Integer-Forcing Linear Receivers

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***Abstract*—Linear receivers are often used to reduce the implementation complexity of multiple antenna systems. In a traditional linear receiver architecture, the receive antennas are used to separate out the codewords sent by each transmit antenna, which can then be decoded individually. Although easy to implement, this approach can be highly sub-optimal when the channel matrix is near singular. In this paper, we develop a new linear architecture that uses the receive antennas to create an effective channel matrix with integer-valued entries. Instead of attempting to recover a transmitted codeword directly, each decoder recovers a different integer combination of the codewords according to the effective channel matrix. If the effective channel is full rank, these linear equations can be digitally solved for the original codewords. By allowing the receiver to equalize the channel to any matrix with integer entries, this scheme can outperform traditional linear architectures such as decorrelators and MMSE receivers while maintaining a similar complexity. Furthermore, in the case where each transmit antenna encodes an independent data stream, the proposed receiver attains the optimal diversity multiplexing tradeoff.**

I. Introduction

Consider the multiple-input multiple-output (MIMO) channel with M transmit and N receive antennas,

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}. \tag{1}$$

We focus on the case where each transmit antenna encodes an independent data stream (see Figure 1). Thus, each data stream $\mathbf{w}_m$ is encoded separately to form a codeword $\mathbf{x}_m$ of length n. We further assume that channel state information is only available at the receiver.

Clearly, joint decoding across the receive antennas is optimal in terms of both rate and probability of error. However, the cost of jointly processing the data streams is high. In practice, it is difficult to implement this type of receiver in wireless systems when the number of streams is large. Linear receivers such as the decorrelator and MMSE receiver are often used since they allow for single stream decoding.

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In a linear receiver architecture, the channel output is projected onto some matrix $\mathbf{B}$ to get the effective channel

$$\begin{aligned}\tilde{\mathbf{y}} &= \mathbf{B}\mathbf{H}\mathbf{x} + \mathbf{B}\mathbf{z} && (2)\\ &= \mathbf{A}\mathbf{x} + \tilde{\mathbf{z}}. && (3)\end{aligned}$$

where $\mathbf{A}$ is the effective channel matrix. Each element $\tilde{y}_1, ..., \tilde{y}_M$ of the output vector $\tilde{\mathbf{y}}$ is then sent to a separate decoder. In the case of the decorrelator (with $N \geq M$), the goal of the linear projection is to separate the incoming data streams $x_1, ..., x_M$ and cancel the interference due to other streams by choosing $\mathbf{A} = \mathbf{I}$ and $\mathbf{B} = \mathbf{H}^{\dagger}$ where $\mathbf{H}^{\dagger} \triangleq (\mathbf{H}^*\mathbf{H})^{-1}\mathbf{H}^*$ is the pseudoinverse of $\mathbf{H}$. Each decoder then attempts to recover one of the data streams. The decorrelator significantly amplifies the noise when the channel matrix is near singular. At low SNR, this noise amplification can be somewhat mitigated by an MMSE receiver that sets $\mathbf{B} = \mathbf{H}^*(\mathbf{H}\mathbf{H}^* + \frac{1}{\mathsf{SNR}}\mathbf{I})^{-1}$. However, both of these receivers are known to be highly suboptimal in terms of rate and diversity [12], [13].

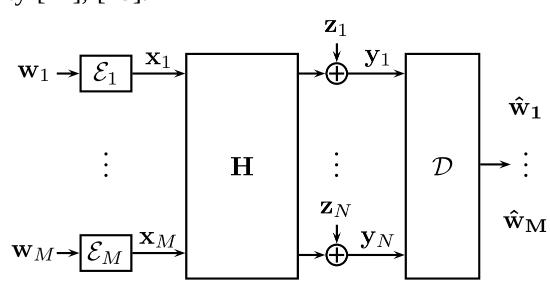

Fig. 1. MIMO Channel with single stream encoding

While using linear front end at the receiver is appealing, we observe that setting the "target" channel matrix $\mathbf{A}$ to be the identity matrix is unnecessarily restrictive when the encoders use the *same linear code*. Consider for example the 2×2 MIMO channel with unit noise variance and channel matrix

