Let E, $F \subseteq \{0,1\}^{\mathbb{N}}$ be sets of positive measure. Then $\{z : z \in \{0,1\}^{\mathbb{N}}, \nu_\omega(E \cap (F + z)) > 0\}$ is open (443C) and not empty (443Da), so meets H in a set of positive measure. Now

$$\begin{aligned}\nu_\omega^2((E\times F)\cap W)&=\nu_\omega^2\{(x,y):x\in E,\,y\in F,\,x-y\in H\}\\&=\nu_\omega^2\{(x,z):x\in E,\,x-z\in F,\,z\in H\}\end{aligned}$$

(because $(x,y)\mapsto(x,x-y)$ is a measure space automorphism for ν_ω^2, as in 255Ae or 443Xa)

$$\begin{aligned}&=\nu_\omega^2\{(x,z):x\in E,\,x\in F+z,\,z\in H\}\\&=\int_H\nu_\omega(E\cap(F+z))\nu_\omega(dz)>0.\end{aligned}$$

Applying the same argument with $\{0,1\}^{\mathbb N}\setminus H$ in the place of H, we see that the same is true of $(E\times F)\setminus W$.

532O Proposition If $A\subseteq\mathfrak B_\omega\setminus\{0\}$ and $\#(A)<\operatorname{cov}\mathcal N_\omega$, then there is a $c\in\mathfrak B_\omega$ such that neither c nor $1\setminus c$ includes any member of A.

proof Take $W\subseteq\{0,1\}^{\mathbb N}\times\{0,1\}^{\mathbb N}$ as in 532N. For $x\in\{0,1\}^{\mathbb N}$, set $c_x=W[\{x\}]^\bullet$ in $\mathfrak B_\omega$. If $a\in A$, then $\{x:a\subseteq c_x\}\in\mathcal N_\omega$. **P** Let $F\in\mathrm T_\omega$ be such that $F^\bullet=a$, and set $E=\{x:a\subseteq c_x\}$. Because $x\mapsto c_x$ is measurable when $\mathfrak B_\omega$ is given its measure-algebra topology (418Ta), $E\in\mathrm T_\omega$. For every $x\in E$, $F\setminus W[\{x\}]$ is negligible, so $(E\times F)\setminus W$ is negligible, by Fubini's theorem (252D). But this means that at least one of E, F must be negligible; since $F^\bullet=a\ne0$, $\nu_\omega E=0$, as required. **Q**

Similarly, $\{x:a\cap c_x=0\}$ is negligible. Since $\{0,1\}^{\mathbb N}$ cannot be covered by $\#(A)$ negligible sets, there is an $x\in\{0,1\}^{\mathbb N}$ such that c_x neither includes, nor is disjoint from, any member of A.

532P Proposition Set $\kappa=\max(\mathfrak d,\operatorname{cov}\mathcal N_\omega)$. If $\mathrm{FN}(\mathcal P\mathbb N)=\omega_1$, then $\omega\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. In particular, if $\mathfrak c=\omega_1$ then $\omega\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\omega_1})$.

proof (a) By 524O(b-ii), $\mathrm{FN}(\mathfrak B_\omega)=\omega_1$; let $f:\mathfrak B_\omega\to[\mathfrak B_\omega]^{\le\omega}$ be a Freese-Nation function. By 532M (if $\kappa=\mathfrak d$) or 532O (if $\kappa=\operatorname{cov}\mathcal N_\omega$), we can choose inductively a family $\langle b_\xi\rangle_{\xi<\kappa}$ in $\mathfrak B_\omega$ such that neither b_ξ nor $1\setminus b_\xi$ includes any nonzero member of $\mathfrak D_\xi$, where $\mathfrak D_\xi$ is the smallest subalgebra of $\mathfrak B_\omega$ including $\{b_\eta:\eta<\xi\}$ and such that $f(d)\subseteq\mathfrak D_\xi$ for every $d\in\mathfrak D_\xi$. Of course this implies that $\langle b_\xi\rangle_{\xi<\kappa}$ is Boolean-independent.

(b) For K, $L\subseteq\kappa$ set $d_{KL}=\inf_{\xi\in K}b_\xi\setminus\sup_{\xi\in L}b_\xi$. For $a\in\mathfrak B_\omega$, set $Q_a=\{(K,L):K,\,L\in[\kappa]^{<\omega}$ are disjoint, $d_{KL}\subseteq a\}$, and let Q'_a be the set of minimal members of Q_a, taking $(K,L)\le(K',L')$ if $K\subseteq K'$ and $L\subseteq L'$. Of course Q_a is well-founded so Q'_a is coinitial with Q_a. Now $R_{an}=\{(K,L):(K,L)\in Q'_a,\,\#(K\cup L)=n\}$ is countable for every $n\in\mathbb N$ and $a\in\mathfrak B_\omega$. **P** Induce on n. If $n=0$ this is trivial. For the inductive step to $n+1$, set $R'_\zeta=\{(K,L):K\cup L\subseteq\zeta,\,(K\cup\{\zeta\},L)\in R_{a,n+1}\}$ for each $\zeta<\kappa$. For $(K,L)\in R'_\zeta$, $b_\zeta\cap d_{KL}=d_{K\cup\{\zeta\},L}$ is included in a, so there is a $c_{KL\zeta}\in f(d_{K\cup\{\zeta\},L})\cap f(a)$ such that $d_{K\cup\{\zeta\},L}\subseteq c_{KL\zeta}\subseteq a$, in which case $b_\zeta\subseteq c_{KL\zeta}\cup(1\setminus d_{KL})$. If $\zeta<\zeta'<\kappa$, $(K,L)\in R'_\zeta$ and $(K',L')\in R'_{\zeta'}$, then $d_{K'L'}\not\subseteq a$ (because $(K'\cup\{\zeta'\},L')$ is a minimal member of Q_a), so $c_{KL\zeta}\cup(1\setminus d_{K'L'})\ne1$; as $c_{KL\zeta}$ and $d_{K'L'}$ both belong to $\mathfrak D_{\zeta'}$, $b_{\zeta'}\not\subseteq c_{KL\zeta}\cup(1\setminus d_{K'L'})$ and $c_{KL\zeta}\ne c_{K'L'\zeta'}$. As $f(a)$ is countable, $A=\{\zeta:R'_\zeta\ne\emptyset\}$ is countable. Next, for any $\zeta\in A$ and $(K,L)\in R'_\zeta$, we see that $d_{KL}\subseteq a\cup(1\setminus b_\zeta)$, and indeed that $(K,L)\in Q'_{a\cup(1\setminus b_\zeta)}$, so that $(K,L)\in R_{a\cup(1\setminus b_\zeta),n}$. By the inductive hypothesis, R'_ζ is countable.

This shows that $\{(K,L,\zeta):K\cup L\subseteq\zeta,\,(K\cup\{\zeta\},L)\in R_{a,n+1}\}$ is countable. In the same way, applying the ideas above to $1\setminus b_\zeta$ in place of b_ζ, $\{(K,L,\zeta):K\cup L\subseteq\zeta,\,(K,L\cup\{\zeta\})\in R_{a,n+1}\}$ is countable; so $R_{a,n+1}$ is countable and the induction proceeds. **Q**

It follows that Q'_a is countable for every $a\in\mathfrak B_\omega$.

(c) Now take any $a\in\mathfrak B_\omega$ and let $J\subseteq\kappa$ be a countable set such that $K\cup L\subseteq J$ whenever $(K,L)\in Q'_a\cup Q'_{1\setminus a}$. **?** Suppose, if possible, that the algebras $\mathfrak C_1$, $\mathfrak C_2$ generated by $\{a\}\cup\{b_\xi:\xi\in J\}$ and $\{b_\eta:\eta\in\kappa\setminus J\}$ are not Boolean-independent. Then there must be finite subsets K, L, K' and L' of κ such that $K\cup L\subseteq J$, $K'\cup L'\subseteq\kappa\setminus J$, $d_{K'L'}\ne0$, and either

$$d_{KL}\cap a\ne0,\ d_{K'L'}\cap d_{KL}\cap a=0$$

or

$$d_{KL}\setminus a\ne0,\ d_{K'L'}\cap d_{KL}\setminus a=0.$$

Suppose the former. Then $(K\cup K',L\cup L')\in Q_{1\setminus a}$ so there is a $(K'',L'')\in Q'_{1\setminus a}$ such that $K''\subseteq K\cup K'$ and $L''\subseteq L\cup L'$; in which case $K''\cup L''\subseteq J$ so in fact $K''\subseteq K$, $L''\subseteq L$ and $d_{KL}\cap a\subseteq d_{K''L''}\cap a=0$, which is impossible. Replacing a by $1\setminus a$ we get a similar contradiction in the second case. **X** So $\mathfrak C_1$ and $\mathfrak C_2$ are Boolean-independent.

(d) As a is arbitrary, (c) shows that $\langle b_\xi\rangle_{\xi<\kappa}$ satisfies the conditions of 532I(iii), so that ω belongs to $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$, as claimed.

532Q Proposition Suppose that $\mathrm{add}\,\mathcal{N}_\omega>\omega_1$.

(a) $\lambda\notin\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ whenever $\lambda\ge\omega$ and $\max(\omega,\mathrm{cf}[\lambda]^{\le\omega})<\kappa$.

(b) If $\omega_1\le\kappa\le\omega_\omega$ then $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)=\{\kappa\}$.

proof (a) ? If $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$, set $\kappa'=(\max(\omega,\mathrm{cf}[\lambda]^{\le\omega}))^+$; then $\lambda<\kappa'$ so $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\kappa'})$ (532K). As $\mathrm{cf}[\lambda]^{\le\omega}<\mathrm{cf}\,\kappa'$, ω belongs to $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\lambda^+})$ (532L) and therefore to $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\omega_1})$ (532K again).

Let $\langle b_\xi\rangle_{\xi<\omega_1}$ be a family in $\mathfrak{B}_\omega$ satisfying the conditions of 532I(iii). By 524Mb, $\omega_1<\mathrm{wdistr}(\mathfrak{B}_\omega)$; by 514K, there is a countable $C\subseteq\mathfrak{B}_\omega\setminus\{0\}$ such that for every $\xi<\omega_1$ there is a $c\in C$ such that $c\subseteq b_\xi$. Let $a\in C$ be such that $\{\xi:\xi<\omega_1,\,a\subseteq b_\xi\}$ is uncountable. There is supposed to be a countable $J\subseteq\omega_1$ such that the subalgebras generated by $\{a\}$ and $\{b_\xi:\xi\in\omega_1\setminus J\}$ are Boolean-independent; but then $\{\xi:a\subseteq b_\xi\}\subseteq J$, which is impossible. **X**

This shows that (a) is true.

(b) If $\omega\le\lambda<\kappa\le\omega_\omega$, then $\mathrm{cf}[\lambda]^{\le\omega}\le\lambda<\kappa$ (5A1E(e-iv)), so (a) tells us that $\lambda\notin\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. From 532C we see that $\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ must be $\{\kappa\}$ exactly.

532R Two combinatorial principles already used in 524O are relevant to the questions treated here.

Proposition Suppose that λ is an uncountable cardinal with countable cofinality such that $\square_\lambda$ (definition: 5A6D(a-ii)) is true. Set $\kappa=\lambda^+$. Then $\lambda\in\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$.

proof (a) Let $\langle I_\xi\rangle_{\xi<\kappa}$ be a family of countably infinite subsets of λ as in 5A6E. For each $\xi<\kappa$, let $\langle I_{\xi n}\rangle_{n\in\mathbb{N}}$, $\langle\alpha_{\xi n}\rangle_{n\in\mathbb{N}}$ be such that $\langle I_{\xi n}\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of subsets of I_ξ with $\#(I_{\xi n})=n$ for each n and $\langle\alpha_{\xi n}\rangle_{n\in\mathbb{N}}$ is a sequence of distinct points in $I_\xi\setminus\bigcup_{n\in\mathbb{N}}I_{\xi n}$. Set

$$U_{\xi n}=\{x:x\in\{0,1\}^\lambda,\,x(\eta)=0\text{ for every }\eta\in I_{\xi n}\},$$

$$V_{\xi n}=\{x:x\in U_{\xi n}\setminus\bigcup\nolimits_{m>n}U_{\xi m},\,x(\alpha_{\xi n})=1\},$$

$$\tilde V_{\xi n}=\{x:x\in U_{\xi n}\setminus\bigcup\nolimits_{m>n}U_{\xi m},\,x(\alpha_{\xi n})=0\}$$

for $n\in\mathbb{N}$. Note that as $\nu_\kappa U_{\xi m}=2^{-m}$ for each n, $V_{\xi n}$ and $\tilde V_{\xi n}$ are non-negligible, while both are determined by coordinates in $\{\alpha_{\xi n}\}\cup\bigcup_{m\ge n}I_{\xi m}\subseteq I_\xi$. Set

$$F_\xi=\bigcup\nolimits_{n\in\mathbb{N}}V_{\xi n},\quad b_\xi=F_\xi^\bullet\in\mathfrak{B}_\lambda.$$

Note that $F_\xi\cap\tilde V_{\xi n}=\emptyset$ for every n.

(b) Take any $a\in\mathfrak{B}_\lambda$. Then we can express a as $E^\bullet$ where $E\subseteq\{0,1\}^\lambda$ is a Baire set; let $I\subseteq\lambda$ be a countable set such that E is determined by coordinates in I. By the choice of $\langle I_\xi\rangle_{\xi<\kappa}$ there is a countable set $J\subseteq\kappa$ such that $I\cap I_\xi$ is finite for every $\xi\in\kappa\setminus J$. Let $\mathfrak{D}_1$, $\mathfrak{D}_2$ be the subalgebras of $\mathfrak{B}_\lambda$ generated by $\{a\}\cup\{b_\xi:\xi\in J\}$ and $\{b_\xi:\xi\in\kappa\setminus J\}$ respectively. Then $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are Boolean-independent. **P** If $d_1\in\mathfrak{D}_1$ and $d_2\in\mathfrak{D}_2$ are non-zero, we can express d_1 as $H_1^\bullet$ where $H_1\subseteq\{0,1\}^\lambda$ is a Baire set determined by coordinates in $L=I\cup\bigcup_{\xi\in K}I_\xi$ for some finite $K\subseteq J$. Next, we can find disjoint finite sets K', $K''\subseteq\kappa\setminus J$ such that $d_2\supseteq\inf_{\xi\in K'}b_\xi\setminus\sup_{\xi\in K''}b_\xi$. Because all the sets $I_\xi\cap I_\eta$, for distinct ξ, $\eta<\kappa$, and also the sets $I\cap I_\xi$, for $\xi\in\kappa\setminus J$, are finite, there is an $m\in\mathbb{N}$ such that all the sets $J_\xi=\{\alpha_{\xi m}\}\cup\bigcup_{n\ge m}I_{\xi n}$, for $\xi\in K'\cup K''$, are disjoint from each other and from I. Look at the sets $V_{\xi m}$, for $\xi\in K'$, and $\tilde V_{\xi m}$, for $\xi\in K''$. Set $H_2=\{0,1\}^\lambda\cap\bigcap_{\xi\in K'}V_{\xi m}\cap\bigcap_{\xi\in K''}\tilde V_{\xi m}$. Then $H_2^\bullet\subseteq d_2$. But observe now that all the $V_{\xi m}$ and $\tilde V_{\xi m}$ are non-negligible and that $V_{\xi m}$, $\tilde V_{\xi m}$ are determined by coordinates in J_ξ for each $\xi\in K'\cup K''$. So the sets H_1, $V_{\xi m}$ (for $\xi\in K'$) and $\tilde V_{\xi m}$ (for $\xi\in K''$) are stochastically independent, and

$$\bar\nu_\lambda(d_1\cap d_2)\ge\nu_\lambda(H_1\cap H_2)=\nu_\lambda H_1\cdot\prod\nolimits_{\xi\in K'}\nu_\lambda V_{\xi m}\cdot\prod\nolimits_{\xi\in K''}\nu_\lambda\tilde V_{\xi m}>0.$$

Thus $d_1\cap d_2\ne0$; as d_1 and d_2 are arbitrary, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are stochastically independent. **Q**

(c) The argument of (b) works equally well with $I=\emptyset$ and J an arbitrary finite subset of κ to show that $\langle b_\xi\rangle_{\xi<\kappa}$ is Boolean-independent. So the conditions of 532I(iii) are satisfied and $\kappa\in\mathrm{Mah}_{\mathrm{crR}}(\lambda)$, as claimed.

532S Proposition Suppose that $\mathrm{add}\,\mathcal{N}_\omega>\omega_1$ and that λ is an infinite cardinal such that $\mathrm{CTP}(\lambda^+,\lambda)$ (definition: 5A6Fa) is true. Then $\lambda\notin\mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$ for any $\kappa>\lambda$.

proof By 532K, it is enough to consider the case $\kappa = \lambda^+$. **?** Suppose, if possible, that there is a family $\langle b_\xi \rangle_{\xi<\kappa}$ in $\mathfrak{B}_\lambda$ satisfying the conditions of 532I(iii). Let $\langle e_\eta \rangle_{\eta<\lambda}$ be the standard generating family in $\mathfrak{B}_\lambda$. Then for each $\xi < \kappa$ we have a countable set $I_\xi \subseteq \lambda$ such that b_ξ belongs to the closed subalgebra of $\mathfrak{B}_\lambda$ generated by $\{e_\eta : \eta \in I_\xi\}$. Because $\mathrm{CTP}(\kappa, \lambda)$ is true, there is an uncountable set $A \subseteq \kappa$ such that $J = \bigcup_{\xi \in A} I_\xi$ is countable (5A6F(b-ii)). Now the closed subalgebra $\mathfrak{C}_J$ generated by $\{e_\eta : \eta \in J\}$ is isomorphic to $\mathfrak{B}_\omega$, so $\langle b_\xi \rangle_{\xi \in A}$ witnesses that $\omega \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\omega_1})$; but this contradicts 532Qa. **X**

532X Basic exercises (a) Let X be a normal Hausdorff space and $Y \subseteq X$ a zero set. Show that $\mathrm{Mah}_{\mathrm{crR}}(Y) \subseteq \mathrm{Mah}_{\mathrm{crR}}(X)$.

(b) Let $\beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$. (i) Show that $\mathrm{Mah}_{\mathrm{crR}}(\beta\mathbb{N}) = \{0\}$. (*Hint*: non-empty zero sets in $\beta\mathbb{N} \setminus \mathbb{N}$ are never ccc.) (ii) Give an example of a non-empty compact Hausdorff space X such that $\mathrm{Mah}_{\mathrm{crR}}(X) = \emptyset$.

(c) Let X and Y be compact Hausdorff spaces. Show that $\mathrm{Mah}_{\mathrm{crR}}(X \times Y) \subseteq \mathrm{Mah}_{\mathrm{crR}}(X) \cup \mathrm{Mah}_{\mathrm{crR}}(Y)$. (*Hint*: 454T.)

(d) Let λ and κ be infinite cardinals such that $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^\kappa)$. (i) Show that there is a strictly positive Maharam-type-homogeneous completion regular Radon probability measure on $\{0,1\}^\kappa$ with Maharam type λ. (ii) Suppose that λ is uncountable and that $H \subseteq \{0,1\}^\kappa$ is a non-empty G_δ set. Show that $\lambda \in \mathrm{Mah}_{\mathrm{crR}}(H)$.

(e) Find a proof of 532E which does not rely on 532D. (*Hint*: 415E.)

(f) Let $\langle (X_i, \mu_i) \rangle_{i \in I}$ be a family of quasi-dyadic spaces with strictly positive completion regular topological probability measures. Show that the ordinary product measure on $\prod_{i \in I} X_i$ is a strictly positive completion regular τ-additive topological probability measure.

532Y Further exercises (a) Let Z be the Stone space of $\mathfrak{B}_\lambda$, where $\lambda \geq \omega$. (i) Show that if $F \subseteq Z$ is a non-empty nowhere dense zero set then it is not ccc. (ii) Show that $\mathrm{Mah}_{\mathrm{crR}}(Z) = \{\lambda\}$. (iii) Show that $\mathrm{Mah}_{\mathrm{crR}}(Z \times Z) = \emptyset$.

(b) Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with countable networks, and Y any topological space. Suppose that we are given a strictly positive topological probability measure μ_i on each X_i, and a τ-additive topological probability measure ν on Y. Show that the ordinary product measure on $\prod_{i \in I} X_i \times Y$ is a topological measure.

(c) Suppose that $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$. Show that there are a Hausdorff space X and a completion regular Radon measure μ on X such that the Maharam type of μ is ω, but the Maharam type of $\mu \restriction \mathcal{B}(X)$ is ω_1. (*Hint*: 419C.)

532Z Problems (a) In 532P, can we take $\kappa = \mathrm{cf}\,\mathcal{N}_\omega$?

(b) We have $\omega \in \mathrm{Mah}_{\mathrm{crR}}(\{0,1\}^{\omega_1})$ if $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$ (532P, 532K) and not if $\mathrm{add}\,\mathcal{N}_\omega > \omega_1$ (532Q). Can we narrow the gap?

(c) For a Hausdorff space X let $\mathrm{Mah}_{\mathrm{spcrR}}(X)$ be the set of Maharam types of strictly positive Maharam homogeneous completion regular Radon measures on X. Describe the sets Γ of cardinals for which there are compact Hausdorff spaces X such that $\mathrm{Mah}_{\mathrm{spcrR}}(X) = \Gamma$.

532 Notes and comments I have spent a good many pages on a rather specialized topic. But I think the patterns here are instructive. When looking at $\mathrm{Mah}_{\mathrm{R}}(X)$, as in §531, we quickly come to feel that it is a measure of a certain kind of complexity; the richer the space X, the larger $\mathrm{Mah}_{\mathrm{R}}(X)$ will be. 531Eb and 531Ed are direct manifestations of this, and 531G develops the theme. $\mathrm{Mah}_{\mathrm{crR}}(X)$ can sometimes tell us more about X; knowing $\mathrm{Mah}_{\mathrm{crR}}(X)$ we may have a lower bound on the complexity of X as well as an upper bound. (On the other hand, $\mathrm{Mah}_{\mathrm{crR}}(X)$ can evaporate for non-trivial reasons, as in 532Xb and 532Ya, and leave us with very little idea of what X might be like.) In place of the straightforward facts in 531E, we have the relatively complex and partial results in 532G and 532K. As soon as we leave the constrained context of powers of $\{0,1\}$, the most natural questions seem to be obscure (532Zc).

However, if we follow the paths which are open, rather than those we might otherwise have chosen, we come to some interesting ideas, starting with 532I. Here, as happened in §531, we see that a proper understanding of the measure algebras $\mathfrak{B}_\lambda$ will take us a long way; and once again we find that this understanding has to be conditional on the model of set theory we are working in. Even to decide which powers of $\{0,1\}$ carry completion regular Radon measures with countable Maharam type we need to examine some new aspects of the Lebesgue measure algebra (532M-532O). Moreover, as well as the familiar cardinals of Cichoń's diagram, we have to look at the Freese-Nation number of $\mathcal{P}\mathbb{N}$ (532P). For larger Maharam types, in a way that we are becoming accustomed to, other combinatorial principles become relevant (532R, 532S).

533 Special topics

I present notes on certain questions which can be answered if we make particular assumptions concerning values of the cardinals considered in §§523-524. The first cluster (533A-533E) looks at Radon and quasi-Radon measures in contexts in which the additivity of Lebesgue measure is large compared with other cardinals of the structures considered. Developing ideas which arose in the course of §531, I discuss 'uniform regularity' in perfectly normal and first-countable spaces (533H). We also have a complete description of the cardinals κ for which $\mathbb{R}^\kappa$ is measure-compact (533J).

As previously, I write $\mathcal{N}(\mu)$ for the null ideal of a measure μ; ν_κ will be the usual measure on $\{0,1\}^\kappa$ and $\mathcal{N}_\kappa = \mathcal{N}(\nu_\kappa)$ its null ideal.

533A Lemma Let (X,Σ,μ) be a semi-finite measure space with measure algebra $(\mathfrak{A},\bar\mu)$. If $\langle\mathcal{K}_\xi\rangle_{\xi<\kappa}$ is a family of ideals in Σ such that μ is inner regular with respect to every $\mathcal{K}_\xi$ and $\kappa < \min(\operatorname{add}\mathcal{N}(\mu),\operatorname{wdistr}(\mathfrak{A}))$, then μ is inner regular with respect to $\bigcap_{\xi<\kappa}\mathcal{K}_\xi$.

proof Take $E\in\Sigma$ and $\gamma<\mu E$. Then there is an $E_1\in\Sigma$ such that $E_1\subseteq E$ and $\gamma<\mu E_1<\infty$. For $\xi<\kappa$, $D_\xi=\{K^\bullet : K\in\mathcal{K}_\xi\}$ is closed under finite unions and is order-dense in $\mathfrak{A}$, so includes a partition of unity A_ξ. Now there is a partition B of unity in $\mathfrak{A}$ such that $\{a : a\in A_\xi,\, a\cap b\ne 0\}$ is finite for every $b\in B$ and $\xi<\kappa$. Let $B'\subseteq B$ be a finite set such that $\bar\mu(E_1^\bullet\cap\sup B')\ge\gamma$, and let $E_2\subseteq E_1$ be such that $E_2^\bullet=E_1^\bullet\cap\sup B'$. For any $\xi<\kappa$,

$$A'_\xi=\{a : a\in A_\xi,\, a\cap E_2^\bullet\ne 0\}\subseteq\bigcup_{b\in B'}\{a : a\in A_\xi,\, a\cap b\ne 0\}$$

is finite, so $\sup A'_\xi$ belongs to D_ξ and can be expressed as $K_\xi^\bullet$ for some $K_\xi\in\mathcal{K}_\xi$. Now $E_2^\bullet\subseteq\sup A'_\xi$ so $E_2\setminus K_\xi$ is negligible. As $\kappa<\operatorname{add}\mathcal{N}(\mu)$, we have a negligible $H\in\Sigma$ including $\bigcup_{\xi<\kappa}E_2\setminus K_\xi$; now $E'=E_2\setminus H\subseteq E$, $\mu E'\ge\gamma$ and $E'\in\bigcap_{\xi<\kappa}\mathcal{K}_\xi$. As E and γ are arbitrary, μ is inner regular with respect to $\bigcap_{\xi<\kappa}\mathcal{K}_\xi$.

Remark Of course this result is covered by 412Ac unless $\operatorname{wdistr}(\mathfrak{A})>\omega_1$, which nearly forces $\mathfrak{A}$ to have countable Maharam type (524Mb).

533B Corollary Let (X,Σ,μ) be a totally finite measure space with countable Maharam type. If $\mathcal{E}\subseteq\Sigma$, $\#(\mathcal{E})<\min(\operatorname{add}\mathcal{N}_\omega,\operatorname{add}\mathcal{N}(\mu))$ and $\epsilon>0$, there is a set $F\in\Sigma$ such that $\mu(X\setminus F)\le\epsilon$ and $\{E\cap F : E\in\mathcal{E}\}$ is countable.

proof Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ. Then $\mathfrak{A}$ is separable in its measure-algebra topology (521Ea). Let $\mathcal{H}\subseteq\Sigma$ be a countable set such that $\{H^\bullet : H\in\mathcal{H}\}$ is dense in $\mathfrak{A}$. For $E\in\mathcal{E}$ and $n\in\mathbb{N}$ choose $H_{En}\in\mathcal{H}$ such that $\mu(E\triangle H_{En})\le 2^{-n}$; let $\mathcal{K}_E$ be the family of measurable sets K such that K is disjoint from $\bigcup_{i\ge n}E\triangle H_{Ei}$ for some n. Then μ is inner regular with respect to $\mathcal{K}_E$. Because $\#(\mathcal{E})<\min(\operatorname{wdistr}(\mathfrak{A}),\operatorname{add}\mathcal{N}(\mu))$ (524Mb), μ is inner regular with respect to $\bigcap_{E\in\mathcal{E}}\mathcal{K}_E$ (533A) and there is an $F\in\bigcap\mathcal{K}_E$ such that $\mu F\ge\mu X-\epsilon$. If $E\in\mathcal{E}$, there is an $n\in\mathbb{N}$ such that $F\cap(E\triangle H_{En})=\emptyset$, that is, $F\cap E=F\cap H_{En}$; so $\{F\cap E : E\in\mathcal{E}\}\subseteq\{F\cap H : H\in\mathcal{H}\}$ is countable.

533C Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space with countable Maharam type.

(a) If $w(X)<\operatorname{add}\mathcal{N}_\omega$, then μ is inner regular with respect to the second-countable subsets of X; if moreover $\mathfrak{T}$ is regular and Hausdorff, then μ is inner regular with respect to the metrizable subsets of X.

(b) If Y is a topological space of weight less than $\operatorname{add}\mathcal{N}_\omega$, then any measurable function $f : X\to Y$ is almost continuous.

(c) If $\langle Y_i\rangle_{i\in I}$ is a family of topological spaces, with $\#(I)<\operatorname{add}\mathcal{N}_\omega$, and $f_i : X\to Y_i$ is almost continuous for every i, then $x\mapsto f(x)=\langle f_i(x)\rangle_{i\in I} : X\to\prod_{i\in I}Y_i$ is almost continuous.

proof Note first that $\operatorname{add}\mathcal{N}(\mu)\ge\operatorname{add}\mathcal{N}_\omega$, by 524Ta.

(a) Let $\mathcal{U}$ be a base for $\mathfrak{T}$ with $\#(\mathcal{U})<\operatorname{add}\mathcal{N}_\omega$. Set

$$\mathcal{F} = \{F : F \subseteq X,\ \{F \cap U : U \in \mathcal{U}\} \text{ is countable}\}.$$

Then μ is inner regular with respect to $\mathcal{F}$. **P** If $E \in \Sigma$ and $\gamma < \mu E$, let $H \in \Sigma$ be such that $H \subseteq E$ and $\gamma < \mu H < \infty$. Then the subspace measure μ_H still has countable Maharam type (use 322I and 514Ed) and

$$\operatorname{add}\mathcal{N}(\mu_H) \geq \operatorname{add}\mathcal{N}(\mu) \geq \operatorname{add}\mathcal{N}_\omega > \#(\{H \cap U : U \in \mathcal{U}\}).$$

By 533B, there is an $F \in \operatorname{dom}\mu_H$ such that $\mu_H F \geq \gamma$ and $\{F \cap H \cap U : U \in \mathcal{U}\}$ is countable; now $F \in \mathcal{F}$, $F \subseteq E$ and $\mu F \geq \gamma$. **Q** But every member of $\mathcal{F}$ is second-countable (use 4A2B(a-vi)). If $\mathfrak{T}$ is regular and Hausdorff, then every member of $\mathcal{F}$ is separable and metrizable (4A2Pb).

(b) If $f : X \to Y$ is measurable, let $\mathcal{V}$ be a base for the topology of Y with $\#(\mathcal{V}) < \operatorname{add}\mathcal{N}_\omega$. Suppose that $E \in \Sigma$ and $\gamma < \mu E$. By 533B, there is an $F \in \Sigma$ such that $F \subseteq E$, $\gamma < \mu F < \infty$ and $\{F \cap f^{-1}[V] : V \in \mathcal{V}\}$ is countable. It follows that $\{f[F] \cap V : V \in \mathcal{V}\}$ is countable, so that the subspace topology on $f[F]$ is second-countable (4A2B(a-vi) again). Giving F its subspace topology $\mathfrak{T}_F$ and measure μ_F, μ_F is inner regular with respect to the closed sets (412Pc). If $H \subseteq f[F]$ is relatively open in $f[F]$, it is of the form $G \cap f[F]$ where G is an open subset of Y, so that $(f\restriction F)^{-1}[H] = F \cap f^{-1}[G]$ is measured by μ_F; thus $f\restriction F : F \to f[F]$ is measurable. By 418J, $f\restriction F$ is almost continuous, and there is a $K \in \Sigma$ such that $K \subseteq F$, $\mu K \geq \gamma$ and $f\restriction K$ is continuous.

As E and γ are arbitrary, f is almost continuous.

(c) For each $i \in I$, set $\mathcal{K}_i = \{K : K \in \Sigma,\ f_i\restriction K \text{ is continuous}\}$. Then $\mathcal{K}_i$ is an ideal in Σ and μ is inner regular with respect to $\mathcal{K}_i$. Also, as in 533B, $\#(I) < \operatorname{wdistr}(\mathfrak{A})$, where $\mathfrak{A}$ is the measure algebra of μ. So μ is inner regular with respect to $\mathcal{K} = \bigcap_{i\in I}\mathcal{K}_i$, by 533A. But $f\restriction K$ is continuous for every $K \in \mathcal{K}$, so f is almost continuous.

533D Proposition Let $(X, \mathfrak{T})$ be a first-countable compact Hausdorff space such that $\operatorname{cf}[w(X)]^{\leq\omega} < \operatorname{add}\mathcal{N}_\omega$, and μ a Radon measure on X with countable Maharam type. Then μ is inner regular with respect to the metrizable zero sets.

proof Set $\kappa = w(X)$. Then there is an injective continuous function $f : X \to [0,1]^\kappa$ (5A4Cc). Let $\mathcal{I}$ be a cofinal subset of $[\kappa]^{\leq\omega}$ with $\#(\mathcal{I}) < \operatorname{add}\mathcal{N}_\omega$. By 524Pa, $\operatorname{add}\mu \geq \operatorname{add}\mathcal{N}_\omega$.

For $I \in \mathcal{I}$ and $x \in X$ set $f_I(x) = f(x)\restriction I$. We need to know that for every $x \in X$ there is an $I \in \mathcal{I}$ such that $\{x\} = f_I^{-1}[f_I[\{x\}]]$. **P** Set $F_I = f_I^{-1}[f_I[\{x\}]]$ for each I. Because $\mathcal{I}$ is upwards-directed, $\langle F_I\rangle_{I\in\mathcal{I}}$ is downwards-directed. Because f is injective and $\bigcup\mathcal{I} = \kappa$, $\bigcap_{I\in\mathcal{I}} F_I = \{x\}$. Let $\mathcal{V}$ be a countable base of open neighbourhoods of x. For each $V \in \mathcal{V}$ there is an $I_V \in \mathcal{I}$ such that $F_{I_V} \cap (X \setminus V) = \emptyset$. Let $I \in \mathcal{I}$ be such that $\bigcup_{V\in\mathcal{V}} I_V \subseteq I$; then $F_I = \{x\}$. **Q**

For $I \in \mathcal{I}$, let λ_I be the image measure μf_I^{-1} on $[0,1]^I$; note that λ_I is a Radon measure (418I). Of course $\operatorname{add}\lambda_I$ is also at least $\operatorname{add}\mathcal{N}_\omega$, and in particular is greater than κ. If $G \subseteq X$ is open, then G and $f_I[G]$ are expressible as unions of at most κ compact sets, so λ_I measures $f_I[G]$.

There is an $I \in \mathcal{I}$ such that $\mu f_I^{-1}[f_I[G]] = \mu G$ for every open set $G \subseteq X$. **P?** Suppose, if possible, otherwise. For each $I \in \mathcal{I}$ choose an open set $G_I \subseteq X$ such that $E_I = f_I^{-1}[f_I[G_I]] \setminus G_I$ is non-negligible; because λ_I measures $f_I[G_I]$, μ measures E_I. Set $E'_I = \bigcup_{J\in\mathcal{I}, J\supseteq I} E_J$ for each $I \in \mathcal{I}$; because $\#(\mathcal{I}) < \operatorname{add}\mu$, μ measures E'_I. Note that $E'_I \subseteq E'_J$ whenever $J \subseteq I$ in $\mathcal{I}$; moreover, any sequence in $\mathcal{I}$ has an upper bound in $\mathcal{I}$. There is therefore an $M \in \mathcal{I}$ such that $E'_M \setminus E'_I$ is negligible for every $I \in \mathcal{I}$. Again because $\#(\mathcal{I}) < \operatorname{add}\mu$, $E'_M \setminus \bigcap_{I\in\mathcal{I}} E'_I$ is negligible; as E'_M is not negligible, there is an $x \in \bigcap_{I\in\mathcal{I}} E'_I$. But there is an $I \in \mathcal{I}$ such that $\{x\} = f_I^{-1}[f_I[\{x\}]]$, so $x \notin E_J$ for any $J \supseteq I$. **XQ**

Let $\mathcal{U}$ be a base for the topology of X with $\#(\mathcal{U}) = \kappa$. Then $\bigcup_{U\in\mathcal{U}} f_I^{-1}[f_I[U]] \setminus U$ is μ-negligible; let Y be its complement. If $x \in X$ and $y \in Y$ and $x \neq y$, there is a $U \in \mathcal{U}$ containing x but not y, so $f_I^{-1}[f_I[U]]$ contains x and not y and $f(x) \neq f(y)$. If $F \subseteq Y$ is compact, then F is homeomorphic to the metrizable $f_I[F]$, so is metrizable, and $F = f_I^{-1}[f_I[F]]$ is a zero set. As μ is surely inner regular with respect to the compact subsets of the conegligible set Y, it is inner regular with respect to the metrizable zero sets.

533E Corollary Suppose that $\operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1$. Let $(X, \mathfrak{T})$ be a first-countable K-analytic Hausdorff space such that $\operatorname{cf}[w(X)]^{\leq\omega} < \operatorname{add}\mathcal{N}_\omega$. Then X is a Radon space.

proof Let μ be a totally finite Borel measure on X, $E \subseteq X$ a Borel set and $\gamma < \mu E$. Because X is K-analytic, there is a compact set $K \subseteq X$ such that $\mu(E \cap K) > \gamma$ (apply 432B to the measure $\mu\llcorner E$). Let λ be the Radon measure on K defined by saying that $\int f d\lambda = \int_K f d\mu$ for every $f \in C(K)$ (using the Riesz Representation Theorem, 436J/436K). Because $\operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1$, ω_1 is a precaliber of every measurable algebra (525J); as K is first-countable, $\omega_1 \notin \operatorname{Mah}_{\mathrm{R}}(K)$ (531P) and λ must have countable Maharam type (531Ef). By 533D, λ is completion regular. But if $F \subseteq K$ is a zero set (for the subspace topology of K), there is a non-increasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $C(K)$ with infimum χF, so

$$\lambda F = \lim_{n\to\infty} \int f_n d\lambda = \lim_{n\to\infty} \int_K f_n d\mu = \mu F.$$

Accordingly

$$\lambda H = \sup\{\lambda F : F \subseteq H \text{ is a zero set}\} = \sup\{\mu F : F \subseteq H \text{ is a zero set}\} \le \mu H$$

for every Borel set $H \subseteq K$. As $\lambda K = \mu K$, λ agrees with μ on the Borel subsets of K. In particular, $\lambda(E \cap K) > \gamma$; now there is a compact set $L \subseteq E \cap K$ such that $\gamma \le \lambda L = \mu L$.

As E and γ are arbitrary, μ is tight; as μ is arbitrary, X is a Radon space.

533F Definition Let X be a topological space and μ a topological measure on X. I will say that μ is **uniformly regular** if there is a countable family $\mathcal{V}$ of open sets in X such that $G \setminus \bigcup\{V : V \in \mathcal{V},\, V \subseteq G\}$ is negligible for every open set $G \subseteq X$.

533G Lemma Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a compact Radon measure space.

(a) The following are equiveridical:

(i) μ is uniformly regular;

(ii) there are a metrizable space Z and a continuous function $f : X \to Z$ such that $\mu f^{-1}[f[F]] = \mu F$ for every closed $F \subseteq X$;

(iii) there is a countable family $\mathcal{H}$ of cozero sets in X such that $\mu G = \sup\{\mu H : H \in \mathcal{H},\, H \subseteq G\}$ for every open set $G \subseteq X$;

(iv) there is a countable family $\mathcal{E}$ of zero sets in X such that $\mu G = \sup\{\mu E : E \in \mathcal{E},\, E \subseteq G\}$ for every open set $G \subseteq X$.

(b) If $\mathfrak{T}$ is perfectly normal, the following are equiveridical:

(i) μ is uniformly regular;

(ii) there are a metrizable space Z and a continuous function $f : X \to Z$ such that $\mu f^{-1}[f[E]] = \mu E$ for every $E \in \Sigma$;

(iii) there are a metrizable space Z and a continuous function $f : X \to Z$ such that $f[G] \ne f[X]$ whenever $G \subseteq X$ is open and $\mu G < \mu X$;

(iv) there is a countable family $\mathcal{E}$ of closed sets in X such that $\mu G = \sup\{\mu E : E \in \mathcal{E},\, E \subseteq G\}$ for every open set $G \subseteq X$.

proof (a)(i)⇒(iii) Given $\mathcal{V}$ as in 533F, then for each $V \in \mathcal{V}$ there is a cozero set $H_V \subseteq V$ of the same measure. **P** $\mathfrak{T}$ is completely regular, so $\mathcal{H}_V = \{H : H \subseteq V \text{ is a cozero set}\}$ has union V; μ is τ-additive, so there is a sequence $\langle H_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{H}_V$ such that $\mu V = \mu(\bigcup_{n\in\mathbb{N}} H_n)$; set $H_V = \bigcup_{n\in\mathbb{N}} H_n$; by 4A2C(b-iii), H_V is a cozero set. **Q** Now $\mathcal{H} = \{H_V : V \in \mathcal{V}\}$ witnesses that (iii) is true.

(iii)⇒(iv) Given $\mathcal{H}$ as in (iii), then for each $H \in \mathcal{H}$ let $\langle F_n(H) \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of zero sets with union H (4A2C(b-vi)). Set $\mathcal{E} = \{F_n(H) : H \in \mathcal{H},\, n \in \mathbb{N}\}$, so that $\mathcal{E}$ is a countable family of zero sets. If $G \subseteq X$ is open,

$$\mu G = \sup_{H\in\mathcal{H}, H\subseteq G} \mu H = \sup_{H\in\mathcal{H}, H\subseteq G, n\in\mathbb{N}} \mu F_n(H) \le \sup_{E\in\mathcal{E}, E\subseteq G} \mu E \le \mu G,$$

so $\mathcal{E}$ witnesses that (iv) is true.

(iv)⇒(ii) Given $\mathcal{E}$ as in (iv), then for each $E \in \mathcal{E}$ choose a continuous $f_E : X \to \mathbb{R}$ such that $E = f_E^{-1}[\{0\}]$, and set $f(x) = \langle f_E(x) \rangle_{E\in\mathcal{E}}$ for $x \in X$. Then $f : X \to Z = \mathbb{R}^{\mathcal{E}}$ is continuous and Z is metrizable and $f^{-1}[f[E]] = E$ for every $E \in \mathcal{E}$. If $F \subseteq X$ is closed, set $\mathcal{E}_0 = \{E : E \in \mathcal{E},\, E \cap F = \emptyset\}$. Then $\bigcup \mathcal{E}_0$ has the same measure as $X \setminus F$ and does not meet $f^{-1}[f[F]]$, so $\mu f^{-1}[f[F]] = \mu F$. As F is arbitrary, f and Z witness that μ satisfies (ii).

(ii)⇒(i) Take Z and $f : X \to Z$ as in (ii). Replacing Z by $f[X]$ if necessary, we may suppose that f is surjective, so that Z is compact, therefore second-countable (4A2P(a-ii)). Let $\mathcal{U}$ be a countable base for the topology of Z closed under finite unions, and set $\mathcal{V} = \{f^{-1}[U] : U \in \mathcal{U}\}$, so that $\mathcal{V}$ is a countable family of open sets in X. If $G \subseteq X$ is open, set $F = X \setminus G$, $\mathcal{U}_0 = \{U : U \in \mathcal{U},\, U \cap f[F] = \emptyset\}$, $\mathcal{V}_0 = \{f^{-1}[U] : U \in \mathcal{U}_0\}$. Then $Z \setminus f[F] = \bigcup \mathcal{U}_0$ so $X \setminus f^{-1}[f[F]] = \bigcup \mathcal{V}_0$ and (because $\mathcal{U}_0$ and $\mathcal{V}_0$ are closed under finite unions)

$$\begin{aligned}\sup\{\mu V : V \in \mathcal{V},\, V \subseteq G\} &\ge \sup_{V\in\mathcal{V}_0} \mu V = \mu(X \setminus f^{-1}[f[F]])\\ &= \mu X - \mu f^{-1}[f[F]] = \mu X - \mu F = \mu G.\end{aligned}$$

Thus $\mathcal{V}$ witnesses that μ is uniformly regular.

(b)(i)⇒(iii) If μ is uniformly regular, then by (a-ii) there are a metrizable space Z and a continuous function $f : X \to Z$ such that $\mu f^{-1}[f[F]] = \mu F$ for every closed $F \subseteq X$. If now $G \subseteq X$ is open and $\mu G < \mu X$, there is a sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ of closed sets with union G, because $\mathfrak{T}$ is perfectly normal. In this case $f^{-1}[f[G]] = \bigcup_{n\in\mathbb{N}} f^{-1}[f[F_n]]$ has the same measure as G, so is not the whole of X, and $f[G] \neq f[X]$. Thus f and Z witness that (iii) is true.

(iii)⇒(ii) Take Z and f from (iii). Let ν be the image measure μf^{-1} on Z; then μ is a Radon measure (418I again). **?** If $E \in \Sigma$ and $\mu^* f^{-1}[f[E]] > \mu E$, let $E' \supseteq E$ be a Borel set such that $\mu E' = \mu E$. Because X is perfectly normal, E' belongs to the Baire σ-algebra of X (4A3Kb), so is Souslin-F (421L), therefore K-analytic (422Hb); consequently $f[E']$ is K-analytic (422Gd) therefore measured by ν (432A). This means that $f^{-1}[f[E']] \in \Sigma$, and of course

$$\mu f^{-1}[f[E']] \geq \mu^* f^{-1}[f[E]] > \mu E = \mu E'.$$

We can therefore find open sets $G \supseteq E'$ and $G' \supseteq X \setminus f^{-1}[f[E']]$ such that $\mu G + \mu G' < \mu X$. But now $G \cup G'$ is an open set of measure less than μX and $f[G \cup G'] = f[X]$, which is supposed to be impossible. **X**

Thus, for any $E \in \Sigma$, we have $\mu^* f^{-1}[f[E]] = \mu E$; of course it follows at once that $f^{-1}[f[E]]$ is measurable, with the same measure as E, as required by (ii).

(ii)⇒(i)⇔(iv) These follow immediately from (a), because all closed sets in X are zero sets.

533H Theorem (a) Suppose that $\operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1$. Let X be a perfectly normal compact Hausdorff space. Then every Radon measure on X is uniformly regular.

(b) (Plebanek 00) Suppose that $\operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1 = \operatorname{non}\mathcal{N}_\omega$. Let X be a first-countable compact Hausdorff space. Then every Radon measure on X is uniformly regular.

proof (a) Let μ be a Radon measure on X. **?** If μ is not uniformly regular, then we can choose $\langle g_\xi \rangle_{\xi<\omega_1}$ and $\langle G_\xi \rangle_{\xi<\omega_1}$ inductively, as follows. Given that $g_\eta : X \to \mathbb{R}$ is continuous for every $\eta < \xi$, set $f_\xi(x) = \langle g_\eta(x) \rangle_{\eta<\xi}$ for $x \in X$, so that $f_\xi : X \to \mathbb{R}^\xi$ is continuous. By 533G(b-iii), there is an open set G_ξ such that $\mu G_\xi < \mu X$ and $f_\xi[G_\xi] = f_\xi[X]$; now G_ξ is a cozero set and there is a continuous function $g_\xi : X \to \mathbb{R}$ such that $G_\xi = \{x : g_\xi(x) \neq 0\}$. Continue.

At the end of the induction, we have a continuous function $f_{\omega_1} : X \to \mathbb{R}^{\omega_1}$, setting $f_{\omega_1}(x) = \langle g_\xi(x) \rangle_{\xi<\omega_1}$ for each x. Now ω_1 is a precaliber of every measurable algebra (525J again), and $\mu(X \setminus G_\xi) > 0$ for each ξ, so there is an $x \in X$ such that $A = \{\xi : x \notin G_\xi\}$ is uncountable (525Ca). Set $H = \{y : f_{\omega_1}(y) \neq f_{\omega_1}(x)\}$; then H is an open set, so expressible as $\bigcup_{n\in\mathbb{N}} K_n$ where each K_n is compact. For each $\xi \in A$ there is an $x_\xi \in G_\xi$ such that $f_\xi(x_\xi) = f_\xi(x)$. As $g_\xi(x_\xi) \neq 0 = g_\xi(x)$, $x_\xi \in H$. Let $n \in \mathbb{N}$ be such that $A' = \{\xi : \xi \in A,\, x_\xi \in K_n\}$ is uncountable. Then

$$f_{\omega_1}(x) \in \overline{\{f_{\omega_1}(x_\xi) : \xi \in A'\}} \subseteq f_{\omega_1}[K_n];$$

but this is impossible, because $K_n \subseteq H$. **X**

So μ must be uniformly regular, as required.

(b) Let μ be a Radon measure on X. If $\mu X = 0$ then of course μ is uniformly regular; suppose $\mu X > 0$. As in (a) and the proof of 533E, the Maharam type of μ is countable. Let $\mathfrak{A}$ be the measure algebra of μ; then $d(\mathfrak{A}) \leq \operatorname{non}\mathcal{N}_\omega$ (524Me), so there is a set $A \subseteq X$, of full outer measure, with $\#(A) \leq \omega_1$ (521Lc). For each $x \in X$, let $\langle V_{xn} \rangle_{n\in\mathbb{N}}$ run over a base of neighbourhoods of x. Let $\mathcal{H}$ be the family of sets expressible as finite unions of V_{xn} for $x \in A$ and $n \in \mathbb{N}$, so that $\mathcal{H}$ is a family of open sets in X and $\#(\mathcal{H}) \leq \omega_1$.

For any open $G \subseteq X$, $\mu G = \sup\{\mu H : H \in \mathcal{H},\, H \subseteq G\}$. **P** Set $H^* = \bigcup\{H : H \in \mathcal{H},\, H \subseteq G\}$. For any $x \in A \cap G$, there is an $n \in \mathbb{N}$ such that $V_{xn} \subseteq G$, and now $V_{xn} \in \mathcal{H}$, so $x \in H^*$. Thus $G \setminus H^*$ does not meet A; as A has full outer measure,

$$\mu G = \mu H^* = \sup\{\mu H : H \in \mathcal{H},\, H \subseteq G\}$$

because $\{H : H \in \mathcal{H},\, H \subseteq G\}$ is closed under finite unions. **Q** So there is a countable $\mathcal{H}' \subseteq \{H : H \in \mathcal{H},\, H \subseteq G\}$ such that $\mu G = \sup_{H\in\mathcal{H}'} \mu H$.

Let $\langle H_\xi \rangle_{\xi<\omega_1}$ run over $\mathcal{H}$. For $\xi < \omega_1$, set

$$\mathcal{G}_\xi = \{G : G \subseteq X \text{ is open, } \mu G = \sup\{\mu H_\eta : \eta \leq \xi,\, H_\eta \subseteq G\}\}.$$

Then $\bigcup_{\xi<\omega_1} \mathcal{G}_\xi = \mathfrak{T}$. For each $\xi < \omega_1$, set

$$Y_\xi = \{y : y \in X,\, V_{yn} \in \mathcal{G}_\xi \text{ for every } n \in \mathbb{N}\};$$

then $X = \bigcup_{\xi<\omega_1} Y_\xi$. Now there is a $\xi < \omega_1$ such that Y_ξ has full outer measure. **P** Let ξ be such that $\mu^* Y_\xi = \mu^* Y_\eta$ for every $\eta \ge \xi$. **?** If $\mu^* Y_\xi < \mu X$, let $K \subseteq X \setminus Y_\xi$ be a non-negligible measurable set. Then the subspace measure μ_K is a Radon measure with countable Maharam type, so

$$\operatorname{cov}\mathcal{N}(\mu_K) \ge \operatorname{cov}\mathcal{N}_\omega \ge \operatorname{cov}\mathcal{N}_{\omega_1} > \omega_1.$$

Since $K \subseteq \bigcup_{\eta<\omega_1} Y_\eta$, there must be some $\eta < \omega_1$ such that $\mu_K^*(K \cap Y_\eta) > 0$; but now $\mu^*(K \cap Y_\eta) > 0$ and $\eta > \xi$ and

$$\mu^* Y_\eta = \mu^*(Y_\eta \setminus K) + \mu^*(Y_\eta \cap K) > \mu^* Y_\xi. \text{ **X**}$$

So Y_ξ has full outer measure. **Q**

Set $\mathcal{H}_\xi = \{H_\eta : \eta \le \xi\}$. If $G \subseteq X$ is open, and $H^* = \bigcup\{H : H \in \mathcal{H}_\xi,\ H \subseteq G\}$, then $G \setminus H^*$ is negligible. **P** Set $\mathcal{V} = \{V_{yn} : y \in Y_\xi,\ n \in \mathbb{N},\ V_{yn} \subseteq G\}$, $H_1^* = \bigcup \mathcal{V}$. Then Y_ξ does not meet $G \setminus H_1^*$, so $\mu H_1^* = \mu G$. Let $\mathcal{V}_0 \subseteq \mathcal{V}$ be a countable set such that $\mu(\bigcup \mathcal{V}_0) = \mu G$. If $V \in \mathcal{V}_0$, then $V \in \mathcal{G}_\xi$ and $V \subseteq G$ so $V \setminus H^*$ is negligible. Accordingly

$$G \setminus H^* \subseteq (G \setminus \bigcup \mathcal{V}_0) \cup \bigcup_{V \in \mathcal{V}_0} (V \setminus H^*)$$

is negligible. **Q** So if we take $\mathcal{H}'$ to be the set of finite unions of members of $\mathcal{H}_\xi$, $\mathcal{H}'$ will be a countable family of open sets and $\mu G = \sup\{\mu H : H \in \mathcal{H}',\ H \subseteq G\}$ for every open $G \subseteq X$. Thus μ is uniformly regular.

533I We know from 435Fb/435H and 439P that $\mathbb{R}^{\mathbb{N}}$ is measure-compact and $\mathbb{R}^{\mathfrak{c}}$ is not. It turns out that we already have a language in which to express a necessary and sufficient condition for $\mathbb{R}^\kappa$ to be measure-compact. To give the result in its full strength I repeat a definition from 435Xk.

Definition A completely regular space X is **strongly measure-compact** if $\mu X = \sup\{\mu^* K : K \subseteq X \text{ is compact}\}$ for every totally finite Baire measure μ on X.

Remark For the elementary properties of these spaces, see 435Xk. I repeat one here: a completely regular space X is strongly measure-compact iff it is measure-compact and pre-Radon. **P(i)** Suppose that X is measure-compact and pre-Radon and that μ is a totally finite Baire measure on X. Because X is measure-compact, μ has an extension to a quasi-Radon measure $\tilde\mu$ (435D); because X is pre-Radon, $\tilde\mu$ is Radon (434Jb) and

$$\begin{aligned}\mu X = \tilde\mu X &= \sup_{K\subseteq X \text{ is compact}} \tilde\mu K \\ &= \sup_{K\subseteq X \text{ is compact}} \tilde\mu^* K \le \sup_{K\subseteq X \text{ is compact}} \mu^* K \le \mu X.\end{aligned}$$

As μ is arbitrary, X is strongly measure-compact. **(ii)** Suppose that X is strongly measure-compact. (α) Let μ be a Baire probability measure on X. Then there is a non-negligible compact set, so X cannot be covered by the negligible open sets; by 435Fa, this is enough to ensure that X is measure-compact. (β) Now let μ be a totally finite τ-additive Borel measure on X. Write ν for the restriction of μ to the Baire σ-algebra of X. Then there is a compact set $K \subseteq X$ which is not ν-negligible. **?** If $\mu(X \setminus K) = \mu X$, then, because μ is τ-additive and X is regular, there is a closed set $F \subseteq X \setminus K$ such that $\mu F + \nu^* K > \mu X$. Because X is completely regular, there is a zero set G including K and disjoint from F, in which case $\nu^* K > \mu G = \nu G$, which is impossible. **X** So $\mu K > 0$; by 434J(a-iii), this tells us that X is pre-Radon. **Q**

533J Theorem (see Fremlin 77) Let κ be a cardinal. Then the following are equiveridical:

(i) $\mathbb{R}^\kappa$ is measure-compact;

(ii) if $\langle X_\xi\rangle_{\xi<\kappa}$ is a family of strongly measure-compact completely regular Hausdorff spaces then $\prod_{\xi<\kappa} X_\xi$ is measure-compact;

(iii) whenever X is a compact Hausdorff space and $\langle G_\xi\rangle_{\xi<\kappa}$ is a family of cozero sets in X, then $X \cap \bigcap_{\xi<\kappa} G_\xi$ is measure-compact;

(iv) for any Radon measure, the union of κ or fewer closed negligible sets has inner measure zero;

(v) for any Radon measure, the union of κ or fewer negligible sets has inner measure zero;

(vi) $\kappa < \operatorname{cov}\mathcal{N}(\mu)$ for any Radon measure μ;

(vii) $\kappa < \operatorname{cov}\mathcal{N}_\kappa$;

(viii) $\kappa < \mathfrak{m}(\mathfrak{A})$ for every measurable algebra $\mathfrak{A}$.

proof not-(iv)⇒not-(i) Suppose that X is a Hausdorff space, μ is a Radon measure on X and $\langle F_\xi\rangle_{\xi<\kappa}$ is a family of closed μ-negligible subsets of X such that $\mu_*(\bigcup_{\xi<\kappa} F_\xi) > 0$. Then there is a compact set $K \subseteq \bigcup_{\xi<\kappa} F_\xi$ such that $\mu K > 0$.

For each $\xi < \kappa$, there is a continuous $g_\xi : K \to [0,1[$ such that $g_\xi(z) = 0$ for $z \in K \cap F_\xi$ and $g_\xi^{-1}[\{0\}]$ is negligible. **P** For each $n \in \mathbb{N}$, there is a compact set $L_n \subseteq K \setminus F_\xi$ such that $\mu L_n \geq \mu K - 2^{-n}$; there is a continuous $f_n : K \to [0,1]$ such that $f_n(z) = 0$ for $z \in K \cap F_\xi$, 1 for $z \in L_n$; set $g_\xi = \sum_{n=0}^{\infty} 2^{-n-2} f_n$. **Q** Set $g(z) = \langle g_\xi(z)\rangle_{\xi<\kappa}$ for $z \in K$, so that $g : K \to [0,1[^\kappa$ is continuous.

Let ν be the Baire measure on $[0,1]^\kappa$ defined by setting $\nu H = \mu g^{-1}[H]$ for every Baire set $H \subseteq [0,1]^\kappa$. Then $]0,1[^\kappa$ has full outer measure for ν. **P** If $H \subseteq [0,1]^\kappa$ is a Baire set including $]0,1[^\kappa$, then H is determined by coordinates in some countable subset I of κ (4A3Mb). If $z \in K$ and $g_\xi(z) > 0$ for every $\xi \in I$, then $g(z)\restriction I \in\,]0,1[^I$ is equal to $w\restriction I$ for some $w \in H$, so $g(z) \in H$. Thus $g^{-1}[H]$ includes $\{z : z \in K,\ g_\xi(z) > 0 \text{ for every } \xi \in I\}$ and

$$\nu H = \mu g^{-1}[H] \geq \mu\{z : g_\xi(z) > 0 \text{ for every } \xi \in I\} = \mu K = \nu[0,1]^\kappa. \textbf{ Q}$$

On the other hand, every point y of $]0,1[^\kappa$ belongs to a ν-negligible cozero set. **P** $g[K]$ is a compact set not containing y, so there is a cozero set W containing y and disjoint from $g[K]$, and now $\nu W = 0$. **Q**

Let ν_0 be the subspace measure on $]0,1[^\kappa$. By 4A3Nd, ν_0 is a Baire measure on $]0,1[^\kappa$. If $y \in\,]0,1[^\kappa$ it belongs to a ν-negligible cozero set $W \subseteq [0,1]^\kappa$, and now $W \cap\,]0,1[^\kappa$ is a ν_0-negligible cozero set in $]0,1[^\kappa$ containing y. At the same time,

$$\nu_0\,]0,1[^\kappa = \nu[0,1]^\kappa = \mu K > 0.$$

So ν_0 witnesses that $]0,1[^\kappa$ is not measure-compact; as $\mathbb{R}^\kappa$ is homeomorphic to $]0,1[^\kappa$, it also is not measure-compact.

(iv)$\Rightarrow$(iii) Suppose that (iv) is true and that we have X and $\langle G_\xi\rangle_{\xi<\kappa}$, as in (iii), with a Baire probability measure μ on $Y = X \cap \bigcap_{\xi<\kappa} G_\xi$. Let ν be the Radon probability measure on X defined by saying that $\int f d\nu = \int (f\restriction Y) d\mu$ for every $f \in C(X)$ (436J/436K again). Then $\nu G_\xi = 1$ for each $\xi < \kappa$. **P** Let $f : X \to \mathbb{R}$ be a continuous function such that $G_\xi = \{x : x \in X,\ f(x) \neq 0\}$. Set $f_n = n|f| \wedge \chi X$ for each n. Then $\lim_{n\to\infty} f_n = \chi G_\xi$, so

$$\nu G_\xi = \lim_{n\to\infty} \int f_n d\nu = \lim_{n\to\infty} \int (f_n\restriction Y) d\mu = \mu Y = 1. \textbf{ Q}$$

By (iv), $\nu_*(\bigcup_{\xi<\kappa}(X \setminus G_\xi)) = 0$, that is, Y has full outer measure. In particular, Y must meet the support of ν; take any z in the intersection. If U is a cozero set in Y containing z, there is an open set $G \subseteq X$ such that $U = G \cap Y$; now there is a continuous $f : X \to [0,1]$ such that $f(z) = 1$ and $f(x) = 0$ for $x \in X \setminus G$; in this case

$$\mu U \geq \int (f\restriction Y) d\mu = \int f d\nu > 0$$

because $\{x : f(x) > 0\}$ is an open set meeting the support of ν. This shows that Y is not covered by the μ-negligible relatively cozero sets; as μ is arbitrary, Y is measure-compact (435Fa).

(iii)$\Rightarrow$(i) We can express $\mathbb{R}^\kappa$ in the form of (iii) by taking $X = [-\infty,\infty]^\kappa$ and $G_\xi = \{x : x(\xi) \text{ is finite}\}$ for each ξ.

(iv)$\Rightarrow$(vii) Let Z be the Stone space of the measure algebra of ν_κ, and λ its usual measure. If $\langle E_\xi\rangle_{\xi<\kappa}$ is a family of λ-negligible sets, then, because λ is inner regular with respect to the open-and-closed sets, we can find negligible zero sets $F_\xi \supseteq E_\xi$ for each ξ. By (iv), $\{F_\xi : \xi < \kappa\}$ cannot cover Z, so the same is true of $\{E_\xi : \xi < \kappa\}$. Thus $\operatorname{cov}\mathcal{N}(\lambda) > \kappa$. By 524Jb, $\operatorname{cov}\mathcal{N}_\kappa > \kappa$.

(vii)$\Rightarrow$(vi) Let θ be $\min\{\operatorname{cov}\mathcal{N}(\nu) : \nu \text{ is a non-zero Radon measure}\}$. By 524Pc, there is an infinite cardinal κ' such that $\theta = \operatorname{cov}\mathcal{N}_{\kappa'}$; by 523F, $\theta = \operatorname{cov}\mathcal{N}_\theta$. **?** If $\theta \leq \kappa$, then 523B tells us that

$$\kappa < \operatorname{cov}\mathcal{N}_\kappa \leq \operatorname{cov}\mathcal{N}_\theta = \theta. \textbf{ X}$$

So $\theta > \kappa$, as required.

(vi)$\Rightarrow$(v) If (vi) is true, (X,μ) is a Radon measure space, $\langle F_\xi\rangle_{\xi<\kappa}$ is a family of negligible sets, and $E \subseteq \bigcup_{\xi<\kappa} F_\xi$ is a measurable set, then the subspace measure μ_E is a Radon measure (416Rb), while E can be covered by κ negligible sets; by (vi), $\mu E = 0$; as E is arbitrary, $\mu_*(\bigcup_{\xi<\kappa} F_\xi) = 0$.

(v)$\Rightarrow$(ii) Suppose that (v) is true, that $\langle X_\xi\rangle_{\xi<\kappa}$ is a family of strongly measure-compact completely regular Hausdorff spaces with product X, and that μ is a Baire probability measure on X. For each $\xi < \kappa$ let Z_ξ be the Stone-Čech compactification of X_ξ; set $Z = \prod_{\xi<\kappa} Z_\xi$, and $\pi_\xi(z) = z(\xi)$ for $z \in Z$, $\xi < \kappa$. Then we have a Radon probability measure λ on Z defined by saying that $\int g\, d\lambda = \int_X (g\restriction X) d\mu$ for every $g \in C(Z)$. Note that if $W \subseteq Z$ is a zero set, there is a non-increasing sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in $C(Z)$ with infimum χW, so that

$$\lambda W = \inf_{n\in\mathbb{N}} \int g_n d\lambda = \inf_{n\in\mathbb{N}} \int_X (g_n\restriction X) d\mu = \mu(W \cap X).$$

Now $\lambda\pi_\xi^{-1}[X_\xi] = 1$ for each ξ. **P** Let $\epsilon > 0$. We have a Baire probability measure μ_ξ on X_ξ defined by setting $\mu_\xi E = \mu(X \cap \pi_\xi^{-1}[E])$ for every Baire set $E \subseteq X_\xi$, and a Radon measure $\lambda_\xi = \lambda\pi_\xi^{-1}$ on Z_ξ. Because X_ξ is strongly measure-compact, there is a compact set $K \subseteq X_\xi$ such that $\mu_\xi^* K \ge 1-\epsilon$. Now K is still compact when regarded as a subset of Z_ξ, so there is a zero set $F \subseteq Z_\xi$, including K, such that $\lambda_\xi F = \lambda_\xi K$. In this case, $F \cap X_\xi$ is a zero set in X_ξ including K, so

$$\begin{aligned}\lambda_* \pi_\xi^{-1}[X_\xi] &\ge \lambda\pi_\xi^{-1}[K] = \lambda_\xi K = \lambda_\xi F = \lambda\pi_\xi^{-1}[F] \\ &= \mu(X \cap \pi_\xi^{-1}[F]) = \mu_\xi(F \cap X_\xi) \ge \mu_\xi^* K \ge 1-\epsilon.\end{aligned}$$

As ϵ is arbitrary, we have the result. **Q**

By (v), $X = Z \cap \bigcap_{\xi<\kappa} \pi_\xi^{-1}[X_\xi]$ has full outer measure for λ. Let $\mathcal{G}$ be the family of μ-negligible cozero sets in X and $\mathcal{H}$ the family of λ-negligible open sets in Z. If $x \in G \in \mathcal{G}$, then there is a continuous function $g : Z \to [0,1]$ such that $g(x) = 1$ and $H = \{y : y \in X,\, g(y) > 0\}$ is included in G; now $\int g\,d\lambda = \int (g{\restriction}X) d\mu = 0$, so $\lambda H = 0$. This shows that $\bigcup \mathcal{G} \subseteq \bigcup \mathcal{H}$ is λ-negligible, and, in particular, is not the whole of X. By 435Fa as usual, this is enough to show that X is measure-compact, as required.

(ii)⇒(i) is elementary, because $\mathbb{R}$ is certainly strongly measure-compact.

(vi)⇒(viii)⇒(vii) are immediate from 524Md.

533X Basic exercises (a) Describe a family $\langle \mathcal{K}_t\rangle_{t\in\mathbb{R}}$ such that every $\mathcal{K}_t$ consists of compact sets, Lebesgue measure on $\mathbb{R}$ is inner regular with respect to every $\mathcal{K}_t$, but $\bigcap_{t\in\mathbb{R}} \mathcal{K}_t = \emptyset$.

(b) Let μ be a uniformly regular topological measure on a topological space X. (i) Show that if $A \subseteq X$ then the subspace measure on A is uniformly regular. (ii) Show that any indefinite-integral measure over μ is uniformly regular. (iii) Show that if Y is another topological space and $f : X \to Y$ is a continuous open map, then the image measure μf^{-1} is uniformly regular.

(c) Show that any Radon measure on the split interval is uniformly regular. (*Hint*: 419L.)

(d) (Babiker 76) Let X and Y be compact Hausdorff spaces, μ a Radon measure on X, $f : X \to Y$ a continuous surjection and $\nu = \mu f^{-1}$ the image measure on Y. Show that the following are equiveridical: (i) $\nu f[F] = \mu F$ for every closed $F \subseteq X$; (ii) $\int g\,d\mu = \inf\{\int h\,d\nu : h \in C(Y),\, hf \ge g\}$ for every $g \in C(X)$; (iii) for every $g \in C(X)$, $\{y : g$ is constant on $f^{-1}[\{y\}]\}$ is ν-conegligible.

(e) Show that any uniformly regular Borel measure has countable Maharam type.

(f) Let $\langle X_i\rangle_{i\in I}$ be a countable family of topological spaces with product X, and μ a τ-additive topological measure on X. Suppose that the marginal measure of μ on X_i is uniformly regular for every $i \in I$. Show that μ is uniformly regular.

(g) Let X be $[0,1] \times \{0,1\}$ with the topology generated by

$$\begin{aligned}&\{G \times \{0,1\} : G \subseteq [0,1] \text{ is relatively open for the usual topology}\} \\ &\qquad \cup \{\{(t,1)\} : t \in [0,1]\} \cup \{X \setminus \{(t,1)\} : t \in [0,1]\}.\end{aligned}$$

Show that X is compact and Hausdorff. Let μ be the Radon measure on X which is the image of Lebesgue measure on $[0,1]$ under the map $t \mapsto (t,0)$. Show that μ is uniformly regular but not completion regular.

(h) Let X be a topological space and μ a uniformly regular topological probability measure on X. Show that there is an equidistributed sequence in X.

(i) Show that there is a first-countable compact Hausdorff space with a uniformly regular topological probability measure, inner regular with respect to the closed sets, which is not τ-additive. (*Hint*: 439K.)

533Y Further exercises (a) (Pol 82) Let X be a compact Hausdorff space and μ a uniformly regular Radon measure on X. Show that if we give the space M_{R}^+ of Radon measures on X its narrow topology (437Jd) then $\chi(\mu, M_{\mathrm{R}}^+) \le \omega$.

(b) For a topological measure μ on a space X, write $\operatorname{ureg}(\mu)$ for the smallest size of any family $\mathcal{V}$ of open subsets of X such that $G \setminus \bigcup\{V : V \in \mathcal{V},\, V \subseteq G\}$ is negligible for every open $G \subseteq X$. (i) Show that if μ is inner regular with respect to the Borel sets then the Maharam type $\tau(\mu)$ of μ is at most $\operatorname{ureg}(\mu)$. (ii) Show that if X is compact and Hausdorff and μ is a Radon measure, then $\operatorname{ureg}(\mu) \le \max(\operatorname{non}\mathcal{N}_{\tau(\mu)}, \chi(X))$. (iii) Show that if X is compact and Hausdorff, μ is a Radon probability measure and $\operatorname{cov}\mathcal{N}_{\tau(\mu)} > \operatorname{ureg}(\mu)$, then μ has an equidistributed sequence.

(c) (Plebanek 00) Suppose that κ is a regular infinite cardinal such that $\operatorname{non}\mathcal{N}_\kappa < \operatorname{cov}\mathcal{N}_\kappa = \kappa$. Let (X,μ) be a Radon probability space such that $\chi(X) < \kappa$. Show that μ has an equidistributed sequence.

(d) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space with countable Maharam type, $\mathcal{A} \subseteq \Sigma$ a set with cardinal less than $\operatorname{add}\mathcal{N}_\omega$, and $\mathfrak{S}$ the topology on X generated by $\mathfrak{T} \cup \mathcal{A}$. Show that μ is $\mathfrak{S}$-Radon.

533Z Problem For which cardinals κ is $\mathbb{R}^\kappa$ Borel-measure-compact?

533 Notes and comments I suppose that from the standpoint of measure theory the most fundamental of all the properties of ω is the fact that the union of countably many Lebesgue negligible sets is again Lebesgue negligible; this is of course shared by every $\kappa < \operatorname{add}\mathcal{N}_\omega$ (which is in effect the definition of $\operatorname{add}\mathcal{N}_\omega$). In 533A-533E and 533J we have results showing that uncountable cardinals can be 'almost countable' in other ways. In each case the fact that ω has the property examined is either trivial (as in 533B) or a basic result from Volume 4 (as in 533Cb, 533Cc and 533E). Similarly, the fact that $\mathfrak{c}$ does *not* have any of these properties is attested by classical examples. If you are familiar with Martin's axiom you will not be surprised to observe that everything here is sorted out if we assume that $\mathfrak{m} = \mathfrak{c}$.

533H does not quite fit this pattern, and the hypothesis in 533Hb definitely contradicts Martin's axiom. 'Uniformly regular' measures got squeezed out of §434 by shortage of space; in the exercises 533Xb-533Xi I sketch some of what was missed. Here I mention them just to show that there is more to say on the subject of first-countable and perfectly normal spaces than I put into 531P and 531Q. Another phenomenon of interest is the occurrence of measures which are inner regular with respect to a family of compact metrizable sets (462J, 533Ca, 533D).

534 Hausdorff measures and strong measure zero

In this section I look at constructions which are primarily metric rather than topological. I start with a note on Hausdorff measures, spelling out connexions between Hausdorff r-dimensional measure on a separable metric space and the basic σ-ideal $\mathcal{N}$ (534B).

The main part of the section section is a brief introduction to a class of ideals which are of great interest in set-theoretic analysis. While the most important ones are based on separable metric spaces, some of the ideas can be expressed in more general contexts, and I give the definition and most elementary properties in terms of uniformities (534C-534D). An associated topological notion is what I call 'Rothberger's property' (534E-534G). A famous characterization of sets of strong measure zero in $\mathbb{R}$ in terms of translations of meager sets can also be represented as a theorem about σ-compact groups (534H). There are few elementary results describing the cardinal functions of strong measure zero ideals, but I give some information on their uniformities (534I) and additivities (534K). There seem to be some interesting questions concerning spaces with isomorphic strong measure zero ideals, which I consider in 534L-534N. A particularly important question, from the very beginning of the topic in Borel 1919, concerns the possible cardinals of sets of strong measure zero; in 534O-534P I give some sample facts and a pair of illustrative examples.

534A An elementary lemma will be useful.

Lemma Let (X,ρ) be a separable metric space. Then there is a countable family $\mathcal{C}$ of subsets of X such that whenever $A \subseteq X$ has finite diameter and $\eta > 0$ then there is a $C \in \mathcal{C}$ such that $A \subseteq C$ and $\operatorname{diam} C \le \eta + 2\operatorname{diam} A$.

proof Let D be a countable dense subset of X and set $\mathcal{C} = \{\emptyset\} \cup \{B(x,q) : x \in D,\ q \in \mathbb{Q},\ q \ge 0\}$. If $A \subseteq X$ has finite diameter and $\eta > 0$, then if $A = \emptyset$ we can take $C = \emptyset$. Otherwise, take $y \in A$ and $q \in \mathbb{Q}$ such that $\operatorname{diam} A + \frac{1}{4}\eta \le q \le \operatorname{diam} A + \frac{1}{2}\eta$. Let $x \in D$ be such that $\rho(x,y) \le \frac{1}{4}\eta$; then $C = B(x,q) \in \mathcal{C}$, $A \subseteq B(y,\operatorname{diam} A) \subseteq C$ and $\operatorname{diam} C \le 2q \le \eta + 2\operatorname{diam} A$.

534B Hausdorff measures There are difficult questions concerning the cardinals associated with even the most familiar Hausdorff measures. However we do have some easy results.

Theorem Let (X,ρ) be a metric space and $r > 0$. Write μ_{Hr} for r-dimensional Hausdorff measure on X, $\mathcal{N}(\mu_{Hr})$ for its null ideal, $\mathcal{N}$ for the null ideal of Lebesgue measure on $\mathbb{R}$ and $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$.

(a) $\operatorname{add}\mu_{Hr} = \operatorname{add}\mathcal{N}(\mu_{Hr})$.

(b) If X is separable, $\mathcal{N}(\mu_{Hr}) \preccurlyeq_{\mathrm{T}} \mathcal{N}$, so that $\operatorname{add}\mu_{Hr} \ge \operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}(\mu_{Hr}) \le \operatorname{cf}\mathcal{N}$.

(c) If X is separable, $(X,\in,\mathcal{N}(\mu_{Hr}))\preccurlyeq_{\mathrm{GT}}(\mathcal{M},\not\ni,\mathbb{R})$, so that $\operatorname{cov}\mathcal{N}(\mu_{Hr})\le\operatorname{non}\mathcal{M}$ and $\mathfrak{m}_{\mathrm{countable}}\le\operatorname{non}\mathcal{N}(\mu_{Hr})$.

(d) If X is analytic and $\mu_{Hr}X>0$, then $\operatorname{add}\mu_{Hr}=\operatorname{add}\mathcal{N}$, $\operatorname{cf}\mathcal{N}(\mu_{Hr})=\operatorname{cf}\mathcal{N}$, $\operatorname{non}\mathcal{N}(\mu_{Hr})\le\operatorname{non}\mathcal{N}$ and $\operatorname{cov}\mathcal{N}(\mu_{Hr})\ge\operatorname{cov}\mathcal{N}$.

proof (a) 521Ac.

(b)(i) Let $\mathcal{C}$ be a countable family of subsets of X such that whenever $A\subseteq X$ has finite diameter and $\eta>0$ there is a $C\in\mathcal{C}$ such that $A\subseteq C$ and $\operatorname{diam}C\le\eta+2\operatorname{diam}A$ (534A).

If $A\subseteq X$, then $A\in\mathcal{N}(\mu_{Hr})$ iff for every $\epsilon>0$ there is a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{C}$ such that $A\subseteq\bigcup_{n\in\mathbb{N}}C_n$ and $\sum_{n=0}^{\infty}(\operatorname{diam}C_n)^r\le\epsilon$. **P** If A is negligible and $\epsilon>0$, then (by the definition in 471A) there must be a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X such that $A\subseteq\bigcup_{n\in\mathbb{N}}A_n$ and $\sum_{n=0}^{\infty}(\operatorname{diam}A_n)^r<2^{-r}\epsilon$. Let $\langle\eta_n\rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{n=0}^{\infty}(\eta_n+2\operatorname{diam}A_n)^r\le\epsilon$. For each n we can find $C_n\in\mathcal{C}_n$ such that $A_n\subseteq C_n$ and $\operatorname{diam}C_n\le\eta_n+2\operatorname{diam}A_n$, so that $\sum_{n=0}^{\infty}(\operatorname{diam}C_n)^r\le\epsilon$, while $A\subseteq\bigcup_{n\in\mathbb{N}}C_n$.

On the other hand, if A satisfies the condition, then for every ϵ, $\delta>0$ there is a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of subsets of X such that $A\subseteq\bigcup_{n\in\mathbb{N}}C_n$ and $\sum_{n=0}^{\infty}(\operatorname{diam}C_n)^r\le\min(\epsilon,\delta^r)$. In this case, $\operatorname{diam}C_n\le\delta$ for every n, so $\theta_{r\delta}A$, as defined in 471A, is at most ϵ. As ϵ is arbitrary, $\theta_{r\delta}A=0$; as δ is arbitrary, A is μ_{Hr}-negligible. **Q**

(ii) It follows that $(\mathcal{N}(\mu_{Hr}),\subseteq,\mathcal{N}(\mu_{Hr}))\preccurlyeq_{\mathrm{GT}}(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$, where $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$ is the $\mathbb{N}$-localization relation (522K).

P$(\boldsymbol{\alpha})$ For each $n\in\mathbb{N}$, let $\mathcal{I}_n$ be the family of finite subsets I of $\mathcal{C}$ such that $\sum_{C\in I}(\operatorname{diam}C)^r\le 4^{-n}$. Let $\langle I_{nj}\rangle_{j\in\mathbb{N}}$ be a sequence running over $\mathcal{I}_n$. Now, given $A\in\mathcal{N}(\mu_{Hr})$, then for each $n\in\mathbb{N}$ let $\langle C_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence in $\mathcal{C}$, covering A, such that $\sum_{i=0}^{\infty}(\operatorname{diam}C_{ni})^r\le 2^{-n-1}$. Let $\langle C_i\rangle_{i\in\mathbb{N}}$ be a re-indexing of the family $\langle C_{ni}\rangle_{n,i\in\mathbb{N}}$, so that $\langle C_i\rangle_{i\in\mathbb{N}}$ is a sequence in $\mathcal{C}$, $\sum_{i=0}^{\infty}(\operatorname{diam}C_i)^r\le 1$, and $A\subseteq\bigcap_{m\ge n}\bigcup_{i\ge m}C_i$. Let $\langle k(n)\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $k(0)=0$ and $\sum_{i=k(n)}^{\infty}(\operatorname{diam}C_i)^r\le 4^{-n}$ for every n. Now, for $n\in\mathbb{N}$, let $\phi(A)(n)$ be such that $\{C_i:k(n)\le i<k(n+1)\}=I_{n,\phi(A)(n)}$.

This process defines a function $\phi:\mathcal{N}(\mu_{Hr})\to\mathbb{N}^{\mathbb{N}}$ such that

$$A\subseteq\bigcap_{m\in\mathbb{N}}\bigcup_{n\ge m}\bigcup I_{n,\phi(A)(n)}$$

for every $A\in\mathcal{N}(\mu_{Hr})$.

