Normal Distribution

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January 13, 2021

Abstract

The hand book of normal distribution. How it is formulated, what it can predict and why we use it.

Contents

1 Normal Distribution in Classic View

1.1 Binominal Distribution

Perform experiment for n times, we assume the trials are independent and follow the same distribution. The output of the experiment is noted as 1 or 0, with no vague. The probability of observing the 1 output is noted as p. Then, we have

$$P(n,m) = (n,m) \cdot p^m \cdot (1-p)^{n-m} \tag{1}$$

where m refers the fact that we observe 1 output for m times.

It is easy to see that the $P(n,m)$ produces a distribution since

$$\sum_{i=0}^{n} P(n,i) = 1, i \in \mathcal{N}$$

Use the computation of **expectation and Variation** of the random variable, we have

$$\mathcal{E}(m) = n \cdot p$$
$$\mathcal{V}(m) = n \cdot p \cdot (1-p)$$

1.2 Poisson Distribution

The poisson distribution is the infinity binominal distribution (see (1)), when p is **small** and n is **large**. It is defined as

$$P(k) = \frac{\lambda^k}{k!} \cdot e^{-\lambda}, \lambda = np \tag{2}$$

Proof. Use the equation of $\lambda = np$, we can rewrite (1) as

$$P(n,k) = (n,k)(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k}$$

Since p is small and n is large, we have k is relatively small compared to n. As a result, in infinity case,

$$\lim_{n\to\infty} \frac{(n,k)}{n^k} = \frac{1}{k!}$$
$$\lim_{n\to\infty} (1-\frac{\lambda}{n})^{n-k} = e^{-\lambda}$$

Hence proved. □

In practice, we require $\lambda < 1$ to produce a valid approximation.

To compute the expectation and variation of the poisson distribution, we use the taylor series of Exp function

$$e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}, t_0 = 0$$

it naturally guarantees the property of PDF that

$$\sum_{k=0}^{\infty} P(k) = 1$$

Use the definition of the expectation, we have

$$\mathcal{E}(k) = \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \cdot e^{-\lambda}$$
$$\mathcal{E}(k) = \lambda$$

Use the definition of the variation, we have

$$\mathcal{E}(k^2) = \sum_{k=1}^{\infty} \frac{k\lambda^k}{(k-1)!} \cdot e^{-\lambda}$$
$$\mathcal{E}(k^2) = \lambda + \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!} \cdot e^{-\lambda}$$
$$\mathcal{E}(k^2) = \lambda + \lambda^2$$

where we used the idea of $k = 1 + (k-1)$. Thus, the variation is

$$\mathcal{V}(k) = \lambda$$

1.3 Normal Distribution

When n is large and p is not so small, the poisson distribution fails on approximate the binominal distribution. The normal distribution is used as a more general replacement.

Basically, when n, np and nq are large, the binominal distribution is well approximated by the normal distribution

$$p(x) = (n,x)p^x q^{n-x} \approx \frac{1}{\sqrt{2\pi npq}} e^{-(x-np)^2/2npq}$$

where $p + q = 1$. See the website [1] for detail.

And they are linked based on Sterling's formula,

$$n! = n^n e^{-n} \sqrt{2\pi n}[1 + \mathcal{O}(1/n)] \tag{3}$$

See the website [2] for detail.

Formally, the normal distribution is expressed as

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{4}$$

where expectation $\mu = \lim_{n\to\infty} np$ and variation $\sigma^2 = \lim_{n\to\infty} npq$.

2 Family members

3 Examples

[1] http://scipp.ucsc.edu/~haber/ph116C/NormalApprox.pdf

[2] https://www.researchgate.net/publication/237571154_A_Very_Short_Proof_of_Stirling's_Formula