$$\mathbf{H} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \tag{4}$$

If we force the effective channel matrix to identity by choosing the linear projection matrix $\mathbf{B} = \mathbf{H}^{-1}$, the effective noise variances are $\sigma^2_{\tilde{z},1} = 2$ and $\sigma^2_{\tilde{z},2} = 5$. However, if we leave the channel as is, that is let $\mathbf{A} = \mathbf{H}$, then the linear combinations $2\mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{x}_1 + \mathbf{x}_2$ can be decoded individually, leaving the noise unchanged. Afterwards, these linear combinations can be solved for $\mathbf{x}_1$ and $\mathbf{x}_2$.

Inspired by the example above, in this paper we develop a new linear receiver architecture that provides non-trivial rate and diversity gains over traditional architectures (see Figure 2). First, we have each transmit antenna encode its data with the same linear code. At the receiver, we preprocess the received output to get the effective channel in (3). However, instead of restricting $\mathbf{A}$ to the identity matrix, we allow $\mathbf{A}$ to be any full rank matrix with integer coefficients. Now, since an integer combination of codewords from a linear code is itself a codeword, we can separately recover equations of the transmitted codewords with coefficients given by $\mathbf{A}$. Finally, these equations are digitally solved for the desired codewords. Our architecture decouples the steps of linear preprocessing, denoising (i.e. decoding) of channel outputs, and inverting the effective channel matrix to recover the codewords. For the case of single-stream encoding, our receiver architecture can achieve the optimal diversity-multiplexing tradeoff corresponding to joint decoding across the receive antennas.

Although our architecture performs nearly as well as a joint receiver, it has a significantly lower implementation complexity. The ideal joint receiver aggregates the time and space dimensions and then employs a joint maximum likelihood (ML) decoder. As a result, its complexity is exponential in the product of the blocklength and the number of data streams. Our architecture decouples the time and space dimensions by allowing for single-stream decoding. First, we search for the best integer matrix $\mathbf{A}$, which has an exponential complexity in the number of data streams in the worst case. For slow fading channels, this search is only needed once per data frame. Afterwards, our receiver recovers M linearly independent equations of codewords according to $\mathbf{A}$ and then solves these for the original codewords. This step is polynomial in the number of data streams and exponential in the blocklength for an ML decoder. Of course, in practice the complexity can be furthered reduced by replacing the ML decoder with a more efficient decoder (at the expense of some performance).

Our analysis uses nested lattice codes originally developed to approach the capacity of point-to-point AWGN and dirty-paper channels [4] and for which practical implementations were presented in [6] and subsequent works. In recent work, we have shown how to employ these codes to efficiently and reliably compute linear functions over AWGN networks [1], [2].

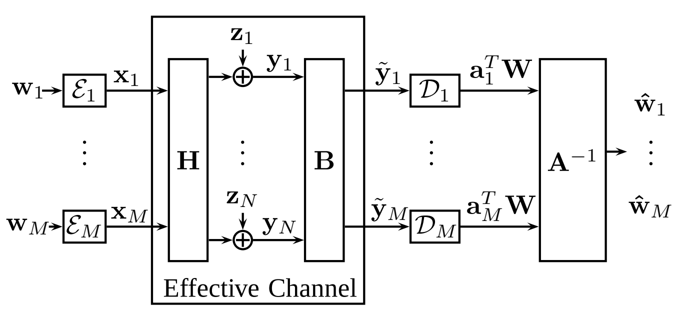

Fig. 2. MIMO channel with an integer-forcing linear receiver.