$(\boldsymbol{\beta})$ For $S\in\mathcal{S}$, set

$$\psi(S)=\bigcap_{m\in\mathbb{N}}\bigcup_{n\ge m}\bigcup_{i\in S[\{n\}]}\bigcup I_{ni}\subseteq X.$$

If $n\in\mathbb{N}$, then

$$\sum\{(\operatorname{diam}C)^r:C\in\bigcup_{i\in S[\{n\}]}I_{ni}\}\le 2^n\cdot 4^{-n}=2^{-n},$$

because $\#(S[\{n\}])\le 2^n$ and $\sum\{(\operatorname{diam}C)^r:C\in I_{ni}\}\le 4^{-n}$ for every i. But this means that, for any $m\in\mathbb{N}$,

$$\sum\{(\operatorname{diam}C)^r:C\in\bigcup_{n\ge m}\bigcup_{i\in S[\{n\}]}I_{ni}\}\le 2^{-m+1},$$

while

$$\psi(S)\subseteq\bigcup(\bigcup_{n\ge m}\bigcup_{i\in S[\{n\}]}I_{ni}).$$

So $\psi(S)\in\mathcal{N}(\mu_{Hr})$ for every $S\in\mathcal{S}$.

$(\boldsymbol{\gamma})$ Suppose that $A\in\mathcal{N}(\mu_{Hr})$ and $\phi(A)\subseteq^* S\in\mathcal{S}$. Then there is some $m_0\in\mathbb{N}$ such that $(n,\phi(A)(n))\in S$ for every $n\ge m_0$. Now, for any $m\in\mathbb{N}$, we have

$$\begin{aligned}A&\subseteq\bigcup_{n\ge\max(m,m_0)}\bigcup I_{n,\phi(A)(n)}\\&\subseteq\bigcup_{n\ge\max(m,m_0)}\bigcup_{i\in S[\{n\}]}\bigcup I_{ni}\subseteq\bigcup_{n\ge m}\bigcup_{i\in S[\{n\}]}\bigcup I_{ni},\end{aligned}$$

so $A\subseteq\psi(S)$. This shows that (ϕ,ψ) is a Galois-Tukey connection from $(\mathcal{N}(\mu_{Hr}),\subseteq,\mathcal{N}(\mu_{Hr}))$ to $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$, and $(\mathcal{N}(\mu_{Hr}),\subseteq,\mathcal{N}(\mu_{Hr}))\preccurlyeq_{\mathrm{GT}}(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$. **Q**

(iii) Since $(\mathcal{N},\subseteq,\mathcal{N})\equiv_{\mathrm{GT}}(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})$ (522M), $(\mathcal{N}(\mu_{Hr}),\subseteq,\mathcal{N}(\mu_{Hr}))\preccurlyeq_{\mathrm{GT}}(\mathcal{N},\subseteq,\mathcal{N})$, that is, $\mathcal{N}(\mu_{Hr})\preccurlyeq_{\mathrm{T}}\mathcal{N}$.

(iv) By 513Ee, as usual, we can conclude that $\operatorname{add}\mathcal{N}(\mu_{Hr})\ge\operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}(\mu_{Hr})\le\operatorname{cf}\mathcal{N}$.

(c)(i) If $\mu_{Hr}X=0$, the result is trivial. **P** Set $\phi(x)=\emptyset$ for $x\in X$, $\psi(t)=X$ for $t\in\mathbb{R}$; then (ϕ,ψ) is a Galois-Tukey connection from $(X,\in,\mathcal{N}(\mu_{Hr}))$ to $(\mathcal{M},\not\ni,\mathbb{R})$. **Q** So let us suppose that X is infinite.

(ii) Let F be the set of 1-Lipschitz functions $f:X\to[0,1]$. Define $T:X\to\ell^\infty(F)$ by setting $(Tx)(f)=f(x)$ for $f\in F$ and $x\in X$. Then

$$\|Tx-Ty\|_\infty=\sup_{f\in F}|f(x)-f(y)|=\min(1,\rho(x,y))$$

for all x, $y\in X$. **P** Of course $\sup_{f\in F}|f(x)-f(y)|\le\min(1,\rho(x,y))$, by the definition of F. On the other hand, given $x\in X$, we can set $f(z)=\min(1,\rho(z,x))$ for every $z\in X$; then $f\in F$ and $|f(x)-f(y)|=\min(1,\rho(x,y))$. So we have equality. **Q** Thus T is 1-Lipschitz for ρ and the usual metric on $\ell^\infty(F)$, and $T[X]$ is a separable subset of $\ell^\infty(F)$ (4A2B(e-iii)). Let V be the closed linear subspace of $\ell^\infty(F)$ generated by $T[X]$; then V is separable (4A4Bg). Being a closed subset of the complete metric space $\ell^\infty(F)$, V is a Polish space. Since X has more than one point, and T is injective, V is non-empty and has no isolated points.

Let $\langle v_n\rangle_{n\in\mathbb{N}}$ enumerate a dense subset of V. Set

$$E=\bigcap_{n\in\mathbb{N}}\bigcup_{i\ge n}U(v_i,2^{-i-1})$$

where $U(v,\delta)=\{u:u\in V,\ \|u-v\|_\infty<\delta\}$ for $v\in V$, $\delta>0$. Then E is the intersection of a sequence of dense open sets in V, so is comeager, and $M=V\setminus E$ belongs to the ideal $\mathcal{M}(V)$ of meager subsets of V. For any $v\in V$, the map $u\mapsto u-v:V\to V$ is a homeomorphism, so $M-v\in\mathcal{M}(V)$. Define $\phi:X\to\mathcal{M}(V)$ by setting $\phi(x)=M-Tx$ for $x\in X$.

In the other direction, define $\psi:V\to\mathcal{P}X$ by setting $\psi(v)=T^{-1}[E-v]$ for $v\in V$. Then $\psi(v)\in\mathcal{N}(\mu_{Hr})$ for every $v\in V$. **P** If $v\in V$ and $\delta\le\frac{1}{2}$, then $\|u-u'\|_\infty<1$ for all u, $u'\in U(v,\delta)$, so $\rho(x,x')\le\|Tx-Tx'\|_\infty$ whenever x, $x'\in T^{-1}[U(v,\delta)]$. Accordingly $\operatorname{diam}T^{-1}[U(v_i-v,2^{-i-1})]\le 2^{-i}$ for every $i\in\mathbb{N}$. This means that

$$\begin{aligned}\mu_{Hr}^*T^{-1}[E-v]&=\mu_{Hr}^*(\bigcap_{n\in\mathbb{N}}\bigcup_{i\ge n}T^{-1}[U(v_i-v,2^{-i-1})]\\&\le\inf_{n\in\mathbb{N}}\sum_{i=n}^\infty(2^{-i})^r=0.\ \textbf{Q}\end{aligned}$$

So ψ is a function from V to $\mathcal{N}(\mu_{Hr})$. We now see that

$$\begin{aligned}\phi(x)\not\ni v&\Longrightarrow v\notin M-Tx\Longrightarrow Tx\notin M-v\\&\Longrightarrow Tx\in E-v\Longrightarrow x\in\psi(v).\end{aligned}$$

Thus (ϕ,ψ) is a Galois-Tukey connection from $(X,\in,\mathcal{N}(\mu_{Hr}))$ to $(\mathcal{M}(V),\not\ni,V)$ and

$$(X,\in,\mathcal{N}(\mu_{Hr}))\preccurlyeq_{\mathrm{GT}}(\mathcal{M}(V),\not\ni,V)\cong(\mathcal{M},\not\ni,\mathbb{R})$$

(522Wb).

(iii) Now

$$\operatorname{cov}\mathcal{N}(\mu_{Hr})=\operatorname{cov}(X,\in,\mathcal{N}(\mu_{Hr}))\le\operatorname{cov}(\mathcal{M},\not\ni,\mathbb{R})=\operatorname{non}\mathcal{M},$$

$$\operatorname{non}\mathcal{N}(\mu_{Hr})=\operatorname{add}(X,\in,\mathcal{N}(\mu_{Hr}))\ge\operatorname{add}(\mathcal{M},\not\ni,\mathbb{R})=\operatorname{cov}\mathcal{M}=\mathfrak{m}_{\mathrm{countable}}$$

(512D, 512Ed, 522Sa).

(d) If X is analytic and $\mu_{Hr}X>0$, then by Howroyd's theorem (471S) there is a compact set $K\subseteq X$ such that $0<\mu_{Hr}K<\infty$. Now the subspace measure $\mu_{Hr}^{(K)}$ on K is an atomless Radon measure (471E, 471Dg, 471F) on a compact metric space, so

$$\operatorname{add}\mathcal{N}\le\operatorname{add}\mathcal{N}(\mu_{Hr})\le\operatorname{add}\mathcal{N}(\mu_{Hr}^{(K)})=\operatorname{add}\mathcal{N},$$

$$\operatorname{cf}\mathcal{N}\ge\operatorname{cf}(X,\mathcal{N}(\mu_{Hr}))\ge\operatorname{cf}(K,\mathcal{N}(\mu_{Hr}^{(K)}))=\operatorname{cf}\mathcal{N},$$

$$\operatorname{non}\mathcal{N}(\mu_{Hr})\le\operatorname{non}\mathcal{N}(\mu_{Hr}^{(K)})=\operatorname{non}\mathcal{N},$$

$$\operatorname{cov}(X,\mathcal{N}(\mu_{Hr}))\ge\operatorname{cov}(K,\mathcal{N}(\mu_{Hr}^{(K)}))=\operatorname{cov}\mathcal{N}$$

by (b) above, 521F and 522Wa.

534C Strong measure zero Let $(X,\mathcal{W})$ be a uniform space. I say that X has **strong measure zero** or **property C** if for any sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}$ there is a cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of X such that $A_n\times A_n\subseteq W_n$ for every $n\in\mathbb{N}$. A subset A of X has strong measure zero if it has strong measure zero in its subspace uniformity, that is, if for any sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}$ there is a cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of A such that $A_n\times A_n\subseteq W_n$ for every $n\in\mathbb{N}$. If (X,ρ) is a metric space, it has strong measure zero if it has strong measure zero for the uniformity associated with the metric (3A4B), that is, for any sequence $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ of strictly positive real numbers there is a cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of X such that $\operatorname{diam}A_n\le\epsilon_n$ for every $n\in\mathbb{N}$.

I will write $\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ or $\mathcal{S}\mathfrak{mz}(X,\rho)$ for the family of sets of strong measure zero in a uniform space $(X,\mathcal{W})$ or a metric space (X,ρ).

534D Proposition (a) If $(X,\mathcal{W})$ is a uniform space, then $\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ is a σ-ideal containing all the countable subsets of X.

(b) If $(X,\mathcal{W})$ is a uniform space with strong measure zero, $(Y,\mathcal{V})$ is a uniform space, and $f:X\to Y$ is uniformly continuous, then $f[X]\in\mathcal{S}\mathfrak{mz}(Y,\mathcal{V})$.

(c) Let $(X,\mathcal{W})$ be a uniform space and $A\subseteq X$. Then $A\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ iff $f[A]\in\mathcal{S}\mathfrak{mz}(Y,\rho)$ whenever (Y,ρ) is a metric space and $f:X\to Y$ is uniformly continuous.

(d) Let $(X,\mathcal{W})$ be a uniform space and $A\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$. If $B\subseteq X$ is such that $B\setminus G\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ whenever G is an open set including A, then $B\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$.

(e) If (X,ρ) is a metric space with strong measure zero, then X is separable and universally negligible.

proof (a) It is immediate from the definition that any subset of a set in $\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ belongs to $\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$, and so does any countable set. Now suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$. Let $\langle W_n\rangle_{n\in\mathbb{N}}$ be any sequence in $\mathcal{W}$. For each $k\in\mathbb{N}$, $\langle W_{2^k(2i+1)}\rangle_{i\in\mathbb{N}}$ is a sequence in $\mathcal{W}$, so there is a sequence $\langle A_{ki}\rangle_{i\in\mathbb{N}}$, covering A_k, such that $A_{ki}\times A_{ki}\subseteq W_{2^k(2i+1)}$ for every i. Set $B_0=\emptyset$, $B_n=A_{ki}$ if $n=2^k(2i+1)$ where k, $i\in\mathbb{N}$; then $A\subseteq\bigcup_{n\in\mathbb{N}}B_n$ and $B_n\times B_n\subseteq W_n$ for every n. As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, A has strong measure zero; as $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ is a σ-ideal.

(b) Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{V}$. For each $n\in\mathbb{N}$, there is a $W_n\in\mathcal{W}$ such that $(f(x),f(x'))\in V_n$ whenever $(x,x')\in W_n$. Because $X\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$, there is a cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of X such that $A_n\times A_n\subseteq W_n$ for every n; now $f[A_n]\times f[A_n]\subseteq V_n$ for every n and $\bigcup_{n\in\mathbb{N}}f[A_n]=f[X]$. As $\langle V_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $f[X]\in\mathcal{S}\mathfrak{mz}(Y,\mathcal{V})$.

(c) If A has strong measure zero, then of course $f[A]$ has strong measure zero for any uniformly continuous function f from X to a metric space, by (b) applied to $f\restriction A$. Now suppose that A satisfies the condition, and that $\langle W_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}$. Then there is a pseudometric ρ on X, compatible with the uniformity in the sense that $\{(x,y):\rho(x,y)\le\epsilon\}\in\mathcal{W}$ for every $\epsilon>0$, such that $\{(x,y):\rho(x,y)<2^{-n}\}\subseteq W_n$ for every n (4A2Ja). Set $\sim=\{(x,y):\rho(x,y)=0\}$. Then $\sim$ is an equivalence relation on X. If Y is the set of equivalence classes, we have a metric $\tilde\rho$ on Y defined by setting $\tilde\rho(x^\bullet,y^\bullet)=\rho(x,y)$ for all x, $y\in X$. Setting $f(x)=x^\bullet$ for $x\in X$, $f:X\to Y$ is uniformly continuous. So $f[A]\in\mathcal{S}\mathfrak{mz}(Y,\tilde\rho)$. Let $\langle B_n\rangle_{n\in\mathbb{N}}$ be a cover of $f[A]$ such that $\operatorname{diam}B_n\le 2^{-n-1}$ for every n, and set $A_n=f^{-1}[B_n]$ for each n. Then $\langle A_n\rangle_{n\in\mathbb{N}}$ is a cover of A. If $n\in\mathbb{N}$ and x, $y\in A_n$, then $\rho(x,y)=\tilde\rho(f(x),f(y))\le 2^{-n-1}$, so $(x,y)\in W_n$. Thus $A_n\times A_n\subseteq W_n$. As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $A\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$.

(d) Let $\langle W_n\rangle_{n\in\mathbb{N}}$ be any sequence in $\mathcal{W}$. For each $n\in\mathbb{N}$, let $V_n\in\mathcal{W}$ be such that $V_n\circ V_n\circ V_n^{-1}\subseteq W_{2n}$. Then there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$, covering A, such that $A_n\times A_n\subseteq V_n$ for every n. Set $B_{2n}=\operatorname{int}V_n[A_n]$ for each n, and $G=\bigcup_{n\in\mathbb{N}}B_{2n}$; then $B_{2n}\times B_{2n}\subseteq W_{2n}$ for every n and G is an open set including A. Accordingly $B\setminus G\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ and there is a sequence $\langle B_{2n+1}\rangle_{n\in\mathbb{N}}$, covering $B\setminus G$, such that $B_{2n+1}\times B_{2n+1}\subseteq W_{2n+1}$ for every n. Now $\langle B_n\rangle_{n\in\mathbb{N}}$ covers B and $B_n\times B_n\subseteq W_n$ for every n. As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, B has strong measure zero.

(e) ? If X is not separable, there is an uncountable $A\subseteq X$ such that $\epsilon=\inf_{x,y\in A,x\ne y}\rho(x,y)$ is greater than 0 (5A4B(h-iii)). Now there can be no cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of X by sets of diameter less than ϵ. **X** Thus X is separable.

Now suppose that μ is a Borel probability measure on X. Then there is a $\delta>0$ such that for every $n\in\mathbb{N}$ there is a Borel set E_n with $\operatorname{diam}E_n\le 2^{-n}$ and $\mu E_n\ge\delta$. **P?** Otherwise, we can find for each $n\in\mathbb{N}$ an $\epsilon_n>0$ such that $\mu E\le 2^{-n-2}$ whenever E is a Borel set and $\operatorname{diam}E\le\epsilon_n$. Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a cover of X such that $\operatorname{diam}A_n\le\epsilon_n$ for every n; then $\operatorname{diam}\overline{A}_n\le\epsilon_n$, so $\mu\overline{A}_n\le 2^{-n-2}$ for every n, and

$$\mu X\le\sum_{n=0}^\infty\mu\overline{A}_n<1.\ \mathbf{XQ}$$

Now consider $E=\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}E_m$. Since $\mu E\ge\delta>0$, there is an $x\in E$. For any $n\in\mathbb{N}$, there is an $m\ge n$ such that

$$x\in E_m\subseteq B(x,2^{-m})\subseteq B(x,2^{-n}),$$

so

$$\mu\{x\} = \inf_{n\in\mathbb{N}} \mu B(x, 2^{-n}) \ge \delta > 0.$$

As μ is arbitrary, this shows that X is universally negligible.

534E Rothberger's property 'Strong measure zero' is a uniform-space property. The topological notion closest to it seems to be the following. If X is a topological space and A is a subset of X, I will say that A has **Rothberger's property in** X if for every sequence $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ of non-empty open covers of X there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ such that $G_n \in \mathcal{G}_n$ for every $n \in \mathbb{N}$ and $A \subseteq \bigcup_{n\in\mathbb{N}} G_n$. Note that this is a relative property of the pair (A, X), not an intrinsic property of the set A (see 534Xe, 534Xo). I will write $\mathcal{R}\mathfrak{bg}(X)$ for the family of subsets of X with Rothberger's property.

534F Proposition Let X be a topological space.

(a) $\mathcal{R}\mathfrak{bg}(X)$ is a σ-ideal containing all the countable subsets of X.

(b) If Y is another topological space, $f : X \to Y$ is continuous and $A \in \mathcal{R}\mathfrak{bg}(X)$, then $f[A] \in \mathcal{R}\mathfrak{bg}(Y)$.

(c) If X is completely regular, and $\mathcal{W}$ is any uniformity on X compatible with its topology, then $\mathcal{R}\mathfrak{bg}(X) \subseteq \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$.

(d) If X is σ-compact and completely regular, and $\mathcal{W}$ is any uniformity on X compatible with its topology, then $\mathcal{R}\mathfrak{bg}(X) = \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$.

proof (a) (Cf. 534Da.) It is immediate from the definition that any subset of a set in $\mathcal{R}\mathfrak{bg}(X)$, and any countable subset of X, belong to $\mathcal{R}\mathfrak{bg}(X)$. Now suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{R}\mathfrak{bg}(X)$, with union A. Let $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ be any sequence of non-empty open covers of X. For each $k \in \mathbb{N}$, $\langle\mathcal{G}_{2^k(2i+1)}\rangle_{i\in\mathbb{N}}$ is a sequence of open covers of X, so there is a sequence $\langle G_{ki}\rangle_{i\in\mathbb{N}}$, covering A_k, such that $G_{ki} \in \mathcal{G}_{2^k(2i+1)}$ for every i. Take H_0 to be any member of $\mathcal{G}_0$, and set $H_n = G_{ki}$ if $n = 2^k(2i+1)$ where k, $i \in \mathbb{N}$; then $A \subseteq \bigcup_{n\in\mathbb{N}} H_n$ and $H_n \in \mathcal{G}_n$ for every n. As $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, A has Rothberger's property in X.

(b) (Cf. 534Db.) Let $\langle\mathcal{H}_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty open covers of Y. For each $n \in \mathbb{N}$, set $\mathcal{G}_n = \{f^{-1}[H] : H \in \mathcal{H}_n\}$; then $\mathcal{G}_n$ is a non-empty open cover of X. Because $A \in \mathcal{R}\mathfrak{bg}(X)$, there is a cover $\langle G_n\rangle_{n\in\mathbb{N}}$ of A such that $G_n \in \mathcal{G}_n$ for every $n \in \mathbb{N}$. Express G_n as $f^{-1}[H_n]$, where $H_n \in \mathcal{H}_n$, for each $n \in \mathbb{N}$; then $f[A] \subseteq \bigcup_{n\in\mathbb{N}} H_n$. As $\langle\mathcal{H}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $f[A]$ has Rothberger's property in Y.

(c) Suppose that $A \in \mathcal{R}\mathfrak{bg}(X)$, and that $\langle W_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{W}$. For each $n \in \mathbb{N}$, set $\mathcal{G}_n = \{G : G \subseteq X$ is open, $G \times G \subseteq W_n\}$; then $\mathcal{G}_n$ is a non-empty open cover of X. So we can find a cover $\langle G_n\rangle_{n\in\mathbb{N}}$ of A such that $G_n \in \mathcal{G}_n$, that is, $G_n \times G_n \subseteq W_n$, for each n. As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $A \in \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$.

(d)(i) Let $K \subseteq X$ be compact and $\mathcal{G}$ an open cover of X. Then there is a $W \in \mathcal{W}$ such that whenever $x \in K$ there is a $G \in \mathcal{G}$ such that $W[\{x\}] \subseteq G$. **P** (Cf. 2A2Ed.) Set

$$Q = \{(x, V) : x \in X,\ V \in \mathcal{W},\ V[V[\{x\}]] \subseteq G \text{ for some } G \in \mathcal{G}\}.$$

Then for every $x \in X$ there are a $G \in \mathcal{G}$ such that $x \in G$ and a $V \in \mathcal{W}$ such that $V[V[\{x\}]] \subseteq G$, and in this case $(x, V) \in Q$ and $x \in \operatorname{int} V[\{x\}]$. So $\{\operatorname{int} V[\{x\}] : (x, V) \in Q\}$ is an open cover of X and there is a finite set $Q_0 \subseteq Q$ such that $K \subseteq \bigcup\{\operatorname{int} V[\{x\}] : (x, V) \in Q_0\}$. Let $W \in \mathcal{W}$ be such that $W \subseteq V$ whenever $(x, V) \in Q_0$. If $x \in K$, there is an $(x', V) \in Q_0$ such that $x \in V[\{x'\}]$; and now there is a $G \in \mathcal{G}$ including $V[V[\{x'\}]] \supseteq W[\{x\}]$. **Q**

(ii) Let $K \subseteq X$ be compact and $A \in \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$. Then $A \cap K \in \mathcal{R}\mathfrak{bg}(X)$. **P** Let $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty open covers of X. For each $n \in \mathbb{N}$ let $W_n \in \mathcal{W}$ be such that $\{W[\{x\}] : x \in K\}$ refines $\mathcal{G}_n$. Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a cover of A such that $A_n \times A_n \subseteq W_n$ for every n. If $n \in \mathbb{N}$ and $A_n \cap K = \emptyset$, take any $G_n \in \mathcal{G}_n$. Otherwise, take $x_n \in A_n \cap K$ and $G_n \in \mathcal{G}_n$ such that $W_n[\{x_n\}] \subseteq G_n$. If $x \in A \cap K$, there is an $n \in \mathbb{N}$ such that $x \in A_n$; now $(x_n, x) \in A_n \times A_n \subseteq W_n$ and $x \in W_n[\{x_n\}] \subseteq G_n$. As x is arbitrary, $A \cap K \subseteq \bigcup_{n\in\mathbb{N}} G_n$. As $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $A \cap K$ has Rothberger's property in X. **Q**

(iii) $\mathcal{S}\mathfrak{mz}(X, \mathcal{W}) \subseteq \mathcal{R}\mathfrak{bg}(X)$. **P** If $A \in \mathcal{S}\mathfrak{mz}(X, \mathcal{W})$, let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a sequence of compact subsets of X covering X. By (iii), $A \cap K_n \in \mathcal{R}\mathfrak{bg}(X)$ for each n; by (a) above, $A \in \mathcal{R}\mathfrak{bg}(X)$. **Q** By (c), we have equality.

534G Corollary If X is a σ-compact completely regular topological space, then any uniformity on X compatible with its topology gives the same sets of strong measure zero.

proof Immediate from 534Fd.

534H Theorem Let X be a σ-compact locally compact Hausdorff topological group and A a subset of X. Then the following are equiveridical:

(i) A has Rothberger's property in X;

(ii) for any sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of the identity e of X, there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X such that $A\subseteq\bigcup_{n\in\mathbb{N}}U_nx_n$;

(iii) $FA\ne X$ for any nowhere dense set $F\subseteq X$;

(iv) $EA\ne X$ for any meager set $E\subseteq X$;

(v) $AF\ne X$ for any nowhere dense set $F\subseteq X$;

(vi) $AE\ne X$ for any meager set $E\subseteq X$.

Remark For the general theory of topological groups see §4A5 and Chapter 44. Readers unfamiliar with this theory, or impatient with the extra discipline needed to deal with non-commutative groups, may prefer to start by assuming that $X=\mathbb{R}^2$, so that every xU becomes $x+U$, and every $V^{-1}V$ becomes $V-V$.

proof (i)⇒(ii) Suppose that (i) is true, and that $\langle U_n\rangle_{n\in\mathbb{N}}$ is any sequence of neighbourhoods of e. Then $\mathcal{G}_n=\{\operatorname{int}U_nx:x\in X\}$ is an open cover of X, so there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ such that $A\subseteq\bigcup_{n\in\mathbb{N}}U_nx_n$.

(ii)⇒(i) Suppose that (ii) is true, and that $\langle W_n\rangle_{n\in\mathbb{N}}$ is any sequence in the right uniformity of X. Then for each $n\in\mathbb{N}$ there is a neighbourhood U_n of e such that $W_n\supseteq\{(x,y):xy^{-1}\in U_n\}$; let V_n be a neighbourhood of e such that $V_nV_n^{-1}\subseteq U_n$. By (ii), there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ such that $A\subseteq\bigcup_{n\in\mathbb{N}}V_nx_n$. Set $A_n=V_nx_n$ for each n. Then $A_nA_n^{-1}=V_nV_n^{-1}\subseteq U_n$, so $A_n\times A_n\subseteq W_n$, for each n, while $\langle A_n\rangle_{n\in\mathbb{N}}$ covers A. As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, A has strong measure zero. By 534Fd, $A\in\mathcal{R}\mathrm{bg}(X)$.

(ii)⇒(iv) Suppose that A satisfies (ii), and that $E\subseteq X$ is meager.

(α) If $K\subseteq X$ is compact and nowhere dense, then there is a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of e such that $K'=\bigcap_{n\in\mathbb{N}}U_nK$ is still compact and nowhere dense. **P** By 443N(ii), there is a nowhere dense zero set $F\supseteq K$. Now F is a G_δ set; suppose that $F=\bigcap_{n\in\mathbb{N}}G_n$ where G_n is open for each n. As $K\subseteq G_n$, the open set $U_n'=\{x:xK\subseteq G\}$ (4A5Ei) contains e; let U_n be a compact neighbourhood of e included in U_n'. Then $U_nK\subseteq G_n$ for every n, so $K'=\bigcap_{n\in\mathbb{N}}U_nK\subseteq F$ is nowhere dense, while K' is compact (use 4A5Ef). **Q**

(β) Let $K\subseteq X$ be compact and nowhere dense and U a neighbourhood of e. Then there is a neighbourhood V of e such that for every $x\in X$ there is an $x'\in Ux$ such that $Vx'\cap K=\emptyset$. **P** Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence of neighbourhoods of e such that $K'=\bigcap_{n\in\mathbb{N}}U_nK$ is compact and nowhere dense ((α) above). Choose a sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ of compact neighbourhoods of e such that $V_0\subseteq U$ and $V_{n+1}V_{n+1}^{-1}\subseteq U_n\cap V_n$ for each $n\in\mathbb{N}$. Then $Y=\bigcap_{n\in\mathbb{N}}V_n$ is a compact subgroup of X (see the proof of 4A5S), and $YK=\bigcap_{n\in\mathbb{N}}V_nK$ (4A5Eh). **?** If for every $n\in\mathbb{N}$ there is an $x_n\in X$ such that $V_n^{-1}x'\cap K\ne\emptyset$ for every $x'\in Ux_n$, then, in particular, $V_n^{-1}x_n\cap K\ne\emptyset$, so $x_n\in V_nK$. Since $\langle V_nK\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of compact sets, $\langle x_n\rangle_{n\in\mathbb{N}}$ has a cluster point

$$x^*\in\bigcap_{n\in\mathbb{N}}V_nK=YK\subseteq K'.$$

Because K' is nowhere dense, $V_1x^*\not\subseteq K'$; take $x\in V_1x^*\setminus K'$. Let W be an open neighbourhood of e such that $Wx\cap K'=\emptyset$. Then Wx is disjoint from $YK=Y^{-1}YK$ so $YWx\cap YK=\emptyset$. Now YW is an open set including $Y=\bigcap_{n\in\mathbb{N}}V_n$, and all the V_n are compact, so there is an $m\ge1$ such that $V_m\subseteq YW$ and $V_mx\cap YK=\emptyset$.

But observe that there is an $n>m$ such that $x_n\in V_1x^*$, so that

$$x\in V_1V_1^{-1}x_n\subseteq V_0x_n\subseteq Ux_n,$$

while $V_n^{-1}x\cap K\subseteq V_mx\cap YK$ is empty. **X**

Thus we can take $V=V_n^{-1}$ for some n. **Q**

(γ) Because X is σ-compact, any F_σ set in X is actually K_σ, and there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of nowhere dense compact sets covering E; we can suppose that $\langle K_n\rangle_{n\in\mathbb{N}}$ is non-decreasing. Choose inductively sequences $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$, $\langle V_n'\rangle_{n\in\mathbb{N}}$ and $\langle V_n''\rangle_{n\in\mathbb{N}}$ of neighbourhoods of e such that

U_0 is any compact neighbourhood of e,

given U_n, V_n is to be a neighbourhood of e such that $V_nV_n\subseteq U_n$,

given V_n, V_n' is to be a neighbourhood of e such that for every $y\in X$ there is a $z\in V_ny$ such that $V_n'z\cap K_{n+1}=\emptyset$

(using (β)),

given V_n', V_n'' is to be an open neighbourhood of e such that $(V_n'')^{-1}V_n''\subseteq V_n'$, given V_n'', U_{n+1} is to be a compact neighbourhood of e, included in $V_n \cap V_n''$, such that $K_{n+1}U_{n+1} \subseteq V_n''K_{n+1}$.

(This last is possible by 4A5Ei, because $V_n''K_{n+1}$ is an open set including K_{n+1}, so $\{x : K_{n+1}x \subseteq V_n''K_{n+1}\}$ is an open set containing e.)

(δ) For each $k \in \mathbb{N}$, $\langle U_{2^k(2i+1)}\rangle_{i\in\mathbb{N}}$ is a sequence of neighbourhoods of e, so there must be a sequence $\langle x_i^{(k)}\rangle_{i\in\mathbb{N}}$ such that $A \subseteq \bigcup_{i\in\mathbb{N}} U_{2^k(2i+1)}x_i^{(k)}$. Set $x_0 = e$ and $x_n = x_i^{(k)}$ if $n = 2^k(2i+1)$. For any $k \in \mathbb{N}$,

$$A \subseteq \bigcup_{i\in\mathbb{N}} A_{ki} \subseteq \bigcup_{i\in\mathbb{N}} U_{2^k(2i+1)}x_i^{(k)} \subseteq \bigcup_{n\ge 2^k} U_nx_n \subseteq \bigcup_{n\ge k} U_nx_n.$$

This means that $EA \subseteq \bigcup_{n\ge1} K_nU_nx_n$. **P** If $z \in EA$, we can express it as xy where $x \in E$ and $y \in A$. There are a $k \ge 1$ such that $x \in K_k$ and an $n \ge k$ such that $y \in U_nx_n$, in which case $z \in K_nU_nx_n$. **Q**

(ϵ) Now choose $\langle y_n\rangle_{n\in\mathbb{N}}$, $\langle z_n\rangle_{n\in\mathbb{N}}$ as follows. Start from $y_0 = e$. Given y_n, let $z_n \in V_ny_nx_{n+1}^{-1}$ be such that $V_n'z_n \cap K_{n+1} = \emptyset$; this is possible by the choice of V_n'. Now set $y_{n+1} = z_nx_{n+1}$, and continue.

For each n,

$$U_{n+1}y_{n+1} \subseteq V_ny_{n+1}$$

(by the choice of U_{n+1})

$$= V_nz_nx_{n+1} \subseteq V_nV_ny_nx_{n+1}^{-1}x_{n+1}$$

(by the choice of z_n)

$$\subseteq U_ny_n$$

by the choice of V_n. Consequently, $U_{n+1}y_{n+1} \cap K_{n+1}U_{n+1}x_{n+1} = \emptyset$. **P** We chose z_n such that $V_n'z_n \cap K_{n+1} = \emptyset$. Because $(V_n'')^{-1}V_n'' \subseteq V_n'$, $V_n''z_n \cap V_n''K_{n+1} = \emptyset$. Because $K_{n+1}U_{n+1} \subseteq V_n''K_{n+1}$ and $U_{n+1} \subseteq V_n''$, $U_{n+1}z_n \cap K_{n+1}U_{n+1} = \emptyset$, that is, $U_{n+1}y_{n+1} \cap K_{n+1}U_{n+1}x_{n+1} = \emptyset$. **Q**

(ζ) From (ϵ) we see that $\langle U_ny_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of compact sets, so has non-empty intersection. Take any $x \in \bigcap_{n\in\mathbb{N}} U_ny_n$. Then $x \notin K_{n+1}U_{n+1}x_{n+1}$ for any n, so $x \notin \bigcup_{n\ge1} K_nU_nx_n \supseteq EA$. Thus $EA \ne X$. As E is arbitrary, (iv) is true.

(iv)⇒(iii) is trivial.

(iii)⇒(ii) Suppose that (iii) is true. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be any sequence of open neighbourhoods of e. Then there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X such that $G = \bigcup_{n\in\mathbb{N}} x_nU_n^{-1}$ is dense. **P** Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a sequence of neighbourhoods of e such that $V_{n+1}V_{n+1}^{-1} \subseteq V_n \cap U_n^{-1}$ for every $n \in \mathbb{N}$. Then there is a compact normal subgroup Y of X such that $Y \subseteq \bigcap_{n\in\mathbb{N}} V_n$ and X/Y is metrizable (4A5S). The canonical map $x \mapsto x^\bullet : X \to X/Y$ is continuous, so X/Y is σ-compact, therefore separable (4A2P(a-ii)). Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a sequence in X such that $\{x_n^\bullet : n \in \mathbb{N}\}$ is dense in X/Y. Set $G_0 = \bigcup_{n\in\mathbb{N}} x_nV_{n+1}Y$. **?** If $H = X \setminus \overline{G}_0$ is non-empty, then $\{x^\bullet : x \in H\}$ is open (4A5Ja) so contains $x_n^\bullet$ for some n. But $x_nY \subseteq x_nV_{n+1}Y \subseteq G_0$, so there can be no $x \in H$ such that $x^\bullet = x_n^\bullet$. **X** Thus G_0 is dense. But, for any $n \in \mathbb{N}$, $Y \subseteq V_{n+1}^{-1}$ so $V_{n+1}Y \subseteq U_n^{-1}$, and $G = \bigcup_{n\in\mathbb{N}} x_nU_n^{-1}$ includes G_0. Thus G is dense, as required. **Q**

Accordingly $F = X \setminus G$ is nowhere dense, and $FA \ne X$; suppose $x \in X \setminus FA$. Then $F \cap xA^{-1} = \emptyset$, that is, $xA^{-1} \subseteq \bigcup_{n\in\mathbb{N}} x_nU_n^{-1}$, that is, $A^{-1} \subseteq \bigcup_{n\in\mathbb{N}} x^{-1}x_nU_n^{-1}$, that is, $A \subseteq \bigcup_{n\in\mathbb{N}} U_nx_n^{-1}x$. As $\langle U_n\rangle_{n\in\mathbb{N}}$ is arbitrary, (ii) is true.

(i)⇔(v)⇔(vi) Because $x \mapsto x^{-1}$ is a homeomorphism,

$$A \in \mathcal{R}\mathrm{bg}(X) \iff A^{-1} \in \mathcal{R}\mathrm{bg}(X)$$
$$\implies EA^{-1} \ne X \text{ whenever } E \subseteq X \text{ is meager}$$
$$\implies E^{-1}A^{-1} \ne X \text{ whenever } E \subseteq X \text{ is meager}$$

(because E^{-1} is meager if E is)

$$\iff AE \ne X \text{ whenever } E \subseteq X \text{ is meager}$$
$$\implies AF^{-1} \ne X \text{ whenever } F \subseteq X \text{ is nowhere dense}$$

(because F^{-1} is nowhere dense if F is)

$$\Longrightarrow FA^{-1} \neq X \text{ whenever } F \subseteq X \text{ is nowhere dense}$$
$$\Longrightarrow A^{-1} \in \mathcal{R}\mathfrak{bg}(X).$$

Remark The case $X = \mathbb{R}$ is due to Galvin Mycielski & Solovay 79.

534I Proposition (a) Let X be a Lindelöf space. Then $\operatorname{non}\mathcal{R}\mathfrak{bg}(X) \geq \mathfrak{m}_{\text{countable}}$.

(b) (Fremlin & Miller 88) Give $\mathbb{N}^{\mathbb{N}}$ the metric ρ defined by setting

$$\rho(x, y) = \inf\{2^{-n} : n \in \mathbb{N},\, x{\restriction}n = y{\restriction}n\}.$$

Then $\operatorname{non}\mathcal{S}\mathfrak{mz}(\mathbb{N}^{\mathbb{N}}, \rho) = \mathfrak{m}_{\text{countable}}$.

proof (a) Suppose that $A \subseteq X$ and $\#(A) < \mathfrak{m}_{\text{countable}}$. Let $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ be a sequence of non-empty open covers of X. Because X is Lindelöf, we can choose for each n a non-empty countable $\mathcal{G}'_n \subseteq \mathcal{G}_n$ covering X. Let P be the set of finite sequences $p = \langle p(i) \rangle_{i<n}$ such that $p(i) \in \mathcal{G}'_i$ for every $i < n$; say that $p \leq q$ in P if q extends P. Then P is a countable partially ordered set. For each $x \in A$, the set $Q_x = \{p : x \in p(i) \text{ for some } i < \#(p)\}$ is cofinal with P. **P** Given $p \in P$, set $n = \#(p)$; let $G \in \mathcal{G}'_n$ be such that $x \in G$; set $q = p \cup \{(n, G)\}$; then $p \leq q \in Q_x$. **Q**

Because $\#(A) < \mathfrak{m}_{\text{countable}} \leq \mathfrak{m}^{\uparrow}(P)$ (517Pc), there is an upwards-directed family $R \subseteq P$ meeting every Q_x (517B(iv)). Now $p^* = \bigcup R$ is a function; its domain I is either $\mathbb{N}$ or a natural number; in either case, $A \subseteq \bigcup_{i\in I} p^*(i)$ and $p^*(i) \in \mathcal{G}_i$ for every $i \in I$. As $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ is arbitrary, A has Rothberger's property in X; as A is arbitrary, $\operatorname{non}\mathcal{R}\mathfrak{bg}(X) \geq \mathfrak{m}_{\text{countable}}$.

(b) By (a) and 534Fc, $\operatorname{non}\mathcal{S}\mathfrak{mz}(\mathbb{N}^{\mathbb{N}}, \rho) \geq \mathfrak{m}_{\text{countable}}$. By 522Sb, there is a set $A \subseteq \mathbb{N}^{\mathbb{N}}$, with cardinal $\mathfrak{m}_{\text{countable}}$, such that for every $y \in \mathbb{N}^{\mathbb{N}}$ there is an $x \in A$ such that $x(n) \neq y(n)$ for every n. **?** If $A \in \mathcal{S}\mathfrak{mz}(\mathbb{N}^{\mathbb{N}}, \rho)$, take a sequence $\langle y_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} B(y_n, 2^{-n-1})$. Set $y(n) = y_n(n)$ for every n. Then there is an $x \in A$ such that $x(n) \neq y(n)$ for every n. But in this case $x(n) \neq y_n(n)$ and $x{\restriction}n+1 \neq y_n{\restriction}n+1$ and $x \notin B(y_n, 2^{-n-1})$ for every n. **X**

Thus A witnesses that $\operatorname{non}\mathcal{S}\mathfrak{mz}(\mathbb{N}^{\mathbb{N}}, \rho) \leq \mathfrak{m}_{\text{countable}}$.

Remark Observe that $(\mathbb{N}^{\mathbb{N}}, \rho)$, as described in (b) above, is isometric with $\mathbb{Z}^{\mathbb{N}}$ with the metric defined by the same formula, and that this metric is translation-invariant, so induces the topological group uniformity of $\mathbb{Z}^{\mathbb{N}}$ (4A5He).

534J Proposition (Fremlin 91) Let (X, ρ) be a separable metric space. Then $\mathcal{S}\mathfrak{mz}(X, \rho) \preccurlyeq_{\text{T}} \mathcal{N}^{\mathfrak{d}}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$ and $\mathfrak{d}$ is the dominating number (522A).

proof (a) By 534A, there is a countable family $\mathcal{C}$ of subsets of X such that whenever $A \subseteq X$ has finite diameter and $\eta > 0$, there is a $C \in \mathcal{C}$ such that $A \subseteq C$ and $\operatorname{diam} C \leq \eta + 2 \operatorname{diam} A$. For each $i \in \mathbb{N}$, let $\langle C_{ij} \rangle_{j\in\mathbb{N}}$ be a sequence running over $\{C : C \in \mathcal{C},\, \operatorname{diam} C \leq 2^{-i}\}$. Let $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ be the $\mathbb{N}$-localization relation.

(b) Let $D \subseteq \mathbb{N}^{\mathbb{N}}$ be a cofinal set with cardinal $\mathfrak{d}$. For each $d \in D$ we can find a function $\phi_d : \mathcal{S}\mathfrak{mz}(X, \rho) \to \mathbb{N}^{\mathbb{N}}$ such that $A \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{i\geq n} C_{d(i),\phi_d(A)(i)}$ for every $A \in \mathcal{S}\mathfrak{mz}(X, \rho)$. **P** For $A \in \mathcal{S}\mathfrak{mz}(X, \rho)$ and $k \in \mathbb{N}$, choose a sequence $\langle A_{ki} \rangle_{i\in\mathbb{N}}$ of sets covering A such that $2 \operatorname{diam} A_{ki} < 2^{-d(2^k(2i+1))}$ for every $i \in \mathbb{N}$. For $n = 2^k(2i+1)$, let $A_n \in \mathcal{C}$ be such that $A_{ki} \subseteq A_n$ and $\operatorname{diam} A_n \leq 2^{-d(n)}$; choose $\phi_d(A)(n)$ such that $A_n = C_{d(n),\phi_d(A)(n)}$. **Q** Define $\phi : \mathcal{S}\mathfrak{mz}(X, \rho) \to (\mathbb{N}^{\mathbb{N}})^D$ by setting $\phi(A) = \langle \phi_d(A) \rangle_{d\in D}$ for $A \in \mathcal{S}\mathfrak{mz}(X, \rho)$.

(c) For $S \in \mathcal{S}$ and $d \in D$, define

$$\psi_d(S) = \bigcap_{n\in\mathbb{N}} \bigcup_{i\geq n} \bigcup_{j\in S[\{i\}]} C_{d(i),j} \subseteq X.$$

For $\langle S_d \rangle_{d\in D} \in \mathcal{S}^D$ set $\psi(\langle S_d \rangle_{d\in D}) = \bigcap_{d\in D} \psi_d(S_d)$. Then $A = \psi(\langle S_d \rangle_{d\in D})$ has strong measure zero. **P** Let $\langle \epsilon_i \rangle_{i\in\mathbb{N}}$ be any family of strictly positive real numbers. Let $d \in D$ be such that $2^{-d(k)} \leq \epsilon_i$ whenever $k \in \mathbb{N}$ and $i < 2^{k+1}$. For each $k \in \mathbb{N}$, $\#(S_d[\{k\}]) \leq 2^k$, so we can find a sequence $\langle A_i \rangle_{i\in\mathbb{N}}$ such that $\langle A_i \rangle_{2^k\leq i<2^{k+1}}$ is a re-enumeration of $\langle C_{d(k),j} \rangle_{j\in S[\{k\}]}$ supplemented by empty sets if necessary. This will ensure that if $2^k \leq i < 2^{k+1}$ then $\operatorname{diam} A_i \leq 2^{-d(k)} \leq \epsilon_i$, while

$$A \subseteq \psi_d(S_d) \subseteq \bigcup_{(k,j)\in S_d} C_{d(k),j} = \bigcup_{i\in\mathbb{N}} A_i.$$

As $\langle \epsilon_i \rangle_{i\in\mathbb{N}}$ is arbitrary, $A \in \mathcal{S}\mathfrak{mz}(X, \rho)$. **Q**

(d) Now (ϕ, ψ) is a Galois-Tukey connection from $(\mathcal{S}\mathfrak{mz}(X, \rho), \subseteq, \mathcal{S}\mathfrak{mz}(X, \rho))$ to $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})^D$, that is, $((\mathbb{N}^{\mathbb{N}})^D, T, \mathcal{S}^D)$, where T is the simple product relation as defined in 512H. **P** $\phi : \mathcal{S}\mathfrak{mz}(X, \rho) \to (\mathbb{N}^{\mathbb{N}})^D$ and $\psi : \mathcal{S}^D \to \mathcal{S}\mathfrak{mz}(X, \rho)$

are functions. Suppose that $A \in \mathcal{S}\mathfrak{mz}(X,\rho)$ and $\langle S_d\rangle_{d\in D}$ are such that $(\phi(A), \langle S_d\rangle_{d\in D}) \in T$, that is, $\phi_d(A) \subseteq^* S_d$ for every d. Fix $d \in D$ for the moment. Then there is an $n \in \mathbb{N}$ such that $(i, \phi_d(A)(i)) \in S_d$ for every $i \ge n$. Now, for any $m \ge n$,

$$A \subseteq \bigcup_{i\ge m} C_{d(i),\phi_d(A)(i)} \subseteq \bigcup_{i\ge m}\bigcup_{j\in S_d[\{i\}]} C_{d(i),j}.$$

Thus

$$A \subseteq \bigcap_{m\in\mathbb{N}}\bigcup_{i\ge m}\bigcup_{j\in S_d[\{i\}]} C_{d(i),j)} = \psi_d(S_d).$$

This is true for every d, so $A \subseteq \psi(\langle S_d\rangle_{d\in D})$. As A and $\langle S_d\rangle_{d\in D}$ are arbitrary, (ϕ,ψ) is a Galois-Tukey connection. **Q**

(e) Thus $(\mathcal{S}\mathfrak{mz}(X,\rho),\subseteq,\mathcal{S}\mathfrak{mz}(X,\rho)) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})^D$. But $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S}) \equiv_{\mathrm{GT}} (\mathcal{N},\subseteq,\mathcal{N})$ (522M), so $(\mathbb{N}^{\mathbb{N}},\subseteq^*,\mathcal{S})^D \equiv_{\mathrm{GT}} (\mathcal{N},\subseteq,\mathcal{N})^D$ (512Hb) and

$$(\mathcal{S}\mathfrak{mz}(X,\rho),\subseteq,\mathcal{S}\mathfrak{mz}(X,\rho)) \preccurlyeq_{\mathrm{GT}} (\mathcal{N},\subseteq,\mathcal{N})^D = (\mathcal{N}^D,\le,\mathcal{N}^D)$$

where $\le$ is the natural partial order of the product partially ordered set $\mathcal{N}^D$. Accordingly $\mathcal{S}\mathfrak{mz}(X,\rho) \preccurlyeq_{\mathrm{T}} \mathcal{N}^D \cong \mathcal{N}^{\mathfrak{d}}$, as claimed.

534K Corollary Let $(X,\mathcal{W})$ be a Lindelöf uniform space. Then $\operatorname{add}\mathcal{S}\mathfrak{mz}(X,\mathcal{W}) \ge \operatorname{add}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.

proof (a) Suppose to begin with that $\mathcal{W}$ is metrizable. Then X is separable (4A2Pd), so $\mathcal{S}\mathfrak{mz}(X,\rho) \preccurlyeq_{\mathrm{T}} \mathcal{N}^{\mathfrak{d}}$, and $\operatorname{add}\mathcal{N} = \operatorname{add}\mathcal{N}^{\mathfrak{d}} \le \operatorname{add}\mathcal{S}\mathfrak{mz}(X,\rho)$ by 513E(e-ii).

(b) For the general case, suppose that $\mathcal{A} \subseteq \mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ and $\#(\mathcal{A}) < \operatorname{add}\mathcal{N}$, and set $A^* = \bigcup\mathcal{A}$. Let f be a uniformly continuous function from X to a metric space Y. Then $f[X]$ is Lindelöf (5A4Bc), and $f[A]$ has strong measure zero in $f[X]$ for every $A \in \mathcal{A}$ (534Db), so $f[A^*] = \bigcup_{A\in\mathcal{A}} f[A]$ has strong measure zero, by (a). As f is arbitrary, A^* has strong measure zero, by 534Dc; as $\mathcal{A}$ is arbitrary, $\operatorname{add}\mathcal{S}\mathfrak{mz}(X,\mathcal{W}) \ge \operatorname{add}\mathcal{N}$.

534L $\mathcal{S}\mathfrak{mz}$-equivalence (a) If $(X,\mathcal{V})$ and $(Y,\mathcal{W})$ are uniform spaces, I say that they are **$\mathcal{S}\mathfrak{mz}$-equivalent** if there is a bijection $f : X \to Y$ such that a set $A \subseteq X$ has strong measure zero in X iff $f[A]$ has strong measure zero in Y. Of course this is an equivalence relation on the class of uniform spaces.

(b) If $(X,\mathcal{V})$ and $(Y,\mathcal{W})$ are uniform spaces, I say that X is **$\mathcal{S}\mathfrak{mz}$-embeddable** in Y if it is $\mathcal{S}\mathfrak{mz}$-equivalent to a subspace of Y (with the subspace uniformity, of course). Evidently this is transitive in the sense that if X is $\mathcal{S}\mathfrak{mz}$-embeddable in Y and Y is $\mathcal{S}\mathfrak{mz}$-embeddable in Z then X is $\mathcal{S}\mathfrak{mz}$-embeddable in Z.

(c) When X and Y are topological spaces, I will say that they are **$\mathcal{R}\mathfrak{bg}$-equivalent** if there is a bijection $f : X \to Y$ such that $A \subseteq X$ has Rothberger's property in X iff $f[A]$ has Rothberger's property in Y.

534M Lemma (a) Suppose that $(X,\mathcal{V})$ and $(Y,\mathcal{W})$ are uniform spaces, and that $\langle X_n\rangle_{n\in\mathbb{N}}$, $\langle Y_n\rangle_{n\in\mathbb{N}}$ are partitions of X, Y respectively such that X_n is $\mathcal{S}\mathfrak{mz}$-equivalent to Y_n for every n. Then X is $\mathcal{S}\mathfrak{mz}$-equivalent to Y.

(b) Suppose that $(X,\mathcal{V})$ and $(Y,\mathcal{W})$ are uniform spaces, and that X is $\mathcal{S}\mathfrak{mz}$-embeddable in Y and Y is $\mathcal{S}\mathfrak{mz}$-embeddable in X. Then X and Y are $\mathcal{S}\mathfrak{mz}$-equivalent.

proof (a) For each $n \in \mathbb{N}$, let $f_n : X_n \to Y_n$ be a bijection identifying the ideals of sets with strong measure zero. Then $f = \bigcup_{n\in\mathbb{N}} f_n$ is a bijection identifying $\mathcal{S}\mathfrak{mz}(X,\mathcal{V})$ and $\mathcal{S}\mathfrak{mz}(Y,\mathcal{W})$.

(b) (Compare 344D.) Let $X_1 \subseteq X$ and $Y_1 \subseteq Y$ be $\mathcal{S}\mathfrak{mz}$-equivalent to Y, X respectively; let $f : X \to Y_1$ and $g : Y \to X_1$ be bijections identifying the ideals of strong measure zero in each pair. Set $X_0 = X$, $Y_0 = Y$, $X_{n+1} = g[Y_n]$ and $Y_{n+1} = f[X_n]$ for each $n \in \mathbb{N}$; then $\langle X_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of X and $\langle Y_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of Y. Set $X_\infty = \bigcap_{n\in\mathbb{N}} X_n$, $Y_\infty = \bigcap_{n\in\mathbb{N}} Y_n$. Then $f\restriction X_{2k}\setminus X_{2k+1}$ is an $\mathcal{S}\mathfrak{mz}$-equivalence between $X_{2k}\setminus X_{2k+1}$ and $Y_{2k+1}\setminus Y_{2k+2}$, while $g\restriction Y_{2k}\setminus Y_{2k+1}$ is an $\mathcal{S}\mathfrak{mz}$-equivalence between $Y_{2k}\setminus Y_{2k+1}$ and $X_{2k+1}\setminus X_{2k+2}$; and $g\restriction Y_\infty$ is an $\mathcal{S}\mathfrak{mz}$-equivalence between Y_∞ and X_∞. So (a) gives the required $\mathcal{S}\mathfrak{mz}$-equivalence between X and Y.

534N Proposition $\mathbb{R}^r$, $[0,1]^r$ and $\{0,1\}^{\mathbb{N}}$ are $\mathcal{R}\mathfrak{bg}$-equivalent for every integer $r \ge 1$.

proof (a) Give $\mathbb{R}$ its usual uniformity. Of course the identity map is an $\mathcal{S}\mathfrak{mz}$-embedding of $[0,1]$ in $\mathbb{R}$. In the other direction, any homeomorphism from $\mathbb{R}$ to $]0,1[$ is an $\mathcal{R}\mathfrak{bg}$-equivalence between $\mathbb{R}$ and $]0,1[$ and therefore, because $\mathbb{R}$ and $]0,1[$ are σ-compact, an $\mathcal{S}\mathfrak{mz}$-embedding of $\mathbb{R}$ in $[0,1]$ (534Fd). By 534Mb, $\mathbb{R}$ and $[0,1]$ and $]0,1[$ are $\mathcal{S}\mathfrak{mz}$-equivalent.

(b) Give $\{0,1\}^{\mathbb{N}}$ the metric ρ defined by saying that

$$\rho(x,y)=\inf\{2^{-n}:n\in\mathbb{N},\,x\restriction n=y\restriction n\}$$

for x, $y\in\{0,1\}^{\mathbb{N}}$. Define $f:\{0,1\}^{\mathbb{N}}\to[0,1]$ by setting $f(x)=\sum_{n=0}^{\infty}2^{-n-1}x(n)$ for $x\in\{0,1\}^{\mathbb{N}}$. Then f is continuous, therefore uniformly continuous, so $f[A]$ has strong measure zero in $[0,1]$ whenever $A\subseteq\{0,1\}^{\mathbb{N}}$ has strong measure zero in $\{0,1\}^{\mathbb{N}}$. It is also the case that $f^{-1}[B]$ has strong measure zero whenever $B\subseteq[0,1]$ does. **P** Let $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ be any sequence of strictly positive numbers. Then there is a sequence $\langle B_n\rangle_{n\in\mathbb{N}}$, covering B, such that $\operatorname{diam}B_n<\frac{1}{2}\min(1,\epsilon_{2n},\epsilon_{2n+1})$ for every n. Fix n for the moment and consider $f^{-1}[B_n]$. If k is such that $2^{-k-1}\le\operatorname{diam}B_n<2^{-k}$, then B_n can meet at most two intervals of the type $I_{ki}=[2^{-k}i,2^{-k}(i+1)]$. So $f^{-1}[B_n]$ can meet at most two sets of the type $\{x:x\restriction k=z\}$, and we can express it as $A_{2n}\cup A_{2n+1}$ where

$$\max(\operatorname{diam}A_{2n},\operatorname{diam}A_{2n+1})\le2^{-k}\le2\operatorname{diam}B_n\le\min(\epsilon_{2n},\epsilon_{2n+1}).$$

Putting these together, we have a cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of $\bigcup_{n\in\mathbb{N}}f^{-1}[B_n]\supseteq f^{-1}[B]$ such that $\operatorname{diam}A_n\le\epsilon_n$ for every n; as $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $f^{-1}[B]$ has strong measure zero. **Q**

Of course f is not a bijection, so it is not in itself an $\mathcal{S}\mathfrak{mz}$-equivalence. But if we set

$$D_1=\{x:x\in\{0,1\}^{\mathbb{N}},\,x\text{ is eventually constant}\},$$

$$D_2=\{2^{-k}i:k\in\mathbb{N},\,i\le2^k\},$$

then $D_1\subseteq\{0,1\}^{\mathbb{N}}$ and $D_2\subseteq[0,1]$ are countably infinite, and $f\restriction\{0,1\}^{\mathbb{N}}\setminus D_1$ is an $\mathcal{S}\mathfrak{mz}$-equivalence between $\{0,1\}^{\mathbb{N}}\setminus D_1$ and $[0,1]\setminus D_2$. Putting this together with any bijection between D_1 and D_2, we have an $\mathcal{S}\mathfrak{mz}$-equivalence between $\{0,1\}^{\mathbb{N}}$ and $[0,1]$.

(c)(i) I show by induction on r that $[0,1]^r$ is $\mathcal{S}\mathfrak{mz}$-equivalent to $\mathbb{R}$ and therefore to $[0,1]$. The case $r=1$ is covered by (a). For the inductive step to $r\ge2$, I adapt the method of (b). Give $\{0,1\}^{\mathbb{N}\times r}$ the metric ρ defined by setting

$$\rho(x,y)=\inf\{2^{-n}:n\in\mathbb{N},\,x\restriction(n\times r)=y\restriction(n\times r)\}$$

for x, $y\in\{0,1\}^{\mathbb{N}\times r}$. Define $f:\{0,1\}^{\mathbb{N}\times r}\to[0,1]^r$ by setting

$$f(x)=\langle\textstyle\sum_{i=0}^{\infty}2^{-i-1}x(i,j)\rangle_{j<r}$$

for $x\in\{0,1\}^{\mathbb{N}\times r}$. Then f is uniformly continuous, so $f[A]$ has strong measure zero in $[0,1]^r$ whenever A has strong measure zero in $\{0,1\}^{\mathbb{N}\times r}$. Moreover, we find once again that $f^{-1}[B]$ has strong measure zero whenever $B\subseteq[0,1]^r$ has strong measure zero. **P** Let $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers. This time, set $m=2^r$ and let $\langle B_n\rangle_{n\in\mathbb{N}}$ be a cover of B such that $\operatorname{diam}B_n<\frac{1}{2}\min(1,\inf_{mn\le i<mn+m}\epsilon_i)$ for every n. (For definiteness, let me say that I am giving $[0,1]^r$ its Euclidean metric.) In this case, if $2^{-k-1}\le\operatorname{diam}B_n<2^{-k}$, B_n can meet at most 2^r intervals of the form $[2^{-k}\boldsymbol{n},2^{-k}(\boldsymbol{n}+\mathbf{1})]$ where $\boldsymbol{n}\in\mathbb{N}^r$ and $\mathbf{1}=(1,\dots,1)$. So $f^{-1}[B_n]$ can meet at most $2^r=m$ sets of the form $\{x:x\restriction(k\times r)=z\}$, and can be covered by m sets $\langle A_j\rangle_{mn\le j<mn+m}$ where

$$\operatorname{diam}A_j\le2^{-k}\le2\operatorname{diam}B_n\le\epsilon_j$$

for every j. Putting these together, we have a cover $\langle A_j\rangle_{j\in\mathbb{N}}$ of $f^{-1}[B]$ such that $\operatorname{diam}A_j\le\epsilon_j$ for every j; as $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $f^{-1}[B]$ has strong measure zero. **Q**

The function f here is very far from being one-to-one. But if we set

$$D_1^*=\textstyle\bigcup_{j<r}\{x:x\in\{0,1\}^{\mathbb{N}\times r},\,\langle x(i,j)\rangle_{i\in\mathbb{N}}\in D_1\},$$

$$D_2^*=\textstyle\bigcup_{j<r}\{z:z\in[0,1]^r,\,z(j)\in D_2\},$$

where $D_1\subseteq\{0,1\}^{\mathbb{N}}$, $D_2\subseteq[0,1]$ are defined as in the proof of (b), then f is a bijection between $\{0,1\}^{\mathbb{N}\times r}\setminus D_1^*$ and $[0,1]^r\setminus D_2^*$, so is an $\mathcal{S}\mathfrak{mz}$-equivalence between these. Accordingly $[0,1]^r\setminus D_2^*$ is $\mathcal{S}\mathfrak{mz}$-embeddable in $\{0,1\}^{\mathbb{N}\times r}$, which is homeomorphic, therefore uniformly equivalent, to $\{0,1\}^{\mathbb{N}}$, which is in turn $\mathcal{S}\mathfrak{mz}$-equivalent to $]0,1[$; so $[0,1]^r\setminus D_2^*$ is $\mathcal{S}\mathfrak{mz}$-embeddable in $]0,1[$.

Now consider D_2^*. This is a countable union of sets which are isometric, therefore $\mathcal{S}\mathfrak{mz}$-equivalent, to $[0,1]^{r-1}$ and therefore to $]0,1[$, by the inductive hypothesis. We can therefore express D_2^* as $\bigcup_{n\in\mathbb{N}}X_n$ where $\langle X_n\rangle_{n\in\mathbb{N}}$ is disjoint and every X_n is $\mathcal{S}\mathfrak{mz}$-embeddable in $]0,1[$ and therefore in $]n+1,n+2[$. Assembling these with the $\mathcal{S}\mathfrak{mz}$-equivalence between $[0,1]^r \setminus D_2^*$ and $]0,1[$ we have already found, we have a $\mathcal{S}\mathfrak{mz}$-embedding from $[0,1]^r$ to $\mathbb{R}$. In the other direction, we certainly have an isometric embedding of $[0,1]$ in $[0,1]^r$ and therefore a $\mathcal{S}\mathfrak{mz}$-embedding of $\mathbb{R}$ in $[0,1]^r$; so $\mathbb{R}$ and $[0,1]^r$ are $\mathcal{S}\mathfrak{mz}$-equivalent. Thus the induction proceeds.

(ii) As for $\mathbb{R}^r$, we have a homeomorphism between $\mathbb{R}^r$ and $]0,1[^r$, which (because these again are σ-compact) is a $\mathcal{S}\mathfrak{mz}$-equivalence and therefore a $\mathcal{S}\mathfrak{mz}$-embedding of $\mathbb{R}^r$ in $[0,1]^r$. So 534Mb, once more, tells us that $\mathbb{R}^r$ and $[0,1]^r$ and $[0,1]$ are $\mathcal{S}\mathfrak{mz}$-equivalent.

(d) So $\mathbb{R}^r$, $[0,1]^r$ and $\{0,1\}^{\mathbb{N}}$ are $\mathcal{S}\mathfrak{mz}$-equivalent, for their usual uniformities. Now 534Fd tells us that they are also $\mathcal{R}\mathfrak{bg}$-equivalent.

534O Large sets with strong measure zero It is a remarkable fact that it is relatively consistent with ZFC to suppose that the only subsets of $\mathbb{R}$ with strong measure zero are the countable sets (Laver 76, Ihoda 88 or Bartoszyński & Judah 95, §8.3). We therefore find ourselves investigating constructions of non-trivial sets with strong measure zero under special axioms. I start by clearing the ground a little.

Proposition (a) If (X,ρ) is a separable metric space and $A \subseteq X$ has cardinal less than $\mathfrak{c}$, there is a Lipschitz function $f : X \to \mathbb{R}$ such that $f{\restriction}A$ is injective.

(b) (Carlson 93) For any cardinal $\kappa < \mathfrak{c}$, if there is any separable metric space with a set of size κ which is of strong measure zero, then there is a subset of $\mathbb{R}$ of size κ which has strong measure zero.

(c)(i) If $\mathrm{cf}(\mathfrak{m}_{\mathrm{countable}}) = \mathfrak{b}$ there is a subset of $\mathbb{R}$ of size $\mathfrak{m}_{\mathrm{countable}}$ which has strong measure zero.

(ii) If $\mathfrak{m}_{\mathrm{countable}} = \mathfrak{d}$ there is a subset of $\mathbb{R}$ of size $\mathfrak{m}_{\mathrm{countable}}$ which has strong measure zero.

(iii) (Rothberger 41) If $\mathfrak{b} = \omega_1$ there is a subset of $\mathbb{R}$ of size ω_1 which has strong measure zero.

proof (a) If $X = \emptyset$ this is trivial. Otherwise, let $\langle x_n\rangle_{n\in\mathbb{N}}$ run over a dense sequence in X, and for $x \in X$ define $g_x : \mathbb{R} \to \mathbb{R}$ by setting

$$g_x(t) = \sum_{n=0}^{\infty} \frac{\min(1,\rho(x,x_n))}{n!} t^n$$

for $t \in \mathbb{R}$. Then g_x is a real-entire function (5A5A). If x, $y \in X$ are distinct, then there must be some n such that $\min(1,\rho(x,x_n)) \ne \min(1,\rho(y,x_n))$, so that one of the coefficients of $g_x - g_y$ is non-zero, and $\{t : g_x(t) = g_y(t)\}$ is countable (5A5A). So if $A \subseteq X$ and $\#(A) < \mathfrak{c}$, we can find a $t \ge 0$ such that $g_x(t) \ne g_y(t)$ for all distinct x, $y \in A$. Set $f(x) = g_x(t)$ for $x \in X$; then $f : X \to \mathbb{R}$ is a function such that $f{\restriction}A$ is injective. If x, $y \in X$ then

$$\begin{aligned} |f(x) - f(y)| &= |\sum_{n=0}^{\infty}(\min(1,\rho(x,x_n)) - \min(1,\rho(y,y_n))\frac{t^n}{n!}| \\ &\le \exp(t)\sup_{n\in\mathbb{N}}|\rho(x,x_n) - \rho(y,y_n)| \le \exp(t)\rho(x,y), \end{aligned}$$

so that f is Lipschitz.

(b) Let (X,ρ) be a separable metric space with a set $A \in [X]^\kappa$ of strong measure zero. Then (a) tells us that we have a uniformly continuous function $f : X \to \mathbb{R}$ which is injective on A, so that $f[A] \in [\mathbb{R}]^\kappa$ has strong measure zero (534Db).

(c)(i) Let $\langle x_\xi\rangle_{\xi<\mathfrak{b}}$ be a family in $\mathbb{N}^{\mathbb{N}}$ which is increasing and unbounded for the pre-order $\le^*$ of 522C(i). Let $C \subseteq \mathfrak{m}_{\mathrm{countable}}$ be a closed cofinal set with cardinal $\mathfrak{b}$ (5A1Ad), and $\langle\zeta_\xi\rangle_{\xi<\mathfrak{b}}$ the increasing enumeration of C; let $\langle y_\eta\rangle_{\eta<\mathfrak{m}_{\mathrm{countable}}}$ be a family of distinct elements of $\mathbb{N}^{\mathbb{N}}$ such that $y_\eta \ge x_\xi$ whenever $\xi < \mathfrak{b}$ and $\zeta_\xi \le \eta < \zeta_{\xi+1}$.

If $K \subseteq \mathbb{N}^{\mathbb{N}}$ is compact, then $\{\eta : y_\eta \in K\}$ has cardinal strictly less than $\mathfrak{m}_{\mathrm{countable}}$. **P** Set $x(n) = \sup_{y\in K} y(n)$ for each $n \in \mathbb{N}$ (I pass over the trivial case $K = \emptyset$). Then there is a $\xi < \mathfrak{b}$ such that $x_\xi \not\le^* x$. If $\zeta_\xi \le \eta < \mathfrak{m}_{\mathrm{countable}}$, there is a $\xi' \ge \xi$ such that $\zeta_{\xi'} \le \eta < \zeta_{\xi'+1}$ (this is where we need to know that C is closed), and now

$$y_\eta \ge x_{\xi'} \ge^* x_\xi, \quad y_\eta \not\le x, \quad y_\eta \notin K.$$

So $\{\eta : y_\eta \in K\} \subseteq \zeta_\xi$ has cardinal less than $\mathfrak{m}_{\mathrm{countable}}$. **Q**

Let $f : \mathbb{N}^{\mathbb{N}} \to [0,1] \setminus \mathbb{Q}$ be any homeomorphism (4A2Ub), and consider $A = \{f(y_\eta) : \eta < \mathfrak{m}_{\mathrm{countable}}\}$. Then $\#(A) = \mathfrak{m}_{\mathrm{countable}}$. Also A has strong measure zero. **P** Of course $\mathbb{Q}$, being countable, has strong measure zero. Let $G \subseteq \mathbb{R}$ be an open set including $\mathbb{Q}$. Then $[0,1] \setminus G$ and $K = f^{-1}[\,[0,1] \setminus G]$ are compact. Now $\#(A \setminus G) = \#(\{\eta : y_\eta \in K\}) < \mathfrak{m}_{\mathrm{countable}}$, so $A \setminus G$ has Rothberger's property (534Ia) and strong measure zero (534Fc). By 534Dd, this is enough to show that A has strong measure zero. **Q**

Thus we have a set of the required kind.

(ii) The argument is similar. This time, let $\langle x_\xi\rangle_{\xi<\mathfrak{d}}$ be a cofinal family in $\mathbb{N}^{\mathbb{N}}$. For each $\xi<\mathfrak{d}$, let $y_\xi\in\mathbb{N}^{\mathbb{N}}$ be such that $y_\xi\ge x_\xi$ and $y_\xi\not\le x_\eta$ for any $\eta<\xi$. Again, if $K\subseteq\mathbb{N}^{\mathbb{N}}$ is compact, then $\{\eta:y_\eta\in K\}$ has cardinal strictly less than $\mathfrak{m}_{\text{countable}}$. **P** Taking $x=\sup K$ as before, there is a $\xi<\mathfrak{d}=\mathfrak{m}_{\text{countable}}$ such that $x\le x_\xi$; now for any $\eta>\xi$ we know that $y_\eta\not\le x_\xi$ so $y_\eta\notin K$. **Q** The rest of the proof proceeds as before. (The set $\{y_\eta:\eta<\mathfrak{d}\}$ has cardinal $\mathfrak{d}$ because it is cofinal with $\mathbb{N}^{\mathbb{N}}$.)

(iii) If $\mathfrak{m}_{\text{countable}}=\omega_1$, this is immediate from (i); otherwise it is a consequence of 534Ia and 534Fc, as used in the arguments above.

534P Subject to the continuum hypothesis we have many ways of building sets with strong measure zero, in addition to that in 534Oc. I give one example to suggest what can be done with a weak form of Martin's axiom.

Example Suppose that $\mathfrak{m}_{\text{countable}}=\mathfrak{c}$. Then there is a set $A\subseteq\mathbb{R}\setminus\mathbb{Q}$, with Rothberger's property in $\mathbb{R}$, such that

(i) $A+A=\mathbb{R}$,

(ii) $A\times A\notin\mathcal{R}\mathsf{bg}(\mathbb{R}^2)$,

(iii) there is a continuous surjection from A onto $\mathbb{R}$,

(iv) $A\notin\mathcal{R}\mathsf{bg}(\mathbb{R}\setminus\mathbb{Q})$.

proof (a) For $x\in\mathbb{N}^{\mathbb{N}}$, define $\psi(x)\in\{0,1\}^{\mathbb{N}}$ by setting $\psi(x)(n)=0$ if $x(n)$ is even, 1 if $x(n)$ is odd. Then $\psi:\mathbb{N}^{\mathbb{N}}\to\{0,1\}^{\mathbb{N}}$ is a continuous surjection. Let $\phi:\mathbb{N}^{\mathbb{N}}\to[0,1]\setminus\mathbb{Q}$ be a homeomorphism (4A2Ub again). Enumerate $\mathbb{N}^{\mathbb{N}}$ as $\langle x_\xi\rangle_{\xi<\mathfrak{c}}$ and $\mathbb{R}$ as $\langle t_\xi\rangle_{\xi<\mathfrak{c}}$ and $\mathbb{Q}$ as $\langle q_n\rangle_{n\in\mathbb{N}}$. For $\xi\le\mathfrak{C}$, set $K_\xi=\{x:x\in\mathbb{N}^{\mathbb{N}},\,x\le x_\xi\}$, so that K_ξ is compact, and $\phi[K_\eta]$ is a compact subset of $[0,1]\setminus\mathbb{Q}$, therefore nowhere dense in $\mathbb{R}$. Write $\mathcal{M}$ for the ideal of meager subsets of $\mathbb{R}$, as in §522.

Choose $\langle a_\xi\rangle_{\xi<\mathfrak{c}}$, $\langle b_\xi\rangle_{\xi<\mathfrak{c}}$ and $\langle c_\xi\rangle_{\xi<\mathfrak{c}}$ as follows. For each $\xi<\mathfrak{c}$, $\{x_\eta:\eta\le\xi\}$ is not cofinal with $\mathbb{N}^{\mathbb{N}}$, because

$$\operatorname{cf}\mathbb{N}^{\mathbb{N}}=\mathfrak{d}\ge\operatorname{cov}\mathcal{M}=\mathfrak{m}_{\text{countable}}=\mathfrak{c}$$

(522I, 522Sa), so we can find a $y_\xi\in\mathbb{N}^{\mathbb{N}}$ such that $y_\xi\not\le x_\eta$ for any $\eta\le\xi$; raising y_ξ if need be, we can arrange that $\psi(y_\xi)=\psi(x_\xi)$. Set $a_\xi=\phi(y_\xi)$. Consider

$$\mathcal{E}_\xi=\{\phi[K_\eta]+\mathbb{Z}:\eta\le\xi\}\cup\{t_\xi-\phi[K_\eta]+\mathbb{Z}:\eta\le\xi\}\cup\{\mathbb{Q}\}\cup\{t_\xi-\mathbb{Q}\}.$$

This is a family of fewer than $\mathfrak{c}=\mathfrak{m}_{\text{countable}}$ meager subsets of $\mathbb{R}$, so does not cover $\mathbb{R}$ (522Sa once more). Take any $b_\xi\in\mathbb{R}\setminus\bigcup\mathcal{E}_\xi$; then neither b_ξ nor $c_\xi=t_\xi-b_\xi$ belongs to $\mathbb{Q}\cup\bigcup_{\eta\le\xi}(\mathbb{Z}+\phi[K_\eta])$.

(b) At the end of the process, set

$$A_0=\{a_\xi:\xi<\mathfrak{c}\}\cup\{b_\xi:\xi<\mathfrak{c}\}\cup\{c_\xi:\xi<\mathfrak{c}\},\qquad A=A_0+\mathbb{Z}.$$

Then $A\cap[0,1]$ has strong measure zero. **P** Let $G\subseteq\mathbb{R}$ be any open set including $\mathbb{Q}$. Then $[0,1]\setminus G$ is a compact subset of $[0,1]\setminus\mathbb{Q}$, and $K=\phi^{-1}[[0,1]\setminus G]$ is a compact subset of $\mathbb{N}^{\mathbb{N}}$. There is therefore some $\xi<\mathfrak{c}$ such that $K\subseteq\{x:x\le x_\xi\}$ and $[0,1]\setminus G\subseteq\phi[K_\xi]$. Now if $\eta\ge\xi$, neither b_η nor c_η belongs to $\phi[K_\xi]+\mathbb{Z}$; and also $a_\eta\in\,]0,1[\,\setminus\phi[K_\xi]$, so a_η also does not belong to $\phi[K_\xi]+\mathbb{Z}$. What this means is that

$$A\cap[0,1]\setminus G\subseteq(\{a_\eta:\eta\le\xi\}\cup\{b_\eta:\eta\le\xi\}\cup\{c_\eta:\eta\le\xi\})+\mathbb{Z}$$

has cardinal less than $\mathfrak{c}=\mathfrak{m}_{\text{countable}}$. By 534Ia and 534Fc, it has strong measure zero. As G is arbitrary, $A\cap[0,1]$ has strong measure zero, by 534Dd. **Q**

Because $A+\mathbb{Z}=A$,

$$A=\bigcup_{n\in\mathbb{Z}}(A\cap[0,1])+n$$

also has strong measure zero, therefore has Rothberger's property (534Fd).

(c) Let us consider the properties (i)-(iv). Because no a_ξ, b_ξ or c_ξ belongs to $\mathbb{Q}$, $A\cap\mathbb{Q}=\emptyset$. For every $\xi<\mathfrak{c}$, $t_\xi=b_\xi+c_\xi\in A+A$; as $\langle t_\xi\rangle_{\xi<\mathfrak{c}}$ is an enumeration of $\mathbb{R}$, $A+A=\mathbb{R}$. Since $+:\mathbb{R}^2\to\mathbb{R}$ is continuous, and $+[A\times A]=A+A$ does not have Rothberger's property, nor does $A\times A$ (534Fb). Thus we have (i) and (ii).

Let $h:\{0,1\}^{\mathbb{N}}\to[0,1]$ be a continuous surjection, and let $f:\mathbb{R}\setminus\mathbb{Q}\to\mathbb{R}$ be the continuous function defined by setting $f(a)=h\psi\phi^{-1}(a-n)+n$ if $a\in\,]n,n+1[\,\setminus\mathbb{Q}$ where $n\in\mathbb{Z}$. For every $\xi<\mathfrak{c}$ such that $x_\xi\in\{0,1\}^{\mathbb{N}}$,

$$f(a_\xi)=h\psi\phi^{-1}(a_\xi)=h(x_\xi);$$

because $\{x_\xi:\xi<\mathfrak{c}\}=\mathbb{N}^{\mathbb{N}}\supseteq\{0,1\}^{\mathbb{N}}$, $f[A]=[0,1]$.

Thus A satisfies (iii). As for (iv), define a metric ρ on $\mathbb{R}\setminus\mathbb{Q}$ by setting $\rho(s,t)=|s-t|+|f(s)-f(t)|$ for s, $t\in\mathbb{R}\setminus\mathbb{Q}$. Because f is continuous, this defines the usual topology of $\mathbb{R}\setminus\mathbb{Q}$. But f is uniformly continuous for ρ and the usual metric of $\mathbb{R}$, and $f[A]$ does not have strong measure zero, so A cannot have strong measure zero for the metric ρ (534Db), and does not have Rothberger's property in $\mathbb{R}\setminus\mathbb{Q}$ (534Fc). This completes the proof.

534X Basic exercises (a)(i) Let (X,ρ) be a metric space, $r>0$ and $A\subseteq X$ a set with strong measure zero. Show that A has zero Hausdorff r-dimensional measure. (ii) Find a subset of $\mathbb{R}^2$ which is universally negligible but does not have strong measure zero (for the usual metric on $\mathbb{R}^2$). (*Hint*: 439H.) (iii) Find a subset of $\{0,1\}^{\mathbb{N}}$ which is universally negligible but does not have strong measure zero for the metric of 534Ib.

(b) Let r, $s\ge 1$ be integers. Let $A\subseteq\mathbb{R}^r$ be a set with strong measure zero, and $f:A\to\mathbb{R}^s$ a function which is differentiable relative to its domain at every point of A. Show that $f[A]$ has strong measure zero. (*Hint*: 262N.)

(c) Let $(X,\mathcal{W})$ be a Hausdorff uniform space with strong measure zero. Show that X is universally negligible iff it is a Radon space.

(d) Give ω_1+1 its order topology and the corresponding uniformity (4A2Jg). Show that it has strong measure zero but is not universally negligible.

(e)(i) Show that a topological space which has Rothberger's property in itself must be Lindelöf. (ii) Give $X=\omega_1+1$ its order topology. Show that ω_1 has Rothberger's property in ω_1+1 but not in itself.

(f) Let X be a topological space and $A\subseteq X$. Show that $A\in\mathcal{R}\mathfrak{bg}(X)$ iff $A\in\mathcal{R}\mathfrak{bg}(\overline{A})$.

(g) Let X be a σ-compact topological space which is either Hausdorff or regular, and $A\subseteq X$. Show that $A\in\mathcal{R}\mathfrak{bg}(X)$ iff for every sequence $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ of finite open covers of X, there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$, covering A, such that $G_n\in\mathcal{G}_n$ for every n.

(h) Let X be a topological space and $A\in\mathcal{R}\mathfrak{bg}(X)$. Suppose that $B\subseteq X$ is such that $B\setminus G\in\mathcal{R}\mathfrak{bg}(X)$ for every open subset G of X including A. Show that $B\in\mathcal{R}\mathfrak{bg}(X)$.

(i) Let X be a paracompact Hausdorff space, and A a subset of X. Show that the following are equiveridical: (i) $A\in\mathcal{R}\mathfrak{bg}(X)$; (ii) $f[A]\in\mathcal{R}\mathfrak{bg}(Y)$ whenever Y is a metrizable space and $f:X\to Y$ is continuous; (iii) $f[A]\in\mathcal{S}\mathfrak{mz}(Y,\rho)$ whenever (Y,ρ) is a metric space and $f:X\to Y$ is continuous; (iv) $A\in\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ whenever $\mathcal{W}$ is a uniformity on X inducing the topology of X. (*Hint*: 5A4Fb.)

(j)(i) Let X be a Hausdorff topological space. Show that if $A\in\mathcal{R}\mathfrak{bg}(X)$ then A is universally τ-negligible (definition: 439Xh). (ii) Let X be a Hausdorff uniform space. Show that if X has strong measure zero then it is universally τ-negligible.

(k) Let X be a locally compact Hausdorff topological group. Show that a subset of X has Rothberger's property in X iff it has strong measure zero for the right uniformity of X iff it has strong measure zero for the bilateral uniformity of X.

(l) Show that $\operatorname{non}\mathcal{R}\mathfrak{bg}(\mathbb{N}^{\mathbb{N}})=\mathfrak{m}_{\text{countable}}$.

(m) Let $(X,\mathcal{W})$ be a Lindelöf uniform space. Show that there is some κ such that $\mathcal{S}\mathfrak{mz}(X,\mathcal{W})\preccurlyeq_{\mathrm{T}}\mathcal{N}^\kappa$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.

(n) Show that every separable metric space (X,ρ) is uniformly equivalent to a subspace of $[0,1]^{\mathbb{N}}$ and is therefore $\mathcal{S}\mathfrak{mz}$-embeddable in $[0,1]^{\mathbb{N}}$.

(o) Suppose that $\mathfrak{d}=\omega_1$. Show that there is a set $A\subseteq\mathbb{R}\setminus\mathbb{Q}$ such that A has Rothberger's property in $\mathbb{R}$ but not in $\mathbb{R}\setminus\mathbb{Q}$.

