

Basic of Distribution

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Abstract

A family of *normal* distributions, like Normal, Chi-squared and Student's t-distribution. The Normal Distribution is the *core* conception, all others are derived from it.

This article begins with Gamma and Beta function. Since they are useful for computing the *moments* the normal distribution family.

Then the distributions are described one by one.

The Appendix provides some necessary proofs.

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1 Pre-knowledge

1.1 Gamma and Beta function

An infinity integral is called as *Gamma* (Γ) *function*

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx \quad (1)$$

The *Beta (B) function* is a two-factor function, derived from Γ function

$$B(\alpha, \beta) = \frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (2)$$

1.2 Important equations

Proposition 1.1. *Some very important equations.*

The value of $\Gamma(\frac{1}{2})$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

The recursive of $\Gamma(n)$, the general situation,

$$\begin{aligned} \Gamma(1 + z) &= z\Gamma(z) \\ \Gamma(1 - z) &= -z\Gamma(-z) \end{aligned}$$

The integer situation,

$$\Gamma(n) = (n - 1)! \quad \forall n \in \mathcal{N}^+$$

The relationship between Γ and e^{x^2}

$$\Gamma(z) = 2 \int_0^\infty x^{2z-1} e^{-x^2} dx$$

The relationship between Γ and B Function

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$$

See A.1 for proof.

2 Normal distribution

2.1 Definition

It is hard to say normal distribution is what, since almost every thing follows it.

The Probability Distribution Function (*pdf*) of normal distribution is

$$p(x) = \frac{1}{\sqrt{2\pi}\delta} \exp\left(-\frac{(x-\mu)^2}{2\delta^2}\right), -\infty < x < \infty \quad (3)$$

the symbolic notion is $p(x) \sim N(\mu, \delta^2)$. When $\mu = 0$ and $\delta^2 = 1$, it is called standard normal distribution.

2.2 Mean and Variance

The mean and variance of the normal distribution is

$$\begin{aligned} \text{Mean} &\triangleq E(x) = \mu \\ \text{Variance} &\triangleq E(x^2) - E^2(x) = \delta^2 \end{aligned}$$

it is easy to proof using Proposition 1.1.

3 Chi-squared distribution

3.1 Definition

If $Y_i \sim N(0, 1)$, then

$$\chi^2 \equiv \sum_{i=1}^r Y_i^2 \quad (4)$$

is distributed as Chi-squared χ^2 distribution with r degrees of freedom. The symbolic notion is $p_r(x) \sim \chi^2(r)$.

The pdf of Chi-squared distribution is

$$P_r(x) = \frac{x^{r/2-1} e^{-x/2}}{\Gamma(r/2) 2^{r/2}}, 0 < x < \infty \quad (5)$$

3.2 Mean and Variance

The mean and variance of the chi-squared distribution is

$$\begin{aligned} \text{Mean} &\triangleq E(x) = r \\ \text{Variance} &\triangleq E(x^2) - E^2(x) = 2r \end{aligned}$$

it is easy to proof using Proposition 1.1.

3.3 Relationship with Normal Distribution

The Chi-squared distribution is derived from Normal Distribution.

4 Student's t-distribution

4.1 Definition

The probability distribution of a random variable T , of the form

$$T = \frac{\bar{x} - m}{s/\sqrt{N}}$$

where \bar{x} is the sample mean value of all N samples, m is the population mean value and s is the population standard deviation.

Or, in a more formal one

$$T = \frac{X}{\sqrt{Y/r}} \quad (6)$$

where $X \sim N(0, 1)$ and $Y \sim \chi_r^2$.

The pdf of Student's t-distribution is

$$t_r(x) = \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2})\sqrt{r\pi}} \left(1 + \frac{x^2}{r}\right)^{-\frac{r+1}{2}}, -\infty < x < \infty \quad (7)$$

it is easy to proof the pdf is a pdf using A.2.

4.2 Relationship with Normal Distribution

It is easy to see that $\lim_{r \rightarrow \infty} t_r(x) \sim N(0, 1)$. It demonstrates that when r is large enough, the Student's t-distribution is equalize to Normal Distribution.

4.3 Mean and Variance

The mean and variance of the Student's t-distribution is

$$\begin{aligned} \text{Mean} &\triangleq E(x) = 0 \\ \text{Variance} &\triangleq E(x^2) - E^2(x) = \frac{r}{r-2} \end{aligned}$$

A Appendix

A.1 The relationship between Γ and $B(\alpha, \beta)$

It is essential to know that

$$\Gamma(m)\Gamma(n) = B(m, n)\Gamma(m+n)$$

Proof. One can write

$$\Gamma(m)\Gamma(n) = \int_0^\infty x^{m-1}e^{-x}dx \int_0^\infty y^{n-1}e^{-y}dy$$

Then rewrite it as a double integral

$$\Gamma(m)\Gamma(n) = \int_0^\infty \int_0^\infty x^{m-1}y^{n-1}e^{-x-y}dxdy$$

Applying the substitution $x = vt$ and $y = v(1-t)$, we have

$$\Gamma(m)\Gamma(n) = \int_0^1 t^{m-1}(1-t)^{n-1}dt \int_0^\infty v^{m+n-1}e^{-v}dv$$

Using the definitions of Γ and Beta functions, we have

$$\Gamma(m)\Gamma(n) = B(m, n)\Gamma(m+n)$$

Hence proved. □

A.2 The pdf of Student's t-distribution is a pdf

The values of $t_r(x)$ is positive and the integral is 1.

$$\int_{-\infty}^\infty t_r(x) dx = 1$$

Proof. Consider the variable part of Student's t-distribution

$$f(x) = \left(1 + \frac{x^2}{r}\right)^{-\frac{r+1}{2}}, -\infty < x < \infty$$

use a replacement as following

$$x^2 = \frac{y}{1-y}$$

it is easy to see that $\lim_{y \rightarrow 0} x = 0$ and $\lim_{y \rightarrow 1} x = \infty$. Additionally, the x^2 is even function. Thus we can write the integral of $f(x)$

$$\int_{-\infty}^\infty f(x) dx = 2\sqrt{r} \int_0^1 \left(\frac{1}{1-y}\right)^{-\frac{r+1}{2}} d\left(\frac{y}{1-y}\right)^{\frac{1}{2}}$$

it is not hard to find out that the integral may end up with

$$\sqrt{r} \int_0^1 (1-y)^{\frac{r}{2}-1} y^{\frac{1}{2}-1} dy = \sqrt{r} B(\frac{r}{2}, \frac{1}{2})$$

Finally the normalization factor has to be

$$\frac{\Gamma(\frac{r+1}{2})}{\sqrt{r}\Gamma(\frac{r}{2})\Gamma(\frac{1}{2})}$$

which makes the integral of $t_r(x)$ is 1.

□