A. Related Work

Lattice codes for MIMO channels have been studied from several perspectives in the literature. For instance, lattice reduction can be used to improve the performance of the decorrelator when the channel matrix is near singular [11] and can achieve the receive diversity [10]. For lattice reduction, the matrix $\mathbf{A}$ is restricted to be unimodular[1]. To see the advantage of non-unimodular forcing, consider the $M \times M$ channel matrix $\mathbf{H}$ with columns $\mathbf{h}_1 = [1\ 0\ 0\ 0, ...,\ 0]^T, \mathbf{h}_2 = [0\ 1\ 0\ 0, ...,\ 0]^T, ..., \mathbf{h}_{M-1} = [0\ 0, ..., 0,\ 1,\ 0]^T, \mathbf{h}_M = [-1\ \ -1\ \ -1\ \ ....\ \ -1\ \ 2]^T$. If we take $\mathbf{B} = \mathbf{A}\mathbf{H}^\dagger$, then the optimal matrix of equations for lattice reduction is given by $\mathbf{A}_{\text{unimodular}} = \mathbf{I}$ while the optimal integer matrix is $\mathbf{A}_{\text{integer}} = \mathbf{H}$. The effective noise variances are $\sigma^2_{\text{unimodular}} = \max\{M/4, 1\}$ and $\sigma^2_{\text{integer}} = 1$. In this example, the channel computes a set of noisy linear equations with integer coefficients. The unimodular constraint does not allow for the use of those equations, but rather inverts the channel at the cost of noise amplification. Other works have developed lattice architectures for joint decoding that can achieve the optimal diversity multiplexing tradeoff [3], [7].

B. Channel Model

Recall that any equation of the form $\mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{z}$ over the complex field can be represented by its real-valued decomposition:

$$\begin{bmatrix}\mathsf{Re}(\mathbf{y})\\ \mathsf{Im}(\mathbf{y})\end{bmatrix} = \begin{bmatrix}\mathsf{Re}(\mathbf{G}) & -\mathsf{Im}(\mathbf{G})\\ \mathsf{Im}(\mathbf{G}) & \mathsf{Re}(\mathbf{G})\end{bmatrix}\begin{bmatrix}\mathsf{Re}(\mathbf{x})\\ \mathsf{Im}(\mathbf{x})\end{bmatrix} + \begin{bmatrix}\mathsf{Re}(\mathbf{z})\\ \mathsf{Im}(\mathbf{z})\end{bmatrix} \quad (5)$$

For ease of analysis, we will work with the real-valued decomposition of the channel matrix and, for notational convenience, we will henceforth refer to this $2M \times 2N$ matrix as $\mathbf{H}$. We will use $2M$ independent encoders and $2M$ independent decoders for the resulting real-valued transmit and receive antennas.[2]

Each antenna has a message vector $\mathbf{w}_m$ drawn independently and uniformly from $\mathcal{W} = \{0, 1, 2, \ldots, p-1\}^k$ and is equipped with an encoder $\mathcal{E}_m : \mathcal{W} \rightarrow \mathbb{R}^n$ that maps its message into n channel uses. We assume that each encoder operates at the same rate, $R_{\text{TX}} = \frac{k}{n}\log_2 p$.

Let $x_m[i] \in \mathbb{R}$ denote the input symbol at antenna m at time i and $\mathbf{x}[i] = [x_1[i], ..., x_{2M}[i]]^T$ denote the transmitted vector. We assume a uniform power allocation across the transmit antennas, $\frac{1}{n}\sum_{i=1}^{n} |x_m[i]|^2 \leq \mathsf{SNR}$. The received vector $\mathbf{y}[i] \in \mathbb{R}^{2N}$ is given by

$$\mathbf{y}[i] = \mathbf{H}\mathbf{x}[i] + \mathbf{z}[i] \quad (6)$$

where $\mathbf{H} \in \mathbb{R}^{2N \times 2M}$ is the channel matrix and $\mathbf{z}[i]$ is i.i.d Gaussian noise, $\mathbf{z}[i] \sim \mathcal{N}(\mathbf{0}, \mathbb{I}^{2N\times 2N})$. We assume that the channel realization $\mathbf{H}$ is known to the receiver but is unknown to the transmitter.

At the receiver, the messages from each of the antennas are decoded:

$$\mathcal{D} : \mathbb{R}^{n2N} \rightarrow \mathcal{W}^{2M} \quad (7)$$

$$(\hat{w}_1, ..., \hat{w}_{2M}) = \mathcal{D}(\mathbf{y}). \quad (8)$$

[1] A matrix is *unimodular* if has integer entries and its inverse has integer entries.

