Concepts

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Abstract

Useful concepts of probability and statistics.

Contents

1 Concepts

1.1 Law of total probability

It is common practice to compute the sum of total probability of all available options.

Thinking **forwardly**. Which means starting from the **reason** to the **result**.

Theorem 1.1. *Law of total probability*

For random variables A and B, we have

$$P(A) = \sum P(A|B_i) \cdot P(B_i), \forall B_i \in B$$

It is automatically accepted that all the B_is are all separable, and mutually exclusive with each other, which means

$$\begin{aligned} P(B_i, B_j) &= P(B_i) \cdot P(B_j), i \neq j \\ P(B_i, B_i) &= P(B_i) \end{aligned}$$

It is a prior rule to be accepted, and we will accept it if not specified.

Thinking **backwardly**. If we have already known that A only has one option (noted as a), which is also inevitable ($P(A = a) = 1$).

Proposition 1.1. *Sum of probability of every options is ONE*

Proof. We have $P(a) = 1$ and $P(a|B_i) = 1, \forall B_i \in B$. Thus,

$$1 = \sum P(B_i), \forall B_i \in B$$

Since a can be something that naturally happens regardless of the choice of B, the proposition may not affected by the choice of a. Thus, the total probability of all options of B is 1. □

1.2 Definition of e

The constant e is defined as the infinity integral of

$$e = \lim_{x \to \infty} (1 + \frac{1}{x})^x \tag{1}$$

the very existence of e is based on the fact that integrand function is monotone bounded.

Proposition 1.2. *The function $f(x)$ is **monotone bounded***

$$f(x) = (1 + \frac{1}{x})^x, x \in \mathcal{R}$$

Proof. Use its discrete sequence version

$$g_n = (1 + \frac{1}{n})^n, n \in \mathcal{N}$$

by expanding it, we have

$$g_n = \sum_{i=0}^{n} (n, i) \cdot \frac{1}{n^i}$$

$$g_{n+1} = \sum_{i=0}^{n+1} (n+1, i) \cdot \frac{1}{(n+1)^i}$$

$$g_{n+1} = \sum_{i=0}^{n} \frac{(n+1, i)}{n+1} \cdot \frac{1}{n^i} + c$$

where $c > 0$. For every i, we have $\frac{(n+1,i)}{n+1} > (n, i)$. Thus, the g_n is **monotone increasing**.

To find a valid upper bound, we enlarge the g_n.

$$g_n < \sum_{i=0}^{n} \frac{1}{i!}$$

the enlarge is based on the idea of $(n, i) < \frac{n^i}{i!}$. Further,

$$g_n < c + \sum_{i=4}^{n} \frac{1}{2^i}$$

where constant $c > 0$. It uses the idea of $2^i < i!, i \geq 4$. Use the sum of geometric series, we can say the g_n is **upper bounded**.

Back to $f(x)$, use **Squeeze Theorem**

$$(1 + \frac{1}{n+1})^n < f(x) < (1 + \frac{1}{n})^{n+1}$$

where $n = \lfloor x \rfloor$. On the left hand, it equals to $g_{n+1}/(1 + \frac{1}{n+1})$; On the right hand, it equals to $g_n * (1 + \frac{1}{n})$. Based on the analysis above, they are of the same value which is defined as e.

Hence proved.

$\square$

1.3 Gamma function

The Gamma function is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, Re(z) > 0 \tag{2}$$

The Gamma function has useful properties

$$\begin{aligned}
\Gamma(1) &= 1 \\
\Gamma(\frac{1}{2}) &= \sqrt{\pi} \\
\Gamma(z) &= z \cdot \Gamma(z-1) \\
\Gamma(n) &= n!, n \in \mathcal{N} \\
\Gamma(z) &= 2 \int_0^\infty t^{2z-1} e^{-t^2} dt
\end{aligned}$$