(p) Let A be the set constructed in 534P on the assumption that $\mathfrak{m}_{\text{countable}}=\mathfrak{c}$. Show that $A\cup\mathbb{Q}$ has Rothberger's property in itself, but A does not have Rothberger's property in itself.

534Y Further exercises (a) Let (X,ρ) be an analytic metric space and μ_{Hr} Hausdorff r-dimensional measure on X, where $r>0$; suppose that $\mu_{Hr}X>0$. Let $\mathcal{I}$ be the σ-ideal of subsets of X generated by $\{A:\mu^*_{Hr}A<\infty\}$. Show that

$$\begin{aligned}\operatorname{non}\mathcal{N}(\mu_{Hr})=\min(\operatorname{non}\mathcal{N},\operatorname{non}\mathcal{I})&=\operatorname{non}\mathcal{N}\text{ if }\mu_{Hr}\text{ is }\sigma\text{-finite},\\&=\operatorname{non}\mathcal{I}\text{ otherwise.}\end{aligned}$$

(b) (i) Set $\mathcal{I}=\{[4^{-m}i,4^{-m}(i+1)[\,:m\in\mathbb{N},\,i\in\mathbb{Z}\}$. For $A\subseteq\mathbb{R}$ set $\theta(A)=\inf\{\sum_{I\in\mathcal{I}'}\sqrt{\operatorname{diam}I}:\mathcal{I}'\subseteq\mathcal{I}$ covers $A\}$. Show that if $\mu^{(1)}_{H,1/2}$ is Hausdorff $\frac{1}{2}$-dimensional measure on $\mathbb{R}$, then $\mu^{(1)}_{H,1/2}(A)=0$ iff $\theta(A)=0$. (ii) Set $\mathcal{J}=\{[2^{-m}i,2^{-m}(i+1)[\times[2^{-m}j,2^{-m}(j+1)[\,:m\in\mathbb{N},\,i,\,j\in\mathbb{Z}\}$, and for $A\subseteq\mathbb{R}^2$ set $\theta'(A)=\inf\{\sum_{J\in\mathcal{J}'}\operatorname{diam}J:\mathcal{J}'\subseteq\mathcal{J}$ covers $A\}$. Show that if $\mu^{(2)}_{H1}$ is Hausdorff 1-dimensional measure on $\mathbb{R}^2$, then $\mu^{(2)}_{H1}(A)=0$ iff $\theta'(A)=0$. (iii) Show that the null ideals $\mathcal{N}(\mu^{(1)}_{H,1/2})$ and $\mathcal{N}(\mu^{(2)}_{H1})$ are isomorphic.

(c) Show that if *either* $\operatorname{non}\mathcal{N}=\operatorname{cf}\mathcal{N}$ *or* $\operatorname{non}\mathcal{N}<\operatorname{cov}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$, then Hausdorff one-dimensional measure on $\mathbb{R}^2$ does not have the measurable envelope property.

(d) Let X be a Hausdorff space. Show that a compact subset of X has Rothberger's property in X iff it is scattered.

(e) Suppose that $\mathfrak{m}_{\text{countable}}=\mathfrak{c}$. Let X be the group of all permutations of $\mathbb{N}$, regarded as the isometry group of $\mathbb{N}$ with its $\{0,1\}$-valued metric, so that X is a Polish group (441Xp-441Xq). Show that there is a subset A of X such that A has strong measure zero for the right uniformity of X but A^{-1} does not.

(f) Let $\mathfrak{G}$ be a collection of families of sets. Let us say that a set A has the **$\mathfrak{G}$-Rothberger property** if for every sequence $\langle\mathcal{G}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{G}$ there is a cover $\langle G_n\rangle_{n\in\mathbb{N}}$ of A such that $G_n\in\mathcal{G}_n$ for every $n\in\mathbb{N}$. (i) Show that the family $\mathcal{I}$ of sets with the $\mathfrak{G}$-Rothberger property is a σ-ideal of sets containing every countable subset of $\bigcap_{\mathcal{G}\in\mathfrak{G}}\bigcup\mathcal{G}$. (ii) Show that if $\mathfrak{H}$ is another collection of families of sets, and f is a function such that for every $\mathcal{H}\in\mathfrak{H}$ there is a member of $\mathfrak{G}$ refining $\{f^{-1}[H]:H\in\mathcal{H}\}$, then $f[A]$ has the $\mathfrak{H}$-Rothberger property whenever $A\in\mathcal{I}$. (iii) Suppose that $\mathfrak{G}$ is a collection of families of open subsets of a topological space X, that $A\in\mathcal{I}$ has the $\mathfrak{G}$-Rothberger property, and that $B\subseteq X$ is such that $B\setminus G\in\mathcal{I}$ for every open set $G\supseteq A$. Show that $B\in\mathcal{I}$. (iv) Suppose that $X=\bigcup\mathcal{G}$ for every $\mathcal{G}\in\mathfrak{G}$, and that every member of $\mathfrak{G}$ is countable. Show that $\operatorname{non}(\mathcal{I},X)\ge\mathfrak{m}_{\text{countable}}$.

(g) Let X be a Lindelöf topological space. Show that $\operatorname{add}\mathcal{R}\mathsf{bg}(X)\ge\operatorname{add}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$.

534Z Problems (a) Let $\mu^{(2)}_{H1}$ be one-dimensional Hausdorff measure on $\mathbb{R}^2$. Is the covering number $\operatorname{cov}\mathcal{N}(\mu^{(2)}_{H1})$ necessarily equal to $\operatorname{cov}\mathcal{N}$? As observed in 534Bc-534Bd, we have $\operatorname{cov}\mathcal{N}\le\operatorname{cov}\mathcal{N}(\mu^{(2)}_{H1})\le\operatorname{non}\mathcal{M}$. We can ask the same question for r-dimensional Hausdorff measure on $\mathbb{R}^n$ whenever $0<r<n$; in particular, for r-dimensional Hausdorff measure on $[0,1]$, where $0<r<1$, and these questions are strongly connected (534Yb). Shelah & Steprāns 05 show that $\operatorname{non}\mathcal{N}(\mu^{(2)}_{H1})$ can be less than $\operatorname{non}\mathcal{N}$; of course this is possible only because $\mu^{(2)}_{H1}$ is not semi-finite (439H, 521Xg).

(b) Can $\operatorname{cf}\mathcal{R}\mathsf{bg}(\mathbb{R})$ be ω_1?

(c) How many types of Polish spaces under $\mathcal{S}\mathfrak{mz}$-equivalence can there be? If we give $\mathbb{N}^{\mathbb{N}}$ the metric of 534Ib, can it fail to be $\mathcal{S}\mathfrak{mz}$-equivalent to $[0,1]^{\mathbb{N}}$?

(d) Suppose that there is a separable metric space of size $\mathfrak{c}$ with strong measure zero. Must there be a subset of $\mathbb{R}$ of size $\mathfrak{c}$ with strong measure zero in $\mathbb{R}$?

534 Notes and comments I have very little to say about Hausdorff measures, and 534B is here only because it would seem even lonelier in a section by itself. All I have tried to do is to run through the obvious questions connecting §471 with Chapter 52. But at the next level there is surely much more to be done (534Za).

'Strong measure zero' has attracted a great deal of attention, starting with the work of E.Borel, who suggested that every subset of $\mathbb{R}$ with strong measure zero must be countable; this is the **Borel conjecture**. It turns out that this is undecidable in ZFC, and that if the Borel conjecture is true then there are no uncountable sets of strong measure zero in any separable metric space (534O). So we have some questions of a new kind: in the ideals $\mathcal{S}\mathfrak{mz}(X,\mathcal{W})$ of sets of strong measure zero, in addition to the standard cardinals add, non, cov and cf, we find ourselves asking for the possible cardinals of sets belonging to the ideal.

The next point is that strong measure zero is not (or rather, not always) either a topological property or a metric property; it is a property of uniform spaces. We must therefore be prepared to examine uniformities, even if we are happy to stay with metrizable ones. In 534P(iv), using an axiom which is a consequence of the continuum hypothesis, I show that we can have a set which has strong measure zero for one of two equivalent metrics and not for the other. Goldstern Judah & Shelah 93 describe a model in which the ideal $\mathcal{R}\mathfrak{bg}(\mathbb{R})$ has $\operatorname{add}\mathcal{R}\mathfrak{bg}(\mathbb{R})=\operatorname{non}\mathcal{R}\mathfrak{bg}(\mathbb{R})=\omega_2$ while $\mathfrak{m}_{\text{countable}}=\omega_1$. So in this case 534Ib tells us that $\mathbb{N}^{\mathbb{N}}$, with the metric described there, is not even $\mathcal{S}\mathfrak{mz}$-embeddable in $\mathbb{R}$. Of course in models of set theory in which the Borel conjecture is true we do have topologically determined structures, for trivial reasons.

Note that for any uncountable complete separable metric space X, there is a subset of X homeomorphic to $\{0,1\}^{\mathbb{N}}$ (423J), and the homeomorphism must be a uniform equivalence; so that $\{0,1\}^{\mathbb{N}}$ and its companions $[0,1]^r$, $\mathbb{R}^r$ must be $\mathcal{S}\mathfrak{mz}$-embeddable in X. In this sense they are the 'simplest' uncountable complete metric spaces. In the same sense, $[0,1]^{\mathbb{N}}$ is the most complex (534Xn).

For σ-compact spaces, strong measure zero becomes a topological property (534Fd, 534G), corresponding to what I call 'Rothberger's property' (534E). Rothberger 38b investigated subsets of $\mathbb{R}$ which have Rothberger's property in themselves, under the name 'property C$''$'. The ideas of 534D and 534J-534K can all be re-presented as theorems about Rothberger's property (534F, 534Xh, 534Xl, 534Yg); the machinery of 534Yf is supposed to suggest a reason for this. It is natural to be attracted to a topological concept, but there is a difficulty in that Rothberger's property is not hereditary in the usual way (534Xe, 534Xo, 534Xp). I note that while 534N is stated in terms of $\mathcal{R}\mathfrak{bg}$-equivalence, isomorphism of the ideals of sets with the appropriate Rothberger's property, the concept of strong measure zero seems to be necessary in the Schröder-Bernstein arguments based on 534M.

For a fuller discussion of strong measure zero in $\mathbb{R}$, see Bartoszyński & Judah 95, chap. 8, from which much of the material of this section is taken.

535 Liftings

I introduced the Lifting Theorem (§341) as one of the fundamental facts about complete strictly localizable measure spaces. Of course we can always complete a measure space and thereby in effect obtain a lifting for any σ-finite measure. For the applications of the Lifting Theorem in §§452-453 this procedure is natural and effective; and generally in this treatise I have taken the view that one should work with completed measures unless there is some strong reason not to. But I have also embraced the principle of maximal convenient generality, seeking formulations which will exhibit the full force of each idea in the context appropriate to that idea, uncluttered by the special features of intended applications. So the question of when, and why, liftings for incomplete measures can be found is one which automatically arises. It turns out to be a fruitful question, in the sense that it leads us to new arguments, even though the answers so far available are unsatisfying.

As usual, much of what we want to know depends on the behaviour of the usual measures on powers of $\{0,1\}$ (535B). An old argument relying on the continuum hypothesis shows that Lebesgue measure can have a Borel lifting; this has been usefully refined, and I give a strong version in 535D-535E. We know that we cannot expect to have translation-invariant Borel liftings (345F), but strong Borel liftings are possible (535H-535I), and in some cases can be built from Borel liftings (535J-535N).

For certain applications in functional analysis, we are more interested in liftings for L^∞ spaces than in liftings for measure algebras; and it is sometimes sufficient to have a 'linear lifting', not necessarily corresponding to a lifting in the strict sense (535O, 535P). I give a couple of paragraphs to linear liftings because in some ways they are easier to handle and it is conceivable that they are relevant to the main outstanding problem (535Zf).

535A Notation (a) The most interesting questions to be examined in this section can be phrased in the following language. If (X,Σ,μ) is a measure space and $\mathfrak{T}$ a topology on X, I will say that a **Borel lifting** of μ is a lifting which takes values in the Borel σ-algebra $\mathcal{B}(X)$ of X. (As usual, I will use the word 'lifting' indifferently for homomorphisms from Σ to itself, or from $\mathfrak{A}$ to Σ, where $\mathfrak{A}$ is the measure algebra of μ. Of course a homomorphism $\theta:\mathfrak{A}\to\Sigma$ is a Borel lifting iff the corresponding homomorphism $E\mapsto\theta E^\bullet:\Sigma\to\Sigma$ is a Borel lifting.) Similarly, a **Baire lifting** of μ is a lifting which takes values in the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$ of X.

(b) I remark at once that if $(X,\mathfrak{T},\Sigma,\mu)$ is a topological measure space and $\phi:\Sigma\to\mathcal{B}(X)$ is a Borel lifting for μ, then $\phi\restriction\mathcal{B}(X)$ is a lifting for the Borel measure $\mu\restriction\mathcal{B}(X)$. Conversely, if $\phi':\mathcal{B}(X)\to\mathcal{B}(X)$ is a lifting for $\mu\restriction\mathcal{B}(X)$, and if for every $E\in\Sigma$ there is a Borel set E' such that $E\triangle E'$ is negligible, then ϕ' extends uniquely to a Borel lifting ϕ of μ.

In the same way, any Baire lifting for a measure μ which measures every zero set will give us a lifting for $\mu\restriction\mathcal{B}\mathfrak{a}(X)$; and a lifting for $\mu\restriction\mathcal{B}\mathfrak{a}(X)$ will correspond to a Baire lifting for μ if, for instance, μ is completion regular, as in 535B below.

(c) As in Chapter 52, I will say that, for any set I, ν_I is the usual measure on $\{0,1\}^I$ and $\mathfrak{B}_I$ its measure algebra.

535B Proposition Let (X,Σ,μ) be a strictly localizable measure space with non-zero measure. Suppose that ν_κ has a Baire lifting (that is, $\nu_\kappa\restriction\mathcal{B}\mathfrak{a}(\{0,1\}^\kappa)$ has a lifting) for every infinite cardinal κ such that the Maharam-type-κ component of the measure algebra of μ is non-zero. Then μ has a lifting.

proof Write $(\mathfrak{A},\bar\mu)$ for the measure algebra of μ.

(a) Suppose first that μ is a Maharam-type-homogeneous probability measure. In this case $\mathfrak{A}$ is either $\{0,1\}$ or isomorphic to $\mathfrak{B}_\kappa$ for some infinite κ. The case $\mathfrak{A}=\{0,1\}$ is trivial, as we can set $\phi E=\emptyset$ if $E\in\Sigma$ is negligible, $\phi E=X$ if $E\in\Sigma$ is conegligible. Otherwise, $\mathfrak{A}$ is τ-generated by a stochastically independent family $\langle e_\xi\rangle_{\xi<\kappa}$ of elements of measure $\frac{1}{2}$. For each $\xi<\kappa$, choose $E_\xi\in\Sigma$ such that $E_\xi^\bullet=e_\xi$, and define $f:X\to\{0,1\}^\kappa$ by setting $f(x)(\xi)=\chi E_\xi(x)$ for $x\in X$ and $\xi<\kappa$. Then $\{F:F\subseteq\{0,1\}^\kappa,\ \nu F$ and $\mu f^{-1}[F]$ are defined and equal$\}$ is a Dynkin class containing all the measurable cylinders in $\{0,1\}^\kappa$, so includes $\mathcal{B}\mathfrak{a}_\kappa=\mathcal{B}\mathfrak{a}(\{0,1\}^\kappa)$, and f is inverse-measure-preserving for μ and $\nu'_\kappa=\nu_\kappa\restriction\mathcal{B}\mathfrak{a}_\kappa$. Note that $\mathfrak{B}_\kappa$ can be identified with the measure algebra of ν'_κ (put 415E and 322Da together, or see 415Xp). So we have an induced measure-preserving Boolean homomorphism $\pi:\mathfrak{B}_\kappa\to\mathfrak{A}$ defined by setting $\pi F^\bullet=f^{-1}[F]^\bullet$ for every $F\in\mathcal{B}\mathfrak{a}_\kappa$. Since $\pi[\mathfrak{B}_\kappa]$ is an order-closed subalgebra of $\mathfrak{A}$ (324Kb) containing every e_ξ, it is the whole of $\mathfrak{A}$.

We are supposing that there is a lifting $\theta:\mathfrak{B}_\kappa\to\mathcal{B}\mathfrak{a}_\kappa$ of ν_κ. Define $\theta_1:\mathfrak{A}\to\Sigma$ by setting $\theta_1 a=f^{-1}[\theta\pi^{-1}a]$ for every $a\in\mathfrak{A}$; then θ_1 is a Boolean homomorphism because θ and π^{-1} are, and

$$(\theta_1 a)^\bullet=\pi((\theta\pi^{-1}a)^\bullet)=\pi\pi^{-1}a=a$$

for every $a\in\mathfrak{A}$, so θ_1 is a lifting for μ.

(b) It follows at once that if μ is any non-zero totally finite Maharam-type-homogeneous measure, then it will have a lifting, as we can apply (a) to a scalar multiple of μ. Now consider the general case. Let $\mathcal{K}$ be the family of measurable subsets K of X such that the subspace measure μ_K is non-zero, totally finite and Maharam-type-homogeneous. Then μ is inner regular with respect to $\mathcal{K}$, by Maharam's theorem (332B). By 412Ia, there is a decomposition $\langle X_i\rangle_{i\in I}$ of X such that at most one X_i does not belong to $\mathcal{K}$, and that exceptional one, if any, is negligible; adding a trivial element $X_k=\emptyset$ if necessary, we may suppose that there is exactly one $k\in I$ such that $\mu X_k=\emptyset$. For each $i\in I\setminus\{k\}$, let μ_i be the subspace measure on X_i, and Σ_i its domain; then μ_i has a lifting $\phi_i:\Sigma_i\to\Sigma_i$. (The point is that if the Maharam type κ of μ_i is infinite, then the Maharam-type-κ component of $\mathfrak{A}$ includes $X_i^\bullet$ and is non-zero, so our hypothesis tells us that ν_κ has a Baire lifting.) At this point, recall that we are also supposing that $\mu X>0$, so there is some $j\in I\setminus\{k\}$; fix $z\in X_j$, and define $\phi:\Sigma\to\mathcal{P}X$ by setting

$$\begin{aligned}\phi E&=\bigcup_{i\in I\setminus\{k\}}\phi_i(E\cap X_i)\text{ if }z\notin\phi_j(E\cap X_j),\\&=X_k\cup\bigcup_{i\in I\setminus\{k\}}\phi_i(E\cap X_i)\text{ if }z\in\phi_j(E\cap X_j).\end{aligned}$$

Then ϕ is a lifting for μ. **P** It is a Boolean homomorphism because every ϕ_i is. If $E\in\Sigma$, then $X_i\cap\phi E=\phi_i(E\cap X_i)$ if $i\in I\setminus\{k\}$, and is either X_k or $\emptyset$ if $i=k$; in any case, it belongs to Σ_i; as $\langle X_i\rangle_{i\in I}$ is a decomposition for μ, $\phi E\in\Sigma$. Also

$$\mu(E\triangle\phi E)\le\mu X_k+\textstyle\sum_{i\in I\setminus\{k\}}\mu_i((E\cap X_i)\triangle\phi_i(E\cap X_i))=0.$$

Finally, if $\mu E=0$, then $\mu_i(E\cap X_i)=0$ and $\phi_i(E\cap X_i)=\emptyset$ for every $i\in I\setminus\{k\}$, so $\phi E=\emptyset$. **Q**

535C Proposition If λ and κ are cardinals with $\lambda=\lambda^\omega\le\kappa$, and ν_κ has a Baire lifting, then ν_λ has a Baire lifting.

proof If λ is finite, the result is trivial, so we may suppose that $\lambda \ge \omega$ (and therefore that $\lambda \ge \mathfrak{c}$). For $I \subseteq \kappa$, write $\mathcal{B}\mathfrak{a}_I$ for the Baire σ-algebra of $\{0,1\}^I$ and T_I for the family of those $E \in \mathcal{B}\mathfrak{a}_\kappa$ which are determined by coordinates in I. Set $\pi_I(x) = x{\restriction}I$ for every $x \in \{0,1\}^\kappa$; then $H \mapsto \pi_I^{-1}[H]$ is a Boolean isomorphism between $\mathcal{B}\mathfrak{a}_I$ and T_I, with inverse $E \mapsto \pi_I[E]$. **P** Because π_I is continuous, $\pi_I^{-1}[H] \in \mathcal{B}\mathfrak{a}_\kappa$ for every $H \in \mathcal{B}\mathfrak{a}_I$. Of course $H \mapsto \pi_I^{-1}[H]$ is a Boolean homomorphism, and it is injective because π_I is surjective. Identifying $\{0,1\}^\kappa$ with $\{0,1\}^I \times \{0,1\}^{\kappa\setminus I}$, we have a function $h : \{0,1\}^I \to \{0,1\}^\kappa$ defined by setting $h(v) = (v, \mathbf{0})$ for $v \in \{0,1\}^I$. This is continuous, therefore $(\mathcal{B}\mathfrak{a}_I, \mathcal{B}\mathfrak{a}_\kappa)$-measurable. If $E \in \mathrm{T}_I$, then $E = \pi_I^{-1}[\pi_I[E]] = \pi_I^{-1}[h^{-1}[E]]$; so $H \mapsto \pi_I^{-1}[H]$ is surjective and is an isomorphism. **Q**

Consequently $\#(\mathrm{T}_I) \le \mathfrak{c}$ for every countable $I \subseteq \kappa$ (4A1O, because $\mathcal{B}\mathfrak{a}_I$ is σ-generated by the cylinder sets, by 4A3Na). For any I, $\mathrm{T}_I = \bigcup_{J\in[I]^{\le\omega}} \mathrm{T}_J$, because every member of $\mathcal{B}\mathfrak{a}_I$ is determined by coordinates in a countable set (4A3Nb). So $\#(\mathrm{T}_I) \le \max(\mathfrak{c}, \#(I)^\omega) = \lambda$ whenever $I \subseteq \kappa$ and $\#(I) = \lambda$.

Let ϕ be a Baire lifting for ν_κ. Choose a non-decreasing family $\langle J_\xi\rangle_{\xi<\omega_1}$ in $[\kappa]^\lambda$ such that $J_0 = \lambda$ and $\phi E \in \mathrm{T}_{J_{\xi+1}}$ whenever $\xi < \omega_1$ and $E \in \mathrm{T}_{J_\xi}$. Set $J = \bigcup_{\xi<\omega_1} J_\xi$; then $\mathrm{T}_J = \bigcup_{\xi<\omega_1} \mathrm{T}_{J_\xi}$, so $\phi E \in \mathrm{T}_J$ for every $E \in \mathrm{T}_J$.

We therefore have a Boolean homomorphism $\phi_1 : \mathcal{B}\mathfrak{a}_J \to \mathcal{B}\mathfrak{a}_J$ defined by setting $\phi_1 H = \pi_J[\phi(\pi_J^{-1}[H])]$ for every $H \in \mathcal{B}\mathfrak{a}_J$. If $\nu_J H = 0$, then $\nu_\kappa \pi_J^{-1}[H] = 0$ and $\phi_1 H = \phi(\pi_J^{-1}[H]) = 0$. For any $H \in \mathcal{B}\mathfrak{a}_J$,

$$\pi_J^{-1}[H \triangle \phi_1 H] = \pi_J^{-1}[H] \triangle \phi(\pi_J^{-1}[H])$$

is ν_κ-negligible, so $H \triangle \phi_1 H$ is ν_J-negligible. Thus ϕ_1 is a lifting for $\nu_J{\restriction}\mathcal{B}\mathfrak{a}_J$. As $\nu_J{\restriction}\mathcal{B}\mathfrak{a}_J$ is isomorphic to $\nu_\lambda{\restriction}\mathcal{B}\mathfrak{a}_\lambda$, the latter also has a lifting. As ν_λ is completion regular (416U), the measure algebra of $\nu_\lambda{\restriction}\mathcal{B}\mathfrak{a}_\lambda$ can be identified with $\mathfrak{B}_\lambda$, and we can interpret a lifting for $\nu_\lambda{\restriction}\mathcal{B}\mathfrak{a}_\lambda$ as a Baire lifting for its completion ν_λ.

535D The following result covers most of the cases in which non-complete probability measures are known to have liftings.

Theorem Let (X, Σ, μ) be a measure space such that $\mu X > 0$, and suppose that its measure algebra is tightly ω_1-filtered (definition: 511Di). Then μ has a lifting.

proof This is a special case of 518L.

535E Proposition Suppose that $\mathfrak{c} \le \omega_2$ and the Freese-Nation number $\mathrm{FN}(\mathcal{P}\mathbb{N})$ is ω_1.

(a) If $\mathfrak{A}$ is a measurable algebra with cardinal at most ω_2, it is tightly ω_1-filtered.

(b) (Mokobodzki 7?) Let (X, Σ, μ) be a σ-finite measure space with non-zero measure and Maharam type at most ω_2.

(i) μ has a lifting.

(ii) If $\mathfrak{T}$ is a topology on X such that μ is inner regular with respect to the Borel sets, then μ has a Borel lifting.

(iii) If $\mathfrak{T}$ is a topology on X such that μ is inner regular with respect to the zero sets, then μ has a Baire lifting.

proof (a) By 524O(b-iii), $\mathrm{FN}(\mathfrak{A}) \le \omega_1$, so 518M gives the result.

(b)(i) By 514De, the measure algebra of μ has cardinal at most

$$\omega_2^\omega = \max(\mathfrak{c}, \omega_2) \le \omega_2$$

(5A1E(e-iii)). So we can put (a) and 535D together.

(ii) Because μ is σ-finite and inner regular with respect to the Borel sets, every measurable set can be expressed as the union of a Borel set and a negligible set. By (i), $\mu{\restriction}\mathcal{B}(X)$ has a lifting, which can be interpreted as a Borel lifting for μ, as in 535Ab.

(iii) As (ii), but with $\mathcal{B}\mathfrak{a}(X)$ in place of $\mathcal{B}(X)$.

535F Using the continuum hypothesis, we can go a little farther with ideas from 341J.

Proposition Let (X, Σ, μ) be a measure space such that $\mu X > 0$ and $\#(\mathfrak{A}) \le \omega_1$, where $\mathfrak{A}$ is the measure algebra of μ, and suppose that $\underline{\theta} : \mathfrak{A} \to \Sigma$ is such that

$$\underline{\theta}0 = \emptyset, \quad \underline{\theta}(a \cap b) = \underline{\theta}a \cap \underline{\theta}b \text{ for all } a,\, b \in \mathfrak{A}, \quad (\underline{\theta}a)^\bullet \subseteq a \text{ for every } a \in \mathfrak{A}.$$

Then μ has a lifting $\theta : \mathfrak{A} \to \Sigma$ such that $\theta E^\bullet \supseteq \underline{\theta}E$ for every $E \in \Sigma$.

proof (a) Adjusting $\underline{\theta}1$ if necessary, we can suppose that $\underline{\theta}1 = X$. Note that $\underline{\theta}a \subseteq \underline{\theta}b$ whenever $a \subseteq b$ in $\mathfrak{A}$. Let $\langle a_\xi\rangle_{\xi<\omega_1}$ be a family running over $\mathfrak{A}$, and for $\alpha \le \omega_1$ let $\mathfrak{C}_\alpha$ be the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \alpha\}$. Define

Boolean homomorphisms $\theta_\alpha : \mathfrak{C}_\alpha \to \Sigma$ inductively, as follows. The inductive hypothesis will be that $(\theta_\alpha c)^\bullet = c$ and $\theta_\alpha c \supseteq \underline{\theta} c$ for every $c \in \mathfrak{C}_\alpha$, while θ_α extends θ_β for every $\beta \le \alpha$. Start with $\theta_0 0 = \emptyset$, $\theta_0 1 = X$.

(b) Given θ_α, where $\alpha < \omega_1$, set

$$F = \bigcup\{\underline{\theta}(c \cup a_\alpha) \setminus \theta_\alpha c : c \in \mathfrak{C}_\alpha\},$$

$$G = \bigcup\{\underline{\theta}(c \cup (1 \setminus a_\alpha)) \setminus \theta_\alpha c : c \in \mathfrak{C}_\alpha\}.$$

Because $\mathfrak{C}_\alpha$ is countable, F and G belong to Σ. If $c \in \mathfrak{C}_\alpha$, then

$$(\underline{\theta}(c \cup a_\alpha) \setminus \theta_\alpha c)^\bullet = \underline{\theta}(c \cup a_\alpha)^\bullet \setminus c \subseteq (c \cup a_\alpha) \setminus c \subseteq a_\alpha,$$

so $F^\bullet \subseteq a_\alpha$; similarly, $G^\bullet \subseteq 1 \setminus a_\alpha$. Next, $F \cap G = \emptyset$. **P** If b, $c \in \mathfrak{C}_\alpha$, then

$$\begin{aligned}(\underline{\theta}(b \cup a_\alpha) \setminus \theta_\alpha b) \cap (\underline{\theta}(c \cup (1 \setminus a_\alpha)) \setminus \theta_\alpha c) &= \underline{\theta}((b \cup a_\alpha) \cap (c \cup (1 \setminus a_\alpha))) \setminus (\theta_\alpha b \cup \theta_\alpha c) \\ &\subseteq \underline{\theta}(b \cup c) \setminus \theta_\alpha(b \cup c) = \emptyset. \text{ } \mathbf{Q}\end{aligned}$$

Choose any $E \in \Sigma$ such that $E^\bullet = a_\alpha$ and set $E_\alpha = (E \cup F) \setminus G$; then $E_\alpha^\bullet = a_\alpha$, $F \subseteq E_\alpha$ and $G \cap E_\alpha = \emptyset$.

If $c \in \mathfrak{C}_\alpha$ and $c \subseteq a_\alpha$, then $\underline{\theta}((1 \setminus c) \cup a_\alpha) = \underline{\theta}1 = X$, so

$$\theta_\alpha c = \underline{\theta}((1 \setminus c) \cup a_\alpha) \setminus \theta_\alpha(1 \setminus c) \subseteq F \subseteq E_\alpha.$$

Similarly, if $c \in \mathfrak{C}_\alpha$ and $c \cap a_\alpha = 0$, then

$$\theta_\alpha c = \underline{\theta}((1 \setminus c) \cup (1 \setminus a_\alpha)) \setminus \theta_\alpha(1 \setminus c) \subseteq G$$

is disjoint from E_α. We can therefore define a Boolean homomorphism $\theta_{\alpha+1} : \mathfrak{C}_{\alpha+1} \to \Sigma$ by setting

$$\theta_{\alpha+1}((b \cap a_\alpha) \cup (c \setminus a_\alpha)) = (\theta_\alpha b \cap E_\alpha) \cup (\theta_\alpha c \setminus E_\alpha)$$

for all b, $c \in \mathfrak{C}_\alpha$ (312O), and $\theta_{\alpha+1}$ will extend θ_β for every $\beta \le \alpha+1$. Because $(\theta_{\alpha+1} a_\alpha)^\bullet = E_\alpha^\bullet = a_\alpha$ and $\theta_{\alpha+1} c = \theta_\alpha c$ for every $c \in \mathfrak{C}_\alpha$, $(\theta_{\alpha+1} a)^\bullet = a$ for every $a \in \mathfrak{C}_{\alpha+1}$.

I have still to check the other part of the inductive hypothesis. If b, $c \in \mathfrak{C}_\alpha$, then

$$\begin{aligned}\underline{\theta}((b \cap a_\alpha) \cup (c \setminus a_\alpha)) &= \underline{\theta}((b \cup c) \cap (c \cup a_\alpha) \cap (b \cup (1 \setminus a_\alpha))) \\ &= \underline{\theta}(b \cup c) \cap \underline{\theta}(c \cup a_\alpha) \cap \underline{\theta}(b \cup (1 \setminus a_\alpha)) \\ &\subseteq \theta_\alpha(b \cup c) \cap (F \cup \theta_\alpha c) \cap (G \cup \theta_\alpha b) \\ &\subseteq \theta_{\alpha+1}(b \cup c) \cap (\theta_{\alpha+1} a_\alpha \cup \theta_{\alpha+1} c) \cap (\theta_{\alpha+1}(1 \setminus a_\alpha) \cup \theta_{\alpha+1} b) \\ &= \theta_{\alpha+1}((b \cap a_\alpha) \cup (c \setminus a_\alpha)),\end{aligned}$$

which is what we need to know.

(c) For non-zero limit ordinals $\alpha \le \omega_1$, we have $\mathfrak{C}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{C}_\beta$ so we can, and must, take $\theta_\alpha = \bigcup_{\beta<\alpha} \theta_\beta$.

At the end of the induction, $\theta_{\omega_1} : \mathfrak{A} \to \Sigma$ is an appropriate lifting.

535G Corollary (see Neumann 31) Suppose that $\mathfrak{c} = \omega_1$. Then for any integer $r \ge 1$ there is a Borel lifting θ of Lebesgue measure on $\mathbb{R}^r$ such that $x \in \theta E^\bullet$ whenever $E \subseteq \mathbb{R}^r$ is a Borel set and x is a density point of E.

proof In 535F, let $\underline{\theta}$ be lower Lebesgue density (341E), interpreted as a function from the Lebesgue measure algebra to the Borel σ-algebra. We need to check that $\underline{\theta} E^\bullet$ is indeed always a Borel set; this is because

$$\underline{\theta} E^\bullet = \operatorname{int}^* E = \{x : \lim_{n\to\infty} \frac{\mu(E \cap B(x, 2^{-n}))}{\mu B(x, 2^{-n})} = 1\}$$

and the functions $x \mapsto \mu(E \cap B(x, 2^{-n}))$ are all continuous (use 443B).

535H Again using the continuum hypothesis, we have some results on 'strong' liftings, as described in §453.

Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space with measure algebra $\mathfrak{A}$. If $\#(\mathfrak{A}) \le \operatorname{add} \mu$ and μ is strictly positive, then μ has a strong lifting.

proof (a) For each $a \in \mathfrak{A}$, set

$$\overline{a} = \bigcap\{F : F \subseteq X \text{ is closed}, F^\bullet \supseteq a\}.$$

Then $\overline{a}$ is closed and $\overline{a}^\bullet \supseteq a$ (414Ac). If a, $b \in \mathfrak{A}$, then $\overline{a \cup b} = \overline{a} \cup \overline{b}$. **P** Of course $\overline{a \cup b} \supseteq \overline{a} \cup \overline{b}$, because the operation $^-$ is order-preserving. On the other hand, $\overline{a} \cup \overline{b}$ is a closed set and $(\overline{a} \cup \overline{b})^\bullet \supseteq a \cup b$, so $\overline{a} \cup \overline{b} \supseteq \overline{a \cup b}$. **Q**

For a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, say that a function $\theta : \mathfrak{B} \to \Sigma$ is 'potentially a strong lifting' if it is a Boolean homomorphism and $(\theta b)^\bullet = b$ and $\theta b \subseteq \overline{b}$ for every $b \in \mathfrak{B}$.

(b) (The key.) Suppose that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, with cardinal less than $\operatorname{add}\mu$, and $c \in \mathfrak{A}$; let $\mathfrak{B}_1$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{c\}$. If $\theta : \mathfrak{B} \to \Sigma$ is potentially a strong lifting, then it has an extension $\theta_1 : \mathfrak{B}_1 \to \Sigma$ which is also potentially a strong lifting.

P Set

$$C_0 = \bigcup\{\theta a : a \in \mathfrak{B},\, a \subseteq c\},$$

$$D_0 = \bigcap\{\theta b : b \in \mathfrak{B},\, c \subseteq b\},$$

$$C_1 = \bigcup\{\theta a \setminus \overline{a \setminus c} : a \in \mathfrak{B}\},$$

$$D_1 = \bigcap\{(X \setminus \theta b) \cup \overline{b \cap c} : b \in \mathfrak{B}\}.$$

Fix $E_0 \in \Sigma$ such that $E_0^\bullet = c$.

If a, a', b, $b' \in \mathfrak{B}$ and $a' \subseteq c \subseteq b'$, then

$$a' \subseteq b', \text{ so } \theta a' \subseteq \theta b';$$

$$\theta a' \cap \theta b = \theta(a' \cap b) \subseteq \overline{a' \cap b} \subseteq \overline{b \cap c}, \text{ so } \theta a' \subseteq (X \setminus \theta b) \cup \overline{b \cap c};$$

$$\theta a \setminus \theta b' = \theta(a \setminus b') \subseteq \overline{a \setminus b'} \subseteq \overline{a \setminus c}, \text{ so } \theta a \setminus \overline{a \setminus c} \subseteq \theta b';$$

$$\theta a \cap \theta b = \theta(a \cap b) \subseteq \overline{a \cap b} = \overline{a \cap b \cap c} \cup \overline{a \cap b \setminus c} \subseteq \overline{a \setminus c} \cup \overline{b \cap c},$$

so

$$\theta a \setminus \overline{a \setminus c} \subseteq (X \setminus \theta b) \cup \overline{b \cap c}.$$

This shows that $C_0 \cup C_1 \subseteq D_0 \cap D_1$. At the same time,

$$E_0^\bullet = c \supseteq a', \text{ so } \theta a' \setminus E_0 \text{ is negligible;}$$

$$E_0^\bullet = c \subseteq b', \text{ so } E_0 \setminus \theta b' \text{ is negligible;}$$

$$(E_0 \cup \overline{a \setminus c})^\bullet \supseteq c \cup (a \setminus c) \supseteq a = (\theta a)^\bullet$$

so $(\theta a \setminus \overline{a \cap c}) \setminus E_0$ is negligible;

$$E_0^\bullet = c \subseteq (1 \setminus b) \cup (b \cap c) \subseteq (X \setminus \theta b)^\bullet \cup \overline{b \cap c}^\bullet,$$

so $E_0 \setminus ((X \setminus \theta b) \cup \overline{b \cap c})$ is negligible. Because $\#(\mathfrak{B}) < \operatorname{add}\mu$, $(C_0 \cup C_1) \setminus E_0$ and $E_0 \setminus (D_0 \cap D_1)$ are measurable and negligible.

If we set

$$E = (E_0 \cup C_0 \cup C_1) \cap (D_0 \cap D_1),$$

then $E \in \Sigma$, $E^\bullet = c$ and $C_0 \cup C_1 \subseteq E_0 \subseteq D_0 \cap D_1$. So we can set $\theta_1 c = E$ to define a homomorphism from $\mathfrak{B}_1$ to Σ (312O again), and we shall have $(\theta_1 d)^\bullet = d$ for every $d \in \mathfrak{B}_1$.

We must check that $\theta_1 d \subseteq \overline{d}$ for every $d \in \mathfrak{B}_1$. Now d is expressible as $(b \cap c) \cup (a \setminus c)$ for some a, $b \in \mathfrak{B}$, and in this case

$$\theta b \cap E \subseteq \theta b \cap ((X \setminus \theta b) \cup \overline{b \cap c}) \subseteq \overline{b \cap c},$$

$$\theta a \setminus E \subseteq \theta a \setminus (\theta a \setminus \overline{a \setminus c}) \subseteq \overline{a \setminus c},$$

so

$$\theta_1 d = (\theta b \cap E) \cup (\theta a \setminus E) \subseteq \overline{b \cap c} \cup \overline{a \setminus c} = \overline{d}.$$

So θ_1 is a potential strong lifting, as required. **Q**

(c) Enumerate $\mathfrak{A}$ as $\langle a_\xi \rangle_{\xi\in\kappa}$ where $\kappa \le \operatorname{add}\mu$, and for $\alpha \le \kappa$ let $\mathfrak{B}_\alpha$ be the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi : \xi < \alpha\}$. Then (b) tells us that we can choose inductively a family $\langle \theta_\alpha \rangle_{\alpha<\kappa}$ such that $\theta_\alpha : \mathfrak{B}_\alpha \to \Sigma$ is a potential strong lifting and $\theta_{\alpha+1}$ extends θ_α for each $\alpha < \kappa$. (At non-zero limit ordinals α, $\mathfrak{B}_\alpha = \bigcup_{\xi<\alpha} \mathfrak{B}_\xi$ so we can take θ_α to be the common extension of $\bigcup_{\xi<\alpha} \theta_\xi$. We need to know that μ is strictly positive in order to be sure that $\overline{1} = X$, so that we can take $\theta_0 1 = X$.) In this way we obtain a lifting $\theta = \theta_\kappa$ of μ. Also $\theta a \subseteq \overline{a}$ for every $a \in \mathfrak{A}$. Looking at this from the other side, if $F \subseteq X$ is closed then $\overline{F^\bullet} \subseteq F$ so $\theta(F^\bullet) \subseteq F$, and θ is a strong lifting.

535I Corollary (see Mokobodzki 75) Suppose that $\mathfrak{c} = \omega_1$. Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a strictly positive σ-finite quasi-Radon measure space with Maharam type at most $\omega_1 = \mathfrak{c}$. Then μ has a strong Borel lifting.

proof Because μ is σ-finite, its measure algebra $\mathfrak{A}$ is ccc, and has size at most $\mathfrak{c}^\omega = \omega_1$; so we can apply 535H to $\mu \restriction \mathcal{B}(X)$.

535J Under certain conditions, we can deduce the existence of a strong lifting from the existence of a lifting. The basic case is the following.

Lemma Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a completely regular totally finite topological measure space with a Borel lifting ϕ. Suppose that $K \subseteq X$ is a self-supporting set of non-zero measure, homeomorphic to $\{0,1\}^{\mathbb{N}}$, such that $K \cap G \subseteq \phi G$ for every open set $G \subseteq X$. Then the subspace measure μ_K has a strong Borel lifting.

proof (a) Taking $\mathcal{E}$ to be the algebra of relatively open-and-closed subsets of K, we have a Boolean homomorphism $\psi_0 : \mathcal{E} \to \mathcal{B}(X)$ such that $E \subseteq \operatorname{int} \psi_0 E$ for every $E \in \mathcal{E}$. **P** We have a Boolean-independent sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ which generates $\mathcal{E}$ and separates the points of K. Because every member of $\mathcal{E}$ is compact, we can choose for each $n \in \mathbb{N}$ an open $H_n \subseteq X$ such that $E_n = K \cap H_n = K \cap \overline{H}_n$. Define $h : X \to K$ by saying that, for every $n \in \mathbb{N}$ and $x \in X$, $h(x) \in E_n$ iff $x \in H_n$. Define $\psi_0 : \mathcal{E} \to \mathcal{B}(X)$ by setting $\psi_0 E = h^{-1}[E]$ for $E \in \mathcal{E}$. Then ψ_0 is a Boolean homomorphism. The set

$$\{E : E \in \mathcal{E},\, E \subseteq \operatorname{int} \psi_0 E,\, K \setminus E \subseteq \operatorname{int} \psi_0(K \setminus E)\}$$

is a subalgebra of $\mathcal{E}$ containing every E_n, so is the whole of $\mathcal{E}$, and ψ_0 has the required property. **Q**

(b) Let $\mathfrak{A}$ be the measure algebra of μ, and $\theta : \mathfrak{A} \to \mathcal{B}(X)$ the lifting corresponding to ϕ. Set $\psi_1 E = (\psi_0 E)^\bullet$ for $E \in \mathcal{E}$, so that $\psi_1 : \mathcal{E} \to \mathfrak{A}$ is a Boolean homomorphism. Let $\mathcal{I}$ be the null ideal of μ_K. Because K is self-supporting, $\mathcal{E} \cap \mathcal{I} = \{\emptyset\}$. Taking $\mathcal{E}' = \{E \triangle F : E \in \mathcal{E},\, F \in \mathcal{I}\}$, $\mathcal{E}'$ is a subalgebra of $\mathcal{P}K$, and we have a Boolean homomorphism $\psi' : \mathcal{E}' \to \mathcal{E}$ defined by setting $\psi'(E \triangle F) = E$ whenever $E \in \mathcal{E}$ and $F \in \mathcal{I}$; set $\psi_1' = \psi_1 \psi'$, so that $\psi_1' : \mathcal{E}' \to \mathfrak{A}$ is a Boolean homomorphism extending ψ_1, and $\psi_1' F = 0$ whenever $F \in \mathcal{I}$. Because μ is totally finite, $\mathfrak{A}$ is Dedekind complete, and there is a Boolean homomorphism $\tilde{\psi}_1 : \mathcal{P}K \to \mathfrak{A}$ extending ψ_1' (314K). Now set

$$\phi_1 E = K \cap (\phi E \cup (\theta \tilde{\psi}_1 E \setminus \phi K))$$

for every measurable $E \subseteq K$. Then ϕ_1 is a strong lifting for μ_K. **P** $\phi \restriction \Sigma_K$ is a Boolean homomorphism from the domain Σ_K of μ_K to $\mathcal{B}(\phi K)$, while $E \mapsto \theta \tilde{\psi}_1 E \setminus \phi K$ is a Boolean homomorphism from Σ_K to $\mathcal{B}(X \setminus \phi K)$; putting these together, ϕ_1 is a Boolean homomorphism from Σ_K to $\mathcal{B}(K)$. If $E \in \Sigma_K$, then $E \triangle (K \cap \phi E)$ and $K \setminus \phi K$ are negligible, so $E \triangle \phi_1 E$ is negligible. If $E \in \Sigma_K$ is negligible, then $\phi E = \emptyset$, $\psi_1' E = 0$ and $\phi_1 E$ is empty. Thus ϕ_1 is a lifting for μ_K. Morover, if $E \in \mathcal{E}$, set $G = \operatorname{int} \psi_0 E$, so that $E = K \cap G$. In this case,

$$E \subseteq \phi G = \theta G^\bullet \subseteq \theta(\psi_0 E)^\bullet = \theta \psi_1 E = \theta \tilde{\psi}_1 E,$$

while

$$E \cap \phi K \subseteq \phi G \cap \phi K = \phi E;$$

so $E \subseteq \phi_1 E$. So if $V \subseteq K$ is relatively open,

$$V = \bigcup\{E : E \in \mathcal{E},\, E \subseteq V\} \subseteq \bigcup\{\phi_1 E : E \in \mathcal{E},\, E \subseteq V\} \subseteq \phi_1 V.$$

Thus ϕ_1 is strong. **Q**

535K Lemma Let X be a metrizable space, μ an atomless Radon measure on X and ν an atomless strictly positive Radon measure on $\{0,1\}^{\mathbb{N}}$. Let $\mathcal{K}$ be the family of those subsets K of X such that K, with the subspace topology and measure, is isomorphic to $\{0,1\}^{\mathbb{N}}$ with its usual topology and a scalar multiple of ν. Then μ is inner regular with respect to $\mathcal{K}$.

proof (a) It will be helpful to note that if $E \in \operatorname{dom}\mu$ and $\gamma < \mu E$ there is a compact set $K \subseteq E$ such that $\mu K = \gamma$. **P** Let $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ be a strictly decreasing sequence with $\gamma_0 < \mu E$ and $\inf_{n\in\mathbb{N}}\gamma_n = \gamma$. Choose $\langle K_n\rangle_{n\in\mathbb{N}}$, $\langle E_n\rangle_{n\in\mathbb{N}}$ inductively as follows. $E_0 = E$. Given that $\mu E_n > \gamma_n$, let $K_n \subseteq E_n$ be a compact set such that $\mu K_n \ge \gamma_n$; now let E_{n+1} be a measurable set with measure γ_n (215D, because μ is atomless). At the end of the induction, set $K = \bigcap_{n\in\mathbb{N}} K_n$. **Q**

(b) Now for the main argument. Suppose that $0 \le \gamma < \mu E$. Let $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ be a strictly decreasing sequence with $\gamma_0 < \mu E$ and $\inf_{n\in\mathbb{N}}\gamma_n = \gamma$. Set $\gamma'_n = \frac{1}{2}(\gamma_n + \gamma_{n+1})$ for each n. For $\sigma \in \bigcup_{n\in\mathbb{N}}\{0,1\}^n$, set $I_\sigma = \{z : \sigma \subseteq z \in \{0,1\}^{\mathbb{N}}\}$. Let K_0 be a compact subset of E of measure γ_0; because X is metrizable, K_0 is second-countable; let $\langle V_n\rangle_{n\in\mathbb{N}}$ run over a base for the topology of K_0. Choose $\langle m(n)\rangle_{n\in\mathbb{N}}$ and L_σ, for $\sigma \in \{0,1\}^{m(n)}$, as follows. Start with $m(0) = 0$ and $L_\emptyset = K_0$. Given that $\langle L_\sigma\rangle_{\sigma\in\{0,1\}^{m(n)}}$ is a disjoint family of compact subsets of X with $\mu L_\sigma = \gamma_n \nu I_\sigma$ for every $\sigma \in \{0,1\}^{m(n)}$, let $m(n+1) > m(n)$ be so large that $\gamma_{n+1}\nu I_\tau < (\gamma_n - \gamma_{n+1})\nu I_\sigma$ whenever $\sigma \in \{0,1\}^{m(n)}$ and $\tau \in \{0,1\}^{m(n+1)}$. (This is where we need to know that ν is atomless and strictly positive.) Now, for each $\sigma \in \{0,1\}^{m(n)}$, enumerate $\{\tau : \sigma \subseteq \tau \in \{0,1\}^{m(n+1)}\}$ as $\langle\tau(\sigma,i)\rangle_{i<2^{m(n+1)-m(n)}}$. Choose inductively disjoint compact sets $L_{\tau(\sigma,i)} \subseteq L_\sigma$, for $i < 2^{m(n+1)-m(n)}$, in such a way that $\mu L_{\tau(\sigma,i)} = \gamma_{n+1}\nu I_{\tau(\sigma,i)}$ and $L_{\tau(\sigma,i)}$ is always either included in V_n or disjoint from it; this will be possible because when we come to choose $L_{\tau(\sigma,i)}$, the measure of the set $F = L_\sigma \setminus \bigcup_{j<i} L_{\tau(\sigma,j)}$ available will be

$$\begin{aligned}\gamma_n\nu I_\sigma - \sum_{j<i}\gamma_{n+1}\nu I_{\tau(\sigma,j)} &\ge (\gamma_n - \gamma_{n+1})\nu I_\sigma + \gamma_{n+1}\nu I_{\tau(\sigma,i)} \\ &> 2\gamma_{n+1}\nu I_{\tau(\sigma,i)},\end{aligned}$$

so at least one of $F \cap V_n$, $F \setminus V_n$ will be of measure greater than $\gamma_{n+1}\nu I_{\tau(\sigma,i)}$. Continue.

Set $K_n = \bigcup\{L_\sigma : \sigma \in \{0,1\}^{m(n)}\}$ for each $n \in \mathbb{N}$, and $K = \bigcap_{n\in\mathbb{N}} K_n$. The construction ensures that whenever $n \le k$, $\sigma \in \{0,1\}^{m(n)}$, $\tau \in \{0,1\}^{m(k)}$ and $\sigma \subseteq \tau$, then $L_\tau \subseteq L_\sigma$. We therefore have a function $f : K \to \{0,1\}^{\mathbb{N}}$ defined by saying that $f(x)\restriction m(n) = \sigma$ whenever $n \in \mathbb{N}$, $\sigma \in \{0,1\}^{m(n)}$ and $x \in K \cap L_\sigma$. Because all the L_σ are compact, f is continuous. But it is also injective. **P** If x, $y \in K$ are different, there is an $n \in \mathbb{N}$ such that $x \in V_n$ and $y \notin V_n$; now $f(x)\restriction m(n+1) \ne f(y)\restriction m(n+1)$. **Q**

For any $n \in \mathbb{N}$, $\sigma \in \{0,1\}^{m(n)}$ and $k \ge n$,

$$\mu(\bigcup\{L_\tau : \sigma \subseteq \tau \in \{0,1\}^{m(k)}\}) = \textstyle\sum_{\sigma\subseteq\tau\in\{0,1\}^{m(k)}} \gamma_k\nu I_\tau = \gamma_k\nu I_\sigma.$$

So

$$\mu(f^{-1}[I_\sigma]) = \textstyle\inf_{k\ge n}\gamma_k\nu I_\sigma = \gamma\nu I_\sigma.$$

Thus the Radon measure μf^{-1} on $\{0,1\}^{\mathbb{N}}$ agrees with the Radon measure $\gamma\nu$ on $\{I_\sigma : \sigma \in \bigcup_{n\in\mathbb{N}}\{0,1\}^{m(n)}\}$; as this is a base for the topology of $\{0,1\}^{\mathbb{N}}$ closed under finite intersections, μf^{-1} and $\gamma\nu$ are identical (415H(v)). Once again because ν is strictly positive, f is surjective and is a homeomorphism. So f witnesses that $K \in \mathcal{K}$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$.

535L Lemma (a) If $(X,\mathfrak{T})$ is a separable metrizable space, there is a zero-dimensional separable metrizable topology $\mathfrak{S}$ on X, finer than $\mathfrak{T}$, with the same Borel sets as $\mathfrak{T}$, such that $\mathfrak{T}$ is a π-base for $\mathfrak{S}$.

(b) If X is a non-empty zero-dimensional separable metrizable space without isolated points, it is homeomorphic to a dense subset of $\{0,1\}^{\mathbb{N}}$.

(c) Any completely regular space of size less than $\mathfrak{c}$ is zero-dimensional.

proof (a) Enumerate a countable base for $\mathfrak{T}$ as $\langle U_n\rangle_{n\in\mathbb{N}}$. Define a sequence $\langle\mathfrak{S}_n\rangle_{n\in\mathbb{N}}$ of topologies on X by saying that $\mathfrak{S}_0 = \mathfrak{T}$ and that $\mathfrak{S}_{n+1}$ is the topology on X generated by $\mathfrak{S}_n \cup \{V_n\}$, where V_n is the closure of U_n for $\mathfrak{S}_n$. Inducing on n, we see that $\mathfrak{S}_n$ is second-countable and has the same Borel sets as $\mathfrak{T}$, for every n. So taking $\mathfrak{S}$ to be the topology generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{S}_n$ (that is, the topology generated by $\{U_n : n \in \mathbb{N}\} \cup \{V_n : n \in \mathbb{N}\}$), this also is second-countable and has the same Borel sets as $\mathfrak{T}$. Each V_n is open for $\mathfrak{S}_{n+1}$ and closed for $\mathfrak{S}_n$, so is open-and-closed for $\mathfrak{S}$. Moreover, since

$$U_n = \bigcup\{U_m : m \in \mathbb{N}, \overline{U}_m^{\mathfrak{T}} \subseteq U_n\} = \bigcup\{V_m : m \in \mathbb{N}, V_m \subseteq U_n\}$$

for each n, $\{V_n : n \in \mathbb{N}\}$ is a base for $\mathfrak{S}$ consisting of open-and-closed sets for $\mathfrak{S}$, and $\mathfrak{S}$ is zero-dimensional. Finally, observe that if V_n is not empty, then $V_n \supseteq U_n \ne \emptyset$, so $\mathfrak{T} \supseteq \{U_n : n \in \mathbb{N}\}$ is a π-base for $\mathfrak{S}$.

(b) The family $\mathcal{E}_0$ of open-and-closed subsets of X is a base for the topology of X, so includes a countable base $\mathcal{U}$ (4A2P(a-iii)). Because X has no isolated points, the subalgebra $\mathcal{E}_1$ of $\mathcal{E}_0$ generated by $\mathcal{U}$ is countable, atomless and non-trivial, and must be isomorphic to the algebra $\mathcal{E}$ of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ (316M). Let $\pi : \mathcal{E} \to \mathcal{E}_1$ be an isomorphism. Then we have a function $f : X \to \{0,1\}^{\mathbb{N}}$ defined by saying that, for $E \in \mathcal{E}$, $f(x) \in E$ iff $x \in \pi E$. Because $\pi E \neq \emptyset$ for every non-empty $E \in \mathcal{E}$, $f[X]$ is dense in $\{0,1\}^{\mathbb{N}}$. Because $\{f^{-1}[E] : E \in \mathcal{E}\} = \mathcal{E}_1 \supseteq \mathcal{U}$ is a base for the topology of X, f is a homeomorphism between X and $f[X]$.

(c) If X is a completely regular space and $\#(X) < \mathfrak{c}$, $G \subseteq X$ is open and $x \in G$, let $f : X \to [0,1]$ be a continuous function such that $f(x) = 1$ and $f(y) = 0$ for $y \in X \setminus G$. Because $\#(X) < \mathfrak{c}$, there is an $\alpha \in [0,1] \setminus f[X]$, and now $\{y : f(x) > \alpha\} = \{y : f(x) \geq \alpha\}$ is an open-and-closed set containing x and included in G. As x and G are arbitrary, X is zero-dimensional.

535M Lemma Suppose that there is a Borel probability measure on $\{0,1\}^{\mathbb{N}}$ with a strong lifting. Then whenever X is a separable metrizable space and $D \subseteq X$ is a dense set, there is a Boolean homomorphism ϕ from $\mathcal{P}D$ to the Borel σ-algebra $\mathcal{B}(X)$ of X such that $\phi A \subseteq \overline{A}$ for every $A \subseteq D$.

proof case 1 Suppose that X is countable. Then it is zero-dimensional (535Lc), so has a base $\mathcal{U}$ consisting of open-and-closed sets; let $\mathcal{E}$ be the algebra of sets generated by $\mathcal{U}$. For $E \in \mathcal{E}$ set $\pi E = E \cap D$; then π is an isomorphism between $\mathcal{E}$ and a subalgebra $\mathcal{E}'$ of $\mathcal{P}D$. Because $\mathcal{B}(X) = \mathcal{P}X$ is Dedekind complete, the Boolean homomorphism $\pi^{-1} : \mathcal{E}' \to \mathcal{E}$ extends to a Boolean homomorphism $\phi : \mathcal{P}D \to \mathcal{P}X = \mathcal{B}(X)$ (314K again). If $A \subseteq D$ and $x \in X \setminus \overline{A}$, then there is a $U \in \mathcal{U}$ such that $x \in U$ and $A \cap U = \emptyset$, in which case

$$\phi A \subseteq \pi^{-1}(D \setminus U) = X \setminus U$$

does not contain x. As x is arbitrary, $\phi A \subseteq \overline{A}$; as A is arbitrary, ϕ has the required property.

case 2 Suppose that X is zero-dimensional and has no isolated points. If X is empty the result is trivial; otherwise, by 535Lb, we may suppose that X is a dense subset of $\{0,1\}^{\mathbb{N}}$. This time, let $\mathcal{E}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. For $E \in \mathcal{E}$, set $\pi E = E \cap D$. Because D is dense in X and therefore in $\{0,1\}^{\mathbb{N}}$, π is an isomorphism between $\mathcal{E}$ and a subalgebra $\mathcal{E}'$ of $\mathcal{P}D$. Fix a Borel probability measure μ on $\{0,1\}^{\mathbb{N}}$ with a strong lifting θ, and let $\mathfrak{A}$ be the measure algebra of μ. Then $A \mapsto (\pi^{-1}A)^{\bullet}$ is a Boolean homomorphism from $\mathcal{E}'$ to $\mathfrak{A}$; because $\mathfrak{A}$ is Dedekind complete, it extends to a Boolean homomorphism $\psi : \mathcal{P}D \to \mathfrak{A}$. For $E \subseteq \{0,1\}^{\mathbb{N}}$, set $\tilde{\pi}E = E \cap X$. Then $\phi = \tilde{\pi}\theta\psi$ is a Boolean homomorphism from $\mathcal{P}D$ to $\mathcal{B}(X)$. If $A \subseteq D$ and $x \in X \setminus \overline{A}$, then there is an $E \in \mathcal{E}$ such that $x \in E$ and $A \cap E = \emptyset$, in which case

$$\phi A \subseteq \theta\psi A \subseteq \theta\psi(D \setminus E) = \theta(\{0,1\}^{\mathbb{N}} \setminus E)^{\bullet} = \{0,1\}^{\mathbb{N}} \setminus E,$$

and $x \notin \phi A$. As x and A are arbitrary, ϕ is a suitable homomorphism.

case 3 Suppose that X has no isolated points. Write $\mathfrak{T}$ for the given topology on X. By 535La, there is a finer zero-dimensional separable metrizable topology $\mathfrak{S}$ on X, with the same Borel sets, such that $\mathfrak{T}$ is a π-base for $\mathfrak{S}$. If $V \in \mathfrak{S}$ is non-empty, there is a non-empty $U \in \mathfrak{T}$ such that $U \subseteq V$, and $D \cap V \supseteq D \cap U$ is non-empty; so D is $\mathfrak{S}$-dense. By case 2, there is a Boolean homomorphism $\phi : \mathcal{P}D \to \mathcal{B}(X, \mathfrak{S})$ such that $\phi A \subseteq \overline{A}^{\mathfrak{S}}$ for every $A \subseteq D$. As $\mathcal{B}(X, \mathfrak{S}) = \mathcal{B}(X, \mathfrak{T})$, and $\overline{A}^{\mathfrak{S}} \subseteq \overline{A}^{\mathfrak{T}}$ for every $A \subseteq X$, this ϕ satisfies the conditions required.

general case In general, let $\mathcal{G}$ be the family of countable open subsets of X, and $G_0 = \bigcup \mathcal{G}$; because X is separable and metrizable, therefore hereditarily Lindelöf, G_0 is countable. Set $Z = X \setminus G_0$, and let D_0 be a countable dense subset of Z; set $Y = D \cup G_0 \cup D_0$. By case 1, there is a Boolean homomorphism $\phi_0 : \mathcal{P}D \to \mathcal{P}Y$ such that $\phi_0 A \subseteq \overline{A}$ for every $A \subseteq D$. By case 3, there is a Boolean homomorphism $\phi_1 : \mathcal{P}(Y \cap Z) \to \mathcal{B}(Z)$ such that $\phi_1 B \subseteq \overline{B}$ for every $B \subseteq Y \cap Z$. Now set

$$\phi A = (\phi_0 A \setminus Z) \cup \phi_1(Z \cap \phi_0 A)$$

for every $A \subseteq D$. Then ϕ is a Boolean homomorphism from $\mathcal{P}D$ to $\mathcal{B}(X)$; and if $A \subseteq D$, then

$$\phi A \subseteq \phi_0 A \cup \phi_1(Z \cap \phi_0 A) \subseteq \overline{A} \cup \overline{Z \cap \overline{A}} = \overline{A},$$

so in this case also we have a homomorphism of the kind we need.

535N Theorem Suppose there is a metrizable space X with a non-zero atomless semi-finite tight Borel measure μ which has a lifting. Then whenever Y is a metrizable space and ν is a strictly positive σ-finite Borel measure on Y, ν has a strong lifting.

proof (a) Let ϕ be a lifting for μ. Then there is a Borel set $E \subseteq X$, of non-zero finite measure, such that $E \cap G \subseteq \phi G$ for every open $G \subseteq X$. **P** Let $L_0 \subseteq X$ be a compact set of non-zero measure; then L_0 has a countable base $\mathcal{U}$; set $E = L_0 \cap \phi L_0 \setminus \bigcup_{U \in \mathcal{U}}(U \triangle \phi U)$, so that $\mu E = \mu L_0 \in]0, \infty[$. If $G \subseteq X$ is open and $x \in E \cap G$, then there is a $U \in \mathcal{U}$ such that $x \in U \subseteq G$. Since $x \in E \cap U$, $x \in \phi U \subseteq \phi G$. As x and G are arbitrary, we have an appropriate E. **Q**

(b) Let λ be any strictly positive atomless Radon measure on $\{0,1\}^{\mathbb{N}}$. There is a compact set $K \subseteq E$ such that K, with its induced topology and measure, is isomorphic to $\{0,1\}^{\mathbb{N}}$ with its usual topology and a non-zero multiple of λ, by 535K. In particular, K is self-supporting. By 535J, the subspace measure on K has a strong Borel lifting. It follows at once that λ has a strong Borel lifting.

(c) Refining (b) slightly, we see that if $Y \subseteq \{0,1\}^{\mathbb{N}}$ is a dense set and λ is a strictly positive atomless totally finite Borel measure on Y, then λ has a strong lifting. **P** There is a Radon measure ν on $\{0,1\}^{\mathbb{N}}$ such that $\nu E = \lambda(Y \cap E)$ for every Borel set $E \subseteq \{0,1\}^{\mathbb{N}}$ (416F); because λ is atomless, so is ν; because λ is strictly positive and Y is dense, ν is strictly positive. So ν has a strong Borel lifting ψ_0 say. If E, $F \in \mathcal{B}(\{0,1\}^{\mathbb{N}})$ and $E \cap Y = F \cap Y$, then $\nu(E \triangle F) = 0$ and $\psi_0 E = \psi_0 F$; we therefore have a Boolean homomorphism $\psi : \mathcal{B}(Y) \to \mathcal{B}(Y)$ defined by setting $\psi(E \cap Y) = Y \cap \psi_0 E$ for every Borel set $E \subseteq \{0,1\}^{\mathbb{N}}$. It is easy to check that ψ is a lifting for λ, and it is strong because if $G \subseteq \{0,1\}^{\mathbb{N}}$ is open then $\psi(Y \cap G) = Y \cap \psi_0 G \subseteq Y \cap G$. **Q**

(d) If $(Y, \mathfrak{S})$ is a separable metrizable space with a strictly positive atomless totally finite Borel measure ν, then ν has a strong lifting. **P** If $Y = \emptyset$ the result is trivial. Otherwise, by 535La, there is a finer separable metrizable topology $\mathfrak{S}'$ on Y with the same Borel sets such that $\mathfrak{S}$ is a π-base for $\mathfrak{S}'$. Because $\mathfrak{S}$ and $\mathfrak{S}'$ have the same Borel sets, ν is a Borel measure for $\mathfrak{S}'$; because every non-empty $\mathfrak{S}'$-open set includes a non-empty $\mathfrak{S}$-open set, ν is strictly positive for $\mathfrak{S}'$; because ν is atomless, Y has no $\mathfrak{S}'$-isolated points. By 535Lb, $(Y, \mathfrak{S}')$ is homeomorphic to a dense subset of $\{0,1\}^{\mathbb{N}}$; by (c) above, ν has a lifting ϕ which is strong with respect to the topology $\mathfrak{S}'$. But now ϕ is still strong with respect to the coarser topology $\mathfrak{S}$. **Q**

(e) Now suppose that Y is a separable metrizable space with a strictly positive totally finite Borel measure ν. Then ν has a strong lifting. **P** The set $D = \{y : \nu\{y\} > 0\}$ is countable. If D is empty, then the result is immediate from (d) applied to a scalar multiple of ν. (If $\nu Y = 0$ then $Y = \emptyset$ and the result is trivial.) Otherwise, let $\nu_{Y \setminus D}$ be the subspace measure; then $\nu_{Y \setminus D}$ is a totally finite Borel measure on $Y \setminus D$, and is zero on singletons, so must be atomless. Because $Y \setminus D$ is hereditarily Lindelöf, $\nu_{Y \setminus D}$ is τ-additive; let Z be its support, and ν_Z the subspace measure on Z. Then ν_Z has a strong Borel lifting ψ_0, by (d) again. Next, Z is relatively closed in $Y \setminus D$, so is expressible as $F \setminus D$ for some closed set $F \subseteq Y$. If $x \in Y \setminus F$ and G is an open set containing x, then $G' = G \setminus F$ is a non-empty open set, so has non-zero measure, while $\nu_{Y \setminus D}(G' \setminus D) = 0$; accordingly $G' \cap D \neq \emptyset$. This shows that $Y \setminus F \subseteq \overline{D}$ so D is dense in $Y \setminus Z$. Now 535M (with (b) above) tells us that there is a Boolean homomorphism $\psi_1 : \mathcal{P}D \to \mathcal{B}(Y \setminus Z)$ such that $\psi_1 A \subseteq \overline{A}$ for every $A \subseteq D$. Define $\psi : \mathcal{B}(Y) \to \mathcal{B}(Y)$ by setting

$$\psi E = \psi_0(E \cap Z) \cup (E \cap D) \cup (\psi_1(E \cap D) \setminus D)$$

for every Borel set $E \subseteq Y$. ψ is a Boolean homomorphism because ψ_0 and ψ_1 are. If $\nu E = 0$, then $\nu_Z(E \cap Z) = 0$ and $E \cap D = \emptyset$, so $\psi E = \emptyset$. For any $E \in \mathcal{B}(Y)$, $\psi_0(E \cap Z) \triangle (E \cap Z)$ and $Y \setminus (D \cup Z)$ are negligible, so $E \triangle \psi E$ is negligible. Thus ψ is a lifting for ν. Finally, for any E,

$$\psi E \subseteq \psi_0(E \cap Z) \cup (E \cap D) \cup \psi_1(E \cap D) \subseteq \overline{E},$$

so ψ is a strong lifting. **Q**

(f) Finally, if Y is a metrizable space and ν is a strictly positive σ-finite Borel measure on Y, then Y must be ccc, therefore separable; and there is a totally finite Borel measure ν' with the same null ideal as ν, so that ν' has a strong lifting, by (e), which is also a strong lifting for ν.

535O Linear liftings Let (X, Σ, μ) be a measure space, with measure algebra $\mathfrak{A}$. Write $\mathcal{L}^\infty(\Sigma)$ for the space of bounded Σ-measurable real-valued functions on X. A **linear lifting** for μ is

either a positive linear operator $T : L^\infty(\mu) \to \mathcal{L}^\infty(\Sigma)$ such that $T(\chi X^\bullet) = \chi X$ and $(Tu)^\bullet = u$ for every $u \in L^\infty(\mu)$

or a positive linear operator $S : \mathcal{L}^\infty(\Sigma) \to \mathcal{L}^\infty(\Sigma)$ such that $S(\chi X) = \chi X$, $Sf = 0$ whenever $f = 0$ a.e. and $Sf =_{\text{a.e.}} f$ for every $f \in \mathcal{L}^\infty(\Sigma)$.

As with liftings (see 341A-341B) we have a direct correspondence between the two kinds of linear operator; given T as in the first formulation, we can set $Sf = T(f^\bullet)$ for every $f \in \mathcal{L}^\infty(\Sigma)$; given S as in the second formulation, we can set $T(f^\bullet) = Sf$ for every $f \in \mathcal{L}^\infty(\Sigma)$.

If $\theta : \mathfrak{A} \to \Sigma$ is a lifting for μ, then we have a corresponding Riesz homomorphism $T : L^\infty(\mathfrak{A}) \to \mathcal{L}^\infty(\Sigma)$ such that $T(\chi a) = \chi(\theta a)$ for every $a \in \mathfrak{A}$, by 363F. Identifying $L^\infty(\mathfrak{A})$ with $L^\infty(\mu)$ as in 363I, we see that T can be regarded as a linear lifting. Of course the associated linear operator from $\mathcal{L}^\infty(\Sigma)$ to itself is the operator derived by the process of 363F from the Boolean homomorphism $E \mapsto \theta E^\bullet : \Sigma \to \Sigma$.

As in 535Aa, I will say that a **Borel linear lifting** is a linear lifting such that all its values are Borel measurable functions; similarly, a **Baire linear lifting** is a linear lifting such that all its values are Baire measurable functions.

535P I give a sample result to show that for some purposes linear liftings are adequate.

Proposition Let (X, Σ, μ) be a countably compact measure space such that Σ is countably generated, (Y, T, ν) a σ-finite measure space with a linear lifting, and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν, consistent with f, such that $y \mapsto \mu_y E$ is a T-measurable function for every $E \in \Sigma$.

proof I use the method of 452H-452I.

(a) Suppose first that μ and ν are probability measures. Let $S : L^\infty(\nu) \to \mathcal{L}^\infty(\mathrm{T})$ be a linear lifting for ν. Let $T : L^\infty(\mu) \to L^\infty(\nu)$ be the positive linear operator defined by saying that $\int_F Tu = \int_{f^{-1}[F]} u$ whenever $u \in L^\infty(\mu)$ and $F \in \mathrm{T}$ (as in part (a) of the proof of 452I). For $y \in Y$ and $E \in \Sigma$, set

$$\psi_y E = (ST(\chi E^\bullet))(y)$$

as in part (b) of the proof of 452H. Because μ is countably compact, we can use the argument of 452H to see that we have a family $\langle \mu'_y \rangle_{y \in Y}$ of totally finite measures on X such that, for any $E \in \Sigma$, $\mu'_y E = \psi_y E$ for almost every $y \in Y$.

Let $\mathcal{H}$ be a countable subalgebra of Σ such that Σ is the σ-algebra of sets generated by $\mathcal{H}$. Set $Y_0 = \{y : \mu'_y H = \psi_y H$ for every $H \in \mathcal{H}\}$, so that Y_0 is conegligible; let $Y_1 \subseteq Y_0$ be a measurable conegligible set; set $\mu_y = \mu'_y$ for $y \in Y_1$, and take μ_y to be the zero measure on X for $y \in Y \setminus Y_1$. If $H \in \mathcal{H}$, then

$$\mu_y H = \psi_y H = ST(\chi H^\bullet)(y)$$

for every $y \in Y_1$, so $y \mapsto \mu_y H$ is T-measurable; also, of course,

$$\int_F \mu_y H \nu(dy) = \int_F ST(\chi H^\bullet) d\nu = \int_F T(\chi H^\bullet) = \int_{f^{-1}[F]} \chi H^\bullet = \mu(H \cap f^{-1}[F]).$$

Now consider the family $\mathcal{E}$ of those $E \in \Sigma$ such that $y \mapsto \mu_y E$ is T-measurable and $\int_F \mu_y E \nu(dy) = \mu(E \cap f^{-1}[F])$ for every $F \in \mathrm{T}$. This is a Dynkin class including $\mathcal{H}$, so is the whole of Σ; which is what we need to know.

(b) In general, if $\nu Y = 0$, the result is trivial. Otherwise, apply (a) to a suitable pair of indefinite-integral measures over μ and ν, as in part (c) of the proof of 452I.

535Q Proposition Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and λ the c.l.d. product measure on $X \times Y$. Suppose that $\lambda \restriction \Sigma \widehat{\otimes} \mathrm{T}$ has a linear lifting. Then μ has a linear lifting.

proof Let $S : \mathcal{L}^\infty(\Sigma \widehat{\otimes} \mathrm{T}) \to \mathcal{L}^\infty(\Sigma \widehat{\otimes} \mathrm{T})$ be a linear lifting for $\lambda \restriction \Sigma \widehat{\otimes} \mathrm{T}$. For $h \in \mathcal{L}^\infty(\Sigma \widehat{\otimes} \mathrm{T})$, set $(Uh)(x) = \int h(x, y) \nu(dy)$ for every $x \in X$; by 252P, Uh is well-defined and is Σ-measurable. Now U is a positive linear operator from $\mathcal{L}^\infty(\Sigma \widehat{\otimes} \mathrm{T})$ to $\mathcal{L}^\infty(\Sigma)$, and $U(\chi(X \times Y)) = \chi X$, because $\nu Y = 1$. Note that

$$\int |Uh| d\mu \le \int U|h| d\mu = \iint |h(x, y)| \nu(dy) \mu(dx) = \int |h| d\lambda$$

for every $h \in \mathcal{L}^\infty(\Sigma \widehat{\otimes} \mathrm{T})$ (252P again). Next, for $f \in \mathcal{L}^\infty(\Sigma)$ set $(Vf)(x, y) = f(x)$ for every $x \in X$ and $y \in Y$, so that V is a positive linear operator from $\mathcal{L}^\infty(\Sigma)$ to $\mathcal{L}^\infty(\Sigma \widehat{\otimes} \mathrm{T})$.

Consider $S_1 = USV : \mathcal{L}^\infty(\Sigma) \to \mathcal{L}^\infty(\Sigma)$. This is a positive linear operator and $S_1(\chi X) = \chi X$. If $f \in \mathcal{L}^\infty(\Sigma)$ and $f = 0$ μ-a.e., then $Vf = 0$ λ-a.e. and $SVf = 0$, so $S_1 f = 0$. For any $f \in \mathcal{L}^\infty(\Sigma)$,

$$\int |f - S_1 f| d\mu = \int |f - USVf| d\mu = \int |UVf - USVf| d\mu \le \int |Vf - SVf| d\lambda = 0,$$

so $f =_{\text{a.e.}} S_1 f$; thus S_1 is a linear lifting for μ.

535R Proposition Write ν_ω^2 for the usual measure on $(\{0,1\}^\omega)^2$, and $\mathrm{T}_\omega^{(2)}$ for its domain. Suppose that ν_κ has a Baire linear lifting for some $\kappa \ge \mathfrak{c}^{++}$. Then there is a Borel linear lifting S for ν_ω^2 which respects coordinates in the sense that if $f \in \mathcal{L}^\infty(\mathrm{T}_\omega^{(2)})$ is determined by a single coordinate, then Sf is determined by the same coordinate.

proof Because $(\{0,1\}^\kappa,\nu_\kappa)$ is isomorphic, as topological measure space, to $(\{0,1\}^{\kappa\times\omega},\nu_{\kappa\times\omega})$, the latter has a Baire linear lifting S_0 say. For $I\subseteq\kappa$, let T_I be the σ-algebra of Baire subsets of $\{0,1\}^{\kappa\times\omega}$ determined by coordinates in $I\times\omega$. Then $\#(\mathrm{T}_I)\le\mathfrak{c}$ whenever $\#(I)\le\mathfrak{c}$. Also $\mathcal{B}\mathfrak{a}(\{0,1\}^{\kappa\times\omega})=\bigcup\{\mathrm{T}_I:I\in[\kappa]^{\le\omega}\}$ (4A3N). It follows that for every $\xi<\kappa$ there is a set $I_\xi\subseteq\kappa$, of size at most $\mathfrak{c}$, such that $\xi\in I_\xi$ and $S_0(\chi E)$ is T_{I_ξ}-measurable whenever $E\in\mathrm{T}_{I_\xi}$; so that S_0f is T_{I_ξ}-measurable whenever $f:\{0,1\}^{\kappa\times\omega}\to\mathbb{R}$ is bounded and T_{I_ξ}-measurable.

Because $\kappa\ge\mathfrak{c}^{++}$, there are ξ, $\eta<\kappa$ such that $\xi\notin I_\eta$ and $\eta\notin I_\xi$ (5A1I(a-iii)). Set $J=\{\xi\}\times\omega$, $K=\{\eta\}\times\omega$ and $L=(\kappa\times\omega)\setminus(J\cup K)$, so that $\{0,1\}^{\kappa\times\omega}$ can be identified with $\{0,1\}^{J\cup K}\times\{0,1\}^L$ and $\mathcal{B}\mathfrak{a}(\{0,1\}^{\kappa\times\omega})$ with $\mathcal{B}\mathfrak{a}(\{0,1\}^{J\cup K})\widehat{\otimes}\mathcal{B}\mathfrak{a}(\{0,1\}^L)$. Set $(Vf)(w,z)=f(w)$ when $f:\{0,1\}^{J\cup K}\to\mathbb{R}$ is a function, $w\in\{0,1\}^{J\cup K}$ and $z\in\{0,1\}^L$; and $(Uh)(w)=\int h(w,z)\nu_L(dz)$ when $h:\{0,1\}^{\kappa\times\omega}\to\mathbb{R}$ is a bounded Baire measurable function and $w\in\{0,1\}^{J\cup K}$. Then $S_1=US_0V$ is a Baire linear lifting for $\nu_{J\cup K}$, just as in 535Q. Moreover, if $f:\{0,1\}^{J\cup K}\to\mathbb{R}$ is a bounded Baire measurable function determined by coordinates in J, in the sense that $f(x,y)=f(x,y')$ whenever $x\in\{0,1\}^J$ and y, $y'\in\{0,1\}^K$, then S_1f is determined by coordinates in J. **P** Vf is determined by coordinates in J, so S_0Vf is determined by coordinates in $I_\xi\times\omega$; since $K\cap(I_\xi\times\omega)$ is empty, $S_0Vf(x,y,z)=S_0Vf(x,y',z)$ for all $x\in\{0,1\}^J$, $z\in\{0,1\}^L$ and y, $y'\in\{0,1\}^K$. It follows at once that

$$S_1f(x,y)=\int S_0Vf(x,y,z)\nu_L(dz)=\int S_0Vf(x,y',z)\nu_L(dz)=S_1f(x,y')$$

whenever $x\in\{0,1\}^J$ and y, $y'\in\{0,1\}^K$. **Q** Similarly, if $f:\{0,1\}^{J\cup K}\to\mathbb{R}$ is a bounded Baire measurable function determined by coordinates in K, then S_1f is determined by coordinates in K.

Now we can transfer S_1 from $\{0,1\}^{J\cup K}\cong\{0,1\}^J\times\{0,1\}^K$ to $(\{0,1\}^\omega)^2$, and we shall obtain a Baire (or Borel) linear lifting S for ν_ω^2 which respects coordinates.

535X Basic exercises (a) Let (X,Σ,μ) be a measure space with a lifting, and A any subset of X. Show that if A has a measurable envelope then the subspace measure μ_A has a lifting. (*Hint*: 322I.)

(b) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, with $\mu_iX_i>0$ for every $i\in I$, and (X,Σ,μ) their direct sum. Show that μ has a lifting iff every μ_i has a lifting.

(c) Let $\mathfrak{A}$ be a Boolean algebra and I a proper ideal of $\mathfrak{A}$. Suppose that $\sup A$ is defined in $\mathfrak{A}$ and belongs to I whenever $A\subseteq I$ and $\#(A)<\#(\mathfrak{A})$. Show that there is a Boolean homomorphism $\theta:\mathfrak{A}/I\to\mathfrak{A}$ such that $(\theta b)^\bullet=b$ for every $b\in\mathfrak{A}/I$. (*Hint*: enumerate $\mathfrak{A}$ as $\{a_\xi:\xi<\kappa\}$; let $\mathfrak{C}_\xi$ be the subalgebra of $\mathfrak{A}/I$ generated by $\{a_\eta^\bullet:\eta<\xi\}$; construct $\theta\restriction\mathfrak{C}_\xi$ inductively by choosing $\theta a_\xi^\bullet$ appropriately.)