[2] The implementation complexity of our scheme can be decreased slightly by specializing it to the complex field using the techniques in [1]. Owing to space limitations, we focus solely on the real-valued decomposition.

We say that the *total rate* $R_{\text{TOTAL}} = 2MR_{\text{TX}}$ is achievable if for all $\epsilon > 0$ and n large enough, there are rate R_{TX} encoders and decoders such that $Pr\left((\hat{w}_1, ..., \hat{w}_{2M}) \neq (w_1, ..., w_{2M})\right) \leq \epsilon$.

II. Receiver Architecture

Prior to decoding, our receiver projects the channel output using the $2M \times 2N$ matrix $\mathbf{B}$ to get the effective channel

$$\widetilde{\mathbf{y}}[i] = \mathbf{B}\mathbf{y}[i] = \mathbf{B}\mathbf{H}\mathbf{x}[i] + \mathbf{B}\mathbf{z}[i]. \tag{9}$$

Each preprocessed output $\tilde{y}_m[i]$ is then passed into a separate decoder $\mathcal{D}_m : \mathbb{R}^n \rightarrow \mathcal{W}$. Decoder m attempts to recover a linear equation of the messages

$$\mathbf{u}_m = \left[\sum_{\ell=1}^{2M} a_{m\ell}\mathbf{w}_\ell\right] \mod p \tag{10}$$

for some $a_{m\ell} \in \mathbb{Z}$. Let $\mathbf{a}_m$ denote the vector of desired coefficients for decoder m, $\mathbf{a}_m = [a_{m1} \ a_{m2} \ \cdots \ a_{m2M}]^T$. We choose $\mathbf{a}_1, ..., \mathbf{a}_{2M}$ to be linearly independent.[3]

We say that the *computation rate* $R(\mathbf{H}, \mathbf{a})$ is achievable if for any $\epsilon > 0$ and n large enough, there exist encoders and decoders $\mathcal{E}_1, \ldots, \mathcal{E}_{2M}, \mathcal{D}_1, ..., \mathcal{D}_{2M}$, such that for any channel matrix $\mathbf{H}$ and coefficient vectors $\mathbf{a}_1, \ldots, \mathbf{a}_{2M}$ all decoders can recover their equations with total probability of error ϵ so long as $R_{\text{TX}} < R(\mathbf{H}, \mathbf{a_m})$ for $m = 1, 2, \ldots, 2M$.

Let $\mathbf{A} = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_{2M}]^T$ denote the matrix of integer equation coefficients. The original messages can be recovered from the set of linear equations by a simple inverse operation:

$$\mathbf{W} = [\mathbf{w}_1 \ \cdots \ \mathbf{w}_{2M}]^T = \mathbf{A}^{-1}[\mathbf{u}_1 \ \cdots \ \mathbf{u}_{2M}]^T. \tag{11}$$

A. Nested Lattices

For our analysis, we use nested lattice codes since they have a linear structure and have provably good performance for AWGN channels. Recall that an n-dimensional *lattice*, Λ, is a discrete subgroup of $\mathbb{R}^n$. This means that if $\mathbf{e}_1, \mathbf{e}_2 \in \Lambda$, then $\mathbf{e}_1 + \mathbf{e}_2 \in \Lambda$, and if $\mathbf{e} \in \Lambda$, then $-\mathbf{e} \in \Lambda$. A pair of lattices Λ, Λ_1 is said to be *nested* if $\Lambda \subset \Lambda_1$. A nested lattice code $\mathcal{C}$ is formed by taking all of the points of the fine lattice Λ_1 in the fundamental Voronoi region $\mathcal{V}$ of the coarse lattice Λ, $\mathcal{C} = \Lambda_1 \cap \mathcal{V}$.

From [4], [5], there exist good nested lattice codes that can approach the capacity of an AWGN channel. These nested lattice codes have been used to develop a "compute-and-forward" framework for efficient and reliable computation over AWGN networks [1], [2]. In this framework, messages are mapped onto fine lattice points and transmitted over the channel. The receiver then recovers an integer combination of the fine lattice points (modulo the coarse lattice) and maps this back to a linear combination of messages (see [1] for details). We will derive the performance of our receiver architecture by making use of the achievable computation rate results from [2].

