Maxwell Equations

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Scalar and vector field

A function of space is known as a field. Let an arbitrary 3-D coordinate system be given.

Scalar field

If to each position $x = (x_1, x_2, x_3)$ of a region in space, it corresponds a scalar $\phi(x_1, x_2, x_3)$, then ϕ is called a *scalar field*. Like *density* field.

Vector field

If to each position $x = (x_1, x_2, x_3)$ of a region in space, it corresponds a vector $\vec{a}(x_1, x_2, x_3)$, then $\vec{a}$ is called a *vector field*. Like *velocity* field.

How does scalar or vector field change along spatial dimensions?

Scalar and vector field

An example of scalar (*temperature*) and vector (*heat transfer*) field.

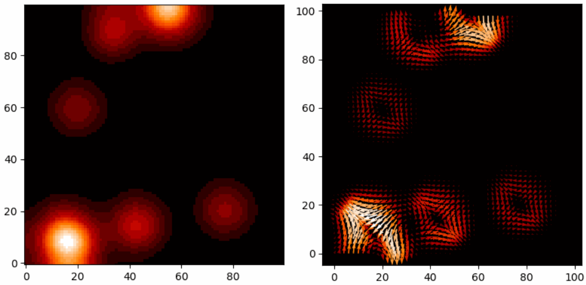

Figure: Heat Map. left: temperature, right: heat transfer

Nabla operator ∇

Partial derivatives is an useful tool to measure the changes along spatial dimensions.
For scalar field

$$\partial_i \phi = \frac{\partial}{\partial x_i} \phi \tag{1}$$

For vector field

$$(\partial_i \vec{a})(\vec{r}) = \lim_{\Delta x_i \to 0} \frac{\vec{a}(\vec{r} + \Delta x_i \vec{e}_i) - \vec{a}(\vec{r})}{\Delta x_i} \tag{2}$$

To simplify the partial derivatives, we imply *Nabla operator*.

Nabla operator

$$\nabla(\cdot) = \sum_i \vec{e}_i \partial_i (\cdot) \tag{3}$$

Gradient, diverence, curl

Gradient

$$grad\phi = \nabla\phi = \sum_i \vec{e}_i \partial_i \phi \tag{4}$$

Divergence

$$div\vec{a} = \nabla\vec{a} = \sum_i \partial_i \vec{a} \tag{5}$$

Curl

$$curl\vec{a} = \nabla \times \vec{a} = det \begin{pmatrix} \vec{e}_1 & \vec{e}_1 & \vec{e}_1 \\ \partial_1 & \partial_1 & \partial_1 \\ \vec{a}_1 & \vec{a}_1 & \vec{a}_1 \end{pmatrix} \tag{6}$$

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Definition

Maxwell Equations describe the dynamic of Electromagnetic Wave.

$$\nabla \cdot \vec{D} = \rho \tag{7}$$

$$\nabla \cdot \vec{B} = 0 \tag{8}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{9}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \tag{10}$$

Table: Meaning

$\vec{D}$	Electric Flux Density
$\vec{B}$	Magnetic Flux Density
$\vec{E}$	Electric Field
$\vec{H}$	Magnetic Field
$\vec{J}$	Current Density
ρ	Electric Charge Density

where $\vec{B} = \mu \vec{H}$, $\vec{D} = \epsilon \vec{E}$.

Example

In vacuum space, the Maxwell Equations are re-written as

$$\nabla \cdot \vec{E} = 0 \tag{11}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{12}$$

$$\nabla \cdot \vec{B} = 0 \tag{13}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{14}$$

Example

$\vec{E}$ and $\vec{B}$ is almost symmetry.

$$\nabla \times (\nabla \times \vec{E}) = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} \tag{15}$$

$$\nabla \times (\nabla \times \vec{B}) = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{B} \tag{16}$$

use

$$\nabla \times (\nabla \times \vec{X}) = \nabla \cdot (\nabla \cdot \vec{X}) - \nabla^2 \vec{X} \tag{17}$$

we have

$$\nabla^2 \vec{X} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{X} \tag{18}$$

Solve the wave function, the wave speed is $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$.[1]

[1] $\mu_0 = 8.854187817 \times 10^{-12} F/m$, $\epsilon_0 = 4\pi \times 10^{-7} N/A^2$

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Forward process

Volume conductor

- Scalp
- Skull
- CSF
- Grey matter
- White matter
- Air pockets
- Conductivity tensor

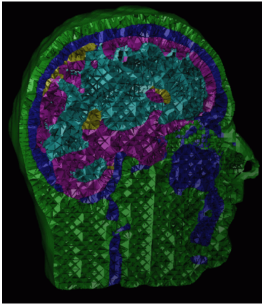


Figure: Skull

Forward process

Source model

- Neurons – brain cells, building blocks
- Pyramidal cells $10^{-5} permm$ in cortex
- Send electric impulse (action potential $20fA$ per synapse)
- $40mm^2$ of active cortex
- Current $I_0 = 10nA$
- Source decays $d = 0.1mm$
- Current dipole $P = I0 * d$

