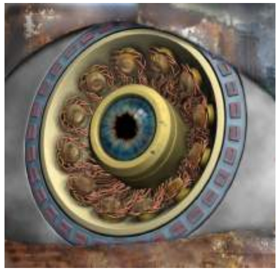

Visión por Computador

Repaso de álgebra

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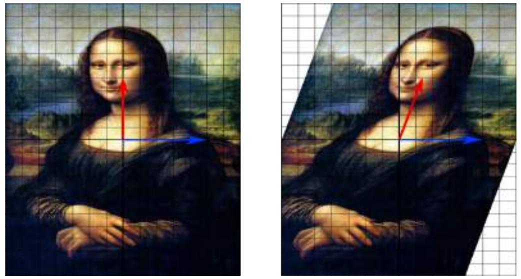
Eigenvalues and eigenvectors

The **eigenvector**, *v*, of a square matrix, A, is a vector that does not change its direction under A,

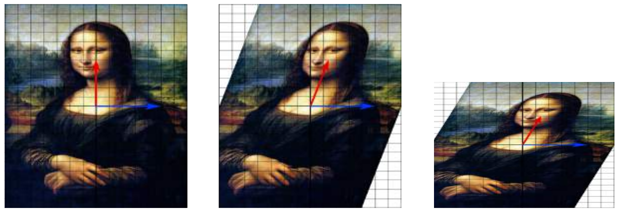
$$Av = v\lambda$$

where $\lambda \in \mathbb{R}$ is the **eigenvalue** associated to v.



Eigenvalues and eigenvectors

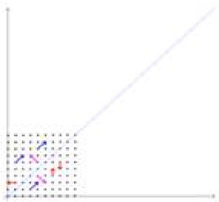
The **eigenvector**, *v*, points in a direction stretched by A, the eigenvalue, λ, is the factor by which it is stretched.



For matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$,

$v_1 = (1,1)^\top; v_2 = (1,-1)^\top$

$\lambda_1 = 3; \lambda_2 = 1$



Eigenvalues and eigenvectors

Let $\lambda_1, \ldots, \lambda_n$ and $v_1, \ldots, v_n$ be the eigenvalues and eigenvectors of A.

We can build matrices

$$Q = [v_1 | \ldots | v_n] \qquad \Lambda = \begin{bmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n \end{bmatrix}$$

such that $AQ = Q\Lambda$ hence,

$Q^{-1}AQ = \Lambda$, so A is diagonalizable.

Also, $A = Q\Lambda Q^{-1}$ is the **eigendecomposition** of A.

Matrix derivatives

Let a and x be a vector of coefficients and variables respectively.

$$\frac{\partial(a^\top x)}{\partial x} = \frac{\partial(x^\top a)}{\partial x} = x$$

$$\frac{\partial(x^\top x)}{\partial x} = 2x$$

Let A be a symmetric coefficient matrix and x a vector of variables

$$\frac{\partial(x^\top A x)}{\partial x} = 2Ax$$

Least-squares solution of linear equation

Let A_{mxn} and b_{mx1} be respectively a matrix and vector of coefficients and x_{nx1} a vector of variables, where $m > n$.

The least-squares solution to the system of equations

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} x = \begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$

is the minimum of $J(x) = (Ax - b)^\top (Ax - b)$, and

using the matrix derivation rules $\boxed{x = (A^\top A)^{-1} A^\top b}$

Least-squares solution of linear equation

Let A_{mxn} and b_{mx1} be respectively a matrix and vector of coefficients and 0_{nx1} a vector of zeros, where $m > n$.

The least-squares solution of the system of equations

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} x = \begin{bmatrix} \\ 0 \\ \\ \end{bmatrix}$$

is the minimum of $J(x) = \dfrac{x^\top (A^\top A) x}{x^\top x}$, and using the matrix derivation rules:

The eigenvector associated with the smallest eigenvalue of $A^\top A$

Maximum likelihood estimation

Let $\mathcal{D} = \{x_1, x_2, \ldots, x_n\}$ be a set of i.i.d. samples, their **likelihood**

$$p(\mathcal{D}|\theta) = \prod_{k=1}^{n} p(x_i|\theta)$$

their *loglikelihood*

$$l(\theta) = \log p(\mathcal{D}|\theta) = \sum_{k=1}^{n} p(x_i|\theta)$$

so $\hat{\theta} = \arg\max_{\theta}\{l(\theta)\}$

Hence, a set of equations to estimate θ are

$$\nabla_\theta l(\theta) = \mathbf{0}$$