We also note that practical implementations of nested lattice codes with near-capacity performance and linear encoding and decoding complexity (in the blocklength) have been presented in [6] and subsequent works. In such constructions the fine lattice is generated by "lifting" a (binary or p-ary) linear code using Construction A. The underlying code may, for instance, be a p-ary LDPC code or RA code (e.g., in [6] it is a binary RA code), thus inducing a low-density "parity" check lattice. We also note that the coarse lattice may simply be taken as the scaled version of $\mathbb{Z}^n$ at the price of losing the shaping gain. In this respect the encoding and decoding burden of a single stream, when using such a nested lattice scheme, is roughly the same as when using the same underlying linear code for coding over an AWGN scalar channel.

B. Achievable Rates

Let $\mathbf{B} = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_{2M}]^T$ denote the preprocessing matrix. From [2], the following computation rate is achievable:

$$R(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m) = \frac{1}{2}\log\left(\frac{\mathsf{SNR}}{\|\mathbf{b}_m\|^2 + \mathsf{SNR}\|\mathbf{H}^T\mathbf{b}_m - \mathbf{a}_m\|^2}\right).$$

Thus, all $2M$ equations can be decoded successfully if the rate per transmit antenna satisfies

$$R_{\text{TX}} < \min_m R(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m). \tag{12}$$

Since the matrix of desired coefficient vectors $\mathbf{A} = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_{2M}]^T$ is chosen to be full rank, the achievable total rate of our scheme is

$$R(\mathbf{H}, \mathbf{A}, \mathbf{B}) \triangleq 2M \min_m R(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m). \tag{13}$$

In the simplest case, the linear projection matrix is given by $\mathbf{B} = \mathbf{A}\mathbf{H}^\dagger$. If we set $\mathbf{A} = \mathbf{I}$, then our scheme reduces to the decorrelator. For a fixed $\mathbf{A}$, we showed in [2] that to maximize the total rate in (13) we should set $\mathbf{B} = \mathbf{A}\mathbf{H}^T\left(\mathbf{H}\mathbf{H}^T + \frac{1}{\mathsf{SNR}}\mathbf{I}\right)^{-1}$. Plugging in for the optimal $\mathbf{B}$, we get

$$\max_{\mathbf{B}} R(\mathbf{H}, \mathbf{A}, \mathbf{B}) = \min_m \left(-\frac{M}{2}\log \mathbf{a}_m^T\mathbf{V}\mathbf{D}\mathbf{V}^T\mathbf{a}_m\right) \tag{14}$$

where $\mathbf{V} \in \mathbb{R}^{2M\times 2M}$ is the right eigenmatrix of $\mathbf{H}$ and $\mathbf{D} \in \mathbb{R}^{2M\times 2M}$ is a diagonal matrix with elements

$$\mathbf{D}_{i,i} = \begin{cases} \frac{1}{\mathsf{SNR}\sigma_i + 1} & i \leq \text{rank}(\mathbf{H}) \\ 1 & i > \text{rank}(\mathbf{H}) \end{cases} \tag{15}$$

where σ_i is the i-th eigenvalue of $\mathbf{H}^T\mathbf{H}$ (see [2] for a proof).

C. Equation Selection

We are free to choose the set of full rank equations that achieves the highest computation rate and, in turn, the highest total rate. Note that we do not have to search over the entire space of integer vectors since any coefficient vector with $\|\mathbf{a}_m\|^2 > 1 + (\max_i \sigma_i)\mathsf{SNR}$ yields zero rate. This means that an exhaustive search only needs to check roughly SNR^M possibilities. Maximizing (14) over the set of all full rank matrices $\mathbf{A}$ with $\|\mathbf{a}_m\|^2 \leq 1 + (\max_i \sigma_i)\mathsf{SNR}$ yields that the following total rate is achievable:

$$\max_{\substack{|\mathbf{A}|\neq 0 \\ \|\mathbf{a}_m\|^2 \leq 1 + (\max_i \sigma_i)\mathsf{SNR}}} \min_m \left(-\frac{M}{2}\log \mathbf{a}_m^T\mathbf{V}\mathbf{D}\mathbf{V}^T\mathbf{a}_m\right). \tag{16}$$

[3] It is sufficient to consider matrices $\mathbf{B}$ and desired coefficient vectors $\mathbf{a}_m$ that are real-valued decompositions of a complex matrix or vector.