(d) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and I a proper ideal of $\mathfrak{A}$. Show that if the quotient Boolean algebra $\mathfrak{A}/I$ is tightly ω_1-filtered, then there is a Boolean homomorphism $\theta:\mathfrak{A}/I\to\mathfrak{A}$ such that $(\theta b)^\bullet=b$ for every $b\in\mathfrak{A}/I$.

(e) Let $\mathfrak{A}$ be a tightly ω_1-filtered Boolean algebra, $\mathfrak{B}$ a Dedekind σ-complete Boolean algebra and $\mathfrak{A}_0$ a countable subalgebra of $\mathfrak{A}$. Show that every Boolean homomorphism from $\mathfrak{A}_0$ to $\mathfrak{B}$ extends to a Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(f) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras such that $\sup A$ is defined in $\mathfrak{A}$ whenever $A\subseteq\mathfrak{A}$ and $\#(A)<\#(\mathfrak{B})$, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a surjective Boolean homomorphism. Suppose that $\underline{\theta}:\mathfrak{B}\to\mathfrak{A}$ is such that $\underline{\theta}0=0$, $\pi\underline{\theta}b\subseteq b$ for every $b\in\mathfrak{B}$ and $\underline{\theta}(b\cap c)=\underline{\theta}b\cap\underline{\theta}c$ for all b, $c\in\mathfrak{B}$. Show that there is a Boolean homomorphism $\theta:\mathfrak{B}\to\mathfrak{A}$ such that $\underline{\theta}b\subseteq\theta b$ and $\pi\theta b=b$ for every $b\in\mathfrak{B}$.

(g) Suppose that $\mathfrak{c}\le\omega_2$ and $\mathrm{FN}(\mathcal{P}\mathbb{N})=\omega_1$. Show that ν_κ has a strong Baire lifting whenever $\kappa\le\omega_2$. (*Hint*: let $\langle e_\xi\rangle_{\xi<\kappa}$ be the standard generating family for $\mathfrak{B}_\kappa$. Show that there is a tight ω_1-filtration $\langle a_\eta\rangle_{\eta<\zeta}$ of $\mathfrak{B}_\kappa$ such that for every $\xi<\kappa$ there is an $\eta<\zeta$ such that the closed subalgebras generated by $\{e_\delta:\delta<\xi\}$ and $\{a_\delta:\delta<\eta\}$ are the same and $e_\xi=a_\eta$.)

(h) Suppose that $\mathfrak{c}\le\omega_2$ and $\mathrm{FN}(\mathcal{P}\mathbb{N})=\omega_1$. Show that whenever X is a separable metrizable space and $D\subseteq X$ is a dense set, there is a Boolean homomorphism $\phi:\mathcal{P}D\to\mathcal{B}(X)$ such that $\phi A\subseteq\overline{A}$ for every $A\subseteq D$.

(i) Let (X,Σ,μ) be a measure space. Show that a linear lifting $S:\mathcal{L}^\infty(\Sigma)\to\mathcal{L}^\infty(\Sigma)$ of μ corresponds to a lifting iff it is 'multiplicative', that is, $S(f\times g)=Sf\times Sg$ for all f, $g\in\mathcal{L}^\infty(\Sigma)$.

(j) Let (X,Σ,μ) be a strictly localizable measure space with non-zero measure. Suppose that ν_κ has a Baire linear lifting for every infinite cardinal κ such that the Maharam-type-κ component of the measure algebra of μ is non-zero. Show that μ has a linear lifting.

(k) Let (X,Σ,μ) be a probability space such that whenever $\mathcal{E}\subseteq\Sigma$, $\#(\mathcal{E})\le\mathfrak{c}$ and $\bigcup\mathcal{E}$ is negligible, then $\bigcup\mathcal{E}\in\Sigma$. Show that μ has a linear lifting. (*Hint*: 363Yf.)

(l) Let (Y,T,ν) be a σ-finite measure space with a linear lifting, Z a set, Υ a countably generated σ-algebra of subsets of Z, and μ a measure with domain $\mathrm{T}\widehat{\otimes}\Upsilon$ such that ν is the marginal measure of μ on Y and the marginal measure of μ on Z is countably compact. Show that there is a family $\langle\mu_y\rangle_{y\in Y}$ of measures with domain Υ such that $y\mapsto\mu_yH$ is a T-measurable function for every $H\in\Upsilon$ and $\mu W=\int\mu_yW[\{y\}]\nu(dy)$ for every $W\in\mathrm{T}\widehat{\otimes}\Upsilon$.

(m) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be τ-additive topological probability spaces, and λ the τ-additive product measure on $X\times Y$ (417G). Suppose that λ has a Borel linear lifting and that μ is inner regular with respect to the Borel sets. Show that μ has a Borel linear lifting.

535Y Further exercises (a) Suppose that we are provided with a bijection between $\mathcal{B}(\mathbb{R})$ and ω_1, but are otherwise not permitted to use the axiom of choice. Show that we can construct a Borel lifting for Lebesgue measure.

(b) Suppose that for every cardinal κ there is a Baire linear lifting for ν_κ. Show that for every $n\in\mathbb{N}$ there is a Borel linear lifting S for Lebesgue measure on $[0,1]^n$ which (α) respects coordinates in the sense that if $f:[0,1]^n\to\mathbb{R}$ is a bounded measurable function determined by coordinates in $I\subseteq n$, then Sf also is determined by coordinates in I (β) is symmetric in the sense that if $\rho:n\to n$ is any permutation and $(\hat\rho f)(x)=f(x\rho)$ for $x\in[0,1]^n$ and $f:[0,1]^n\to\mathbb{R}$, then S commutes with $\hat\rho$. (*Hint*: 5A1Ib.)

(c) Let (X,Σ,μ) be a countably compact measure space, (Y,T,ν) a σ-finite measure space with a linear lifting, and $f:X\to Y$ an inverse-measure-preserving function. Suppose there is a family $\mathcal{H}\subseteq\Sigma$ such that Σ is the σ-algebra of sets generated by $\mathcal{H}$ and $\#(\mathcal{H})<\operatorname{add}\nu$. Show that there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν, consistent with f, such that $y\mapsto\mu_yE$ is a T-measurable function for every $E\in\Sigma$.

(d) (Törnquist 11) Let (X,Σ,μ) be a countably separated perfect complete strictly localizable measure space, $\mathfrak{A}$ its measure algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ of cardinal at most $\min(\operatorname{add}\mathcal{N},\mathfrak{p})$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$. Show that there is an action $\bullet$ of G on X such that $\pi\bullet E=\{\pi\bullet x:x\in E\}$ belongs to Σ and $(\pi\bullet E)^\bullet=\pi(E^\bullet)$ whenever $\pi\in G$ and $E\in\Sigma$. (*Hint*: 344C, 425Ya.)

535Z Problems (a) Can it be that every probability space has a lifting?

By 535B, it is enough to consider $(\{0,1\}^\kappa,\mathcal{B}\mathfrak{a}(\{0,1\}^\kappa),\nu_\kappa\restriction\mathcal{B}\mathfrak{a}(\{0,1\}^\kappa))$ where κ is a cardinal. Since Mokobodzki's theorem (535Eb) deals with $\kappa\le\omega_2$ when $\mathfrak{c}=\omega_1$, the key case to consider seems to be $\kappa=\omega_3$.

(b) Suppose that $\mathfrak{c}\ge\omega_3$. Does ν_ω have a Borel lifting?

It is known to be relatively consistent with ZFC to suppose that $\mathfrak{c}=\omega_2$ and that $\mathrm{FN}(\mathcal{P}\mathbb{N})=\omega_1$ (554G-554H). In this case ν_ω has a Borel lifting (535E(b-ii)). But if $\mathfrak{c}\ge\omega_3$ then $\mathfrak{B}_\omega$ is not tightly ω_1-filtered (518S).

(c) (A.H.Stone) Can there be a countable ordinal ζ and a lifting ϕ of ν_ω such that ϕE is a Borel set, with Baire class at most ζ, for every Borel set $E\subseteq\{0,1\}^\omega$?

The point of this question is that while, subject to the continuum hypothesis, we can almost write down a formula for a Borel lifting for Lebesgue measure (535Ya), the method gives no control over the Baire classes of the sets constructed.

(d) Can there be a strictly positive Radon probability measure of countable Maharam type which does not have a strong lifting? (See 453G, 453N, 535I, 535Xg.)

(e) Is there a probability space which has a linear lifting but no lifting?

(f) Can there be a Borel linear lifting for the usual measure on $(\{0,1\}^\omega)^2$ which respects coordinates in the sense of 535R?

It seems possible that there is a proof in ZFC that there is no such lifting; in which case 535R shows that we should have a negative answer to (a).

535 Notes and comments For a fuller account of this topic, see Burke 93.

Neumann & Stone 35 used a direct construction along the lines of 535Xc to show that if the continuum hypothesis is true then Lebesgue measure has a Borel lifting. The method works equally well for ν_{ω_1}, but for ν_{ω_2} we need a further idea from Mokobodzki 7?; the version I give here is based on Geschke 02, itself derived at some remove from Carlson Frankiewicz & Zbierski 94, who showed that we could have a Borel lifting for Lebesgue measure in a model in which the continuum hypothesis is false (554I).

It is not a surprise that there should be a model of set theory in which Lebesgue measure has no Borel lifting. Nor is it a surprise that the first such model should have been found by S.Shelah (Shelah 83). What does remain surprising is that in most of the vast number of models of set theory which have been studied, we do not know whether there is such a lifting. Only in the familiar case $\mathfrak{c} = \omega_1$, the special combination $\mathfrak{c} = \omega_2 = \mathrm{FN}(\mathcal{P}\mathbb{N})^+$ (535E), and in variations of Shelah's model, do we have definite information. It remains possible that in any model in which $\mathfrak{m} > \omega_1$ or $\mathfrak{c} = \omega_3$ there is no Borel lifting for Lebesgue measure. When we leave the real line, the position is even more open; conceivably it is relatively consistent with ZFC to suppose that every probability space has a lifting, and at least equally believably it is a theorem of ZFC that ν_{ω_3} does not have a Baire lifting.

From 535I we see that ω_2 appears in Losert's example (453N) for a good reason. Once again, it seems to be unknown whether it is consistent to suppose that there is a (completed) strictly positive Radon probability measure with countable Maharam type which has no strong lifting (535Zd). When we come to look for strong Borel liftings, we have some useful information in the separable metrizable case (535N). The result is natural enough. We are used to supposing that Polish spaces are all very much the same, and that point-supported measures are trivial. But because the concept of 'strong' lifting is topological, and cannot easily be reduced to the Borel structure, we have to work a bit; and it seems also that point-supported measures need care (535M).

'Linear liftings' (535O-535R) remain poor relations. I give them house room here partly for completeness and partly because of a slender hope that they will lead us to a solution of 535Za. Of course the match between ω_3 in 535Za and $\mathfrak{c}^{++}$ in 535R may show only a temporarily coincidental frontier of ignorance. Burke & Shelah 92 have shown that it is relatively consistent with ZFC to suppose that ν_ω has no Borel linear lifting.

536 Alexandra Bellow's problem

In 463Za I mentioned a curious problem concerning pointwise compact sets of continuous functions. This problem is known to be soluble if we are allowed to assume the continuum hypothesis, for instance. Here I present the relevant arguments.

536A The problem I recall some ideas from §463. Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ the space of all Σ-measurable functions from X to $\mathbb{R}$, so that $\mathcal{L}^0$ is a linear subspace of $\mathbb{R}^X$. On $\mathcal{L}^0$ we have the linear space topologies $\mathfrak{T}_p$ and $\mathfrak{T}_m$ of pointwise convergence and convergence in measure (462Ab, 245Ab). $\mathfrak{T}_p$ is Hausdorff and locally convex; if μ is σ-finite, $\mathfrak{T}_m$ is pseudometrizable. The question, already asked in 463Za, is this: suppose that $K \subseteq \mathcal{L}^0$ is compact for $\mathfrak{T}_p$, and that $\mathfrak{T}_m$ is Hausdorff on K. Does it follow that $\mathfrak{T}_p$ and $\mathfrak{T}_m$ agree on K?

536B Known cases Let (X, Σ, μ) be a σ-finite measure space. Given that $K \subseteq \mathcal{L}^0$ is compact for $\mathfrak{T}_p$, and $\mathfrak{T}_m$ is Hausdorff on K, and

either K is sequentially compact for $\mathfrak{T}_p$

or K is countably tight for $\mathfrak{T}_p$

or K is convex

or X has a topology for which $K \subseteq C(X)$, μ is a strictly positive topological measure, and every function $h \in \mathbb{R}^X$ which is continuous on every relatively countably compact set is continuous

or μ is perfect,

then K is metrizable for $\mathfrak{T}_p$, and $\mathfrak{T}_p$ and $\mathfrak{T}_m$ agree on K (463Cd, 463F, 463G, 463H, 463Lc).

Now for the new results.

536C Proposition (see Talagrand 84, 9-3-3.) Let (X, Σ, μ) be a probability space such that the π-weight $\pi(\mu)$ of μ is at most $\mathfrak{p}$. If $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-compact then it is $\mathfrak{T}_m$-compact.

proof (a) For the time being (down to the end of (d) below), suppose that $|f| \le \chi X$ for every $f \in K$. Let $\langle f_i \rangle_{i \in \mathbb{N}}$ be any sequence in K.

(b) For $I \in [\mathbb{N}]^\omega$, write $\limsup_{i\to I} f_i$ for $\inf_{n\in\mathbb{N}} \sup_{i\in I\setminus n} f_i$ and $\liminf_{i\to I} f_i$ for $\sup_{n\in\mathbb{N}} \inf_{i\in I\setminus n} f_i$. Then there is an $I \in [\mathbb{N}]^\omega$ such that $\liminf_{i\to J} f_i =_{\text{a.e.}} \liminf_{i\to I} f_i$ and $\limsup_{i\to J} f_i =_{\text{a.e.}} \limsup_{i\to I} f_i$ for every $J \in [I]^\omega$. **P** (See the proof of 463D.) For $I, J \in [\mathbb{N}]^\omega$ set $\Delta(I) = \int \limsup_{i\to I} f_i - \liminf_{i\to I} f_i$ and say that $J \preccurlyeq I$ if either $J \subseteq I$ or $J \setminus I$ is finite and $I \setminus J$ is infinite. Then $\Delta(J) \le \Delta(I)$ whenever $J \preccurlyeq I$, and any non-increasing sequence in $[\mathbb{N}]^\omega$ has a $\preccurlyeq$-lower bound in $[\mathbb{N}]^\omega$. By 513P, inverted, there is an $I \in [\mathbb{N}]^\omega$ such that $\Delta(J) = \Delta(I)$ whenever $J \preccurlyeq I$, and this I will serve. **Q**

Set $g = \liminf_{i\to I} f_i$ and $h = \limsup_{i\to I} f_i$.

(c) ? Suppose, if possible, that $E = \{x : g(x) < h(x)\}$ is not negligible. Let $\mathcal{H}$ be a coinitial subset of $\Sigma \setminus \mathcal{N}(\mu)$, where $\mathcal{N}(\mu)$ is the null ideal of μ, with cardinal $\pi(\mu) \le \mathfrak{p}$, and $\langle H_\xi\rangle_{\xi<\mathfrak{p}}$ a family running over $\{H : H \in \mathcal{H}, H \subseteq E\}$. Choose $\langle I_\xi\rangle_{\xi<\mathfrak{p}}$, $\langle x_\xi\rangle_{\xi<\mathfrak{p}}$ and $\langle y_\xi\rangle_{\xi<\mathfrak{p}}$ inductively, as follows. The inductive hypothesis will be that, for any $\xi < \mathfrak{p}$, $\langle I_\eta\rangle_{\eta<\xi}$ is a family of infinite subsets of $\mathbb{N}$ such that $I_\eta \setminus I_\zeta$ is finite whenever $\zeta \le \eta < \xi$. Start with $I_0 = I$. For the inductive step to $\xi + 1$, where $\xi < \mathfrak{p}$, since $g =_{\text{a.e.}} \liminf_{i\to I_\xi} f_i$, there must be an $x_\xi \in H_\xi \cap E$ such that $g(x_\xi) = \liminf_{i\to I_\xi} f_i(x)$. Let $J \in [I_\xi]^\omega$ be such that $\lim_{i\to J} f_i(x_\xi) = g(x_\xi)$. Now $\limsup_{i\to J} f_i =_{\text{a.e.}} h$, so we can find a $y_\xi \in E \cap H_\xi$ such that $\limsup_{i\to J} f_i(y_\xi) = h(y_\xi)$ and an $I_{\xi+1} \in [J]^{<\omega}$ such that $\lim_{i\to I_{\xi+1}} f_i(y_\xi) = h(y_\xi)$.

For non-zero limit ordinals $\xi < \mathfrak{p}$, let I_ξ be an infinite subset of I such that $I_\xi \setminus I_\eta$ is finite for every $\eta < \xi$.

At the end of the induction, there will be a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ containing I_ξ for every $\xi < \mathfrak{p}$. Set $f = \lim_{i\to\mathcal{F}} f_i$. Because K is $\mathfrak{T}_p$-compact, $f \in K \subseteq \mathcal{L}^0$. So at least one of the measurable sets $E' = \{x : x \in E, g(x) < f(x)\}$ and $E'' = \{x : x \in E, f(x) < h(x)\}$ is non-negligible and contains H_ξ for some $\xi < \mathfrak{p}$. Now $I_{\xi+1} \in \mathcal{F}$, so $f(x_\xi) = \lim_{i\to I_{\xi+1}} f_i(x_\xi) = g(x_\xi)$ and $f(y_\xi) = h(y_\xi)$. But this means that $x_\xi \in H_\xi \setminus E''$ and $y_\xi \in H_\xi \setminus E'$, so H_ξ cannot be included in either E' or E''. **X**

(d) So $g =_{\text{a.e.}} h$ and $\{x : g(x) = \lim_{i\to I} f_i(x)\}$ includes the conegligible set $\{x : g(x) = h(x)\}$. We also have a $g_0 \in K$ which is a $\mathfrak{T}_p$-cluster point of $\langle f_i\rangle_{i\in I}$. Of course $g \le g_0 \le h$, and all three must be equal μ-a.e. But this means that $\langle f_i\rangle_{i\in I}$ converges almost everywhere to g_0, and therefore converges in measure to g_0 (245Ec). Now recall that $\langle f_i\rangle_{i\in\mathbb{N}}$ was an arbitrary sequence in K. So we see that every sequence in K has a subsequence which is $\mathfrak{T}_m$-convergent to a point of K. As $\mathfrak{T}_m$ is pseudometrizable, K is $\mathfrak{T}_m$-compact (4A2Le).

(e) This concludes the proof when $|f| \le \chi X$ for every $f \in K$. For the general case, let $\phi : \mathbb{R} \to \,]-1, 1[$ be a homeomorphism, and consider $K' = \{\phi f : f \in K\}$. Since $f \mapsto \phi f$ is a $\mathfrak{T}_p$-continuous function from $\mathcal{L}^0$ to itself, K' is $\mathfrak{T}_p$-compact, therefore $\mathfrak{T}_m$-compact, by (a)-(c). Next, $f \mapsto \phi^{-1} f : K' \to K$ is $\mathfrak{T}_m$-continuous. **P** If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in K' which is $\mathfrak{T}_m$-convergent to $f \in K'$, and $\langle g_n\rangle_{n\in\mathbb{N}}$ is a subsequence of $\langle f_n\rangle_{n\in\mathbb{N}}$, then $\langle g_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence $\langle h_n\rangle_{n\in\mathbb{N}}$ converging a.e. to f (245Ka); now $\phi^{-1} h_n$ converges a.e. to $\phi^{-1} f \in K$, so converges in measure to $\phi^{-1} f$. As $\langle g_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\langle \phi^{-1} f_n\rangle_{n\in\mathbb{N}}$ converges in measure to $\phi^{-1} f$. Thus $f \mapsto \phi^{-1} f$ is sequentially continuous for $\mathfrak{T}_m$, therefore continuous (4A2Ld). **Q** So $K = \{\phi^{-1} f : f \in K'\}$ is $\mathfrak{T}_m$-compact, as claimed.

536D Theorem Let (X, Σ, μ) be a probability space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on $\mathcal{L}^0$. Suppose that $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-compact and that $\mu\{x : f(x) \ne g(x)\} > 0$ for any distinct $f, g \in K$, but that K is not $\mathfrak{T}_p$-metrizable.

(a) Every infinite Hausdorff space which is a continuous image of a closed subset of K has a non-trivial convergent sequence.

(b) There is a continuous surjection from a closed subset of K onto $\{0, 1\}^{\omega_1}$.

(c) Every infinite compact Hausdorff space of weight at most ω_1 has a non-trivial convergent sequence.

(d) $\mathfrak{c} > \omega_1$.

(e) The Maharam type of μ is at least 2^{ω_1}.

(f) There is a non-negligible measurable set in Σ which can be covered by ω_1 negligible sets.

(g) $\pi(\mu) > \mathfrak{p}$.

(h) $\mathfrak{m}_{\text{countable}} = \omega_1$.

proof For $f, g \in \mathcal{L}^0$ set $\rho(f, g) = \int \min(1, |f - g|)$; then ρ is a pseudometric on $\mathcal{L}^0$ defining $\mathfrak{T}_m$, and $\rho\restriction K \times K$ is a metric on K. Set $\Delta(\emptyset) = 0$, and for non-empty $A \subseteq \mathcal{L}^0$ set $\Delta(A) = \sup\{\rho(\inf L, \sup L) : \emptyset \ne L \in [A]^{<\omega}\}$. Note that if $A \subseteq K$ has more than one member then $\Delta(A) > 0$, and that $\Delta(A) \le \Delta(B)$ whenever $A \subseteq B$.

(a)(i) ? Suppose, if possible, that Z is an infinite Hausdorff space, $K_0 \subseteq K$ is closed, $\phi : K_0 \to Z$ is a continuous surjection and there is no non-trivial convergent sequence in Z. Write $\mathcal{L}$ for the family of closed subsets L of K_0 such that $\phi[L]$ is infinite. Then $L = \bigcap_{n\in\mathbb{N}} L_n$ belongs to $\mathcal{L}$ for every non-increasing sequence $\langle L_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}$. **P** $\langle\phi[L_n]\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of infinite closed subsets of Z; because Z is supposed to have no non-trivial convergent sequence, $M = \bigcap_{n\in\mathbb{N}} \phi[L_n]$ is infinite (4A2G(h-i)). Since $\phi[L] = M$ (5A4Cf), $L \in \mathcal{L}$. **Q** By 513P again, there is a $K_1 \in \mathcal{L}$ such that $\Delta(L) = \Delta(K_1)$ for every $L \in \mathcal{L}$ such that $L \subseteq K_1$.

(ii) Now there is no non-trivial convergent sequence in $\phi[K_1]$, so $\phi[K_1]$ cannot be scattered (4A2G(h-ii)), and there is a continuous surjection $\psi : \phi[K_1] \to [0,1]$ (4A2G(j-iv)). Let $M \subseteq \phi[K_1]$ be a closed set such that $\psi[M] = [0,1]$ and $\psi{\restriction}M$ is irreducible (4A2G(i-i)). Then M is infinite, has a countable π-base and no isolated points (4A2G(i-ii)). Let $K_2 \subseteq \phi^{-1}[M]$ be a closed set such that $\phi[K_2] = M$ and $\phi{\restriction}K_2$ is irreducible. Then K_2 has a countable π-base, and $\phi[K_2]$ is infinite, so $\Delta[K_2] = \Delta[K_1]$.

Let $\mathcal{V}$ be a countable π-base for the topology of K_2, not containing $\emptyset$. For each $V \in \mathcal{V}$, choose $h_V \in V$. Set $g_0 = \inf_{V\in\mathcal{V}} h_V$, $g_1 = \sup_{V\in\mathcal{V}} h_V$ in $\mathbb{R}^X$. Then g_0 and g_1 are measurable, and

$$\int g_1 - g_0 \ge \Delta(K_2) = \Delta(K_1) > 0.$$

Set $g(x) = \max(\frac{1}{2}(g_0(x) + g_1(x)), g_1(x) - \frac{1}{2})$ for $x \in X$, and

$$E = \{x : g_0(x) < g_1(x)\} = \{x : g(x) < g_1(x)\} = \{x : g_0(x) < g(x)\},$$

so that $\mu E > 0$. For $x \in E$, the set $F_x = \{f : f \in K_2,\ f(x) \le g(x)\}$ is a proper closed subset of K_2, so there is some $V \in \mathcal{V}$ such that $V \cap F_x = \emptyset$. Because $\mathcal{V}$ is countable, there is a $V \in \mathcal{V}$ such that $D = \{x : x \in E,\ V \cap F_x = \emptyset\}$ is non-negligible. But now observe that $f(x) > g(x)$ whenever $f \in V$ and $x \in D$, so $h_U(x) > g(x)$ whenever $U \in \mathcal{V}$, $U \subseteq V$ and $x \in D$. Set $\mathcal{V}' = \{U : U \in \mathcal{V},\ U \subseteq V\}$, $g_0' = \inf_{U\in\mathcal{V}'} h_U$ and $L = \{f : f \in K_2,\ g_0' \le f \le g_1\}$. Then $g \le g_0'$ and

$$\{x : x \in X,\ g_1(x) - g_0'(x) < \min(1, g_1(x) - g_0(x))\} \supseteq D$$

is non-negligible, so

$$\Delta(L) \le \int \min(1, g_1 - g_0') < \int \min(1, g_1 - g_0) = \Delta(K_1).$$

On the other hand, L meets every member of $\mathcal{V}'$, so $L \cap V$ is dense in V and L includes V. Because $\phi{\restriction}K_2$ is irreducible, $\phi[K_2 \setminus V] \ne M$ and $\phi[L]$ includes the non-empty open subset $M \setminus \phi[K_2 \setminus V]$ of M, which is infinite because M has no isolated points. So $\Delta(L)$ ought to be equal to $\Delta(K_1)$, by the choice of K_1. **X**

Thus (a) is true.

(b) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in K which converges at almost every point of X, then any two $\mathfrak{T}_p$-cluster points of $\langle f_n\rangle_{n\in\mathbb{N}}$ must be equal a.e. and therefore equal, so $\langle f_n\rangle_{n\in\mathbb{N}}$ is $\mathfrak{T}_p$-convergent (5A4Ce).

? Suppose, if possible, that there is no continuous surjection from a closed subset of K onto $\{0,1\}^{\omega_1}$. Then 463D tells us that every sequence in K has a subsequence which is convergent almost everywhere, therefore convergent. So K is sequentially compact, which is impossible, as noted in 536B. **X**

(c) Since $[0,1]$ is a continuous image of $\{0,1\}^{\mathbb{N}}$, $[0,1]^{\omega_1}$ is a continuous image of $\{0,1\}^{\omega_1\times\mathbb{N}} \cong \{0,1\}^{\omega_1}$ and therefore of a closed subset of K. If Z is an infinite compact Hausdorff space of weight at most ω_1, it is homeomorphic to a closed subset of $[0,1]^{\omega_1}$ (5A4Cc) and therefore to a continuous image of a closed subset of K. By (a), Z must have a non-trivial convergent sequence.

(d) Since $\beta\mathbb{N}$ has weight $\mathfrak{c}$ (5A4Ia), is infinite, but has no non-trivial convergent subsequence (4A2I(b-v)), we must have $\omega_1 < \mathfrak{c}$.

(e)(i) If F_1, F_2 are disjoint non-empty $\mathfrak{T}_p$-closed subsets of K, then $\rho(F_1, F_2) > 0$. **P?** Otherwise, there are sequences $\langle f_n\rangle_{n\in\mathbb{N}}$ in F_1, $\langle g_n\rangle_{n\in\mathbb{N}}$ in F_2 such that $\rho(f_n, g_n) \le 2^{-n}$ for every $n \in \mathbb{N}$. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$ and set $f = \lim_{n\to\mathcal{F}} f_n$, $g = \lim_{n\to\mathcal{F}} g_n$, taking the limits in K for the topology $\mathfrak{T}_p$. Then, for any $n \in \mathbb{N}$,

$$\{x : |f(x) - g(x)| > 2^{-n}\} \subseteq \bigcup_{i\ge 2n} \{x : |f_i(x) - g_i(x)| > 2^{-n}\}$$

has measure at most $\sum_{i=2n}^{\infty} 2^{-i+n} = 2^{-n+1}$, so $f =_{\text{a.e.}} g$ and $f = g$; but $f \in F_1$ and $g \in F_2$, so this is impossible. **X** **Q**

(ii) By (b), there are a closed subset K_0 of K and a continuous surjection $\psi : K_0 \to \{0,1\}^{\omega_1}$. For $\xi < \omega_1$, set $F_\xi = \{f : f \in K_0,\ \psi(f)(\xi) = 0\}$, $F_\xi' = \{f : f \in K_0,\ \psi(f)(\xi) = 1\}$; then $\rho(F_\xi, F_\xi') > 0$. There must therefore be a $\delta > 0$ such that $C = \{\xi : \rho(F_\xi, F_\xi') \ge \delta\}$ is uncountable. For each $D \subseteq C$, choose $h_D \in K_0$ such that $\psi(h_D){\restriction}C = \chi D$. Then $\rho(h_D, h_{D'}) \ge \delta$ for all distinct D, $D' \subseteq C$. Thus $A = \{h_D^\bullet : D \subseteq C\}$ is a subset of $L^0 = L^0(\mu)$, of cardinal 2^{ω_1}, such that any two members of A are distance at least δ apart for the metric on L^0 corresponding to ρ. Accordingly the cellularity and topological density of L^0 are at least 2^{ω_1}; by 529Bb, the Maharam type of μ is at least 2^{ω_1}.

(f)(i) By (b), there is a continuous surjection $\psi : K_0 \to \{0,1\}^{\omega_1}$ where $K_0 \subseteq K$ is closed. Let Q be the set of pairs (F,C) such that $F \subseteq K_0$ is closed, $C \subseteq \omega_1$ is closed and cofinal and $\{\psi(f)\restriction C : f \in F\} = \{0,1\}^C$. If $\langle (F_n, C_n)\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Q, then it has a lower bound in Q. **P** Set $F = \bigcap_{n\in\mathbb{N}} F_n$ and $C = \bigcap_{n\in\mathbb{N}} C_n$. Then for any $z \in \{0,1\}^C$ and $n \in \mathbb{N}$ there is an $f_n \in F_n$ such that $\psi(f_n)\restriction C = z$; now take a $\mathfrak{T}_p$-cluster point f of $\langle f_n\rangle_{n\in\mathbb{N}}$, and see that $f \in F$ and that $\psi(f)\restriction C = z$. As z is arbitrary, $(F,C) \in Q$. **Q** By 513P once more, there is a member (K_1, C^*) of Q such that $\Delta(F) = \Delta(K_1)$ whenever $(F,C) \in Q$, $F \subseteq K_1$ and $C \subseteq C^*$. Now C^* is order-isomorphic to ω_1 and its order topology agrees with the subspace topology induced by the order topology of ω_1 (4A2Rm). Let $\theta : \omega_1 \to C^*$ be an order-isomorphism and set $\psi_1(f) = \psi(f)\theta$ for $f \in K_1$. Then $\psi_1 : K_1 \to \{0,1\}^{\omega_1}$ is a continuous surjection, and if $F \subseteq K_1$ is closed, $C \subseteq \omega_1$ is closed and cofinal and $\{\psi_1(f)\restriction C : f \in F\} = \{0,1\}^C$, then $(F, \theta[C]) \in Q$ so $\Delta(F) = \Delta(K_1)$.

(ii) Let $K_2 \subseteq K_1$ be a compact set such that $\psi_1\restriction K_2$ is an irreducible surjection onto $\{0,1\}^{\omega_1}$ (4A2G(i-i) again). Because $\{0,1\}^{\omega_1}$ is separable (4A2B(e-ii)), so is K_2 (5A4C(d-i)). Let $\langle f_n\rangle_{n\in\mathbb{N}}$ enumerate a dense subset of K_2. Because K_2 is compact in $\mathbb{R}^X$, $h_1 = \sup_{n\in\mathbb{N}} f_n$ and $h_0 = \inf_{n\in\mathbb{N}} f_n$ are defined in $\mathbb{R}^X$, and of course they belong to $\mathcal{L}^0$. If $f \in K_2$, then

$$f(x) \in \overline{\{f_n(x) : n \in \mathbb{N}\}} \subseteq [h_0(x), h_1(x)]$$

for every x, and $h_0 \le f \le h_1$. Accordingly we have

$$\Delta(K_2) \le \rho(h_0,h_1) = \sup_{n\in\mathbb{N}} \rho(\inf_{i\le n} f_i, \sup_{i\le n} f_i) \le \Delta(K_2).$$

Let $\mathcal{U}$ be the family of non-empty cylinder sets in $\{0,1\}^{\omega_1}$. For $U \in \mathcal{U}$ set $I_U = \{n : n \in \mathbb{N},\ \psi_1(f_n) \in U\}$ and $g_U = \inf\{f_n : n \in I_U\}$. Observe that $F_U = \{f : f \in K_2,\ g_U \le f \le h_1\}$ is a closed subset of K_1 and that $F_U \cap \psi_1^{-1}[U]$ is dense in $\psi_1^{-1}[U]$, so $U \cap \psi_1[F_U]$ must be dense in U and $U \subseteq \psi_1[F_U]$. There is a finite set $I \subseteq \omega_1$ such that U is determined by coordinates in I; in this case, $C = \omega_1 \setminus I$ is closed and cofinal in ω_1, and $\{z\restriction C : z \in U\} = \{0,1\}^C$. By the choice of K_1, $\Delta(F_U) = \Delta(K_1)$. As $F_U \subseteq [g_U, h_1]$ in $\mathcal{L}^0$, $\rho(g_U,h_1) = \Delta(K_1) = \rho(h_0,h_1)$, and $\min(1, h_1 - g_U) =_{\text{a.e.}} \min(1, h_1 - h_0)$.

Set $h(x) = \max(\frac{1}{2}(h_0(x) + h_1(x)), h_1(x) - \frac{1}{2})$ for $x \in X$, and $E = \{x : h_0(x) < h_1(x)\} = \{x : h(x) < h_1(x)\}$, so that E is measurable and not negligible. If $U \in \mathcal{U}$, then

$$\begin{aligned} E_U &= \{x : x \in E,\ h(x) \le g_U(x)\} \\ &\subseteq \{x : x \in E,\ h_1(x) - g_U(x) < \min(1, h_1(x) - h_0(x))\} \end{aligned}$$

is negligible.

For every $x \in E$, $F'_x = \{f : f \in K_2,\ f(x) \le h(x)\}$ is a proper closed subset of K_2, so $\psi_1[F'_x] \ne \{0,1\}^{\omega_1}$ and there is some $U \in \mathcal{U}$ such that $U \cap \psi_1[F'_x] = \emptyset$. In this case $f_n \notin F'_x$, that is, $f_n(x) > h(x)$, for every $n \in I_U$, so $g_U(x) \ge h(x)$. Thus $E = \bigcup_{U\in\mathcal{U}} E_U$ is a non-negligible measurable set covered by ω_1 negligible sets.

(g) This is immediate from 536C.

(h) Continuing the argument from (f), define $\phi : X \to \mathbb{R}^{\mathbb{N}}$ by setting $\phi(x) = \langle f_n(x)\rangle_{n\in\mathbb{N}}$ for $x \in X$. Then ϕ is measurable (418Bd), so we have a non-zero totally finite Borel measure ν on $\mathbb{R}^{\mathbb{N}}$ defined by setting $\nu H = \mu(E \cap \phi^{-1}[H])$ for every Borel set $H \subseteq \mathbb{R}^{\mathbb{N}}$. Note that $\phi[X] \subseteq \ell^\infty$ and that $\ell^\infty = \bigcup_{n\in\mathbb{N}} \bigcap_{i\in\mathbb{N}} \{w : |w(i)| \le n\}$ is an F_σ set in $\mathbb{R}^{\mathbb{N}}$. Now set

$$h'_1(w) = \sup_{n\in\mathbb{N}} w(n), \quad h'_0(w) = \inf_{n\in\mathbb{N}} w(n),$$

$$h'(w) = \max(\tfrac{1}{2}(h'_0(w) + h'_1(w)), h'_1(w) - \tfrac{1}{2})$$

for $w \in \ell^\infty$, so that $h_1 = h'_1\phi$, $h_0 = h'_0\phi$ and $h = h'\phi$; for $U \in \mathcal{U}$, set

$$E'_U = \{w : w \in \ell^\infty,\ h'(w) \le \inf_{n\in I_U} w(n)\}$$

so that E'_U is an F_σ set and $E_U = E \cap \phi^{-1}[E'_U]$; accordingly $\nu E'_U = 0$. Because $E \subseteq \bigcup_{U\in\mathcal{U}} E_U$, $\phi[E] \subseteq \bigcup_{U\in\mathcal{U}} E'_U$.

Thus we have a non-negligible subset of $\mathbb{R}^{\mathbb{N}}$ which is covered by ω_1 negligible F_σ sets and therefore by ω_1 closed negligible sets. By 526M, $\mathfrak{m}_{\text{countable}} = \omega_1$.

536X Basic exercises (a) Let (X,Σ,μ) be a complete measure space, with null ideal $\mathcal{N}(\mu)$. Suppose that $\operatorname{add}\mathcal{N}(\mu) = \operatorname{cov}\mathcal{N}(\mu)$. Show that there is a $\mathfrak{T}_p$-compact $\mathfrak{T}_m$-compact $K \subseteq \mathcal{L}^0(\Sigma)$ such that the identity map on K is not $(\mathfrak{T}_p, \mathfrak{T}_m)$-continuous.

(b) Let (X, Σ, μ) be a perfect measure space. For $E \subseteq X$, write $\mathcal{N}(\mu_E)$ for the null ideal of the subspace measure on E. Suppose that $\mathrm{non}(E, \mathcal{N}(\mu_E)) < \mathrm{cov}(E, \mathcal{N}(\mu_E))$ for every non-negligible measurable set E of finite measure. Show that if $K \subseteq \mathcal{L}^0(\Sigma)$ is $\mathfrak{T}_p$-compact, then the identity map on K is $(\mathfrak{T}_p, \mathfrak{T}_m)$-continuous.

536Y Further exercises (a) Suppose that the additivity and covering number of the Lebesgue null ideal are equal. Find a strictly localizable perfect measure space (X, Σ, μ) and a $\mathfrak{T}_p$-compact $K \subseteq \mathcal{L}^0(\Sigma)$ such that $\mathfrak{T}_m$ is Hausdorff on K but K is not $\mathfrak{T}_m$-compact.

536 Notes and comments The methods here are derived from ideas of M.Talagrand. They seem frustratingly close to delivering an answer to the original question. But it seems clear that even if a positive answer – every $\mathfrak{T}_p$-compact $\mathfrak{T}_m$-separated set is metrizable – is true in ZFC, some further idea will be needed in the proof. On the other side, while it may well be that in some familiar model of set theory there is a negative answer, parts (c), (d) and (g) of 536D give simple tests to rule out many candidates.

537 Sierpiński sets, shrinking numbers and strong Fubini theorems

W.Sierpiński observed that if the continuum hypothesis is true then there are uncountable subsets of $\mathbb{R}$ which have no uncountable negligible subsets, and that such sets lead to curious phenomena; he also observed that, again assuming the continuum hypothesis, there would be a (non-measurable) function $f : [0,1]^2 \to \{0,1\}$ for which Fubini's theorem failed radically, in the sense that

$$\iint f(x,y)dxdy = 0, \quad \iint f(x,y)dydx = 1.$$

In this section I set out to explore these two insights in the light of the concepts introduced in Chapter 52. I start with definitions of 'Sierpiński' and 'strongly Sierpiński' set (537A), with elementary facts and an excursion into the theory of 'entangled' sets (537C-537G). Turning to repeated integration, I look at three interesting cases in which, for different reasons, some form of separate measurability is enough to ensure equality of repeated integrals (537I, 537L, 537S). Working a bit harder, we find that there can be valid non-trivial inequalities of the form $\overline{\int}\underline{\int} dxdy \le \overline{\int}\,\overline{\int} dydx$ (537N-537Q).

As elsewhere, I will write $\mathcal{N}(\mu)$ for the null ideal of a measure μ.

537A Definitions (a) If (X, Σ, μ) is a measure space, a subset of X is a **Sierpiński set** if it is uncountable but meets every negligible set in a countable set.

(b) If (X, Σ, μ) is a measure space, a subset A of X is a **strongly Sierpiński set** if it is uncountable and for every $n \ge 1$ and for every set $W \subseteq X^n$ which is negligible for the (c.l.d.) product measure on X^n, the set $\{u : u \in A^n \cap W, u(i) \ne u(j) \text{ for } i < j < n\}$ is countable.

537B Proposition (a) Let (X, Σ, μ) be a measure space and $A \subseteq X$ a Sierpiński set.

(i) $\mathrm{add}\,\mathcal{N}(\mu) = \mathrm{non}\,\mathcal{N}(\mu) = \omega_1$ and $\mathrm{cov}\,\mathcal{N}(\mu) \ge \#(A)$.

(ii) If $\{x\}$ is negligible for every $x \in A$, then $\mathrm{cf}\,\mathcal{N}(\mu) \ge \mathrm{cf}([\#(A)]^{\le\omega})$.

(b) Suppose that (X, Σ, μ) and (Y, T, ν) are measure spaces such that singleton subsets of Y are negligible. Let $f : X \to Y$ be an inverse-measure-preserving function.

(i) If $A \subseteq X$ is a Sierpiński set, then $f[A]$ is a Sierpiński set in Y and $\#(f[A]) = \#(A)$.

(ii) Now suppose that ν is σ-finite. If $A \subseteq X$ is a strongly Sierpiński set, then $f[A]$ is a strongly Sierpiński set in Y.

(c) Suppose that λ and κ are infinite cardinals and that (X, Σ, μ) is a locally compact semi-finite measure space of Maharam type at most λ in which singletons are negligible and $\mu X > 0$. Give $\{0,1\}^\lambda$ its usual measure.

(i) If $\{0,1\}^\lambda$ has a Sierpiński subset of size κ, then X has a Sierpiński subset of size κ.

(ii) If $\{0,1\}^\lambda$ has a strongly Sierpiński subset of size κ, then X has a strongly Sierpiński subset of size κ.

proof (a)(i) We are told that A is uncountable; now any subset of A with ω_1 members witnesses that $\mathrm{non}\,\mathcal{N}(\nu) \le \omega_1$. On the other hand, if $\mathcal{E}$ is a family of negligible sets covering X, then $\#(A) \le \max(\omega, \#(\mathcal{E}))$, so $\#(\mathcal{E}) \ge \#(A)$; as $\mathcal{E}$ is arbitrary, $\mathrm{cov}\,\mathcal{N}(\mu) \ge \#(A)$.

(ii) If $\{x\}$ is negligible for every $x \in A$, then $[A]^{\le\omega} \subseteq \mathcal{N}(\mu)$, and the identity function is a Tukey function from $[A]^{\le\omega}$ to $\mathcal{N}(\mu)$; so $\mathrm{cf}[A]^{\le\omega} \le \mathrm{cf}\,\mathcal{N}(\mu)$.

(b)(i) If $y \in Y$, then $\{y\}$ and $f^{-1}[\{y\}]$ are negligible, so $A \cap f^{-1}[\{y\}]$ is countable; consequently $\#(A) \le \max(\omega, \#(f[A]))$ and $\#(f[A]) = \#(A)$. If $F \subseteq Y$ is negligible, then $f^{-1}[F]$ is negligible so $A \cap f^{-1}[F]$ and $f[A] \cap F$ are countable. So $f[A]$ is a Sierpiński set.

(ii) Let $W \subseteq Y^n$ be a negligible set for the product measure λ' on Y^n, where $n \ge 1$. Define $\boldsymbol{f} : X^n \to Y^n$ by saying that $\boldsymbol{f}(x_0, \dots, x_{n-1}) = (f(x_0), \dots, f(x_{n-1}))$ for $x_0, \dots, x_{n-1} \in X$. Because ν is σ-finite, $\boldsymbol{f}$ is inverse-measure-preserving for λ and λ' (251Wp). If W is λ'-negligible, then $\boldsymbol{f}^{-1}[W]$ is λ-negligible, and $B = \{u : u \in A^n \cap \boldsymbol{f}^{-1}[W],\ u(i) \ne u(j) \text{ for } i < j < n\}$ is countable. Consequently

$$\{v : v \in f[A]^n \cap W,\ v(i) \ne v(j) \text{ for } i < j < n\} \subseteq \boldsymbol{f}[B]$$

is countable.

(c) Take any set $E \subseteq X$ of non-zero finite measure, and give E its normalized subspace measure $\mu'_E = (\mu E)^{-1}\mu_E$. Then there is an $f : \{0,1\}^\lambda \to E$ which is inverse-measure-preserving for ν_λ and μ'_E (343Cd). So (b) tells us that E has a subset A of size κ which is Sierpiński or strongly Sierpiński for μ'_E. But now A is still Sierpiński or strongly Sierpiński for μ.

537C Entangled sets (a) Definition If X is a totally ordered set, then X is **ω_1-entangled** if whenever $n \ge 1$, $I \subseteq n$ and $\langle x_{\xi i}\rangle_{\xi<\omega_1, i<n}$ is a family of distinct elements of X, then there are distinct ξ, $\eta < \omega_1$ such that $I = \{i : i < n,\ x_{\xi i} \le x_{\eta i}\}$.

(b) Give $\{0,1\}^{\mathbb{N}}$ its lexicographic ordering, that is,

$$x \le y \text{ iff either } x = y \text{ or there is an } n \in \mathbb{N} \text{ such that } x\restriction n = y\restriction n \text{ and } x(n) < y(n).$$

Then the map $x \mapsto \frac{2}{3}\sum_{n=0}^{\infty} 3^{-n}x(n) : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is an order-isomorphism between $\{0,1\}^{\mathbb{N}}$ and the Cantor set, so any ω_1-entangled subset of $\{0,1\}^{\mathbb{N}}$ can be transferred to an ω_1-entangled subset of $\mathbb{R}$.

537D Lemma Let X be an ω_1-entangled totally ordered set.

(a) There is a countable set $D \subseteq X$ which meets $[x, y]$ whenever $x < y$ in X.

(b) Whenever $n \ge 1$, $I \subseteq n$ and $\langle x_{\xi i}\rangle_{\xi<\omega_1, i<n}$ is a family of distinct elements of X, there are $\xi < \eta < \omega_1$ such that $I = \{i : i < n,\ x_{\xi i} \le x_{\eta i}\}$.

proof (a)(i) There is a countable set $D_0 \subseteq X$ which meets $[x, z]$ whenever $x < y < z$ in X. **P?** Otherwise, choose $\langle x_{\xi i}\rangle_{\xi<\omega_1, i<3}$ inductively so that $x_{\xi 0} < x_{\xi 1} < x_{\xi 2}$ and $[x_{\xi 0}, x_{\xi 2}]$ does not meet $\{x_{\eta i} : \eta < \xi,\ i < 3\}$. Now, if ξ, $\eta < \omega_1$ are different, we cannot have

$$x_{\xi 0} \le x_{\eta 0}, \quad x_{\xi 1} > x_{\eta 1}, \quad x_{\xi 2} \le x_{\eta 2}.$$

So $\langle x_{\xi i}\rangle_{\xi<\omega_1, i<3}$ witnesses that X is not ω_1-entangled. **XQ**

(ii) Set $A = \{(x, y) : x < y,\ [x, y] \cap D_0 = \emptyset\}$. Note that if (x, y), $(x', y') \in A$ are distinct, then $[x, y] \cap [x', y'] = \emptyset$, since otherwise $[\min(x, x'), \max(y, y')]$ would be an interval disjoint from D_0 with at least three elements. It follows that A is countable. **P?** Otherwise, let $\langle(x_{\xi 0}, x_{\xi 1})\rangle_{\xi<\omega_1}$ be a family of distinct elements of A. Then all the $x_{\xi i}$ are distinct. But if ξ, $\eta < \omega_1$ are different, we cannot have

$$x_{\xi 0} \le x_{\eta 0}, \quad x_{\xi 1} > x_{\eta 1}.$$

So $\langle x_{\xi i}\rangle_{\xi<\omega_1, i<2}$ witnesses that X is not entangled. **XQ**

(iii) So if we set $D = D_0 \cup \{x : (x, y) \in A\}$ we shall have a suitable countable set.

(b) For $i < n$ write $\le_i = \le$ if $i \in I$, $\le_i = \ge$ if $i \in n \setminus I$; we are seeking $\xi < \eta$ such that $x_{\xi i} \le_i x_{\eta i}$ for every $i < n$. For each family $\boldsymbol{d} = \langle d_i\rangle_{i<n}$ in D, set $A_{\boldsymbol{d}} = \{\xi : x_{\xi i} \le_i d_i \text{ for each } i < n\}$. Let $\zeta < \omega_1$ be such that $A_{\boldsymbol{d}} \cap \zeta \ne \emptyset$ whenever $\boldsymbol{d} \in D^n$ and $A_{\boldsymbol{d}} \ne \emptyset$. Now there are distinct ξ', $\eta \in \omega_1 \setminus \zeta$ such that $x_{\xi' i} \le_i x_{\eta i}$ for every $i < n$. For each $i < n$, there is a $d_i \in D$ such that $x_{\xi' i} \le_i d_i \le_i x_{\eta i}$. Set $\boldsymbol{d} = \langle d_i\rangle_{i<n}$; then $\xi' \in A_{\boldsymbol{d}}$ so there is a $\xi \in \zeta \cap A_{\boldsymbol{d}}$. Now $\xi < \eta$ and $x_{\xi i} \le_i x_{\eta i}$ for every i, as required.

537E Lemma Suppose that $n \ge 1$, $I \subseteq n$ and that $A \subseteq (\{0,1\}^{\mathbb{N}})^n$ is non-negligible for the usual product measure $\nu_{\mathbb{N}}^n$ on $(\{0,1\}^{\mathbb{N}})^n$. Let $\le$ be the lexicographic ordering of $\{0,1\}^{\mathbb{N}}$. Then there are v, $w \in A$ such that $v(i) \ne w(i)$ for every $i < n$ and $\{i : i < n,\ v(i) \le w(i)\} = I$.

proof For each $k \in \mathbb{N}$ let Σ_k be the algebra of subsets of $X = (\{0,1\}^{\mathbb{N}})^n$ generated by sets of the form $\{v : v \in X, v(i)(j) = 1\}$ for $i < n$ and $j < k$. Then $\langle \Sigma_k \rangle_{k \in \mathbb{N}}$ is a non-decreasing sequence of finite algebras and the σ-algebra generated by $\bigcup_{k \in \mathbb{N}} \Sigma_k$ is the Borel σ-algebra $\mathcal{B}(X)$ of X. Let $E \in \mathcal{B}(X)$ be a measurable envelope of A for $\nu_{\mathbb{N}}^n$. For each $k \in \mathbb{N}$, let f_k be the conditional expectation of χE on Σ_k, that is,

$$f_k(u) = 2^{kn} \nu_{\mathbb{N}}^n \{v : v \in E,\, v(i) \restriction k = u(i) \restriction k \text{ for every } i < n\}$$

for $u \in X$. By Lévy's martingale theorem (275I), $\chi E =_{\text{a.e.}} \lim_{k \to \infty} f_k$. In particular, there are a $u \in A$ and a $k \in \mathbb{N}$ such that $f_k(u) > 1 - 2^{-n}$. But this means that

$$F = \{v : v \in E,\, v(i) \restriction k = u(i) \restriction k \text{ for every } i < n\}$$

has measure greater than $2^{-kn}(1 - 2^{-n})$, and both the sets

$$F' = \{v : v \in F,\, v(i)(k) = 0 \text{ for } i \in I,\, v(i)(k) = 1 \text{ for } i \in n \setminus I\},$$

$$F'' = \{w : w \in F,\, w(i)(k) = 1 \text{ for } i \in I,\, w(i)(k) = 0 \text{ for } i \in n \setminus I\},$$

must have positive measure. Accordingly we can find $v \in A \cap F'$ and $w \in A \cap F''$, and these will serve.

537F Corollary Suppose that $A \subseteq \{0,1\}^{\mathbb{N}}$ is strongly Sierpiński for the usual measure on $\{0,1\}^{\mathbb{N}}$. Then A is ω_1-entangled for the lexicographic ordering of $\{0,1\}^{\mathbb{N}}$.

proof Let $\langle x_{\xi i} \rangle_{\xi < \omega_1, i < n}$ be a family of distinct points in A, where $n \geq 1$, and I a subset of n. Then $x_\xi = \langle x_{\xi i} \rangle_{i < n}$ belongs to A^n, and has no two coordinates the same, for every $\xi < \omega_1$. So $D = \{x_\xi : \xi < \omega_1\}$ cannot be negligible. By 537E, there are distinct $\xi, \eta < \omega_1$ such that $I = \{i : x_{\xi i} \leq x_{\eta i}\}$.

537G Theorem (Todorčević 85) Suppose that there is an ω_1-entangled totally ordered set X of size $\kappa \geq \omega_1$. Then there are two upwards-ccc partially ordered sets P, Q such that $c^{\uparrow}(P \times Q) \geq \kappa$.

proof (a) Let $Y \subseteq X$ be a set such that $\#(Y) = \#(X \setminus Y) = \kappa$, and $f : Y \to X \setminus Y$ an injective function. Set

$$P = \{I : I \in [Y]^{<\omega},\, f \restriction I \text{ is order-preserving}\},$$

$$Q = \{I : I \in [Y]^{<\omega},\, f \restriction I \text{ is order-reversing}\},$$

both ordered by $\subseteq$. Then $\{(\{y\}, \{y\}) : y \in Y\}$ is an up-antichain in $P \times Q$, so $c^{\uparrow}(P \times Q) \geq \kappa$.

(b) P is upwards-ccc. **P** Let $\langle I_\alpha \rangle_{\alpha < \omega_1}$ be a family in P. By the Δ-system Lemma (4A1Db), there is an uncountable set $A \subseteq \omega_1$ such that $\langle I_\alpha \rangle_{\alpha \in A}$ is a Δ-system with root I say; now there is an $n \in \mathbb{N}$ such that $B = \{\alpha : \alpha \in A, \#(I_\alpha \setminus I) = n\}$ is uncountable. If $n = 0$ then $I_\alpha = I_\beta$ are upwards-compatible for any $\alpha, \beta \in B$ and we can stop.

If $n \geq 1$, enumerate $I_\alpha \setminus I$ in increasing order as $\langle x_{\alpha i} \rangle_{i < n}$, for each $\alpha \in B$. Let $D \subseteq X$ be a countable set such that D meets every interval in X with more than one member (537Da). For $i < j < n$ and $\alpha \in B$ let $d_{\alpha ij}$, $d'_{\alpha ij} \in D$ be such that $x_{\alpha i} \leq d_{\alpha ij} \leq x_{\alpha j}$ and $f(x_{\alpha i}) \leq d'_{\alpha ij} \leq f(x_{\alpha j})$. (Because $I_\alpha \in P$, $f \restriction I_\alpha$ is order-preserving so $f(x_{\alpha i}) < f(x_{\alpha j})$.) Let $\langle d_{ij} \rangle_{i < j < n}$, $\langle d'_{ij} \rangle_{i < j < n}$ be such that

$$C = \{\alpha : \alpha \in B,\, d_{\alpha ij} = d_{ij} \text{ and } d'_{\alpha ij} = d'_{ij} \text{ whenever } i < j < n\}$$

is uncountable.

Consider the family $\langle y_{\alpha i} \rangle_{\alpha \in C, i < 2n}$ where $y_{\alpha i} = x_{\alpha i}$ and $y_{\alpha, i+n} = f(x_{\alpha i})$ if $i < n$. Because X is entangled, there must be distinct α, $\beta \in C$ such that $y_{\alpha i} \leq y_{\beta i}$ for every $i < 2n$, that is, $x_{\alpha i} \leq x_{\beta i}$ and $f(x_{\alpha i}) \leq f(x_{\beta i})$ for every $i < n$. But now examine $I = I_\alpha \cup I_\beta$. If x, $x' \in I$ and $x \leq x'$,

either both x and x' belong to I_α and $f(x) \leq f(x')$ because $I_\alpha \in P$,

or both x and x' belong to I_β and $f(x) \leq f(x')$,

or $x = x_{\alpha i}$ and $x' = x_{\beta j}$ where $i < j < n$, so that

$$f(x) = f(x_{\alpha i}) \leq d'_{ij} \leq f(x_{\beta_j}) = f(x'),$$

or $x = x_{\beta i}$ and $x' = x_{\alpha j}$ where $i < j < n$, so that $f(x) \leq f(x')$,

or $x = x_{\alpha i}$ and $x' = x_{\beta i}$ where $i < n$, so that $f(x) = f(x_{\alpha i}) \leq f(x_{\beta i}) = f(x')$.

(Note that we cannot have $x = x_{\alpha i}$ and $x' = x_{\beta j}$ with $j < i$, because in this case $x_{\beta j} \leq d_{ji} \leq x_{\alpha i}$ while $x_{\beta j} \neq x_{\alpha i}$; nor can we have $x = x_{\beta i} < x' = x_{\alpha i}$ with $i < n$.) So $f \restriction I$ is order-preserving and $I \in P$ witnesses that I_α and I_β are upwards-compatible in P. As $\langle I_\alpha \rangle_{\alpha < \omega_1}$ is arbitrary, P is upwards-ccc. **Q**

(c) Similarly, Q is upwards-ccc. **P** The principal changes needed in the argument above are

— in the choice of the $d'_{\alpha ij}$, we need to write '$f(x_{\alpha i}) \ge d'_{\alpha ij} \ge f(x_{\alpha j})$';

— in the choice of particular α and β in the set C, we need to write '$y_{\alpha i} \le y_{\beta i}$ for $i < n$ and $y_{\alpha i} \ge y_{\beta i}$ for $n \le i < 2n$'. **Q**

So P and Q satisfy our requirements.

537H Scalarly measurable functions (a) Definition Let X be a set, Σ a σ-algebra of subsets of X and U a linear topological space. A function $\phi : X \to U$ is **scalarly (Σ-)measurable** if $f\phi : X \to \mathbb{R}$ is (Σ-)measurable for every $f \in U^*$.

(b) If $\phi : X \to U$ is scalarly measurable, V is another linear topological space and $T : U \to V$ is a continuous linear operator, then $T\phi : X \to V$ is scalarly measurable, because $hT \in U^*$ for every $h \in V^*$.

(c) If U is a separable metrizable locally convex space and $\phi : X \to U$ is scalarly measurable, then it is measurable. **P** $\mathrm{T} = \{F : F \subseteq U, \phi^{-1}[F] \in \Sigma\}$ includes the cylindrical σ-algebra of U (4A3T), which is the Borel σ-algebra (4A3V). **Q**

537I Proposition Let (X, Σ, μ) and (Y, T, ν) be probability spaces and U a reflexive Banach space. Suppose that $x \mapsto u_x : X \to U$ and $y \mapsto f_y : Y \to U^*$ are bounded scalarly measurable functions. Then $\iint f_y(u_x)\mu(dx)\nu(dy)$ and $\iint f_y(u_x)\nu(dy)\mu(dx)$ are defined and equal.

proof (a)(i) Recall from 467Hc that if $V \subseteq U$ and $W \subseteq U^*$ are closed linear subspaces, I call them a 'projection pair' if $U = V \oplus W^\circ$ and $\|v + v'\| \ge \|v\|$ for all $v \in V$ and $v' \in W^\circ$. We need to know that this is symmetric; that is, that in this case

$$U^* = W \oplus V^\circ, \quad \|g + g'\| \ge \|g\| \text{ for all } g \in W,\ g' \in V^\circ.$$

P Note first that if $g \in W \cap V^\circ$, then $g(u) = 0$ for every $u \in W^\circ + V$, that is, $g = 0$. Now take any $f \in U^*$. Define $g : U \to \mathbb{R}$ by saying that $g(v + v') = f(v)$ for $v \in V$, $v' \in W^\circ$. Then g is linear and continuous and $\|g\| \le \|f\|$. Now $g(v') = 0$ for every $v' \in W^\circ$, that is, $g \in W^{\circ\circ}$, which is the weak*-closure of W (4A4Eg); but as U and U^* are reflexive, this is just the norm-closure of W, which is equal to W. Set $g' = f - g$. Then $g' \in V^\circ$. This shows that $f \in W + V^\circ$; as f is arbitrary, $U^* = W \oplus V^\circ$. Finally, I remarked in the course of the argument that $\|g\| \le \|f\|$, which is what we need to know to check that $\|g\| \le \|g + g'\|$ whenever $g \in W$ and $g' \in V^\circ$. **Q**

(ii) Because U is reflexive, its unit ball is weakly compact, so U is surely weakly compactly generated, therefore weakly K-countably determined (467M). Now turn to Lemma 467J. This tells us that there is a family $\mathcal{M}$ of subsets of $U \cup U^*$ such that

for every $B \subseteq X \cup X^*$ there is an $M \in \mathcal{M}$ such that $B \subseteq M$ and $\#(M) \le \max(\omega, \#(B))$;

whenever $\mathcal{M}' \subseteq \mathcal{M}$ is upwards-directed, then $\bigcup \mathcal{M}' \in \mathcal{M}$;

whenever $M \in \mathcal{M}$ then (V_M, W_M) is a projection pair of subspaces of U and U^*,

where I write $V_M = \overline{M \cap U}$ and $W_M = \overline{M \cap U^*}$. For $M \in \mathcal{M}$,

$$U = V_M \oplus W_M^\circ, \quad U^* = W_M \oplus V_M^\circ;$$

let $P_M : U \to V_M$ and $Q_M : U^* \to W_M$ be the corresponding projections. Since $\|v\| \le \|v + v'\|$ whenever $v \in V_M$ and $v' \in W_M^\circ$, $\|P_M\| \le 1$; similarly, $\|Q_M\| \le 1$.

If $u \in U$, $f \in U^*$ and $M \in \mathcal{M}$, then

$$f(P_M u) = (Q_M f)(u) = (Q_M f)(P_M u).$$

P Express u as $v + v'$ and f as $g + g'$, where $v \in V_M$, $v' \in W_M^\circ$, $g \in W_M$ and $g' \in V_M^\circ$. Then

$$f(v) = g(v) = g(u),$$

that is,

$$f(P_M u) = (Q_M f)(P_M u) = (Q_M f)(u). \ \mathbf{Q}$$

(iii) If M_0, $M_1 \in \mathcal{M}$ and $M_0 \subseteq M_1$ then $P_{M_0} = P_{M_0}P_{M_1} = P_{M_1}P_{M_0}$. **P** If $u \in U$, express it as $v_0 + v_0'$ where $v_0 \in V_{M_0}$ and $v_0' \in W_{M_0}^\circ$; now express v_0' as $v_1 + v_1'$ where $v_1 \in V_{M_1}$ and $v_1' \in W_{M_1}^\circ$. Then

$$P_{M_0} u = v_0 \in V_{M_1},$$

so $P_{M_1}P_{M_0}u = P_{M_0}u$. On the other hand, $u = v_0 + v_1 + v_1'$ where $v_0 + v_1 \in V_{M_1}$ and $v_1' \in W^\circ_{M_1}$, so $P_{M_1}u = v_0 + v_1$; and as $v_1 = v_0' - v_1'$ belongs to $W^\circ_{M_0}$, $P_{M_0}P_{M_1}u = v_0 = P_{M_0}u$. **Q**

(iv) If $\langle M_\xi\rangle_{\xi<\zeta}$ is a non-decreasing family in $\mathcal{M}$, where ζ is a non-zero limit ordinal, then we know that $M = \bigcup_{\xi<\zeta} M_\xi$ belongs to $\mathcal{M}$. Now

$$P_M u = \lim_{\xi\uparrow\zeta} P_{M_\xi}u$$

for every $u \in U$, the limit being for the norm topology on U. **P** Let $\epsilon > 0$. We know that $P_M u \in V_M = \overline{M \cap U}$, so there is a $u' \in M \cap U$ such that $\|u' - P_M u\| \le \frac{1}{2}\epsilon$. Let $\xi < \zeta$ be such that $u' \in M_\xi$. If $\xi \le \eta < \zeta$, then

$$\|P_{M_\eta}u - P_M u\| = \|P_{M_\eta}(P_M u - u') + P_M(u' - P_M u)\|$$

(because $u' \in V_{M_\eta}$, so $P_M u' = P_{M_\eta}u' = u'$)

$$\le 2\|P_M u - u'\| \le \epsilon. \text{ **Q**}$$

(v) Similarly,

$$Q_{M_0} = Q_{M_0}Q_{M_1} = Q_{M_1}Q_{M_0}$$

whenever M_0, $M_1 \in \mathcal{M}$ and $M_0 \subseteq M_1$, and

$$Q_M f = \lim_{\xi\uparrow\zeta} Q_{M_\xi} f$$

whenever $f \in U^*$ and ζ is a non-zero limit ordinal and $\langle M_\xi\rangle_{\xi<\zeta}$ is a non-decreasing family in $\mathcal{M}$ with union M.

(b) Now let $\mathcal{M}_0$ be $\{M : M \in \mathcal{M}, \#(M) \le \omega\}$. Then there is an $M_0 \in \mathcal{M}_0$ such that

$$P_{M_0}(u_x) = P_M(u_x)\ \mu\text{-a.e.}(x)$$

whenever $M_0 \subseteq M \in \mathcal{M}_0$.

P? Suppose, if possible, otherwise. Then we can choose inductively an increasing family $\langle M_\xi\rangle_{\xi<\omega_1}$ in $\mathcal{M}_0$ such that

$$\mu\{x : P_{M_{\xi+1}}(u_x) \ne P_{M_\xi}(u_x)\} > 0 \text{ for every } \xi < \omega_1,$$

$$M_\xi = \bigcup_{\eta<\xi} M_\eta \text{ whenever } \xi < \omega_1 \text{ is a non-zero countable limit ordinal.}$$

(The set of x for which $P_{M_{\xi+1}}(u_x) \ne P_{M_\xi}(u_x)$ is necessarily measurable because $x \mapsto P_{M_{\xi+1}}u_x - P_{M_\xi}u_x$ is scalarly measurable, by 537Hb, therefore measurable for the norm topology, by 537Hc, since $V_{M_{\xi+1}}$ is separable.) Now there must be a $\delta > 0$ such that

$$A = \{\xi : \xi < \omega_1,\ \mu E_\xi \ge \delta\}$$

is infinite, where

$$E_\xi = \{x : \|P_{M_{\xi+1}}(u_x) - P_{M_\xi}(u_x)\| \ge \delta\}$$

for each $\xi < \omega_1$. But in this case there must be an $x \in X$ such that

$$A' = \{\xi : \xi \in A,\ x \in E_\xi\}$$

is infinite. (Take a sequence $\langle\xi_n\rangle_{n\in\mathbb{N}}$ of distinct points in A, and $x \in \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n} E_{\xi_m}$.) Let ζ be any cluster point of A' in ω_1. Then

$$P_{M_\zeta}(u_x) = \lim_{\xi\uparrow\zeta} P_{M_\xi}(u_x)$$

((a-iv) above), which is impossible. **XQ**

(c) Similarly, there is an $M_1 \in \mathcal{M}_0$ such that $M_1 \supseteq M_0$ and

$$P_{M_1}(f_y) = P_M(f_y)\ \nu\text{-a.e.}(y)$$

whenever $M_1 \subseteq M \in \mathcal{M}_0$. Because $x \mapsto P_{M_1}(u_x)$ and $y \mapsto Q_{M_1}(f_y)$ are scalarly measurable maps to norm-separable spaces, they are norm-measurable; again because V_{M_1} and W_{M_1} are separable, $(x,y) \to (P_{M_1}u_x, Q_{M_1}f_y) : X \times Y \to V_{M_1} \times W_{M_1}$ is $\Sigma\widehat{\otimes}\mathrm{T}$-measurable (418Bd). Because $(f,x) \mapsto f(x) : U^* \times U \to \mathbb{R}$ is norm-continuous, $(x,y) \mapsto (Q_{M_1}f_y)(P_{M_1}u_x)$ is $\Sigma\widehat{\otimes}\mathrm{T}$-measurable, and

$$\iint (Q_{M_1} f_y)(P_{M_1} u_x)\mu(dx)\nu(dy) = \iint (Q_{M_1} f_y)(P_{M_1} u_x)\nu(dy)\mu(dx)$$

by Fubini's theorem (252C).

Now observe that if $y \in Y$ there is an $M \in \mathcal{M}_0$ such that $M_1 \subseteq M$ and $f_y \in M$. So

$$\begin{aligned}\int f_y(u_x)\mu(dx) &= \int (Q_M f_y)(u_x)\mu(dx) = \int f_y(P_M u_x)\mu(dx)\\ &= \int f_y(P_{M_1} u_x)\mu(dx) = \int (Q_{M_1} f_y)(P_{M_1} u_x)\mu(dx).\end{aligned}$$

This is true for every y. So $\iint f_y(u_x)\mu(dx)\nu(dy)$ is defined and equal to $\iint (Q_{M_1} f_y)(P_{M_1} u_x)\mu(dx)\nu(dy)$. Similarly,

$$\iint f_y(u_x)\nu(dy)\mu(dx) = \iint (Q_{M_1} f_y)(P_{M_1} u_x)\nu(dy)\mu(dx).$$

Putting these together, we have the result.

537J Corollary Let (X,Σ,μ), (Y,T,ν) and (Z,Λ,σ) be probability spaces. Let $x \mapsto U_x : X \to \Lambda$ and $y \mapsto V_y : Y \to \Lambda$ be functions such that

$$x \mapsto \sigma(U_x \cap W), \quad y \mapsto \sigma(V_y \cap W)$$

are measurable for every $W \in \Lambda$. Then $\iint \sigma(U_x \cap V_y)\mu(dx)\nu(dy)$ and $\iint \sigma(U_x \cap V_y)\nu(dy)\mu(dx)$ are defined and equal.

proof (a) For $x \in X$ set $u_x = (\chi U_x)^\bullet$ in $L^2(\sigma)$. Then $x \mapsto u_x$ is scalarly measurable. **P** If $f \in U^*$, there is a $v \in L^2(\sigma)$ such that $f(u) = \int u \times v$ for every $u \in L^2(\sigma)$ (244K). Let $\epsilon > 0$. Then there are $W_0,\dots,W_n \in \Lambda$ and $\alpha_0,\dots,\alpha_n \in \mathbb{R}$ such that $\|v - \sum_{i=0}^n \alpha_i(\chi W_i)^\bullet\|_2 \le \epsilon$ (244Ha), so that

$$|f(u_x) - \sum_{i=0}^n \alpha_i\sigma(U_x \cap W_i)| = |\int u_x \times v - \int u_x \times \sum_{i=0}^n \alpha_i(\chi W_i)^\bullet| \le \epsilon\|u_x\|_2 \le \epsilon$$

for every $x \in X$. Now the function $x \mapsto \sum_{i=0}^n \alpha_i\sigma(U_x \cap W_i)$ is Σ-measurable. So we see that the function $x \mapsto f(u_x)$ is uniformly approximated by Σ-measurable functions and is itself Σ-measurable. As f is arbitrary, $x \mapsto u_x$ is scalarly measurable. **Q**

(b) Similarly, setting $v_y = (\chi V_y)^\bullet$ for $y \in Y$, $y \mapsto v_y : Y \to L^2(\sigma)$ is scalarly measurable. Identifying $L^2(\sigma)$ with its dual, 537I tells us that

$$\iint (u_x|v_y)\mu(dx)\nu(dy) = \iint (u_x|v_y)\nu(dy)\mu(dx),$$

that is, that

$$\iint \sigma(U_x \cap V_y)\mu(dx)\nu(dy) = \iint \sigma(U_x \cap V_y)\nu(dy)\mu(dx).$$

537K The next few paragraphs will be concerned with upper and lower integrals. For the basic theory of these, see §133 and 214J.

Theorem (Freiling 86, Shipman 90) Let $\langle(X_j,\Sigma_j,\mu_j)\rangle_{j\le m}$ be a finite sequence of probability spaces and $\langle\kappa_j\rangle_{j\le m}$ a sequence of cardinals such that $X_j^{\mathbb{N}}$, with its product measure $\mu_j^{\mathbb{N}}$, has a subset with cardinal κ_j which is not covered by κ_{j-1} negligible sets (if $j \ge 1$) and is not negligible (if $j = 0$). Let $f : \prod_{j\le m} X_j \to \mathbb{R}$ be a bounded function, and suppose that $\sigma : m+1 \to m+1$ and $\tau : m+1 \to m+1$ are permutations. Set

$$I = \underline{\int} \dots \underline{\int} f(x_0,\dots,x_m)dx_{\sigma(m)}\dots dx_{\sigma(0)},$$

$$I' = \overline{\int} \dots \overline{\int} f(x_0,\dots,x_m)dx_{\tau(m)}\dots dx_{\tau(0)}.$$

Then $I \le I'$.

proof Let $M \ge 0$ be such that $|f(x_0,\dots,x_m)| \le M$ for all $x_0,\dots,x_m$.

(a) Set $Z = \prod_{j\le m} X_j^{\mathbb{N}}$. The key fact is that we can find negligible sets $W(\boldsymbol{u}) \subseteq X_k^{\mathbb{N}}$, for $k \le m$ and $\boldsymbol{u} \in \prod_{j\le m, j\ne k} X_j^{\mathbb{N}}$, such that

$$I \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(t_{0i},\dots,t_{mi})$$

whenever $\langle t_j\rangle_{j\le m} = \langle\langle t_{ji}\rangle_{i\in\mathbb{N}}\rangle_{j\le m}$ is such that $t_k \notin W(t_0,\dots,t_{k-1},t_{k+1},\dots,t_m)$ for every k. **P** Because the formula

$$\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}f(t_{0i},\dots,t_{mi})$$

is tolerant of permutations of the coordinates $0,\dots,m$, it is enough to consider the case $\sigma(j)=j$ for $j\le m$, so that

$$I=\underline{\int}\dots\underline{\int}f(x_0,\dots,x_m)dx_m\dots dx_0.$$

(i) Define $D_0,\dots,D_{m+1}$ as follows. $D_0=\{\emptyset\}=\prod_{j<0}X_j^{\mathbb{N}}$. For $0<k\le m$ let D_k be the set of those $(t_0,\dots,t_{k-1})\in\prod_{j<k}X_j^{\mathbb{N}}$ such that

$$I\le\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\underline{\int}\dots\underline{\int}f(t_{0i},\dots,t_{k-1,i},x_k,\dots,x_m)dx_m\dots dx_k,$$

where $t_j=\langle t_{ji}\rangle_{i\in\mathbb{N}}$ for $j<k$. For $k<m$ and $\boldsymbol{u}=(u_0,\dots,u_{k-1},u_{k+1},\dots,u_m)$ in $\prod_{j\le m,j\ne k}X_j^{\mathbb{N}}$, set

$$\begin{aligned}W(\boldsymbol{u})&=\emptyset\text{ if }(u_0,\dots,u_{k-1})\notin D_k,\\&=\{t:t\in X_k^{\mathbb{N}},\ (u_0,\dots,u_{k-1},t)\notin D_{k+1}\}\text{ otherwise.}\end{aligned}$$

(ii) $W(\boldsymbol{u})\subseteq X_k^{\mathbb{N}}$ is negligible. To see this, we need consider only the case in which $(u_0,\dots,u_{k-1})$ belongs to D_k. Express u_j as $\langle u_{ji}\rangle_{i\in\mathbb{N}}$ for $j<k$, and for $i\in\mathbb{N}$ define $h_i:X_k\to\mathbb{R}$ by setting

$$h_i(x)=\underline{\int}\dots\underline{\int}f(u_{0i},\dots,u_{k-1,i},x,x_{k+1},\dots,x_m)dx_m\dots dx_{k+1}$$

for $x\in X_k$. Now the definition of D_k tells us just that

$$I\le\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\underline{\int}\dots\underline{\int}f(u_{0i},\dots,u_{k-1,i},x_k,\dots,x_m)dx_m\dots dx_k,$$

that is, that

$$I\le\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\underline{\int}h_i(x)dx.$$

For each $i\in\mathbb{N}$ let $g_i:X_k\to[-M,M]$ be a measurable function such that $g_i(x)\le h_i(x)$ for every x and $\int g_id\mu_k=\underline{\int}h_id\mu_k$. Now consider the functions $\tilde g_i:X_k^{\mathbb{N}}\to\mathbb{R}$ defined by setting $\tilde g_i(t)=g_i(t_i)$ for $t=\langle t_i\rangle_{i\in\mathbb{N}}\in X_k^{\mathbb{N}}$. We have $\int\tilde g_id\mu_k^{\mathbb{N}}=\underline{\int}h_id\mu_k$ for each i, while $\langle\tilde g_i\rangle_{i\in\mathbb{N}}$ is a uniformly bounded independent sequence of random variables. By the strong law of large numbers in the form 273H,

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}(\tilde g_i(t)-\int\tilde g_id\mu_k^{\mathbb{N}})=0$$

for almost every $t\in X_k^{\mathbb{N}}$. Since

$$\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\int\tilde g_id\mu_k^{\mathbb{N}}=\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\underline{\int}h_id\mu_k\ge I,$$

we have

$$\begin{aligned}I&\le\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\tilde g_i(t)\le\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}h_i(t_i)\\&=\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\underline{\int}\dots\underline{\int}f(u_{0i},\dots,u_{k-1,i},t_i,x_{k+1},\dots,x_m)dx_m\dots dx_{k+1}\end{aligned}$$

for almost every $t=\langle t_i\rangle_{i\in\mathbb{N}}\in X_k^{\mathbb{N}}$, that is, $(u_0,\dots,u_{k-1},t)\in D_{k+1}$ for almost every $t\in X_k^{\mathbb{N}}$, that is, $W(\boldsymbol{u})$ is negligible, as required.

(iii) Suppose that $\boldsymbol{t}=(t_0,\dots,t_m)\in Z$ is such that $t_k\notin W(t_0,\dots,t_{k-1},t_{k+1},\dots,t_m)$ for every $k<m$. Then $(t_0,\dots,t_k)\in D_{k+1}$ for every k; in particular, $\boldsymbol{t}\in D_{m+1}$ and, writing $t_j=\langle t_{ji}\rangle_{i\in\mathbb{N}}$ for each j,

$$I\le\liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}f(t_{0i},\dots,t_{mi}).\ \mathbf{Q}$$

(b) Similarly, or applying the argument above to $-f$, we have negligible sets $W'(\boldsymbol{u})\subseteq X_k^{\mathbb{N}}$, for $k\le m$ and $\boldsymbol{u}\in\prod_{j\le m,j\ne k}X_j^{\mathbb{N}}$, such that

$$I' \ge \limsup_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(t_{0i},\dots,t_{mi})$$

whenever $\langle t_j\rangle_{j\le m} = \langle\langle t_{ji}\rangle_{i\in\mathbb{N}}\rangle_{j\le m}$ is such that $t_k \notin W'(t_0,\dots,t_{k-1},t_{k+1},\dots,t_m)$ for every k. Enlarging the $W'(\boldsymbol{u})$ if necessary, we may suppose that $W'(\boldsymbol{u}) \supseteq W(\boldsymbol{u})$ for every $\boldsymbol{u}$.

(c) Now the point of the construction is that we can find a $\boldsymbol{t} = (t_0,\dots,t_m) \in Z$ such that $t_k \notin W'(t_0,\dots,t_{k-1},t_{k+1},\dots,t_m)$ for every k. **P** For each $k \le m$ let $A_k \subseteq X_k^{\mathbb{N}}$ be a non-negligible set of size κ_k which (if $k \ge 1$) cannot be covered by κ_{k-1} negligible sets. Choose t_m, $t_{m-1},\dots,t_0$ in such a way that

$$t_k \in A_k, \quad t_k \notin W(\boldsymbol{u}) \text{ whenever } \boldsymbol{u} \in \prod_{j<k} A_j \times \prod_{k<j\le m}\{t_j\};$$

this is always possible because $\#(A_0\times\dots\times A_{k-1}) = \kappa_{k-1}$ if $k \ge 1$. **Q**

So we get

$$\begin{aligned} I &\le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(t_{0i},\dots,t_{mi}) \\ &\le \limsup_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(t_{0i},\dots,t_{mi}) \le I', \end{aligned}$$

as claimed.

537L Corollary Let $\langle(X_j,\Sigma_j,\mu_j)\rangle_{j\le m}$ be a finite sequence of probability spaces such that $X_j^{\mathbb{N}}$, with its product measure $\mu_j^{\mathbb{N}}$, has a Sierpiński set with cardinal ω_{j+1} for each $j \le m$. Let $f : \prod_{j\le m} X_j \to \mathbb{R}$ be a bounded function, and suppose that $\sigma : m+1 \to m+1$ and $\tau : m+1 \to m+1$ are permutations such that the two repeated integrals

$$I = \int\dots\int f(x_0,\dots,x_m)dx_{\sigma(m)}\dots dx_{\sigma(0)},$$

$$I' = \int\dots\int f(x_0,\dots,x_m)dx_{\tau(m)}\dots dx_{\tau(0)},$$

are both defined. Then $I = I'$.

proof Apply 537K in both directions.

537M A pair of simple facts which I never got round to spelling out will be useful below.

Lemma Suppose that (X,Σ,μ) is a totally finite measure space and f is a $[0,\infty]$-valued function defined almost everywhere in X.

(a) If $\gamma < \overline{\int} f$, then there is a measurable integrable function $g : X \to [0,\infty[$ such that $\int g \ge \gamma$ and $\{x : x \in \operatorname{dom} f, g(x) \le f(x)\}$ has full outer measure in X.

