Geometric Basic

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Abstract

Not done yet.

Contents

1 Prepare knowledge

1.1 Space

A *space* is a collection $\mathfrak{C}$ meeting following conditions:

$$\begin{aligned} f(\mathbf{a}, \mathbf{b}) \in \mathfrak{C} \quad \forall \mathbf{a}, \mathbf{b} \in \mathfrak{C} \\ f(\mathbf{a}, \mathbf{o}) = \mathbf{a} \quad \exists \mathbf{o} \in \mathfrak{C} \\ f(a \cdot \mathbf{a} + b \cdot \mathbf{b}) = af(\mathbf{a}) + bf(\mathbf{b}) \end{aligned} \tag{1}$$

The 3-D space is a classic space, each node in 3-D space can be expressed as $\vec{x} = (x_1, x_2, x_3) \in \mathfrak{R}^3$ with a set of axes $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. To make sure the system can represent every node in the 3-D space, it requires that $det[\vec{e}_1, \vec{e}_2, \vec{e}_3] \neq 0$.

1.2 Scalar and vector field

A function of space is known as a field. Let an arbitrary 3-D coordinate system be given.

If to each position $\vec{x} = (x_1, x_2, x_3)$ of a region in space, it corresponds a scalar $\phi(x_1, x_2, x_3)$, then ϕ is called a *scalar field.* Like *density* field.

If to each position $\vec{x} = (x_1, x_2, x_3)$ of a region in space, it corresponds a vector $\vec{a}(x_1, x_2, x_3)$, then $\vec{a}$ is called a *vector field.* Like *velocity* field.

1.3 Nabla operator ∇

Partial derivatives is an useful tool to measure the changes along axes.

For scalar field

$$\partial_i \phi = \frac{\partial}{\partial x_i} \phi \tag{2}$$

For vector field

$$(\partial_i \vec{a})(\vec{r}) = \lim_{\Delta x_i \to 0} \frac{\vec{a}(\vec{r} + \Delta x_i \vec{e}_i) - \vec{a}(\vec{r})}{\Delta x_i} \tag{3}$$

To simplify the partial derivatives, we imply *Nabla operator*.

$$\nabla(\cdot) = \sum_i \vec{e}_i \partial_i(\cdot) \tag{4}$$

1.4 Gradient, Divergence and Curl

Gradient, Divergence and Curl are three basic and important measurement of spatial changes of the field.

Gradient

$$grad\phi = \nabla\phi = \sum_i \vec{e}_i \partial_i \phi \tag{5}$$

Divergence

$$div\vec{a} = \nabla \cdot \vec{a} = \sum_i \partial_i \vec{a} \tag{6}$$

Curl

$$curl\vec{a} = \nabla \times \vec{a} = det \begin{pmatrix} \vec{e}_1 & \vec{e}_1 & \vec{e}_1 \\ \partial_1 & \partial_1 & \partial_1 \\ \vec{a}_1 & \vec{a}_1 & \vec{a}_1 \end{pmatrix} \tag{7}$$

Theorem 1.1. *Gradient, Divergence and Curl following the equations.*

$$\begin{aligned} \nabla(\phi + \psi) &= \nabla\phi + \nabla\psi \\ \nabla \cdot (\vec{a} + \vec{b}) &= \nabla \cdot \vec{a} + \nabla \cdot \vec{b} \\ \nabla \times (\vec{a} + \vec{b}) &= \nabla \times \vec{a} + \nabla \times \vec{b} \\ \nabla \cdot (\phi\vec{a}) &= \phi\nabla \cdot \vec{a} + \vec{a} \cdot \nabla\phi \\ \nabla \cdot (\vec{a} \times \vec{b}) &= \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b}) \\ \nabla \times (\nabla \times \vec{a}) &= \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a} \\ \nabla \cdot (\nabla \times \vec{a}) &= 0 \\ \nabla \times \nabla\phi &= 0 \end{aligned} \tag{8}$$

The interesting things are the divergence of curl is zero, and the curl of gradient is zero.