This representation suggests that we should choose equations $\mathbf{a}_1, ..., \mathbf{a}_{2M}$ to be in the direction of maximum right eigenvector.

III. EXAMPLE: FIXED CHANNEL

Consider the 2×2 real MIMO channel with channel matrix

$$\mathbf{H} = \begin{bmatrix} 0.7 & 1.3 \\ 0.8 & 1.5 \end{bmatrix}. \tag{17}$$

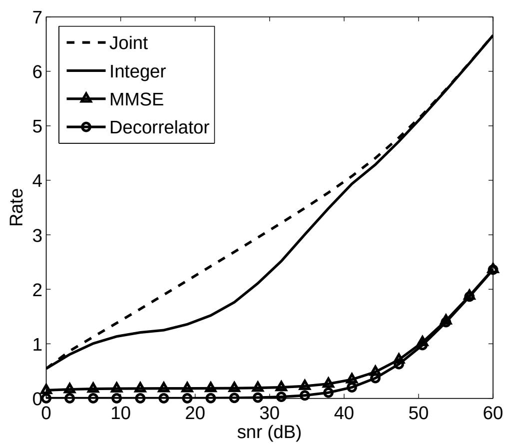

Fig. 3. Achievable rates for the 2×2 real-valued MIMO channel with fixed channel matrix from (17).

Figure 3 shows the performance of the different architectures. The decorrelator and the MMSE estimator aim to separate the data streams and cancel the interference from the other stream. However, this is difficult since the columns of the channel matrix are far from orthogonal. Our architecture attempts to exploit the interference by decoding two linearly independent equations in the direction of the maximum eigenvector $\mathbf{v}_{\text{MAX}} = [0.47 \;\; 0.88]^T$. For example, at $\mathsf{SNR} = 30dB$, we choose equations $\mathbf{a}_1 = [1 \;\; 2]^T$ and $\mathbf{a}_2 = [6 \;\; 11]^T$. Note that for different values of SNR, the optimal equation coefficients may change.

IV. RAYLEIGH FADING CHANNEL

In this section, we adopt the standard quasi-static Rayleigh fading model and work with real-valued decomposition of the resulting matrix.

A. Outage Rates

In a slow fading scenario, we must accept some probability of outage since the transmitter does not have access to the channel realization. The outage probability for our scheme is:

$$P_{\text{OUT}} = \Pr\left(R(\mathbf{H}, \mathbf{A}, \mathbf{B}) < R_{\text{TOTAL}}\right) \tag{18}$$

An outage rate is a rate that is achievable for some fixed P_{OUT}. Figure 4 shows the outage rates for $P_{\text{OUT}} = 0.02$. Our architecture nearly matches the rate of the joint decoder while traditional linear architectures are highly suboptimal.

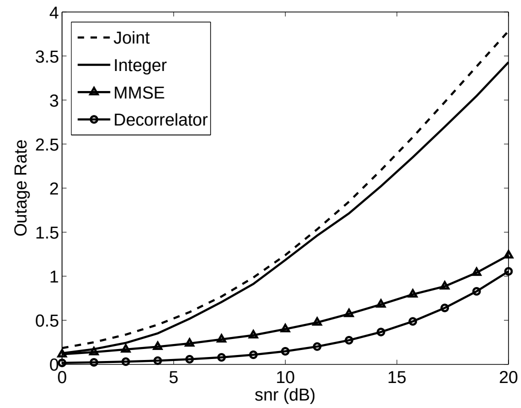

Fig. 4. 2 percent outage rates for the 2×2 complex-valued MIMO channel with Rayleigh fading.