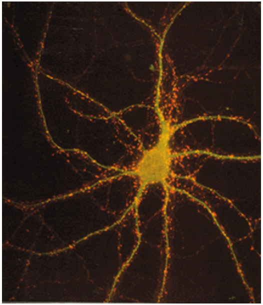


Figure: Source model

Forward process

Physical model

- Head is a volume conductor
- Concentrated current source (intra-cellular currents)
- Passive return volume currents (extra-cellular currents)

$$\vec{J} = \vec{J_i} + \vec{J_e}$$

- No flux boundary conditions

$$(\vec{J_1} - \vec{J_2}) \cdot \vec{n} = 0$$

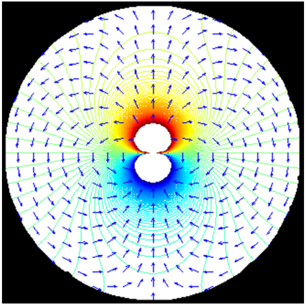


Figure: Physical model

Forward process

Forward problem solution

- Estimate the potential on the skull surface
- Knowing the currents beneath the skull

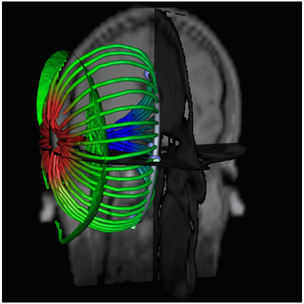


Figure: Forward solution

Mathematical model

Poisson equation

$$\nabla \cdot (\sigma \nabla \Phi) = \sum \vec{I_s}, \vec{I_s} in \Omega \tag{19}$$

Neumann boundary conditions

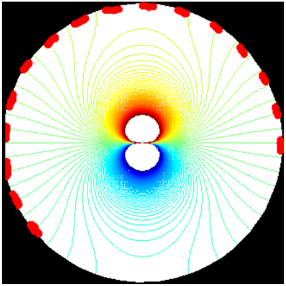
$$\sigma(\nabla \Phi) \cdot n = 0 \tag{20}$$

Dipole current source

$$I_s(\vec{r}) = \lim_{d \to 0} \vec{I_0}(\delta(\vec{r} - \vec{r_s} - \frac{d}{2}) - \delta(\vec{r} - \vec{r_s} + \frac{d}{2})) \tag{21}$$

Forward estimation

The forward solution is exampled as following



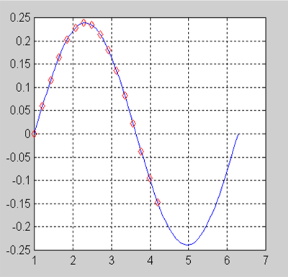


Figure: Forward estimation

Backward process

The reverse process of forward process is Backward process.
EEG measures the potential on the skull surface

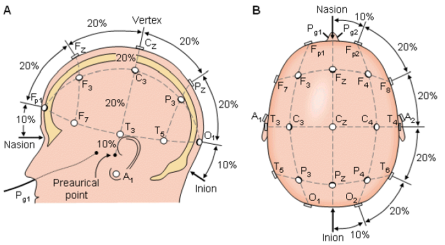


Figure: EEG hat

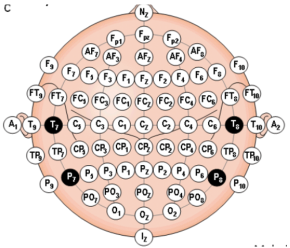


Figure: EEG layout

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Co-register

The first thing is to co-register the two coordinate

- The cortex coordinate, where the source currents are
    - Sphere model
    - Structure MRI model
    - ...
- The EEG cap coordinate, where the EEG sensors are

Gain matrix

Calculate a *Gain Matrix* using the forward model

$$GainMatrix \quad G \in \mathfrak{R}^{m,n} \tag{22}$$

where m is the number of EEG sensors, n is the number of source currents.
$G_{i,j}$ refers the $\vec{E}$ detected by i^{th} EEG sensor caused by j^{th} source current.
In another word, the EEG signal were considered as a *linear combination* of source currents.

Solution

The *Source location* problem can be formulated as

$$\arg\min_{\vec{W}} \left\| \vec{Y} - G \cdot \vec{W} \right\|_2^2 + loss(\vec{W}) \tag{23}$$

where $\vec{Y} \in \mathfrak{R}^{m,1}$ is the EEG signal, $\vec{W} \in \mathfrak{R}^{n,1}$ is the strong of source currents.
The problem is $\vec{G}$ has more rows than columns, it makes the solution can be variable.
So, using proper *loss* function, like *sparse loss* is necessary.