(b) If $\underline{\int} f < \gamma$, then there is a measurable integrable function $g : X \to [0,\infty[$ such that $\int g \le \gamma$ and $\{x : x \in \operatorname{dom} f, f(x) \le g(x)\}$ has full outer measure in X.

proof (a) By 135H(b-i),

$$\overline{\int} f = \sup_{k\in\mathbb{N}} \overline{\int} \min(f(x),k)\mu(dx);$$

let $k \in \mathbb{N}$ be such that $\overline{\int} f_k > \gamma$, where $f_k(x) = \min(f(x),k)$ for $x \in \operatorname{dom} f$. Because $\mu X < \infty$, $\overline{\int} f_k$ is finite. By 133J(a-i), there is an integrable h such that $\int h = \overline{\int} f_k$ and $f_k \le_{\text{a.e.}} h$; adjusting h on a negligible set if necessary, we can arrange that h is defined (and finite) everywhere on X and is measurable. Set $\epsilon = (\int h - \gamma)/(1+\mu X)$, and $g = h - \epsilon\chi X$; then by the last part of 133J(a-i),

$$\{x : x \in \operatorname{dom} f,\ g(x) \le f(x)\} = \{x : x \in \operatorname{dom} f,\ h(x) \le f(x) + \epsilon\}$$

has full outer measure in X, while $\int g \ge \gamma$.

(b) By 135Ha, there is a measurable $h : X \to [0,\infty]$ such that $h \le_{\text{a.e.}} f$ and $\int h = \underline{\int} f$; as $\int h$ is finite, h is finite a.e. and can be adjusted to be finite everywhere. Set $\epsilon = (\gamma - \int h)/(1+\mu X)$, and $g = h + \epsilon\chi X$; then $\int g \le \gamma$ and $\{x : f(x) \le g(x)\}$ has full outer measure.

537N For ordinary two-variable repeated integrals we can squeeze a little bit more out than is given by 537K.

Proposition Let (X,Σ,μ) be a semi-finite measure space, (Y,T,ν) a probability space, and $\nu^{\mathbb{N}}$ the product measure on $Y^{\mathbb{N}}$. If $\mathrm{non}(E,\mathcal{N}(\mu)) < \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}})$ for every $E \in \Sigma\setminus\mathcal{N}(\mu)$, then

$$\underline{\int}\underline{\int} f(x,y)\nu(dy)\mu(dx) \le \overline{\int}\overline{\int} f(x,y)\mu(dx)\nu(dy)$$

for every function $f : X\times Y \to [0,\infty]$.

proof (a) To begin with, suppose that $\mu X < \infty$ and $\#(X) < \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}})$. For each $y \in Y$, let $h_y : X \to [0,\infty]$ be a measurable function such that $f(x,y) \le h_y(x)$ for every $x \in X$ and $\int h_y d\mu = \overline{\int} f(x,y)\mu(dx)$; let $v : Y \to [0,\infty]$ be a measurable function such that $\int h_y d\mu \le v(y)$ for every $y \in Y$ and $\int v\,d\nu = \overline{\int}\,\overline{\int} f(x,y)\mu(dx)\nu(dy)$. If this is infinite, we can stop. Otherwise, for each $x \in X$ let $g_x : Y \to [0,\infty]$ be a measurable function such that $g_x(y) \le f(x,y)$ for every $y \in Y$ and $\int g_x d\nu = \underline{\int} f(x,y)\nu(dy)$, and let $u : X \to [0,\infty]$ be a measurable function such that $u(x) \le \int g_x\,d\nu$ for every x and $\int u\,d\mu = \underline{\int}\,\underline{\int} f(x,y)\nu(dy)\mu(dx)$.

As $\#(X) < \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}})$, we can find a sequence $\langle y_i\rangle_{i\in\mathbb{N}}$ in Y such that

$$\int v\,d\nu = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n v(y_i)$$

and

$$\int g_x d\nu = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n g_x(y_i)$$

for every $x \in X$. (For by 273J, the set of such sequences is the intersection of fewer than $\mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}})$ conegligible sets in $Y^{\mathbb{N}}$, and cannot be empty.) If $x \in X$, then

$$u(x) \le \int g_x d\nu = \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n g_x(y_i) \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n h_{y_i}(x).$$

So

$$\underline{\int}\,\underline{\int} f(x,y)\nu(dy)\mu(dx) = \int u\,d\mu \le \liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n \int h_{y_i} d\mu$$

(by Fatou's Lemma)

$$\le \liminf_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n v(y_i) = \int v\,d\nu$$

$$= \overline{\int}\,\overline{\int} f(x,y)\mu(dx)\nu(dy),$$

as required.

(b) Now suppose that μ is totally finite and that X has a subset A of full outer measure with $\#(A) < \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}})$. Let μ_A be the subspace measure on A. Then for any $q : X \to [0,\infty]$ we have

$$\underline{\int} q\,d\mu \le \underline{\int}(q{\restriction}A)d\mu_A \le \overline{\int}(q{\restriction}A)d\mu_A \le \overline{\int} q\,d\mu$$

(214J). So, writing f_A for the restriction of f to $A\times Y$,

$$\underline{\int}\,\underline{\int} f(x,y)\nu(dy)\mu(dx) \le \underline{\int}\,\underline{\int} f_A(x,y)\nu(dy)\mu_A(dx)$$

$$\le \overline{\int}\,\overline{\int} f_A(x,y)\mu_A(dx)\nu(dy)$$

(by (a))

$$\le \overline{\int}\,\overline{\int} f(x,y)\mu(dx)\nu(dy).$$

(c) For the general case, let $u : X \to [0,\infty]$ be a measurable function such that $u(x) \le \underline{\int} f(x,y)\nu(dy)$ for every $x \in X$ and $\int u\,d\mu = \underline{\int}\,\underline{\int} f(x,y)\nu(dy)\mu(dx)$. Take any $\gamma < \int u\,d\mu$. Because μ is semi-finite, there is a non-empty set

$F \in \Sigma$ of finite measure such that $\int_F u\,d\mu > \gamma$. Now let $\mathcal{E}$ be the family of measurable sets $E \subseteq F$ of finite measure for which there is a non-empty set $A \subseteq E$, with cardinal less than $\operatorname{cov}\mathcal{N}(\nu^{\mathbb{N}})$, such that $\mu^*A = \mu E$, that is, A has full outer measure for the subspace measure μ_E, that is, E is a measurable envelope of A. Then $\mathcal{E}$ is closed under finite unions and every non-empty member of Σ includes a member of $\mathcal{E}$. So there is a non-decreasing sequence $\langle E_k\rangle_{k\in\mathbb{N}}$ in $\mathcal{E}$ such that $\bigcup_{k\in\mathbb{N}} E_k \subseteq F$ and $F \setminus \bigcup_{k\in\mathbb{N}} E_k$ is negligible. In this case, $\gamma < \int_F u\,d\mu = \lim_{k\to\infty}\int_{E_k} u\,d\mu$, so there is a $k \in \mathbb{N}$ such that $\gamma \le \int_{E_k} u\,d\mu$. Set $E = E_k$.

Consider the restriction f_E of f to $E \times Y$ and the subspace measure μ_E on E. We have

$$\gamma \le \int_E u\,d\mu = \int (u\restriction E)d\mu_E \le \underline{\int}\underline{\int} f_E(x,y)\nu(dy)\mu_E(dx)$$

$$\le \overline{\int}\overline{\int} f_E(x,y)\mu_E(dx)\nu(dy)$$

(because $E \in \mathcal{E}$, so we can use (b))

$$\le \overline{\int}\overline{\int} f(x,y)\mu(dx)\nu(dy)$$

because $\overline{\int} f_E(x,y)\mu_E(dx) \le \overline{\int} f(x,y)\mu(dx)$ for every y, by 214Ja or otherwise. Since γ is arbitrary,

$$\underline{\int}\underline{\int} f(x,y)\nu(dy)\mu(dx) \le \overline{\int}\overline{\int} f(x,y)\mu(dx)\nu(dy)$$

in this case also.

537O Corollary Let (X,Σ,μ) and (Y,T,ν) be probability spaces, and $\nu^{\mathbb{N}}$ the product measure on $Y^{\mathbb{N}}$. If $\operatorname{shr}^+\mathcal{N}(\mu) \le \operatorname{cov}\mathcal{N}(\nu^{\mathbb{N}})$ then

$$\overline{\int}\underline{\int} f(x,y)\nu(dy)\mu(dx) \le \overline{\int}\overline{\int} f(x,y)\mu(dx)\nu(dy)$$

for every function $f : X \times Y \to [0,\infty[$.

proof Take any $\gamma < \overline{\int}\underline{\int} f(x,y)\nu(dy)\mu(dx)$. By 537Ma, there are a measurable function $u : X \to [0,\infty[$ and a set A of full outer measure in X such that $\int u\,d\mu \ge \gamma$ and $u(x) \le \underline{\int} f(x,y)\nu(dy)\mu(dx)$ for every $x \in A$. Let μ_A be the subspace measure on A, and f_A the restriction of f to $A \times Y$. If $B \subseteq A$ is any non-negligible relatively measurable set, there is a non-negligible $D \subseteq B$ such that $\#(D) < \operatorname{shr}^+\mathcal{N}(\mu)$, so

$$\operatorname{non}(B,\mathcal{N}(\mu_A)) = \operatorname{non}(B,\mathcal{N}(\mu)) \le \#(D) < \operatorname{cov}\mathcal{N}(\nu^{\mathbb{N}}).$$

So

$$\gamma \le \int u\,d\mu = \int (u\restriction A)d\mu_A \le \underline{\int}\underline{\int} f_A(x,y)\nu(dy)\mu_A(dx)$$

(because $u\restriction A$ is measurable and $(u\restriction A)(x) \le \underline{\int} f_A(x,y)\nu(dy)$ for every $x \in A$)

$$\le \overline{\int}\overline{\int} f_A(x,y)\mu_A(dx)\nu(dy)$$

(by 537N)

$$\le \overline{\int}\overline{\int} f(x,y)\mu(dx)\nu(dy)$$

because $\overline{\int} f_A(x,y)\mu_A(dx) \le \overline{\int} f(x,y)\mu(dx)$ for every y, by 214J again. As γ is arbitrary, we have the result.

Remark There is a similar inequality, under different hypotheses, in 543C below.

537P Corollary Let (X,Σ,μ) and (Y,T,ν) be probability spaces, and $\nu^{\mathbb{N}}$ the product measure on $Y^{\mathbb{N}}$; suppose that $\operatorname{shr}^+\mathcal{N}(\mu) \le \operatorname{cov}\mathcal{N}(\nu^{\mathbb{N}})$, and that $f : X \times Y \to \mathbb{R}$ is bounded.

(a)

$$\overline{\int}\underline{\int} f(x,y)\nu(dy)\mu(dx) \le \overline{\int}\,\overline{\int} f(x,y)\mu(dx)\nu(dy),$$

$$\underline{\int}\,\underline{\int} f(x,y)\mu(dx)\nu(dy) \le \underline{\int}\overline{\int} f(x,y)\nu(dy)\mu(dx).$$

(b) If $\iint f(x,y)\mu(dx)\nu(dy)$ is defined, and $\int f(x,y)\nu(dy)$ is defined for almost every x, then the other repeated integral $\iint f(x,y)\nu(dy)\mu(dx)$ is defined and equal to $\iint f(x,y)\mu(dx)\nu(dy)$.

proof (a) Apply 537O to the functions $(x,y) \mapsto M + f(x,y)$, $(x,y) \mapsto M - f(x,y)$ for suitable M.

(b) By (a),

$$\begin{aligned}\iint f(x,y)\mu(dx)\nu(dy) &\le \underline{\int}\int f(x,y)\nu(dy)\mu(dx)\\ &\le \overline{\int}\int f(x,y)\nu(dy)\mu(dx) \le \iint f(x,y)\mu(dx)\nu(dy).\end{aligned}$$

537Q We can extend the second part of 537Pa, as well as the first, to unbounded functions, if we strengthen the set-theoretic hypothesis.

Proposition (Humke & Laczkovich 05) Let (X,Σ,ν) and (Y,T,μ) be probability spaces, and $\mu^{\mathbb{N}}$, $\nu^{\mathbb{N}}$ the product measures on $X^{\mathbb{N}}$, $Y^{\mathbb{N}}$ respectively. If $\operatorname{shr}^+\mathcal{N}(\mu^{\mathbb{N}}) \le \operatorname{cov}\mathcal{N}(\nu^{\mathbb{N}})$ then $\underline{\int}\,\underline{\int} f(x,y)\mu(dx)\nu(dy) \le \underline{\int}\overline{\int} f(x,y)\nu(dy)\mu(dx)$ for every function $f : X \times Y \to [0,\infty[$.

proof ? Suppose, if possible, otherwise.

(a) There is a measurable function $u : Y \to [0,\infty[$ such that

$$u(y) \le \underline{\int} f(x,y)\mu(dx) \text{ for every } y, \quad \underline{\int}\overline{\int} f(x,y)\nu(dy)\mu(dx) < \int u\,d\nu.$$

Since $\int u\,d\nu$ is the supremum of the integrals of the non-negative simple functions dominated by u, we may suppose that u itself is a simple function; express it as $\sum_{j=0}^{m} \alpha_j \chi F_j$ where $\alpha_j \ge 0$ for each i and $(F_0,\dots,F_m)$ is a partition of Y into measurable sets. Now

$$\sum_{j=0}^{m} \underline{\int}\overline{\int} f(x,y)\chi F_j(y)\nu(dy)\mu(dx) \le \underline{\int}\sum_{j=0}^{m}\overline{\int} f(x,y)\chi F_j(y)\nu(dy)\mu(dx)$$

(133J(b-v))

$$\le \underline{\int}\overline{\int} f(x,y)\nu(dy)\mu(dx)$$

(because if $x \in X$ and $q : Y \to [0,\infty]$ is measurable and $f(x,y) \le q(y)$ for every y, then $\sum_{j=0}^{m} \overline{\int} f(x,y)\chi F_j(y)\nu(dy)$ is at most $\sum_{j=0}^{m} \int q \times \chi F_j d\nu = \int q\,d\nu$)

$$< \int u\,d\nu = \sum_{j=0}^{m} \alpha_j \nu F_j.$$

There are therefore a $j \le m$ and a $\gamma < 1$ such that

$$\underline{\int}\overline{\int} f(x,y)\chi F_j(y)\nu(dy)\mu(dx) < \gamma\alpha_j\nu F_j.$$

Now there is a measurable function $v : X \to [0,\infty[$ such that $\int v\,d\mu \le \gamma\alpha_j\nu F_j$ and

$$D = \{x : x \in X,\ \overline{\int} f(x,y)\chi F_j(y)\nu(dy) \le v(x)\}$$

has full outer measure in X, by 537Mb.

(b) For $y \in Y$ and $\boldsymbol{x} = \langle x_i\rangle_{i\in\mathbb{N}} \in X^{\mathbb{N}}$, set $h(\boldsymbol{x},y) = \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i,y)$. If $y \in Y$, then $\underline{\int} f(x,y)\mu(dx) \le h(\boldsymbol{x},y)$ for $\mu^{\mathbb{N}}$-almost every $\boldsymbol{x}$. **P** We have a measurable function $q : X \to [0,\infty[$ such that $q(x) \le f(x,y)$ for every x and

$$\underline{\int} f(x,y)\mu(dx) = \int q\,d\mu \le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} q(x_i)$$

$$\le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i,y) = h(\boldsymbol{x},y)$$

for almost every $\boldsymbol{x} = \langle x_i\rangle_{i\in\mathbb{N}}$. **Q** At the same time,

$$V = \{\langle x_i\rangle_{i\in\mathbb{N}} : \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} v(x_i) \le \gamma\alpha_j\nu F_j\}$$

is conegligible in $X^{\mathbb{N}}$, because $\int v\,d\mu \le \gamma\alpha_j\nu F_j$.

(c) Set

$$W = \{(\boldsymbol{x},y) : \boldsymbol{x} \in X^{\mathbb{N}},\ y \in F_j,\ h(\boldsymbol{x},y) \ge \alpha_j\}$$

and consider the function $\chi W : X^{\mathbb{N}} \times Y \to \{0,1\}$. If $y \in F_j$ then $\underline{\int} f(x,y)\mu(dx) \ge \alpha_j$ so $W^{-1}[\{y\}]$ is conegligible in $X^{\mathbb{N}}$. On the other hand, if $\boldsymbol{x} = \langle x_i\rangle_{i\in\mathbb{N}}$ belongs to $V \cap D^{\mathbb{N}}$,

$$\overline{\int} \alpha_j \chi W(\boldsymbol{x},y)\nu(dy) \le \overline{\int} h(\boldsymbol{x},y)\chi F_j(y)\nu(dy)$$

$$= \overline{\int} \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i,y)\chi F_j(y)\nu(dy)$$

$$\le \liminf_{n\to\infty} \overline{\int} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i,y)\chi F_j(y)\nu(dy)$$

(133Kb)

$$\le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \overline{\int} f(x_i,y)\chi F_j(y)\nu(dy)$$

(133J(b-ii))

$$\le \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} v(x_i) \le \gamma\alpha_j\nu F_j.$$

(d) As V is conegligible and $D^{\mathbb{N}}$ has full outer measure (254Lb),

$$\underline{\int}\overline{\int} \chi W(\boldsymbol{x},y)\nu(dy)\mu^{\mathbb{N}}(d\boldsymbol{x}) \le \gamma\nu F_j < \nu F_j = \iint \chi W(\boldsymbol{x},y)\mu^{\mathbb{N}}(d\boldsymbol{x})\nu(dy)$$

$$= \underline{\int}\underline{\int} \chi W(\boldsymbol{x},y)\mu^{\mathbb{N}}(d\boldsymbol{x})\nu(dy).$$

But we are supposing that $\operatorname{shr}^+\mathcal{N}(\mu^{\mathbb{N}}) \le \operatorname{cov}\mathcal{N}(\nu^{\mathbb{N}})$, so this contradicts 537P. **X**

So we have the result.

537R Lemma Let (X,Σ,μ) be a complete probability space and (Y,T,ν) a probability space such that $\operatorname{shr}^+\mathcal{N}(\mu) \le \operatorname{cov}\mathcal{N}(\nu^{\mathbb{N}})$, where $\nu^{\mathbb{N}}$ is the product measure on $Y^{\mathbb{N}}$. Let $f : X\times Y \to \mathbb{R}$ be a bounded function which is measurable in each variable separately, and set $u(x) = \int f(x,y)\nu(dy)$ for $x \in X$. Then $u : X \to \mathbb{R}$ is measurable.

proof ? Otherwise, there are a non-negligible measurable set $E \subseteq X$ and α, $\beta \in \mathbb{R}$ such that $\alpha < \beta$ and

$$\mu^*\{x : x \in E,\ u(x) \le \alpha\} = \mu^*\{x : x \in E,\ u(x) \ge \beta\} = \mu E$$

(413G). Let $A \subseteq \{x : x \in E,\ u(x) \le \alpha\}$ and $B \subseteq \{x : x \in E,\ u(x) \ge \beta\}$ be sets with cardinal less than $\operatorname{shr}^+\mathcal{N}(\mu)$ and outer measure greater than $\frac{1}{2}\mu E$ (521Ca). Let $\langle y_i\rangle_{i\in\mathbb{N}}$ be a sequence in Y such that

$$u(x) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x,y_i)$$

for every $x \in A \cup B$. Because $x \mapsto f(x,y_i)$ is measurable for each i, $u\restriction A\cup B$ is measurable; but this means that A and B can be separated by measurable sets, which is impossible, because $\mu^*A + \mu^*B > \mu E$. **X**

537S Proposition Let (X, Σ, μ) and (Y, T, ν) be probability spaces such that

$$\mathrm{shr}^{+}\mathcal{N}(\mu) \le \mathrm{cov}\,\mathcal{N}(\nu^{\mathbb{N}}),$$

where $\nu^{\mathbb{N}}$ is the product measure on $Y^{\mathbb{N}}$, and

$$\mathrm{cf}([\tau(\nu)]^{\le\omega}) < \mathrm{cov}(E, \mathcal{N}(\mu)) \text{ for every } E \in \Sigma \setminus \mathcal{N}(\mu),$$

where $\tau(\nu)$ is the Maharam type of ν. Let $f : X \times Y \to [0, \infty[$ be a function which is measurable in each variable separately. Then $\iint f(x, y)\mu(dx)\nu(dy)$ and $\iint f(x, y)\nu(dy)\mu(dx)$ exist and are equal.

proof (a) Let $\tilde{\Lambda} \supseteq \Sigma\widehat{\otimes}\mathrm{T}$ be the σ-algebra of sets $W \subseteq X \times Y$ such that all the vertical and horizontal sections of W are measurable. If $W \in \tilde{\Lambda}$, then $x \mapsto \nu W[\{x\}] : X \to [0, 1]$ is measurable, by 537R. If $W \in \tilde{\Lambda}$ and almost every horizontal section of W is negligible, then

$$\begin{aligned}\overline{\int}\nu W[\{x\}]\mu(dx) &= \overline{\int}\underline{\int}\chi W(x, y)\nu(dy)\mu(dx)\\ &\le \overline{\int}\,\overline{\int}\chi W(x, y)\mu(dx)\nu(dy) = 0\end{aligned}$$

by 537Pa, so almost every vertical section of W is negligible.

(b) Let $(\mathfrak{B}, \bar{\nu})$ be the measure algebra of (Y, T, ν). If $W \in \tilde{\Lambda}$ and there is a metrically separable subalgebra $\mathfrak{C}$ of $\mathfrak{B}$ containing $W[\{x\}]^{\bullet}$ for every $x \in X$, then there is a $W' \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $W[\{x\}]\triangle W'[\{x\}]$ is negligible for almost every x. **P** Note first that for every $F \in \mathrm{T}$ the map

$$x \mapsto \nu(W[\{x\}]\triangle F) = \nu((W\triangle(X \times F))[\{x\}]$$

is measurable, by (a). So $x \mapsto W[\{x\}]^{\bullet} : X \to \mathfrak{C}$ is measurable, by 418Bc. By 418T(b-ii), there is a $W' \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $W[\{x\}]^{\bullet} = W'[\{x\}]^{\bullet}$ for almost every x. **Q**

(c) In fact we find that for any $W \in \tilde{\Lambda}$ there is a $W' \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $W[\{x\}]\triangle W'[\{x\}]$ is negligible for almost every x. **P** Set $\kappa = \tau(\nu) = \tau(\mathfrak{B})$, and let $\langle e_\xi\rangle_{\xi<\kappa}$ generate $\mathfrak{B}$. Let $\mathcal{K} \subseteq [\kappa]^{\le\omega}$ be a cofinal set of size $\mathrm{cf}[\kappa]^{\le\omega}$. For $K \in \mathcal{K}$, let $\mathfrak{B}_K$ be the closed subalgebra of $\mathfrak{B}$ generated by $\{e_\xi : \xi \in K\}$ and A_K the set $\{x : x \in X, W[\{x\}]^{\bullet} \in \mathfrak{B}_K\}$. Note that $K \mapsto A_K$ is non-decreasing and that the union of any sequence in $\mathcal{K}$ is included in a member of $\mathcal{K}$. So there is a $K_0 \in \mathcal{K}$ such that $\mu^*A_{K_0} = \sup_{K\in\mathcal{K}}\mu^*A_K$.

If E is a measurable envelope of A_{K_0}, then $\{A_K \setminus E : K \in \mathcal{K}\}$ is a cover of $X \setminus E$ by negligible sets. So $\mathrm{cov}(X \setminus E, \mathcal{N}(\mu)) \le \mathrm{cf}[\kappa]^{\le\omega}$ and $X \setminus E$ must be negligible, that is, A_{K_0} has full outer measure.

Taking a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in T such that $\{F_n^{\bullet} : n \in \mathbb{N}\}$ is dense in $\mathfrak{B}_{K_0}$, we see from (a) that the function $x \mapsto \inf_{n\in\mathbb{N}}\nu(W[\{x\}]\triangle F_n)$ is measurable, while it is zero on A_{K_0}. So $W[\{x\}]^{\bullet} \in \mathfrak{B}_{K_0}$ for almost every $x \in X$, that is, A_{K_0} is actually conegligible. Taking a measurable conegligible set $E' \subseteq A_{K_0}$ and applying (b) to $W \cap (E' \times Y)$, we see that there is a $W' \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $W[\{x\}]\triangle W'[\{x\}]$ is negligible for almost every $x \in X$. **Q**

(d) Now turn to the function f under consideration. For $q \in \mathbb{Q}$ set $W_q = \{(x, y) : f(x, y) \ge q\} \in \tilde{\Lambda}$. By (c), we have $V_q \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $V_q[\{x\}]\triangle W_q[\{x\}]$ is ν-negligible for μ-almost every x, and therefore $W_q^{-1}[\{y\}]\triangle V_q^{-1}[\{y\}]$ is μ-negligible for ν-almost every y, by (a). If $q \le q'$ then $W_{q'} \setminus W_q$ is empty, so $V_{q'}[\{x\}] \setminus V_q[\{x\}]$ is ν-negligible for μ-almost every x, and $V_{q'} \setminus V_q$ is $(\mu \times \nu)$-negligible, where $\mu \times \nu$ is the product measure on $X \times Y$. Similarly, $\bigcap_{q'<q} V_{q'} \setminus V_q$ is negligible for every q. Moreover, writing V_∞ for $\bigcap_{q\in\mathbb{Q}} V_q$, $V_\infty[\{x\}]$ is ν-negligible for μ-almost every x, so $(\mu \times \nu)V_\infty = 0$; similarly, $(\mu \times \nu)V_0 = 1$. There is therefore a $\Sigma\widehat{\otimes}\mathrm{T}$-measurable $g : X \times Y \to [0, \infty[$ such that $V_q\triangle\{(x, y) : g(x, y) \ge q\}$ is $(\mu \times \nu)$-negligible for every $q \in \mathbb{Q}$. In this case,

$$\{x : f(x, y) \ne g(x, y)\} \text{ is } \mu\text{-negligible for } \nu\text{-almost every } y,$$

$$\{y : f(x, y) \ne g(x, y)\} \text{ is } \nu\text{-negligible for } \mu\text{-almost every } x,$$

and

$$\begin{aligned}\iint f(x, y)\mu(dx)\nu(dy) &= \iint g(x, y)\mu(dx)\nu(dy)\\ &= \iint g(x, y)\nu(dy)\mu(dx) = \iint f(x, y)\nu(dy)\mu(dx)\end{aligned}$$

by 252H.

(e) Finally, if f is unbounded, set $f_k(x,y)=\min(f(x,y),k)$ for each $k\in\mathbb{N}$. Then

$$\begin{aligned}\iint f(x,y)\mu(dx)\nu(dy)&=\lim_{k\to\infty}\iint f_k(x,y)\mu(dx)\nu(dy)\\&=\lim_{k\to\infty}\iint f_k(x,y)\nu(dy)\mu(dx)=\iint f(x,y)\nu(dy)\mu(dx).\end{aligned}$$

537X Basic exercises (a)(i) Let (X,Σ,μ) be a measure space such that singletons are negligible and $\operatorname{cf}\mathcal{N}(\mu)=\omega_1$. Show that there is a Sierpiński subset of X. (ii) Show that if μ is Lebesgue measure on $\mathbb{R}$ and $\operatorname{cf}\mathcal{N}(\mu)=\omega_1$, then there is a strongly Sierpiński subset of $\mathbb{R}$.

(b) Show that for any uncountable cardinal κ there is a purely atomic probability space with a strongly Sierpiński set of size κ.

(c) Let (X,Σ,μ) be a measure space. Show that the union of any sequence of Sierpiński sets in X is again a Sierpiński set in X.

(d) Let (X,Σ,μ) be a measure space and Y any subspace of X. Show that a subset of Y is a Sierpiński set for the subspace measure on Y iff it is a Sierpiński set for μ.

(e) Suppose that λ is an infinite cardinal and the usual measure ν_λ on $\{0,1\}^\lambda$ has a Sierpiński set of size κ. Show that ν_λ has a Sierpiński set A such that $\#(A\cap E)=\kappa$ whenever $\nu_\lambda E>0$.

(f) Let (X,ρ) be a non-separable metric space with r-dimensional Hausdorff measure, where $r>0$. Show that X has a Sierpiński subset of size equal to the topological density of X.

>(g) Suppose that $\operatorname{non}\mathcal{N}<\operatorname{cov}\mathcal{N}$, where $\mathcal{N}$ is the null ideal of Lebesgue measure on $\mathbb{R}$. Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon probability spaces of countable Maharam type, and $f:X\times Y\to[0,\infty[$ a function such that $I=\iint f(x,y)\mu(dx)\nu(dy)$ and $I'=\iint f(x,y)\nu(dy)\mu(dx)$ are both defined. Show that $I=I'$.

>(h) Let (X,Σ,μ) be a probability space in which there is a well-ordered family in $\mathcal{N}(\mu)$ with union X; e.g., because $\operatorname{non}\mathcal{N}(\mu)=\#(X)$ or $\operatorname{add}\mathcal{N}(\mu)=\operatorname{cov}\mathcal{N}(\mu)$. Show that there is a function $f:X\times X\to[0,1]$ such that $\int f(x,y)\mu(dx)=0$ for every $y\in X$ and $\int f(x,y)\mu(dy)=1$ for every $x\in X$.

>(i) (In this exercise, all integrals are to be taken with respect to one-dimensional Lebesgue measure μ.) (i) Find a function $f:[0,1]^2\to\{0,1\}$ such that $\int\overline{\int}f(x,y)dxdy=1$ but $\iint f(x,y)dydx=0$. (*Hint*: there is a disjoint family $\langle A_y\rangle_{y\in[0,1]}$ of sets of full outer measure.) (ii) Find a function $f:[0,1]^2\to\{0,1\}$ such that $\iint f(x,y)dxdy=1$ but $\int\underline{\int}f(x,y)dydx=0$. (iii) Find a function $f:[0,1]^2\to\{0,1\}$ such that $\overline{\int}\int f(x,y)dxdy=1$ but $\underline{\int}\int f(x,y)dydx=0$. (*Hint*: enumerate $[0,1]$ as $\langle x_\xi\rangle_{\xi<\mathfrak{c}}$ in such a way that $\{x_\xi:\xi<\operatorname{non}\mathcal{N}(\mu)\}$ has full outer measure; set $f(x_\xi,x_\eta)=1$ if $\eta<\xi$.)

537Z Problems (a) Is it relatively consistent with ZFC to suppose that $\mathbb{R}$, with Lebesgue measure, has a Sierpiński subset but no strongly Sierpiński subset?

(b) Is it relatively consistent with ZFC to suppose that there is a probability space (X,μ) such that (X,μ) has a Sierpiński set but its power $(X^{\mathbb{N}},\mu^{\mathbb{N}})$ does not?

537 Notes and comments It is easy to see that if $\mathfrak{c}=\omega_1$ then there is a strongly Sierpiński set of size ω_1 for Lebesgue measure (537Xa). Countable-cocountable measures have strongly Sierpiński sets for trivial reasons. To eliminate all Sierpiński sets (on the definition of 537A) from atomless complete locally determined measure spaces, it is enough to ensure that the uniformity of Lebesgue measure is greater than ω_1 (537Bb). For the simplest models with non-trivial Sierpiński sets of size greater than ω_1, see 552E below.

The 'entangled sets' of 537C-537G belong rather to combinatorics than to measure theory; I go as far as I do into this theory because it is interesting in view of 552E. But it includes a proof that if the continuum hypothesis is true then there are two ccc partially ordered sets whose product is not ccc, which in its own context is of great importance.

Fubini's theorem is so important in measure theory that exploration of its boundaries has been a perennial challenge. I gave elementary examples in 252Xf-252Xg to show that as soon as we abandon the requirement that $\iint |f(x,y)|dxdy < \infty$ our repeated integrals can be expected to be unreliable. But for non-negative functions f on σ-finite spaces, measurability is enough to ensure that repeated integrals are equal (252H). In this section I look for results which will be valid for non-measurable functions. In 537I-537J we have a rather esoteric example – or, some would say, an example from a topic which I have neglected in this book – which is unusual in that it is a theorem of ZFC; for a note on its ancestry see Fremlin 93, 5L. In 537K-537L we see that, in the presence of a sufficient supply of Sierpiński sets, for instance, we must have $\iint f(x,y)dxdy = \iint f(x,y)dydx$ for ordinary bounded real-valued functions on the product of probability spaces, as long as both repeated integrals are defined. The argument here depends on using the strong law of large numbers to replace an integral $\int f(x,y)dx$ by the limit of a sequence of averages of values $f(x_i,y)$. This is why the Sierpiński sets must be available not in the original probability spaces $X_0,\dots,X_m$ but in their powers $X_j^{\mathbb{N}}$. Of course for our favourite spaces, starting with $[0,1]$, $(X^{\mathbb{N}},\mu^{\mathbb{N}})$ is isomorphic to (X,μ), so this does not seem too large a step; but it begs an obvious question (537Zb). For any result of this kind we certainly need some special axiom (537Xh).

In 537L the hypothesis includes strong 'separate measurability' conditions; we need not only separate measurability, but measurability of the functions $x \mapsto \int f(x,y)dy$ and $y \mapsto \int f(y,x)dx$. With a different set-theoretic hypothesis we can relax these (537S). I approach this form through ideas from Humke & Laczkovich 05, where there is a careful analysis of repeated integrals of the form $\underline{\int}\overline{\int}$, etc. My own version is in 537N-537Q. At every step there are ZFC examples to show that we cannot change the formulae involving $\underline{\int}$, $\overline{\int}$ without disaster (537Xi); but it is not so clear that the set-theoretic hypotheses offered are unimprovable.

538 Filters and limits

A great many special types of filter have been studied. In this section I look at some which are particularly interesting from the point of view of measure theory: Ramsey ultrafilters, measure-converging filters and filters with the Fatou property. About half the section is directed towards Benedikt's theorem (538M) on extensions of perfect probability measures; on the way we need to look at measure-centering ultrafilters (538G-538K) and iterated products of filters (538E, 538L). The second major topic here is a study of 'medial limits' (538P-538S); these are Banach limits of a very special type. In between, the measure-converging property (538N) and the Fatou property (538O) offer some intriguing patterns.

538A Filters For ease of reference, I begin the section with a list of the special types of filter on $\mathbb{N}$ which we shall be looking at later.

Definitions Let $\mathcal{F}$ be a filter on $\mathbb{N}$.

(a) $\mathcal{F}$ is **free** if it contains every cofinite subset of $\mathbb{N}$, that is, includes the Fréchet filter.

(b) $\mathcal{F}$ is a p**-point filter** if it is free and for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ there is an $A\in\mathcal{F}$ such that $A\setminus A_n$ is finite for every $n\in\mathbb{N}$. (Compare 5A6Ga.)

(c) $\mathcal{F}$ is **Ramsey** or **selective** if it is free and for every $f:[\mathbb{N}]^2\to\{0,1\}$ there is an $A\in\mathcal{F}$ such that f is constant on $[A]^2$.

(d) $\mathcal{F}$ is **rapid** if it is free and for every sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ of real numbers which converges to 0, there is an $A\in\mathcal{F}$ such that $\sum_{n\in A}|t_n|$ is finite. Note that a free filter $\mathcal{F}$ on $\mathbb{N}$ is rapid iff for every $f\in\mathbb{N}^{\mathbb{N}}$ there is an $A\in\mathcal{F}$ such that $\#(A\cap f(k))\le k$ for every $k\in\mathbb{N}$. **P** (i) If $\mathcal{F}$ is rapid and $f\in\mathbb{N}^{\mathbb{N}}$, let $g\in\mathbb{N}^{\mathbb{N}}$ be a strictly increasing sequence such that $f\le g$. Set $t_i=2$ if $i<g(0)$, $\frac{1}{k+1}$ if $g(k)\le i<g(k+1)$; then there is an $A\in\mathcal{F}$ such that $\sum_{i\in A}t_i$ is finite; as $\mathcal{F}$ is free, there is an $A\in\mathcal{F}$ such that $\sum_{i\in A}t_i\le 1$, in which case $\#(A\cap f(k))\le\#(A\cap g(k))\le k$ for every $k\in\mathbb{N}$. (ii) If $\mathcal{F}$ satisfies the condition and $\langle t_i\rangle_{i\in\mathbb{N}}\to 0$, take a strictly increasing $f\in\mathbb{N}^{\mathbb{N}}$ such that $|t_i|\le 2^{-k}$ whenever $k\in\mathbb{N}$ and $i\ge f(k)$; let $A\in\mathcal{F}$ be such that $\#(A\cap f(k))\le k$ for every k; then $\sum_{i\in A}|t_i|\le\sum_{k=0}^{\infty}2^{-k}\#(A\cap f(k+1)\setminus f(k))$ is finite. **Q**

(e) $\mathcal{F}$ is **nowhere dense** if for every sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$ there is an $A\in\mathcal{F}$ such that $\{t_n : n\in A\}$ is nowhere dense.

(f) $\mathcal{F}$ is **measure-centering** or has **property M** if whenever $\mathfrak{A}$ is a Boolean algebra, $\nu : \mathfrak{A} \to [0, \infty[$ is an additive functional, and $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$, there is an $A \in \mathcal{F}$ such that $\{a_n : n \in A\}$ is centered.

(g) $\mathcal{F}$ is **measure-converging** if whenever (X, Σ, μ) is a probability space, $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $\lim_{n\to\infty} \mu E_n = 1$, then $\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} E_n$ is conegligible.

(h) $\mathcal{F}$ has the **Fatou property** if whenever (X, Σ, μ) is a probability space, $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $X = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} E_n$, then $\lim_{n\to\mathcal{F}} \mu E_n$ is defined and equal to 1.

(i) For any countably infinite set I, I will say that a filter $\mathcal{F}$ on I is free, or a p-point filter, or Ramsey, etc., if it is isomorphic to such a filter on $\mathbb{N}$. Of course this usage is possible only because every property here is invariant under permutations of $\mathbb{N}$. For 'rapid' and 'measure-converging' filters, we need an appropriate translation of 'sequence converging to 0'; the corresponding notion on an arbitrary index set I is a function $u \in \boldsymbol{c}_0(I)$, that is, a real-valued function u on I such that $\{i : i \in I, |u(i)| \ge \epsilon\}$ is finite for every $\epsilon > 0$; if we give I its discrete topology, $c_0(I)$ is $C_0(I)$ as defined in 436I.

538B We need a number of basic ideas which can profitably be examined in a rather more general context. I start with a fundamental pre-order on the class of all filters.

The Rudin-Keisler ordering If $\mathcal{F}$, $\mathcal{G}$ are filters on sets I, J respectively, I will say that $\mathcal{F} \le_{\mathrm{RK}} \mathcal{G}$ if there is a function $f : J \to I$ such that

$$\mathcal{F} = f[[\mathcal{G}]] = \{A : A \subseteq I,\ f^{-1}[A] \in \mathcal{G}\},$$

the filter on I generated by $\{f[B] : B \in \mathcal{G}\}$. (I ought to remark that while this is a standard idea for ultrafilters, in the case of general filters the terminology is not well established.) Of course $\le_{\mathrm{RK}}$ is reflexive and transitive. If $\mathcal{F} \le_{\mathrm{RK}} \mathcal{G}$ and $\mathcal{G}$ is an ultrafilter, then $\mathcal{F}$ is an ultrafilter (2A1N). If $\mathcal{F}$ is a principal ultrafilter then $\mathcal{F} \le_{\mathrm{RK}} \mathcal{G}$ for every filter $\mathcal{G}$.

538C Lemma (a) If I is a set, $\mathcal{F}$ is an ultrafilter on I and $f : I \to I$ is a function such that $f[[\mathcal{F}]] = \mathcal{F}$, then $\{i : f(i) = i\} \in \mathcal{F}$.

(b) If I is a set, $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters on I, $\mathcal{F} \le_{\mathrm{RK}} \mathcal{G}$ and $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F}$, then there is a permutation $h : I \to I$ such that $h[[\mathcal{F}]] = \mathcal{G}$; that is, $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.

proof (a) It is enough to consider the case in which $I = \kappa$ is a cardinal.

(i) $\{\xi : \xi < \kappa,\ \xi \le f(\xi)\} \in \mathcal{F}$. **P** Define $\langle D_n \rangle_{n\in\mathbb{N}}$, $\langle E_n \rangle_{n\in\mathbb{N}}$ by saying that

$$D_0 = \kappa, \quad D_{n+1} = \{\xi : \xi \in D_n,\ f(\xi) \in D_n,\ f(\xi) < \xi\}, \quad E_n = D_n \setminus D_{n+1}$$

for $n \in \mathbb{N}$. If $\xi \in D_n$ then $\xi > f(\xi) > \ldots > f^n(\xi)$, so $\bigcap_{n\in\mathbb{N}} D_n = \emptyset$ and $\langle E_n \rangle_{n\in\mathbb{N}}$ is a partition of κ. If $\xi \in E_{n+1}$ then $f^{n+1}(\xi) < f^n(\xi) < \ldots < \xi$, $f^{n+1}(\xi) \le f^{n+2}(\xi)$, so $f(\xi) \in E_n$. Set $E = \bigcup_{n\ge1} E_{2n}$, $E' = \bigcup_{n\in\mathbb{N}} E_{2n+1}$; then $f[E] \subseteq E'$ is disjoint from E, so $E \notin \mathcal{F}$. Also $f[E'] \subseteq E \cup E_0$ is disjoint from E', so $E' \notin \mathcal{F}$. Because $\mathcal{F}$ is an ultrafilter, $E_0 \in \mathcal{F}$, as claimed. **Q**

(ii) If $A \subseteq I$ and $A \notin \mathcal{F}$ then $B = \bigcup_{n\in\mathbb{N}} (f^n)^{-1}[A]$ does not belong to $\mathcal{F}$. **P** For $\xi \in B$ set $m(\xi) = \min\{n : n \in \mathbb{N},\ f^n(\xi) \in A\}$. If $m(\xi) > 0$ then $m(f(\xi)) = m(\xi) - 1$. So setting $C = \{\xi : m(\xi)$ is even and not $0\}$, $C' = \{\xi : m(\xi)$ is odd$\}$ we have $f[C] \cap C = \emptyset$, $f[C'] \cap C' = \emptyset$ and $B \subseteq A \cup C \cup C'$; so $B \notin \mathcal{F}$. **Q**

Turning this round, if $A \in \mathcal{F}$ then $\bigcup_{n\in\mathbb{N}} (f^n)^{-1}[\kappa \setminus A] \notin \mathcal{F}$ and $\bigcap_{n\in\mathbb{N}} (f^n)^{-1}[A] \in \mathcal{F}$.

(iii) For $\xi < \kappa$ set

$$g(\xi) = \min\{\zeta : \text{there is some } n \in \mathbb{N} \text{ such that } f^n(\zeta) = \xi\}.$$

Then $g[[\mathcal{F}]] = \mathcal{F}$. **P** If $A \in \mathcal{F}$ then $\mathcal{F}$ contains $\bigcap_{n\in\mathbb{N}} (f^n)^{-1}[A] \subseteq g^{-1}[A]$, so $g^{-1}[A] \in \mathcal{F}$. Thus $\mathcal{F} \subseteq g[[\mathcal{F}]]$; as $\mathcal{F}$ is an ultrafilter, $\mathcal{F} = g[[\mathcal{F}]]$. **Q**

Now $g(\xi) \le \xi$ for every $\xi < \kappa$; applying (i) to g, we see that $G = \{\xi : g(\xi) = \xi\} \in \mathcal{F}$. But consider $H = \{\xi : \xi < f(\xi)\}$. Then $g(\eta) < \eta$ for every $\eta \in f[H]$, so $f[H] \notin \mathcal{F}$ and $H \notin \mathcal{F}$. Since we already know that $\{\xi : \xi \le f(\xi)\} \in \mathcal{F}$, we see that $\{\xi : f(\xi) = \xi\}$ belongs to $\mathcal{F}$, as claimed.

(b) Let f, $g : I \to I$ be such that $f[[\mathcal{F}]] = \mathcal{G}$ and $g[[\mathcal{G}]] = \mathcal{F}$. Then $(gf)[[\mathcal{F}]] = g[[f[[\mathcal{F}]]\,]] = \mathcal{F}$, so $J_0 = \{i : g(f(i)) = i\} \in \mathcal{F}$, by (a). Similarly, $J_1 = \{i : f(g(i)) = i\}$ belongs to $\mathcal{G}$. Set $J = J_0 \cap f^{-1}[J_1] \in \mathcal{F}$; then $g(f(i)) = i$

for every $i \in J$ and $f(g(j)) = j$ for every $j \in f[J]$, so $f{\restriction}J$ and $g{\restriction}f[J]$ are inverse bijections between $J \in \mathcal{F}$ and $f[J] \in \mathcal{G}$. If J is finite, then certainly $\#(I \setminus J) = \#(I \setminus f[J])$ and there is an extension of $f{\restriction}J$ to a permutation of I. If J is infinite, let $J' \subseteq J$ be a set such that $\#(J') = \#(J \setminus J') = \#(J)$ and $J' \in \mathcal{F}$; then $\#(I \setminus J') = \#(I \setminus f[J']) = \#(I)$ so there is an extension of $f{\restriction}J'$ to a permutation of I.

Thus in either case we have a permutation $h : I \to I$ and a $K \in \mathcal{F}$ such that $K \subseteq J$ and $h{\restriction}K = f{\restriction}K$. But now $h[[\mathcal{F}]] = \mathcal{G}$ and h is an isomorphism between $(I, \mathcal{F})$ and $(I, \mathcal{G})$.

538D Finite products of filters (a) Suppose that $\mathcal{F}$, $\mathcal{G}$ are filters on sets I, J respectively. I will write $\mathcal{F} \ltimes \mathcal{G}$ for

$$\{A : A \subseteq I \times J,\ \{i : A[\{i\}] \in \mathcal{G}\} \in \mathcal{F}\}.$$

It is easy to check that $\mathcal{F} \ltimes \mathcal{G}$ is a filter. (Compare the skew product $\mathcal{I} \ltimes \mathcal{J}$ of ideals defined in 527Ba.)

(b) If $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters, so is $\mathcal{F} \ltimes \mathcal{G}$. **P** If $A \subseteq I \times J$ and $A \notin \mathcal{F} \ltimes \mathcal{G}$, then $\{i : A[\{i\}] \in \mathcal{G}\} \notin \mathcal{F}\}$ and

$$\{i : ((I \times J) \setminus A)[\{i\}] \in \mathcal{G}\} = \{i : i \in I,\ J \setminus A[\{i\}] \in \mathcal{G}\} = I \setminus \{i : A[\{i\}] \in \mathcal{G}\} \in \mathcal{F},$$

so $(I \times J) \setminus A \in \mathcal{F} \ltimes \mathcal{G}$. **Q**

(c) If $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are filters on I, J, K respectively, then the natural bijection between $(I \times J) \times K$ and $I \times (J \times K)$ is an isomorphism between $(\mathcal{F} \ltimes \mathcal{G}) \ltimes \mathcal{H}$ and $\mathcal{F} \ltimes (\mathcal{G} \ltimes \mathcal{H})$. **P** If $A \subseteq I \times (J \times K)$ and $B = \{((i,j),k) : (i,(j,k)) \in A\}$, then

$$\begin{aligned} A \in \mathcal{F} \ltimes (\mathcal{G} \ltimes \mathcal{H}) &\iff \{i : A[\{i\}] \in \mathcal{G} \times \mathcal{H}\} \in \mathcal{F} \\ &\iff \{i : \{j : (A[\{i\}])[\{j\}] \in \mathcal{H}\} \in \mathcal{G}\} \in \mathcal{F} \\ &\iff \{(i,j) : (A[\{i\}])[\{j\}] \in \mathcal{H}\} \in \mathcal{F} \ltimes \mathcal{G} \\ &\iff \{(i,j) : B[\{(i,j)\}] \in \mathcal{H}\} \in \mathcal{F} \ltimes \mathcal{G} \\ &\iff B \in (\mathcal{F} \ltimes \mathcal{G}) \ltimes \mathcal{H}. \textbf{ Q} \end{aligned}$$

(d) It follows that we can define a product $\mathcal{F}_0 \ltimes \ldots \ltimes \mathcal{F}_n$ of any finite string $\mathcal{F}_0, \ldots, \mathcal{F}_n$ of filters, and under the natural identifications of the base sets we shall have $(\mathcal{F}_0 \ltimes \ldots \ltimes \mathcal{F}_n) \ltimes (\mathcal{F}_{n+1} \ltimes \ldots \ltimes \mathcal{F}_m)$ identified with $\mathcal{F}_0 \ltimes \ldots \ltimes \mathcal{F}_m$ whenever $\mathcal{F}_0, \ldots, \mathcal{F}_n, \ldots, \mathcal{F}_m$ are filters.

(e) For any filters $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F} \le_{\mathrm{RK}} \mathcal{F} \ltimes \mathcal{G}$ and $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F} \ltimes \mathcal{G}$. **P** Taking the base sets to be I, J respectively and $f(i,j) = i$, $g(i,j) = j$ for $i \in I$ and $j \in J$, we have $\mathcal{F} = f[[\mathcal{F} \ltimes \mathcal{G}]]$ and $\mathcal{G} = g[[\mathcal{F} \ltimes \mathcal{G}]]$. **Q**

Inducing on n, we see that $\mathcal{F}_n \le_{\mathrm{RK}} \mathcal{F}_0 \ltimes \ldots \ltimes \mathcal{F}_n$ whenever $\mathcal{F}_0, \ldots, \mathcal{F}_n$ are filters; consequently $\mathcal{F}_m \le_{\mathrm{RK}} \mathcal{F}_0 \ltimes \ldots \ltimes \mathcal{F}_n$ whenever $\mathcal{F}_0, \ldots, \mathcal{F}_n$ are filters and $m \le n$.

(f) If $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{G}$ and $\mathcal{G}'$ are filters, with $\mathcal{F} \le_{\mathrm{RK}} \mathcal{F}'$ and $\mathcal{G} \le_{\mathrm{RK}} \mathcal{G}'$, then $\mathcal{F} \ltimes \mathcal{G} \le_{\mathrm{RK}} \mathcal{F}' \ltimes \mathcal{G}'$. **P** Let the base sets of the filters be I, I', J and J', and let $f : I' \to I$ and $g : J' \to J$ be such that $\mathcal{F} = f[[\mathcal{F}']]$ and $\mathcal{G} = g[[\mathcal{G}']]$. Set $h(i,j) = (f(i), g(j))$ for $i \in I$ and $j \in J$. If $A \subseteq I \times J$, then

$$\begin{aligned} h^{-1}[A] \in \mathcal{F}' \ltimes \mathcal{G}' &\iff \{i : (h^{-1}[A])[\{i\}] \in \mathcal{G}'\} \in \mathcal{F}' \\ &\iff \{i : g^{-1}[A[\{f(i)\}]] \in \mathcal{G}'\} \in \mathcal{F}' \\ &\iff \{i : A[\{f(i)\}] \in \mathcal{G}\} \in \mathcal{F}' \\ &\iff \{i : A[\{i\}] \in \mathcal{G}\} \in \mathcal{F} \iff A \in \mathcal{F} \ltimes \mathcal{G}. \end{aligned}$$

So $\mathcal{F} \ltimes \mathcal{G} = h[[\mathcal{F}' \ltimes \mathcal{G}']]$ and $\mathcal{F} \ltimes \mathcal{G} \le_{\mathrm{RK}} \mathcal{F}' \ltimes \mathcal{G}'$. **Q**

Accordingly $\mathcal{F}_0 \ltimes \ldots \ltimes \mathcal{F}_n \le_{\mathrm{RK}} \mathcal{G}_0 \ltimes \ldots \ltimes \mathcal{G}_n$ whenever $\mathcal{F}_i \le_{\mathrm{RK}} \mathcal{G}_i$ for every $i \le n$.

(g) It follows that if $\mathcal{F}_0, \ldots, \mathcal{F}_n$ are filters and $k_0 < \ldots < k_m \le n$, then $\mathcal{F}_{k_0} \ltimes \ldots \ltimes \mathcal{F}_{k_m} \le_{\mathrm{RK}} \mathcal{F}_0 \ltimes \ldots \ltimes \mathcal{F}_n$. **P** Induce on m to see that $\mathcal{F}_{k_0} \ltimes \ldots \ltimes \mathcal{F}_{k_m} \le_{\mathrm{RK}} \mathcal{F}_0 \ltimes \ldots \ltimes \mathcal{F}_{k_m}$. **Q**

538E There are many variations on the construction here. A fairly elaborate extension will be needed in 538L below.

Iterated products of filters (a) First, a scrap of notation for the rest of the first half of this section (down to 538M). Set $S = \bigcup_{i\in\mathbb{N}} \mathbb{N}^i$. If $i, j \in \mathbb{N}$, $\sigma \in \mathbb{N}^i$ and $\tau \in \mathbb{N}^j$, define $\sigma^\frown\tau \in \mathbb{N}^{i+j}$ by setting

$$\begin{aligned}(\sigma^\frown\tau)(k) &= \sigma(k) \text{ if } k < i,\\ &= \tau(k-i) \text{ if } i \le k < i+j.\end{aligned}$$

For $k \in \mathbb{N}$ write $<\!k\!>$ for the member of $\mathbb{N}^1$ with value k.

Fix on a family $\langle\theta(\xi,k)\rangle_{1\le\xi<\omega_1,k\in\mathbb{N}}$ such that each $\langle\theta(\xi,k)\rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence running over a cofinal subset of ξ. (You will probably prefer to suppose that when $\xi = \eta+1$ is a successor ordinal, then $\theta(\xi,k) = \eta$ for every $k \in \mathbb{N}$.)

(b) Now suppose that ζ is a non-zero countable ordinal. Let $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$ be a family of filters on $\mathbb{N}$. For $\xi \le \zeta$, define $\mathcal{G}_\xi \subseteq \mathcal{P}S$ as follows. Start by taking $\mathcal{G}_0$ to be the principal filter generated by $\{\emptyset\}$. For $1 \le \xi \le \zeta$, set

$$\mathcal{G}_\xi = \{A : A \subseteq S,\ \{k : k \in \mathbb{N},\ \{\tau : <\!k\!>^\frown\tau \in A\} \in \mathcal{G}_{\theta(\xi,k)}\} \in \mathcal{F}_\xi\}.$$

It is elementary to check that every $\mathcal{G}_\xi$ is a filter, and that if every $\mathcal{F}_\xi$ is free, so is every $\mathcal{G}_\xi$. Moreover, if every $\mathcal{F}_\xi$ is an ultrafilter, so is every $\mathcal{G}_\xi$.

(c) Continuing from (b), we find that $\mathcal{F}_\xi \le_{\mathrm{RK}} \mathcal{G}_\xi$ whenever $1 \le \xi \le \zeta$ and $\mathcal{G}_\eta \le_{\mathrm{RK}} \mathcal{G}_\xi$ whenever $0 \le \eta \le \xi \le \zeta$. **P** Induce on ξ. (i) If $\xi \ge 1$, define $f : S \to \mathbb{N}$ by setting $f(\tau) = \tau(0)$ if $\tau \ne \emptyset$, $f(\emptyset) = 0$. Then, for $B \subseteq \mathbb{N}$,

$$\begin{aligned}f^{-1}[B] \in \mathcal{G}_\xi &\iff \{k : \{\tau : <\!k\!>^\frown\tau \in f^{-1}[B]\} \in \mathcal{G}_{\theta(\xi,k)}\} \in \mathcal{F}_\xi\\ &\iff B \in \mathcal{F}_\xi,\end{aligned}$$

so $\mathcal{F}_\xi = f[[\mathcal{G}_\xi]] \le_{\mathrm{RK}} \mathcal{G}_\xi$. (ii) If $\eta = \xi \le \zeta$ then of course $\mathcal{G}_\eta \le_{\mathrm{RK}} \mathcal{G}_\xi$. (iii) If $0 \le \eta < \xi$ then there is a k_0 such that $\eta \le \theta(\xi,k)$ for $k \ge k_0$. For $k \ge k_0$, $\mathcal{G}_\eta \le_{\mathrm{RK}} \mathcal{G}_{\theta(\xi,k)}$ by the inductive hypothesis; let $g_k : S \to S$ be such that $\mathcal{G}_\eta = g_k[[\mathcal{G}_{\theta(\xi,k)}]]$. Now define $g : S \to S$ by setting

$$\begin{aligned}g(\tau) &= g_k(\sigma) \text{ if } k \ge k_0 \text{ and } \tau = <\!k\!>^\frown\sigma,\\ &= \emptyset \text{ otherwise.}\end{aligned}$$

For $B \subseteq S$,

$$\begin{aligned}g^{-1}[B] \in \mathcal{G}_\xi &\iff \{k : \{\sigma : <\!k\!>^\frown\sigma \in g^{-1}[B]\} \in \mathcal{G}_{\theta(\xi,k)}\} \in \mathcal{F}_\xi\\ &\iff \{k : k \ge k_0,\ \{\sigma : g(<\!k\!>^\frown\sigma) \in B\} \in \mathcal{G}_{\theta(\xi,k)}\} \in \mathcal{F}_\xi\end{aligned}$$

(because $\mathcal{F}_\xi$ is free)

$$\begin{aligned}&\iff \{k : k \ge k_0,\ \{\sigma : g_k(\sigma) \in B\} \in \mathcal{G}_{\theta(\xi,k)}\} \in \mathcal{F}_\xi\\ &\iff \{k : k \ge k_0,\ B \in \mathcal{G}_\eta\} \in \mathcal{F}_\xi \iff B \in \mathcal{G}_\eta,\end{aligned}$$

so $\mathcal{G}_\eta = g[[\mathcal{G}_\xi]] \le_{\mathrm{RK}} \mathcal{G}_\xi$. **Q**

(d) It follows that if $1 \le \xi_0 < \ldots < \xi_n \le \zeta$ then $\mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_{\xi_n}$. **P** Induce on the pair (ξ_n, n). If $\xi_n = 1$ then $n = 0$ and we just have $\mathcal{F}_1 \le_{\mathrm{RK}} \mathcal{G}_1$, as in part (i) of the proof of (c). For the inductive step to $\xi_n = \xi > 1$, if $n = 0$ then again we need only note that $\mathcal{F}_{\xi_0} = \mathcal{F}_\xi \le_{\mathrm{RK}} \mathcal{G}_\xi$. If $n > 0$, let $k_0 \ge 1$ be such that $\xi_{n-1} \le \theta(\xi,k)$ for every $k \ge k_0$. For $k \ge k_0$,

$$\mathcal{F}_{\xi_{n-1}} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_{\xi_{n-1}} \le \mathcal{G}_{\theta(\xi,k)}$$

by the inductive hypothesis, so we have a function $g_k : S \to \mathbb{N}^n$ such that $\mathcal{F}_{\xi_{n-1}} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} = g_k[[\mathcal{G}_{\theta(\xi,k)}]]$. Define $g : S \to \mathbb{N}^{n+1}$ by setting

$$\begin{aligned}g(\tau) &= <\!k\!>^\frown g_k(\sigma) \text{ if } k \ge k_0 \text{ and } \tau = <\!k\!>^\frown\sigma,\\ &= \text{the constant function with value 0 otherwise.}\end{aligned}$$

Then, for $B \subseteq \mathbb{N}^{n+1}$,

$$\begin{aligned} g^{-1}[B] \in \mathcal{G}_\xi &\iff \{k : \{\sigma : g(<\!k\!>^\frown\sigma) \in B\} \in \mathcal{G}_{\theta(\xi,k)}\} \in \mathcal{F}_\xi \\ &\iff \{k : k \ge k_0,\, \{\sigma : <\!k\!>^\frown g_k(\sigma) \in B\} \in \mathcal{G}_{\theta(\xi,k)}\} \in \mathcal{F}_\xi \\ &\iff \{k : k \ge k_0,\, \{\sigma : g_k(\sigma) \in B_k\} \in \mathcal{G}_{\theta(\xi,k)}\} \in \mathcal{F}_\xi \end{aligned}$$

(writing $B_k = \{\sigma : <\!k\!>^\frown\sigma \in B\} \subseteq \mathbb{N}^n$ for $k \in \mathbb{N}$)

$$\begin{aligned} &\iff \{k : k \ge k_0,\, B_k \in \mathcal{F}_{\xi_{n-1}} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0}\} \in \mathcal{F}_\xi \\ &\iff \{k : k \in \mathbb{N},\, B_k \in \mathcal{F}_{\xi_{n-1}} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0}\} \in \mathcal{F}_\xi \\ &\iff B \in \mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0}. \end{aligned}$$

Thus g witnesses that $\mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_{\xi_n}$, and the induction proceeds. **Q**

Consequently $\mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_\zeta$ whenever $1 \le \xi_0 < \ldots < \xi_n \le \zeta$.

(e) The following special remark will be useful in Theorem 538L. Suppose that we are given $A_\xi \in \mathcal{F}_\xi$ for each $\xi \in [1,\zeta]$. Define $T \subseteq S$ and $\alpha : T \to [0,\zeta]$ as follows. Start by saying that $\emptyset \in T$ and $\alpha(\emptyset) = \zeta$. Having determined $T \cap \mathbb{N}^n$ and $\alpha{\restriction}T \cap \mathbb{N}^n$, where $n \in \mathbb{N}$, then for $\tau \in \mathbb{N}^{n+1}$ say that $\tau \in T$ iff τ is of the form $\sigma^\frown<\!k\!>$ where

$$\sigma \in T \cap \mathbb{N}^n, \quad \alpha(\sigma) > 0, \quad k \in A_{\alpha(\sigma)}, \quad \sigma(i) < k \text{ for every } i < n,$$

and in this case set $\alpha(\tau) = \theta(\alpha(\sigma), k)$. Continue. Observe that $\alpha(\tau) < \alpha(\sigma)$ whenever σ, $\tau \in T$ and τ properly extends σ.

Suppose that $D \in \bigcap_{1\le\xi\le\zeta} \mathcal{F}_\xi$. Then $T_D^* = \{\tau : \tau \in T \cap \bigcup_{n\in\mathbb{N}} D^n,\, \alpha(\tau) = 0\}$ belongs to $\mathcal{G}_\zeta$. **P** I aim to show by induction on ξ that if $\tau \in T \cap \bigcup_{n\in\mathbb{N}} D^n$ and $\alpha(\tau) = \xi$ then $\{\sigma : \tau^\frown\sigma \in T_D^*\}$ belongs to $\mathcal{G}_\xi$. If $\xi = 0$ then of course $\{\sigma : \tau^\frown\sigma \in T_D^*\} = \{\emptyset\} \in \mathcal{G}_0$. For the inductive step to $\xi > 0$,

$$\begin{aligned} &\{k : \{\sigma : \tau^\frown<\!k\!>^\frown\sigma \in T_D^*\} \in \mathcal{G}_{\theta(\xi,k)}\} \\ &\qquad\qquad \supseteq \{k : k \in D,\, \tau^\frown<\!k\!> \in T,\, \alpha(\tau^\frown<\!k\!>) = \theta(\xi,k)\} \end{aligned}$$

(by the inductive hypothesis)

$$\begin{aligned} &\supseteq \{k : k \in A_\xi \cap D,\, \tau(i) < k \text{ for every } i < \operatorname{dom}\tau\} \\ &\in \mathcal{F}_\xi, \end{aligned}$$

so $\{\sigma : \tau^\frown\sigma \in T_D^*\} \in \mathcal{G}_\xi$. At the end of the induction, we can apply this to $\tau = \emptyset$ and $\xi = \zeta$. **Q**

538F Ramsey filters There is an extensive and fascinating theory of Ramsey filters; see, for instance, Comfort & Negrepontis 74. Here, however, I will give only those fragments which are directly relevant to the other work of this section.

Proposition (a) A Ramsey filter on $\mathbb{N}$ is a rapid p-point ultrafilter.

(b) If $\mathcal{F}$ is a Ramsey ultrafilter on $\mathbb{N}$, $\mathcal{G}$ is a non-principal ultrafilter on $\mathbb{N}$, and $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F}$, then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic and $\mathcal{G}$ is a Ramsey ultrafilter.

(c) Let $\mathcal{F}$ be a Ramsey filter on $\mathbb{N}$. Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{F}$. Then there is an $A \in \mathcal{F}$ such that $n \in A_m$ whenever m, $n \in A$ and $m < n$.

(d) Let $\mathcal{F}$ be a Ramsey filter on $\mathbb{N}$. Let $\mathcal{S} \subseteq [\mathbb{N}]^{<\omega}$ be such that $\emptyset \in \mathcal{S}$ and $\{n : I \cup \{n\} \in \mathcal{S}\} \in \mathcal{F}$ for every $I \in \mathcal{S}$. Then there is an $A \in \mathcal{F}$ such that $[A]^{<\omega} \subseteq \mathcal{S}$.

(e) If $\mathfrak{F}$ is a countable family of distinct Ramsey filters on $\mathbb{N}$, there is a disjoint family $\langle A_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathfrak{F}}$ of subsets of $\mathbb{N}$ such that $A_{\mathcal{F}} \in \mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}$.

(f) Let $\mathfrak{F}$ be a countable family of non-isomorphic Ramsey ultrafilters on $\mathbb{N}$, and $\mathfrak{h} : \mathbb{N} \to [\mathfrak{F}]^{<\omega}$ a function. Suppose that we are given an $A_{\mathcal{F}} \in \mathcal{F}$ for each $\mathcal{F} \in \mathfrak{F}$. Then there is an $A \in \bigcap\mathfrak{F}$ such that whenever i, $j \in A$, $\mathcal{F} \in \mathfrak{h}(i)$ and $i < j$, there is a $k \in A_{\mathcal{F}}$ such that $i < k < j$.

(g) If $\mathfrak{m}_{\mathrm{countable}} = \mathfrak{c}$, there is a Ramsey ultrafilter on $\mathbb{N}$.

proof (a) Let $\mathcal{F}$ be a Ramsey filter on $\mathbb{N}$.

(i) $\mathcal{F}$ is an ultrafilter. **P** Let A be any subset of $\mathbb{N}$. Define $f:[\mathbb{N}]^2\to\{0,1\}$ by setting $f(I)=1$ if $\#(I\cap A)=1$, 0 otherwise. Then we have an $I\in\mathcal{F}$ such that f is constant on $[I]^2$. As $\mathcal{F}$ is free, $\#(I)\ge 3$ and the constant value of f cannot be 1. So either $I\subseteq A$ and $A\in\mathcal{F}$, or $I\cap A=\emptyset$ and $\mathbb{N}\setminus A\in\mathcal{F}$. As A is arbitrary, $\mathcal{F}$ is an ultrafilter. **Q**

(ii) $\mathcal{F}$ is a p-point filter. **P** Let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$. Set $K_n=(\mathbb{N}\setminus n)\cap\bigcap_{i<n}I_i$, $J_n=K_n\setminus K_{n+1}$ for each n; then $\langle J_n\rangle_{n\in\mathbb{N}}$ is a partition of $\mathbb{N}$. Define $f:[\mathbb{N}]^2\to\{0,1\}$ by setting $f(a)=0$ if there is an $n\in\mathbb{N}$ such that $a\subseteq J_n$, 1 otherwise. Let $I\in\mathcal{F}$ be such that f is constant on $[I]^2$.

Since $\mathbb{N}\setminus J_n\in\mathcal{F}$ for every n, there must be two points in I belonging to different J_n; so that the constant value of f must be 1, and no two points of I belong to the same J_n. In particular, $I\cap J_n$ is always finite, and $I\setminus I_n\subseteq\bigcup_{i\le n}I\cap J_i$ is always finite. As $\langle I_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{F}$ is a p-point filter. **Q**

(iii) $\mathcal{F}$ is rapid. **P** Let $\langle t_n\rangle_{n\in\mathbb{N}}$ be a sequence converging to 0. For each n, set $I_n=\{i:|t_i|\le 2^{-n}\}$; as $\mathcal{F}$ is free, $I_n\in\mathcal{F}$. Looking again at the proof of (ii) above, we see that the construction there gives us an $I\in\mathcal{F}$ such that $\#(I\setminus I_n)\le n+1$ for every n. We can therefore enumerate I as $\langle k_n\rangle_{n\in\mathbb{N}}$ in such a way that $k_{n+1}\in I_n$ for every n. But this means that

$$\textstyle\sum_{i\in I}|t_i|=\sum_{n=0}^\infty|t_{k_n}|\le|t_{k_0}|+\sum_{n=1}^\infty 2^{-n+1}<\infty.$$

As $\langle t_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{F}$ is rapid. **Q**

(b) Let $f:\mathbb{N}\to\mathbb{N}$ be such that $f[[\mathcal{F}]]=\mathcal{G}$. For $K\in[\mathbb{N}]^2$, set $h(K)=0$ if $f{\restriction}K$ is constant, 1 otherwise. Then there is an $A\in\mathcal{F}$ such that h is constant on $[A]^2$, that is, f is either constant or injective on A. Since $f[A]\in\mathcal{G}$, $f[A]$ is infinite, so f is injective on A. Let $g:\mathbb{N}\to\mathbb{N}$ be any function extending $(f{\restriction}A)^{-1}$; then $gf(n)=n$ for every $n\in A$, so

$$(gf)[[\mathcal{F}]]=\{I:(gf)^{-1}[I]\in\mathcal{F}\}=\{I:A\cap(gf)^{-1}[I]\in\mathcal{F}\}=\{I:A\cap I\in\mathcal{F}\}=\mathcal{F}.$$

But this means that $g[[\mathcal{G}]]=\mathcal{F}$ and $\mathcal{F}\le_{\mathrm{RK}}\mathcal{G}$.

By 538Cb, $\mathcal{F}$ and $\mathcal{G}$ are isomorphic, so $\mathcal{G}$ also must be a Ramsey ultrafilter.

(c) For $m<n$ in $\mathbb{N}$, set $h(\{m,n\})=1$ if $n\in A_m$, 0 otherwise. Then there is an $A\in\mathcal{F}$ such that $h{\restriction}[A]^2$ is constant. Setting $k=\min A$, A meets $A_k\setminus(k+1)$, so h takes the value 1 on $[A]^2$; consequently $n\in A_m$ whenever m, $n\in A$ and $m<n$.

(d) For $n\in\mathbb{N}$, set

$$A_n=\{i:I\cup\{i\}\in\mathcal{S}\text{ whenever }I\subseteq n+1\text{ and }I\in\mathcal{S}\}\in\mathcal{F}.$$

By (c), there is an $A\in\mathcal{F}$ such that $n\in A_m$ whenever m, $n\in A$ and $m<n$; and we can suppose that $A\subseteq A_0$, so that $\{n\}\in\mathcal{S}$ for every $n\in A$. Now an easy induction on n shows that $\mathcal{P}(A\cap n)\subseteq\mathcal{S}$ for every n, so $[A]^{<\omega}\subseteq\mathcal{S}$.

(e) Enumerate $\mathfrak{F}$ as $\langle\mathcal{F}_n\rangle_{n<\#(\mathfrak{F})}$. For distinct m, $n<\#(\mathfrak{F})$ there is a $B_{mn}\in\mathcal{F}_m\setminus\mathcal{F}_n$. **P** We know that there is a set in $\mathcal{F}_m\triangle\mathcal{F}_n$; now either this set or its complement will serve for B_{mn}. **Q** Because every member of $\mathfrak{F}$ is a p-point filter ((a) above), we can find for each $n<\#(\mathfrak{F})$ a set $C_n\in\mathcal{F}_n$ such that $C_n\setminus(B_{nm}\setminus B_{mn})$ is finite for every $m<\#(\mathfrak{F})$ such that $m\ne n$. Set $A_{\mathcal{F}_n}=C_n\setminus\bigcup_{m<n}C_m$ for $n<\#(\mathfrak{F})$; then $\langle A_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathfrak{F}}$ is disjoint. Since

$$C_m\cap C_n\subseteq(C_m\setminus B_{mn})\cup(C_n\cap B_{mn})$$

is finite whenever $m\ne n$, $C_n\setminus A_{\mathcal{F}_n}$ is finite and $A_{\mathcal{F}_n}\in\mathcal{F}_n$ for each $n<\#(\mathfrak{F})$.

(f)(i) We can suppose that $\mathfrak{h}(i)\subseteq\mathfrak{h}(j)$ whenever $i\le j$, and that $\mathfrak{F}=\bigcup_{i\in\mathbb{N}}\mathfrak{h}(i)$. Let $g:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function such that $g(0)>0$ and whenever $i\in\mathbb{N}$ and $\mathcal{F}\in\mathfrak{h}(i)$, there is a $k\in A_{\mathcal{F}}$ such that $i<k<g(i)$. Set $l_m=g^m(0)$ and $J_m=l_{m+1}\setminus l_m$ for each m, so that $\langle J_m\rangle_{m\in\mathbb{N}}$ is a partition of $\mathbb{N}$. Let $\langle a_\xi\rangle_{\xi<\omega_1}$ be a family of infinite subsets of $\mathbb{N}$, all containing 0, such that $a_\xi\cap a_\eta$ is finite for all distinct ξ, $\eta<\omega_1$ (5A1Fa), and set $M_\xi=\bigcup_{m\in a_\xi}J_m$ for each ξ; then $M_\xi\cap M_\eta$ is finite for all distinct ξ, $\eta<\omega_1$. It follows that each member of $\mathfrak{F}$ can contain at most one M_ξ, and there is a $\xi<\omega_1$ such that M_ξ does not belong to any member of $\mathfrak{F}$, that is, $M=\mathbb{N}\setminus M_\xi$ belongs to $\bigcap\mathfrak{F}$.

(ii) Define $f:\mathbb{N}\to\mathbb{N}$ by setting $f(n)=\max\{m:m\in a_\xi,\ l_m\le n\}$ for $n\in\mathbb{N}$. For each $\mathcal{F}\in\mathfrak{F}$, $f[[\mathcal{F}]]$ is isomorphic to $\mathcal{F}$, by (b). It follows that if $\mathcal{F}$, $\mathcal{F}'$ are distinct members of $\mathfrak{F}$, $f[[\mathcal{F}]]\ne f[[\mathcal{F}']]$. Because $\mathfrak{F}$ is countable, there is a disjoint family $\langle K_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathfrak{F}}$ of sets such that $K_{\mathcal{F}}\in f[[\mathcal{F}]]$ for every $\mathcal{F}\in\mathfrak{F}$ ((e) above). Set $L_{\mathcal{F}}=f^{-1}[K_{\mathcal{F}}]\in\mathcal{F}$ for each $\mathcal{F}\in\mathfrak{F}$.

(iii) For $i < j$ in $\mathbb{N}$, set $h(\{i,j\}) = 1$ if $j < g(i)$, 0 otherwise. $\mathcal{F} \in \mathfrak{F}$, there is an $L'_{\mathcal{F}} \in \mathcal{F}$ such that $L'_{\mathcal{F}} \subseteq L_{\mathcal{F}}$ and h is constant on $[L'_{\mathcal{F}}]^2$. As $L'_{\mathcal{F}}$ is infinite, the constant value cannot be 1 and must be 0, that is, $g(i) \le j$ whenever $i, j \in L'_{\mathcal{F}}$ and $i < j$.

(iv) Consider $A = \bigcup_{\mathcal{F} \in \mathfrak{F}} L'_{\mathcal{F}} \cap M$. Then $A \in \bigcap \mathfrak{F}$. Suppose that $i, j \in A$ and $i < j$; then $g(i) \le j$. **P** Let $\mathcal{F}$, $\mathcal{F}' \in \mathfrak{F}$ be such that $i \in L'_{\mathcal{F}}$ and $j \in L'_{\mathcal{F}'}$.

case 1 If $\mathcal{F} = \mathcal{F}'$, then both i and j belong to $L'_{\mathcal{F}}$, so $g(i) \le j$ by (iii).

case 2 If $\mathcal{F} \ne \mathcal{F}'$, then $i \in L_{\mathcal{F}}$ and $j \in L_{\mathcal{F}'}$, so $f(i) \in K_{\mathcal{F}}$ and $f(j) \in K_{\mathcal{F}'}$ and $f(i) \ne f(j)$. Let $m, n \in \mathbb{N}$ be such that $i \in J_m$ and $j \in J_n$; since $j \notin M_\xi$, $n \notin a_\xi$ and $f(j) < n$. As $K_{\mathcal{F}}$ and $K_{\mathcal{F}'}$ are disjoint, $f(i) < f(j)$. It follows that $m < f(j) < n$, so

$$g(i) \le g(l_{m+1}) \le g(l_{f(j)}) \le l_n \le j$$

and $g(i) \le j$ in this case also. **Q**

By the choice of g, this means that if $\mathcal{F} \in \mathfrak{h}(i)$ there must be a $k \in A_{\mathcal{F}}$ such that $i < k < j$, as required.

(g)(i) Suppose that $\mathcal{E} \subseteq \mathcal{P}\mathbb{N}$ is a filter base, containing $\mathbb{N} \setminus n$ for every $n \in \mathbb{N}$, and of size less than $\mathfrak{m}_{\text{countable}}$. Let $f : [\mathbb{N}]^2 \to \{0,1\}$ be a function. Then there is an $F \subseteq \mathbb{N}$ such that f is constant on $[F]^2$ and F meets every member of $\mathcal{E}$. **P** Set

$$\mathcal{E}^+ = \{J : J \subseteq \mathbb{N},\, J \cap E \ne \emptyset \text{ for every } E \in \mathcal{E}\},$$

$$S_n = \{n\} \cup \{i : i \in \mathbb{N} \setminus \{n\},\, f(\{i,n\}) = 1\},$$

$$S'_n = \{n\} \cup \{i : i \in \mathbb{N} \setminus \{n\},\, f(\{i,n\}) = 0\}$$

for $n \in \mathbb{N}$.

case 1 Suppose that $\{n : n \in J,\, J \cap S_n \in \mathcal{E}^+\}$ belongs to $\mathcal{E}^+$ for every $J \in \mathcal{E}^+$. Set

$$\mathcal{I} = \{I : I \in [\mathbb{N}]^{<\omega},\, f(K) = 1 \text{ for every } K \in [I]^2,\, \mathbb{N} \cap \textstyle\bigcap_{i \in I} S_i \in \mathcal{E}^+\}.$$

If $I \in \mathcal{I}$, $J = \mathbb{N} \cap \bigcap_{i \in I} S_i$ and $E \in \mathcal{E}$, then $J \in \mathcal{E}^+$; because $\mathcal{E}$ is a filter base, $J \cap E \in \mathcal{E}^+$; by hypothesis, $\{n : n \in J \cap E,\, J \cap E \cap S_n \in \mathcal{E}^+\}$ belongs to $\mathcal{E}^+$ and is not empty. There is therefore some $n \in J \cap E$ such that $J \cap S_n \in \mathcal{E}^+$, in which case $I \cup \{n\} \in \mathcal{I}$.

In particular, there is some $k \in \mathbb{N}$ such that $\{k\} \in \mathcal{I}$. Set

$$C = \{\alpha : \alpha \in \mathbb{N}^{\mathbb{N}},\, \{\alpha(i) : i < m\} \in \mathcal{I} \text{ for every } m \in \mathbb{N}\}.$$

Then C is compact, and it is non-empty because the constant function with value k belongs to C. Moreover, if $\alpha \in C$ and $m \in \mathbb{N}$ and $E \in \mathcal{E}$, there is an $n \in E$ such that $\{\alpha(i) : i < m\} \cup \{n\} \in \mathcal{I}$, so there is a $\beta \in C$ such that $\beta(i) = \alpha(i)$ for $i < m$ and $\beta(m) = n$. Thus $\{\beta : \beta \in C,\, E \cap \beta[\mathbb{N}] \ne \emptyset\}$ is a dense open subset of C. Writing $\mathcal{M}(C)$ for the ideal of meager subsets of C, $\operatorname{cov} \mathcal{M}(C)$ is either ∞ (if C has an isolated point) or $\operatorname{cov} \mathcal{M}(\mathbb{R}) = \mathfrak{m}_{\text{countable}}$, by 522Wb and 522Sb; in either case, it is greater than $\#(\mathcal{E})$. There is therefore some $\alpha \in C$ such that $F = \alpha[\mathbb{N}]$ meets every member of $\mathcal{E}$; in this case, f is equal to 1 everywhere in $[F]^2$, so we have an appropriate F.

case 2 Otherwise, there is a $K \in \mathcal{E}^+$ such that $\{n : n \in K,\, K \cap S_n \in \mathcal{E}^+\}$ does not belong to $\mathcal{E}^+$. Let $E_0 \in \mathcal{E}$ be disjoint from $\{n : n \in K,\, K \cap S_n \in \mathcal{E}^+\}$. Set $\mathcal{G} = \mathcal{E} \cup \{K \cap E : E \in \mathcal{E}\}$, so that $\mathcal{G}$ is a filter base and $\#(\mathcal{G}) < \mathfrak{m}_{\text{countable}}$. If $n \in E_0$ then there is an $E'_n \in \mathcal{E}$ disjoint from $K \cap S_n$. So if $J \in \mathcal{G}^+$, $J \cap S'_n \supseteq (J \cap K \cap E'_n) \setminus \{n\}$ belongs to $\mathcal{G}^+$ for every $n \in E_0$; accordingly $\{n : n \in J,\, J \cap S'_n \in \mathcal{G}^+\} \supseteq J \cap E_0$ belongs to $\mathcal{G}^+$.

We can therefore apply the argument of case 1 to $\mathcal{G}$ and the function $1 - f$ to see that there is an $F \subseteq \mathbb{N}$, meeting every member of $\mathcal{G} \supseteq \mathcal{E}$, such that $f = 0$ on $[F]^2$. **Q**

(ii) Enumerate the set of functions from $[\mathbb{N}]^2$ to $\{0,1\}$ as $\langle f_\xi \rangle_{\xi < \mathfrak{c}}$. Choose a non-decreasing family $\langle \mathcal{E}_\xi \rangle_{\xi \le \mathfrak{c}}$ inductively, as follows; the inductive hypothesis will be that $\mathcal{E}_\xi \subseteq \mathcal{P}\mathbb{N}$ is a filter base with cardinal at most $\max(\omega, \#(\xi))$. Start with $\mathcal{E}_0 = \{\mathbb{N} \setminus n : n \in \mathbb{N}\}$. Given $\mathcal{E}_\xi$, where $\xi < \mathfrak{c} = \mathfrak{m}_{\text{countable}}$, use (i) to find a set F_ξ, meeting every member of $\mathcal{E}_\xi$, such that f_ξ is constant on $[F_\xi]^2$; take $\mathcal{E}_{\xi+1} = \mathcal{E}_\xi \cup \{E \cap F_\xi : E \in \mathcal{E}_\xi\}$. Given $\langle \mathcal{E}_\eta \rangle_{\eta < \xi}$, where $\xi \le \mathfrak{c}$ is a non-zero limit ordinal, set $\mathcal{E}_\xi = \bigcup_{\eta < \xi} \mathcal{E}_\eta$.