B. Diversity Multiplexing Tradeoff

The diversity multiplexing tradeoff (DMT) provides a rough characterization of the performance of a MIMO transmission scheme at high SNR [12]. A family of codes is said to achieve spatial multiplexing gain r and diversity gain d if the total data rate and the average probability of error satisfy

$$\lim_{\mathsf{SNR}\to\infty} \frac{R(\mathsf{SNR})}{\log \mathsf{SNR}} \geq r, \qquad \lim_{\mathsf{SNR}\to\infty} \frac{\log P_e(\mathsf{SNR})}{\log \mathsf{SNR}} \leq -d. \tag{19}$$

In the case where each transmit antenna encodes an independent data stream[4], the optimal DMT is

$$d_{\text{JOINT}}(r) = N\left(1 - \frac{r}{M}\right) \tag{20}$$

where $r \in [0, M]$ and can be achieved by joint decoding. The diversity achieved by the decorrelator is

$$d_{\text{DECORR}}(r) = \left(1 - \frac{r}{M}\right). \tag{21}$$

In the case of the decorrelator, the goal of the linear projection is to invert the channel to produce an effective channel that is interference free. As a result, the noise is heavily amplified when the channel matrix is near singular and the performance is limited by the minimum singular value of the channel matrix. We now show that allowing the linear projection to steer towards any integer matrix is sufficient to recover the optimal DMT.

Theorem 1: For a MIMO channel with M transmit, $N \geq M$ receive antennas, and Rayleigh fading, the achievable diversity multiplexing tradeoff for the integer-forcing receiver is given by

$$d_{\text{INTEGER}}(r) = N\left(1 - \frac{r}{M}\right) \tag{22}$$

where $r \in [0, M]$.

Proof: Let $R = r \log \mathsf{SNR}$ be the target rate where $r \in [0, M]$. For a fixed set of equations $\mathbf{A} = [\mathbf{a}_1 \cdots \mathbf{a}_{2M}]^T$ and

[4] If joint encoding across the antennas is permitted, then a better DMT is achievable. See [12] for more details.

a fixed preprocessing matrix $\mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_{2M}]^T$, the outage probability is

$$\begin{aligned} P_{\text{OUT}}(r, \mathbf{A}, \mathbf{B}) &= \Pr\left(R(\mathbf{H}, \mathbf{A}, \mathbf{B}) < r \log \mathsf{SNR}\right) \\ &= \Pr\left(\min_m R(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m) < \frac{r}{2M} \log \mathsf{SNR}\right) \\ &= \Pr\left(\max_m \|\mathbf{b}_m\|^2 + \mathsf{SNR}\|\mathbf{H}^T\mathbf{b}_m - \mathbf{a}_m\|^2 > \mathsf{SNR}^{1-\frac{r}{M}}\right) \end{aligned}$$

We are free to choose any projection matrix $\mathbf{B}$, resulting in the following bound:

$$\begin{aligned} P_{\text{OUT}}(r, \mathbf{A}) &= \min_{\mathbf{B}} P_{\text{OUT}}(r, \mathbf{A}, \mathbf{B}) \\ &\leq P_{\text{OUT}}(r, \mathbf{A}, \mathbf{A}\mathbf{H}^\dagger) \\ &= \Pr\left(\max_m \left\|\left(\mathbf{H}^T\right)^\dagger \mathbf{a}_m\right\|^2 > \mathsf{SNR}^{1-\frac{r}{M}}\right) \end{aligned}$$

We then choose the best set of full rank equations by optimizing over all $\mathbf{A} \in \mathbb{Z}^{2M\times 2M}$ with non-zero determinant:

$$P_{\text{OUT}}(r) \leq \min_{\mathbf{A}:|\mathbf{A}|>0} \Pr\left(\max_m \left\|\left(\mathbf{H}^T\right)^\dagger \mathbf{a}_m\right\|^2 > \mathsf{SNR}^{1-\frac{r}{M}}\right) \tag{23}$$

We use properties of lattices to further bound the outage probability. Let $\Lambda_{\text{CHANNEL}}, \Lambda_{\text{DUAL}}$ be lattices generated by $\mathbf{H}$ and $\left(\mathbf{H}^T\right)^\dagger$, respectively:

$$\Lambda_{\text{CHANNEL}} = \left\{\mathbf{H}\mathbf{d} : \mathbf{d} \in \mathbb{Z}^{2M}\right\} \tag{24}$$

$$\Lambda_{\text{DUAL}} = \left\{\left(\mathbf{H}^T\right)^\dagger \mathbf{d} : \mathbf{d} \in \mathbb{Z}^{2M}\right\} \tag{25}$$

For $i = 1, ..., 2M$, the i^{th} successive minima $\lambda_i(\Lambda)$ of the lattice Λ is defined as (see, for instance, [8])

$$\begin{aligned} \lambda_i(\Lambda) = \min\{\lambda > 0 : \lambda\mathcal{B} \cap \Lambda \text{ contains } i \\ \text{linearly independent points}\} \end{aligned} \tag{26}$$

where $\mathcal{B} = \left\{x \in \mathbb{R}^{2M} : \|x\| \leq 1\right\}$ is the unit ball.

In (23), the goal is to find $2M$ linearly independent points on Λ_{DUAL} that are closest to the origin. Using the definition of successive minima, (23) can be rewritten in terms of the successive minima of Λ_{DUAL}:

$$P_{\text{OUT}}(r) \leq \Pr\left(\lambda_{2M}^2(\Lambda_{\text{DUAL}}) > \mathsf{SNR}^{1-\frac{r}{M}}\right) \tag{27}$$

We note that Λ_{DUAL} is the dual lattice of Λ_{CHANNEL} (see [8]) so we can bound the successive minima of Λ_{DUAL} in terms of the successive minima of Λ_{CHANNEL} using the following lemma from [8].

Lemma 1: Let $\Lambda \subset \mathbb{R}^n$ be an arbitrary lattice with a rank ℓ generator matrix and Λ^* be its dual. The successive minima of the lattices satisfy:

$$\lambda_i^2(\Lambda^*)\lambda_1^2(\Lambda) \leq \frac{\ell^2(\ell+3)}{4} \text{ for } i = 1, ..., \ell \tag{28}$$

Proof: See Proposition 3.3 in [8] ■

Applying Lemma 1 to Λ_{DUAL} and Λ_{CHANNEL}, we have that:

$$\lambda_{2M}^2(\Lambda_{\text{DUAL}}) \leq \frac{2M^3 + 3M^2}{\lambda_1^2(\Lambda_{\text{CHANNEL}})} \tag{29}$$

Combining equations (27) and (29),

$$P_{\text{OUT}}(r) \leq \Pr\left(\lambda_1^2(\Lambda_{\text{CHANNEL}}) < \frac{2M^3 + 3M^2}{\mathsf{SNR}^{1-\frac{r}{M}}}\right) \tag{30}$$

We bound the probability distribution of $\lambda_1(\Lambda_{\text{CHANNEL}})$ using the following lemma from [9].

Lemma 2: Let $\mathbf{H} \in \mathbb{R}^{2N\times 2M}$ be the real-valued decomposition of a $N \times M$ complex Gaussian matrix with i.i.d. Rayleigh entries. Let $\Lambda = \left\{\mathbf{H}\mathbf{d} : \mathbf{d} \in \mathbb{Z}^{2M}\right\}$ be the lattice generated by $\mathbf{H}$. Then

$$\Pr(\lambda_1(\Lambda) \leq s) = \begin{cases} \gamma s^{2N}, & M < N, \\ \delta s^{2N} \max\left\{-(\ln s)^{N+1}, 1\right\}, & M = N. \end{cases}$$

where γ and δ are constants independent of s.

Proof: See Lemma 3 in [9]. ■

Applying Lemma 2, it follows that for large SNR

$$\begin{aligned} &\Pr\left(\lambda_1^2(\Lambda_{\text{CHANNEL}}) < \frac{2M^3 + 3M^2}{\mathsf{SNR}^{1-\frac{r}{M}}}\right) \\ &\quad \leq \frac{\max\{\gamma, \delta\}\left(2M^3 + 3M^2\right)^N (\ln \mathsf{SNR})^{N+1}}{\mathsf{SNR}^{N\left(1-\frac{r}{M}\right)}}. \end{aligned}$$

Plugging this into the definition of diversity yields that $d = N(1 - \frac{r}{M})$ as desired. ■

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