At the end of the induction, let $\mathcal{F}$ be the filter generated by $\mathcal{E}_{\mathfrak{c}}$; then $\mathcal{F}$ is a Ramsey filter.

538G Measure-centering filters: Theorem Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Write ν_ω for the usual measure on $\{0,1\}^{\mathbb{N}}$, T_ω for its domain and $(\mathfrak{B}_\omega, \bar\nu_\omega)$ for its measure algebra. Then the following are equiveridical:

(i) $\mathcal{F}$ is measure-centering;

(ii) whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}_\omega$ such that $\inf_{n\in\mathbb{N}} \bar\nu_\omega a_n > 0$, there is an $A \in \mathcal{F}$ such that $\{a_n : n \in A\}$ is centered;

(iii) whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in T_ω such that $\inf_{n\in\mathbb{N}} \nu_\omega F_n > 0$, there is an $A \in \mathcal{F}$ such that $\bigcap_{n\in A} F_n \ne \emptyset$;

(iv) whenever (X, Σ, μ) is a perfect totally finite measure space and $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, $\mu^*(\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n) \ge \liminf_{n\to\mathcal{F}} \mu F_n$;

(v) whenever μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, then $\mu^*\mathcal{F} \ge \liminf_{n\to\mathcal{F}} \mu E_n$, where $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ for each n.

proof (i)⇒(ii) is trivial.

not-(iv)⇒not-(ii) Suppose there are a perfect totally finite measure space (X, Σ, μ) and a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in Σ such that $\liminf_{n\in\mathbb{N}} \mu F_n > \mu^*(\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n)$. Let F be a measurable envelope of $\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n$. Let T be the σ-subalgebra of Σ generated by $\{F\} \cup \{F_n : n \in \mathbb{N}\}$; then $\mu\restriction\mathrm{T}$ is a compact measure (451F). Let ν be its normalization $\frac{1}{\mu X}\mu\restriction\mathrm{T}$; then ν is a compact probability measure. We see that $\liminf_{n\to\mathcal{F}} \nu F_n > \nu F$; take γ such that $\nu F < \gamma < \liminf_{n\to\mathcal{F}} \nu F_n$, and set $C = \{n : \nu F_n > \gamma\}$, so that $C \in \mathcal{F}$.

Let $\mathcal{K}$ be a compact class such that ν is inner regular with respect to $\mathcal{K}$. For $n \in C$, let $K_n \in \mathcal{K} \cap \mathrm{T}$ be such that $K_n \subseteq F_n \setminus F$ and $\nu K_n \ge \gamma - \nu F$; for $n \in \mathbb{N} \setminus C$ set $K_n = X$.

The measure algebra $(\mathfrak{B}, \bar\nu)$ of ν is a probability algebra with countable Maharam type, so there is a measure-preserving Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{B}_\omega$ (332P or 333D). Set $a_n = \pi K_n^\bullet$ for each n. Then

$$\bar\nu_\omega a_n = \nu K_n \ge \gamma - \nu F > 0$$

for every n. On the other hand, if $A \in \mathcal{F}$, then $A \cap C \in \mathcal{F}$ so $\bigcap_{n\in A\cap C} K_n \subseteq \bigcap_{n\in A\cap C} F_n \setminus F$ is empty. As K_n belongs to the compact class $\mathcal{K}$ for every $n \in A \cap C$, there must be a finite set $I \subseteq A \cap C$ such that $\bigcap_{n\in I} K_n = \emptyset$, in which case $\inf_{n\in I} a_n = \pi(\bigcap_{n\in I} K_n)^\bullet = 0$. This shows that $\{a_n : n \in A\}$ is not centered. So $\langle a_n\rangle_{n\in\mathbb{N}}$ witnesses that (ii) is false.

(iv)⇒(i) Suppose that (iv) is true. Take a Boolean algebra $\mathfrak{A}$, an additive functional $\nu : \mathfrak{A} \to [0, \infty[$ and a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$. By 311E and 311H, we can suppose that $\mathfrak{A}$ is the algebra of open-and-closed subsets of a compact zero-dimensional space Z. In this case, there is a Radon measure μ on Z extending ν (416Qa). Of course μ is perfect (416Wa), and $\liminf_{n\to\mathcal{F}} \mu a_n \ge \inf_{n\in\mathbb{N}} \nu a_n > 0$, so (iv) tells us that there is an $A \in \mathcal{F}$ such that $\bigcap_{n\in A} a_n \ne \emptyset$, in which case $\{a_n : n \in \mathbb{N}\}$ is centered in $\mathfrak{A}$. As $\mathfrak{A}$, ν and $\langle a_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{F}$ is measure-centering.

(iv)⇒(v) The point is simply that μ is perfect (416Wa again) and that

$$\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} E_n = \bigcup_{A\in\mathcal{F}} \{a : A \subseteq a \subseteq \mathbb{N}\} = \mathcal{F}.$$

(v)⇒(iii) Suppose that (v) is true, and that $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in T_ω such that $\inf_{n\in\mathbb{N}} \nu_\omega F_n > 0$. Define $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{P}\mathbb{N}$ by setting $\phi(x) = \{n : x \in F_n\}$ for each n. Then ϕ is almost continuous (418J), so the image measure $\mu = \nu_\omega \phi^{-1}$ is a Radon probability measure on $\mathcal{P}\mathbb{N}$ (418I). Defining E_n as in (v), we have

$$\mu E_n = \nu_\omega \phi^{-1}[E_n] = \nu_\omega F_n$$

for every $n \in \mathbb{N}$, so

$$0 < \inf_{n\in\mathbb{N}} \bar\nu_\omega a_n \le \liminf_{n\to\mathcal{F}} \mu E_n \le \mu^*\mathcal{F} = \nu_\omega^* \phi^{-1}[\mathcal{F}]$$

(451Pc). In particular, there must be an $x \in \phi^{-1}[\mathcal{F}]$, so that $A = \{n : x \in F_n\}$ belongs to $\mathcal{F}$, and $\bigcap_{n\in A} F_n$ is non-empty.

(iii)⇒(ii) Assume (iii). Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{B}_\omega$ such that $\inf_{n\in\mathbb{N}} \bar\nu_\omega a_n > 0$. Let $\theta : \mathfrak{B}_\omega \to \mathrm{T}_\omega$ be a lifting (341K), and set $F_n = \theta a_n$ for each n. Then $\nu_\omega F_n = \bar\nu_\omega a_n$ for every n, so (iii) tells us that there is an $A \in \mathcal{F}$ such that $\bigcap_{n\in A} F_n \ne \emptyset$. In this case, $\theta(\inf_{n\in I} a_n) = \bigcap_{n\in I} F_n \ne \emptyset$ for every non-empty finite $I \subseteq A$, so $\{a_n : n \in A\}$ is centered.

538H Proposition (a) Any measure-centering filter on $\mathbb{N}$ is an ultrafilter.

(b) If $\mathcal{F}$ is a measure-centering ultrafilter on $\mathbb{N}$ and $\mathcal{G}$ is a filter on $\mathbb{N}$ such that $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F}$, then $\mathcal{G}$ is measure-centering.

(c) Every Ramsey ultrafilter on $\mathbb{N}$ is measure-centering.

(d) (Shelah 98b) Every measure-centering ultrafilter on $\mathbb{N}$ is a nowhere dense ultrafilter.

(e) (Benedikt 99) If $\operatorname{cov}\mathcal{N} = \mathfrak{c}$, where $\mathcal{N}$ is the Lebesgue null ideal, then there is a measure-centering ultrafilter on $\mathbb{N}$.

proof (a) Let a, b be disjoint non-zero elements of $\mathfrak{B}_\omega$, where $(\mathfrak{B}_\omega, \bar\nu_\omega)$ is the measure algebra of the usual measure on $\{0,1\}^{\mathbb{N}}$, as in 538G. Given $I \subseteq \mathbb{N}$, set $a_n = a$ if $n \in I$, b if $n \in \mathbb{N} \setminus I$. Then $\inf_{n\in\mathbb{N}} \bar\nu_\omega a_n > 0$, so there is a $J \in \mathcal{F}$ such that $\{a_n : n \in J\}$ is centered, in which case either $J \subseteq I$ or $J \cap I = \emptyset$; so that one of I, $\mathbb{N} \setminus I$ must belong to $\mathcal{F}$.

(b) Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f[[\mathcal{F}]] = \mathcal{G}$. Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ with $\inf_{n\in\mathbb{N}} \bar\mu a_n > 0$. Then $\langle a_{f(n)}\rangle_{n\in\mathbb{N}}$ has the same property, so there is an $A \in \mathcal{F}$ such that $\{a_{f(n)} : n \in A\}$ is centered. Now $f[A] \in \mathcal{G}$ and $\{a_m : m \in f[A]\}$ is centered.

(c) Let $\mathcal{F}$ be a Ramsey ultrafilter and $\langle b_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{B}_\omega$ such that $\gamma = \inf_{n\in\mathbb{N}} \bar\nu_\omega b_n$ is greater than 0. Set $b = \inf_{A\in\mathcal{F}} \sup_{n\in A} b_n$; then $\bar\nu_\omega b \ge \gamma$. Set $\mathcal{S} = \{I : I \in [\mathbb{N}]^{<\omega},\ b \cap \inf_{n\in I} b_n \ne 0\}$. Then $\emptyset \in \mathcal{S}$. If $I \in \mathcal{S}$, set $c = b \cap \inf_{n\in I} b_n$ and $C = \{n : c \cap b_n = 0\}$. Then $\sup_{n\in C} b_n$ does not meet c so does not include b, and $C \notin \mathcal{F}$. Accordingly

$$\{n : I \cup \{n\} \in \mathcal{S}\} = \mathbb{N} \setminus C \in \mathcal{F}.$$

By 538Fd, there is an $A \in \mathcal{F}$ such that $[A]^{<\omega} \subseteq \mathcal{S}$, in which case $\{b_n : n \in A\}$ is centered. As $\langle b_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{F}$ is measure-centering.

(d) Let $\mathcal{F}$ be a measure-centering ultrafilter, and $\langle t_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathbb{R}$. Let $F \subseteq [0,1[$ be a nowhere dense set with non-zero Lebesgue measure, and set $H = \bigcup_{k\in\mathbb{Z}} F + k$, so that H is nowhere dense in $\mathbb{R}$; let μ be Lebesgue measure on $[0,1]$. For $n \in \mathbb{N}$ set

$$E_n = \{x : x \in [0,1],\ x + t_n \in H\} = [0,1] \cap \bigcup\nolimits_{k\in\mathbb{Z}} F - t_n + k,$$

so that $\mu E_n = \mu F > 0$. By 538G(iv), there is an $A \in \mathcal{F}$ such that $\bigcap_{n\in A} E_n$ is non-empty; take $x \in \bigcap_{n\in A} E_n$, so that $t_n \in H - x$ for every $n \in A$, and $\{t_n : n \in A\}$ is nowhere dense. As $\langle t_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{F}$ is a nowhere dense filter.

(e)(i) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{B}_\omega$ such that $\inf_{n\in\mathbb{N}} \bar\nu_\omega a_n > 0$, and $\mathcal{C} \subseteq \mathcal{P}\mathbb{N}$ a filter base such that $\#(\mathcal{C}) < \operatorname{cov}\mathcal{N}$. Then there is an $A \subseteq \mathbb{N}$ such that A meets every member of C and $\{a_n : n \in A\}$ is centered. **P** Set $\epsilon = \inf_{n\in\mathbb{N}} \bar\nu_\omega a_n$. For $C \in \mathcal{C}$ set $b_C = \sup_{n\in C} a_n$; because $C \ne \emptyset$, $\bar\nu_\omega b_C \ge \epsilon$. Set $b = \inf_{C\in\mathcal{C}} b_C$; because $\mathcal{C}$ is downwards-directed, $\bar\nu_\omega b \ge \epsilon$ (321F) and $b \ne 0$.

Let $\theta : \mathfrak{B}_\omega \to \mathrm{T}_\omega$ be a lifting (341K). For $C \in \mathcal{C}$, set $F_C = \bigcup_{n\in C} \theta a_n$; then

$$F_C^\bullet = b_C \supseteq b,$$

so $\theta b \setminus F_C$ is negligible. Because $b \ne 0$, θb is not negligible; because $\#(\mathcal{C}) < \operatorname{cov}\mathcal{N}$, $\theta b \cap \bigcap_{C\in\mathcal{C}} F_C$ is non-empty (apply 522Wa to the subspace measure on θb). Take any x in the intersection, and set $A = \{n : x \in \theta a_n\}$. For every $C \in \mathcal{C}$, there is an $n \in C$ such that $x \in \theta a_n$, so $A \cap C \ne \emptyset$. If $I \subseteq A$ is finite and not empty, then $\theta(\inf_{n\in I} a_n) = \bigcap_{n\in I} \theta a_n$ contains x, so $\inf_{n\in I} a_n \ne 0$; thus $\{a_n : n \in A\}$ is centered. **Q**

(ii) Since $\#(\mathfrak{B}_\omega) = \mathfrak{c}$ (524Ma), we can enumerate as $\langle\langle a_{\xi n}\rangle_{n\in\mathbb{N}}\rangle_{\xi<\mathfrak{c}}$ the family of all sequences $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}_\omega$ such that $\inf_{n\in\mathbb{N}} \bar\nu_\omega a_n > 0$. Choose $\langle\mathcal{C}_\xi\rangle_{\xi<\mathfrak{c}}$ inductively, as follows. The inductive hypothesis will be that $\mathcal{C}_\xi \subseteq \mathcal{P}\mathbb{N}$ is a filter base and $\#(\mathcal{C}_\xi) \le \max(\omega, \#(\xi))$. Start with $\mathcal{C}_0 = \{\mathbb{N} \setminus n : n \in \mathbb{N}\}$. Given $\mathcal{C}_\xi$, where $\xi < \mathfrak{c}$, such that

$$\#(\mathcal{C}_\xi) \le \max(\omega, \#(\xi)) < \mathfrak{c} = \operatorname{cov}\mathcal{N},$$

(i) tells us that there is an $A_\xi \subseteq \mathbb{N}$, meeting every member of $\mathcal{C}_\xi$, such that $\{a_{\xi n} : n \in A_\xi\}$ is centered; set

$$\mathcal{C}_{\xi+1} = \mathcal{C}_\xi \cup \{C \cap A_\xi : C \in \mathcal{C}_\xi\}.$$

For a non-zero limit ordinal $\xi \le \mathfrak{c}$, set $\mathcal{C}_\xi = \bigcup_{\eta<\xi} \mathcal{C}_\eta$. Let $\mathcal{F}$ be the filter generated by $\mathcal{C}_\mathfrak{c}$; then $\mathcal{F}$ is a free filter satisfying 538G(ii), so is measure-centering.

538I Theorem Suppose that $\mathcal{F}$ is a measure-centering ultrafilter on $\mathbb{N}$, and that (X, Σ, μ) is a perfect probability space. Let $\mathcal{A}$ be the family of all sets of the form $\lim_{n\to\mathcal{F}} E_n$ where $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ. Then there is a unique complete measure λ on X such that λ is inner regular with respect to $\mathcal{A}$ and $\lambda(\lim_{n\to\mathcal{F}} E_n) = \lim_{n\to\mathcal{F}} \mu E_n$ for every sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ; and λ extends μ.

Remark By '$\lim_{n\to\mathcal{F}} E_n$' I mean the limit in the compact Hausdorff space $\mathcal{P}X$, that is,

$$\{x : \{n : x \in E_n\} \in \mathcal{F}\} = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} E_n = \bigcap_{A\in\mathcal{F}} \bigcup_{n\in A} E_n.$$

proof (a) $\mathcal{A}$ is an algebra of subsets of X. **P** If $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ, then

$$\lim_{n\to\mathcal{F}}(E_n \cap F_n) = (\lim_{n\to\mathcal{F}} E_n) \cap (\lim_{n\to\mathcal{F}} F_n),$$

$$\lim_{n\to\mathcal{F}}(E_n \triangle F_n) = (\lim_{n\to\mathcal{F}} E_n) \triangle (\lim_{n\to\mathcal{F}} F_n)$$

because $\mathcal{F}$ is an ultrafilter. **Q** Of course $\Sigma \subseteq \mathcal{A}$, because if $E_n = E$ for every n then $\lim_{n\to\mathcal{F}} E_n = E$.

(b) If $\langle E_n\rangle_{n\in\mathbb{N}}$ and $\langle F_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ and $\lim_{n\to\mathcal{F}} E_n = \lim_{n\to\mathcal{F}} F_n$, then $\lim_{n\to\mathcal{F}} \mu E_n = \lim_{n\to\mathcal{F}} \mu F_n$. **P**

$$|\lim_{n\to\mathcal{F}} \mu E_n - \lim_{n\to\mathcal{F}} \mu F_n| = \lim_{n\to\mathcal{F}} |\mu E_n - \mu F_n| \le \lim_{n\to\mathcal{F}} \mu(E_n \triangle F_n) \le \mu^*(\lim_{n\to\mathcal{F}} E_n \triangle F_n)$$

(538G(iv))

$$= \mu^*(\lim_{n\to\mathcal{F}} E_n \triangle \lim_{n\to\mathcal{F}} F_n) = \mu^*\emptyset = 0. \text{ **Q**}$$

(c) We therefore have a functional $\phi : \mathcal{A} \to [0,1]$ defined by setting $\phi(\lim_{n\to\mathcal{F}} E_n) = \lim_{n\to\mathcal{F}} \mu E_n$ for every sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ. Clearly ϕ extends μ. Also ϕ is additive. **P** If $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ such that $\lim_{n\to\mathcal{F}} E_n$ and $\lim_{n\to\mathcal{F}} F_n$ are disjoint, then

$$\begin{aligned}
\phi(\lim_{n\to\mathcal{F}} E_n \cup \lim_{n\to\mathcal{F}} F_n) &= \phi(\lim_{n\to\mathcal{F}} E_n \cup F_n) = \lim_{n\to\mathcal{F}} \mu(E_n \cup F_n) \\
&= \lim_{n\to\mathcal{F}} \mu E_n + \mu F_n - \mu(E_n \cap F_n) \\
&= \lim_{n\to\mathcal{F}} \mu E_n + \lim_{n\to\mathcal{F}} \mu F_n - \lim_{n\to\mathcal{F}} \mu(E_n \cap F_n) \\
&= \phi(\lim_{n\to\mathcal{F}} E_n) + \phi(\lim_{n\to\mathcal{F}} F_n) - \phi(\lim_{n\to\mathcal{F}} E_n \cap F_n) \\
&= \phi(\lim_{n\to\mathcal{F}} E_n) + \phi(\lim_{n\to\mathcal{F}} F_n). \text{ **Q**}
\end{aligned}$$

(d) Next, if $\langle A_m\rangle_{m\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{A}$, and $0 \le \gamma < \inf_{m\in\mathbb{N}} \phi A_m$, there is an $A \in \mathcal{A}$ such that $A \subseteq \bigcap_{m\in\mathbb{N}} A_m$ and $\phi A \ge \gamma$. **P** We can suppose that $A_0 = X$. For each $m \in \mathbb{N}$, let $\langle E_{mn}\rangle_{n\in\mathbb{N}}$ be a sequence in Σ such that $A_m = \lim_{n\to\mathcal{F}} E_{mn}$, starting with $E_{0n} = X$ for every n. For $m \in \mathbb{N}$, set $E'_{mn} = \bigcap_{i\le m} E_{in}$ for $n \in \mathbb{N}$; then

$$A_m = \bigcap_{i\le m} A_i = \lim_{n\to\mathcal{F}} E'_{mn};$$

set $I_m = \{n : n \in \mathbb{N},\ \mu E'_{mn} \ge \gamma\}$. Since $\lim_{n\to\mathcal{F}} \mu E'_{mn} = \phi A_m > \gamma$, $I_m \in \mathcal{F}$. For $n \in \mathbb{N}$, set $F_n = \bigcap\{E'_{mn} : m \in \mathbb{N},\ \mu E'_{mn} \ge \gamma\}$; set $A = \lim_{n\to\mathcal{F}} F_n$. Then $\mu F_n \ge \gamma$ for every n, so $\phi A \ge \gamma$. Also, for $m \in \mathbb{N}$, $F_n \subseteq E'_{mn}$ whenever $n \in I_m$, so $A \subseteq \lim_{n\to\mathcal{F}} E'_{mn} = A_m$. **Q**

(e) In particular, $\inf_{m\in\mathbb{N}} \phi A_m$ must be 0 whenever $\langle A_m\rangle_{m\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{A}$ with empty intersection. By 413J, there is a complete measure λ on X extending ϕ and inner regular with respect to $\mathcal{A}_\delta$, the set of intersections of sequences in $\mathcal{A}$. But $\lambda C = \sup\{\lambda A : A \in \mathcal{A},\ A \subseteq C\}$ for every $C \in \mathcal{A}_\delta$. **P** Suppose that $0 \le \gamma < \lambda C$. There is a sequence $\langle A_m\rangle_{m\in\mathbb{N}}$ in $\mathcal{A}$ with intersection C; because $\mathcal{A}$ is an algebra of sets, we can suppose that $\langle A_m\rangle_{m\in\mathbb{N}}$ is non-increasing. Now

$$\gamma < \lambda C = \inf_{m\in\mathbb{N}} \lambda A_m = \inf_{m\in\mathbb{N}} \phi A_m,$$

so (d) tells us that there is an $A \in \mathcal{A}$ such that $A \subseteq C$ and $\gamma \le \phi A = \lambda A$. **Q** It follows at once that λ is inner regular with respect to $\mathcal{A}$.

(f) If $E \in \Sigma$ and we set $E_n = E$ for every $n \in \mathbb{N}$, then $E = \lim_{n\to\mathcal{F}} E_n$ belongs to $\mathcal{A}$ and

$$\lambda E = \phi E = \lim_{n\to\mathcal{F}} \mu E_n = \mu E.$$

So λ extends μ. Finally, we see from 412L, as usual, that λ is uniquely defined.

Notation In this context, I will call λ the **$\mathcal{F}$-extension** of μ.

538J Proposition Let $\mathcal{F}$ be a measure-centering ultrafilter on $\mathbb{N}$ and (X, Σ, μ) a perfect probability space; let λ be the $\mathcal{F}$-extension of μ as defined in 538I.

(a) Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of μ, $(\mathfrak{B},\bar{\lambda})$ the measure algebra of λ, and $(\mathfrak{C},\bar{\nu})$ the probability algebra reduced power $(\mathfrak{A},\bar{\mu})^{\mathbb{N}}|\mathcal{F}$ (328C). Then we have a measure-preserving isomorphism $\pi:\mathfrak{B}\to\mathfrak{C}$ defined by saying that

$$\pi((\lim_{n\to\mathcal{F}}E_n)^\bullet)=\langle E_n^\bullet\rangle_{n\in\mathbb{N}}^\bullet$$

for every sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ.

(b) Let (X',Σ',μ') be another perfect probability space, and $\phi:X\to X'$ an inverse-measure-preserving function. Let λ' be the $\mathcal{F}$-extension of μ'. Then ϕ is inverse-measure-preserving for λ and λ'.

(c) Let $\mathcal{F}'$ be a filter on $\mathbb{N}$ such that $\mathcal{F}'\le_{\mathrm{RK}}\mathcal{F}$, and λ' the $\mathcal{F}'$-extension of μ. Then λ extends λ'.

proof (a)(i) I had better check first that the formula for π defines a function. If $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ such that $(\lim_{n\to\mathcal{F}}E_n)^\bullet=(\lim_{n\to\mathcal{F}}F_n)^\bullet$ in $\mathfrak{B}$, then

$$\begin{aligned}0&=\lambda(\lim_{n\to\mathcal{F}}E_n\triangle\lim_{n\to\mathcal{F}}F_n)=\lim_{n\to\mathcal{F}}\mu(E_n\triangle F_n)\\&=\lim_{n\to\mathcal{F}}\bar{\mu}(E_n^\bullet\triangle F_n^\bullet)=\bar{\nu}(\langle E_n^\bullet\rangle_{n\in\mathbb{N}}^\bullet\triangle\langle F_n^\bullet\rangle_{n\in\mathbb{N}}^\bullet),\end{aligned}$$

so $\langle E_n^\bullet\rangle_{n\in\mathbb{N}}^\bullet=\langle F_n^\bullet\rangle_{n\in\mathbb{N}}^\bullet$ in $\mathfrak{C}$.

(ii) Setting $\mathfrak{B}_0=\{E^\bullet:E\in\mathcal{A}\}$, where $\mathcal{A}$ is as in 538I, it is now routine to check that $\pi:\mathfrak{B}_0\to\mathfrak{C}$ is a surjective measure-preserving Boolean homomorphism. (Recall that $\mathfrak{C}$ is, by definition, the quotient of $\mathfrak{A}^{\mathbb{N}}$ by the ideal $\{\langle a_n\rangle_{n\in\mathbb{N}}:\lim_{n\to\mathcal{F}}\bar{\mu}a_n=0\}$.) But of course this means that $\mathfrak{B}_0$ is isomorphic to $\mathfrak{C}$, therefore Dedekind complete. Since λ is inner regular with respect to $\mathcal{A}$ (538I), $\mathfrak{B}_0$ is order-dense in $\mathfrak{B}$, and must be the whole of $\mathfrak{B}$.

(b) Setting

$$\mathcal{A}=\{\lim_{n\to\mathcal{F}}E_n:E_n\in\Sigma\ \forall\,n\in\mathbb{N}\},\quad\mathcal{A}'=\{\lim_{n\to\mathcal{F}}F_n:F_n\in\Sigma'\ \forall\,n\in\mathbb{N}\}$$

as in 538I, $\phi^{-1}[C]\in\mathcal{A}$ and $\lambda\phi^{-1}[C]=\lambda'C$ for every $C\in\mathcal{A}'$. **P** Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ' such that $C=\lim_{n\to\mathcal{F}}F_n$; then

$$\begin{aligned}\lambda\phi^{-1}[C]&=\lambda\phi^{-1}[\lim_{n\to\mathcal{F}}F_n]=\lambda(\lim_{n\to\mathcal{F}}\phi^{-1}[F_n])\\&=\lim_{n\to\mathcal{F}}\mu\phi^{-1}[F_n]=\lim_{n\to\mathcal{F}}\mu'F_n=\lambda'C.\ \mathbf{Q}\end{aligned}$$

By 412K, ϕ is inverse-measure-preserving for λ and λ'.

(c) By 538Hb, $\mathcal{F}'$ is measure-centering. Let $f:\mathbb{N}\to\mathbb{N}$ be such that $\mathcal{F}'=f[[\mathcal{F}]]$. Setting

$$\mathcal{A}=\{\lim_{n\to\mathcal{F}}E_n:E_n\in\Sigma\ \forall\,n\in\mathbb{N}\},\quad\mathcal{A}'=\{\lim_{n\to\mathcal{F}'}E_n:E_n\in\Sigma\ \forall\,n\in\mathbb{N}\},$$

$\mathcal{A}'\subseteq\mathcal{A}$ and $\lambda A=\lambda'A$ for every $A\in\mathcal{A}'$. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ such that $A=\lim_{n\to\mathcal{F}'}E_n$; then $A=\lim_{n\to\mathcal{F}}E_{f(n)}$, so

$$\lambda A=\lim_{n\to\mathcal{F}}\mu E_{f(n)}=\lim_{n\to\mathcal{F}'}\mu E_n=\lambda'A.\ \mathbf{Q}$$

By 412K again, the identity map from X to itself is inverse-measure-preserving for λ and λ', that is, λ extends λ'.

538K Having identified the measure algebra of a measure-centering-ultrafilter extension λ as a probability algebra reduced product (538Ja), we are in a position to apply the results of §377.

Proposition Let (X,Σ,μ) be a perfect probability space, $\mathcal{F}$ a measure-centering ultrafilter on $\mathbb{N}$ and λ the $\mathcal{F}$-extension of μ as constructed in 538I.

(a)(i) Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^0(\mu)$ such that $\{f_n^\bullet:n\in\mathbb{N}\}$ is bounded in the linear topological space $L^0(\mu)$. Then

(α) $f(x)=\lim_{n\to\mathcal{F}}f_n(x)$ is defined in $\mathbb{R}$ for λ-almost every $x\in X$;

(β) $f\in\mathcal{L}^0(\lambda)$.

(ii) For every $f\in\mathcal{L}^0(\lambda)$ there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0(\mu)$, bounded in the sense of (i), such that $f=\lim_{n\to\mathcal{F}}f_n$ λ-a.e.

(b) Suppose that $1\le p\le\infty$, and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^p(\mu)$ such that $\sup_{n\in\mathbb{N}}\|f_n\|_p$ is finite. Set $f(x)=\lim_{n\to\mathcal{F}}f_n(x)$ whenever this is defined in $\mathbb{R}$.

(i)(α) $f\in\mathcal{L}^p(\lambda)$;

(β) $\|f\|_p\le\lim_{n\to\mathcal{F}}\|f_n\|_p$.

(ii) Let g be a conditional expectation of f on Σ.

(α) If $p=1$ and $\{f_n:n\in\mathbb{N}\}$ is uniformly integrable, then then $\|f\|_1=\lim_{n\to\mathcal{F}}\|f_n\|_1$ and $g^\bullet=\lim_{n\to\mathcal{F}}f_n^\bullet$ for the weak topology of $L^1(\mu)$.

(β) If $1<p<\infty$, then $g^\bullet=\lim_{n\to\mathcal{F}}f_n^\bullet$ for the weak topology of $L^p(\mu)$.

(c) Suppose that $1\le p\le\infty$ and $f\in\mathcal{L}^p(\lambda)$.

(i) There is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^p(\mu)$ such that $f=\lim_{n\to\mathcal{F}}f_n$ λ-a.e. and $\|f\|_p=\sup_{n\in\mathbb{N}}\|f_n\|_p$.

(ii) If $p=1$, we can arrange that $\langle f_n\rangle_{n\in\mathbb{N}}$ should be uniformly integrable.

(d) Let (X',Σ',μ') be another perfect measure space, and λ' the $\mathcal{F}$-extension of μ'. Let $S:L^1(\mu)\to L^1(\mu')$ be a bounded linear operator.

(i) There is a unique bounded linear operator $\hat S:L^1(\lambda)\to L^1(\lambda')$ such that $\hat Sf^\bullet=g^\bullet$ whenever $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ are uniformly integrable sequences in $\mathcal{L}^1(\mu)$, $\mathcal{L}^1(\nu)$ respectively, $f=\lim_{n\to\mathcal{F}}f_n$ λ-a.e., $g=\lim_{n\to\mathcal{F}}g_n$ λ'-a.e., and $g_n^\bullet=Sf_n^\bullet$ for every $n\in\mathbb{N}$.

(ii) The map $S\mapsto\hat S:\mathsf{B}(L^1(\mu);L^1(\mu'))\to\mathsf{B}(L^1(\lambda);L^1(\lambda'))$ is a norm-preserving Riesz homomorphism.

proof We shall find that most of the work for this result has been done in §377. The only new step is in (a)(i), but we shall have some checking to do.

(a)(i) Let $\langle\tilde f_n\rangle_{n\in\mathbb{N}}$ be a sequence of Σ-measurable functions from X to $\mathbb{R}$ such that $\tilde f_n=f_n$ μ-a.e. for every $n\in\mathbb{N}$.

(α) Let $\epsilon>0$. Applying 367Rd[1] to $\{\tilde f_n^\bullet:n\in\mathbb{N}\}=\{f_n^\bullet:n\in\mathbb{N}\}$, there is a $\gamma>0$ such that $\mu E_n\le\epsilon$ for every $n\in\mathbb{N}$, where $E_n=\{x:|\tilde f_n(x)|\ge\gamma\}$. Set $E=\lim_{n\to\mathcal{F}}E_n$, so that $\lambda E\le\epsilon$. For $x\in X\setminus E$, $\{n:|\tilde f_n(x)|\le\gamma\}\in\mathcal{F}$, so $\lim_{n\to\mathcal{F}}\tilde f_n(x)$ is defined in $\mathbb{R}$. As ϵ is arbitrary, $\lim_{n\to\mathcal{F}}\tilde f_n(x)$ is defined in $\mathbb{R}$ for λ-almost every x. Since

$$\{x:x\in\operatorname{dom}f_n\text{ and }f_n(x)=\tilde f_n(x)\text{ for every }n\in\mathbb{N}\}$$

is μ-conegligible, therefore λ-conegligible, $\lim_{n\to\mathcal{F}}f_n$ is defined in $\mathbb{R}$ λ-a.e.

(β) For any $\alpha\in\mathbb{R}$,

$$\{x:\lim_{n\to\mathcal{F}}\tilde f_n(x)>\alpha\}=\bigcup_{k\in\mathbb{N}}\lim_{n\to\mathcal{F}}\{x:f_n(x)\ge\alpha+2^{-k}\}\in\operatorname{dom}\lambda.$$

So $f=_{\text{a.e.}}\lim_{n\to\mathcal{F}}\tilde f_n$ belongs to $\mathcal{L}^0(\lambda)$.

(ii) At this point I seek to import the machinery of §377.

(α) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\lambda)$ be the measure algebras of μ, λ respectively; recall that we can identify $L^0(\mu)$ and $L^0(\lambda)$ with $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ (364Ic). Write $(\mathfrak{C},\bar\nu)$ for the probability algebra reduced power $(\mathfrak{A},\bar\mu)^{\mathbb{N}}|\mathcal{F}$; let $\phi:\mathfrak{A}^{\mathbb{N}}\to\mathfrak{C}$ be the canonical surjection, and $\pi:\mathfrak{B}\to\mathfrak{C}$ the isomorphism of 538Ja; set $\psi=\pi^{-1}\phi:\mathfrak{A}^{\mathbb{N}}\to\mathfrak{B}$. Then $\bar\lambda\psi(\langle a_n\rangle_{n\in\mathbb{N}})=\lim_{n\to\mathcal{F}}\bar\mu a_n$ for every sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, and ψ is surjective.

(β) Let $W_0\subseteq L^0(\mathfrak{A})^{\mathbb{N}}$ be the set of sequences bounded for the topology of convergence in measure, and $\mathcal{W}_0\subseteq\mathcal{L}^0(\mu)^{\mathbb{N}}$ the set of sequences $\langle f_n\rangle_{n\in\mathbb{N}}$ such that $\langle f_n^\bullet\rangle_{n\in\mathbb{N}}\in W_0$. Then we have a Riesz homomorphism $T:W^0\to L^0(\mathfrak{B})$ defined by saying that $T(\langle f_n^\bullet\rangle_{n\in\mathbb{N}})=(\lim_{n\to\mathcal{F}}f_n)^\bullet$ whenever $\langle f_n\rangle_{n\in\mathbb{N}}\in\mathcal{W}_0$. **P** We know from (i) that $(\lim_{n\to\mathcal{F}}f_n)^\bullet$ is defined in $L^0(\lambda)\cong L^0(\mathfrak{B})$ whenever $\langle f_n\rangle_{n\in\mathbb{N}}\in\mathcal{W}_0$. (I am taking the domain of $\lim_{n\to\mathcal{F}}f_n$ to be $\{x:\lim_{n\to\mathcal{F}}f_n(x)\text{ is defined in }\mathbb{R}\}$.) Since

$$\lim_{n\to\mathcal{F}}f_n=_{\text{a.e.}}\lim_{n\to\mathcal{F}}g_n$$

whenever $f_n=_{\text{a.e.}}g_n$ for every n, T is well-defined. Since

$$\lim_{n\to\mathcal{F}}f_n+g_n=_{\text{a.e.}}\lim_{n\to\mathcal{F}}f_n+\lim_{n\to\mathcal{F}}g_n,$$

$$\lim_{n\to\mathcal{F}}\alpha f_n=_{\text{a.e.}}\alpha\lim_{n\to\mathcal{F}}f_n,\quad\lim_{n\to\mathcal{F}}|f_n|=_{\text{a.e.}}|\lim_{n\to\mathcal{F}}f_n|$$

whenever $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}\in\mathcal{W}_0$ and $\alpha\in\mathbb{R}$, T is a Riesz homomorphism. **Q**

(γ) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{A}$, $T(\langle\chi a_n\rangle_{n\in\mathbb{N}})=\chi\psi(\langle a_n\rangle_{n\in\mathbb{N}})$. **P** Express each a_n as $E_n^\bullet$, where $E_n\in\Sigma$, and set $F=\lim_{n\to\mathcal{F}}E_n$. In the language of 538Ja,

$$\psi(\langle a_n\rangle_{n\in\mathbb{N}})=\pi^{-1}\phi(\langle a_n\rangle_{n\in\mathbb{N}})=\pi^{-1}(\langle a_n\rangle^\bullet_{n\in\mathbb{N}})=F^\bullet,$$

so

$$T(\langle\chi a_n\rangle_{n\in\mathbb{N}})=(\lim_{n\to\mathcal{F}}\chi E_n)^\bullet=(\chi F)^\bullet=\chi(F^\bullet)=\chi\psi(\langle a_n\rangle_{n\in\mathbb{N}}).\ \mathbf{Q}$$

[1] Later editions only.

(δ) Recalling that W_0 is just the family of sequences $\langle u_n\rangle_{n\in\mathbb{N}}$ in L^0 such that $\inf_{k\in\mathbb{N}}\sup_{n\in\mathbb{N}}\bar\mu[\![|u_n|>k]\!]=0$ (367Rd again), (γ) means that we can identify $T:W_0\to L^0(\mathfrak{B})$ with the Riesz homomorphism described in 377B. By 377D(d-i), $T[W_0]=L^0(\mathfrak{B})$, which is what we need to prove the immediate result here.

(b)(i) As in part (a) of the proof of 377C, we see that a $\|\,\|_p$-bounded sequence in $\mathcal{L}^p(\mu)$ will belong to $\mathcal{W}_0$. So we can use 377Db.

(ii) Use 377Ec.

(c) Use 377Dd.

(d) Use 377F.

538L Theorem Suppose that ζ is a non-zero countable ordinal and $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$ is a family of Ramsey ultrafilters on $\mathbb{N}$, no two isomorphic. Let $\langle\mathcal{G}_\xi\rangle_{\xi\le\zeta}$ be the corresponding iterated product system, as described in 538E. Then $\mathcal{G}_\zeta$ is a measure-centering ultrafilter.

proof (a) Define $\langle(\mathfrak{A}_\xi,\bar\mu_\xi)\rangle_{\xi\le\zeta}$ inductively, as follows. $(\mathfrak{A}_0,\bar\mu_0)=(\mathfrak{B}_\omega,\bar\nu_\omega)$ is to be the measure algebra of the usual measure on $\{0,1\}^{\mathbb{N}}$. Given $\langle(\mathfrak{A}_\eta,\bar\mu_\eta)\rangle_{\eta<\xi}$, where $0<\xi\le\zeta$, let $(\mathfrak{A}_\xi,\bar\mu_\xi)$ be the probability algebra reduced product $\prod_{k\in\mathbb{N}}(\mathfrak{A}_{\theta(\xi,k)},\bar\mu_{\theta(\xi,k)})|\mathcal{F}_\xi$, as described in 328A-328C. At the end of the induction, write $(\mathfrak{C},\bar\nu)$ for $(\mathfrak{A}_\zeta,\bar\mu_\zeta)$.

(b) We have a family $\langle\phi_{\xi\eta}\rangle_{\eta\le\xi\le\zeta}$ defined by induction on ξ, as follows. The inductive hypothesis will be that $\phi_{\eta'\eta}$ is a measure-preserving Boolean homomorphism from $\mathfrak{A}_\eta$ to $\mathfrak{A}_{\eta'}$, and that $\phi_{\eta''\eta}=\phi_{\eta''\eta'}\phi_{\eta'\eta}$ whenever $\eta\le\eta'\le\eta''<\xi$. For the inductive step to ξ, take $\phi_{\xi\xi}$ to be the identity map on $\mathfrak{A}_\xi$. If $\xi>0$, set $\tilde\phi_{kj}=\phi_{\theta(\xi,k),\theta(\xi,j)}$ for $j\le k$ in $\mathbb{N}$; then 328Ea tells us that we have measure-preserving Boolean homomorphisms $\tilde\phi_k:\mathfrak{A}_{\theta(\xi,k)}\to\mathfrak{A}_\xi$ such that $\tilde\phi_j=\tilde\phi_k\tilde\phi_{kj}$ for $j\le k$. If $j\le k$ and $\eta\le\theta(\xi,j)$, then

$$\tilde\phi_k\phi_{\theta(\xi,k),\eta}=\tilde\phi_k\tilde\phi_{kj}\phi_{\theta(\xi,j),\eta}=\tilde\phi_j\phi_{\theta(\xi,j),\eta}$$

whenever $k\ge j$; so we can take this common value for $\phi_{\xi\eta}$. If $\eta\le\eta'<\xi$, then take k such that $\eta'\le\theta(\xi,k)$, and see that

$$\phi_{\xi\eta'}\phi_{\eta'\eta}=\tilde\phi_k\phi_{\theta(\xi,k),\eta'}\phi_{\eta'\eta}=\tilde\phi_k\phi_{\theta(\xi,k),\eta}=\phi_{\xi\eta},$$

so the induction proceeds.

For each $\xi\le\zeta$, write π_ξ for $\phi_{\zeta\xi}:\mathfrak{A}_\xi\to\mathfrak{C}$, and $\mathfrak{C}_\xi$ for the subalgebra $\pi_\xi[\mathfrak{A}_\xi]$. Of course $\pi_\xi\phi_{\xi\eta}=\pi_\eta$, so that $\mathfrak{C}_\eta\subseteq\mathfrak{C}_\xi$, whenever $\eta\le\xi\le\zeta$.

(c) For each $\xi>0$, we have a canonical map $\langle a_k\rangle_{k\in\mathbb{N}}\to\langle a_k\rangle^\bullet_{k\in\mathbb{N}}:\prod_{k\in\mathbb{N}}\mathfrak{A}_{\theta(\xi,k)}\to\mathfrak{A}_\xi$. Since every $\pi_\xi:\mathfrak{A}_\xi\to\mathfrak{C}_\xi$ is a measure-preserving isomorphism, we have a corresponding map $\psi_\xi:\prod_{k\in\mathbb{N}}\mathfrak{C}_{\theta(\xi,k)}\to\mathfrak{C}_\xi$. Reading off the basic facts of 328Ab and 328Eb, we see that

— $\bar\nu\psi_\xi(\langle c_k\rangle_{k\in\mathbb{N}})=\lim_{k\to\mathcal{F}_\xi}\bar\nu c_k$ for every sequence $\langle c_k\rangle_{k\in\mathbb{N}}\in\prod_{k\in\mathbb{N}}\mathfrak{C}_{\theta(\xi,k)}$,

— $\psi_\xi(\langle c_k\rangle_{k\in\mathbb{N}})\subseteq\sup_{k\in A}c_k$ whenever $\langle c_k\rangle_{k\in\mathbb{N}}\in\prod_{k\in\mathbb{N}}\mathfrak{C}_{\theta(\xi,k)}$ and $A\in\mathcal{F}_\xi$

(we can take the supremum in $\mathfrak{C}$ because $\mathfrak{C}_\xi$ is regularly embedded in $\mathfrak{C}$, as noted in 314Ga).

(d) Let $\langle a_\tau\rangle_{\tau\in S}$ be a family in $\mathfrak{A}_0=\mathfrak{B}_\omega$ such that $\gamma=\inf_{\tau\in S}\bar\mu a_\tau$ is non-zero. By 538Fe, we can find a disjoint family $\langle A_\xi\rangle_{1\le\xi\le\zeta}$ of subsets of $\mathbb{N}$ such that $A_\xi\in\mathcal{F}_\xi$ for every ξ. Use these to define $T\subseteq S$ and $\alpha:T\to[0,\zeta]$ as in 538Ee. Set $c_\tau=0$ for $\tau\in S\setminus T$. For $\tau\in T$ define $c_\tau\in\mathfrak{C}_{\alpha(\tau)}$ by induction on $\alpha(\tau)$, as follows. If $\alpha(\tau)=0$, set $c_\tau=\pi_0a_\tau$. For the inductive step to $\alpha(\tau)=\xi>0$, we know that $\tau^\frown{<}k{>}\in T$ and $\alpha(\tau^\frown{<}k{>})=\theta(\xi,k)$ whenever $k\in A_\xi$ and $\tau(i)<k$ for every $i<\operatorname{dom}\tau$; for other k, $\tau^\frown{<}k{>}\notin T$ so $c_{\tau^\frown<k>}=0\in\mathfrak{C}_{\theta(\xi,k)}$. Thus $c_{\tau^\frown<k>}\in\mathfrak{C}_{\theta(\xi,k)}$ for every k, and $\psi_\xi(\langle c_{\tau^\frown<k>}\rangle_{k\in\mathbb{N}})\in\mathfrak{C}_\xi$; take this for c_τ. Note that

$$\bar\nu c_\tau=\lim_{k\to\mathcal{F}_\xi}\bar\nu c_{\tau^\frown<k>}\ge\inf\{\bar\nu c_{\tau^\frown<k>}:k\in\mathbb{N},\,\tau^\frown{<}k{>}\in T\}.$$

Inducing on $\alpha(\tau)$, we see that $\bar\nu c_\tau\ge\gamma$ for every $\tau\in T$. In particular, $\bar\nu c_\emptyset\ge\gamma$.

(e) For $I\subseteq\mathbb{N}$, set $T_I=T\cap\bigcup_{n\in\mathbb{N}}I^n$ and $e_I=\inf_{\tau\in T_I}c_\tau$; let $\mathcal{S}$ be the family of those finite subsets I of $\mathbb{N}$ such that $e_I\ne\emptyset$. Then $T_\emptyset=\{\emptyset\}$, $e_\emptyset=c_\emptyset$ and $\emptyset\in\mathcal{S}$. Moreover, if $I\in\mathcal{S}$ and $1\le\xi\le\zeta$, then $\{k:I\cup\{k\}\in\mathcal{S}\}\in\mathcal{F}_\xi$. **P** Set $k_0=\sup I+1$. If $k\in A_\xi$ and $k\ge k_0$, set

$$d_k=\inf\{c_{\tau^\frown<k>}:\tau\in T_I,\,\alpha(\tau)=\xi\}.$$

Set $B=\{k:k\in A_\xi,\,k\ge k_0,\,d_k\cap e_I\ne 0\}$. If $k\in B$, then

$$T_{I\cup\{k\}}=T_I\cup\{\tau^\frown{<}k{>}:\tau\in T_I,\,\alpha(\tau)=\xi\},$$

because every member of T is strictly increasing and $\tau^\frown{<}k{>}$ can belong to T only when $k \in A_{\alpha(\tau)}$, that is, $\alpha(\tau) = \xi$. So $e_{I\cup\{k\}} = d_k \cap e_I \neq 0$ and $I \cup \{k\} \in \mathcal{S}$.

? If $B \notin \mathcal{F}_\xi$, then $B' = \{k : k \in A_\xi,\, k \geq k_0,\, d_k \cap e_I = 0\}$ belongs to $\mathcal{F}_\xi$. So

$$\begin{aligned} e_I &\subseteq \inf\{c_\tau : \tau \in T_I,\, \alpha(\tau) = \xi\} \\ &= \inf_{\substack{\tau\in T_I\\ \alpha(\tau)=\xi}} \psi_\xi(\langle c_{\tau^\frown<k>}\rangle_{k\in\mathbb{N}}) = \psi_\xi(\langle \inf_{\substack{\tau\in T_I\\ \alpha(\tau)=\xi}} c_{\tau^\frown<k>}\rangle_{k\in\mathbb{N}}) \end{aligned}$$

(because ψ_ξ is a Boolean homomorphism and T_I is finite)

$$\subseteq \sup_{k\in B'} \inf_{\substack{\tau\in T_I\\ \alpha(\tau)=\xi}} c_{\tau^\frown<k>}$$

(by (c))

$$= \sup_{k\in B'} d_k.$$

But $e_I \cap d_k = 0$ for every $k \in B'$ and $e_I \neq 0$. **X**

Thus $\{k : I \cup \{k\} \in \mathcal{S}\} \supseteq B \in \mathcal{F}_\xi$. **Q**

(f) For $i \in \mathbb{N}$ set

$$C_i = \{k : I \cup \{k\} \in \mathcal{S} \text{ whenever } I \in \mathcal{S} \text{ and } I \subseteq i\},$$

so that $C_i \in \mathcal{F}_\xi$ for every $\xi \in [1,\zeta]$. At this point, recall that every $\mathcal{F}_\xi$ is supposed to be a Ramsey ultrafilter. So for each $\xi \in [1,\zeta]$ we have an $A'_\xi \in \mathcal{F}_\xi$ such that $A'_\xi \subseteq A_\xi \cap C_0$ and $j \in C_i$ whenever $i, j \in A'_\xi$ and $i < j$ (538Fc). Next, for $i \in \mathbb{N}$ set $M_i = \{\alpha(\tau) : \tau \in T,\, \tau(j) \leq i$ whenever $j < \operatorname{dom}\tau\}$; then M_i is finite, so there is a $D \in \bigcap_{1\leq\xi\leq\zeta} \mathcal{F}_\xi$ such that whenever $i, j \in D$, $i < j$ and $\xi \in M_i$, there is a $k \in A'_\xi$ such that $i < k < j$ (538Ff). Of course we can suppose that $D \subseteq \bigcup_{1\leq\xi\leq\zeta} A'_\xi$, so that $D \cap A_\xi = D \cap A'_\xi$ for every ξ.

(g) $J \in \mathcal{S}$ for every finite subset J of D. **P** Induce on $\#(J)$. We know that $\emptyset \in \mathcal{S}$. If $i \in D$, then $\{i\} \in \mathcal{S}$ because $D \subseteq C_0$. For the inductive step to $\#(J) \geq 2$, set $j = \max J$, $I = J \setminus \{j\}$ and $i = \max I$. Then $I \in \mathcal{S}$, by the inductive hypothesis; so if $T_J = T_I$, we certainly have $J \in \mathcal{S}$. Otherwise, there is a member of $T_J \setminus T_I$, and this must be of the form $\tau^\frown{<}j{>}$ where $\tau \in T_I$ and $j \in A_{\alpha(\tau)}$; as $j \in D$, $j \in A'_{\alpha(\tau)}$. But this means that $\alpha(\tau) \in M_i$ and there is a $k \in A'_{\alpha(\tau)}$ such that $i < k < j$. In this case, $j \in C_k$ and $I \subseteq k$, so $J = I \cup \{k\}$ belongs to $\mathcal{S}$, and the induction proceeds. **Q**

(h) Thus $\{c_\tau : \tau \in T_D\}$ is centered; setting $T^*_D = \{\tau : \tau \in T_D,\, \alpha(\tau) = 0\}$, $\{c_\tau : \tau \in T^*_D\}$ and therefore $\{a_\tau : \tau \in T^*_D\}$ are centered. But T^*_D belongs to $\mathcal{G}_\zeta$, by 538Ee.

Since $\langle a_\tau\rangle_{\tau\in S}$ was chosen arbitrarily in (d) above, $\mathcal{G}_\zeta$ satisfies the condition of 538G(ii), translated to the countably infinite set S, and is measure-centering.

538M Benedikt's theorem (Benedikt 98) Let (X,Σ,μ) be a perfect probability space. Then there is a measure λ on X, extending μ, such that $\lambda(\lim_{n\to\mathcal{F}} E_n)$ is defined and equal to $\lim_{n\to\mathcal{F}} \mu E_n$ for every sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ and every Ramsey filter $\mathcal{F}$ on $\mathbb{N}$.

proof (a) If there are no Ramsey filters, we can take $\lambda = \mu$ and stop; so let us suppose that there is at least one Ramsey filter. Let $\mathfrak{F}$ be a family of Ramsey filters consisting of just one member of each isomorphism class, so that every Ramsey filter is isomorphic to some member of $\mathfrak{F}$, and no two members of $\mathfrak{F}$ are isomorphic. Fix a well-ordering $\preccurlyeq$ of $\mathfrak{F}$ with a greatest member $\mathcal{F}^*$ and a family $\langle\theta(\xi,k)\rangle_{1\leq\xi<\omega_1,k\in\mathbb{N}}$ such that $\langle\theta(\xi,k)\rangle_{k\in\mathbb{N}}$ is always a non-decreasing sequence of ordinals less than ξ such that $\{\theta(\xi,k) : k \in \mathbb{N}\}$ is cofinal with ξ.

(b)(i) For any non-empty countable set $D \subseteq \mathfrak{F}$ containing $\mathcal{F}^*$, enumerate it in $\preccurlyeq$-increasing order as $\langle\mathcal{F}_\xi\rangle_{1\leq\xi\leq\zeta}$, and let $\mathcal{G}_D$ be the measure-centering ultrafilter constructed from $\langle\mathcal{F}_\xi\rangle_{1\leq\xi\leq\zeta}$ and $\langle\theta(\xi,k)\rangle_{1\leq\xi\leq\zeta,k\in\mathbb{N}}$ by the method of 538E-538L; let λ_D be the $\mathcal{G}_D$-extension of μ as defined in 538I.

(ii) For any non-empty finite set $I \subseteq \mathfrak{F}$, list it in $\preccurlyeq$-increasing order as $\mathcal{F}_0, \dots, \mathcal{F}_n$, and set $\mathcal{H}_I = \mathcal{F}_n \ltimes \dots \ltimes \mathcal{F}_0$ as defined in 538D. By 538Ed, or otherwise, $\mathcal{H}_I \leq_{\text{RK}} \mathcal{G}_I$, so $\mathcal{H}_I$ is measure-centering (538Hb); let λ'_I be the $\mathcal{H}_I$-extension of μ.

(c) If $\emptyset \neq I \subseteq J \in [\mathfrak{F}]^{<\omega}$, then $\mathcal{H}_I \leq_{\text{RK}} \mathcal{H}_J$, by 538Dg, and λ'_J extends λ'_I, by 538Jc. Thus $\langle\lambda'_I\rangle_{\emptyset\neq I\in[\mathfrak{F}]^{<\omega}}$ is an upwards-directed family of probability measures on X.

If $\mathcal{I} \subseteq [\mathfrak{F}]^{<\omega} \setminus \{\emptyset\}$ is countable, we have a non-empty countable set $D \subseteq \mathfrak{F}$ including $\{\mathcal{F}^*\} \cup \bigcup \mathcal{I}$. Now 538Ed tells us that $\mathcal{H}_I \leq_{\mathrm{RK}} \mathcal{G}_D$ for every $I \in \mathcal{I}$, so that λ_D extends λ'_I for every $I \in \mathcal{I}$ (538Jc again). Thus for every countable subset of $\{\lambda'_I : I \in [\mathfrak{F}]^{<\omega} \setminus \{\emptyset\}\}$ there is a measure on X extending them all. By 457G, there is a measure λ on X extending every λ'_I.

(d) If $\mathcal{F}$ is a Ramsey ultrafilter and $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in Σ, there is an $\mathcal{F}' \in \mathfrak{F}$ such that $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic. In particular, $\mathcal{F} \leq_{\mathrm{RK}} \mathcal{F}'$, so $\tilde\lambda_{\mathcal{F}'}$ extends $\tilde\lambda_{\mathcal{F}}$, where $\tilde\lambda_{\mathcal{F}}$, $\tilde\lambda_{\mathcal{F}'}$ are the $\mathcal{F}$-extension and $\mathcal{F}'$-extension of μ. But λ extends $\lambda'_{\{\mathcal{F}'\}} = \tilde\lambda_{\mathcal{F}'}$ and therefore extends $\tilde\lambda_{\mathcal{F}}$. Accordingly $\lambda(\lim_{n\to\mathcal{F}} E_n)$ is defined and equal to $\tilde\lambda_{\mathcal{F}}(\lim_{n\to\mathcal{F}} E_n) = \lim_{n\to\mathcal{F}} \mu E_n$, as required.

538N Measure-converging filters: Proposition (a) Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Let ν_ω be the usual measure on $\{0,1\}^{\mathbb{N}}$, and T_ω its domain. Then the following are equiveridical:

(i) $\mathcal{F}$ is measure-converging;

(ii) whenever $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence in T_ω and $\lim_{n\to\infty} \nu_\omega F_n = 1$, then $\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n$ is conegligible;

(iii) whenever (X, Σ, μ) is a measure space with locally determined negligible sets (definition: 213I), and $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ which converges in measure to $f \in \mathcal{L}^0$, then $\lim_{n\to\mathcal{F}} f_n =_{\text{a.e.}} f$;

(iv) whenever μ is a Radon measure on $\mathcal{P}\mathbb{N}$ such that $\lim_{n\to\infty} \mu E_n = 1$, where $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ for each n, then $\mu\mathcal{F} = 1$.

(b) Every measure-converging filter is free.

(c) Suppose that $\mathcal{F}$ is a measure-converging filter.

(i) If $\mathcal{G}$ is a filter on $\mathbb{N}$ including $\mathcal{F}$, then $\mathcal{G}$ is measure-converging.

(ii) If $\mathcal{G}$ is a filter on $\mathbb{N}$ and $\mathcal{G} \leq_{\mathrm{RB}} \mathcal{F}$ (definition: 5A6Ic), then $\mathcal{G}$ is measure-converging.

(d) (M.Foreman) Every rapid filter is measure-converging.

(e) (M.Talagrand) If there is a measure-converging filter, there is a measure-converging filter which is not rapid.

(f) Let $\mathcal{F}$ be a measure-converging filter on $\mathbb{N}$ and $\mathcal{G}$ any filter on $\mathbb{N}$. Then $\mathcal{G} \ltimes \mathcal{F}$ is measure-converging.

(g) If $\mathfrak{m}_{\text{countable}} = \mathfrak{d}$, there is a rapid filter.

proof (a)(i)$\Rightarrow$(iii) Suppose that $\mathcal{F}$ is measure-converging, and that (X, Σ, μ), $\langle f_n \rangle_{n\in\mathbb{N}}$ and f are as in (iii). Let $H \in \Sigma$ be a conegligible set such that $H \subseteq \operatorname{dom} f \cap \operatorname{dom} f_n$ and $f{\restriction}H$ and $f_n{\restriction}H$ are measurable for every $n \in \mathbb{N}$. Let $k \in \mathbb{N}$; set $H_k = \{x : x \in H, \limsup_{n\to\mathcal{F}} |f_n(x) - f(x)| > 2^{-k}\}$. Then $H_k \cap F$ is negligible whenever $F \in \Sigma$ and $\mu F < \infty$. **P** If $\mu F = 0$ this is trivial. Otherwise, let $\nu = \frac{1}{\mu F}\mu_F$ be the normalized subspace measure on F. For each $n \in \mathbb{N}$, set $F_n = \{x : x \in F \cap H, |f_n(x) - f(x)| \leq 2^{-k}\}$. Then

$$\lim_{n\to\infty} \nu(F \setminus F_n) \leq \frac{2^k}{\mu F} \lim_{n\to\infty} \int \min(|f_n - f|, \chi F) d\mu = 0$$

because $\langle f_n \rangle_{n\in\mathbb{N}} \to f$ in measure. So $\lim_{n\to\infty} \nu F_n = 1$ and $H' = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n$ is ν-conegligible. But $H' \cap H_k = \emptyset$, so $\mu^*(H_k \cap F) = \nu^*(H_k \cap F) = 0$. **Q**

Since μ has locally determined negligible sets, H_k is negligible. This is true for every $k \in \mathbb{N}$, so $H \setminus \bigcup_{k\in\mathbb{N}} H_k$ is conegligible; and $\lim_{n\to\mathcal{F}} f_n(x) = f(x)$ for every $x \in H \setminus \bigcup_{k\in\mathbb{N}} H_k$, so $\lim_{n\to\mathcal{F}} f_n = f$ a.e., as required.

(iii)$\Rightarrow$(iv) Assuming (iii), let μ and $\langle E_n \rangle_{n\in\mathbb{N}}$ be as in (iv). Set $f_n = \chi(\mathcal{P}\mathbb{N} \setminus E_n)$ for each n; then $\lim_{n\to\infty} \int f_n d\mu = 0$, so $\langle f_n \rangle_{n\in\mathbb{N}} \to 0$ in measure, and $H = \{a : \lim_{n\to\mathcal{F}} f_n(a) = 0\}$ is conegligible. But for any $a \in H$,

$$a = \{n : a \in E_n\} = \{n : f_n(x) \leq \tfrac{1}{2}\}$$

belongs to $\mathcal{F}$, so $H \subseteq \mathcal{F}$ and $\mu\mathcal{F} = 1$.

(iv)$\Rightarrow$(ii) Assume (iv), and let $\langle F_n \rangle_{n\in\mathbb{N}}$ be as in (ii). Define $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{P}\mathbb{N}$ by setting $\phi(x) = \{n : x \in F_n\}$ for $x \in \{0,1\}^{\mathbb{N}}$. Then ϕ is almost continuous (418J), so the image measure $\mu = \nu_\omega \phi^{-1}$ on $\mathcal{P}\mathbb{N}$ is a Radon measure (418I). Since $F_n = \phi^{-1}[E_n]$ for each n, $\lim_{n\to\infty} \mu E_n = 1$ and $1 = \mu\mathcal{F} = \nu_\omega \phi^{-1}[\mathcal{F}]$. But now

$$\bigcup_{a\in\mathcal{F}} \bigcap_{n\in a} F_n = \bigcup_{a\in\mathcal{F}} \{x : a \subseteq \phi(x)\} = \phi^{-1}[\mathcal{F}]$$

is ν_ω-conegligible, as required.

(ii)$\Rightarrow$(i) Assume (ii), and take a probability space (X, Σ, μ) and a sequence $\langle H_n \rangle_{n\in\mathbb{N}}$ in Σ such that $\lim_{n\to\infty} \mu H_n = 1$; set $G = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} H_n$.

Let λ be the c.l.d. product measure on $X \times \{0,1\}^{\mathbb{N}}$, and set

$$W_n = H_n \times \{0,1\}^{\mathbb{N}}, \quad V_n = \{(x,y) : x \in X,\, y \in \{0,1\}^{\mathbb{N}},\, y(n) = 1\}$$

for $n \in \mathbb{N}$. Let Λ_1 be the σ-algebra of subsets of $X \times \{0,1\}^{\mathbb{N}}$ generated by $\{W_n : n \in \mathbb{N}\} \cup \{V_n : n \in \mathbb{N}\}$, and λ_1 the completion of the restriction $\lambda\restriction\Lambda_1$. Note that as the identity map from $X \times \{0,1\}^{\mathbb{N}}$ is inverse-measure-preserving for λ and $\lambda\restriction\Lambda_1$, it is inverse-measure-preserving for their completions (234Ba); but λ is complete, so this just means that λ extends λ_1. Then λ_1 is a complete atomless probability measure with countable Maharam type. Its measure algebra $\mathfrak{C}$ is therefore isomorphic, as measure algebra, to the measure algebra $\mathfrak{B}_\omega$ of ν_ω; let $\pi : \mathfrak{B}_\omega \to \mathfrak{C}$ be a measure-preserving isomorphism. By 343B, or otherwise, there is a realization $\phi : X \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, inverse-measure-preserving for λ_1 and ν_ω, such that $\phi^{-1}[F]^\bullet = \pi F^\bullet$ in $\mathfrak{C}$ for every $F \in \mathrm{T}_\omega$. Because π is surjective, there is for each $n \in \mathbb{N}$ an $F_n \in \mathrm{T}_\omega$ such that $\phi^{-1}[F_n] \triangle W_n$ is λ_1-negligible.

Since

$$\lim_{n\to\infty} \nu_\omega F_n = \lim_{n\to\infty} \lambda_1 W_n = \lim_{n\to\infty} \lambda W_n = \lim_{n\to\infty} \mu H_n = 1,$$

$F = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n$ is ν_ω-conegligible, and $\phi^{-1}[F] = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} \phi^{-1}[F_n]$ is λ_1-conegligible. We have $G \times \{0,1\}^{\mathbb{N}} = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} W_n$, so

$$(G \times \{0,1\}^{\mathbb{N}}) \triangle \phi^{-1}[F] \subseteq \bigcup_{n\in\mathbb{N}} W_n \triangle \phi^{-1}[F_n]$$

is λ_1-negligible. Thus $G \times \{0,1\}^{\mathbb{N}}$ is λ_1-conegligible, therefore λ-conegligible. But this means that G is μ-conegligible, by 252D applied to $G \times \{0,1\}^{\mathbb{N}}$; and this is what we needed to know.

(b) Let $\mathcal{F}$ be a measure-converging filter and $m \in \mathbb{N}$. Take a singleton set $X = \{x\}$ and the probability measure μ on X; set $E_i = \emptyset$ for $i < n$, X for $i \geq n$. Then $\lim_{i\to\infty} \mu E_i = 1$, so there is an $A \in \mathcal{F}$ such that $\bigcap_{i\in A} E_i$ is non-empty. Now $\mathbb{N} \setminus n \supseteq A$ belongs to $\mathcal{F}$; as n is arbitrary, $\mathcal{F}$ is free.

(c)(i) Immediate from the definition in 538Ag.

(ii) Let $f : \mathbb{N} \to \mathbb{N}$ be a finite-to-one function such that $\mathcal{G} = f[[\mathcal{F}]]$. Let (X, Σ, μ) be a probability space and $\langle E_n \rangle_{n\in\mathbb{N}}$ a sequence in Σ such that $\lim_{n\to\infty} \mu E_n = 1$. Set $F_n = E_{f(n)}$ for $n \in \mathbb{N}$; because f is finite-to-one, $\lim_{n\to\infty} \mu F_n = 1$. So $H = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n$ is conegligible. If $x \in H$, set $A_x = \{n : x \in E_n\}$; then

$$f^{-1}[A_x] = \{n : x \in E_{f(n)}\} = \{n : x \in F_n\}$$

belongs to $\mathcal{F}$ so $A_x \in f[[\mathcal{F}]]$ and $x \in \bigcup_{B\in f[[\mathcal{F}]]} \bigcap_{n\in B} E_n$. Thus $\bigcup_{B\in f[[\mathcal{F}]]} \bigcap_{n\in B} E_n \supseteq H$ is conegligible. As (X, Σ, μ) and $\langle E_n \rangle_{n\in\mathbb{N}}$ are arbitrary, $f[[\mathcal{F}]]$ is measure-converging.

(d) Let $\mathcal{F}$ be a rapid filter on $\mathbb{N}$, and $\langle H_n \rangle_{n\in\mathbb{N}}$ a sequence in T_ω such that $\lim_{n\to\infty} \nu_\omega H_n = 1$. Set $G = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} H_n$. Since $\lim_{n\to\infty}(1 - \nu_\omega H_n) = 0$, there is an $A \in \mathcal{F}$ such that $\sum_{n\in A} 1 - \nu_\omega H_n < \infty$; set $H = \bigcup_{m\in\mathbb{N}} \bigcap_{n\in A\setminus m} H_n \subseteq G$. Then

$$\nu_\omega H \geq \sup_{m\in\mathbb{N}} 1 - \sum_{n\in A\setminus m} (1 - \nu_\omega H_n) = 1,$$

so G is conegligible. Thus $\mathcal{F}$ satisfies (a-ii) and is measure-converging.

(e) Let $\mathcal{F}$ be a measure-converging filter. Let $\langle I_n \rangle_{n\in\mathbb{N}}$ be a sequence of non-empty finite subsets of $\mathbb{N}$ such that $\lim_{n\to\infty} \#(I_n) = \infty$. Let $\mathcal{G}$ be

$$\{A : A \subseteq \mathbb{N},\, \lim_{n\to\mathcal{F}} \frac{1}{\#(I_n)} \#(A \cap I_n) = 1\}.$$

Then $\mathcal{G}$ is a filter on $\mathbb{N}$.

(i) $\mathcal{G}$ is measure-converging. **P** Let $\langle H_i \rangle_{i\in\mathbb{N}}$ be a sequence in T_ω such that $\lim_{i\to\infty} \nu_\omega H_i = 1$, and set $G = \bigcup_{A\in\mathcal{G}} \bigcap_{i\in A} H_i$. Set $g_n = \frac{1}{\#(I_n)} \sum_{i\in I_n} \chi H_i$ for each n; then

$$\lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \frac{1}{\#(I_n)} \sum_{i\in I_n} \nu_\omega H_i = 1$$

because $\lim_{n\to\infty} \#(I_n) = \infty$ and $\lim_{i\to\infty} \nu_\omega H_i = 1$. Since $0 \leq g_n \leq \chi\{0,1\}^{\mathbb{N}}$ for every n, $\langle g_n \rangle_{n\in\mathbb{N}} \to \chi\{0,1\}^{\mathbb{N}}$ in measure. By (a-iii) above, $H = \{x : \lim_{n\to\mathcal{F}} g_n(x) = 1\}$ is conegligible.

For $x \in H$, set $A_x = \{i : x \in H_i\}$. Then

$$\frac{1}{\#(I_n)} \#(I_n \cap A_x) = g_n(x) \to 1$$

as $n \to \mathcal{F}$, so $A_x \in \mathcal{G}$ and $x \in G$. Accordingly $G \supseteq H$ is conegligible. As $\langle H_i \rangle_{i\in\mathbb{N}}$ is arbitrary, $\mathcal{G}$ is measure-converging. **Q**

(ii) $\mathcal{G}$ is not rapid. **P** Define $\langle t_i \rangle_{i\in\mathbb{N}}$ by saying that

$$t_i = \sup\{\frac{1}{\#(I_n)} : n \in \mathbb{N},\, i \in I_n\}$$

for $i \in \mathbb{N}$, counting $\sup\emptyset$ as 0. Then $\lim_{i\to\infty} t_i = 0$. If $A \in \mathcal{G}$ and $m \in \mathbb{N}$, then $B = \{n : \#(A \cap I_n) \ge \frac{2}{3}\#(I_n)\}$ belongs to $\mathcal{F}$, and must be infinite, by (b) above. So there is an $n \in B$ such that $\#(I_n) \ge 3m$, and now

$$\sum_{i\in A\setminus m} t_i \ge \#(A \cap I_n \setminus m) \cdot \frac{1}{\#(I_n)} \ge \frac{1}{3}.$$

As m is arbitrary, $\sum_{i\in A} t_i = \infty$; as A is arbitrary, $\mathcal{G}$ is not rapid. **Q**

(f) Let $\langle E_{ij} \rangle_{i,j\in\mathbb{N}}$ be a family in T_ω such that $\langle \nu_\omega E_{i_n j_n} \rangle_{n\in\mathbb{N}} \to 1$ for some, or any, enumeration $\langle (i_n, j_n) \rangle_{n\in\mathbb{N}}$ of $\mathbb{N} \times \mathbb{N}$. Set $G = \bigcup_{C\in\mathcal{G}\ltimes\mathcal{F}} \bigcap_{(i,j)\in C} E_{ij}$. For each $i \in \mathbb{N}$, $\lim_{j\to\infty} \nu_\omega E_{ij} = 1$, so $G_i = \bigcup_{A\in\mathcal{F}} \bigcap_{j\in A} E_{ij}$ is conegligible; set $H = \bigcap_{i\in\mathbb{N}} G_i$. For $x \in H$, set $A_x = \{(i,j) : x \in E_{ij}\}$. As $x \in G_i$, $A_x[\{i\}] \in \mathcal{F}$ for every $i \in \mathbb{N}$; thus $A_x \in \mathcal{G} \ltimes \mathcal{F}$ and $x \in G$. So G includes the conegligible set H, and is itself conegligible. As $\langle E_{ij} \rangle_{i,j\in\mathbb{N}}$ is arbitrary, $\mathcal{G}$ is measure-converging.

(g)(i) Suppose that $\mathcal{E} \subseteq [\mathbb{N}]^\omega$ is a family with $\#(\mathcal{E}) < \mathfrak{m}_{\mathrm{countable}}$, and that $f \in \mathbb{N}^\mathbb{N}$ is non-decreasing. Then there is an $A \subseteq \mathbb{N}$, meeting every member of $\mathcal{E}$, such that $\#(A \cap f(n)) \le n$ for every $n \in \mathbb{N}$. **P** Consider $X = \prod_{n\in\mathbb{N}} \mathbb{N} \setminus f(n)$. Then X is a closed subset of $\mathbb{N}^\mathbb{N}$, homeomorphic to $\mathbb{N}^\mathbb{N}$. For $E \in \mathcal{E}$, set

$$G_E = \{x : x \in X,\, E \cap x[\mathbb{N}] \ne \emptyset\};$$

then G_E is a dense open subset of X. Writing $\mathcal{M}(X)$ for the ideal of meager subsets of X, $\#(\mathcal{E}) < \mathfrak{m}_{\mathrm{countable}} = \operatorname{cov}\mathcal{M}(X)$, so there is an $x \in X \cap \bigcap_{E\in\mathcal{E}} G_E$; set $A = x[\mathbb{N}]$. **Q**

(ii) Let $\langle f_\xi \rangle_{\xi<\mathfrak{d}}$ be a cofinal family in $\mathbb{N}^\mathbb{N}$; we may suppose that every f_ξ is strictly increasing. Choose a non-decreasing family $\langle \mathcal{E}_\xi \rangle_{\xi\le\mathfrak{d}}$ inductively, as follows. $\mathcal{E}_0 = \{\mathbb{N} \setminus n : n \in \mathbb{N}\}$. Given that $\xi < \mathfrak{d} = \mathfrak{m}_{\mathrm{countable}}$ and that $\mathcal{E}_\xi \subseteq [\mathbb{N}]^{<\omega}$ is a filter base with cardinal at most $\max(\omega, \#(\xi))$, use (i) to find a set $A_\xi \subseteq \mathbb{N}$, meeting every member of $\mathcal{E}_\xi$, such that $\#(A_\xi \cap f_\xi(n)) \le n$ for every n; set

$$\mathcal{E}_{\xi+1} = \mathcal{E}_\xi \cup \{A_\xi \cap E : E \in \mathcal{E}_\xi\}.$$

For non-zero limit ordinals $\xi \le \mathfrak{d}$ set $\mathcal{E}_\xi = \bigcup_{\eta<\xi} \mathcal{E}_\eta$.

At the end of the induction, let $\mathcal{F}$ be the filter on $\mathbb{N}$ generated by $\mathcal{E}_\mathfrak{d}$. Then $\mathcal{F}$ is rapid. **P** It is free because $\mathcal{E}_0 \subseteq \mathcal{F}$. If $\langle t_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ converging to 0, let $f \in \mathbb{N}^\mathbb{N}$ be such that $|t_i| \le 2^{-n}$ whenever $n \in \mathbb{N}$ and $i \ge f(n)$, and let $\xi < \mathfrak{d}$ be such that $f \le f_\xi$. Then $A_\xi \in \mathcal{F}$ and

$$\sum_{i\in A_\xi} |t_i| \le \sum_{n=0}^\infty 2^{-n}\#(A_\xi \cap f_\xi(n+1) \setminus f_\xi(n)) \le \sum_{n=0}^\infty 2^{-n}(n+1)$$

is finite. **Q**

538O The Fatou property: Proposition (a) Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Let ν_ω be the usual measure on $\{0,1\}^\mathbb{N}$, and T_ω its domain. Then the following are equiveridical:

(i) $\mathcal{F}$ has the Fatou property;

(ii) whenever $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence in T_ω and $\nu_\omega^*(\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n) = 1$, then $\lim_{n\to\mathcal{F}} \nu_\omega F_n = 1$;

(iii) whenever (X, Σ, μ) is a measure space and $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence of non-negative functions in $\mathcal{L}^0(\mu)$, then $\overline{\int} \liminf_{n\to\mathcal{F}} f_n d\mu \le \liminf_{n\to\mathcal{F}} \int f_n d\mu$;

(iv) whenever μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, then $\mu^*\mathcal{F} \le \liminf_{n\to\mathcal{F}} \mu E_n$, where $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ for each $n \in \mathbb{N}$.

(b) If $\mathcal{F}$ and $\mathcal{G}$ are filters on $\mathbb{N}$, $\mathcal{G} \le_{\mathrm{RK}} \mathcal{F}$ and $\mathcal{F}$ has the Fatou property, then $\mathcal{G}$ has the Fatou property.

(c) If $\mathcal{F}$ and $\mathcal{G}$ are filters on $\mathbb{N}$ with the Fatou property, then $\mathcal{F} \ltimes \mathcal{G}$ has the Fatou property.

proof (a) not-(iii)⇒not-(i) Suppose that (X, Σ, μ) is a measure space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of non-negative functions in $\mathcal{L}^0$ such that $\overline{\int} \liminf_{n\to\mathcal{F}} f_n d\mu > \liminf_{n\to\mathcal{F}} \int f_n d\mu$. Changing the f_n on negligible sets does not change either $\overline{\int} \liminf_{n\to\mathcal{F}} f_n d\mu$ or $\overline{\int} \liminf_{n\to\mathcal{F}} f_n d\mu$, so we may assume that every f_n is defined everywhere in X and is Σ-measurable. Take α such that $\overline{\int} \liminf_{n\to\mathcal{F}} f_n d\mu > \alpha > \liminf_{n\to\mathcal{F}} \int f_n d\mu$; set $A = \{n : \int f_n d\mu \le \alpha\}$;

then A meets every member of $\mathcal{F}$. Since f_n is integrable for every $n \in A$, the set $G = \{x : \sup_{n\in A} f_n(x) > 0\}$ is a countable union of sets of finite measure.

Let λ be the c.l.d. product measure on $G\times\mathbb{R}$, and consider the ordinate sets $W_n = \{(x,\beta) : x \in G,\, 0 \le \beta < f_n(x)\}$ for $n \in A$. Set $W = \bigcup_{C\in\mathcal{F}} \bigcap_{n\in C\cap A} W_n$; writing g for $\liminf_{n\to\mathcal{F}} f_n$,

$$\{(x,\beta) : x \in G,\, 0 \le \beta < g(x)\} \subseteq W.$$

Since λ is a product of two σ-finite measures it is σ-finite, and W has a measurable envelope $\tilde W$ say. Now $\lambda^*W > \alpha$. **P?** Otherwise, $\lambda\tilde W \le \alpha$. Writing μ_L for Lebesgue measure on $\mathbb{R}$,

$$\alpha \ge \lambda\tilde W = \int_G \mu_L\tilde W[\{x\}]\mu(dx)$$

(252D)

$$\ge \overline{\int}_G g\,d\mu > \alpha. \text{ **XQ**}$$

There is therefore a set $V \subseteq \tilde W$ such that $\alpha < \lambda V < \infty$, and now $\lambda^*(V\cap W) > \alpha$. Let ν be the subspace measure on $V\cap W$. Set

$$\begin{aligned} V_n &= V\cap W\cap W_n \text{ if } n\in A,\\ &= V\cap W \text{ if } n\in\mathbb{N}\setminus A. \end{aligned}$$

Then

$$\begin{aligned} \liminf_{n\to\mathcal{F}} \nu V_n &= \sup_{C\in\mathcal{F}} \inf_{n\in C} \nu V_n \le \sup_{n\in A} \nu V_n\\ &\le \sup_{n\in A} \lambda W_n = \sup_{n\in A} \int f_n d\mu \le \alpha. \end{aligned}$$

On the other hand,

$$\textstyle\bigcup_{C\in\mathcal{F}} \bigcap_{n\in C} V_n = \bigcup_{C\in\mathcal{F}} \bigcap_{n\in C\cap A} V\cap W\cap W_n = V\cap W$$

and $\nu(V\cap W) = \lambda^*(V\cap W) > \alpha$. Moving to a normalization of ν, we see that (i) is false.

(iii)⇒(iv) If $\mathcal{F}$ satisfies (iii) and μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, set $g = \liminf_{n\to\mathcal{F}} \chi E_n$. If $a \in \mathcal{F}$, then $\{n : \chi E_n(a) = 1\} = a \in \mathcal{F}$, so $g(a) = 1$; thus

$$\mu^*\mathcal{F} = \overline{\int} \chi\mathcal{F} d\mu$$

(133Je)

$$\le \overline{\int} g\,d\mu \le \liminf_{n\to\mathcal{F}} \int \chi E_n = \liminf_{n\to\mathcal{F}} \mu E_n,$$

as required.

(iv)⇒(ii) Given (iv), suppose that $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in T_ω and $\nu_\omega^*(\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n) = 1$. As in the corresponding part of the argument for 538Na, define $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{P}\mathbb{N}$ by setting $\phi(x) = \{n : x \in F_n\}$, and let μ be the Radon measure $\nu_\omega\phi^{-1}$. Then

$$\mu^*\mathcal{F} = \nu_\omega^*\phi^{-1}[\mathcal{F}] = \nu_\omega^*(\textstyle\bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} F_n) = 1$$

(using 451Pc again for the first equality), so $\lim_{n\to\mathcal{F}} \nu_\omega F_n = \lim_{n\to\mathcal{F}} \mu E_n = 1$.

(ii)⇒(i) Assume (ii), and take a probability space (X,Σ,μ) and a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ in Σ such that $X = \bigcup_{A\in\mathcal{F}} \bigcap_{n\in A} H_n$.

As in the corresponding part of the argument for 538Na, let λ be the c.l.d. product measure on $X\times\{0,1\}^{\mathbb{N}}$, and set

$$W_n = H_n \times \{0,1\}^{\mathbb{N}}, \quad V_n = \{(x,y) : x \in X,\, y \in \{0,1\}^{\mathbb{N}},\, y(n) = 1\}$$

for $n \in \mathbb{N}$. Let Λ_1 be the σ-algebra of subsets of $X \times \{0,1\}^{\mathbb{N}}$ generated by $\{W_n : n \in \mathbb{N}\} \cup \{V_n : n \in \mathbb{N}\}$, and λ_1 the completion of the restriction $\lambda\restriction\Lambda_1$. As before, there is a function $\phi : X \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, inverse-measure-preserving for λ_1 and ν_ω, such that there is for each $n \in \mathbb{N}$ an $F_n \in \mathrm{T}_\omega$ such that $\phi^{-1}[F_n]\triangle W_n$ is λ_1-negligible. Set $G = \bigcup_{A\in\mathcal{F}}\bigcap_{n\in A} F_n$.

Since $X = \bigcup_{A\in\mathcal{F}}\bigcap_{n\in A} H_n$, $X \times \{0,1\}^{\mathbb{N}} = \bigcup_{A\in\mathcal{F}}\bigcap_{n\in A} W_n$ and

$$(X \times \{0,1\}^{\mathbb{N}}) \setminus \phi^{-1}[G] \subseteq \bigcup_{n\in\mathbb{N}} \phi^{-1}[F_n]\triangle W_n$$

is λ_1-negligible. By 413Eh,

$$\nu_\omega^* G \ge \lambda_1\phi^{-1}[G] = 1.$$

By (ii), $\lim_{n\to\mathcal{F}} \nu_\omega F_n = 1$. But

$$\nu_\omega F_n = \lambda_1\phi^{-1}[F_n] = \lambda_1 W_n = \lambda W_n = \mu H_n$$

for each n, so $\lim_{n\to\mathcal{F}} \mu H_n = 1$. As (X, Σ, μ) and $\langle H_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{F}$ has the Fatou property.

(b) Let $h : \mathbb{N} \to \mathbb{N}$ be such that $\mathcal{G} = h[[\mathcal{F}]]$. Let (X, Σ, μ) be a probability space and $\langle H_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ such that

$$\begin{aligned} X &= \bigcup_{A\in\mathcal{G}}\bigcap_{n\in A} H_n \\ &= \bigcup_{A\subseteq\mathbb{N},h^{-1}[A]\in\mathcal{F}}\bigcap_{n\in A} H_n = \bigcup_{A\in\mathcal{F}}\bigcap_{n\in A} H_{h(n)}. \end{aligned}$$

Then

$$\begin{aligned} 1 &= \liminf_{n\to\mathcal{F}} \mu H_{h(n)} = \sup_{A\in\mathcal{F}}\inf_{n\in A} \mu H_{h(n)} \\ &\le \sup_{A\in\mathcal{G}}\inf_{n\in A} \mu H_n = \liminf_{n\to\mathcal{G}} \mu H_n. \end{aligned}$$

As (X, Σ, μ) and $\langle H_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{G}$ has the Fatou property.

(c) Let (X, Σ, μ) be a probability space and $\langle E_{ij}\rangle_{i,j\in\mathbb{N}}$ a family in Σ such that $X = \bigcup_{C\in\mathcal{F}\ltimes\mathcal{G}}\bigcap_{(i,j)\in C} E_{ij}$. For each $i \in \mathbb{N}$, set $F_i = \bigcup_{B\in\mathcal{G}}\bigcap_{j\in B} E_{ij}$, and let $G_i \in \Sigma$ be a measurable envelope of F_i. Then $\bigcup_{A\in\mathcal{F}}\bigcap_{i\in A} G_i = X$. **P** If $x \in X$, there is a $C \in \mathcal{F}\ltimes\mathcal{G}$ such that $x \in E_{ij}$ whenever $(i,j) \in C$. Set $A = \{i : C[\{i\}] \in \mathcal{G}\} \in \mathcal{F}$. If $i \in A$, then

$$x \in \bigcap_{j\in C[\{i\}]} E_{ij} \subseteq F_i \subseteq G_i,$$

so $x \in \bigcap_{i\in A} G_i$. **Q**

Accordingly $\lim_{i\to\mathcal{F}} \mu G_i = 1$. Take $\epsilon > 0$; then $A = \{i : \mu G_i \ge 1-\epsilon\}$ belongs to $\mathcal{F}$. For each $i \in A$,

$$1-\epsilon \le \mu G_i = \mu^* F_i = \overline{\int}\chi F_i = \overline{\int}\liminf_{j\to\mathcal{G}} \chi E_{ij} \le \liminf_{j\to\mathcal{G}} \int\chi E_{ij}$$

(by (a-iii) above)

$$= \liminf_{j\to\mathcal{G}} \mu E_{ij},$$

so $\{j : \mu E_{ij} \ge 1-2\epsilon\} \in \mathcal{G}$. But this means that $\{(i,j) : \mu E_{ij} \ge 1-2\epsilon\} \in \mathcal{F}\ltimes\mathcal{G}$. As ϵ is arbitrary, $\lim_{(i,j)\to\mathcal{F}\ltimes\mathcal{G}} \mu E_{ij} = 1$. As (X, Σ, μ) and $\langle E_{ij}\rangle_{i,j\in\mathbb{N}}$ are arbitrary, $\mathcal{F}\ltimes\mathcal{G}$ has the Fatou property.

538P Theorem Let $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ be a bounded finitely additive functional. Write $⨍ \ldots d\nu$ for the associated linear functional on ℓ^∞ (see 363L), and set $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ for each $n \in \mathbb{N}$. Then the following are equiveridical:

(i) whenever μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, $\int \nu(a)\mu(da)$ is defined and equal to $⨍\mu E_n\nu(dn)$;

(ii) whenever μ is a Radon probability measure on $[0,1]^{\mathbb{N}}$, $\int⨍ x\,d\nu\,\mu(dx)$ is defined and equal to $⨍\int x(n)\mu(dx)\nu(dn)$;

(iii) whenever (X, Σ, μ) is a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of measurable real-valued functions on X, then $\int⨍ f_n(x)\nu(dn)\mu(dx)$ is defined and equal to $⨍\int f_n d\mu\,\nu(dn)$;

(iv) whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence of Borel subsets of $\{0,1\}^{\mathbb{N}}$, $\int⨍\chi F_n(x)\nu(dn)\nu_\omega(dx)$ is defined and equal to $⨍\nu_\omega F_n\nu(dn)$, where ν_ω is the usual measure on $\{0,1\}^{\mathbb{N}}$.

proof (i)⇒(ii)(α) For $t \in [0,1]$ define $h_t : [0,1]^{\mathbb{N}} \to \mathcal{P}\mathbb{N}$ by setting $h_t(x) = \{n : x(n) \ge t\}$ for $x \in [0,1]^{\mathbb{N}}$, and let $\mu_t = \mu h_t^{-1}$ be the image measure on $\mathcal{P}\mathbb{N}$. Then μ_t is a Radon measure for each t. **P** Because h_t is Borel measurable and $\mathcal{P}\mathbb{N}$ is metrizable, h_t is almost continuous (418J), so μ_t is a Radon measure (418I). **Q**

(β) For $m \in \mathbb{N}$ define $v_m \in [0,1]^{\mathbb{N}}$ by setting

$$v_m(n) = 2^{-m}\textstyle\sum_{k=1}^{2^m} \mu\{x : x(n) \ge 2^{-m}k\}.$$

Then $\|v_{m+1} - v_m\|_\infty \le 2^{-m-1}$. **P** For any $n \in \mathbb{N}$,

$$\begin{aligned} v_m(n) - v_{m+1}(n) &= 2^{-m}\sum_{k=1}^{2^m} \mu\{x : x(n) \ge 2^{-m}k\} - 2^{-m-1}\sum_{k=1}^{2^{m+1}} \mu\{x : x(n) \ge 2^{-m-1}k\} \\ &= 2^{-m-1}\sum_{k=1}^{2^m}(2\mu\{x : x(n) \ge 2^{-m}k\} - \mu\{x : x(n) \ge 2^{-m}k\} \\ &\qquad\qquad - \mu\{x : x(n) \ge 2^{-m-1}(2k+1)\}) \\ &= 2^{-m-1}\sum_{k=1}^{2^m} \mu\{x : 2^{-m}k \le x(n) < 2^{-m-1}(2k+1)\} \le 2^{-m-1}. \ \mathbf{Q} \end{aligned}$$

So $v = \lim_{m\to\infty} v_m$ is defined in ℓ^∞ and $⨍ v\,d\nu = \lim_{m\to\infty} ⨍ v_m d\nu$. Also $v(n) = \int x(n)\mu(dx)$ for every $n \in \mathbb{N}$, so $⨍\int x(n)\mu(dx)\nu(dn) = ⨍ v\,d\nu$.

(γ) Set

$$f(t) = ⨍ \mu_t E_n \nu(dn) = \int \nu(a)\mu_t(da)$$

for each $t \in [0,1]$ (using (i)). Then, for any $m \in \mathbb{N}$,

$$\begin{aligned} ⨍ v_m d\nu &= 2^{-m}\sum_{k=1}^{2^m} ⨍ \mu\{x : x(n) \ge 2^{-m}k\}\nu(dn) \\ &= 2^{-m}\sum_{k=1}^{2^m} ⨍ \mu\{x : h_{2^{-m}k}(x) \in E_n\}\nu(dn) \\ &= 2^{-m}\sum_{k=1}^{2^m} ⨍ \mu_{2^{-m}k}E_n\nu(dn) = 2^{-m}\sum_{k=1}^{2^m} f(2^{-m}k). \end{aligned}$$

(δ) Next, for $m \in \mathbb{N}$ and $x \in [0,1]^{\mathbb{N}}$, set $q_m(x) = 2^{-m}\sum_{k=1}^{2^m} \chi h_{2^{-m}k}(x)$, so that $\langle q_m(x)\rangle_{m\in\mathbb{N}}$ is non-decreasing and $\|x - q_m(x)\|_\infty \le 2^{-m}$ for each m, while $q_m : [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ is Borel measurable. Now

$$\begin{aligned} ⨍\!\!\int q_m(x)d\nu\,\mu(dx) &= 2^{-m}\sum_{k=1}^{2^m}\int \nu(h_{2^{-m}k}(x))\mu(dx) \\ &= 2^{-m}\sum_{k=1}^{2^m}\int \nu(a)\mu_{2^{-m}k}(da) = 2^{-m}\sum_{k=1}^{2^m} f(2^{-m}k). \end{aligned}$$

Also $\langle ⨍ q_m(x)d\nu\rangle_{m\in\mathbb{N}} \to ⨍ x\,d\nu$ uniformly for $x \in [0,1]^{\mathbb{N}}$, so $\int⨍ x\,d\nu\,\mu(dx)$ is defined and equal to

$$\begin{aligned} \lim_{m\to\infty}\int⨍ q_m(x)d\nu\,\mu(dx) &= \lim_{m\to\infty} 2^{-m}\sum_{k=1}^{2^m} f(2^{-m}k) = \lim_{m\to\infty}⨍ v_m d\nu \\ &= ⨍ v\,d\nu = ⨍\!\!\int x(n)\mu(dx)\nu(dn). \end{aligned}$$

As μ is arbitrary, (ii) is true.

(ii)⇒(iii) Assume (ii), and let (X,Σ,μ) be a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a uniformly bounded sequence of measurable real-valued functions on X. As completing μ does not affect the integral $\int\ldots d\mu$ (212Fb), we may suppose that μ is complete. Let $\gamma>0$ be such that $|f_n(x)|\le\gamma$ for every $n\in\mathbb{N}$ and $x\in X$, and set $q(x)(n)=\frac{1}{2\gamma}(\gamma+f_n(x))$ for all n and x. Then $q:X\to[0,1]^{\mathbb{N}}$ is measurable, so there is a Radon probability measure λ on $[0,1]^{\mathbb{N}}$ such that q is inverse-measure-preserving for μ and λ. **P** Taking $\lambda_0E=\mu q^{-1}[E]$ for Borel sets $E\subseteq[0,1]^{\mathbb{N}}$, q is inverse-measure-preserving for μ and λ_0; taking λ to be the completion of λ_0, q is inverse-measure-preserving for μ and λ, by 234Ba; and λ is a Radon measure by 433Cb. **Q** Now

$$\fint\int f_n d\mu\,\nu(dn)=2\gamma\fint\int q(x)(n)\mu(dx)\nu(dn)-\gamma$$

$$=2\gamma\fint\int z(n)\lambda(dz)\nu(dn)-\gamma$$

(235Gc)

$$=2\gamma\int\fint z(n)\nu(dn)\lambda(dz)-\gamma$$

(by (ii))

$$=2\gamma\int\fint q(x)(n)\nu(dn)\mu(dx)-\gamma=\int\fint f_n(x)\nu(dn)\mu(dx).$$

As μ and $\langle f_n\rangle_{n\in\mathbb{N}}$ are arbitrary, (iii) is true.

(iii)⇒(iv) is elementary, taking $f_n=\chi F_n$ and $\mu=\nu_\omega$.

(iv)⇒(i) If (iv) is true and μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, there is an inverse-measure-preserving function ϕ from $(\{0,1\}^{\mathbb{N}},\nu_\omega)$ to $(\mathcal{P}\mathbb{N},\mu)$ (343Cd). For each $n\in\mathbb{N}$, set $F_n=\phi^{-1}[E_n]$ for each n and choose a Borel set $F'_n\subseteq\{0,1\}^{\mathbb{N}}$ such that $\nu_\omega(F'_n\triangle F_n)=0$. Then $\int\fint\chi F'_n(x)\nu(dn)\nu_\omega(dx)$ is defined and equal to

$$\fint\nu_\omega F'_n\nu(dn)=\fint\nu_\omega F_n\nu(dn)=\fint\mu E_n\nu(dn).$$

Now

$$\fint\mu E_n\nu(dn)=\int\fint\chi F'_n(x)\nu(dn)\nu_\omega(dx)=\int\fint\chi F_n(x)\nu(dn)\nu_\omega(dx)$$

(because for almost every x, $\chi F_n(x)=\chi F'_n(x)$ for every n)

$$=\int\fint\chi E_n(\phi(x))\nu(dn)\nu_\omega(dx)=\int\fint\chi E_n(a)\nu(dn)\mu(da)$$

(235Gc again)

$$=\int\fint\chi a(n)\nu(dn)\mu(da)=\int\nu(a)\,\mu(da).$$

As μ is arbitrary, (i) is true.

538Q Definition I will say that a bounded finitely additive functional ν satisfying (i)-(iv) of 538P is a **medial functional**; if, in addition, ν is non-negative, $\nu a=0$ for every finite set $a\subseteq\mathbb{N}$ and $\nu\mathbb{N}=1$, I will call ν a **medial limit**. I should remark that the term 'medial limit' is normally used for the associated linear functional $\fint\ldots d\nu$ on ℓ^∞, rather than the additive functional ν on $\mathcal{P}\mathbb{N}$; thus $h\in(\ell^\infty)^*$ is a medial limit if $h\ge0$, $h(w)=\lim_{n\to\infty}w(n)$ for every convergent sequence $w\in\ell^\infty$ and $\int h(\langle f_n(x)\rangle_{n\in\mathbb{N}})\mu(dx)$ is defined and equal to $h(\langle\int f_nd\mu\rangle_{n\in\mathbb{N}})$ whenever (X,Σ,μ) is a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of measurable real-valued functions on X.

Note that 538P(i) tells us that a medial limit $\nu:\mathcal{P}\mathbb{N}\to\mathbb{R}$ is universally Radon-measurable (definition: 434Ec), therefore universally measurable (434Fc).

538R Proposition Let $M \cong (\ell^\infty)^*$ be the L-space of bounded finitely additive functionals on $\mathcal{P}\mathbb{N}$, and $M_{\text{med}} \subseteq M$ the set of medial functionals.

(a) M_{med} is a band in M, and if $T \in L^\times(\ell^\infty;\ell^\infty)$ (definition: 355G) and $T' : M \to M$ is its adjoint, then $T'\nu \in M_{\text{med}}$ for every $\nu \in M_{\text{med}}$.

(b) Taking M_τ to be the band of completely additive functionals on $\mathcal{P}\mathbb{N}$ and M_{m} the band of measurable functionals, as described in §464, $M_\tau \subseteq M_{\text{med}} \subseteq M_{\text{m}}$.

(c) Suppose that $\langle\nu_k\rangle_{k\in\mathbb{N}}$ is a norm-bounded sequence in M_{med}, and that $\nu \in M_{\text{med}}$. Set $\tilde\nu(a) = \fint \nu_k(a)\nu(dk)$ for $a \subseteq \mathbb{N}$. Then $\tilde\nu \in M_{\text{med}}$.

(d) Suppose that $\nu \in M$ is a medial limit, and set $\mathcal{F} = \{a : a \subseteq \mathbb{N},\ \nu(a) = 1\}$. Then $\mathcal{F}$ is a measure-converging filter with the Fatou property.

(e) Let (X,Σ,μ) and (Y,T,λ) be probability spaces, and $T \in L^\times(L^\infty(\mu);L^\infty(\lambda))$. Let $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ be sequences in $\mathcal{L}^\infty(\mu)$, $\mathcal{L}^\infty(\nu)$ respectively such that $Tf_n^\bullet = g_n^\bullet$ for every n and $\langle f_n^\bullet\rangle_{n\in\mathbb{N}}$ is norm-bounded in $L^\infty(\mu)$. Let $\nu \in M$ be a medial functional. Then $f(x) = \fint f_n(x)\nu(dn)$ and $g(y) = \fint g_n(y)\nu(dn)$ are defined for almost every $x \in X$ and $y \in Y$; moreover, $f \in \mathcal{L}^\infty(\mu)$, $g \in \mathcal{L}^\infty(\lambda)$ and $Tf^\bullet = g^\bullet$.

proof (a)(i) Any of the four conditions of 538P makes it clear that M_{med} is a linear subspace of M.

We see also that M_{med} is norm-closed in M. **P** Let $\langle\nu_n\rangle_{n\in\mathbb{N}}$ be a sequence in M_{med} which is norm-convergent to $\nu \in M$. If μ is a Radon probability measure on $[0,1]^{\mathbb{N}}$, then $\langle\fint x\,d\nu_n\rangle_{n\in\mathbb{N}} \to \fint x\,d\nu$ uniformly for $x \in [0,1]^{\mathbb{N}}$, so

$$\begin{aligned}\int\fint x\,d\nu\,\mu(dx) &= \lim_{n\to\infty}\int\fint x\,d\nu_n\mu(dx)\\ &= \lim_{n\to\infty}\fint\int x(i)\mu(dx)\nu_n(di) = \fint\int x(i)\,\mu(dx)\nu(di).\end{aligned}$$

As μ is arbitrary, $\nu \in M_{\text{med}}$. **Q**

(ii) Before completing the proof that M_{med} is a band, I deal with the second clause of (a).

(α) Recall from §355 that $L^\times(\ell^\infty;\ell^\infty)$ is the set of differences of order-continuous positive linear operators from ℓ^∞ to itself. Since M can be identified with $(\ell^\infty)^*$, any $T \in L^\times(\ell^\infty;\ell^\infty)$ has an adjoint $T' : M \to M$ defined by saying that $(T'\nu)(a) = \fint T(\chi a)d\nu$ for every $a \subseteq \mathbb{N}$. Since $x \mapsto \fint Tx\,d\nu$ and $x \mapsto \fint x\,d(T'\nu)$ both belong to $(\ell^\infty)^*$ and agree on $\{\chi a : a \subseteq \mathbb{N}\}$, they are equal, that is, $\fint Tx\,d\nu = \fint x\,d(T'\nu)$ for every $x \in \ell^\infty$.

(β) If $T : \ell^\infty \to \ell^\infty$ is an order-continuous positive linear operator, it is a norm-bounded linear operator (355C), and all the functionals $x \mapsto (Tx)(n)$ are order-continuous, therefore represented by members of ℓ^1; that is, we have a family $\langle\alpha_{ni}\rangle_{n,i\in\mathbb{N}}$ in $[0,\infty[$ such that

$$(Tx)(n) = \textstyle\sum_{i=0}^\infty \alpha_{ni}x(i) \text{ whenever } x \in \ell^\infty \text{ and } n \in \mathbb{N},$$

$$\textstyle\sup_{n\in\mathbb{N}}\sum_{i=0}^\infty \alpha_{ni} = \|T\| \text{ is finite.}$$

In this case, if $\nu \in M$ and $\nu' = T'\nu$ in M,

$$\fint x\,d\nu' = \fint (Tx)(n)\nu(dn) = \fint \textstyle\sum_{i=0}^\infty \alpha_{ni}x(i)\nu(dn)$$

for every $x \in \ell^\infty$.

Now suppose that that $\|T\| \le 1$, so that $\sum_{i=0}^\infty \alpha_{ni} \le 1$ for every n. Consider the function $\phi = T\restriction[0,1]^{\mathbb{N}}$. This is a function from $[0,1]^{\mathbb{N}}$ to itself, and it is continuous for the product topology on $\mathbb{N}$.

Take any $\nu \in M$ and Radon probability measure μ on $[0,1]^{\mathbb{N}}$; then the image measure $\mu_1 = \mu\phi^{-1}$ on $[0,1]^{\mathbb{N}}$ is a Radon probability measure (418I), and $\int f(\phi(x))\mu(dx) = \int f(x)\mu_1(dx)$ for any μ_1-integrable function f. In particular, setting $f(x) = \fint x\,d\nu$,

$$\textstyle\int\fint \phi(x)\,d\nu\,\mu(dx) = \int\fint x\,d\nu\,\mu_1(dx) = \fint\int x(n)\mu_1(dx)\nu(dn)$$

because $\nu \in M_{\text{med}}$.

Set $\nu' = T'\nu$. Then we can calculate

$$\fint\int x(n)\mu(dx)\nu'(dn) = \fint\sum_{i=0}^\infty \alpha_{ni}\int x(i)\mu(dx)\nu(dn) = \fint\int\sum_{i=0}^\infty \alpha_{ni}x(i)\mu(dx)\nu(dn)$$

(the inner integral is with respect to a genuine σ-additive measure, so we have B.Levi's theorem)

$$= \int\!\!\!\!\!\!-\!\!\int \phi(x)(n)\mu(dx)\nu(dn) = \int\!\!\!\!\!\!-\!\!\int x(n)\mu_1(dx)\nu(dn)$$
$$= \int\!\!\int\!\!\!\!\!\!- \phi(x)\,d\nu\,\mu(dx) = \int\!\!\int\!\!\!\!\!\!- Tx\,d\nu\,\mu(dx) = \int\!\!\int\!\!\!\!\!\!- x\,d\nu'\,\mu(dx).$$

As μ is arbitrary, ν' satisfies 538P(ii), and is a medial functional.

(γ) Thus $T'\nu \in M_{\text{med}}$ whenever $\nu \in M_{\text{med}}$ and $T : \ell^\infty \to \ell^\infty$ is positive, order-continuous and of norm at most 1. As M_{med} is a linear subspace of M, the same is true for every positive order-continuous T and for differences of such operators, that is, for every $T \in L^\times(\ell^\infty;\ell^\infty)$, as claimed.

(iii) I now return to the question of showing that M_{med} is a band. The point is that if ν is a medial functional and $b \subseteq \mathbb{N}$, then ν_b is a medial functional, where $\nu_b(a) = \nu(a \cap b)$ for every $a \subseteq \mathbb{N}$. **P** Define $T : \ell^\infty \to \ell^\infty$ by setting $Tx = x \times \chi b$ for $x \in \ell^\infty$. Then T is a positive order-continuous operator, and $T'\nu \in M_{\text{med}}$, by (iii) above. But

$$(T'\nu)(a) = \int\!\!\!\!\!\!- T(\chi a)d\nu = \int\!\!\!\!\!\!- \chi(a \cap b)d\nu = \nu(a \cap b) = \nu_b(a)$$

for every $a \subseteq \mathbb{N}$, so $\nu_b = T'\nu$ is a medial functional. **Q**

By 436M, this is enough to ensure that M_{med} is a band in M.

(b)(i) Recall that an additive functional on $\mathcal{P}\mathbb{N}$ is completely additive iff it corresponds to an element of ℓ^1, that is, belongs to the band generated by the elementary functionals δ_k where $\delta_k(a) = \chi a(k)$ for $k \in \mathbb{N}$ and $a \subseteq \mathbb{N}$. To see that δ_k belongs to M_{med}, all we have to do is to note that $\delta_k = \chi E_k$ where E_k is defined as in 538P; so if μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, we shall have

$$\int \delta_k d\mu = \mu E_k = \int\!\!\!\!\!\!- \mu E_n \delta_k(dn).$$

Since M_{med} is a band, it must include M_τ.

(ii) On the other side, 538P(i) tells us that every member of M_{med} is universally measurable, and therefore belongs to M_{m}, which is just the set of bounded additive functionals which are Σ-measurable, where Σ is the domain of the usual measure on $\mathcal{P}\mathbb{N}$.

(c)(i) Because $\langle \nu_k \rangle_{k\in\mathbb{N}}$ is norm-bounded, $\tilde{\nu}$ is well-defined and additive; also it is bounded. **P** If γ is such that $\|\nu\| \le \gamma$ and $\|\nu_k\| \le \gamma$ for every k, then

$$|\tilde{\nu}(a)| \le \gamma \sup_{k\in\mathbb{N}} |\nu_k(a)| \le \gamma^2$$

for every $a \subseteq \mathbb{N}$. **Q**

Note that

$$\int\!\!\!\!\!\!- \chi a\,d\tilde{\nu} = \tilde{\nu}(a) = \int\!\!\!\!\!\!- \nu_k(a)\nu(dk) = \int\!\!\!\!\!\!-\!\!\int\!\!\!\!\!\!- \chi a\,d\nu_k \nu(dk)$$

for every $a \subseteq \mathbb{N}$, so that

$$\int\!\!\!\!\!\!- x\,d\tilde{\nu} = \int\!\!\!\!\!\!- x(n)\,\tilde{\nu}(dn) = \int\!\!\!\!\!\!-\!\!\int\!\!\!\!\!\!- x(n)\,\nu_k(dn)\nu(dk) = \int\!\!\!\!\!\!-\!\!\int\!\!\!\!\!\!- x\,d\nu_k\nu(dk)$$

whenever $x \in \ell^\infty$ is a linear combination of indicator functions, and therefore for every $x \in \ell^\infty$.

(ii) Now suppose that (X, Σ, μ) is a probability space and that $\langle f_n \rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of measurable real-valued functions on X. Let $(X, \hat{\Sigma}, \hat{\mu})$ be the completion of (X, Σ, μ). For $k \in \mathbb{N}$ and $x \in X$ set $g_k(x) = \int\!\!\!\!\!\!- f_n(x)\nu_k(dn)$; because ν_k is a medial functional, we know that $\int g_k d\mu = \int\!\!\!\!\!\!-\!\!\int f_n(x)\mu(dx)\nu_k(dn)$ is defined, so g_k is $\hat{\Sigma}$-measurable. Consequently $\int\!\!\int\!\!\!\!\!\!- g_k(x)\nu(dk)\hat{\mu}(dx)$ is defined and equal to $\int\!\!\!\!\!\!-\!\!\int g_k(x)\hat{\mu}(dx)\nu(dk)$. It follows that

$$\int\!\!\!\!\!\!-\!\!\int f_n(x)\mu(dx)\tilde{\nu}(dn) = \int\!\!\!\!\!\!-\!\!\int\!\!\!\!\!\!-\!\!\int f_n(x)\mu(dx)\nu_k(dn)\nu(dk)$$
$$= \int\!\!\!\!\!\!-\!\!\int\!\!\int\!\!\!\!\!\!- f_n(x)\nu_k(dn)\mu(dx)\nu(dk) = \int\!\!\!\!\!\!-\!\!\int g_k(x)\hat{\mu}(dx)\nu(dk)$$
$$= \int\!\!\int\!\!\!\!\!\!- g_k(x)\nu(dk)\hat{\mu}(dx) = \int\!\!\int\!\!\!\!\!\!-\!\!\int\!\!\!\!\!\!- f_n(x)\nu_k(dn)\nu(dk)\hat{\mu}(dx)$$
$$= \int\!\!\int\!\!\!\!\!\!- f_n(x)\tilde{\nu}(dn)\hat{\mu}(dx) = \int\!\!\int\!\!\!\!\!\!- f_n(x)\tilde{\nu}(dn)\mu(dx).$$

(Recall that μ and $\hat{\mu}$ give rise to the same integrals, by 212Fb again.) As (X, Σ, μ) and $\langle f_n \rangle_{n\in\mathbb{N}}$ are arbitrary, $\tilde{\nu} \in M_{\text{med}}$.

(d) Of course $\mathcal{F}=\{\mathbb{N}\setminus a:\nu(a)=0\}$ is a filter.

(i) If (X,Σ,μ) is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $\lim_{n\to\infty}\mu E_n=1$, then

$$\int\!\!⨍\chi E_n(x)\nu(dn)\mu(dx)=⨍\!\!\int\chi E_n d\mu\,\nu(dn)=⨍\mu E_n\nu(dn)=1.$$

So $E=\{x:⨍\chi E_n(x)\nu(dn)=1\}$ is μ-conegligible. But if $x\in E$ and $a=\{n:x\in E_n\}$, then $\nu a=⨍\chi E_n(x)\nu(dn)=1$ and $a\in\mathcal{F}$ and $x\in\bigcap_{n\in a}E_n$. Thus $\bigcup_{a\in\mathcal{F}}\bigcap_{n\in a}E_n\supseteq E$ is conegligible. As (X,Σ,μ) and $\langle E_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{F}$ is measure-converging.

(ii) If (X,Σ,μ) is a probability space, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, and $X=\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A}E_n$, then $\{n:x\in E_n\}\in\mathcal{F}$ for every $x\in X$, and

$$⨍\mu E_n\nu(dn)=\int\!\!⨍\chi E_n(x)\nu(dn)\mu(dx)=\int\nu\{n:x\in E_n\}\mu(dx)=1.$$

So for any $\epsilon>0$, $\nu\{n:\mu E_n\le 1-\epsilon\}=0$ and $\{n:\mu E_n\ge 1-\epsilon\}\in\mathcal{F}$; accordingly $\lim_{n\to\mathcal{F}}\mu E_n=1$. As (X,Σ,μ) and $\langle E_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $\mathcal{F}$ has the Fatou property.

(e)(i) For each $n\in\mathbb{N}$, we can find a Σ-measurable function $f_n':X\to\mathbb{R}$, equal almost everywhere to f_n, and such that $\sup_{x\in X}|f_n'(x)|=\operatorname{ess\,sup}|f_n|$. Now $\langle f_n'\rangle_{n\in\mathbb{N}}$ is uniformly bounded, so $f'(x)=⨍f_n'(x)\nu(dn)$ is defined for every $x\in X$; and $f(x)$ is defined and equal to $f'(x)$ for μ-almost every x. Since f' is integrable, f' and f are μ-virtually measurable and essentially bounded, and $f\in\mathcal{L}^\infty(\mu)$. Similarly, $g\in\mathcal{L}^\infty(\lambda)$.

(ii) If $h\in\mathcal{L}^1(\mu)$, then $\int f\times h\,d\mu=⨍\!\int f_n\times h\,d\mu\,\nu(dn)$. **P** (α) If h is defined everywhere, measurable and bounded, then, taking f_n' and f' as in (i), $(f'\times h)(x)=⨍f_n'(x)h(x)\nu(dn)$ for every $x\in X$, so

$$\begin{aligned}\int f\times h\,d\mu=\int f'\times h\,d\mu&=\int\!\!⨍(f_n'\times h)(x)\nu(dn)\mu(dx)\\&=⨍\!\!\int f_n'\times h\,d\mu\,\nu(dn)=⨍\!\!\int f_n\times h\,d\mu\,\nu(dn).\end{aligned}$$

(β) In general, set $\gamma=\sup_{n\in\mathbb{N}}\operatorname{ess\,sup}f_n$. Given $\epsilon>0$, there is a simple function h' such that $\|h-h'\|_1\le\epsilon$, and now

$$\begin{aligned}&|\int f\times h\,d\mu-⨍\!\!\int f_n\times h\,d\mu\,\nu(dn)|\\&\quad\le|\int f\times h\,d\mu-\int f\times h'\,d\mu|+|\int f\times h'\,d\mu-⨍\!\!\int f_n\times h'\,d\mu\,\nu(dn)|\\&\qquad\qquad+|⨍\!\!\int f_n\times h'\,d\mu\,\nu(dn)-⨍\!\!\int f_n\times h\,d\mu\,\nu(dn)|\\&\quad\le\|f\|_\infty\|h-h'\|_1+\sup_{n\in\mathbb{N}}|\int f_n\times h'\,d\mu-\int f_n\times h\,d\mu|\le 2\epsilon\gamma.\end{aligned}$$

As ϵ is arbitrary, $\int f\times h\,d\mu=⨍\!\int f_n\times h\,d\mu\,\nu(dn)$. **Q**

Similarly, $\int g\times h\,d\lambda=⨍\!\int g_n\times h\,d\lambda\,\nu(dn)$ for every λ-integrable h.

(iii) If $h\in\mathcal{L}^1(\lambda)$ there is an $\tilde h\in\mathcal{L}^1(\mu)$ such that $\int\tilde h^\bullet\times v=\int h^\bullet\times Tv$ for every $v\in L^\infty(\mu)$. **P** Recall that $L^1(\mu)$, $L^1(\lambda)$ can be identified with $L^\infty(\mu)^\times$ and $L^\infty(\nu)^\times$ (365Lb[2]); perhaps I should remark that the formulae $\int\tilde h^\bullet\times v$, $\int h^\bullet\times Tv$ represent abstract integrals taken in $L^1(\mu)$, $L^1(\lambda)$ respectively (242B). Setting $\phi(w)=\int h^\bullet\times w$ for $w\in L^\infty(\lambda)$, $\phi\in L^\infty(\lambda)^\times$, so $\phi T\in L^\infty(\mu)^\times$ (355G) and there is an $\tilde h\in\mathcal{L}^1(\mu)$ such that

$$\int\tilde h^\bullet\times v=\phi(Tv)=\int h^\bullet\times Tv$$

for every $v\in L^\infty(\mu)$. **Q**

(iv) Take h and $\tilde h$ as in (iii), and consider

$$\int h^\bullet\times g^\bullet=\int h\times g\,d\lambda=⨍\!\!\int h\times g_n\,d\lambda\,\nu(dn)$$

(by (ii))

[2]Formerly 365Mb.

$$= \fint (\int h^\bullet \times g_n^\bullet)\nu(dn) = \fint (\int h^\bullet \times Tf_n^\bullet)\nu(dn)$$

$$= \fint (\int \tilde{h}^\bullet \times f_n^\bullet)\nu(dn) = \fint\!\!\int \tilde{h} \times f_n d\mu\, \nu(dn) = \int \tilde{h} \times f d\mu$$

(by (ii) again)

$$= \int \tilde{h}^\bullet \times f^\bullet = \int h^\bullet \times Tf^\bullet.$$

As h is arbitrary, and the duality between $L^\infty(\mu)$ and $L^1(\lambda)$ is separating, $Tf^\bullet = g^\bullet$, as required.

538S Theorem (a) If $\mathfrak{m}_{\text{countable}} = \mathfrak{c}$, there is a medial limit.

(b) (Larson 09) Suppose that the filter dichotomy (5A6Id) is true. If I is any set and ν is a finitely additive real-valued functional on $\mathcal{P}I$ which is universally measurable for the usual topology on $\mathcal{P}I$, then ν is completely additive.[3] Consequently there is no medial limit.

proof (a)(i) Let M be the L-space of bounded additive functionals on $\mathcal{P}\mathbb{N}$. Let us say that a subset C of M is **rationally convex** if $\alpha\nu + (1-\alpha)\nu' \in C$ whenever ν, $\nu' \in C$ and $\alpha \in [0,1] \cap \mathbb{Q}$; for $A \subseteq M$, write $\Gamma_{\mathbb{Q}}(A)$ for the smallest rationally convex set including A. Set $Q = \Gamma_{\mathbb{Q}}(\{\delta_n : n \in \mathbb{N}\})$ where $\delta_n(a) = \chi a(n)$ for $a \subseteq \mathbb{N}$ and $n \in \mathbb{N}$. In the language of 538Rb, $Q \subseteq M_\tau \subseteq M_{\text{med}}$, so 538P(i) tells us that $\int \nu\, d\mu = \fint \mu E_n \nu(dn)$ for every $\nu \in Q$, where $E_n = \{a : n \in a \subseteq \mathbb{N}\}$ as usual.

(ii) Suppose that $\mathcal{F}$ is a filter base on Q, consisting of rationally convex sets, with cardinal less than $\mathfrak{m}_{\text{countable}}$. Let μ be a Radon probability measure on $\mathcal{P}\mathbb{N}$. Then there is a sequence $\langle \nu_k \rangle_{k\in\mathbb{N}}$ in Q such that

$\sum_{k=0}^\infty \int |\nu_{k+1}(a) - \nu_k(a)|\mu(da) < \infty$,

$\{k : k \in \mathbb{N},\, \nu_k \in F\}$ is infinite for every $F \in \mathcal{F}$.

P Each $\nu \in Q$ is a bounded Borel measurable real-valued function on $\mathcal{P}\mathbb{N}$; let $u \in L^2 = L^2(\mu)$ be a $\mathfrak{T}_s(L^2, L^2)$-cluster point of $\langle \nu^\bullet \rangle_{\nu \in Q}$ along the filter generated by $\mathcal{F}$. For any $F \in \mathcal{F}$, the $\|\,\|_2$-closure of the rationally convex set $\{\nu^\bullet : \nu \in F\} \subseteq L^2$ is convex, so includes the weak closure of $\{\nu^\bullet : \nu \in F\}$ and therefore contains u. So $\{\nu^\bullet : \nu \in F\}$ meets $\{v : v \in L^2,\, \|v - u\|_2 \le \epsilon\}$ for every $\epsilon > 0$.

Set $H_k = \{\nu : \nu \in Q,\, \|\nu^\bullet - u\|_2 \le 2^{-k}\}$ for each $k \in \mathbb{N}$; then every H_k meets every member of $\mathcal{F}$. If we give each H_k its discrete topology, and take H to be the product $\prod_{k\in\mathbb{N}} H_k$, then H is homeomorphic to $\mathbb{N}^\mathbb{N}$. Writing $\mathcal{M}(H)$ for the ideal of meager subsets of H, $\operatorname{cov}\mathcal{M}(H) = \mathfrak{m}_{\text{countable}} > \#(\mathcal{F})$, while

$$\bigcup_{k\ge n}\{\alpha : \alpha \in H,\, \alpha(k) \in F\}$$

is a dense open subset of H for every $F \in \mathcal{F}$ and $n \in \mathbb{N}$. There is therefore an $\alpha \in H$ such that $\{k : \alpha(k) \in F\}$ is infinite for every $F \in \mathcal{F}$; take $\nu_k = \alpha(k)$ for each k. Since μ is a probability measure,

$$\int |\nu_{k+1} - \nu_k| d\mu \le \|\nu_{k+1}^\bullet - \nu_k^\bullet\|_2$$

(244E; see 244Xd)

$$\le 2^{-k-1} + 2^{-k}$$

for every k, and $\sum_{k=0}^\infty \int |\nu_{k+1} - \nu_k| d\mu$ is finite. **Q**

(iii) Because a Radon probability measure on $\mathcal{P}\mathbb{N}$ is defined by its values on the countable algebra $\mathfrak{B}$ of open-and-closed sets, the number of such measures is at most $\#(\mathbb{R}^\mathfrak{B}) = \mathfrak{c}$. Enumerate them as $\langle \mu_\xi \rangle_{\xi<\mathfrak{c}}$. Choose a non-decreasing family $\langle \mathcal{F}_\xi \rangle_{\xi\le\mathfrak{c}}$ of filter bases on Q, as follows. The inductive hypothesis will be that $\mathcal{F}_\xi$ has cardinal at most $\max(\omega, \#(\xi))$ and consists of rationally convex sets. Start with $\mathcal{F}_0 = \{F_n : n \in \mathbb{N}\}$ where $F_n = \Gamma_{\mathbb{Q}}(\{\delta_i : i \ge n\})$ for each n. Given $\mathcal{F}_\xi$ where $\xi < \mathfrak{c}$, apply (ii) with $\mu = \mu_\xi$ to see that there is a sequence $\langle \nu_{\xi k} \rangle_{k\in\mathbb{N}}$ in Q such that

$\sum_{k\in\mathbb{N}} \int |\nu_{\xi,k+1} - \nu_{\xi k}| d\mu_\xi < \infty$,

$\{k : \nu_{\xi k} \in F\}$ is infinite for every $F \in \mathcal{F}_\xi$.

Let $\mathcal{F}_{\xi+1}$ be

$$\mathcal{F}_\xi \cup \{F \cap \Gamma_{\mathbb{Q}}(\{\nu_{\xi k} : k \ge l\}) : F \in \mathcal{F}_\xi,\, l \in \mathbb{N}\}.$$

[3] The result developed into this form in the course of correspondence with J.Pachl.

For non-zero limit ordinals $\xi \le \mathfrak{c}$, set $\mathcal{F}_\xi = \bigcup_{\eta<\xi} \mathcal{F}_\eta$.

(iv) At the end of the induction, let $\mathcal{F}$ be the filter on $M \cong (\ell^\infty)^*$ generated by $\mathcal{F}_\mathfrak{c}$, and let θ be a cluster point of $\mathcal{F}$ for the weak* topology of $(\ell^\infty)^*$. Then θ is a medial limit. **P** If μ is a Radon probability measure on $\mathcal{P}\mathbb{N}$, take $\xi < \mathfrak{c}$ such that $\mu = \mu_\xi$. Because $\Gamma_\mathbb{Q}(\{\nu_{\xi k} : k \ge l\})$ belongs to $\mathcal{F}$ for every $l \in \mathbb{N}$, $⨍ u(n)\theta(dn) = \lim_{k\to\infty} ⨍ u(n)\nu_{\xi k}(dn)$ for every $u \in \ell^\infty$ for which the limit is defined. In particular, $\theta(a) = \lim_{k\to\infty} \nu_{\xi k}(a)$ whenever $a \subseteq \mathbb{N}$ is such that the limit is defined. Because $\sum_{k\in\mathbb{N}} \int |\nu_{\xi,k+1} - \nu_{\xi k}| d\mu$ is finite, this is the case for μ-almost every a, so

$$\int \theta(a)\mu(da) = \lim_{k\to\infty} \int \nu_{\xi k}(a)\mu(da) = \lim_{k\to\infty} ⨍ \mu E_n \nu_{\xi k}(dn);$$

and because the latter limit is defined it is equal to $⨍ \mu E_n \theta(dn)$. As μ is arbitrary, θ satisfies condition (i) of 538P, and is a medial functional; because $Q \in \mathcal{F}$, $\theta\mathbb{N} = 1$; and because $\mathcal{F}_0 \subseteq \mathcal{F}$, $\theta(a) = 0$ for every finite $a \subseteq \mathbb{N}$, so θ is a medial limit. **Q**

(b)(i) The key is the following. Suppose that $\nu : \mathcal{P}I \to \mathbb{R}$ is a universally measurable additive functional.

(α) For every set J and function $\phi : I \to J$, $\nu\phi^{-1}$ is universally measurable, where $(\nu\phi^{-1})(b) = \nu(\phi^{-1}[b])$ for every $b \subseteq J$. **P** We have only to observe that $b \mapsto \phi^{-1}[b] : \mathcal{P}J \to \mathcal{P}I$ is continuous, and apply 434Df. **Q**

(β) ν is bounded. **P?** Otherwise, there is a disjoint sequence $\langle c_k\rangle_{k\in\mathbb{N}}$ of subsets of I such that $\lim_{k\to\infty} |\nu c_k| = \infty$ (326D(ii)). Enlarging c_0 if necessary, we can suppose that $\bigcup_{k\in\mathbb{N}} c_k = I$. Set $\phi(i) = k$ for $k \in \mathbb{N}$ and $i \in c_k$. Then $\nu\phi^{-1}[\{k\}] \to \infty$ as $k \to \infty$. But $\nu' = \nu\phi^{-1}$ is universally measurable, therefore $\mathrm{T}_\mathbb{N}$-measurable, where $\mathrm{T}_\mathbb{N}$ is the domain of the usual measure $\lambda_\mathbb{N}$ on $\mathcal{P}\mathbb{N}$. Let M be such that $\lambda_\mathbb{N} E > 0$ where $E = \{a : |\nu' a| \le M\}$. Then there are an $n \in \mathbb{N}$ such that for every $k \ge n$ there are a, $b \in E$ such that $a\triangle b = \{k\}$ (345E; recall that the natural bijection $a \to \chi a : \mathcal{P}\mathbb{N} \to \{0,1\}^\mathbb{N}$ identifies $\lambda_\mathbb{N}$ with the usual measure on $\{0,1\}^\mathbb{N}$). In this case, k belongs to exactly one of a, b; suppose that $k \in a \setminus b$; then $|\nu'\{k\}| = |\nu a - \nu' b| \le 2M$. This is supposed to be true for every $k \ge n$, so $\limsup_{k\to\infty} |\nu'\{k\}| \le 2M$. **XQ**

(γ) $|\nu|$ is universally measurable. **P** As in part (b-i) of the proof of 464K, there is a sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{P}I$ such that $\nu^+ a = \lim_{n\to\infty} \nu(a \cap c_n)$ for every $a \subseteq I$. Since all the functions $a \mapsto a \cap c_n$ are continuous, $a \mapsto \nu(a \cap c_n)$ is universally measurable for every n, and ν^+ is universally measurable (use 418C). Consequently $|\nu| = 2\nu^+ - \nu$ is universally measurable. **Q**

(ii) If $\nu : \mathcal{P}\mathbb{N} \to [0,\infty[$ is a universally measurable additive functional and $\nu\{n\} = 0$ for every $n \in \mathbb{N}$, then $\nu = 0$. **P?** Otherwise, consider $\mathcal{F} = \{a : \nu a = \nu\mathbb{N}\}$. This is a filter on $\mathbb{N}$ containing every cofinite set. Let $\phi : \mathbb{N} \to \mathbb{N}$ be finite-to-one, and write ν' for $\nu\phi^{-1}$. Setting $\mathcal{I} = \{a : \nu' a = 0\}$, we have a strictly positive additive functional on the quotient algebra $\mathcal{P}\mathbb{N}/\mathcal{I}$, so $\mathcal{P}\mathbb{N}/\mathcal{I}$ is ccc and $\mathcal{I}$ cannot be $[\mathbb{N}]^{<\omega}$, that is, $\phi[[\mathcal{F}]]$ is not the Fréchet filter. On the other hand, ν' is universally measurable, by (i-α), so

$$\phi[[\mathcal{F}]] = \{a : \phi^{-1}[a] \in \mathcal{F}\} = \{a : \nu' a = \nu'\mathbb{N}\}$$

is a universally measurable subset of $\mathcal{P}\mathbb{N}$, and cannot be an ultrafilter (464Ca). Thus $\mathcal{F}$ witnesses that the filter dichotomy is false. **XQ**

(iii) Returning to the general case of a universally measurable additive functional $\nu : \mathcal{P}I \to \mathbb{R}$, set $\gamma_i = \nu\{i\}$ for $i \in I$. By (i-β), $\sup_{J\in[I]^{<\omega}} |\sum_{j\in J} \gamma_j| = \sup_{J\in[I]^{<\omega}} |\nu J|$ is finite, so $\sum_{i\in I} |\gamma_i| < \infty$, and we have a functional $\nu_1 : \mathcal{P}I \to \mathbb{R}$ defined by setting $\nu_1 a = \sum_{i\in a} \gamma_i$ for every $a \subseteq I$. ν_1 is continuous for the topology of $\mathcal{P}I$, so $\nu_2 = \nu - \nu_1$ is universally measurable, and $\nu' = |\nu_2|$ is universally measurable, by (i-γ).

$\nu' J = 0$ for every countable set $J \subseteq I$. **P** If J is finite, this is trivial, because

$$|\nu_2|\{i\} = |\nu_2\{i\}| = |\nu\{i\} - \nu_1\{i\}| = |\gamma_i - \gamma_i| = 0$$

for every $i \in I$. If J is countably infinite, then the embedding $\mathcal{P}J \subsetneqq \mathcal{P}I$ is continuous, so $\nu' \restriction \mathcal{P}J$ is universally measurable for the usual topology on $\mathcal{P}J$; also it is still zero on singletons, so (ii) tells us that it is zero on the whole of $\mathcal{P}J$. **Q**

It follows that ν' is zero everywhere. **P** Take $c \subseteq I$ and $\epsilon > 0$. ν' must be T_I-measurable, where T_I is the domain of the usual measure λ_I on $\mathcal{P}I$. Since λ_I is a completion regular Radon measure (416Ub), there must be a non-negligible zero set $K \subseteq \mathcal{P}I$ such that $|\nu' a - \nu' b| \le \epsilon$ for all a, $b \in K$; and there is a countable set $J \subseteq I$ such that K is determined by coordinates in J (4A3Nc, applied to $\{0,1\}^I \cong \mathcal{P}I$). Take any $a \in K$. Then $c_1 = (c \setminus J) \cup (a \cap J)$ and $a \cap J$ both belong to K. But as $\nu'(c \cap J) = 0$,

$$|\nu' c| = |\nu' c_1 - \nu'(a \cap J)| \le \epsilon.$$

As c and ϵ are arbitrary, $\nu' = 0$. **Q**

Accordingly $\nu_2 = 0$ and $\nu = \nu_1$. But of course ν_1 is completely additive.

(iv) Finally, a medial limit would be a non-zero additive functional from $\mathcal{P}\mathbb{N}$ to $[0,1]$ which was universally measurable, as noted in 538Q, and zero on singletons; and this has already been ruled out by (ii).

Remark It is possible to have medial limits when $\mathfrak{m}_{\text{countable}} \ll \mathfrak{c}$; see 553N.

538X Basic exercises (a) Let $\mathcal{F}$ be a filter on $\mathbb{N}$, and I an infinite subset of $\mathbb{N}$ such that $\mathbb{N} \setminus I \notin \mathcal{F}$; write $\mathcal{F}\lceil I$ for the filter $\{A \cap I : A \in \mathcal{F}\}$. Show that if $\mathcal{F}$ is free, or a p-point filter, or Ramsey, or rapid, or nowhere dense, or measure-centering, or measure-converging, or with the Fatou property, then so is $\mathcal{F}\lceil I$.

(b) For $A \in [\mathbb{N}]^\omega$ let $f_A : \mathbb{N} \to A$ be the increasing enumeration of A. Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Show that $\mathcal{F}$ is rapid iff $\{f_A : A \in \mathcal{F}\}$ is cofinal with $\mathbb{N}^{\mathbb{N}}$.

(c) Let $\mathcal{F}$ be a filter which is universally measurable (regarded as a subset of $\mathcal{P}(\bigcup \mathcal{F})$ with its usual topology), and $\mathcal{G}$ another filter such that $\mathcal{G} \leq_{\text{RK}} \mathcal{F}$. Show that $\mathcal{G}$ is universally measurable.

(d) Let $\mathcal{F}_{\text{Fr}}$ be the Fréchet filter and $\mathcal{F}_d$ the asymptotic density filter, the filter of subsets of $\mathbb{N}$ with asymptotic density 1. (i) Show that $\mathcal{F}_{\text{Fr}}$ and $\mathcal{F}_d$ are p-point filters. (ii) Show that $\mathcal{F}_{\text{Fr}} \leq_{\text{RB}} \mathcal{F}_d$ but that $\mathcal{F}_{\text{Fr}} \ltimes \mathcal{F}_{\text{Fr}} \not\leq_{\text{RK}} \mathcal{F}_d$.

(e)(i) Let $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ be a sequence of filters on $\mathbb{N}$, and $\mathcal{F}$ a filter on $\mathbb{N}$. Write $\lim_{n\to\mathcal{F}} \mathcal{F}_n$ for the filter $\{A : A \subseteq \mathbb{N}, \{n : n \in \mathbb{N}, A \in \mathcal{F}_n\} \in \mathcal{F}\}$. Show that if every $\mathcal{F}_n$ is rapid, then $\lim_{n\to\mathcal{F}} \mathcal{F}_n$ is rapid. (ii) Let $\mathcal{F}$ be a rapid filter, and $\mathcal{G}$ any filter on $\mathbb{N}$. Show that $\mathcal{G} \ltimes \mathcal{F}$ is rapid. (iii) In 538E, suppose that $\mathcal{F}_1$ is rapid. Show that $\mathcal{G}_\xi$ is rapid for every $\xi \geq 1$.

(f)(i) Let $\mathcal{F}$ be a nowhere dense filter, and $\mathcal{G}$ a filter on $\mathbb{N}$ such that $\mathcal{G} \leq_{\text{RK}} \mathcal{F}$. Show that $\mathcal{G}$ is nowhere dense. (ii) Show that a p-point ultrafilter is nowhere dense. (iii) In 538E, show that if every $\mathcal{F}_\xi$ is a nowhere dense ultrafilter, then $\mathcal{G}_\zeta$ is a nowhere dense ultrafilter.

>(g) Let $\mathcal{F}$ be a free filter on $\mathbb{N}$. Show that the following are equiveridical: (i) $\mathcal{F}$ is a Ramsey filter; (ii) whenever K is finite, $k \in \mathbb{N}$ and $f : [\mathbb{N}]^k \to K$ is a function, there is an $F \in \mathcal{F}$ such that f is constant on $[F]^k$; (iii) $\mathcal{F}$ is a p-point filter and whenever $\langle E_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $[\mathbb{N}]^{<\omega}$, there is an $F \in \mathcal{F}$ such that $\#(F \cap E_n) \leq 1$ for every n; (iv) whenever $\langle E_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{P}\mathbb{N} \setminus \mathcal{F}$, there is an $F \in \mathcal{F}$ such that $\#(F \cap E_n) \leq 1$ for every n.

(h) Let $\mathfrak{F}$ be a countable family of distinct p-point ultrafilters on $\mathbb{N}$. Show that there is a disjoint family $\langle A_{\mathcal{F}} \rangle_{\mathcal{F}\in\mathfrak{F}}$ of subsets of $\mathbb{N}$ such that $A_{\mathcal{F}} \in \mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}$.

(i) Let (X, Σ, μ) be a complete perfect probability space, $(Y, \mathfrak{S})$ a perfectly normal compact Hausdorff space, $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of measurable functions from X to Y, $\mathcal{F}$ a measure-centering ultrafilter on $\mathbb{N}$ and λ the $\mathcal{F}$-extension of μ. (i) Setting $f(x) = \lim_{n\to\mathcal{F}} f_n(x)$ for $x \in X$, show that f is $\operatorname{dom}\lambda$-measurable. (ii) For each $n \in \mathbb{N}$, show that there is a unique Radon measure ν_n on Y such that f_n is inverse-measure-preserving for μ and ν_n. (iii) Let ν be the limit $\lim_{n\to\mathcal{F}} \nu_n$ for the narrow topology on the space of Radon probability measures on Y (437Jd). Show that f is inverse-measure-preserving for λ and ν. (*Hint*: look at the Radon measure associated with the image measure λf^{-1}. You may prefer to begin with metrizable Y.)

(j) Let $\langle (\mathfrak{A}_i, \bar{\mu}_i) \rangle_{i\in I}$ be a family of probability algebras, $\mathcal{F}$ an ultrafilter on I, and $(\mathfrak{A}, \bar{\mu})$ the probability algebra reduced product of $\prod_{i\in I}(\mathfrak{A}_i, \bar{\mu}_i)|\mathcal{F}$. For each $i \in I$, let $\subseteq_i$ be the order relation on $\mathfrak{A}_i$; set $P = \prod_{i\in I} \mathfrak{A}_i$ and let $P|\mathcal{F}$ be the partial order reduced product of $\langle (\mathfrak{A}_i, \subseteq_i) \rangle_{i\in I}$ modulo $\mathcal{F}$ as defined in 5A2A. Describe a canonical order-preserving map from $P|\mathcal{F}$ to $\mathfrak{A}$.

(k)(i) Let $(\mathfrak{A}, \bar{\mu})$ be a homogeneous probability algebra with Maharam type κ, I a non-empty set, $\mathcal{F}$ an ultrafilter on I and $(\mathfrak{C}, \bar{\nu})$ the probability algebra reduced power $(\mathfrak{A}, \bar{\mu})^I|\mathcal{F}$. Show that $\mathfrak{C}$ is homogeneous, with Maharam type the transversal number $\operatorname{Tr}_{\mathcal{I}}(I; \kappa)$ (definition: 5A1L), where $\mathcal{I} = \{I \setminus A : A \in \mathcal{F}\}$. (*Hint*: 5A1Md, 521Eb.) (ii) Show that if $(\mathfrak{A}, \bar{\mu})$ is any probability algebra and $\mathcal{F}$ and $\mathcal{G}$ are non-principal ultrafilters on $\mathbb{N}$, then the probability algebra reduced powers $(\mathfrak{A}, \bar{\mu})^{\mathbb{N}}|\mathcal{F}$ and $(\mathfrak{A}, \bar{\mu})^{\mathbb{N}}|\mathcal{G}$ are isomorphic.

(l) Let (X, Σ, μ) be a perfect probability space and μ' an indefinite-integral measure over μ which is also a probability measure. Let $\mathcal{F}$ be a measure-centering ultrafilter on $\mathbb{N}$ and λ, λ' the $\mathcal{F}$-extensions of μ and μ'. Show that λ' is an indefinite-integral measure over λ.

>(m) (Benedikt 98) (i) Let $\mathcal{F}$ be any free filter on $\mathbb{N}$. Show that $\mathcal{F} \ltimes \mathcal{F}$ is not measure-centering. (*Hint*: let $\langle e_n \rangle_{n\in\mathbb{N}}$ be the standard generating family in $\mathfrak{B}_\omega$, and consider $a_{mn} = e_m \setminus e_n$ if $m < n$, 1 otherwise.) (ii) Let $\mathcal{F}$ be a measure-centering ultrafilter on $\mathbb{N}$. Show that if f, $g \in \mathbb{N}^\mathbb{N}$ and $\{n : f(n) \ne g(n)\} \in \mathcal{F}$, then $f[[\mathcal{F}]] \ne g[[\mathcal{F}]]$. (*Hint*: consider $a_n = e_{f(n)} \setminus e_{g(n)}$ if $f(n) \ne g(n)$.)

(n) Let X be a locally compact Hausdorff topological group, and μ a left Haar measure on X. Show that there is a complete locally determined left-translation-invariant measure λ on X such that $\lambda(\lim_{n\to\mathcal{F}} E_n)$ is defined and equal to $\sup_{K\subseteq X \text{ is compact}} \lim_{n\to\mathcal{F}} \mu(E_n \cap K)$ whenever $\mathcal{F}$ is a Ramsey ultrafilter on $\mathbb{N}$ and $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence of Haar measurable subsets of X.

(o)(i) Let $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ be a sequence of measure-converging filters on $\mathbb{N}$. Show that $\bigcap_{n\in\mathbb{N}} \mathcal{F}_n$ is measure-converging, so that $\lim_{n\to\mathcal{F}} \mathcal{F}_n$ (538Xe) is measure-converging for any filter $\mathcal{F}$ on $\mathbb{N}$. (ii) In 538E, suppose that $\mathcal{F}_1$ is measure-converging. Show that $\mathcal{G}_\xi$ is measure-converging for every $\xi \in [1, \zeta]$.

(p) Suppose that $\langle \mathcal{F}_\xi \rangle_{\xi<\kappa}$ is a family of measure-converging filters, where κ is non-zero and less than the additivity $\operatorname{add}\mathcal{N}$ of Lebesgue measure. Show that $\bigcap_{\xi<\kappa} \mathcal{F}_\xi$ is measure-converging.

(q)(i) Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Show that $\mathcal{F}$ has the Fatou property iff $\int f d\mu$ and $\lim_{n\to\mathcal{F}} \int f_n d\mu$ are defined and equal whenever (X, Σ, μ) is a measure space, $g : X \to [0, \infty[$ is an integrable function and $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence of measurable functions on X such that $|f_n| \le_{\text{a.e.}} g$ for every n and $\lim_{n\to\mathcal{F}} f_n =_{\text{a.e.}} f$. (ii) Show that a non-principal ultrafilter on $\mathbb{N}$ cannot have the Fatou property. (*Hint*: 464Ca.)

(r) Show that the asymptotic density filter (538Xd) has the Fatou property.

(s)(i) Let $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ be a sequence of filters with the Fatou property, and $\mathcal{F}$ a filter with the Fatou property. Show that $\lim_{n\to\mathcal{F}} \mathcal{F}_n$ (538Xe) has the Fatou property. (ii) In 538E, suppose that $\mathcal{F}_\xi$ has the Fatou property for every $\xi \in [1, \zeta]$. Show that $\mathcal{G}_\xi$ has the Fatou property for every $\xi \le \zeta$.

(t) Let $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ be a bounded additive functional. (i) Show that ν is a medial functional iff $\int \nu\{n : x \in E_n\}\mu(dx)$ is defined and equal to $⨍ \mu E_n \nu(dn)$ whenever (X, Σ, μ) is a probability space and $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in Σ. (ii) Show that in this case $a \mapsto \nu\phi^{-1}[a]$ is a medial functional for any $\phi : \mathbb{N} \to \mathbb{N}$.

>(u) Let (X, Σ, μ) be a probability space, and T a σ-subalgebra of Σ. Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^\infty(\mu)$ such that $\sup_{n\in\mathbb{N}} \operatorname{ess\,sup} |f_n|$ is finite, and for each $n \in \mathbb{N}$ let g_n be a conditional expectation of f_n on T. Suppose that ν is a medial functional. Show that $f(x) = ⨍ f_n(x)\nu(dn)$ and $g(x) = ⨍ g_n(x)\nu(dn)$ are defined for almost every x, that $f \in \mathcal{L}^\infty(\mu)$, and that g is a conditional expectation of f on T.

(v) (V.Bergelson) Show that there are a probability algebra $(\mathfrak{A}, \bar\mu)$ and a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \bar\mu a_n > 0$ but $a_m \cap a_n \cap a_{m+n} = 0$ whenever m, $n > 0$. (*Hint*: for $n \ge 1$, set $E_n = \{x : x \in [0, 1], \lfloor 3nx \rfloor \equiv 1 \bmod 3\}$.)

538Y Further exercises (a) Show that if $\mathcal{F}$ and $\mathcal{G}$ are filters and $\mathcal{F} \le_{\text{RK}} \mathcal{G}$, then, in the language of 512A, $(\mathcal{F}, \supseteq, \mathcal{F}) \preccurlyeq_{\text{GT}} (\mathcal{G}, \supseteq, \mathcal{G})$, so that $\operatorname{ci}\mathcal{F} \le \operatorname{ci}\mathcal{G}$ and $\mathcal{F}$ is κ-complete whenever κ is a cardinal and $\mathcal{G}$ is κ-complete.

(b) Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$, and suppose that whenever $\mathcal{G}$ is a free filter on $\mathbb{N}$ and $\mathcal{G} \le_{\text{RK}} \mathcal{F}$, then $\mathcal{F} \le_{\text{RK}} \mathcal{G}$. Show that $\mathcal{F}$ is a Ramsey ultrafilter. (*Hint*: Comfort & Negrepontis 74.)

(c) Show that if $\mathfrak{p} = \mathfrak{c}$ then there are $2^\mathfrak{c}$ Ramsey ultrafilters on $\mathbb{N}$, and therefore $2^\mathfrak{c}$ isomorphism classes of Ramsey ultrafilters.

(d) Let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$. Show that $\mathcal{F}$ is measure-centering iff whenever $\mathfrak{A}$ is a Boolean algebra, $D \subseteq \mathfrak{A} \setminus \{0\}$ has intersection number greater than 0 (definition: 391H) and $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in D, then there is an $A \in \mathcal{F}$ such that $\{a_n : n \in A\}$ is centered.

(e)(i) Show that if $\operatorname{cov}\mathcal{N} = \mathfrak{c}$, there is a measure-centering ultrafilter on $\mathbb{N}$ including the asymptotic density filter (538Xd). (ii) Show that an ultrafilter on $\mathbb{N}$ including the asymptotic density filter cannot be a p-point filter. (iii) Show that a filter on $\mathbb{N}$ including the asymptotic density filter cannot be a rapid filter.

(f)(i) Let $\mathcal{F}$, $\mathcal{G}$ be free filters on $\mathbb{N}$ such that $\mathcal{F} \ltimes \mathcal{G}$ is measure-centering. Show that there is no free filter $\mathcal{H}$ such that $\mathcal{H} \leq_{\text{RK}} \mathcal{F}$ and $\mathcal{H} \leq_{\text{RK}} \mathcal{G}$. (ii) Show that if there are two non-isomorphic Ramsey ultrafilters on $\mathbb{N}$, then there are two non-isomorphic measure-centering ultrafilters $\mathcal{F}$, $\mathcal{G}$ on $\mathbb{N}$ such that $\mathcal{F} \ltimes \mathcal{G}$ is not measure-centering.

(g) For an uncountable set I, let us say that a filter $\mathcal{F}$ on I is **uniform and measure-centering** if $\#(A) = \#(I)$ for every $A \in \mathcal{F}$ and whenever $\mathfrak{A}$ is a Boolean algebra, $\nu : \mathfrak{A} \to [0, \infty[$ is an additive functional, and $\langle a_i \rangle_{i \in I}$ is a family in $\mathfrak{A}$ with $\inf_{i \in I} \nu a_i > 0$, there is an $A \in \mathcal{F}$ such that $\{a_i : i \in A\}$ is centered. (i) State and prove a result corresponding to 538G for such filters. (*Hint*: in the part corresponding to 538G(iv), use 'compact' measures rather than 'perfect' measures.) (ii) State and prove a result corresponding to 538H. (*Hint*: set $\kappa = \#(I)$. In the part corresponding to 538Hc, suppose that you have a κ-complete ultrafilter on I, rather than a Ramsey ultrafilter; see 4A1L. In the part corresponding to 538He, suppose that κ is regular and that $\operatorname{cov} \mathcal{N}_\kappa = 2^\kappa$, where $\mathcal{N}_\kappa$ is the null ideal of the usual measure on $\{0,1\}^\kappa$.) (iii) State and prove results corresponding to 538I-538K. (iv) State and prove results corresponding to 538L-538M, but with 'normal ultrafilters' in place of 'Ramsey ultrafilters'.

(h) Show that if $\mathcal{F}$ and $\mathcal{G}$ are filters on $\mathbb{N}$, $\mathcal{F}$ is rapid and $\mathcal{G} \leq_{\text{RB}} \mathcal{F}$, then $\mathcal{G}$ is rapid.

(i) Give an example of filters $\mathcal{F}$, $\mathcal{G}$ on $\mathbb{N}$ such that $\mathcal{F}$ has the Fatou property, $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{G}$ does not have the Fatou property.

(j)(i) Let $\mathcal{F}$ be a nowhere dense filter on $\mathbb{N}$, and $\mathcal{I}$ the ideal $\{\mathbb{N} \setminus A : A \in \mathcal{F}\}$. Show that $\mathcal{P}\mathbb{N}/\mathcal{I}$ is finite. (ii) Show that a free filter with the Fatou property cannot be nowhere dense.

(k) Let (X, Σ, μ) be a probability space and $\langle f_m \rangle_{m \in \mathbb{N}}$, $\langle g_n \rangle_{n \in \mathbb{N}}$ two uniformly bounded sequences of real-valued measurable functions defined on X. Let ν, $\nu' : \mathcal{P}\mathbb{N} \to \mathbb{R}$ be bounded additive functionals. Show that $⨍⨍\int f_m \times g_n d\mu\, \nu(dm)\nu'(dn) = ⨍⨍\int f_m \times g_n d\mu\, \nu'(dn)\nu(dm)$.

(l) (Meyer 73) Let ν be a medial limit. Write U for the set of sequences $u \in \mathbb{R}^{\mathbb{N}}$ such that $\sup\{⨍ v\, d\nu : v \in \ell^\infty, v \leq |u|\}$ is finite; for $u \in U$, write $⨍ u\, d\nu$ for $\lim_{m \to \infty} ⨍ \operatorname{med}(-m, u(n), m)\nu(dn)$ (see 364Xj). Suppose that (X, Σ, μ) is a probability space and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of μ-integrable real-valued functions on X such that $\langle \int |f_n| d\mu \rangle_{n \in \mathbb{N}} \in U$. (i) Show that $\langle f_n(x) \rangle_{n \in \mathbb{N}} \in U$ for μ-almost every $x \in X$. Set $f(x) = ⨍ f_n(x)\nu(dn)$ whenever $\langle f_n(x) \rangle_{n \in \mathbb{N}} \in U$. (ii) Show that if every f_n is non-negative then $\int f d\mu \leq ⨍\int f_n d\mu\, \nu(dn)$. (iii) Show that if $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable then $\int f d\mu = ⨍\int f_n d\mu\, \nu(dn)$. (iv) Show that if $\langle f_n^\bullet \rangle_{n \in \mathbb{N}}$ is weakly convergent to 0 in $L^1(\mu)$, then $f =_{\text{a.e.}} 0$. (v) Suppose that $\langle f_n \rangle_{n \in \mathbb{N}}$ is uniformly integrable. Let T be a σ-subalgebra of Σ, and for each $n \in \mathbb{N}$ let g_n be a conditional expectation of f_n on T; set $g(x) = ⨍ g_n(x)\nu(dn)$ whenever $\langle g_n(x) \rangle_{n \in \mathbb{N}} \in U$. Show that g is a conditional expectation of f on T.

(m) Suppose that $\mathcal{F}$ is a filter on $\mathbb{N}$ with the Fatou property, and $\langle \nu_n \rangle_{n \in \mathbb{N}}$ a sequence of medial limits. Set $\mathcal{G} = \{A : A \subseteq \mathbb{N}, \lim_{n \to \mathcal{F}} \nu_n A = 1\}$. Show that $\mathcal{G}$ is a filter with the Fatou property.

(n) Show that $\mathfrak{u} \geq \mathfrak{r}(\omega, \omega) \geq \max(\operatorname{cov} \mathcal{N}, \mathfrak{m}_{\text{countable}})$ (definitions: 5A6Ia, 529G).

(o)(i) Show that if $\mathcal{F}$ is a rapid filter on $\mathbb{N}$, then $\operatorname{ci} \mathcal{F} \geq \mathfrak{d}$. (ii) Show that $\mathfrak{d} \geq \mathfrak{g}$ (definition: 5A6I(b-ii)). (iii) Show that if $\mathfrak{u} < \mathfrak{g}$ there are no rapid filters on $\mathbb{N}$, and if there is a measure-converging filter there is a measure-converging ultrafilter with coinitiality $\mathfrak{u}$.

(p) Suppose that the filter dichotomy is true. (i) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that if $\nu : \mathfrak{A} \to \mathbb{R}$ is an additive functional which is universally measurable for the order-sequential topology of $\mathfrak{A}$, then ν is countably additive. (ii) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Show that if $\nu : \mathfrak{A} \to \mathbb{R}$ is an additive functional which is universally measurable for the measure-algebra topology on $\mathfrak{A}$, then it is continuous.

(q)(i) Show that there is a semigroup operation $\dot{+}$ on the set $\beta\mathbb{N}$ of ultrafilters on $\mathbb{N}$ defined by saying that $\mathcal{F} \dot{+} \mathcal{G} = +[[\mathcal{F} \ltimes \mathcal{G}]]$ for all $\mathcal{F}$, $\mathcal{G} \in \beta\mathbb{N}$, where $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is addition. (ii) Show that if we identify $\beta\mathbb{N}$ with the Stone-Čech compactification of $\mathbb{N}$ (4A2I(b-i)), then $\dot{+}$ is continuous in the first variable. (iii) Show that there is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ which is **idempotent**, that is, $\mathcal{F} \dot{+} \mathcal{F} = \mathcal{F}$. (*Hint*: consider a minimal closed sub-semigroup of the set of non-principal ultrafilters.) (iv) For any function $f \in \mathbb{N}^{\mathbb{N}}$, write $\mathrm{FS}(f)$ for $\{\sum_{n \in K} f(n) : K \in [\mathbb{N}]^{<\omega}\}$; say a **finite sum set** is a set of the form $\mathrm{FS}(f)$ for some strictly increasing function $f \in \mathbb{N}^{\mathbb{N}}$. Show that if $\mathcal{F}$ is a non-principal idempotent ultrafilter on $\mathbb{N}$ and $I \in \mathcal{F}$, then I includes a finite sum set. (This is a version of **Hindman's theorem**.) (v) Show that if $I \subseteq \mathbb{N}$ is a finite sum set there is an idempotent ultrafilter containing

I. (vi) Suppose that $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a measure-preserving Boolean homomorphism. (α) Show that if $\mathcal{F}$ is an idempotent ultrafilter on $\mathbb{N}$, then $\lim_{n\to\mathcal{F}}\mu(a\cap\pi^n a)\ge(\mu a)^2$ for every $a\in\mathfrak{A}$ (β) Show that there is a finite sum set $I\subseteq\mathbb{N}$ such that $\{\pi^n a:n\in I\}$ is centered. (vii) Show that no idempotent ultrafilter is measure-centering. (*Hint*: 538Xv.) (viii) Show that if $\mathcal{F}$ is a p-point ultrafilter then $\mathcal{F}\dot{+}\mathcal{F}$ is isomorphic to $\mathcal{F}\ltimes\mathcal{F}$ and is not measure-centering. (ix) Repeat, as far as possible, for semigroups other than $(\mathbb{N},+)$.

(r) (V.Bergelson-M.Talagrand) Show that there are a probability algebra $(\mathfrak{A},\bar\mu)$ and a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\bar\mu a_n=\frac{1}{2}$ for every $n\in\mathbb{N}$ but $\inf_{m,n\in I}\bar\mu(a_m\cap a_n)=0$ whenever $I\subseteq\mathbb{N}$ does not have asymptotic density 0.

538Z Problem Show that it is relatively consistent with ZFC to suppose that there are no measure-converging filters on $\mathbb{N}$.

538 Notes and comments This is a long section, and rather a lot of ideas are crowded into it, starting with the list in 538A. If you have looked at ultrafilters on $\mathbb{N}$ at all, you are likely to have encountered 'p-point', 'rapid' and 'Ramsey' ultrafilters, and most of 538B-538D and 538F will probably be familiar. The 'iterated products' of 538E will also be a matter of adapting known concepts to my particular formulation.

Some of the slightly contorted language of 538Fe and 538Ff (with references to '$\#(\mathfrak{F})$') is there because we do not know how many isomorphism classes of Ramsey filters there are. If there are none (as in random real models, see 553H), or one (Shelah 82, §VI.5), then things are very simple. If there are infinitely many then we could rephrase 538Ff in terms of sequences of non-isomorphic filters. But it is possible that there should be two, or seventeen (Shelah 98a, p. 335).

In 538H-538M I try to set out, and expand, some of the principal ideas of Benedikt 98. The starting point is the observation that a Ramsey ultrafilter gives us an extension of Lebesgue measure on $[0,1]$, indeed of any perfect probability measure. Observing that this property is preserved by iterations, we are led to 'measure-centering' ultrafilters. Once we have the idea of measure-centering-ultrafilter extension of a perfect probability measure, we can set out to look at its properties in terms of the (by now very extensive) general theory of this treatise. The first step has to be the identification of its measure algebra (538Ja, 538Xk), followed, if possible, by the identification of the corresponding Banach function spaces. It turns out that these can be reached by an alternative route *not* involving special properties of the ultrafilter or the probability space, which I have expressed in general forms in §§328 and 377. This gives a long list of facts, which I have written out in 538Ja and 538K. Minor variations of the measure and the filter are straightforward (538Jb, 538Jc, 538Xl). For iterated products of filters we have more work to do (538L), especially if we are to express them in a form adequate for the objective, the universal-extension result of 538M.

You will have noticed that in the statement of 538G I speak of '$\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A}F_n$' and '$\liminf_{n\to\mathcal{F}}\mu F_n$'. Something of the sort is necessary since in that theorem I do not insist from the outset that $\mathcal{F}$ should be an ultrafilter. Of course only ultrafilters are of interest in this context, by 538Ha, and for these we have $\bigcup_{A\in\mathcal{F}}\bigcap_{n\in A}F_n=\lim_{n\to\mathcal{F}}F_n$ and $\liminf_{n\to\mathcal{F}}\mu F_n=\lim_{n\to\mathcal{F}}\mu F_n$, as in 538I.

For most of this section I have kept firmly to the study of filters on $\mathbb{N}$. For measure-centering filters, at least, there are interesting extensions to filters on uncountable sets, which I mention in 538Yg. We can do a good deal with the ideas of 538G-538K on cardinals less than $\mathfrak{c}$ in the presence of (for instance) Martin's axiom; but for anything corresponding to 538L-538M it seems that we must use a two-valued-measurable cardinal (541M below).

Measure-converging filters (538N) and filters with the Fatou property (538O) form an oddly complementary pair. I have tried to emphasize the correspondence in the characterizations 538Na and 538Oa (compare 538G(v), 538Na(iv) and 538Oa(iv)), but after this they seem to diverge. The phrase 'Fatou property' comes from 538O(a-iii); if you like, Fatou's Lemma says that the Fréchet filter has the Fatou property. From 538Xq(i) I see that I could just as well have called it the 'Lebesgue property'. Note that any filter larger than a measure-converging filter is again measure-converging, so that if there is a measure-converging filter there is a measure-converging ultrafilter; but that no non-principal ultrafilter can have the Fatou property (538Xq(ii)). On the other hand, there are many free filters with the Fatou property, but it is not known for sure whether there have to be measure-converging filters. It is possible for a measure-converging filter to have the Fatou property (538Rd).

In the last part of the section I look at a different kind of limit. A 'Banach limit' is an extension to ℓ^∞ of the ordinary limit regarded as a linear functional on the closed subspace of convergent sequences; a 'medial limit' is a Banach limit which commutes with integration in appropriate settings. To study these I use the formulae of repeated integration to do some surprising things. In 363L I tried to explain what I meant by the formula '$\fint\ldots d\nu$' for a *finitely* additive functional ν. This defines linear functionals which are positive for non-negative ν. In 'repeated integrals' like $⨍\int f_n(x)\mu(dx)\nu(dn)$ (538P(iii)), we must interpret the formula as $⨍\bigl(\int f_n(x)\mu(dx)\bigr)\nu(dn)$; the 'inner integral' is an ordinary integral with respect to the countably additive measure μ, and the 'outer integral' is a name for a linear functional. In the integral $⨍\ldots d\nu$ we have no problem with measurability, though we must check that the integrand $n\mapsto\int f_n d\mu$ is bounded (or, at least, satisfies the condition in 538Yl); but when we look at the other repeated integrals, $\int\nu(a)\mu(da)$ or $\int⨍x\,d\nu\,\mu(dx)$ or $\int⨍f_n(x)\nu(dn)\mu(dx)$, the conditions of 538P must explicitly assert that the outer integrals are defined.

Because we don't need to consider measurability, the 'finitely additive integrals' here are in some ways easy to deal with; 'disintegrations' like $\tilde\nu=⨍\nu_k\,\nu(dk)$ (538Rc) slide past all the usual questions. However we must always be vigilant against the temptations of limiting processes. As with the Riemann integral, of course, we can integrate the limit of a uniformly convergent sequence of functions. But see the manoeuvres of part (a-iii) of the proof of 538R, where the sums $\sum_{i=0}^\infty\alpha_{ni}\ldots$ demand different treatments at different points. And Fubini's theorem nearly disappears; the point of 'medial functionals' is that something extraordinary has to happen before we can expect to change the order of integration.

I have used the language of Volume 3 to express 538Re in a general form. Of course by far the most important example is when the operator T is a conditional expectation operator (538Xu). For more examples of operators in $L^\times(L^\infty;L^\infty)$, see §§373-374.

For most of the classes of filter here, there is a question concerning their existence. Subject to the continuum hypothesis, there are many Ramsey ultrafilters, and refining the argument we find that the same is true if $\mathfrak p=\mathfrak c$ (538Yc). There are many ways of forcing non-existence of Ramsey ultrafilters, of which one of the simplest is in 553H below. With more difficulty, we can eliminate p-point ultrafilters (Wimmers 82) or rapid filters (Miller 80) or nowhere dense filters and therefore measure-centering ultrafilters (538Hd, Shelah 98b). It is not known for sure that we can eliminate measure-converging filters (538Z).

539 Maharam submeasures

Continuing the work of §§392-394 and 496, I return to Maharam submeasures and the forms taken by the ideas of the present volume in this context. At least for countably generated algebras, and in some cases more generally, many of the methods of Chapter 52 can be applied (539B-539K). In 539L-539N I give the main result of Balcar Jech & Pazak 05 and Veličković 05: it is consistent to suppose that every Dedekind complete ccc weakly (σ,∞)-distributive Boolean algebra is a Maharam algebra. In 539R-539U I introduce the idea of 'exhaustivity rank' of an exhaustive submeasure.

539A The story so far As submeasures have hardly appeared before in this volume, I begin by repeating some of the essential ideas.

(a) If $\mathfrak A$ is a Boolean algebra, a **submeasure** on $\mathfrak A$ is a functional $\nu:\mathfrak A\to[0,\infty]$ such that $\nu 0=0$, $\nu a\le\nu b$ whenever $a\subseteq b$, and $\nu(a\cup b)\le\nu a+\nu b$ for all a, $b\in\mathfrak B$ (392A); it is **totally finite** if $\nu 1<\infty$. If ν is a submeasure defined on an algebra of subsets of a set X, I say that the **null ideal** of ν is the ideal $\mathcal N(\nu)$ of subsets of X generated by $\{E:\nu E=0\}$ (496Bc). A submeasure ν on a Boolean algebra $\mathfrak A$ is **exhaustive** if $\lim_{n\to\infty}\nu a_n=0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb N}$ in $\mathfrak A$; it is **uniformly exhaustive** if for every $\epsilon>0$ there is an $n\in\mathbb N$ such that there is no disjoint family $a_0,\ldots,a_n$ with $\nu a_i\ge\epsilon$ for every $i\le n$ (392Bc). A **Maharam submeasure** is a totally finite sequentially order-continuous submeasure (393A); a Maharam submeasure on a Dedekind σ-complete Boolean algebra is exhaustive (393Bc).

(b) A **Maharam algebra** is a Dedekind σ-complete Boolean algebra with a strictly positive Maharam submeasure. Any Maharam algebra is ccc and weakly (σ,∞)-distributive (393Eb). A Maharam algebra is measurable iff it carries a strictly positive uniformly exhaustive submeasure (393D). If ν is any Maharam submeasure on a Dedekind σ-complete Boolean algebra $\mathfrak A$, its Maharam algebra is the quotient $\mathfrak A/\{a:\nu a=0\}$ (496Ba).

(c) If ν is any strictly positive totally finite submeasure on a Boolean algebra $\mathfrak A$, there is an associated metric $(a,b)\mapsto\nu(a\,\triangle\,b)$ on $\mathfrak A$; the completion $\widehat{\mathfrak A}$ of $\mathfrak A$ under this metric is a Boolean algebra (392Hc). If ν is exhaustive, then $\widehat{\mathfrak A}$ is a Maharam algebra (393H). If ν and ν' are both strictly positive Maharam submeasures on the same Maharam algebra $\mathfrak A$, ν is absolutely continuous with respect to ν' (393F). Consequently the associated metrics are uniformly equivalent, and $\mathfrak A$ has a canonical topology and uniformity, its **Maharam-algebra topology** and **Maharam-algebra uniformity** (393G).

(d) Let $\mathfrak{A}$ be a Boolean algebra.

(i) A sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ order*-converges to $a\in\mathfrak{A}$ (definition: 367A) iff there is a partition B of unity in $\mathfrak{A}$ such that $\{n : b\cap(a_n\triangle a)\ne 0\}$ is finite for every $b\in B$ (393Ma).

(ii) The **order-sequential topology** on $\mathfrak{A}$ is the topology for which the closed sets are just the sets closed under order*-convergence (393L).

(iii) If $\mathfrak{A}$ is ccc and Dedekind σ-complete, a subalgebra of $\mathfrak{A}$ is order-closed iff it is closed for the order-sequential topology (393O).

(iv) If $\mathfrak{A}$ is ccc and weakly (σ,∞)-distributive, then the closure of a set $A\subseteq\mathfrak{A}$ for the order-sequential topology is the set of order*-limits of sequences in A (393Pb).

(v) If $\mathfrak{A}$ is a Maharam algebra, then its Maharam-algebra topology is its order-sequential topology (393N).

(vi) If $\mathfrak{A}$ is a Dedekind σ-complete ccc weakly (σ,∞)-distributive Boolean algebra, and $\{0\}$ is a G_δ set for the order-sequential topology, then $\mathfrak{A}$ is a Maharam algebra (393Q).

(e) It was a long-outstanding problem (the 'Control Measure Problem') whether every Maharam algebra is in fact a measurable algebra; this was solved by a counterexample in Talagrand 08, described in §394.

(f) If X is a Hausdorff space, a **totally finite Radon submeasure** on X is a totally finite submeasure ν defined on a σ-algebra Σ of subsets of X such that (i) if $E\subseteq F\in\Sigma$ and $\nu F=0$ then $E\in\Sigma$ (ii) every open set belongs to Σ (iii) if $E\in\Sigma$ and $\epsilon>0$ there is a compact set $K\subseteq E$ such that $\nu(E\setminus K)\le\epsilon$ (496C). Every totally finite Radon submeasure is a Maharam submeasure (496Da). If X is a Hausdorff space and ν is a totally finite Radon submeasure on X, a set $E\in\operatorname{dom}\nu$ is **self-supporting** if $\nu(E\cap G)>0$ whenever $G\subseteq X$ is an open set meeting E. If $E\in\operatorname{dom}\nu$ and $\epsilon>0$, there is a compact self-supporting $K\subseteq E$ such that $\nu(E\setminus K)\le\epsilon$ (496Dd).

Let ν be a strictly positive Maharam submeasure on a Dedekind σ-complete Boolean algebra $\mathfrak{A}$. Let Z be the Stone space of $\mathfrak{A}$, and write $\widehat{a}$ for the open-and-closed subset of Z corresponding to each $a\in\mathfrak{A}$. Then there is a unique totally finite Radon submeasure ν' on Z such that $\nu'\widehat{a}=\nu a$ for every $a\in\mathfrak{A}$; the null ideal of ν' is the nowhere dense ideal of Z (496G).

(g) For a cardinal κ, I write $\mathcal{N}_\kappa$ for the null ideal of the usual measure on $\{0,1\}^\kappa$; $\mathcal{N}\cong\mathcal{N}_\omega$ will be the null ideal of Lebesgue measure on $\mathbb{R}$, and $\mathcal{M}$ the meager ideal of $\mathbb{R}$.

539B Proposition Let $\mathfrak{A}$ be a Maharam algebra, $\tau(\mathfrak{A})$ its Maharam type and $d_{\mathfrak{T}}(\mathfrak{A})$ its topological density for its Maharam-algebra topology. Then $\tau(\mathfrak{A})\le d_{\mathfrak{T}}(\mathfrak{A})\le\max(\omega,\tau(\mathfrak{A}))$.

proof Recall that the Maharam-algebra topology is the order-sequential topology (539A(d-v)). $\mathfrak{A}$ is ccc and weakly (σ,∞)-distributive (539Ab), so if $D\subseteq\mathfrak{A}$ is topologically dense, then every element of $\mathfrak{A}$ is expressible as the order*-limit $\inf_{n\in\mathbb{N}}\sup_{m\ge n}a_m$ of some sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in D (539A(d-iv)). In this case D τ-generates $\mathfrak{A}$ and $\tau(\mathfrak{A})\le\#(D)$; accordingly $\tau(\mathfrak{A})\le d_{\mathfrak{T}}(\mathfrak{A})$. If $D\subseteq\mathfrak{A}$ τ-generates $\mathfrak{A}$, let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by D and $\overline{\mathfrak{B}}$ its topological closure. Then $\overline{\mathfrak{B}}$ is order-closed (because $\mathfrak{A}$ is ccc), so is the whole of $\mathfrak{A}$, and $d_{\mathfrak{T}}(\mathfrak{A})\le\#(\mathfrak{B})\le\max(\omega,\#(D))$; accordingly $d_{\mathfrak{T}}(\mathfrak{A})\le\max(\omega,\tau(\mathfrak{A}))$.

539C Theorem Let $\mathfrak{A}$ be a Maharam algebra.

(a)

$$(\mathfrak{A}^+,\supseteq',[\mathfrak{A}^+]^{\le\max(\omega,\tau(\mathfrak{A}))})\preccurlyeq_{\mathrm{GT}}(\mathrm{Pou}(\mathfrak{A}),\sqsubseteq^*,\mathrm{Pou}(\mathfrak{A})),$$

where $\mathfrak{A}^+=\mathfrak{A}\setminus\{0\}$, $(\mathfrak{A}^+,\supseteq',[\mathfrak{A}^+]^{\le\kappa})$ is defined as in 512F, and $(\mathrm{Pou}(\mathfrak{A}),\sqsubseteq^*)$ as in 512Ee.

(b) $\mathrm{Pou}(\mathfrak{A})\preccurlyeq_{\mathrm{T}}\mathcal{N}_{\tau(\mathfrak{A})}$.

proof If $\mathfrak{A}=\{0\}$ these are both trivial; suppose otherwise. Fix a strictly positive Maharam submeasure ν on $\mathfrak{A}$ such that $\nu 1=1$. Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$ which is dense in $\mathfrak{A}$ for the metric $(a,b)\mapsto\nu(a\triangle b)$ and has cardinal at most $\kappa=\max(\omega,\tau(\mathfrak{A}))$ (539B).

(a)(i) For $a\in\mathfrak{A}^+$ choose $\phi(a)\in\mathrm{Pou}(\mathfrak{A})$ as follows. Start by taking $d_n\in\mathfrak{B}$, for $n\in\mathbb{N}$, such that $\nu(d_n\triangle(1\setminus a))\le 2^{-n-2}\nu a$ for each n; set $b_n=d_n\setminus\sup_{i<n}b_i$ for $n\in\mathbb{N}$, $a'=1\setminus\sup_{n\in\mathbb{N}}b_n=1\setminus\sup_{n\in\mathbb{N}}d_n$; then every b_n belongs to $\mathfrak{B}$,

$$\nu(a' \setminus a) \le \inf_{n\in\mathbb{N}} \nu((1 \setminus d_n) \setminus a) \le \inf_{n\in\mathbb{N}} \nu(d_n \triangle (1 \setminus a)) = 0,$$

$$\nu(a \setminus a') \le \sum_{n=0}^{\infty} \nu(a \cap d_n) < \nu a,$$

so $0 \ne a' \subseteq a$. Now set $\phi(a) = \{a'\} \cup \{b_n : n \in \mathbb{N}\}$.

(ii) For $C \in \mathrm{Pou}(\mathfrak{A})$, set

$$\psi(C) = \{c \cap b : c \in C,\, b \in \mathfrak{B}\} \setminus \{0\} \in [\mathfrak{A}^+]^{\le\kappa}.$$

(iii) Suppose that $a \in \mathfrak{A}^+$, $C \in \mathrm{Pou}(\mathfrak{A})$ and $\phi(a) \sqsubseteq^* C$. Then there is a $b \in \psi(C)$ such that $b \subseteq a$. **P** Let $c \in C$ be such that $c \cap a' \ne 0$, where a' is defined as in (i) above. Then $B = \{b : b \in \phi(a) \setminus \{a'\},\, c \cap b \ne 0\}$ is a finite subset of $\mathfrak{B}$, so $\sup B \in \mathfrak{B}$ and $c \setminus \sup B \in \psi(C)$. But $c \setminus \sup B = c \cap a' \subseteq a$. **Q** Thus $a \supseteq' \psi(C)$.

As a is arbitrary, (ϕ, ψ) is a Galois-Tukey connection from $(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\kappa})$ to $(\mathrm{Pou}(\mathfrak{A}), \sqsubseteq^*, \mathrm{Pou}(\mathfrak{A})$, and $(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\kappa}) \preccurlyeq_{\mathrm{GT}} (\mathrm{Pou}(\mathfrak{A}), \sqsubseteq^*, \mathrm{Pou}(\mathfrak{A})$.

(b)(i) If $\tau(\mathfrak{A})$ is finite, then $\mathfrak{A}$ is purely atomic and $\mathrm{Pou}(\mathfrak{A})$ has an upper bound in itself, as does $\mathcal{N}_\kappa$; so the result is trivial. Accordingly we may suppose henceforth that $\tau(\mathfrak{A}) = \kappa$ is infinite.

(ii) If $C \in \mathrm{Pou}(\mathfrak{A})$, there is a sequence $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}$ such that $\nu b_n \le 4^{-n}$ for every $n \in \mathbb{N}$ and $\{c : c \in C,\, c \not\subseteq \sup_{i\ge n} b_i\}$ is finite for every $n \in \mathbb{N}$. **P** If C is finite this is trivial. Otherwise, set $\epsilon_n = 4^{-n}/(n+2)$ for each $n \in \mathbb{N}$, and enumerate C as $\langle c_n \rangle_{n\in\mathbb{N}}$. Let $\langle k(n) \rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence such that $\nu c'_n \le \epsilon_n$ for every n, where $c'_n = \sup_{i\ge k(n)} c_i$; choose $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}$ inductively so that

$$\nu(b_n \triangle \sup_{j\le n}(c'_j \setminus \sup_{j\le i<n} b_i)) \le \epsilon_{n+1}$$

for each $n \in \mathbb{N}$. Then we see by induction on n that

$$\nu(c'_j \setminus \sup_{j\le i<n} b_i) \le \epsilon_n$$

whenever $j \le n$ in $\mathbb{N}$, and therefore that

$$\nu b_n \le \epsilon_{n+1} + (n+1)\epsilon_n \le 4^{-n}$$

for each n; while $c'_j \subseteq \sup_{i\ge j} b_i$ for every j, so

$$1 \setminus \sup_{i\ge n} b_i \subseteq 1 \setminus c'_n = \sup_{i<k(n)} c_i$$

meets only finitely many members of C, for every n. **Q**

(iii) Now fix on an enumeration $\langle b_\xi \rangle_{\xi<\kappa}$ of $\mathfrak{B}$. Consider the κ-localization relation $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa)$ (522K). We see from (ii) that we can find a function $\phi : \mathrm{Pou}(\mathfrak{A}) \to \kappa^{\mathbb{N}}$ such that

$$\nu b_{\phi(C)(n)} \le 4^{-n} \text{ for every } n \in \mathbb{N},$$

$$1 \setminus \sup_{i\ge n} b_{\phi(C)(i)} \text{ meets only finitely many members of } C, \text{ for every } n \in \mathbb{N}.$$

Next, define $\psi : \mathcal{S}_\kappa \to \mathrm{Pou}(\mathfrak{A})$ as follows. Given $S \in \mathcal{S}_\kappa$, set $a_0(S) = 1$,

$$a_{n+1}(S) = \sup_{m\ge n} \sup\{b_\xi : (m, \xi) \in S,\, \nu b_\xi \le 4^{-m}\}$$

for each n; then $\nu a_{n+1}(S) \le \sum_{m=n}^{\infty} 2^{-m} = 2^{-n+1}$ for every n, so $\psi(S) = \{a_n(S) \setminus a_{n+1}(S) : n \in \mathbb{N}\}$ is a partition of unity in $\mathfrak{A}$.

(iv) Suppose that $C \in \mathrm{Pou}(\mathfrak{A})$ and $S \in \mathcal{S}_\kappa$ are such that $\phi(C) \subseteq^* S$. In this case there is an $m \in \mathbb{N}$ such that $(n, \phi(C)(n)) \in S$ for every $n \ge m$. Since $\nu b_{\phi(C)(n)} \le 4^{-n}$ for every n, $\sup_{i\ge n} b_{\phi(C)(i)} \subseteq a_{n+1}(S)$ and $1 \setminus a_{n+1}(S)$ meets only finitely many members of C, for every $n \ge m$. Thus every member of $\psi(S)$ meets only finitely many members of C, and $C \sqsubseteq^* \psi(S)$.

This shows that (ϕ, ψ) is a Galois-Tukey connection from $(\mathrm{Pou}(\mathfrak{A}), \sqsubseteq^*, \mathrm{Pou}(\mathfrak{A}))$ to $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa)$, and $(\mathrm{Pou}(\mathfrak{A}), \sqsubseteq^*, \mathrm{Pou}(\mathfrak{A})) \preccurlyeq_{\mathrm{GT}} (\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa)$. On the other side, we know already that $(\kappa^{\mathbb{N}}, \subseteq^*, \mathcal{S}_\kappa) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}_\kappa, \subseteq, \mathcal{N}_\kappa)$ (524G); so $(\mathrm{Pou}(\mathfrak{A}), \sqsubseteq^*, \mathrm{Pou}(\mathfrak{A})) \preccurlyeq_{\mathrm{GT}} (\mathcal{N}_\kappa, \subseteq, \mathcal{N}_\kappa)$, that is, $\mathrm{Pou}(\mathfrak{A}) \preccurlyeq_{\mathrm{T}} \mathcal{N}_\kappa$.

539D Corollary Let $\mathfrak{A}$ be a Maharam algebra.

(a) $\pi(\mathfrak{A}) \le \max(\mathrm{cf}[\tau(\mathfrak{A})]^{\le\omega}, \mathrm{cf}\,\mathcal{N})$.

(b) If $\tau(\mathfrak{A}) \le \omega$, then $\mathrm{wdistr}(\mathfrak{A}) \ge \mathrm{add}\,\mathcal{N}$.

proof Set $\kappa = \tau(\mathfrak{A})$.

(a) If $\pi(\mathfrak{A})$ is countable, or $\pi(\mathfrak{A}) \le \mathrm{cf}[\kappa]^{\le\omega}$, we can stop. Otherwise, κ is infinite and

$$\max(\omega,\kappa) \le \max(\omega, \mathrm{cf}[\kappa]^{\le\omega}) < \pi(\mathfrak{A})$$
$$= \mathrm{cov}(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) \le \max(\omega, \kappa, \mathrm{cov}(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\kappa}))$$

(512Gf), so

$$\pi(\mathfrak{A}) \le \mathrm{cov}(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\kappa}) \le \mathrm{cov}(\mathrm{Pou}(\mathfrak{A}), \sqsubseteq^*, \mathrm{Pou}(\mathfrak{A}))$$

(539Ca, 512Da)

$$= \mathrm{cf}\,\mathrm{Pou}(\mathfrak{A}) \le \mathrm{cf}\,\mathcal{N}_\kappa$$

(539Cb, 513E(e-i))

$$= \max(\mathrm{cf}[\kappa]^{\le\omega}, \mathrm{cf}\,\mathcal{N})$$

(523N).

(b) If κ is finite, $\mathrm{wdistr}(\mathfrak{A}) = \infty$ and we can stop. Otherwise, $\kappa = \omega$ and

$$\mathrm{wdistr}(\mathfrak{A}) = \mathrm{add}\,\mathrm{Pou}(\mathfrak{A})$$

(512Ee)

$$\ge \mathrm{add}\,\mathcal{N}_\kappa$$

(539Cb, 513E(e-ii))

$$= \mathrm{add}\,\mathcal{N}.$$

539E Proposition (VELIČKOVIĆ 05, BALCAR JECH & PAZÁK 05) If $\mathfrak{A}$ is an atomless Maharam algebra, not $\{0\}$, there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\sup_{n\in I} a_n = 1$ and $\inf_{n\in I} a_n = 0$ for every infinite $I \subseteq \mathbb{N}$.

proof Fix a strictly positive Maharam submeasure ν on $\mathfrak{A}$.

(a) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\delta = \inf_{n\in\mathbb{N}} \nu a_n$ is greater than 0, there are a non-zero $d \in \mathfrak{A}$ and an infinite $I \subseteq \mathbb{N}$ such that $d \subseteq \sup_{i\in J} a_i$ for every infinite $J \subseteq I$. **P?** Otherwise, set $b_J = \sup_{i\in J} a_i$ for $J \subseteq \mathbb{N}$. Choose $\langle I_\xi\rangle_{\xi<\omega_1}$, $\langle c_\xi\rangle_{\xi<\omega_1}$ and $\langle d_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. $I_0 = \mathbb{N}$. The inductive hypothesis will be that I_ξ is an infinite subset of $\mathbb{N}$, $I_\xi \setminus I_\eta$ is finite whenever $\eta \le \xi$, and $c_\xi \cap b_{I_{\xi+1}} = 0$ for every $\xi < \omega_1$. Given $\langle I_\eta\rangle_{\eta\le\xi}$ where $\xi < \omega_1$, set $d_\xi = \inf_{n\in\mathbb{N}} b_{I_\xi\setminus n}$. Since $\nu b_J \ge \delta$ for every non-empty $J \subseteq \mathbb{N}$, $\nu d_\xi \ge \delta$ and $d_\xi \ne 0$. By hypothesis, there is an infinite $I_{\xi+1} \subseteq I_\xi$ such that $c_\xi = d_\xi \setminus b_{I_{\xi+1}}$ is non-zero. Given $\langle I_\eta\rangle_{\eta<\xi}$ where $\xi < \omega_1$ is a non-zero limit ordinal, let I_ξ be an infinite set such that $I_\xi \setminus I_\eta$ is finite for every $\eta < \xi$, and continue.

Now observe that if $\eta < \xi < \omega_1$, $I_\xi \setminus I_\eta$ is finite, so that there is an $n \in \mathbb{N}$ such that $I_\xi \setminus n \subseteq I_{\eta+1}$, and

$$c_\xi \subseteq d_\xi \subseteq b_{I_\xi\setminus n} \subseteq b_{I_{\eta+1}}$$

is disjoint from c_η. But this means that $\langle c_\xi\rangle_{\xi<\omega_1}$ is disjoint, which is impossible, because $\mathfrak{A}$ is ccc. **XQ**

(b) Let us say that a Boolean algebra $\mathfrak{B}$ **splits reals** if there is a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}$ such that $\sup_{n\in I} b_n = 1$ and $\inf_{n\in I} b_n = 0$ for every infinite $I \subseteq \mathbb{N}$. Now the set of those $d \in \mathfrak{A}$ such that the principal ideal $\mathfrak{A}_d$ generated by d splits reals is order-dense in $\mathfrak{A}$. **P** Let $a \in \mathfrak{A}^+$.

case 1 If $\nu{\restriction}\mathfrak{A}_a$ is uniformly exhaustive, then $\mathfrak{A}_a$ is measurable (539Ab). Let $\bar\mu$ be a probability measure on $\mathfrak{A}_a$; because $\mathfrak{A}_a$, like $\mathfrak{A}$, is atomless, there is a stochastically independent family $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_a$ with $\bar\mu a_n = \frac{1}{2}$ for every n, and now $\langle a_n\rangle_{n\in\mathbb{N}}$ witnesses that $\mathfrak{A}_a$ splits reals.

case 2 If $\nu{\restriction}\mathfrak{A}_a$ is not uniformly exhaustive, let $\langle b_{ni}\rangle_{i\le n\in\mathbb{N}}$ be a family of elements of $\mathfrak{A}_a$ such that $\langle b_{ni}\rangle_{i\le n}$ is disjoint for each n and $\epsilon = \inf_{i\le n\in\mathbb{N}} \nu b_{ni}$ is greater than 0. There is a family $\langle f_\xi\rangle_{\xi<\omega_1}$ in $\prod_{n\in\mathbb{N}}\{0,\dots,n\}$ such that $\{n : f_\xi(n) = f_\eta(n)\}$ is finite whenever $\eta < \xi < \omega_1$. (For each $\xi < \omega_1$ let $\theta_\xi : \xi \to \mathbb{N}$ be injective. Now define $\langle f_\xi\rangle_{\xi<\omega_1}$ inductively by saying that

$$f_\xi(n) = \min(\mathbb{N} \setminus \{f_\eta(n) : \eta < \xi,\ \theta_\xi(\eta) < n\})$$

for every $\xi<\omega_1$ and $n\in\mathbb{N}$.)

? If for every $\xi<\omega_1$ and $I\in[\mathbb{N}]^\omega$ there is a $J\in[I]^\omega$ such that $\inf_{i\in J}b_{i,f_\xi(i)}\ne 0$, choose $\langle I_\xi\rangle_{\xi<\omega_1}$ inductively so that $I_\xi\in[\mathbb{N}]^\omega$, $I_\xi\setminus I_\eta$ is finite for every $\eta<\xi$, and $c_\xi=\inf_{i\in I_\xi}b_{i,f_\xi(i)}$ is non-zero for every $\xi<\omega_1$. Then whenever $\eta<\xi$ the set $I_\xi\cap I_\eta$ is infinite, so there is an $i\in I_\xi\cap I_\eta$ such that $f_\xi(i)\ne f_\eta(i)$; now $c_\xi\cap c_\eta\subseteq b_{i,f_\xi(i)}\cap b_{i,f_\eta(i)}=0$. But this means that we have an uncountable disjoint family in $\mathfrak{A}_a$, which is impossible, because $\mathfrak{A}$ is ccc. **X**

Thus we have a $\xi<\omega_1$ and an infinite $I\subseteq\mathbb{N}$ such that $\inf_{i\in J}d_i=0$ for every infinite $J\subseteq I$, where $d_i=b_{i,f_\xi(i)}$ for $i\in I$. Next, applying (a) to $\langle d_i\rangle_{i\in I}$, we have an infinite $K\subseteq I$ and a $d\ne 0$ such that $d=\sup_{i\in J}d_i$ for every infinite $J\subseteq K$. But this means that $\langle d\cap d_i\rangle_{i\in K}$ witnesses that $\mathfrak{A}_d$ splits reals; while $d\subseteq a$.

As a is arbitrary, we have the result. **Q**

(c) By 313K, there is a partition D of unity in $\mathfrak{A}$ such that $\mathfrak{A}_d$ splits reals for every $d\in D$; choose a sequence $\langle a_{dn}\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_d$ witnessing this for each $d\in D$. Set $a_n=\sup_{d\in D}a_{dn}$ for each n. If $I\subseteq\mathbb{N}$ is infinite, then

$$\sup_{n\in I}a_n=\sup_{d\in D}\sup_{n\in I}a_{dn}=\sup D=1,$$

while

$$d\cap\inf_{n\in I}a_n=\inf_{n\in I}a_{dn}=0$$

for every $d\in D$, so $\inf_{n\in I}a_n=0$. Thus $\langle a_n\rangle_{n\in\mathbb{N}}$ witnesses that $\mathfrak{A}$ splits reals, as claimed.

539F Definition For the next result I need a name for one more cardinal between ω_1 and $\mathfrak{c}$. The **splitting number** $\mathfrak{s}$ is the least cardinal of any family $\mathcal{A}\subseteq\mathcal{P}\mathbb{N}$ such that for every infinite $I\subseteq\mathbb{N}$ there is an $A\in\mathcal{A}$ such that $I\cap A$ and $I\setminus A$ are both infinite.

539G Proposition Let X be a set, Σ a σ-algebra of subsets of X, and ν an atomless Maharam submeasure on Σ. Let $\mathcal{M}$ be the ideal of meager subsets of $\mathbb{R}$.

(a) $\operatorname{non}\mathcal{N}(\nu)\ge\max(\mathfrak{s},\mathfrak{m}_{\text{countable}})$.

(b) $\operatorname{cov}\mathcal{N}(\nu)\le\operatorname{non}\mathcal{M}$.

proof If $\nu X=0$, these are both trivial; suppose otherwise.

(a)(i) Suppose that $D\subseteq X$ and $\#(D)<\mathfrak{m}_{\text{countable}}$. For any $\epsilon>0$, there is an $F\in\Sigma$ such that $D\subseteq F$ and $\nu F\le\epsilon$. **P** By 393I, there is for each $n\in\mathbb{N}$ a finite partition $\mathcal{E}_n$ of X into members of Σ such that $\nu E\le 2^{-n-1}\epsilon$ for each $E\in\mathcal{E}_n$. Express each $\mathcal{E}_n$ as $\{E_{ni}:i<k(n)\}$. For $x\in D$, let $f_x\in\prod_{n\in\mathbb{N}}k(n)$ be such that $x\in E_{n,f_x(n)}$ for every n. Because $\#(D)<\mathfrak{m}_{\text{countable}}$, there is an $f\in\mathbb{N}^\mathbb{N}$ such that $f\cap f_x\ne\emptyset$ for every $x\in D$ (522Sb); we may suppose that $f(n)<k(n)$ for every n. Set $F=\bigcup_{n\in\mathbb{N}}E_{n,f(n)}$; this works. **Q**

Applying this repeatedly, we get a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in Σ such that $D\subseteq F_n$ and $\nu F_n\le 2^{-n}$ for every n; now $F=\bigcap_{n\in\mathbb{N}}F_n$ includes D and belongs to $\mathcal{N}(\nu)$. As D is arbitrary, $\operatorname{non}\mathcal{N}(\nu)\ge\mathfrak{m}_{\text{countable}}$.

(ii) Set $\mathfrak{A}=\Sigma/\Sigma\cap\mathcal{N}(\nu)$, and define $\bar\nu:\mathfrak{A}\to[0,\infty[$ by setting $\bar\nu E^\bullet=\nu E$ for every $E\in\Sigma$. Then $\bar\nu$ is a strictly positive atomless Maharam submeasure on $\mathfrak{A}$. By 539E, there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\sup_{n\in I}a_n=1$ and $\inf_{n\in I}a_n=0$ for every infinite $I\subseteq\mathbb{N}$. For $n\in\mathbb{N}$, let $E_n\in\Sigma$ be such that $E_n^\bullet=a_n$.

Suppose that $D\subseteq X$ and $\#(D)<\mathfrak{s}$. For $x\in D$, set $A_x=\{n:x\in E_n\}$. Because $\#(D)<\mathfrak{s}$, there is an infinite $I\subseteq\mathbb{N}$ such that one of $I\cap A_x$, $I\setminus A_x$ is finite for every $x\in D$. Set

$$F=\bigcup_{m\in\mathbb{N}}\big((X\setminus\bigcup_{n\in I\setminus m}E_n)\cup(\bigcap_{n\in I\setminus m}E_n)\big);$$

then

$$F^\bullet=\sup_{m\in\mathbb{N}}\big((1\setminus\sup_{n\in I\setminus m}a_n)\cup(\inf_{n\in I\setminus m}a_n)\big)=0,$$

so $F\in\mathcal{N}(\nu)$, while $D\subseteq F$. As D is arbitrary, $\operatorname{non}\mathcal{N}(\nu)\ge\mathfrak{s}$.

(b) Let $\langle k(n)\rangle_{n\in\mathbb{N}}$, $\langle E_{ni}\rangle_{i<k(n)}$ and $\langle f_x\rangle_{x\in X}$ be as in (a-i) above, with $\epsilon=1$. Give $Z=\prod_{n\in\mathbb{N}}k(n)$ its compact metrizable product topology. By 522Wb, there is a family $\langle g_\xi\rangle_{\xi<\operatorname{non}\mathcal{M}}$ in Z such that $\{g_\xi:\xi<\operatorname{non}\mathcal{M}\}$ is non-meager. For each $f\in Z$, the set

$$H(f)=\bigcap_{m\in\mathbb{N}}\bigcup_{n\ge m}\{g:g\in Z,\,g(n)=f(n)\}$$

is comeager in Z, so contains some g_ξ; turning this round, $Z=\bigcup_{\xi<\operatorname{non}\mathcal{M}}H(g_\xi)$. Consider the sets $F_\xi=\{x:x\in X,\ f_x\in H(g_\xi)\}$; then $X=\bigcup_{\xi<\operatorname{non}\mathcal{M}}F_\xi$, while

$$\nu F_\xi\le\inf_{m\in\mathbb{N}}\sum_{n=m}^\infty\nu E_{n,g_\xi(n)}=0$$

for every ξ. So $\operatorname{cov}\mathcal{N}(\nu)\le\operatorname{non}\mathcal{M}$.

539H Corollary Let $\mathfrak{A}$ be an atomless Maharam algebra, not $\{0\}$. Then $d(\mathfrak{A}) \ge \max(\mathfrak{s}, \mathfrak{m}_{\mathrm{countable}})$.

proof Let Z be the Stone space of $\mathfrak{A}$ and ν' the totally finite Radon submeasure on Z corresponding to a strictly positive Maharam submeasure ν on $\mathfrak{A}$ (539Af), so that $\mathcal{N}(\nu')$ is the ideal of meager subsets of Z. Note that the meager sets of Z are all nowhere dense, because $\mathfrak{A}$ is weakly (σ,∞)-distributive (316I). Because $\mathfrak{A}$ is atomless, so are ν and ν'. As every meager subset of Z is nowhere dense (and $Z \ne \emptyset$), no dense set can be meager, and

$$d(\mathfrak{A}) = d(Z)$$

(514Bd)

$$\ge \operatorname{non}\mathcal{N}(\nu') \ge \max(\mathfrak{s}, \mathfrak{m}_{\mathrm{countable}})$$

by 539Ga.

539I Corollary Suppose that $\#(X) < \max(\mathfrak{s}, \mathfrak{m}_{\mathrm{countable}})$, where $\mathfrak{s}$ is the splitting number. Let Σ be a σ-algebra of subsets of X such that (X,Σ) is countably separated, in the sense that there is a sequence in Σ separating the points of X, and $\mathcal{I}$ a σ-ideal of Σ containing singletons. Then there is no non-zero Maharam submeasure on $\Sigma/\mathcal{I}$.

proof (a) Let μ be a Maharam submeasure on $\Sigma/\mathcal{I}$. Then we have a Maharam submeasure ν on Σ defined by setting $\nu E = \mu E^\bullet$ for every $E \in \Sigma$, and $\nu\{x\} = 0$ for every $x \in X$.

(b) ν is atomless. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ separating the points of X, and $F \in \Sigma$ such that $\nu F > 0$. Choose $\langle F_n\rangle_{n\in\mathbb{N}}$ inductively so that $F_0 = F$ and, given that $\nu F_n > 0$, F_{n+1} is either $F_n \cap E_n$ or $F_n \setminus E_n$ and $\nu F_{n+1} > 0$. Then $\bigcap_{n\in\mathbb{N}} F_n$ has at most one member, so $\lim_{n\to\infty} \nu F_n = 0$, and there is an n such that $\nu F_n = \nu(F \cap F_n)$ and $\nu(F \setminus F_n)$ are non-zero. **Q**

(c) By 539Ga,

$$\operatorname{non}\mathcal{N}(\nu) \ge \max(\mathfrak{s}, \mathfrak{m}_{\mathrm{countable}}) > \#(X)$$

and $\nu X = 0$, so μ is identically 0.

539J Theorem (a) Let ν be a totally finite Radon submeasure on a Hausdorff space X (539Af) and $\mathfrak{A}$ its Maharam algebra. Then $\mathcal{N}(\nu) \preccurlyeq_{\mathrm{T}} \mathrm{Pou}(\mathfrak{A})$.

(b) Let ν be a totally finite Radon submeasure on a Hausdorff space X and $\mathfrak{A}$ its Maharam algebra.

(i) $\mathrm{wdistr}(\mathfrak{A}) \le \operatorname{add}\mathcal{N}(\nu)$.

(ii) $\tau(\mathfrak{A}) \le w(X)$.

(iii) $\operatorname{cf}\mathcal{N}(\nu) \le \max(\operatorname{cf}[\tau(\mathfrak{A})]^{\le\omega}, \operatorname{cf}\mathcal{N})$.

(iv) If $\tau(\mathfrak{A}) \le \omega$ (e.g., because X is second-countable), then $\operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}(\nu) \le \operatorname{cf}\mathcal{N}$.

proof (a) For $E \in \mathcal{N}(\nu)$, let $\mathcal{K}_E$ be a maximal disjoint family of compact sets of non-zero submeasure disjoint from E, and set $C_E = \{K^\bullet : K \in \mathcal{K}_E\}$. Because ν is inner regular with respect to the compact sets, $C_E \in \mathrm{Pou}(\mathfrak{A})$. Now $E \mapsto C_E : \mathcal{N}(\nu) \to \mathrm{Pou}(\mathfrak{A})$ is a Tukey function. **P** Suppose that $\mathcal{E} \subseteq \mathcal{N}(\nu)$ and $D \in \mathrm{Pou}(\mathfrak{A})$ are such that $C_E \sqsubseteq^* D$ for every $E \in \mathcal{E}$; take any $\epsilon > 0$. Because D is countable, we have a countable partition $\mathcal{H}$ of X into measurable sets such that $D = \{H^\bullet : H \in \mathcal{H}\}$. Because ν is inner regular with respect to the self-supporting compact sets (539Af), we can find a self-supporting compact set $K \subseteq X$ such that $\nu(X \setminus K) \le \epsilon$ and K is covered by finitely many members of $\mathcal{H}$; consequently $K^\bullet$ meets only finitely many members of D.

If $E \in \mathcal{E}$, then $K^\bullet$ meets only finitely many members of C_E, so there is a finite set $\mathcal{K}'_E \subseteq \mathcal{K}_E$ such that $K \setminus K_E$ is negligible, where $K_E = \bigcup \mathcal{K}'_E$. But K_E is compact and K is self-supporting, so $K \subseteq K_E$ and $K \cap E = \emptyset$.

This means that $\bigcup \mathcal{E} \subseteq X \setminus K$ is included in an open set of submeasure at most ϵ. This is true for every $\epsilon > 0$, so $\bigcup \mathcal{E}$ is included in a negligible G_δ set and belongs to $\mathcal{N}(\nu)$; that is, $\mathcal{E}$ is bounded above in $\mathcal{N}(\nu)$. As $\mathcal{E}$ is arbitrary, $E \mapsto C_E$ is a Tukey function. **Q**

(b)(i) Putting (a) and 513E(e-ii) together,

$$\mathrm{wdistr}(\mathfrak{A}) = \operatorname{add}\mathrm{Pou}(\mathfrak{A}) \le \operatorname{add}\mathcal{N}(\nu).$$

(ii) If $\mathcal{U}$ is a base for the topology of X with $\#(\mathcal{U}) = w(X)$, consider $D = \{U^\bullet : U \in \mathcal{U}\}$ and the order-closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by D; note that $\mathfrak{B}$ is closed for the order-sequential (or Maharam-algebra) topology of $\mathfrak{A}$ (539Ad). Let $\mathcal{E}$ be the algebra of sets generated by $\mathcal{U}$. If $F \in \operatorname{dom}\nu$ and $\epsilon > 0$, there are compact sets $K \subseteq F$,

$L \subseteq X \setminus F$ such that $\nu(X \setminus (K \cup L)) \le \epsilon$. There is an $E \in \mathcal{E}$ such that $K \subseteq E \subseteq X \setminus L$, so $\nu(E \triangle F) \le \epsilon$. Now $E^\bullet \in \mathfrak{B}$ and $\bar\nu(F^\bullet \triangle E^\bullet) \le \epsilon$; as ϵ is arbitrary, $F^\bullet \in \mathfrak{B}$; as F is arbitrary, $\mathfrak{B} = \mathfrak{A}$ and $\mathfrak{A}$ is τ-generated by D. This means that $\tau(\mathfrak{A}) \le \#(D) \le w(X)$, as required.

(iii) Setting $\kappa = \tau(\mathfrak{A})$, (a) and 539Cb tell us that $\mathcal{N}(\nu) \preccurlyeq_{\mathrm{T}} \mathcal{N}_\kappa$, where $\mathcal{N}_\kappa$ is the null ideal of the usual measure on $\{0,1\}^\kappa$. So $\operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}_\kappa$ and

$$\operatorname{cf}\mathcal{N}(\nu) \le \operatorname{cf}\mathcal{N}_\kappa \le \max(\operatorname{cf}[\kappa]^{\le\omega}, \operatorname{cf}\mathcal{N})$$

(513E(e-i), 523N).

(iv) If $\kappa \le \omega$ then $\mathcal{N}_\kappa \preccurlyeq_{\mathrm{T}} \mathcal{N}$ so $\operatorname{add}\mathcal{N}(\nu) \ge \operatorname{add}\mathcal{N}$ and $\operatorname{cf}\mathcal{N}(\nu) \le \operatorname{cf}\mathcal{N}$.

539K We can approach precalibers by some of the same combinatorial methods as before.

Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$.

(a) Let $\langle a_i\rangle_{i\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\inf_{i\in\mathbb{N}} \nu a_i > 0$.

(i) There is an infinite $I \subseteq \mathbb{N}$ such that $\{a_i : i \in I\}$ is centered.

(ii) For every $k \in \mathbb{N}$ there are an $S \in [\mathbb{N}]^\omega$ and a $\delta > 0$ such that $\nu(\inf_{i\in J} a_i) \ge \delta$ for every $J \in [S]^k$.

(b) Suppose that $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa} \nu a_\xi > 0$, where κ is a regular uncountable cardinal. Then for every $k \in \mathbb{N}$ there are a stationary set $S \subseteq \kappa$ and a $\delta > 0$ such that $\nu(\inf_{i\in J} a_i) \ge \delta$ for every $J \in [S]^k$.

(c) If ν is strictly positive, then (κ, κ, k) is a precaliber triple of $\mathfrak{A}$ for every regular uncountable cardinal κ and every $k \in \mathbb{N}$; in particular, $\mathfrak{A}$ satisfies Knaster's condition.

proof (a)(i) This is 392J.

(ii) Induce on k. The cases $k = 0$, $k = 1$ are trivial. For the inductive step to $k+1$, let $M \in [\mathbb{N}]^\omega$ and $\delta > 0$ be such that $\nu(\inf_{i\in J} a_i) \ge \delta$ for every $J \in [M]^k$. **?** Suppose, if possible, that for every $S \in [M]^\omega$ and $\gamma > 0$ there is a $J \in [S]^{k+1}$ such that $\nu(\inf_{i\in J} a_i) < \gamma$. Using Ramsey's theorem (4A1G) repeatedly, we can find $\langle I_n\rangle_{n\in\mathbb{N}}$ such that $I_0 \in [M]^\omega$, $I_{n+1} \in [I_n]^\omega$, $r_n = \min I_n \notin I_{n+1}$ and $\nu(\inf_{i\in J} a_i) \le 2^{-n-2}\delta$ for every $n \in \mathbb{N}$ and $J \in [I_n]^{k+1}$. Set $S = \{r_n : n \in \mathbb{N}\}$. If $J \in [S]^k$ and $\min J = r_n$, then $J \cup \{r_m\} \in [I_m]^{k+1}$, so $\nu(\inf_{i\in J} a_i \cap a_{r_m}) \le 2^{-m-2}\delta$, for every $m < n$. It follows that $\nu(\inf_{i\in J} a_i \cap \sup_{m<n} a_{r_m}) \le \frac{1}{2}\delta$ and $\nu(\inf_{i\in J} a_i \setminus \sup_{m<n} a_{r_m}) \ge \frac{1}{2}\delta$. But this means that $\nu c_n \ge \frac{1}{2}\delta$ where $c_n = a_{r_n} \setminus \sup_{m<n} a_{r_m}$ for each n. As $\langle c_n\rangle_{n\in\mathbb{N}}$ is disjoint, this is impossible. **X**

Thus we can find $\gamma > 0$ and $S \in [M]^\omega$ such that $\nu(\inf_{i\in J} a_i) \ge \gamma$ for every $J \in [S]^{k+1}$, and the induction continues.

(b) Again induce on k. The cases $k = 0$, $k = 1$ are trivial. For the inductive step to $k+1 \ge 2$, write $c_J = \inf_{i\in J} a_i$ for $J \in [\kappa]^{<\omega}$. We know from the inductive hypothesis that there are a stationary set $S \subseteq \kappa$ and a $\delta > 0$ such that $\nu c_J \ge 3\delta$ for every $J \in [S]^k$. For each $\xi \in S$, choose $m(\xi) \in \mathbb{N}$ and $\langle J_{\xi i}\rangle_{i<m(\xi)}$ as follows. Given $\langle J_{\xi i}\rangle_{i<j}$, where $j \in \mathbb{N}$, choose, if possible, $J_{\xi j} \in [S \cap \xi]^k$ such that $\nu(c_{J_{\xi j}} \cap c_{J_{\xi i}}) \le 2^{-i}\delta$ for every $i < j$ and $\nu(a_\xi \cap c_{J_{\xi j}}) \le 2^{-j}\delta$; if this is not possible, set $m(\xi) = j$ and stop. Now the point is that we always do have to stop. **P?** Otherwise, set $d_i = c_{J_{\xi i}}$ for each $i \in \mathbb{N}$. Because $J_{\xi i} \in [S]^k$, $\nu d_i \ge 3\delta$ for each i; also $\nu(d_i \cap d_j) \le 2^{-i}\delta$ for $i < j$; so $\nu d'_j \ge \delta$, where $d'_j = d_j \setminus \sup_{i<j} d_i$ for each j. But now $\langle d'_j\rangle_{j\in\mathbb{N}}$ is disjoint and ν is not exhaustive. **XQ**

At the end of the process, we have $m(\xi)$ and $\langle J_{\xi i}\rangle_{i<m(\xi)}$ for each $\xi \in S$. By the Pressing-Down Lemma (4A1Cc), there are $\tilde m$ and $\langle \tilde J_i\rangle_{i<\tilde m}$ such that $S' = \{\xi : \xi \in S,\ m(\xi) = \tilde m,\ J_{\xi i} = \tilde J_i \text{ for every } i < \tilde m\}$ is stationary in κ. **?** Suppose, if possible, that $I \in [S']^{k+1}$ and $\nu c_I \le 2^{-\tilde m}\delta$. Set $\xi = \max I$, $J = I \setminus \{\xi\}$, $\eta = \min I \in J$. Then $J \in [S \cap \xi]^k$. For each $i < \tilde m = m(\xi)$,

$$\nu(c_J \cap c_{J_{\xi i}}) \le \nu(a_\eta \cap c_{J_{\xi i}}) = \nu(a_\eta \cap c_{J_{\eta i}}) \le 2^{-i}\delta,$$

while

$$\nu(a_\xi \cap c_J) = \nu c_I \le 2^{-\tilde m}\delta.$$

But this means that we could have extended the sequence $\langle J_{\xi i}\rangle_{i<\tilde m}$ by setting $J_{\xi\tilde m} = J$. **X**

So S' and $2^{-\tilde m}\delta$ provide the next step in the induction.

(c) This is now immediate from (b).

539L I come now to the work of Balcar Jech & Pazák 05, based on the characterizations of Maharam algebras set out in §393.

Lemma (Quickert 02) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathcal{I}$ the family of countable subsets I of $\mathfrak{A}$ for which there is a partition C of unity such that $\{a : a \in I,\, a \cap c \neq 0\}$ is finite for every $c \in C$.

(a) $\mathcal{I}$ is an ideal of $\mathcal{P}\mathfrak{A}$ including $[\mathfrak{A}]^{<\omega}$.

(b) If $A \subseteq \mathfrak{A}^+$ is such that $A \cap I$ is finite for every $I \in \mathcal{I}$, and $B = \{b : b \supseteq a \text{ for some } a \in A\}$, then $B \cap I$ is finite for every $I \in \mathcal{I}$.

(c) If $\mathfrak{A}$ is ccc, then there is no uncountable $B \subseteq \mathfrak{A}$ such that $[B]^{\le\omega} \subseteq \mathcal{I}$.

(d) If $\mathfrak{A}$ is ccc and weakly (σ, ∞)-distributive, $\mathcal{I}$ is a p-ideal (definition: 5A6Ga).

proof (a) Of course every finite subset of $\mathfrak{A}$ belongs to $\mathcal{I}$. If $I_0,\, I_1 \in \mathcal{I}$ and $J \subseteq I_0 \cup I_1$, then $J \in [\mathfrak{A}]^{\le\omega}$. For each j, we have a partition C_j of unity in $\mathfrak{A}$ such that $\{a : a \in I_j,\, a \cap c \neq 0\}$ is finite for every $c \in C_j$. Set $C = \{c_0 \cap c_1 : c_0 \in C_0,\, c_1 \in C_1\}$; then C is a partition of unity in $\mathfrak{A}$ and $\{a : a \in J,\, a \cap c \neq 0\}$ is finite for every $c \in C$.

(b) Take $I \in \mathcal{I}$. Set $J = B \cap I$. For each $b \in J$, let $a_b \in A$ be such that $a_b \subseteq b$. Let C be a partition of unity such that $\{b : b \in I,\, b \cap c \neq 0\}$ is finite for every $c \in C$; then $\{a_b : b \in J,\, a_b \cap c \neq 0\}$ is finite for every $c \in C$, so $\{a_b : b \in J\}$ belongs to $\mathcal{I}$ and must be finite. **?** If J is infinite, there is an $a \in A$ such that $K = \{b : b \in J,\, a = a_b\}$ is infinite; but in this case there is a $c \in C$ such that $a \cap c \neq 0$ and $b \cap c \neq 0$ for every $b \in K$. **X** So J is finite, as claimed.

(c) Let $\widehat{\mathfrak{A}}$ be the Dedekind completion of $\mathfrak{A}$ (314U). Let $B \subseteq \mathfrak{A}$ be an uncountable set, and $\langle b_\xi \rangle_{\xi<\omega_1}$ a family of distinct elements of B. Set $d = \inf_{\xi<\omega_1} \sup_{\xi\le\eta<\omega_1} b_\eta$, taken in $\widehat{\mathfrak{A}}$. Then (because $\widehat{\mathfrak{A}}$ is ccc, by 514Ee) $d = \sup_{\xi\le\eta<\omega_1} b_\eta$ for some ξ (316E); in particular, $d \neq 0$. Next, we can find a strictly increasing sequence $\langle \xi_n \rangle_{n\in\mathbb{N}}$ in ω_1 such that $d \subseteq \sup_{\xi_n\le\eta<\xi_{n+1}} b_\eta$ for every $n \in \mathbb{N}$. Set $I = \{b_\eta : \eta < \sup_{n\in\mathbb{N}} \xi_n\} \in [B]^{\le\omega}$. If C is any partition of unity in $\mathfrak{A}$, there must be some $c \in C$ such that $c \cap d \neq 0$, and now $\{a : a \in I,\, a \cap c \neq 0\}$ is infinite. So $I \notin \mathcal{I}$.

(d) Let $\langle I_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{I}$. For each $n \in \mathbb{N}$, let C_n be a partition of unity such that $\{a : a \in I_n,\, a \cap c \neq 0\}$ is finite for every $c \in C_n$. Let D be a partition of unity such that $\{c : c \in C_n,\, c \cap d \neq 0\}$ is finite for every $d \in D$ and $n \in \mathbb{N}$. Then

$$\{a : a \in I_n,\, a \cap d \neq 0\} \subseteq \bigcup_{c\in C_n, c\cap d\neq 0} \{a : a \in I_n,\, a \cap c \neq 0\}$$

is finite for every $d \in D$ and $n \in \mathbb{N}$. Let $\langle d_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $D \cup \{\emptyset\}$ and set $I = \bigcup_{n\in\mathbb{N}} \{a : a \in I_n,\, a \cap d_i = 0 \text{ for every } i \le n\}$. Then

$$I_n \setminus I \subseteq \bigcup_{i\le n} \{a : a \in I_n,\, a \cap d_i \neq \emptyset\}$$

is finite for each n. Also

$$\{a : a \in I,\, a \cap d_n \neq 0\} \subseteq \bigcup_{i<n} \{a : a \in I_i,\, a \cap d_n \neq 0\}$$

is finite for each n, so $I \in \mathcal{I}$.

Remark In this context, $\mathcal{I}$ is called **Quickert's ideal**.

539M Lemma Let $\mathfrak{A}$ be a weakly (σ, ∞)-distributive ccc Dedekind σ-complete Boolean algebra, and suppose that $\mathfrak{A}^+$ is expressible as $\bigcup_{k\in\mathbb{N}} D_k$ where no infinite subset of any D_k belongs to Quickert's ideal $\mathcal{I}$. Then $\mathfrak{A}$ is a Maharam algebra.

proof The point is that if $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ which order*-converges to 0, then $\{a_n : n \in \mathbb{N}\} \in \mathcal{I}$ (539A(d-i)). So no sequence in any D_k can order*-converge to 0. Because $\mathfrak{A}$ is weakly (σ, ∞)-distributive and ccc, 0 does not belong to the closure $\overline{D}_k$ of D_k for the order-sequential topology on $\mathfrak{A}$ (539A(d-iv)). So $\mathfrak{A}^+ = \bigcup_{k\in\mathbb{N}} \overline{D}_k$ is F_σ and $\{0\}$ is G_δ for the order-sequential topology. It follows that $\mathfrak{A}$ is a Maharam algebra (539A(d-vi)).

539N Theorem (Balcar Jech & Pazák 05, Veličković 05) Suppose that Todorčević's p-ideal dichotomy (5A6Gb) is true. Then every Dedekind σ-complete ccc weakly (σ, ∞)-distributive Boolean algebra is a Maharam algebra.

proof Let $\mathfrak{A}$ be a Dedekind σ-complete ccc weakly (σ, ∞)-distributive Boolean algebra. Let $\mathcal{I}$ be Quickert's ideal on $\mathfrak{A}$; then $\mathcal{I}$ is a p-ideal (539Ld). By 539Lc, there is no $B \in [\mathfrak{A}]^{\omega_1}$ such that $[B]^{\le\omega} \subseteq \mathcal{I}$. We are assuming that Todorčević's p-ideal dichotomy is true; so $\mathfrak{A}$ must be expressible as $\bigcup_{n\in\mathbb{N}} D_n$ where no infinite subset of any D_n belongs to $\mathcal{I}$. By 539M, $\mathfrak{A}$ is a Maharam algebra.

539O Corollary Suppose that Todorčević's p-ideal dichotomy is true. Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that every countably generated order-closed subalgebra of $\mathfrak{A}$ is a measurable algebra. Then $\mathfrak{A}$ is a measurable algebra.

proof (a) $\mathfrak{A}$ is ccc. **P?** Otherwise, let $\langle a_\xi\rangle_{\xi<\omega_1}$ be a disjoint family of non-zero elements of $\mathfrak{A}$. Let $f:\omega_1\to\{0,1\}^{\mathbb{N}}$ be an injective function, and set $b_n=\sup\{a_\xi:\xi<\omega_1,\ f_\xi(n)=1\}$ for each n; let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{b_n:n\in\mathbb{N}\}\cup\{\sup_{\xi<\omega_1}a_\xi\}$. Then $a_\xi\in\mathfrak{B}$ for every $\xi<\omega_1$, so $\mathfrak{B}$ is not ccc; but $\mathfrak{B}$ is supposed to be measurable. **XQ**

(b) $\mathfrak{A}$ is weakly (σ,∞)-distributive. **P** Let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence of partitions of unity in $\mathfrak{A}$. As $\mathfrak{A}$ is ccc, every C_n is countable; let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}}C_n$. Then $\mathfrak{B}$ is measurable, therefore weakly (σ,∞)-distributive, and there is a partition D of unity in $\mathfrak{B}$ such that $\{c:c\in C_n,\ c\cap d\ne 0\}$ is finite for every $n\in\mathbb{N}$ and $d\in D$. As $\mathfrak{B}$ is order-closed, D is still a partition of unity in $\mathfrak{A}$. As $\langle C_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive. **Q**

(c) By 539N, $\mathfrak{A}$ is a Maharam algebra; let ν be a strictly positive Maharam submeasure on $\mathfrak{A}$. Now ν is uniformly exhaustive. **P?** Otherwise, there are $\epsilon>0$ and a family $\langle a_{ni}\rangle_{i\le n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\langle a_{ni}\rangle_{i\le n}$ is disjoint for every $n\in\mathbb{N}$ and $\nu a_{ni}\ge\epsilon$ whenever $i\le n\in\mathbb{N}$. Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{a_{ni}:i\le n\in\mathbb{N}\}$. Then $\mathfrak{B}$ is a measurable algebra; let $\bar\mu$ be a functional such that $(\mathfrak{B},\bar\mu)$ is a totally finite measure algebra. Since $\bar\mu$ and $\nu\restriction B$ are both strictly positive Maharam submeasures on $\mathfrak{B}$, ν is absolutely continuous with respect to $\bar\mu$ (539Ac). But $\nu a_{ni}\ge\epsilon$ for every n and i, while $\inf_{i\le n\in\mathbb{N}}\bar\mu a_{ni}$ must be zero. **XQ**

(d) So $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra with a strictly positive uniformly exhaustive Maharam submeasure, and is a measurable algebra (539Ab).

539P I should say at once that 539N-539O really do need some special axiom. In fact the following example was found at the very beginning of the study of Maharam algebras.

Souslin algebras: Proposition Suppose that T is a well-pruned Souslin tree (554Yc, 5A1Dd), and set $\mathfrak{A}=\mathrm{RO}^{\uparrow}(T)$.

(a) $\mathfrak{A}$ is Dedekind complete, ccc and weakly (σ,∞)-distributive.

(b) If $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ and $\tau(\mathfrak{B})\le\omega$, then $\mathfrak{B}\cong\mathcal{P}I$ for some countable set I; in particular, $\mathfrak{B}$ is a measurable algebra.

(c) (Maharam 47) The only Maharam submeasure on $\mathfrak{A}$ is identically zero.

proof (a)(i) $\mathfrak{A}$ is Dedekind complete just because it is a regular open algebra.

(ii) T is upwards-ccc, so $\mathfrak{A}$ is ccc, by 514Nc.

(iii) For $t\in T$, set $\widehat{t}=\operatorname{int}\overline{[t,\infty[}\in\mathfrak{A}$; then $\{\widehat{t}:t\in T\}$ is order-dense in $\mathfrak{A}$. Let $r:T\to\mathrm{On}$ be the rank function of T (5A1Da). For each $\xi<\omega_1$, $A_\xi=\{\widehat{t}:t\in T,\ r(t)=\xi\}$ is a partition of unity in $\mathfrak{A}$. **P** If $r(t)=r(t')$ and $t\ne t'$ then $[t,\infty[\,\cap\,[t',\infty[\,=\emptyset$ so $\widehat{t}\cap\widehat{t'}=0$ in $\mathfrak{A}$; thus A_ξ is disjoint. If $a\in\mathfrak{A}\setminus\{0\}$, there is an $s\in T$ such that $\widehat{s}\subseteq a$; if $r(s)\ge\xi$, there is a $t\le s$ such that $r(t)=\xi$, and $a\cap\widehat{t}\ne 0$; if $r(s)<\xi$, there is a $t\ge s$ such that $r(t)=\xi$ (because T is well-pruned), and $\widehat{t}\subseteq a$. Thus $\sup A_\xi=1$ in $\mathfrak{A}$. **Q**

If $A\subseteq\mathfrak{A}$ is a partition of unity, there is a $\xi<\omega_1$ such that A_ξ refines A in the sense that every member of A_ξ is included in some member of A (see 311Ge). **P** $B=\{\widehat{t}:t\in T,\ \widehat{t}\subseteq a$ for some $a\in A\}$ is order-dense in $\mathfrak{A}$, so there is a partition C of unity included in B; C is countable; let $D\subseteq T$ be a countable set such that $C=\{\widehat{t}:t\in D\}$; set $\xi=\sup_{t\in D}r(t)$. **Q**

Of course A_η refines A_ξ whenever $\xi\le\eta<\omega_1$. So if $\langle C_n\rangle_{n\in\mathbb{N}}$ is a sequence of partitions of unity in $\mathfrak{A}$, there is a $\xi<\omega_1$ such that A_ξ refines C_n for every $n\in\mathbb{N}$, and then $\{c:c\in C_n,\ a\cap c\ne 0\}$ has just one member for every $a\in A_\xi$ and $n\in\mathbb{N}$. As $\langle C_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive.

(b) If $B\subseteq\mathfrak{A}$ is a countable set τ-generating $\mathfrak{B}$, there is a countable set $D\subseteq T$ such that $b=\sup\{\widehat{t}:t\in D,\ \widehat{t}\subseteq b\}$ for every $b\in B$. Now $\xi=\sup\{r(t):t\in D\}$ is countable, and $b=\sup\{a:a\in A_\xi,\ a\subseteq b\}$ for every $b\in B$, so $\mathfrak{B}$ is included in the order-closed subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by A_ξ. Of course A_ξ is order-dense in $\mathfrak{C}$. For $a\in A_\xi$, set $b_a=\inf\{b:b\in\mathfrak{B},\ b\supseteq a\}$; then every b_a is an atom in $\mathfrak{B}$ and $\{b_a:a\in A_\xi\}$ is order-dense in $\mathfrak{B}$, so $\mathfrak{B}$ is purely atomic. As $\mathfrak{B}$ is ccc, the set I of its atoms is countable; being Dedekind complete, $\mathfrak{B}$ is isomophic to $\mathcal{P}I$.

(c) Let ν be a Maharam submeasure on $\mathfrak{A}$. Then for every $\epsilon>0$ there is a $\xi<\omega_1$ such that $\nu a\le\epsilon$ for every $a\in A_\xi$. **P** Set

$$T'=\{t:\nu\widehat{t}\ge\epsilon\}.$$

Then T' is a subtree of T and $\{t : t \in T',\, r(t) = \xi\}$ is finite for every $\xi < \omega_1$, because ν is exhaustive. Also T', like T, can have no uncountable branches. It follows that the height of T' is countable (5A1D(b-i)), that is, that there is a $\xi < \omega_1$ such that $r(t) < \xi$ for every $t \in T'$ and $\nu a \le \epsilon$ for every $a \in A_\xi$. **Q**

As this is true for every $\epsilon > 0$, there is actually a $\xi < \omega_1$ such that $\nu a = 0$ for every $a \in A_\xi$. But as A_ξ is a countable partition of unity and ν is a Maharam submeasure, $\nu 1 = 0$ and ν is identically zero.

539Q Reflection principles In 539O, we have a theorem of the type 'if every small subalgebra of $\mathfrak{A}$ is ..., then $\mathfrak{A}$ is ...'. There was a similar result in 518I, and we shall have another in 545G. Here I collect some simple facts which are relevant to the present discussion.

(a) If $\mathfrak{A}$ is a Boolean algebra and every subset of $\mathfrak{A}$ of cardinal ω_1 is included in a ccc subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ is ccc. (For there can be no disjoint set with cardinal ω_1.)

(b) If $\mathfrak{A}$ is ccc and every countable subset of $\mathfrak{A}$ is included in a weakly (σ,∞)-distributive subalgebra of $\mathfrak{A}$, then $\mathfrak{A}$ is weakly (σ,∞)-distributive. **P** If C_n is a partition of unity in $\mathfrak{A}$ for every n, set

$$D = \{d : \{c : c \in C_n,\, c \cap d \ne 0\} \text{ is finite for every } n \in \mathbb{N}\}.$$

? If D is not order-dense in $\mathfrak{A}$, take $a \in \mathfrak{A}^+$ such that $d \not\subseteq a$ for every $d \in D$. Let $\mathfrak{B}$ be a weakly (σ,∞)-distributive subalgebra of $\mathfrak{A}$ including $\{a\} \cup \bigcup_{n\in\mathbb{N}} C_n$. Then every C_n is a partition of unity in $\mathfrak{B}$, so there is a partition B of unity in $\mathfrak{B}$ such that $B \subseteq D$. But now $a \in \mathfrak{B}^+$ so there is a $b \in B$ such that $a \cap b \ne 0$ and $a \cap b \in D$. **X**

So D is order-dense in $\mathfrak{A}$ and includes a partition of unity in $\mathfrak{A}$. As $\langle C_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive. **Q**

(c) If every countable subset of $\mathfrak{A}$ is included in a subalgebra of $\mathfrak{A}$ with the σ-interpolation property, then $\mathfrak{A}$ has the σ-interpolation property. **P** If A, $B \subseteq \mathfrak{A}$ are countable and $a \subseteq b$ whenever $a \in A$ and $b \in B$, let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$, including $A \cup B$, with the σ-interpolation property; then there is a $c \in \mathfrak{B}$ such that $a \subseteq c \subseteq b$ for every $a \in A$ and $b \in B$. **Q**

(d) If $\mathfrak{A}$ is a Maharam algebra and every countably generated closed subalgebra of $\mathfrak{A}$ is a measurable algebra, then $\mathfrak{A}$ is measurable. (This is part (c) of the proof of 539O.)

(e) Suppose that Todorčević's p-ideal dichotomy is true. Let $\mathfrak{A}$ be a Boolean algebra such that every subset of $\mathfrak{A}$ of cardinal at most ω_1 is included in a subalgebra of $\mathfrak{A}$ which is a Maharam algebra. Then $\mathfrak{A}$ is a Maharam algebra. **P** By (a), $\mathfrak{A}$ is ccc; by (c), $\mathfrak{A}$ is Dedekind complete; by (b), $\mathfrak{A}$ is weakly (σ,∞)-distributive; by 539N, $\mathfrak{A}$ is a Maharam algebra. **Q**

(f) Suppose that Todorčević's p-ideal dichotomy is true. Let $\mathfrak{A}$ be a Boolean algebra such that every subset of $\mathfrak{A}$ of cardinal at most $\mathfrak{c}$ is included in a subalgebra of $\mathfrak{A}$ which is a measurable algebra. Then $\mathfrak{A}$ is measurable. **P** By (a), $\mathfrak{A}$ is ccc. So if $\mathfrak{B}$ is a countably generated order-closed subalgebra, it has cardinal $\mathfrak{c}$, and is included in a measurable subalgebra $\mathfrak{C}$ of $\mathfrak{A}$. Now $\mathfrak{B}$ is order-closed in $\mathfrak{C}$, so is itself a measurable algebra. By 539O, $\mathfrak{A}$ also is measurable. **Q**

(g) On the other hand, Farah & Veličković 06 show that if κ is an infinite cardinal such that $2^\kappa = \kappa^+$, $\square_\kappa$ (5A6D) is true and the cardinal power κ^ω is equal to κ, then there is a Dedekind complete Boolean algebra $\mathfrak{A}$, with cardinal κ^+, such that every order-closed subalgebra of $\mathfrak{A}$ with cardinal at most κ is a measurable algebra, but $\mathfrak{A}$ is not a measurable algebra (and therefore is not a Maharam algebra, by (d) above). In particular, this can easily be the case with $\kappa = \mathfrak{c}$.

539R Exhaustivity rank While we now know that there are non-measurable Maharam algebras, we know practically nothing about their structure. I introduce the following idea as a possible tool for investigation.

Definitions Suppose that $\mathfrak{A}$ is a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$. For $\epsilon > 0$, say that $a \preccurlyeq_\epsilon b$ if either $a = b$ or $a \subseteq b$ and $\nu(b \setminus a) > \epsilon$. Then $\preccurlyeq_\epsilon$ is a well-founded partial order on $\mathfrak{A}$ (use 5A1Cc; if $\langle a_n\rangle_{n\in\mathbb{N}}$ were strictly decreasing for $\preccurlyeq_\epsilon$, then $\langle a_n \setminus a_{n+1}\rangle_{n\in\mathbb{N}}$ would be disjoint, with $\nu(a_n \setminus a_{n+1}) \ge \epsilon$ for every n). Let $r_\epsilon : \mathfrak{A} \to \mathrm{On}$ be the corresponding rank function, so that

$$r_\epsilon(a) = \sup\{r_\epsilon(b) + 1 : b \subseteq a,\, \nu(a \setminus b) > \epsilon\}$$

for every $a \in \mathfrak{A}$ (5A1Cb). Now the **exhaustivity rank** of ν is $\sup_{\epsilon>0} r_\epsilon(1)$.

539S Elementary facts Let $\mathfrak{A}$ be a Boolean algebra with an exhaustive submeasure ν and associated rank functions r_ϵ for $\epsilon > 0$.

(a) $r_\delta(a) \le r_\epsilon(b)$ whenever $\nu(a \setminus b) \le \delta - \epsilon$. **P** Induce on $r_\epsilon(b)$. If $r_\epsilon(b) = 0$, then $\nu b \le \epsilon$ so $\nu a \le \delta$ and $r_\delta(a) = 0$. For the inductive step to $r_\epsilon(b) = \xi$, if $c \subseteq a$ and $\nu(a \setminus c) > \delta$ then $\nu(b \setminus c) > \epsilon$ and $r_\epsilon(b \cap c) < \xi$. Also $\nu(c \setminus b) \le \delta - \epsilon$ so, by the inductive hypothesis, $r_\delta(c) \le r_\delta(b \cap c) < \xi$; as c is arbitrary, $r_\delta(a) \le \xi$ and the induction continues. **Q** In particular,

$$r_\epsilon(a) \le r_\epsilon(b) \text{ if } a \subseteq b, \quad r_\delta(a) \le r_\epsilon(a) \text{ if } \epsilon \le \delta.$$

(b) If a, $b \in \mathfrak{A}$ are disjoint and $\epsilon > 0$, then $r_\epsilon(a \cup b)$ is at least the ordinal sum $r_\epsilon(a) + r_\epsilon(b)$. **P** Induce on $r_\epsilon(b)$. If $r_\epsilon(b) = 0$, the result is immediate from (a) above. For the inductive step to $r_\epsilon(b) = \xi$, we have for any $\eta < \xi$ a $c \subseteq b$ such that $\nu(b \setminus c) > \epsilon$ and $\eta \le r_\epsilon(c) < \xi$. Now $r_\epsilon(a \cup c) \ge r_\epsilon(a) + \eta$, by the inductive hypothesis, and $\nu((a \cup b) \setminus (a \cup c)) > \epsilon$, so $r_\epsilon(a \cup b) > r_\epsilon(a) + \eta$; as η is arbitrary, $r_\epsilon(a \cup b) \ge r_\epsilon(a) + \xi$ and the induction continues. **Q**

539T The rank of a Maharam algebra Note that the rank function r_ϵ associated with an exhaustive submeasure ν depends only on the set $\{a : \nu a > \epsilon\}$. In particular, if ν and ν' are exhaustive submeasures on a Boolean algebra $\mathfrak{A}$ and $\nu a \le \epsilon$ whenever $\nu' a \le \delta$, then $r_\epsilon^{(\nu)}(a) \le r_\delta^{(\nu')}(a)$ for every $a \in \mathfrak{A}$. If $\mathfrak{A}$ is a Maharam algebra, then any two Maharam submeasures on $\mathfrak{A}$ are mutually absolutely continuous (539Ac), so have the same exhaustivity rank; I will call this the **Maharam submeasure rank** of $\mathfrak{A}$, $\mathrm{Mhsr}(\mathfrak{A})$. Note that if $a \in \mathfrak{A}$ then $\mathrm{Mhsr}(\mathfrak{A}_a) \le \mathrm{Mhsr}(\mathfrak{A})$.

If $\mathfrak{A}$ is a measurable algebra, $\mathrm{Mhsr}(\mathfrak{A}) \le \omega$, because if μ is an additive functional and $\epsilon > 0$, then $\mu a > \epsilon r_\epsilon^{(\mu)}(a)$ for every $a \in \mathfrak{A}$. More generally, for any uniformly exhaustive submeasure ν and $\epsilon > 0$, $r_\epsilon^{(\nu)}(a)$ is finite, being the maximal size of any disjoint set consisting of elements, included in a, of submeasure greater than ϵ.

539U Theorem Suppose that $\mathfrak{A}$ is a non-measurable Maharam algebra. Then $\mathrm{Mhsr}(\mathfrak{A})$ is at least the ordinal power ω^ω.

proof Let ν be a strictly positive Maharam submeasure on $\mathfrak{A}$.

(a) For the time being (down to the end of (d) below), assume that $\mathfrak{A}$ is nowhere measurable (definition: 391Bc). For $a \in \mathfrak{A}$, set

$$\check{\nu} a = \inf_{n\in\mathbb{N}} \sup\{\min_{i\le n} \nu a_i : a_0, \dots , a_n \subseteq a \text{ are disjoint}\}.$$

Then $\check{\nu}$ is a Maharam submeasure. **P** Of course $\check{\nu}0 = 0$ and $\check{\nu}a \le \check{\nu}b$ whenever $a \subseteq b$. If a, $b \in \mathfrak{A}$ and $\epsilon > 0$, then there are n_0, $n_1 \in \mathbb{N}$ such that whenever $\langle c_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$, then $\#(\{i : \nu(c_i \cap a) \ge \check{\nu}a + \epsilon\}) \le n_0$ and $\#(\{i : \nu(c_i \cap b) \ge \check{\nu}b + \epsilon\}) \le n_1$. So

$$\#(\{i : \nu(c_i \cap (a \cup b)) \ge \check{\nu}a + \check{\nu}b + 2\epsilon\}) \le n_0 + n_1.$$

It follows that $\check{\nu}(a \cup b) \le \check{\nu}a + \check{\nu}b + 2\epsilon$; as ϵ, a and b are arbitrary, $\check{\nu}$ is a submeasure. Because $\check{\nu} \le \nu$, $\check{\nu}$ is a Maharam submeasure. **Q**

(b) Because $\mathfrak{A}$ is nowhere measurable, $\check{\nu}$ is strictly positive. **P** If $a \in \mathfrak{A} \setminus \{0\}$, the principal ideal $\mathfrak{A}_a$ is not measurable, so the Maharam submeasure $\nu\restriction\mathfrak{A}_a$ cannot be uniformly exhaustive; that is, there is an $\epsilon > 0$ such that there are arbitrarily long disjoint strings $\langle a_i\rangle_{i\le n}$ in $\mathfrak{A}_a$ with $\nu a_i \ge \epsilon$ for every $i \le n$. But this means that $\check{\nu}a \ge \epsilon > 0$. **Q**

(c) Let r_ϵ, $\check{r}_\epsilon$ be the rank functions associated with ν and $\check{\nu}$. Then $r_\epsilon(a)$ is at least the ordinal product $\omega \cdot \check{r}_\epsilon(a)$ whenever $a \in \mathfrak{A}$ and $\epsilon > 0$. **P** Induce on $\check{r}_\epsilon(a)$. If $\check{r}_\epsilon(a) = 0$, the result is trivial. For the inductive step to $\check{r}_\epsilon(a) = \xi+1$, take $b \subseteq a$ such that $\check{\nu}b > \epsilon$ and $\check{r}_\epsilon(a \setminus b) = \xi$. Then for every $n \in \mathbb{N}$ there are disjoint $b_0, \dots , b_n \subseteq b$ such that $\nu b_i > \epsilon$ for every i, and $r_\epsilon(b) \ge \omega$; by the inductive hypothesis, $r_\epsilon(a \setminus b) \ge \omega \cdot \xi$; by 539Sb, $r_\epsilon(a) \ge \omega \cdot \xi + \omega = \omega \cdot (\xi + 1)$, and the induction proceeds. The inductive step to non-zero limit ξ is elementary. **Q**

(d) Now

$$\mathrm{Mhsr}(\mathfrak{A}) = \sup_{\epsilon>0} r_\epsilon(1) \ge \sup_{\epsilon>0} \omega \cdot \check{r}_\epsilon(1) = \omega \cdot \sup_{\epsilon>0} \check{r}_\epsilon(1)$$

(5A1Bb)

$$= \omega \cdot \mathrm{Mhsr}(\mathfrak{A});$$

as $\mathrm{Mhsr}(\mathfrak{A}) > 0$, $\mathrm{Mhsr}(\mathfrak{A}) \ge \omega^\omega$ (5A1Bc).

(e) For the general case, let $a \in \mathfrak{A}^+$ be such that the principal ideal $\mathfrak{A}_a$ is nowhere measurable. Then $\mathrm{Mhsr}(\mathfrak{A}) \ge \mathrm{Mhsr}(\mathfrak{A}_a) \ge \omega^\omega$.

539X Basic exercises (a) Let $\mathfrak{A}$ be a Maharam algebra. Show that $\mathrm{link}_n(\mathfrak{A}) \le \max(\omega, \tau(\mathfrak{A}))$ for every $n \ge 2$.

(b) Show that, in the language of §522, $\mathfrak{p} \le \mathfrak{s} \le \min(\mathrm{non}\,\mathcal{N}, \mathrm{non}\,\mathcal{M}, \mathfrak{d})$.

(c) Let $\mathfrak{A}$ be a Maharam algebra. (i) Show that if
(α) $\mathrm{cf}[\lambda]^{\le\omega} \le \lambda^+$ for every cardinal $\lambda \le \tau(\mathfrak{A})$,
(β) $\square_\lambda$ is true for every uncountable cardinal $\lambda \le \tau(\mathfrak{A})$ of countable cofinality,
then $\mathrm{FN}(\mathfrak{A}) \le \mathrm{FN}(\mathcal{P}\mathbb{N})$, with equality unless $\mathfrak{A}$ is finite. (*Hint*: 518D, 518I.) (ii) Show that if $\#(\mathfrak{A}) \le \omega_2$ and $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$, then $\mathfrak{A}$ is tightly ω_1-filtered. (*Hint*: 518M.)

(d) Let X be a set, Σ a σ-algebra of subsets of X, and $\nu : \Sigma \to [0, \infty[$ a non-zero Maharam submeasure; set $\mathcal{I} = \{E : E \in \Sigma, \nu E = 0\}$ and $\mathfrak{A} = \Sigma/\mathcal{I}$. Suppose that $\#(\mathfrak{A}) \le \omega_2$ and $\mathrm{FN}(\mathcal{P}\mathbb{N}) = \omega_1$. Show that there is a lifting for ν, that is, a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$. (*Hint*: 518L.)

(e) Let $\mathfrak{A}$ be a Boolean algebra, ν an exhaustive submeasure on $\mathfrak{A}$, and $\langle a_i \rangle_{i\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ such that $\inf_{i\in\mathbb{N}} \nu a_i > 0$. Let $\mathcal{F}$ be a Ramsey ultrafilter on $\mathbb{N}$. (i) Show that there is an $I \in \mathcal{F}$ such $\inf_{i,j\in I} \nu(a_i \cap a_j) > 0$. (ii) Show that for every $k \in \mathbb{N}$ there is an $I \in \mathcal{F}$ such that $\inf\{\nu(\inf_{i\in K} a_i) : K \in [I]^k\} > 0$. (iii) Show that there is an $I \in \mathcal{F}$ such that $\{a_i : i \in I\}$ is centered. (*Hint*: 538Hc.)

(f) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I} \triangleleft \mathcal{P}X$ a σ-ideal; suppose that $\Sigma/\Sigma \cap \mathcal{I}$ is ccc. Let Y be a set, T a σ-algebra of subsets of Y, and $\nu : \mathrm{T} \to [0, \infty[$ a Maharam submeasure; let $\mathcal{I} \ltimes \mathcal{N}(\nu)$ be the skew product as defined in 527B. Show that $(\Sigma\widehat{\otimes}\mathrm{T})/(\Sigma\widehat{\otimes}\mathrm{T}) \cap (\mathcal{I} \ltimes \mathcal{N}(\nu))$ is ccc. (*Hint*: 527L.)

539Y Further exercises (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with a countable σ-generating set (331E), and ν a Maharam submeasure on $\mathfrak{A}$. Set $\mathcal{I} = \{a : \nu a = 0\}$. Show that $\mathcal{I} \preccurlyeq_{\mathrm{T}} \mathcal{N}$.

(b) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a proper σ-ideal of subsets of X generated by $\Sigma \cap \mathcal{I}$; let Σ_L be the algebra of Lebesgue measurable subsets of $\mathbb{R}$. Write $\mathfrak{A}$ for $\Sigma/\Sigma \cap \mathcal{I}$, $\mathcal{L}$ for $(\Sigma\widehat{\otimes}\Sigma_L) \cap (\mathcal{I} \ltimes \mathcal{N})$ and $\mathfrak{C}$ for $\Sigma\widehat{\otimes}\Sigma_L/\mathcal{L}$. (i) Show that $c(\mathfrak{C}) = \max(\omega, c(\mathfrak{A}))$ and $\tau(\mathfrak{C}) = \max(\omega, \tau(\mathfrak{A}))$. (ii) Show that $\mathfrak{C}$ is weakly (σ, ∞)-distributive iff $\mathfrak{A}$ is. (iii) Show that $\mathfrak{C}$ is measurable iff $\mathfrak{A}$ is. (iv) Show that $\mathfrak{C}$ is a Maharam algebra iff $\mathfrak{A}$ is.

(c) Let $\mathfrak{A}$ be a Boolean algebra with a strictly positive Maharam submeasure $\hat\nu$, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ which is dense for the associated metric (539Ac); set $\nu = \hat\nu \restriction \mathfrak{B}$, so that ν is an exhaustive submeasure on $\mathfrak{B}$. For $\epsilon > 0$ let $r_\epsilon : \mathfrak{B} \to \mathrm{On}$ and $\hat r_\epsilon : \mathfrak{A} \to \mathrm{On}$ be the rank functions associated with ν and $\hat\nu$ respectively. Show that

$$r_\delta(b) \le \hat r_\delta(b) \le r_\epsilon(b)$$

whenever $b \in \mathfrak{B}$ and $0 < \epsilon < \delta$.

(d) Let $\mathfrak{A}$ be an infinite Maharam algebra. Show that $\mathrm{Mhsr}(\mathfrak{A}) < \tau(\mathfrak{A})^+$.

(e) (J.Kupka) Let ν be a totally finite submeasure on a Boolean algebra $\mathfrak{A}$, and set

$$\check\nu a = \inf_{n\in\mathbb{N}} \sup\{\min_{i\le n} \nu a_i : a_0, \dots, a_n \subseteq a \text{ are disjoint}\}.$$

for $a \in \mathfrak{A}$, as in the proof of 539U. Show that *either* $\check\nu \ge \frac{1}{3}\nu$ *or* there is a non-zero additive $\mu : \mathfrak{A} \to [0, \infty[$ such that $\mu a \le \nu a$ for every $a \in \mathfrak{A}$. (*Hint*: 392D.)

(f) Show that the exhaustive submeasures constructed by Talagrand's method, as described in §394, have exhaustivity rank at most the ordinal power ω^{ω^2}.

(g) Suppose that $\mathfrak{A}$ is a non-measurable Maharam algebra. Show that $\mathrm{Mhsr}(\mathfrak{A}) = \omega \cdot \mathrm{Mhsr}(\mathfrak{A})$.

539Z Problems (a) Let ν be a non-zero totally finite Radon submeasure on a Hausdorff space X. Must there be a lifting for ν? that is, writing Σ for the domain of ν, must there be a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that $\nu(E \triangle \phi E) = 0$ for every $E \in \Sigma$ and $\phi E = \emptyset$ whenever $\nu E = 0$?

(b) Can a non-measurable Maharam algebra $\mathfrak{A}$ have Maharam submeasure rank different from ω^{ω^2}?

539 Notes and comments During the growth of this treatise, the sections on Maharam submeasures were twice transformed by new discoveries, and I naturally hope that the work I have just presented will be similarly outdated before too long. In the pages above I have tried in the first place to show how the cardinal functions of chapters 51 and 52 can be applied in this more general context. With minor refinements of technique, we can go a fair way. Because we know we have at least two non-trivial atomless Maharam algebras of countable type, we are led to a more detailed analysis, as in 539Ca and 539J.

Equally instructive are the apparent limits to what the methods can achieve, which mostly point to remaining areas of obscurity. I say 'remaining'; but what is most conspicuous about the present situation is our nearly total ignorance concerning the structure of non-measurable Maharam algebras. Talagrand's construction, as described in §394, gives us a family of such algebras, but so far we can answer hardly any of the most elementary questions about them (539Yf, 394Z).

The message of Balcar Jech & Pazák 05 is that a Dedekind complete, ccc, weakly (σ,∞)-distributive Boolean algebra is 'nearly' a Maharam algebra. Any further condition (e.g., the σ-finite chain condition, as in 393S) is likely to render it a Maharam algebra; and with a little help from an extra axiom of set theory, it is already necessarily a Maharam algebra (539N). Similarly, much of the work of the last sixty years on submeasures suggests that exhaustive submeasures are 'nearly' uniformly exhaustive, and that an extra condition (e.g., sub- or super-modularity) is enough to tip the balance (413Yf). At both boundaries, there are few examples to limit conjectures about further conditions on which such results might be based. Besides 539P and Talagrand's examples, we have a further important possibility of a not-quite-Maharam algebra in 555K below.

Concordance to chapters 51-53

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to these chapters, and which have since been changed.

521Q, 521S Measurable additive functionals The result referred to in the 2003 edition of Volume 4 as to be expected in §531, in the 2006 edition as 521Q, and in the 2013 edition as 521S, is now 521T.